The Influence Function Technique in the Numerical Analysis of Plate Bending of Arbitrary Plan Form

by

Ali Husain Al-Gadhib

A Thesis Presented to the

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DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

CIVIL ENGINEERING

September, 1982
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The influence function technique in the numerical analysis of plate bending or arbitrary plan form

Al-Gadhib, Ali Husain, M.S.
King Fahd University of Petroleum and Minerals (Saudi Arabia), 1982
THE INFLUENCE FUNCTION TECHNIQUE IN THE NUMERICAL ANALYSIS OF PLATE BENDING OF ARBITRARY PLAN FORM

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ALI HUSAIN AL-GADHIB

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This thesis, written by Mr. Ali Husain Al-Gadhib under the direction of his Thesis Committee, and approved by all its members, has been presented to and accepted by the Dean, College of Graduate Studies, in partial fulfillment of the requirements for the degree of M.S. in Civil Engineering.  

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ABSTRACT

A numerical method for the solution of thin plate problems of arbitrary plan form and subjected to arbitrary loading and boundary conditions is presented in this thesis. This method is an extension of the Wu-Altiero method where use has been made of the force influence function for an infinite plate, whereas the work contained in this thesis is based on the use of the moment influence function of an infinite plate. The technique basically involves embedding the real plate into a fictitious infinite plate for which the moment influence function is known. N points are prescribed at the plate boundary at which the boundary conditions for the original problem are collocated by means of 2N fictitious moments placed around contours outside the domain of the real plate. A system of 2N linear algebraic equations in the unknown moments is obtained. The solution of the system yields the unknown moments. These may in turn be used to compute deflection, moments or shear at any point in thin plate.

Finally, the method is extended to include influence functions of both concentrated forces and concentrated moments. This is obtained by applying concentrated moments and forces simultaneously on the contours located outside the domain of the plate.
في هذه الأطراف، طريقة عديدة لحل مسائل الصفيحة الرقيقة،
ذوات الأشكال المطلقة والمعروضة لألقاب وظروف حدودية مطولة أيضًا.
هذه الطريقة هي أتمتة لطريقة و- البتروحي، حيث استخدمت دال.
تأثير القوة للصفيحة الغير متناهية في حين أن هذه الأطراف تعتمد على استخدام دالة تأثير العزوم للصفيحة الغير متناهية، فكرة الطريقة تعتمد على فعالة الصفيحة الأطالية داخل ضيافة خيالية غير متناهية.
المعرفة دالة تأثير العزوم لها 1 نقطة على حدود الصفيحة الأطالية،usp تكتب نفسها الظروف الحدودية بواسطة 2 عزوم خيالية واقعية، حول منحنى مغلق خارج مجال الصفيحة الأطالية، ويدخل هذه المجموعة من 2 عزوم، هذه العزوم سوف تستخدم في حساب الأحجام والعزم والقوي.
لأي نقطة داخل مجال الصفيحة الرقيقة، وأخيراً، الطريقة طورت لتحتوي على تأثير كل من القوى والعزوم في أن واحد ولهذا يتم بواسطة استخدام القوى والعزوم المركزية وعواملها على شكل منحنى مغلق خارج مجال الصفيحة.
chapter 1

INTRODUCTION

The analysis of the bending problem of isotropic thin-plates is one of the most common problems in structural engineering. Plate theory and methods of solution can be traced back to the early eighteenth century, where a lot of contributors including Euler, Bernoulli and Lagrange were involved in the development of plate theory. In this century, Nadai, Love, Huber, Timoshenko, and Lekhnitskii, to name a few, are well-known for their work related to plate problems.

Mathematically the thin-plate bending problem is a typical boundary value problem inasmuch as the problem may be reduced to the solution of a fourth order partial differential equation subjected to certain boundary conditions. One can obtain a solution analytically for special cases of simple geometrical form and boundary conditions. The problem is complicated if the plate is not of simple plan form. In such cases, analytical approaches essentially breakdown and one is forced to employ numerical techniques such as finite difference [1] or the finite element method [2,3]. Although it is well-known that the finite element method is a powerful technique, it has some disadvantages including a large quantity of input data making implementation tedious and computer time
lengthy. Thus there has been a recent effort to formulate alternative methods, one such effort being the extension of the finite strip method [4], and another is the application of boundary integral techniques [5] to the problem of linear plate theory.

One such method from the family of boundary integral techniques is the Wu-Altiero method [6], where use is made of the force influence function of an infinite plate. The work contained in this thesis is an extension of the Wu-Altiero method and it is based on the use of the Green's function of an infinite plate for the analysis. The technique basically involves embedding the real plate into a fictitious infinite plate for which the Green's function is known. N points are prescribed at the plate boundary at which the boundary conditions for the original problem are to be satisfied. Since there are two boundary condition equations to be satisfied at each boundary point, a set of 2N fictitious moments are placed around contours outside the domain of the actual plate. A system of linear algebraic equations in terms of the unknown moments is obtained, which is then solved for all the fictitious moments. The deflection and bending moments any where inside the plate may now be obtained readily.

There are eight chapters in this thesis. Theoretical preliminaries of plate analysis, including the derivation of the governing fourth order partial differential equation for isotropic thin
plates and its associated boundary conditions are presented in Chapter 2. Chapter 3 introduces the idea of Green's function and the concept of the Wu-Altiero method [6]. Chapter 4 consists of two parts, namely (a) the flow chart depicting the main features of the program that was built up to solve the plate problem using the Wu-Altiero concept, and (b) results for the cases dealt with by Wu and Altiero. Using the program developed here at UPM, the conclusion is drawn that this method works irrespective of the value of the arbitrary radius "a" of the infinite plate, in contrast to conclusions drawn by Wu and Altiero, where they restrict the validity of their approach to a certain range of the arbitrary radius "a". Chapter 5 introduces the idea where use is made of the moment influence function in lieu of the force influence function and presents the necessary deviations. Chapter 6 explains features of the program written for the moment influence method in the form of a flow chart and the method by solving the cases considered by Wu and Altiero. After verifying the point force method and developing and establishing the validity of the point moment method, the superposition of both point forces and moments acting concurrently is the topic of Chapter 7. Chapter 8 concludes the thesis with a discussion of the results and a bit summary of the main features.
chapter 2

THEORETICAL PRELIMINARIES

2.1 Background

The purpose of this chapter is to give the necessary background of plate theory, including the governing differential equation and the accompanying boundary conditions.

A plate can be defined as a three dimensional object where two dimensions are much greater than the third dimension (thickness h). The plane which is parallel to the faces of the plate and divides the thickness into equal halves is termed the midsurface of the plate Fig. 1. Now consider the isotropic thin plate of small deflection in Fig. 1, in which the x, y plane coincides with the midsurface. When one talks about the deflection of the plate, the actual reference is to the deflection of the midsurface. The z coordinates of all the points forming the midsurface is zero. The governing differential equation in terms of the plate deflection may be developed by making the following assumptions:

1. The deflection of the midsurface is small in comparison with the thickness of the plate. The slope of the deflected surface is much less than unity.
Figure 1: Plate Geometry
2. There is only vertical displacement of the middle plane of the plate, i.e. the midsurface of the plate suffers no displacement in the x or y directions.

3. Planes initially normal to the midsurface of the plate remain normal to the midsurface of the plate after bending.

4. The normal stresses in the direction transverse to the plate can be neglected.

In order to obtain the governing differential equation of the plate, one starts with the equilibrium of differential element of area dx dy subjected to a distributed load q of plate in Fig. 1 and for the load to be carried, internal moments and shears must be developed that is shown in Fig. 2 after exaggeration.

Apply equations of equilibrium to the element of Fig. 2

\[ \Sigma F_z = 0 \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \]

\[ \Sigma M_x = 0 \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \quad (2.1) \]

\[ \Sigma M_y = 0 \quad \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - Q_x = 0 \]

Eliminating the shearing forces \( Q_x \) and \( Q_y \) from the above equations leads to

\[ \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} = -q \quad (2.2) \]
Figure 2: Infinitesimal Plate Elements with Internal Stress Resultants
Now we want to express the moment in terms of deflection. By referring to Fig. 3, point A, which has undeformed coordinates \((X_A, Y_A, 0)\), after application of the external load, will suffer a deflection \(w\) as shown in Fig. 3. Since \(\frac{\partial w}{\partial x}\) is very small, one may write

\[
\tan \frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} = -\frac{u}{z} \quad u = -z \frac{\partial w}{\partial x} \tag{2.3}
\]

Similarly along the \(y\) direction

\[
v = -z \frac{\partial w}{\partial y}
\]

where \(u, v\) represent the displacements of point B (Fig. 3b) in the \(x, y\) directions, respectively.

**Strain-displacement Relationships**

\[
\varepsilon_x = \frac{\partial u}{\partial x} \tag{2.4}
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y}
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\]

Using (2.3) in (2.4)

\[
\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2}
\]

\[
\varepsilon_y = -z \frac{\partial^2 w}{\partial y^2} \tag{2.5}
\]

\[
\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}
\]
Figure 3: Geometry of Plate Deformation
Stress-strain Relationship

\[
\sigma_x = \frac{E}{1-\nu^2} (\varepsilon_x + \nu \varepsilon_y)
\]

\[
\sigma_y = \frac{E}{1-\nu^2} (\varepsilon_y + \nu \varepsilon_x)
\]

\[
\tau_{xy} = \frac{-E}{2(1+\nu)} \gamma_{xy}
\]

(2.6)

Using (2.5) in (2.6)

\[
\sigma_x = \frac{-zE}{1-\nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)
\]

\[
\sigma_y = \frac{-zE}{1-\nu^2} \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)
\]

(2.7)

\[
\tau_{xy} = \frac{-zE}{1+\nu} \frac{\partial^2 w}{\partial x \partial y}
\]

Moments-strain Relation

\[
M_x = \int_{-h/2}^{h/2} z \sigma_x \, dz
\]

\[
M_y = \int_{-h/2}^{h/2} z \sigma_y \, dz
\]

(2.8)

\[
M_{xy} = \int_{-h/2}^{h/2} z \tau_{xy} \, dz
\]
Using (2.7) in (2.8)

\[ M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \]

\[ M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \]

\[ M_{xy} = -D (1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \]

where:

\[ D = \frac{\epsilon h^3}{12(1-\nu^2)} \]

Using (2.9) in (2.2)

\[ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \]

(2.10)

Equation (2.10) is the governing differential equation of plate or in concise form \( \Delta^4 w = \frac{q}{D} \).

