Upper Semicontinuous Decomposition of $E^3$ into Subarc’s of Bing’s Sling and Points

by

Mohammad Showkat Rahim Chowdury

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DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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In

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Upper semicontinuous decompositions of $\mathbb{E}^3$ into subarcs of Bing's sling and points

Chowdhury, Mohammad Showkat Rahim, M.S.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1988
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This thesis, written by

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Date: June 5, 1988
This thesis is dedicated to my parents and my wife.
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"الخلاصة"

معالج "بيتش" هو متخيّل يُسيط مغلق معرف على الفضاء الثلاثي E^3 حيث يوجد (هومومورفيزم) من h إلى نفسه الذي E^3 من مساحة "بيتش" المنحنى إلى دائرة. يدعى معلاج "بيتش" منحنى يسيط جامع أي قانون من معالج "بيتش" يكون خليلاً بمثابة أي منطقة محيطة بالقيمة الجزيئية A تحتوي على خليج ثلاثية تكون A بداخلها تمامًا.

في هذا البحث سندرس تحليل شبه المتصل العلوي للفضاء الثلاثي E^3 إلى أقواس جزيئية من معالج "بيتش" غير متقطعة مع بعضها البعض، وسوف نثبت أن هذا التحليل ينتج عنه دائماً فضاءات متصلة (هومومورفية) إلى E^3.

في الباب الأول سنستعرض بعض المفاهيم والنتائج من نظرية تحليل الفضاء والتي تحتاجها في البابين اللاحقين في الباب الثاني ندرس تركيب معالج "بيتش" وفي الباب الثالث ندرس نوعاً خاصاً من تحليل الفضاء E^3 المذكور سابقاً.

* * *

(iv)
ABSTRACT

Bing's sling is a simple closed curve in Euclidean 3-space $E^3$ for which there is no homeomorphism $h$ from $E^3$ onto itself taking it to a circle. We say that Bing's sling is a wild simple closed curve. Any subarc $A$ of Bing's sling is cellular that is, each neighborhood of $A$ contains a 3-cell which contains $A$ in its interior.

We study upper semicontinuous decompositions of Euclidean 3-space $E^3$ into points and pairwise disjoint subarcs of Bing's sling. We prove that such decompositions always yield decomposition spaces that are homeomorphic to $E^3$.

In chapter 1 we review some basic concepts and results from decomposition space theory needed in the following chapters. Chapter 2 studies the construction of Bing's sling and chapter 3 studies the special type of decomposition space of $E^3$ mentioned above.
# CONTENTS

<table>
<thead>
<tr>
<th>CONTENT</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>(ii)</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>(v)</td>
</tr>
<tr>
<td>CHAPTER 1: INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Decomposition Spaces</td>
<td>1</td>
</tr>
<tr>
<td>1.2 O-dimensional Spaces</td>
<td>9</td>
</tr>
<tr>
<td>1.3 Embedding in $E^3$</td>
<td>11</td>
</tr>
<tr>
<td>CHAPTER 2: THE CONSTRUCTION OF BING'S SLING</td>
<td>14</td>
</tr>
<tr>
<td>CHAPTER 3: UPPER SEMICONTINUOUS DECOMPOSITIONS OF $E^3$ INTO SUBARCS OF BING'S SLING AND POINTS</td>
<td>19</td>
</tr>
<tr>
<td>3.1 $H$ is Countable</td>
<td>19</td>
</tr>
<tr>
<td>3.2 The Diameter of $g_1$ Approaches Zero as $i$ Tends to Infinity</td>
<td>20</td>
</tr>
<tr>
<td>3.3 $G$ is an Upper Semicontinuous Decomposition of $E^3$</td>
<td>20</td>
</tr>
<tr>
<td>3.4 Basic Lemma</td>
<td>22</td>
</tr>
<tr>
<td>3.5 $E^3/G$ is Homeomorphic to $E^3$</td>
<td>37</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>44</td>
</tr>
</tbody>
</table>

(vi)
CHAPTER ONE

INTRODUCTION

In this chapter, we present some basic definitions and theorems in decomposition space theory which will be needed in the following chapters. We assume familiarity with basic results from general topology.

1.1 Decomposition Spaces

Definition 1.1.1

Let $X$ be a topological space. Let $G$ be a collection of pairwise disjoint non-empty subsets of $X$ whose union is $X$. $G$ is said to be a decomposition of $X$. We use $H$ for the set of all non-degenerate elements in $G$, that is, those with more than one point.

Definition 1.1.2

If $G$ is a decomposition of $X$, then the map $P$ from $X$ to $G$ taking each $x$ in $X$ to the unique element $P(x)$ in $G$ which contains $x$ is called the projection map.

Note that in our terminology maps need not be continuous.

Definition 1.1.3

Let $X$ and $Y$ be topological spaces. A map $f$ from $X$ onto $Y$ is a quotient map if a subset $U$ of $Y$ is
open in \( Y \) iff \( f^{-1}(U) \) is open in \( X \).

**Definition 1.1.4**

If \( X \) is a space and \( G \) is a set and if \( f \) is a surjective map from \( X \) onto \( G \), then the topology on \( G \) relative to which \( f \) is a quotient map is called the quotient topology. It is an easy exercise to verify the existence and uniqueness of quotient topologies.

**Definition 1.1.5**

Let \( G \) be a decomposition of a space \( X \). The decomposition space or quotient space of \( X \) induced by \( G \) is the set \( G \) together with the quotient topology induced by the projection map \( P \) from \( X \) onto \( G \). We denote this space by \( X/G \).

**Theorem 1.1.1**

A closed map or an open map is a quotient map if it is surjective and continuous.

The proof follows directly from the definition.

**Theorem 1.1.2**

Let \( X/G \) be a decomposition space of \( X \) with projection map \( P \). Then,

1. \( P \) is continuous.
2. If \( Y \) is a space and \( f \) from \( X/G \) to \( Y \) a function,
then $f$ is continuous iff $f \circ P$ is continuous.

For proof of this theorem, we refer to [6, page 103].

**Theorem 1.1.3**

Let $f$ be a quotient map from the space $X$ to the space $Y$ and let $G = \{f^{-1}(y) | y \in Y\}$. Then there is a homeomorphism $\phi$ from $X/G$ onto $Y$ such that $\phi \circ P = f$, where $P$ is the projection map from $X$ onto $G$ as defined before.

For proof we refer to [6, page 106].

**Theorem 1.1.4**

Let $X/G$ be a decomposition space of the space $X$.

1. If $X$ is compact, then so is $X/G$.
2. If $X$ is connected, then so is $X/G$.
3. If $X$ is separable, then so is $X/G$.
4. If $X$ is locally connected, then so is $X/G$.

For proof we refer to [6, page 104].

**Definition 1.1.6**

A decomposition $G$ of $X$ is said to be upper semi-continuous if for each $g \in G$ and for each open set $U$
containing \( g \), \( \{g' \in G | g' \subset U \} \) is open in \( X \).

**Theorem 1.1.5**

A decomposition \( G \) of a space \( X \) is upper semicontinuous iff the projection map \( P: X \to X/G \) is closed.

For proof we refer to [6, page 105].

**Theorem 1.1.6**

Let \( G \) be an upper semicontinuous decomposition of a space \( X \) such that the elements of \( G \) are compact sets.

1. If \( X \) is second countable, then so is \( X/G \).
2. If \( X \) is separable metrizable, then so is \( X/G \).

For proof we refer to [6, page 106].

Our main objects of study are upper semicontinuous decompositions of \( E^3 \) into arcs and points. The following theorems and examples are some of the relevant previous research in this area.

**Theorem 1.1.7**

Suppose \( G \) is an upper semicontinuous decomposition of \( E^3 \) such that each \( g \) in \( G \) is cellular [see page 12 for the definition], \( G \) has only a countable number of non-degenerate elements, and their union is a \( G_\delta \) set. Then
the decomposition space $E^3/G$ is homeomorphic to $E^3$
[Theorem 1 of [1]].

**Theorem 1.1.8**

Suppose $G$ is an upper semicontinuous decomposition of $E^3$ such that $G$ has only a countable number of non-degenerate elements and each is a tame arc [see page 11 for the definition]. Then $E^3/G$ is homeomorphic to $E^3$.
[Theorem 3 of [1]].

**Example 1.1.1**

There is an upper semicontinuous decomposition $G$ of $E^3$ such that $H$ consists of uncountably many tame arcs and $uH$ is a compact set but $E^3/G$ is not homeomorphic to $E^3$. The space $E^3/G$ is called Bing's Dogbone space. The details are in [2].

**Example 1.1.2**

There is an upper semicontinuous decomposition of the 3-sphere $S^3$ whose non-degenerate elements are all contained in a cellular arc, but the decomposition space $S^3/G$ is not homeomorphic to $S^3$. Details are in [5].

We now state and prove the following two theorems which are general topology results. These theorems will be used in Chapter 3. There we will have a sequence $\langle h_1 \rangle$ of
homeomorphisms from $E^3$ onto itself which shrink nondegenerate elements into sets of arbitrarily small diameters. By applying the following two theorems to the sequence $<h_i>$ it will be shown that $E^3/G$ is homeomorphic to $E^3$.

Throughout this thesis we will use the usual metric $d$ on $E^3$.

**Theorem 1.1.9**

Suppose that $<f_i>$ is a sequence of homeomorphisms of $E^3$ onto itself. Further, suppose that there is a sequence $<V_i>$ of open sets such that

(a) $V_i \supset V_{i+1}$,

(b) $f_{i+1} = f_i$ on $E^3 \setminus V_i$, and

(c) each component of $f_i(V_i)$ has diameter less than $1/i$.

Then $<f_i>$ converges uniformly to a continuous function $f$.

