On the blow-up by Comparison

by

Husain Salem Al-Attas

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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In

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Summary

Chapter 1 contains the concepts of the blow-up and the local existence of the solution of the nonlinear reaction-diffusion equation. We shall briefly discuss the two methods, the maximum principle and the semigroup approach.

Chapter 2 is a summary of the Chipot and Weissler work

$$u_t = \Delta u - |\nabla u|^q + |u|^{p-1}u \quad x \in \Omega \subset R^n, \ t > 0, \ n \geq 1$$

Chapter 3 deals with the one-dimensional nonlinear reaction-diffusion equation of the form $u_t = u_{xx} + u_x^2 \frac{s''(u)}{s'(u)} - \frac{\lambda g(x, s(u), s(u)_x)}{s'(u)}$ where $s(u)$ is a certain positive function of regular variation. If we set $v = s(u)$, then the solution of the v-equation $v_t = \Delta v + f(x, v, \nabla v)$. will converge to a steady state equation under certain conditions. Thus, the solution of the u-equation will blow-up, if the solution of the v-equation or the steady state equation changes sign.

Chapter 4 deals with the general form of the n-dimensional case

$$u_t = \Delta u + f(x, u, \nabla u).$$

In this case, the v-equation will contain a singular term, namely $\frac{|\nabla v|^2}{v}$. By a regularization method we shall study the v-equation and see when its solution has a zero.

خلاصة الرسالة

اسم الطالب الكامل : حسين سالم العطاس

عنوان الرسالة : الانفجار بواسطة المقارنه

التخصص : رياضيات

تاريخ الشهادة : محرم ١٤١٥هـ

من المعروف صعوبة تخمين وقت الانفجار بالنسبه للحلول الموجبه في المعادلات التفاضلية المتفاعله اللاخطيه $U_t = \Delta U + F(X, U, \nabla U)$. الصعوبه تكمن في انه كلما كان الحل غير محدود فان الطرق المعروفه مثل طريقة المبدء الاقصى غير ممكن تطبيقها . لمعالجة هذه الحاله سوف نحاول تحويل المعادله بحيث تكون مجموعة الانفجار مكافئه لمجموعة الاصفار بالنسبه للحل في المعادله الجديده ، ومن ثم بواسطة النظر عند حالة الاستقرار فانه يكون لدينا المقدره على تحديد وجود الانفجار أو عدم وجوده بالنسبه للمعادله الاصليه .

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THESIS ABSTRACT

FULL NAME OF STUDENT	HUSAIN SALEM AL-ATTAS
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It is known that it is difficult to estimate the blow-up time of positive solutions of nonlinear reaction-diffusion equations $u_t = \Delta u + f(x, u, \nabla u)$. The main difficulty lies in the fact that as the solution becomes unbounded, known methods such as the maximum principle method are inapplicable. To remedy this situation, we shall transform the equation so that the blow-up set is equivalent to the set of zeros of a solution of a new problem. Then by looking at the steady state, we shall be able to decide whether we have a blow-up or not for the original equation.

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CHAPTER 1

INTRODUCTION

In applied sciences and in particular mathematics, differential equations play an increasingly important role. In this study, we shall be concerned with a class of reaction–diffusion equations of the form $u_t = \Delta u + f(x, u, \nabla u)$. This will find useful applications in various disciplines such as the nerve impulse and the evolution of a chemical reactor in which reaction and diffusion take place. This can also be generalized to systems of such equations (see [4]). We would like the solution to be bounded in order to control the process and to avoid blow–up from taking place (see [2]). Thus, we need to study the blow–up of the reaction–diffusion equation of the above form.

§1. Blow–Up

Let Ω be an open bounded set in R^n.

Definition (1.1)

A point $(x, T) \in \Omega \times R_+$ is called a finite time blow–up point if there exists a sequence (x_n, t_n) such that $(x_n, t_n) \to (x, T)$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$ (see [6]).

Examples

(1.). Consider the nonlinear ODE

$$u' = u^2$$

$$u(0) = \frac{1}{a} \qquad \text{where} \quad a > 0 \tag{1.1}$$

where the solution is $u = \dfrac{1}{a-x}$. So, $u \to \infty$ as $x \to a$, which means u will blow up as $x \to a$. Hence, we can obtain a local solution on $(0, a)$.

(2). Consider the Chipot–Weissler equation

$$u_t = \Delta u - |\nabla u|^q + |u|^{p-1}u \quad t > 0, \ x \in \Omega$$

$$u(t,x) = 0, \qquad t > 0, \quad x \in \Gamma$$

$$u(0,x) = \phi(x), \qquad x \in \Omega$$

where they prove, for certain p and q, the blow–up occurs (see [4]).

Unfortunately, there are no methods that can be used for the more general parabolic equations of the form $u_t = \Delta u + f(x, u, \nabla u)$. (see [4]). So, we will try to study the equation in its general form by a new and simple method.

§2. Existence

To prove the existence of the solution, we have, in general, two methods: one is the maximum principle by using upper and lower solutions; and the second is based on the semigroup for linear operators.

2.1 The First Method "Maximum Principle"

We shall begin by defining the upper and lower solution of the equation

$$\begin{aligned} u_t - \Delta u &= f(x,u) \quad \text{in} \quad D_T \\ Bu \equiv \alpha_0 \frac{\partial u}{\partial v} + \beta_0 u &= h(x) \quad \text{on } S_T \\ u(0,x) &= u_0(x) \quad \text{in} \quad \Omega \end{aligned} \tag{1.2}$$

where $D_T = (0,T] \times \Omega, \quad S_T = (0,T] \times \partial\Omega.$

Definition (1.2)

A function $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ is called an upper solution of (1.2) if it satisfies the inequalities

$$\begin{aligned} \overline{u}_t - \Delta\overline{u} &\geq f(x,\overline{u}) \quad \text{in } D_T \\ B\overline{u} &\geq h(x) \quad \text{on } S_T \\ \overline{u}(0,x) &\geq u_0(x) \quad \text{in} \quad \Omega \end{aligned} \tag{1.3}$$

Similarly, $\underline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ is called a lower solution if it satisfies all the reversed inequalities in (1.3).

Result

Let $\overline{u}, \underline{u}$ be ordered upper and lower solutions of (1.2) and let

$$-\underline{c}(u_1 - u_2) \leq f(x, u_1) - f(x, u_2) \leq \overline{c}(u_1 - u_2)$$

$$\text{for } \underline{u} \leq u_2 \leq u_1 \leq \overline{u} \quad ((t,x) \in D_T) \tag{1.4}$$

Then the sequences $\{\overline{u}^{(k)}\}, \{\underline{u}^{(k)}\}$ converge monotonically to a unique solution u of (1.2) and $\underline{u} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1)} \leq u \leq \overline{u}^{(k+1)} \leq \overline{u}^{(k)} \leq \overline{u}$ in $\overline{D}_T$.

Proof: See [11]. In our case $u_t = \Delta u - f(x, u, \nabla u)$, we will have the existence by the maximum principle if $|f(x, u, \nabla u)| \leq c(1 + |\nabla u|^2)$ where c is a bounded function (see [1]).

2.2 The Second Method "Semigroups"

There is a second method for obtaining existence of local solution. We shall first define the sectorial operator.

Definition (1.3)

A sectorial operator A, is a linear operator acting in a Banach Space X and

satisfying the following conditions:

1. A is a closed and densely defined operator in X.

2. For some ϕ in $(0, \pi/2)$ and some $M \geq 1$ and real a, the sector

$$S_{a,\phi} = \{\lambda | \phi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\}$$

is in the resolvent set of A, i.e. $(\lambda I - A)^{-1}$ is bounded.

3. $\|(\lambda - A)^{-1}\| \leq M/|\lambda - a|$ for all $\lambda \in S_{a,\phi}$.

Examples

1. A bounded linear operator on a Banach space is sectorial.

2. A is sectorial if it satisfies the following conditions:

   (a) A is a self-adjoint densely defined operator in a Hilbert space.

   (b) A is bounded below.

Theorem (1.1)

If A is a sectorial operator satisfying $\|A(\lambda - A)^{-1}\| \leq c$ for $|\arg \lambda| \geq \phi_0$, $|\lambda| \geq R_0$ for some positive constants R_0, c, and $\phi_0 < \pi/2$ and if B is a linear operator with $D(A) \subset D(B)$ with $\|Bx\| \leq \epsilon\|Ax\| + k\|x\|$ for all $x \in D(A)$ and ϵ, k are positive constants with $\epsilon c < 1$, then $A + B$ is sectorial.

Proof: We want to prove $\|\{\lambda-(A+B)\}^{-1}\| \leq \frac{\text{constant}}{|\lambda|}$. We note that

$$\|B(\lambda-A)^{-1}\| \leq \epsilon\|(A(\lambda-A)^{-1}\| + k\|(\lambda-A)^{-1}\| \leq \epsilon c + \frac{K(1+c)}{|\lambda|}$$

for $|\arg\ \lambda| \geq \phi_0$, $|\lambda| \geq R_0$. So, this implies

$$\|\{\lambda-(A+B)\}^{-1} = \|(\lambda-A)^{-1}\{I-B(\lambda-A)^{-1}\}^{-1}\| \leq \frac{1+c}{|\lambda|}\{1-\epsilon c-\frac{k(1+c)}{|\lambda|}\}^{-1}$$

$$\leq \frac{\text{constant}}{|\lambda|}$$

for $|\arg\ \lambda| \geq \phi_0$ and $|\lambda|$ sufficiently large. So $A+B$ is sectorial.

Definition (1.4)

A family of continuous linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space X is called an analytic semigroup if:

(a) $T(0) = I, T(t)T(s) = T(t+s)$ for $t \geq 0, s \geq 0$

(b) $T(t)x \to x$ as $t \to 0^+$, and for each $x \in X$.

(c) $t \to T(t)x$ is real analytic in t on $0 < t < \infty$ and for each $x \in X$.

Definition (1.5)

The infinitesimal generator of the analytical semigroup $\{T(t)\}$ is defined by $Lx = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x)$ and its domain $D(L)$ consisting of all $x \in X$ for which this limit (in X) exists. We usually write $T(t) = e^{Lt}$.

