Analytical Calculation of Bow Shock Waves From Delta Shaped Supersonic Lifting Wings

by

Daoud Yacoub Shehadeh Abdel-Nabi

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

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In Partial Fulfillment of the
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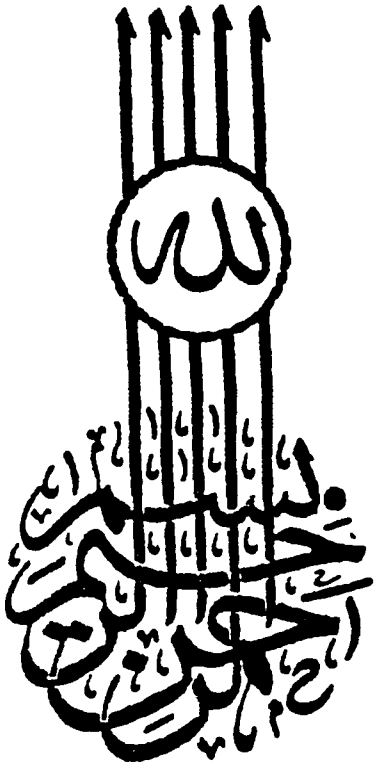

In the Name of ALLAH,
the Most Merciful and Most Gracious.

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This thesis, written by

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Member (Dr. Muhammad U. Budair)

To My

Parents

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Praise and thanks be to Almighty *Allah* first and last who made every difficult thing easy for us.

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TABLE OF CONTENTS

Page #

LIST OF FIGURES

NOMENCLATURE

A	$A = \tan\alpha_2 \tan\theta_2$ in Chapter 3 cross-sectional area in Chapter 4
A_C, A_D	total area of the delta wing with curved and straight leading edges respectively
$A(M_o)$	parameter defined in equation (5.35)
A_1	parameter defined in equation (4.56)
a	speed of sound
B	$B = \sqrt{\frac{1 + T_S}{2T_S}}$
B_1	parameter defined in equation (4.56)
$\vec{b}$	acceleration vector
C	$1 - \frac{E^2}{2}$ in Chapter 3
C_1	parameter defined in equation (4.56)
C_L	lift coefficient
C_p	specific heat at constant pressure
D	parameter defined in equation (5.42)
D_1	parameter defined in equation (4.56)
D_S	shock wave drag
E	$E = \beta_o \tan\alpha_1$ in Chapter 4
E_1	parameter defined in equation (4.56)
$\vec{f}$	force vector
$f(x)$	distribution of sources as a function of x
$f(\tau)$	doublet strength as a function of the retarted time
$\vec{F}$	force field
$F(\xi)$	Whitham F-function

$F_c(\xi)$	Whitham F-function in the conical region
$F_e(\xi)$	Whitham F-function in the expansion region
F_t	total lift
g^*	potential energy
$G(t)$	function of time
h	specific enthalpy
H	parameter defined in equation (4.72)
H_1	parameter defined in equation (4.78)
I	parameter defined in equation (3.66)
J	parameter defined in equation (3.73)
K	$K = \frac{E^2}{2}$ in Chapter 3
K_o, K_1, K_2, K_3	parameters defined in equations (2.33), (4.39) and (4.54) respectively
$K_1(\xi), K_2(\eta)$	integration functions, equation (5.38)
L_A	power of the shock wave
l	length of the wing
$l_i(\xi)$	length distribution of lift
M	Mach number
M_{obs}	observer Mach number
N_1	$N_1 = 1 - \beta_o^2 \tan^2\alpha_1 \sin^2\theta_n$
N_2	$N_2 = 1 - \beta_o^2 \tan^2\alpha_2 \sin^2\theta_n$
p	local pressure
$\underline{P}$	pressure function, $\underline{P} = \int \frac{dp}{\rho}$
Δp	$\Delta p = p - p_o$
$\Delta p'$	pressure difference across the wing
q	flow velocity

q^*	heat transfer per unit mass
Q	source strength
r	radial distance of the observer
r_o	$r_o = \sqrt{y_o^2 + z_o^2}$ in first order approximation
r_{ph}	phase radius, $r_{ph} = \frac{M_o \cdot x + r}{\beta_o}$
s	streamwise coordinate
S_1	parameter defined in equation (5.42)
S_w	wing total area
$S(\xi)$	cross-sectional area as a function of ξ
S_c	parameter defined by equation (4.12)
S_o	parameter defined in equation (4.22)
t	bicharacteristic parameter in Chapter 2
t	time parameter in Chapter 3
$\bar{t}$	time in the moving frame
T	temperature
$T(x)$	thickness of two dimensional body
T_s	Tsien-parameter, $T_s = \tan\nu_o \cot\alpha_o$
u,v,w	dimensionless velocity in the x-, y- and z-directions respectively
u',v',w'	velocity perturbations in the x-, y- and z-directions respectively
$\delta u_1, \delta v_1$	parameters defined in equation (5.35)
U,V,W	velocity components in the x-, y- and z-directions respectively
U^*	force potential
w^*	work per unit mass
x,y,z	tri-rectangular coordinate system in the physical space

$\bar{x},\bar{y},\bar{z}$	coordinates in the moving frame
α	local Mach angle in Chapters 3, 4 and 5
α_1	half-apex angle of the delta wing with straight leading edges
α_2	half-apex angle of the delta wing with curved leading edges.
β	$\beta = \sqrt{M^2 - 1}$
β_s	$\beta_s = \frac{M_o - 1}{M_o + 1}$
γ	ratio of specific heats
γ_n	$\gamma_n = 2\theta_n$
δ	parameter defined in equation (4.16)
ε	expansion parameter
η	free stream characteristic variable, $\eta = x + \beta_o \cdot y$
η_a	parameter defined by equation (4.26)
ζ_l	characteristic coordinate used in Chapter 3
ζ	characteristic variable
θ	flow angle relative to x, z-plane
θ_1,θ_2	inclination angle of the observer's plane in the lateral direction for the delta wing with straight and curved leading edges respectively
θ_n	inclination angle related to θ_1 by equation (3.71)
ξ	free stream characteristic variable, $\xi = x - \beta_o \cdot y$
ξ_l	characteristic coordinate used in Chapter 3
ξ_n	neutral Mach line
ξ'	parameter defined in equation (3.48)
ρ	fluid density
Λ	parameter defined in equation (4.17)

Λ_a	parameter defined by equation (4.32)
μ	Mach angle, used in Chapter 2, $\mu = \sin^{-1}\left(\frac{1}{M}\right)$
τ	retarded time, $\tau = t - \frac{r_{oh}}{a}$
ν	the angle of the streamline relative to x-direction
ν_o	half-apex angle of the cone
ν_1	dimensionless quantity, $\nu_1 = \frac{V}{U_o}$
ψ	acceleration potential
Φ	velocity potential
ϕ	perturbed velocity potential, $\phi = \Phi - Ux$

Subscripts

o	parameter is related to the undisturbed free stream
0,1,2,...	order of approximation
x,y,z,ξ,η, S,t, ...	partial derivatives with respect to x, y, z, ξ, η, S, t, ...
ℓ	upper value (used in Chapter 5)
u	lower value (used in Chapter 5)

Abbreviations

AMC	Analytical Method of Characteristics
MCS	Method of Characteristic Surfaces
PLK	Poincaré-Lighthill-Kuo Method.

ABSTRACT

In this thesis an acceleration potential (source) is used to investigate the effect of force distribution on the generation and decay of the bow shock waves due to delta wings with straight and curved leading edges in a supersonic flow field. Formulas for the bow shock strength are derived for an observer perpendicular below the wing and for the lateral direction as well.

It is concluded that the equivalent body of revolution is valid throughout the supersonic far flow field. The decay of the shock waves is predicted for the observer in the plane vertical to the wing.

Also the position of the characteristics is corrected up to first order.

ملخص البحث

حساب الموجات الصدمية الأمامية للأجنحة المثلثية الرافعة ذات السرعات فوق الصوتية

في هذا البحث تم استخدام جهد التسارع (Accelaration Potential) لدراسة تأثير اختلاف توزيع القوى على تكوّن وتلاشي الموجات الصدمية الأمامية - المتسببة عن الأجنحة المثلثية الرافعة ذات الحواف المستقيمة والمنحنية وذات السرعات فوق الصوتية - في المسافات الطويلة ، وبذلك تم ايجاد صيغ رياضية تصف قوة وشكل هذه الموجات الصدمية بالنسبة لمراقب ليس فقط في المستوى العمودي على الجناح بل في أي اتجاه جانبي آخر.

وباستخدام الجسم الدوراني المكافئ لهذه الأجنحة فقد تم حساب قوة وشكل هذه الموجات الصدمية في المستويات العمودية على الجناح.

كما أنه تم تصحيح موقع خطوط مسار الاضطرابات " Characteristics " في المسافات البعيدة حتى رتبة العظم الأولى.

CHAPTER 1

INTRODUCTION

An aircraft moving through the atmosphere at a supersonic speed creates an acoustic wave. This wave propagates through the atmosphere, undergoes some nonlinear distortion and eventually strikes the ground or is dissipated in the upper atmosphere. An observer on the ground usually hears a sonic boom (acoustic phenomenon associated with supersonic flight) as a sharp double report. Sonic boom is a noxious phenomenon — it is annoying and can do some physical damage. The sonic boom presence and apparant unavoidability impose important limitations on the operation of supersonic transport (SST). Present design and operational planning for a supersonic transport has been based on the assumption that all overland flights would be at subsonic speeds. The unacceptability of sonic boom is largely connected with the presence of the shock waves. Bangless boom or sonic boom signal without shock waves, would likely have a much higher degree of acceptability. There is a need for analytical methods that are capable of accurately predicting the flow field pressures for design optimization in connection with the sonic boom as well as the performance of the vehicle.

A method particularly well suited for such problems is

the Analytical Method of Characteristics.

LITERATURE REVIEW

From its development during the 1950s, the theory of Whitham and Walkden [59, 62, 63] has gained great confidence for the prediction of sonic boom pressure signatures, generated by three-dimensional lifting or non-lifting configurations. This confidence is rather due to a good agreement of the obtained results with experimental data [5, 53, 64] than to a thorough understanding of the flow phenomena, described by the theory.

In fact, considering the simplicity of the calculation technique, the correctness of its results is rather astonishing. In order to calculate the flow perturbations at large distances (far field) from any configuration, this configuration is first schematized by the so-called 'Supersonic area rule'. According to this rule, the flow in any plane through the axis of a wing body combination is, at large distances, equivalent to the flow past a certain body of revolution.

The present supersonic area rule is an extension of the original supersonic area rule of Hayes [14] and Ward [60] and the transonic area rule of Whitcomb [61] and Oswatitsch [41]. In these older formulations the area distribution of the equivalent body of revolution is taken equal to the area

distribution of the original configuration. Also in these formulations, the presence of any lift forces does not result in an additional equivalent area distribution because those papers were concerned with the effect of thickness. The shape of the equivalent body of revolution is found as the sum of two separate contributions. The first one is due to the distribution of 'frontal area' of the intersections of the configuration with oblique planes, forming the Mach angle (μ) with the configuration's axis.

The second contribution is obtained from the distribution of 'lift forces', parallel to the chosen meridional plane [37].

Accepting the supersonic area rule, the far field pressure signatures may be calculated from the equivalent body by the asymptotic procedure of Whitham [62] which is valid for axisymmetric bodies only. The basis of the procedure is the hypothesis, "the conventional linearized theory gives a good approximation of the flow perturbations in an entire supersonic flow field, provided that the undisturbed, acoustical characteristics in the solution are replaced by more exact ones".

Although Whitham could not fully prove his hypothesis, he made it acceptable by examining the approximations, incorporated in linearized theory. His argumentation was as follows:

In linearized theory disturbances are supposed to propagate through the entire flow field at constant speed which is the speed of sound in the undisturbed, uniform flow. The theory does not account for the variation of the propagation speed due to the presence of the perturbations themselves. As a result of this approximation the wave fronts (the characteristics) are found to be straight and parallel.

If interest is only in a description of the supersonic flow at small distances (near field) from a slender body, the effect of the approximation is negligible up to first order. With growing distance to the body the neglection of the variation of the propagation speed introduces a cumulative error in the location of the characteristics. This will no longer be of second or higher order.

It may be expected that the magnitude of the flow perturbations is not affected by the cumulative error. Then the cumulative second order effect only causes a shift of the linear flow perturbations from the straight and parallel characteristics (contained in linearized theory) to a more exact representation of the characteristics. Due to the good agreement of the predictions by the Whitham-Walkden theory with experimental data for a wide variety of configurations the technique has been generally accepted for many years. However, the physical significance, in particular of the area rule concept, has never been fully understood (Sears 51).

That is because already at relatively small distances, the supersonic flow in each meridional plane through a slender wing-body combination can be described as a flow past a certain body of revolution. Especially the fact that the equivalent body does not change with increasing distance seems rather peculiar. In fact this means that the flow in one meridional plane develops independently of the flow in neighbouring planes. Sears formulate this phenomena as, "In such (nearly cylindrical acoustical flow) fields there is a rapid meridional adjustment near the axis, but negligible further meridional adjustment at large distances".

In 1972, however, for the first time the supersonic area rule was seriously questioned by Oswatitisch and Sun [42]. The authors pointed out that an approximately plane expansion wave eminates from the straight trailing edge of a supersonic delta-wing. It seems unlikely that the plane expansion wave can correctly be accounted for by the area rule, which, due to the equivalent body of revolution concept, deals only with cylindrical waves. In order to circumvent the area rule, Oswatitsch and Sun calculated the wave formation in the vertical plane of symmetry of the delta wing by the Analytical Method of Characteristics (AMC). This method was proposed by Oswatitsch in 1962 [43], as an extension of the characteristic perturbation theory of C.C.Lin [36] who made the first treatise on the analytical method of characteristics. In a selected reference plane the first order flow

perturbations are shifted from straight and parallel linear characteristics to the generally curved first order characteristics. In contrast to Whitham-Walkden theory the first order characteristics are calculated by means of the unmodified linearized flow perturbations. In the AMC, therefore, no supplementary approximation is introduced by the supersonic area rule concept.

As a striking result, Oswatitsch and Sun concluded that the bow shock wave in the vertical plane of symmetry of the delta wing vanishes at a finite distance from the wing due to the strong interaction of the conical shock wave and the plane trailing edge expansion wave. In contradiction to the result of the Whitham-Walkden theory, no classical N-shaped pressure signature occurs.

In 1975, Hendriks, T.P.M., studied the characteristic surfaces which envelop the domains of dependence in the flow field near the trailing edge expansion fan [18]. In 1978, Kluwick,A., developed the employed geometric considerations into an analytical procedure [25]. The investigations reveal a rapid three-dimensional spreading of the waves generated by the configuration. The spreading in its turn causes the meridional adjustment of the flow perturbations, which opens the way for an application of the supersonic area rule. Therefore, it is concluded qualitatively that the N-wave from the Whitham-Walkden theory is more likely than the pressure found

from the formally more accurate AMC.

In [18] the transition of the complex flow near the delta wing to the simple flow past a set of equivalent bodies of revolution (one for each meridional plane) could only be described qualitatively.

In 1981, Hendriks [19] presented a Method of Characteristic Surfaces (MCS). The method allows the quantitative calculation of the transition of the three-dimensional near field flow to the quasi-axisymmetric far field flow. The MCS is a three-dimensional analogue to the two-dimensional AMC.

In the AMC the solution from the linearized theory is corrected for the variation of the propagation speed of the flow perturbation due to the presence of the perturbations themselves.

In two-dimensional supersonic flow fields this correction consists of the replacement of the linear wave fronts (linear characteristics) by the more exact first order wave fronts (first order characteristics). While in three-dimensional flow fields the wave fronts are equivalent to the charecteristic surfaces.

For the mathematical background of the AMC it is appropriate to go back to the year 1949. In that year, Lighthill suggested a technique for rendering approximate solutions for physical problems uniformly valid [35]. The procedure is

useful in solving non-linear differential equations, which may be linearized everywhere, except in a limited region of the solution area. This method of Lighthill involves the introduction of a new set of independent variables. With respect to these new variables the original independent variables are expanded in a power series of a small parameter. A successive approximation results in a straining of the original variables. The straining is chosen in such a way, that, with respect to the strained coordinates, the linearization of the differential equation is approximately valid throughout the solution area.

Next to Lighthill the names of Poincaré and Kuo must be mentioned in relation to the method described above. Poincaré used a similar technique already in 1893 to describe nonlinear oscillations [47]. Kuo extended the procedure substantially by the introduction of the boundary layer concept, both in the mathematical and physical sense [27, 28]. In view of these contributions the procedure is generally known as the PLK-method. In some literature, however, it is denoted as the Lighthill method or the method of strained coordinates.

In several investigations the PLK-method has proven to be very useful in describing flow problem in two independent variables. In 1949, Lighthill himself applied the technique to the shock formation problems in supersonic conical

fields [35]. At that point some insight is gained in the effects of the three-dimensionality of the problem. It appears that for a correct calculation of the wave formation these spatial effects may not be neglected.

Having collaborated with Lighthill in developing the Lighthill part of the PLK-method, Whitham applied the technique to the supersonic flow past slender bodies of revolution, far from the axis [64]. The calculation results in a very simple recipe for gaining correct far field perturbations from the classical linearized theory. In its usual form, the linearized theory is only valid in the very near surroundings of slender supersonic configurations.

Whitham, however, found that the magnitude of the flow perturbations in the far field is correctly predicted by linearized theory. To achieve a correct distribution at large distances, the perturbations have to be shifted from the straight and parallel characteristics in the linearized theory towards the corresponding first order characteristics.

The extension of the PLK-calculation to the entire flow field was found to be impossible. Therefore, Whitham [61] formulated a hypothesis, that the above displacement of the linearized flow solution results in a uniform first order description not only in the far field, but also at small and moderate distances from the body. This hypothesis forms the

basis of the well-known Whitham-Walkden theory. In a somewhat modified form it reappears as the starting point of Oswatitsch Analytical Method of Characteristics (AMC).

In his method, Oswatitsch used the characreristic manifolds (bicharacteristics) as independent variables. This insures the correct inclusion of the ranges of influence and domains of dependence. The physical coordinates including the time as well as the velocity and the thermodynamic quantities are expressed by power series expansions. These series are then substituted into the hyperbolic system under consideration. The physical coordinates (now dependent variables) are obtained by integration along bicharacteristic lines. The dependent variables for a given boundary and initial conditions (written in the characteristic space) are found from the compatibility conditions. This procedure can be repeated in an iterative manner until the desired accuracy is reached. The characteristic space indicated by the coordinates x_o, y_o, z_o and t_o coincides with the physical space if there is no perturbations. The geometry of wave fronts and rays in the characteristic space has to be known before applying the Analytical Method of Characteristics (AMC).