**Proof:** For $k \geq 1$ we have $f_{i+k} = f_i$ on $E^3 \setminus V_i$, so for $p \notin V_i$, $f_{i+k}(p) = f_i(p)$. If $p \in V_i$, then let $D$ be the component of $V_i$ containing $p$. By condition (c), $D$ is not all of $E^3$. Now $D$ is open in $E^3$; it cannot be closed in $E^3$, because then $E^3 \setminus D$ should be also closed and $E^3$ would be disconnected which is false. Hence $D \setminus D \neq \emptyset$. 
We have $\overline{D \setminus D} \subset E^3 \setminus V_1$. So, $f_{i+k} = f_i$ on $\overline{D \setminus D}$ and therefore $f_i(\overline{D}) \cap f_{i+k}(\overline{D}) \neq \emptyset$ ... (1). Now for all $k \geq 1$, we have $f_i(V_1) = f_{i+k}(V_1)$. Since the $f_i$'s are homeomorphisms and $D$ is a component of $V_1$, $f_{i+k}(D)$ is a component of $f_i(V_1)$. So, there is a component $D^K$ of $V_1$ such that $f_i(D^K) = f_{i+k}(D)$. Then $f_i(D^K) \subset f_{i+k}(\overline{D})$. From (1) we get $f_i(D^K) \cap f_i(\overline{D}) \neq \emptyset$. Hence, $D^K \cap \overline{D} \neq \emptyset$ ... (2).

Now $f_i(D^K)$ and $f_i(D)$ are components of $f_i(V_1)$. So, $\text{diam } f_i(D^K) < 1/1$ and $\text{diam } f_i(D) < 1/1$. Now, $f_{i+k}(p) \in f_i(D^K)$, so there is a $y \in D^K$ such that $f_i(y) = f_{i+k}(p)$. Then $d(f_i(p), f_{i+k}(p)) = d(f_i(p), f_i(y))$. Since $D^K \cap \overline{D} \neq \emptyset$, we have $d(f_i(p), f_i(y)) \leq \text{diam } f_i(D^K) + \text{diam } f_i(\overline{D}) < 1/1 + 1/1 = 2/1$. We conclude that $d(f_i(p), f_{i+k}(p)) < 2/1$ for all $p$ in $E^3$.

For any $\varepsilon > 0$, there is an integer $i$ with $2/1 < \varepsilon/2$. Then, for $j, k \geq 1$ and $p \in E^3$, $d(f_k(p), f_j(p)) \leq d(f_k(p), f_i(p)) + d(f_i(p), f_j(p)) < 2/1 + 2/1 < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $\langle f_i \rangle$ converges uniformly to a function $f$. Since the $f_i$'s are continuous, $f$ is also continuous.

**Theorem 1.1.10**

Suppose a sequence $\langle f_i \rangle$ of homeomorphisms of $E^3$ onto itself converges uniformly to a function $f$. Then $f$ is onto $E^3$ and $f$ is closed.
Proof: Let $A$ be a closed set in $E^3$. We need to show that $f(A)$ is closed in $E^3$. To do this, let $\langle f(a_n) \rangle$ be a sequence in $f(A)$ so that $\langle f(a_n) \rangle$ converges to $x_0$, for some $x_0$ in $E^3$. We shall show that $x_0 \in f(A)$. Now, $f_\perp$ converges to $f$ uniformly. So, given $\epsilon > 0$, there is $N_0$ such that for $i \geq N_0$, $d(f_\perp(a_i), x_0) < \epsilon$. Thus, $\langle f_\perp(a_i) \rangle$ converges to $x_0$.

Now, $\exists m_0$ such that for $m, n \geq m_0$ we have, $d(f_m(a_m), x_0) < 1$ and $d(f_m(y), f_n(y)) < 1$ for each $y$. Let $i \geq m_0$. Then $d(f_\perp(a_i), x_0) < 1$ and $d(f_\perp(a_i), f_{m_0}(a_i)) < 1$, so, $d(f_{m_0}(a_i), x_0) < 2$. So, for $i \geq m_0$ we have $a_i \in f_{m_0}^{-1}(B(x_0, 2))$, where $B(x_0, 2)$ is the open ball with center $x_0$ and radius 2. But $f_{m_0}^{-1}(\overline{B(x_0, 2)})$ is compact. Hence there is a subsequence $\langle a_{i_j} \rangle$ of $\langle a_i \rangle$ which converges to some $x$ in $f_{m_0}^{-1}(\overline{B(x_0, 2)})$. Since $A$ is closed, $x \in A$. Then $\langle f(a_{i_j}) \rangle$ converges to $f(x)$, since $f$ is continuous. But $\langle f(a_{i_j}) \rangle$ also converges to $x_0$, so that $f(x) = x_0$ and hence $x_0 \in f(A)$.

We now show that $f$ is onto. Let $y \in E^3$. There is a sequence $n_1 < n_2 < n_3 < \ldots$ such that for $m, n \geq n_k$ we have $d(f_m(x), f_n(x)) < 1/k$, for each $x \in E^3$. Now suppose $k \geq j$. Then $n_k \geq n_j$ so that $d(y, f_{n_j}(f_k^{-1}(y))) = d(f_{n_k}(f_k^{-1}(y)), f_{n_j}(f_k^{-1}(y))) < 1/j$. Hence $f_{n_k}^{-1}(y) \in f_{n_j}^{-1}(B(y, 1/j))$. 
Now \( \overline{B(y, l/j)} \) is compact and hence so is \( f_{n_j}^{-1}(B(y, l/j)) = \overline{f_{n_j}^{-1}(B(y, l/j))} \). Then the sequence \( <f_{n_k}^{-1}(y)>_{k \geq j} \) is in \( f_{n_j}^{-1}(B(y, l/j)) \), for each \( j \). Hence there is a subsequence \( <f_{n_{k_i}}^{-1}(y)> \) converging to \( x \) and \( x \in f_{n_j}^{-1}(B(y, l/j)) = \overline{f_{n_j}^{-1}(B(y, l/j))} \), for each \( j \). Hence \( f_{n_j}(x) \in B(y, l/j) \) for each \( j \). i.e. \( d(f_{n_j}(x), y) < 1/j \), for each \( j \).

Now, given \( \varepsilon > 0 \), there exists \( N_0 \) such that \( 1/N_0 < \varepsilon/2 \). Then for each \( j \geq N_0 \), \( d(f_{n_j}(x), y) < 1/j \leq 1/N_0 < \varepsilon/2 \). Also, for each \( j \geq N_0 \) and for each \( n, n_j \geq n_0 \),

\[
d(f_n(x), f_{n_j}(x)) < 1/N_0 < \varepsilon/2.
\]

Then \( d(f_n(x), y) \leq d(f_n(x), f_{n_j}(x)) + d(f_{n_j}(x), y) < \varepsilon/2 + \varepsilon/2 = \varepsilon \). So, \( f_n(x) \) converges to \( y \). But \( f_n(x) \) converges to \( f(x) \) since \( f_n \) converges to \( f \) uniformly. Since the limit is unique, we have \( f(x) = y \). Hence, \( f \) is onto \( E^3 \).

1.2 0-dimensional Spaces

**Definition 1.2.1**

A space \( X \) has dimension 0 at a point \( p \) if \( p \) has arbitrarily small neighborhoods with empty boundaries, i.e. if for each neighborhood \( U \) of \( p \) there exists a neighborhood \( V \) of \( p \) such that \( V \subset U \) and \( \text{Bd}V = \phi \).
Definition 1.2.2

A nonempty space $X$ has dimension 0, written $\dim X = 0$, if $X$ has dimension 0 at each of its points. We define the dimension of the empty set to be $-1$.

Remark 1.2.1

A 0-dimensional space can also be defined as a non-empty space in which there is a basis for the open sets made up of sets which are at the same time open and closed.

Example 1.2.1

Every non-empty finite or countable metrizable space $X$ is 0-dimensional [4, page 10].

Theorem 1.2.1

A non-empty subset of a 0-dimensional space is 0-dimensional.

For proof we refer to [4, page 13].

Theorem 1.2.2

Let $X$ be a metrizable space and $M$ a subset of $X$ of dimension less than or equal to 0. Suppose $U_1$ and $U_2$ are two open sets of $X$ which cover $M$. Then there exist two open sets $V_1$ and $V_2$ which cover $M$ and satisfy
$V_1 \subset U_1$, $V_2 \subset U_2$ and $V_1 \cap V_2 = \emptyset$.

For proof we refer to [4, page 53].

**Theorem 1.2.3**

Let $X$ be a metrizable space, $M$ a subset of $X$ of dimension $\leq 0$ and $\{U_1 | i = 1,2, \ldots \}$ a covering of $M$ with each $U_i$ open in $X$. Then there is a covering $\{V_i | i = 1,2, \ldots \}$ of $M$ such that $V_i \subset U_i$, $i = 1,2, \ldots$, $V_i \cap V_j = \emptyset$ for $i \neq j$ and each $V_i$ is open in $X$.

For proof we refer to [4, page 54].

1.3 Embeddings in $E^3$

**Definition 1.3.1**

Let $M$ be an arc in $E^3$. If there is a homeomorphism $\phi$ from $E^3$ onto itself such that $\phi(M)$ is the closed unit interval $I = \{(x,y,z)|0 \leq x \leq 1, \ y = z = 0\}$ in $E^3$, then $M$ is said to be tamely embedded or just tame. If $M$ is a simple closed curve in $E^3$ and $\phi$ is a homeomorphism from $E^3$ onto itself such that $\phi(M)$ is a polygonal simple closed curve in $E^3$, then $M$ is said to be tamely embedded. In either case, if $M$ is not tame, then it is said to be wild.
Definition 1.3.2.