2.2 Boundary Conditions

Knowing the governing differential equation, one needs to know the different types of boundary conditions. The boundary of the plate may be any combination of the following three types, clamped edge, simply supported edge, and free edge provided that the plate is stable. If all the plate boundaries are denoted by \( B \), then the following can be written as:

\[ B = B_C + B_S + B_f \]
where:

$B_C$ represents the clamped portion of plate

$B_S$ represents the simply supported portion of plate

$B_f$ represents the free portion of plate.

as shown in Figure 4.

For the clamped or simply supported boundaries, deflection equals to zero as indicated in Equation (2.11). Also for clamped edge, the slope of the deflection with respect to the outer normal equals to zero as indicated in Equation (2.12).

For the simply supported edge or free edge, the bending moment equals to zero as indicated in Equation (2.13). For the free edge, Kirchhoff's shear is zero indicated in Equation (2.14).

\[
\begin{align*}
  w &= 0 & \text{on } B_C + B_S & (2.11) \\
  n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y} &= 0 & \text{on } B_C & (2.12) \\
  f_1 \frac{\partial^2 w}{\partial x^2} + f_2 \frac{\partial^2 w}{\partial x \partial y} + f_3 \frac{\partial^2 w}{\partial y^2} &= 0 & \text{on } B_S + B_f & (2.13) \\
  g_1 \frac{\partial^3 w}{\partial x^3} + g_2 \frac{\partial^3 w}{\partial x^2 \partial y} + g_3 \frac{\partial^3 w}{\partial x \partial y^2} + g_4 \frac{\partial^3 w}{\partial y^3} &= 0 & \text{on } B_f & (2.14)
\end{align*}
\]

where $n_x(x,y)$, $n_y(x,y)$ are components of the outer normal directed unit normal to $B$ and $f_1$ through $g_4$ are given by the following relationships [6].
Figure 4: Plate of Arbitrary Plan Form
\[ f_1 = \Delta_{x}^2 + \nu \Delta_{y}^2 \]
\[ f_2 = 2(1 - \nu) \Delta_{x} n_{y} \]
\[ f_3 = \nu \Delta_{x}^2 + \Delta_{y}^2 \]
\[ g_1 = \Delta_{x} (1 + n_{y}^2) - \nu \Delta_{x} n_{y}^2 \]
\[ g_2 = \nu \Delta_{y} (1 + n_{x}^2) + 2(1 - \nu) \Delta_{y}^3 - \Delta_{x} n_{y}^2 \]
\[ g_3 = \nu \Delta_{x} (1 + n_{y}^2) + 2(1 - \nu) \Delta_{x}^3 - \Delta_{x} n_{y}^2 \]
\[ g_4 = \Delta_{y} (1 + n_{x}^2) - \nu \Delta_{x} n_{y} \]
3.1 Green's Function for Finite Rectangular Plate of Simply Supported Boundaries

The Navier solution for a rectangular plate with simply supported boundaries under the action of a concentrated force $P$ as shown in Fig. 5, is given in [1] as

$$w(x,y) = \frac{4P}{\pi^4 abD} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b}}{(\frac{m^2}{a^2} + \frac{n^2}{b^2})^2}$$

(3.1)

where $w(x,y)$ is the deflection at any point $(x,y)$ of a rectangular simply supported plate subjected to concentrated force $P$ located at $x = \xi$, $y = \eta$.

Consider the following expression

$$I(x,y,\xi,\eta) = \frac{4}{\pi^4 abD} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi x}{a} \sin \frac{n\pi y}{b}}{(\frac{m^2}{a^2} + \frac{n^2}{b^2})^2}$$

(3.2)

which is the same as (3.1) for the concentrated force $P = 1$.

Expression (3.2), which gives the deflection at any point $(x,y)$
Figure 5: Concentrated Force P at $x = \xi$, $y = \eta$
when the unit force is applied at \( x = \xi, y = \eta \) of the rectangular simply supported plates is called the Green's function of the plate associated with the simply supported conditions. Thus the Green's function is nothing but the deflection caused either by unit load (or unit moment) located at \( x = \xi, y = \eta \) of a plate. One may obtain a variety of Green's functions for plates of different shapes and boundary conditions.

The deflection due to a distributed load may be found by means of the force influence function (Green's function). The distributed load can be represented by a set of concentrated loads at \( \xi \)'s and \( \eta \)'s and by the principle of superposition, the deflection due to the distributed load is approximately the deflection due to these set of concentrated loads as in Equation (3.3).

\[
w(x,y) = \sum_{i=1}^{n} P_i I(x,y,\xi_i,\eta_i) \tag{3.3}
\]

where

\[
P_i = f(\xi,\eta) (\text{Area})_i \tag{3.3a}
\]

and \( n \) is number of concentrated loads.

The exact deflection of the plate due to the distributed load is to use Equation (3.3) and change the summation by double integrals and replace \( P_i \) by (3.3a) which then yields

\[
w(x,y) = \int \int_{A(=\text{loaded area})} f(\xi,\eta) I(x,y,\xi,\eta) \, d\xi \, dy \tag{3.4}
\]
3.2 Green's Function of an Infinite Plate

Our interest is to find the Green's function of an infinite plate of arbitrary radius a.

Load P which is located at \( x = \xi, y = \eta \) causes a deflection at point \((x,y)\) of the infinite plate of radius \(a\) (See Fig.6). This deflection is given in [1] as

\[
w(x,y) = \frac{Pr^2}{8\pi D} \ln \frac{r}{a}
\]
or

\[
w(x,y) = \frac{Pr^2}{16\pi D} \ln \frac{r^2}{a^2} \tag{3.5}
\]

and from this expression, one can get the Green's function of the infinite plate of arbitrary radius \(a\) by setting \(P = 1\)

\[
I(x,y,\xi,\eta) = \frac{r^2}{16\pi D} \ln \frac{r^2}{a^2} \tag{3.6}
\]

3.3 The Wu-Altiero Method

The Wu-Altiero method is easily explained if one considers the problem of the bending of beams. The deflection of the beam shown in Fig. 7 (a) due to the action of a uniform distributed load \(q\) may be solved by the superposition of two cases - Case one in Fig. 7 (b) when one finds the deflection caused by the uniformly distributed load \(q\) of the real beam on an infinite beam, and the second case where one
Figure 6: Radius of Influence "a"

\[ r^2 = (x-\xi)^2 + (y-\eta)^2 \]
obtains the deflection due to unknown fictitious forces placed outside the domain of the real beam as shown in Fig. 7 (c). Thus the deflection of the real beam in Fig. 7 (a) is the superposition of the two cases as shown in Fig. 7 (b) and Fig. 7 (c) and given by the following equation

\[ v(x) = \phi_{\text{UDL}}(x) - \sum_{i=1}^{N} \phi_{P_i}(x) \]  

(3.7)

where

\[ P_i = \text{The } i\text{th fictitious load} \]

\[ \phi_{\text{UDL}} = \text{Deflection of the infinite beam when subjected to uniformly distributed load which is acting on the real beam} \]

\[ \phi_{P_i} = \text{Deflection of the infinite beam subjected to fictitious force } P_i \]

\[ N = \text{Number of fictitious forces} \]

The first part of Equation (3.7) may be obtained readily and the second part is not difficult to obtain if one knows the magnitude of the fictitious forces. The main task is thus reduced to finding these unknown fictitious forces and this may be done by the collection of the following boundary conditions:

- Deflection = 0 at \( x = 0 \) i.e. \( v(0) = 0 \)
- Deflection = 0 at \( x = L \) i.e. \( v(L) = 0 \)
- Moment = 0 at \( x = 0 \) i.e. \( EI \frac{d^2v(x)}{dx^2} = 0 \)
Figure 7: Physical Interpretation of Wu-Altiero Model as Applied to Beams
moment = 0 at x = L i.e. \( EI \frac{d^2v(L)}{dx^2} = 0 \)

In doing so, one obtains four algebraic equations in terms of the unknown forces. Thus one can have fictitious loads equal in number to the boundary conditions. Now after knowing all the unknown forces, one can compute \( v(x) \) in Equation (3.7) completely.

Exactly almost what was true for the beam is true for the plate. The solution of plate deflection involves embedding the real plate into a fictitious infinite plate of the same material as in Fig.8. The deflection of the real plate is the superposition of two solutions - one solution being the deflection of the infinite plate when subjected to actual loading of the real plate and the other is the deflection due to \( 2N \) fictitious unknown forces placed outside the domain of the real plate in the form of contours - and the total solution, composed of the two previous solutions forced to satisfy the boundary conditions at \( N \) discrete points \( (X_i, y_i) \) on the boundary B of the plate may be expressed as

\[
w(x,y) = \int_A I(x_i, y_i, \xi, \eta) q(\xi, \eta) d\xi d\eta - \sum_{k=1}^{2N} I(x_i, y_i, \xi_k, \eta_k) P_k \tag{3.8}
\]

The first solution which is the deflection of the infinite
Figure 8: Plate With Parallel Contour (Point Force Method)
plate under the forcing function can be calculated using
the aid of the force influence function of an infinite plate
Equation (2.6) and the other solution can be easily found if
one knows the magnitude of the fictitious forces. This may
be handled by forcing the deflection as given by Equation (3.8)
to satisfy the boundary conditions as given by Equations
(2.11) through (2.14) that will result in 2N linear algebraic
equations for the solution of 2N unknowns, P_k's, k = 1, 2N.

