

**REDUCTIVE LIE GROUPS AND
MOSTOW FIBRATION**

BY

SALEH ALI AL-BASHRAWI

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES
**KING FAHD UNIVERSITY
OF PETROLEUM AND MINERALS**
DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCES

In

MATHEMATICAL SCIENCES


December 2004


KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

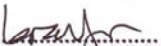
DEANSHIP OF GRADUATE STUDIES


This thesis, written by Al-Bashrawi, Saleh under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICAL SCIENCES.

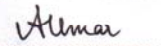
Thesis Committee


.....
Dr. Azad, Hassan
(Thesis Advisor)


.....
Dr. Al-Shakhs, Adnan
(Member)



.....
Dr. Laradji, Abdallah
(Member)


.....
Dr. Al-Bar, Mohammed
(Member)



.....
Dr. Umar, Abdullahi
(Member)


.....

Dr. Khaled M. Furati
Department Chairman


.....

Dr. Mohammad A. Al-Ohali
Dean of Graduate Studies


.....
Date 2-4-2005



Acknowledgement

All praise be to God, Allah the Almighty Lord, without whose mercy and clemency nothing would have been possible. I am deeply obliged to Doctor Hassan Azad, my thesis supervisor. Without him this work would not have been accomplished.

I wish to express my thanks and gratitude to my thesis committee members: Dr. Adnan Al-Shaks, Dr. Abdullah Laradji, Dr. Mohamed Al-Bar, Dr. Abdullah Umar.

I extend my thanks to my family, my parents, my brothers and sisters and my friends specially Ali Hayyan and Fadhel Abbas, and my wife.

THESIS ABSTRACT

Full Name of Student: Saleh Ali Jawad Al-Bashrawi

TITLE OF STUDY: Reductive Lie Groups and Mostow fibration

MAJOR FIELD: Mathematics

DATE OF DEGREE: December 2004

This thesis discusses reductive Lie groups and Mostow fibration structure.

A Lie group is first of all a group, and in addition it has the structure of a smooth manifold: these two structures are related in that multiplication and inversion are required to be smooth mappings. Each Lie group has a Lie algebra which can be defined as a collection of Left invariant vector fields and this Lie algebra defines the kind of the Lie group and its properties. The most famous examples of Lie groups are the linear Lie groups which are groups of nonsingular matrices.

A connected Lie subgroup of $GL(n, \mathbb{R})$ is called reductive if its Lie algebra can be decomposed into two spaces satisfying certain conditions and its complexification is a Lie algebra of a compact Lie subgroup of $GL(n, \mathbb{C})$. Examples include semisimple Lie groups with finite center, any connected closed linear group closed under conjugate transpose, and the centralizer in a reductive Lie group of a θ stable abelian subalgebra of the Lie algebra. The involution θ , which is called the “Cartan involution” of the Lie algebra, is the differential of a global Cartan involution Θ of G .

The Mostow fibration is a continuous mapping between two homogeneous spaces showing that any non-compact space can be deformed to a compact space.

MASTER OF SCIENCE DEGREE
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
Dhahran, Saudi Arabia
December, 2004

خلاصة الرسالة

اسم الطالب: صالح علي جواد البشراوي

عنوان الدراسة: زمر لي المتقلصة و انتشار موستاو

التخصص: رياضيات

تاريخ الشهادة: ديسمبر 2004 – ذو القعدة 1425

هذه الرسالة عبارة عن محاولة جادة للتعرف على زمر لي المتقلصة و دراسة خصائصها عن طريق إعطاء بعض الحقائق المتعلقة و التعاريف الأساسية. كما أنها تشرح آلية بناء حزمة موستاو.

زمرة لي عبارة عن زمرة في المقام الأول بالإضافة إلى كونها تتكون من مطوية متصلة. هذين التركيبين مرتبطين عن طريق كون دوال الضرب و النظير الخاصة بالزمرة مطلوبة لتكون متصلة. كل زمر من زمر لي تحتوي على فضاء خاص يعرف بـ (جبر لي) و يمكن تعريف هذا الجبر على أنه مجموعة من حقول المتجهات غير المتغيرة يسارياً، و هذا الجبر يحدد نوع زمرة لي و خصائصها. و أكثر الأمثلة شهرةً لزمر لي هي زمر لي الخطية و التي عبارة عن زمر للمصفوفات غير الوحيدة.

زمرة لي المتجزئة عبارة عن زمرة يمكن تجزئة جبرها إلى جزأين بحيث يحققان شروط خاصة و يرتبطان بكيفية خاصة بزمرة مركبة متضامة.

درجة الماجستير في العلوم الرياضية

جامعة الملك فهد للبترول و المعادن

الظهران – المملكة العربية السعودية

التاريخ: 2004 / 12 / 26

Contents

CHAPTER I : LIE GROUPS

1. Introduction	7
2. Topological Groups	8
3. Homogeneous Spaces of Topological Groups, Locally Compact Groups	9
4. Lie Group and Lie algebra	10
5. One-parameter Subgroups and Exponential map	16
6. Complex Lie Groups and Complex Lie algebra	25
7. Lie Subgroup of a Lie Group	27
8. Ideals	31
9. Killing Form	35
10. Automorphisms and Derivations	37
11. Linear Lie Groups	41
12. Compact Lie Algebra	49
13. Cartan Decomposition on the Lie algebra level	50

CHAPTER II : REDUCTIVE LIE GROUPS

58

CHAPTER III : MOSTOW FIBRATION.....

64

References

80

CHAPTER I

LIE GROUPS

A Lie group is a *differentiable manifold* obeying the group properties and that satisfies the additional condition that the group operations are continuous.

The simplest examples of Lie groups are one-dimensional. Under addition, the real line is a Lie group. After picking a specific point to be the identity element, the circle is also a Lie group. Another point on the circle at angle θ from the identity then acts by rotating the circle by the angle θ . In general, a Lie group may have a more complicated group structure, such as the orthogonal group $O(n)$ (i.e., the $n \times n$ orthogonal matrices), or the general linear group $GL(n)$ (i.e., the $n \times n$ invertible matrices).

The tangent space at the identity of a Lie group always has the structure of a Lie algebra, and this Lie algebra determines the local structure of the Lie group via the exponential map. For example, the function e^{it} gives the exponential map from the circle's tangent space (i.e., the reals), to the circle, thought of as the unit circle in \mathbf{C} . A more difficult example is the exponential map e^A from skew symmetric $n \times n$ matrices to the special orthogonal group $SO(n)$, the subset of $O(n)$ with determinant 1.

The topology of a Lie group is fairly restricted. For example, there always exists a nonvanishing vector field. This structure has allowed complete classification of the finite dimensional semisimple Lie groups and their representations.

1. Introduction:

Definition (Homeomorphism) 1.1: A homeomorphism f between two topological spaces M and N is a bijective, continuous map $f : M \rightarrow N$ with a continuous inverse $f^{-1} : N \rightarrow M$. [3]

Definition (Manifold) 1.2: An n -manifold is a topological space M with the property that, if $x \in M$, then there is some neighborhood U of x such that U is homeomorphic to \mathbb{R}^n . [5]

Definition (Chart, Atlas, Maximal Atlas) 1.3: A chart (φ, U) on an n -manifold M is an open set U of M and a homeomorphism $\varphi : U \rightarrow \mathbb{R}^n$. Two charts, (φ, U) and (ϕ, V) , are said to have smooth overlap if the maps $\phi^{-1} \circ \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\varphi^{-1} \circ \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are smooth. A family of charts which covers M and whose members have smooth overlap is called an atlas. A maximal atlas for M is an atlas which contains the maximum number of charts. [3]

Definition (Differentiable Manifold) 1.4: A differentiable manifold is a manifold with an associated maximal atlas. [3]

Definition (Exponential) 1.5: The matrix exponential function, $\exp : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$, is defined in terms of the Taylor series expansion of the exponential:

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots . [16]$$

2. Topological Groups:

Definition 1.6: If a set G has the following properties, G is called a topological group:

- 1- The set G is a group and a topological space simultaneously.
- 2- The map $(x, y) \rightarrow xy$ from the direct product space $G \times G$ to G is continuous. Here xy denotes the product of two elements x and y of the group G .
- 3- The map $x \rightarrow x^{-1}$ from G to G is continuous. Here x^{-1} is the inverse element of x in G . [13]

Let G be a topological group, and $g \in G$. Define the maps L_g and R_g from G to G by :

$$L_g(x) = gx, \quad R_g(x) = xg \quad (x \in G).$$

Then L_g and R_g are homeomorphisms of G . In fact, by the definition of a topological group, L_g and R_g are both 1:1 continuous maps from G onto G , and their inverse maps are $L_{g^{-1}}$ and $R_{g^{-1}}$ respectively. The inverse maps are also continuous, so L_g and R_g are homeomorphisms of G . The maps L_g and R_g are called the *left translation* and the *right translation* of G , respectively, by the element g of G .

Examples:

- 1) \mathbb{R}^n is a topological group with respect to addition.
- 2) Let $GL(n, \mathbb{R})$ be the set of nonsingular $n \times n$ matrices with real entries. $GL(n, \mathbb{R})$ is a group with respect to multiplication of matrices. The $n \times n$ unit matrix is the identity element, and the inverse of a is the inverse matrix of a . On the other hand, we can identify the set of all $n \times n$ real matrices with \mathbb{R}^{n^2} . The determinant ($\det a$) of a matrix a is a continuous function of $a \in \mathbb{R}^{n^2}$. We have

$GL(n, R) = \{ a \in R^{n^2} \mid \det a \neq 0 \}$. Hence $GL(n, R)$ is an open set of R^{n^2} , and hence can be considered to be a topological space. The group $GL(n, R)$ is a topological group with respect to this topology. We call $GL(n, R)$ the real general linear group of degree n .

3) Let G be an arbitrary group. If we define any subset of G to be an open set, then G becomes a topological space (This topology is called the discrete topology of G .) G is a topological group with respect to the discrete topology.[8]

3. Homogeneous Spaces of Topological Groups, Locally Compact Groups:

Definition 1.7: Let G be a topological group and X a topological space. If G and X satisfy the following condition, then G is called a topological *transformation group* on X . A continuous map φ from $G \times X$ to X is defined, for which, writing $\varphi(g, x) = g \cdot x$, the following holds:

- 1) $(g \cdot h) \cdot x = g \cdot (h \cdot x)$ for $g, h \in G, x \in X$,
- 2) For the identity element e of G and all $x \in X$, $e \cdot x = x$. [10]

If this is the case, $x \rightarrow g \cdot x$ is a continuous map from X to X , and by (1), (2) we have :

$$g^{-1} \cdot (g \cdot x) = (g \cdot g^{-1}) \cdot x = e \cdot x = x.$$

From this, we see that $x \rightarrow g \cdot x$ is a homeomorphism of X .

If the only element g of G satisfying $g \cdot x = x$ for all $x \in X$ is the identity element e , then G is said to act *effectively* on X .

If for any two points x, y of X , there is an element g of G satisfying $g \cdot x = y$, then G is said to act *transitively* on X . If G acts transitively on X , then X is called a *homogeneous space* of the topological group G .

Definition 1.8: Let G be a Hausdorff topological group.

- 1) If G is locally compact as a topological space, then G is called a *locally compact group*.
- 2) If G is compact as a topological space, then G is called a *compact group*. [9]

4. Lie Groups and Lie Algebras:

Definition 1.9: Let G be a group which is at the same time a differentiable manifold. For $x, y \in G$ let xy denote their product and x^{-1} the inverse of x . Now, G is a *Lie group* provided that the mapping of $G \times G \rightarrow G$ defined by $(x, y) \rightarrow xy$ and the mapping of $G \rightarrow G$ defined by $x \rightarrow x^{-1}$ are both C^∞ mappings. [16]

Let L_g and R_g denote the left and right translation, respectively, by an element g of a Lie group G . Then L_g and R_g are diffeomorphisms of G . From the definition of L_g and R_g , we have :

$$L_g \cdot L_h = L_{gh}, \quad R_g \cdot R_h = R_{hg},$$

$$L_{g^{-1}} = L_g^{-1}, \quad R_{g^{-1}} = R_g^{-1}, \quad L_g \cdot R_h = R_h \cdot L_g. \quad (1.10)$$

For $g \in G$, set

$$A_g = L_g \cdot R_{g^{-1}}. \quad (1.11)$$

The transformation A_g is a diffeomorphism of G and, by definition, we have $A_g(x) = gxg^{-1}$. Hence for two elements x, y of G we have $A_g(xy) = A_g(x)A_g(y)$. We call A_g the *inner automorphism* of G by the element g of G .

Examples of Lie groups:

1) For two elements $x = (x^1, \dots, x^n), y = (y^1, \dots, y^n)$ of \mathbb{R}^n , define the sum

$x + y = (x^1 + y^1, \dots, x^n + y^n)$. Then \mathbb{R}^n is a commutative group. With this group structure, and with its differentiable structure as an affine space, \mathbb{R}^n becomes a Lie group. Since \mathbb{R}^n is commutative, $L_g = R_g$, and L_g is nothing but the parallel translation $x \rightarrow x + g$.

2) Let G, G' be two groups. If we define the product of two elements $(a, b), (a', b')$ of the direct product set $G \times G'$ by $(a, b)(a', b') = (aa', bb')$, then $G \times G'$ becomes a group. If G, G' are Lie groups, then $G \times G'$ is a manifold as a direct product manifold of G and G' , and $G \times G'$ is a Lie group with respect to the group structure mentioned above and this differentiable structure. This Lie group is called the direct product of the Lie group G and the Lie group G' . Similarly, we can define the direct product $G_1 \times \dots \times G_r$ of r Lie groups G_1, \dots, G_r . [11]

3) The direct product $T^r = T^l \times \dots \times T^l$ of r copies of the Lie group T^l is called the r -dimensional torus.[16]

4) The general linear group $GL(n, R)$ can be considered to be an open set of R^n , so $GL(n, R)$ has a structure of a manifold. If $a = (a^i_j)_{i,j=1,\dots,n}$ is a matrix in $GL(n, R)$, set

$$x^i_j(a) = a^i_j \quad (i, j = 1, \dots, n).$$

Then (x^i_j) is a coordinate system on $GL(n, R)$. The map $\varphi : (a, b) \rightarrow a \cdot b$ from $GL(n, R) \times GL(n, R)$ to $GL(n, R)$ and the map $\psi : a \rightarrow a^{-1}$ from $GL(n, R)$ to $GL(n, R)$ are both differentiable. In fact, if we set $x^i_j \circ \varphi = \varphi^i_j$, then $\varphi^i_j(a, b)$ is the (i, j) -th entry of matrix $a \cdot b$. Hence we have

$$\varphi^i_j(a, b) = \sum_{k=1}^n x_k^i(a) \cdot x_j^k(b),$$

which shows that $\varphi^i_j(a, b)$ is a polynomial in the coordinates of a and b . Hence φ^i_j is of class C^∞ , and the map φ is of class C^∞ . Similarly, if we set $x^i_j \circ \psi = \psi_j^i$, then $\psi_j^i(a)$ is the (i, j) th entry of the inverse matrix a^{-1} of a . By Cramer's rule, we find that (i, j) th entry of a^{-1} to be of the form :

$$\psi_j^i(a) = f^i_j(a) / \det a,$$

where the denominator and the numerator are both polynomials in the coordinates $x^i_j(a)$ of a , and the denominator is not zero at any point of $GL(n, R)$. Hence ψ_j^i is of class C^∞ ,

and hence ψ is also of class C^∞ . Thus we have shown that $GL(n, R)$ is an n^2 -dimensional Lie group.[2]

Now let us consider the complex general linear group $GL(n, C)$. As in the case of $GL(n, R)$, we can consider $GL(n, C)$ as an open set of C^{n^2} . Hence, as an open submanifold of C^{n^2} , $GL(n, C)$ is a complex n^2 -dimensional complex manifold. We can show, as in the case of $GL(n, R)$, that the maps $(a, b) \rightarrow a \cdot b$ and $a \rightarrow a^{-1}$ are holomorphic maps. Hence, $GL(n, C)$ is a $2n^2$ -dimensional Lie group.

If X is a vector field on a Lie group G satisfying

$$(L_g)^*X = X$$

for all $g \in G$, then X is called a left invariant vector field. If, instead, X satisfies

$$(R_g)^*X = X$$

for all $g \in G$, then X is called a right invariant vector field.

Definition 1.12: Let \mathfrak{g} be an algebra over G . For any two elements X, Y of \mathfrak{g} , we denote the product of X and Y by $[X, Y]$. If, for any $X, Y, Z \in \mathfrak{g}$, the following conditions (1) and (2) are satisfied, then we call \mathfrak{g} a *Lie algebra* over G :

$$(1) [X, Y] = -[Y, X],$$

$$(2) [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.[4]$$

If a nonsingular linear transformation θ of \mathfrak{g} satisfies $\theta[X, Y] = [\theta(X), \theta(Y)]$ for arbitrary $X, Y \in \mathfrak{g}$, then θ is called an *automorphism* of the Lie algebra \mathfrak{g} .

Definition 1.13: The Lie algebra \mathfrak{g} formed by the set of all invariant vector fields on G is called the Lie algebra of the Lie group G .[4]

Theorem 1.14: *If a Lie group G has dimension n , then the Lie algebra \mathfrak{g} of G also has dimension n .*

Proof:

To $X \in \mathfrak{g}$, we assign X_e , the value of X at the identity element e , and obtain a linear map from \mathfrak{g} to the tangent space $T_e(G)$ of G at e . Denote this linear map by α . If we can show that α is 1:1 map from \mathfrak{g} onto $T_e(G)$, then, since the tangent space $T_e(G)$ has dimension n , \mathfrak{g} will also have dimension n . First, let us show that α is 1:1. Let $X, Y \in \mathfrak{g}$, and

$$X_g = ((L_g)^*X)_g = (L_g)^*X_e, \quad \text{and similarly}$$

$$Y_g = (L_g)^*Y_e.$$

Since, $X_e = Y_e$ by assumption, we get $X_g = Y_g$ for all $g \in G$. That is, we get $X = Y$. Hence α is 1:1. Next we shall see that α is a map from \mathfrak{g} onto $T_e(G)$. For this it suffices to show that for any given $\nu \in T_e(G)$, there is an $X \in \mathfrak{g}$ such that $X_e = \nu$. If we define a vector field $X: \mathfrak{g} \rightarrow X_g$ by

$$X_g = (L_g)^* \nu,$$

then clearly $X_e = \nu$. We must show that this vector field X is of class C^∞ and left invariant.

