Vibration Analysis of Point and Column Supported Mindlin Plates

by

Mamdouh B. Felemban

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES
KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the Requirements for the Degree of

MASTER OF SCIENCE

In

MECHANICAL ENGINEERING

June, 1989
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Felemban, Mamdouh B., M.S.
King Fahd University of Petroleum and Minerals (Saudi Arabia), 1989
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This thesis, written by

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under the direction of his Thesis Advisor and approved by his Thesis committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

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Date: June 12, 1989
بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ
المحرر للدروسب الفضائيين والمعدلة والسلامة جهاء رسول الله ﷺ
لنشرة غيرالتكبير والإغاثة والإعمار للmanuel إلى التوجيه
للتعاميم هذه الرسالة
To

Whom I love
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All praises and glory to Almighty Allah without Whose help no work can be accomplished.

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# LIST OF EQUIPMENT

[ 1 ]: Exciter Control  
   type B&K  1047

[ 2 ]: Power Amplifier  
   type B&K  2712

[ 3 ]: Vibration Exciter  
   type B&K  4808

[ 4 ]: Accelerometer  
   type B&K  4343

[ 5 ]: Conditioning Amplifier  
   type B&K  2625

[ 6 ]: Two Channel Tracking Filter  
   type B&K  5716 / WHO255

[ 7 ]: Low Frequency Mounting Table  
   M.E. Workshop (U.P.M.)

[ 8 ]: Accelerometer  
   type B&K  4344

[ 9 ]: Charge Amplifier  
   type B&K  2635

[10 ]: Phase Meter  
   type B&K  2971

[11 ]: Voltmeter  
   type B&K  2432

[12 ]: Cross Spectrum Unit  
   type B&K  5748

[13 ]: X-Y Recorder  
   type B&K  2308

[14 ]: Two Channel Level Recorder  
   type B&K  2309
NOMENCLATURE

A : area
A-A : Antisymmetric-Antisymmetric
[A] : coefficient matrix
a : length of a rectangular plate
a : length of and width of a square plate
b : width of a rectangular plate
a_{xx}, a_{yy} : directional cosines
D : flexural rigidity \( \equiv Eh^3/(1-v^2) \)
D_x : flexural rigidity in the x-direction
D_y : flexural rigidity in the y-direction
E : Young's modulus of elasticity
F : quadrature function in discrete displacement and rotational components.
f : radian frequency
f_1, f_2, f_3 : strain energy sub-functions
f_4 : kinetic energy sub-functions
G : shear modulus
H : linear function of the square of discrete velocities
h : plate thickness
i, j : node indices
k : shear coefficient \( \pi^2/12 \)
M_{x}, M_{y} : bending moment per unit length of sections of a plate perpendicular to x and y axes respectively
M_{xy} : twisting moment per unit length of sections of a plate perpendicular to x axis
\( M_v \): bending moment per unit length of sections of a plate perpendicular to \( v \) axis

\( M_w \): twisting moment per unit length of sections of a plate perpendicular to \( v \) axis

\( M,N \): number of rows and columns in the node set, respectively

\( m \): number of rows and columns in the node set of a square plate

\( 90^\circ \): phase shift of 90 degrees

\( n_{AA}n_{SA}n_{SS} \): total number of equations for \( AA, SA \) and \( SS \) mode, respectively

\( Q_x, Q_y \): shear forces parallel to \( z \)-axis per unit length of section of a plate perpendicular to \( x \) and \( y \) axes, respectively

\( q_1, q_2, q_3 \): generalized coordinate

\( q \): external force (or load)

\( S-A \): Symmetric-Antisymmetric

\( S-S \): Symmetric-Symmetric

\( t \): time

\( T_v \): kinetic energy per unit volume

\( T_A \): kinetic energy per unit area

\( T \): total kinetic energy

\( \delta T \): Virtual kinetic energy

\( U_v \): strain energy per unit volume

\( U_A \): strain energy per unit area

\( U \): total strain energy

\( \delta U \): Virtual strain energy

\( U_1, U_2, U_3 \): sub-strain energies

\( u \): displacement field in \( x \) - direction

\( V \): total potential energy
v : displacement field in y - direction
w : displacement field in z - direction
u_1 , u_2 , u_3 : displacement field
v_i , s : transformation coordinate
\bar{w} : average displacement components
w : non-dimensional component
W : displacement component independent of time
x, y, z : rectangular coordinate
[X] : eigenvector
L : lagrangian = U + V
α : position of the point support
β : thickness ratio (h/a)
γ : aspect ratio (a/b)
γ_x , γ_y : angles of distortion due to shear
γ_{xy} , γ_{yz} , γ_{zx} : unit shearing strains
ε_{xx}, ε_{yy}, ε_{zz} : strain field
ε_{xy}, ε_{yz}, ε_{zx} : strain field
ε_x , ε_y , ε_z : unit elongation in x, y and z directions
σ_x, σ_y, σ_z : normal stresses parallel to X, Y and Z axes respectively
τ : shear stress
τ_{xy}, τ_{yz}, τ_{zx} : unit elongation in X, Y and Z directions
V : differential operator = ∂^2/∂x^2 + 2∂^2/∂x∂y + ∂^2/∂y^2
V^4 : differential operator = ∂^4/∂x^4 + 4∂^4/∂x^2∂y^2 + ∂^4/∂y^4
ζ, η : non-dimensional coordinate
ΔX , ΔY : finite difference increments
$\Delta \zeta, \Delta \eta$ : non-dimensional finite difference increments

$\Omega$ : eigen value $= \omega h \sqrt{\rho (1 - \nu^2) / E}$

$\lambda$ : frequency parameter (point support) $= \omega a^2 \sqrt{\rho h / D}$

$\lambda_c$ : frequency parameter (rigid support) $= \omega a^2 \sqrt{\rho h / D}$

$\delta$ : virtual displacement

$\omega$ : natural frequency (rad/sec)

$\xi$ : finite area of rigid support

$\nu$ : poisson's ratio

$\rho$ : density

$\varphi_x, \varphi_y$ : angles of rotations due to bending in x and y directions respectively

$\varphi_v, \varphi_s$ : angles of rotations due to bending in v and s directions respectively

$\Phi_X, \Phi_Y$ : rotational components independent of time

$\pi$ : summation of the total strain and potential energies
THESIS ABSTRACT

NAME : MAMDOUGH B. FELEMBAN

TITLE : VIBRATION ANALYSIS OF POINT AND COLUMN SUPPORTED MINDLIN PLATES

MAJOR : MECHANICAL ENGINEERING

DATE : 31st MAY, 1989

Free vibration characteristic of an isotropic, homogeneous point supported square plate with no initial deflection and no initial stresses have been critically examined by a method based on the application of variation principle to the energy expressions in conjunction with Finite Difference methods with interlacing grids. Using the symmetry of the geometry of the plate and the symmetry in the distribution of the point supports, only one quarter of the plate was examined. Three mode types were developed to account for the full plate.

The effect of transverse shear deformation and rotary inertia have been considered and the effect of finite area of the point support has also been analyzed. The produced mode shapes and nodal patterns were presented. Experiments were conducted to give an insight to the theoretical predictions. The natural frequencies were obtained from the frequency response of the Real and the Imaginary part. Nyquist (Argand) plots of the corner supported case were also presented.

The convergence of the present method to the theoretical exact value of the frequency parameter were found out to be either from above or below. It has been shown by the theoretical investigation that the effect of transverse shear deformation and rotary inertia were more pronounced in higher modes. It was observed that the fundamental frequency parameter increases as the point support moves towards the center and the maximum occurs at $\alpha = 0.23$. The radian natural frequency obtained from the frequency response showed that the theoretical predictions were considerably accurate with maximum discrepancy of 2.48 percentage.

The result of the studies could be applied in many practical application such as the design of large shakers capable of supplying yaw as well as thrust motion to the test specimens.

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لقد استخدمت نظرية تعمدة على تطبيق المبادئ التقليدية على دوال الطاقة بالاتجاه مع أسلوب الفرق المحدد بشبكة متشابكة وذلك لكشف خواص الاهتزاز الحر للوح مرن أوموجو الخواص ومتفاوت ومتناقص بدون ازاحة أو شد أولي. وباستخدام خواص التشابة في شكل الربح والتوازن في توزيع نقل التثبيت أمكن اختيار ربع اللوح فقط ولذلك طورت ثلاثة اشكال تركيبية حتى تغطي الشكل الثقافي للوح الكامل.

لقد أشتمل البحث على دراسة تأثير تضمين التشوه القصي المستعرض والصور ذاتي الدوران وكذلك تأثير المساحة المحصنة لنقطة التثبيت أيضا تقدم الإشكال التركيبية ورسومات الخطوط العقلية. كذلك اشتملت الدراسة على إجراء تجربة عملية لإعطاء فكره أوضح من التقديرات النظرية للترددات الطبيعية. أمكن الحصول على الترددات الطبيعية عن طريق القسم الحقيقي والخيالي لرسومات الذبذبات. كذلك تم تقديم رسومات ناىكوسنت (أرقان) لهذه الترددات.

لقد استنتج من هذا البحث ان خاصية الميل للالتباس بين النظرية الحالية والقيم الحقيقية النظرية للاهتزازات يمكن ان تكون من أعلى أو من أسفل. وأستنتج كذلك خلال الدراسة ان تأثير التشوه القصي المستعرض والصور ذاتي الدوران يكون بدرجة كبيرة للترددات العالية، ولان المطارد الأولي يزيد مع حركة النقطة باتجاه المركز تصل لاي قيمة عند 320. ووضعت النتائج أن الترددات الطبيعية (رانيان) تقارب بدرجة كبيرة التقديرات النظرية بفرق أعلى 2%. ويمكن استخدام نتائج هذا البحث في عدة تطبيقات عملية مثل تصميم الهيازاالت الكبيرة القادرة على تقديم الانبعاث بالإضافة إلى حركة الدفع للقطع تحت الاختبار.

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CHAPTER 1

INTRODUCTION

The problem of conducting free vibration analysis of rectangular plates resting on point supports has received considerable attention during recent years. A thorough survey of the literature indicates that most of these analyses have been confined to the vibration of square plates with point supports located on the diagonals. A number of papers concerned with the free vibration of square or rectangular plates on four-point supports have already appeared. Theoretical studies based on different methods were applied, namely, Finite Difference method, Rayleigh-Ritz method, Finite Element method, Lagrangian Multiplier method, Finite Element Displacement method, Finite Strip method, Modal Constraint method, Series solution, Superposition method and others. In all of the methods mentioned above, the numerical study was based on thin plate theory where the effect of rotary inertia and transverse shear deformation are neglected.

A limited amount of experimental data has been reported by various researchers. Experimental verification of the results has been reported in some references. Some worked with a plate supported by four bolts which are used as point supports and the plate in turn was mounted on a shaker. The effect of transverse shear deformation and rotary inertia were not investigated since the experiment were conducted on plates with small thickness/dimension ratio ($\beta$ less than 0.1).
In the present study, the use of variational principles applied to the energy expressions in conjunction with the finite difference technique with interlacing girds is applied as a method for determination of vibration characteristics, mode shapes and nodal patterns of elastic, isotropic and homogeneous corner and point supported square plates. Expressing the strain and the kinetic energy equations by the finite difference equations with equal intervals, these expressions are obtained in terms of discrete displacement and rotational components. Harmonic motion is assumed to eliminate the time dependence and the energy functional is minimized with respect to discretized components. The natural frequency parameters with mode shapes are determined as the solutions of a linear algebraic eigenvalue problem.

The concept of interlacing grids and nodal sub-domains are applied for the strain energy expression and modified finite difference method is used in the formulation. The strain energy functional is divided into sub-functions. The sub-strain energies are computed as the summation of strain energies summed over the set of non-overlapping sub-domains. Although the total number of nodes remains unchanged, employing interlacing grids technique provides a finite difference formulation with reduced node intervals and results in reduced discretization error.

The use of variational principles in conjunction with finite difference method provides the following advantages:

1. Only first order finite difference approximation is required.
2. First order derivatives provide the interlacing grid technique to be used which results in high degree of accuracy.
3. The stress boundary conditions are satisfied automatically through the minimization process and only geometric boundary conditions are considered.

4. Automatically generated band matrix saves the computation time and computer core storage.

In this study, the dynamic behavior of relatively thick plates (Mindlin) is analyzed including the effect of transverse shear deformation and rotary inertia by applying the finite difference technique. The mode types are divided into three categories namely, Antisymmetric-Antisymmetric, Symmetric-Antisymmetric, and Symmetric-Symmetric, since quarter of the plate is considered. Also the numerical analysis further extended to study the effect of rigid support and the effect of increasing the finite area of the rigid point support.

Experiment was conducted to give insight into the numerical prediction. Three steel plates of different thickness ratios, $\beta$, were tested for two different location of the point support namely corner position $\alpha = 0.0$ and point support $\alpha = 0.1$. The frequency response was plotted for the real and imaginary parts. Nyquist plot of the corner supported case is also presented.

A comprehensive survey of literature was conducted and a summary of all related studies is presented in Chapter 2. In Chapter 3, the theory which the numerical results are based on, is explained. The derivation of the energy expressions is presented and the use of the variational principle is explained. Finite difference formulation is presented in short in Chapter 4. Only the inter-lacing sub-domains are given and for further details references are cited. A description of the experiment, test equipment, instrumentation and test proce-
procedure is presented in Chapter 5. The discussion of results, Chapter 6, is divided into four parts. In part one, the convergence study is discussed based on the numerical values obtained for different mesh sizes. Results of corner and point supported case are given in part two. Part three is concerned with the experimental results. Finally, discussion of column supported plate is given in part four. Conclusions and recommendations are presented in Chapter 7. All of the figures related to the convergence study, corner and point supported square plates, and all tables, frequency response, Nyquist plots, mode shapes, nodal patterns and the computer program are given in the appendices.
CHAPTER 2

LITERATURE SURVEY

Considerable interest has been shown in the determination of the vibration characteristics of point-supported plates. Such an investigation finds its application either in the design of large shakers capable of supplying yaw as well as thrust motions to the test specimens, or in the determination of dynamic response of column-supported slabs which experience oscillatory loads from the machinery resting on them.

A large amount of work has been done on the vibration of elastic thin plates of uniform thickness point supported along the diagonals, but none (up to the knowledge of the author) has considered thick plate. Thin plate is considered when the thickness to the plate dimension ratio, $\beta$, is less than 0.02 (or 1:50).

The first paper published concerned about the fundamental frequency parameter of uniform isotropic rectangular plate that have free edges and pinpoint supports at the four corners was in 1960 [5] by Cox, H. and Boxer, J. They have used Finite Difference expressions, which simplify the treatment of the free boundaries for definite values of Poisson’s ratio, in conjunction with extrapolation procedure to obtain the approximate solutions. They had neglected the effect of rotary inertia and transverse shear deformation. They have considered square plate as a special case of rectangular plate. They have presented the first five mode shape of corner supported square plate.
In 1962 [6] Kirk, C. determined the frequency expression and mode shape for a square plate vibrating at the lowest natural frequency by considerations of energy. He stated that the lowest natural frequency of a square plate that is point-supported at the corners may be determined approximately by equating the maximum strain energy of the plate to the maximum kinetic energy. He used the classical theory of plate where the effect rotary inertia and transverse shear deformation were neglected.

In 1965 [7] Reed, R. compared some of the methods in calculating frequencies of corner-supported rectangular plates. He had used two methods namely the Ritz method and the Series solution to the classical differential equation of motion of the plate. The series method showed fast convergence over the Ritz method. He also carried out a short experimental program to see the correlation with the theoretical results. He used a 30-watt sound speaker that excited the plate (β = 1/93) through an attached string. The speaker was driven by variable frequency oscillator and the motion of the plate was measured by a capacitance-type displacement instrument. He visualized the mode shape by sprinkling carborundum chips on the plate during excitation.

In 1966 [8] Tso, W. used Rayleigh-Ritz method with three different assumed deflection to obtain the fundamental frequency. He was the first to study the effect of point location along the diagonals of a square plate to the fundamental frequency. He also conducted an experiment where the plate (β = 0.0035) was supported by four bolts mounted on a shaker, and the fundamental frequency of the plate for various locations of the four support points were determined. To compare his experimental results with the theoretical calculations, the experimental value of the nondimensional frequency parameter for
the corner supported case \((\alpha = 0.0)\) was made to coincide with the value given by H. Cox and J. Boxer [5]. Then the experimental results of the other point locations were reduced by the same factor.

In 1969 [10] Johns, D.J. and Nagaraj, V.T. used an alternative finite difference formulation of the governing differential equation of motion and an energy-type analysis involving the assumption of modal forms to determine the fundamental frequency of a square plate symmetrically supported at four points. Because of the symmetry (assumed) of the fundamental mode about the diagonals and about the co-ordinate axes, only a triangular portion of the plate was considered with different mesh sizes.

In 1969 [11] Leissa, A. has presented an extensive review of published literature on vibration of plate up to 1965. Leissa's monograph deals with the classical theory of plates, anisotropy, in-plane force and variable thickness plates. Since the classical thin plate theory assumes that the effects of transverse shear deformation in conjunction with rotary inertia are neglected, the assumption can lead to substantial error for the case of thick plates. The effects of transverse shear deformation and rotary inertia become increasingly important with the increase in plate thickness relative to plate length \((\beta)\).

In 1971 [12] Dowell, E.H., described a method for the analysis of the free vibration of a linear structure supported in an arbitrary way. It was presented based upon the use of the normal modes of the unsupported or unconstrained structure in Rayleigh-Ritz analysis with the support or constrained conditions enforced by means by Lagrange Multipliers.
In 1971 [13] Mirza, W.H. and Petyt, M. presented some of the results obtained by analyzing the problem of vibration of point-supported plate using finite element technique. Two displacement functions incorporating three degree of freedom per node \((\text{viz, } w, \partial w/\partial x, \partial w/\partial y)\) and four degree of freedom per node \((\text{viz, } w, \partial w/\partial x, \partial w/\partial y, \partial^2 w/\partial x \partial y)\), respectively, were used. They have applied this technique for two different mesh size (Quarter of the plate were analyzed) namely \(2 \times 2\) and \(4 \times 4\). The first five frequencies of the corner supported square plate were given and also the variation of the fundamental frequency with the point location.

In 1971 [14] Damle, S.K. and Fesser, L. J., used finite element method to find the fundamental frequencies and mode shapes for an elastic thin square plate supported symmetrically at four points on the diagonals of the plate. They have compared their results with Tso, W.K. (1966) [8] which were based on numerical procedures and also experimental results. All the three method (F.E. & Rayleigh Ritz & Experiment seemed to agree with good accuracy up to a value of \(\alpha = 0.32\).

In 1972 [15] Petyt, M. and Mirza, W.H., used finite element displacement method of analysis to determine the vibration characteristics of floor slabs resting on four column supports. They have compared their results with other theoretical solutions and also experimental measurements. They also investigated the effect of rigidity and finite area of the column supports. Finally, they have considered the vibration characteristics of various arrangements of slabs on many supports. They were the first to discuss and investigate the effect of joint rigidity. The finite area of the columns, assumed square represented by the four cor-
ner elements.

In 1972 [16] Johns, D.J. and Nataraja, R. presented an alternative analysis of the vibration of a square plate symmetrically supported at four points, and discussion of the corresponding results were given. They have re-examined their previous results. \{ 1969 [10] Johns, D.J. and Nagaraj, V.T. \} and corrected their previous assumption that the fundamental mode shape were symmetric about the diagonals for all diagonal supports points.

In 1973 [17] Venkateswara Rao, G., Raju, I.S. and Amba-Rao, C.L. tried to get a very accurate upper bound for the first few frequencies by using Finite Element Method. They have used double precision arithmetic. For the first three natural frequency of corner supported square plate, Finite Element Method seemed to give upper bound and Finite Difference gave lower bound.

In 1974 [20] Sadasiva Rao, Y.V.K., Venkateswara Rao, G. and Amba Rao, C.L. conducted an experimental study of vibration of a four point supported square plate. An aluminium plate of dimensions 555 by 555 by 1.5 mm (β = 1/370) thick is used to obtain the fundamental frequencies at different locations of the supports along the diagonal. The test setup consisted of a thick horizontal rectangular bottom plate at the corners of which four vertical channels of mild steel of very high stiffness were welded. Two thick steel beams with central longitudinal grooves were welded to each other so as to form a frame of plan form "X". The specimen was supported at four points by means of two screws for each support point. The plate was excited by means of an electromagnetic shaker at the center for all support points except center support where the excitation was in another arbitrary location. They have compared their results

In 1974 [21] Leuner, T.R. performed an experimental and theoretical study to investigate the effect of varying the stiffness of corner elastic point supports on plate vibration. An experiment was conducted in which the bending stiffness of horizontal beams was used to support a square plate at its four corners. He observed that the stiffness of these supports can be varied over such a range that the plate fundamental frequency was lowered up to 40 percentage.

In 1974 [22] Dowell, E.H. commented on the disagreement of the comparison between the results obtained by Venkateswara, G., Raju, I.S. and Amba. Rao, C.L. (1973) [17] and the previous results obtained by Dowell, E.H. (1971) [12] for some certain range at support location. He pointed out that the difference was due to the fact that the lowest frequency mode changes with support positions.

In 1975 [24] Srinivasan, R.S. and Munaswamy, K. were the first to study Finite Strip with a non-uniform support conditions along the edges. They have accomplished the free vibration analysis of skew orthotropic plate with point support by using higher degree skew finite strip. They have derived the expressions of strain energy and kinetic energy by applying small deflection theory. The displacement function for the strip was assumed as a series with polynomials in one direction and beam function in the other direction. Their results can be compared with others for the case of free vibration of square plate symmetrically supported along the diagonals when the skew angle is 0 degree.
In 1975 [25] Venkateswara, Rao, G., Amba Rao, C.L. and Murthy, T.V.G. commented on the comments given by Dowell, E.H.(1974) [22] where Dowell, E.H. stated that the frequency of point supported square plate with $\alpha = 0.2$ was the fundamental frequency. They have emphasized (at the risk of annoying the reader) that 'fundamental' implies the lowest.

In 1977 [29] Leissa, A.W. summarized all known results for the vibration frequencies, mode shapes and nodal patterns. A thorough search revealed approximately 500 references. In this paper which is part I of a two-part review of literature published over the period 1973-1976 that deals with free, undamped vibrations of plate. This part is limited to problems governed by the classical theory of plates.

In 1978 [32] Leissa, A.W., published part II of his search of sources related to plate vibration. It dealt with complicating effects of free, undamped vibration of plates that appeared from 1973-1975 and in part of 1976. Recent research dealing with the complicating effects of anisotropy, inplane forces, variable thickness, surrounding media, large deflections, shear deformation, rotary inertia, and non-homogeneity were summerized. For the case of the vibration of plates including the shear deformation rotary inertia, none of the reviewed paper applied it to point supported rectangular or square plate.

In 1979 [33] Kerstens, J.G.M., described a method for establishing the natural frequencies of a rectangular plate supported at an arbitrary number of points and locations. The method is based on some extensions to the intermediate problem technique of Aronszjan and Weinstein (1941,1961,1972) through the use of finite sets of constraints which is called "Modal Constraint method". The
merit of this method lies in the fact that the eigen values and eigen functions of a completely free vibrating rectangular plate are used as the reference structure and the boundary conditions stemming from the point support are imposed on that reference structure. The modifications associated with the point supports are taken into account by Lagrangian generalized forces of constraint acting on the reference structure. In this paper, Kerstens, tabulated the results for free plate by the mode type, i.e. Antisymmetric-Antisymmetric, Symmetric-Antisymmetric, and Symmetric-Symmetric.

In 1979 [34] Gorman, D.J. developed a Levy type solution for the vibratory response of simply supported rectangular plate subjected to a harmonic force distributed along the diagonal. He extended the solution to determine the free vibration response of the same rectangular plate with inelastic lateral support on the diagonal.