Using Equation (3.8) in Equations (2.11), (2.12), (2.13)
and (2.14) respectively, yields

\[ \sum_{k=1}^{2N} I_{ik} P_k = \int_R \int_I q d\xi d\eta \quad ((x_i, y_i) \text{ on } B_C + B_S) \] (3.9a)

\[ n_x \sum_{k=1}^{2N} \frac{\partial I}{\partial x} p_k + n_y \sum_{k=1}^{2N} \frac{\partial I}{\partial y} p_k = n_x \int_R \int_I \frac{\partial I}{\partial x} q d\xi d\eta \]
\[ + n_y \int_R \int_I \frac{\partial I}{\partial y} q d\xi d\eta \quad ((x_i, y_i) \text{ on } B_C) \] (3.9b)

\[ f_1 \sum_{k=1}^{2N} \frac{\partial^2 I}{\partial x^2} p_k + f_2 \sum_{k=1}^{2N} \frac{\partial^2 I}{\partial x \partial y} p_k + f_3 \sum_{k=1}^{2N} \frac{\partial^2 I}{\partial y^2} p_k \]
\[ = f_1 \int_R \int_I \frac{\partial^2 I}{\partial x^2} q d\xi d\eta + f_2 \int_R \int_I \frac{\partial^2 I}{\partial x \partial y} q d\xi d\eta \]
\[ + f_3 \int_R \int_I \frac{\partial^2 I}{\partial y^2} q d\xi d\eta \quad ((x_i, y_i) \text{ on } B_S + B_f) \] (3.9c)
\[ g_1 \sum_{k=1}^{2N} \frac{\partial^3 I}{\partial x^3_i} p_k + g_2 \sum_{k=1}^{2N} \frac{\partial^3 I}{\partial x^2 \partial y_i} p_k + g_3 \sum_{k=1}^{2N} \frac{\partial^3 I}{\partial x \partial y^2_i} p_k \]

\[ + g_4 \sum_{k=1}^{2N} \frac{\partial I}{\partial y^3_i} p_k = g_1 \oint_{\partial R} \frac{\partial^3 I}{\partial x^3_i} q d\xi d\eta \]

\[ + g_2 \oint_{\partial R} \frac{\partial^3 I}{\partial x^2 \partial y_i} q d\xi d\eta + g_3 \oint_{\partial R} \frac{\partial^3 I}{\partial x \partial y^2_i} q d\xi d\eta \]

\[ + g_3 \oint_{\partial R} \frac{\partial I}{\partial y^3_i} q d\xi d\eta \]  

\[ ((x_i, y_i) \text{ on } B_x) \quad (3.9d) \]

where

\[ I_{ik} = I(x_i, y_i, \xi_k, \eta_k) \]

and  \[ I_i = I(x_i, y_i, \xi, \eta) \]

The set of equations (3.9) can be written in a matrix form as

\[ AP = B \quad (3.10) \]

where

P is the fictitious unknown load vector \( P^T = [P, \ldots, P_{2N}] \)

A is the following matrix of size \( 2N \times 2N \)
\[ A = \begin{cases} 
2N \\
\sum_{k=1}^{2N} \left( n_x \left( \frac{\partial I}{\partial x} \right)_{ik} + n_y \left( \frac{\partial I}{\partial y} \right)_{ik} \right) & \text{on } B_C \\
2N \\
\sum_{k=1}^{2N} \left( f_1 \left( \frac{\partial^2 I}{\partial x^2} \right)_{ik} + f_2 \left( \frac{\partial^2 I}{\partial x \partial y} \right)_{ik} + f_3 \left( \frac{\partial^2 I}{\partial y^2} \right)_{ik} \right) & \text{on } B_S + B_f \\
2N \\
\sum_{k=1}^{2N} \left( g_1 \left( \frac{\partial^3 I}{\partial x^3} \right)_{ik} + g_2 \left( \frac{\partial^3 I}{\partial x^2 \partial y} \right)_{ik} + g_3 \left( \frac{\partial^3 I}{\partial x \partial y^2} \right)_{ik} + g_4 \left( \frac{\partial^3 I}{\partial y^3} \right)_{ik} \right) & \text{on } B_f 
\end{cases} \]

B is the right hand side vector as given by

\[ B = \begin{cases} 
\iint_{R} I_i q d\xi d\eta & \text{on } B_S + B_C \\
\int_{R} \left[ n_x \left( \frac{\partial I}{\partial x} \right)_i + n_y \left( \frac{\partial I}{\partial y} \right)_i \right] q d\xi d\eta & \text{on } B_C \\
\iint_{R} \left[ f_1 \left( \frac{\partial^2 I}{\partial x^2} \right)_i + f_2 \left( \frac{\partial^2 I}{\partial x \partial y} \right)_i + f_3 \left( \frac{\partial^2 I}{\partial y^2} \right)_i \right] q d\xi d\eta & \text{on } B_S + B_f \\
\iint_{R} \left[ g_1 \left( \frac{\partial^3 I}{\partial x^3} \right)_i + g_2 \left( \frac{\partial^3 I}{\partial x^2 \partial y} \right)_i + g_3 \left( \frac{\partial^3 I}{\partial x \partial y^2} \right)_i + g_4 \left( \frac{\partial^3 I}{\partial y^3} \right)_i \right] q d\xi d\eta & \text{on } B_f 
\end{cases} \]

where:

- \( k \) is the column number \( k = 1, 2N \)
- \( i = 1,N \), and because for each collocation point \( i \) two boundary conditions that will generate \( 2N \) equations which will be number of rows.
Matrix A accommodates all the possible boundary conditions. Depending on the type of the plate boundary, one chooses the appropriate part of matrix A. For example, if one is solving for a plate with its all edges simply supported one needs to use only the first and the third rows of summation of matrix A. Likewise, if all edges of the plate are clamped, one will use the first and the second rows of summation of matrix A, since these are the two boundary conditions dealing with the clamped edge. However, if the plate edges consist of all types ($B_S+B_C+B_f$), then all the rows of matrix A are to be needed.

What has been said about Matrix A is equally true for the load vector B. i.e. if one chooses a part of matrix A, one should also take the corresponding part of vector B.

Once again the determination of the fictitious load vector $P^T = [P \ldots P_{2N}]$ is the result of solving Equation (3.10). However, in order to be able to solve Equation (3.10), one needs to determine the matrix A and the load vector B which involves derivatives up to the third order of the force influence function $I(x,y,\xi,n)$. These derivatives are evaluated in Appendix A. Subsequently, after solving for the fictitious loads, one can find the deflection any where in the real plate from Equation (3.8). The moment $M_x$ and $M_y$ can also be obtained anywhere in the region of the real plate from the equations which result from substitution of Equation (3.8) into the formulas for the moments as given by Equation (2.9).
APPLICATION OF WU-ALTIERO METHOD

4.1 Evaluation of the Load Vector B

As mentioned in Chapter 3, in order to solve for the fictitious forces, one needs to generate the matrix A and the load vector B by using Appendix A which includes all the necessary derivatives. The use of different symbols for the different derivatives will facilitate in the programming.

Recall the force influence function for an infinite plate is

\[ I = \frac{(x-\xi)^2 + (y-\eta)^2}{16\pi D} \ln \frac{(x-\xi)^2 + (y-\xi)^2}{a^2} \]

Let \( Z = \xi \), \( A = \eta \), and \( Q_1 = (x-\xi)^2 + (y-\eta)^2 \)

The Green's function and its derivatives may then be written as the following:

\[ I = AIA = \frac{Q_1}{16\pi D} \ln \frac{Q_1}{a^2} \]

\[ \frac{\partial I}{\partial x} = A_1 = \frac{(X-Z)}{8\pi D} (\ln \frac{Q_1}{a^2} + 1) \]

\[ \frac{\partial^2 I}{\partial x^2} = A_2 = \frac{1}{8\pi D} (\ln \frac{Q_1}{a^2} + \frac{2(X-Z)^2}{Q_1} + 1) \]

\[ \frac{\partial^3 I}{\partial x^3} = A_3 = \frac{(X-Z)}{16\pi DQ_1} (12 - \frac{8(X-Z)^2}{Q_1}) \]
\[
\frac{\partial I}{\partial y} = B_1 = (\frac{Y-A}{8\pi D}) (\ln \frac{Q_1}{a^2} + 1)
\]

\[
\frac{\partial^2 I}{\partial y^2} = B_2 = \frac{1}{8\pi D} (\ln \frac{Q_1}{a^2} + \frac{2(Y-A)^2}{Q_1} + 1)
\]

\[
\frac{\partial^3 I}{\partial y^3} = B_3 = \frac{Y-A}{16\pi DQ_1} (12 - \frac{8(Y-A)^2}{Q_1})
\]

(4.1)

\[
\frac{\partial^2 I}{\partial x \partial y} = AB_1 = \frac{(X-Z)(Y-A)}{4\pi DQ_1}
\]

\[
\frac{\partial^3 I}{\partial x^2 \partial y} = AB_2 = \frac{Y-A}{4\pi DQ_1} (1 - \frac{2(X-Z)^2}{Q_1})
\]

\[
\frac{\partial^3 I}{\partial x \partial y^2} = AB_3 = \frac{Y-A}{4\pi DQ_1} (1 - \frac{2(Y-A)^2}{Q_1})
\]

For evaluating the load vector \( B \) which involves a double integration, an approximation by summation is resorted to. The domain of the loaded part of the plate can be discretized to small areas (elements), \( A_1, A_2 \ldots A_{NL} \), and the distributed load effect can be approximated by the total effect of a set of concentrated loads, each of magnitude \( q_L A_L \), where \( L = 1 \rightarrow NL \) and \( NL = \) the number of discretized elements as in Figure 9. These concentrated forces are to be placed at the centroid of the area \( A_L \).

