IMPULSIVE DYNAMIC BOUNDARY VALUE PROBLEMS ON TIME SCALES

BY

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DEDICATION

This Thesis is Dedicated to

Abdulwahhab, Maryam, Ahmad and Zainab.

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ABSTRACT

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We shall consider dynamic equations on time scales subject to two-point boundary conditions and impulsive effects. The approach is based on converting a dynamic first order system with impulses into an augmented system without impulse through a suitable transformation, solve these equations and use the inverse transformation to obtain the intended results with impulses.

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Chapter 1

Introduction

The theory of dynamic equations on time scales, introduced in 1988 by Stefan Hilger, unifies and extends continuous and discrete analyses and also provides a framework for studying general class of dynamic equations. For example, differential equations and difference equations are unified, a long awaited objective [12], [13].

To start with, what is a time scale? A time scale, denoted by $\mathbb{T}$, is an arbitrary non empty closed subset of $\mathbb{R}$, the set of real numbers. Thus $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $q^{\mathbb{Z}}$, $[0,1] \cup \mathbb{N}$ are examples of time scales whereas $\mathbb{Q}$, $\mathbb{C}$, $(0,1)$, $\mathbb{R} \backslash \mathbb{Q}$ are not.

A new derivative, called the delta derivative, is introduced that reduces to the ordinary derivative if $\mathbb{T} = \mathbb{R}$ and to the forward difference if $\mathbb{T} = \mathbb{Z}$. A corresponding integral is also defined.

The theory of time scales has received a lot of attention in recent years and many classical results in ordinary differential equations have been generalized into this setting.

- First Order Linear Equations

  ∘The Exponential Function

- Initial Value Problems

- Second Order Equations

  - Trigonometric and Hyperbolic Functions
  - Redection of Order
  - Cauchy-Euler Equation
  - Variation of Parameters
  - Laplace Transform (unifies the classical Laplace transforms and $\mathbb{Z}$– transforms)

....to mention but a few. For more on the subject and its ramifications see the monographs [3], [5], [17] and the special issue [1] and the references therein.

On the other hand, impulsive differential equations are attracting many researchers and many theoretical results exits [2], [9], [16], [18], [20]. However, when it comes to exhibiting solutions to such systems using simple mathematics, not many papers are available except in particular [15] whose method shall be adapted to our situation.

The outline of the thesis is as follows.

In Chapter II we present the relevant results about Time Scale Calculus and the idea of differential equations on time scales and show how it particularises to differential equations when the time scale is $\mathbb{R}$, and to difference equation when the time scale is $\mathbb{Z}$.

In Chapter III, we shall present briefly the idea of differential equations with impulses and outline a method recently introduced in [15] by L. I. Karandzhulav, a method which will be adapted for our own work. Our main result concerning Boundary Value Problems on Time Scales with impulses will be presented in Chapter IV with conclusion and suggestion on further research work in chapter V.

Chapter 2

The Time Scales Calculus

The theory of dynamic equations on time scales, introduced in 1988 by Stefan Hilger, unifies and extends continuous and discrete analyses and also provides a framework for studying general class of dynamic equations. For example, differential equations and difference equations are unified, along awaited objective. We shall present in this chapter the basic elements of Time Scale Calculus relevant to our work. For proof of the theorems, we may consult [1],[3], [4] from which we have taken these materials.

Definition 2.1 *A time scale, denoted by* $\mathbb{T}$*, is an arbitrary non-empty closed subset of* $\mathbb{R}$*, the set of real numbers.*

Thus $\mathbb{R}, \mathbb{Z}, \mathbb{N}, q^{\mathbb{Z}}, [0,1] \cup \mathbb{N}$ *are time scales while* $\mathbb{Q}, \mathbb{C}, (0,1), \mathbb{R} \backslash \mathbb{Q}$ *are not.*

Definition 2.2 *For* $t \in \mathbb{T}$*, the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by* $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$*and the backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by* $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

Definition 2.3 *A point $t \in \mathbb{T}$ is said to be right-scattered, right dense, left-scattered, left dense if $\sigma(t) > t, \sigma(t) = t, \rho(t) < t, \rho(t) = t$ holds respectively. Points that are right-scattered and left-scattered at the same time are said to be isolated while those that are right-dense and left-dense at the same time are called dense. Also, the graininess function $\mu : \mathbb{T} \to [0, \infty]$ is defined by $\mu(t) = \sigma(t) - t$.*

The following examples illustrate the above definitions.

Example 2.4 *If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$*

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t \text{ and}$$
$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t$$

So every point $t \in \mathbb{R}$ is dense and the graininess function $\mu(t) = 0$ for all $t \in \mathbb{R}$.

Example 2.5 *If $\mathbb{T} = \mathbb{Z}$, then for any $t \in \mathbb{Z}$*

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf(t+1, t+2, ...) = t+1 \text{ and}$$
$$\rho(t) = \sup\{s \in \mathbb{Z} : s < t\} = \sup(...., t-2, t-1) = t-1$$

Thus every point $t \in \mathbb{Z}$ is isolated and the graininess function $\mu(t) = 1$ for all $t \in \mathbb{Z}$.

Example 2.6 *Let $\mathbb{T} = \{\frac{1}{n} : n \in N\} \cup \{0\}$. We have $\sigma(\frac{1}{n}) = \frac{1}{n-1}$, $n \neq 1$. Let $t = \frac{1}{n}$, then $\sigma(t) = \frac{t}{1-t}, t \neq 1$. Similarly, $\rho(t) = \frac{t}{1+t}$.*

Obviously $\sigma(0) = \rho(0) = 0$. Thus 0 is a dense point and all other points are isolated.