Integration along the characteristics gives the correct position of the characteristics. Shocks are then received in areas of overlapping of characteristics in front of and

behind the shock. Prandtl-Meyer expansions are received as well as regions where the characteristics are divergent. Weak shocks are found from the condition that they are in first order bisectors of the characteristic slopes in front of and behind the shock. This is known as the bisector rule for shock waves.

The main difference between Oswatitsch's method and Whitham method is that Oswatitsch uses several sets of independent variables (characteristics), while Whitham uses one set of independent variables only. By Whitham's method, one can calculate the characteristics running in the shock direction of a projectile but not the other set of characteristics which intersects the first set. Both sets can be easily calculated by Oswatitsch technique, which is applicable to the flow in the near field, mid field, and the far field as well.

Most of the previous work uses a velocity potential which is convenient for calculating the effect of thickness. However, to account for the effects of forces such as lift and drag an acceleration potential may be more convenient. The acceleration potential (Prandtl's acceleration potential) existance is assured for <u>barotropic</u> flows. The term '<u>barotropic</u>' implies a unique pressure-density relation throughout the entire flow field.

In the present thesis, in order to gain some understanding of the relation between geometric and mathematical properties the linearized theory, the PLK-method and the AMC are described rather extensively in Chapter 2.

In Chapter 3 the 'Acceleration Potential' is presented to calculate the pressure jump along the characteristics not only in the planes normal to the wing but also in other inclined planes, for different force distributions.

In Chapter 4 the shock strength and geometry for delta wings with straight and curved leading edges is calculated analytically in the plane perpendicular to the wing using the equivalent body of revolution. Next the shock drag is calculated for both wings.

In Chapter 5 using the AMC the formula for calculating the first order perturbed coordinates is derived. By integrating the first order velocity perturbations along the characteristics, the first order perturbed coordinates can be calculated which corrects the position of the characteristics.

OBJECTIVES

1. Presentation of an Acceleration Potential to investigate the effect of different force distributions on

the generation and decay of shock waves in a far supersonic flow field.

2. Calculation of the bow shock wave strength for a delta wing with straight leading edges and a one with curved leading edges in the vertical plane.

3. Correction of the characteristics position up to first order.

CHAPTER 2

THE LINEARIZED THEORY, POINCARE-LIGHTHILL-KUO METHOD, AND THE ANALYTICAL METHOD OF CHARACTERISTICS.

In the following sections these three methods are compared by discussing the calculation of the supersonic flow past thin two-dimensional bodies.

First the classical linearized theory is outlined. It is demonstrated that the obtained solution is only valid in the vicinity of the perturbing body.

Subsequently the Poincare-Lighthill-Kuo (PLK) method and the Analytical Method of Characteristics (AMC) are described. Both methods eliminate the non-uniformity in the linearized solution by a straining of the spatial coordinates. Non-uniformity means the solution is not uniformly valid; it is only valid near the body.

In the PLK-method the appropriate straining is found from the condition that the contributions of second order terms must be finite throughout the flow field. This leads to the displacement of the linearized flow solution from the acoustical characteristics to the first order ones.

In the AMC, in a sense the reverse procedure is applied. The calculation starts with the displacement of the linearized

flow solution in the above manner. Only as a second step the uniformity of the resulting first order description is derived from an estimation of the second order perturbations of the flow quantities.

Because in the AMC the straining of the spatial coordinates is a priori derived from a first order calculation of the characteristics, the method can easily be applied to problems which are inaccessible for the PLK-method. Even the description of flow problems, which are beyond the scope of the procedure, can often be realized by the AMC.

2.1 : THE LINEARIZED THEORY

In the inviscid, isentropic, steady, adiabatic flow of a compressible fluid in the absence of body forces and external work the following governing linear partial differential equation can be derived from basic laws of motion:

$$(a^2-U^2)\Phi_{xx}+(a^2-V^2)\Phi_{yy}+(a^2-W^2)\Phi_{zz}-2UV\Phi_{xy}-2UW\Phi_{xz}-2VW\Phi_{yz}=0 \tag{2.1}$$

For the flow past a two-dimensional body, all derivatives in the lateral z- direction are identical zero. Then equation (2.1) is reduced to:

$$(a^2-U^2)\Phi_{xx}+(a^2-V^2)\Phi_{yy}-2UV\Phi_{xy}=0 \tag{2.2}$$

Herein Φ denotes the velocity potential, whereas $U = \Phi_x$ and $V = \Phi_y$, (a) is the local speed of sound, U and V are the velocity components in x- and y-directions respectively. Introducing the perturbation potential ϕ as $\Phi = U_o(x + \phi)$, then:

$$\Phi_x = U = U_o(1 + \phi_x),$$

$$\Phi_{xx} = U_o \phi_{xx},$$

$$\Phi_y = V = U_o \phi_y,$$

$$\Phi_{yy} = U_o \phi_{yy},$$

and $$\Phi_{xy} = U_o \phi_{xy}.$$

Substituting these values in equation (2.2) above yields:

$$[a^2-(U_o+U_o\phi_x)^2]U_o\phi_{xx}+(a^2-U_o^2\phi_y^2)U_o\phi_{yy}-2(U_o+U_o\phi_x)(U_o\phi_y)U_o\cdot\phi_{xy} = 0 \tag{2.3}$$

Simplifying equation (2.3) one can get:

$$a^2(\phi_{xx}+\phi_{yy}) = U_o^2\{\phi_{xx}+(2\phi_x+\phi_x^2)\phi_{xx}+2(1+\phi_x)\phi_{xy}\phi_y+\phi_y^2\cdot\phi_{yy}\} \tag{2.4}$$

to simplify this form we have to express the variation of (a) in terms of ϕ_x and ϕ_y:

The energy equation for adiabatic flow in terms of a, a_o and q is given by:

$$a^2 = a_o^2 - \frac{\gamma - 1}{2} \cdot q^2 \tag{2.5}$$

where q is the flow velocity, i.e.,

$$q^2 = U^2 + V^2 \tag{2.6}$$

but $\quad U = \Phi_x$

$\quad\quad = U_o(1 + \phi_x)$

and $\quad V = \Phi_y$

$\quad\quad = U_o \phi_y$

Substituting these values of U and V in equation (2.6), then substituting the resulting equation in equation (2.5) yields:

$$a^2 = a_o^2 - \frac{\gamma - 1}{2} \cdot U_o^2 \{ 1 + 2\phi_x + \phi_x^2 + \phi_y^2 \} \tag{2.7}$$

Substituting equation (2.7) in equation (2.4) yields:

$$a_o^2(\phi_{xx}+\phi_{yy}) = \frac{\gamma - 1}{2} U_o^2\{2\phi_x+\phi_x^2+\phi_y^2\}\cdot(\phi_{xx}+\phi_{yy})+U_o^2\{\phi_{xx}+(2\phi_x+\phi_x^2)\phi_{xx} + 2(1+\phi_x)\phi_{xy}\phi_y+\phi_y^2\phi_{yy}\}$$

Simplifying more one can get:

$$-\beta_o^2 \phi_{xx}+\phi_{yy} = M_o^2 \left\{ \frac{\gamma - 1}{2} (2\phi_x+\phi_x^2+\phi_y^2)(\phi_{xx} + \phi_{yy})+(2\phi_x+\phi_x^2)\phi_{xx} +2(1+\phi_x)\phi_{xy}\phi_y+\phi_y^2\phi_{yy} \right\} \tag{2.8}$$

where $M_o = \frac{U_o}{a_o}$, $\beta_o^2 = M_o^2-1$ and γ is the specific heat ratio. The index "o" refers to uniform flow quantities. If a perturbation of the uniform flow is caused by a slender body which is described as $y = \varepsilon T(x)$ with $\varepsilon << 1$, where $T(x)$ is the thickness distribution of the body, the potential ϕ can be expanded in a power series of ε:

$$\phi(x,y) = \varepsilon\phi_1(x,y) + \varepsilon^2\phi_2(x,y) + \varepsilon^3\phi_3(x,y) + \ldots\ldots \tag{2.9}$$

Substituting equation (2.9) in equation (2.8) and equating terms having equal powers of ε results in the following set of relations:

$$-\beta_o^2\ \phi_{1xx} + \phi_{1yy} = 0 \tag{2.10a}$$

$$-\beta_o^2\ \phi_{2xx} + \phi_{2yy} = M_o^2\{(\gamma-1)\phi_{1x}\cdot(\phi_{1xx}+\phi_{1yy}) + 2\phi_{1x}\phi_{1xx} + 2\phi_{1y}\phi_{1xy} \tag{2.10b}$$

$$-\beta_o^2\ \phi_{3xx} + \phi_{3yy} = \ldots\ldots \tag{2.10c}$$

Using linearized theory means solving only equation (2.10a) which contains the lowest power terms of ε. For the upper half plane the solution of equation (2.10a) is given as:

$$\left.\begin{aligned} \phi_{1x} &= -\frac{T'(\xi)}{\beta_o} \\ \phi_{1y} &= T'(\xi) \end{aligned}\right\} \tag{2.11}$$

where ξ is the free stream characteristic variable, $\xi = x - \beta_o y$ and $T' = \frac{dT}{dx} = \left.\frac{dT}{d\xi}\right|_{y=0}$.

According to this result the flow perturbations are constant along the charcteristics ξ = constant of the undisturbed uniform flow. The geometry of the chracteristics have not been affected by the presence of the body (Fig. 2.1).

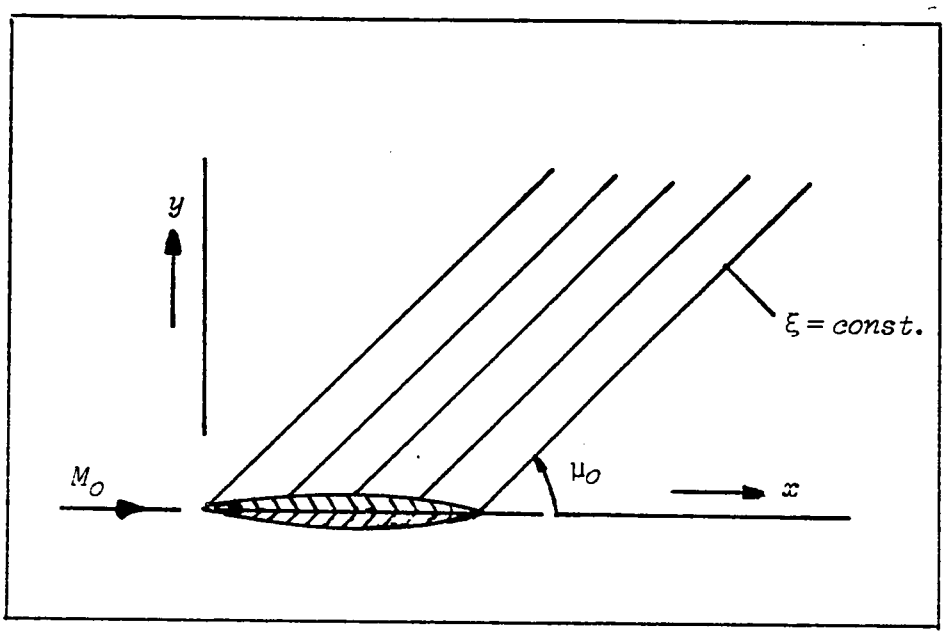

Fig. 2.1. The undisturbed characteristics ξ = constant, as predicted by the linearized theory.

The linear solution (2.11) is valid in the very near vicinity of the disturbing body only. With growing distance, terms which are locally of second order accumulate and give rise to a contribution which is no longer negligible up to first order. These terms have a so-called cumulative effect (Hayes[15]). The presence of such terms can be shown by solving equation (2.10b) for the ε^2-term of the potential equation. The right hand side of this equation is a known function of x and y due to equation (2.11). As given by Van Dyke [55] the result has the form:

$$\phi_{2x} = \frac{1}{\beta_0^2}\left(1 - \frac{\gamma+1}{4}\frac{M_0^4}{\beta_0^2}\right)T'^2(\xi) - \frac{\gamma+1}{2}\frac{M_0^4}{\beta_0^3}\, yT'(\xi)\cdot T''(\xi) - T''(\xi)\cdot T(\xi) \qquad (2.12)$$

The cumulative effect is contained in the y-dependent second term. Due to this the second order contribution ϕ_2 in the perturbation potential ϕ grows unlimited with increasing distance to the body. At large distances ϕ_2 will no longer be small compared to the first order potential ϕ_1. Therefore, the linear solution ϕ_1 is not suitable for decreasing the perturbations in that part of the flow field.

The accumulation of second order terms in the solution of equation (2.10) may be ascribed to the incorrect approximation of the characteristic directions. For the full perturbation equation (2.8), the characteristic directions up to first order are given as:

$$\left.\frac{dx}{dy}\right|_{\text{charact.}} = \pm \left\{\beta_o \quad \frac{\gamma+1 \cdot M_o^4}{2\beta_o} \cdot \phi_x\right\} - M_o^2(\phi_y \pm \beta_o \, \phi_x) \qquad (2.13)$$

In each level of approximation in the system (2.10), however, the characteristics remain unchanged and identical to the characteristics of the undisturbed free stream. The direction of these characteristics satisfy:

$$\left.\frac{dx}{dy}\right|_{\text{charact.}} = \pm \beta_o \qquad (2.14)$$

Realizing that characteristics from the boundaries of the domain of influence of any point in the flow field, it may be clear that the approximation of equation (2.8) by the system (2.10) cannot lead to a correct description of an entire supersonic flow field, especially at large distances from a disturbing body, the differences in the domain of dependence can be essential (Fig. 2.2).

In the following paragraphs two techniques for correcting the non-uniformity in the linear solution are outlined. Although the approach of the problem in the PLK-method is rather different from the Analytical Method of Characteristics, both techniques are essentially identical.

2.2 : THE POINCARÉ-LIGHTHILL-KUO METHOD

As early as 1893 Poincaré used a technique of straining an independent variable of a differential equation in order to circumvent a non-uniformity in calculating oscillatory

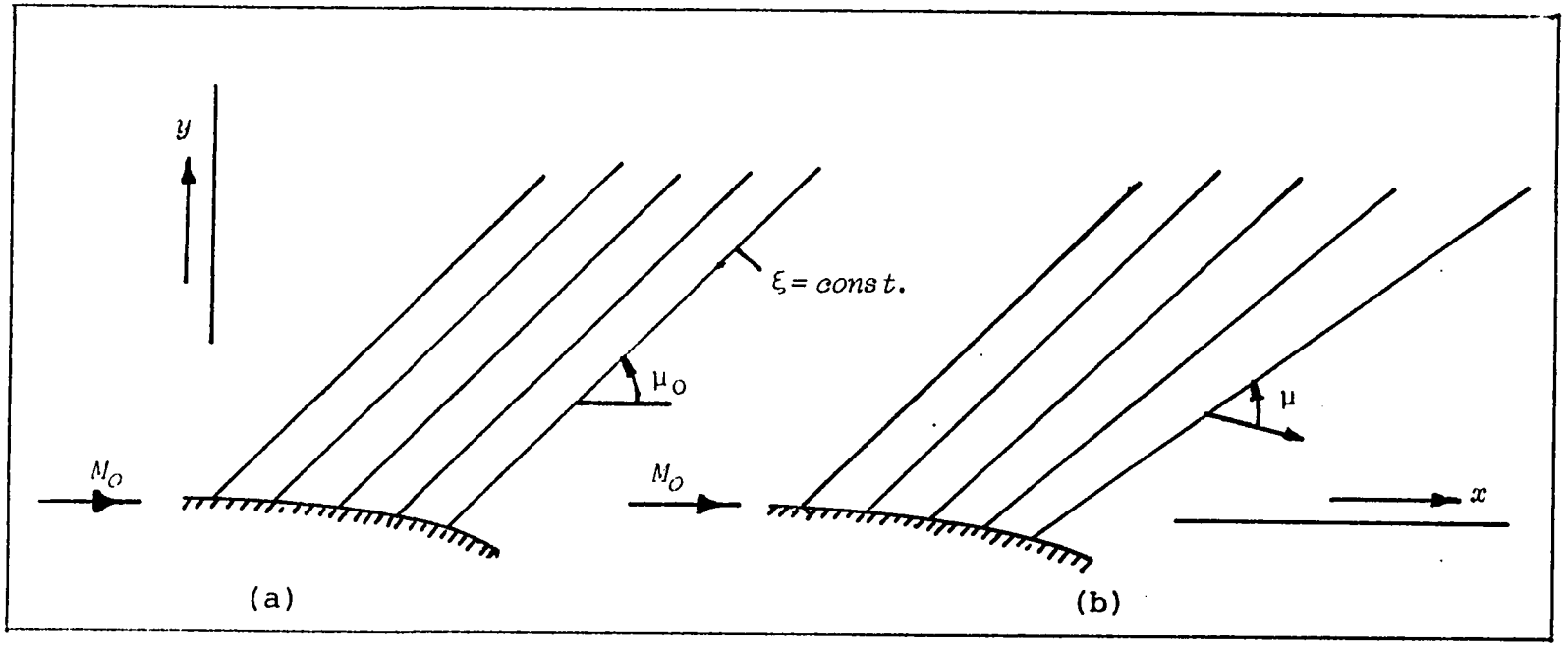

Fig. 2.2

(a) Linear characteristics, equation (2.14).

(b) First order characteristics, equation (2.13)

systems [47]. In 1949 Lighthill presented the technique as a more general procedure for treating non-linear differential equations [34]. The technique is capable of removing non-uniformities from an approximate linear solution, provided that the non-uniformities are confined to a limited part of the solution area.