Let W be an open set of G , (y^1, \dots, y^n) a local coordinate system of G on W , and $g_0 \in W$.

The map $(g, h) \rightarrow \varphi(g, h) = g \cdot h$ from $G \times G$ to G is differentiable and, since we can

write $g_0 = g_0 \cdot e$, there is a sufficiently small neighborhood W_0 of g_0 ($W_0 \subset W$) and a

sufficiently small neighborhood U of e such that $W_0 \cdot U \subset W$. Take a local coordinate

system (x^1, \dots, x^n) on U , where $x^i(e) = 0$ ($i = 1, \dots, n$). We have

$$y^i(\varphi(g, s)) = \varphi^i(y^1(g), \dots, y^n(g); x^1(s), \dots, x^n(s)) \quad (g \in W_0, s \in U).$$

Here $\varphi^i(u^1, \dots, u^n, v^1, \dots, v^n)$ is a C^∞ function of $2n$ variables. Now let

$$v = \sum_{i=1}^n a^i (\partial/\partial x^i)_e \quad \text{and} \quad X = \sum_{i=1}^n \xi^i \partial/\partial y^i \quad \text{on } W. \quad \text{Since } \varphi(g, s) = L_g(s), \text{ we have}$$

$$\xi^i(g) = \sum \frac{\partial \varphi^i}{\partial v^j} (y^1(g), \dots, y^n(g), 0, \dots, 0) a^j.$$

Hence ξ^i is of class C^∞ , and thus X is of class C^∞ . On the other hand, for arbitrary $s, g \in G$, we have

$$((L_s)^* X)_g = (L_s)^* X_{s^{-1}g} = (L_s)^* (L_{s^{-1}g})^* v.$$

But $L_s L_{s^{-1}g} = L_g$, so that $(L_s)^* (L_{s^{-1}g})^* v = (L_g)^* v$. Hence, from the equation above and the definition of X_g , we have $((L_s)^* X)_g = X_g$. Hence $(L_s)^* X = X$ holds for all $s \in G$, and X is left invariant. This proves the theorem.[8]

Definition 1.15: Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras over a field K . If a linear map α from \mathfrak{g}_1 into \mathfrak{g}_2 satisfies

$$\alpha([X, Y]) = [\alpha(X), \alpha(Y)],$$

for arbitrary $X, Y \in \mathfrak{g}_1$, then α is called a *homomorphism* from \mathfrak{g}_1 to \mathfrak{g}_2 . If $\alpha(\mathfrak{g}_1) = \mathfrak{g}_2$, then α is called a homomorphism from \mathfrak{g}_1 *onto* \mathfrak{g}_2 . If the homomorphism α is 1:1 map, then α is called an *isomorphism* from \mathfrak{g}_1 into \mathfrak{g}_2 ; if furthermore $\alpha(\mathfrak{g}_1) = \mathfrak{g}_2$, then α is

called an isomorphism from \mathfrak{g}_1 onto \mathfrak{g}_2 . If there is an isomorphism from \mathfrak{g}_1 onto \mathfrak{g}_2 , then \mathfrak{g}_1 and \mathfrak{g}_2 are said to be isomorphic, and we write $\mathfrak{g}_1 \cong \mathfrak{g}_2$. An isomorphism from a Lie algebra \mathfrak{g} onto itself is called an automorphism of \mathfrak{g} . [4]

5. One-Parameter Subgroups and Exponential Map:

Let G be a Lie group, and $a : t \rightarrow a(t)$ a differentiable curve of G defined on $(-\infty, +\infty)$.

If, for any $s, t \in \mathbb{R}$, we have

$$a(s) \cdot a(t) = a(s+t), \quad (1.16)$$

then $\{a(t) \mid t \in \mathbb{R}\}$ is called a *one-parameter subgroup* of G . By (1.16), we have

$$a(0)a(t) = a(t),$$

so that multiplying by the inverse element of $a(t)$ on the right, we have

$$a(0) = e.$$

Also, since

$$a(t)a(-t) = a(-t)a(t) = a(0) = e, \text{ we have } a(t)^{-1} = a(-t).$$

Furthermore, since $a(s)a(t) = a(s+t) = a(t)a(s)$, $a(s)$ and $a(t)$ commute. Hence a one-parameter subgroup of G is a commutative subgroup of G .

Now, for a one-parameter group of transformations $\{a(t) \mid t \in \mathbb{R}\}$, we can define a vector field X on G by

$$X_g f = \left[\frac{df(a_t(g))}{dt} \right]_{t=0}.$$

This vector field X is called the *infinitesimal transformation* of the given one-parameter group of transformations

$\{L_{a(t)}; t \in \mathbb{R}\}$, $\{R_{a(t)}; t \in \mathbb{R}\}$ are both one-parameter groups of transformations of G , and the orbits of the identity element e by these transformation groups coincide with $a(t)$. For the infinitesimal transformations of these one-parameter groups of transformations, the following lemma holds.

Lemma 1.17: Let X be the infinitesimal transformation of $R_{a(t)}$, and let Y be the infinitesimal transformation of $L_{a(t)}$. Then X is left invariant and Y is right invariant, and $X_e = Y_e = a'(0)$ holds. Here $a'(t)$ denotes the tangent vector to the curve a at $a(t)$.

Proof: If f is a C^∞ function on a neighborhood of a point h of G , then

$$(L_g^* X)_h f = X_{g^{-1}h} (f \circ L_g).$$

On the other hand, by the definition of X and by the commutativity of L_g and $R_{a(t)}$, we have

$$\begin{aligned} X_{g^{-1}h} (f \circ L_g) &= \lim_{t \rightarrow 0} \frac{1}{t} [(f \circ L_g)(R_{a(t)} g^{-1}h) - (f \circ L_g)(g^{-1}h)] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} [f(R_{a(t)}h) - f(h)] = X_h f. \end{aligned}$$

Hence $(L_g^* X)_h = X_h$ holds at each point h of G , and X is left invariant. Similarly, we can show that Y is right invariant. Since $R_{a(t)}(e) = a(t)$, we have that $a(t)$ is an integral curve of X , and hence $X_{a(t)} = a'(t)$ holds. In particular, we have $X_e = a'(0)$. Similarly, we have $Y_e = a'(0)$. [8]

Lemma 1.18: Let $\{\varphi_t ; t \in \mathbf{R}\}$ be a one-parameter group of transformations of G , and set $\varphi_t(e) = a(t)$. If $\varphi_t \cdot L_g = L_g \cdot \varphi_t$ holds for all $g \in G$ and for all $t \in \mathbf{R}$, then $a(t)$ is a one-parameter subgroup of G , and $\varphi_t = R_{a(t)}$ holds for all $t \in \mathbf{R}$. If $\varphi_t \cdot R_g = R_g \cdot \varphi_t$ holds for all $g \in G$ and $t \in \mathbf{R}$, then $a(t)$ is a one-parameter subgroup of G , and $\varphi_t = L_{a(t)}$ holds for all $t \in \mathbf{R}$.

Proof: The map $t \rightarrow a(t)$ is differentiable and, moreover,

$$a(s+t) = \varphi_{t+s}(e) = \varphi_t(\varphi_s(e)) = \varphi_t(L_{a(s)}(\varphi_t(e))) = a(s)a(t).$$

Hence $a(t)$ is a one-parameter subgroup of G . On the other hand, for any $g \in G$, we have $\varphi_t(g) = \varphi_t(L_g(e)) = L_g(\varphi_t(e)) = g \cdot a(t) = R_{a(t)}(g)$. Hence $\varphi_t = R_{a(t)}$. We can argue similarly for the case $\varphi_t \cdot R_g = R_g \cdot \varphi_t$.

If $a(t)$ is one-parameter subgroup of G , then there is an $X \in \mathfrak{g}$ such that $a(t) = (\text{Exp } tX)(e)$; for an arbitrary $X \in \mathfrak{g}$, $(\text{Exp } tX)(e) = a(t)$ is a one-parameter subgroup of G , and $\text{Exp } tX = R_{a(t)}$. [8]

Proposition 1.19: Let $a:(-\varepsilon, \varepsilon) \rightarrow G$ be a one-parameter subgroup in G . Then for every $t \in (-\varepsilon, \varepsilon)$, a is differentiable at t and

$$a'(t) = a'(0)a(t) = a(t)a'(0).$$

Proof: For small $h \in \mathbf{R}$,

$$a(h)a(t) = a(h+t) = a(t+h) = a(t)a(h).$$

Hence

$$\begin{aligned}
a'(t) &= \lim_{h \rightarrow 0} \frac{1}{h} (a(t+h) - a(t)) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} (a(h) - I)a(t) \\
&= a'(0)a(t),
\end{aligned}$$

and similarly $a'(t) = a(t)a'(0)$. [2]

Theorem 1.20: Let $a: \mathbf{R} \rightarrow G$ be a one-parameter group in G . Then it has the form

$$a(t) = \exp(tX)$$

for some $X \in M_n(K)$, where $K = \mathbf{R}$ or \mathbf{C} .

Proof:

Let $X = a'(0)$. By the above proposition, a satisfies the differential equation

$$a'(t) = Xa(t), \quad a(0) = I.$$

This equation has the unique solution $a(t) = \exp(tX)$. [2]

Definition 1.21: For $X \in \mathfrak{g}$, set $\exp tX = (\text{Exp } tX)(e)$. The map $X \rightarrow \exp X$ is a map from \mathfrak{g} to G , and is called the *exponential map*. [14]

By definition, $\exp tX$ is a one-parameter subgroup of G , and we have

$$\text{Exp}(t+s)X = \exp(tX)\exp(sX)$$

$$R_{\exp tX} = \text{Exp } tX.$$

So for $X, Y \in \mathfrak{g}$, we have

$$[X, Y]_{\mathfrak{g}} = \lim_{t \rightarrow 0} \frac{1}{t} \{Y_{\mathfrak{g}} - ((R_{\exp tX})_* Y)_{\mathfrak{g}}\}. \quad (1.22)$$

Since, for an arbitrary element g of G , we have $A_g = R_{g^{-1}} \cdot L_g$, it follows that for $Y \in \mathfrak{g}$, we have $A_g * Y = R_{g^{-1}} * Y$.

However, since L_h and $R_{g^{-1}}$ commute, we have

$$L_h * (R_{g^{-1}} * Y) = R_{g^{-1}} * (L_h * Y) = R_{g^{-1}} * Y,$$

so that $R_{g^{-1}} * Y \in \mathfrak{g}$. Hence, for $g \in G$, we have $A_g * Y \in \mathfrak{g}$. The map $Y \rightarrow A_g * Y$ is a linear transformation of the vector space \mathfrak{g} , and we denote this linear transformation by $\text{Ad}(g)$. That is,

$$\text{Ad}(g)Y = A_g * Y = R_{g^{-1}} * Y \quad (g \in G, Y \in \mathfrak{g}).$$

Furthermore, since $A_{gh} = A_g A_h$, we have

$$\text{Ad}(gh) = \text{Ad}(g)\text{Ad}(h)$$

for any two elements g, h of G . In particular, it is clear from the definition that $\text{Ad}(e)$ is the identity transformation 1 of the vector space \mathfrak{g} . Hence we have $\text{Ad}(g^{-1}) \cdot \text{Ad}(g) = 1$.

Hence $\text{Ad}(g)$ is a nonsingular linear transformation of \mathfrak{g} , and

$$\text{Ad}(g)^{-1} = \text{Ad}(g^{-1})$$

holds. The map $g \rightarrow \text{Ad}(g)$ is called the *adjoint representation* of the Lie group G on \mathfrak{g} . When we want to emphasize the space on which $\text{Ad}(x)$ operates, we write $\text{Ad}_{\mathfrak{g}}(x)$ for the linear transformation.

Since $A_g * [X, Y] = [A_g * X, A_g * Y]$, we have

$$\text{Ad}(\mathfrak{g})[X, Y] = [\text{Ad}(\mathfrak{g})X, \text{Ad}(\mathfrak{g})Y] \quad (X, Y \in \mathfrak{g}).$$

That is, $\text{Ad}(\mathfrak{g})$ is an automorphism of the Lie algebra \mathfrak{g} . If we let

$$A_x(t) = \text{Ad}(\exp tX) \quad (X \in \mathfrak{g}),$$

then we have $A_x(t + s) = A_x(t)A_x(s)$. That is, $A_x(t)$ is a one-parameter group of linear transformations of the vector space \mathfrak{g} . If we set

$$C_x = \left[\frac{d}{dt} A_x(t) \right]_{t=0}$$

then we have

$$A_x(t) = \exp tC_x.$$

In fact from $A_x(t + s) = A_x(t)A_x(s)$, we obtain

$$\frac{d}{dt} A_x(t) = C_x A_x(t),$$

$$A_x(0) = 1,$$

which shows that $A_x(t)$ is a solution to a system of differential equations and satisfies a given initial condition. However, clearly $\exp tC_x$ satisfies the same system of differential equations and the same initial condition, hence by uniqueness of solutions, we conclude that $A_x(t) = \exp tC_x$.

On the other hand, from (1.22), in \mathfrak{g} we have

$$\begin{aligned} [X, Y] &= \lim_{t \rightarrow 0} \frac{1}{t} \{Y - \text{Ad}(\exp(-tX)) \cdot Y\} \\ &= - \left[\frac{d}{dt} A_x(-t) \right]_{t=0} \cdot Y \\ &= C_x \cdot Y. \end{aligned}$$

Hence C_x is equal to the linear transformation $\text{ad}(X)$ of \mathfrak{g} defined by $Y \rightarrow [X, Y]$. That is, if we set

$$\text{Ad}(X)Y = [X, Y] \quad (X, Y \in \mathfrak{g}),$$

then $A_x(t) = \exp t \text{ad}(X)$. Hence

$$\text{Ad}(\exp tX) = \exp t \text{ad}(X) \quad (1.23)$$

holds.

The map $X \rightarrow \text{ad}(X)$ is called the *adjoint representation* of the Lie algebra \mathfrak{g} . From the Jacobi identity for Lie algebra, we have

$$\text{Ad}(X)[Y, Z] = [\text{ad}(X)Y, Z] + [Y, \text{ad}(X)Z].$$

That is, $\text{ad}(X)$ is a derivation of the Lie algebra \mathfrak{g} .

Let us now investigate the Lie algebra of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$.

Let $A(t)$ be a one-parameter subgroup of $GL(n, \mathbb{R})$. We can write $A(t) = \exp tC$, where C is determined uniquely by

$$C = \left[\frac{dA(t)}{dt} \right]_{t=0}.$$

Conversely, if C is an arbitrary $n \times n$ real matrix, then the exponential function $\exp tC$ is a one-parameter subgroup of $GL(n, \mathbb{R})$.

Now let \mathfrak{g} be the Lie algebra of $GL(n, \mathbb{R})$. For $X \in \mathfrak{g}$, consider the one-parameter subgroup $\exp tX$ of $GL(n, \mathbb{R})$. Then there is an $n \times n$ matrix $C(X)$ such that

$$\text{Exp } tX = \exp tC(X).$$

Applying

$$X_a f = \lim_{t \rightarrow 0} \frac{1}{t} [f(\text{Exp } tX(a)) - f(a)]$$

to $f = x_j^i$, and using $\text{Exp } tX = R_{\text{exp } tX} = R_{\text{exp } tC(X)}$, we get for the matrix $(X_a x_j^i)$:

$$\begin{aligned} (X_a x_j^i) &= \lim_{t \rightarrow 0} \frac{1}{t} [a \exp tC(X) - a] \\ &= a \cdot C(X). \end{aligned}$$

That is, if we set $C(X) = (c_j^i(X))$, $a = (a_j^i)$,

then we have

$$X_a x_j^i = \sum_{k=1}^n a_k^i c_j^k(X).$$

Hence the vector field X is expressed as

$$X = \sum_{i,j=1}^n \left(\sum_{k=1}^n (x_k^i c_j^k(X)) \right) \frac{\partial}{\partial x_j^i} \quad (1.24)$$

with respect to the coordinate system (x_j^i) . Again, if we compute $[X, Y] x_j^i$, we see that it

is equal to

$$\sum_{k=1}^n x_k^i \left(\sum_{t=1}^n (c_t^k(X) c_j^t(Y) - c_t^k(Y) c_j^t(X)) \right).$$

Hence we have

$$[X, Y] = \sum_{i,j=1}^n \left(\sum_{k=1}^n x_j^k \left(\sum_{t=1}^n (c_t^k(X)c_j^t(Y) - c_t^k(Y)c_j^t(X)) \right) \right) \frac{\partial}{\partial x_j^i},$$

and we obtain

$$c_j^i([X, Y]) = \sum_{i=1}^n (c_i^i(X)c_j^i(Y) - c_i^i(Y)c_j^i(X)). \quad (1.25)$$

Now, if we define the commutator product $[A, B]$ in the associative algebra of all $n \times n$ real matrices to be

$$[A, B] = AB - BA,$$

then we obtain a Lie algebra, which will be denoted by $\mathfrak{gl}(n, \mathbb{R})$. The formula (1.25) then becomes: $C([X, Y]) = [C(X), C(Y)]$.