Example 2.7 *Let $\mathbb{T} = \{\frac{n}{2} : n \in N_0\}$. We have $\sigma(\frac{n}{2}) = \frac{n+1}{2} = \frac{n}{2} + \frac{1}{2}$. Let $t = \frac{n}{2}$, so $\sigma(t) = t + \frac{1}{2}$.*

Similarly, $\rho(t) = t - \frac{1}{2}$. Hence every point $t \in \mathbb{T}$ is isolated.

2.1 Differentiation

We now define the delta derivative of f at a point $t \in \mathbb{T}^k$ where $\mathbb{T}^k$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum.

Definition 2.8 *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Define $f^{\Delta}(t)$ to be the number with the property that given any $\varepsilon > 0$, there exists a neighborhood V of t such that $|$ $[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \mid \leq \varepsilon \mid \sigma(t) - s] \mid$ for all $s \in V$. f^{Δ} is called the delta derivative of f at t.*

We shall present next some properties of delta-differentiability.

Theorem 2.9 *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^k$. Then we have the following.*

(a) If f is (delta) differentiable at t, then f is continuous at t.

(b) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

(c) if t is right-dense, then f is differentiable with

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)) - f(s)}{t - s}$$

provided the limit on the right hand side exists as a finite number.

(d) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$$

Theorem 2.10 *Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^k$. Then*

(a) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$$

(b) *For any constant c, $cf : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(cf)^{\Delta}(t) = cf^{\Delta}(t)$$

(c) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\ fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$$

(d) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$(\tfrac{1}{f})^{\Delta}(t) = -\tfrac{f^{\Delta}(t)}{f(t)f(\sigma(t))}$$

(e) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with

$$(\tfrac{f}{g})^{\Delta} = \tfrac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}$$

Example 2.11 *Obviously, If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = c$ for all $t \in \mathbb{T}$, where c is constant, then $f^{\Delta}(t) = 0$. This is obvious since for $\epsilon > 0$,*

$| \ [f(\sigma(t)) - f(s)] - 0.[\sigma(t) - s] \ | = |c - c| = 0 \leq \varepsilon \mid \sigma(t) - s] \mid$*holds for all $s \in \mathbb{T}$.*

Also, if $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) = 1$ since for $\epsilon > 0$,

$| \ [f(\sigma(t)) - f(s)] - 1.[\sigma(t) - s] \ | = |(\sigma(t) - s) - (\sigma(t) - s)| = 0 \leq \varepsilon \mid \sigma(t) - s] \mid$*holds for all $s \in \mathbb{T}$.*

Example 2.12 *Let α be a constant and $m \in N$.*

(a) If f is a function defined by $f(t) = (t-\alpha)^m$ then

$$f^{\Delta}(t) = \sum_{k=0}^{m-1} [\sigma(t) - \alpha)]^k (t-\alpha)^{m-1-k}$$

(b) If g is a function defined by $(t-\alpha)^{-m}$ then

$$g^{\Delta}(t) = -\sum_{k=0}^{m-1} [\sigma(t) - \alpha)]^{-m+k} (t-\alpha)^{-k-1}$$

provided that $[\sigma(t) - \alpha)](t-\alpha) \neq 0.$

Example 2.13 *Let $f(t) = t^3, t \in \mathbb{T}$. For $\sigma(t) > t$, we have from theorem 2.9(b)*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{(\sigma(t))^3 - \ t^3}{\sigma(t) - t} = (\sigma(t))^2 + t\sigma(t) + t^2.$$

We observe that $f^{\Delta}(t) = 3t^2$ for $\mathbb{T} = \mathbb{R}$ and $f^{\Delta}(t) = 3t^2 + 3t + 1$ for $\mathbb{T} = \mathbb{Z}$.

Example 2.14 *$f(t) = t^2$ where $\mathbb{T} = \{\frac{1}{n} : n \in N\} \cup \{0\}$.*

We know from example 2.6 that $\sigma(t)$ is $\frac{t}{1-t}$. Thus

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{(\frac{t}{1-t})^2 - t^2}{\frac{t}{1-t} - t} = \frac{2t - t^2}{1-t}.$$

2.2 Integration

Definition 2.15 *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be regulated if its right-sided limits exist(finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at all left dense points in $\mathbb{T}$.*

Definition 2.16 *A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at all right-dense points and its left-sided limit exists at all left-dense points.*

The set of rd-continuous functions is denoted by C_{rd}.

Theorem 2.17 *Let $f : \mathbb{T} \to \mathbb{R}$.*

(a) If f is continuous, then f is rd-continuous.

(b) If f is rd-continuous, then f is regulated.

(c) The jump operator σ is rd-continuous.

(d) If f is regulated or rd-continuous, then so is f^{σ}.

(e) Suppose f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then so is fog.

Definition 2.18 *A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if*

$$F^{\Delta}(t) = f(t)$$

holds for all $t \in \mathbb{T}^k$. The definite integral is defined by

$$\int_a^b f(\tau)\ \Delta\tau = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}$$

Theorem 2.19 *(Existence of antiderivative). Every rd-continuous function has an antiderivative.*

In particular if $t_0 \in \mathbb{T}$, *then* F *defined by*

$$F(t) = \int_{t_0}^{t} f(\tau)\ \Delta\tau \quad \textit{for } t \in \mathbb{T},$$

is an antiderivative of f.