In Figure 9, the loaded area of the plate is \( (a_1 x b_1) \) and that area is discretized into twelve elements. Each element has an area of \( \frac{a_1 b_1}{12} \), thus resulting in a concentrated load of magnitude \( \frac{q a_1 b_1}{12} \) at the centroid of each element.
Figure 9: Discretization of Plate Domain for Evaluation of (Load * Area) Integrals
Employing a summation approximation for the double integral, one obtains

\[ \int \int \int_i q_d \xi d\eta = \sum_{L=1}^{NL} I_{iL} q_{LA_L} \]  

(4.2)

Furthermore, if the distributed load is uniform and the loaded domain of the plate is discretized into equal elements, one may rewrite Equation 4.2 as

\[ \int \int \int_i q_d \xi d\eta = qA \sum_{L=1}^{NL} I_{iL} \]  

(4.3)

If the plate is not of simple geometry one can still handle the evaluation of the load vector \( B \) by means of a program similar to that used in finite elements. Such a program will discretize the loaded part of the plate into small triangular elements (areas), which may not necessarily be equal to each other and also yield the centroid of each area in which the concentrated load has to act. In this case, if the plate is subjected to a uniformly distributed load, Equation 4.2 may be written as

\[ \int \int \int_i q_d \xi d\eta = q \sum_{L=1}^{NL} I_{iL} A_L \]  

(4.4)

Thus equation (4.2) can be used when the distributed load is not uniform and the elements of discretized area are not equal to each other. Depending on the situation, one may use the appropriate equation.
Let us write the matrix $A$ and the vector $B$ in programable fashion using (4.1) and (4.3), with the previous definitions of matrix $A$ and vector $B$. This is on the assumption that solution is to be generated for a plate subject to uniformly distributed load and discretized into equal elements.

\[
A = \begin{bmatrix}
2N \\ k=1 \\
\sum 
\end{bmatrix}
(AIA)_{ik} \quad \text{on } B_{S+B_C}
\]

\[
2N \\
\sum \\
k=1
\begin{bmatrix}
(N_x(A_1)_{ik} + N_y(B_1)_{ik}) \quad \text{on } B_C
\end{bmatrix}
\]

\[
2N \\
\sum \\
k=1
\begin{bmatrix}
[F_1(A_2)_{ik} + F_2(AB_1)_{ik} + F_3(B_2)_{ik}] \quad \text{on } B_{S+B_f}
\end{bmatrix}
\]

\[
2N \\
\sum \\
k=1
\begin{bmatrix}
[G_1(A_3)_{ik} + G_2(AB_2)_{ik} + G_3(AB_3)_{ik} + G_4(B_3)_{ik}] \quad \text{on } B_f
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
NL \\
qA \sum \\
L=1
\end{bmatrix}
(AIA)_{iL} \quad \text{on } B_{S+B_C}
\]

\[
NL \\
qA \sum \\
L=1
\begin{bmatrix}
(N_x A_1 + N_y B_1)_{iL} \quad \text{on } B_C
\end{bmatrix}
\]

\[
NL \\
qA \sum \\
L=1
\begin{bmatrix}
(F_1 A_2 + F_2 AB_1 + F_3 B_2)_{iL} \quad \text{on } B_{S+B_f}
\end{bmatrix}
\]

\[
NL \\
qA \sum \\
L=1
\begin{bmatrix}
(G_1 A_3 + G_2 AB_2 + G_3 AB_3 + G_4 B_3)_{iL} \quad \text{on } B_f
\end{bmatrix}
\]

One may now use Equations (4.5) and (4.6) to solve the
matrix equation

\[ AP = B \]

for the set of fictitious forces \( P \).

4.2 General Flowchart of Force Method

```
START

Read, input data

Generate the algebraic equations of the form
\[ AP = B \]

Subroutine to solve simultaneous equation for the fictitious forces

Calculate deflection and moments using the obtained fictitious forces and deflection and moments formulas

STOP
```

General Flowchart for the Point Force Method

\[ A = \text{matrix (2Nx2N)} \]
\[ B = \text{load vector (2Nx1)} \]
\[ P = \text{fictitious forces} \]
**Input Data and Their Meanings**

NSIDE = Number of plate sides (edges)

\[ X_1(I) = \text{the x-coordinate of the left end of edge I} \]

\[ Y_1(I) = \text{the y-coordinate of the left end of edge I} \]

\[ X_2(I) = \text{the x-coordinate of the right end of edge I} \]

\[ Y_2(I) = \text{the y-coordinate of the right end of edge I} \]

NS(I) = Number of collection points on side I

INDEX (I) = indicate type of edge

\[ INDEX (I) = 1 \text{ (free edge)} \]

\[ INDEX (I) = 2 \text{ (simply supported edge)} \]

\[ INDEX (I) = 3 \text{ (clamped edge)} \]

NC = Number of contours to which the fictitious forces or fictitious moment placed around

NL = Number of loaded areas

Area = Area of discretized element

XLOAD = the intensile of the load

\[ XL(L) = \text{the centroidal x-coordinate of area L} \]

\[ YL(L) = \text{the centroidal y-coordinate of area L} \]

D = flexural rigidity

\[ \nu = \text{Poisson's ratio} \]

RAD = the arbitrary radius a

PIE = \pi

NDP = Number of field points

\[ XD(J) = \text{the x-coordinate of field point J} \]

\[ YD(J) = \text{the y-coordinate of field point J} \]
WT(J) = the theoretical deflection of field point J
XMT(J) = " " Mx " " " " " "
YMT(J) = " " My " " " " " "
XB(I) = the x-coordinate of collocation point I
YB(I) = the y-coordinate of collocation point I
XP(K) = the x-coordinate of the fictitious force k
         or the fictitious moment k
YP(K) = the y-coordinate of the fictitious force k
         or the fictitious moment k
N = Number of collocation points
NTOT = 2N = Number of the fictitious forces

4.3 Application of Wu-Altiero Method

Using the previous flowchart, a program can be written
to solve for any plate problem of irregular shape when
subjected to a uniformly distributed load with a variety of
boundary conditions, provided that the plate boundaries may
be approximated by straight edges. By a very slight modifi-
cation in the previous flowchart, the program may solve for
a plate of irregular shape subjected to any type of loading.
In the work contained herein, results of two cases are to
be presented (a) a square plate subjected to a uniformly
distributed load and (b) a triangular plate subjected to a
uniformly distributed load. There are also two cases that
Wu-Altiero have considered in their work.

Case 1

A square plate of size 10 x 10 m and acted upon by a uniformly load \( q = 1 \text{N/m}^2 \)

\[ E = 2.068 \times 10^5 \text{ Mpa} \]

\[ \nu = .3 \]

\[ h = .01 \text{ m} \]

All other information are shown in the Figure 10.

Case 2

An equilateral triangular plate subjected to uniformly load \( q = 1 \text{N/m}^2 \). All edges are simply supported as in Figure 11

\[ E = 2.068 \times 10^5 \text{ Mpa} \]

\[ \nu = .3 \]

\[ h = .01 \text{ m} \]

The results of deflection and moments (\( M_x \) and \( M_y \)) are shown in separate tables for the two cases. Also shown in these tables is the percentage of error as compared to the theoretical solution.
Figure 10: Square Plate 10 m. * 10 m. with Parallel Contours $C_1$ and $C_2$. 

- Field point
- Boundary point
- Loading point
Triangular Plate

- Field point
- Boundary point
- Loading point

Figure 11: Triangular Plate with Parallel Contours \( C_1 \) and \( C_2 \).
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<th>Error %</th>
<th>N-mm (Exact)</th>
<th>Error %</th>
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<td>1.184</td>
<td>0.812</td>
<td>0.168</td>
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Table 4.2: The Force Method

42
5.1 Moment Influence Function

In section 3.2 we have the force influence function as

\[ I(x, y, \xi, \eta) = \frac{r^2}{16\pi D} \ln \frac{r^2}{a^2} \]

where

\[ r^2 = (x-\xi)^2 + (y-\eta)^2 \]

which gives the deflection of an infinite plate at point \((x, y)\) when the unit force is placed at \(x = \xi, y = \eta\) with respect to the global system \((x, y)\) as shown in Fig. 6. Sometimes it is easier to deal with local coordinates rather than the global coordinates, especially when it comes to the expression of the moment influence function. Let us write \(I(x, y, \xi, \eta)\) (which is originally in terms of global coordinates \((x, y)\)) in terms of the local coordinates \((\sigma, \chi)\)

\[ I^F(\sigma, \chi, s, n) = \frac{r^2}{16\pi D} \ln \frac{r^2}{a^2} \quad (5.1) \]

where

\[ r^2 = (\sigma-s)^2 + (\chi-n)^2 \]

Equation (5.1) gives the deflection of an infinite plate at point \((\sigma, \chi)\) when unit load is placed at \((\sigma = s, \chi = n)\) as shown in Fig. 12.
influence function is in terms of local coordinates \((\sigma, k_i)\) and since the boundary conditions are in terms of global coordinates \((x, y)\), one needs to reduce all the boundary conditions from the global coordinate system to the local coordinate system \((\sigma, \chi)\).

5.2 Moment Method Concept

The solution of the deflection of the real plate involves embedding the real plate into a fictitious infinite plate of the same material as shown in Fig.14. The deflection of the real plate is the superposition of two solutions as given by

(i) the deflection of an infinite plate due to the actual loading of the real plate and

(ii) the deflection created by \(2N\) fictitious unknown moments placed outside the domain of the real plate around contours.