From (1.24), it is clear that the correspondence $X \rightarrow C(X)$ is 1:1 and onto, and that $C(\lambda X) = \lambda C(X)$ for $\lambda \in \mathbb{R}$, and $C(X + Y) = C(X) + C(Y)$. Hence, the map $X \rightarrow C(X)$ is an isomorphism from the Lie algebra \mathfrak{g} of $GL(n, \mathbb{R})$ onto the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$. We shall identify \mathfrak{g} and $\mathfrak{gl}(n, \mathbb{R})$ by this isomorphism from now on. Then, the exponential map

$$\text{Exp} : \mathfrak{gl}(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$$

is nothing but the exponential function, which assigns to each matrix X , belonging to $\mathfrak{gl}(n, \mathbb{R})$, the value $\exp X$. [8]

Lemma 1.26 : If X is an infinitesimal transformation of G , then so is θ^*X where θ is a transformation on G , and we have

$$\text{Exp } t \theta^*X = \theta(\text{Exp } tX) \theta^{-1}. [8]$$

If we let $\theta = A_a$, then $\text{Exp } t(\text{Ad}(a)X) = A_a \cdot (\text{Exp } tX) \cdot A_a^{-1}$.

So, we have

$$\exp t(\text{Ad}(a)X) = a(\exp tX)a^{-1} \quad (X \in \mathfrak{gl}(n, \mathbb{R}), a \in GL(n, \mathbb{R})).$$

Differentiating both sides with respect to t and setting $t = 0$, we obtain

$$\text{Ad}(a)X = a \cdot X \cdot a^{-1}. \quad (1.27)$$

That is, if we consider $\mathfrak{gl}(n, \mathbb{R})$ to be the Lie algebra of $GL(n, \mathbb{R})$, then the adjoint representation of $GL(n, \mathbb{R})$ is given by (1.27).

Similarly, the set $\mathfrak{gl}(n, \mathbb{C})$ of all $n \times n$ complex matrices is a Lie algebra with respect to the commutator product $[A, B] = AB - BA$.

6. Complex Lie Groups and Complex Lie Algebra:

In the theory of Lie groups, as in the theory of differentiable manifolds, we can also consider complex Lie groups in a natural way.

Definition 1.28: If a group G has the structure of a complex manifold with a countable base, and if the map $(x, y) \rightarrow x \cdot y$ from $G \times G$ to G and the map $x \rightarrow x^{-1}$ from G to G are both holomorphic (analytic), then we call G a *complex Lie group*. The complex dimension of the complex manifold G is called the complex dimension of the complex Lie group G .

Complex manifolds of complex dimension n are differentiable manifolds of (real) dimension $2n$, and holomorphic maps are differentiable. Hence complex Lie groups are certainly Lie groups in sense of definition in *sec1*. Sometimes we shall call a Lie group, in the sense of the definition in *sec1*, a real Lie group, as opposed to a complex Lie group. A complex Lie group of complex dimension n is a real Lie group of dimension $2n$.

Definition 1.29: A Lie algebra over the complex number field \mathbf{C} is called a *complex Lie algebra*.

We sometimes call a Lie algebra over the real number field \mathbf{R} a real Lie algebra.

If \mathfrak{g} is an n -dimensional complex Lie algebra, then \mathfrak{g} is also a Lie algebra over the real numbers, and the dimension of \mathfrak{g} over \mathbf{R} is $2n$. That is, an n -dimensional complex Lie algebra can be considered as a $2n$ -dimensional real Lie algebra. Considering \mathfrak{g} as a $2n$ -dimensional real vector space, denote the linear transformation of \mathfrak{g} , given by $X \rightarrow iX$ (i is the imaginary unit), by I . Since \mathfrak{g} is a Lie algebra over \mathbf{C} , we have

$$[X, IY] = I[X, Y] \quad (1.30)$$

for arbitrary $X, Y \in \mathfrak{g}$. That is,

$$ad(X) \cdot I = I \cdot ad(X) \quad (1.30')$$

holds for all $X \in \mathfrak{g}$.

Conversely, let \mathfrak{g} be a $2n$ -dimensional Lie algebra over \mathbf{R} . Let I be a complex structure of the real vector space \mathfrak{g} . For any $X \in \mathfrak{g}$, let $I \cdot X = iX$, and consider \mathfrak{g} as an n -dimensional vector space over \mathbf{C} . Suppose I satisfies (1.30') for all elements X of \mathfrak{g} . Then, for an arbitrary complex number $\lambda = a + ib$ ($a, b \in \mathbf{R}$), we have

$$\begin{aligned}
[X, \lambda Y] &= [X, aY + bI \cdot Y] \\
&= a[X, Y] + b[X, I \cdot Y] \\
&= a[X, Y] + bI[X, Y] = \lambda[X, Y].
\end{aligned}$$

We also have $[\lambda X, Y] = -[Y, \lambda X] = \lambda[Y, X]$. Hence \mathfrak{g} is a Lie algebra over \mathbf{C} , i.e., \mathfrak{g} is a complex Lie algebra. Thus we have shown that a Lie algebra over \mathbf{R} is a complex Lie algebra if and only if it admits a complex structure satisfying (1.30').

Examples:

- 1) $GL(n, \mathbf{C})$ is a complex Lie group.
- 2) \mathbf{C}^n is a complex Lie group with respect to addition .
- 3) Direct products of complex Lie groups are complex Lie groups.

7. Lie Subgroup of a Lie Group:

Definition 1.31: Let G be a Lie group. A Lie group H is called a Lie subgroup of G if it has the following two properties:

- (1) H is a submanifold of the manifold G .
- (2) H is a subgroup of the group G . [8]

In particular, if H is connected as a manifold, then H is called a connected Lie subgroup. If H is a closed submanifold of G , then H is called a closed Lie subgroup of G .

Definition 1.32: Let \mathfrak{g} be a Lie algebra over a field \mathbf{K} . A subset \mathfrak{h} of \mathfrak{g} is called a Lie subalgebra of \mathfrak{g} if it has the following two properties:

- (1) The set \mathfrak{h} is a subspace of \mathfrak{g} , i.e., if $X, Y \in \mathfrak{h}$, and $\lambda, \mu \in \mathbf{K}$, then $\lambda X + \mu Y \in \mathfrak{h}$.

(2) If $X, Y \in \mathfrak{h}$, then $[X, Y] \in \mathfrak{h}$. [16]

It is clear that a Lie subalgebra of a Lie algebra is itself a Lie algebra. We shall show that the Lie algebra \mathfrak{h} of a Lie subgroup H of a Lie group G can be considered as a Lie subalgebra of the Lie algebra \mathfrak{g} of G , and conversely, that to each Lie subalgebra \mathfrak{h} of \mathfrak{g} , there corresponds a unique connected Lie subgroup of G .

Let G be a Lie group, H a Lie subgroup of G , and i the injection map from H to G . By condition (1) for a Lie subgroup, i is differentiable, and, at each point h of H , the differential i^* of i is an isomorphism from $T_h(H)$ into $T_{i(h)}(G)$. From condition (2), we have $i(xy) = i(x)i(y)$ ($x, y \in H$). Hence, for $h \in H$, we have

$$i \cdot L_h = L_{i(h)} \cdot i. \quad (1.33)$$

For an element X of the Lie algebra \mathfrak{h} of H , there is a unique element X' of \mathfrak{g} such that $(i^*)_e X_e = X'_e$. Moreover, X and X' are i -related, and we have $i^*X = X'$. In fact, by (1.33), we have

$$i^*X_h = i^*(L_h)^* X_e = (L_{i(h)})^* X'_e = X'_{i(h)}.$$

Conversely, if $X' \in \mathfrak{g}$, and $X'_e \in i^*T_e(H)$, then there is a unique $X \in \mathfrak{h}$ such that $i^*X = X'$. Now for $X, Y \in \mathfrak{h}$, we have

$$i^*[X, Y] = [i^*X, i^*Y].$$

Hence, if \mathfrak{h}' is the set of all X' of \mathfrak{g} such that $X'_e \in i^*T_e(H)$, then \mathfrak{h}' is a Lie subalgebra of \mathfrak{g} , and \mathfrak{h} and \mathfrak{h}' are isomorphic by the map $X \rightarrow i^*X$. We identify the Lie algebra \mathfrak{h} of

H with a Lie subalgebra of the Lie algebra \mathfrak{g} of G by this isomorphism. We also identify the tangent vector space $T_h(H)$ of H with the subspace $i^*T_h(H)$ of the tangent vector space $T_h(G)$. Then the Lie algebra of H is the set of elements X of \mathfrak{g} satisfying $X_e \in T_e(H)$.

Conversely, let \mathfrak{h} be an arbitrary Lie subalgebra of \mathfrak{g} . For an arbitrary element g of G, set $\mathfrak{h}_g = \{X_g \mid X \in \mathfrak{h}\}$. Then $D: \mathfrak{g} \rightarrow \mathfrak{h}_g$ is a completely integrable differential system on G. In fact, if $\{Y_1, \dots, Y_r\}$ is a basis of \mathfrak{h} , then $\{Y_1, \dots, Y_r\}$ is a local basis for this differential system on G. Moreover, since \mathfrak{h} is a Lie algebra, we have

$$[Y_i, Y_j] = \sum_{k=1}^r c_{ij}^k Y_k, \quad c_{ij}^k \in R \quad (i, j = 1, \dots, r)$$

Hence, the differential system D is completely integrable. Let H be the maximal connected integral manifold of D passing through e . If $a \in G$, since $(L_a)_* X_g = X_{ag}$ ($X \in \mathfrak{h}$), we have $(L_a)_* \mathfrak{h}_g = \mathfrak{h}_{ag}$. That is, L_a leaves the differential system D invariant. Hence L_a transforms a maximal connected integral manifold to a maximal connected integral manifold. Hence, if $h \in H$, then $L_{h^{-1}}H$ is also a maximal connected integral manifold passing through e . That is, we have $L_{h^{-1}}H = H$ ($h \in H$). From this, we see that H is a subgroup of G. Now let us show that the map from $H \times H$ to H, given by $(a, b) \rightarrow a \cdot b$, and the map from H to H, given by $a \rightarrow a^{-1}$, are both differentiable. Since H is a submanifold of G, and G is a Lie group, it is clear that $\varphi: (a, b) \rightarrow a \cdot b$ is differentiable as a map from $H \times H$ to G. But $\varphi(H \times H) \subseteq H$, H is a maximal connected integral manifold of D, and G has a countable base, so φ is differentiable also as a map from $H \times H$ to H. Similarly, the map $a \rightarrow a^{-1}$ from H to H is also differentiable. Thus we have proved that H is a connected Lie

subgroup of G , and, from the construction of H , it is clear that the Lie algebra of H coincides with the given Lie subalgebra \mathfrak{h} of \mathfrak{g} . [8]

Theorem 1.34: *Let G be a Lie group, \mathfrak{g} the Lie algebra of G . If H is a Lie subgroup of G , then the Lie algebra \mathfrak{h} of H can be regarded as a Lie subalgebra of \mathfrak{g} . Conversely, if \mathfrak{h} is a Lie subalgebra of \mathfrak{g} , then there is a unique connected Lie subgroup of G whose Lie algebra is \mathfrak{h} . [8]*

Now let us consider the case when G is a complex Lie group.

Definition 1.35: Let G be a complex Lie group. A complex Lie group H is called a complex Lie subgroup of G if the following two conditions are satisfied:

- (1) The complex manifold H is a complex submanifold of G .
- (2) The group H is a subgroup of G . [8]

A complex Lie subgroup of G is of course a Lie subgroup of G . We also have the following theorem.

Theorem 1.36: *Let \mathfrak{g} be the Lie algebra of a complex Lie group G . If H is a Lie subgroup of G , then H is a complex Lie subgroup of G if and only if the Lie algebra \mathfrak{h} of H is a complex Lie subalgebra of \mathfrak{g} .*

8. Ideals:

Definition 1.37: Let \mathfrak{h} be a Lie subalgebra of the Lie algebra \mathfrak{g} . If, for all $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$, we have $[X, Y] \in \mathfrak{h}$, then \mathfrak{h} is called an ideal of \mathfrak{g} .

Proposition 1.38: If \mathfrak{a} and \mathfrak{b} are ideals in a Lie algebra, then so are $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $[\mathfrak{a}, \mathfrak{b}]$.

Proof : The conclusions for $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are obvious. In the case of $[\mathfrak{a}, \mathfrak{b}]$, we have

$$\begin{aligned} [\mathfrak{g}, [\mathfrak{a}, \mathfrak{b}]] &\subseteq [[\mathfrak{g}, \mathfrak{a}], \mathfrak{b}] + [\mathfrak{a}, [\mathfrak{g}, \mathfrak{b}]] \\ &\subseteq [\mathfrak{a}, \mathfrak{b}] + [\mathfrak{a}, \mathfrak{b}] \\ &\subseteq [\mathfrak{a}, \mathfrak{b}]. \end{aligned} \quad [6]$$

We say that the Lie algebra \mathfrak{g} is *reductive* if to each ideal \mathfrak{a} in \mathfrak{g} corresponds an ideal \mathfrak{b} in \mathfrak{g} with $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$

Examples:

- 1) $Z_{\mathfrak{g}}$ (center of \mathfrak{g}) = $\{ X \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g} \}$. This is the centralizer of \mathfrak{g} in \mathfrak{g} .
- 2) $[\mathfrak{g}, \mathfrak{g}] = \text{commutator ideal}$. This is an ideal by the above proposition.
- 3) $\ker \pi$ whenever $\pi : \mathfrak{g} \rightarrow \hat{\mathfrak{h}}$ is a homomorphism of Lie algebras.

Let \mathfrak{g} be a Lie algebra. Each $\text{ad } X$ for $X \in \mathfrak{g}$ is a member of $\text{End}_k \mathfrak{g}$, and these members satisfy

$$\text{ad}[X, Y] = \text{ad } X \text{ ad } Y - \text{ad } Y \text{ ad } X \quad (1.39)$$

As a consequence of the Jacobi identity.

Let \mathfrak{a} be an ideal in the Lie algebra \mathfrak{g} . Then $\mathfrak{g}/\mathfrak{a}$ as a vector space becomes a Lie algebra under the definition $[X + \mathfrak{a}, Y + \mathfrak{a}] = [X, Y] + \mathfrak{a}$ and is called the *quotient algebra* of \mathfrak{g} and \mathfrak{a} . Checking that this bracket operation is independent of the choices uses that \mathfrak{a} is an ideal, and then the defining properties of the bracket operation of a Lie algebra follow from the corresponding properties in \mathfrak{g} . The quotient map $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{a}$ is a homomorphism of Lie algebras, by definition, and hence every ideal is the kernel of a homomorphism.

Let \mathfrak{g} denote a finite-dimensional Lie algebra. We define recursively

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j].$$

Then the decreasing sequence: $\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \mathfrak{g}^2 \supseteq \dots$

is called the *commutator series* for \mathfrak{g} . Each \mathfrak{g}^j is an ideal in \mathfrak{g} , by the above proposition and induction. We say that \mathfrak{g} is *solvable* if $\mathfrak{g}^j = 0$ for some j . A nonzero solvable \mathfrak{g} has a nonzero abelian ideal, namely the last nonzero \mathfrak{g}^j .

Next we define recursively

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j].$$

Then the decreasing sequence: $\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots$

is called the *lower central series* for \mathfrak{g} . Each \mathfrak{g}_j is an ideal in \mathfrak{g} , by the above proposition and induction. We say that \mathfrak{g} is *nilpotent* if $\mathfrak{g}_j = 0$ for some j . A nonzero nilpotent \mathfrak{g} has nonzero center, the last nonzero \mathfrak{g}_j being in the center. Inductively we see that $\mathfrak{g}^j \subseteq \mathfrak{g}_j$, and it follows that nilpotent implies solvable.

Below are the standard examples of solvable and nilpotent Lie algebras.

Examples:

1) The Lie algebra $\mathfrak{g} = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \\ 0 & & & 0 \end{pmatrix}$ is solvable.

2) The Lie algebra $\mathfrak{g} = \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \\ 0 & & & 0 \end{pmatrix}$ is nilpotent.

Proposition 1.40: Any subalgebra or quotient algebra of a solvable Lie algebra is solvable. Similarly any subalgebra or quotient algebra of a nilpotent Lie algebra is nilpotent.

Proof: If \mathfrak{h} is a subalgebra of \mathfrak{g} , then induction gives $\mathfrak{h}^k \subseteq \mathfrak{g}^k$. Hence \mathfrak{g} solvable implies \mathfrak{h} solvable. If $\pi: \mathfrak{g} \rightarrow \mathfrak{h}$ is a homomorphism of the Lie algebra \mathfrak{g} onto the Lie algebra \mathfrak{h} , then $\pi(\mathfrak{g}^k) = \mathfrak{h}^k$. Hence \mathfrak{g} solvable implies \mathfrak{h} solvable. The arguments in the nilpotent case are similar.[6]

Proposition 1.41: If $\hat{\mathfrak{a}}$ is a solvable ideal in \mathfrak{g} and if $\mathfrak{g} / \hat{\mathfrak{a}}$ is solvable, then \mathfrak{g} is solvable.

Proof: Let $\pi: \mathfrak{g} \rightarrow \mathfrak{g} / \hat{\mathfrak{a}}$ be the quotient homomorphism, and suppose that $(\mathfrak{g} / \hat{\mathfrak{a}})^k = 0$. Since $\pi(\mathfrak{g}) = \mathfrak{g} / \hat{\mathfrak{a}}$, $\pi(\mathfrak{g}^j) = (\mathfrak{g} / \hat{\mathfrak{a}})^j$ for all j . Thus $\pi(\mathfrak{g}^k) = 0$, and we conclude that $\mathfrak{g}^k \subseteq \hat{\mathfrak{a}}$. By assumption $\hat{\mathfrak{a}}^l = 0$, and \mathfrak{g} is solvable.[6]

Proposition 1.42: If \mathfrak{g} is finite-dimensional Lie algebra, then there exists a unique solvable ideal τ of \mathfrak{g} containing all solvable ideals in \mathfrak{g} .