Theorem (1.2)

If A is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$, where

$$e^{-At} = \frac{1}{2\pi i}\int_\Gamma (\lambda + A)^{-1} e^{\lambda t} d\lambda$$

where Γ is a contour in $\rho(-A)$ with $\arg(\lambda) \to \pm\theta$ as $|\lambda| \to \infty$ for some θ in $(\pi/2, \pi)$.

Also, e^{-At} can be continued analytically into a sector $\{t \neq 0 : |\arg\, t| < \epsilon\}$ containing the positive real axis, and if Re $\sigma(A) > a$, i.e. if Re $\lambda > a$ whenever $\lambda \in \sigma(A)$, then for $t > 0$

$$\|e^{-At}\| \leq Ce^{-at}, \|Ae^{-At}\| \leq \frac{C}{t}e^{-at}$$

for some constant C.

Finally, $\frac{d}{dt}e^{-At} = -Ae^{-At}$ for $t > 0$.

Proof: Without loss of generality, assume $a = 0$ and $\|(\lambda + A)^{-1}\| \leq \frac{M}{|\lambda|} + \delta$ for $|\pi - \arg\lambda| \geq \phi$ for some constants $\delta > 0$, $M > 0$ and $\phi \in (0, \pi/2)$; otherwise replace A by $A - aI$.

Choose θ in $\pi/2 < \theta < \pi - \phi$. Define e^{-At} by the above integral, and note that the integral converges absolutely if $t > 0$. By Cauchy's theorem, the integral

is unchanged when the contour Γ is shifted to the right a small distance: call the shifted contour Γ'. Then for $t \geq 0, s > 0$

$$e^{-At}e^{-As} = (2\pi i)^{-2} \int_\Gamma \int_{\Gamma'} e^{\lambda t}(\lambda I + A)^{-1} e^{\mu s}(\mu I + A)^{-1} d\mu d\lambda$$

$$= (2\pi i)^{-2} \int_\Gamma \int_{\Gamma'} e^{\lambda t + \mu s}(\mu - \lambda)^{-1}\{(\lambda I + A)^{-1} - (\mu I + A)^{-1}\} d\mu d\lambda$$

By using the resolvent identity. But for $\lambda \in \Gamma$, $\mu \in \Gamma'$,

$$\int_\Gamma e^{\lambda t}(\mu - \lambda)^{-1} d\lambda = 0, \ \int_{\Gamma'} e^{\mu s}(\mu - \lambda)^{-1} d\mu = 2\pi i e^{\lambda s}$$

so

$$e^{-At}e^{-As} = (2\pi i)^{-1} \int_\Gamma e^{\lambda(t+s)}(\lambda I + A)^{-1} d\lambda = e^{-A(t+s)}$$

and $\{e^{-At}\}_{t \geq 0}$ is a semigroup.

In fact, for $0 < \epsilon < \theta - \frac{\pi}{2}$, the integral converges uniformly in any compact set for $\{|\arg t| < \epsilon\}$, and this proves the semigroup is analytic.

Also, putting $\mu = \lambda t$ in the integral (with $t > 0$)

$$\|e^{-At}\| = \left\| \frac{1}{2\pi i} \int_\Gamma e^{\mu} \left(\frac{\mu}{t} + A \right)^{-1} \frac{d\mu}{t} \right\|$$
$$\geq \frac{M}{2\pi} \int_\Gamma |e^{\mu}| \frac{|d\mu|}{|\mu|}$$

and

$$\|Ae^{-At}\| \leq \frac{1}{2\pi} \frac{M}{\delta} \int_\Gamma |e^{\mu}| \frac{|d\mu|}{|\mu|} \cdot \frac{1}{t} = \frac{\text{constant}}{t}$$

So, we prove $e^{-At}x \to x$ as $t \to 0^+$ for each $x \in X$. It is sufficient to prove this for $x \in D(A)$, a dense set, since $\|e^{-At}\| \leq C$ for all $t \geq 0$. If $x \in D(A)$ and $t > 0$,

$$e^{-At}x - x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}[(\lambda + A)^{-1} - \lambda^{-1}]x d\lambda$$
$$= -\frac{-1}{2\pi i}\int_\Gamma \lambda^{-1}e^{\lambda t}(\lambda I + A)^{-1}x d\lambda$$

so $\|e^{-At}x - x\| \leq$ constant $\|Ax\|t$. Thus $\{e^{-At}\}_{t\geq 0}$ is a strongly continuous semigroup which extends to an analytic semigroup in $|\arg t| < \epsilon$. If $x \in D(A)$, $t > 0$, then

$$\frac{d}{dt}e^{-At}x + Ae^{-At}x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + A)(\lambda + A)^{-1}x d\lambda = 0$$

So, if $x \in D(A)$, as $t \to 0^+$

$$\frac{1}{t}(e^{-At}x - x) = \frac{-1}{t}\int_0^t e^{-As}Ax ds \to -Ax$$

so $-A$ is contained in the generator G of the semigroup.

To see that $-A$ actually is the generator, define for $\lambda \geq 0$

$$R(\lambda)x = \int_0^\infty e^{-\lambda t}e^{-At}x dt.$$

For any x, $e^{-At}x \in D(A)$ for $t > 0$, and for $\delta > 0$

$$A\int_\delta^\infty e^{-\lambda t}e^{-At}x dt = e^{-\lambda\delta}e^{-A\delta}x - \lambda\int_\delta^\infty e^{-\lambda t}e^{-At}x dt$$

By closedness of A, it follows that

$$R(\lambda)x \in D(A) \subset D(G) \text{ for every } \lambda \geq 0, \quad x \in X.$$

But, if $x \in D(G)$, then $e^{-At}x \in D(G)$ for all $t \geq 0$ and $Ge^{-At}x = \frac{d}{dt}e^{-At}x = e^{-At}Gx$, and a similar argument shows

$$R(\lambda)(\lambda - G)x = x \text{ for } x \in D(G).$$

Thus $F(G) \subset$ range of $R(\lambda) \subset D(A)$, hence $-A = G$ as claimed.

Remark

The converse is also true: if $-A$ generates an analytic semigroup, then A is sectorial.

Definition (1.6)

Suppose A is a sectorial operator and Re $\sigma(A) > 0$; then for any $\alpha > 0$

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt$$

where Γ is the usual Gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{for} \quad x > 0.$$

Examples

(1) If A is a positive scalar ($X = R^1$), then $A^{-\alpha}$ as defined is the usual $(-\alpha)$ power of A.

(2) If A is a positive definite self-adjoint operator in a Hilbert space with a spectral representation $A = \int_0^\infty \lambda dE(\lambda)$, then

$$A^{-\alpha} = \int_0^\infty \lambda^{-\alpha} dE(\lambda).$$

(3) If $A = I + B$ where $\|B\| \leq 1$, then $A^{-\alpha}$ as defined above agrees with the usual power series representation: $(I+B)^{-\alpha} = \sum_{n=0}^{\infty} \binom{-\alpha}{n} B^n$, where $\binom{-\alpha}{n} = (-1)^n \frac{\Gamma(\alpha+n)}{n!\Gamma(\alpha)} = (-1)^n \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}$

(4) A^{-1} (the case $\alpha = 1$) is the inverse of A.

Theorem 1.3

If A is a sectorial in X with $\operatorname{Re}\sigma(A) > 0$, then for any $\alpha > 0$, $A^{-\alpha}$ is a bounded linear operator on X which is one-one and satisfies $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ whenever $\alpha > 0, \beta > 0$. Also, for $0 < \alpha < 1$

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1} d\lambda.$$

Proof. For some $\delta > 0$, $\operatorname{Re}\sigma(A) > \delta$, so by Theorem 1.2,

$$\|e^{-At}\| \leq Ce^{-\delta t} \text{ for } t > 0.$$

So

$$\|A^{-\alpha}x\| \leq \frac{1}{\Gamma(\alpha)} \int_0^\infty ct^{\alpha-1}e^{-\delta t}\, dt \|x\|,$$

and $A^{-\alpha}$ is bounded when $\alpha > 0$. Also, for $\alpha > 0, \beta > 0$

$$\begin{aligned} A^{-\alpha}A^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty\int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-A(t+s)}dsdt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty d\mu\int_0^t t^{\alpha-1}(\mu-t)^{\beta-1}e^{-A\mu}dt \\ &= A^{-(\alpha+\beta)}, \text{ using } \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

Also, if $A^{-\alpha}x = 0$ for some $\alpha > 0$, then for integer $n > \alpha$, $A^{-n}x = A^{-(n-\alpha)}A^{-\alpha}x = 0$; but A^{-1} is one–one, so $A^{-n} = n$–th power of A^{-1} is also one-one, so $x = 0$.

Finally,

$$(\lambda + A)^{-1} = \int_0^\infty e^{-At}e^{-\lambda t}dt \text{ for } \lambda \geq 0,$$

So,

$$\begin{aligned} \int_0^\infty \lambda^{-\alpha}(\lambda+A)^{-1}d\lambda &= \int_0^\infty e^{-At}\left(\int_0^\infty \lambda^{-\alpha}e^{-\lambda t}d\lambda\right)dt \\ &= \int_0^\infty e^{-At}t^{\alpha-1}\Gamma(1-\alpha)dt = \frac{\pi}{\sin\pi\alpha}A^{-\alpha} \end{aligned}$$

Using the fact $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin\pi\alpha}$ for $0 < \alpha < 1$, (see 9).

Theorem 1.4

Suppose A is a sectorial and Re $\sigma(A) > \delta > 0$. For $\alpha \geq 0$ there exists $C_\alpha < \infty$ such that $\|A^\alpha e^{-At}\| \leq C_\alpha t^{-\alpha}e^{-\delta t}$ for $t > 0$, and if $0 < \alpha \leq 1$, $x \in D(A^\alpha)$,

$$\|(e^{-At} - 1)x\| \leq \frac{1}{\alpha}C_{1-\alpha}t^\alpha\|A^\alpha x\|$$

Also, C_α is bounded for α in any compact interval of $(0, \infty)$.