Theorem 2.20 *If* $f \in C_{rd}$ *and* $t \in \mathbb{T}^k$, *then*

$$\int_{t}^{\sigma(t)} f(\tau)\ \Delta\tau = \mu(t) f(t)$$

Theorem 2.21 *If* $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$ *and* $f, g \in C_{rd}$ *then*

(a) $\int_a^b [f(t) + g(t)]\ \Delta t = \int_a^b f(t)\ \Delta t + \int_a^b g(t)\ \Delta t$

(b) $\int_a^b [\alpha f](t)\ \Delta t = \alpha \int_a^b f(t)\ \Delta t$

(c) $\int_a^b f(t)\ \Delta t = -\int_b^a f(t)\ \Delta t$

(d) $\int_a^b f(t)\ \Delta t = \int_a^c f(t)\ \Delta t + \int_c^b f(t)\ \Delta t$

(e) $\int_a^b f(\sigma(t)) g^{\Delta}(t)\ \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(t)\ \Delta t$

(f) $\int_a^b f(t) g^{\Delta}(t)\ \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g(\sigma(t))\ \Delta t$

(g) $\int_a^a f(t)\ \Delta t = 0$

(h) If $\mid f(t) \mid \leq g(t)$ *on [a,b), then*

$$\mid \int_a^b f(t)\Delta t \mid \leq \int_a^b g(t)\Delta t$$

(i) If $f(t) \geq 0$ *for all* $a \leq t \leq b$, *then*

$$\int_a^b f(t)\Delta t \geq 0$$

Theorem 2.22 *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(a) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt$$

where the integral on the right is the normal Riemann integral.

(b) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & if \quad a < b \\ -\sum_{t=b}^{a-1} f(t) & if \quad a > b \end{cases}$$

Definition 2.23 *If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and if f is rd-continuous on $[a, \infty)$, then the improper integral is defined by*

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided the limit exists.

2.3 First order linear differential equations on time scale

Having introduced the delta-differential and integral calculi in the previous sections, we are now ready to introduce the concept of differential equations on time scales.

Definition 2.24 *Assume $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation*

$$y^{\Delta} = f(t, y, y^{\sigma}), \text{ where } y^{\sigma} = y\left(\sigma\left(t\right)\right)$$

is called a first order dynamic equation and it is linear if

$$f(t, y, y^{\sigma}) = f_1(t)y + f_2(t) \text{ or } f(t, y, y^{\sigma}) = f_1(t)y^{\sigma} + f_2(t)$$

where f_1 and f_2 are functions.

Definition 2.25 *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if*

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } \ t \in \mathbb{T}.$$

Definition 2.26 *If $t_0 \in \mathbb{T}$ and p is rd-continuous and regressive, then the only solution of the initial value problem*

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1$$

is given by $e_p(., t_0)$ where

$$e_p(t, s) = \exp[\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau] \quad s, t \in \mathbb{T}$$

is the exponential function with

$$\xi_\mu(z) = \begin{cases} \frac{1}{\mu}\log(1+\mu z) & \text{if } \mu \neq 0 \\ z & \text{if } \mu = 0 \end{cases}$$

being the cylinder transformation.

Example 2.27 *If $p(t) = c$ (a constant), then the solution to the initial value problem*

$$y^\Delta = cy, \quad y(0) = 1$$

is given by

$$e_c(t,0) = \exp[\int_0^t \xi_{\mu(\tau)}(c))\Delta\tau].$$

If $\mathbb{T} = \mathbb{R}$, then

$$e_c(t,0) = \exp\left(\int_0^t \xi_0(c)\, d\tau\right) = \exp\left(\int_0^t c d\tau\right) = e^{ct}.$$

If $\mathbb{T} = \mathbb{Z}$, then

$$e_c(t,0) = \exp\left(\int_0^t \log(1+c)\,\Delta\tau\right) = \exp\left(\sum_{n=0}^{t-1} \log(1+c)\right) = (1+c)^t$$

Example 2.28 *If $p(t) = t$, then the solution to the initial value problem*

$$y^\Delta = ty, \quad y(0) = 1$$

is given by

$$e_t(t,0) = \exp[\int_0^t \xi_{\mu(\tau)}(t)\,\Delta\tau]$$

If $\mathbb{T} = \mathbb{R}$, then

$$e_t(t,0) = e^{\frac{t^2}{2}}.$$

If $\mathbb{T} = \mathbb{Z}$, then

$$e_t(t,0) = \prod_{n=0}^{t-1} (1+n) = t!$$

Definition 2.29 *Suppose $p, q \in \Re$, the set of all regressive and rd-continuous functions. Then the circle plus '$\oplus$' and circle minus '$\ominus$' on $\Re$ are respectively defined by*

$$p \oplus q = p + q + \mu pq \quad \text{and } p \ominus q = p \oplus (\ominus q)$$

where $\ominus q = -\frac{q}{1+\mu q}$.

We now consider a theorem that illustrates some properties of the exponential function.