Proof: By finite-dimensionality it suffices to show that the sum of two solvable ideals, which is an ideal by Proposition 1.33, is solvable. Thus let $\hat{\mathfrak{a}}$ and \mathfrak{B} be solvable ideals and let $\mathfrak{h} = \hat{\mathfrak{a}} + \mathfrak{B}$. Then $\hat{\mathfrak{a}}$ is a solvable ideal in \mathfrak{h} , and *second Isomorphism Theorem* gives

$$\mathfrak{h} / \hat{\mathfrak{a}} = (\hat{\mathfrak{a}} + \mathfrak{B}) / \hat{\mathfrak{a}} \cong \mathfrak{B} / (\hat{\mathfrak{a}} \cap \mathfrak{B}).$$

This is solvable by Proposition 1.40 since \mathfrak{B} is solvable. Hence \mathfrak{h} is solvable by Proposition 1.41.[6]

The ideal τ is called the *radical* of \mathfrak{g} and is denoted $\text{rad } \mathfrak{g}$.

A finite-dimensional Lie algebra \mathfrak{g} is *simple* if \mathfrak{g} is nonabelian and \mathfrak{g} has no proper nonzero ideal. A finite-dimensional Lie algebra \mathfrak{g} is *semisimple* if \mathfrak{g} has no nonzero solvable ideals, i.e., if $\text{rad } \mathfrak{g} = 0$.

Proposition 1.43: In a simple Lie algebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Every simple Lie algebra is semisimple. Every semisimple Lie algebra has 0 center.

Proof: Let \mathfrak{g} be simple. The commutator $[\mathfrak{g}, \mathfrak{g}]$ is an ideal and hence is 0 or \mathfrak{g} . It cannot be 0 since \mathfrak{g} is nonabelian. So it is \mathfrak{g} . This proves the first statement. For the second statement, $\text{rad } \mathfrak{g}$ is an ideal and so is 0 or \mathfrak{g} . If $\text{rad } \mathfrak{g} = \mathfrak{g}$, then \mathfrak{g} is solvable and $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, contradiction. So $\text{rad } \mathfrak{g} = 0$, and \mathfrak{g} is semisimple. For the third statement, $Z_{\mathfrak{g}}$ is an abelian ideal and must be 0 , by definition of semisimplicity.[6]

Proposition 1.44: If \mathfrak{g} is a finite-dimensional Lie algebra, then $\mathfrak{g} / \text{rad } \mathfrak{g}$ is semisimple.

Proof: Let $\pi : \mathfrak{g} \rightarrow \mathfrak{g} / \text{rad } \mathfrak{g}$ be the quotient homomorphism, and let \mathfrak{h} be a solvable ideal in $\mathfrak{g} / \text{rad } \mathfrak{g}$. Form the ideal $\check{\mathfrak{a}} = \pi^{-1}(\mathfrak{h}) \subseteq \mathfrak{g}$. Then $\pi(\check{\mathfrak{a}}) = \mathfrak{h}$ is solvable, and $\ker \pi|_{\check{\mathfrak{a}}}$ is solvable, being in $\text{rad } \mathfrak{g}$. So $\check{\mathfrak{a}}$ is solvable by Proposition 2.38. Hence $\check{\mathfrak{a}} \subseteq \text{rad } \mathfrak{g}$ and $\mathfrak{h} = 0$. Therefore $\mathfrak{g} / \text{rad } \mathfrak{g}$ is semisimple.[6]

Example: Any 3-dimensional Lie algebra \mathfrak{g} is either solvable or simple. If \mathfrak{g} is not simple, then it has a nontrivial ideal \mathfrak{h} . This \mathfrak{h} is solvable, and so is $\mathfrak{g} / \mathfrak{h}$. Hence \mathfrak{g} is solvable by Proposition 1.41.

To decide whether such a \mathfrak{g} is solvable or simple, we have only to compute $[\mathfrak{g}, \mathfrak{g}]$, if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then \mathfrak{g} is simple (because the commutator series cannot end in 0), while if $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$, then \mathfrak{g} is solvable (because $[\mathfrak{g}, \mathfrak{g}]$ has dimension at most 2 and is therefore solvable).

9. Killing Form:

Let \mathbf{K} be a field, and let \mathbf{K} be an extension field. If $\mathbf{K} = \mathbf{R}$ and $\mathbf{K} = \mathbf{C}$ and V is a real vector space, the complex vector space $V^{\mathbf{C}}$ is called the *complexification* of V . If W is complex vector space, then $W^{\mathbf{R}}$ is W regarded as a real vector space. The operation $(\cdot)^{\mathbf{C}}$

and $(\cdot)^{\mathbf{R}}$ are not inverse to each other : $(V^{\mathbf{C}})^{\mathbf{R}}$ has twice the real dimension of V , and $(W^{\mathbf{R}})^{\mathbf{C}}$ has twice the complex dimension of W . More precisely :

$$(V^{\mathbf{C}})^{\mathbf{R}} = V \oplus iV \quad (1.45a)$$

as real vector spaces, where V means $V \otimes 1$ in $V \otimes_{\mathbf{R}} \mathbf{C}$ and the i refers to the real linear transformation multiplication-by- i . Often we abbreviate (1.45a) simply as

$$V^{\mathbf{C}} = V \oplus iV \quad (1.45b).$$

Formula (1.45a) says that any real vector space is a real form of its complexification. In (1.45b) the \mathbf{R} linear map that is 1 on V and -1 on iV is called the *conjugation* of the complex vector space $V^{\mathbf{C}}$ with respect to the real form V .

In the special case that $\mathbf{K} = \mathbf{R}$ and $\mathbf{K} = \mathbf{C}$ and \mathfrak{g}_0 is a real Lie algebra, the complex Lie algebra $(\mathfrak{g}_0)^{\mathbf{C}}$ is called the *complexification* of \mathfrak{g}_0 . Similarly when a complex Lie algebra \mathfrak{g} and a real Lie algebra \mathfrak{g}_0 are related as vector spaces over \mathbf{R} by

$$\mathfrak{g}^{\mathbf{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0, \quad (1.46)$$

we say that \mathfrak{g}_0 is a *real form* of the complex Lie algebra \mathfrak{g} . Any real Lie algebra is a real form of its complexification. The conjugation of a complex Lie algebra \mathfrak{g} with respect to a real form is a Lie algebra isomorphism of $\mathfrak{g}^{\mathbf{R}}$ with itself.

Now let \mathfrak{g} be a finite-dimensional Lie algebra over K . If X and Y are in \mathfrak{g} , then $\text{ad } X$ and $\text{ad } Y$ is a linear transformation from \mathfrak{g} to itself, and it is meaningful to define

$$B(X, Y) = \text{Tr}(\text{ad } X \text{ ad } Y). \quad (1.47)$$

Then B is a symmetric bilinear form on \mathfrak{g} known as the *Killing form* of \mathfrak{g} after the person who introduced it. The Killing form is *invariant* in the sense that

$$B((\text{ad } X)Y, Z) = -B(Y, (\text{ad } X)Z) \quad (1.48)$$

for all X, Y , and Z in \mathfrak{g} .

10. Automorphisms and Derivations:

Definition 1.49: An automorphism of a Lie algebra is an invertible linear map L that preserves brackets: $[L(X), L(Y)] = L[X, Y]$. For example if \mathfrak{g} is the (real) Lie algebra of a Lie group G and if g is in G , then $\text{Ad}(g)$ is an automorphism of \mathfrak{g} . [6]

If \mathfrak{g} is real, let $\text{Aut}_{\mathbf{R}} \mathfrak{g} \subseteq \text{GL}_{\mathbf{R}}(\mathfrak{g})$ be the subgroup of \mathbf{R} linear automorphisms of \mathfrak{g} . This is a closed subgroup of a general linear group, hence a Lie group. If \mathfrak{g} is complex, we can regard

$$\text{Aut}_{\mathbf{C}} \mathfrak{g} \subseteq \text{GL}_{\mathbf{C}}(\mathfrak{g}) \subseteq \text{GL}_{\mathbf{R}}(\mathfrak{g}^{\mathbf{R}}),$$

the subscript \mathbf{C} referring to complex-linearity and $\mathfrak{g}^{\mathbf{R}}$ denoting the underlying real Lie algebra of \mathfrak{g} . But also we have the option of regarding \mathfrak{g} as the real Lie algebra $\mathfrak{g}^{\mathbf{R}}$ directly. Then we have

$$\text{Aut}_{\mathbb{C}} \mathfrak{g} \subseteq \text{Aut}_{\mathbb{R}} \mathfrak{g}^{\mathbb{R}} \subseteq \text{GL}_{\mathbb{R}}(\mathfrak{g}^{\mathbb{R}}).$$

Proposition 1.50: If G is a Lie group and \mathfrak{g} is its Lie algebra, then Ad is a smooth homomorphism from G into $\text{GL}(\mathfrak{g})$.

Lemma 1.51: If a is an automorphism of \mathfrak{g} and if X is in \mathfrak{g} , then $\text{ad}(aX) = a(\text{ad } X) a^{-1}$.

Proof: We have $\text{ad}(aX)Y = [aX, Y] = a[X, a^{-1}Y] = (a(\text{ad } X)a^{-1})Y$. [6]

Proposition 1.52: If B is the Killing form of \mathfrak{g} and if a is an automorphism of \mathfrak{g} , then $B(aX, aY) = B(X, Y)$ for all X and Y in \mathfrak{g} .

Proof: By the above lemma we have

$$\begin{aligned} B(aX, aY) &= \text{Tr}(\text{ad}(aX)\text{ad}(aY)) \\ &= \text{Tr}(a(\text{ad } X)a^{-1}a(\text{ad } Y)a^{-1}) \\ &= \text{Tr}((\text{ad } X)(\text{ad } Y)) \\ &= B(X, Y), \end{aligned}$$

as required. [6]

Now we define a *derivation* to be any D in $\text{End}_{\mathbb{R}} \mathfrak{g}$ for which

$$D[X, Y] = [X, DY] + [DX, Y].$$

Let $\text{Der}_{\mathbb{R}} \mathfrak{g}$ be the vector space of all derivations of \mathfrak{g} .

Proposition 1.53: If \mathfrak{g} is real, the Lie algebra of $\text{Aut}_{\mathbb{R}} \mathfrak{g}$ is $\text{Der}_{\mathbb{R}} \mathfrak{g}$. If \mathfrak{g} is complex, the Lie algebra of $\text{Aut}_{\mathbb{C}} \mathfrak{g}$ is $\text{Der}_{\mathbb{C}} \mathfrak{g}$. In either case the Lie algebra contains $\text{ad } \mathfrak{g}$.

Proof: First let \mathfrak{g} be real. If $c(t)$ is a curve of automorphism from 1 with $c'(0) = l$, then $c(t)[X, Y] = [c(t)X, c(t)Y]$ implies $l[X, Y] = [l(X), Y] + [X, l(Y)]$. Hence the Lie algebra in question is a Lie subalgebra of $\text{Der}_{\mathbf{R}} \mathfrak{g}$. For the reverse direction, we show that $l \in \text{Der}_{\mathbf{R}} \mathfrak{g}$ implies that e^{tl} is in $\text{Aut}_{\mathbf{R}} \mathfrak{g}$, so that $\text{Der}_{\mathbf{R}} \mathfrak{g}$ is a Lie subalgebra of the Lie algebra in question. Thus consider

$$y_1(t) = e^{tl} [X, Y] \quad \text{and} \quad y_2(t) = [e^{tl} X, e^{tl} Y]$$

as two curves in the real vector space \mathfrak{g} with value $[X, Y]$ at $t = 0$. For any t we have

$$y_1'(t) = l e^{tl} [X, Y] = l y_1(t)$$

and

$$\begin{aligned} y_2'(t) &= [l e^{tl} X, e^{tl} Y] + [e^{tl} X, l e^{tl} Y] \\ &= l[e^{tl} X, e^{tl} Y] && \text{by the derivation property} \\ &= l y_2(t). \end{aligned}$$

Then $e^{tl} [X, Y] = [e^{tl} X, e^{tl} Y]$ by the uniqueness theorem for linear systems of ordinary differential equations.[6]

If \mathfrak{g} is complex, then the Lie algebra of $\text{Aut}_{\mathbf{C}} \mathfrak{g}$ is contained in $\text{Der}_{\mathbf{R}} \mathfrak{g}^{\mathbf{R}}$ by the above, and it is contained in $\text{End}_{\mathbf{C}} \mathfrak{g}$, which is the Lie algebra of $\text{GL}_{\mathbf{C}}(\mathfrak{g})$. Hence the Lie algebra in question is contained in their intersection, which is $\text{Der}_{\mathbf{C}} \mathfrak{g}$. In the reverse direction, if l is in $\text{Der}_{\mathbf{C}} \mathfrak{g}$, then e^{tl} is contained in $\text{Aut}_{\mathbf{R}} \mathfrak{g}^{\mathbf{R}}$ by the above, and it is contained in $\text{GL}_{\mathbf{C}}(\mathfrak{g})$ also. Hence it is contained in the intersection, which is $\text{Aut}_{\mathbf{C}} \mathfrak{g}$.

Finally $\text{ad } \mathfrak{g}$ is a Lie subalgebra of the Lie algebra of derivation as a consequence of the Jacobi identity.

Define $\text{Int } \mathfrak{g}$ to be the analytic subgroup of $\text{Aut}_{\mathbf{R}} \mathfrak{g}$ with Lie algebra $\text{ad } \mathfrak{g}$. If \mathfrak{g} is complex, the definition is unaffected by using $\text{Aut}_{\mathbf{C}} \mathfrak{g}$ instead of $\text{Aut}_{\mathbf{R}} \mathfrak{g}^{\mathbf{R}}$ as the ambient group, since $\text{ad } \mathfrak{g}$ is the same set of transformations as $\text{ad } \mathfrak{g}^{\mathbf{R}}$.

The analytic group $\text{Int } \mathfrak{g}$ is a universal version of the group of inner automorphisms. To be more precise, let us think of \mathfrak{g} as real. Suppose \mathfrak{g} is the Lie algebra of a Lie group G . As usual, we define $\text{Ad}(g)$ to be the differential at the identity of the inner automorphism $x \mapsto gxg^{-1}$. Then Proposition 1.50 shows that $g \mapsto \text{Ad}(g)$ is a smooth homomorphism of G into $\text{Aut}_{\mathbf{R}} \mathfrak{g}$, and we may regard $\text{Ad}(G)$ as a Lie subgroup of $\text{Aut}_{\mathbf{R}} \mathfrak{g}$. As such, its Lie algebra is $\text{ad } \mathfrak{g}$. By definition the analytic subgroup of $\text{Aut}_{\mathbf{R}} \mathfrak{g}$ with Lie algebra $\text{ad } \mathfrak{g}$ is $\text{Int } \mathfrak{g}$. Thus $\text{Int } \mathfrak{g}$ is the identity component of $\text{Ad}(G)$ and equals $\text{Ad}(G)$ if G is connected. In this sense $\text{Int } \mathfrak{g}$ is a universal version of $\text{Ad}(G)$ that can be defined without reference to a particular group G .

Example: If $\mathfrak{g} = \mathbf{R}^2$, then $\text{Aut}_{\mathbf{R}} \mathfrak{g} = \text{GL}_{\mathbf{R}}(\mathfrak{g})$ and $\text{Der}_{\mathbf{R}} \mathfrak{g} = \text{End}_{\mathbf{R}} \mathfrak{g}$. Also $\text{ad } \mathfrak{g} = 0$, and so $\text{Int } \mathfrak{g} = \{1\}$. In particular $\text{Int } \mathfrak{g}$ is strictly smaller than the identity component of $\text{Aut}_{\mathbf{R}} \mathfrak{g}$ for this example.

Proposition 1.54: If \mathfrak{g} is semisimple (real or complex), then $\text{Der } \mathfrak{g} = \text{ad } \mathfrak{g}$.

Proof: Let D be a derivation of \mathfrak{g} . The Killing form is nondegenerate. Thus we can find X in \mathfrak{g} with $\text{Tr}(D \text{ ad } Y) = B(X, Y)$ for all $Y \in \mathfrak{g}$. The derivation property

$$[DY, Z] = D[Y, Z] - [Y, DZ]$$

can be written as

$$\text{ad}(DY) = [D, \text{ad } Y].$$

Therefore

$$\begin{aligned} B(DY, Z) &= \text{Tr}(\text{ad}(DY)\text{ad } Z) \\ &= \text{Tr}([D, \text{ad } Y]\text{ad } Z) \\ &= \text{Tr}(D \text{ad}[Y, Z]) \quad \text{by expanding both sides} \\ &= B(X, [Y, Z]) \quad \text{by definition of } X \\ &= B([X, Y], Z) \quad \text{by invariance of } B \text{ as in (1.48)} \end{aligned}$$

By a second application of nondegeneracy of B , $DY = [X, Y]$. Thus $D = \text{ad } X$. [6]

11. Linear Lie Groups:

A closed subgroup G of nonsingular real or complex matrices will be called a *closed linear group*. Such a group has canonically the structure of a Lie group. Let us carry out the details of this identification, referring to the Lie algebra of matrices eventually as the “linear Lie algebra” of G . For a closed linear group G , we define

$$\mathfrak{g} = \left\{ c'(0) \mid c : \mathbb{R} \rightarrow G \text{ is a curve with } c(0) = 1 \text{ that is smooth as a function into matrices} \right\}. \quad (1.55)$$

Use of $t \mapsto c(kt)$ shows that \mathfrak{g} is closed under multiplication by the real number k , and use of $t \mapsto c(t)b(t)$ shows that \mathfrak{g} is closed under addition. Use of the curve $t \mapsto gc(t)g^{-1}$, for $g \in G$, then shows that \mathfrak{g} is closed under the operation $\text{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}$ given by

$$\text{Ad}(g)X = gXg^{-1}. \quad (1.56)$$

To see that \mathfrak{g} is closed under the bracket operation on matrices, we combine three facts:

- (i) \mathfrak{g} contains $\text{Ad}(c(t))X$ if $c(t)$ is a curve as in (1.55) and X is in \mathfrak{g}
- (ii) \mathfrak{g} is topologically closed (being a vector subspace)
- (iii) $\frac{d}{dt} \text{Ad}(c(t))X = c'(t)Xc(t)^{-1} - c(t)Xc(t)^{-1}c'(t)c(t)^{-1}$.