Proof. See [9, p. 26].

Theorem 1.5.

If $0 \leq \alpha \leq 1$, $x \in D(A)$, then $\|A^\alpha x\| \leq C\|Ax\|^\alpha \, \|x\|^{1-\alpha}$, i.e.

$$\|A^\alpha x\| \leq \epsilon\|Ax\| + C' \, \epsilon^{-\alpha/(1-\alpha)}\|x\| \text{ for all } \epsilon > 0.$$

Proof. See [9, p.. 26–27].

Definition (1.7)

A function f in a domain D is called uniformly Holder continuous in D with exponent $\alpha, 0 < \alpha \leq 1$, if for all $x, y \in D$

$$|f(x) - f(y)| \leq k|x - y|^\alpha \text{ for some } k > 0$$

If $\alpha = 1$ we also say that f is Lipschitz continuous (see 5).

Moreover, f is locally Holder continuous in t and locally Lipschitz in x on U, if f satisfies the following condition:

If $(t_1, x_1) \in U$, there exists a neighborhood $V \subset U$ of (t_1, x_1) such that for $(t, x) \in V, (s, y) \in V$,

$$\|f(t, x) - f(s, y)| \leq L(|t - s|^\theta + \|x - y\|_\alpha),$$

for some constants $L > 0$, $\theta > 0$.

Lemma 1.6

Let $f : (0,T) \to X$ be locally Holder continuous with $\int_0^\rho \|f(s)\|ds < \infty$ for some $\rho > 0$. For $0 \le t \le T$, define

$$F(t) = \int_0^t e^{-A(t-s)} f(s)ds.$$

Then $F(\cdot)$ is continuous on $[0,T)$, continuously differentiable on $(0,T)$, with $F(t) \in D(A)$ for $0 < t < T$, and $\frac{dF(t)}{dt} + AF(t) = f(t)$ on $0 < t < T$, $F(t) \to 0$ in X as $t \to 0^+$.

Proof. For small $\rho > 0$, define

$$F_\rho(t) = \int_0^{t-\rho} e^{-A(t-s)} f(s)ds, \quad \rho \le t < T,$$

with $F_\rho(t) = 0$ for $0 \le t \le \rho$.

Then (setting $f(s) = 0$ for $s < 0$)

$$\|F(t) - F_\rho(t)\| \le \int_{t-\rho}^t \|e^{-A(t-s)}\| \, \|f(s)\|ds$$

which tends to 0 as $\rho \to 0^+$, uniformly in $0 \le t \le t_0$ for any $t_0 < T$. Also, F_ρ is continuous, since

$$F_\rho(t+h) - F_\rho(t) = (e^{-Ah} - I) \int_0^{t-\rho} e^{-A(t-s)} f(s)ds + \int_{t-\rho}^{t+h-\rho} e^{-A(t+h-s)} f(s)ds$$

$(0 \leq t \leq t+h \leq t_0)$, which tends to zero as $h \to 0$.

Therefore, F is continuous on $[0,T)$ into X, and

$$\|F(t)\| \leq \int_0^t \|e^{-A(t-s)}\| \, \|f(s)\| ds \to 0 \quad \text{as } t \to 0^+$$

Also, if $0 \leq s < t$, then $e^{-A(t-s)}f(s)$ is in $D(A)$, so the Riemann sums for $F_\rho(t)$, $\sum_{t-s_j \geq \rho} e^{-A(t-s_j)}f(s_j)\Delta s_j$, are in $D(A)$ and

$$\lim_{\Delta s \to 0} A \sum_{s \leq t-\rho} e^{-A(t-s)}f(s)\Delta s = \int_0^{t-\rho} e^{-A(t-s)}f(s)ds.$$

So, by closedness of $A, F_\rho(t) \in D(A)$ and

$$AF_\rho(t) = \int_0^{t-\rho} Ae^{-A(t-s)}f(s)ds = \int_0^{t-\rho} Ae^{-A(t-s)}\{f(s)-f(t)\}ds + \{e^{-A\rho}-e^{-At}\}f(t)$$

Now $\|Ae^{-A(t-s)}\| = O((t-s)^{-1}), \|f(s)-f(t)\| = O(|t-s|^\theta)$ for some $\theta > 0$ as $s \to t^-$, hence as $\rho \to 0^+$,

$$AF_\rho(t) \to \int_0^t Ae^{-A(t-s)}\{f(s)-f(t)\}ds + \{I - e^{-At}\}f(t).$$

Thus, again by closedness of $A, F(t) \in D(A)$ for $0 < t < T$.

Consider any strictly interior interval $[t_0, t_1]$, $0 < t_0 < t_1 < T$; then $AF_\rho(t) \to AF(t)$ uniform'y on $t_0 \leq t \leq t_1$, since $\|f(t)-f(s)\| \leq k|t-s|^\theta$ for t, s in $[t_0, t_1]$ and some $\theta > 0$, so

$$\|AF_\rho(t) - AF(t)\| = \|\{-I + e^{-A\rho}\}f(t) + \int_{t-\rho}^t Ae^{-A(t-s)}\{f(s)-f(t)\}ds\|$$

$$\leq \|\{e^{-A\rho} - I\}f(t)\| + C\int_{t-\rho}^t (t-s)^{-1+\theta}ds \to 0$$

as $\rho \to 0^+$, uniformly in $t_0 \leq t \leq t_1$.

Finally, $F_\rho(t)$ is differentiable when $t < \rho$, with $\frac{dF_\rho(t)}{dt} = -AF_\rho(t) + e^{-A\rho} f(t-\rho)^\rho, \quad \rho < t < T$. The rightside converges uniformly to $-AF(t) + f(t)$ on $t_0 \leq t \leq t, \quad (0 < t_0 < t_1 < T)$ as $\rho \to 0^+$, so F is continuously differentiable on the open interval $(0, T)$, with $\frac{dF}{dt} + AF = f(t)$.

Theorem 1.7

Suppose A is a sectorial operator in X, $x_0 \in X$, $f : (0,T) \to X$ is locally Holder continuous and $\int_0^\rho \|f(t)\| dt < \infty$ for some $\rho > 0$; then there exists a unique (strong) solution $x(\cdot)$ of

$$\frac{dx}{dt} + Ax = f(t), \quad 0 < t < T; x(0) = x_0,$$

namely

$$x(t) = e^{-At} x_0 + \int_0^t e^{-A(t-s)} f(s) ds.$$

Now consider the nonlinear equation

$$\frac{dx}{dt} + Ax = f(t, x), \quad t > t_0 \qquad x(t_0) = x_0. \tag{1.5}$$

where we assume A is a sectorial operator so that the fractional powers of $A_1 = A + aI$ are well defined, and the spaces $X^\alpha = D(A_1^\alpha)$ with the graph norm $\|x\|_\alpha = \|A_1^\alpha x\|$ are defined for $\alpha \geq 0$. We assume f maps some open set U in $R \times X^\alpha$ into

X, for some α in $0 \leq \alpha < 1$, and f is locally Holder continuous in t and locally Lipschitz in X on U.

Definition (1.8).

A solution of the above problem on (t_0, t_1) is a continuous function $x : [t_0, t_1) \to X$ such that $x(t_0) = x_0$ and on (t_0, t_1) we have $(t, x(t)) \in U, x(t) \in D(A), \frac{dx}{dt}(t)$ exists, $t \to f(t, x(t))$ is locally Holder continuous, and $\int_{t_0}^{t_0+\rho} \|f(t, x(t))\| dt < \infty$ for some $\rho > 0$, and the differential equation (1.5) is satisfied on (t_0, t_1).

Lemma 1.8

If x is a solution of (1.5) on

$$(t_0, t_1), \text{ then } x(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^{t} e^{-A(t-s)} f(s, x(s)) ds. \tag{1.6}$$

Conversely, if x is a continuous function from (t_0, t_1) into X^α, and $\int_{t_0}^{t_0+\rho} \|f(s, x(s))\| ds < \infty$ for some $\rho > 0$, and if the integral equation (1.6) holds for $t_0 < t < t_1$, then $x(\cdot)$ is a solution of the differential equation (1.5) on (t_0, t_1).

Proof. The first claim is immediate from the definition of the solution and Theorem 1.7. Suppose x is a solution of the integral equation (1.6) and $x \in C((t_0, t_1); X^\alpha)$. We first prove x is locally Holder continuous from (t_0, t_1) to X^α.

If $t_0 < t < t+h < t_1$, then

$$x(t+h) - x(t) = (e^{-Ah} - I)e^{-A(t-t_0)}x_0 + \int_{t_0}^{t}(e^{-Ah} - I)e^{-A(t-s)}f(s,x(s))ds$$
$$+ \int_{t}^{t+h} e^{-A(t+h-s)}f(s,x(s))ds.$$

Now if $0 < \delta < 1-\alpha$, then for any $z \in X$,

$$\|(e^{-Ah} - I)e^{-A(t-s)}z\|_\alpha \leq C(t-s)^{-(\alpha+\delta)}h^\delta e^{a(t-s)}\|z\|.$$

(Theorem 1.4) hence for $t \in [t_0^*, t_1^*] \subset (t_0, t_1)$,

$$\|x(t+h) - x(t)\|_\alpha \leq \text{ constant } h^\alpha.$$

It follows that $t \to f(t,x(t))$ is locally Holder continuous on (t_0, t_1), so by Th. 1.7 x solves the linear equation

$$\frac{dv}{dt} + Ay = f(t,x(t)) \text{ on } t_0 < t < t_1, y(t_0) = x_0$$

hence x is also a solution of (1.5) on (t_0, t_1).

Theorem 1.9

Assume A is a sectorial operator, $0 \leq \alpha < 1$, and $f : U \to X$, U an open subset of $R \times X^\alpha$, $f(t,x)$ is locally Holder continuous in t, locally Lipschitzian in x; then for any $(t_0, x_0) \in U$ there exists $T = T(t_0, x_0) > 0$ such that (1.5) has a unique solution x on $(t_0, t_0 + T)$ with initial value $x(t_0) = x_0$.

Proof. See [9, p. 54]. Also, for more information, see [9].