In [34] Lighthill estimated the integrated effect of various second order terms in general quasi-linear hyperbolic partial differential equations. Due to the general character of the analysis, it is rather abstract and global. The essential points in the PLK-method can better be illustrated by the more specific example of a supersonic flow past a thin two-dimensional body. For this problem a solution in closed form for the coordinate straining can be obtained. The non-uniformity in the linear solution, described in the foregoing paragraph, arises as $y \to \infty$ along the free stream characteristic ξ = constant. Following Lighthill the 'regular' coordinate ξ should be strained in order to achieve a non-diverging second order solution. Firstly, equation (2.8) for the perturbation potential ϕ is transformed into the linear characteristic coordinates $\xi = x - \beta_o y$ and $\eta = x + \beta_o y$. Neglecting the ϕ^3-terms, equation (2.8) takes the form:

$$-4\beta_o^2\phi_{\xi\eta} = M_o^2 \{ (\gamma+1)M_o^2(\phi_\xi\phi_{\xi\xi} + \phi_\eta\phi_{\eta\eta}) + (\gamma M_o^2 + 1 - 3\beta_o^2)(\phi_\xi\phi_{\eta\eta} + \phi_\eta\phi_{\xi\xi})$$

$$+ 2(M_o^2 - \beta_o^2\gamma + \gamma)(\phi_\xi + \phi_\eta)\phi_{\xi\eta} \} \qquad (2.15)$$

Lighthill argues that the only accumulating ϕ^2-term in equation (2.15) is the $\phi_\xi \phi_{\xi\xi}$ term. All other terms on the right hand side have time second order influence. In order to obtain a uniform first order solution it is therefore sufficient to deal with the equation

$$\phi_{\xi\eta} + \frac{(\gamma+1)M_o^4}{4\beta_o^2}\,\phi_\xi \phi_{\xi\xi} = 0 \tag{2.16}$$

Substituting $u = \phi_\xi = \frac{1}{2}\,\phi_x - \frac{1}{2\beta_o}\,\phi_y$ results in:

$$u_\eta + \frac{(\gamma+1)M_o^4}{4\beta_o^2} \cdot u \cdot u_\xi = 0 \tag{2.17}$$

where u is dimensionless x-velocity and

η is the free stream characteristic variable,
$\eta = x + \beta_o y$.

At this point the straining of ξ is initiated. The equation (2.17) is solved using a power series in ε, not only for the dependent parameter u but also for the independent variable ξ. A new set of independent variables (s,t) is introduced, so that

$$\left.\begin{aligned} u(s,t) &= \varepsilon\, u_1(s,t) + \varepsilon^2 u_2(s,t) + \ldots\ldots \\ s &= \xi(s,t) + \varepsilon\, \xi_1(s,t) + \ldots\ldots \\ t &= \eta \end{aligned}\right\} \tag{2.18}$$

where s is a streamwise coordinate, and t is a bicharacteristic parameter.

Equating equal power terms in ε yields the relations:

$$u_{1_t} = 0 \tag{2.19a}$$

$$u_{2_t} + \left(\xi_{1_t} + \frac{(\gamma+1)M_o^4}{4\beta_o^2} u_1\right) u_{1_s} = 0 \tag{2.19b}$$

where the subscripts 0, 1, 2, ...represent the order of approximation. Solving equation (2.19) with the boundary condition $u_1(s,0) = \frac{-T'(s)}{\beta_o}$ yields:

$$u_1(s,t) = \frac{-T'(s)}{\beta_o} \tag{2.20}$$

which corresponds to the linearized solution (2.11) except for the replacement of ξ by s.

The presence of the coordinate straining ξ_1 in the second order equation (2.19b) offers the possibility to prevent unlimited growing of the contribution u_2. The most simple way to achieve this is to choose:

$$\xi_{1t} = -\frac{(\gamma+1)M_o^4}{4\beta_o^2} u_1 \tag{2.21}$$

Then for the second order perturbation velocity u_2 there remains a relation, similar to (2.19a):

$$u_{2t} = 0.$$

Any divergence, as in the second order contribution (2.12) can not be generated by this equation.

Integration of the coordinate straining equation (2.21) and substituting into equation (2.18) leads to:

$$s = \xi - \varepsilon \frac{(\gamma + 1) M_o^4}{4\beta_o} u_{1t} \tag{2.22}$$

Due to the non-divergence of u_2, the equations (2.20) and (2.22) form a solution of (2.17), which is uniform up to first order. The expansion (2.18) is thus proved to be valid. The geometric meaning of the solution (2.20, 2.22) becomes clear if the direction of the new coordinate curves s = constant is calculated and found as:

$$\left.\frac{dx}{dy}\right|_s = \beta_o + \frac{\varepsilon(\gamma + 1) \cdot M_o^4}{2\beta_o} u_1(s) \tag{2.23}$$

This relation corresponds to the first order approximation (2.13) of the characteristic direction of the full perturbation equation (2.8).

The equivalence is easily seen when remembering that

$$u_1 = \phi_{1\xi} = \frac{1}{2}\left(\phi_{1x} - \frac{1}{\beta_o}\phi_{1y}\right),$$

while $\phi_{1y} = -\beta_o \phi_{1x}$ for the first order solution of a simple wave flow.

The solution (2.20) for the first order flow perturbation u_1 is similar to the corresponding result (2.10) from the linearized theory. The only difference is that, in equation (2.15) the undisturbed characteristic ξ = constant is

replaced by the new coordinate curve s = const. (Fig. 2.3). Consequently the following geometric conclusion can be formulated:

> "The linear solution for a plane supersonic flow field is uniformly valid, provided that the linear downstream characteristics ξ = const. in the solution are replaced by at least a first order approximation of the exact characteristics."

This simply means that the solution on the characteristics is correct but its position has to be corrected.

The result of the PLK-method for the supersonic flow past two-dimensional bodies is confirmed and extended by Lin [36] and later adopted by Oswatitsch [44] as the basis of the AMC. AMC technique can be used for calculating those flow fields in two independent variables, which can not easily be handled by the more complex PLK-method.

2.3 : THE ANALYTICAL METHOD OF CHARACTERISTICS

Small-amplitude waves which are governed by nonlinear partial differential equations of hyperbolic type are of importance in many engineering applications. Although the smallness of the wave amplitude results in a weak nonlinearity it is commonly found that the effect may be cumulative and thus must not be neglected if the approximation to wave motion

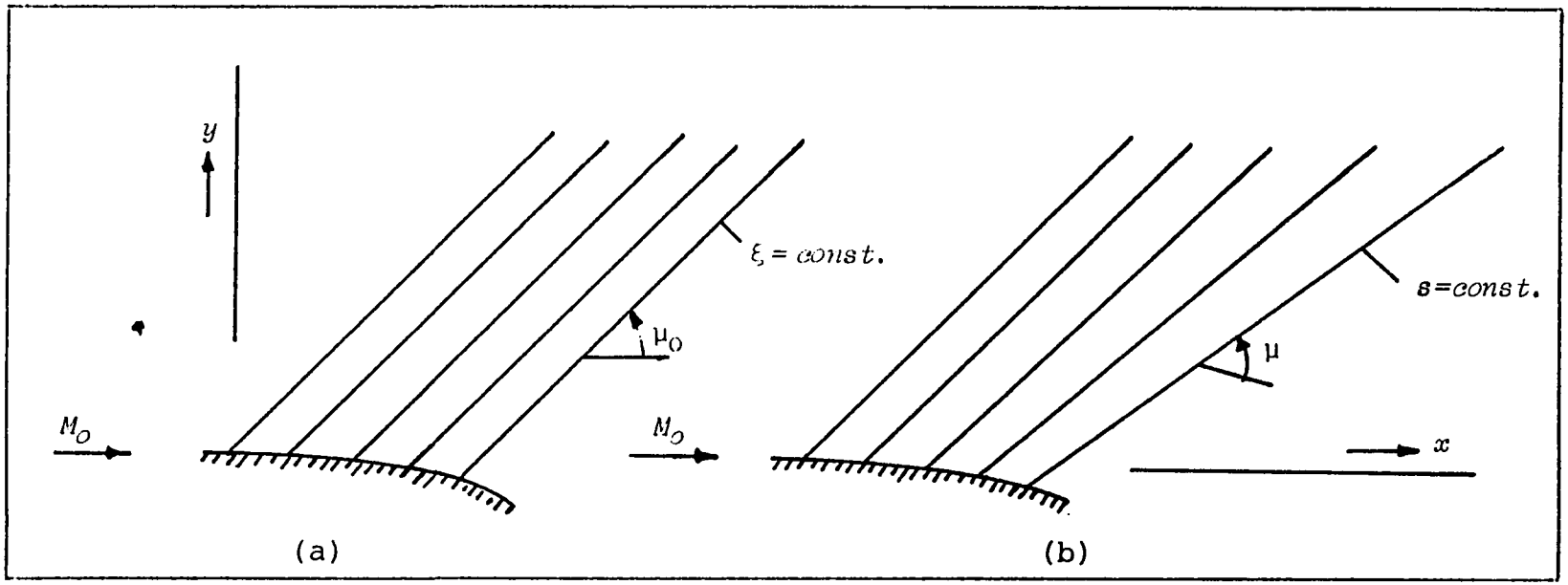

Fig. 2.3. In the PLK-solution the undisturbed characteristics ξ = const. are replaced by the curves s = const.

(a) Linear characteristics

(b) First order characteristics, s = const.

under consideration is to be uniformly valid. A method particularly well suited for this type of problem is the Analytical Method of Characteristics. This method is applicable to the wave propagation in fluids as well as the investigation of nonlinear small-amplitude waves in elastic solids. The concept of *characteristics* may be introduced from several points of view:

1. From a physical point of view, a characteristic curve, for brevity termed a characteristic, is defined as the path of propagation of a physical disturbance. In a supersonic flow field, disturbances are propagated along the Mach lines for the flow. The Mach lines can be shown to be the characteristics for a supersonic flow.

2. From a purely heuristic point of view, a characteristic is defined as a curve along which the governing partial differential equations can be manipulated into total differential equations.

3. From a more rigorous mathematical point of view, a characteristic is a curve across which the derivatives of a physical property may be discontinuous, while the property itself remains constant. Thus regions of flow having continuous properties and derivatives within each region, but a discontinuity in derivatives at their interface, may be joined together along a characteristic.

4. From the most rigorous mathematical point of view, a characteristic is defined as a curve

along which the governing partial differential equations reduced to an interior operator, that is, a total differential equation. Along a characteristic the dependent variables must satisfy the compatibility equation.

The first treatise on the Analytical Method of Characteristics was published by C.C.Lin [35] when he presented a perturbation theory based on the method of characteristics. The theory offers a uniform first order solution for any two dimensional supersonic flow problem. If the flow is just a simple wave the procedure confirms the result of the Poincaré-Lighthill-Kuo method, that the linearized flow solution should be shifted from the undisturbed free stream characteristics ξ = constant to the corresponding first order characteristics.

Compared to the Poincaré-Lighthill-Kuo technique, the procedure of C.C.Lin is so much simpler, that even the calculation of the non-simple wave flow can easily be performed. This computation shows that, for this class of two dimensional flow problems, the linearized solution can be made uniformly valid too. To achieve this, both families of free stream characteristics, ξ = const. and η = const. must be replaced by their first order approximation. In this way a very simple recipe is obtained for the calculation of the intersection of two supersonic wave systems. The computation of such a flow field by the PLK-technique can hardly be performed.

In 1962 Oswatitsch [44] reformulated the theory of Lin to cover not only two dimensional flows, but all supersonic flow problems, expressible in 2, 3 or 4 independent variables. Unlike the original theory of Lin, the uniformity of the obtained first order solution cannot be proved in general. To circumvent this problem, Oswatitsch chooses the following hypothesis as the starting point of the AMC:

> "The linearized solution for a supersonic flow in two independent variables is transformed into a uniformly valid one by replacing both families of free stream characteristics by the corresponding first order characteristics."

That is, the position of the chracteristics is corrected. Besides the fact that this statement is a simple extrapolation of Lin's results, some physical arguments can be given in order to make it acceptable. These arguments were generally gathered by Whitham [59], who used a similar hypothesis to calculate the supersonic flow past a slender body of revolution. A rather extensive review of this argumentation is presented in the introduction of this theory. At this point only the headlines are briefly recalled:

The main reason for the failure of the linearized theory at large distances (far field) from a disturbing body is due to the approximation of geometrical acoustics in undisturbed flow in the propagation speed of the flow distur-

bances. Due to this assumption the chracteristics in the linearized flow solution are identical to the free stream characteristics. Replacing the latter by a more exact approximation, the failure in the linear solution should be repaired, so that a uniformly valid first order description of the flow field is obtained.

The AMC-solution for a simple wave flow is equivalent to the PLK-result. However, the efforts, necessary for the application of the techniques, are mostly rather divergent. In the PLK-technique the complete second order solution must be calculated formally. The appropriate coordinate straining is then prescribed by the demand, that this second order solution is finite throughout the flow field.

Conversly, in the AMC the coordinate straining is determined a priori from the hypothesis given above. The uniformity of the first order solution must be proved afterwards by showing all second order terms non-cumulative. Mostly, this can be done by a simple estimation of their contributions after integration along the characteristics.

In the following,such a uniform proof is outlined for the simple wave flow past a thin two-dimensional body.

Different from the PLK-calculation, the AMC is not based on a successive approximation of the potential equation (2.15) written in terms of the linear characteristics

$\xi = x - \beta_o y$ and $\eta = x + \beta_o y$. Instead the calculation implies a step by step approximation of the inviscid flow equations in characteristic form. In these relations the exact characteristic parameters ξ, η occurs as independent variables. From the gas dynamic equation together with the equation of irrotationality the well-known equation of compatibility is obtained:

$$\left.\begin{aligned} \frac{\beta}{q}\,\frac{dq}{d\xi} - \frac{\partial\theta}{\partial\xi} &= 0 \\ \frac{\beta}{q}\,\frac{\partial q}{\partial\eta} + \frac{\partial\theta}{\partial\eta} &= 0 \end{aligned}\right\} \qquad (2.24)$$

where $\beta = \sqrt{M^2 - 1}$, q is the flow velocity and θ is the flow angle.

Here ξ = const. and η = const. denote the left and right-running Mach lines (characteristics) respectively.

The compatibility equation (2.24) gives the relation between the variation of the flow angle θ and the variation of the value of the flow velocity q along the characteristics. The direction of the ξ- and η-characteristics is described by the characteristic relation (2.25):

$$\left.\begin{aligned} \frac{dx}{d\xi} - \frac{dy}{d\xi}\cot(\theta + \mu) &= 0 \\ \frac{dx}{d\eta} - \frac{dy}{d\eta}\cot(\theta - \mu) &= 0 \end{aligned}\right\} \qquad (2.25)$$

where μ is the Mach angle.

In order to approximate the system (2.24) and (2.25), both the flow parameters θ, q, β, μ and the spatial coordinates x and y are expanded in a power series. In this expansion the characteristics variables ξ, η are used as the true independent variables in the flow problem, so that:

$$\left.\begin{aligned}
q(\xi,\eta) &= q_0 + q_1(\xi,\eta) + q_2(\xi,\eta) + \ldots\ldots \\
\theta(\xi,\eta) &= \theta_1(\xi,\eta) + \theta_2(\xi,\eta) + \ldots\ldots \\
\beta(\xi,\eta) &= \beta_0 + \beta_1(\xi,\eta) + \beta_2(\xi,\eta) + \ldots\ldots \\
\mu(\xi,\eta) &= \mu_0 + \mu_1(\xi,\eta) + \mu_2(\xi,\eta) + \ldots\ldots \\
x(\xi,\eta) &= x_0(\xi,\eta) + x_1(\xi,\eta) + x_2(\xi,\eta) + \ldots\ldots \\
y(\xi,\eta) &= y_0(\xi,\eta) + y_1(\xi,\eta) + y_2(\xi,\eta) + \ldots\ldots
\end{aligned}\right\} \qquad (2.26)$$

The parameters β_1 and μ_1 can directly be expressed in terms of q_1 (Hendricks 19):

$$\left.\begin{aligned}
\beta_1 &= \left[\frac{\gamma+1}{2\beta_0} M_0^4 - \beta_0 M_0^2\right]\frac{q_1}{q_0} \\
\mu_1 &= \frac{-1}{M_0^2}\beta_1
\end{aligned}\right\} \qquad (2.27)$$

For the zeroth order characteristics, the compatibility equation (2.24) becomes meaningless, which leaves

$$\left.\begin{aligned}
\frac{dx_0}{d\xi} - \beta_0 \frac{dy_0}{d\xi} &= 0 \\
\frac{dx_0}{d\eta} + \beta_0 \frac{dy_0}{d\eta} &= 0
\end{aligned}\right\} \qquad (2.28)$$

The expression describes the direction of the characteristics with respect to the undisturbed coordinates x_0, y_0. The corresponding characteristic picture is identical to the picture for the uniform free stream in the absence of any disturbances. The same characteristic network occurs in the linearized theory.

For the higher order characteristics, equation (2.24) yields:

$$\left.\begin{aligned} \frac{\beta_0}{q_0}\,\frac{dq_1}{d\xi} - \frac{d\theta_1}{d\xi} &= 0 \\ \frac{\beta_0}{q_0}\,\frac{dq_1}{d\eta} + \frac{d\theta_1}{d\eta} &= 0 \end{aligned}\right\} \tag{2.29}$$

and equation (2.25) yields:

$$\left.\begin{aligned} \frac{dx_1}{d\xi} - \beta_0\,\frac{dy_1}{d\xi} &= +\frac{dy_0}{d\xi}\left\{\frac{(\gamma+1)M_0^4}{2\beta_0}\cdot\frac{q_1}{q_0} - \frac{M_0^2}{\beta_0}\left(\beta_0\,\frac{q_1}{q_0} + \theta_1\right)\right\} \\ \frac{dx_1}{d\eta} + \beta_0\,\frac{dy_1}{d\eta} &= -\frac{dy_0}{d\eta}\left\{\frac{(\gamma+1)M_0^4}{2\beta_0}\cdot\frac{q_1}{q_0} - \frac{M_0^2}{\beta_0}\left(\beta_0\,\frac{q_1}{q_0} - \theta_1\right)\right\} \end{aligned}\right\} \tag{2.30}$$

Since the first order compatibility equation (2.29) is independent of the spatial coordinates x and y, this relation can be integrated without any knowledge of the characteristic curves. The integration yields the first order flow perturbations θ_1 and q_1 as functions of the chracteristic variables ξ, η. With $\theta_1(\xi,\eta)$ and $q_1(\xi,\eta)$ two different solutions for the flow field can be composed. In the most simple one θ_1

and q_1 are placed on the characteristic network described by (2.28). This combination of the first order flow quantities and the zeroth order (undisturbed) characteristic network is completely equivalent to the solution of the linearized potential equation;

$$-\beta_0^2 \phi_{1xx} + \phi_{1yy} = 0 \tag{2.10a}$$

This description of the flow field is therefore only valid at small distances from the disturbing body.