The first two facts are clear, and the third follows from (1.56) and the formula

$$\frac{d}{dt} c(t)^{-1} = -c(t)^{-1}c'(t)c(t)^{-1},$$

which in turn follows by applying the product rule for differentiation to the identity $c(t)c(t)^{-1} = 1$. Now let us combine (i), (ii), and (iii). By (i) and (ii), $\frac{d}{dt} \text{Ad}(c(t))X$ is in \mathfrak{g} for all t . Putting $t = 0$ in (iii), we see that $c'(0)X - Xc'(0)$ is in \mathfrak{g} . Consequently \mathfrak{g} is closed under the bracket operation $[X, Y] = XY - YX$ and is a Lie algebra of matrices. We call \mathfrak{g} the *linear Lie algebra* of G .

Still with G as a closed linear group, we work toward seeing that G is a Lie group and exhibiting an isomorphism of the linear Lie algebra of G and the Lie algebra of G . We recall the exponential mapping for matrices, defined by

$$e^X = \sum \frac{1}{n!} X^n. \quad (1.57)$$

Part of the relevance of the matrix exponential mapping for matrices is that it provides specific curves in G of the kind (1.55). According to the following proposition, e^{tX} is such a curve if X is in the linear Lie algebra.

Proposition 1.58: If G is a closed linear group and \mathfrak{g} is its linear Lie algebra, then the matrix exponential function $e^{(\cdot)}$ carries \mathfrak{g} into G . Consequently

$$\mathfrak{g} = \{ X \in \mathfrak{gl}(n, \mathbb{C}) \mid e^{tX} \text{ is in } G \text{ for all real } t \}. [6]$$

Proposition 1.59: If G is a closed linear group, then G (with its relative topology) becomes a Lie group in a unique way such that

- a) the restrictions from $GL(n, \mathbb{C})$ to G of the real and imaginary parts of each entry function are smooth and
- b) whenever $\Phi : M \rightarrow GL(n, \mathbb{C})$ is a smooth function on a smooth manifold M such that $\Phi(M) \subseteq G$, then $\Phi : M \rightarrow G$ is smooth. [6]

Moreover, the dimension of the linear Lie algebra \mathfrak{g} equals the dimension of the manifold G . And, in addition, there exist open neighborhoods U of 0 in \mathfrak{g} and V of 1 in G such that $e^{(\cdot)} : U \rightarrow V$ is a homeomorphism onto and such that $(V, (e^{(\cdot)})^{-1})$ is a compatible chart.

A Lie subgroup of $GL(n, \mathbb{R})$ or $GL(n, \mathbb{C})$ is called a linear Lie group. We shall study various examples of linear Lie groups in the following.

(1) The special linear groups $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$: We denote the set of all $n \times n$ real (complex) matrices of determinant 1 by $SL(n, \mathbb{R})$ ($SL(n, \mathbb{C})$), and call it the real (complex) special linear group. The group $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$. For $x \in GL(n, \mathbb{R})$ satisfying $f(x) - 1 = 0$, we have

$$df = \sum_{k=1}^n \begin{vmatrix} x_1^1 & \dots & dx_k^1 & \dots & x_n^1 \\ x_1^2 & \dots & dx_k^2 & \dots & x_n^2 \\ \cdot & & & & \\ \cdot & & & & \\ x_1^n & \dots & dx_k^n & \dots & x_n^n \end{vmatrix}$$

Hence $(df)_x$ is not 0 at any $x \in SL(n, \mathbb{R})$. Hence, $SL(n, \mathbb{R})$ is an $(n^2 - 1)$ -dimensional closed submanifold of $GL(n, \mathbb{R})$. Hence, the maps $(a, b) \rightarrow a \cdot b$ and $b \rightarrow b^{-1}$ are differentiable. So, $SL(n, \mathbb{R})$ is a closed Lie subgroup of $GL(n, \mathbb{R})$. Now let $\exp tX$ ($X \in \mathfrak{gl}(n, \mathbb{R})$) be a one-parameter subgroup of $GL(n, \mathbb{R})$. Then we have

Lemma 1.60:

$$\det(\exp tX) = \exp t(\operatorname{tr} X). \quad \text{for } X \in M_n(\mathbb{R}).$$

Proof:

Consider the curve

$$a: \mathbb{R} \rightarrow GL(n, \mathbb{R}); \quad a(t) = \det \exp(tX).$$

Then for small $h \in \mathbb{R}$ we have

$$\begin{aligned}
a'(t) &= \lim_{h \rightarrow 0} \frac{1}{h} (\det \exp((t+h)X) - \det \exp(tX)) \\
&= \det \exp(tX) \lim_{h \rightarrow 0} \frac{1}{h} (\det \exp(hX) - 1) \\
&= \det \exp(tX) \operatorname{tr} X \\
&= a(t) \operatorname{tr} X
\end{aligned}$$

So a satisfies the same differential equation and initial condition as the curve $t \mapsto e^{t \operatorname{tr} X}$.

Hence

$$a(t) = \det \exp(tX) = e^{t \operatorname{tr} X}$$

by the uniqueness theorem.[2]

Hence $\exp tX$ is a one-parameter subgroup of $\mathrm{SL}(n, \mathbf{R})$ if and only if $\operatorname{tr} X = 0$. Set

$$\mathfrak{sl}(n, \mathbf{R}) = \{ X \in \mathfrak{gl}(n, \mathbf{R}) \mid \operatorname{tr} X = 0 \}.$$

Then $\mathfrak{sl}(n, \mathbf{R})$ is an $(n^2 - 1)$ -dimensional subalgebra of $\mathfrak{gl}(n, \mathbf{R})$, and $\mathfrak{sl}(n, \mathbf{R})$ is the Lie algebra of $\mathrm{SL}(n, \mathbf{R})$.

Let \mathbf{R}^+ be the Lie group formed by the set of all positive real numbers with respect to multiplication. Define a map φ from $\mathbf{R}^+ \times \mathrm{SL}(n, \mathbf{R})$ to $\mathrm{GL}^+(n, \mathbf{R})$ by

$\varphi(\lambda, a) = \lambda \cdot a$ ($\lambda \in \mathbf{R}^+$, $a \in \mathrm{SL}(n, \mathbf{R})$). Then φ is 1:1 homomorphism from $\mathbf{R}^+ \times \mathrm{SL}(n, \mathbf{R})$ onto $\mathrm{GL}^+(n, \mathbf{R})$. Hence, φ is an isomorphism from $\mathbf{R}^+ \times \mathrm{SL}(n, \mathbf{R})$ onto $\mathrm{GL}^+(n, \mathbf{R})$.

Similarly, we can prove that $\mathrm{SL}(n, \mathbf{C})$ is a closed complex Lie subgroup of $\mathrm{GL}(n, \mathbf{C})$ of complex dimension $n^2 - 1$.

(2) *The orthogonal group $O(n)$ and the special orthogonal group $SO(n)$:* Let $O(n)$ be the set of all $n \times n$ orthogonal matrices, and set $SO(n) = O(n) \cap SL(n, \mathbf{R})$. The set $O(n)$ and $SO(n)$ are both subgroups of $GL(n, \mathbf{R})$. We call $O(n)$ the orthogonal group, and $SO(n)$ the special orthogonal group.[7]

If $\mathbf{x} = (x_j^i)$ is an element of $GL(n, \mathbf{R})$, define the $\frac{1}{2}n(n+1)$ functions f_{ik} ($1 \leq i \leq k \leq n$) by

$$f_{ik}(\mathbf{x}) = \sum_{j=1}^n x_i^j x_k^j \quad (1 \leq i \leq k \leq n).$$

Then we have

$$O(n) = \{a \in GL(n, \mathbf{R}) \mid f_{ik}(a) - \delta_{ik} = 0, 1 \leq i \leq k \leq n\}.$$

We can easily check by computation that the df_{ik} 's are linearly independent at each point of $O(n)$. Then, $O(n)$ is a $\frac{1}{2}n(n-1)$ -dimensional closed submanifold of $GL(n, \mathbf{R})$. Hence

$O(n)$ is a closed Lie subgroup of $GL(n, \mathbf{R})$.

If $\exp tX$ ($X \in \mathfrak{gl}(n, \mathbf{R})$) is a one-parameter subgroup of $GL(n, \mathbf{R})$, then, since $(\exp tX)^t = \exp tX^t$ and $(\exp tX)^{-1} = \exp(-tX)$, $\exp tX$ belong to $O(n)$ if and only if

$$X^t + X = 0.$$

An $n \times n$ matrix satisfying this condition is called a *skew-symmetric matrix*. The set of all $n \times n$ skew-symmetric matrices is denoted by $\mathfrak{o}(n)$. It is a subalgebra of $\mathfrak{gl}(n, \mathbf{R})$ and the Lie algebra of $O(n)$ is $\mathfrak{o}(n)$.

(3) *The complex orthogonal group $O(n, \mathbf{C})$:* The set of all $n \times n$ complex matrices satisfying $a^t a = 1_n$ (1_n is the $n \times n$ unit matrix) is denoted by $O(n, \mathbf{C})$. The set $O(n, \mathbf{C})$ is a

complex Lie subgroup of $GL(n, \mathbf{C})$, and has complex dimension $\frac{1}{2}n(n-1)$. The Lie algebra of $O(n, \mathbf{C})$ is the subalgebra $\mathfrak{o}(n, \mathbf{C})$ of $\mathfrak{gl}(n, \mathbf{C})$ formed by all $n \times n$ skew symmetric complex matrices.

These facts can be all verified using arguments similar to those used for $O(n)$. The group $O(n, \mathbf{C})$ is called the complex orthogonal group.

(4) *The unitary group $U(n)$ and the special unitary group $SU(n)$* : Let $U(n)$ denote the set of all $n \times n$ unitary matrices, and let $SU(n)$ denote the set of all $n \times n$ unitary matrices with determinant 1. The sets $U(n)$ and $SU(n)$ are subgroups of $GL(n, \mathbf{C})$. Since, for

$$a \in GL(n, \mathbf{C}),$$

$$a \in U(n) \Leftrightarrow {}^t \bar{a} a = 1_n,$$

the entries of the matrices belonging to $U(n)$ are bounded, and $U(n)$ is a closed set of $GL(n, \mathbf{C})$. Hence $U(n)$ is compact. Similarly, $SU(n)$ is also compact. Hence these are closed subgroups of $GL(n, \mathbf{C})$. The groups $U(n)$ and $SU(n)$ are called the unitary group and the special group, respectively.

Now let $\exp tX$ ($X \in \mathfrak{gl}(n, \mathbf{C})$) be a one-parameter subgroup of $GL(n, \mathbf{C})$. Since $(\overline{\exp tX})^t = \exp t \bar{X}^t$ and $(\exp tX)^{-1} = \exp(-tX)$, we have $\exp tX \in U(n)$ if and only if

$$\bar{X}^t + X = 0.$$

An $n \times n$ complex matrix X satisfying this condition is called an $n \times n$ *skew-Hermitian matrix*. The set of all $n \times n$ skew-Hermitian matrices is denoted by $\mathfrak{u}(n)$. The set $\mathfrak{u}(n)$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbf{C})$, and the Lie algebra of $U(n)$ is $\mathfrak{u}(n)$. Similarly, the set $\mathfrak{su}(n)$ of all $n \times n$ skew-Hermitian matrices whose traces are 0 is also a Lie subalgebra of $\mathfrak{gl}(n, \mathbf{C})$, and the Lie algebra of $SU(n)$ is $\mathfrak{su}(n)$. [13]

Theorem 1.61: *Every closed subgroup of a Lie group is a closed Lie subgroup.*

Theorem 1.62: *Let H be a Lie subgroup of a Lie group G . If the topology of H is the induced topology, then H is closed.*[8]

To prove this theorem, we need the following elementary lemma.

Lemma 1.63: Let M be a C^∞ manifold, and N a regular submanifold of M such that $\dim N < \dim M$. Then the closure \bar{N} of N is different from M .

Proof: For each point p of N , there exists a neighborhood U of p in M , and r functions f_1, \dots, f_r on U ($r = \dim M - \dim N$), such that df_1, \dots, df_r are linearly independent at each point of $N \cap U$, and such that

$$N \cap U = \{q \in U \mid f_1(q) = \dots = f_r(q) = 0\}.$$

If $\bar{N} = M$, then we have $f_1(q) = \dots = f_r(q) = 0$ for all $q \in U$, and this is impossible because df_1, \dots, df_r are linearly independent at q . Therefore $\bar{N} \neq M$. [8]

Proof of Theorem 1.62: Let \bar{H} be the closure of H . Then \bar{H} is a closed Lie subgroup of G , by Theorem 1.56. The identity map $i: H \rightarrow G$ is differentiable, and $i(H) \subset \bar{H}$. Since \bar{H} is a regular submanifold of G , then i is a differentiable map from H into \bar{H} . Since H also has the topology induced from G by the hypothesis, this shows that H is a regular submanifold of \bar{H} . But then, by the above lemma, we get that $\dim H = \dim \bar{H}$, and we conclude that $H = \bar{H}$. [8]

12. Compact Lie Algebras:

Let \mathfrak{g} be a real Lie algebra. We say that \mathfrak{g} is a *compact Lie algebra* if the group $\text{Int } \mathfrak{g}$ is compact. More generally let \mathfrak{r} be a Lie subalgebra of \mathfrak{g} , and let $\text{Int}_{\mathfrak{g}}(\mathfrak{r})$ be the analytic subgroup of $\text{GL}(\mathfrak{g})$ with Lie algebra $\text{ad}_{\mathfrak{g}}(\mathfrak{r})$. We say that \mathfrak{r} is *compactly embedded* in \mathfrak{g} if $\text{Int}_{\mathfrak{g}}(\mathfrak{r})$ is compact.

Proposition 1.64: If G is a Lie group with Lie algebra \mathfrak{g} and if K is a compact subgroup with corresponding Lie subalgebra \mathfrak{r} , then \mathfrak{r} is a compactly embedded subalgebra of \mathfrak{g} . In particular, the Lie algebra of a compact Lie group is always a compact Lie algebra.

Proof: Since K is compact, so is the identity component K_0 . Then $\text{Ad}_{\mathfrak{g}}(K_0)$ must be compact, being the continuous image of a compact group. The groups $\text{Ad}_{\mathfrak{g}}(K_0)$ and $\text{Int}_{\mathfrak{g}}(\mathfrak{r})$ are both analytic subgroups of $\text{GL}(\mathfrak{g})$ with Lie algebra $\text{Int}_{\mathfrak{g}}(\mathfrak{r})$ and hence are isomorphic as Lie groups. Therefore $\text{int}_{\mathfrak{g}}(\mathfrak{r})$ is compact.[6]

The next proposition and its two corollaries give properties of compact Lie algebras.

Proposition 1.65: Let G be a compact Lie group, and let \mathfrak{g} be its Lie algebra. Then the real vector space \mathfrak{g} admits an inner product (\cdot, \cdot) that is invariant under $\text{Ad}(G)$: $(\text{Ad}(g)u, \text{Ad}(g)v) = (u, v)$. Relative to this inner product the members of $\text{Ad}(G)$ act by orthogonal transformations, and the members of $\text{ad } \mathfrak{g}$ act by skew-symmetric transformations.[6]

Corollary 1.66: Let G be a compact Lie group, let \mathfrak{g} be its Lie algebra. Then \mathfrak{g} is reductive, and hence $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}_{\mathfrak{g}}$ is the center and where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.

Corollary 1.67: If G is a compact Lie group with Lie algebra \mathfrak{g} , then the Killing form of \mathfrak{g} is negative semidefinite.

Proof: Define (\cdot, \cdot) as in Proposition 1.65. By the proposition, $\text{ad } X$ is skew symmetric for $X \in \mathfrak{g}$. The eigenvalues of $\text{ad } X$ are therefore purely imaginary, and the eigenvalues of $(\text{ad } X)^2$ must be ≤ 0 . If B is the Killing form, then it follows that $B(X, X) = \text{Tr}((\text{ad } X)^2)$ is ≤ 0 . [6]

13. Cartan Decomposition on the Lie Algebra Level:

The map $\theta(X) = -X^*$ ($*$ means conjugate) is an *involution* i.e., an automorphism of the Lie algebra with square equal to the identity. To see that θ respects brackets, we just write

$$\theta[X, Y] = -[X, Y]^* = -[Y^*, X^*] = [-X^*, -Y^*] = [\theta(X), \theta(Y)].$$

Let B be the Killing form. The involution θ has the property that

$$B_\theta(X, Y) = -B(X, \theta Y)$$

is symmetric and positive definite because Proposition 1.54 gives :

$$\begin{aligned} B_\theta(X, Y) &= -B(X, \theta Y) = -B(\theta X, \theta^2 Y) \\ &= -B(\theta X, Y) = -B(Y, \theta X) = B_\theta(Y, X) \end{aligned}$$

and by $(\text{ad } y)^* = \text{ad } (y^*)$ we have

$$\begin{aligned}
B_\theta(X, Y) &= -B(X, \theta Y) = -\text{Tr}((\text{ad } X)(\text{ad } \theta Y)) \\
&= \text{Tr}((\text{ad } X)(\text{ad } X^*)) = \text{Tr}(\text{ad } X)(\text{ad } X)^* \geq 0.
\end{aligned}$$

An involution θ of a real semisimple Lie algebra \mathfrak{g}_0 such that the symmetric bilinear form

$$B_\theta(X, Y) = -B(X, \theta Y) \quad (1.68)$$

is positive definite is called a *Cartan involution*. We shall see that any real semisimple Lie algebra has a Cartan involution and that the Cartan involution is unique up to inner automorphism.