Theorem 2.30 *Let $p, q : \mathbb{T} \to \mathbb{R}$ be regressive and rd-continuous, then*

(a) $e_0(t,s) = 1$ and $e_p(t,t) = 1$

(b) $e_p(\sigma(t), s) = (1 + \mu(t)p(t)) \;\; e_p(t,s)$

(c) $\frac{1}{e_p(t,s)} = \; e_{\ominus p}(t,s)$

(d) $e_p(t,s) = \frac{1}{e_p(s,t)} = \; e_{\ominus p}(s,t)$

(e) $e_p(t,s)e_p(s,r) = e_p(t,r)$

(f) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$

(g) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$

(i) $(\frac{1}{e_p(.,s)})^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(s,t)}$

Note that the above theorem recovers all the known results about the exponential functions when the time scale is $\mathbb{R}$.

Theorem 2.31 *Let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and regressive. Assume $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$.*

Then the unique solution of the initial non-homogeneous value problem

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = \ e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau \tag{2.1}$$

2.4 First order linear system on time scale

We shall consider, in this section, a system of the form

$$y^{\Delta} = A(t)y(t) \tag{2.2}$$

where A is an $n \times n$ matrix valued function on $\mathbb{T}$. If $y(t)$ satisfies (2.2) for all $t \in \mathbb{T}^k$ then y is said to be a solution to the system.

Definition 2.32 *An $n \times n$ matrix valued function A on a time scale $\mathbb{T}$ is called regressive provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$, and the class of all such regressive and rd-continuous function is also denoted by $\Re$. (2.2) is said to be regressive if $A \in \Re$.*

For the existence and uniqueness of solution to initial value problems, we have

Theorem 2.33 *(Existence and Uniqueness Theorem). Let $A \in \Re$ be an $n \times n$ matrix valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\Delta} = A(t)y + f(t), \quad y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$

Definition 2.34 *Let A and B be regressive $n \times n$ matrix valued functions on $\mathbb{T}$. Then $A \oplus B$ is defined by*

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)\, A(t)B(t) \ for\ all\ t \in \mathbb{T}^k,$$

and $\ominus A$ *is defined by*

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) \ for \ all \ t \in \mathbb{T}^k.$$

Definition 2.35 *(Matrix Exponential Function). Let* $t_0 \in \mathbb{T}$ *and assume that* $A \in \Re$ *is an* $n \times n$ *matrix valued function. The unique matrix valued solution of the initial value problem*

$$Y^{\Delta} = A(t)Y, \quad Y(t_0) = I,$$

where I *denotes the* $n \times n$ *identity matrix , is known as the matrix exponential function at* t_0 *and is denoted by* $e_A(., t_0)$.

Theorem 2.36 *(Variation of Parameters). Let* $A \in \Re$ *be an* $n \times n$ *matrix valued function on* $\mathbb{T}$ *and suppose that* $f : \mathbb{T} \to \mathbb{R}^n$ *is rd-continuous. Let* $t_0 \in \mathbb{T}$ *and* $y_0 \in \mathbb{R}^n$. *Then the initial value problem*

$$y^{\Delta} = A(t)y + f(t), \quad y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$ *given by*

$$y(t) = \ e_A(t, t_0)y_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Next, we shall solve two examples of differential equations on time scale starting with the case when A is a constant matrix and then consider a simple non-constant matrix A.

When the matrix A is constant, we have the following theorem.

Theorem 2.37 *If* (λ_0, ξ) *is an eigenpair for the constant* $n \times n$ *matrix* A, *then*

$$y(t) = e_{\lambda_0}(t, t_0)\xi$$

is a solution of $y^{\Delta} = Ax$ *on* $\mathbb{T}$.

Example 2.38 *Solve the vector dynamic equation*

$$y^{\Delta} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} y.$$

Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$. The eigenvalues are solution to $\mid A - \lambda I \mid = 0$, thus $(3-\lambda)(2-\lambda) - 2 = 0$, that is $\lambda^2 - 5\lambda + 4 = 0$, from which we get $\lambda_1 = 1$ and $\lambda_2 = 4$. Eigenvectors corresponding to λ_1 and λ_2 are $\xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ respectively. So

$$y(t) = C_1 e_1(t, t_0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e_4(t, t_0) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

is the general solution for any time scale $\mathbb{T}$. For a specific time scale, we need only to interprete $e_1(t, t_0)$ and $e_4(t, t_0)$.

Example 2.39 *Solve the vector initial value problem*

$$y^{\Delta} = \begin{pmatrix} 1 & 1 \\ 0 & t \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This can be written as

$$y_1^{\Delta} = y_1 + y_2, \qquad y_1(0) = 1 \tag{2.3}$$

$$y_2^{\Delta} = ty_2, \qquad y_2(0) = 1 \tag{2.4}$$

>From (2.4) and for $\mu \neq 0$, we have

$$y_2(t) = e_t(t,0) = \exp\left(\int_0^t \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)\tau)\,\Delta\tau\right)$$

Thus (2.3) becomes

$$y_1^{\Delta} = y_1 + \exp\left(\int_0^t \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)\tau)\,\Delta\tau\right), \qquad y_1(0) = 1$$

Using (2.1)

$$y_1(t) = e_1(t,0)\,y_1(0) + \int_0^t e_1(t,\sigma(s))\,y_2(s)\,\Delta s$$

$$y_1(t) = e_1(t,0) + \int_0^t \frac{e_1(t,s)}{1+\mu(s)} \exp\left(\int_0^s \frac{1}{\mu(\tau)} \log(1 + \mu(\tau)\tau)\,\Delta\tau\right)\Delta s.$$

For $\mu = 0$, the solution reduces to $y_1(t) = e^t + \int\limits_0^t e^{(t-s)+\frac{s^2}{2}}\Delta s, \quad y_2(t) = e^{\frac{t^2}{2}}$.