A 3-cell $D$ is a space homeomorphic to the closed unit ball $\{(x, y, z)|x^2 + y^2 + z^2 \leq 1\}$ in $E^3$. If $\phi$ is a homeomorphism from $D$ to the closed unit ball, then $\text{Int } D = \phi^{-1}(\{(x, y, z)|x^2 + y^2 + z^2 < 1\})$. It can be shown that $\text{Int } D$ is well-defined.

Definition 1.3.3

A subset $X$ of $E^3$ is cellular if $X = \bigcup_{i=1}^{\infty} D_i$, where $D_i$ is a 3-cell and $\text{Int } D_i \supset D_{i+1}$, for all $i$.

Proposition 1.3.1

A tame arc is cellular but the converse is not true.

Proof: Since $M$ is tame, there is a homeomorphism $\phi$ from $E^3$ onto itself such that $\phi(M) = I$. It can be shown that $I$ is cellular. To do this, let $D_1 = [0 - \frac{1}{4}, 1 + \frac{1}{4}] \times [-\frac{1}{4}, \frac{1}{4}]$. Then $D_1$ is a 3-cell and $\text{Int } D_1 \supset D_{1+1}$ for all $i$. Also, $\bigcup_{i=1}^{\infty} D_i = [0,1] \times \{0\} \times \{0\} = I$. Hence $I$ is cellular.

Now, let $\phi^{-1}(D_i) = E_i$, for all $i$. Then the $E_i$'s are 3-cells, because $\phi^{-1}$ is a homeomorphism. Since $\text{Int } D_i \supset D_{i+1}$, $\text{Int } \phi^{-1}(D_i) = \phi^{-1}(\text{Int } D_i) \supset \phi^{-1}(D_{i+1})$, so that $\text{Int } E_i \supset E_{i+1}$. $\text{Int } E_i \supset M$ for all $i$, as
Int $D_i \supset I$ for all $i$. Moreover, $\quad_1^\infty E_1 = _1^\infty \phi^{-1}(D_1) = \phi^{-1}(\_1^\infty D_1) = \phi^{-1}(I) = M$. Hence, $M$ is cellular.

A cellular set need not be tame, for example, that each subarc of Bing's sling is cellular can be proved from the work in Chapter 3 while the fact that they are not tame is verified in [3].
CHAPTER TWO

THE CONSTRUCTION OF BING'S SLING

Bing's sling is a certain wild simple closed curve in $E^3$ for which any subarc is cellular. We will use subarcs of Bing's sling for the non-degenerate elements in the decompositions of $E^3$ that we study. The construction of Bing's sling was first given in [3].

To construct Bing's sling we take first a solid torus $T_1$. A solid torus is a torus with its interior, that is, it is homeomorphic to the product of a circle and a 2-dimensional disk. Then we divide $T_1$ into 12 nonoverlapping cylindrical blocks $C_1, C_2, \ldots, C_{12}$, that is, each $C_i$ is homeomorphic to the product of a line segment and a 2-dimensional disk.

In each of these blocks we put 3 solid tubes as shown in the figure below.

![Figure 1](image)

The tubes inside the blocks $C_i$ fit together to make a solid torus $T_2$ in $\text{Int} \ T_1$. $T_2$ has many knots.
$T_2 \subset \text{Int } T_1$

Figure 2
In each component of $T_2 \cap C_1$ we put 4 copies of the block with tubes. The tubes inside these smaller cylinders fit together to make a thinner solid torus $T_3$.

We continue this process, to get solid tori

$T_1 \supset \text{Int } T_1 \supset T_2 \supset \text{Int } T_2 \supset T_3 \supset \ldots$, so that the diameters of the blocks of $T_i$ approach 0 as $i$ tends to $\infty$, and so that $T_1$ has $12^i$ blocks.

![Diagram of T1](image)

Figure 3

Now with each block $C_i$ of $T_1$ we associate 12 blocks $C_{i1}, C_{i2}, \ldots, C_{i12}$ of $T_2$, indexed as shown. Similarly, with each block $C_{i1i_2 \ldots i_r}$ of $T_r$ we associate 12 blocks $C_{i1i_2 \ldots i_{r1}}, C_{i1i_2 \ldots i_{r2}}, \ldots, C_{i1i_2 \ldots i_{r12}}$ of $T_{r+1}$ lying in $C_{i1i_2 \ldots i_r}$ and the block following $C_{i1i_2 \ldots i_r}$ in $T_r$.

The following figure gives some idea about the construction of Bing's sling.
Let $J = \bigcap_{i=1}^{n} T_i$. Then $J$ is a wild simple closed curve that does not pierce any disk, that is, $J$ does not pass through any disk without intersecting the disk in more than one point. This simple closed curve is called Bing's sling.

We briefly indicate how to define a homeomorphism $h$ from a circle $\Sigma$ onto $J = \bigcap_{i=1}^{n} T_i$. Let $S_i$ be the center simple closed curve of $T_i$. Divide $\Sigma$ into 12 equal arcs $J_1, \ldots, J_{12}$ and let $h_1$ be a homeomorphism from $\Sigma$ onto $S_1$ taking $J_1$ to $S_1 \cap C_1$. Then divide each $J_i$ into 12 equal arcs. Let $h_2$ be a homeomorphism from $\Sigma$ onto $S_2$ taking each subsegment $J_{ij}$ of $J_i$ to $C_{ij} \cap S_2$ for $i,j = 1,2,\ldots,12$. We continue this process indefinitely and get a sequence $\left<h_1\right>$ of homeomorphisms from $\Sigma$ into $E^3$. The sequence $\left<h_1\right>$ converges uniformly to a homeomorphism $h$ from $\Sigma$ onto $J$. This proves that $J$ is a simple closed curve.
CHAPTER THREE

UPPER SEMICONTINUOUS DECOMPOSITIONS OF $E^3$
INTO SUBARCS OF BING'S SLING AND POINTS

In this chapter we will study upper semicontinuous
decomposition spaces of $E^3$ where the set $H$ of non-degen-
erate elements in the decomposition is a collection of pair-
wise disjoint subarcs of Bing's sling, so throughout this
chapter $G$ and $H$ will have the appropriate meaning.

3.1 $H$ is Countable

Let $J$ be Bing's sling. $J$ is a separable metric
space, so $J$ has a countable basis $B$.

Suppose $H$ is uncountable. Now the non-degenerate
elements in $H$ are compact sets. Since each
$g \in H$ is a subarc of the simple closed curve $J$,
each $g \in H$ has interior points in $J$. Thus, there is a
nonempty basis element $B_g$ such that $B_g \subset g$. Since the
elements of $H$ are pairwise disjoint, $\{B_g | g \in H\}$ is
a collection of pairwise disjoint sets. Hence $\{B_g | g \in H\}$
is an uncountable subset of the basis for $J$. This is a
contradiction because the basis $B$ for $J$ is countable.
Hence $H$ is countable. We write $H = \{g_i | i = 1, 2, \ldots\}$.

### 3.2 The Diameter of $g_i$ Approaches Zero as $i$ Tends to Infinity

Suppose there exists $\varepsilon > 0$ such that $\text{diam } g_i \geq \varepsilon$ for infinitely many $i$'s, say for $i_1, i_2, \ldots$. There exist $a_{i_j}, b_{i_j}$ in $g_{i_j}$ with $d(a_{i_j}, b_{i_j}) > \varepsilon/2$. Since $J$ is compact, there exist subsequences $\langle a_{i_{j_k}} \rangle, \langle b_{i_{j_k}} \rangle$ that converge to $a, b$ respectively. Since $d(a_{i_{j_k}}, b_{i_{j_k}}) > \varepsilon/2$, $d(a, b) > \varepsilon/2$, so $a \neq b$. But now the $g_{i_{j_k}}$'s are pairwise disjoint subarcs of the simple closed curve $J$ and for any large $k$, $g_{i_{j_k}}$ contains a point near $a$ and a point near $b$, which is impossible from the geometry of a simple closed curve.

### 3.3 $G$ is an Upper Semicontinuous Decomposition of $E^3$

Given $g \in G$, let $U$ be an open set in $E^3$ with $g \subseteq U$. Consider $A = \bigcup \{g' \in G | g' \cap (E^3 \setminus U) \neq \emptyset\}$. We want to show that $A$ is closed, so that $E^3 \setminus A = \bigcup \{g' \in G | g' \subset U\}$ will be open in $E^3$, thus showing that $G$ is upper semicontinuous. Any convergent sequence of points from $E^3 \setminus U$ converges to a point in $E^3 \setminus U$, but $A$ may contain points
from $U$.

Suppose $\langle a_n \rangle$ is a convergent sequence of points in $A$ with $a_n \in U$. Then each $a_n$ belongs to some non-degenerate element $g_{i_n}$ which meets $E^3 \setminus U$. But if $\{g_{i_n} | a_n \in U\}$ were finite, then $\cup \{g_{i_n} | a_n \in U\}$ would be a closed set and the sequence $\langle a_n \rangle$ would converge to some point in $\cup \{g_{i_n} | a_n \in U\}$ which is contained in $A$, because $g_{i_n} \cap (E^3 \setminus U) \neq \emptyset$, so that $g_{i_n} \subset A$ from each $i_n$.

So, suppose $\{g_{i_n} | a_n \in U\}$ is countably infinite. But $\langle a_n \rangle$ converges to $a_0$ for some $a_0 \in E^3$. If $a_0$ belongs to some non-degenerate element $g_{i_n}$ to which $a_n$ belongs then $a_0 \in A$ since $g_{i_n} \subset A$.

Suppose $a_0$ does not belong to any non-degenerate element to which some $a_n$ belongs and that $a_0 \in U$. Let $d(a_0, E^3 \setminus U) = \varepsilon > 0$. Since $a_n$ converges to $a_0$, there exists $n_0$ such that for each $n \geq n_0$, $d(a_0, a_n) < \varepsilon/2$.