Lemma 1.69: Let \mathfrak{g}_0 be a real finite-dimensional Lie algebra, and let ρ be an automorphism of \mathfrak{g}_0 that is diagonalizable with positive eigenvalues d_1, \dots, d_m and corresponding eigenspaces $(\mathfrak{g}_0)_{d_j}$. For $-\infty < r < \infty$, define ρ^r to be the linear transformation on \mathfrak{g}_0 that is d_j^r on $(\mathfrak{g}_0)_{d_j}$. Then $\{\rho^r\}$ is a one-parameter group in $\text{Aut } \mathfrak{g}_0$.

If \mathfrak{g}_0 is semisimple, then ρ^r lies in $\text{Int } \mathfrak{g}_0$.

Proof:

If X is in $(\mathfrak{g}_0)_{d_i}$ and Y is in $(\mathfrak{g}_0)_{d_j}$, then

$$\rho[X, Y] = [\rho X, \rho Y] = d_i d_j [X, Y]$$

since ρ is an automorphism. Hence $[X, Y]$ is in $(\mathfrak{g}_0)_{d_i d_j}$, and we obtain

$$\rho^r[X, Y] = (d_i d_j)^r [X, Y] = [d_i^r X, d_j^r Y] = [\rho^r X, \rho^r Y].$$

Consequently ρ^r is an automorphism. Therefore $\{\rho^r\}$ is a one-parameter group in $\text{Aut } \mathfrak{g}_0$, hence in the identity component $(\text{Aut } \mathfrak{g}_0)_0$. If \mathfrak{g}_0 is semisimple, then Propositions 1.48 and 1.49 show that $(\text{Aut } \mathfrak{g}_0)_0 = \text{Int } \mathfrak{g}_0$, and the lemma follows.[6]

Theorem 1.70: *Let \mathfrak{g}_0 be a real semisimple Lie algebra, let θ be a Cartan involution, and let σ be any involution. Then there exists $\varphi \in \text{Int } \mathfrak{g}_0$ such that $\varphi \theta \varphi^{-1}$ commutes with σ .*

Proof: Since θ is given as a Cartan involution, B_θ is an inner product for \mathfrak{g}_0 . Put $\omega = \sigma\theta$.

This is an automorphism of \mathfrak{g}_0 , and Proposition 1.52 shows that it leaves B invariant.

From $\sigma^2 = \theta^2 = 1$, we have

$$B(\omega X, \theta Y) = B(X, \omega^{-1}\theta Y) = B(X, \theta\omega Y)$$

and hence

$$B_\theta(\omega X, Y) = B_\theta(X, \omega Y).$$

Thus ω is symmetric, and its square $\rho = \omega^2$ is positive definite. Write ρ^r for the positive-definite r^{th} power of ρ , $-\infty < r < \infty$. Lemma 1.69 shows that ρ^r is a one-parameter group in $\text{Int } \mathfrak{g}_0$.

Now

$$\rho\theta = \omega^2\theta = \sigma\theta\sigma\theta\theta = \sigma\theta\sigma = \theta\theta\sigma\theta\sigma = \theta\omega^{-2} = \theta\rho^{-1}.$$

In terms of a basis of \mathfrak{g}_0 that diagonalizes ρ , the matrix form of this equation is

$$\rho_{ii} \theta_{ij} = \theta_{ij} \rho_{jj}^{-1} \quad \text{for all } i \text{ and } j.$$

Considering separately the cases $\theta_{ij} = 0$ and $\theta_{ij} \neq 0$, we see that

$$\rho_{ii}^r \theta_{ij} = \theta_{ij} \rho_{jj}^{-r}$$

and therefore that

$$\rho^r \theta = \theta \rho^{-r}. \quad (1.71)$$

Put $\varphi = \rho^{\frac{1}{4}}$. Then two applications of (1.71) give

$$\begin{aligned} (\varphi \theta \varphi^{-1}) \sigma &= \rho^{\frac{1}{4}} \theta \rho^{-\frac{1}{4}} \sigma = \rho^{\frac{1}{2}} \theta \sigma \\ &= \rho^{\frac{1}{2}} \omega^{-1} = \rho^{-\frac{1}{2}} \rho \omega^{-1} \\ &= \rho^{-\frac{1}{2}} \omega = \omega \rho^{\frac{1}{2}} \\ &= \sigma \theta \rho^{-\frac{1}{2}} = \sigma \rho^{\frac{1}{4}} \theta \rho^{-\frac{1}{4}} = \sigma(\varphi \theta \varphi^{-1}), \end{aligned}$$

as required.[6]

Proposition 1.72: Let \mathfrak{g} be a complex semisimple Lie algebra, let \mathfrak{u}_0 be a compact real form of \mathfrak{g} , and let τ be the conjugation of \mathfrak{g} with respect to \mathfrak{u}_0 . If \mathfrak{g} is regarded as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$, then τ is a Cartan involution.[6]

Corollary 1.73: If \mathfrak{g}_0 is a real semisimple Lie algebra, then \mathfrak{g}_0 has a Cartan involution.

Proof: Let \mathfrak{g} be the complexification of \mathfrak{g}_0 , and choose a compact real form \mathfrak{u}_0 of \mathfrak{g} . Let σ and τ be the conjugations of \mathfrak{g} with respect to \mathfrak{g}_0 and \mathfrak{u}_0 . If we regard \mathfrak{g} as a real Lie algebra $\mathfrak{g}^{\mathbb{R}}$, then σ and τ are involutions of $\mathfrak{g}^{\mathbb{R}}$, and Proposition 1.72 shows that τ is a Cartan involution. By Theorem 1.70 we can find $\varphi \in \text{Int}(\mathfrak{g}^{\mathbb{R}}) = \text{Int } \mathfrak{g}$ such that $\varphi \tau \varphi^{-1}$ commutes with σ .

Here $\varphi \tau \varphi^{-1}$ is the conjugation of \mathfrak{g} with respect to $\varphi(\mathfrak{u}_0)$, which is another compact real form of \mathfrak{g} . Thus

$$(B_{\mathfrak{g}^{\mathbb{R}}})_{\varphi\tau\varphi^{-1}}(Z_1, Z_2) = -2\operatorname{Re}B_{\mathfrak{g}}(Z_1, \varphi\tau\varphi^{-1}Z_2)$$

is positive definite on $\mathfrak{g}^{\mathbb{R}}$.

The Lie algebra \mathfrak{g}_0 is characterized as the fixed set of σ . If $\sigma X = X$, then

$$\sigma(\varphi\tau\varphi^{-1}X) = \varphi\tau\varphi^{-1}\sigma X = \varphi\tau\varphi^{-1}X.$$

Hence $\varphi\tau\varphi^{-1}$ restricts to an involution θ of \mathfrak{g}_0 . We have

$$B_{\theta}(X, Y) = -B_{\mathfrak{g}_0}(X, \theta Y) = -B_{\mathfrak{g}}(X, \varphi\tau\varphi^{-1}Y) = \frac{1}{2}(B_{\mathfrak{g}^{\mathbb{R}}})_{\varphi\tau\varphi^{-1}}(X, Y).$$

Thus B_{θ} is positive definite on \mathfrak{g}_0 , and θ is a Cartan involution.[6]

A Cartan involution θ of \mathfrak{g}_0 yields an eigenspace decomposition

$$\mathfrak{g}_0 = \mathfrak{r}_0 \oplus \mathfrak{p}_0 \quad (1.74)$$

of \mathfrak{g}_0 into +1 and -1 eigenspaces, and these must bracket according to the rules

$$[\mathfrak{r}_0, \mathfrak{r}_0] \subseteq \mathfrak{r}_0, \quad [\mathfrak{r}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{r}_0 \quad (1.75)$$

since θ is an involution. From (1.74) and (1.75) it follows that

$$\mathfrak{r}_0 \text{ and } \mathfrak{p}_0 \text{ are orthogonal under } B_{\mathfrak{g}_0} \text{ and under } B_{\theta}. \quad (1.76)$$

In fact, if X is in \mathfrak{r}_0 and Y is in \mathfrak{p}_0 , then $\operatorname{ad} X \operatorname{ad} Y$ carries \mathfrak{r}_0 to \mathfrak{p}_0 and \mathfrak{p}_0 to \mathfrak{r}_0 . Thus it has trace 0, and $B_{\mathfrak{g}_0}(X, Y) = 0$; since $\theta Y = -Y$, $B_{\theta}(X, Y) = 0$ also.

Since B_θ is positive definite, the eigenspaces \mathfrak{r}_θ and \mathfrak{p}_θ in (1.74) have the property that

$$B_{\mathfrak{g}_\theta} \text{ is } \begin{cases} \text{negative definite on } \mathfrak{r}_\theta \\ \text{positive definite on } \mathfrak{p}_\theta. \end{cases} \quad (1.77)$$

A decomposition (1.74) of \mathfrak{g}_θ that satisfies (1.75) and (1.77) is called a *Cartan decomposition* of \mathfrak{g}_θ .

Conversely a Cartan decomposition determines a Cartan involution θ by the formula

$$\theta = \begin{cases} +1 & \text{on } \mathfrak{r}_\theta \\ -1 & \text{on } \mathfrak{p}_\theta. \end{cases}$$

If $\mathfrak{g}_\theta = \mathfrak{r}_\theta \oplus \mathfrak{p}_\theta$ is a Cartan decomposition of \mathfrak{g}_θ , then $\mathfrak{r}_\theta \oplus i\mathfrak{p}_\theta$ is a compact real form of $\mathfrak{g} = (\mathfrak{g}_\theta)^{\mathbb{C}}$.

Definition 1.78: A Lie group automorphism Θ of a semisimple Lie group G with differential θ , $\Theta^2 = 1$ is called the *global Cartan involution*, and the onto diffeomorphism mapping

$K \times \mathfrak{p}_\theta \rightarrow G$ given by $(k, X) \rightarrow k \exp X$, where K is the subgroup of G fixed by Θ , is called the *global Cartan decomposition*. [6]

Theorem 1.79: Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra \mathfrak{g}_0 , let $\mathfrak{g}_0 = \mathfrak{r}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition, and let K be the analytic subgroup of G with Lie algebra \mathfrak{r}_0 . Then

- (a) there exists a Lie group automorphism Θ of G with differential θ , and Θ has $\Theta^2 = 1$
- (b) the subgroup of G fixed by Θ is K
- (c) the mapping $K \times \mathfrak{p}_0 \rightarrow G$ given by $(k, X) \rightarrow k \exp X$ is a diffeomorphism onto
- (d) K is closed
- (e) K contains the center Z of G
- (f) K is compact if and only if Z is finite
- (g) when Z is finite, K is a maximal compact subgroup of G .

Proof: see [Knapp].[6]

Proposition 1.80: Let \mathfrak{g} be a real Lie algebra of matrices over \mathbf{R} or \mathbf{C} . If \mathfrak{g} is closed under the operation conjugate transpose, then \mathfrak{g} is reductive.

Proof:

For matrices X and Y , define $\langle X, Y \rangle = \operatorname{Re} \operatorname{Tr}(XY^*)$. This is a real inner product on \mathfrak{g} , the symmetry following from the form $(\operatorname{Re} x\bar{y} = \operatorname{Re} y\bar{x})$. Let a be an ideal in \mathfrak{g} , and let a^\perp be the orthogonal complement of a in \mathfrak{g} . Then $\mathfrak{g} = a \oplus a^\perp$ as vector spaces. To see that a^\perp is an ideal in \mathfrak{g} , let X be in a^\perp , let Y be in \mathfrak{g} , and let Z be in a . Then

$$\begin{aligned} \langle [X, Y], Z \rangle &= \operatorname{Re} \operatorname{Tr}(XYZ^* - YXZ^*) \\ &= -\operatorname{Re} \operatorname{Tr}(XZ^*Y - XYZ^*) \\ &= -\operatorname{Re} \operatorname{Tr}(X(Y^*Z)^* - X(ZY^*)^*) \\ &= -\langle X, [Y^*, Z] \rangle. \end{aligned}$$

Since Y^* is in \mathfrak{g} , $[Y^*, Z]$ is in a . Thus the right side is 0 for all Z , and $[X, Y]$ is in a^\perp .

Hence a^\perp is an ideal, and \mathfrak{g} is reductive.[6]

Corollary 1.81: If \mathfrak{g}_0 is a real semisimple Lie algebra and θ is a Cartan involution, then any θ stable subalgebra $\hat{\mathfrak{s}}_0$ of \mathfrak{g}_0 is reductive.

Proof : see [Knapp].[6]

CHAPTER II

REDUCTIVE LIE GROUPS

Definition 2.1: A connected subgroup G of $GL(n, \mathbb{R})$ is reductive if its Lie algebra \mathfrak{g} has a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where

(i) $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$

(ii) the Lie group \tilde{K} of $GL(n, \mathbb{C})$ whose Lie algebra is $\tilde{\mathfrak{k}} = \mathfrak{k} \oplus i\mathfrak{p}$ is compact.[1]

Examples:

1) The group $SO(n, \mathbb{R}) = \mathfrak{k}$ is reductive with $\mathfrak{p} = 0$.

2) The group $SL(n, \mathbb{R})$ is reductive : $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} is the Lie algebra of skew

symmetric, \mathfrak{p} the space of symmetric

matrices: here $\mathfrak{k} \oplus i\mathfrak{p}$ is the Lie algebra of skew hermitian matrices of trace 0, so

it is the Lie algebra of the compact group $SU(n)$.

(3) The group $GL(n, \mathbb{C})$ is reductive:

We have Lie algebra of $(GL(n, \mathbb{C})) = \mathfrak{k} \oplus i\mathfrak{k}$, where \mathfrak{k} is the Lie algebra of unitary matrices.

Proposition 2.2: Let G be an analytic subgroup of real or complex matrices whose Lie algebra \mathfrak{g}_0 is semisimple. Then G has finite center and is a closed linear group.[6]

Corollary 2.3: Let G be an analytic subgroup of real or complex matrices whose Lie algebra \mathfrak{g}_0 is reductive, and suppose that the identity component of the center of G is compact. Then G is a closed linear group.

Remark: If G is an analytic group whose Lie algebra \mathfrak{g}_0 is a direct sum $\mathfrak{g}_0 = \hat{\mathfrak{a}} \oplus \mathfrak{b}$ of ideals and if A and B are the analytic subgroups corresponding to $\hat{\mathfrak{a}}$ and \mathfrak{b} , then G is a commuting product $G = AB$.

Proof: Write $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$. The analytic subgroup of G corresponding to $Z_{\mathfrak{g}_0}$ is $(Z_G)_0$, and we let G_{ss} (semisimple subgroup) be the analytic subgroup corresponding to $[\mathfrak{g}_0, \mathfrak{g}_0]$. By the remark before the proof, G is the commuting product $(Z_G)_0 G_{ss}$. The group G_{ss} is closed as a group of matrices, and $(Z_G)_0$ is compact by assumption. Hence the set of products, which is G , is closed.[6]

Corollary 2.4: Let G be a connected closed linear group whose Lie algebra \mathfrak{g}_0 is reductive. Then the analytic subgroup G_{ss} of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ is closed, and G is the commuting product $G = (Z_G)_0 G_{ss}$.

Lemma 2.5: On matrices let Θ be conjugate transpose inverse, and let θ be negative conjugate transpose. Let G be a connected abelian closed linear group that is stable under Θ , and let \mathfrak{g}_0 be its linear Lie algebra, stable under θ . Let $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of \mathfrak{g}_0 into +1 and -1 eigenspaces under θ , and let $K = \{x \in G \mid \Theta x = x\}$. Then the map $K \times \mathfrak{p}_0 \rightarrow G$ given by $(k, X) \rightarrow k \exp X$ is a Lie group isomorphism.

Proof: The group K is a closed subgroup of the unitary group and is compact with Lie algebra k . Since p is abelian, $(\exp p)$ is the analytic subgroup of G with Lie algebra p . By the remark after Corollary 2.3, $G = K \exp p$. The smooth map $K \times p \rightarrow G$ is compatible with the polar decomposition of matrices and is therefore one-one. It is a Lie group homomorphism since G and p are abelian. Its inverse is smooth since the inverse of the polar decomposition of matrices is smooth.[6]

Proposition 2.6: On matrices let Θ be conjugate transpose inverse, and let θ be negative conjugate transpose. Let G be a connected closed linear group that is stable under Θ , and let \mathfrak{g}_0 be its linear Lie algebra, stable under θ . Let $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of \mathfrak{g}_0 into $+1$ and -1 eigenspaces under θ , and let $K = \{x \in G \mid \Theta x = x\}$. Then the map $K \times \mathfrak{p}_0 \rightarrow G$ given by $(k, X) \rightarrow k \exp X$ is a diffeomorphism onto.[6]

Proposition 2.7: If G is a reductive Lie group, then

- a) K is a maximal compact subgroup of G
- b) K meets every component of G , i.e., $G = KG_0$
- c) each member of $\text{Ad}(K)$ leaves k and p stable and therefore commutes with θ
- d) $(\text{ad } X)^* = -\text{ad } \theta X$ relative to B_θ if X is in \mathfrak{g}_0
- e) θ leaves $Z_{\mathfrak{g}_0}$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ stable, and the restriction of θ to $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a Cartan decomposition
- f) the identity component G_0 is a reductive Lie group (with maximal compact subgroup obtained by intersection and with Cartan involution and invariant form unchanged).

Proof: For (a) assumes the contrary, and let K_1 be a compact subgroup of G properly containing K . If k_1 is in K_1 but not in K , write $k_1 = k \exp X$. Then $\exp X$ is in K_1 . By

compactness of K_1 , $(\exp X)^n = \exp nX$ has a convergent subsequence in G , but this contradicts the homeomorphism in proposition 2.6.