Chapter 3

Impulsive First Order Differential Equation

In a recent paper [15], the author has shown with simple mathematics how we can get a solution to an impulsive differential equation. More specifically, he considered a two point-boundary value problem with generalized impulse conditions at finite number of points

$$\begin{cases} \frac{dy}{dx} = A(x)\,y + f(x)\,,\ x \in [a,b]\,,\ x \neq \tau_i \\ By(a) + Cy(b) = d \qquad\qquad\qquad\qquad , i = 1, 2, ..., p \\ N_i y(\tau_i + 0) + M_i y(\tau_i - 0) = v_{i,} \end{cases} \tag{3.1}$$

$$a = \tau_0 < \tau_1 < ..... < \tau_p < \tau_{p+1} = b$$

where the coefficients of the system satisfy the following conditions.

(A1) A is an $n \times n$ matrix with continuous elements on $[a, b]$, f is an n-dimensional piecewise continuous vector function with break points at τ_i : $f(x) = f_1(x)$, $x \in [a, \tau_1]$, $f(x) = f_i(x)$, $x \in (\tau_i, \tau_{i+1}]$, $i = 1, 2, ..., p$ where $f_i(x)$ is continuous in $(\tau_i, \tau_{i+1}]$, $i = 1, 2, ..., p$

(A2) B and C are $k \times n$ constant matrices, $d \in \mathbb{R}^k$ and M_i, N_i, $i = 1, 2, ..., p$, are $s \times n$ constant matrices, $v_i \in \mathbb{R}^s$.

The author gave conditions for the existence of a solution $y(.) \in C^1([a, b] \setminus \{\tau_1, ..., \tau_p\})$ of the impulsive boundary-value problem (3.1) and constructed it.

Let $x_i = \tau_i$, $i = 0, 1, 2, .., p + 1$. The key idea is to consider the system over each one of the intervals $[x_{i-1}, x_i]$ and use a linear change of variable

$$x = x_{i-1} + th_i \text{ where } h_i = x_i - x_{i-1},\ t \in [0, 1]. \tag{3.2}$$

The original system becomes

$$\frac{dz_i(t)}{dt} = A_i(t) z_i(t) + g_i(t),\ i = 1, 2, ..., p + 1,\ t \in [0, 1] \tag{3.3}$$

where $z_i(t) = y(x_{i-1} + th_i)$, $A_i(t) = h_i A(x_{i-1} + th_i)$, $g_i(t) = h_i f(x_{i-1} + th_i)$. It is not difficult to see that $z_1(0) = y(a)$, $z_{p+1}(1) = y(b)$ so that

$$Bz_1(0) + Cz_{p+1}(1) = d \tag{3.4}$$

Since the solution y of the problem (3.1) is continuous on every interval $[a, x_1], (x_{i-1}, x_i]$, $i = 2, .., p + 1$, then as $\lim_{x \to \tau_i + 0} y(x) = y(\tau_i + 0)$,

we obtain

$$y(\tau_i - 0) = z_i(1)\,,\; y(\tau_i + 0) = z_{i+1}(0)\,.$$

Therefore

$$M_i z_i(1) + N_i z_{i+1}(0) = v_i, 1, 2, .., p \tag{3.5}$$

Defining the $(p+1)\,n-$dimensional vectors z and g in an obvious manner and the matrices $\overline{A}, \overline{B}, \overline{C}$ and the vector $\overline{d}$ accordingly, we obtain the generalized two-point boundary-value problem

$$\dot{z}(t) = \bar{A}(t)\,z(t) + g(t)\,,\;\; t \in [0,1] \tag{3.6}$$

$$\bar{B}z(0) + \bar{C}z(1) = \bar{d} \tag{3.7}$$

Let $\bar{\Phi}(t)\,,\; \bar{\Phi}(0) = E_{(p+1)n}$ be the fundamental matrix of $\dot{z}(t) = \bar{A}(t)\,z$. The generalized solution of the system (3.6) takes the form

$$z(t) = \bar{\Phi}(t)c + \eta(t),\;\; c \in \mathbb{R}^{(p+1)n} \tag{3.8}$$

where $\eta(t) = \int\limits_0^t \bar{\Phi}(t)\;\bar{\Phi}^{-1}(s)g(s)ds$ is a particular solution of (3.6) . Substituting (3.8) into the boundary condition (3.7) bearing in mind that $\bar{\Phi}(0) = E_{(p+1)n}$, $\eta(0) = 0$, one obtains

$$Qc = \bar{d} - \bar{C}\eta(1), \tag{3.9}$$

where $Q = \bar{B} + \bar{C}\;\bar{\Phi}(1)$ is $(k+ps) \times (p+1)n$ matrix.

Denoting by Q^+ the $(p+1)n \times (k+ps)$ pseudo-inverse matrix of the matrix Q, by P_Q and P_{Q^*} the ortho-projectors $P_Q : \mathbb{R}^{(p+1)n} \to \ker(Q)$, $P_{Q^*} : \mathbb{R}^{k+ps} \to \ker(Q^*)$, $Q^* = Q^T$. For information regarding pseudo-inverse and ortho-projectors, see [6],[10], [14], [19]. The author derives the following theorem and corollaries.