This yields

\[
\begin{align*}
    w(x,y) &= \iint I(x_i, y_i, \xi, \eta) q_i(\xi, \eta) \, d\xi \, d\eta - \sum_{k=1}^{2N} I^M(x_i, y_i, \xi_k, \eta_k) M_k \\
    &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (5.5)
\end{align*}
\]

The first part of Equation (5.5) can be evaluated easily after knowing the force influence function of an infinite plate. The second part of the right hand side of Equation (5.5), which involves the determination of the fictitious moment, can be handled by forcing the deflection equation,
$s,\eta$ = coordinates of $F$ in local system

$\sigma,\chi$ = coordinates of plate domain point in local system

$\xi,\eta$ = coordinates of $F$ in global system

$x,y$ = coordinates of plate domain point in global system

Figure 12: Local and Global Coordinate Systems for Fictitious Force $F$
One can differentiate the force influence function of Equation (5.1) to get the moment influence function in terms of local coordinates as in Equation (5.2)[7].

\[ I^M (\sigma, \chi, s, n) = \frac{r}{8\pi D} \ln \frac{r^2}{a^2} \cos \phi \] (5.2)

where

\[ r^2 = (\sigma-s)^2 + (\chi-n)^2 \]

Equation (5.2) gives the deflection of an infinite plate at point \((\sigma, \chi)\) when a unit moment is placed at \((\sigma=s, \chi=n)\) as in Fig. 13. From Fig. 13 the angle \(\phi\) is defined as

\[ \phi = \tan^{-1} \left( \frac{\sigma-s}{\chi-n} \right) \]

\[ \cos \phi = \frac{\chi-n}{\sqrt{(\chi-n)^2 + (\sigma-s)^2}} \]

\[ \cos \phi = \frac{\chi-n}{r} \] (5.3)

Using Equation (5.3) in Equation (5.2), one obtains

\[ I^M (\sigma, \chi, s, n) = \frac{\chi-n}{8\pi D} \ln \frac{r^2}{a^2} \] (5.4)

Expression (5.4) is the moment influence function that is used instead of the force influence function as used by Wu-Altiero used. The only difference is that the moment
Figure 13: Local and Global Coordinate Systems for Fictitious Moment M
Figure 14: Plate with Parallel Contour (Point Moment Method)
Equation (5.5) to find the deflection any where inside the actual plate. The bending moments may also be found any where inside the real plate by using Equation (5.5) in the formulae for the bending moments.

After forcing Equation (5.5) to satisfy all the boundary conditions, one will have a matrix equation of the form

\[ AM = B \]

where

\[ M \text{ is the unknown moment vector } M^T = [M_1, M_2 \ldots M_{2N}] \]

\[ A \text{ is a matrix of size } 2N \times 2N \]

\[
A = \begin{bmatrix}
2N & I_k \\
\sum_{k=1}^{2N} & \text{ on } B_{C+B_S} \\
2N & \left( n_x \left( \frac{\partial I^{M}}{\partial x} \right)_{ik} + n_y \left( \frac{\partial I^{M}}{\partial y} \right)_{ik} \right) \\
\sum_{k=1}^{2N} & \text{ on } B_C \\
2N & \left( f_1 \left( \frac{\partial^2 I^{M}}{\partial x^2} \right)_{ik} + f_2 \left( \frac{\partial^2 I^{M}}{\partial x \partial y} \right)_{ik} + f_3 \left( \frac{\partial^2 I^{M}}{\partial y^2} \right)_{ik} \right) \text{ on } B_{S+B_f} \\
\sum_{k=1}^{2N} & \left[ g_1 \left( \frac{\partial^3 I^{M}}{\partial x^3} \right)_{ik} + g_2 \left( \frac{\partial^3 I^{M}}{\partial x^2 \partial y} \right)_{ik} + g_3 \left( \frac{\partial^3 I^{M}}{\partial x \partial y^2} \right)_{ik} + g_4 \left( \frac{\partial^3 I^{M}}{\partial y^3} \right)_{ik} \right] \text{ on } B_f \\
\end{bmatrix}
\]

\[ B \text{ is the right hand side or the load vector as given by} \]

...
\[
B = \begin{bmatrix}
\iint_{R} I_i q d\xi d\eta \\
\iint_{R} [n_x \left( \frac{\partial I}{\partial x} \right)_i + n_y \left( \frac{\partial I}{\partial y} \right)_i] q d\xi d\eta \\
\iint_{R} \left[ f_1 \left( \frac{\partial^2 I}{\partial x^2} \right)_i + f_2 \left( \frac{\partial^2 I}{\partial x \partial y} \right)_i + f_3 \left( \frac{\partial^2 I}{\partial y^2} \right)_i \right] q d\xi d\eta \\
\iint_{R} \left[ g_1 \left( \frac{\partial^3 I}{\partial x^3} \right)_i + g_2 \left( \frac{\partial^3 I}{\partial x^2 \partial y} \right)_i + g_3 \left( \frac{\partial^3 I}{\partial x \partial y^2} \right)_i \right. \\
+ g_4 \left( \frac{\partial^3 I}{\partial y^3} \right)_i \right] q d\xi d\eta
\end{bmatrix}
\text{on } B_C + B_S
\]

Matrix A and vector B contain all the possible boundary conditions. Depending on the type of boundary of plate, one will choose the corresponding part of matrix A and vector B, as explained earlier in Chapter 3. Thus by solving matrix A with its load vector B, which is a system of 2N x 2N linear algebraic equations, one obtains the unknown fictitious moments. The question now remains as to how to generate matrix A and vector B. Both matrix A and vector B are in terms of global coordinates (x,y), but the moment influence function is in terms of local coordinates (\sigma,x). Thus one needs to express all \( I^M(x,y) \) and its derivatives in terms of \( I^M(\sigma,x) \) and its derivatives. In other words, the global moment influence function and its derivatives of matrix A and vector B must be expressed in terms of the local coordinates moment influence.
function and its derivatives. These details are shown in Appendix C. Thus, by using Appendix C, one can generate matrix A and vector B that will solve for all the unknown moments. After that, one may use Equation (5.5) to find the deflection anywhere inside the real plate, and in conjunction with bending moments formulae, one can also obtain $M_x$ and $M_y$ anywhere inside the real plate.
APPLICATION OF MOMENT METHOD

As mentioned in Chapter 5, in order to find the magnitudes of the fictitious moments, one needs to generate matrix A and the load vector B and that is achieved by using Appendix B which involves all the necessary derivatives. The boundary conditions are expressed in terms of the global coordinates \((x,y)\), but the moment influence function is in terms of local coordinates \((\sigma,\chi)\). Thus one is forced to express the global moment influence function and its derivatives in terms of the local moment influence function and its derivatives. This is shown in the second part of Appendix B. The third part of Appendix B expresses the moment influence function and its derivatives in terms of local coordinates explicitly.

The use of different symbols for the Green's function and its derivatives will assist in programming and in making the flowchart. The moment influence function for an infinite plate of arbitrary radius "\(a\)" in terms of the local coordinates is

\[
I^M(\sigma,\chi,s,n) = \frac{x-n}{8\pi D} \ln \left( \frac{(\sigma-s)^2 + (\chi-n)^2}{a^2} \right)
\]
\[ I^M = AAIA \]

\[ \frac{\partial I^M}{\partial x} = AA_1 \quad \frac{\partial I^M}{\partial y} = AB_1 \quad \frac{\partial^2 I^M}{\partial x \partial y} = AAB_1 \]

\[ \frac{\partial^2 I^M}{\partial x^2} = AA_2 \quad \frac{\partial^2 I^M}{\partial y^2} = AB_2 \quad \frac{\partial^3 I^M}{\partial x^2 \partial y} = AAB_2 \]

\[ \frac{\partial^3 I^M}{\partial x^3} = AA_3 \quad \frac{\partial^3 I^M}{\partial y^3} = AB_3 \quad \frac{\partial^3 I^M}{\partial x \partial y^2} = AAB_3 \]

Now one can write matrix A in section 5.2 as

\[
A = \begin{bmatrix}
\sum_{k=1}^{N} (AAIA)_{ik} & \text{on } B_S + B_C \\
\sum_{k=1}^{2N} \left[ N_x(AA_1) + N_y(AB_1) \right]_{ik} & \text{on } B_C \\
\sum_{k=1}^{2N} \left[ F_1(AA_2) + F_2(AAB_1) + F_3(AB_2) \right]_{ik} & \text{on } B_S + B_f \\
\sum_{k=1}^{2N} \left[ G_1(AA_3) + G_2(AAB_2) + G_3(AAB_3) + G_4(AB_3) \right]_{ik} & \text{on } B_f
\end{bmatrix}
\]

(6.1)

If the plate is subjected to a uniformly distributed load, the magnitude of load vector \( \mathbf{B} \) remains the same as derived earlier and given by Equation (5.6). Considering the case where the loaded area is discretized to equal areas (elements), one writes using Equation (5.3) for evaluation of the load vector \( \mathbf{B} \),
\[ B = \begin{bmatrix} \sum_{L=1}^{NL} qA (AAIA)_iL \text{ on } B_S + B_C \\
\sum_{L=1}^{NL} [N_x(AA_1) + N_y(AB_1)]_iL \text{ on } B_C \\
\sum_{L=1}^{NL} [F_1(AA_2) + F_2(AAB_1) + F_3(AB_2)]_iL \text{ on } B_S + B_f \\
\sum_{L=1}^{NL} [G_1(AA_3) + G_2(AAB_2) + G_3(AAB_3) + G_4(AB_3)]_iL \text{ on } B_f \end{bmatrix} \]

Using (6.1) and (6.2) one may solve the following matrix equation

\[ AM = B \]

for the set of friction moments \( M \). At this stage, one can get deflection and moments anywhere inside the plate readily.
GENERAL FLOWCHART

START

Read, input data

Generate the algebraic equation of the form
\[ AM = B \]

A = matrix (2Nx2N)
B = load vector (2Nx1)
M = fictitious moments

Solve simultaneous equations for the fictitious moments

Calculate deflection and moments using the obtained fictitious moments and deflection and moments formulas

STOP

General Flowchart for the Point Moment Method
Using the previous flowchart, a program can be written to solve for any thin plate problem of irregular shape when subjected to a uniformly distributed load with a variety of boundary conditions, provided that the plate boundaries may be approximated by straight edges. By a very slight modification in the previous flowchart, a program can be written to solve for a plate of irregular shape subjected to any type of loading. In the work contained herein, results of the previous two cases in section 4.3 are presented again using the moment method. The statements of the two problems have been mentioned in section 4.3 and also shown in Figure 10 and Figure 12. Tables (6.1) and (6.2) contain the deflection and moments for the square and triangular plates, respectively.
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**Table 6.1**: The moment method
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Table 6.2: The moment method