In 1980 [35] Gorman, D.J., introduced a highly accurate mathematical technique for establishing the free vibration eigenvalues and mode shapes of rectangular plate with symmetrically distributed point supports along the edges. The method is based on the principle of superposition. An exact delineation is made between those modes which are fully symmetric, fully antisymmetric or symmetric-antisymmetric with respect to the plate central axes.

In 1981 [38] Gorman, D.J. expanded and continued his previous work (1980) [35] to include the free vibration eigenvalues and mode shapes of rectangular plate with symmetrically distributed point support along the diagonal. The first four frequencies in each mode of fully symmetric, fully antisymmetric and symmetric-antisymmetric were presented. The range of the point support
covers from \( \alpha = 0.0 \) (corner support) up to \( \alpha = 0.4 \) with increment of 0.05.

In 1981 [39] Leissa, A.W. summarized the researches in free, transverse vibrations of plate in the period of 1976-1980. This part covers the classical theory of plates, i.e. homogeneous, isotropic, thin, constant thickness, no inplane initial forces, small transverse displacement, vibrating in a vacuum, etc.

Later in 1981 [40] Leissa, A.W. published the second part of his search of all the researches conducted during 1976-1980 for the plate vibration considering the complicating effects. Complicating effects are those which directly affect the governing differential equation of motion of a plate. Transverse shear deformation and rotary inertia are among those effects. None of the reviewed papers in that research revealed the titled problem (free vibration of corner and point support plate) including the effect of transverse shear deformation and rotary inertia since all the theoretical studies were based on the classical theory of plates.

In 1983 [43] Raju, I.S. and Amba-Rao, C.L. presented the first six symmetric-symmetric, antisymmetric-symmetric and antisymmetric-antisymmetric natural frequencies for a square plate resting on four point supports along the diagonals as obtained by using finite element analysis. They have compared their result with Gorman, D.J. (1981) [38] and explained the discrepancies. Quarter of the plate were analyzed with Poisson’s ratio \( \nu = 1/3 \).

In 1984 [45] Narita, Y. and Leissa, A.W. as a part of their research of vibration of corner supported shallow shells of rectangular plane form, determined the frequency parameter for corner supported flat plate having square planeform with \( (\nu = 0.3) \). They have used Ritz method, with algebraic
polynomials forming the set of trail functions.

In 1984 [46] Fan, S.C. and Cheung, Y.K. introduced the spline finite strip method and applied it to the study of flexural free vibration response of thin rectangular plates with complex support conditions. The natural frequencies parameter for the square plate with corner supports were given.

In 1984 [48] Narita, Y. used Ritz method, with a trial function expressed in terms of double power series to solve the problem of the free vibrations of point supported rectangular plates. The constraint conditions of the support were taken into account by Lagrange multipliers. The method applied for orthotropic plate and as a special case for isotropic plate (i.e. the flexural rigidity in the X-direction equals the flexural rigidity in the Y-direction; \( D_x = D_y \)).

In 1986 [54] Aksu, G. described a method of determining the dynamic characteristics of a four-point supported square plate with free edge using sweep sine-wave testing. The idea behind this method is to make use of free vibration time-response data such as acceleration to determine the natural frequencies and the associated mode shapes. Detailed experimental results have been obtained for various support locations lying at specified positions along the plate diagonals, to compare with numerical results obtained by others. It was found that experimental results were generally in reasonable agreement with numerical ones. A steel plate of dimensions 250 by 250 by 1.5 mm thick was used (\( \beta = 0.006 \)).

In 1987 [56] Mizusawa, T. used spline element method to deal with the vibration of skew plates resting on point supports. It can be regarded as an alternative form of the displacement formulation of the finite element procedure.
in that the minimum total potential energy theorem is used to develop the relationship between the unknown parameters and the applied loading. As a special case of skew plate, (skew angle $\varphi = 0$), the results of corner supported square plate can be deduced.

In 1987, Leissa, A.W. presented his third search of all recent studies in the period of 1981-1985 on both classical theory [57] and complicating effect [58] of vibration of plate. As his search was extensive, it is clear up to the knowledge of the author that no study has been done on the vibration of a moderately thick isotropic square plate point supported along the diagonals that included the effect of transverse shear deformation and rotary inertia.
CHAPTER 3

THEORY

Theoretical studies based upon classical theory of plates are limited to those solving the governing differential equation of motion of thin, elastic, homogeneous and isotropic plate is given by the following equation:

\[ D V^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \]  \hspace{1cm} (3.1)

Where,

- \( D \) : Flexural rigidity
- \( V \) : Differential operator
- \( w \) : Transverse displacement
- \( \rho \) : Density of plate
- \( h \) : Plate thickness

To include the effect of transverse shear deformation and rotary inertia, Mindlin plate theory is used. This theory is well explained in Timoshenko [63]. It was established by Mindlin, R.D. [4] who published the first paper about the influence of rotary inertia and shear on flexural motions of isotropic elastic plates.

The derivation of the equation of motion of plate including the transverse shear deformation and rotary inertia will be summarized and presented.
GENERAL ASSUMPTIONS

The present study will be mainly based on the following assumptions:

1. The material of the plate is elastic, homogeneous, and isotropic.
2. The plate is initially flat (i.e. no initial stresses).
3. The deformation is initially straight line normal to the middle surface, and remain straight line but no longer normal to the middle surface. This means that the deformation due to transverse shear will be considered.
4. The deflections are small compared to the plate thickness. i.e high amplitude vibration will not be included.

3.1 STRESS-STRAIN RELATION:

Figure 3.1 shows a cross section element of a uniform plate before and after deflection. Starting in the usual way by proposing displacement field and then deleting stretching effects,

\[ u_1 = u(x,y,z,t) = z\Phi_x(x,y,t) = z\Phi_x \]
\[ u_2 = v(x,y,z,t) = z\Phi_y(x,y,t) = z\Phi_y \]
\[ u_3 = w(x,y,z,t) = \bar{w}(x,y,t) = w \] \hspace{1cm} (3.2)

where \( \Phi_x \) and \( \Phi_y \) are body element rotation angles in \( x \) and \( y \) directions respectively. They can be expressed as :

\[ \Phi_x(x,y,t) = -\left\{ \frac{\partial w}{\partial x} + \gamma_x(x,y,t) \right\} \] \hspace{1cm} (3.3)
\[ \Phi(x,y,t) = - \left\{ \frac{\partial w}{\partial y} + \gamma_y(x,y,t) \right\} \] (3.4)

where \( \gamma_x \) and \( \gamma_y \) are angles of distortion due to shear. The strain field for the assumed displacement follows directly as:

\[ \varepsilon_{xx} = z \frac{\partial \varphi_x}{\partial x} \]

\[ \varepsilon_{yy} = z \frac{\partial \varphi_y}{\partial y} \]

\[ \varepsilon_{zz} = 0 \]

\[ \varepsilon_{xy} = \frac{1}{2} z \left\{ \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right\} \]

\[ \varepsilon_{yx} = \frac{1}{2} \left\{ \varphi_y + \frac{\partial w}{\partial y} \right\} \]

\[ \varepsilon_{zx} = \frac{1}{2} \left\{ \varphi_x + \frac{\partial w}{\partial x} \right\} \] (3.5)

and the unit shearing strains \( \gamma_{xy} \), \( \gamma_{xz} \), and \( \gamma_{yz} \)

\[ \gamma_{xy} = z \left\{ \frac{\partial \varphi_y}{\partial x} + \frac{\partial \varphi_x}{\partial y} \right\} \]

\[ \gamma_{yz} = - \gamma_y = \varphi_y + \frac{\partial w}{\partial y} \]

\[ \gamma_{zx} = - \gamma_x = \varphi_x + \frac{\partial w}{\partial x} \] (3.6)
Since assumed plane stress distribution employed, i.e the material is elastic and isotropic, the use of the two dimensional Hooke's law is permitted.

\[
\sigma_x = E\varepsilon_{xx} + \nu\sigma_y
\]

\[
\sigma_y = E\varepsilon_{yy} + \nu\sigma_x
\]

(3.7)

where \(E\) is modulus of elasticity and \(\nu\) is Poisson's ratio, which relates stress and strain in a plane element. Now using Eq.(3.7) in combination with Eq.(3.5) leads to:

\[
\sigma_x = \frac{Ez}{1-\nu^2}\left\{\frac{\partial \varepsilon_x}{\partial x} + \nu \frac{\partial \varepsilon_y}{\partial y}\right\}
\]

\[
\sigma_y = \frac{Ez}{1-\nu^2}\left\{\frac{\partial \varepsilon_y}{\partial y} + \nu \frac{\partial \varepsilon_x}{\partial x}\right\}
\]

\[
\sigma_z = 0
\]

(3.8)

also the shear stress \(\tau\) and shear strain \(\gamma\) can be related by Hooke's law

\[
\tau = G\gamma
\]

(3.9)

The shear stress can be obtained by combination of Eq.(3.9) and Eq.(3.6)

\[
\tau_{xy} = Gz\left\{\frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y}\right\}
\]

\[
\tau_{xz} = G\left\{\varepsilon_x + \frac{\partial w}{\partial x}\right\}
\]

\[
\tau_{yz} = G\left\{\varepsilon_y + \frac{\partial w}{\partial y}\right\}
\]

(3.10)
Now the resultant intensity function can be obtained using Eq.(3.8) and Eq.(3.10) in combination with the following equation:

\[
M = \int_{-h/2}^{h/2} t z \, dz
\]  
\[
(3.11)
\]

where \(h\) is the plate thickness.

Thus for \(M_x\):

\[
M_x = \int_{-h/2}^{h/2} \tau_{xx} z \, dz
\]

\[
= 2 \int_0^{h/2} \sigma_x z \, dz
\]

\[
= \frac{2E}{(1-v^2)} \int_0^{h/2} z^2 \left\{ \frac{\partial \phi_x}{\partial x} + v \frac{\partial \phi_y}{\partial y} \right\} \, dz
\]

\[
= \frac{E h^3}{12(1-v^2)} \left\{ \frac{\partial \phi_x}{\partial x} + v \frac{\partial \phi_y}{\partial y} \right\}
\]

\[
= D \left\{ \frac{\partial \phi_x}{\partial x} + v \frac{\partial \phi_y}{\partial y} \right\}
\]  
\[
(3.12)
\]

where \(D\) is the flexural rigidity defined as:

\[
D = \frac{E h^3}{12(1-v^2)}
\]  
\[
(3.13)
\]

Following the same procedure,
\[ M_y = \frac{E h^3}{12(1-\nu^2)} \left\{ \frac{\partial \phi_y}{\partial y} + \nu \frac{\partial \phi_x}{\partial x} \right\} \]  

(3.14)

and,

\[ M_{xy} = \frac{G h^3}{12} \left\{ \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_x}{\partial y} \right\} \]  

(3.15)

where \( G \) is the shear modulus and defined as:

\[ G = \frac{E}{2(1+\nu)} \]  

(3.16)

Now we consider the resultant shear force intensity function, for \( Q_x \)

\[ Q_x = h^2 \int_{-h/2}^{h/2} \tau_{xz} \, dz = \tau_{xz} \cdot h \]

Since there is an error stemming from Eq.(3.5) that shear strain \( \varepsilon_{xz} \) is constant over the thickness of the plate (i.e. \( \varepsilon_{xz} \neq \varepsilon_{xz}(z) \)), a correction factor should be employed to overcome this error. This factor \( k \) is called shear coefficient \( \{ \text{usually taken as} \quad \pi^2/12 \} \).

Hence,

\[ Q_x = k \tau_{xz} \cdot h \]  

(3.17)

Now combining Eq.(3.13) and Eq.(3.10), we have for \( Q_x \):

\[ Q_x = k h G \left\{ \Phi_x + \frac{\partial w}{\partial x} \right\} \]  

(3.18)
Similarly we get for $Q_y$:

$$Q_y = khG\left\{\Phi_y + \frac{\partial w}{\partial y}\right\}$$  \hspace{1cm} (3.19)

Figure 3.2 represent a three-dimensional element cut out of an isotropic elastic rectangular plate showing the stress distribution. In Figure 3.3 the shear forces are shown for the same element.
Figure 3.1: Section before and after deflection
Figure 3.2: Stresses on a plate element

Figure 3.3: Shear forces and moments
3.2 STRAIN ENERGY:

The strain energy function per unit volume, in the three dimensional theory is given by:

\[ U_v = \sigma_x \varepsilon_{xx} + \sigma_y \varepsilon_{yy} + \sigma_z \varepsilon_{zz} + 2\tau_{xy} \varepsilon_{xy} + 2\tau_{yz} \varepsilon_{yz} + 2\tau_{zx} \varepsilon_{zx} \]  \hspace{1cm} (3.20)

Since plane stresses are assumed (i.e. dropping \( \sigma_z \)), and employing Hooke's law Eq.(3.7) and Eq.(3.9) for \( \sigma_x, \sigma_y, \) and \( \tau_{xy}, \) Eq.(3.20) will be:

\[ U_v = \frac{E}{(1-v^2)}(\varepsilon_{xx} + \nu \varepsilon_{yy})\varepsilon_{xx} + \frac{E}{(1-v^2)}(\varepsilon_{yy} + \nu \varepsilon_{xx})\varepsilon_{yy} \]

\[ + 4G\varepsilon_{xy}^2 + 2\tau_{xx} \varepsilon_{xx} + 2\tau_{yz} \varepsilon_{yz} \] \hspace{1cm} (3.21)

Now using Eq.(3.5) to replace the strains in Eq.(3.21) and then integrating over the thickness, the strain energy per unit area will be deduced as follows:

\[ U_A = \frac{1}{2} \frac{Eh^3}{12(1-v^2)}[\left(\frac{\partial \phi_x}{\partial x}\right)^2 + (\frac{\partial \phi_y}{\partial y})^2 + 2\nu(\frac{\partial \phi_x}{\partial x})(\frac{\partial \phi_y}{\partial y})] \]

\[ + \frac{1}{2} \frac{Gh^3}{12} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)^2 + \tau_{xx}(\phi_x + \frac{\partial w}{\partial x}) + \tau_{yz}(\phi_y + \frac{\partial w}{\partial y}) \] \hspace{1cm} (3.22)

Now, the total strain energy can be obtained by integrating Eq.(3.22) over the entire region (area),

\[ U = \frac{1}{2} \iint_R \left\{ \frac{Eh^3}{12(1-v^2)}[\left(\frac{\partial \phi_x}{\partial x}\right)^2 + (\frac{\partial \phi_y}{\partial y})^2 + 2\nu(\frac{\partial \phi_x}{\partial x})(\frac{\partial \phi_y}{\partial y})] \right\} dA \] \hspace{1cm} (3.23)
Now, introducing D and replacing $\tau_{xx}$ and $\tau_{yx}$ and by using Hooke's Law and Eq.(3.5), the total strain energy will be:

$$\begin{align*}
U &= \frac{1}{2} \int_{\Omega} \left\{ D \left[ \left( \frac{\partial \phi_x}{\partial x} \right)^2 + \left( \frac{\partial \phi_y}{\partial y} \right)^2 + 2\nu \left( \frac{\partial \phi_x}{\partial x} \right) \left( \frac{\partial \phi_y}{\partial y} \right) \right] \\
&\quad + \frac{G h^3}{12} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right)^2 \\
&\quad + k h G \left[ \left( \frac{\partial w}{\partial x} + \phi_x \right)^2 + \left( \frac{\partial w}{\partial y} + \phi_y \right)^2 \right] \right\} \, dA
\end{align*}$$

(3.24)

### 3.3 KINETIC ENERGY:

The kinetic energy per unit volume, according to the general linear theory is given by the following equation:

$$T_v = \frac{\rho}{2} \left\{ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right\}$$

(3.25)

using Eq.(3.2) with the above equation

$$T_v = \frac{\rho}{2} \left\{ z^2 \left( \frac{\partial \phi_x}{\partial t} \right)^2 + z^2 \left( \frac{\partial \phi_y}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right\}$$

(3.26)

integrating Eq.(3.26) with respect to the plate thickness, it becomes:

$$T_a = \int_{-h/2}^{h/2} \frac{\rho}{2} \left\{ z^2 \left( \frac{\partial \phi_x}{\partial t} \right)^2 + z^2 \left( \frac{\partial \phi_y}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right\} \, dz$$

$$= \frac{1}{2} \frac{\rho h^3}{12} \left\{ \left( \frac{\partial \phi_x}{\partial t} \right)^2 + \left( \frac{\partial \phi_y}{\partial t} \right)^2 \right\} + \frac{\rho h}{2} \left( \frac{\partial w}{\partial t} \right)^2$$

(3.27)
Hence, the total kinetic energy will be obtained by integrating Eq.(3.27) (Kinetic energy per unit length) over the whole region (area).

\[ T = \frac{1}{2} \int \int \left\{ \frac{\rho h^3}{12} \left[ \left( \frac{\partial \Phi_x}{\partial t} \right)^2 + \left( \frac{\partial \Phi_y}{\partial t} \right)^2 \right] + \rho h \left( \frac{\partial w}{\partial t} \right)^2 \right\} \ dA \]  

(3.28)

3.4 EQUATIONS OF MOTION:

Equations of motion can be obtained by applying the principle of virtual work to the equation of the total kinetic energy Eq.(3.23) and the total strain energy Eq.(3.28).

First, for the kinetic energy:

\[ \delta T = \frac{1}{2} \int \int_R \left\{ \frac{\rho h^3}{12} \left\{ 2 \frac{\partial \Phi_x}{\partial t} \frac{\partial \delta \Phi_x}{\partial t} + 2 \frac{\partial \Phi_y}{\partial t} \frac{\partial \delta \Phi_y}{\partial t} \right\} + 2 \rho h \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} \right\} \ dA \]  

(3.29)

Also, for the strain energy:

\[ \delta U = \frac{1}{2} \int \int \left\{ D \left[ 2 \frac{\partial \Phi_x}{\partial x} \frac{\partial \delta \Phi_x}{\partial x} + 2 \frac{\partial \Phi_y}{\partial y} \frac{\partial \delta \Phi_y}{\partial y} + 2v \frac{\partial \Phi_x}{\partial x} \frac{\partial \delta \Phi_y}{\partial y} + 2v \frac{\partial \Phi_y}{\partial y} \frac{\partial \delta \Phi_x}{\partial x} \right] + 2 \frac{\partial \Phi_x}{\partial y} \frac{\partial \delta \Phi_y}{\partial x} \right\} \]

\[ + \frac{G h^3}{12} \left[ \left( \frac{\partial \Phi_x}{\partial y} + \frac{\partial \Phi_y}{\partial x} \right) \left( \frac{\partial \delta \Phi_x}{\partial y} + \frac{\partial \delta \Phi_y}{\partial x} \right) \right] \]
\[ + khG \left[ 2 \left( \frac{\partial w}{\partial x} + \phi_x \right) \left( \frac{\partial \delta w}{\partial x} + \delta \phi_x \right) \right. \\
\left. + 2 \left( \frac{\partial w}{\partial y} + \phi_y \right) \left( \frac{\partial \delta w}{\partial y} + \delta \phi_y \right) \right] \, dA \]  \tag{3.30}

Now introducing Eq.(3.29) and Eq.(3.30) in the variational principle:

\[ \int_{t_1}^{t_2} (\delta T - \delta \pi) \, dt = 0 \]  \tag{3.31}

where \( \pi \) is the summation of the total strain and potential energy (i.e. \( \pi = U + V \)) and since there is no external load (or force), (i.e \( V = \iint_R wq \, dA = 0 \)) and hence \( \pi = U \), and Eq.(3.31) will be:

\[ \int_{t_1}^{t_2} \left\{ \left[ - \rho h^3 \left( \frac{\partial \phi_x}{\partial t} \frac{\partial \delta \phi_x}{\partial t} + \frac{\partial \phi_x}{\partial t} \frac{\partial \delta \phi_x}{\partial t} \right) \right. \right. \]
\[ - \left. \left. \left. \left[ D \left\{ \frac{\partial \phi_x}{\partial x} \frac{\partial \delta \phi_x}{\partial x} + \frac{\partial \phi_x}{\partial y} \frac{\partial \delta \phi_y}{\partial y} + v \frac{\partial \phi_x}{\partial x} \frac{\partial \delta \phi_y}{\partial y} + v \frac{\partial \phi_y}{\partial y} \frac{\partial \delta \phi_x}{\partial x} \right\} \right. \right. \right. \]
\[ + \frac{G}{12} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \left( \frac{\partial \delta \phi_x}{\partial y} + \frac{\partial \delta \phi_y}{\partial x} \right) \]
\[ + khG \left( \frac{\partial w}{\partial x} + \phi_x \right) \left( \frac{\partial \delta w}{\partial x} + \delta \phi_x \right) \]
\[ + \left( \frac{\partial w}{\partial y} + \phi_y \right) \left( \frac{\partial \delta w}{\partial y} + \delta \phi_y \right) \right] \, dA \, dt = 0 \]  \tag{3.32}

Now, integrating by parts with respect to time for the first three terms keeping in mind that all variations at the limits (i.e at \( t = t_1 \) and \( t = t_2 \) are equal
\[
\iint_R G \frac{\partial H}{\partial x} \, dx \, dy = \iint_R H \frac{\partial G}{\partial x} \, dx \, dy + \oint_R GH \, dy
\]
and,
\[
\iint_R G \frac{\partial H}{\partial y} \, dx \, dy = -\iint_R H \frac{\partial G}{\partial x} \, dx \, dy - \oint_R GH \, dx
\]  \hspace{1cm} (3.33)

Now, Eq.(3.32) will have the form:

\[
\iint_R \left\{ - \left[ \frac{\rho h^3}{12} \frac{\partial^2 \phi_x}{\partial t^2} \delta \phi_x + \frac{\rho h^3}{12} \frac{\partial^2 \phi_y}{\partial t^2} \delta \phi_y + \rho h \frac{\partial^2 w}{\partial t^2} \delta w \right] + D \left[ \frac{\partial^2 \phi_x}{\partial x^2} \delta \phi_x + \frac{\partial^2 \phi_y}{\partial y^2} \delta \phi_y + v \frac{\partial^2 \phi_x}{\partial x \partial y} \delta \phi_y + v \frac{\partial^2 \phi_y}{\partial x \partial y} \delta \phi_x \right] + \frac{\rho h}{12} \left[ \frac{\partial^2 \phi_x}{\partial y \partial x} \delta \phi_x + \frac{\partial^2 \phi_y}{\partial x \partial y} \delta \phi_y + \frac{\partial^2 \phi_x}{\partial x^2} \delta \phi_x + \frac{\partial^2 \phi_y}{\partial x^2} \delta \phi_y \right] + KGH \left[ \frac{\partial^2 w}{\partial x^2} \delta w + \frac{\partial w}{\partial x} \delta \phi_x - \frac{\partial \phi_x}{\partial x} \delta w - \phi_x \delta \phi_x \right. \\
\left. + \frac{\partial^2 w}{\partial y^2} \delta w + \frac{\partial w}{\partial y} \delta \phi_y - \frac{\partial \phi_y}{\partial y} \delta w - \phi_y \delta \phi_y \right] \right\} \, dA \, dt
\]

\[
+ \oint_{\partial R} \left\{ - D \left[ \frac{\partial \phi_x}{\partial x} \delta \phi_x \, dy - \frac{\partial \phi_y}{\partial y} \delta \phi_y \, dx - v \frac{\partial \phi_x}{\partial x} \, dx + v \frac{\partial \phi_y}{\partial y} \delta \phi_y \, dy \right] - \frac{\rho h^3}{12} \left[ - \frac{\partial \phi_x}{\partial y} \delta \phi_x \, dy + \frac{\partial \phi_y}{\partial y} \delta \phi_y \, dx - \frac{\partial \phi_x}{\partial x} \delta \phi_x \, dx + \frac{\partial \phi_y}{\partial x} \delta \phi_y \, dy \right] \right\}
\]
\[ - \text{K} \text{G} \text{h} \left[ \frac{\partial w}{\partial x} \delta wdy - \varphi_x \delta wdy - \frac{\partial w}{\partial y} \delta wdx + \varphi_y \delta wdx \right] \right\} \right] dt = 0 \quad (3.34) \]