CHAPTER 2

THE CHIPOT AND WEISSLER PROBLEM

§1. Introduction

Several treatments of the semilinear parabolic equation were concerned with obtaining conditions for the blow-up of positive solution. The main method used to study the blow-up is the classical energy method, which basically means estimating the norm of the solution as time evolves. In this chapter, we shall recall a well-known treatment by Chipot and Weissler concerning the semi-linear heat equation of the form

$$\begin{aligned} u_t &= \Delta u - |\nabla u|^q + |u|^{p-1}u, \quad t > 0,\ x \in \Omega \\ u(t,x) &= 0, \quad t > 0,\ x \in \Gamma \\ u(0,x) &= \phi(x) \quad x \in \Omega \end{aligned} \tag{2.1}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary Γ, $u = u(t,x)$. It is proven that if $1 < q < \dfrac{2p}{p+1}$, then there exists a suitable initial value ϕ so that the corresponding solution of (2.1) blows up in finite time. Although the method we develop is more general, we shall work with the Chipot–Weissler example (in chapter 4) to test and verify our method.

§2. The Energy Method

We want to deal with the equation of Chipot and Weissler $u_t = \Delta u - |\nabla u|^q + |u|^{p-1}u$. Let $J_1(u) = -|\nabla u|^q, J_2(u) = |u|^{p-1}u$ and if we let $J(u) = J_1(u) + J_2(u)$, then (2.1) will become

$$u_t - \Delta u = J(u).$$

Then by the variation of parameters,

$$u(t) = e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta}\, J(u(s))ds$$

where the domain of the generators of $e^{t\Delta}$ in L^s is

$$D_s(\Delta) = W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega).$$

We will construct a local theory for (2.1) in the Banach space $W_0^{1,s}(\Omega)$ by using a fixed point argument.

Note: We want $J_1 : W_0^{1,s} \to L^{r_1}$, so we choose r_1 to be equal to $\frac{s}{q}$. Also, we want $J_2 : W_0^{1,s} \to L^{r_2}$, so we choose r_2 such that $W_0^{1,s}$ is embedded in $L^{r_2 p}$ so the spaces will be sufficient because $u \in W_0^{1,s}$. Then $J(u)$ will bring it to L^r where $1 \le r \le s < \infty$ and either r_1 or r_2, then $e^{t\Delta}$ will bring us back to $L^r \to W_0^{1,s}$. In this case $u : W_0^{1,s} \xrightarrow{J} L^r \xrightarrow{e^{t\Delta}} W_0^{1,s}$. Now let us return to the equation (2.1)

$$u_t = \Delta u - |\nabla u|^q + |u|^{p-1}u$$

and multiply both sides by u and integrate over Ω to obtain

$$\int_\Omega u_t u dx = \int_\Omega (u\Delta u - |\nabla u|^q u + |u|^{p-1}u^2)dx$$

or

$$\frac{1}{2}\frac{d}{dt}\int u^2 dx = \int_\Omega (u\Delta u - |\nabla u|^q u + |u|^{p-1}u^2)dx.$$

Now we will try to simplify the right hand side by taking the first part of it, which is the integral

$$\int_\Omega u\Delta u dx$$

Note:

$$\begin{aligned}
\int_\Omega v\Delta u dx &= \sum_{i=1}^n \int_\Omega u_{x_i x_i} v dx_1 dx_2 \cdots dx_i \cdots dx_n \\
&= \sum_{i=1}^n \int_\Omega v du_{x_i} dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\
&= \sum_{i=1}^n \int_\Omega d(u_{x_i} v) dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\
&\quad - \int u_{x_i} dv dx_1 dx_2 \cdots dx_{i-1} dx_{i+1} \cdots dx_n
\end{aligned}$$

Then by Stoke's theorem

$$\begin{aligned}
\int_\Omega v\Delta u dx &= \sum_{i=1}^n \left(\int_{\partial\Omega} u_{x_i} v dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n - \int_\Omega u_{x_i} dv dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \right) \\
&= \int_{\partial\Omega} v \nabla u ds - \sum_{i=1}^n \int_\Omega u_{x_i} v_{x_i} dx_1 \cdots dx_i \cdots dx_n
\end{aligned}$$

where $dv = \sum v_{x_i} dx_i$.

Then

$$\int_\Omega v\Delta u dx = \int_{\partial\Omega} v \nabla u ds - \int_\Omega \nabla u \nabla v ds.$$

So,

$$\int_\Omega u\Delta u dx = \int_{\partial\Omega} u \nabla u ds - \int_\Omega |\nabla u|^2 dx.$$

From

$$u|_{\partial\Omega} = 0, \int_{\partial\Omega} u \nabla u ds = 0.$$

So equation (2.1) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx = \int_\Omega (-|\nabla u|^2 - |\nabla u|^q u + |u|^{p-1}u^2) dx.$$

Or we can write it

$$\frac{d}{dt}\|u\|_2^2 = -2\|\nabla u\|_2^2 - 2\langle u, |\nabla u|^q\rangle + 2\|u\|_{p+1}^{p+1}. \tag{2.2}$$

Now, let us define the energy of the solution $u(t)$ as

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1}.$$

Lemma 2.1

The energy of the solution is a non-increasing function t of $[0, T_\phi]$ where T_ϕ is the existence time of the solution starting at ϕ.

Proof: Since $u \in C^1([0, T_\phi), W_0^{1,s})$ and because $W_0^{1,s} \subset H_0^1$ and $W_0^{1,s} \subset L^\infty$, then

it is easily shown that $E(u(t))$ is a C^1 function of $t \in [0, T_\phi]$, and also

$$\frac{d}{dt}E(u(t)) = \langle -\Delta u(t), u'(t)\rangle - \langle u(t)^p, u'(t)\rangle$$

$$= -\langle u'(t) + |\nabla u(t)|^q, u'(t)\rangle \leq 0$$

Lemma 2.2.

Suppose $E(\phi) = E(u(0)) \leq 0$ and $q \leq \dfrac{2p}{p+1}$. Then for all $t \in [0, T_\infty)$

$$|\langle u(t), |\nabla u(t)|^q\rangle| \leq \left(\frac{2}{p+1}\right)^{q/2} C(p,q)\|u(t)\|_{p+1}^{p+1-\alpha}$$

where $\alpha = p - \dfrac{q(p+1)}{2} \geq 0$ and $C(p,q) = 1$ if $q = \frac{2p}{p+1}$.

Proof. See ([4]).

Theorem 2.1.

Let $1 < q \leq \dfrac{2p}{p+1}$ and let $\phi \in W^{3,s}(\Omega)$ for s sufficiently large, ϕ not identically zero. Also, suppose the following

(i) $\phi = 0$ on Γ.

(ii) $\Delta\phi - |\nabla \phi|^q + |\phi|^{p-1}\phi = 0$ on Γ.

(iii) $\phi \geq 0$ in Ω.

(iv) $\Delta\phi - |\nabla \phi|^q + \phi^p \geq 0$ in Ω.

(v) $E(\phi) \leq 0$.

(vi) If $q < \dfrac{2p}{p+1}$, then $\|\phi\|_{p+1}$ is sufficiently large.

(vii) If $q = \dfrac{2p}{p+1}$, then p is sufficiently large.

Then the corresponding solution of (2.1) blows up in finite time, in the L^∞ norm.

Proof. From equation (2.2), we can do the following

$$\begin{aligned}
\frac{d}{dt}\|u\|_2^2 &= -2\| \nabla u\|_2^2 - 2\langle u, | \nabla u|^q\rangle + \frac{2p+2}{p+1}\|u\|_{p+1}^{p+1} \\
&= -2\| \nabla u\|_2^2 + \frac{u}{p+1}\|u\|_{p+1}^{p+1} + 2\left(\frac{p-1}{p+1}\right)\|u\|_{p+1}^{p+1} - 2\langle u, | \nabla u|^q\rangle \\
&= -4E(t) + 2\left(\frac{p-1}{p+1}\right)\|u\|_{p+1}^{p+1} - 2\langle u, | \nabla u|^q\rangle.
\end{aligned}$$

Then by Lemmas 2.1 and 2.2

$$\begin{aligned}
\frac{d}{dt}\|u\|_2^2 &\geq 2\left(\frac{p-1}{p+1}\right)\|u\|_{p+1}^{p+1} - 2\left(\frac{2}{p+1}\right)^{q/2}\|u(t)\|_{p+1}^{p+1-\alpha} \\
&\geq 2\|u\|_{p+1}^{p+1}\left[\left(\frac{p-1}{p+1}\right) - \left(\frac{2}{p+1}\right)^{q/2} C(p,q)\|u\|_{p+1}^{-\alpha}\right]. \qquad (2.3)
\end{aligned}$$

Let $F(t) = \|u\|_2^2$. Then (2.3) will become

$$F'(t) \geq CF(t)^{\frac{p+1}{2}}\left[\left(\frac{p-1}{p+1}\right) - \left(\frac{2}{p+1}\right)^{q/2} C(p,q)\|u(t)\|_{p+1}^{-\alpha}\right].$$

If $q < \dfrac{2p}{p+1}$ and $\|\phi\|_{p+1}$ is sufficiently large, then

$$\left(\frac{p-1}{p+1}\right) - \left(\frac{2}{p+1}\right)^{q/2} C(p,q)\|\phi\|_{p+1}^{-\alpha} \equiv k > 0.$$

Then because $u'(t) \geq 0$

$$\|\phi\|_{p+1}^{-\alpha} \geq \|u(t)\|_{p+1}^{-\alpha}$$

and this implies

$$F'(t) \geq CF(t)^{\frac{p+1}{2}} \left[\left(\frac{p-1}{p+1} \right) - \left(\frac{2}{p+1} \right)^{q/2} C(p,q) \|\phi\|_{p+1}^{-\alpha} \right]$$

So $F'(t) \geq CkF(t)^{\frac{p+1}{2}}$.