To extend the validity of the solution of $\theta_1(\xi,\eta)$ and $q_1(\xi,\eta)$, the AMC adds to the characteristic network the correction obtained from the integration of (2.30). If this procedure is applied to a simple wave flow, the linearized flow solution is displaced from the characteristics $\xi = \text{const.}$ toward the corresponding first order curves (fig. 2.4).

The essential differences between the linearized theory and the AMC can clearly be observed from the following diagram. In this diagram the successive approximation of the flow equations in characteristic form is outlined schematically:

Zeroth order characteristics equation (2.28)	
	First order flow quantities equation (2.20)
First order characteristics equation (2.30)	
	Second order flow quantities
Second order characteristic	
. . . . .	. . . . .

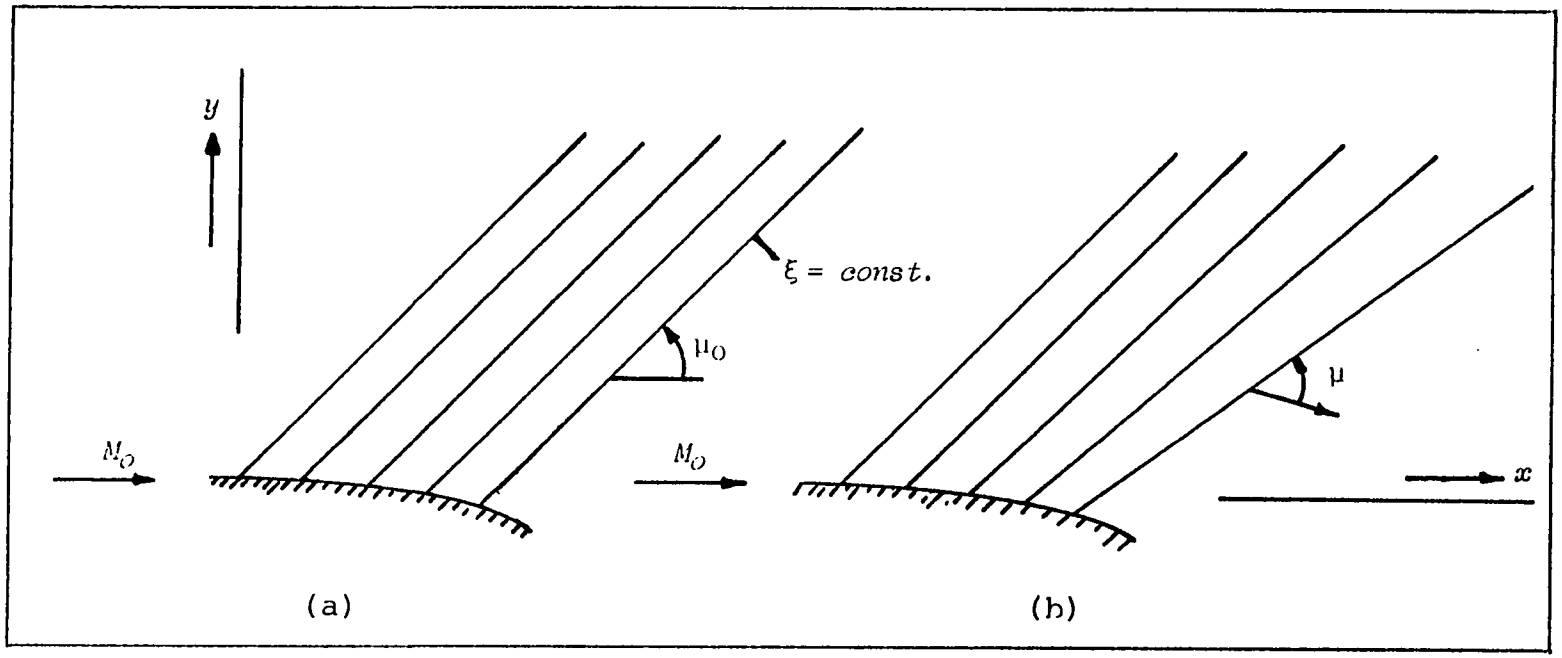

Fig. 2.4. The AMC displaces the linearized flow solution from linear to the first order characteristics.

(a) Linear characteristics.

(b) First order characteristics.

Linearized theory implies the upper line of the above diagram only, since it calculates the first order flow perturbations on the zeroth order characteristics. The effect of the perturbations on the shape of the characteristics is not taken into account.

In the AMC the characteristics are corrected by the first order relation (2.30). In this way the characteristic equation (2.25) as well as the compatibility equation (2.24) are approximated to the same order of accuracy. The consistency of this description of the flow field seems increased in comparison with the result of the linearized theory.

The solution of (2.28) and (2.29, 2.30), according to the AMC may be said to be uniformly valid, if the second order flow quantities θ_2 and q_2 can be shown to be finite throughout the flow field. These parameters result from the following equation of the second order terms of the compatibility relation (2.24):

$$\left.\begin{aligned} \frac{\beta_0}{q_0}\frac{dq_2}{d\xi} - \frac{d\theta_2}{d\xi} &= \frac{(\beta_0 q_1 - \beta_1 q_0)}{q_0^2}\frac{dq_1}{d\xi} \\ \frac{\beta_0}{q_0}\frac{dq_2}{d\eta} + \frac{d\theta_2}{d\eta} &= \frac{(\beta_0 q_1 - \beta_1 q_0)}{q_0^2}\frac{dq_1}{d\eta} \end{aligned}\right\} \qquad (2.31)$$

Since the left hand side of (2.31) is similar to (2.29) non-uniformities in θ_2, q_2 can only arise from cumulative contributions from the term at the right hand side of (2.31).

The absence of such irregularities can be shown by an integration of the right hand side of eqn.(2.31) along both characteristics ξ and η.

For a point P in a supersonic flow between two disturbing bodies (Fig. 2.5) the integrations take the form:

$$\int_{\xi_1}^{\xi_P} \frac{(\beta_0 q_1 - \beta_1 q_0)}{q_0^2} \frac{dq_1}{d\xi} \cdot d\xi = \frac{\beta_0 - K_0}{q_0^2} \int_{q_1(\xi_1,\eta_P)}^{q_1(\xi_P,\eta_P)} q_1 dq_1$$

$$= \frac{\beta_0 - K_0}{2q_0^2} [q_1^2(\xi_P,\eta_P) - q_1^2(\xi_1,\eta_P)] \qquad (2.32)$$

and

$$\int_{\eta_1}^{\eta_P} \frac{\beta_0 q_1 - \beta_1 q_0}{q_0^2} \frac{dq_1}{d\eta} d\eta = \frac{\beta_0 - K_0}{2q_0^2} [q_1^2(\xi_P,\eta_P) - q_1^2(\xi_P,\eta_1)] \qquad (2.33)$$

where

$$K_0 = \frac{\gamma+1}{2\beta_0} M_0^4 - \beta_0 M_0^2$$

In these relations K_0 is used to express β_1 in terms of q_1 in accordance with (2.27).

Since the first order solution $q_1(\xi,\eta) = 0(1)$, the second order contributions (2.32, 2.33) are finite throughout the flow field. This proves, that the successive approximation of the flow equations in characteristics form (2.24, 2.25) does not result in an intolerable accumulation of second order terms. The description of the flow by equations (2.28) and (2.29, 2.30) as proposed in the AMC, is therefore, not only valid at small distances from a disturbing body, but also at moderate and large distances.

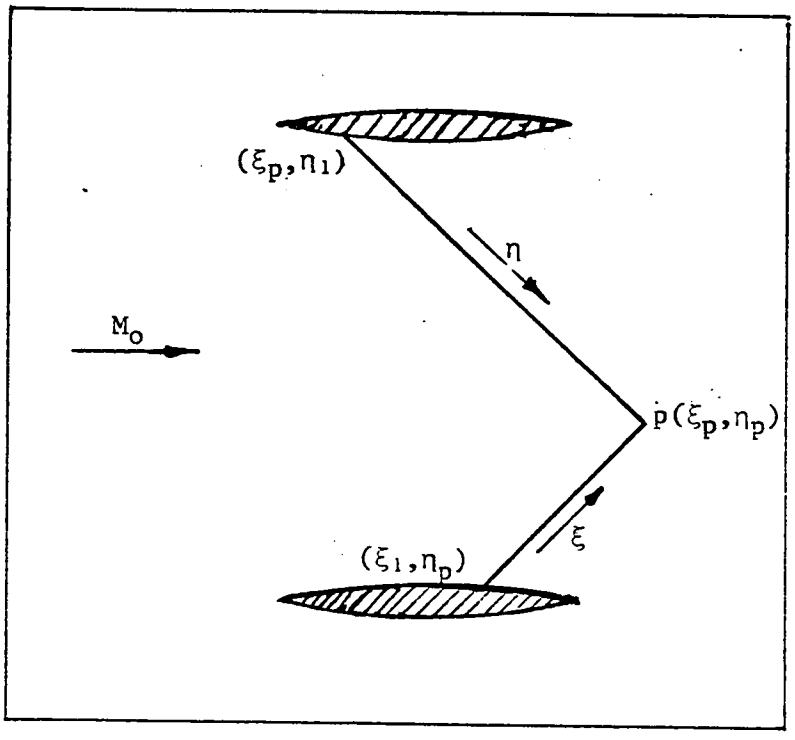

Fig. 2.5. The flow quantities in 'P' result from an integration of compatibility equations along the characteristics ξ, η.

2.4 : The Analytical Method of Characteristics in Non-Plane Flow Fields

Although the AMC is very suitable for the calculation of the plane supersonic flow past two-dimensional bodies, it has primarily been developed for the description of more complicated flow fields. In fact all supersonic flows, provided that they can be written in two independent variables, may be tackled with the procedure. The uniformity of the found solution may be demonstrated by showing the contribution of the second order terms non-cumulative.

The simplest example of a non-plane two-dimensional flow is the supersonic flow past a pointed body of revolution at zero incidence (axisymmetric flows). For such a flow the theory of Whitham [62] was already presented in 1952. In fact, this theory may be considered as a special case of the AMC dedicated to this special class of flow problems.

At the presentation of this theory, Whitham gives a number of physical arguments, why the linearized solution must become uniformly valid after a recalculation of the relevant characteristics. Although the argumentation is very plausible, there always has been a faint desire for more formal evidence. Suprisingly such evidence can be obtained very simply from a successive approximation of the governing flow equations in characteristic form.

The most intersting applications of the AMC are concerned with quasi two-dimensional flow problems, embedded in flow fields that as a whole are three-dimensional (axisymmetric flows). A good example is the description of the supersonic flow in the plane of a thin conical delta wing (Fig. 2.6), having subsonic or near sonic leading edges [55, 56, 12].

In this flow problem the location and the strength of the bow shock wave can easily be obtained.

The AMC is not limited to two dimensional problems (plane or axisymmetric). It can also be used to solve three-dimensional problems with four independent variables.

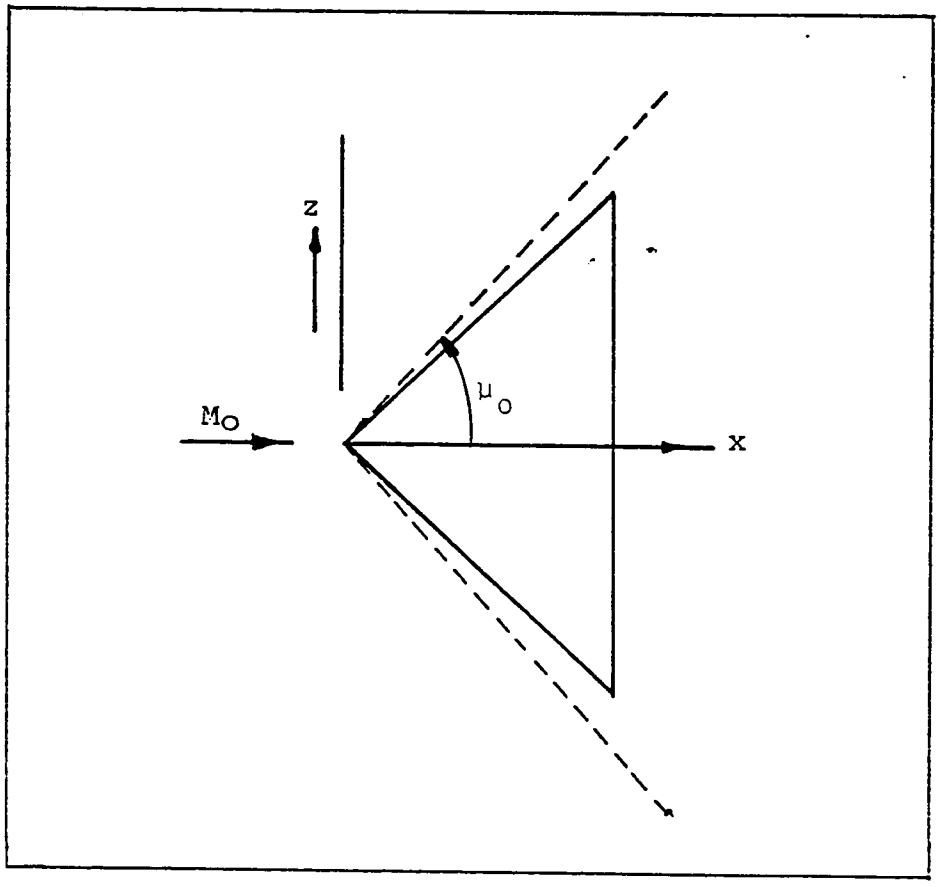

Fig. 2.6. Conical delta wing having nearly sonic leading edges.

CHAPTER 3

ACCELERATION POTENTIAL

3.1 IRROTATIONAL FORCE FIELD

A force field is said to be irrotational in a certain region of space when in that region curl F vanishes. Further, one can then represent the force field as the gradient of a scaler field. (Kramcheti 21):

$$\vec{F} = \text{grad } U^* \tag{3.1}$$

In irrotational force field the scaler function U^* is physically equivalent to the work done by the force field. In problems of mechanics the negative of the work done by an irrotational force field is defined as the *"potential energy"* or the *"potential"* of the system in which the force acts. According to this definition a scaler function that represents an irrotational force field is essentially the negative of the potential energy. (If one had initially written $\vec{F} = -\text{grad } U^*$, instead of $\vec{F} = \text{grad } U^*$, U^* would have become identical with the potential energy.)

It is customary to refer to a scaler function, such as U^* in equation (3.1), as a *'force potential'* or simply a *'potential'*. This nomenclature is further extended to all irrotational vector fields, and it is common to talk about 'velocity potential',

'acceleration potential', etc. On the same basis one refers to irrotational vector fields as 'potential fields'.

In the following discussion irrotational flows will be considered.

3.2 Derivation of the Acceleration Potential in Compressible Flows

The momentum equation describing an inviscid, compressible fluid through a rigid non-accelerating control volume, in the absence of body forces reads:

$$\frac{D\vec{q}}{Dt} + \frac{1}{\rho} \text{ grad } p = 0 \tag{3.2}$$

This equation can be written as:

$$\frac{D\vec{q}}{Dt} = \frac{-1}{\rho} \text{ grad } p \equiv \vec{b} \tag{3.3}$$

where $\vec{b}$ is the acceleration vector, and
$\vec{q}$ is the velocity vector.

For a barotropic fluid in an absent body force field, equation (3.3) can be written as:

$$\vec{b} \equiv \frac{D\vec{q}}{Dt} = -\text{grad}\left(\int \frac{dp}{\rho}\right) \tag{3.4}$$

The term 'barotropic' implies a unique pressure-density relation throughout the entire flow field.

From equation (3.4) it follows immediately that:

$$\nabla \times \vec{b} = 0 \tag{3.5}$$

from which one can conclude that:

$$\vec{b} = \text{grad } \psi \tag{3.6}$$

where $\psi(\vec{r}, t)$ is a scaler function. This scaler function (ψ) is called the acceleration potential.

Let

$$\underline{P} = \int \frac{dp}{\rho} \tag{3.7}$$

Substituting equations (3.7) and (3.6) in equation (3.4) yields:

$$\text{grad } \psi + \text{grad } \underline{P} = 0 \tag{3.8}$$

Integrating this equation one can get:

$$\psi + \underline{P} = G(t) \tag{3.9}$$

where G(t) is a function of time. In most practical problems G(t) reduces to a constant.

If the perturbation velocities u', v', w' are considered to be small in comparison with the fundamental velocity, then from the momentum equation in the x-direction one can get:

$$U \frac{dU}{dx} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \tag{3.10}$$

for small perturbations:

$$U = U_o + u' \tag{3.11}$$

Substituting equation (3.11) into equation (3.10) yields:

$$U_o \frac{du'}{dx} = -\frac{1}{\rho} \frac{\partial p}{\partial x} = b_x \tag{3.12}$$

where b_x is the acceleration in the x-direction. From equation (3.6):

$$b_x = \frac{\partial \psi}{\partial x} \tag{3.13}$$

Substituting equation (3.13) in equation (3.12) yields:

$$\psi = U_o \cdot u' \tag{3.14}$$

As will be shown in the next section the acceleration potential satisfies the following equation (in an air moving coordinate system):

$$\beta_o^2 \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{3.15}$$

where $\beta_o^2 = M_o^2 - 1$.

The acceleration potential becomes practically useful in compressible flows when small disturbances occurs, equation (3.7) reduces to:

$$\int_{p_o}^{p} \frac{dp}{\rho} \simeq \frac{p - p_o}{\rho_o} \tag{3.16}$$

From equations (3.4), (3.6) and (3.16) one can get:

$$\psi = \frac{p - p_o}{\rho_o} \tag{3.17}$$

Here, ψ differs only by a constant factor from the disturbance pressure $(p - p_o)$. It is often referred to as the pressure function.

Doublets of acceleration potential (ψ) prove to be a useful tool for representing lifting surfaces.

3.3 Derivation of the Wave Equation in Terms of the Acceleration Potential ψ

3.3.1 : IN AN AIR FIXED COORDINATE SYSTEM

The momentum and continuity equations describing an inviscid, compressible fluid through a rigid non-accelerating control volume, in the absence of body forces are:

$$\frac{D\vec{q}}{Dt} + \frac{1}{\rho} \nabla p = 0 \tag{3.2}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{q} = 0 \tag{3.18}$$

where $\frac{D}{Dt}$ is the material (substantial) derivative.