But diam $g_{i_n}$ approaches 0 as $n$ tends to infinity. So, for $\varepsilon/2 > 0$, there exists $m_0$ such that for each $n \geq m_0$, diam $g_{i_n} < \varepsilon/2$. Let $N_0 = \max\{n_0, m_0\}$. Then for each $n \geq N_0$, and for each $x \in g_{i_n}$, $d(a_0, x) \leq d(a_0, a_n) + d(a_n, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. So, $g_{i_n} \subset U$. Thus, for each
n ≥ N_0, \( g_n \subseteq U \), which is a contradiction because the \( g_n \)'s meet \( E^3 \setminus U \). Hence, \( a_0 \subseteq E^3 \setminus U \subseteq A \).

Finally, let \( <a_n> \) be a convergent sequence of points of \( A \) for which \( a_i \subseteq E^3 \setminus U \) for infinitely many \( i \) and \( a_i \subseteq U \) for infinitely many \( i \). Now, the subsequence \( <a_{n_i}> \) with \( a_{n_i} \subseteq E^3 \setminus U \) converges to some point in \( E^3 \setminus U \) since \( E^3 \setminus U \) is closed. Since \( <a_n> \) is a convergent sequence, the subsequence and the sequence converge to the same point in \( E^3 \setminus U \subseteq A \). Hence \( A \) is closed.

Consequently, \( E^3 / G \) is an upper semicontinuous decomposition of \( E^3 \).

3.4 Basic Lemma

In this section we will construct homeomorphisms of \( E^3 \) onto itself which shrink each non-degenerate element of \( G \) into a set of small diameter. With this end in mind, we state and prove the following basic lemma.

**Basic Lemma:**

Given a positive \( \varepsilon \) and an open set \( U \) containing \( \bigcup H \), there exists a homeomorphism \( h_\varepsilon \) of \( E^3 \) onto itself which shrinks each \( g \subseteq G \) into a set \( h_\varepsilon(g) \) of diameter less than \( \varepsilon \), is identity on \( E^3 \setminus U \) and takes each component of \( U \) onto itself.
Proof: Let $\varepsilon > 0$ be given and let $U$ be an open set such that $U \supset \varepsilon H$. 

Let $H_\varepsilon = \{g \in H \mid \text{diam } g \geq \varepsilon / 2 \}$. Now, $H_\varepsilon$ is finite by the result in Section 3.2. If $H_\varepsilon = \emptyset$, then we need only take $h_\varepsilon$ to be the identity, so suppose $H_\varepsilon \neq \emptyset$. Consider any $g$ in $H_\varepsilon$. Fix a point $q$ in $J \setminus \varepsilon H_\varepsilon$ and let $O_g = \{ x \mid d(x, g) < \frac{1}{2} d(g, (\varepsilon H_\varepsilon \cup \{q\}) \setminus g) \}$. Then $O_g$ is an open set in $E^3$ containing $g$ and not meeting any other element of $H_\varepsilon$ or the point $q$.

Let $V_g = \bigcup \{ g' \in G \mid g' \subset O_g \cap U \}$. Then $V_g$ is open since $G$ is an upper semicontinuous decomposition of $E^3$. Further, by the definition of $O_g$, $\{ V_g \mid g \in H_\varepsilon \}$ is a collection of pairwise disjoint open sets. Also, for each $g \in H_\varepsilon$, $g \subset V_g \subset U$, for any element $g' \in G$ either $g' \cap V_g = \emptyset$ for each $g \in H_\varepsilon$ or $g' \subset V_g$ for some $g \in H_\varepsilon$ and $q \notin V_g$ for all $g \in H_\varepsilon$. Now let us fix a $g$ in $H_\varepsilon$ and let $r$ be the minimum of $\frac{1}{2} d(g, E^3 \setminus V_g)$ and $\varepsilon / 4$. There exists a positive integer $n_1$ such that the solid torus $T_{n_1}$ has blocks of diameter less than $r$.

Now, let $B_1$ be the union of all blocks of $T_{n_1}$ meeting $g$. Then $B_1$ is connected since $g$ and the blocks are all connected. $B_1 \subset V_g$, by the choice of $r$, so not all blocks of $T_{n_1}$ lie in $B_1$ because the point $q$ lies in $J$, hence in a block of $T_{n_1}$, but $q \notin V_g$. Thus, $B_1$
is a connected union of some but not all blocks of $T_{n_1}$ and so is a 3-cell.

Now the blocks of $T_{n_1}$ meeting $g$ are ordered in the same way as ordered in $T_{n_1}$. Let them be $C_1, C_2, \ldots, C_{p_1}^{n_1}$ in this ordering, so that $B_1 = \bigcup_{i=1}^{p_1} C_1$.

Since $g \subset \text{Int } T_{n_1}$ and every block of $T_{n_1}$ meeting $g$ lies in $B_1$, we have $g \subset \text{Int } B_1 \subset B_1$. Since $B_1 \subset V_g$, the only element of $G$ of diameter greater than or equal to $\varepsilon/2$ meeting $B_1$ is $g$ itself.

Now, $g \subset \text{Int } B_1$ and $\text{Int } B_1$ is open. Since $G$ is an upper semicontinuous decomposition of $E^3$, $W_1 = \cup \{ g' \in G \mid g' \subset \text{Int } B_1 \}$ is open in $E^3$. Then $g \subset W_1 \subset \text{Int } B_1$ and there exists a positive integer $m_1 > n_1$ such that each block of the solid torus $T_{m_1}$ has diameter less than $\frac{1}{2} d(g, E^3 \setminus W_1)$.

As before, the union of all blocks of $T_{m_1}$ meeting $g$ is a 3-cell $D_1$. Since $m_1 > n_1$, all blocks of $T_{m_1}$ have diameter less than $\varepsilon/4$. Also, $D_1 \subset W_1$ by properties of the diameters of blocks of $T_{m_1}$. Also, just as before $g \subset \text{Int } D_1$. Thus $g \subset \text{Int } D_1 \subset D_1 \subset W_1 \subset \text{Int } B_1 \subset B_1 \subset V_g$.

Consider the 3-cell $C_1 \cup C_2$. Now, let $h_1$ be a homeomorphism from $E^3$ onto itself such that $h_1 = \text{Id}$ (the
identity) on $E^3 \setminus \text{Int}(C_1^n \cup C_2^n)$, $h_1$ takes all the blocks of $D_1$ meeting $C_1^n$ into the interior of $C_2^n$, and $h_1$ takes all blocks of $D_1$ lying in $C_1^n \cup C_2^n$ into $C_2^n$. The reason that such an $h_1$ exists is because of the special construction of the solid tori, in particular, when the sequence of blocks of $T_{m_1}$ leaves $C_1^n$, it does not go beyond $C_2^n$ before returning. Some other blocks of $T_{m_1}$ not lying in $D_1$ but contained in $C_1^n$ may be pulled into the interior of $C_2^n$ or may be stretched by $h_1$ so that their images meet the interior of $C_2^n$.

Thus $h_1(g) \in \text{Int}(C_2^n \cup C_3^n \cup \ldots \cup C_{p_1}^n)$, so that $g \in h_1^{-1}(\text{Int}(C_2^n \cup C_3^n \cup \ldots \cup C_{p_1}^n))$. Because $h_1^{-1}(\text{Int}(C_2^n \cup C_3^n \cup \ldots \cup C_{p_1}^n))$ is open in $E^3$, $W_2 = \cup \{g' \in G | g' \in h_1^{-1}(\text{Int}(C_2^n \cup C_3^n \cup \ldots \cup C_{p_1}^n)) \}$ is open in $E^3$ and $g \in W_2 \subset h_1^{-1}(\text{Int}(C_2^n \cup \ldots \cup C_{p_1}^n))$. Then there exists a positive integer $n_2 > m_1$ such that each block of the solid torus $T_{n_2}$ has diameter less than $\frac{1}{2} d(g, E^3 \setminus W_2)$. Since $T_{n_2} \subset \text{Int} T_{m_1}$, the diameter of each block of $T_{n_2}$ is less than $\varepsilon/4$.

Now, consider the 3-cell $B_2 = \bigcup_{i=1}^{p_2} h_1(C_i^n)$, where $n_2, n_2, \ldots, C_{p_2}$ are all the blocks of $T_{n_2}$ meeting $g$ which are ordered in the same way as ordered in $T_{n_2}$. Then $B_2 \subset h_1(W_2)$, because the blocks of $T_{n_2}$ meeting $g$ lie in $W_2$. Moreover, $h_1(g) \subset \text{Int} B_2$, because
\[ g \in \text{Int}(C_1 \cup C_2 \cup \ldots \cup C_{p_2}^{n_2}). \] Thus, 
\[ h_1(g) \in \text{Int} B_2 \subset B_2 \subset h_1(W_2) \subset \text{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}) \subset \text{Int} B_1 \subset B_1 \subset V_g. \]

Let 
\[ W_2' = \{ g' \in G | g' \in \text{Int} \bigcup_{i=1}^{p_2} C_1^{n_2} \}. \] Then 
\[ g \in W_2' \subset \text{Int} \bigcup_{i=1}^{p_2} C_1^{n_2} \] so 
\[ h_1(g) \subset h_1(W_2') \subset \text{Int} B_2. \] There exists a positive integer \( m_2 > n_2 \) such that each block of the solid torus \( T_{m_2} \) has diameter less than \( \frac{1}{2} d(g, E^3 \setminus W_2') \). Since 
\[ T_{m_2} \subset \text{Int} T_{n_2}, \] all blocks of \( T_{m_2} \) have diameter less than \( \varepsilon/4 \).