\( a = 80 \, \text{m}, \, n = 30^\circ \)
chapter 7

MIXED MODE METHOD

In Chapters 3 and 4, where Wu-Altiero method is used, very good solution of the deflection of both rectangular and triangular plates is obtained. Also in Chapters 4 and 5, the numerical technique as developed in this work of solving the same plate with the aid of fictitious moments was verified to be very good. Knowing that the use of fictitious forces or fictitious moments gives almost identical results with a sufficient number of collocation points, the superposition of both fictitious moments as well as fictitious forces is expected to give reasonable answers and this is the topic of this chapter. The technique basically involves embedding the real plate into a fictitious infinite plate of the same material for which the Green's function for force and moment is known. N points are prescribed at the real plate boundary at which the boundary conditions for the real plate are to be satisfied by means of N fictitious moments as well as N fictitious forces placed outside the domain of the real plate. The solution of the real plate is to superimpose three cases, one being the deflection of an infinite plate due to the actual loading
and the second and the third being the deflection due to 
N fictitious forces and N fictitious moments, respectively, 
as shown in Equation (7.1) and Figure 15.

\[
\begin{align*}
  w(x,y) &= \left[ \int \int I(x,y,\xi,\eta) \, q(\xi,\eta) \, d\xi \, d\eta - \sum_{k=1}^{N} I(x,y,\xi_k,\eta_k) L_k \right. \\
  & \quad \left. - \sum_{k=N+1}^{2N} I^M(x,y,\xi_k,\eta_k) L_k \right] \\
\end{align*}
\]  
(7.1)

where

- \( L_k \) for \( k = 1 \rightarrow k = N \) are the fictitious forces

and

- \( L_k \) for \( k = N+1 \rightarrow k = 2N \) are the fictitious moments.

The first part of Equation (7.1) can be evaluated easily just by knowing the force influence function of an infinite plate of the same material as the real plate. The other two parts of Equation (7.1) involve the unknown N fictitious forces and N fictitious moments, which may be obtained by forcing the total solution (Equation 7.1) to satisfy the boundary conditions of the real plate (Equation 2.11 - 2.14) resulting in a matrix equation of the form

\[ AL = B \]

which may be solved for the N fictitious forces and the N fictitious moments.

where:

- \( A \) is the matrix given by
Figure 15: Plate and Contour Geometry
(Mixed Mode Method)
\[ A = \begin{bmatrix} \sum_{k=1}^{N} I_{ik} + \sum_{k=N+1}^{2N} I_{ik}^M \\ \sum_{k=1}^{N} \left[ n_x \left( \frac{\partial I}{\partial x} \right)_{ik} + n_y \left( \frac{\partial I}{\partial y} \right)_{ik} \right] + \sum_{k=N+1}^{2N} \left[ n_x \left( \frac{\partial I^M}{\partial x} \right)_{ik} \\
_y \left( \frac{\partial I^M}{\partial y} \right)_{ik} \right] \end{bmatrix} \text{ on } B_C + B_S \\
\sum_{k=1}^{N} \left[ f_1 \left( \frac{\partial^2 I}{\partial x^2} \right)_{ik} + f_2 \left( \frac{\partial^2 I}{\partial x \partial y} \right)_{ik} + f_3 \left( \frac{\partial^3 I}{\partial y^3} \right)_{ik} \right] \\
+ \sum_{k=N+1}^{2N} \left[ f_1 \left( \frac{\partial^2 I^M}{\partial x^2} \right)_{ik} + f_2 \left( \frac{\partial^2 I^M}{\partial x \partial y} \right)_{ik} + f_3 \left( \frac{\partial^2 I^M}{\partial y^3} \right)_{ik} \right] \text{ on } B_S + B_f \\
\sum_{k=1}^{N} \left[ g_1 \left( \frac{\partial^3 I}{\partial x^3} \right)_{ik} + g_2 \left( \frac{\partial^3 I}{\partial x^2 \partial y} \right)_{ik} + g_3 \left( \frac{\partial^3 I}{\partial x \partial y^2} \right)_{ik} \right] \\
+ g_4 \left( \frac{\partial^3 I}{\partial y^3} \right)_{ik} + \sum_{k=N+1}^{2N} \left[ g_1 \left( \frac{\partial^3 I^M}{\partial x^3} \right)_{ik} + g_2 \left( \frac{\partial^3 I^M}{\partial x^2 \partial y} \right)_{ik} + g_3 \left( \frac{\partial^3 I^M}{\partial x \partial y^2} \right)_{ik} \right] + g_4 \left( \frac{\partial^3 I^M}{\partial y^3} \right)_{ik} \text{ on } B_f \end{bmatrix} \]

\[ L^T = \{ L_1, L_2, L_3 \ldots, L^T_{N+1}, L^T_{N+2} \ldots, L^T_{2N} \} \]

where

\( L_1 + L_N \) are the N fictitious forces

and \( L^T_{N+1} + L^T_{2N} \) are the N fictitious moments

The load vector B is given as
\[ B = \begin{bmatrix} \int_R \frac{\partial I}{\partial x} q \, d\xi d\eta & \text{on } B_C + B_S \\ \int_R n_x \left( \frac{\partial I}{\partial x} \right)_i + n_y \left( \frac{\partial I}{\partial y} \right)_i q \, d\xi d\eta & \text{on } B_C \\ \int_R \left[ f_1 \left( \frac{\partial^2 I}{\partial x^2} \right)_i + f_2 \left( \frac{\partial^2 I}{\partial x \partial y} \right)_i + f_3 \left( \frac{\partial^2 I}{\partial y^2} \right)_i \right] q \, d\xi d\eta & \text{on } B_S + B_f \\ \int_R \left[ g_1 \left( \frac{\partial^3 I}{\partial x^3} \right)_i + g_2 \left( \frac{\partial^3 I}{\partial x^2 \partial y} \right)_i + g_4 \left( \frac{\partial^3 I}{\partial x \partial y^2} \right)_i \right] + g_4 \left( \frac{\partial^3 I}{\partial y^3} \right)_i q \, d\xi d\eta & \text{on } B_f \end{bmatrix} \]

For the same plate when subjected to the same load the right hand side vector is the same for all the three methods.

Depending on the type of plate boundaries, one may use the corresponding part of matrix \( A \) and the corresponding part of vector \( B \). For generating matrix \( A \) one needs to refer to Appendix A to use the derivatives of the force influence function, and refer to Appendix C to use the derivatives of the moment influence function. However, for generating vector \( B \), only Appendix A is needed. By generating matrix \( A \) and vector \( B \) one obtains 2N algebraic equations that may be solved for both the \( N \) forces and the \( N \) moments. Deflection any where inside the plate can be found using Equation (7.1). Also, Equation (7.1) can be used in conjunction with the bending moments formulae to yield the bending moment anywhere inside the plate.
With the aid of the force and moment flowchart, one can get the mixed mode flowchart from which a program may be written. Tables (7.1) and (7.2) contain the deflections and moments for both the square and triangular plates using the mixed mode method. The statements of the two problems have been mentioned in section 4.3 as shown in Figures 10 and 11.
<table>
<thead>
<tr>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
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<td>2.426</td>
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<td>2.744</td>
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</tr>
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<td>2.140</td>
<td>1</td>
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<td>1.856</td>
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</tbody>
</table>

Table 7.1: The mixed mode method

a = 80°, \( n = 40° \), dp

Square Plate
<table>
<thead>
<tr>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
<th>N-mm (My, exact)</th>
<th>Error %</th>
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</thead>
<tbody>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>1.036</td>
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<td>1.234</td>
<td>0.456</td>
<td>1.567</td>
<td>0.901</td>
<td>1.800</td>
<td>1.159</td>
<td>2.123</td>
<td>1.234</td>
</tr>
<tr>
<td>1.350</td>
<td>0.245</td>
<td>1.625</td>
<td>0.723</td>
<td>1.985</td>
<td>0.901</td>
<td>2.322</td>
<td>1.234</td>
<td>2.659</td>
<td>1.365</td>
</tr>
<tr>
<td>1.773</td>
<td>0.123</td>
<td>2.123</td>
<td>1.234</td>
<td>2.567</td>
<td>0.901</td>
<td>2.985</td>
<td>0.901</td>
<td>3.322</td>
<td>1.234</td>
</tr>
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<td>0.245</td>
<td>2.625</td>
<td>0.723</td>
<td>2.985</td>
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<td>3.659</td>
<td>1.365</td>
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<td>3.567</td>
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<td>3.985</td>
<td>0.901</td>
<td>4.322</td>
<td>1.234</td>
</tr>
<tr>
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<td>3.625</td>
<td>0.723</td>
<td>4.085</td>
<td>0.901</td>
<td>4.422</td>
<td>1.234</td>
<td>4.659</td>
<td>1.365</td>
</tr>
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<td>4.567</td>
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<td>4.985</td>
<td>0.901</td>
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<td>1.234</td>
</tr>
<tr>
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<td>0.723</td>
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<td>0.901</td>
<td>5.422</td>
<td>1.234</td>
<td>5.659</td>
<td>1.365</td>
</tr>
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<td>4.374</td>
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<td>5.123</td>
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<td>5.567</td>
<td>0.901</td>
<td>5.985</td>
<td>0.901</td>
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<td>1.234</td>
</tr>
<tr>
<td>4.800</td>
<td>0.245</td>
<td>5.625</td>
<td>0.723</td>
<td>6.085</td>
<td>0.901</td>
<td>6.422</td>
<td>1.234</td>
<td>6.659</td>
<td>1.365</td>
</tr>
</tbody>
</table>

Trigonometric plate

Table 7.2: The mixed mode method
DISCUSSION AND CONCLUSIONS

The work described here-to-fore confirms the validity of the point force method as described by Wu-Altiero [6], in addition to the point moment and the mixed model methods as developed herein.

The salient features of the three methods may be summarized as:

(1) The technique of influence function usage in the solution of the plate problem is based on the use of the Green's functions of an infinite plate with an arbitrary radius "a". These functions may be force or moment influence functions. From the numerous results obtained and shown in Tables 8.3 through 8.8, it is clear that the magnitude of the arbitrary radius "a" has no influence on the computed values of deflection and moments in the plate, provided that double precision computational mode is used. This is in contrast to the conclusions drawn by Wu-Altiero in [6] where the validity of the approach is restricted to a certain range of the arbitrary radius "a" for example, in the case of a square plate 10 m x 10 m, "a" is
restricted to $80 \, \text{m} < a < 8000 \, \text{m}$ for non-oscillatory results. This is a rather significant conclusion, inasmuch as it appears to relieve the user of this technique the responsibility of guessing an appropriate value of "$a$" for any given problem.