If we let $\beta = \frac{p+1}{2}$, ten $\dfrac{F'(t)}{F(t)\beta} \geq B$ where $B = ck$ and by taking integration of both sides

$$\frac{1}{1-\beta} F(t)^{1-\beta} - \frac{1}{1-\beta} (F(0))^{1-\beta} \geq Bt.$$

If we let $a = \frac{1}{1-\beta}(F(0))^{1-\beta}$, then

$$(\beta - 1)(a - Bt) \geq F^{(1-\beta)}.$$

So,

$$F \geq \left(\frac{1}{(\beta-1)(a-Bt)} \right)^{\frac{1}{\beta-1}}$$

which means $\exists\, t = \frac{a}{B}$ such that $F \to \infty$, and this implies that $\exists\, t$ such that the solution will blow up in finite time.

CHAPTER 3

THE ONE–DIMENSIONAL CASE

In chapters 3 and 4, we shall discuss a new method that helps to show the existence of the blow–up of the solution. In this chapter, we will be concerned with the equation of the form

$$u_t = u_{xx} - \frac{\lambda g(x, s(u), s(u)_x)}{s'(u)} + u_x^2 \frac{s''(u)}{s'(u)} \tag{3.1}$$

where $s(u)$ is a certain positive function to be defined later. However, sometimes it is not possible to choose a suitable function $s(u)$ to write the equation in the form (3.1). So, we will deal with the equation $u_t = u_{xx} + \lambda f(x, u, u_x)$ and we will discuss this case in chapter 4 with the n–dimensional equation of the form

$$u_t = \Delta u + f(x, u, \nabla u)$$

In both forms, we will study necessary and sufficient conditions for the blow–up to exist.

As we shall see, the method comes because of the difficulty, sometimes, in estimating the norm of the solution as time evolves. So, we use a suitable change of variables that will exchange infinity with zero. Thus, we will have a blow–up if the solution changes sign which can be easily determined.

§1. The Method

For the sake of simplicity, we shall explain the method by discussing the one-dimensional case. In this case, the equation reads

$$
\begin{aligned}
u_t &= u_{xx} - \frac{\lambda g(x, s(u), s(u)_x)}{s'(u)} + u_x^2 \frac{s''(u)}{s'(u)} \qquad -1 < x < 1, \ \ t > 0 \\
u(x,0) &= u_0(x) \\
u(-1,t) &= u(1,t) = 0
\end{aligned} \tag{3.2}
$$

which we call the u–equation. The method needs several steps. In the first step we transform the u–equation by a suitable substitution $u = s(v)$. After that, we study the existence of a steady state solution. If the solution of the steady state equation exists and changes sign, then we will have a blow–up in the solution of the u–equation. The above discussion is concerned with the method briefly. Now, we deal with the method in more details.

1. Substitution

Assume that $u(x,t)$ is a positive solution of the u–equation and let $v(x,t) = s(u(x,t))$ where $s(u)$ is a certain decreasing differentiable positive function from R_+ to R_+ such that $s(\infty) = 0$ and $s(0) = 1$. Then $v(x,t)$ is a solution of the equation

$$v_t = v_{xx} - \lambda g(x, v, v_x)$$

$$v(-1,t) = v(1,t) = 1$$

$$v(x,0) = h(x) = s(u(x,0)) > 0. \tag{3.3}$$

We shall refer to (3.3) as the v–equation

2. Existence

We shall use Theorem 3.1 to prove the existence of the solution of u– and v–equations.

Theorem 3.1

If $h(x)$ is an L^∞–function on $[a,b]$ and g is a abounded function on $(a,b) \times S$ where S is compact in R whose sup–norm satisfies:

$$\sup\{|g(x,u,p)| \;\; |x \in [a,b],\; |u| \leq N, \quad p \in R\} \leq M_{S,N}(1+|p|)^{2\beta})$$

for some $M_{S,N} > 0$ depending on S and N for some $\beta < 1$. Moreover, F is continuous with respect to u and p, uniformly in the following sense: for every $\epsilon > 0$ and every compact set S in R, there is a $\delta > 0$ such that for all $(u,p) \in S$ and $(v,q) \in S$ with $|u-v| + |p-q| < \delta$.

$$\sup_{x\in[a,b]} |g(x,u,p) - g(x,u,q)| \leq t^{-\beta}\epsilon$$

Then the problems

$$u_t(x,t) = u_{xx}(x,t) + g(x,u(x,t),u_x(x,t)) \quad a < x < b$$

$$u(0) = h(x)$$

$$u(a,t) = u(b,t) = 0$$

has a generalized solution on $[a,b] \times [0,s]$ for some $s < T$.

Proof. See Theorem 3.4 in [5].

3. Steady State

We would like now to find sufficient conditions so that a solution to the v–equation converges to a steady state solution.

Theorem 3.2

If

1. $g(x,v,w)$ is locally Lipshitz in v and w.

2. $h''(x) - g(x,h,h') \leq 0$.

3. $\inf v(x,t) > -\infty$.

4. $\sup\ |v_x(x,t)| < \infty$ or

$$g(x,y,w) > -r(y)q(w^2)$$

where $r(u)$ and $q(u_x^2)$ are continuous and $\int^{\infty} \frac{ds}{q(s)} = \infty$.

Then the solution converges uniformly to the steady state $y(x)$, i.e.

$$y''(x) = \lambda g(x, y(x), y'(x))$$

$$y(-1) = y(1) = 1.$$

Proof. See the Chan–Kwong paper [3].

To prove the existence of the solution $y(x)$ of the steady state equation, we will use the following theorem.

Theorem 3.3

If

a) $g(x, R, 0) > 0$ and $g(x, -R, 0) < 0$ for some $R > 0$.

b) There exists a positive function $\psi \in C^1[0, \infty)$ such that $\int_0^\infty \frac{p}{\psi(p)} dp = \infty$ and $|g(x, y, y')| < \psi(|y'|)$ for $|y| \leq R$. Then

$$y'' = g(x, y, y')$$

$$y(a) = 0 = y(b)$$

has at least one solution $y(x)$ satisfying $|y(x)| \leq R$.

Proof. See Theorem V.10 in [7].

To complete our work, we want to discuss when the solution $y(x)$ will change sign.

4. Change of Sign

Let $\phi(x) = 1 - ew(x)$ where

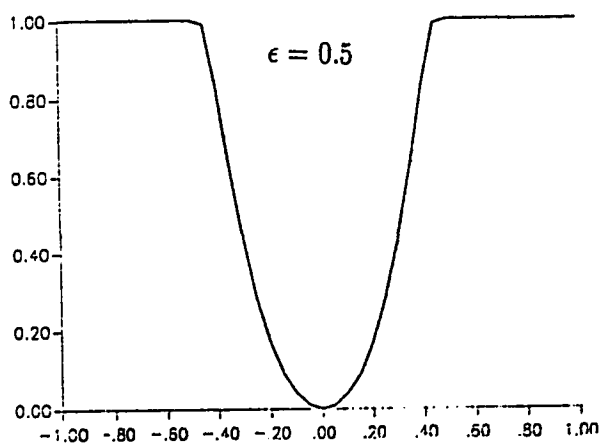
$$w(x) = \begin{cases} e^{\frac{-\epsilon^2}{\epsilon^2 - x^2}} & |x| \le \epsilon \\ 0 & |x| > \epsilon \end{cases} \quad \text{where } 0 < \epsilon < 1$$

and note

$$\phi''(x) = e^{\frac{-x^2}{\epsilon^2 - x^2}} \left[\frac{-2\epsilon^2[x\epsilon^2 - x^2 + \epsilon^2 - 5x^2]}{(\epsilon^2 - x^2)^3} \right] \quad \text{if } |x| \le \epsilon,$$

$$\phi(1) = \phi(-1) = 1 \quad \text{and} \quad \phi(0) = 0,$$

and the graph of $\phi(x)$ as follows:



For $w(x)$ see [13].

Theorem 3.4

If $y(x)$ is a steady state solution and $\phi''(x) \leq g(x, \phi(x), \phi'(x))$, then $y(x)$ will change sign.

Proof. $\phi(x)$ is an upper solution since

$$\phi''(x) - g(x, \phi(x), \phi'(x)) \leq 0 = y''(x) - g(x, y, y')$$

Also,

$$\phi(-1) = \phi(1) = 1 \geq 1 = y(-1) = y(1)$$

and this implies

$$y(x) < \phi(x) \quad \forall\, x$$

But since $\phi(0) = 0$, then $y(0) < 0$ and this implies that $y(x)$ will change sign.

5. Blow–up

When $v(x,t) > 0$ in the v–equation, then $u(x,t) = s^{-1}(v)$ is a bounded solution of the u–equation. Also, the blow–up is possible in the u–equation when the steady state solution changes sign.

Because the solution of the v–equation converges to the steady state, there exists $T > 0$ such that $v(x,t) = 0$ before it becomes negative. Furthermore, the solution of the u–equation will become $u(x,t) = s^{-1}(0) = \infty$ which means that

T is the blow-up time for the u-equation and this will complete our work for the one-dimensional case.

§2. Examples

In this section, we will try to discuss some examples to test the method.

First Example

Let the u-equation be given by

$$u_t = u_{xx} + \lambda(1+e^{-u})^{\alpha}e^{u} - u_x^2 \qquad -1 < x < 1$$

$$u(-1,t) = u(1,t) = 0$$

$$u(x,0) = -\log(h(x))$$

where $\alpha > 0$

Note:

$$\frac{g(x, s(u), s(u)_x)}{s'(u)} = -\lambda(1+e^{-u})^{\alpha}e^{u}. \tag{3.4}$$

Also

$$\frac{s''(u)}{s'(u)} = -1 \tag{3.5}$$

From (3.5), $s(u) = c_1 + c_2e^{-u}$, but we know that $s(0) = 1$ and $s(\infty) = 0$. So $s(u) = e^{-u}$. Let

$$s(u) = v(x,t) = e^{-u(x,t)} \tag{3.6}$$

Then from (3.4) $g(x, v, v_x) = \lambda(1+v)^\alpha$ and from (3.6) $u(x,t) = -\ln v(x,t)$. So, by the change of variables $u(x,t) = -\ln u(x,t)$ we will transform the u–equation into the v–equation

$$v_t = v_{xx} - \lambda(1+v)^\alpha$$

$$v(-1,t) = v(1,t) = 1$$

$$v(x,0) = h(x) > 0$$

Now, we will try to satisfy the conditions for the convergence to the steady state equation by using Theorem 2.