Collecting terms in the above formulation with respect to variation terms, we have:

\[
\int_{R}^{b} \left\{ \left[ - \frac{\rho h^3}{12} \frac{\partial^2 \varphi_x}{\partial t^2} + D \left( \frac{\partial^2 \varphi_x}{\partial x^2} + v \frac{\partial^2 \varphi_y}{\partial x \partial y} \right) + \frac{\text{G} h^3}{12} \left( \frac{\partial^2 \varphi_x}{\partial y^2} + \frac{\partial^2 \varphi_y}{\partial x \partial y} \right) \right. \\
+ \text{K} \text{G} \text{h} \left( \frac{\partial w}{\partial x} - \varphi_x \right) \delta \varphi_x \\
+ \left[ - \frac{\rho h^3}{12} \frac{\partial^2 \varphi_x}{\partial t^2} + D \left( \frac{\partial^2 \varphi_x}{\partial x^2} + v \frac{\partial^2 \varphi_y}{\partial x \partial y} \right) + \frac{\text{G} h^3}{12} \left( \frac{\partial^2 \varphi_x}{\partial y^2} + \frac{\partial^2 \varphi_y}{\partial x \partial y} \right) \right. \\
+ \text{K} \text{G} \text{h} \left( \frac{\partial w}{\partial y} - \varphi_y \right) \delta \varphi_y \\
+ \left[ - \rho \frac{\partial w}{\partial t} + \text{K} \text{G} \text{h} \left( \frac{\partial w}{\partial x} - \frac{\partial \varphi_x}{\partial x} + \frac{\partial w}{\partial y} - \frac{\partial \varphi_y}{\partial y} \right) \right] \delta w \} \right\} dA \ dt \\
+ \int_{R}^{b} \left\{ \left[ - D \left( \frac{\partial \varphi_x}{\partial x} + v \frac{\partial \varphi_y}{\partial y} \right) dy + \frac{\text{G} h^3}{12} \left( \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right) dx \right] \delta \varphi_x \\
+ \left[ D \left( \frac{\partial \varphi_x}{\partial y} + v \frac{\partial \varphi_y}{\partial x} \right) dx - \frac{\text{G} h^3}{12} \left( \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right) dy \right] \delta \varphi_y \\
- \text{K} \text{G} \text{h} \left[ \frac{\partial w}{\partial x} dy - \varphi_x dy - \frac{\partial w}{\partial y} dx + \varphi_y dx \right] \delta w \} \right\} dt = 0 \quad (3.35) \]

In accordance with previous remarks it is clear that each of the coefficients in the integral \( \int_{R}^{b} \) for the variations must be zero.
We consider the coefficient for $\delta\varphi_x$ now. We have then:

\[-\frac{\rho h^3}{12} \frac{\partial^2 \varphi_x}{\partial t^2} + D \left( \frac{\partial^2 \varphi_x}{\partial x^2} + v \frac{\partial^2 \varphi_y}{\partial x \partial y} \right) + \frac{G h^3}{12} \left( \frac{\partial^2 \varphi_x}{\partial y^2} + \frac{\partial^2 \varphi_y}{\partial x \partial y} \right) \]

\[+ K Gh (\frac{\partial w}{\partial x} - \varphi_x) = 0 \tag{3.36} \]

and similarly for $\delta \varphi_x$ and $\delta w$, we have respectively,

\[-\frac{\rho h^3}{12} \frac{\partial^2 \varphi_y}{\partial t^2} + D \left( \frac{\partial^2 \varphi_y}{\partial x^2} + v \frac{\partial^2 \varphi_x}{\partial x \partial y} \right) + \frac{G h^3}{12} \left( \frac{\partial^2 \varphi_y}{\partial x^2} + \frac{\partial^2 \varphi_x}{\partial x \partial y} \right) \]

\[+ K Gh (\frac{\partial w}{\partial y} - \varphi_y) = 0 \tag{3.37} \]

and

\[-\rho h \frac{\partial^2 w}{\partial t^2} + K Gh (\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi_x}{\partial x} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial \varphi_y}{\partial y}) = 0 \tag{3.38} \]

Eqs. (3.36), (3.37) and (3.38) are the equations of motion.

Noting that $D = \frac{E h^3}{12(1-\nu^2)}$ and $G = \frac{E}{2(1+\nu)}$, we may replace the term $\frac{G h^3}{12}$

by $\frac{D}{2(1-\nu)}$ in the above formulation.

Hence we have the equation of motion in these forms:

\[D \left\{ \frac{\partial^2 \varphi_x}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 \varphi_x}{\partial y^2} + \frac{1+\nu}{2} \frac{\partial^2 \varphi_y}{\partial x \partial y} \right\} \]
\[ + \text{K} \text{G} \text{h} \left( \frac{\partial w}{\partial x} - \varphi_x \right) - \frac{\rho h^3}{12} \frac{\partial^2 \varphi_x}{\partial t^2} = 0 \] (3.39)

\[ \text{D} \left\{ \frac{\partial^2 \varphi_x}{\partial y^2} + \frac{1-v}{2} \frac{\partial^2 \varphi_x}{\partial x^2} + \frac{1+v}{2} \frac{\partial^2 \varphi_x}{\partial x \partial y} \right\} + \text{K} \text{G} \text{h} \left( \frac{\partial w}{\partial y} - \varphi_y \right) - \frac{\rho h^3}{12} \frac{\partial^2 \varphi_y}{\partial t^2} = 0 \] (3.40)

\[ - \text{K} \text{G} \text{h} \left\{ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial \varphi_x}{\partial x} - \frac{\partial \varphi_y}{\partial y} \right\} + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \] (3.41)

Now, the equations of motion are given in Eqs. (3.39), (3.40) and (3.41) and by eliminating \( \varphi_x \) and \( \varphi_y \) we can have the equation of motion in terms of only the deflection of the middle plane (Dynamic Equilibrium Equation of Motion):

\[ \left\{ \nabla^2 - \frac{\rho}{K \text{G}} \frac{\partial^2}{\partial t^2} \right\} \left\{ \text{D} \nabla^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \right\} w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \] (3.42)

If only the rotary inertia is deleted from Equation 3.42 we set \( \frac{\rho h^3}{12} \frac{\partial^2 w}{\partial t^2} = 0 \)

We have:

\[ \text{D} \left\{ \nabla^2 - \frac{\rho}{K \text{G}} \frac{\partial^2}{\partial t^2} \right\} \nabla^2 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \] (3.43)

Also, if only the effect of transverse shear deformation is deleted by setting \( 1/k \text{G} \rightarrow 0 \) we have:
\[ \left\{ D V^2 - \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \right\} V^2 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \]  \( (3.44) \)

Finally, if both of rotary inertia and transverse shear deformation are to be deleted we can have the classical plate equation (see Equation 3.1)

\[ D V^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \]  \( (3.45) \)

### 3.5 Boundary Conditions:

Going back to the line integral in Eq.(3.35) and by arranging the terms we have:

\[ \int_0^l \int_0^b \{ \delta \phi_x dy \left[ - D \left( \frac{\partial \phi_x}{\partial x} + v \frac{\partial \phi_y}{\partial y} \right) \right] - \delta \phi_x dx \left[ - \frac{G h^3}{12} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \right] \]

\[ + \delta \phi_x dy \left[ - \frac{G h^3}{12} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \right] - \delta \phi_y dx \left[ - D \left( \frac{\partial \phi_y}{\partial y} + v \frac{\partial \phi_x}{\partial x} \right) \right] \]

\[ + \delta w dy \left[ - K G h \left( \frac{\partial w}{\partial x} - \phi_x \right) \right] - \delta w dx \left[ - K G h \left( \frac{\partial w}{\partial x} - \phi_y \right) \right] \} dt = 0 \]  \( (3.46) \)

Now using Eqs. (3.12), (3.14), (3.15), (3.18) and (3.19) we have:

\[ \int_0^l \int_0^b \{ M_x \delta \phi_x dy - M_y \delta \phi_y dx + M_{xy} \delta \phi_y dx - M_{xy} \delta \phi_x dx \]

\[ - Q_x \delta w dy + Q_y \delta w dx \} dt = 0 \]  \( (3.47) \)
Introducing transformation coordinate in accordance with Figure 3.4, we can replace \((dx)\) by \(-a_{vy} ds\) and \((dy)\) by \(a_{vx} ds\) to get:

\[
\int_{\Gamma} \left\{ - M_x a_{vx} \delta \varphi_x + M_y a_{vy} \delta \varphi_y + M_{xy} a_{vx} + M_{xy} a_{vy} \delta \varphi_x - Q_x a_{vx} \delta w - Q_y a_{vy} \delta w \right\} ds \, dt = 0
\]

(3.48)

Using \(v\) and \(s\) as coordinates we can state:

\[
\begin{align*}
\varphi_x &= a_{vx} \varphi_v - a_{vy} \varphi_s \\
\varphi_y &= a_{vy} \varphi_v - a_{vx} \varphi_s
\end{align*}
\]

(3.49)

Solving for \(\varphi_v\) and \(\varphi_s\) from Eq.(3.49), we have:

\[
\begin{align*}
\varphi_v &= a_{vx} \varphi_x + a_{vy} \varphi_y \\
\varphi_s &= a_{vx} \varphi_y - a_{vy} \varphi_x
\end{align*}
\]

(3.50)

Now, substituting in Eq.(3.48) and collecting terms:

\[
\int_{\Gamma} \delta \left\{ \left[ \frac{a_{vx}^2}{a_{vy}^2} M_x + 2a_{vx}a_{vy}M_{xy} + a_{vy}^2 M_y \right] \delta \varphi_v \\
+ \left[ - a_{vx} a_{vy} M_x + a_{vx} a_{vy} M_y + (a_{vx}^2 - a_{vy}^2) M_{xy} \right] \delta \varphi_s \\
- \left[ a_{vx} Q_x + a_{vy} Q_y \right] \delta w \right\} ds \, dt = 0
\]

(3.51)

Now, using the following suitable moment transformation equation (described in details in reference [59]):
\[ M_v = a_v^2 M_x + 2a_v a_y M_{xy} + a_y^2 M_y \]

\[ M_{vs} = a_v a_y (M_y - M_x) + (a_v^2 - a_y^2) M_{xy} \]

\[ Q_v = a_v Q_x + a_y Q_y \quad (3.52) \]

Hence, Eq.(3.51) will be as follows:

\[ \int_{t_1}^{t_2} \left\{ M_v \delta \varphi_v + M_{vs} \delta \varphi_s - Q_v \delta w \right\} \, ds \, dt \quad (3.53) \]

Now, we can obtain the boundary condition from Eq.(3.53):

Either \( M_v = 0 \) \quad OR \quad \varphi_v \) is specified \quad (a)

Either \( M_{vs} = 0 \) \quad OR \quad \varphi_s \) is specified \quad (b)

Either \( Q_v = 0 \) \quad OR \quad w \) is specified \quad (c) \quad (3.54)

Note that we get three boundary conditions on the edge of the plate as opposed to the two boundary conditions in classical plate theory.
\[ dy = ds \cos \alpha = ds \ a_{\nu x} \]
\[ dx = -ds \cos \beta = -ds \ a_{\nu y} \]

Figure 3.4: Transformation coordinate
CHAPTER 4

FINITE DIFFERENCE FORMULATION

The strain and kinetic energy equations previously obtained [Eqs.(3.24) and (3.28)] are expressed in dimensionless forms. Then by applying the finite difference method, they are expressed in terms of discrete displacement and rotational components. Harmonic motion is assumed to eliminate time dependence. Euler's necessary condition is applied to minimize the total energy of the system and finally an algebraic eigenvalue problem is obtained.

4.1 DIMENSIONAL ANALYSIS:

Consider a rectangular plate with dimensions shown in Figure 4.1 in which the position coordinates are x, y and z respectively. In accordance with Figure 4.1, the strain and kinetic energy expressions [Eqs.(3.24) and (3.28)] can be expressed as:

\[
U = \frac{1}{2} \iiint_\Omega \left\{ D \left[ \left( \frac{\partial \phi_x}{\partial x} \right)^2 + \left( \frac{\partial \phi_y}{\partial y} \right)^2 + 2\nu \left( \frac{\partial \phi_x}{\partial x} \right) \left( \frac{\partial \phi_y}{\partial y} \right) \right] \\
+ \frac{Gh^3}{12} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right)^2 \\
+ khG \left[ \left( \phi_x + \frac{\partial w}{\partial x} \right)^2 + \left( \phi_y + \frac{\partial w}{\partial y} \right)^2 \right] \right\} \, dy \, dx
\]

(4.1)
\[ T = \frac{1}{2} \int_{0}^{b} \left\{ \frac{\rho h}{12} \left[ (\frac{\partial \varphi}{\partial t})^2 + (\frac{\partial \varphi}{\partial t})^2 \right] + \rho h (\frac{\partial \bar{w}}{\partial t})^2 \right\} \text{dy dx} \] (4.2)

Using the following dimensional parameters in accordance with Figure 4.2 the strain and kinetic energy expressions [Eqs.(4.1) and (4.2)] can be expressed in non-dimensional forms:

\[ \zeta = \frac{x}{a}, \quad \partial x = a \partial \zeta \]

\[ \eta = \frac{y}{a}, \quad \partial y = a \partial \eta \]

\[ w = \frac{\bar{w}}{a}, \quad \partial \bar{w} = a \partial w \]

\[ \gamma = \frac{b}{a}, \quad \beta = \frac{h}{a} \] (4.3)

and,

\[ \Delta x = \frac{a}{M-1}, \quad \Delta \zeta = \frac{1}{M-1} \]

\[ \Delta y = \frac{b}{N-1}, \quad \Delta \eta = \frac{1}{N-1} \] (4.4)

where \( \gamma \) is the aspect ratio, \( \beta \) is the plate thickness/length ratio, and \( M \) and \( N \) are numbers of nodes per length and width respectively.

Noting that \( D = \frac{Eh^3}{12(1-v^2)} \), the strain energy will be:
\[
U = \frac{1}{2} \int_0^1 \left\{ \frac{Eh^3}{12(1-\nu)a^2} \left[ (\frac{\partial \varphi}{\partial \zeta})^2 + (\frac{\partial \varphi}{\partial \eta})^2 + 2\nu(\frac{\partial \varphi}{\partial \zeta})(\frac{\partial \varphi}{\partial \eta}) \right] \\
+ \frac{Gh^3}{12a^2} \left( \frac{\partial \varphi}{\partial \zeta} + \frac{\partial \varphi}{\partial \eta} \right)^2 \\
+ khG \left[ (\varphi + \frac{\partial w}{\partial \zeta})^2 + (\varphi + \frac{\partial w}{\partial \eta})^2 \right] \right\} a^2 \, d\eta \, d\zeta \quad (4.5)
\]

Noting that \( G = \frac{E}{2(1+\nu)} \) and \( \beta = \frac{h}{a} \) Eq.(4.5) will be

\[
U = \frac{1}{2} \frac{Eha^2}{(1-\nu)^2} \int_0^1 \left\{ \frac{\beta^2}{12} \left[ (\frac{\partial \varphi}{\partial \zeta})^2 + (\frac{\partial \varphi}{\partial \eta})^2 + 2\nu(\frac{\partial \varphi}{\partial \zeta})(\frac{\partial \varphi}{\partial \eta}) \right] \\
+ \frac{\beta^2}{12} \frac{1-\nu}{2} \left( \frac{\partial \varphi}{\partial \zeta} + \frac{\partial \varphi}{\partial \eta} \right)^2 \\
+ \frac{k(1-\nu)}{2} \left[ (\varphi + \frac{\partial w}{\partial \zeta})^2 + (\varphi + \frac{\partial w}{\partial \eta})^2 \right] \right\} a^2 \, d\eta \, d\zeta \quad (4.6)
\]

Similarly, for the kinetic energy expression, Eq.(4.2) will be:

\[
T = \frac{1}{2} \int_0^1 \left\{ \frac{\rho h^3}{12} \left[ (\frac{\partial \varphi}{\partial \zeta})^2 + (\frac{\partial \varphi}{\partial \eta})^2 \right] + \rho ha^2 \left( \frac{\partial w}{\partial t} \right)^2 \right\} a^2 \, d\eta \, d\zeta \quad (4.7)
\]

and by arranging,

\[
T = \frac{1}{2} \rho h^3 a^2 \int_0^1 \left\{ \frac{1}{12} \left[ (\frac{\partial \varphi}{\partial \zeta})^2 + (\frac{\partial \varphi}{\partial \eta})^2 \right] + \frac{1}{\beta^2} \left( \frac{\partial w}{\partial t} \right)^2 \right\} \, d\eta \, d\zeta \quad (4.8)
\]
Figure 4.1: Node set for rectangular plate

Figure 4.2: Node set for dimensionless rectangular plate
4.2 EULER’S EQUATION:

The integral in the strain energy expression is replaced by finite approximation summation based on the mesh covering the plate. Subsequently, with the use of standard finite difference formulas, the total strain energy of the plate can be expressed as a quadratic form in discrete displacement and rotational components:

\[ F\{\phi_{li}, \phi_{ni}, w_{ij}\} \approx U \]  \hspace{1cm} (4.9)

Similar approximation reduce the kinetic energy to a linear function \( H \) of the squares of discrete velocities:

\[ H\{(\frac{\partial q_{li}}{\partial t})^2, (\frac{\partial q_{ni}}{\partial t})^2, (\frac{\partial w_{ij}}{\partial t})^2\} \approx T \]  \hspace{1cm} (4.10)

Application of Euler’s necessary condition to minimize the total energy of the system gives the relations:

\[ \frac{d}{dt} \{ \frac{\partial L}{\partial q_k} \} - \frac{\partial L}{\partial q_k} = 0 \hspace{1cm} k = 1,2,3 \]  \hspace{1cm} (4.11)

where \( L \) is the Lagrangian defined as

\[ L = T - U \]  \hspace{1cm} (4.12)

Equation (4.11) is known as Lagrange’s Equation. The solution of harmonic motion may be assumed in the form of the product:

\[ \phi_\xi(\zeta, \eta, t) = (A \cos \omega t + B \sin \omega t) \Psi_\xi(\zeta, \eta) \]
\[ \varphi_c(\zeta, \eta, t) = (A \cos \omega t + B \sin \omega t) \psi_c(\zeta, \eta) \]

\[ \varphi_n(\zeta, \eta, t) = (A \cos \omega t + B \sin \omega t) w(\zeta, \eta) \]

and the accelerations at the nodes of the mesh are obtained as:

\[ \{\ddot{\varphi}_c(\zeta, \eta, t)\}_{ij} = -\omega^2\{\dot{\varphi}_c(\zeta, \eta, t)\}_{ij} (A \cos \omega t + B \sin \omega t) \]

\[ \{\ddot{\varphi}_n(\zeta, \eta, t)\}_{ij} = -\omega^2\{\dot{\varphi}_n(\zeta, \eta, t)\}_{ij} (A \cos \omega t + B \sin \omega t) \]

\[ \{\dddot{w}(\zeta, \eta, t)\}_{ij} = -\omega^2\{\dot{w}(\zeta, \eta, t)\}_{ij} (A \cos \omega t + B \sin \omega t) \] (4.14)

Now applying Eqs.(4.13) and (4.14) to Lagrange's Equation [Eq.(4.11)], the result will be a matrix eigenvalue problem as follows:

\[ [A] \{X\} = \Omega \{X\} \] (4.15)

Where \( \Omega \) is the frequency parameter and can be obtained from Eqs.(4.6) and (4.8) by:

\[ \Omega = \omega h \sqrt{\frac{\rho(1-\nu^2)}{E}} \] (4.16)

A computer program is developed for this analysis. It carries out the substitution of finite difference equation, performs the partial differentiation required by Eq.(4.11), calculates the natural frequencies and prints out the mode shapes and the nodal patterns.

Note that the formulation was carried for rectangular plate and since this thesis is concerned only about the square plate, the formulation will be the
same but with some minor changes, which are \( a = b \), \( \Delta x = \Delta y \), \( M = N \), \( \Delta \zeta = \Delta \eta \), and the aspect ratio \( \gamma = 1 \), where \( \gamma = b/a \).

The total strain energy of the plate is computed as the summation of the strain energies of the nodal sub-domains obtained by using interlacing grids. The integrals in the strain and kinetic energy expressions are replaced by finite approximation summation and applying Euler's necessary condition to minimize the total energy of the system. Finite difference equations are obtained for each of the displacement and rotational components associated with each node.

### 4.3 Concept of Interlacing Grids & Nodal Sub-Domains:

For the application of the interlacing grids method, the strain energy function given by Eq.(4.6) is divided into sub functions as follows:

\[
U = \frac{1}{2} \frac{E_h a^2}{(1 - \nu^2)} \iint_\Omega \left\{ f_1 + f_2 + f_3 \right\} \, d\eta \, d\zeta
\]  

(4.17)

Where,

\[
f_1 = \frac{\beta^2}{12} \left( \frac{\partial \Phi}{\partial \zeta} \right)^2 + \frac{k(1 - \nu)}{2} \left( \frac{\partial w}{\partial \zeta} \right)^2
\]  

(4.18)

\[
f_2 = \frac{\beta^2}{12} \left( \frac{\partial \Phi}{\partial \eta} \right)^2 + \frac{k(1 - \nu)}{2} \left( \frac{\partial w}{\partial \eta} \right)^2
\]  

(4.19)

\[
f_3 = \frac{\beta^2}{12} \frac{1 - \nu}{2} \left( \frac{\partial \Phi}{\partial \eta} + \frac{\partial \Phi}{\partial \zeta} \right)^2 + 2\nu \left( \frac{\partial \Phi}{\partial \zeta} \right) \left( \frac{\partial \Phi}{\partial \eta} \right)
\]  

(4.20)
The nodal sub-domains used to determine the sub-strain energies for each sub-function \( f_1, f_2, f_3 \) are shown in Figures 4.3, 4.4 and 4.5. Note that equal intervals are used. The sub-strain energies are computed as the sum of strain energies summed over the set of non-overlapping sub-domains. It is better to mention that employing two sets of interlacing grids provides a finite difference formulation with a reduced mesh size and results in reduced discretization error.

The node set associated with kinetic energy function \( f_4 \) is shown in Figure 4.6.