Now, let \( D_2 \) be the image under \( h_1 \) of the union of all blocks of \( T_{m_2} \) meeting \( g \). The interior of this union contains \( g \) and because the blocks of \( T_{m_2} \) meeting \( g \) lie in \( W_2' \), the union lies in \( W_2' \), so that 
\[ h_1(g) \subset \text{Int} D_2 \subset h_1(W_2') \subset \text{Int} B_2 \subset B_2 \subset h_1(W_2) \subset \text{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1}) \subset \text{Int} B_1 \subset B_1 \subset V_g. \]

Consider the 3-cell 
\[ h_1(C_1^{n_2} \cup C_2^{n_2} \cup \ldots \cup C_{k_2}^{n_2}), \] where 
\[ k_2 < p_2, \quad h_1(C_{k_2}^{n_2}) \subset \text{Int} C_3^{n_1} \text{ and } h_1(C_{k_2}^{n_2} \cup C_{k_2+1}^{n_2} \cup \ldots \cup C_{p_2}^{n_2}) \subset \text{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1}). \] There is a homeomorphism \( h_2 \) from \( E^3 \) onto itself such that \( h_2 = \text{Id} \) on \( E^3 \setminus \text{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{k_2}^{n_2}) \), \( h_2 \) takes all the blocks of \( D_2 \) meeting \( h_1(C_1^{n_2} \cup \ldots \cup C_{k_2}^{n_2}) \) into the interior of \( h_1(C_{k_2}^{n_2}) \), and \( h_2 \) takes all blocks of \( D_2 \) lying in \( h_1(C_1^{n_2} \cup \ldots \cup C_{k_2}^{n_2}) \) into
$h_1(C_{k_2})$. The reason that such an $h_2$ exists is because of the special construction of the solid tori, in particular, when the sequence of blocks of $h_1(T_{m_2})$ leaves $h_1(C_1 \cup \ldots \cup C_{k_2-1})$, it does not go beyond $h_1(C_{k_2})$ before returning. Some other blocks of $h_1(T_{m_2})$ not lying in $D_2$ but contained in $h_1(C_1)$ may be pulled into the interior of $h_1(C_{k_2})$ or may be stretched by $h_2$ into $h_1(C_1 \cup C_2 \cup \ldots \cup C_{k_2})$. Certainly, $h_2(h_1(g)) \subseteq \text{Int}(C_1 \cup \ldots \cup C_{p_1})$.

Continuing inductively in this way, we get the following:

(1) integers $n_1 < m_1 < n_2 < m_2 < n_3 < m_3 < \ldots < n_j < m_j < \ldots < n_{p_1-2} < m_{p_1-2} < n_{p_1-1} < m_{p_1-1}$

(2) homeomorphisms $h_1, h_2, \ldots, h_{p_1-1}$ of $E^3$ onto itself, and for $j = 2, 3, \ldots, p_1$ putting

\[ \psi_{j-1} = h_{j-1} \circ h_{j-2} \circ \ldots \circ h_2 \circ h_1, \]

(3) open sets $W_j = u \{ g' \in G | \psi_{j-1}(g') \subseteq \text{Int}(C_{j} \cup \ldots \cup C_{p_1}) \}$,

(4) 3-cells $B_j = \bigcup_{i=1}^{p_j} \psi_{j-1}(C_1^j)$, where $C_1^j, C_2^j, \ldots, C_{p_j}^j$ are the blocks of $T_{n_j}$ meeting $g$ which are ordered in the same way as ordered in $T_{n_j}$.
(5) open sets \( W_j' = \{ g' \in G \mid \psi_{j-1}(g') \in \text{Int } B_j \} \),

(6) 3-cells \( D_j \) which are the image under \( \psi_{j-1} \) of the union of all blocks of \( T_{m_j} \) meeting \( g \).

(7) 3-cells \( \bigcup_{i=1}^{n_j} C_i \), where \( \ell_j < p_j \).

These integers, homeomorphisms, open sets and 3-cells satisfy the following conditions:

(a) Each block of the solid torus \( T_{n_j} \) has diameter less than \( \frac{1}{2} d(g, E^3 \setminus W_j) \),

(b) Each block of the solid torus \( T_{m_j} \) has diameter less than \( \frac{1}{2} d(g, E^3 \setminus W_j') \),

(c) \( h_j \) takes \( \psi_{j-1}(C_{1}^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j}) \) onto itself,

(d) \( h_j \) shrinks the part of \( D_j \) meeting \( \psi_{j-1}(C_{1}^{n_j} \cup \ldots \cup C_{\ell_j-1}^{n_j}) \) into the interior of \( \psi_{j-1}(C_{\ell_j}^{n_j}) \),

(e) \( h_j \) takes the part of \( D_j \) lying in \( \psi_{j-1}(C_{1}^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j}) \) into \( \psi_{j-1}(C_{\ell_j}^{n_j}) \),

(f) \( h_j = \text{Id} \) on \( E^3 \setminus \text{Int } \psi_{j-1}(C_{1}^{n_j} \cup \ldots \cup C_{\ell_j}^{n_j}) \),

(g) \( \psi_{j-1}(C_{i}^{n_j}) \subset \text{Int } (C_{j}^{n_1} \cup C_{j+1}^{n_1}) \),

(h) \( \psi_{j-1}(C_{\ell_j}^{n_j}) \subset \text{Int } C_{j+1}^{n_1} \),

(i) \( \psi_{j-1}(C_{\ell_j}^{n_j} \cup C_{\ell_j+1}^{n_1} \cup \ldots \cup C_{p_j}^{n_j}) \subset \text{Int } (C_{j+1}^{n_1} \cup C_{j+2}^{n_1} \cup \ldots \cup C_{p_1}^{n_1}) \).
From the definition of the 3-cells $D_j$ and conditions (d), (e), (h) and (i) we have $\psi_j(g) \in \text{Int}(C_{j+1}^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$. Thus $\psi_{p_1-1}(g) \in \text{Int}(C_{p_1}^{n_1})$ and so has diameter less than $\varepsilon/4$. Let $h_g = \psi_{p_1-1}$. Then $h_g$ is a homeomorphism from $E^3$ onto itself such that $h_g = \text{Id}$ on $E^3 \setminus V_g$ and the diameter of $h_g(g)$ is less than $\varepsilon/4$.

Now, we want to prove that $h_g$ maps any other $g'$ in $H$ meeting $B_1 \subset V_g$ into a set of diameter less than $\varepsilon$. So, suppose that $g' \in H$ and that $g'$ meets $C_1^{n_1} \cup C_2^{n_1}$ but that $g'$ does not meet $W_2$. Then by definition (2) we have $h_1(g')$ does not meet $h_1(W_2)$. Now, by definitions (4) and (7) and condition (a) we have

$h_1(C_1^{n_2} \cup \ldots \cup C_{p_2}^{n_2}) \subset h_1(W_2)$. Since, $h_1(g')$ does not meet $h_1(W_2)$, it does not meet $\text{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{p_2}^{n_2})$.

Now, by condition (f) in the construction of the homeomorphism $h_g$, we get $h_2 = \text{Id}$ on $E^3 \setminus \text{Int} h_1(C_1^{n_2} \cup \ldots \cup C_{p_2}^{n_2})$. Hence, $h_2$ does not move any point of $h_1(g')$. So, $h_2(h_1(g')) = h_1(g')$.

By definition (3), we have $W_3 = \{g' \in G | h_2(h_1(g')) \in \text{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})\}$ and $W_2 = \{g' \in G | h_1(g') \in \text{Int}(C_2^{n_1} \cup \ldots \cup C_{p_1}^{n_1})\}$. Consider any $g' \in W_3$. Then $h_2(h_1(g')) \in \text{Int}(C_3^{n_1} \cup \ldots \cup C_{p_1}^{n_1})$. Suppose that
$h_1(g') \notin \text{Int}(C_{2}^{n1} \cup \ldots \cup C_{p1}^{n1})$. Then $h_1(g') \cap h_1(W_2) = \emptyset$, by definition (3). By condition (f) we have $h_2 = \text{Id}$ on $E^3 \setminus \text{Int} (h_1(C_{1}^{n2} \cup \ldots \cup C_{\ell_2}^{n2}))$. Again by definitions (4) and (7) and condition (a) we have $\text{Int} h_1(C_{1}^{n2} \cup \ldots \cup C_{\ell_2}^{n2}) \subset h_1(W_2)$ so $E^3 \setminus \text{Int} h_1(C_{1}^{n2} \cup \ldots \cup C_{\ell_2}^{n2}) = E^3 \setminus h_1(W_2)$. Thus $h_2(h_1(W_2)) = h_1(W_2)$ so $h_1(g') \cap h_2(h_1(W_2)) = h_1(g') \cap h_1(W_2) = \emptyset$ and hence $h_2$ does not move any point of $h_1(g')$. Then $h_2(h_1(g')) = h_1(g') \notin \text{Int}(C_{2}^{n1} \cup \ldots \cup C_{p1}^{n1})$ and thus $h_1(g') = h_2(h_1(g')) \notin \text{Int}(C_{3}^{n1} \cup \ldots \cup C_{p1}^{n1})$, which is a contradiction because $g' \subset W_3$. Hence $h_1(g') \subset \text{Int}(C_{2}^{n1} \cup \ldots \cup C_{p1}^{n1})$ and thus $g' \subset W_2$ and $W_3 \subset W_2$. So $h_2(h_1(W_3)) = h_2(h_1(W_2))$. Since $h_1(g') \cap h_2(h_1(W_2)) = \emptyset$, we have $h_1(g') \cap h_2(h_1(W_3)) = \emptyset$.