(1) $g(x, v, w)$ is locally Lipshitz in v and w.

$$\begin{aligned} |g(x,v_1,w) - g(x,v_2,w)| &= \lambda|(1+v_1)^\alpha - (1+v_2)^\alpha| \\ &< M|(1+v_1-(1+v_2))| \\ &< M|v_1 - v_2| \end{aligned}$$

where $M > 0$.

Also, $|g(x,u,w_1) - g(x,v,w_2)| = 0 < |w_1 - w_2|$.

(2) We assume $h''(x) - g(x,h,h') \leq 0$ so that $v_t(x,0) \leq 0$.

(3) Note $w = -1$ is a lower solution and this implies $v(x,t)$ is bounded from below; so $\inf v(x,t) > -\infty$.

(4) $g(x,y,w) = (1+y)^\alpha$

$r(y) \quad = -(1+y)^\alpha$

$q(w^2) \quad = 1$

so, $g(x,y,w) > -r(y)q(w^2)$. Then the solution will converge uniformly to the steady state solution $y(x)$ (i.e. as $t \to \infty$, $\sup_x |v(x,t) - y(x)| \to 0$). So

$$y''(x) = \lambda(1+y)^\alpha$$

$$y(-1) = y(1) = 1 \tag{3.7}$$

Note: (3.7) has at least one solution.

For $\lambda > 0$ large enough the solution $y(x)$ will change sign. For example if we take $\alpha = \frac{1}{2}$, then $y(x) = 2x^4 - 1$ where $\lambda = \frac{24}{\sqrt{2}}$, and clearly the steady state changes its sign.

Also, for the boundary condition, we can take $h(x) = 1 - \frac{1}{2}\cos(\pi x)$ so that $h''(x) - \lambda g(x,h,h') < 0$. Finally, the u–equation exhibits a blow–up point in finite time.

Second Example

Consider the u–equation

$$u_t = u_{xx} - u_x^2 + \frac{1}{2}e^u - 1 \quad t > 0, \;\; 0 < x < \pi$$

$$u(0,t) = u(\pi,t) = \log 2$$

$$u(x,0) = -\log(h(x))$$

Note:

$$\frac{g(x, s(u), s(u)_x)}{s'(u)} = \frac{1}{2}e^u - 1 \tag{3.8}$$

Also,

$$\frac{s''(u)}{s'(u)} = -1 \tag{3.9}$$

and this implies $s(u) = e^{-u}$.

Let

$$s(u) = v(x,t) = e^{-u(x,t)} \tag{3.10}$$

Then from (3.8), $g(x, v, v_x) = -\frac{1}{2} + v$ and from (3.10) $u(x,t) = -\ln v(x,t)$. So, by the change of variables $u(x,t) = -\ln v(x,t)$, we will transform the u–equation into the v–equation

$$v_t = v_{xx} + v - \frac{1}{2} \quad t > 0, \;\; 0 < x < \pi$$

$$v(0,t) = v(\pi,t) = \frac{1}{2}$$

$$v(x,0) = h(x) > 0$$

Now, we will try to satisfy the conditions for the convergence to the steady state equation by using Theorem 2.

1. $g(x,v,w)$ is locally Lipshitz in v and w.

   (a) $|g(x,v_1,w) - g(x,v_2,w)| = \left|-\frac{1}{2} + v_1 + \frac{1}{2} - v_2\right|$
   $= |v_1 - v_2|$.

   (b) Also $|g(x,v,w_1) - g(x,v,w_2)| = 0 < |w_1 - w_2|$.

2. We assume $h''(x) - g(x,h,h') \leq 0$.

3. Note $v = \frac{1}{2}$ is a lower solution and this implies $v(x,t)$ is bounded from below.

   So, $\inf v(x,t) > -\infty$.

4. $g(x,y,w) = -\dfrac{1}{2} + y$
   $r(y) \quad = -\left(\dfrac{-3}{2} + y\right)$
   $q(w^2) \quad = 1$
   So, $g(x,y,w) > -r(y)q(w^2)$. Then the solution will converge uniformly to the steady state solution $y(x)$. So,

$$y''(x) = \frac{1}{2} - y$$
$$y(0) = y(\pi) = \frac{1}{2} \tag{3.11}$$

   Note (3.11) has at least one solution. Also, note $y(x) = c\sin x + \frac{1}{2}$ where c is an arbitrary constant.

In order for the steady state to change its sign, we only need to require that $c < -\frac{1}{2}$.

Finally, the u–equation exhibits a blow–up point in finite time.

CHAPTER 4

THE N–DIMENSIONAL CASE

In general, the equation $u_t = \Delta u + f(x, u, \nabla u)$ cannot be reduced to the form $u_t = u_{xx} - \dfrac{\lambda g(x, s(u), s(u)_x)}{s'(u)} + u_x^2 \dfrac{s''(u)}{s'(u)}$ where $s(u)$ is a suitable function as in Chapter 3. So, we will try to discuss the heat equation in its general form with an emphasis on the n–dimensional case $u_t \equiv \Delta u + \lambda f(x, u, \nabla u)$.

Let $u(x,t)$ be a positive solution of the equation

$$\begin{aligned} u_t &= \Delta u + \lambda f(x, u, \nabla u) \\ u(x,0) &= h(x) \geq 0 \\ u(\partial\Omega, t) &= 0 \end{aligned} \tag{4.1}$$

which we will call the u–equation.

To prove the existence of the blow–up, many steps should be performed. The first step, as in chapter 3, will change equation (4.1) by the change of variables $u(x,t) = s(v(x,t))$ to another equation, namely the v–equation. So, the u–equation has a blow–up if the v–equation changes sign.

1. Substitution

If $u(x,t) = s(v(x,t))$, then equation (4.1) can be transformed into the equation

$$v_t = \Delta v - \frac{|\nabla v|^2}{v} A(v) + \lambda g(x, u, \nabla v) \tag{4.2}$$

where

$$A(v) = -\frac{v s''(v)}{s'(v)} \text{ and } g(x, v, \nabla v) = \frac{f(x, v, \nabla u)}{s'(v)}$$

Notes:

1. We should set a condition on $s'(v) \neq 0$ to make $s(v)$ a one–to–one function.

2. If $s''(v)$ is of a regular variation i.e., $s''(v) \in RV_\alpha^0$, then the function $A(v)$ is bounded since the $\lim_{v \to 0} A(v)$ exists (see [8]).

2. Regularization

To solve equation (4.2), we need to regularize the singular term $\frac{|\nabla v|^2}{v}$, so equation (4.2) will become

$$\begin{aligned} v_t^\epsilon &= \Delta v^\epsilon - |\nabla v^\epsilon|^2 \frac{v^\epsilon A(v^\epsilon)}{(v^\epsilon)^2 + \epsilon^2} + g(x, v^\epsilon, \nabla v^\epsilon) \\ v(x,0) &= k(x), \quad v(\partial\Omega, t) = 1 \end{aligned} \tag{4.3}$$

where $k(x) = s^{-1}(h(x)) \leq 1$.

Note: Since the term $-|\nabla v^\epsilon|^2 \frac{v^\epsilon A(v^\epsilon)}{v_\epsilon^2 + \epsilon^2}$ comes from the Laplacian, it should not affect the blow–up. So, we can consider a simpler equation comparable to (4.3) by

letting $\epsilon \to \infty$, and we shall call it the w-equation

$$w_t = \Delta w + \lambda \tilde{g}(x, w, , \nabla w)$$

$$w(x,0) = k(x), w(\partial\Omega, t) = 1 \tag{4.4}$$

Now, we will consider two cases for the nonlinear term $\tilde{g}(x, v, p)$. The first case is concerned with finding $s(v)$. So $\tilde{g}(x, v, p)$ has a C^2 extension $\tilde{g}(x, v, p) < 0$ for $v < 0, v > 1$ and $\forall x, p$, where this simple condition will help to obtain global existence of the solution for the regularized equation and the second case when $\tilde{g}$ cannot be defined or extended for changing sign functions.

In both cases, we shall put the following conditions on $s(v)$.

C1

1. $s : (0,1] \to R_+$ and s is a decreasing function.

2. $\lim_{v \to 0} s(v) = \infty$, $s(v) \in C^2$.

3. $0 < \dfrac{vs''(v)}{-s'(v)} \leq M \leq \infty$ or $s'(v)$ is of regular variation, i.e. $s'(v) \in RV_\alpha^0$.

§1. The First Case

1. Method

Theorem 1

If condition C1 holds and

1. Ω is a bounded domain in R^n, with a smooth boundary, $\partial\Omega \in C^{2+\beta}$, say.

2. There exists $A \in R^n$ and $b \in R$ such that $\forall x \in \Omega$, $t > 0$.

$$|A|^2 M \geq (A \cdot x + b)\lambda\tilde{g}(x, A \cdot x + b, A)$$

$$A \cdot x + b \leq 0$$

$$\tilde{g}(x, v, p) \leq 0 \tag{4.5}$$

   where $M = \sup A(v)$.

3. There exists $c(m)$ such that $|\tilde{g}(x, v, p)| < c(m)(1+|p|^2)$ for $v \in [-m, m]$, $\forall\, p \in R^n$ where $c(m)$ is locally bounded.

4. $\tilde{g}(x, v, p)$ and $A(v)$ are Hölder continuous with respect to x, v, p with exponent β.

5. $\dfrac{\partial\tilde{g}}{\partial v}, \dfrac{\partial\tilde{g}}{\partial p_i}, \dfrac{\partial A}{\partial v}$ are bounded if v, p are bounded.

6. $\Delta k(x) + \lambda\tilde{g}(x, 1, 0) = 0$ and $\nabla k(x) = 0$ for $x \in \partial\Omega$, then $\forall\epsilon > 0$, there exists

a unique solution of the equation $v^\epsilon(x,t) \in H_{Q_T}^{2+\beta,1+\frac{\beta}{2}}$ for all $T > 0$ and such that $A \cdot x + b \leq v^\epsilon \leq 1$ where $Q_T = \Omega \times [0,T]$.