The total strain energy \( U \) can be separated into three sub-strain energies \( U_1, U_2 \) and \( U_3 \) that associated with \( f_1, f_2 \) and \( f_3 \) respectively as follows.

\[
U_1 = \frac{1}{2} \frac{E h^2}{(1 - \nu^2)} \int_{\eta_0}^{\eta_1} \int_{\zeta_0}^{\zeta_1} f_1 \, d\eta \, d\zeta
\]

\[
U_2 = \frac{1}{2} \frac{E h^2}{(1 - \nu^2)} \int_{\eta_0}^{\eta_1} \int_{\zeta_0}^{\zeta_1} f_2 \, d\eta \, d\zeta
\]

\[
U_3 = \frac{1}{2} \frac{E h^2}{(1 - \nu^2)} \int_{\eta_0}^{\eta_1} \int_{\zeta_0}^{\zeta_1} f_3 \, d\eta \, d\zeta
\]

By taking small variation in the sub-strain energies, Eqs.(4.21), (4.22) and (4.23) can be written as:

\[
(\Delta U_1)_{i,j+1/2} = \frac{1}{2} \frac{E h^2}{(1 - \nu^2)} \left\{ (f_1)_{i,j+1/2} \Delta \zeta \Delta \eta \right\} \Delta \eta \Delta \zeta
\]

\[
(\Delta U_2)_{i+1/2,j} = \frac{1}{2} \frac{E h^2}{(1 - \nu^2)} \left\{ (f_2)_{i+1/2,j} \Delta \eta \Delta \zeta \right\} \Delta \eta \Delta \zeta
\]
\[(\Delta U_3)_{i+1/2,j+1/2} = \frac{1}{2} \frac{Eha^2}{(1 - \nu^2)} \left\{ (f_y)_{i+1/2,j+1/2} \Delta \eta \Delta \zeta \right\} \Delta \eta \Delta \zeta \quad (4.26)\]

Now, the total sub-strain energies can be computed by the summation over the set of non-overlapping sub-domains as follows:

\[(\Delta U_1) \approx \frac{1}{2} \frac{Eha^2}{(1 - \nu^2)} \sum_{i-j}^{N} \sum_{j-l}^{M} \left\{ (f_y)_{i,j} \right\} (\Delta \eta \Delta \zeta)^2 \quad (4.27)\]

\[(\Delta U_2) \approx \frac{1}{2} \frac{Eha^2}{(1 - \nu^2)} \sum_{i-l}^{N} \sum_{j-j}^{M} \left\{ (f_y)_{i,j} \right\} (\Delta \eta \Delta \zeta)^2 \quad (4.28)\]

\[(\Delta U_3) \approx \frac{1}{2} \frac{Eha^2}{(1 - \nu^2)} \sum_{i-j}^{N} \sum_{j-j}^{M} \left\{ (f_y)_{i,j} \right\} (\Delta \eta \Delta \zeta)^2 \quad (4.29)\]

and, hence the total strain energy will be:

\[U = U_1 + U_2 + U_3 \quad (4.30)\]

Similarly, by referring to the total kinetic energy Eq.(4.8), the kinetic energy sub-function can be expressed as:

\[f_4 = \frac{1}{12} \left[ (\frac{\partial \Phi_x}{\partial t})^2 + (\frac{\partial \Phi_y}{\partial t})^2 \right] + \frac{1}{\beta^2} (\frac{\partial w}{\partial t})^2 \quad (4.31)\]

and finally, the total kinetic energy can be presented in the form:

\[T = \frac{1}{2} \rho h^3 a^2 \sum_{i-l}^{N} \sum_{j-l}^{M} \frac{1}{4} \left\{ (f_y)_{i,j} + (f_y)_{i+1,j} + (f_y)_{i,j+1} + (f_y)_{i+1,j+1} \right\} (\Delta \eta \Delta \zeta)^2 \quad (4.32)\]

Application of the finite difference formulation for the interior node on the
Lagrangian Equation [Eq.(4.11)] and the corresponding formulation is well explained in details in reference [60].
Figure 4.3: Nodal sub-domain for $f_1$

Figure 4.4: Nodal sub-domain for $f_2$
Figure 4.5: Nodal sub-domain for $f_3$

Figure 4.6: Nodal sub-domain for $f_4$
CHAPTER 5

EXPERIMENT

The experimental study of structural vibration has always provided a major contribution to our efforts to understand and to control the many vibration phenomena encountered in practice. Since the very early days of awareness of vibrations, experimental observations have been made for the two major objectives of:

a) determining the nature and extent of vibration response levels.

b) verifying theoretical models and predictions.

Today, structural vibration problems present a major hazard and design limitation for a very wide range of engineering products. First, there are a number of structures, from turbine blades to suspension bridges, for which structural integrity is of paramount concern, and for which a thorough and precise knowledge of the dynamic characteristics is essential. Then, there is an even wider set of components or assemblies for which vibration is directly related to performance, either by virtue of causing temporary malfunction during excessive motion or by creating disturbance or discomfort and noise. For all these examples, it is important that the vibration levels encountered in service or operation be anticipated and brought under satisfactory control.

The two vibration measurement objectives indicated above represent two corresponding types of test. The first is one where vibration forces or, more usually, responses are measured during operation of the machine or structure under
study, while the second is a test where the structure component is vibrated with a known excitation, often out of its normal service environment. This second type of test is generally made under much more closely controlled conditions than the former and consequently yields more accurate data acquisition and its subsequent analysis is nowadays called 'Modal Testing'.

5.1 TEST SPECIMEN AND BASE FRAME:

Figure 5.1 shows the square plate of length a symmetrically supported on the diagonals at distance \( \alpha \) from the free edges. The case of \( \alpha/a = 0 \) corresponds to corner supported plate and the one for \( \alpha/a = 0.5 \) corresponds to a single point support at the center of the plate. In the present study three different steel plates were used with the following dimensions:

- Plate 1: 200 by 200 by 2 mm (i.e. \( \beta = 0.01 \))
- Plate 2: 200 by 200 by 5 mm (i.e. \( \beta = 0.025 \))
- Plate 3: 7.5 by 7.5 by 0.375 in (i.e. \( \beta = 0.05 \))

The material properties of the plates are taken to be:

- \( E \): Young’s Modulus of Elasticity = \( 2.07 \times 10^{11} \) N/m²
- \( v \): Poisson’s Ratio = 0.3
- \( \rho \): density = \( 7.83 \times 10^3 \) Kg/m³

The initial deflection and initial stresses as well as the anisotropy of the material are assumed to be almost negligible.

The details of the test plate and test rig are shown in Figure 5.2. The test rig consists mainly of a vibration isolation table (Low Frequency Mounting
Table) at the sides of which four vertical channels of mild steel of very high stiffness are fixed by means of bolts and nuts. Two thick mild steel beams having T section are welded to each other so as to form a crossed structural frames of form ' + '. These two frames are fixed to the vertical channels by means of bolts and nuts. The plate is supported at four points with two screws for each support point at the end of which conical shape is formed in order to give perfect point contact. Two sliding frames for each support point having internal threads to accomplish vertical movement of the screws are free to move horizontally on the flange of the crossed structural frame. These sliding frames are guided in order to support the plate at any desired point. The support structure is shown in Figure 5.3. The above support structure dimensions (vertical channels, crossed structural frame, sliding frame) limit the dimension of the test plates. The maximum dimension of a square plate that can be tested is 240 mm. Also, the location of the point support is limited to \( \alpha = 0.3 \). Furthermore modification of the support structure will enable us to perform the test for the entire range of \( \alpha \) (from 0 to 0.5).

The plate is excited via an electromagnetic shaker (Vibration Exciter) which is bolted to the vibration isolation table at the center. By shaking the plate sinusoidally with radian frequency \( \Omega \) and peak acceleration \( a \) (m/s\(^2\)), a uniformly distributed pulsating load is applied to specimen.
Figure 5.1: Test specimen
Figure 5.2: Test plate and test rig
5.2 TEST EQUIPMENT AND INSTRUMENTATION:

Although techniques used in the field of vibration testing have become more complicated during the past few years. Vibration testing (Modal Testing) has been performed, the most extensively used technique, however, is still the sweeping sine wave test. The reason for the popularity of this test is that it requires fairly simple and inexpensive electronic equipment, and for laboratory work it is a highly efficient tool for evaluating experimental models for frequency response.

Test will be conducted by exciting the test plate periodically and measuring the absolute acceleration by a piezoelectric accelerometer. The mounting of the test plate, the excitation system and the data recording system arrangements are shown in the block diagram of Figure 5.4. A piezoelectric accelerometer is used to monitor the acceleration of the exciter head.

Most of the equipment and instruments are Brue & Kjaer products. For reference, a photograph of the actual experimental lay-out is shown in Figure 5.5. The apparatus used can be divided into two separate systems namely, the excitation system and the data recording system.

5.2.a Excitation system:

The set-up numbered [1] to [6] in Figure 5.4 is the instrumentation related to the excitation system. In the figure, the Exciter Control type B&K 1047 [1] is used to generate sine signal which covers 5 Hz to 10 kHz in one continuous range. This signal is amplified by using a 180 VA Power Amplifier type B&K 2712 [2], which has low distortion over wide frequency range. The amplified signal is used to drive the Vibration Exciter type B&K 4808 [3]. The
maximum dynamic force of the Vibration Exciter is about 187 N with assisted air cooling. The Accelerometer type B&K 4343 (charge sensitivity of 1.022 pC/ms \(^{-2}\), voltage sensitivity of 0.922 mV/ms \(^{-2}\) and weight 16.3 g) \[4\] is attached to the Vibration Exciter head. Conditioning Amplifier type B&K 2625 \[5\] is used in the feed-back through the Two Channel Tracking Filter type B&K 5716 / WHO255 \[6\] to the Exciter Control \[1\]. With this arrangement \[1\] to \[6\], the compressor or automatic gain section of the Exciter Control regulates the output from the Exciter Control to the Power Amplifier \[2\] according to the feedback signal from the Vibration Exciter \[3\]. Thus the vibration is kept constant at the prescribed level.

5.2.6 Data recording system:

The measuring response arrangement used in the experiment is shown in the block diagram numbered \[8\] to \[13\]. A piezoelectric accelerometer type B&K 4344 (charge sensitivity of 0.310 pC/ms \(^{-2}\), voltage sensitivity of 0.281 mV/ms \(^{-2}\) and weight 2.7 g) \[8\] is used to pick up the response from the vibrating plate \[7\]. When light structures are tested, such as Plate 1, light accelerometer should be used to avoid mass loading problem. Because the output signal from the Accelerometer is small, it is entered into a Charge Amplifier type B&K 2635 \[9\]. The amplified signal is fed to the Two Channel Tracking Filter \[6\]. The frequency range in the Two Channel Tracking Filter is controlled by the Exciter Control \[1\] which (as mentioned before) plays as the feedback. Three output signals can be obtained from the Two Channel Tracking Filter [signal A (the excitation), signal B (response) and signal B \[90^\circ\]] which is used to measure either the phase and the amplitude or the Co-ordinated and the Quadrature components. To measure the phase and the amplitude, the output
signals from the Two Channel Tracking Filter are fed to the Phase Meter type B&K 2971 [10] and the Voltmeter type B&K 2432 [11]. In the second case for measuring the Co-ordinated and the Quadrature components, the three output signals from the Two Channel Tracking Filter are fed to the Cross Spectrum Unit type B&K 5748 [12]. Finally the cross correlated signals are recorded in the X-Y Recorder type B&K 2308 [13] or the Two Channel Level Recorder type B&K 2309 [14].
Figure 5.4: Block diagram of the test arrangement
5.3 EXPERIMENTAL PROCEDURE:

The Electromagnetic Vibration Exciter is located at the central position of the test rig where both symmetric and and antisymmetric modes would be excited. The range of frequencies covered in the test was from 85 to 1850 Hz. However, the natural frequencies and the corresponding mode shapes were examined in the frequency band of interest which included the first lowest seven natural frequencies for Plate 1, the first lowest five natural frequencies for Plate 2 and the first lowest three natural frequencies for Plate 3. The resonance of the test rig and the structure support were observed in the range of frequencies involved in the measurement. However these resonances were notified and examined.

The electromagnetic field created by the Vibration Exciter was sufficient to drive the test plate as a non-contacting exciter. This type of excitation, was used in the vibration measurement for the two examined positions of the point support (i.e. for $\alpha = 0.0$ and for $\alpha = 0.1$).

The mode shape of vibration for a particular natural frequency was obtained by exciting the plate when the Vibration Exciter head was in direct physical contact and the positions of the nodal lines were located by moving the accelerometer with the aid of hand-held probe over a predetermined grid.

The Low Frequency Mounting Table used in this experiment was originally designed by the Research and Development Engineers in tecquipment Company, Nottingham, ENGLAND. The cost of this table, which was about 65 Thousands Saudi Riyals (in 1984), gave us the challenge to manufacture a modified one at the Mechanical Engineering Workshop of King Fahd University of
Petroleum and Minerals. The only part we had to import was the Dunlop Air Spring. The rest of the construction was modified and manufactured at the M.E. Workshop.

We can simply say that the Low Frequency Mounting Table is a heavy box resting on air spring. The object under testing is mounted on that box and by inflating the air spring, the whole box and the object will be isolated from the ground, and as a result the vibration of the surrounding is eliminated.
CHAPTER 6

RESULTS AND DISCUSSION

The examination of the natural frequencies and mode shapes of point supported rectangular plate along the diagonal, can be treated as the free vibration analysis of a completely free rectangular plate except at the given four points distributed symmetrically with respect to the plate central axis. Such a plate is shown in Figure 6.1.

Because of the symmetry in the distribution of the point supports with respect to the plate central axis as well as the symmetry in the geometry of the plate dimensions, the free vibration modes of this plate will fall into four categories:

1. modes symmetric with respect to both the $\zeta$ and $\eta$ axes (i.e. Symmetric-Symmetric mode : $S-S$),
2. modes antisymmetric with respect to both the $\zeta$ and $\eta$ axes (i.e. Antisymmetric-Antisymmetric mode : $A-A$),
3. modes symmetric with respect to the $\zeta$ axis and antisymmetric with respect to the $\eta$ axis (i.e. Symmetric-Antisymmetric mode: $S-A$), and
4. modes antisymmetric with respect to the $\zeta$ axis and symmetric with respect to the $\eta$ axis (i.e. Antisymmetric-Symmetric mode: $A-S$).

Also, because of the symmetry of the geometry of the square plate with respect to the plate diagonals, the above four categories can be reduced to only
three modes. The three type of modes are:

1. Antisymmetric - Antisymmetric,
2. Symmetric - Antisymmetric, and

The Antisymmetric - Symmetric mode will be the same as the Symmetric - Antisymmetric mode with interchanging between the $\zeta$ and $\eta$ axes. Consequently, instead of analyzing the full plate with free edges and point supported at the desired position, one quarter of the plate can be studied in conjunction with imposing the boundary conditions stemming from the symmetry in the distribution of the point supports with respect to the plate central axis as well as the symmetry in the geometry of the plate dimension with respect to the diagonals.

These boundary conditions are summarized as follows (see Figure 6.2)

1. Antisymmetric edge: slope and displacement are zero along the edge.
2. Symmetric edge: normal slope to the edge is zero.

According to the above discussion, the problem was solved three times with A-A, S-A, and S-S boundary conditions on the $\zeta$ and $\eta$ axes. The first lowest six frequencies in each category were obtained.

The results obtained will be divided into four parts. Part one will be concerned about the convergence of the numerical solution used. Four different mesh sizes were used in the analysis of corner supported case (i.e. $\alpha = 0.0$), namely eight by eight, nine by nine, ten by ten and finally eleven by eleven nodes per side.
By studying the effect of increasing the mesh size, in other words, increasing the number of equations in the computation, the convergence was discussed. Since this thesis is concerned about a square plate with equal intervals (see Chapter 4 section 2), the eleven by eleven mesh size was selected over the others to study the vibration of point supported plate because of two reasons:

a) Satisfactory convergence.

b) In order to have the more commonly \( \alpha \) values used in literature (i.e. \( \alpha = 0.1, 0.2, 0.3, 0.4 \) and 0.5).

The results are given in part two. The mode shapes and the nodal patterns are given in the Appendices. Part three is the discussion of the experimental results. The frequency response and the Nyquist circles are given in the Appendices. Part four deals with the effect of rigidity of the point support. The study of increasing the finite area of the support is investigated and discussed.
Figure 6.1: Point supported rectangular plate

Figure 6.2: Boundary condition for the square plate
6.1 CONVERGENCE:

All of the results reported here in this study were computed with a value of Poisson's ratio equal to 0.3. Referring to Chapter 4 where the frequency parameter $\Omega$ was given by Eq. (4.16):

$$\Omega = \omega h \sqrt{\frac{\rho(1-\nu^2)}{E}}$$

(4.16)

In order to compare this parameter with other studies in literature it was multiplied by a non-dimsional factor $\sqrt{12}/\beta^2$ to have it in the most commonly used form of the frequency parameter:

$$\lambda = \omega a^2 \sqrt{\frac{\rho h}{D}}$$

(6.1)

All the results given in the present study are based on the above expression for the frequency parameter $\lambda$ (non-dimensional).

Four different mesh sizes were examined, namely, eight by eight, nine by nine, ten by ten and eleven by eleven nodes per side. Since we have three variables per node (i.e. displacement $w$, rotation $\varphi_1$ and rotation $\varphi_2$), and equal intervals were used the total equations per mode will be three multiplied by the square of the mesh size. Depending on the mode type (i.e. A-A, S-A, or S-S), the total number of the known variables at the boundaries and at the point support are subtracted from the total number of equations.

Eleven different values of $\beta$ were computed for each frequency, in each mode type and for the different mesh size. The values of $\beta$ computed were 0.2,
0.1, 0.05, 0.025, 0.02, 0.0125, 0.01, 0.005, 0.0025, 0.001 and 0.0005.

Referring to Figure 6.2 and the above information, the examined number of equations (i.e. unknown values of \( w, \varphi_t \) and \( \varphi_a \)) for the three types of mode can be found as follows:

\[
\begin{align*}
\text{n}_{\text{AA}} &= 3m^2 - (4m - 1) - k \\
\text{n}_{\text{SA}} &= 3m^2 - (3m - 1) - k \\
\text{n}_{\text{SS}} &= 3m^2 - (2m + 1) - k
\end{align*}
\] (6.2.a) (6.2.b) (6.2.c)

where \( m \): mesh size

\[
k : \text{constant} = \begin{cases} 
1 & , \quad \alpha \neq 0.5 \\
0 & , \quad \alpha = 0.5
\end{cases}
\]

Hence, as an example, the total number of equations in S-A mode with nine by nine mesh size is 216 equations.

When the frequency parameter \( \lambda \) is drawn versus the number of equations used in the computation for some selected values of \( \beta \), the tendency of convergence is obtained. All of the convergence study plots are given in Appendix (A.1). The selected values of \( \beta \) are 0.2, 0.1, 0.05, 0.01 and 0.0005. In fact, at the early stage of this study, full plate was examined and to investigate the convergence of a mesh size of eighteen by eighteen was reached but the accuracy and the convergence were not satisfactory, and then quarter of the plate were analyzed. Also, the frequency parameter \( \lambda \) is plotted versus the inverse of \( \beta \) in logarithmic scale for the four mesh sizes in order to visualize the variation of \( \lambda \) with the \( \beta \) and how the effect of the transverse shear deformation and rotary
inertia change the computed values of $\lambda$. These variation plots are given in Appendix (A.1).

These figures are arranged by ascending order with respect to the frequency parameter and by the mode type. First, the first lowest six Antisymmetric - Antisymmetric modes are given, followed by the first lowest six Symmetric - Antisymmetric modes and then the first lowest six Symmetric - Symmetric modes. For the sake of clarity, the two types of graphs mentioned previously, are given together in one page. (i.e each page will be for a different frequency parameter).

In Figure A.1.1 the convergence of the first A-A mode is presented for the selected values of $\beta$ followed by Figure A.1.2 where the variation of the frequency parameter $\lambda$ of the four mesh sizes computed were presented. As it is clear from Figure A.1.1, the frequency parameter decreases as the mesh size increases. That means, for this particular frequency parameter, the convergence is from above, and with more refinement of the mesh size (i.e. increasing the mesh size) the exact value can be approached from above. Conversely, Figure A.1.3 shows that the exact value of the frequency parameter $\lambda$ of the second A-A mode can be approached from below. The reason of this contrast is due to the following remark:

"It should be remembered that energy methods always overestimate the fundamental frequency, so with more refined analysis the exact value can be approached from above. Conversely the Finite Difference method appears to underestimate the natural frequency and with increasing refinement in the analysis the exact value can be approached from below [10]", and as mentioned in the Introduction, that this study will be based on the variational principle
applied to the energy expression in conjunction with Finite Difference technique, the exact value can be approached either from above (as in A-A 1) or from below (as in A-A 2). It should be noted that, in Figure A.1.4, the comparison of the variation of the frequency parameter with $\beta$ is only for the upper mesh size (i.e. eleven by eleven) and the lower mesh size (i.e. eight by eight). This is intended only for the sake of clarity, since the in-between mesh sizes are always limited between the upper and lower mesh sizes. For the second and third modes (S-A and S-S modes), also the four mesh sizes are given for only the lowest frequency parameter in each type.

6.2 CORNER AND POINT SUPPORTED SQUARE PLATE:

When the solution of the algebraic equations is obtained by the developed program, a subroutine was written in order to arrange the computed values of $w$, $\varphi_c$, and $\varphi_n$. These values are arranged in such a way that the mode shape can be graphed in three dimensional figures. The mode shapes of the first lowest six frequencies for the six selected values of $\alpha$ (i.e $\alpha = 0.0, 0.1, 0.2, 0.3, 0.4, \text{ and } 0.5$) are given in Appendix (A.a). Also the nodal patterns for the first lowest six frequency parameters in each mode type are given for two different values of $\beta$ namely $\beta = 0.2$ and $\beta = 0.0005$ in Appendix (A.b).

This section will be divided into two parts, namely corner and point supported square plate. In part one, the numerical values of the frequency parameter are tabulated in Table A.3.1 to Table A.3.12. Each mesh size is computed for the three types of modes. In the present study the Poisson’s ratio $\nu$ is taken to be 0.3. At the end of each table the values of the frequency parameter
computed by Gorman [38] and Amba-Rao [43] are entered. It should be mentioned that Gorman [38] used a value of Poisson's ratio of 0.333 while Amba-Rao [43] used a value of 1/3.

6.2.a Corner supported square plate:

The computed values of the frequency parameter $\lambda$ for each of the three types of modes for the different mesh size are first tabulated and second, the variation of $\lambda$ with $\beta$ are graphed and third the correction in $\lambda$ are plotted. In all the tables (Table A.3.1 to Table A.3.12) excellent agreement is observed between the present results and those of Gorman [38] and Amba-Rao [43]. The comparison is made for the value of $\beta = 0.0005$ which means the frequency parameter of thin plate, since both of Gorman and Amba-Rao applied their numerical solution to the classical governing differential equation of motion of the plate [see Eqs.(3.1) & (3.45)]. In Figure A.2.1 the variation of $\lambda$ with $\beta$ for the first six A-A modes with mesh size of eight by eight is given. In this figure we can observe that higher modes are more affected by increasing the thickness ratio $\beta$. To have more insight of this effect, the correction in the frequency parameter is graphed versus $\beta$. Figure A.2.2 shows this effect. The frequency parameter at the lowest value of $\beta$ (i.e $\lambda(\beta = 0.0005)$) is considered to be the thin plate solution including the effect of transverse shear deformation and rotary inertia, and then the ratio of the frequency parameter for the other values of $\beta$ to that frequency parameter is graphed (i.e. $\lambda(\beta)/\lambda(\beta = 0.0005)$ v.s $\beta$). The following observations of Tables A.3.1 to A.3.12 and Figures A.2.1 to A.2.24 can be summarized:

1) Higher modes show slow convergence to thin plate solution. This also
means that the effect of transverse shear deformation and rotary inertia is more pronounced in higher modes with the increasing thickness ratio $\beta$.

2) Correction in the frequency parameter increases with higher modes.

3) The S-S modes show faster convergence to thin plate solution either than S-A or A-A modes.

4) The second and fifth S-S modes, as well as the second and fifth A-A modes that characterized by nodal lines along the diagonals [see the Nodal Patterns in Appendix (A.b)] show low correction factors.

5) The first S-A mode will have the line of antisymmetry (when full plate is plotted) along the center line. In all the studies reviewed (Chapter 2) where full plate is analyzed this mode is sometimes have the line of antisymmetry along one of the diagonals. This can explained by the type of solution used. When full plate is considered, the second shape of nodal lines is obtained, while the former shape is obtained when only quarter of the plate is examined. That is due to antisymmetry boundary condition imposed on one of the edges.

6) The correction in $\lambda$ for $\beta = 0.01$ is very small for almost all the mode types and all the frequencies with maximum value of 5.0 percentage. Hence, the approximation of $\beta = 0.01$ to be thin plate is more accurate than 0.02.

6.2.b Point supported square plate:

The eleven by eleven mesh size was selected over the other three mesh sizes in order to have appropriate values of $\alpha$. Again in this section, the computed values of $\lambda$ are tabulated for each mode type for the five locations of
\( \alpha \) i.e. \( \alpha = 0.1, 0.2, 0.3, 0.4 \) and 0.5 \) in Tables A.3.13 to A.3.27 and also the values obtained by Gorman [38] and Amba-Rao [43] are tabulated for comparison. The following observations can be made:

1) Repeated modes are obtained for various location of the point support. These modes are characterized by nodal lines along the diagonals and can be found also for the vibration of completely free plates.