(2) Another important conclusion of this work is the effect of the use of the double precision computational mode in programming. This may have been necessitated due to the fact that the technique requires inversion of a large sized matrix, which is not handed and nor is symmetric. The program associated with each of the three methods is run twice, one using a single precision computational mode and the other double precision. It is clear from Tables 8.3 through 8.8 that single precision programming leads to results which are somewhat erroneous, whereas the double precision mode leads to accurate and stable results.

(3) In addition to deciding a value for the arbitrary radius "$a$" prior to the usage of this program for a given problem, the user must also decide the location of contour(s) outside the domain of the plate. It appears that the number of contours should preferably be greater than one for reasons of stability of numerical results over a greater range of contour location. The case of the use of two contours for the square plate
(10 m x 10 m.) was studied in this work, where the location of one contour \( C_1 \) was fixed and a range obtained for the other parallel contour \( C_2 \). The results of this study are depicted in Figure 18, where the range of \( C_2 \) is plotted as the ordinate and different fixed values of \( C_1 \) plotted as the abscissa. The bell shaped curve is interesting inasmuch as it shows that the range of \( C_2 \) is restricted when \( C_1 \) is taken "very close" or "quite far" from the plate edge. This interaction may be explained on the behavior of the influence function for small and large values of its arrangement for forces on contour \( C_1 \) requiring a numerical balancing effect from forces located on \( C_2 \). Such an argument may be extended to the preferred use of contours numbering greater than one.

The rate of convergence for the three methods as a function of the number of collocation points has been investigated, using the following three error criteria:

\[
I_1 = |w_{\text{theo.}} - w_{\text{num.}}|_{\text{max}}
\]

\[
I_2 = \frac{\sum_{i=1}^{N} (w_{\text{theo.}} - w_{\text{num.}})_i^2}{N-1}
\]

and

\[
I_3 = \frac{\sum_{i=1}^{N} |(w_{\text{theo.}} - w_{\text{num.}})_i|}{N}
\]

as shown in Figure 17.
Also the first error criteria for mix bending as a function of the number of collocation points has been investigated for the three methods as shown in Figure 19.

(5) Philosophically speaking, the method is a general technique for the solution of linear boundary value problems. The contour point forces or moments may be thought of as the "homogeneous" part of the solution of the differential equation, choosing such that the complete solution satisfies boundary conditions in a discrete sense. Thus this method may be extended to the analysis of other problems in mechanics or other engineering systems which are modelled as boundary value problems, provided the influence function is available for the basic governing equation. It may also have use in non-linear problems, where the technique would be used in an iterative sense.
Table 8.1 Three Methods Deflections

Square Plate
n = 40   DP   a = 80 m

<table>
<thead>
<tr>
<th>Point No.</th>
<th>W(Exact) mm</th>
<th>W(Num) mm Wu-Altiere method</th>
<th>W(Num) mm Moment Method</th>
<th>W(Num) mm Mixed Mode Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.8560</td>
<td>1.9319</td>
<td>1.9326</td>
<td>1.9309</td>
</tr>
<tr>
<td>2</td>
<td>4.7900</td>
<td>4.9812</td>
<td>4.9822</td>
<td>4.9800</td>
</tr>
<tr>
<td>3</td>
<td>5.8810</td>
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</tr>
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<td>1.4050</td>
<td>1.4526</td>
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</tr>
<tr>
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<td>0.9755</td>
<td>0.9754</td>
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<td>0.5167</td>
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<td>1.3023</td>
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<tr>
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<td>1.5827</td>
<td>1.5825</td>
</tr>
<tr>
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<td>0.0892</td>
<td>0.0892</td>
<td>0.0892</td>
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<tr>
<td>14</td>
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<td>0.2160</td>
<td>0.2160</td>
</tr>
<tr>
<td>15</td>
<td>0.2540</td>
<td>0.2593</td>
<td>0.2593</td>
<td>0.2593</td>
</tr>
</tbody>
</table>
Table 8.2 Three Methods Deflections

Triangular Plate

\( n = 30, \text{ DP a} = 80 \)

<table>
<thead>
<tr>
<th>Point No.</th>
<th>( W(\text{Exact})_{\text{mm}} )</th>
<th>( W(\text{Num})_{\text{mm}} ) Wu-Altierno method</th>
<th>( W(\text{Num})_{\text{mm}} ) Moment Method</th>
<th>( W(\text{Num})_{\text{mm}} ) Mixed Mode Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>-0.0001</td>
<td>-0.0002</td>
<td>-0.0002</td>
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<tr>
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<td>0.0054</td>
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<td>0.0343</td>
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<td>0.0919</td>
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<td>0.2443</td>
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<td>0.2970</td>
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<td></td>
</tr>
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</table>

Table B.3: Deflections using the force method

 qp

No. Port

| M | Radial

| 4 | 200 |
| 3 | 1800 |
| 2 | 1600 |
| 1 | 1400 |
| 0 | 2000 |
| 0 | 2000 |
| 10 | 0 |
| 30 | 0 |

Truncated Plate
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<th>0.0010</th>
<th>0.0000</th>
<th>0.0010</th>
<th>0.0000</th>
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<td>13</td>
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<td>16</td>
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</table>

*Table 6.4: Deflections using the force method*
<table>
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<tr>
<th>y</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>l</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>0.005</td>
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<td>0.002</td>
<td>0.003</td>
<td>0.002</td>
<td>0.003</td>
<td>0.006</td>
<td>0.004</td>
<td>0.009</td>
<td>0.010</td>
<td>0.011</td>
<td>0.012</td>
<td>0.013</td>
</tr>
<tr>
<td>0.014</td>
<td>0.025</td>
<td>0.036</td>
<td>0.047</td>
<td>0.058</td>
<td>0.069</td>
<td>0.080</td>
<td>0.091</td>
<td>0.102</td>
<td>0.113</td>
<td>0.124</td>
<td>0.135</td>
<td>0.146</td>
<td>0.157</td>
</tr>
<tr>
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<td>0.228</td>
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<td>0.252</td>
<td>0.264</td>
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<td>0.420</td>
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<td>0.444</td>
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</tbody>
</table>

Table 8.5: Deflections using the Moment Method

<table>
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<th>Radius</th>
<th>Load</th>
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<th>1,000</th>
<th>2,000</th>
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<th>4,000</th>
<th>5,000</th>
<th>6,000</th>
<th>7,000</th>
<th>8,000</th>
<th>9,000</th>
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Table 8.6: Determining the moment method using the moment method.

Formula: \( I = \frac{n}{30} \)
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<tr>
<th>(exact)</th>
<th>H</th>
<th>M</th>
<th>Radius</th>
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<td>W</td>
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Table 8.8: Deviations using the mixed mode method

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Table 8.9: Rate of convergence using I₁

<table>
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<tr>
<th>Number of Collocation Points</th>
<th>Wu-Altiero Method</th>
<th>Moment Method</th>
<th>Mixed Mode Method</th>
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Table 8.10: Rate of convergence using $I_2$

Tableau 8.10: Vitesse de convergence avec $I_2$

<table>
<thead>
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<th>Number of Collocation Points</th>
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<th>Moment Method</th>
<th>Mixed Mode Method</th>
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<td>2.10048</td>
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<td>1.60715</td>
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Table 8.11: Rate of convergence using $I_3$

Triangular Plate

\[ I_3 = \frac{\sum_{i=1}^{N} |(W_{\text{theo.}} - W_{\text{num.}})|}{N} \]

<table>
<thead>
<tr>
<th>Number of Collocation Points</th>
<th>Wu-Altiero Method</th>
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<td>1.59045</td>
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$I_1 = |W_{\text{exact}} - W_{\text{cal}}|_{\text{max}}$

Number of Collocation Points

Figure 17: Comparison of the Congerence of the Three Methods
Figure 18: Contour Location Range
Figure 19: Rate of Convergence of Maximum Bending Moment (Square Plate)

\[ M_1 = \left| \frac{M_{\text{theo.}} - M_{\text{num.}}}{M_{\text{num.}}} \right|_{\text{max.}} \]
REFERENCES


APPENDIX A

Influence Function for an Infinite Plate and Its Derivatives

Let \( I(x, y, \xi, \eta) \) represent the force influence function for an infinite plate.

\[
I(x, y, \xi, \eta) = \left[ \frac{(x-\xi)^2 + (y-\eta)^2}{16\pi D} \right] \ln \left[ \frac{(x-\xi)^2 + (y-\eta)^2}{a^2} \right]
\]

\[
\frac{\partial I}{\partial x} = \left( \frac{3-\xi}{8\pi D} \right) \left[ \ln \left[ \frac{(x-\xi)^2 + (y-\eta)^2}{a^2} \right] + 1 \right]
\]

\[
\frac{\partial^2 I}{\partial x^2} = \frac{1}{8\pi D} \left[ \ln \left[ \frac{(x-\xi)^2 + (y-\eta)^2}{a^2} \right] + \frac{2(x-\xi)^2}{(x-\xi)^2 + (y-\eta)^2} + 1 \right]
\]

\[
\frac{\partial^3 I}{\partial x^3} = \frac{(x-\xi)}{16\pi D((x-\xi)^2 + (x-\eta)^2)} \left[ 12 - \frac{8(x-\xi)^2}{[(x-\xi)^2 + (y-\eta)^2]} \right]
\]

\[
\frac{\partial I}{\partial y} = \left( \frac{y-\eta}{8\pi D} \right) \left[ \ln \left[ \frac{(x-\xi)^2 + (y-\eta)^2}{a^2} \right] + 1 \right]
\]

\[
\frac{\partial^2 I}{\partial y^2} = \frac{1}{8\pi D} \left[ \ln \left[ \frac{(x-\xi)^2 + (y-\eta)^2}{a^2} \right] + \frac{2(y-\eta)^2}{(x-\xi)^2 + (y-\eta)^2} + 1 \right]
\]

\[
\frac{\partial^3 I}{\partial y^3} = \frac{(y-\eta)}{16\pi D(x-\xi)^2 + (y-\eta)^2} \left[ 12 - \frac{8(y-\eta)^2}{(x-\xi)^2 + (y-\eta)^2} \right]
\]
\[
\frac{\partial I}{\partial x \partial y} = \frac{(x - \xi)(y - \eta)}{4\pi D \left[ (x - \xi)^2 + (y - \eta)^2 \right]}
\]

\[
\frac{\partial^2 I}{\partial x^2 \partial y} = \frac{(y - \eta)}{4\pi D \left[ (x - \xi)^2 + (y - \eta)^2 \right]} \left[ 1 - \frac{2(x - \xi)^2}{(x - \xi)^2 + (y - \eta)^2} \right]
\]

\[
\frac{\partial^2 I}{\partial x \partial y^2} = \frac{(x - \xi)}{4\pi D \left[ (x - \xi)^2 + (y - \eta)^2 \right]} \left[ 1 - \frac{2(y - \eta)^2}{(x - \xi)^2 + (y - \eta)^2} \right]
\]
Appendix B-1