Now, by definitions (4), (7) and condition (a) we have

$$\text{Int}(h_2 \circ h_1)(C_{1}^{n3} \cup \ldots \cup C_{\ell_3}^{n3}) \subset B_3 = \bigcup_{1=1}^{P_3} (h_2 \circ h_1)(C_{1}^{n3}) = h_2(h_1(W_3)).$$

Hence, $h_1(g') \cap \text{Int}(h_2 \circ h_1)(C_{1}^{n3} \cup \ldots \cup C_{\ell_3}^{n3}) = \emptyset$. Again by condition (f) we have $h_3 = \text{Id}$ on $E^3 \setminus \text{Int}(h_2 \circ h_1)(C_{1}^{n3} \cup \ldots \cup C_{\ell_3}^{n3})$ and thus $h_3(h_1(g')) = h_1(g')$. Similarly, $h_4, h_5, \ldots, h_{p1-1}$ do not move any point of $h_1(g')$. Hence $h_1(g') = h_1(g')$. Since $h_1 = \text{Id}$ on $E^3 \setminus \text{Int}(C_{1}^{n1} \cup C_{2}^{n1})$, $h_1(g') \subset g' \cup C_{1}^{n1} \cup C_{2}^{n1}$, and because $g'$ meets $C_{1}^{n1} \cup C_{2}^{n1}$, $\text{diam}(g' \cup C_{1}^{n1} \cup C_{2}^{n1}) \leq \text{diam } g' + \text{diam}(C_{1}^{n1} \cup C_{2}^{n1}) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence $\text{diam } h_1(g') = \text{diam } h_1(g') < \epsilon$. 

\[\text{diam } h_1(g') = \text{diam } h_1(g') < \epsilon.\]
The construction of $J$ shows that $g$ is the only element of $H$ which meets a block of $T_{n_1}$ between $C_{n_1}^{P_1-1}$ and $C_{n_1}^{P_1}$. To prove this suppose $g' \neq g$ is an element in $H$ which meets $C_{n_1}^{P_1}$. We want to show that $g'$ does not meet $C_{n_1}^{P_1}$. So, if possible suppose that $g'$ meets $C_{n_1}^{P_1}$. Now $g' \cap g = \emptyset$ and both $g$ and $g'$ are compact. Then $d(g,g') > 0$. There exists $j$ such that each block of the solid torus $T_j$ has diameter less than $\frac{1}{2} d(g,g')$ and thus any block of $T_j$ meets at most one of $g'$ and $g$. Since $g'$ meets $C_{n_1}^{P_1}$, there exists a block $C$ of $T_j$ lying in $C_{n_1}^{P_1}$ that meets $g'$ only. But $g$ is connected, lies in $\overset{\circ}{T_{n_1}}$ and so meets all the blocks between $C_{n_1}^{P_1}$ and $C_{n_1}^{P_1}$ of $T_{n_1}$. Therefore $g$ must pass through all the blocks of $T_j$ lying in $C_{n_1}^{P_1}$, in particular $C$. This is a contradiction. Hence $g'$ does not meet $C_{n_1}^{P_1}$. Similarly, any other $g' \neq g$ in $H$ meeting $C_{n_1}^{P_1}$ can not meet $C_{n_1}^{P_1}$. Hence $g$ is the only element of $H$ which meets a block of $T_{n_1}$ between $C_{n_1}^{P_1-1}$ and $C_{n_1}^{P_1}$.

Now suppose that $g' \neq g$ in $H$ is such that it meets $C_{n_1}^{P_1-1} \cup C_{n_1}^{P_1}$. Then $g'$ does not meet $C_{n_1}^{P_1-2}$. Now $g' = (g' \cap (C_{n_1}^{P_1-1} \cup C_{n_1}^{P_1})) \cup (g' \cap (E^3 \setminus \text{Int } B_1))$. From the construction of $h_g$ we see that $h_g(g' \cap (C_{n_1}^{P_1-1} \cup C_{n_1}^{P_1})) \subset C_{n_1}^{P_1}$. So, $\text{diam } h_g(g' \cap (C_{n_1}^{P_1-1} \cup C_{n_1}^{P_1})) \leq \text{diam } C_{n_1}^{P_1} < \varepsilon/4.$
Since $B_1 \subset V_g$ and $g'$ meets $B_1$ we have $g' \subset \tilde{V}_g$ and therefore $\text{diam } g' < \varepsilon/2$. Hence $\text{diam } h_g(g' \cap (E^3 \setminus \text{Int } B_1)) < \varepsilon/2$, because $h_g = \text{Id}$ on $E^3 \setminus \text{Int } B_1$. Consequently,

$$\text{diam } h_g(g') \leq \text{diam } h_g(g' \cap (C_{p_1 - 1}^{n_1} \cup C_{p_1}^{n_1})) + \text{diam } h_g(g' \cap (E^3 \setminus \text{Int } B_1)) \leq \varepsilon/4 + \varepsilon/2 < \varepsilon.$$

Finally, from definitions (3), (4) and (7) and conditions (a) and (c) we see that if $g'$ belongs to $H$ and meets $C_{1}^{n_1} \cup C_{2}^{n_1}$ and $W_2$ then either $g' \cap W_j = \emptyset$ and $g' \subset W_{j-1}$ for some $j$ in $\{3,2,...,p_1-1\}$ or $g' \subset W_j$ for all $j$. So we have the following two general cases, which will complete the proof that $h_g$ takes any $g'$ in $H$ meeting $B_1$ into a set of diameter less than $\varepsilon$.

**General Case 1:** Suppose that $g'$ belongs to $H$, that $g'$ meets $(C_{1}^{n_1} \cup C_{2}^{n_1})$ and $W_2$, that $g'$ does not meet $W_j$ but that $g' \subset W_{j-1}$ for some $j$ in $\{3,2,...,p_1-1\}$. Then by definition (3) we have $\psi_{j-1}(g')$ does not meet $\psi_{j-1}(W_j)$. Now, by definitions (4) and (7) and condition (a) we have $\psi_{j-1}^{n^j_1}(C_{1}^{n_1} \cup ... \cup C_{k_j}^{n_1}) = \psi_{j-1}(W_j)$. Since $\psi_{j-1}(g')$ does not meet $\psi_{j-1}(W_j)$, it does not meet $\text{Int } \psi_{j-1}(C_{1}^{n_1} \cup ... \cup C_{k_j}^{n_1})$. Now, by condition (f) in the construction of the homeomorphism $h_g$, we get $h_j = \text{Id}$ on $E^3 \setminus \text{Int } \psi_{j-1}(C_{1}^{n_1} \cup ... \cup C_{k_j}^{n_1})$. Hence $h_j$ does not
move any point of $\psi_{j-1}(g')$. So, $\psi_j(g') = h_j(\psi_{j-1}(g')) = \psi_{j-1}(g')$.

By definition (3), we have $W_{j+1} = \cup \{g' \in G | \psi_j(g') \in \operatorname{Int}(C_{n1}^{j+1} \cup \ldots \cup C_{n1}^{p_1}) \}$ and $W_j = \cup \{g' \in G | \psi_{j-1}(g') \in \operatorname{Int}(C_{n1}^{j} \cup \ldots \cup C_{n1}^{p_1}) \}$. Consider any $g' \in W_{j+1}$. Then

$\psi_j(g') \in \operatorname{Int}(C_{n1}^{j+1} \cup \ldots \cup C_{n1}^{p_1})$. Suppose that $\psi_{j-1}(g') \notin \operatorname{Int}(C_{n1}^{j} \cup \ldots \cup C_{n1}^{p_1})$. Then $\psi_{j-1}(g') \cap \psi_{j-1}(W_j) = \emptyset$, by definition (3). By condition (f) we have $h_j = \text{Id}$ on $E^3 \setminus \operatorname{Int}(C_{n1}^{j} \cup \ldots \cup C_{n1}^{p_1})$. Again by definitions (4) and (7) and condition (a) we have $\operatorname{Int}(C_{n1}^{j} \cup \ldots \cup C_{n1}^{p_1}) \subset \psi_{j-1}(W_j)$ so $E^3 \setminus \operatorname{Int}(C_{n1}^{j} \cup \ldots \cup C_{n1}^{p_1}) \supset E^3 \setminus \psi_{j-1}(W_j)$.

Thus $h_j(\psi_{j-1}(W_j)) = \psi_{j-1}(W_j)$ so $\psi_{j-1}(g') \cap h_j(\psi_{j-1}(W_j)) = \psi_{j-1}(g') \cap \psi_{j-1}(W_j) = \emptyset$ and hence $h_j$ does not move any point of $\psi_{j-1}(g')$. Then $h_j(\psi_{j-1}(g')) = \psi_{j-1}(g') \notin \operatorname{Int}(C_{n1}^{j} \cup \ldots \cup C_{n1}^{p_1})$ and thus $\psi_{j-1}(g') = h_j(\psi_{j-1}(g')) \notin \operatorname{Int}(C_{n1}^{j+1} \cup \ldots \cup C_{n1}^{p_1})$, which is a contradiction because $g' \in W_{j+1}$. Hence $\psi_{j-1}(g') \subset \operatorname{Int}(C_{n1}^{j} \cup \ldots \cup C_{n1}^{p_1})$ and thus $g' \subset W_j$ and $W_{j+1} \subset W_j$. So, $h_j(\psi_{j-1}(W_{j+1})) \subset h_j(\psi_{j-1}(W_j))$, that is $\psi_j(W_{j+1}) \subset \psi_j(W_j)$. Since $\psi_{j-1}(g') \cap \psi_j(W_j) = \emptyset$, we have $\psi_{j-1}(g') \cap \psi_j(W_{j+1}) = \emptyset$. Definitions (4) and (7) and condition (a) imply that

$\operatorname{Int}(C_1^{j+1} \cup \ldots \cup C_{\ell(j+1)}^{p_{j+1}}) \subset B_{j+1} = \bigcup_{i=1}^{p_{j+1}} \psi_i(C_i^{j+1})$.
$\psi_j(W_{j+1})$. Hence, $\psi_{j-1}(g') \cap \text{Int } \psi_j(c_{1}^{j+1} \cup ... \cup c_{|j+1|}^{j+1}) = \phi$. Again by condition (f) we have $h_{j+1} = \text{Id}$ on $E^3 \setminus \text{Int } \psi_j(c_{1}^{j+1} \cup ... \cup c_{|j+1|}^{j+1})$ and thus $h_{j+1}(\psi_{j-1}(g')) = \psi_{j-1}(g')$. Similarly, $h_{j+2}$, $h_{j+3}$, ..., $h_{p_1-1}$ do not move any point of $\psi_{j-1}(g')$. Hence $h_{g}(g') = \psi_{j-1}(g')$.