2) Most of the modes change slightly their nodal patterns. To visualize these changes the nodal patterns for each mode is given for two values of \( \beta \) namely \( \beta = 0.0005 \), which represent thin plate, and \( \beta = 0.2000 \), which represent Mindlin plate (thick plate).

3) Most of the nodal pattern at \( \beta = 0.2 \) appear to be similar for those associated with \( \beta = 0.0005 \) and exactly the same for repeated modes.

4) Some modes interchange their order. An example is A-A 4 for point supported plate with \( \alpha = 0.3 \) (Table A.3.19).

5) In the A-A case, (Tables A.3.13, A.3.16, A.3.19, A.3.22, and A.3.25), show certain discrepancies between the present results and those of Gorman [38]. For \( \alpha = 0.2, 0.3 \), and 0.4 the present solution gave a lower frequency for the first mode than that reported in [38]. In fact, examination of the associated tables (A.3.16, A.3.19 and A.3.22) suggests that in [38] the first frequency may have been missed for \( \alpha = 0.2, 0.3 \), and 0.4. If one assumes that this is true, excellent agreement will be observed for \( \alpha = 0.2, 0.3 \), and 0.4 between the first, second, third and fourth, fifth modes of the present solution, respectively. Hence, in the tables mentioned, the values obtained by [38] will be arranged according to the above observation for the matter of easy
comparison.

6) For the A-A case, excellent agreement is observed when comparing the present method and that of Amba-Rao [43].

7) For the S-A case, (Tables A.3.14, A.3.17, A.3.20, A.3.23, and A.3.26), excellent agreement is found between the present solution and that of Gorman [38] and Amba-Rao [43] for all modes and for all values of $\alpha$.

8) For the S-S case, (Tables A.3.15, A.3.18, A.3.21, A.3.24 and A.3.27), excellent agreement is noted with the exception of the case of $\alpha = 0.3$. First mode given by [38] and [43] is one of the repeated modes which has two nodal line along the diagonals, while the first mode computed in the present study is not. The nodal pattern for the fundamental frequency in the present method is characterized by a circle passing through the point supports as it is seen in Appendix (A.b). And for the sake of easy comparison the result obtained in this study for the frequency parameter in the case of $\alpha = 0.3$ is tabulated in the 2nd column in Table A.3.21.

It should be kept in mind that the above comparison is performed for $\beta = 0.0005$

6.2.c Fundamental frequency:

The fundamental frequency which implies the lowest was found to be always from the S-S mode. It is plotted versus the location of the point support $\alpha$ for some selected values of $\beta$ (see Figure 6.3). Not only the fundamental frequencies are symmetric about the center lines, but also they are symmetric, about the diagonals except for $\alpha = 0.2$. Johns, D. and Nagran, V. [10]
assumed that the fundamental frequencies were symmetric about the central axes and also symmetric about the diagonals. As a result, they have analyzed only one eighth of the square plate.
Figure 6.3: Fundamental frequency
6.3 EXPERIMENT:

One of the major requirements of the subject of modal testing is a thorough integration of three components:

1) the theoretical basis of vibration,
2) accurate measurement of vibration and
3) realistic and detailed data analysis.

There has been in the past a tendency to regard these as three different specialist areas, with individual experts in each. However, the subject of Modal Testing now demands a high level of understanding and competence in all and cannot reach its full potential without the proper and judicious mixture of the necessary three components.

Before discussing the experimental results, a brief discussion of the theoretical basis of vibration will be introduced. A detailed and thorough account can be found in literature (see reference [62]).

It is very important that a clear distinction is made between the free vibration and the forced vibration analyses. For a single degree of freedom (i.e. only one parameter is needed to define the motion), a free vibration analysis yields its natural frequency and damping factor while a particular type of forced response analysis, assuming a harmonic excitation, leads to the definition of the Frequency Response Function (FRF) such as mobility, the ratio of velocity response to force input. These two types of results are referred to as "modal properties" and "frequency response characteristics" respectively. It is appropriate to consider the form which a plot of a FRF takes.
Three alternative ways to plotting this information are a) Linear, b) Logarithmic, and c) Nyquist plot. Although very few practical structures could realistically be modelled by a single degree of freedom (SDOF) system, the properties of such a system are very important because those for a more complex multi-degree of freedom (MDOF) system can always be represented as the linear superposition of a number of SDOF characteristics.

So far we have defined the mobility frequency response function as the ratio of velocity response to force input. Another FRF is receptance which is the ratio between a harmonic displacement response and the harmonic force. Inertance or sometimes called accelerance is another FRF defined as the ratio of the acceleration response and the input force.

There is an overriding complication to plotting FRF data which derives from the fact that they are complex and thus there are three quantities; frequency plus two parts of the complex function, and these cannot be fully displayed on a standard x-y graph. Because of this, any such single plot can only show two of the three quantities and so there are different possibilities available.

The three most common forms of presentation are:

a) The Bode type of plot consisting of two graphs of Modulus (FRF) versus Frequency and Phase versus Frequency.

b) Two plots consisting of Real part (of FRF) versus Frequency, and Imaginary part versus Frequency.

c) Nyquist plot which is a single graph of Real part of (of FRF) versus Imaginary Part, and this does not explicitly contain frequency information.
In the present study, the experimental results are presented in the last two types with some modification. This modification can be explained in the following paragraph.

The Cross Spectrum Unit type B&K 5748 enables the Two-Channel Tracking Filter type B&K 5716/WHO255 to be used for cross-correlation measurement, which can be useful for identifying the origins of unwanted noises and vibration. It takes three input signals from the Two Channel Tracking Filter [X, Y and Y\textsuperscript{90°}] derived from transducers (or accelerometers) cited at two positions to which the cross correlation is to be measured. It generates four positive direct voltages proportional to the functions g(X), g(Y), C(XY), and Q(XY) defined as follows:

\[
g(X) = \frac{1}{2T} \int_{-T/2}^{T/2} X^2 \, dt ,
\]

\[
g(Y) = \frac{1}{2T} \int_{-T/2}^{T/2} Y^2 \, dt ,
\]

\[
C(XY) = \frac{1}{10T} \int_{-T/2}^{T/2} X \, Y \, dt ,
\]

\[
Q(XY) = \frac{1}{10T} \int_{-T/2}^{T/2} X \, Y \, 90° \, dt ,
\]

and if these output voltages are used to deflect the pen of the Two Channel Level Recorder type B&K 2309 or the X-Y Recorder type B&K 2308 which is synchronized with the sweep of the Two Channel Tracking Filter, C(XY) and Q(XY) will give the Co-ordinated and the Quadrature cross spectra respectively, and g(X) and g(Y) will give the frequency spectra of A and B signals (the two
transducers signals) respectively at the inputs to the Two Channel Tracking Filter (X corresponds to A, Y to B, and Y to B to B on the Two Channel Tracking Filter). The averaging time T for all four outputs may be selected on a switch giving a choice of 0.1, 0.3, and 1.0 second. The averaging characterized by the integration functions is performed with exponential weighting. Because of the lack of the Remote Control instrument which should be used to synchronize the Two Channel Level Recorder or X-Y Recorder with the Two Channel Tracking Filter, the excitation signal (from the Accelerometer block [4] in Figure 5.4) is used as the input signal A. Signal A is now the point Inertance (since the response and the excitation coordinate are the same, if they are different, it will be called Transfer Inertance). By this modification the presentation of the FRF as Real part versus Frequency and Imaginary part versus Frequency will not be as the theoretical plots. The theoretical presentation of Real part versus Frequency, Imaginary part versus Frequency and Nyquist plots for the FRF are given in Figure 6.4 for a Damped SDOF system. The difference between the plots in this study and the theoretical plots will be clear as we proceed in the discussion.

The experiment was conducted on three different steel plates with the dimensions given in Chapter 5 - section 1. In accordance with Eq.(6.1) the computed nondimensional values of the frequency parameter \( \lambda \) for the three \( \beta \) ratios are dimensionalized by the following expression (to obtain the frequency in Hz):

\[
 f = \frac{h}{a^2} \left\{ \frac{1}{2\pi} \sqrt{\frac{E}{12(1-\nu^2)\rho}} \right\} \lambda \quad \text{Hz} \quad (6.3)
\]

where,
h : plate thickness (m).

a : plate dimension (m).

E : Young’s of Elasticity $= 2.07 \times 10^{11}$ (N/m$^2$)

v: poissons’s Ratio $= 0.3$

$\rho$ : Density $7.83 \times 10^{3}$ (Kg/m$^3$)

and hence, for

Plate 1, $f = 12.38178 \lambda$ (Hz)

Plate 2, $f = 30.95445 \lambda$ (Hz)

Plate 3, $f = 64.99622 \lambda$ (Hz) \hspace{1cm} (6.4)

Two positions of the point support were examined - corner supported ($\alpha = 0.0$) and point supported ($\alpha = 0.1$). The experimental results of these two cases are given in Tables A.3.28 to A.3.33. The Co-ordinated and the Quadrature components are presented in Figures A.4.1 to A.4.15. The Nyquist plots are presented for only the first case (corner supported, $\alpha = 0.0$), in Figure A.5.1 to Figure A.5.14.

A view over the above tables will lead to the following observations:

1) Two experimental modes are obtained for each symmetric-Antisymmetric theoretical mode. In theory, the S-A mode will be the same as A-S (as discussed earlier in the introduction of this chapter) and this means that two modes are expected to be excited at the same frequency. In practice, two different frequencies are expected to occur at or close to the predicted frequency. At one of these two frequencies, the maximum discrepancy in frequency spectra is observed, for example in Table A.3.29 the maximum discrepancy is observed for the S-A mode.
(2.33 %) and ( − 2.20 %) for the second mode S-A 1.

2) Symmetric-Symmetric modes show low discrepancy and this is due to the position selected for the response accelerometer. This position was selected by examining the nodal patterns for the frequencies under observation.

3) In plate 1, most of the discrepancies are positive which means that the theoretical values of λ are higher. As β increase the theoretical values are observed to be lower such as in Plate 3.

The values of the frequency in Tables A.3.28 to A.3.33 are obtained by observing the peaks in the frequency response. When the frequency response is plotted for the Co-ordinated and the Quadrature Components, as an overall, the peaks are approximated and by close sweep rate, a more accurate reading can be obtained from the Exciter Control.

Refering to Figures A.4.1 to A.4.15, the following observation can be deduced:

1) The difference between these figures and the theoretical representation of the Real part versus Frequency and the Imaginary part versus Frequency plots for a FRF is quite clear. The Co-ordinated and the Quadrature components given in these figures are interchanging with each other for different modes. This is due to modification of the experiment lay-out. However, since only the frequencies are demanded by this method, it gives quite reasonable agreement with the predicted theoretical values.

2) Test rig vibration exists in the frequency band of interest. This vibration
was reduced by low power excitation.

3) Mode couplings are observed to occur close to the predicted frequencies of the S-A modes. When slow sweep rate is selected more clear picture can be obtained.

From Nyquist plots being shown in Figures A.5.1 to A.5.14, one can deduce the following:

1) Natural frequencies can also be determined by observing the frequencies when the pen of the X-Y Recorder pass by the tangent of the circle. Slow sweep rate of the frequency leads to better chance to observe the natural frequency.

2) A very interesting observation is the mode coupling. Figure A.5.2 shows the mode coupling of the second mode (SA1). The frequency noted at the interconnection between the two circles of the Nyquist plot gives a more close value to the predicted one. As an example, the predicted value of the second mode \(α = 0.0, β = 0.01\) is 194.97 Hz and the frequency at the interconnection of the two circles showed to be 195 Hz.
Figure 6.4: Plots of FRF of a SDOF system

(a) Receptance
(b) Mobility
(c) Inertance
6.4 **COLUMN SUPPORTED PLATE** : 

Column-supported floor slabs are a common feature in contemporary building construction. The columns are generally located in a regular pattern so as to facilitate the partitioning of the floor area for various purposes. The dynamic response of such a configuration is of interest when the slabs experience out-of-plane oscillatory loads from machinery resting upon them. The simplest representation of a column-supported floor slab is a point-supported plate. In this section, an investigation into the effect of the rigidity of the column support is presented. The term **Point-Support** is used to denote a constraint of zero deflection which suggests a simply supported boundary condition. The term "Rigid-Support" is used to constraints of zero deflection and rotation which suggests a clamped supported boundary condition.

Only the first lowest four frequency parameters are investigated. By modification of the developed program, the numerical values of these frequencies are obtained. The only paper found in literature was written in 1972 by Petyt, M. and Mirza, W.H. [12]. They used Finite Element Displacement method of analysis to determine the vibration characteristic of floor slabs resting on four column supports. No numerical values for the frequency parameters were given in this paper, instead a graphical presentation of the ratio of frequency with hinged joint versus the finite area of the column support $\xi$ (see Figure 6.4). The finite area of the column supported plate $\xi$ investigated ranged from zero (correspond to rigid point support) to 0.15 with increment of 0.05. Following the same monograph of Chapter 6-section 2, the computed values of this study will be presented. The variation of $\lambda_c$ with the inverse of $\beta$ is plotted and the correction in
\( \lambda_c \) for each case of \( \xi \) (i.e. for \( \xi = 0.0, 0.05, 0.10 \) and 0.15). These plots being shown in Figures A.6.1 to A.6.8. It can be observed that as the finite area of column support increases the frequency parameter \( \lambda_c \) tends to slow in convergence to thin column support frequency (i.e. \( \lambda_c (\beta = 0.0005) \)) and hence the correction in \( \lambda_c \) tends to increase. To have clear insight of the effect of rigidity, the ratio of the frequency parameter with rigid point (or column) support to the frequency parameter of corner supported plate is plotted versus the finite area of the column support \( \xi \). (i.e. \( \lambda_c / \lambda (\alpha = 0) \)). These plots are given in Figures A.6.9 to A.6.19, and one can observe the following:

1) The effect of rigid support is more pronounced in thin plate with small \( \beta \) ratios. Changing the point support to column support will allow the transverse shear deformation term to increase. The content of this should be more pronounced with large values of \( \beta \), but since only the ratio of \( \lambda_c / \lambda \) is plotted, the percentage increment in thick plate is smaller than the content in thin plate. This is more clear if the variation plots are reviewed.

2) Even for the rigid point support, there is a large increase in the fundamental frequency when \( w, \partial w / \partial x, \) and \( \partial w / \partial y \) are constraints at the joints.
Figure 6.5: Column supported plate
CHAPTER 7

CONCLUSION

Concerning the present method used in this study, the following conclusions are deduced:

1) Although the number of equations used in the present Finite Difference formulation is small, the accuracy maintained is quite good in the comparison of the existing results. The use of interlacing grids in addition to analyzing only one quarter of the plate are the two factors behind this comparatively small number of equations. Solving quarter of the plate with interlacing grids of eight by eight mesh size is like solving full plate with ordinary grids of thirty by thirty mesh size.

2) The effect of transverse shear deformation and rotary inertia are found to be more pronounced in higher modes. This is clear from the correction plots given in the Appendices.

3) The effect of transverse shear deformation and rotary inertia are not included in all (up to the knowledge of the author) the papers found in literature for point supported plate along the diagonals, and only the classical theory of plates were analyzed for this problem. The comparison of the present method is compared with existing studies for the case of small thickness ratio $\beta$.

4) Plates with $\beta = 0.01$ have small discrepancy with $\beta = 0.0005$, which can
be considered as thin plate solution. In the correction plots of $\lambda$ for the three mode types, A-A, S-A and S-S, and for all point position, it can be verified that this $\beta$ ratio has good approximation for thin plate case.

5) Even for the rigid point support, there is a large increase in the fundamental frequency when $w$, $\partial w/\partial x$, and $\partial w/\partial y$ are constraints at the joints.

6) With the available equipment in the Dynamic Labs, the performed experiment shows very reasonable agreement with predicted theoretical values of the frequency parameter. Slow sweep rate over the natural frequency gives very dependable readings. The use of the Real part and the Imaginary Part (or the Co-ordinate and the Quadrature) signals enable us to spot the natural frequency very precisely.

7) Hand pressure at the point support is sufficient to hold the plate and insure a zero deflection of the plate. Increasing the pressure of the point support yields higher frequencies than expected. This is due to the effect of rigidity which moves the point support to rigid support. This effect is more pronounced in the case of a thin plate. It was found out in the experiment that hand pressure will be enough for thin plate, since it permits the transverse shear to occur rather than increasing this term more significantly. Contrarily, it was found out for a large value of $\beta$, the hand pressure is not enough as it yields lower frequencies than those theoretically predicted. Increasing the pressure of the point will permit the transverse deformation to occur and the fre-
frequency obtained will be more accurate and close to theoretical predictions.

8) Even if the increment in increase of the natural frequency for a rigid square plate is more pronounced in small values of $\beta$, the study of the correction shows that the effect of the rigidity of the point support is more pronounced in large values of $\beta$. This contrast is solved by reminding the reader that the Effect of Rigidity plots are given in terms of the ratio between the frequency parameter with column support and the frequency parameter of corner support (i.e. $\lambda_c/\lambda(\alpha=0.0)$).

9) Nyquist plot, where the Real Part and the Imaginary Part of the Frequency Response Function are graphed verses each other as in polar plots, yields a one plot and does not explicitly contain the radian frequency. To calculate the loss factor (or structural damping) from this type of plot needs repeated run over the natural frequency in the range of interest. First when the Nyquist circle is plotted, the X-Y Recorder B&K type 2309 is set on the servo on position and then another run is performed and the point under question is notified.

10) Wide range of frequency could be obtained and investigated if the proper logarithmic paper for the X-Y Recorder is available.

11) Some of the modes interchange their order or their shapes (Nodal Pattern) depending on thickness variations. The nodal lines (characterized by zero deflection) of mode shapes which correspond to Mindlin plate tend to change to the nodal lines of mode shapes correspond to
thin plate in the direction of decreasing $\beta$.
REFERENCES


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32. Leissa, A.W. "Recent Research in Plate Vibrations 1973-1976 Complicating


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APPENDICES
APPENDIX A.1

CONVERGENCE

Legend used in the Convergence Plots:

\[
\begin{align*}
\beta &= 0.0005 & \rule{5cm}{0.4pt} \\
\beta &= 0.0100 & \rule{4cm}{0.4pt} \\
\beta &= 0.0500 & \rule{3cm}{0.4pt} \\
\beta &= 0.1000 & \rule{2cm}{0.4pt} \\
\beta &= 0.2000 & \rule{1cm}{0.4pt}
\end{align*}
\]

Legend used in the Variation Plots:

\[
\begin{align*}
8 \times 8 \text{ Mesh Size} & \rule{5cm}{0.4pt} \\
9 \times 9 \text{ Mesh Size} & \rule{4cm}{0.4pt} \\
10 \times 10 \text{ Mesh Size} & \rule{3cm}{0.4pt} \\
11 \times 11 \text{ Mesh Size} & \rule{1cm}{0.4pt}
\end{align*}
\]
Figure A.1.1: Convergence of the 1st Λ-Λ Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.2: Comparison of $\lambda$ for the 1st Λ-Λ Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.3 : Convergence of the 2nd Λ-Λ Mode. 
Corner Supported : \( \alpha = 0.0 \)

Figure A.1.4 : Comparison of \( \lambda \) for the 2nd Λ-Λ Mode. 
Corner Supported : \( \alpha = 0.0 \)
Figure A.1.5: Convergence of the 3rd A-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.6: Comparison of $\lambda$ for the 3rd A-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.7: Convergence of the 4th A-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.8: Comparison of $\lambda$ for the 4th A-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.9: Convergence of the 5th A-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.10: Comparison of $\lambda$ for the 5th A-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.11: Convergence of the 6th A-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.12: Comparison of $\lambda$ for the 6th A-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.13: Convergence of the 1st S-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.14: Comparison of $\lambda$ for the 1st S-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.15: Convergence of the 2nd S-A Mode. 
Corner Supported: $\alpha = 0.0$

Figure A.1.16: Comparison of $\lambda$ for the 2nd S-A Mode. 
Corner Supported: $\alpha = 0.0$
Figure A.1.17: Convergence of the 3rd S-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.18: Comparison of $\lambda$ for the 3rd S-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.19: Convergence of the 4th S-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.20: Comparison of $\lambda$ for the 4th S-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.21: Convergence of the 5th S-A Mode.
Corner Supported: \( \alpha = 0.0 \)

Figure A.1.22: Comparison of \( \lambda \) for the 5th S-A Mode.
Corner Supported: \( \alpha = 0.0 \)
Figure A.1.23: Convergence of the 6th S-A Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.24: Comparison of $\lambda$ for the 6th S-A Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.25: Convergence of the 1st S-S Mode. Corner Supported: $\alpha = 0.0$.

Figure A.1.26: Comparison of $\lambda$ for the 1st S-S Mode. Corner Supported: $\alpha = 0.0$. 
Figure A.1.27: Convergence of the 2nd S-S Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.28: Comparison of $\lambda$ for the 2nd S-S Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.29: Convergence of the 3rd S-S Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.30: Comparison of $\lambda$ for the 3rd S-S Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.31: Convergence of the 4th S-S Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.32: Comparison of $\lambda$ for the 4th S-S Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.33: Convergence of the 5th S-S Mode.
Corner Supported: $\alpha = 0.0$

Figure A.1.34: Comparison of $\lambda$ for the 5th S-S Mode.
Corner Supported: $\alpha = 0.0$
Figure A.1.35: Convergence of the 6th S-S Mode. 
Corner Supported: $\alpha = 0.0$

Figure A.1.36: Comparison of $\lambda$ for the 6th S-S Mode. 
Corner Supported: $\alpha = 0.0$
APPENDIX A.2

CORNER & POINT SUPPORTED PLATES

Legend used in the Variation Plots:

Mode 1: ........................................
Mode 2: ........................................
Mode 3: ........................................
Mode 4: ........................................
Mode 5: ........................................
Mode 6: ........................................

Legend used in the Correction Plots:

Mode 1: ........................................
Mode 2: ........................................
Mode 3: ........................................
Mode 4: ........................................
Mode 5: ........................................
Mode 6: ........................................
Figure A.2.1: Variation of $\lambda$ with $\beta$ for the First Six $A$-$A$ Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size $8 \times 8$.

Figure A.2.2: Correction in $\lambda$ for the First Six $A$-$A$ Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size $8 \times 8$. 
Figure A.2.3: Variation of $\lambda$ with $\beta$ for the First Six S-A Modes. Corner Supported: $\alpha = 0.0$, Mesh Size 8 x 8.

Figure A.2.4: Correction in $\lambda$ for the First Six S-A Modes. Corner Supported: $\alpha = 0.0$, Mesh Size 8 x 8.
Figure A.2.5: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size 8 x 8.

Figure A.2.6: Correction in $\lambda$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size 8 x 8.
Figure A.2.7: Variation of $\lambda$ with $\beta$ for the First Six $A$-$A$ Modes. Corner Supported: $\alpha = 0.0$, Mesh Size 9 x 9.

Figure A.2.8: Correction in $\lambda$ for the First Six $A$-$A$ Modes. Corner Supported: $\alpha = 0.0$, Mesh Size 9 x 9.
Figure A.2.9: Variation of $\lambda$ with $\beta$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size 9 x 9.

Figure A.2.10: Correction in $\lambda$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size 9 x 9.
Figure A.2.11: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes. 
Corner Supported: $\alpha = 0.0$, Mesh Size 9 x 9.

Figure A.2.12: Correction in $\lambda$ for the First Six S-S Modes. 
Corner Supported: $\alpha = 0.0$, Mesh Size 9 x 9.
Figure A.2.13: Variation of $\lambda$ with $\beta$ for the First Six A-A Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size $10 \times 10$.

Figure A.2.14: Correction in $\lambda$ for the First Six A-A Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size $10 \times 10$. 
Figure A.2.15: Variation of \( \lambda \) with \( \beta \) for the First Six S-A Modes.
Corner Supported: \( \alpha = 0.0 \), Mesh Size 10 x 10.