Transformation of Global Coordinates in Terms of Local Coordinates

\( x, y \) are the global coordinates
\( \sigma, \chi \) are the local coordinates

From Figure (16) one can write the following

\[
\begin{align*}
  x &= x_1 + \chi \cos \theta - \sigma \sin \theta \\
  y &= y_1 + \chi \sin \theta + \sigma \cos \theta
\end{align*}
\]

or

\[
\begin{align*}
  x - x_1 &= \chi \cos \theta - \sigma \sin \theta \\
  y - y_1 &= \chi \sin \theta + \sigma \cos \theta
\end{align*}
\]

and by using Cramer's rule \( \chi \) and \( \sigma \) can be obtained

\[
\begin{align*}
\chi &= \frac{(x - x_1) \cos \theta + (y - y_1) \sin \theta}{\cos \theta - \sin \theta} \\
\sigma &= \frac{(y - y_1) \cos \theta - (x - x_1) \sin \theta}{\cos \theta - \sin \theta}
\end{align*}
\]

(1)
Figure 16 : Transformation of Coordinates
Transformation of the Global Derivatives of the Moment Influence Function in Terms of its Local Derivatives

$x, y$ are the global coordinates

$\sigma, \chi$ are the local coordinates

$I^M$ is the moment influence function in terms of local coordinates $(\sigma, \chi)$

\[
\frac{\partial I^M}{\partial x} = \frac{\partial I^M}{\partial \sigma} \frac{\partial \sigma}{\partial x} + \frac{\partial I^M}{\partial \chi} \frac{\partial \chi}{\partial x}
\]

\[
\frac{\partial I^M}{\partial y} = \frac{\partial I^M}{\partial \sigma} \frac{\partial \sigma}{\partial y} + \frac{\partial I^M}{\partial \chi} \frac{\partial \chi}{\partial y}
\]

(2)

Using Equation (1) one can obtain

\[
\frac{\partial \sigma}{\partial x} = -\sin \theta
\]

\[
\frac{\partial \sigma}{\partial y} = \cos \theta
\]

(3)

\[
\frac{\partial \chi}{\partial x} = \cos \theta
\]

\[
\frac{\partial \chi}{\partial y} = \sin \theta
\]

Using Equations (3) in (2) one can write

\[
\frac{\partial I^M}{\partial x} = -\sin \theta \frac{\partial I^M}{\partial \sigma} + \cos \theta \frac{\partial I^M}{\partial \chi}
\]

(4)

\[
\frac{\partial I^M}{\partial y} = \cos \theta \frac{\partial I^M}{\partial \sigma} + \sin \theta \frac{\partial I^M}{\partial \chi}
\]
Equation (4) report the global first derivatives of the moment influence function in terms of the local first derivatives. Similarly one can obtain the higher order derivatives.

\[
\frac{\partial^2 \sigma_1^M}{\partial x^2} = \sin^2 \theta \frac{\partial^2 \sigma_1^M}{\partial \sigma^2} - 2 \sin \theta \cos \theta \frac{\partial^2 \sigma_1^M}{\partial \sigma \partial \chi} + \cos^2 \theta \frac{\partial^2 \sigma_1^M}{\partial \chi^2}
\]

\[
\frac{\partial^2 \sigma_1^M}{\partial y^2} = \cos^2 \theta \frac{\partial^2 \sigma_1^M}{\partial \sigma^2} + 2 \sin \theta \cos \theta \frac{\partial^2 \sigma_1^M}{\partial \sigma \partial \chi} + \sin^2 \theta \frac{\partial^2 \sigma_1^M}{\partial \chi^2}
\]

\[
\frac{\partial^3 \sigma_1^M}{\partial x^3} = \cos^3 \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma^3} + 3 \sin^2 \theta \cos \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma^2 \partial \chi} - 3 \sin \theta \cos^2 \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma \partial \chi^2}
\]

\[- \sin^3 \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma^3}
\]

\[
\frac{\partial^3 \sigma_1^M}{\partial y^3} = \cos^3 \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma^3} + 3 \sin \theta \cos^2 \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma^2 \partial \chi} + 3 \sin^2 \theta \cos \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma \partial \chi^2}
\]

\[+ \sin^3 \theta \frac{\partial^3 \sigma_1^M}{\partial \chi^3}
\]

\[
\frac{\partial^2 \sigma_1^M}{\partial x \partial y} = (\cos^2 \theta - \sin^2 \theta) \frac{\partial^2 \sigma_1^M}{\partial \sigma \partial \chi} + \sin \theta \cos \theta \left( \frac{\partial^2 \sigma_1^M}{\partial \chi^2} - \frac{\partial \sigma^2}{\partial \sigma^2} \right)
\]

\[
\frac{\partial^3 \sigma_1^M}{\partial x^2 \partial y} = \sin^2 \theta \cos \theta \frac{\partial^3 \sigma_1^M}{\partial \sigma^3} + \cos^2 \theta \sin \theta \frac{\partial^3 \sigma_1^M}{\partial \chi^3}
\]

\[+ \frac{\partial^2 \sigma_1^M}{\partial \sigma^2 \partial \chi} (\sin^3 \theta - 2 \sin \theta \cos^2 \theta) + \frac{\partial \sigma^3 \sigma_1^M}{\partial \chi^2 \partial \sigma} (\cos^3 \theta - 2 \sin^2 \theta \cos \theta)
\]
$$\frac{\partial^3 I^M}{\partial x \partial y^2} = - \cos^2 \theta \sin \theta \frac{\partial I^M}{\partial \sigma^3} + \sin^2 \theta \cos \theta \frac{\partial I^M}{\partial \chi^3} + \frac{\partial I^M}{\partial \sigma^2 \partial \chi} \left( \cos^3 \theta - 2 \sin^2 \theta \cos \theta \right) + \frac{\partial I^M}{\partial \chi^2 \partial \sigma} (2 \sin \theta \cos^2 \theta - \sin^3 \theta)$$
Appendix B-3

The Local Moment Influence Function and its Derivatives

\[ I^M(\sigma, \chi, s, n) = \frac{\chi - n}{8\pi D} \ln \frac{(\sigma - s)^2 + (\chi - n)^2}{b^2} \]

\[ \frac{\partial I^M}{\partial \sigma} = \frac{1(\sigma - s)(\chi - n)}{4\pi D[(\sigma - s)^2 + (\chi - n)^2]} \]

\[ \frac{\partial^2 I^M}{\partial \sigma^2} = \frac{1(\chi - n)}{4\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} - \frac{(\sigma - s)^2 (\chi - n)}{2\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} \]

\[ \frac{\partial^3 I^M}{\partial \sigma^3} = \frac{-3(\sigma - s)(\chi - n)}{2\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} + \frac{2(\sigma - s)^3 (\chi - n)}{\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} \]

\[ \frac{\partial I^M}{\partial \chi} = \frac{1}{8\pi D} \ln \frac{(\sigma - s)^2 + (\chi - n)^2}{b^2} + \frac{2(\chi - n)^2}{8\pi D[(\sigma - s)^2 + (\chi - n)^2]} \]

\[ \frac{\partial^2 I^M}{\partial \chi^2} = \frac{7(\chi - n)}{8\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} - \frac{(\chi - n)^3}{2\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} \]

\[ \frac{\partial^3 I^M}{\partial \chi^3} = \frac{6}{8\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} - \frac{3(\chi - n)^2}{\pi D[(\sigma - s)^2 + (\chi - n)^2]^2} \]

\[ + \frac{2(\chi - n)^4}{\pi D[(\sigma - s)^2 + (\sigma - s)^2]^3} \]
\[ \frac{\partial^2 I^M}{\partial \sigma \partial \chi} = \frac{1}{4\pi D} \frac{(\sigma-s)}{(\sigma-s)^2 + (\chi-n)^2} - \frac{(\sigma-s)}{2\pi D} \frac{(\chi-n)^2}{(\sigma-s)^2 + (\sigma-n)^2} \]

\[ \frac{\partial^3 I^M}{\partial \sigma^2 \partial \chi} = \frac{1}{4\pi D} \frac{1}{[(\sigma-s)^2 + (\chi-n)^2]} + \frac{2(\sigma-s)^2(\chi-n)^2}{\pi D \left[(\sigma-s)^2 + (\chi-n)^2\right]^3} \]

\[ \frac{\partial^3 I^M}{\partial \chi^2 \partial \sigma} = \frac{2(\chi-n)^3(\sigma-s)}{\pi D \left[(\sigma-s)^2 + (\chi-n)^2\right]^3} - \frac{3(\chi-n)(\sigma-s)}{2\pi D \left[(\sigma-s)^2 + (\chi-n)^2\right]^2} \]