For small perturbations, in an air fixed coordinate system:

$$M_{obs.} = 0, \quad U = u', \quad V = v', \quad W = w', \quad P = P_o + \Delta P \quad \text{and} \quad \rho = \rho_o + \Delta\rho.$$

The material (substantial) derivative is expressible as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u' \frac{\partial}{\partial x} + v' \frac{\partial}{\partial y} + w' \frac{\partial}{\partial z} \tag{3.19}$$

For a fixed coordinate system, equation (3.19) reduces to:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} \tag{3.20}$$

Substituting equation (3.20) in equation (3.18) yields:

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{3.21}$$

The speed of sound is expressible as:

$$a^2 = \left(\frac{\partial p}{\partial \rho} \right)_s \tag{3.22}$$

for an isentropic flow (i.e., a flow with constant values of entropy along each of its streamlines) the partial derivative in equation (3.22) may be written as a total derivative. Hence,

$$dp = a^2 d\rho \tag{3.23}$$

along a streamline.

If the time derivatives following a particle are determined, one can obtain the substantial derivative given below:

$$\frac{Dp}{Dt} = a^2 \frac{D\rho}{Dt} \tag{3.24}$$

Equation (3.24) can be used to eliminate the density derivative from the governing equations. Density derivatives appear only in the continuity equation.

Substituting equation (3.24) in equation (3.18) yields the following alternate form for the continuity equation:

$$\frac{Dp}{Dt} + \rho a^2 \nabla \cdot \vec{q} = 0 \tag{3.25}$$

Using equation (3.20), equation (3.25) may be written as:

$$\frac{\partial p}{\partial t} + \rho_o a_o^2 \nabla \cdot \vec{q} = 0 \tag{3.26}$$

but $a_o^2 = \frac{\gamma p_o}{\rho_o}$.

Substituting this value of a_o^2 in equation (3.26) yields the following relation:

$$\frac{1}{\gamma p_o} \frac{\partial p}{\partial t} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \tag{3.27}$$

Using equation (3.20), equation (3.2) can be rewritten as:

$$\frac{\partial q}{\partial t} + \frac{1}{\rho_o} \nabla p = 0 \tag{3.28}$$

or

$$\frac{\partial u'}{\partial t} = - \frac{1}{\rho_o} \frac{\partial p}{\partial x} \tag{3.29a}$$

$$\frac{\partial v'}{\partial t} = - \frac{1}{\rho_o} \frac{\partial p}{\partial y} \tag{3.29b}$$

$$\frac{\partial w'}{\partial t} = - \frac{1}{\rho_o} \frac{\partial p}{\partial z} \tag{3.29c}$$

Differentiating equations (3.29a), (3.29b), and (3.29c) with respect to x, y, z, respectively yields:

$$\frac{\partial^2 u'}{\partial x \partial t} = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial x^2} \tag{3.30a}$$

$$\frac{\partial^2 v'}{\partial y \partial t} = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial y^2} \tag{3.30b}$$

$$\frac{\partial^2 w'}{\partial z \partial t} = -\frac{1}{\rho_o}\frac{\partial^2 p}{\partial z^2} \tag{3.30c}$$

Differentiating equation (3.27) with respect to t yields:

$$\frac{-1}{\gamma p_o}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 u'}{\partial x \partial t} + \frac{\partial^2 v'}{\partial y \partial t} + \frac{\partial^2 w'}{\partial z \partial t} \tag{3.31}$$

Adding the equations (3.30a), (3.30b), and (3.30c) yields:

$$\frac{\partial^2 u'}{\partial x \partial t} + \frac{\partial^2 v'}{\partial y \partial t} + \frac{\partial^2 w'}{\partial z \partial t} = -\frac{1}{\rho_o}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) \tag{3.32}$$

Substituting equation (3.31) in equation (3.32) yields:

$$\frac{1}{\gamma p_o}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\rho_o}\nabla^2 p = 0 \tag{3.33}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Recalling that

$$a_o^2 = \frac{\gamma p_o}{\rho_o}$$

equation (3.33) becomes

$$\frac{-1}{a_o^2}\frac{\partial^2 p}{\partial t^2} + \nabla^2 p = 0 \tag{3.34}$$

If the body-forces are present, equation (3.34) becomes:

$$\frac{-1}{a_o^2}\frac{\partial^2 p}{\partial t^2} + \nabla^2 p = \rho_o \nabla\cdot\vec{f} \tag{3.35}$$

where $\vec{f}$ is the body force vector.

Recalling equations (3.7) and (3.9), equation (3.34) can be written in terms of the stream function ψ as:

$$\frac{-1}{a_o^2}\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = 0. \tag{3.36}$$

3.3.2 : <u>IN A MOVING AIR COORDINATE SYSTEM</u>

In a moving air system:

$M_{obs.} \neq 0$ but $\frac{\partial}{\partial t} = 0$

Let's introduce the following Galilian trnasformations:

$$\left.\begin{aligned} \bar{x} &= x + U_o t \\ \bar{y} &= y \\ \bar{z} &= z \\ \bar{t} &= t \end{aligned}\right\} \tag{3.37}$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates in the moving frame.

The time derivative in the fixed frame is expressible in the moving frame as:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}}\cdot\frac{\partial \bar{t}}{\partial t} + \frac{\partial}{\partial \bar{x}}\cdot\frac{\partial \bar{x}}{\partial t} + \frac{\partial}{\partial \bar{y}}\cdot\frac{\partial \bar{y}}{\partial t} + \frac{\partial}{\partial \bar{z}}\cdot\frac{\partial \bar{z}}{\partial t} \tag{3.38}$$

from equation (3.37) into equation (3.38), one can get

$$\frac{\partial}{\partial t} = U_o \frac{\partial}{\partial x} \tag{3.39}$$

Similarly one can get:

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \bar{x}} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial \bar{y}} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial \bar{z}} \end{aligned}\right\} \tag{3.40}$$

Applying equations (3.34) and (3.39) into equation (3.34), one can get for a supersonic flow in a moving air system in the absence of the body forces the following relation:

$$(M_o^2 - 1) \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} - \frac{\partial^2 p}{\partial z^2} = 0 \tag{3.41}$$

Recalling equations (3.7) and (3.9), equation (3.41) can be written in terms of the acceleration potential (in the absence of body forces) as:

$$(M_o^2 - 1) \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{3.42}$$

which is the same as equation (3.15) mentioned before.

3.4 Pressure Solution Using Distribution of Pressure Doublets

The solution of a moving acceleration monopole positioned at the origin of the coordinate system is given by:

$$\underline{P} = \frac{-Q(\tau)}{4\pi r} \tag{3.42}$$

where

Q is the monopole strength

τ is the retarded time (the time at which the sound should be launched to reach the observer at time, t)

r_{ph} is the phase radius

r is the radial distance of the observer from the source position.

In mathematical notations:

$$\tau = t - r_{ph}/a$$

$$r_{ph} = \frac{M_\infty \cdot x + r}{\beta^2}, \qquad \beta^2 = M^2 - 1$$

and $$r = \sqrt{x^2 - \beta^2(y^2 + z^2)}$$

The doublet solution can be generated by simply differentiating the monopole solution with respect to distance in the direction of the force. Hence, the solution of a pressure doublet positioned at the origin may be expressed as:

$$\underline{P} = \frac{-1}{4\pi} \frac{\partial}{\partial z} \cdot \frac{f(\tau)}{r} \tag{3.43}$$

where $f(\tau)$ is the doublet strength. It has a constant value under the assumption that we have constant pressure difference across the wing. It acts in the z-direction.

In the case of a distribution of pressure doublets over a certain body surface the pressure solution will be

given by integrating the above equation over the surface area.

$$\underline{P} = \frac{-1}{4\pi} \cdot \frac{\partial}{\partial z} \iint_S \frac{f(\tau)}{r} \cdot dA \tag{3.44}$$

In the rest of this chapter two configurations will be considered; a delta wing and a wing with curved leading edges. The delta wing is considered to have two flow regions; a conical region and an expansion region. These two cases will also be discussed extensively in the next chapter using the equivalent body of revolution concept.

3.4.1 : THE DELTA WING (Conical Region)

As mentioned above, the flow over a delta wing forms two regions; the conical region and the expansion region. The expansion region solution can be found from the equivalent body of revolution (see next chapter), so it will not be discussed here. In the following paragraphs the conical region is considered.

Let's consider a distribution of pressure doublets over a delta wing surface up to the expansion region. For an observer at point(0)shown in the figure (Fig. 3.1), the area which contributes to the solution is the hatched one. This area is formed when the Mach cone forms a hyperbola as intersection line with the wing.

For simplicity, the tangent of this hyperbola will be considered to form the boundaries of the area rather than

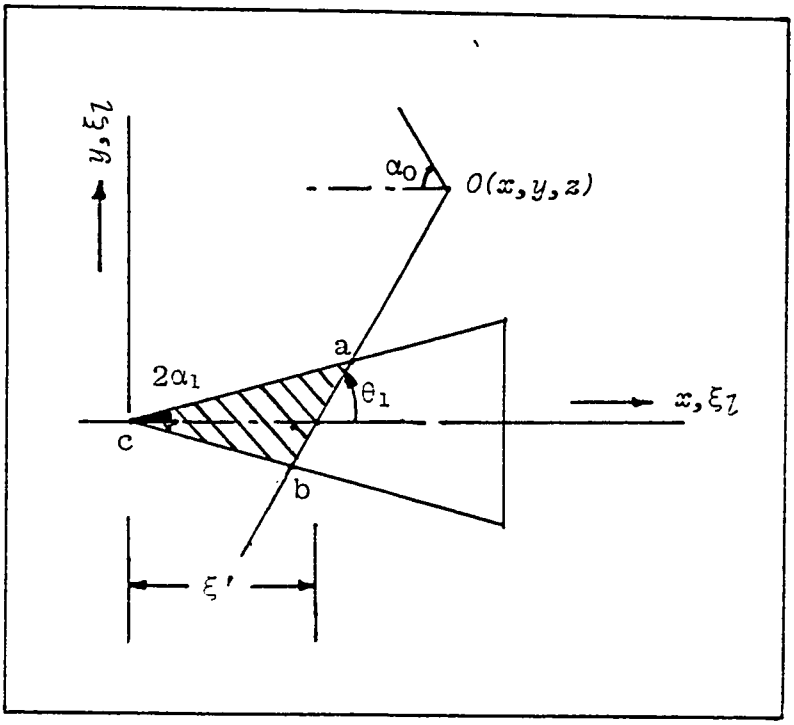

Fig. 3.1. Distribution of pressure doublets over the surface area of a delta wing with straight leading edges.

the hyperbola itself.

For a source having (ξ_o, ζ_o) as its coordinates:

$$r = \sqrt{(x-\xi_o)^2 - \beta^2[(y-\zeta_o)^2 + z^2]} \tag{3.45}$$

$$= \sqrt{x^2 - 2x\xi_o + \xi_o^2 - \beta^2(y^2 - 2y\zeta_o + \zeta_o^2 + z^2)} \tag{3.46}$$

In the regions near the shock, the following approximations can be introduced:

$$\frac{\zeta_o}{x+\beta r_o} << 1 \quad , \quad \frac{\xi_o}{x+\beta r_o} << 1 \ ,$$

then r can be expressed as:

$$r \simeq \sqrt{x+\beta r_o} \cdot \sqrt{(x-\beta r_o) - \frac{2x}{x+\beta r_o}\,\xi_o + \frac{2\beta^2 y}{x+\beta r_o}\,\zeta_o} \tag{3.47}$$

where $r_o = \sqrt{y^2 + z^2}$. (For first order approximations; $r_o = \sqrt{y_o^2 + z_o^2}$).

To evaluate the surface integral in equation (3.20) above, let's introduce the new variable ξ':

$$\xi' = \frac{2x}{x+\beta r_o}\,\xi_o - \frac{2\beta^2 y}{x+\beta r_o}\,\zeta_o \tag{3.48}$$

which can be rewritten as:

$$\xi_o - \frac{\beta^2 y}{x}\,\zeta_o = \xi' \cdot \frac{x+\beta r_o}{2x} \simeq \xi' \tag{3.49}$$

since $x + \beta r_o \simeq 2x$.

In the characteristic plane, (if the left running characteristics are considered) by differentiating equation (3.49)

along these characteristics one can find that:

$$d\xi_o = d\xi' \tag{3.50}$$

Equation (3.49) above resembles the tangent of the hyperbola (the line passing through the points a and b). The slope of this line is given as:

$$\tan\theta_1 = \frac{y\beta^2}{x} \tag{3.51}$$

Now equation (3.44) can be rewritten as

$$\underline{P} = \frac{-f(\tau)}{4\pi} \cdot \frac{\partial}{\partial z} \int_0^{x-\beta r_o} \int_{\zeta_{o1}}^{\zeta_{o2}} \frac{d\zeta_o \, d\xi'}{\sqrt{x-\beta r_o-\xi_o'}\,\sqrt{x+\beta r_o}} \tag{3.52}$$

The limits of the inside integral (ζ_{o1}, ζ_{o2}) can be found by equating the (ξ_o-coordinate) for the points b and a respectively obtained from the straight line passing through a and b and the leading edge equations.

From the wing leading edges equations:

$$\left.\begin{aligned} \zeta_o &= \tan\alpha_1 \cdot \xi_o \\ \zeta_o &= -\tan\alpha_1 \cdot \xi_o \end{aligned}\right\} \tag{3.53}$$

where α_1 is the half apex angle of the delta wing.

The straight line equation passing through the points b and a is given by equation (3.49).

From equations (3.49) and (3.53) one can verify that

$$\zeta_{o1} = \frac{-\xi' \tan\alpha_1}{1 + \tan\theta_1 \cdot \tan\alpha_1} \tag{3.54}$$

$$\zeta_{o2} = \frac{\xi' \tan\alpha_1}{1 - \tan\theta_1 \cdot \tan\alpha_1} \tag{3.55}$$

where ζ_{o1}, ζ_{o2} are the ζ_o-coordinates of the points b and a respectively.

Equation (3.52) can be reduced to:

$$\underline{P} = \frac{-f(\tau)}{4\pi} \frac{\partial}{\partial z} \int_0^{x-\beta r_o} \frac{d\xi'}{\sqrt{x - \beta r_o - \xi'}} \cdot (\zeta_{o2} - \zeta_{o1}) \frac{1}{\sqrt{x_o + \beta r_o}} \tag{3.56}$$

From equations (3.54) and (3.55) above (by subtraction) one can get:

$$\zeta_{o2} - \zeta_{o1} = \frac{2\xi' \cdot \tan\alpha_1}{1 - \tan^2\theta_1 \cdot \tan^2\alpha_1} \tag{3.57}$$

Substituting equation (3.57) in equation (3.56) implies:

$$\int_0^{x-\beta r_o} \frac{d\xi'}{\sqrt{x - \beta r_o - \xi'}} \frac{-(\zeta_{o1} - \zeta_{o2})}{\sqrt{x_o + \beta r_o}} = I = \frac{2 \tan\alpha_1}{\sqrt{x_o + \beta r_o}\,(1 - \tan^2\theta_1 \tan^2\alpha_1)} \cdot \int_0^{x-\beta r_o} \frac{\xi' d\xi'}{\sqrt{x - \beta r_o - \xi'}} \tag{3.58}$$

Evaluating the integral on the right hand side of equation (3.58) yields:

$$\int_0^{x-\beta r_o} \frac{\xi' d\xi'}{\sqrt{x - \beta r_o - \xi'}} = \left\{ \frac{1}{3} (x - \beta r_o - \xi_o') - (x - \beta r_o) \right\} \cdot 2\sqrt{x - \beta r_o - \xi'} \Big|_0^{x-\beta r_o}$$

$$= \frac{3}{4} (x - \beta r_o)^{3/2} \tag{3.59}$$

This form shows that the equivalent body of revolution concept can be used to solve for the pressure variations across the shocks. Let's introduce the following characteristics:

$$\xi = x - \beta r_o \tag{3.60a}$$

$$\eta = x + \beta r_o \tag{3.60b}$$

from which one can verify that

$$x = \frac{1}{2}(\eta + \xi)$$

$$\beta r_o = \frac{1}{2}(\eta - \xi)$$

For first order approximations:

$$x = x_o \quad , \quad y = y_o \quad , \quad \text{and} \quad z = z_o \quad , \quad r_o = \sqrt{y_o^2 + z_o^2}$$

Hence:

$$x_o = \frac{1}{2}(\eta + \xi) \tag{3.61a}$$

$$\beta\sqrt{y_o^2 + z_o^2} = \frac{1}{2}(\eta - \xi) \tag{3.61b}$$

The differentiation with respect to z can be expressed in terms of the characteristics by the chain rule as:

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} + \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial z} \tag{3.62}$$

From equation (3.60b):

$$\frac{\partial \eta}{\partial z} = \beta \cdot \frac{z_o}{\sqrt{y_o^2 + z_o^2}}$$

$$= \beta \cos(\theta_n) \tag{3.63}$$

where $\cos\theta_n = \frac{z_o}{r_o}$.

Similarly, from equation (3.60a):

$$\frac{\partial\xi}{\partial z} = -\beta \cos(\theta_n) \tag{3.64}$$

In terms of the characteristics, equation (3.59) can be written as:

$$\int_0^{X-\beta r_o} \frac{\xi' d\xi'}{\sqrt{x - \beta r_o - \xi'}} = \frac{4}{3} (\xi)^{3/2} \tag{3.65}$$

This integral represents the line distribution of sources along the x-axis.

Substituting equation (3.59) and equation (3.60b) in equation (3.58) yields:

$$I = \frac{8 \tan\alpha_1}{3\sqrt{\eta}} \frac{(\xi)^{3/2}}{(1 - \tan^2\theta_1 \tan^2\alpha_1)} \tag{3.66}$$

$$\frac{\partial I}{\partial z} = \frac{\partial I}{\partial \eta} \cdot \frac{\partial \eta}{\partial z} + \frac{\partial I}{\partial \xi} \cdot \frac{\partial \xi}{\partial z}$$

from equation (3.66) one can get that:

$$\frac{\partial I}{\partial z} = \frac{-8 \tan\alpha_1}{3(1 - \tan^2\theta_1 \tan^2\alpha_1)} \cdot \left\{ \frac{3\beta \cos\theta_n}{2} \frac{\sqrt{\xi}}{\sqrt{\eta}} + \frac{\beta}{2} \cos\theta_n \frac{(\xi)^{3/2}}{(\eta)^{3/2}} \right\} \tag{3.67}$$

For long field approximations near the shock wave $\frac{\xi}{\eta} << 1$; $\frac{\partial I}{\partial \xi} >> \frac{\partial I}{\partial \eta}$, from which $\frac{\partial I}{\partial z} \simeq \frac{\partial I}{\partial \xi}$:

$$\frac{\partial I}{\partial z} \simeq \frac{-4 \tan \alpha_1}{(1 - \tan^2\theta_1 \tan^2\alpha_1)} \cdot \beta \cos\theta_n \cdot \frac{\sqrt{\xi}}{\sqrt{\eta}} \tag{3.68}$$

Substituting equation (3.67) in equation (3.56) yields:

$$\underline{P} = \frac{f(\tau)}{\pi} \cdot \frac{\tan\alpha_1}{1 - \tan^2\theta_1 \ \tan^2\alpha_1} \cdot \beta \ \cos\theta_n \cdot \frac{\sqrt{\xi}}{\sqrt{\eta}} \tag{3.69}$$

For an observer directly under the wing $(y = 0)$ $\cos\theta_n$ is unity and $\tan\theta_1$ is zero. Hence, equation (3.69) reduces to:

$$\underline{P} = \frac{f(\tau)}{\pi} \cdot \tan\alpha_1 \cdot \beta \cdot \frac{\sqrt{\xi}}{\sqrt{\eta}} \tag{3.70}$$

This equation defines the pressure jump along the characteristics. The term $\frac{1}{\sqrt{\eta}}$ is a distance function which can be expressed as a function of $\sqrt{\xi}$ from the shock wave geometry which is presented in the next chapter.