Now, by condition (f), for any $k$ we have $h_{k} = \text{Id}$ on $E^3 \setminus \text{Int } \psi_{k-1}(c_{1}^{k} \cup ... \cup c_{|k|}^{k})$. So, condition (g) implies that $h_{k}$ only moves points lying in $\text{Int}(c_{1}^{1} \cup c_{|k+1|}^{k})$. We proved before that $g$ is the only element of $H$ which meets a block of $T_{n_1}$ between $c_{p_1-1}^{1}$ and $c_{1}^{2}$. Now, $g'$ meets $c_{1}^{1} \cup c_{1}^{2}$ so it can not meet $c_{3}^{1} \cup ... \cup c_{p_1}^{1}$. Since $g' \subset W_{j-1}$, $\psi_{j-2}(g') \subset \text{Int}(c_{j-1}^{1} \cup ... \cup c_{|p_1|}^{1})$ by definition (3). Since $h_{1} = \text{Id}$ on $E^3 \setminus c_{1}^{1} \cup c_{1}^{2}$, by conditions (f) and (g) we see that $\psi_{j-2} = h_{j-2} \circ ... \circ h_{2} \circ h_{1}$ only moves points in $c_{1}^{1} \cup ... \cup c_{j-1}^{1}$ so that $\psi_{j-2}(g') \subset \text{Int } c_{j-1}^{1}$. Since $h_{j-1}$ only moves points in $c_{j-1}^{1} \cup c_{1}^{1}$, we have $h_{g}(g') = \psi_{j-1}(\psi_{j-2}(g')) \subset c_{j-1}^{1} \cup c_{1}^{1}$. Consequently, $\text{diam } h_{g}(g') \leq \text{diam } c_{j-1}^{1} + \text{diam } c_{1}^{1} < \epsilon/4 + \epsilon/4 = \epsilon/2$.

General Case 2: If $g'$ in $H$ is such that $g'$ meets $c_{1}^{1} \cup c_{1}^{2}$ but $\psi_{j-1}(g') \subset \text{Int}(c_{j}^{1} \cup ... \cup c_{|p_1|}^{1})$ for each $j = 2,3,...,p_1-1$, then $h_{g}(g') = \psi_{p_1-1}(g')$ and $\psi_{p_1-2}(g') \subset$
\[ \text{Int}(C_{p_1}^{n_1} \cup C_{p_1}) \cap \text{Int}(C_{p_1}^{n_1} \cup C_{p_1}) \]

Thus \( h_{p_1-1}(\psi_{p_1-2}(g')) \in \text{Int}(C_{p_1}^{n_1} \cup C_{p_1}) \).

Hence \( h_g(g') = \psi_{p_1-1}(g') \in \text{Int}(C_{p_1}^{n_1} \cup C_{p_1}) \). Consequently

\[ \text{diam } h_g(g') < \text{diam } C_{p_1-1}^{n_1} + \text{diam } C_{p_1}^{n_1} < \varepsilon/4 + \varepsilon/4 = \varepsilon/2. \]

Since \( H_\varepsilon \) is finite, we let \( H_\varepsilon = \{g_1, g_2, \ldots, g_r\} \).

Now, let \( h_\varepsilon = h_{g_1} \circ h_{g_2} \circ \ldots \circ h_{g_r} \). Since \( B_1 \) is connected and contained in \( U \) it lies in the component of \( U \) containing \( g \). Then using the conditions (f) and (g) we see that \( h_j \) maps \( B_1 \) onto itself and therefore any component of the open set \( U \) onto itself. Using this property in the construction of \( h_g \) and the definition of \( h_\varepsilon \), it follows that \( h_\varepsilon \) maps any component of the open set \( U \) onto itself.

Consider any \( g' \in g \). If \( g' \) meets an open set \( V_{g_1} \) for some \( g_1 \in H_\varepsilon \), then \( g' \subseteq V_{g_1} \). For \( j \neq i \), \( h_{g_j} = \text{Id on } V_{g_1} \) because \( h_{g_j} = \text{Id on } E^3 \setminus V_{g_j} \) and

\[ V_{g_1} \cap V_{g_j} = \emptyset, \] so \( h_\varepsilon(g') = h_{g_1}(g') \). Since \( h_{g_1} \) shrinks each \( g' \) in \( V_{g_1} \) into a set of diameter less than \( \varepsilon \), we have \( \text{diam } h_{g_1}(g') < \varepsilon \) and hence \( \text{diam } h_\varepsilon(g') < \varepsilon \).

If \( g' \cap V_{g_j} = \emptyset \), for all \( g_j \in H_\varepsilon \) then since \( h_{g_j} = \text{Id on } E^3 \setminus V_{g_j} \), we have \( h_\varepsilon(g') = (h_{g_1} \circ \ldots \circ h_{g_r})(g') = g' \). Hence, \( \text{diam } h_\varepsilon(g') = \text{diam } g' < \varepsilon/2 \), because
each element of $G$ with diameter $> \epsilon/2$ lies in some $V_{g_i}$.

Since for each $j$, $V_{g_j} \subset U$, $h_{g_j} = \text{Id}$ on $E^3 \setminus U$ for each $j$. Hence $h_\epsilon = \text{Id}$ on $E^3 \setminus U$. Consequently, $h_\epsilon$ is a homeomorphism from $E^3$ onto itself which shrinks each $g' \in G$ into a set $h_\epsilon(g')$ of diameter less than $\epsilon$, is $\text{Id}$ on $E^3 \setminus U$ and takes each component of $U$ onto itself. This completes the proof of the basic lemma.

3.5 $E^3/G$ is Homeomorphic to $E^3$

In this section we will prove the following.

Main Theorem. Suppose $G$ is a decomposition of $E^3$ such that the non-degenerate elements of $G$ are pairwise disjoint subarcs of Bing's sling. Then the decomposition space $E^3/G$ is homeomorphic to $E^3$.

The theorem is proved by repeated applications of the Basic Lemma in Section 3.4. We must define a continuous function $h$ from $E^3$ onto itself such that $G = \{h^{-1}(x) | x \in E^3\}$ and $h$ is a quotient function. Then by Theorem 1.1.3 we will prove that $E^3/G$ is homeomorphic to $E^3$. We will obtain $h$ as the uniform limit of a sequence $<h_1>$ of homeomorphisms from $E^3$ onto itself.

In Section 3.1 we proved that the set $H$ of non-
degenerate elements is countable and in Section 3.3 we proved that \( G \) is an upper semicontinuous decomposition of \( E^3 \). We use these results in the following proof of the above theorem.

**Proof.** Let us consider \( \varepsilon = 1 \) and open set \( U = E^3 \). Then by the Basic Lemma in Section 3.4, there is a homeomorphism \( h_1 \) of \( E^3 \) onto itself which takes each \( g' \in G \) into a set \( h_1(g') \) of diameter less than 1.

Before we describe \( h_2, h_3, \ldots \), we shall describe some open sets in the decomposition space \( E^3/G \). Since \( H \) is countable and \( E^3/G \) is a metric space by Theorem 1.1.6, we have by Example 1.2.1, that \( H \) is 0-dimensional in the decomposition space \( E^3/G \). For each \( i \), let us consider open balls \( B(g_i, 1/2m) \) in \( E^3/G \), where \( m = 1, 2, 3, \ldots \). Then, \( \text{diam } B(g_i, 1/2m) \leq 1/m \) and \( g_i \in B(g_i, 1/2m) \), so that \( \{B(g_i, 1/2m) | i = 1, 2, \ldots \} \) is a countable open covering for the 0-dimensional space \( H \) in \( E^3/G \). Then by Theorem 1.2.3 for each \( m \) there exists a collection of open sets \( \{M^m_i | i = 1, 2, \ldots \} \) covering \( H \) such that \( M^m_i \subset B(g_i, 1/2m) \) for each \( i \), and \( M^m_i \cap M^m_j = \emptyset \) for \( i \neq j \). Thus, the diameter of \( M^m_i \) is less than or equal to \( 1/m \). Also \( \bigcup_{i=1}^{\infty} M^m_i \subset \bigcup_{i=1}^{\infty} B(g_i, 1/2m) = B(\{g_i | i = 1, 2, \ldots \}, 1/2m) \).
Since in $E^3$ we have $\bigcup_{i=1}^\infty g_i = J$, the subset $\bigcup_{i=1}^\infty g_i$ is a compact subset of $E^3$. Now, $P(\bigcup_{i=1}^\infty g_i) = \{g_i | i=1, 2, \ldots\}$, because $P$ is closed by Theorem 1.1.5. Since $P$ is continuous, $\{g_i | i=1, 2, \ldots\}$ is compact in $E^3/G$. Now, there is a neighborhood $N_m$ of $\bigcup_{i=1}^\infty g_i$ in $E^3$ such that $N_m$ is compact in $E^3$ and therefore $P(N_m)$ is compact in $E^3/G$ with $\{g_i | i=1, 2, \ldots\} \subset P(N_m) \subset \bigcup_{i=1}^\infty M_i^m$. Let $V_m = P(N_m)$. Since $P^{-1}(V_m) = P^{-1}P(N_m) = N_m$, we have $P(N_m) = V_m$ is open in $E^3/G$ because $P$ is a quotient map. Then $\overline{V}_m = P(\overline{N}_m) = P(\overline{N}_m)$. Since $P(\overline{N}_m)$ is compact, $\overline{V}_m$ is compact in $E^3/G$. Thus, $V_m$ is a bounded open set in the decomposition space $E^3/G$ such that $\bigcup_{i=1}^\infty M_i^m \supset V_m \supset H$.