Theorem 3.1 *Let the conditions (A1), (A2) be satisfied and rank* $Q = n_1 < \min(k+ps, (p+1)n)$. *Then the boundary-value problem (3.6), (3.7) has a one-parametric family of solutions*

$$z(t,\xi) = \Phi(t)P_Q\xi + \bar{z}(t),\ \ z(t) = \Phi(t)Q^+\left(\bar{d} - \bar{C}\eta(1)\right) + \eta(t) \tag{3.10}$$

if and only if $P_{Q^*}(\bar{d} - \bar{C}\eta(1)) = 0$.

Corollary 3.2 *Let the conditions (A1), (A2) be satisfied and* $P_{Q^*} = 0$. *Then the boundary-value problem (3.6), (3.7) has a unique solution of the form (3.10).*

In this case rank $Q = (p+1)n$ *and the system (3.9) is always solvable.*

Corollary 3.3 *Let the conditions (A1), (A2) be satisfied and* $P_Q = 0$. *Then the boundary-value problem (3.6), (3.7) has a unique solution* $z(t) = \bar{z}(t)$ *if and only if* $P_{Q^*}(\bar{d} - \bar{C}\eta(1)) = 0$.

In this case rank $Q = k + ps$.

Corollary 3.4 *Let the conditions (A1), (A2) be satisfied and* $P_Q = 0$, $P_{Q^*} = 0$. *Then the boundary-value problem (3.6), (3.7) has a unique solution* $z(t) = \bar{z}(t)$ *and* $Q^+ = Q^{-1}$.

In this case $k+ps = (p+1)n$ *and* $\det Q \neq 0$. *Here we may supplement a case when* $k = s = n$.

As a final point one has to go back to the original variables to obtain

$$y_1(x) = [\bar{\Phi}(\frac{x-a}{h_1})P_Q]_{n_1}\xi + [\bar{z}(\frac{x-a}{h_1})]_{n_1},\ \ x \in [a, \tau_1]$$

$$y_i(x) = [\bar{\Phi}(\frac{x-\tau_i}{h_i})P_Q]_{n_i}\xi + [\bar{z}(\frac{x-\tau_i}{h_i})]_{n_i},\ x \in [\tau_{i-1}, \tau_i],\ i = 2, ..., p+1$$

where $n_1 + n_2 + ... + n_{p+1} = (p+1)n$, n_1 means the first n in number rows, n_2- the second and etc., $n_{p+1}-$ the last n in the number rows of the matrix $\bar{\Phi}(t)P_Q$ and the vector $\bar{z}(t)$.

We shall adapt the above approach to the time scale setting in the next chapter.

Chapter 4

Boundary Value Problems on Time Scales with Impulse Effects

Let $\mathbb{T}_0$ be a time scale and suppose $0, 1 \in \mathbb{T}_0$. We define the interval $[0,1)_{\mathbb{T}_0}$ by $[0,1)_{\mathbb{T}_0} = \{t \in \mathbb{T}_0 : \ 0 \leq t < 1\}$. Other types of intervals are defined similarly. Let $t_0 < t_1 < \cdots < t_{p+1}$ be in $\mathbb{R}$ and let $\varphi_i : [0,1)_{\mathbb{T}_0} \to [t_{i-1}, t_i)$, $i = 1, \cdots, p+1$, be strictly increasing differentiable functions and onto. Then $\mathbb{T} = \{t_{p+1}\} \cup \left(\cup_{i=1}^{p+1} \varphi_i([0,1)_{\mathbb{T}_0})\right)$ is a time scale. Let $a = t_0$ and $b = \rho(t_{p+1})$, ρ being the backward jump operator on $\mathbb{T}$. We shall adapt the approach presented in [15] to the time scale setting. Note that we cannot simply take an arbitrary time scale $\mathbb{T}$ and map the intervals $[t_{i-1}, t_i]_{\mathbb{T}}$ to $[0,1]_{\mathbb{R}}$ as is done in [15] because $\varphi_i^{-1}([t_{i-1}, t_i]_{\mathbb{T}}) \neq \varphi_j^{-1}([t_{j-1}, t_j]_{\mathbb{T}})$ for $j \neq i$.

Consider the boundary value problem with impulsive effects

$$\begin{cases} y^{\Delta} = A(t)y + f(t) \ , \ t \in [a,b]_{\mathbb{T}} \ , \ t \neq t_i \\ By(a) + Cy(\sigma(b)) = d \\ N_i y(t_i + 0) + M_i y(t_i - 0) = v_i \ , \ i = 1, \cdots, p \end{cases} \tag{4.1}$$

σ being the forward jump operator in $\mathbb{T}$ and the coefficients involved satisfy the following hypotheses,

(H1) A is an $n \times n$ matrix-valued function on $\mathbb{T}$, rd-continuous and regressive and $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous.

(H2) B and C are $k \times n$ constant matrices, d is a constant k-vector, M_i, N_i are constant $m \times n$ matrices for $i = 1, \cdots, p$ and v_i are constant m-vectors, $i = 1, \cdots, p$.