To eliminate one variable (θ_1) from the derived equations,the angle θ_1 (Fig. 3.1) can be expressed in terms of θ_n by equating equation (3.60a) to zero, considering zero order position of the shock waves as envelopes of spheres in undisturbed flow. Hence;

$$x - \beta r_o = 0$$

from which

$$\frac{\beta^2 y}{x} = \frac{\beta y}{r_o}$$

hence;

$$\tan\theta_1 = \beta \ \sin\theta_n \tag{3.71}$$

where $\sin\theta_n = \frac{y}{r_o}$ while $\cos\theta_n = \frac{z}{r_o}$.

Substituting equation (3.71) in equation (3.69) yields:

$$\underline{P} = \frac{f(\tau)}{\pi} \cdot \frac{\tan\alpha_1 \cdot \beta \cos\theta_n}{1-\beta^2\sin^2\theta_n \tan^2\alpha_1} \cdot \frac{\sqrt{\xi}}{\sqrt{\eta}} \tag{3.72}$$

3.4.2 : THE WING WITH CURVED LEADING EDGES

For this configuration one can proceed in a similar fashion as for the delta wing. Let the pressure doublets be distributed all over the wing.

For an observer at point(O) the area which contributes to the solution is the hatched one (Fig. 3.2).

Again the pressure solution is given by equation (3.52) as:

$$\underline{P} = \frac{-f(\tau)}{4\pi} \cdot \frac{\partial}{\partial z} \int_0^{x-\beta r_o} \int_{\zeta_{01}}^{\zeta_{02}} \frac{d\zeta_0 d\xi'}{\sqrt{x-\beta r_o-\xi'}\ \sqrt{x+\beta r_o}} \tag{3.52}$$

Let

$$J = \int_0^{x-\beta r_o} \int_{\zeta_{01}}^{\zeta_{02}} \frac{d\zeta_0 d\xi'}{\sqrt{x-\beta r_o-\xi'}}$$

$$= \int_0^{x-\beta r_o} \frac{(\zeta_{02}-\zeta_{01})d\xi'}{\sqrt{x-\beta r_o-\xi'}} \tag{3.73}$$

Hence, it is needed to find ζ_{02} and ζ_{01}.

The leading edge of the wing is characterized by two parabolas; obc and oad.

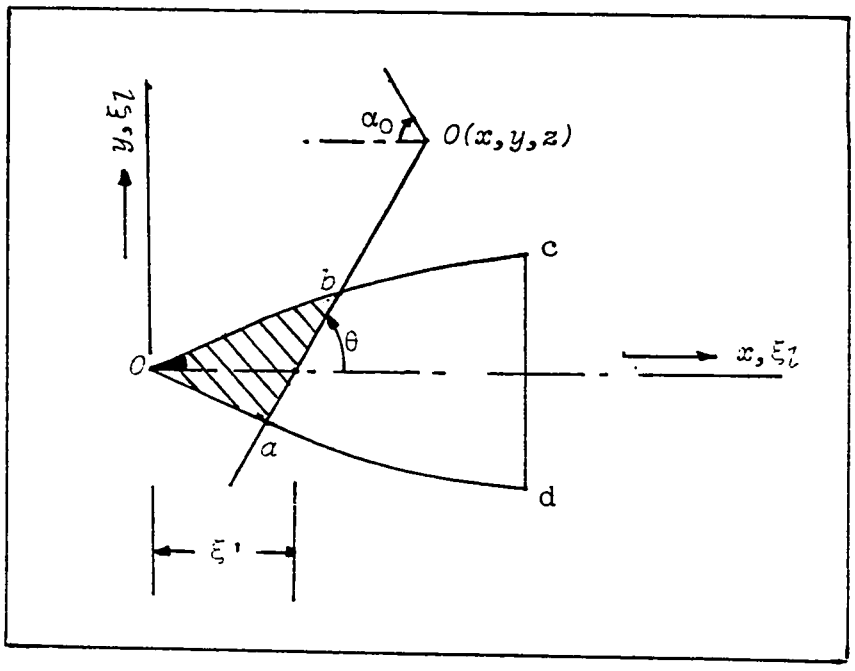

Fig. 3.2. Distribution of pressure doublets over the surface area of a delta wing with curved leading edges.

The equation of the upper leading edge (obc) (Fig. 3.2) is:

$$\zeta_0 = \tan\alpha_2(\xi_0 - \xi_0^2) \tag{3.74}$$

where α_2 is the half apex angle of the wing.

The equation of the line passing through the points a and b (the tangent of the hyperbola) is:

$$\xi_0 = \frac{y\beta^2}{x}\zeta_0 + \xi' \tag{3.75}$$

form which the ξ_0-coordinate of point 'b' is:

$$\xi_{02} = \frac{y\beta^2}{x}\zeta_{02} + \xi' \tag{3.76}$$

but $\frac{y\beta^2}{x} = \tan\theta_2$, hence:

$$\xi_{02} = \tan\theta_2\ \zeta_{02} + \xi' \tag{3.77}$$

$$\xi_{02}^2 = \tan^2\theta_2\ \zeta_{02}^2 + 2\ \tan\theta_2\ \xi'\ \zeta_{02} + \xi'^2 \tag{3.78}$$

From equation (3.74) above:

$$\zeta_{02} = \tan\alpha_2(\xi_{02} - \xi_{02}^2) \tag{3.79}$$

Substituting equation (3.77) and equation (3.78) in equation (3.79) yields:

$$\tan^2\theta_2\ \zeta_{02}^2 + \zeta_{02}[\cot\alpha_2 + \tan\theta_2(2\xi' - 1)] + (\xi'^2 - \xi') = 0 \tag{3.80}$$

from which one can find that:

$$\zeta_{o2} = \frac{1}{2\ \tan^2\theta_2} \cdot \left\{ [\tan\theta_2(1-2\xi') - \cot\alpha_2] + \sqrt{(\cot\alpha_2 - \tan\theta_2)^2 + 4\xi' \cdot \cot\alpha_2 \cdot \tan\theta_2} \right\} \tag{3.81}$$

The negative square root of the solution of equation (3.80) is neglected since it gives the other point at which the line ab crosses the hyperbola passing through obc in the $(\xi_o, -\zeta_o)$ plane.

Equation (3.81) can be rewritten as:

$$\zeta_{o2} = \frac{1}{2} \left\{ \cot\theta_2(1-2\xi') - \cot\alpha_2\ \cot^2\theta_2 + \sqrt{(\cot\alpha_2\ \cot^2\theta_2 - \cot\theta_2)^2 + 4\xi'\ \cot\alpha_2 \cdot \cot^3\theta_2} \right\} \tag{3.82}$$

Consider now the square root in equation (3.82)

$$\sqrt{(\cot\alpha_2 \cot^2\theta_2 - \cot\theta_2)^2 + 4\xi' \cot\alpha_2 \cdot \cot^3\theta_2}$$

$$= (\cot\alpha_2 \cot^2\theta_2 - \cot\theta_2) \sqrt{1 + \frac{4\xi'\ \cot\alpha_2\ \cot^3\theta_2}{(\cot\alpha_2\ \cot^2\theta_2 - \cot\theta_2)^2}} \tag{3.83}$$

Expanding the square root in a Tayler series yields:

$$\sqrt{1 + \frac{4\xi'\ \cot\alpha_2\ \cot^3\theta_2}{(\cot\alpha_2\ \cot^2\theta_2 - \cot\theta_2)^2}} = 1 + \frac{2\xi'\ \cot\alpha_2\ \cot^3\theta_2}{(\cot\alpha_2\ \cot^2\theta_2 - \cot\theta_2)^2} - \frac{2\xi'^2 \cot^2\alpha_2 \cdot \cot^6\theta_2}{(\cot\alpha_2\ \cot^2\theta_2 - \cot\theta_2)^4} + \ldots\ldots \tag{3.84}$$

Neglecting higher order terms in equation (3.84), substituting the resulting equation in equation (3.83), and substituting the new resulting equation in equation (3.82) yields:

$$\zeta_{O2} = \frac{1}{2}\left\{\cot\theta_2(1-2\xi') - \cot\alpha_2 \cot^2\theta_2 + (\cot\alpha_2 \cot^2\theta_2 - \cot\theta_2)\right.$$
$$\left.\left(1 + \frac{2\xi' \cot\alpha_2 \cot^3\theta_2}{(\cot\alpha_2 \cot^2\theta_2 - \cot\theta_2)^2} - \frac{2\xi'^2 \cot^2\alpha_2 \cdot \cot^6\theta_2}{(\cot\alpha_2 \cot^2\theta_2 - \cot\theta_2)^4}\right)\right\} \tag{3.85}$$

From equation (3.85), after some simplification one can get:

$$\zeta_{O2} = \frac{\tan\alpha_2}{1-\tan\alpha_2 \cdot \tan\theta_2}\left(\xi' - \frac{\xi'^2}{(1-\tan\alpha_2 \tan\theta_2)^2}\right) \tag{3.86}$$

In equation (3.86), an interesting case is when $y = 0$, i.e., when the plane of interest is the one normal to the wing. For this case, equation (3.86) reads:

$$\zeta_{O2} = \tan\alpha_2(\xi' - \xi'^2) \tag{3.87}$$

To find ζ_{O1} which is the ζ_O-coordinate of point"a"(Fig. 3.2), the parabola passing through the points d, a, o is described as:

$$\zeta_O = -\tan\alpha_2(\xi' - \xi'^2) \tag{3.88}$$

The equation of the line passing through the points a, b reads:

$$\xi_O - \tan\theta_2 \, \zeta_O = \xi' \tag{3.89}$$

The ζ_O-coordinate of point"a"can be found from equations (3.88) and (3.89) in a similar fashion in which ζ_{O2} is found. ζ_{O1} comes out to be:

$$\zeta_{01} = \frac{1}{2\ \tan^2\theta_2}\left\{\cot\alpha_2 + \tan\theta_2(1-2\xi') - \sqrt{(\cot\alpha_2+\tan\theta_2)^2 - 4\xi'\ \cot\alpha_2 \cdot \tan\theta_2}\right\} \tag{3.90}$$

Again following the same procedure used to simplify ζ_{02}, one can finally get:

$$\zeta_{01} = \frac{\tan\alpha_2}{(1+\tan\alpha_2 \cdot \tan\theta_2)}\left(-\xi' + \frac{\xi'^2}{(1+\tan\alpha_2\ \tan\theta_2)^2}\right) \tag{3.91}$$

Again for the plane of the wing $(y=0)$, equation (3.91) reads:

$$\zeta_{01} = \tan\alpha_2(-\xi' + \xi'^2) \tag{3.92}$$

Subtracting equation (3.91) from equation (3.86) yields:

$$\zeta_{02} - \zeta_{01} = \frac{2\ \tan\alpha_1}{(1-A^2)}\left\{\xi' - \frac{\xi'^2(1+A^2)}{(1-A^2)}\right\} \tag{3.93}$$

where $A = \tan\alpha_2\ \tan\theta_2$.

Substituting equation (3.93) in equation (3.72) yields:

$$J = \frac{2\ \tan\alpha_2}{(1-A^2)}\left\{\int_0^{x-\beta r_o} \frac{\xi'\,d\xi'}{\sqrt{x-\beta r_o-\xi'}} - \frac{(1+A^2)}{(1-A^2)}\int_0^{x-\beta r_o} \frac{\xi'^2 d\xi'}{\sqrt{x-\beta r_o-\xi'}}\right\} \tag{3.94}$$

Evaluating these integrals one can get:

$$J = \frac{2\ \tan\alpha_2}{(1-A^2)}\left\{\frac{4}{3}\cdot(x-\beta r_o)^{3/2} - \frac{16}{15}(x-\beta r_o)^{5/2}\cdot\frac{(1+A^2)}{(1-A^2)}\right\} \tag{3.95}$$

Substituting equation (3.95) in equation (3.52) yields:

$$\underline{P} = \frac{-f(\tau)}{4\pi} \cdot \frac{\partial}{\partial z}\left(J \cdot \frac{1}{\sqrt{x + \beta r_o}}\right)$$

$$= \frac{-f(\tau)}{4\pi} \cdot \frac{2 \tan\alpha_2}{(1 - A^2)} \cdot \frac{\partial}{\partial z} \left\{ \left[\frac{4}{3} (x-\beta r_o)^{3/2} - \frac{16}{15} \cdot \left(\frac{1+A^2}{1-A^2} \right) (x-\beta r_o)^{5/2} \right] \cdot \frac{1}{\sqrt{x + \beta r_o}} \right\} \tag{3.96}$$

In terms of the characteristics ξ and η equation (3.96) becomes:

$$\underline{P} = \frac{-f(\tau)}{2\pi} \cdot \frac{\tan\alpha_2}{(1 - A^2)} \cdot \frac{\partial}{\partial z} \left\{ \frac{4}{3} (\xi)^{3/2} - \frac{16}{15} \left(\frac{1+A^2}{1-A^2} \right) (\xi)^{5/2} \cdot \frac{1}{\sqrt{\eta}} \right\} \tag{3.97}$$

For the regions near the shock wave, $\frac{\xi}{\eta} << 1$ from which it can be deduced that $\frac{\partial}{\partial \xi} >> \frac{\partial}{\partial \eta}$.

Under these assumptions equation (3.97) can be simplified to:

$$\underline{P} = \frac{-f(\tau)}{2\pi \sqrt{\eta}} \frac{\tan\alpha_2}{1 - A^2} \frac{\partial}{\partial z} \left\{ \frac{4}{3} (\xi)^{3/2} - \frac{16}{15} \left(\frac{1+A^2}{1-A^2} \right) (\xi)^{5/2} \right\} \tag{3.98}$$

With the help of equation (3.62) equation (3.98) reads:

$$\underline{P} = \frac{f(\tau)}{\pi \sqrt{\eta}} \cdot \frac{\tan\alpha_2}{(1 - A^2)} \cdot \beta \cos\theta_n \left\{ \sqrt{\xi} - \frac{4}{3} \left(\frac{1+A^2}{1-A^2} \right) (\xi)^{3/2} \right\} \tag{3.99}$$

Equation (3.99) defines the pressure jump along the Characteristics.

For the plane normal to the wing where $y = 0$, equation (3.99) reads:

$$\underline{P} = \frac{f(\tau)}{\pi \sqrt{\eta}} \cdot \tan\alpha_2 \cdot \beta \cdot \left\{ \sqrt{\xi} - \frac{4}{3} (\xi)^{3/2} \right\} \tag{3.100}$$

As mentioned before $f(\tau)$ is constant since a constant

pressure difference across the wing is assumed. This constant is to be determined by comparing the results of the pressure jump along the characteristics obtained in this chapter by that of the corresponding cases obtained by using the concept of the equivalent body of revolution at the tip of the body where $\xi = \eta \simeq 0$.

The angle θ_2 can be expressed in terms of θ_n in a similar way used to find the relation between θ_1 and θ_n:

From equation (3.60a)

$$x = \beta r_o$$

from which

$$\frac{\beta^2 y}{x} = \beta \frac{y}{r_o} = \beta \sin\theta_n ,$$

but

$$\frac{\beta^2 y}{x} = \tan\theta_2 ,$$

from which one gets that $\tan\theta_2 = \beta \sin\theta_n$, hence, the constant 'A' can be expressed as

$$A = \beta \tan\alpha_2 \cdot \sin\theta_n$$

where $\sin\theta_n = \frac{y}{r_o}$, while $\cos\theta_n = \frac{z}{r_o}$.

Equation (3.99) can be rewritten as:

$$\underline{P} = \frac{f(\tau)}{\pi\sqrt{\eta}} \cdot \frac{\tan\alpha_2 \cdot \beta \cos\theta_n}{1-\beta^2\sin^2\theta_n\tan^2\alpha_2} \left\{ \sqrt{\xi} - \frac{4}{3} \frac{1+\beta^2\sin^2\theta_n\tan^2\alpha_2}{1-\beta^2\sin^2\theta_n\tan^2\alpha_2} \cdot (\xi)^{3/2} \right\}$$

(3.101)

This equation represents the shock wave strength $\left(\frac{\Delta p}{p_o}\right)$ in any lateral direction as well as in the vertical plane, where p_o is the pressure in the undisturbed flow region, and Δp is the pressure jump across the shock wave.

CHAPTER 4

SHOCK WAVE CALCULATIONS

A shock wave is a relatively thin region of rapid state variation across which there is a flow of matter. Because the region of variation is thin, it can almost always be idealized as a surface of discontinuity in space. This surface propagates into the fluid and is not necessarily stationary. In general, all fluid properties (pressure, velocity, density, etc.,) are discontinuous across the shock. The treatment of shock waves as discontinuous, or surfaces of zero thickness, is an idealization of inviscid gas dynamics. Physically, shocks are found to have a finite and measurable thickness, commonly of the order 10^{-6} m. Transfer of matter across the shock front must satisfy the conditions of balance for mass, momentum, energy and entropy. The shock strength is defined by the pressure jump across the shock. The shock wave is weak if the pressure jump is small compared to the pressure ahead of the shock and the shock wave is strong if the pressure jump is big compared to the pressure ahead of the shock. For any turning angle, there are too possible distinct solutions, which are called the weak solution and the strong solution. This nomenclature has no direct connection to the weak shock and strong shock.

An aircraft can be represented by a distribution of mass sources and lifting elements. Within the assumption of slender body theory the linear source distribution can be used.