Since the $M_i^m$'s are pairwise disjoint open sets, a component of $V_m$ must be a subset of some $M_i^m$. Hence each component of $V_m$ has diameter less than or equal to $1/m$.

Thus, we get a sequence of open sets $V_1, V_2, V_3, \ldots$, in the decomposition space $E^3/G$ such that for each $m$, $V_m \supset H$, each component of $V_m$ is of diameter less than or equal to $1/m$ in $E^3/G$ and $\overline{V}_m$ is compact.

Let $V_m^* = \{g' \in G | g' \in V_m\}$. Thus $V_m^* \subset E^3$. Now, for any $g_i \in H$, diam $h_1(g_i) < 1$ and so there exists an open set $W_i \supset h_1(g_i)$ such that diam $W_i < 1$. By upper semi-continuity of $G$, $O_i = \{g' \in G | g' \in h_1^{-1}(W_i)\}$ is open in
$E^3$ and $g_i \in O_1 \subset h_i^{-1}(W_1)$. Since $P^{-1}P(O_1) = O_1$,
$\{P(O_1) \mid i = 1,2,\ldots\}$ is an open cover of $H$ in $E^3/G$, so
by 0-dimensionality of $H$ in $E^3/G$ and Theorem 1.2.3,
there exists an open cover $\{C_i\}$ of $H$ in $E^3/G$ such
that the $C_i$'s are pairwise disjoint and $C_i \subset P(O_1)$
for each $i$. Let $U_1 = (\bigcup_{i \in I} P^{-1}(C_i)) \cap V_i^*$. Then $U_1$
is open in $E^3$.

Now, if $K$ is a component of $h_1(U_1)$, then $h_1^{-1}(K)$
is a component of $U_1$. Since the $P^{-1}(C_i)$'s are pairwise
disjoint, $h_1^{-1}(K)$ must lie in some $P^{-1}(C_i)$, hence in
$P^{-1}P(O_1) = O_1 \subset h_1^{-1}(W_1)$, so that $K \subset W_1$, and hence
diam $K \leq$ diam $W_1 < 1$.

Now, $V_1 = P(N_1)$ and $P^{-1}(V_1) = P^{-1}P(N_1) = N_1$. Then
$P^{-1}(V_1) = \overline{N}_1$. Since $\overline{N}_1$ is compact in $E^3$,
we have
$P^{-1}(V_1)$ is compact in $E^3$. But $V_1^* = P^{-1}(V_1)$, so $V_1^* =
\overline{P^{-1}(V_1)}$. Hence $\overline{V}_1^*$ is compact in $E^3$. Since $\overline{U}_1 \subset V_1^*$, it
is compact in $E^3$. Hence, $h_1$ is uniformly continuous on
$U_1$. Let $\delta > 0$ correspond to $\varepsilon = 1/2$ in the definition
of the uniform continuity of $h_1$ on $U_1$. Then, for the
given $\delta$, Basic Lemma says there is a homeomorphism $\psi_\delta$
of $E^3$ onto itself which shrinks each $g'$ in $G$ into a
set of diameter less than $\delta$ and which is the identity on
$E^3 \setminus U_1$. Consider $h_2 = h_1 \circ \psi_\delta$. Then $h_2 = h_1$ on $E^3 \setminus U_1$.
and \( \text{diam } h_2(g') = \text{diam } h_1(\psi_0(g')) < 1/2 \) for each \( g' \in G \), by the uniform continuity of \( h_1 \) on \( U_1 \).

Thus, \( h_2 \) is a homeomorphism from \( E^3 \) onto itself which shrinks each \( g' \) in \( G \) into a set of diameter less than \( 1/2 \) and \( h_2 = h_1 \) on \( E^3 \setminus U_1 \).

Just as we constructed \( U_1 \) and \( h_2 \), we can inductively construct homeomorphisms \( h_3, h_4, \ldots \), and open sets \( U_2 \), \( U_3, U_4, \ldots \), so that \( h_1 \) shrinks each \( g' \) in \( G \) into a set \( h_1(g') \) with \( \text{diam } h_1(g') < 1/i \), \( h_{i+1} = h_i \) on \( E^3 \setminus U_i \), \( U_{i+1} \subset U_i \cap V_i^{\ast} \) and each component of \( h_i(U_i) \) has diameter less than \( 1/i \).

Thus, \( \langle h_i \rangle \) is a sequence of homeomorphisms of \( E^3 \) onto itself. Further, there is a sequence \( \langle U_i \rangle \) of open sets such that (a) \( U_i \supset U_{i+1} \), (b) \( h_{i+1} = h_i \) on \( E^3 \setminus U_i \) and (c) each component of \( h_i(U_i) \) has diameter less than \( 1/i \). Hence, by Theorem 1.1.9 \( \langle h_i \rangle \) converges uniformly to a continuous function \( h \). Also by Theorem 1.1.10, \( h \) is onto \( E^3 \) and \( h \) is closed. Since \( h \) is closed, it is a quotient map by Theorem 1.1.1.

Now, we want to prove that \( G = \{ h^{-1}(x) | x \in E^3 \} \).
Suppose \( p, q \) belong to \( g' \), for some \( g' \) in \( H \). Then for each \( i \), \( g' \) is contained in a component of the open set \( U_i \) so that \( h_i(g') \) is contained in a component of
the open set \( h_i(U_i) \) and hence for each \( i \),
\[ d(h_i(p), h_i(q)) < 1/i. \]

Since \( h_i \) converges uniformly to \( h \), it also converges pointwise to \( h \). Then, \( h_i(p) \) converges to \( h(p) \), \( h_i(q) \) converges to \( h(q) \) as \( i \to \infty \) and therefore
\[ d(h(p), h(q)) \leq 0. \] So, \( d(h(p), h(q)) = 0 \). Hence
\[ h(p) = h(q). \]

Now, suppose \( p \) and \( q \) belong to different elements \( P(p) \) and \( P(q) \) of \( G \). \( E^3/G \) is a metric space and
\[ d(P(p), P(q)) > 0. \] There is a positive integer \( m \) such that
\[ \frac{1}{m} < \frac{1}{3} d(P(p), P(q)). \] Consider the subset \( V_m \) of \( G \). Now \( V_m \supset H \) and each component of \( V_m \) has diameter less than or equal to \( 1/m \). Hence \( P(p) \) and \( P(q) \) do not belong to the same component of \( V_m \cup \{ P(p) \} \cup \{ P(q) \} \) in the decomposition space \( E^3/G \). Hence \( P(p) \) and \( P(q) \) are not subsets of the same component of \( U_m \cup \{ p \} \cup \{ q \} \) in \( E^3 \).

Let \( D_p \) and \( D_q \) be the components of \( U_m \cup \{ p \} \cup \{ q \} \) such that \( P(p) \subset D_p \) and \( P(q) \subset D_q \). Then \( P(D_p) \) and \( P(D_q) \) will be contained in some components of \( V_m \cup \{ P(p) \} \cup \{ P(q) \} \). Since \( d(P(p), P(q)) > 3/m \) and components of \( V_m \cup \{ P(p) \} \cup \{ P(q) \} \) have diameters less than or equal to \( 1/m \), \( d(P(D_p), P(D_q)) \) must be at least \( 1/m \). Hence
\[ P(D_p) \cap P(D_q) = \emptyset, \text{ so } P^{-1}P(D_p) \cap P^{-1}P(D_q) = \emptyset. \] But
\[ \overline{D}_p \subset P^{-1} \overline{F(D_p)} \quad \text{and} \quad \overline{D}_q \subset P^{-1} \overline{F(D_q)} \quad \text{so that} \quad \overline{D}_p \cap \overline{D}_q = \emptyset. \]

Now, \( h_m(p) \in h_m(D_p) \subset h_m(U_m \cup \{p\} \cup \{q\}) \) and\( h_m(q) \in h_m(D_q) \subset h_m(U_m \cup \{p\} \cup \{q\}). \) Also, by the construction of the \( h_i \)'s we see that \( h_m, h_{m+1}, h_{m+2}, \ldots, \) all map \( D_p \) onto \( h_m(D_p) \) and \( D_q \) onto \( h_m(D_q) \). Thus, \( h_m(D_p) = h_{m+1}(D_p) = \ldots = h_{m+k}(D_p) \) and \( h_m(D_q) = h_{m+1}(D_q) = \ldots = h_{m+k}(D_q) \). So, \( h_{m+k}(p) \in h_m(D_p) \) and \( h_{m+k}(q) \in h_m(D_q) \) for all \( k \). Thus, \( h(p) \in \overline{h_m(D_p)} \) and \( h(q) \in \overline{h_m(D_q)} \). Hence, \( h(p) \neq h(q) \) as \( \overline{D}_p \cap \overline{D}_q = \emptyset \).

Consequently, \( G = \{ h^{-1}(x) | x \in E^3 \} \). Since \( h \) is a quotient map, by Theorem 1.1.3 there is a homeomorphism from \( E^3/G \) onto \( E^3 \). Hence, the decomposition space \( E^3/G \) is homeomorphic to \( E^3 \).
REFERENCES


[2] Bing, R.H., A decomposition of $E^3$ into points and tame arcs such that the decomposition space is topologically different from $E^3$, Annals of Mathematics (2)65(1957), pp.484-500.


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