Figure A.2.16: Correction in \( \lambda \) for the First Six S-A Modes.
Corner Supported: \( \alpha = 0.0 \), Mesh Size 10 x 10.
Figure A.2.17: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes. Corner Supported: $\alpha = 0.0$, Mesh Size 10 x 10.

Figure A.2.18: Correction in $\lambda$ for the First Six S-S Modes. Corner Supported: $\alpha = 0.0$, Mesh Size 10 x 10.
Figure A.2.19: Variation of $\lambda$ with $\beta$ for the First Six $A$-$A$ Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size $11 \times 11$.

Figure A.2.20: Correction in $\lambda$ for the First Six $A$-$A$ Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size $11 \times 11$. 
Figure A.2.21 : Variation of $\lambda$ with $\beta$ for the First Six S-A Modes.
Corner Supported : $\alpha = 0.0$, Mesh Size $11 \times 11$.

Figure A.2.22 : Correction in $\lambda$ for the First Six S-A Modes.
Corner Supported : $\alpha = 0.0$, Mesh Size $11 \times 11$. 
Figure A.2.23: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size 11 x 11.

Figure A.2.24: Correction in $\lambda$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.0$, Mesh Size 11 x 11.
Figure A.2.25: Variation of $\lambda$ with $\beta$ for the First Six $\Lambda$-$\Lambda$ Modes. Corner Supported: $\alpha = 0.1$, Mesh Size 11 x 11.

Figure A.2.26: Correction in $\lambda$ for the First Six $\Lambda$-$\Lambda$ Modes. Corner Supported: $\alpha = 0.1$, Mesh Size 11 x 11.
Figure A.2.27: Variation of $\lambda$ with $\beta$ for the First Six S-A Modes. Corner Supported: $\alpha = 0.1$, Mesh Size 11 x 11.

Figure A.2.28: Correction in $\lambda$ for the First Six S-A Modes. Corner Supported: $\alpha = 0.1$, Mesh Size 11 x 11.
Figure A.2.29 : Variation of $\lambda$ with $\beta$ for the First Six S-S Modes.  
Corner Supported : $\alpha = 0.1$, Mesh Size 11 x 11.

Figure A.2.30 : Correction in $\lambda$ for the First Six S-S Modes.  
Corner Supported : $\alpha = 0.1$, Mesh Size 11 x 11.
Figure A.2.31: Variation of $\lambda$ with $\beta$ for the First Six A-A Modes.  
Corner Supported: $\alpha = 0.2$, Mesh Size $11 \times 11$.

Figure A.2.32: Correction in $\lambda$ for the First Six A-A Modes.  
Corner Supported: $\alpha = 0.2$, Mesh Size $11 \times 11$. 
Figure A.2.33: Variation of $\lambda$ with $\beta$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.2$, Mesh Size 11 x 11.

Figure A.2.34: Correction in $\lambda$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.2$, Mesh Size 11 x 11.
Figure A.2.35: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.2$, Mesh Size 11 x 11.

Figure A.2.36: Correction in $\lambda$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.2$, Mesh Size 11 x 11.
Figure A.2.37: Variation of $\lambda$ with $\beta$ for the First Six A-A Modes.
Corner Supported: $\alpha = 0.3$, Mesh Size 11 x 11.

Figure A.2.38: Correction in $\lambda$ for the First Six A-A Modes.
Corner Supported: $\alpha = 0.3$, Mesh Size 11 x 11.
Figure A.2.39: Variation of $\lambda$ with $\beta$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.3$, Mesh Size 11 x 11.

Figure A.2.40: Correction in $\lambda$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.3$, Mesh Size 11 x 11.
Figure A.2.41: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.3$, Mesh Size 11 x 11.

Figure A.2.42: Correction in $\lambda$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.3$, Mesh Size 11 x 11.
Figure A.2.43: Variation of $\lambda$ with $\beta$ for the First Six $A$-$A$ Modes.
Corner Supported: $\alpha = 0.4$, Mesh Size 11 x 11.

Figure A.2.44: Correction in $\lambda$ for the First Six $A$-$A$ Modes.
Corner Supported: $\alpha = 0.4$, Mesh Size 11 x 11.
Figure A.2.45: Variation of $\lambda$ with $\beta$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.4$, Mesh Size 11 x 11.

Figure A.2.46: Correction in $\lambda$ for the First Six S-A Modes.
Corner Supported: $\alpha = 0.4$, Mesh Size 11 x 11.
Figure A.2.47: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes. Corner Supported: $\alpha = 0.4$, Mesh Size 11 x 11.

Figure A.2.48: Correction in $\lambda$ for the First Six S-S Modes. Corner Supported: $\alpha = 0.4$, Mesh Size 11 x 11.
Figure A.2.49: Variation of $\lambda$ with $\beta$ for the First Six $A-A$ Modes.
Corner Supported: $\alpha = 0.5$, Mesh Size 11 x 11.

Figure A.2.50: Correction in $\lambda$ for the First Six $A-A$ Modes.
Corner Supported: $\alpha = 0.5$, Mesh Size 11 x 11.
Figure A.2.51: Variation of $\lambda$ with $\beta$ for the First Six S-A Modes. Corner Supported: $\alpha = 0.5$, Mesh Size 11 x 11.

Figure A.2.52: Correction in $\lambda$ for the First Six S-A Modes. Corner Supported: $\alpha = 0.5$, Mesh Size 11 x 11.
Figure A.2.53: Variation of $\lambda$ with $\beta$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.5$, Mesh Size 11 x 11.

Figure A.2.54: Correction in $\lambda$ for the First Six S-S Modes.
Corner Supported: $\alpha = 0.5$, Mesh Size 11 x 11.
APPENDIX A.3

TABLES
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The Frequency Parameter $\lambda$ for the First Six S-A Modes.

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TABLE A.3.3

The Frequency Parameter $\lambda$ for the First Six S-S Modes.

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TABLE A.3.4

The Frequency Parameter $\lambda$ for the First Six A-A Modes.

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### TABLE A.3.5

The Frequency Parameter $\lambda$ for the First Six S-A Modes.

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TABLE A.3.6
The Frequency Parameter $\lambda$ for the First Six S-S Modes.

$\alpha = 0.0$, Mesh Size $= 9 \times 9$

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TABLE A.3.7

The Frequency Parameter $\lambda$ for the First Six A-A Modes.

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The Frequency Parameter $\lambda$ for the First Six S-A Modes.

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The Frequency Parameter $\lambda$ for the First Six S-S Modes.

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TABLE A.3.12
The Frequency Parameter $\lambda$ for the First Six S-S Modes.
$a = 0.0$, Mesh Size $= 11 \times 11$

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TABLE A.3.13

The Frequency Parameter \( \lambda \) for the First Six A-A Modes.

\( a = 0.1, \) Mesh Size = \( 11 \times 11 \)

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<th>Mode #4</th>
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<td>184.792</td>
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TABLE A.3.14

The Frequency Parameter $\lambda$ for the First Six S-A Modes.

$a = 0.1$, Mesh Size $= 11 \times 11$

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<th>Mode #4</th>
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<th>Mode #6</th>
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TABLE A.3.15
The Frequency Parameter \( \lambda \) for the First Six S-S Modes.
\( \alpha = 0.1 \), Mesh Size = \( 11 \times 11 \)

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<th>Mode #5</th>
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TABLE A.3.16
The Frequency Parameter $\lambda$ for the First Six A-A Modes.

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TABLE A.3.17

The Frequency Parameter $\lambda$ for the First Six S-A Modes.

$\alpha = 0.2$, Mesh Size = $11 \times 11$

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TABLE A.3.18

The Frequency Parameter $\lambda$ for the First Six S-S Modes.

$\alpha = 0.2$, Mesh Size $= 11 \times 11$

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TABLE A.3.19

The Frequency Parameter \( \lambda \) for the First Six A-A Modes.

\[ \alpha = 0.3, \text{ Mesh Size } = 11 \times 11 \]

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<th>Mode #4</th>
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TABLE A.3.20
The Frequency Parameter $\lambda$ for the First Six S-A Modes.

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### TABLE A.3.21

The Frequency Parameter $\lambda$ for the First Six S-S Modes.

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TABLE A.3.22

The Frequency Parameter $\lambda$ for the First Six A-A Modes.

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TABLE A.3.23

The Frequency Parameter $\lambda$ for the First Six S-A Modes.

$\alpha = 0.4$, Mesh Size = $11 \times 11$

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TABLE A.3.24

The Frequency Parameter $\lambda$ for the First Six S-S Modes.

$\alpha = 0.4$, Mesh Size $= 11 \times 11$

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TABLE A.3.25

The Frequency Parameter $\lambda$ for the First Six A-A Modes.

$\alpha = 0.5$, Mesh Size $= 11 \times 11$

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<th>Mode #4</th>
<th>Mode #5</th>
<th>Mode #6</th>
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The Frequency Parameter $\lambda$ for the First Six S-A Modes.

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<th>Mode #5</th>
<th>Mode #6</th>
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- $\alpha = 0.5$, Mesh Size $= 11 \times 11$
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TABLE (A.3.28)
Experimental Results, $\beta = 0.01$  $\alpha = 0.0$

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<td>EXP.</td>
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<td>------</td>
<td>--------</td>
<td>------</td>
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### TABLE (A.3.30)

**Experimental Results, $\beta = 0.025$  $\alpha = 0.0$**

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### TABLE (A.3.31)

**Experimental Results, $\beta = 0.025$  $\alpha = 0.1$**

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TABLE (A.3.32)
Experimental Results, $\beta = 0.05$ $\alpha = 0.0$

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TABLE (A.3.33)
Experimental Results, $\beta = 0.05$ $\alpha = 0.1$

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<th>% DISCREPANCY</th>
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TABLE A.3.34

The Frequency Parameter $\lambda_c$ for the First Four Modes.

$\zeta = 0.00$

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<th>Mode #4</th>
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<td>28.3854</td>
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TABLE A.3.35
The Frequency Parameter $\lambda_c$ for the First Four Modes.
$\xi = 0.05$

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</thead>
<tbody>
<tr>
<td>0.2000</td>
<td>11.2841</td>
<td>18.8408</td>
<td>22.4591</td>
<td>37.7752</td>
</tr>
<tr>
<td>0.1000</td>
<td>13.4253</td>
<td>23.9331</td>
<td>27.3876</td>
<td>51.7686</td>
</tr>
<tr>
<td>0.0500</td>
<td>14.5991</td>
<td>26.5255</td>
<td>30.1413</td>
<td>59.6252</td>
</tr>
<tr>
<td>0.0250</td>
<td>15.3226</td>
<td>27.6687</td>
<td>31.2171</td>
<td>63.0569</td>
</tr>
<tr>
<td>0.0200</td>
<td>15.4651</td>
<td>27.8596</td>
<td>31.3710</td>
<td>63.6266</td>
</tr>
<tr>
<td>0.0125</td>
<td>15.6493</td>
<td>28.0951</td>
<td>31.5507</td>
<td>64.3275</td>
</tr>
<tr>
<td>0.0100</td>
<td>15.6976</td>
<td>28.1554</td>
<td>31.5956</td>
<td>64.5056</td>
</tr>
<tr>
<td>0.0050</td>
<td>15.7662</td>
<td>28.2405</td>
<td>31.6600</td>
<td>64.7550</td>
</tr>
<tr>
<td>0.0025</td>
<td>15.7842</td>
<td>28.2630</td>
<td>31.6778</td>
<td>64.8198</td>
</tr>
<tr>
<td>0.0010</td>
<td>15.7893</td>
<td>28.2694</td>
<td>31.6830</td>
<td>64.8381</td>
</tr>
<tr>
<td>0.0005</td>
<td>15.7900</td>
<td>28.2703</td>
<td>31.6838</td>
<td>64.8407</td>
</tr>
</tbody>
</table>
TABLE A.3.36

The Frequency Parameter $\lambda_c$ for the First Four Modes.
$\xi = 0.10$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Mode #1</th>
<th>Mode #2</th>
<th>Mode #3</th>
<th>Mode #4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2000</td>
<td>14.9734</td>
<td>23.8567</td>
<td>26.8215</td>
<td>46.8794</td>
</tr>
<tr>
<td>0.1000</td>
<td>18.3410</td>
<td>30.9465</td>
<td>34.0218</td>
<td>66.2904</td>
</tr>
<tr>
<td>0.0500</td>
<td>20.2747</td>
<td>34.4960</td>
<td>37.7568</td>
<td>77.9649</td>
</tr>
<tr>
<td>0.0250</td>
<td>21.2316</td>
<td>35.8467</td>
<td>39.1123</td>
<td>82.7667</td>
</tr>
<tr>
<td>0.0200</td>
<td>21.3951</td>
<td>36.0476</td>
<td>39.3026</td>
<td>83.4953</td>
</tr>
<tr>
<td>0.0125</td>
<td>21.6037</td>
<td>36.2883</td>
<td>39.5214</td>
<td>84.3738</td>
</tr>
<tr>
<td>0.0100</td>
<td>21.6597</td>
<td>36.3494</td>
<td>39.5744</td>
<td>84.5980</td>
</tr>
<tr>
<td>0.0050</td>
<td>21.7422</td>
<td>36.4363</td>
<td>39.6471</td>
<td>84.9182</td>
</tr>
<tr>
<td>0.0025</td>
<td>21.7648</td>
<td>36.4593</td>
<td>39.6658</td>
<td>85.0034</td>
</tr>
<tr>
<td>0.0010</td>
<td>21.7713</td>
<td>36.4659</td>
<td>39.6710</td>
<td>85.0277</td>
</tr>
<tr>
<td>0.0005</td>
<td>21.7722</td>
<td>36.4668</td>
<td>39.6718</td>
<td>85.0312</td>
</tr>
</tbody>
</table>
TABLE A.3.37

The Frequency Parameter $\lambda_c$ for the First Four Modes.
$\xi = 0.15$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Mode #1</th>
<th>Mode #2</th>
<th>Mode #3</th>
<th>Mode #4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2000</td>
<td>19.5483</td>
<td>29.8829</td>
<td>32.4588</td>
<td>57.8018</td>
</tr>
<tr>
<td>0.1000</td>
<td>24.7939</td>
<td>39.9307</td>
<td>42.7696</td>
<td>84.9415</td>
</tr>
<tr>
<td>0.0500</td>
<td>27.8081</td>
<td>45.0460</td>
<td>48.1201</td>
<td>102.755</td>
</tr>
<tr>
<td>0.0250</td>
<td>29.1429</td>
<td>46.8906</td>
<td>50.0000</td>
<td>110.239</td>
</tr>
<tr>
<td>0.0200</td>
<td>29.3619</td>
<td>47.1562</td>
<td>50.2541</td>
<td>111.377</td>
</tr>
<tr>
<td>0.0125</td>
<td>29.6448</td>
<td>47.4760</td>
<td>50.5396</td>
<td>112.768</td>
</tr>
<tr>
<td>0.0100</td>
<td>29.7238</td>
<td>47.5595</td>
<td>50.6073</td>
<td>113.134</td>
</tr>
<tr>
<td>0.0050</td>
<td>29.8475</td>
<td>47.6841</td>
<td>50.6998</td>
<td>113.682</td>
</tr>
<tr>
<td>0.0025</td>
<td>29.8841</td>
<td>47.7195</td>
<td>50.7241</td>
<td>113.837</td>
</tr>
<tr>
<td>0.0010</td>
<td>29.8950</td>
<td>47.7300</td>
<td>50.7313</td>
<td>113.882</td>
</tr>
<tr>
<td>0.0005</td>
<td>29.8966</td>
<td>47.7315</td>
<td>50.7324</td>
<td>113.888</td>
</tr>
</tbody>
</table>
APPENDIX A.4

FREQUENCY RESPONSE
Figure A.4.1 Overall frequency response.

\[ \alpha = 0.0, \beta = 0.01 \]
Figure A.4.2 Slow sweep of the 1st, 2nd, 3rd and 4th modes.

$\alpha = 0.0, \beta = 0.01$
Figure A.4.3 Slow sweep of the 5th, 6th and 7th modes.

$a = 0.0$, $\beta = 0.01$
Figure A.4.4 Overall frequency response.

\[ \alpha = 0.1, \ \beta = 0.01 \]
Figure A.4.5 Slow sweep of the 1st, 2nd, 3rd and 4th modes.

$\alpha = 0.1$, $\beta = 0.01$
Figure A.4.6 Slow sweep of the 5th, 6th, and 7th modes.

\( a = 0.1, \beta = 0.01 \)
Figure A.4.7 Overall frequency response.

$\alpha = 0.0, \beta = 0.025$
Figure A.4.8 Slow sweep of the 1st, 2nd and 3rd modes.
\[ \alpha = 0.0, \beta = 0.025 \]
Figure A.4.9 Slow sweep of the 4th and 5th modes.

$\alpha = 0.0, \beta = 0.025$
Figure A.4.10 Overall frequency response.
\[ \alpha = 0.1, \beta = 0.025 \]
Figure A.4.11 Slow sweep of the frequency response.

\( a = 0.1, \beta = 0.025 \)
Figure A.4.12 Overall frequency response.
\[ \alpha = 0.0, \beta = 0.05 \]
Figure A.4.13 Slow sweep of the frequency response.

$\alpha = 0.0, \beta = 0.05$

$= 447.5 \text{ Hz }$ = 972.3 & 1012.5 Hz

$= 1231.2 \text{ Hz}$

Co.

Cuad.
Figure A.4.14 Overall frequency response.

\[ a = 0.1, \beta = 0.05 \]
Figure A.4.15 Slow sweep of the frequency response.

\[ \alpha = 0.1, \beta = 0.05 \]
APPENDIX A.5

NYQUIST PLOTS
Figure A.5.1
Nyquist Plot of the 1st Mode.
\[ a = 0.0, \beta = 0.01 \]
\[ r = 88.7 \text{ Hz} \]
Figure A.5.2
Nyquist Plot of the 2nd Mode.
\[ \alpha = 0.0, \beta = 0.01 \]
\[ \gamma = 192.5 \text{ & } 197.1 \text{ Hz} \]
Figure A.5.3
Nyquist Plot of the 3rd Mode.
\[ \alpha = 0.0, \beta = 0.01 \]
\[ f = 24.3 \text{ Hz} \]
Figure A-5.4
Nyquist Plot of the 4th Mode.
\[ \alpha = 0.0, \beta = 0.01 \]
\[ \Gamma = 47.5 \text{ Hz} \]
Figure A.5.5
Nyquist Plot of the 5th Mode.
\[ \alpha = 0.0, \beta = 0.01 \]
\[ \Gamma = 554.6 \text{ Hz} \]
Figure A.5.7
Nyquist Plot of the 1st Mode.
\[ \alpha = 0.0, \beta = 0.025 \]
\[ \gamma = 216.5 \text{ Hz} \]
Figure A.5.8
Nyquist Plot of the 2nd Mode.
\[ \alpha = 0.0, \beta = 0.025 \]
\[ f = 486.0 \text{ Hz} \]
Figure A.5.9
Nyquist Plot of the 3rd Mode.
\[ \alpha = 0.0, \beta = 0.025 \]
\[ \gamma = 609.3 \text{ Hz} \]
Figure A.5.10
Nyquist Plot of the 4th Mode.
\[ \alpha = 0.0, \beta = 0.025, \gamma = 1190.6 \text{ Hz} \]
Figure A.5.11
Nyquist Plot of the 5th Mode.
α = 0.0, β = 0.025
f = 1319.9 Hz
Figure A.5.12
Nyquist Plot of the 1st Mode.
\[ \alpha = 0.0, \beta = 0.05 \]
\[ f = 447.5 \text{ Hz} \]
Figure A.5.13
Nyquist Plot of the 2nd Mode.
\[ \alpha = 0.0 \text{ , } \beta = 0.05 \]
\[ f = 972.3 \text{ & } 1012.5 \text{ Hz} \]
Figure A.5.14
Nyquist Plot of the 3rd Mode.
\[ \alpha = 0.0, \beta = 0.05 \]
\[ \Gamma = 123.12 \text{ Hz} \]
APPENDIX A.6

COLUMN SUPPORTED PLATES

Legend used in the Variation & Correction and Effect of rigidity Plots:

Mode 1: _______________
Mode 2: _____________
Mode 3: _____________
Mode 4: _______________
Figure A.6.1: Variation of $\lambda_c$ with $\beta$, $\xi = 0.00$

Figure A.6.2: Correction in $\lambda_c$, $\xi = 0.00$
Figure A.6.3: Variation of $\lambda_c$ with $\beta$, $\xi = 0.05$

Figure A.6.4: Correction in $\lambda_c$, $\xi = 0.05$
Figure A.6.5: Variation of $\lambda_c$ with $\beta$, $\xi = 0.10$

Figure A.6.6: Correction in $\lambda_c$, $\xi = 0.10$
Figure A.6.7: Variation of $\lambda_c$ with $\beta$, $\xi = 0.15$

Figure A.6.8: Correction in $\lambda_c$, $\xi = 0.15$
Figure A.6.9: Effect of Rigid Column Support, $\beta = 0.2000$

Figure A.6.10: Effect of Rigid Column Support, $\beta = 0.1000$
Figure A.6.11: Effect of Rigid Column Support. $\beta = 0.0500$

Figure A.6.12: Effect of Rigid Column Support. $\beta = 0.0250$
Figure A.6.13: Effect of Rigid Column Support. $\beta = 0.0200$

Figure A.6.14: Effect of Rigid Column Support. $\beta = 0.0125$
Figure A.6.15: Effect of Rigid Column Support. $\beta = 0.0100$

Figure A.6.16: Effect of Rigid Column Support. $\beta = 0.0050$
Figure A.6.17: Effect of Rigid Column Support. $\beta = 0.0025$

Figure A.6.18: Effect of Rigid Column Support. $\beta = 0.0010$
Figure A.6.19: Effect of Rigid Column Support. $\beta = 0.0005$
APPENDIX A.a

MODE SHAPES
Figure A.a.1: Mode Shapes for the First Frequency
Corner Supported: $\alpha = 0.0$
Figure A.a.2: Mode Shapes for the Second Frequency
Corner Supported: $\alpha = 0.0$
Figure A.3.3: Mode Shapes for the Third Frequency
Corner Supported: $\alpha = 0.0$
Figure A.a.4: Mode Shapes for the Fourth Frequency
Corner Supported: $\alpha = 0.0$
Figure A.a.5: Mode Shapes for the Fifth Frequency
Corner Supported: $\alpha = 0.0$
Figure A.a.6: Mode Shapes for the Sixth Frequency
Corner Supported: $\alpha = 0.0$
Figure A.a.7: Mode Shapes for the First Frequency
Point Supported: $\alpha = 0.1$
Figure A.a.8: Mode Shapes for the Second Frequency
Point Supported: $\alpha = 0.1$
Figure A.a.9: Mode Shapes for the Third Frequency
Point Supported: $\alpha = 0.1$
Figure A.a.10: Mode Shapes for the Fourth Frequency
Point Supported: $\alpha = 0.1$
Figure A.a.11: Mode Shapes for the Fifth Frequency
Point Supported: $\alpha = 0.1$
Figure A.a.12: Mode Shapes for the Sixth Frequency
Point Supported: $\alpha = 0.1$
Figure A.a.13: Mode Shapes for the First Frequency
Point Supported: $\alpha = 0.2$
Figure A.a.14: Mode Shapes for the Second Frequency
Point Supported: $\alpha = 0.2$
Figure A.a.15: Mode Shapes for the Third Frequency
Point Supported: $\alpha = 0.2$
Figure A.a.16: Mode Shapes for the Fourth Frequency
Point Supported: $\alpha = 0.2$
Figure A.a.17: Mode Shapes for the Fifth Frequency
Point Supported: $\alpha = 0.2$
Figure A.a.18: Mode Shapes for the Sixth Frequency
Point Supported: \( \alpha = 0.2 \)
Figure A.a.19: Mode Shapes for the First Frequency
Point Supported: $\alpha = 0.3$
Figure A.a.20: Mode Shapes for the Second Frequency
Point Supported: $\alpha = 0.3$
Figure A.a.21: Mode Shapes for the Third Frequency
Point Supported: $\alpha = 0.3$
Figure A.a.22: Mode Shapes for the Fourth Frequency
Point Supported: $\alpha = 0.3$
Figure A.a.23: Mode Shapes for the Fifth Frequency
Point Supported: $\alpha = 0.3$
Figure A.a.24: Mode Shapes for the Sixth Frequency
Point Supported: $\alpha = 0.3$
Figure A.a.25: Mode Shapes for the First Frequency
Point Supported: $\alpha = 0.4$
Figure A.a.26: Mode Shapes for the Second Frequency
Point Supported: $\alpha = 0.4$
Figure A.a.27: Mode Shapes for the Third Frequency
Point Supported: $\alpha = 0.4$
Figure A.a.28: Mode Shapes for the Fourth Frequency
Point Supported: $\alpha = 0.4$
Figure A.a.29: Mode Shapes for the Fifth Frequency
Point Supported: $\alpha = 0.4$
Figure A.a.30: Mode Shapes for the Sixth Frequency
Point Supported: $\alpha = 0.4$
Figure A.a.31: Mode Shapes for the First Frequency
Point Supported: $\alpha = 0.5$
Figure A.a.32 : Mode Shapes for the Second Frequency
Point Supported : $\alpha = 0.5$
Figure A.a.33: Mode Shapes for the Third Frequency
Point Supported: $\alpha = 0.5$
Figure A.a.34: Mode Shapes for the Fourth Frequency
Point Supported: $\alpha = 0.5$
Figure A.a.35: Mode Shapes for the Fifth Frequency
Point Supported: α = 0.5
Figure A.a.36: Mode Shapes for the Sixth Frequency
Point Supported: $\alpha = 0.5$
APPENDIX A.6