From the point of view of a distant observer all of these singularities lying on oblique Mach planes (or upstream Mach cones) passing near the observer are equivalent, and the calculation is reduced to that for an equivalent body of revolution.

In this chapter, all the calculations are corresponding to the plane directly under the wing, the shock wave is taken in first order as a bisector of the characteristics before and behind the shock (bisector rule), and the calculations of the shock geometry and strength are considered using the equivalent body of revolution (Hendricks 18). These calculations are applied to two configurations; to a delta wing* and to a delta wing with curved leading edges.

4.1 THE DELTA WING

4.1.1 : THE EQUIVALENT BODY OF REVOLUTION

The total lift of a wing is given by:

---

* *Delta wing refers to a delta wing with straight 'leading edges'.*

$$F_t = \frac{1}{2} \rho U^2 \cdot CL\ S_w \tag{4.1}$$

where

CL is the lift coefficient

S_w is the total area of the wing

ρ is the fluid density, and

U is the flow velocity in x-direction.

The length distribution of lift (which is defined as the total lift per unit length) for the delta wing is given as (see Fig. 4.1):

$$l_i(\xi) = 2 \tan\alpha_1 \cdot \xi \cdot \Delta P' \tag{4.2}$$

where $\Delta P'$ is the pressure difference between the upper and lower sides of the wing, and

α_1 is the half apex angle of the delta wing.

The derivative of the cross-sectional area distribution of the body of revolution equivalent to the delta wing in the vertical plane is given as [40]:

$$S'(\xi) = \frac{l_i(\xi)}{\rho U^2} \cdot \sqrt{M_o^2 - 1} \tag{4.3}$$

where M_o is the undisturbed flow Mach number.

Substituting equation (4.2) in equation (4.3), and integrating the resulting equation, the cross-sectional area distribution of the equivalent body of revolution of a delta wing can be easily verified to be:

$$S(\xi) = \tan\alpha_1 \cdot \frac{\Delta P'}{\rho U^2} \cdot \sqrt{M_o^2 - 1} \cdot \xi^2 \tag{4.4}$$

but the lift coefficient CL is defined as:

$$CL = \frac{2\,\Delta P'}{\rho U^2} \tag{4.5}$$

Substituting this value of CL in equation (4.4), it can be rewritten as:

$$S(\xi) = \tan\alpha_1 \cdot \frac{CL}{2}\sqrt{M_o^2 - 1} \cdot \xi^2 \tag{4.6}$$

at $x = l$, equation (4.6) becomes:

$$S(l) = \tan\alpha_1 \cdot \frac{CL}{2}\sqrt{M_o^2 - 1} \cdot l^2 \tag{4.7}$$

For the delta wing shown in Fig. 4.1, the total area of the wing is:

$$S_w = \tan\alpha_1 \cdot l^2 \tag{4.8}$$

and the total lift is

$$\begin{aligned} F_t &= \Delta P' \cdot S_w \\ &= \tan\alpha_1 \cdot l^2 \cdot \Delta P' \end{aligned} \tag{4.9}$$

Equation (3.1) can be rewritten as:

$$S_w\, CL = \frac{2F_t}{\rho U^2} \tag{4.10}$$

The equivalent body of revolution for the delta wing is given by a cone-cylinder configuration, the cross-sectional area distribution of this configuration for $l \leqslant x < \infty$ is given by:

$$S_c = \frac{\beta_o}{2} \cdot CL \cdot S_w \tag{4.11}$$

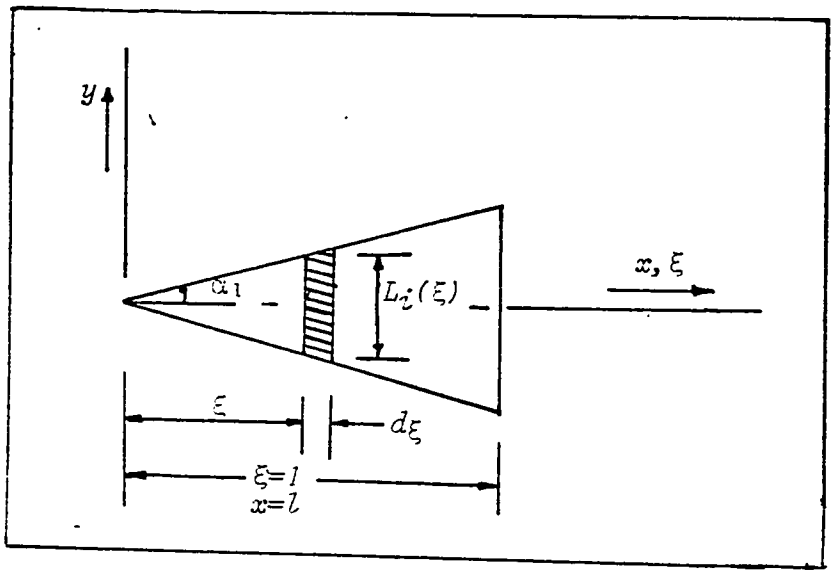

Fig. 4.1. Delta wing with straight leading edges.

where $\beta_o = \sqrt{M_o^2 - 1}$.

Substituting equation (4.9) in equation (4.10) and substituting the resulting equation in equation (4.11), one can get:

$$S_c = \frac{CL}{2} \cdot \sqrt{M_o^2 - 1} \cdot l^2 \tan\alpha_1 \qquad (4.12)$$

Comparing equation (4.12) with equation (4.7), one can see that both equations are equivalent. The problem of a delta wing is reduced to the equivalent body of revolution problem which is a cone-cylinder configuration. The cone has a half apex angle equal to ν_o (Morris 40).

4.1.2 : THE RELATION BETWEEN THE HALF-APEX ANGLES (α_1) AND (ν_o)

As stated above, the delta wing problem with (α_1) as a half-apex angle is reduced to the cone-cylinder configuration problem with ν_o as the half apex angle of the cone.

The cross-sectional area distribution of the cone is given as:

$$S(\xi) = \xi^2 \tan^2\nu_o \cdot \pi \qquad (4.13)$$

The cross-sectional area distribution of the cone should be equivalent to that of the delta wing whose cross-sectional

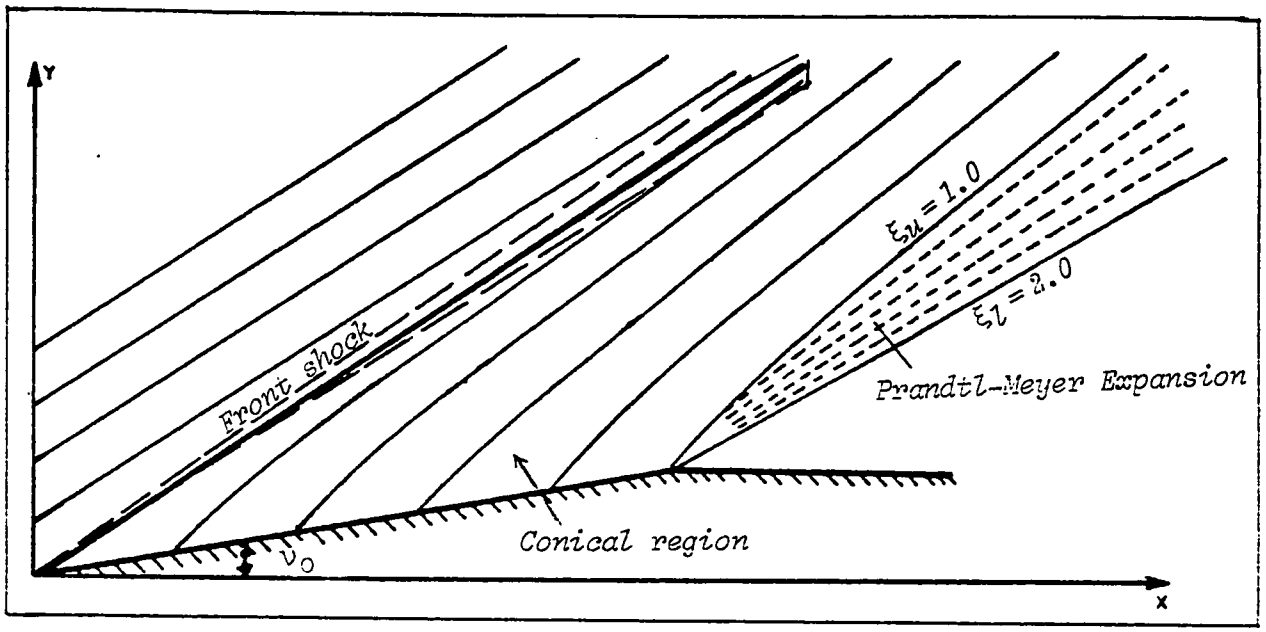

Fig. 4.2. Flow over a cone-cylinder configuration

area distribution is given by equation (4.6) to produce the same lift assuming that $\Delta P'$ is the same for both configurations.

Equating equation (4.6) to equation (4.13), one can get:

$$\tan\nu_o = \left(\frac{\tan\alpha_1}{\pi} \cdot \frac{CL}{2} \cdot \sqrt{M_o^2 - 1}\right)^{\frac{1}{2}} \qquad (4.14)$$

which gives the relation between the half-apex angles (ν_o) and (α_1).

4.1.3 : SHOCK WAVE STRENGTH AND GEOMETRY

As can be seen from Fig. 4.2 an expansion fan (Prandtl-Meyer expansion) will be formed at the end of the cone. The flow over this configuration can be simplified by dividing it into two regions, namely, the conical region and the expansion region. Each of these regions is studied separately.

(A) In The Conical Region

A conical flow is one for which the flow properties (velocity component, pressure, etc.,) are constant along rays through one point (the cone apex). These flows are not realizable in a subsonic stream.

Generally, the shock wave strength whether it is in the conical or expansion region is given by (Oswatitsch 46):

$$\frac{\Delta P}{P_o} = \gamma \cdot M_o^2 \cdot \tan\alpha_o \cdot \delta \cdot \Lambda \qquad (4.15)$$

where γ is the ratio of specific heats, (P_o is the pressure in the undisturbed region), α_o is the flow angle, and the other variables are defined as:

$$\delta = \frac{\pi \tan^2\nu_o}{(1 - \tan^2\nu_o \cdot \cot^2\alpha_o)^{\frac{1}{2}}} \tag{4.16}$$

$$\Lambda = \frac{\cot \alpha_1}{\sqrt{\eta}} \quad F(\xi) \tag{4.17}$$

where $F(\xi)$ is the Witham F-function and is defined as:

$$F(\xi) = \int_o^{\xi} \frac{S''(\bar{x})\,d\bar{x}}{\sqrt{\xi - \bar{x}}} \tag{4.18}$$

where $S''(x)$ is the second derivative of the cross-sectional area distribution of the cone. The Whitham F-Function for a cone can be given as:

$$F(\xi) = \frac{2}{\pi}\sqrt{\xi} \tag{4.19}$$

The relation between the characteristics ξ and η along the shock wave is given by Oswatitsch [46] as:

$$\frac{d}{d\xi}\left[\sqrt{\eta}\ F^2(\xi)\right] = -2r_{o_\xi} \cdot \frac{F(\xi)}{S_o\ \delta} \tag{4.20}$$

In the conical region $r_{o_\xi} = \frac{-\tan\alpha_o}{2}$.

Substituting the value of r_{o_ξ} in equation (4.20) and integrating the resultant equation yields:

$$\sqrt{\eta} = \frac{\tan\alpha_o}{S_o\delta} \cdot \frac{1}{F^2(\xi)} \int_o^{\xi} F(\xi)\,d\xi \tag{4.21}$$

Substituting equation (4.19) in equation (4.21), then after performing the integration one can get:

$$\sqrt{\eta} = \frac{\pi \tan\alpha_o}{3S_o\delta} \sqrt{\xi} \tag{4.22}$$

where $S_o = \frac{\gamma + 1}{4} \cdot \frac{M_o^4}{(M_o^2 - 1)^{3/2}}$

Equation (4.22) indicates the relation between the characteristics ξ and η along the shock wave in the conical region.

Substituting equation (4.17) and equation (4.22) in equation (4.15), the shock strength can be found as:

$$\frac{\Delta P}{P_o} = \frac{6\gamma M_o^2 \cdot \delta^2 S_o}{\pi^2 \tan\alpha_o} \tag{4.23}$$

As can be seen from this equation, the shock strength is constant in the entire conical region, i.e., it does not change with the distance (η). In the conical region the shock strength varies as the Mach number (M), the ratio of specific heats (γ), and the apex angle of the body ($2\alpha_1$) vary.

(B) <u>In The Expansion Region</u>

The Prandtl-Meyer analysis for the steady uniform supersonic flow of a compressible fluid around a sharp convex corner is based on many assumptions and observations, Hoffman [20]. In this expansion it is considered that the

flow is steady two-dimensional and isentropic throughout the flow field. Accordingly, the expansion region associated with the supersonic flow around a convex corner is a simple wave region, hence, each Mach line inside of the expansion fan must be a straight line along which the flow properties are uniform while these Mach lines are slightly curved in the axisymmetric case. Since the disturbance produced by the corner is very weak, a Mach line (characteristic) is propagated radially outward from the corner into the incident supersonic flow (Fig. 4.2).

After crossing the first characteristic $\xi_u = 1$, the flow turns and accelerates, and all of the changes in the static pressure of the fluid are transmitted by a series of Mach lines (characteristics $\xi_u = 1$ to $\xi_l = 2$) eminating outward from the convex corner. After crossing the last of the series of the characteristics ($\xi = 2$) the expansion of the fluid is completed.

The shock wave does not feel the expansion region until the first characteristic eminating from the convex sharp corner hits the shock. The characteristics running into the front (bow) shock extend from $\xi = 1$ upto $\xi = \xi_n$, where ξ_n is the neutral Mach line. The neutral Mach line ξ_n is found from the condition that $F(\xi_n) = 0$ as:

$$\xi_n = 1 + \sqrt{\frac{2T_s}{1 + T_s}} \tag{4.24}$$

where T_s is the Tsien-parameter, $T_s = \tan\nu_o \cot\alpha_o$.

To calculate the shock strength in the expansion region, recall equations (4.15) and (4.17):

$$\frac{\Delta P}{P_o} = \gamma M_o^2 \cdot \tan\alpha_o \cdot \delta \Lambda \qquad (4.15)$$

$$\Lambda = \frac{\cot\alpha_o}{\sqrt{\eta}} F(\xi) \qquad (4.17)$$

The Whitham F-Function for the expansion region is:

$$F(\xi) = \frac{2}{\pi}\left\{1 - \sqrt{\frac{T_s + 1}{2T_s}}\ (\xi - 1)\right\} \text{ for } 1 \leqslant \xi \leqslant 2 \qquad (4.25)$$

The first characteristics runs into the shock hits the shock at point $P_a(1, \eta_a)$ (Fig. 4.3).

The relation between the characteristics ξ and η is given by equation (4.22):

$$\sqrt{\eta} = \frac{\pi \tan\alpha_o}{3\,\delta\, S_o} \sqrt{\xi} \qquad (4.22)$$

Putting $\xi = 1$, yields:

$$\eta_a = \left(\frac{\pi \tan\alpha_o}{3\, S_o\, \delta}\right)^2 \qquad (4.26)$$

Recall equation (4.20):

$$\frac{d}{d\xi}\left[\sqrt{\eta}\ F^2(\xi)\right] = -\frac{2 r_{o\xi} \cdot F(\xi)}{S_o\ \delta} \qquad (4.20)$$

in the expansion region, $r_{o\xi} = 0$.

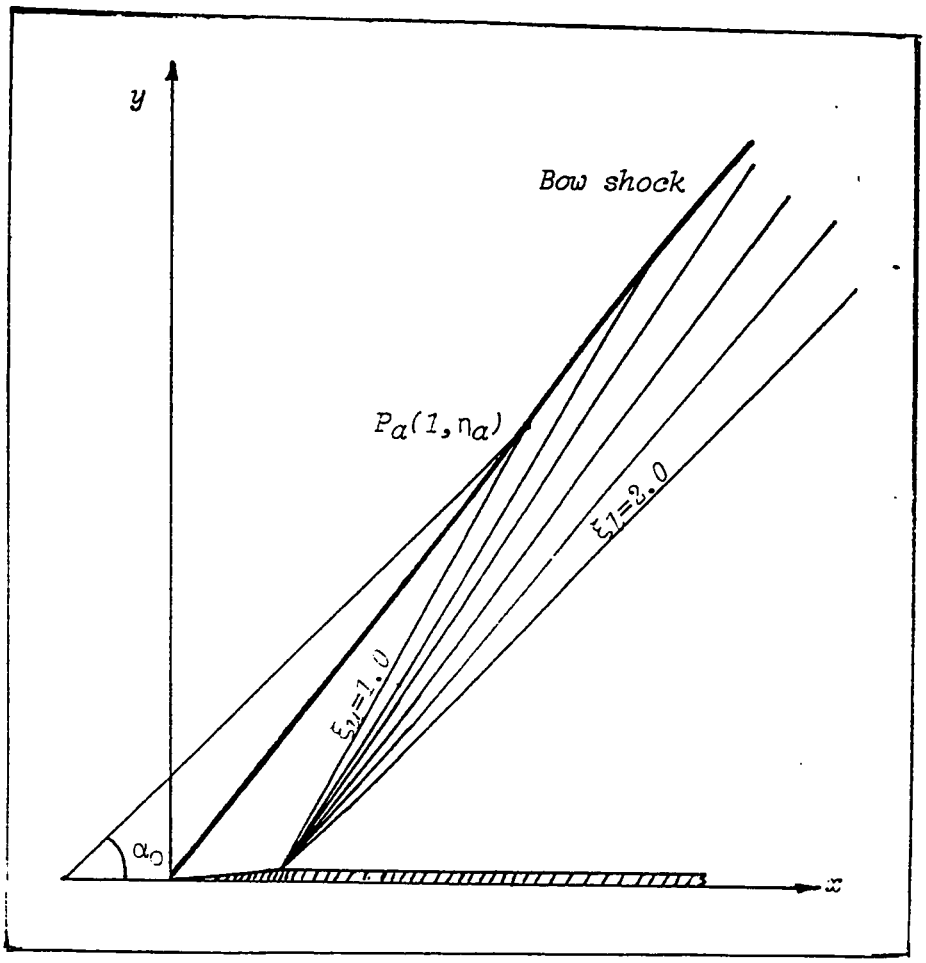

Fig. 4.3. Interaction of the expansion region with the bow shock.