Conclusion (b) is clear. In (c), $\text{Ad}(K)(\mathfrak{k}) \subseteq \mathfrak{k}$ since K has Lie algebra \mathfrak{k} . Since B is $\text{Ad}(K)$ invariant, $\text{Ad}(K)$ leaves stable the subspace of \mathfrak{g}_0 orthogonal to \mathfrak{k} , and this is just \mathfrak{p} .

For (d) we have

$$\begin{aligned} B_\theta((\text{ad } X)Y, Z) &= -B((\text{ad } X)Y, \theta Z) \\ &= B(Y, [X, \theta Z]) = B(Y, \theta [\theta X, Z]) = B_\theta(Y, -(\text{ad } \theta X)Z), \end{aligned}$$

and (d) is proved. Conclusion (e) follows from the facts that θ is an involution and B_θ is positive definite, and conclusion (f) is trivial.[6]

Proposition 2.8: If G is a reductive Lie group, then the function $\Theta : G \rightarrow G$ defined by

$$\Theta(k \exp X) = k \exp(-X) \quad \text{for } k \in K \text{ and } X \in \mathfrak{p}$$

is an automorphism of G and its differential is θ .

Remark: As in the semisimple case, Θ is called the *global Cartan involution*.

Proof: The function Θ is a well defined diffeomorphism. First, consider its restriction to the analytic subgroup G_{ss} with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. By Proposition 2.7e the Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ has a Cartan decomposition

$$[\mathfrak{g}_0, \mathfrak{g}_0] = ([\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{k}) \oplus ([\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{p}).$$

If K_{ss} denotes the analytic subgroup of G_{ss} whose Lie algebra is the first summand on the right side, then Theorem 1.79 shows that G_{ss} consists exactly of the elements in $K_{\text{ss}} \exp([\mathfrak{g}_0, \mathfrak{g}_0] \cap \mathfrak{p})$ and that Θ is an automorphism on G_{ss} with differential θ .

Next consider the restriction of Θ to the analytic subgroup $(Z_{G_0})_0$. By Proposition 2.7e the Lie algebra of this abelian group decomposes as

$$Z_{\mathfrak{g}_0} = (Z_{\mathfrak{g}_0} \cap k) \oplus (Z_{\mathfrak{g}_0} \cap p)$$

contained in $K \exp(p)$. Thus Θ on $(Z_{G_0})_0$ is the lift to the group of θ on the Lie algebra and hence is an automorphism of the subgroup $(Z_{G_0})_0$.

The subgroup G_{ss} and $(Z_{G_0})_0$ commute, and hence Θ is an automorphism of their commuting product, which is G_0 by the remark after corollary 2.3.

Now consider Θ on all of G , where it is given consistently by $\Theta(kg_0) = k \Theta(g_0)$ for $k \in K$ and $g_0 \in G$. By Proposition 2.7c we have $\theta \text{Ad}(k) = \text{Ad}(k)\theta$ on \mathfrak{g}_0 , from which we obtain $\Theta(k \exp X k^{-1}) = k \Theta(\exp X) k^{-1}$ for $k \in K$ and $X \in \mathfrak{g}_0$. Therefore

$$\Theta(kg_0k^{-1}) = k \Theta(g_0)k^{-1} \quad \text{for } k \in K \text{ and } g_0 \in G_0.$$

On the product of two general elements kg_0 and $k' g'_0$ of G , we therefore have

$$\begin{aligned} \Theta(kg_0k' g'_0) &= \Theta(kk' k'^{-1} g_0k' g'_0) = kk' \Theta(k'^{-1}g_0k' g'_0) \\ &= kk' \Theta(k'^{-1}g_0k') \Theta(g'_0) = k \Theta(g_0) k' \Theta(g'_0) = \Theta(kg_0) \Theta(k' g'_0), \end{aligned}$$

as required.[6]

Proposition 2.9: If G is a reductive Lie group and \mathfrak{h}_0 is a θ stable abelian subalgebra of its Lie algebra, then $Z_G(\mathfrak{h}_0)$ is a reductive Lie group.[6]

Here the maximal compact subgroup of $Z_G(\mathfrak{h}_0)$ is given by intersection, and the Cartan involution and invariant form are given by restriction.

Reductive Lie groups are supposed to have all essential structure-theoretic properties of semisimple groups and to be closed under various operations that allow us to prove theorems by induction on the dimension of the group.

The first remark concerns the Cartan decomposition. The decomposition on the Lie algebra level is built into the definition of reductive Lie group, and the property of the global Cartan decomposition are given partly in Proposition 2.9.

CHAPTER III

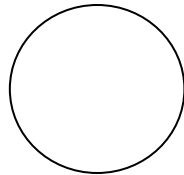
Mostow Fibration

A general problem is to understand the relationship between the real locus of a system of equations and its complex locus . To guess the answer, let us look at some simple equations:

For example, take the equations :

$$\begin{array}{ll} x^2 + y^2 = 1 & z^2 + w^2 = 1 \\ x, y \in \mathbb{R} & z, w \in \mathbb{C} \end{array}$$

the real locus is a circle .



What is the complex locus?

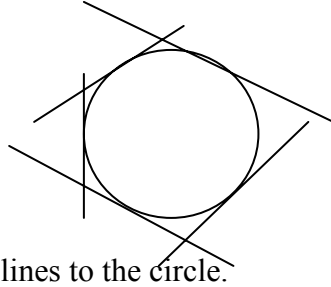
$$z^2 + w^2 = 1$$

$$(z + iw)(z - iw) = 1$$

set $z + iw = z_1$ and $z - iw = z_2$

so $z_1 z_2 = 1$

therefore the complex locus is $C^* = \{z \in \mathbb{C} : z \neq 0\}$



$C^* =$ all tangent lines to the circle.

Similarly we can see that the complex locus of the sphere of dimension n is the tangent bundle ($Q^n = T(S^n)$) that can be deformed into a sphere. [12]

Mostow Fibration explains this situation for homogeneous spaces.

Definition 3.1: If $K \subset \tilde{K}$ and K operates on a manifold F , then K operates on $\tilde{K} \times F$ by

$$k \cdot (x, f) = (xk^{-1}, k \cdot f).$$

The set of orbits of K in $\tilde{K} \times F$ is the space $\tilde{K} \times_K F$. [14]

Theorem (Mostow Fibration) 3.2: Let G be a connected complex reductive Lie group, K a maximal compact subgroup, L closed subgroup of K and $H = L^{\mathbb{C}}$. Let B be an $Ad(k)$ -invariant, positive-definite, symmetric bilinear form then, via the mapping $K \times \mathfrak{p}_0 \rightarrow G$, the homogeneous space G/H can be considered as the K -vector bundle $K \times_L \mathfrak{p}_0 \rightarrow K/L$ over the K -orbit $K_{x_0} = K/L$ of the neutral point. [15]

Let S be the space of symmetric matrices . Define

$$\begin{aligned} \tau_x(y) &= e^{\frac{-x}{2}} \left(\lim_{t \rightarrow 0} \frac{e^{X+tY} - e^x}{t} \right) e^{\frac{-x}{2}} \\ &= e^{\frac{-x}{2}} (d(\exp)_x)(Y) e^{\frac{-x}{2}} \end{aligned}$$

Now if $P =$ space of positive definite matrices and $P(t)$ is any path in P , then

$$P(t) = \exp \log(P(t)), \quad \text{so}$$

$$\dot{P}(t) = (d \exp)_{\log P(t)} [\log P(t)]^\bullet$$

so that

$$\begin{aligned} \text{Tr}[P^{-1} \dot{P}]^2 &= \text{Tr}[P^{\frac{-1}{2}} \dot{P} P^{\frac{-1}{2}}]^2 \\ &= \text{Tr}[e^{\frac{-1}{2} \log P} (d \exp)_{\log P(t)} [\log P(t)]^\bullet e^{\frac{-1}{2} \log P}]^2 \\ &= \text{Tr}[\tau_{\log P(t)} (\log P(t))^\bullet]^2 = \text{Tr}[\tau_{\log} (\log)^\bullet]^2 \end{aligned}$$

This is the reason for the definition of τ_x .

Lemma 3.3: For any $X \in S =$ symmetric matrices,

$$\tau_x = \frac{\text{Sinh}\left(\frac{D_x}{2}\right)}{\left(\frac{D_x}{2}\right)}$$

on S is self-adjoint, where $D_x = R_x - L_x$.

Proof: By definition

$$\begin{aligned} \tau_x(Y) &= e^{\frac{-x}{2}} d(\exp)_{(x)}(Y) e^{\frac{-x}{2}} \\ &= e^{\frac{-x}{2}} \left[\frac{d}{dt} \Big|_{t=0} \exp(X + tY) \right] e^{\frac{-x}{2}}. \end{aligned}$$

Now $(X + tY) \exp(X + tY) = (\exp(X + tY))(X + tY)$.

So differentiating this we obtain:

$$Y \exp(X + tY) + (X + tY)[\exp(X + tY)]^\bullet = [\exp(X + tY)]^\bullet (X + tY) + [\exp(X + tY)]Y$$

So

$$Y \exp(X) + X[d \exp_x(Y)] = [d \exp_x(Y)]X + [\exp(X)]Y$$

$$\therefore Y \exp(X) - \exp(X)Y = [d(\exp)_x(Y)]X - X[d \exp_x(Y)]$$

Hence

$$e^{\frac{-X}{2}} Y \exp(X) e^{\frac{-X}{2}} - e^{\frac{-X}{2}} \exp(X) Y e^{\frac{-X}{2}} = e^{\frac{-X}{2}} d(\exp)_x(Y) e^{\frac{-X}{2}} X - X e^{\frac{-X}{2}} d(\exp)_x(Y) e^{\frac{-X}{2}}$$

or

$$e^{\frac{-X}{2}} Y e^{\frac{X}{2}} - e^{\frac{X}{2}} Y e^{\frac{-X}{2}} = \tau_x(Y) X - X \tau_x(Y)$$

$$\text{So } e^{\frac{-x}{2}} Y e^{\frac{x}{2}} - e^{\frac{x}{2}} Y e^{\frac{-x}{2}} = D_x(\tau_x(y))$$

Since $D_x = R_x - L_x$ we have

$$[\exp(D_x)](Y) = e^{-X} Y e^X$$

So, last equation becomes (considering $D_x = L_x - R_x$)

$$[\exp(\frac{D_x}{2}) - \exp(-\frac{D_x}{2})](Y) = D_x(\tau_x(Y))$$

So

$$D \left(\tau(Y) - \frac{\text{Sinh} \frac{D}{2}}{\frac{D}{2}} \right) (Y) = 0$$

So formally:

$$\frac{\exp(\frac{D}{2}) - \exp(-\frac{D}{2})}{D} = \tau$$

$$\text{or } \frac{\text{Sinh}(\frac{D}{2})}{\frac{D}{2}} = \sum_{n=0}^{\infty} \frac{(\frac{D}{2})^{2n}}{(2n+1)!}$$

Observation 1: D^2 is self-adjoint:

Proof :

$$D_X = (R_X - L_X)$$

$$\text{Tr}[(D_X A)(B)] = \text{Tr}[(AX - XA)B]$$

$$= \text{Tr}(AXB - XAB)$$

$$= \text{Tr}(XBA - XAB) = \text{Tr}(X[B, A])$$

$$\text{Tr}[(AD_X B)] = \text{Tr}(A(BX - XB))$$

$$= \text{Tr}(ABX - AXB) = \text{Tr}(ABX - XBA) = \text{Tr}(X[A, B]).$$

So $\langle D_X A, B \rangle = - \langle A, D_X B \rangle$.

Hence

$$\begin{aligned} \langle D_X^2 A, B \rangle &= - \langle D_X A, D_X B \rangle \\ &= \langle A, D_X^2 B \rangle. \end{aligned}$$

So, D_X^2 is self-adjoint.

Therefore $\frac{\text{Sinh}(\frac{D_X}{2})}{(\frac{D_X}{2})}$ is self-adjoint.

Observation 2:

$$\lim_{t \rightarrow 0} \frac{\exp(X + tA) - \exp(A)}{t} = \lim_{t \rightarrow 0} \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{(X + tA)^n - X^n}{t} \right].$$

Now if $\phi(t) = (X + tA)^n - X^n$, then

$$\phi(t) = \phi(0) + t\phi'(0) + \dots = t\phi'(0) + \dots$$

So $[(X + tA)^n]' = A(X + tA)^{n-1} + (X + tA)A(X + tA)^{n-2} + \dots$

So at $t = 0$

$$\left. \frac{d}{dt} \right|_{t=0} (X + tA)^n = AX^{n-1} + XAX^{n-2}X^2AX^{n-3} + \dots + AX^{n-1}$$

So
$$\lim_{t \rightarrow 0} \left[\frac{\exp(X + tA) - \exp(A)}{t} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\sum_{p+q=n-1} X^p AX^q \right] = d \exp_X(A).$$

$$\begin{aligned} \therefore \text{Tr}[(\tau_X A)(B)] &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p+q=n-1} \text{Tr}(e^{\frac{-X}{2}} X^p AX^q e^{\frac{-X}{2}} B) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p+q=n-1} \text{Tr}(AX^q e^{\frac{-X}{2}} B e^{\frac{-X}{2}} X^p) \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \right) \left(\text{Tr} \sum_{p+q=n-1} A e^{\frac{-X}{2}} X^q B X^p e^{\frac{-X}{2}} \right) = \text{Tr}(A \tau_X B) \end{aligned}$$

So

$\tau = \tau_X$ is self-adjoint on S. Moreover, if $X, Y \in S$, then

$$\begin{aligned} D_X^2(Y) &= D_X(YX - XY) = (YX - XY)X - X(YX - XY) \\ &= YX^2 - XYX - XYX + X^2Y \end{aligned}$$

So D_X^2 maps S into itself and, as already seen, it is self-adjoint. Thus

$$\tau - \frac{\text{Sinh}\left(\frac{D}{2}\right)}{\frac{D}{2}} \text{ is self-adjoint on S.}$$

Let $N = \{ A : A \in S \text{ and } DA = 0 \}$. So, as $D \left[\left(\tau - \frac{\text{Sinh}\frac{D}{2}}{\frac{D}{2}} \right) (Y) \right] = 0$, we see that

$\tau - \frac{(\text{Sinh}(\frac{D}{2}))}{\frac{D}{2}}$ maps S into N. Let N^\perp denote the orthogonal complement of N in S. Then

as $\tau - \frac{(\text{Sinh}(\frac{D}{2}))}{\frac{D}{2}}$ is self-adjoint, $(\tau - \frac{(\text{Sinh}(\frac{D}{2}))}{\frac{D}{2}}) \cap N^\perp \subset N^\perp \cap N = \{0\}$. On the other

hand, if $Y \in N$, then $XY = YX$, $\exp(X + tY) = \exp X \exp tY$. So

$$\tau X = X = \left(\frac{(\text{Sinh}(\frac{D}{2}))}{\frac{D}{2}} \right) (X).$$

Hence, $\tau = \frac{(\text{Sinh}(\frac{D}{2}))}{\frac{D}{2}}$ throughout.

Observation 3: If X is real symmetric with eigenvalues $\lambda_1, \dots, \lambda_n$, then eigenvalues of

$$D_X = -\text{Ad}(X) \text{ are } \lambda_i - \lambda_j, \text{ so eigenvalues of } \tau_X = \frac{(\text{Sinh}(\frac{D}{2}))}{\frac{D}{2}} \text{ are } \frac{\sinh(\lambda_i - \lambda_j)}{(\lambda_i - \lambda_j)}$$

hence as $\frac{\sinh \theta}{\theta} = 1 + \frac{\theta^2}{3!} + \dots$

we see that the eigenvalues of τ_X are all positive and ≥ 1 . So if Z is any symmetric matrix,

then, writing $Z = \sum c_i e_i$, e_i -eigenvectors (orthonormal), we have $\tau Z = \sum c_i \lambda_i e_i$ or

$$\|\tau Z\|^2 = \sum c_i^2 \lambda_i^2 \geq \sum c_i^2 = \|Z\|^2$$

Combining all these facts we obtain the basic result:

Proposition 3.4: Along any differentiable path $p(t)$ in $P =$ positive definite matrices,

$$\text{Tr}([\log p]^\bullet)^2 \leq \text{Tr}\left(p^{-1} \dot{p}\right)^2.$$

Proof :

$P(t) = \exp(\log p(t))$, so $\dot{p}(t) = d(\exp)_{\log p(t)}[\log p(t)]^\bullet$ or briefly $\dot{p} = d(\exp)_{\log p}[\log]^\bullet$.

So $p^{\frac{-1}{2}} \dot{p} p^{\frac{-1}{2}} = e^{\frac{-1}{2} \log p} d(\exp)_{\log p}[\log p]^\bullet e^{\frac{-1}{2} \log p} = \tau_{\log p}[\log p]^\bullet$

Now, writing $\tau = \tau_{\log p}$, $D = D_{\log p}$ we know that $\tau = \frac{(\text{Sinh}(\frac{D}{2}))}{\frac{D}{2}}$ on S , so if the

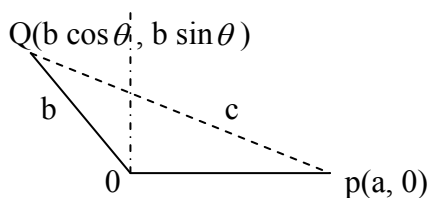
eigenvalues of $\log p$ are $\lambda_1, \dots, \lambda_n$, then those of τ are $\frac{(\text{Sinh}(\frac{\lambda_i - \lambda_j}{2}))}{\frac{\lambda_i - \lambda_j}{2}}$ hence are all ≥ 1 .

Since τ is self-adjoint, by choosing an orthonormal basis of eigenvectors of τ we see that

$$\|[\log]^\bullet\|^2 \leq \|\tau[\log]^\bullet\|^2 = \left\| p^{\frac{-1}{2}} \dot{p} p^{\frac{-1}{2}} \right\|^2$$

i.e. $\text{Tr}([\log p]^\bullet)^2 \leq \text{Tr}\left[p^{\frac{-1}{2}} \dot{p} p^{\frac{-1}{2}}\right]^2 = \text{Tr}\left[p^{-1} \dot{p}\right]^2.$

So the length (Euclidean) of the path $\log p(t)|_{t=a}^{t=b} \leq$ length of the path $p(t)|_{t=a}^{t=b}$.