NODAL PATTERNS
Figure A.b.1 : Nodal pattern
\[ \alpha = 0.0, \beta = 0.2000 \]
Figure A.b.2: Nodal pattern
\( \alpha = 0.0 \), \( \beta = 0.0005 \).
Figure A.3: Nodal pattern
\[ \alpha = 0.1, \beta = 0.2000 \]
Figure A.b.4: Nodal pattern
\( \alpha = 0.1, \beta = 0.0005 \).
Figure A.b.5 : Nodal pattern
\[\alpha = 0.2, \beta = 0.2000\]
Figure A.b.6 : Nodal pattern

$\alpha = 0.2$, $\beta = 0.0005$. 
Figure A.b.7: Nodal pattern
\( \alpha = 0.3, \beta = 0.2000 \).
Figure A.b.8: Nodal pattern

\[ \alpha = 0.3, \beta = 0.0005 \]
Figure A.b.9: Nodal pattern
\[ \alpha = 0.4, \beta = 0.2000 \]
Figure A.b.10: Nodal pattern
\[ \alpha = 0.4, \beta = 0.0005. \]
Figure A.b.11 : Nodal pattern
\[ \alpha = 0.5, \beta = 0.2000 \]
Figure A.b.12: Nodal pattern
$\alpha = 0.5$, $\beta = 0.0005$. 
APPENDIX A.c

COMPUTER PROGRAM
VIBRATION ANALYSIS OF POINT AND COLUMN SUPPORTED
MINDLIN PLATES

IMPLICIT REAL *8 (A-H,O-Z)
INTEGER R,C
REAL *8 LAMBDA,MACHIEP
DIMENSION V(319),U(319),F(319,35),ZL(319,6)
DIMENSION INT(319),L(319),C(319)
DIMENSION LAMBDA(10)
DIMENSION A2(319,71)
COMMON / ARYS / WF(11,11),FZ(11,11),FN(11,11)
COMMON / IWFN / IFZ(121),IFN(121)
COMMON / POSN / IX,IIY,MM,NN,MMF,NNF,LINE
COMMON / STOR / IX,M1,M2,ITS(400)
COMMON / HOLE / NH11,NH12,MI11,MI12
COMMON / ATAR / A(319),TR(319)
COMMON / ICQF / ICQ(6),ICF(6)
COMMON / MESH / XX(15),YY(15)
COMMON / COAR / AI(319,71)
COMMON / ZRQF / ZR(319,6)
COMMON / NODAL / D(10),AA(2),BB(2),CC(2,2),DD(2,2)
COMMON / PONTQ / XPONTQ(121,6),YPONTQ(121,6)
COMMON / PONTF / XPONTF(441,6),YPONTF(441,6)
COMMON / CORDQ / XCORDQ(121),YCORDQ(121)
COMMON / CORDF / XCORDF(441),YCORDF(441)
COMMON / ETAQ / ANQ(121,6),FNQ(11,11,6)
COMMON / ETAQ / AQ(121,6),FZQ(11,11,6)
COMMON / WETAQ / AWQ(121,6),WFQ(11,11,6)
COMMON / WETAQ / ANF(441,6),FNF(21,21,6)
COMMON / ETAF / AZF(441,6),FZF(21,21,6)
COMMON / WETAQ / AWF(441,6),WFF(21,21,6)
COMMON / CODEQ / XNODEQ(11),YNODEQ(11)
COMMON / CODEF / XNODEF(21),YNODEF(21)
COMMON / INTER / P,Q,S,T

PI = DASOS(-1.0)
RK = (PI*PI)/12.0
BETA = 0.2000D0
RMU = 0.3D0
FACTOR = 5.00D4
LINE = 9
ALPHA = (LINE-1.)/20.
BETA = BETA*2.0D0
BP = BETA*BETA/12.0D0
D(1) = BP
D(2) = (RK*(1.00-RMU))/2.0D0
D(3) = BP
D(4) = (RK*(1.00-RMU))/2.0D0
D(5) = 2.00*RMU*BP
D(6) = (BP*(1.00-RMU))/2.0D0
D(7) = 1.00/12.0D0
D(8) = 1.00/(BETA*BETA)
DO 31 I = 1, 6 
31 D(I) = D(I)/(12.D0*BP*BP) 
    M1 = 35 
    M2 = 35 
    M = M1 + M2 + 1 
    GAMA = 1.D0 
    N = 319 
    NS = 319 
    NN = 11 
    MM = 11 
    IM = NN-IMM 
    NE = 6 
    NM = NN*MM 
DO 42 I = 1, NE 
42 LAMBDA(I) = (LAMBDA(I)**2)/(FACTOR*16.D0) 
15 HX = 1.D0/(MM-1) 
    HY = GAMA/(NN-1) 
    RH = HY/IX 
DO 16 I = 1, 15 
    XX(I) = 1.D0 
16 YY(I) = 1.D0 
    MM1 = MM-1 
    DO 17 JJ = 2, MM1 
    WF(1, JJ) = 1.D0 
    WF(NN, JJ) = 0.D0 
    FZ(1, JJ) = 1.D0 
    FZ(NN, JJ) = 0.D0 
    FN(1, JJ) = 1.D0 
17 FN(NN, JJ) = 1.D0 
    NNI = NN-1 
    DO 18 II = 2, NNI 
    WF(II, 1) = 1.D0 
    WF(II, MM) = 0.D0 
    FZ(II, 1) = 1.D0 
    FZ(II, MM) = 1.D0 
    FN(II, 1) = 1.D0 
18 FN(II, MM) = 0.D0 
    WF(1, I) = 1.D0 
    FZ(1, I) = 1.D0 
    FN(1, I) = 1.D0 
    WF(1, MM) = 0.D0 
    FZ(1, MM) = 1.D0 
    FN(1, MM) = 0.D0 
    WF(NN, 1) = 0.D0 
    FZ(NN, 1) = 0.D0 
    FN(NN, 1) = 1.D0 
    WF(NN, MM) = 0.D0 
    FZ(NN, MM) = 0.D0 
    FN(NN, MM) = 0.D0 
    NN1 = NN-1 
    MM1 = MM-1 
    DO 19 II = 2, NNI 
    DO 19 JJ = 2, MM1 
    WF(II, JJ) = 1.D0 
    FZ(II, JJ) = 1.D0
19  FN(I,J,J) = 1.D0
    WF(LINE,LINE) = 0.D0
    MM1 = MM-1
    DO 40 II = 1,NN
    DO 40 JJ = 1,MM
      CALL INTEGR2(I,J,J,IIX,IIY,II,NN,NX,NN,MM)
        NN1 = NN-1
        DO 21 II = 1,NN1
        DO 21 JJ = 1,MM
      CALL INTEGR3(I,J,J,IIX,IIY,II,NN,NX,NN,MM)
        DO 23 II = 1,NN
        DO 23 JJ = 1,MM
      CALL INTEGR4(I,J,J,IIX,IIY,II,NN,MM)
    25  DO 30 I = 1,N
        DO 30 J = 1,M
      A1(I,J) = A1(I,J)/(TR(I)*FACTOR)
        LA = 10
        MACHEP = 1.0D-18
        CALL UNSRAY(N,M1,M2,M,NE,MACHEP,A1,I,A,LAMBDA,L,C,ZR,ZL,
        & A2,V,F,U,INT)
        DO 34 I = 1,NE
        RTLAMD = 4.D0*DSQRT(LAMBDA(I)*FACTOR)
    34  WRITE(6,35) RTLAMD
    35  FORMAT(10X,'RLAMD = ',E24.15)
        DO 43 I = 1,NE
        EIGN = 4.D0*((BETA*BETA)/DSQRT(12.D0))*(DSQRT(LAMBDA(I)*FACTOR))
    43  WRITE(6,44) EIGN
    44  FORMAT(10X,'EIGN = ',E24.15)
      CALL MODE(NM,N,NE)
    STOP
  END

C**********************************************************************
FUNCTION F(I,J,J)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON / POSN / II,IIY,MM,NN,MMF,NNF,LINE
  KII = 0
  KV = 0
  KP = 0
  IF(II.EQ.NN) KII = 2
  IF(JJ.EQ.MM) KV = 1
  IF(II.EQ.LINE.AND.JJ.GE.LINE) KP = 1
  IF(II.GT.LINE) KP = 1
  10  F = (3*MM-2)*(II-1)-KP+JJ*(3-KII)-KV
      RETURN
  END

C**********************************************************************
SUBROUTINE MODE(NM,N,NE)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON / ZRQF / ZRQF(319,6)
COMMON / ICQF / ICQF(6)

COMMON / POSN / IIX, HY, MM, NN, MMF, NNF, LINE
COMMON / IWZN / IWF(121), IFN(121), IFZ(121)
COMMON / ARYS / WF(11,11), FZ(11,11), FN(11,11)
COMMON / PONTQ / XPONTQ(121,6), YPONTQ(121,6)
COMMON / PONTF / XPONTF(441,6), YPONTF(441,6)
COMMON / CORDQ / XCORDQ(121), YCORDQ(121)
COMMON / CORDF / XCORDF(441), YCORDF(441)
COMMON / NETAQ / ANQ(121,6), FNQ(11,11,6)
COMMON / ZETAQ / AZQ(121,6), FZQ(11,11,6)
COMMON / WETAQ / AWQ(121,6), WFQ(11,11,6)
COMMON / NETAF / ANF(441,6), FNF(21,21,6)
COMMON / ZETAF / AZF(441,6), FZF(21,21,6)
COMMON / WETAF / AWF(441,6), WFF(21,21,6)
COMMON / CODEQ / XNODEQ(11), YNODEQ(11)
COMMON / CODEF / XNODEF(21), YNODEF(21)

C-----------------------------------------------------------------------------------

NNF = (2*NN)-1
MMF = (2*MM)-1
DO 10 J = 1, MM
XNODEQ(J) = (J-1)*IIX
10 YNODEQ(J) = (J-1)*I1Y
      K = 1
      DO 11 I = 1, NN
      DO 11 J = 1, MM
         XCORDQ(K) = XNODEQ(J)
         YCORDQ(K) = YNODEQ(I)
11 K = K + 1
      DO 12 J = 1, MMF
         XNODEF(J) = (J-1)*IIX/2.0D0
12 YNODEF(J) = (J-1)*I1Y/2.0D0
      K = 1
      DO 13 I = 1, NNF
      DO 13 J = 1, MMF
         XCORDF(K) = XNODEF(J)
         YCORDF(K) = YNODEF(I)
13 K = K + 1
      DO 14 J1 = 1, NE
14 CALL MODEQ(J1, NE)
      DO 15 J1 = 1, NE
15 CALL MODEF(J1, NF)
RETURN
END

C-----------------------------------------------------------------------------------

SUBROUTINE MODEQ(J1, NE)
IMPLICIT REAL *8 (A-H,O-Z)

ALL COMMON UESD IN SUBROUTINE MODE

IK = 1
DO 11 I = 1, NN
DO 11 J = 1, MM
IFN(IK) = F(I,J)
IFZ(IK) = F(I,J)-1
IWF(IK) = F(I,J) - 2
ANQ(IK,J1) = ZR(IFN(IK),J1)
AZQ(IK,J1) = ZR(IFZ(IK),J1)
IF(IWF(IK),EQ,0) GO TO 12
AWQ(IK,J1) = ZR(IWF(IK),J1)*2*D0
12 IF(FN(I,J),EQ,0) ANQ(IK,J1) = 0.D0
  IF(FZ(I,J),EQ,0) AZQ(IK,J1) = 0.D0
  IF(WF(I,J),EQ,0) AWQ(IK,J1) = 0.D0
11  IK = IK + 1
   IK = 1
   DO 13 I = 1,NN
13   DO 13 J = 1,MM
   WFQ(I,J,J1) = AWQ(IK,J1)
   FZQ(I,J,J1) = AZQ(IK,J1)
   FNQ(I,J,J1) = ANQ(IK,J1)
   CALL NODEQ(J1)
   RETURN
END

C******************************************************************************
SUBROUTINE NODEQ(J1)
IMPLICIT REAL *8 (A-H,I-O-Z)

ALL COMMON USED IN SUBROUTINE MODE

ICQ(J1) = 0
DO 41 I = 1,NN
41   DO 42 J = 1,MM-1,1
   IF(WFQ(I,J+1,J1),EQ,0) GO TO 44
   GO TO 45
44  ICQ(J1) = ICQ(J1) + 1
   XPONTQ(ICQ(J1),J1) = XNODEQ(J1 + 1)
   YPONTQ(ICQ(J1),J1) = XNODEQ(J1)
   GO TO 43
45  IF((WFQ(I,J,J1)*WFQ(I,J+1,J1),LE,0.D0)GO TO 46
   GO TO 43
46  ICQ(J1) = ICQ(J1) + 1
   AR = DABS(WFQ(I,J ,J1))
   AL = DABS(WFQ(I,J+1,J1))
   RATIO = (AR/(AR + AL))*IX
   XPONTQ(ICQ(J1),J1) = XNODEQ(J1) + RATIO
   YPONTQ(ICQ(J1),J1) = XNODEQ(J1)
43  CONTINUE
42  CONTINUE
41  CONTINUE
   DO 51 J = 1,MM
51   DO 52 I = 1,NN-1,1
   IF(WFQ(I+1,J,J1),EQ,0) GO TO 54
   GO TO 55
54  ICQ(J1) = ICQ(J1) + 1
   XPONTQ(ICQ(J1),J1) = XNODEQ(J1)
   YPONTQ(ICQ(J1),J1) = XNODEQ(J1 + 1)
   GO TO 53
55 IF((WFQ(I,J,J1)*WFQ(I+1,J,J1)).LE.0.D0) GO TO 56
   GO TO 53
56 ICQ(J1) = ICQ(J1) + 1
   AR = ABS(WFQ(I,J,J1))
   AL = ABS(WFQ(I+1,J,J1))
   RATIO = (AR/(AR+AL))*IIY
   XPONTQ(ICQ(J1),J1) = XNODEQ(J)
   YPONTQ(ICQ(J1),J1) = XNODEQ(I)+ RATIO
53 CONTINUE
52 CONTINUE
51 CONTINUE
   DO 61 I = 1,ICQ(J1)
   DO 62 J = 1+1,ICQ(J1)
65 IF((DABS(XPONTQ(J,J1)-XPONTQ(I,J1)).LT.1.D-8).AND. 
   .&  (DABS(YPONTQ(J,J1)-YPONTQ(I,J1)).LT.1.D-8)) GO TO 63
   GO TO 62
63 DO 64 JJ = J,ICQ(J1)-1,1
   XPONTQ(J,J1) = XPONTQ(J+1,J1)
   YPONTQ(J,J1) = YPONTQ(J+1,J1)
64 CONTINUE
   ICQ(J1) = ICQ(J1) - 1
   IF(J.LT.ICQ(J1)) GO TO 65
62 CONTINUE
61 CONTINUE
   RETURN
END

C ******************************************************
SUBROUTINE MODEF(J1,NE)
IMPLICIT REAL *8 (A-H,O-Z)

ALL COMMON UESD IN SUBROUTINE MODE

DO 11 I = 1,NN
DO 11 J = 1,MM
J3 = NNF.I + 1
J3 = MMF.J + 1
WFQ(I,J,J1) = WFQ(I,J,J1)+1.D0
WFQ(I,J3,J1) = WFQ(I,J1)*(-1.D0)
WFQ(I3,J,J1) = WFQ(I,J1)*(-1.D0)
WFQ(I3,J3,J1) = WFQ(I,J1)+1.D0
FZF(I,J,J1) = FZF(I,J,J1)+1.D0
FZF(I,J3,J1) = FZF(I,J1)+1.D0
FZF(I3,J,J1) = FZF(I,J1)+1.D0
FZF(I3,J3,J1) = FZF(I,J1)+1.D0
FNF(I,J,J1) = FNQ(I,J,J1)+1.D0
FNF(I,J3,J1) = FNQ(I,J1)+1.D0
FNF(I3,J,J1) = FNQ(I,J1)+1.D0
FNF(I3,J3,J1) = FNQ(I,J1)+1.D0
11 CONTINUE
WRITE(6,10)
   IK = 1
DO 12 I = 1,NNF
DO 12 J = 1,MMF
`AWF(I,K,J) = WFF(I,J,J)`
`AZF(I,K,J) = FZF(I,J,J)`
`ANF(I,K,J) = FNF(I,J,J)`
`WRITE(6,20) I,K,J,AWF(I,K,J),AZF(I,K,J),ANF(I,K,J)`
`& ,XCORDF(IK),YCORDF(IK)`

12 **IK = IK + 1**

13 **WRITE(6,31) J**

14 **DO 15 I = 1,NNF**

15 **WRITE(6,30)(FNF(I,J,J),J = 1,MMF)**

16 **CALL NODF(I)**

10 **CALL(I2X,'NE',2X,'N',3X,'J',8X,'AWI',11X,'AZ',**
`& 11X,'AN',9X,'XCOR',3X,'YCOR'`
`& ,72('..'))`

20 **FORMAT(4(I,J,I3),3(2X,F12.9),2(3X,F4.2))**

30 **FORMAT(1(I2X,F8.5))**

31 **FORMAT(649(' = '),\*AW BY NODE \',49(' = '),\*NE = \',12)**

32 **FORMAT(649(' = '),\*AZ BY NODE \',49(' = '),\*NE = \',12)**

33 **FORMAT(649(' = '),\*AN BY NODE \',49(' = '),\*NE = \',12)**

RETURN

END

```c
 SUBROUTINE NODF(I)
 IMPLICIT REAL *8 (A-H,0-Z)

 ALL COMMON UEDS IN SUBROUTINE MODE

 ICF(J1) = 0
 DO 11 I = 1,ICQ(I1)
 XPONTF(ICF(J1) + 1,J1) = XPONTF(IJ,J1)/2.D0
 YPONTF(ICF(J1) + 1,J1) = YPONTF(IJ,J1)/2.D0
 XPONTF(ICF(J1) + 2,J1) = XPONTF(IJ,J1)/2.D0
 YPONTF(ICF(J1) + 2,J1) = 1.D0-(YPONTF(IJ,J1)/2.D0)
 XPONTF(ICF(J1) + 3,J1) = 1.D0-(XPONTF(IJ,J1)/2.D0)
 YPONTF(ICF(J1) + 3,J1) = YPONTF(IJ,J1)/2.D0
 XPONTF(ICF(J1) + 4,J1) = 1.D0-(XPONTF(IJ,J1)/2.D0)
 YPONTF(ICF(J1) + 4,J1) = 1.D0-(YPONTF(IJ,J1)/2.D0)
 ICF(J1) = ICF(J1) + 4

11 CONTINUE
 DO 21 I = 1,ICF(J1)
 DO 22 J = 1+1,ICF(J1)
 21 IF((DABS(XPONTF(J1,J1)-XPONTF(IJ,J1)),1.T.1.D-8).AND.
`& (DABS(YPONTF(J1,J1)-YPONTF(IJ,J1)),1.T.1.D-8)) GO TO 23`
 22 GO TO 22
 DO 24 I = J,ICF(J1)-1,1
 XPONTF(IJ,J1) = XPONTF(IJ+1,J1)
 YPONTF(IJ,J1) = YPONTF(IJ+1,J1)

24 CONTINUE
```
ICF(J1) = ICF(J1) - 1
IF(J, LE, ICF(J1)) GO TO 25
22 CONTINUE
21 CONTINUE
WRITE(6,30)ICF(J1)
WRITE(6,10)J1
WRITE(6,20)(XPOINT(KL,J1),YPOINT(KL,J1),KL = I,ICF(J1))
10 FORMAT(34("="),/,'NODAL LINE NE = ',I2,'/',34("=")
&'/XPOINT   YPOINT')
20 FORMAT(2(F16.14,2X))
30 FORMAT(,'ICF(J1) = ',I4)
RETURN
END

C ***********************************************
SUBROUTINE DATA(I,J)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / MESH1 / XX(15),YY(15)
COMMON / INTER / P,Q,S,T
I = II + 1
J = JJ + 1
J1 = J - 1
P = XX(J1)
Q = XX(I)
I1 = I - 1
S = YY(I1)
T = YY(I)
RETURN
END

C ***********************************************
SUBROUTINE INTEGRAL(I,J,H,X,RH,N,M,NS,NN,MM)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON / NODAL / D(10),AA(2),BB(2),CC(2,2),DD(2,2)
COMMON / ARYS / WF(11,11),FZ(11,11),FN(11,11)
COMMON / HOLE / NH1,NH2,MH1,MH2
COMMON / ATAR / A(319),TR(319)
COMMON / COAR / A(319,71)
COMMON / INTER / P,Q,S,T
CALL DATA(I,J)
IF(I.GT.1) GO TO 10
TINTEGRAL = (Q*T*RHO*HX*IXX)/2.D0
GO TO 20
10 IF(I.LE.III) GO TO 15
TINTEGRAL = (Q*S*RHO*HX*IX)/2.D0
GO TO 20
15 IF(I.I.EQ.1) GO TO 25
TINTEGRAL = (Q*(S+T)*RHO*HX*IX)/2.D0
20 IZ = 1
I = 1
J1 = JJ + 1
AA(1) = (-1.D0/(Q*IXX))*FZ(I,J1)
AA(2) = (1.D0/(Q*IXX))*FZ(I,J1)
CALL DERIV(I,J,J1,IZ,TINTEGRAL,N,M,NS)
IZ = 2
I = 2
J1 = JJ + 1
AA(1) = 0.5D0*FZ(I, J)
AA(2) = 0.5D0*FZ(I, J)
BB(1) = (1.0D0/(Q*IX))*WF(I, J)
BB(2) = (1.0D0/(Q*IX))*WF(I, J)
CALL DERIV(I, J, IZ, TINTEG, N, M, NS)
30 RETURN
END