Proof. First we want to find an upper and lower solution for the v–equation.

It is easily seen that

(a) $\hat{w} \equiv 1$ is an upper solution because

1. $v^\epsilon|_{\partial\Omega} \leq 1$
2. $\Delta v^\epsilon - v_t^\epsilon \equiv \dfrac{|\nabla v^\epsilon|^2 v^\epsilon A(v^\epsilon)}{(v^\epsilon)^2 + \epsilon^2} - \lambda \tilde{g}(x, v^\epsilon, \nabla v^\epsilon) \geq 0$. So, $\hat{w} \equiv 1$ is an upper solution by the maximum principle.

(b) $\underline{w} \equiv A \cdot x + b$ is a lower solution because

1. $A \cdot x + b \leq 0$.
2. $\Delta \underline{w} - \underline{w}_t - \frac{|\nabla \underline{w}|^2 \underline{w} A(\underline{w})}{\underline{w}^2 + \epsilon^2} + \lambda \tilde{g}(x, \underline{w}, \nabla \underline{w}) \geq 0$ by condition (4.5).

Also, since the set Ω is bounded, we have the solution v^ϵ bounded for $\epsilon > 0$. By the remaining assumptions we have the existence of the solution v^ϵ for each $\epsilon > 0$ by Theorem 6.1 in (10). Observe that T is arbitrary and so we have a global existence.

Remarks

1. From Theorem 6.1 in (10), we can use the condition: $\tilde{g}(x, v, 0) \geq -b_1^2 v^2 - b_2^2$ to show the solution is *a priori* bound.

2. To obtain *a priori* bound for the gradient, we need Condition 3, and so we deduce global existence of the solutions and also the existence of a steady state, see section 2 and Lemma 3.1 in (1).

Theorem 2

If the condition of Theorem 1 is satisfied and $\epsilon_1 \geq \epsilon_2$, then $v^{\epsilon_1} \geq v^{\epsilon_2}$ for $x \in \Omega, t > 0$.

Proof. If $\epsilon_1 \geq \epsilon_2$, then $\Delta v^{\epsilon_1} - v_t^{\epsilon_1} \leq \Delta v^{\epsilon_2} - v^{\epsilon_2}$. So by the maximum principle $v^{\epsilon_1} \geq v^{\epsilon_2}$.

Theorem 3

If for a certain $\epsilon > 0$, $\exists x_0, T$ such that $v^{\epsilon}(x_0, T) = 0$, then $u(x_0, T) = \infty$, i.e. the solution $u(x, t)$ blows up.

Proof. Note that if $u(x, t)$ is bounded for $t < T$, $v^{\epsilon} = s^{-1}(u(x, t))$ exits and is well defined, and this means $v^0 > 0$. By Theorem 2, $v^{\epsilon} > v^0 > 0$ for all $\epsilon > 0$.

Theorem 4

If the conditions of Theorem 1 hold and $\forall\, \epsilon > 0 \quad v^\epsilon(x,t) \geq a > 0\ \forall (x,t)\ Q_T$, then the solution $u(x,t)$ has no blow-up in Q_T.

Proof. We need only to prove that v^0 exists and $v^0 \geq a > 0$. First, let us set $F_\epsilon(v^\epsilon) \equiv |\nabla v^\epsilon|^2 \dfrac{v^\epsilon A(v^\epsilon)}{(v^\epsilon)^2 + \epsilon^2} - \lambda \tilde{g}(x, v^\epsilon, \nabla v^\epsilon)$. Note it is easily seen that $F_\epsilon(v^\epsilon)$ is the nonlinear term in (4.3). If we assume that $1 > v^\epsilon \geq a > 0$ then $F_\epsilon(v^\epsilon)$ satisfies Theorem 6.1 in [10]. Also, by condition C1

$$|\nabla v^\epsilon|^2 \frac{v^\epsilon A(v^\epsilon)}{(v^\epsilon)^2 + \epsilon^2} \leq \frac{M}{a} |\nabla v^\epsilon|^2.$$

So, all the solutions v^ϵ are uniformly bounded in $H_{Q_T}^{2+\beta+1+\frac{\beta}{2}} \subset C_{Q_T}^{1+\frac{\beta}{2}}$ and this implies the set v^ϵ is compact in $C_{Q_T}^{1+v}$ where $v < \beta/2$. So the limit v^0 exists and $v^0 > a > 0$ and the solution $u(x,t) = s(v^0)$ does not blow up.

Theorem 5

If the conditions of Theorem 1 hold and the solution $w(x,t)$ has a zero in Q_T, then the solution $u(x,t)$ blows up in Q_T.

By Theorem 1, the solution $w(x,t)$ exists. Also because

1. $\Delta w - w_t \leq \Delta v^\epsilon - v_t^\epsilon$.

2. $w|_{\partial\Omega} \geq v_\epsilon / \partial\Omega$.

So $w \geq v_\epsilon$ for $t \geq 0$ and $x \in \Omega$. So, if $w(x,t)$ has a zero, then v_ϵ has a zero and by Theorem 4, $u(x,t)$ blows up.

Now, we want to find sufficient conditions concerning the steady state of the w–equation.

Theorem 6

If the conditions of Theorem 1 are satisfied, and

1. $\dfrac{\partial \tilde{g}(x,w,\eta)}{\partial w} \leq 0$ and $w(x,t) \in C^3_{Q_T} \forall T > 0$.
2. $\exists z(x)$ such that $\begin{cases} \Delta z(x) + \lambda \tilde{g}(x,z,\nabla z) < 0 \\ z(\partial\Omega) \geq 1 \end{cases}$.
3. $\exists p \in \Omega$ such that $z(p) = 0$.

Then the blow–up is possible for the solution $u(x,t)$.

Proof. By Theorem 1, we know that there exists a solution $w(x,t)$ and a steady state $S(x)$ say. Also, by Condition 1 and the maximum principle, we deduce the uniqueness of the steady state. Moreover, from Condition 2 $S(x) < Z(x)$ where $Z(x)$ is a steady state and the steady state must change its sign by Condition 3.

Finally, we need to show that there exists a positive initial condition such that the corresponding solution will converge to the steady state. So, $w(x,t)$ has a zero and $u(x,t)$ blows up from Theorem 5.

For the proof of the final result, we will prove it in the following manner. First, we choose $k(x) = z(x) + c\phi(x)$ where $\phi(x)$ is the first positive eigenfunction of the Laplacian

$$-\Delta\phi(x) = \lambda_1\phi(x) \text{ and } \phi(\partial\Omega) = 0.$$

Clearly from

$$w_t(x,0) = \Delta z - \lambda_1\phi(x) + \lambda\tilde{g}(x, z + c\phi, \nabla(z + c\phi))$$

and the smoothness we can see $\exists c_0$ such that $w_t(x,0) < 0$. Also, because $w|_{\partial\Omega} = 0$, we can say that $w_t < 0$ for $(x,t) \in Q_T$ by the maximum principle applied to the equation satisfied by w_t. Because $w_t < 0$, we can say the sequence $w(x,n)$ is decreasing and bounded from below. So, the sequence will converge pointwise to a function $\ell(x)$. By using Dini's theorem, we obtain that $w(x,n)$ will converge uniformly to $\ell(x)$ in Ω. By Theorem 2.2 in [1], we can say $\|w(x,t\|_{C^{1+u,(1+\mu)/2}Q_T} \leq \gamma(\|w\|_{C(Q_T)}, \|h(x)\|_{C^2})$. So the sequence $w(x,n)$ is bounded in $C_\Omega^{1+u/2}$ and so compact in C_Ω^{1+u} where $v < u/2$. Therefore, we have $w(x,n) \to \ell(x)$ and $\nabla w(x,n) \to \Delta\ell(x)$ uniformly in Ω. This will show that $\ell(x)$ is a weak solution of the steady state equation. Moreover, $\ell(x)$ is a strong solution because $\tilde{g}(x, \ell(x), \nabla(\ell(x)))$ is smooth and by looking at the steady state equation as a linear equation. Finally, $S(x) = \ell(x)$ because of the uniqueness and so there exists N such that $w(x,N)$ change sign, i.e. $u(x,t)$ blows up.

Now, we will try to find another way and before that we shall define some functions. Let u be the first eigenvalue of $-\Delta\Phi(x) = u\Phi(x)$, $\Phi(\partial\tilde{\Omega}) = 0$ where $\Omega \subset \tilde{\Omega}$ and $\Phi(x) \neq 0$ if $x \in \tilde{\Omega}$. Also, let $R(c,x)$ represent the real valued function defined by $R(c,x) \equiv \frac{1}{\Phi(x)}\lambda\tilde{g}(x, z + c\Phi, \nabla(z + c\Phi))$ and $v \equiv \sup_{x\in\Omega, 0<\theta<1} \dfrac{\partial R(\theta, x)}{\partial c}$

Theorem 7

If the conditions of Theorem 1 are satisfied and there exists a steady state $S(x)$ that changes sign, and if $u > v$, then blow-up is possible.

Proof. Let $k(x) \equiv S(x) + c(0)\Phi(x)$ and we want to find an upper solution of the form $D(x,t) \equiv S(x) + c(t)\Phi(x)$ where $c(t) > 0$. So we need

$$\Delta(S + c(t)\Phi(x)) + \tilde{g}(x, S + c\Phi, \nabla(S + c\Phi)) - c'(t)\Phi(x) \leq 0$$

or we can say

$$-c' - cu + R(c,x) - R(0,x) \leq 0$$

So

$$-c' - cu + cv \leq 0,$$

and this implies

$$-c' + c(v - u) \leq 0 \tag{4.6}$$

If we take $c(t) = c(0)\exp((v - u(t)$, then we will satisfy (4.6). Also, because $S(x)$

changes sign, there exists x_0 such that $S(x_0) < 0$. So, there exists T such that

$$w(x,t) < S(x_0) + c(0)\exp(v-u)T < 0.$$

Finally, by Theorem 6, we have a blow-up of the solution $u(x,t)$.