For $t \in [t_{i-1}, t_i)_{\mathbb{T}}$, let $x = \varphi_i^{-1}(t)$ and define $z_i(x) = y(\varphi_i(x))$. Let $\widetilde{\Delta}$ denote differentiation in $\mathbb{T}_0$ and σ_0 the corresponding forward jump. Replacing into the differential equation we get,

$$\begin{aligned} z_i^{\widetilde{\Delta}}(x) &= y^{\Delta}(\varphi_i(x))\varphi_i^{\widetilde{\Delta}}(x) \\ &= \varphi_i^{\widetilde{\Delta}}(x)A(\varphi_i(x))y(\varphi_i(x)) + \varphi_i^{\widetilde{\Delta}}(x)f(\varphi_i(x)) \\ &= \varphi_i^{\widetilde{\Delta}}(x)A(\varphi_i(x))z_i(x) + \varphi_i^{\widetilde{\Delta}}(x)f(\varphi_i(x)) \end{aligned}$$

Define

$$A_i(x) = \varphi_i^{\widetilde{\Delta}}(x)A(\varphi_i(x))$$
$$g_i(x) = \varphi_i^{\widetilde{\Delta}}(x)f(\varphi_i(x))$$

We obtain

$$z_i^{\widetilde{\Delta}}(x) = A_i(x)z_i(x) + g_i(x) \ , \ x \in [0,1)_{\mathbb{T}_0}, \ i = 1, \cdots, p.$$

As for the boundary conditions, we get

$$Bz_1(0) + Cz_{p+1}(1) = d$$

Since y is rd-continuous over each interval $[t_{i-1}, t_i)_{\mathbb{T}}$, we have $y(t_i + 0) = z_{i+1}(0)$ and $y(t_i - 0) = z_i(1)$ so that the impulse conditions become

$$N_i z_{i+1}(0) + M_i z_i(1) = v_i \ , \ i = 1, \cdots, p.$$

Now define z, g, $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{d}$ by

$$\begin{aligned}
z &= \begin{pmatrix} z_1 \\ \vdots \\ z_{p+1} \end{pmatrix}, \ g = \begin{pmatrix} g_1 \\ \vdots \\ g_{p+1} \end{pmatrix}, \\
\overline{A} &= \operatorname{diag}(A_1, \cdots, A_{p+1}), \ \overline{B} = \operatorname{diag}(B, N_1, \cdots, N_p) \\
\overline{C} &= \begin{pmatrix} 0 & \cdots & \cdots & 0 & C \\ M_1 & 0 & & & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & 0 & & 0 & \vdots \\ 0 & \cdots & 0 & M_p & 0 \end{pmatrix}, \ \overline{d} = (d, v_1, \cdots, v_p)^T
\end{aligned}$$

with appropriate dimensions. Therefore, (4.1) is transformed into

$$\begin{cases} z^{\tilde{\Delta}}(x) = \overline{A}(x)z(x) + g(x) \ , \ x \in [0,1]_{\mathbb{T}_0} \\ \overline{B}z(0) + \overline{C}z(1) = \overline{d} \end{cases} \tag{4.2}$$

Since $\overline{A}$ is regressive and g is rd-continuous the initial value problem

$$\begin{cases} z^{\tilde{\Delta}}(x) = \overline{A}(x)z(x) + g(x) \ , \ x \in [0,1]_{\mathbb{T}_0} \\ z(0) = \alpha \end{cases} \tag{4.3}$$

has a unique solution given by (Theorem 2.34)

$$z(x) = e_{\overline{A}}(x,0)\alpha + \int_0^x e_{\overline{A}}(x,\sigma_0(\xi))g(\xi)\Delta\xi \ , \ x \in [0,1]_{\mathbb{T}_0}$$

so that

$$z(1) = e_{\overline{A}}(1,0)\alpha + \int_0^1 e_{\overline{A}}(1,\sigma_0(\xi))g(\xi)\Delta\xi$$

and the boundary condition in (4.2) becomes

$$\overline{B}\alpha + \overline{C}\left\{e_{\overline{A}}(1,0)\alpha + \int_0^1 e_{\overline{A}}(1,\sigma_0(\xi))g(\xi)\Delta\xi\right\} = \overline{d}$$

that is

$$Q\alpha = S$$

where $Q = \overline{B} + \overline{C}e_{\overline{A}}(1,0)$ and $S = \overline{d} - \overline{C}\int_0^1 e_{\overline{A}}(1,\sigma_0(\xi))g(\xi)\Delta\xi$.

Denoting by Q^+ the pseudo-inverse of the $(p+1)n$ by $(k+ms)$ matrix Q and by P_Q and P_{Q^*} the ortho-projectors $P_Q : \mathbb{R}^{(p+1)n} \to Ker\ Q$ and $P_{Q^*} : \mathbb{R}^{k+ms} \to Ker\ Q^*$, $Q^* = Q^T$, we obtain the following analogue theorem and corollary of [15].

Theorem 4.1 *Suppose hypotheses (H1) and (H2) are satisfied then the boundary*

value problem on time scale (4.2) has a one-parameter family of solutions

$$z(x,\eta) = e_{\overline{A}}(x,0)\left\{P_Q\eta + Q^+S\right\} + \int_0^x e_{\overline{A}}(x,\sigma_0(\xi))g(\xi)\Delta\xi \ , \ x \in [0,1)_{\mathbb{T}_0}$$

if and only if $P_{Q^*}S = 0$.

Corollary 4.2 *Suppose hypotheses (H1) and (H2) are satisfied and* $P_Q = 0$ *then the boundary value problem on time scale (4.2) has a unique solution*

$$z(x,\eta) = e_{\overline{A}}(x,0)Q^+S + \int_0^x e_{\overline{A}}(x,\sigma_0(\xi))g(\xi)\Delta\xi \ , \ x \in [0,1)_{\mathbb{T}_0}$$

if and only if $P_{Q^*}S = 0$.