Some elementary geometry:

in the shown triangle :

$$\begin{aligned} c^2 &= |QP|^2 = [b \cos \theta - a]^2 + [b \sin \theta]^2 \\ &= b^2 + a^2 - 2ab \cos \theta . \end{aligned}$$

Since $\cos(\theta) = \cos(-\theta) = \cos(2\pi - \theta)$, it does not matter whether we take θ clockwise or counterclockwise. Take θ in the interval $0 \leq \theta \leq \pi$.

Since $c^2 < |b|^2 + |a|^2 + 2ab$, we have $|c| < |a| + |b|$.

Conversely, if we are given positive numbers a, b, c such that

$$a \leq b + c, \quad b \leq a + c, \quad c \leq a + b, \quad \text{then assuming } a \geq b \text{ we have}$$

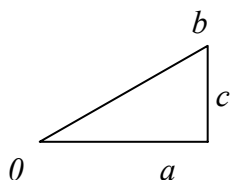
$$a - b \leq c \leq a + b \quad \text{so} \quad a^2 + b^2 - 2ab \leq c^2 \leq a^2 + b^2 + 2ab$$

Then $-1 \leq \frac{c^2 - a^2 - b^2}{2ab} \leq 1$. Hence \exists a unique θ in $[0, \pi]$, such that

$$\frac{c^2 - a^2 - b^2}{2ab} = \cos \theta .$$

Now we take segments OP, OQ of lengths a and b with angle θ between them and

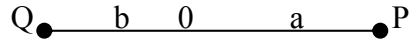
$0 \leq \theta \leq \pi$. Our triangle is



Special Cases: If $|c| = |a| + |b|$, then

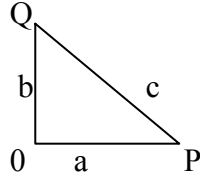
$$|c|^2 = |a|^2 + |b|^2 + 2|a||b| = |a|^2 + |b|^2 - 2|a||b|\cos \theta$$

So $\cos \theta = -1$, hence $\theta = \pi$ and our triangle is



If $|c|^2 = |a|^2 + |b|^2$, then $|c|^2 = |a|^2 + |b|^2 - 2|a||b|\cos \theta \Rightarrow \cos \theta = 0$

So the triangle is



If two angles are $\frac{\pi}{2}$ then $c^2 = a^2 + b^2$, $b^2 = a^2 + c^2$ or $a^2 = c^2 + b^2$. So $a = 0$ or $b = 0$,

hence two of the vertices are coincident.

Proposition 3.5: $e^{t \log p}$ ($0 \leq t \leq 1$) is the unique geodesic in P joining I to $p \in P$. More generally, the geodesic between any two points is unique.

Proof : Let $H = \log p$. Now if $f(t) = e^{tH}$, then

$$\dot{f}(t) = He^{tH}, \text{ so } \left\| \dot{f}(t) \right\|^2 = \text{Tr}(e^{-tH} H e^{tH})^2 = \text{Tr}(H^2)$$

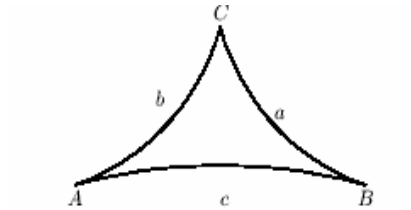
So $\left| \dot{f}(t) \right| = (\text{Tr } H^2)^{\frac{1}{2}} = \|H\|$, then

$$\int_0^1 \left\| \dot{f}(t) \right\| dt = \|H\| = \text{dist}(H, 0) = \text{dist}(\log p, \log I).$$

$\therefore [\text{Tr}(H^2)]^{\frac{1}{2}} \leq \text{length of any path joining } \log p \text{ with } 0 \leq \text{length of any path in } P \text{ joining } p$

with I . So the path $f(t) = e^{tH}$, $0 \leq t \leq 1$ is the unique geodesic joining I with p .

Theorem 3.6: *The riemannian angle between two paths f and g intersecting at I is equal to the Euclidean angle between two paths $\log f$ and $\log g$ intersecting at 0 . Moreover, in any geodesic triangle*



$$c^2 \geq a^2 + b^2 - 2ab \cos \hat{C}$$

Proof:

Notice that if $f(t) = e^{\phi(t)}$, then $\dot{f}(t) = d \exp_{\phi(t)}(\phi'(t))$. So if $f(0) = I$, then $\phi(0) = 0$ and

$\dot{f}(0) = d \exp_0(\phi'(0))$. Therefore, if $\psi(0) = 0$ and $\psi'(0) = g'(0)$ and $\tilde{f}(t) = e^{\psi(t)}$, then

$$\dot{\tilde{f}}(0) = \tilde{f}'(0).$$

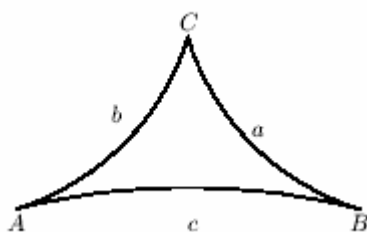
So to compute the angle between curves f, g passing through I , we may replace f by $e^{t f'(0)}$ and g by $e^{t g'(0)}$.

Now

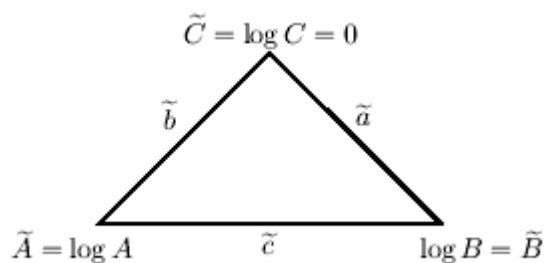
$$\langle f'(0), g'(0) \rangle = \text{Tr}(e^{-1} f'(0) \cdot e^{-1} g'(0)) = \text{Tr}[f'(0) \cdot g'(0)] = \text{Tr}[\phi'(0), \psi'(0)].$$

So, the angle of intersection between f, g at I = angle between $\log f, \log g$ at 0 .

Now suppose ABC is a geodesic triangle, we may suppose that $C = I$.



from here, we get a triangle



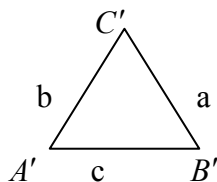
And one has $c^2 \geq (\tilde{c})^2 = (\tilde{a})^2 + (\tilde{b})^2 - 2\tilde{a}\tilde{b} \text{Cos}(\tilde{C})$

But $\tilde{a} = a$, $\tilde{b} = b$ and $\hat{C} = \hat{C}$ by the first part. Hence

$$c^2 \geq a^2 + b^2 - 2ab \text{Cos}(\hat{C})$$

Now construct an euclidean triangle with sides a, b, c

So



$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cos(\hat{C}') \\
&= a^2 + b^2 - 2ab \cos(\hat{C}) \\
\Rightarrow \cos(\hat{C}') &\leq \cos(\hat{C}) \\
\text{So } \hat{C}' &\geq \hat{C}
\end{aligned}$$

Similarly $\hat{A}' \leq \hat{A}$, $\hat{B}' \leq \hat{B}$. Hence, sum of angles in a geodesic triangle $\leq \pi$.

Application to decomposition:

$$g = kp, \quad g^t = p^t k^{-1}, \text{ so } g^t g = p^t p = p^2 \Rightarrow (g^t g)^{\frac{1}{2}}.$$

Now, $k = gp^{-1}$, then $k^{-1} = pg^{-1}$, $k^t = (p^{-1})^t g^t$ so $k^{-1} = k^t \Leftrightarrow pg^{-1} = p^{-1} g^t \Leftrightarrow p^2 = g^t g$.

So $g = kp$ holds. Let $G = G^0$ be a subgroup of $GL(N, R)$ such that

$$\dot{G} = \dot{K} \oplus P, \quad [\dot{K}, P] \subset P, [\dot{K}, \dot{K}] \subset \dot{K}, [P, P] \subset \dot{K} \text{ and } \dot{K} \oplus iP \text{ is a Lie algebra of a}$$

compact group. So $G^0 = G = \langle e^{\dot{K}}, e^P \rangle$ and \dot{K} is represented by skew-symmetric and P by

symmetric matrices. Therefore G is stable under $\sigma = \text{transpose inverse}$. If $g = kp \in G$,

then $\sigma(g) = kp^{-1} \in G$. So $\sigma(g)^{-1} g = p^2 \in G$.

For notational convenience, from now on $\tilde{G} = GL(n, R)$, $\tilde{K} = O(n, R)$. So $\tilde{G} = \tilde{K}\tilde{P}$.

G is a reductive subgroup of \tilde{G} .

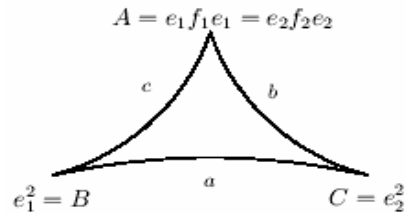
Theorem 3.7: Let $\exp(\tilde{P}) = \text{all positive definite matrices, where } \tilde{p} \text{ is the space of all symmetric matrices and let } F = P^\perp. \text{ Then}$

$$\exp(\tilde{P}) = \{efe : e \in \exp(P), f \in \exp(P^\perp)\}.$$

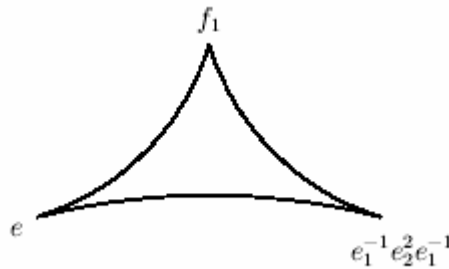
Proof: Consider $\phi : E \times F \rightarrow \exp(\tilde{P})$ such that $\phi(e, f) = efe$, then :

i) ϕ is 1:1

Suppose $e f_1 e_1 = e_2 f_2 e_2$. Consider the geodesic triangle



By the isometry $x \mapsto e_1^{-1} x e_1^{-1}$ this is mapped to



So $\overline{f_1 I} \subset \exp(P^\perp)$ and $\overline{I e_1^{-1} e_2^2 e_1^{-1}}$ is contained in $\exp P$, because $x \mapsto e_1^{-1} x e_1^{-1}$ maps $\exp P$ to $\exp P$. So angle at I = 90 .

\therefore angle at B = 90. Similarly angle at C = 90.

So

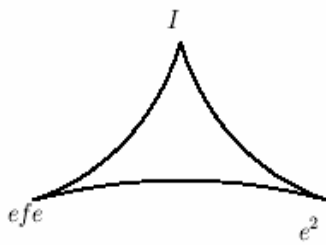
$$\begin{aligned} b^2 &\geq a^2 + c^2 \\ c^2 &\geq b^2 + a^2 \end{aligned}$$

this means that $b^2 = c^2$, $a = 0$. Hence $e_1^2 = e_2^2$, so $e_1 = e_2$.

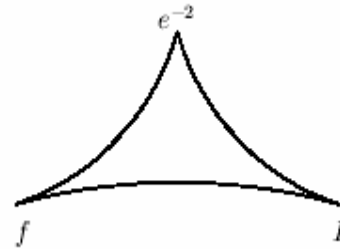
$\therefore f_1 = f_2$ and ϕ is 1:1.

To show closedness, we must estimate $|\phi(e, f)| = d(efe, I)$ in term of $d(e, I)$, $d(f, I)$.

Consider the geodesic triangle



which is isomorphic to



So the angle at $I = 90 \Rightarrow$ the angle at $e^2 = 90$.

$$\begin{aligned} [d(efe, I)]^2 &\geq [d(efe, e^2)]^2 + [d(e^2, I)]^2 \\ &= [d(f, I)]^2 + [2d(e, I)]^2. \end{aligned}$$

So $d(efe, I) \geq \sup \{d(f, I), d(e, I)\}$.

Observation 1: $\phi(E \times F)$ is closed in $\exp \tilde{P}$: If $e_n f_n e_n \rightarrow x \in \exp \tilde{P}$, then

$d(e_n f_n e_n, I) \rightarrow d(x, I)$. So as $d(e_n, I), d(f_n, I) \leq d(e_n f_n e_n, I)$, we see that $\{e_n\}, \{f_n\}$ are bounded, so we can extract convergent subsequences. Hence $e_n f_n e_n$ converges to an element in $\exp \tilde{P}$. So image is closed.

Observation 2: ϕ is an open map:

Since ϕ is continuous and 1:1, and $E \times F$ and P are Euclidean spaces of same dimension, ϕ maps open sets to open sets. So ϕ is open. Also, $\text{im } \phi = P$ as $\text{im } \phi$ is closed. Hence $\phi: E \times F \rightarrow P$ is a homeomorphism.

Theorem 3.8: Any non-singular $n \times n$ matrix can be expressed uniquely and continuously as $k \cdot f \cdot e$ where k is orthogonal, $e \in E, f \in F$.

Proof: Given x , find f such that $x^t x = e f^2 e$, which is possible as $x^t x$ is symmetric and belongs to $\exp(P)$.

Now $x(fe)^{-1}$ is orthogonal for

$$\begin{aligned} (xe^{-1}f^1)^t x(e^{-1}f^1) &= f^1 e^{-1} x^t x e^{-1} f^1 \\ &= f^1 e^{-1} e f^2 e e^{-1} f^1 = id. \end{aligned}$$

So if we set $k = x(fe)^{-1}$, then $x = kfe$, with $e \in E, f \in F, k \in K$ (notice that if G is reductive and $G = \exp(K)\exp(P)$ and $k \in O(n) \cap G$, then $k \in \exp(K)$).

Suppose $x = k_1 f_1 e_1 = k_2 f_2 e_2$ then $x^t = e_1 f_1 k_1^{-1} = e_2 f_2 k_2^{-1}$.

$\therefore xx^t = e_1 f_1^2 e_1 = e_2 f_2^2 e_2$, then as already seen, $e_1 = e_2, f_1^2 = f_2^2$ so $f_1 = f_2$. Hence, uniqueness of decomposition holds. So $(k, f, e) \xrightarrow{\phi} kfe$ is 1:1 onto. In the representation $x = kfe$, e, f depend continuously on $x^t x$, so on x and therefore k also depends continuously on x . Hence ϕ^{-1} is also continuous.

Therefore $\tilde{G} = \tilde{K}FE$. Hence we have a well-defined a map

$$\tilde{K} \times_K F \rightarrow \tilde{G}/G$$

where $G = KP$ is a real subgroup of \tilde{G} , by $\tilde{k} \times f \mapsto \tilde{k}fG$ which is surjective as $E \subset G$.

Suppose $\tilde{k}fG = \tilde{k}_1 f_1 G$, so $\tilde{k}f = \tilde{k}_1 f_1 k e, k \in K, e \in \exp(P)$

$\therefore \tilde{k}f = \tilde{k}_1 k (k^{-1} f_1 k) e$. Hence, by the uniqueness of decomposition

$$\tilde{k} = \tilde{k}_1 k, f = k^{-1} f_1 k, e = I.$$

So $\tilde{k}_1 = \tilde{k}k^{-1}, f_1 = kfk^{-1}$. Hence the map $\tilde{K} \times_K F \rightarrow \tilde{G}/G$ defined as $[\tilde{k} \times f] \mapsto \tilde{k}fG$ is an isomorphism. So $\tilde{G}/G \cong \tilde{K} \times_K F$ even differentiable.

REFERENCES:

- [1] Azad, H and J.J Loeb." Some applications of plurisubharmonic functions to orbits of real reductive groups." Indagationes mathematicae 10 (1999): 473-482.
- [2] Baker, Andrew. Matrix Groups: An introduction to Lie group Theory. Springer-Verlag London 2002.
- [3] Bourbaki, N. Lie Groups and Lie algebras Part 1. Herman Paris 1971.
- [4] De Kerf, B. Lie algebras. Elsevier science Publishers B.V., North-Holand 1990.
- [5] Fleming, W. Functions of Several Variables. Springer-Verlag New York Inc. 1977.
- [6] Knapp, W. Lie groups Beyond an Introduction. Blue Sky Research and Y&Y Inc. 1996.
- [7] Kolman, B. Elementary Linear Algebra. Prentice-Hall International, Inc. 1996.
- [8] Matsushima, Y. Differentiable Manifolds. Marcel Dekker, Inc. New York 1972.
- [9] Munkres, J. Topology a first course. Printece-Hall, Inc Englewood Cliff, New Jersey 1975.
- [10] Spivak, M. A Comprehensive Introduction to Differential Geometry. Publish or Perish, Inc.1970.
- [11] Schwartz, Jacob T. Differential Geometry and Topolgy. Gordon and Breach Sciences Puplishers Inc. 1968.
- [12] Steenrod, Norman. The Topology of Fibre Bundles. Princeton University Press.1951.
- [13] Helgason, Sigurdur. Differential Geometry, Lie Groups, and Symmetric Spaces. American Mathematical Society. 2001.
- [14] Helgason, Sigurdur. Groups and Geometric Analysis: Intgral Geometry, Invariant Differential Operators, and Spherical Functions. Academic Press, Inc. 1984.

[15] Mostow, G. D. "Covariant Fiberrings of Klein spaces." American Journal of mathematics 77 (1955): 247-278.

[16] <http://mathworld.wolfram.com>.

Vita

- Saleh Ali Jawad Al-Bashrawi
- Born in Qatif, Saudi Arabia in 1977
- Received Bachelor of Science (B. S) degree in Mathematical Sciences from King Fahd University of Petroleum and Minerals (KFUPM), Dhahran, Saudi Arabia in 2000
- E-mail: sal_bash@hotmail.com