2. Examples

1. $u_t = \Delta u + \lambda e^u \quad x \in \Omega, |\Omega| < \infty.$

   Let $u = s(v) = -\log(v)$, so the regularized v-equation will become

$$v_t = \Delta v - \frac{|\nabla v|^2 v}{\epsilon^2 + v^2} - \lambda$$

   For the sake of simplicity, let Ω be a simple domain, such as ball, $\Omega = B_a = \{x| \, |x| \leq a\}$. Also choose the upper solution to be $z(x) = sr^\alpha$ where $\alpha \geq 2$ and $r \equiv |x|$. So, the condition for the upper solution in the w-equation (we will consider it for simplicity) will become

$$\begin{cases} \Delta z - \lambda < 0 \\ z(\partial\Omega) = 1 \end{cases}$$

   or equivalently,

$$\begin{cases} s(\alpha + (n-2))\alpha r^{\alpha-2} < \lambda \quad 0 < r < a \\ sa^\alpha = 1 \end{cases}$$

   So finally,

$$\frac{\alpha^2 + (n-2)\alpha}{a^2} \leq \lambda$$

So the condition will satisfy if $\lambda^* \equiv \frac{2n}{a^2} < \lambda$ by choosing $\alpha = 2$. So, there is a $\lambda^*(a)$ such that the blow-up occurs in finite time.

2. $u_t = \Delta u + \lambda |u|^{p-1} u$ where u is a positive function.

Let $u = s(v) = v^{\frac{1}{1-p}}$, so the regularized v–equation will become

$$v_t = \Delta v + \frac{p}{1-p} \frac{|\nabla v|^2 v}{\epsilon^2 + v^2} + (1-p)\lambda.$$

As above, let Ω be balls, and choose the upper solution $z(x) = sr^2$. So, we will need $2sn + (1-p)\lambda < 0$ and $sa^2 = 1$. Thus the blow-up occurs in finite time if there is a $\lambda^*(a)$ such that $\lambda > \lambda^*(a) = \frac{2n}{a^2(p-1)}$.

§2. The Second Case

In this case we will consider $\tilde{g}$ which cannot be defined or extended for changing sign functions. We note there is no global existence and because of that we can assume the local existence of the solution of the v–equation. By using the same notations, we will proceed, in general, as the first case for the proof.

1. Method

Theorem 8

If the conditions 1, 4, 5, 6 of Theorem 1 and

1. $\exists z(x) \in C^2(\Omega)\Delta z(x) + \lambda\tilde{g}(x, z(x), \nabla z(x)) < 0$ and $z(\partial\Omega) \geq 1$.

2. $\exists\, p$ such that $z(p) = 0$.

3. $u > v$.

4. $\exists n(t) > 0$ for $0 < t \leq \delta(a)$ such that $\lambda\tilde{g}(x, n(t), 0) \geq n'(t)$ and $n(0) = a$ for any $a > 0$ and $\forall x \in \Omega$. Then the blow-up for the solution $u(x,t)$ is possible.

Proof. As in the first case, we need to find $k(x)$ such that the corresponding solution reaches $z(x)$ in finite time. We want the form $s(x,t) = z(x) + c(t)\Phi(x)$, where $\Phi(x)$ is the first positive eigenfunction of the Laplacian in a larger domain, to be an upper solution.

Let

$$T(c) \equiv \Delta(z(x) + c(t)\Phi(x)) + \lambda\tilde{g}(x, z(x) + c(t)\Phi(x), \nabla(z(x) + c(t)\phi(x))$$

where $T(c)$ is defined for any $c \geq 0$. Also, $\exists\, c_0 > 0$ such that $T(c_0) \leq \delta_- < 0$ because of the smoothness conditions on $\tilde{g}(x, v, \eta)$. Let $k(x) = z(x) + c(0)\Phi(x)$ and to make $s(x,t)$ to be an upper solution we will need the following conditions

$$\Delta(z(x) + c(t)\Phi(x)) + \lambda\tilde{g}(x, z(x) + c(t)\Phi(x), \nabla(z(x) + c(t)\Phi(x))) \leq c'(t)\Phi(x)$$

So

$$\Delta z(x) - \mu c\Phi(x) + \lambda\tilde{g}(x, z(x) + c(t)\Phi(x), \nabla(z(x) + c(t)\Phi(x))) \leq c'(t)\Phi(x)$$

and this implies

$$c'(t)\Phi(x)+\mu c(t)\Phi(x) \geq \lambda\tilde{g}(x, z(x)+c(t)\Phi(x), \nabla(z(x)+c(t)\Phi(x)))-\lambda\tilde{g}(x, z(x), \nabla z(x))$$

Finally, we will have

$$c'(t) + (u - v)c \geq \delta_-$$

where one solution of this inequality (consider the equal sign) is $c(t) = (c(0) - \frac{\delta_-}{u-v})e^{(v-u)t} + \frac{\delta_-}{u-v}$, where as $t \to \infty$, then $c(t) \to \frac{\delta_-}{u-v}$ and so the upper solution becomes negative.

To prove the existence of the solution, we need a lower solution which can be found by Condition 4. By Condition 4, a local lower solution can be found in the interval $(0, \delta_-)$. If the solution has a zero for $t = \delta_-$, then we are done, otherwise the solution is bounded away from zero, that is $\exists\, x \in \Omega\ w(x, \delta_-) \equiv a_1 > 0$. So, there is a need to obtain a new lower solution defined by $t \geq \delta_-$, $n'(t) \leq \lambda\tilde{g}(x, u(t), 0)$, $\quad n(\delta_-) = a_1$ and repeat the same argument.

Because the upper solution becomes negative, the solution must also have a zero. So by Theorem 5, the solution $u(x,t)$ blows up.

Another Way

We can look for an upper solution for the w-equation of the form $D(x,t) =$

$z(x) + c(t)$. For $D(x,t)$ to be an upper solution, we need

$$\Delta z + \lambda \tilde{g}(x, z + c(t), \nabla(z(x))) \leq c'(t) \quad c(t) \geq 0.$$

or

$$\Delta z + \lambda \tilde{g}(x, z, \nabla z) + \lambda \tilde{g}(x, z + c(t), \nabla z(x)) - \lambda \tilde{g}(x, z, \nabla z) \leq c'(t)$$

and we assume that $\Delta z + \lambda \tilde{g}(x, z, \nabla z) \leq \delta_0 < 0$ and $\sup_{x \in \Omega} \dfrac{\partial \tilde{g}(x, z + c, z)}{\partial c} \equiv d \leq 0$, so we obtain $\delta_0 + dc(t) \leq c'(t)$ where a positive solution of this inequality is

$$c(t) = c(0) + \delta_0 t.$$

So $D(x,t)$ is an upper solution of the w–equation that changes sign and by the same condition as above, we will have the lower condition. Finally, we will have a blow–up for the solution $u(x,t)$. Thus we have shown the following theorem.

Theorem 9

If the Conditions 1, 4, 5, 6 of Theorem 1 hold and

1. $\exists\, z(x) C^2_\Omega \Delta z + \lambda \tilde{g}(x, z, \nabla z) \leq \delta_0 < 0, \quad z(\delta) \geq 1.$
2. $\sup_{x \in \Omega} \dfrac{\partial \tilde{g}(x, z + c, \nabla z)}{\partial c} \equiv d \leq 0$ for $c \in (0, \alpha)$ where $\alpha > 0$.
3. $\exists\, n(t) > 0$ for $0 < t \leq \delta(a)$ such that $\lambda \tilde{g}(x, n(t), 0) \geq n'(t)$, $n(0) = a$ for any $a > 0$ and $x \in \Omega$.

then blow–up is possible for the solution $u(x,t)$.

2. Examples

Example. We need finally to show the blow-up of Chipot and Weissler equation

$$u_t = \Delta u - |\nabla u|^q + |u|^p$$

By using the substitution $s(v) = v^{\frac{1}{1-p}}$, $\quad p > 1$, the u-equation reduces to:

$$v_t = \Delta v + \left(\frac{1}{1-p}\right) \frac{|\nabla v|^2 v}{\epsilon^2 + v^2} - \frac{|\alpha|^q}{\alpha} v^{(q-1)(\alpha-1)} |\nabla v|^q - (p-1)\lambda$$

where $\alpha = \dfrac{1}{1-p}$. So $\tilde{g}(x, v, \nabla v) \equiv |\alpha|^{q-1} v^{\beta} |\nabla v|^q - (p-1)\lambda$ where $\beta = (q-1)(\alpha - 1) < 0$, and this implies that $\tilde{g}$ is of the second case and we shall try to use Theorem 9. Let $z(x) = \frac{r^2}{a^2}$ where $r = |x|$ and $\frac{\gamma}{a} \geq 1$, where we assume $\Omega = B_\gamma = \{x/|x| < \gamma\}$. So, $z(x)$ is an upper solution if

$$\frac{2n}{a^2} + 2^q |\alpha|^{q-1} r^{2\beta+q} a^{-2(\beta+q)} - (p-1)\lambda < 0 \tag{4.7}$$

If $2\beta + q > 0$, then it is possible to find a λ such that (4.7) is satisfied.

However, if λ and p are fixed values, then we can choose a very large such that (4.7) holds; that is

$$\frac{2n}{a^2} + 2^q |\alpha|^{q-1} < (p-1)\lambda \tag{4.8}$$

Also,

$$\frac{\partial g(x, z+c, z)}{\partial c} = \frac{2^q}{a^{2q}} |\alpha|^{q-1} \beta (z+c)^{\beta-1} r^q \leq 0$$

and for the lower solution we can take $n(t) = a - (p-1)\lambda t$ and so $\delta_- = \dfrac{a}{(p-1)\lambda} > 0$. So, the conditions of Theorem 9 hold if $2\beta + q > 0$ and so blow-up occurs if

$1 < q < \dfrac{2p}{p+1}$ and a is large enough; so we get the same result as in the Chipot and Weissler paper.

Finally, I would say that this simple method will open a new survey for the blow-up of the reaction-diffusion equation in its general form

$$u_t = \Delta u + f(x, u, \nabla u)$$

Also, I think we can generalize it to systems of such equations.

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