This leads to the following theorem regarding the existence of a unique solution to the original problem (4.1).

Theorem 4.3 *Suppose hypotheses (H1) and (H2) are satisfied and* $P_Q = 0$ *then the boundary value problem on time scale with impulses (4.1) has a unique solution if and only if* $P_{Q^*}S = 0$.

Remark 4.4

1) If $\mathbb{T}_0 = \mathbb{R}$ then we recover the results of [15] if we define

$$\varphi_i(x) = t_{i-1} + x(t_i - t_{i-1}).$$

Example 4.5 *Let* $\mathbb{T}_0$ *be and arbitrary time scale. Consider* $\varphi_1 : [0,1)_{\mathbb{T}_0} \to \mathbb{R}$, $\varphi_2 : [0,1)_{\mathbb{T}_0} \to \mathbb{R}$ *defined by* $\varphi_1(x) = x$, $\varphi_2(x) = x+1$. *Let* $\mathbb{T} = \{2\} \cup \varphi_1\left([0,1)_{\mathbb{T}_0}\right) \cup \varphi_2\left([0,1)_{\mathbb{T}_0}\right)$. $\mathbb{T}$ *is a time scale.*

Consider the impulsive problem

$$y^{\Delta} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} y, \quad t \in [0,2]_{\mathbb{T}}, t \neq 1 \tag{4.4}$$

$$y(0) + y(2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.5}$$

$$y(1+0) - y(1-0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.6}$$

This can be written as

$$y_1^{\Delta} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} y_1, \quad t \in [0,1)_{\mathbb{T}}$$

$$y_2^{\Delta} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} y_2, \quad t \in (1,2]_{\mathbb{T}}$$

Let $x = \varphi_1^{-1}(t) = t$ *and* $x = \varphi_2^{-1}(t) = t-1$ *be the respective change of variables. Let* $y_1(t) = z_1(x)$, $y_2(t) = z_2(x+1)$ *and for* $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, *we have*

$$z^{\tilde{\Delta}}(x) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} z(x), \ x \in [0,1]_{\mathbb{T}_0} \tag{4.7}$$

$$z(0) + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} z(1) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{4.8}$$

which are solved instead of $(4.4) - (4.6)$.

Let $\overline{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}$, *and suppose that* $\mu \neq 1$ *or* $\frac{1}{2}$ *then* $\overline{A}$ *is regressive.*

The eigenvalues are solution to $\left|\overline{A} - \lambda I\right| = 0$, *that is*

$$\begin{vmatrix} -1-\lambda & 1 & 0 & 0 \\ 0 & -2-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 1 \\ 0 & 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

which gives $(-1-\lambda)^2(-2-\lambda)^2 = 0$, *that is* $\lambda_1 = -1, \lambda_2 = -2$ *(multiplicity 2).*

For $\lambda_1 = -1$,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

giving $\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ *and* $\xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ *as two linearly independent eigenvectors.*

For $\lambda_2 = -2$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

giving $\xi_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ *and* $\xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ *as two linearly independent eigenvectors.*

The general solution is

$$z(x) = E_1 e_{-1}(x,0)\ \xi_1 + E_2 e_{-1}(x,0)\ \xi_2 + E_3 e_{-2}(x,0)\ \xi_3 + E_4 e_{-2}(x,0)\ \xi_4$$

>*From* (4.8), *we have, by putting* $\overline{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

$(E_1\ \xi_1 + E_2\xi_2 + E_3\ \xi_3 + E_4\ \xi_4) +$

$$\overline{C}\left[E_1 e_{-1}(1,0)\ \xi_1 + E_2 e_{-1}(1,0)\ \xi_2 + E_3 e_{-2}(1,0)\ \xi_3 + E_4 e_{-2}(1,0)\ \xi_4\right] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

that is

$$FE = \overline{d}$$

which is a system of four equations and four unknown E_1, E_2, E_3, E_4 *where*

$$F = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{pmatrix} + \overline{C}\begin{pmatrix} e_{-1}(1,0)\ \xi_1 & e_{-1}(1,0)\ \xi_2 & e_{-2}(1,0)\ \xi_3 & e_{-2}(1,0)\ \xi_4 \end{pmatrix},$$

$$E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix} \text{ and } \overline{d} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}.$$

We denote by F^+ the pseudo-inverse of F and by P_F, P_{F^*} the orthoprojectors and get the one-parameter family of solutions

$$z(x,\eta) = e_{\overline{A}}(x,0)\left\{P_F\eta + F^+\overline{d}\right\} \text{ where } P_F\overline{d} = 0$$

To get the solution $y(t,\eta)$ over each interval $[t_{i-1}, t_i)$, we have to use the change of variable $x = \varphi^{-1}(t)$ giving

$$y(t,\eta) = z\left(\varphi^{-1}(t), \eta\right),\ t \in [t_{i-1}, t_i)\ ,\ i = 1, 2.$$

Chapter 5

Conclusion and Further Research

In this work, we have considered boundary value problems on time scales with impulse effects using a technique introduced in [15] in the context of impulsive differential equations over the set of real numbers. We have shown how his work can be extended to time scale settings. We also worked out an example to illustrate the method.

Since our work is based on a time scale having a particular structure, then a possible research work would be to remove this restriction by using a different method. Another research topic would be the numerical analysis of differential equations on time scale so that modeling and simulations of real life problems could be achieved.

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