C ........................................................................
SUBROUTINE INTEGR2(I, JJ, I, IX, RII, N, M, NS, NN, MM)
IMPLICIT REAL *8 (A-H, O-Z)
COMMON / NODAL / D(I0), AA(2), BB(2), CC(2, 2), DD(2, 2)
COMMON / ARYS / WF(I1, I1), FZ(I1, I1), FN(I1, I1)
COMMON / HOLE / NH1, NH2, MII1, MII2
COMMON / ATAR / A(319), TR(319)
COMMON / COAR / A(319, 71)
COMMON / INTER / P, Q, S, T
CALL DATA(I, J)
IF(JJ > 1) GO TO 10
TINTEG = (T*Q*RII*IX*IX)/2.0
GO TO 20
10 IF(JJ > MM) GO TO 15
TINTEG = (T*P*RII*IX*IX)/2.0
GO TO 20
15 IF(I1 > 1) GO TO 25
TINTEG = (T*(P + Q)*RII*IX*IX)/2.0
20 IZ = 3
L = 3
II = II + 1
AA(1) = (-1.0D0/(T*RII*IX))*FN(I, J)
AA(2) = (1.0D0/(T*RII*IX))*FN(I, J)
CALL DERIV(I, J, IZ, TINTEG, N, M, NS)
IZ = 4
L = 4
II = II + 1
AA(1) = 0.5D0*FN(I, J)
AA(2) = 0.5D0*FN(I, J)
BB(1) = (1.0D0/(T*RII*IX))*WF(I, J)
BB(2) = (1.0D0/(T*RII*IX))*WF(I, J)
CALL DERIV(I, J, IZ, TINTEG, N, M, NS)
30 RETURN
END

C ........................................................................
SUBROUTINE INTEGR3(I, JJ, I, IX, RII, N, M, NS, NN, MM)
IMPLICIT REAL *8 (A-H, O-Z)
COMMON / NODAL / D(I0), AA(2), BB(2), CC(2, 2), DD(2, 2)
COMMON / ARYS / WF(I1, I1), FZ(I1, I1), FN(I1, I1)
COMMON / HOLE / NH1, NH2, MII1, MII2
COMMON / ATAR / A(319), TR(319)
COMMON / COAR / A(319, 71)
COMMON / INTER / P, Q, S, T
CALL DATA(I, J)
IF(I1 > 1) GO TO 10
10 TINTEG = Q*T*RII*IX*IX
IZ = 5
L = 5
II = II + 1
J1 = JJ + 1
CC(1,1) = (-0.5D0/(Q*11X))*FZ(I1,J1)
CC(1,2) = (0.5D0/(Q*11X))*FZ(I1,J1)
CC(2,1) = (-0.5D0/(Q*11X))*FZ(I1,J1)
CC(2,2) = (0.5D0/(Q*11X))*FZ(I1,J1)
DD(1,1) = (-0.5D0/(T*R11*11X))*FN(I1,J1)
DD(1,2) = (0.5D0/(T*R11*11X))*FN(I1,J1)
DD(2,1) = (0.5D0/(T*R11*11X))*FN(I1,J1)
DD(2,2) = (0.5D0/(T*R11*11X))*FN(I1,J1)
CALL DERIV(I1,J1,I1Z,TINTEG,N,M,NS)
IZ = 6
L = 6
II = II + 1
J1 = JJ + 1
CC(1,1) = (-0.5D0/(T*R11*11X))*FZ(I1,J1)
CC(1,2) = (0.5D0/(T*R11*11X))*FZ(I1,J1)
CC(2,1) = (0.5D0/(T*R11*11X))*FZ(I1,J1)
CC(2,2) = (-0.5D0/(T*R11*11X))*FZ(I1,J1)
DD(1,1) = (-0.5D0/(Q*11X))*FN(I1,J1)
DD(1,2) = (0.5D0/(Q*11X))*FN(I1,J1)
DD(2,1) = (0.5D0/(Q*11X))*FN(I1,J1)
DD(2,2) = (-0.5D0/(Q*11X))*FN(I1,J1)
CALL DERIV(I1,J1,I1Z,TINTEG,N,M,NS)
15 RETURN
END

C******************************************************************************

SUBROUTINE INTEG4(I1J11X,RI1,NN,MM)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON /NODA1./ D(10),AA(2),BB(2),CC(2),DD(2,2)
COMMON /ARYS/WF(I1,11),FZ(I1,11),FN(I1,11)
COMMON /HOLE/NH1,NH2,MH1,MH2
COMMON /ATAR/A(319),TR(319)
COMMON /COAR/A(319,71)
COMMON /INTER/P,Q,S,T

C

CALL DATA(I1J1)
I = F(I1J1)
II = I-1
J2 = J1-2
IF(J1.EQ.1.OR.I1.EQ.NN) GO TO 3
IF(J1.GT.1.AND.J1.LT.MM) GO TO 4
3 IF(I1.GT.1) GO TO 5
TINTEG = (T*(P+Q)*R11*11X)/4.D0
IF(J1.EQ.1) TINTEG = (T*Q*R11*11X)/4.D0
IF(J1.EQ.MM) TINTEG = (P*T*R11*11X)/4.D0
GO TO 10
5 IF(I1.EQ.NN) GO TO 6
IF(J1.EQ.1) TINTEG = (Q*(S+T)*R11*11X)/4.D0
IF(J1.EQ.MM) TINTEG = (P*(S+T)*R11*11X)/4.D0
GO TO 10
6 TINTEG = (S*(P+Q)*R11*11X)/4.D0
IF(J1.EQ.1) TINTEG = (S*Q*R11*11X)/4.D0
IF(J1.EQ.MM) TINTEG = (P*S*R11*11X)/4.D0
GO TO 10
IF(HL.EQ.1) GO TO 7

TINTEG=((P+Q)*(S+T)*RH*IX*IX)/4.D0

IF(FN(HJ,J).EQ.1.D0) TR(I)=D(7)*TINTEG
IF(FZ(JI,J).EQ.1.D0) TR(J)=D(7)*TINTEG
IF(WF(JI,J).EQ.1.D0) TR(IJ)=D(8)*TINTEG

RETURN

END

C******************************************************************************

SUBROUTINE DERIV(HJ,J,J,JZ,TINTEG,N,M,NS)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON /NODAL/ D(I0),AA(2),BB(2),CC(2,2),DD(2,2)
COMMON /ARYS/ WF(11,11),FZ(11,11),FN(11,11)
COMMON /ATAR/ A(319),TR(319)
COMMON /COAR/ A(319,71)
GO TO (7,8,9,10,11,12),JZ

II=II

DO 15 JI=1,2
IF(AA(JI).EQ.0.D0) GO TO 15
J1=JI+JI-1
I=FI(JI,JI)-1
J=FII(JI,JI)-1
IF(AA(I).EQ.0.D0) GO TO 16
A(I)=TINTEG*D(I)*AA(JI)*AA(I)

16 J2=JI+1
J=FII(JI,J2)-1
IF(AA(2).EQ.0.D0) GO TO 17
A(J)=TINTEG*D(I)*AA(JI)*AA(I)

17 CALL STORAG(I,N,M,NS)
15 CONTINUE

DO 18 I=1,2
AA(I,.I)=.D0
GO TO 20

8 KX=1
KY=2

II=II

DO 21 JI=1,2
IF(AA(JI).EQ.0.D0) GO TO 21
J1=JI+JI-1
I=FI(JI,JI)-KX
DO 22 IK=1,2
J2=JI+IK-1
J=FII(JI,J2)-KX
IF(AA(IK).EQ.0.D0) GO TO 23
A(J)=TINTEG*D(I)*AA(JI)*AA(IK)

22 CONTINUE
CALL STORAG(I,N,M,NS)
21 CONTINUE

IF(KX.EQ.2) GO TO 24
KX=2
KY=1
DO 25 I=1,2
VR=AA(I,.I)

25 CONTINUE
\[ \Lambda(A \cdot L) = BB(LL) \]

25  \[ BB(LL) = VR \]
    GO TO 26

24  DO 27 LL = 1,2
    \[ \Lambda(A \cdot L) = 0.D0 \]
27  \[ BB(LL) = 0.D0 \]
    GO TO 20

9  J1 = JJ
    DO 31 JJ = 1,2
    IF(\[A \cdot A(JJ), EQ, 0.D0\]) GO TO 31
    J1 = J1 + JJ-1
    I1 = F(I1, J1)
    J1 = F(I1, JJ)
    IF(\[A \cdot A(I1), EQ, 0.D0\]) GO TO 33
    \[ A(J) = TINTEG*D(I1)*\Lambda(A(JJ))*\Lambda(A(I1)) \]
33  I2 = I1 + 1
    J = F(I2, JJ)
    IF(\[A \cdot A(J), EQ, 0.D0\]) GO TO 34
    \[ A(J) = TINTEG*D(I1)*\Lambda(A(JJ))*\Lambda(A(J)) \]
34  CALL STORAG(I, N, M, NS)
31  CONTINUE
    DO 28 LL = 1,2
28  \[ \Lambda(A \cdot L) = 0.D0 \]
    GO TO 20
10  KX = 0
    KY = 2

40  J1 = JJ
    DO 35 JJ = 1,2
    IF(\[A \cdot A(JJ), EQ, 0.D0\]) GO TO 35
    J1 = J1 + JJ-1
    I1 = F(I1, J1)-KX
    DO 36 IK = 1,2
    I2 = I1 + IK-1
    J = F(I2, JJ)-KX
    IF(\[A \cdot A(IK), EQ, 0.D0\]) GO TO 37
    \[ A(J) = TINTEG*D(I1)*\Lambda(A(JJ))*\Lambda(A(IK)) \]
37  J = F(I2, JJ)-KY
    IF(\[BB(IK), EQ, 0.D0\]) GO TO 36
    \[ A(J) = TINTEG*D(I1)*\Lambda(A(JJ))*BB(IK) \]
36  CONTINUE
    CALL STORAG(I, N, M, NS)
35  CONTINUE
    IF(\[KX, EQ, 2\]) GO TO 38
    KX = 2
    KY = 0
    DO 39 LL = 1,2
    VR = \[\Lambda(A \cdot L) \]
    \[ \Lambda(A \cdot L) = BB(LL) \]
39  \[ BB(LL) = VR \]
    GO TO 40
38  DO 41 LL = 1,2
    \[ \Lambda(A \cdot L) = 0.D0 \]
41  \[ BB(LL) = 0.D0 \]
    GO TO 20
11  KX = 1
KY = 0
51   DO 47 J = 1, 2
    DO 47 J1 = 1, 2
    IF(CC(I, J, I1, J1), EQ, 0, D0) GO TO 47
    I1 = I1 + 1
    J1 = J1 + 1
    I = F(I1, J1, KX)
    DO 48 IK = 1, 2
    DO 48 JK = 1, 2
    I2 = I2 + 1
    J2 = J2 + 1
    J = F(I2, J2, KY)
    IF(DD(IK, JK), EQ, 0, D0) GO TO 48
    A(J) = TINTEG * D(I, I1) * CC(I1, JI) * DD(IK, JK) / 2, D0
48   CONTINUE
    CALL STORAG(I, N, M, NS)
47   CONTINUE
    IF(KX, EQ, 0) GO TO 49
    KX = 0
    KY = 1
    DO 50 LL = 1, 2
    DO 50 KK = 1, 2
    VR = CC(LL, KK)
    CC(LL, KK) = DD(LL, KK)
50   DD(LL, KK) = VR
    GO TO 51
49   DO 52 LL = 1, 2
    DO 52 KK = 1, 2
    CC(LL, KK) = 0, D0
52   DD(LL, KK) = 0, D0
    GO TO 20
12   KX = 1
    KY = 0
63   DO 58 J = 1, 2
    DO 58 J1 = 1, 2
    IF(CC(I, J, I1, J1), EQ, 0, D0) GO TO 58
    I1 = I1 + 1
    J1 = J1 + 1
    I = F(I1, J1, KX)
    DO 59 IK = 1, 2
    DO 59 JK = 1, 2
    I2 = I2 + 1
    J2 = J2 + 1
    J = F(I2, J2, KY)
    IF(CC(IK, JK), EQ, 0, D0) GO TO 60
    A(J) = TINTEG * D(I, I1) * CC(I1, JI) * CC(IK, JK)
60   J = F(I2, J2, KY)
    IF(DD(IK, JK), EQ, 0, D0) GO TO 59
    A(J) = TINTEG * D(I, I1) * CC(I1, JI) * DD(IK, JK)
59   CONTINUE
    CALL STORAG(I, N, M, NS)
58   CONTINUE
    IF(KX, EQ, 0) GO TO 61
    KX = 0
    KY = 1
DO 62 I.L = 1,2
DO 62 KK = 1,2
VR = CC(I.L,KK)
CC(I.L,KK) = DD(I.L,KK)
62 DD(I.L,KK) = VR
GO TO 63
61 DO 64 LL = 1,2
DO 64 KK = 1,2
CC(I.L,KK) = 0.D0
64 DD(I.L,KK) = 0.D0
20 RETURN
END

SUBROUTINE STORAG(I,N,M,NS)
IMPLICIT REAL *8 (A-H,O-Z)
COMMON / ARYS / WF(11,11), FZ(11,11), FN(11,11)
COMMON / STOR / 1X,M1,M2, ITS(400)
COMMON / ATAR / A(319), TR(319)
COMMON / COAR / A(319,71)
J = 1
10 JJ = M1 + 1
12 IF(J.EQ.1) GO TO 11
   JJ = JJ - 1
   J = 1 - J
   IF(JJ.EQ.1) GO TO 11
   GO TO 12
11 J = 1
   JJ = M1 + 1
13 IF(J.EQ.N) GO TO 7
   J = J + 1
   JJ = JJ + 1
   IF(JJ.EQ.M) GO TO 7
   GO TO 13
7 DO 15 J = 1,N
15 A(J) = 0.D0
RETURN
END

SUBROUTINE UNSRAY(N,M1,M2,MM,R,MACHEP,A,I,A, LAMBDAA,
&     L,C, ZR, ZI, AA, V, F, U, INT)
IMPLICIT REAL *8 (A-H,O-Z)
DOUBLE PRECISION ZS, ZI, Z2, S1, S2
INTEGER R, C, D2, P, Q
REAL *8 MACHEP, LAMBDAA
DIMENSION LAMBDAA(R), A(N,MM), I(N), C(N), ZR(N,R), ZI(N,R),
&     AA(N,MM), V(N), F(N,M1), INT(N)
EPS = 0.D0
DO 101 I = 1,N
X = 0.D0
DO 100 J = 1,MM
100 X = X + DABS(A(I,J))
IF(EPS.LT.X) EPS = X
101 CONTINUE
   EPS = EPS*MACHIEP
   DO 127 K = 1,R
   DO 102 I = 1,N
   V(I) = 1.D0
   DO 102 J = 1,MM
102 AA(I,J) = A(I,J)
   LB = 0
   CALL BANDET(N,M1,M2,MM,1,LAMBDA(K),MACHIEP,AA,D1,D2,F,INT)
   CALL BANSOL(N,M1,M2,MM,1,AA,F,INT,V)
   X = 0.D0
   DO 103 I = 1,N
      IF(DABS(V(I)).LE.DABS(X)) GO TO 103
      X = V(I)
      P = 1
103 CONTINUE
   X = 1.D0/X
   DO 104 I = 1,N
      V(I) = X*V(I)
104 U(I) = V(I)
1 D1 = 0.D0
   LB = LB + 1
   CALL BANSOL(N,M1,M2,MM,0,1,AA,F,INT,V)
   Y = I.D0/V(P)
   X = 0.D0
   DO 105 I = 1,N
      IF(DABS(V(I)).LE.DABS(X)) GO TO 105
      X = V(I)
      P = 1
105 CONTINUE
   X = 1.D0/X
   DO 106 I = 1,N
      R1 = DABS((U(I)-Y*V(I))*X)
      IF(R1.GT.D1) D1 = R1
      V(I) = X*V(I)
106 U(I) = V(I)
   IF(D1.LT.EPS) GO TO 108
   IF(LB.LT.LA) GO TO 1
   LB = LB + 1
108 CONTINUE
   DO 109 I = 1,N
      U(I) = V(I)
109 ZR(I,K) = V(I)
   L(K) = LB
   LAMBDA(K) = LAMBDA(K) + Y
   LB = 0
2 D1 = 0.D0
   LB = LB + 1
   V(I) = V(I)/AA(I,1)
   DO 111 J = 2,N
      X = V(J)
      Q = J-1
      DO 110 I = D2,Q
\text{IN1} = \text{I-1+1}

110 \ X = X - \text{AA}\text{(I,IN1)} \times V(I)
\ V(I) = X / \text{AA}(I,I)
\ \text{IF}(I \text{.GE.} \text{MM}) \text{ D2 = D2 + 1}

111 \text{CONTINUE}
\ J = N

112 \ X = 0.D0

114 \ Q = J + 1
\ D2 = J + M1
\ \text{IF}(D2 \text{.GT.} N) \text{ D2 = N}
\ \text{IF}(Q \text{.GT.} D2) \text{ GO TO 113}
\ \text{DO} 115 \text{ I = Q,D2}
\ \text{IN2} = 1-J

115 \ X = X - F(J,IN2) \times V(I)

113 \text{V(J) = V(J) + X}
\ I = \text{INT}(J)
\ \text{IF}(I \text{.EQ.} J) \text{ GO TO 116}
\ X = V(J)
\ V(J) = V(I)
\ V(I) = X

116 \ J = J-1
\ \text{IF}(J \text{.GE.} 1) \text{ GO TO 112}
\ Y = 1.D0 / V(P)
\ X = 0.D0
\ \text{DO} 117 \text{ I = 1,N}
\ \text{IF}(\text{DABS}(V(I)) \text{.LE.} \text{DABS}(X)) \text{ GO TO 117}
\ X = V(I)
\ P = I

117 \ \text{CONTINUE}
\ X = 1.D0 / X
\ \text{DO} 118 \text{ I = 1,N}
\ R1 = \text{DABS}((U(I)-Y \times V(I)) \times X)
\ \text{IF}(R1 \text{.GT.} D1) \text{ D1 = R1}
\ V(I) = X \times V(I)

118 \text{U(I) = V(I)}
\ \text{IF}(D1 \text{.LE.} \text{EPS}) \text{ GO TO 119}
\ \text{IF}((B1 \text{.LT.} L) \text{ GO TO 2}
\ LB = L + 1

119 \text{C(K) = L.B}
\ D1 = \text{LAMBDA}(K)
\ X = 0.D0
\ D2 = M1
\ \text{DO} 121 \text{ I = 1,M1}
\ S1 = D1
\ S2 = ZR(I,K)
\ S = S1 \times S2
\ Z = 0.D0
\ Q = 2+1 \times M1
\ \text{DO} 120 \text{ J = Q,MM}
\ Z1 = \Lambda(I,J)
\ \text{IN3} = J \times D2
\ Z2 = ZR(IN3,K)

120 \ Z = Z + Z1 \times Z2
\ D2 = D2 + 1
\ Y = \text{SNGL}(Z-S)
121  X = X + Y*V(I)
    Q = M1 + 1
    LB = N-M2
    DO 123 I = Q,LB
    S1 = D1
    S2 = ZR(I,K)
    S = S1*S2
    Z = 0.D0
    DO 122 J = 1,MM
    Z1 = A(I,J)
    IN9 = J + D2
    Z2 = ZR(IN9,K)
122  Z = Z + Z1*Z2
    Y = SNGL(Z,S)
    X = X + Y*V(I)
123  D2 = D2 + 1
    LB = LB + 1
    Q = MM-1
    DO 125 I = LB,N
    S1 = D1
    S2 = ZR(I,K)
    S = S1*S2
    Z = 0.D0
    DO 124 J = 1,Q
    Z1 = A(I,J)
    IN4 = J + D2
    Z2 = ZR(IN4,K)
124  Z = Z + Z1*Z2
    Y = SNGL(Z,S)
    X = X + Y*V(I)
    D2 = D2 + 1
125  Q = Q-1
    Y = 0.D0
    DO 126 I = 1,N
    Y = Y + ZR(I,K)*V(I)
126  ZL(I,K) = V(I)
    LAMBD(K) = LAMBD(K) + X/Y
127  CONTINUE
    RETURN
END

C******************************************************************************
SUBROUTINE BANSOL (N,M1,M2,MM,E,R,A,M,INT,B)
IMPLICIT REAL *8 (A-H,O-Z)
REAL*8 M
INTEGER E,R,W
DIMENSION A(N,MM),INT(N),M(N,M1),B(N)
C
L = M1
IF(E.NE.0) GO TO 204
DO 200 K = 1,N
   I = INT(K)
IF(I.EQ.K) GO TO 202
   X = B(K)
   B(K) = B(I)
   B(I) = X
200  CONTINUE
202  CONTINUE
202 IF(L.LT.N) L = L + 1
    II = K + 1
    IF((II.GT.N)) GO TO 200
    DO 203 I = II, L
    IN7 = I-K
    X = M(K,IN7)
203 B(I) = B(I)*X*B(K)
200 CONTINUE
204 L = -M1
    I = N
205 X = B(I)
    W = M1 + I
    K = I-M1
206 IF(K.GT.L.) GO TO 207
    KK = K + M1 + I
    IN8 = K + W
    X = X-A(I,KK)*B(IN8)
    K = K + 1
    GO TO 206
207 B(I) = X/A(I,1)
    IF(L.LT.M2) L = L + 1
    I = I-1
    IF(I.GE.1) GO TO 205
RETURN
END

C ******************************************************************************
SUBROUTINE BANDET(N,M1,M2,MM,E,LAMBDAM,EWAY,AM,E,D1,D2,M,INT)
IMPLICIT REAL*8 (A-H,0-Z)
REAL*8 LAMBDAM,EWAY,AM,E,NORM
INTEGER D2,E
DIMENSION A(N,MM),M(N,M1),INT(N)

C
NORM = 0.D0
IF(E.EQ.1) GO TO 102
DO 101 I = 1,N
    X = 0.D0
    DO 100 J = 1,MM
100    X = X + DABS(A(I,J))
    IF(NORM.LT.X) NORM = X
101 CONTINUE
102 J = M1 + 1
    DO 103 I = 1,N
103 A(I,J) = A(I,J)-LAMBDAM
    L = M1
    DO 105 I = 1,M1
    JJ = M1 + 2-I
    DO 104 J = JJ,MM
    IN5 = J-L
104 A(I,IN5) = A(I,J)
    L = L-1
    JJ = MM-L
    DO 105 J = JJ,MM
    A(I,J) = 0.D0
105 CONTINUE
    D2 = 0
D1 = 1.D0
L = M1
DO 115 K = 1,N
X = A(K,1)
I = K
IF(L.LT.N) L = L + 1
JJ = K + 1
IF(JJ.GT.N) GO TO 1006
DO 106 J = JJ,L
IF(DABS(A(J,1)).LE.DABS(X)) GO TO 106
X = A(J,1)
I = J
106 CONTINUE
1006 CONTINUE
INT(K) = I
D1 = D1*X
46 IF(X.NE.0.D0) GO TO 107
D2 = 0
IF(E.EQ.1) A(K,1) = NORM*MACHIEP
IF(E.NE.1) RETURN
107 CONTINUE
IF(D1.EQ.0.D0) GO TO 110
IF(DABS(D1).GE.1.D0) GO TO 108
IF(DABS(D1).LT.0.0625D0) GO TO 109
GO TO 110
108 D2 = D2 + 4
D1 = D1*0.0625D0
GO TO 110
109 D2 = D2 - 4
D1 = D1*16.D0
110 CONTINUE
IF(E.EQ.K) GO TO 112
D1 = -D1
DO 111 J = I,MM
X = A(K,J)
A(K,J) = A(I,J)
111 A(I,J) = X
112 JJ = K + 1
IF(JJ.GT.N) GO TO 115
I = JJ
177 X = A(I,1)/A(K,1)
IN6 = I-K
M(K,IN6) = X
DO 113 J = 2,MM
113 A(I,J) = A(I,J) - X*A(K,J)
114 A(I,MM) = 0.D0
I = I + 1
IF(I.LE.E) GO TO 177
115 CONTINUE
RETURN
END
C ********************************************************************
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