Hence, from equation (4.20) one can get:

$$\sqrt{\eta}\ F^2(\xi) = \text{const.} \tag{4.27}$$

or

$$\sqrt{\eta}\ F^2(\xi) = \sqrt{\eta_a}\ F^2(\xi_a) \tag{4.28}$$

From equation (4.17) one can get that:

$$F(\xi) = \Lambda \sqrt{\eta}\ \tan\alpha_o \tag{4.29}$$

$$F(\xi_a) = \Lambda_a \sqrt{\eta_a} \cdot \tan\alpha_o \tag{4.30}$$

Substituting equations (4.29) and (4.30) in equation (4.28), yields:

$$\eta = \eta_a \left(\frac{\Lambda_a}{\Lambda}\right)^{4/3} \tag{4.31}$$

Substituting equations (4.26), (4.17) and (4.22) in equation (4.31) yields:

$$\Lambda_a = \frac{6\ S_o \cdot \delta\ \cot^2\alpha_o}{\pi^2} \tag{4.32}$$

Substituting equations (4.17), (4.25), and (4.32) in equation (4.31) and simplifying the resultant equation one can get:

$$\eta^{1/3} = \left(\frac{\pi\ \tan\alpha_o}{3\,S_o\,\delta}\right)^2 \cdot \left\{\frac{3\ S_o\ \delta\ \cdot\ \cot\alpha_o}{\pi\left(1 - \sqrt{\frac{1+T_s}{2T_s}}\ (\xi - 1)\right)}\right\}^{4/3} \tag{4.33}$$

Substituting equation (4.33) in equation (4.15) and simplifying the resultant equation, one can get that the shock strength in the expansion region is:

$$\frac{\Delta P}{P_o} = \frac{6\ \gamma\ M_o^2 \cot\alpha_o}{\pi^2} \cdot \frac{S_o\ \delta^2}{(\eta)^{3/4}} \cdot \left(\frac{\pi\ \tan\alpha_o}{3\ S_o\ \delta}\right)^{3/2} \tag{4.34}$$

for $\eta \geqslant \eta_a$ and $1 \leqslant \xi \leqslant 2$.

Equation (3.34) can be simplified more to:

$$\frac{\Delta P}{P_o} = \frac{(.65147) \cdot \gamma\ M_o^2}{\eta^{3/4}} \sqrt{\frac{\delta\ \tan\alpha_o}{S_o}} \tag{4.35}$$

(C) Results Obtained

Equation (4.35) describes the shock wave strength in the expansion region while equation (4.23) describes the shock strength in the conical region.

The relation between the shock strength in both regions and "η" is shown in Fig. 4.4.

In the expansion region the shock strength varies also with the Mach number and the half-apex angle (α_1) of the wing:

In Fig. 4.5, the variation of the shock strength with the Mach number is shown with $\alpha_1 = 15^o$ and $\eta = 100$ body length.

In Fig. 4.6, the variation of the shock strength is plotted against (α_1) for $M = \sqrt{2}$ and $\eta = 100$ body length.

The area of the conical region depends on η_a, and η_a is described by equation (4.26) which shows that η_a varies with the Mach number (M) and the half-apex angle of the delta wing (α_1) keeping other parameters constant.

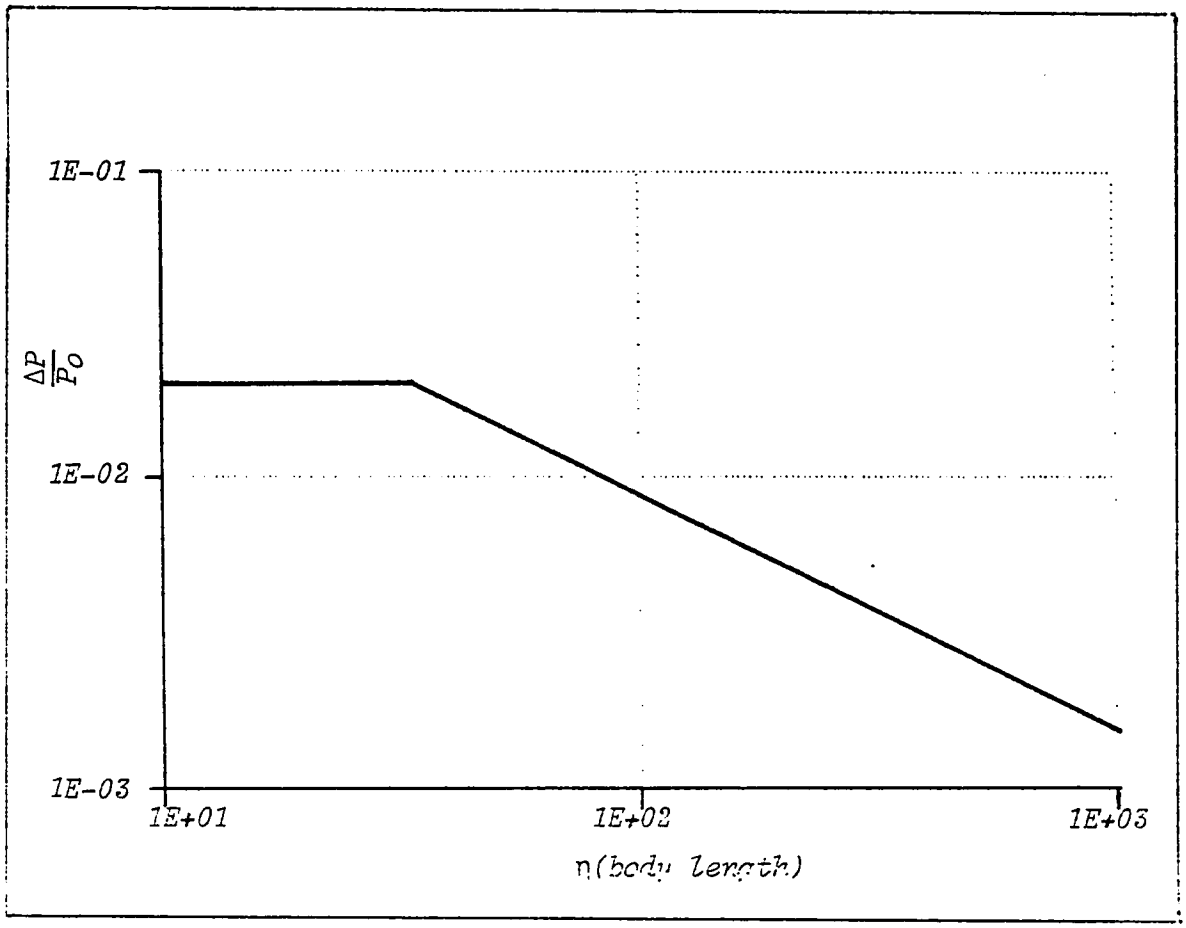

Fig. 4.4. Bow shock wave strength of a delta wing with straight leading edges as it varies with the distance, η ($M = \sqrt{2}$, $\alpha_1 = 15°$).

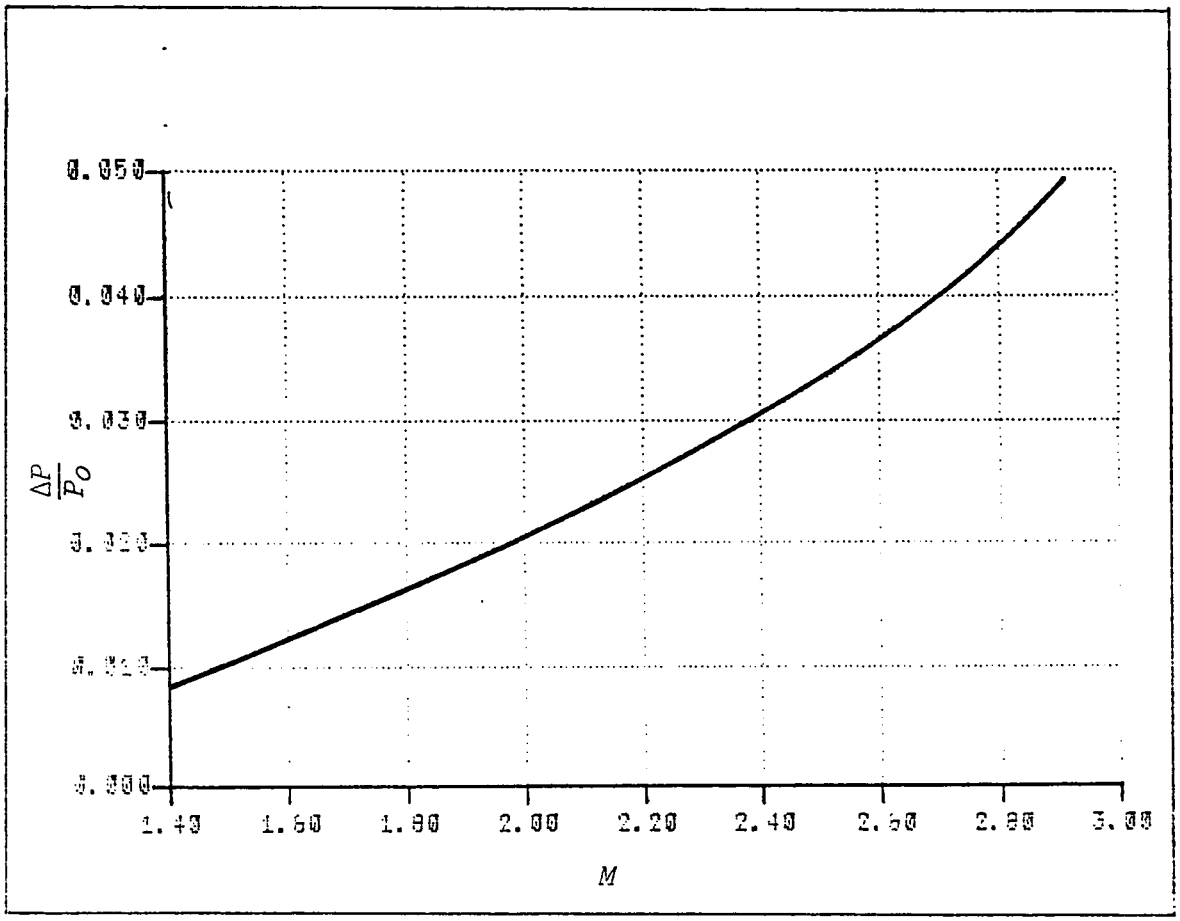

Fig. 4.5. The shock wave strength of a delta wing with straight leading edges as it varies with Mach number (η = 100, α_1 = 15°)(expansion region).

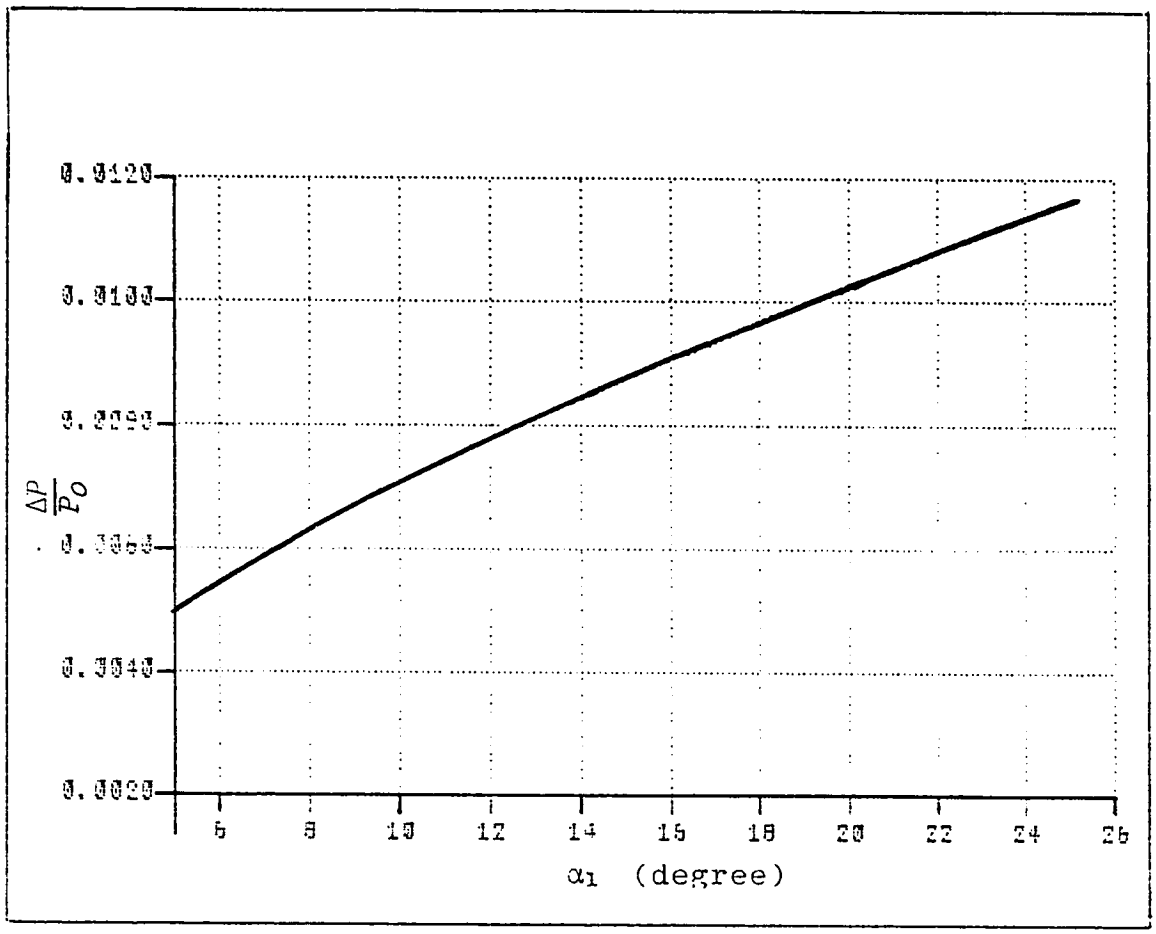

Fig. 4.6. Shock strength of a delta wing with straight leading edges as it varies with (α_1). ($M = \sqrt{2}$, $\eta = 100$)(expansion region)

In Fig. 4.7 the variation of η_a with the Mach number is plotted for $\alpha_1 = 15°$ and $\eta = 100$ body length.

In Fig. 4.8 the variation of η_a with α_1 is plotted for $M = \sqrt{2}$ and $\eta = 100$ body length.

4.2 THE WING WITH CURVED LEADING EDGES:

4.2.1 : THE WHITHAM F-FUNCTION OF THE WING:

The length distribution of lift for the given configuration is given as (see Fig. 4.1):

$$l_i(\xi) = 2(\tan\alpha_2\xi - C\xi^2) \cdot \Delta P' \tag{4.36}$$

where C is an arbitrary constant to be determined, and

α_2 is the half-apex angle of the wing.

The derivative of the cross-sectional area distribution is:

$$S'(\xi) = l_i(\xi) \cdot \sqrt{M_o^2 - 1} / \rho U^2 \tag{4.37}$$

and the second derivative is:

$$S''(\xi) = \frac{\partial\, l_i(\xi)}{\partial\xi} \sqrt{M_o^2 - 1} \;/ \rho U^2$$

$$= 2\,\Delta P' \cdot (\tan\alpha_2 - 2C\xi) \frac{\sqrt{M_o^2 - 1}}{\rho U^2} \tag{4.38}$$

or in a more compact form

$$S''(\xi) = K_1 - K_2\,\xi \tag{4.39}$$

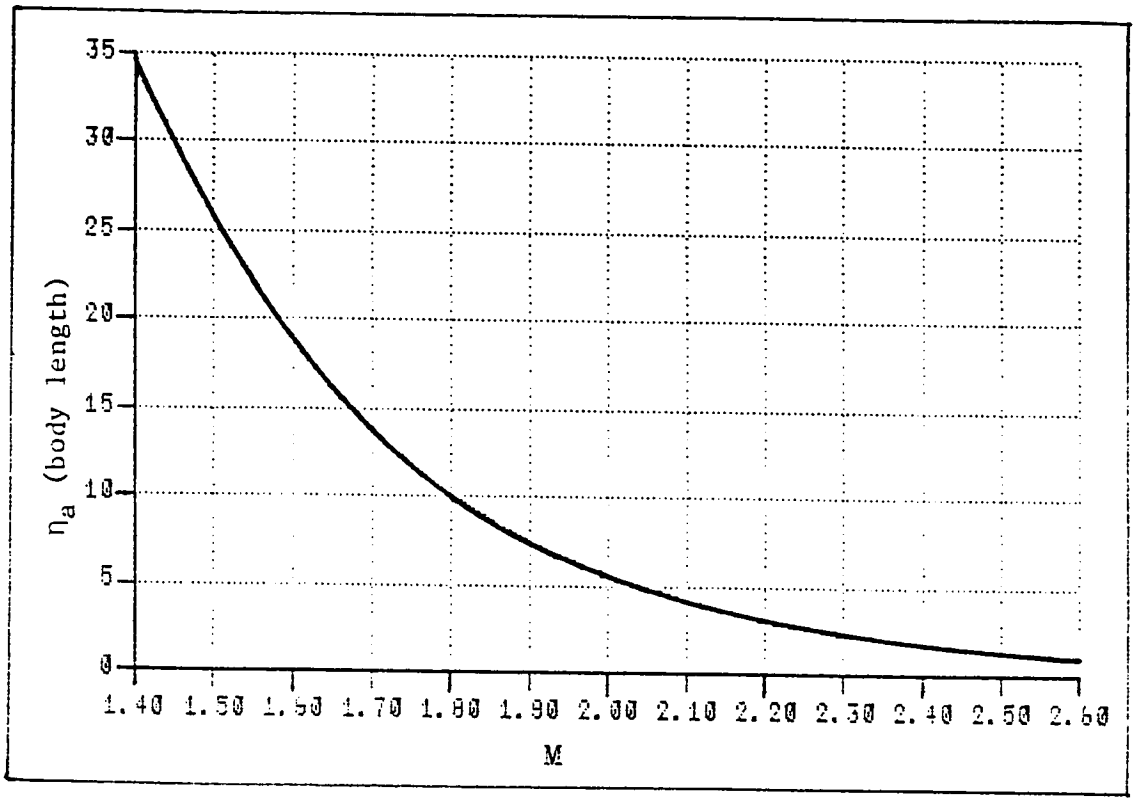

Fig. 4.7. The limit of the conical region (η_a) of a delta wing with straight leading edges as it varies with the Mach number. ($\alpha_1 = 15°$, $\eta = 100$)

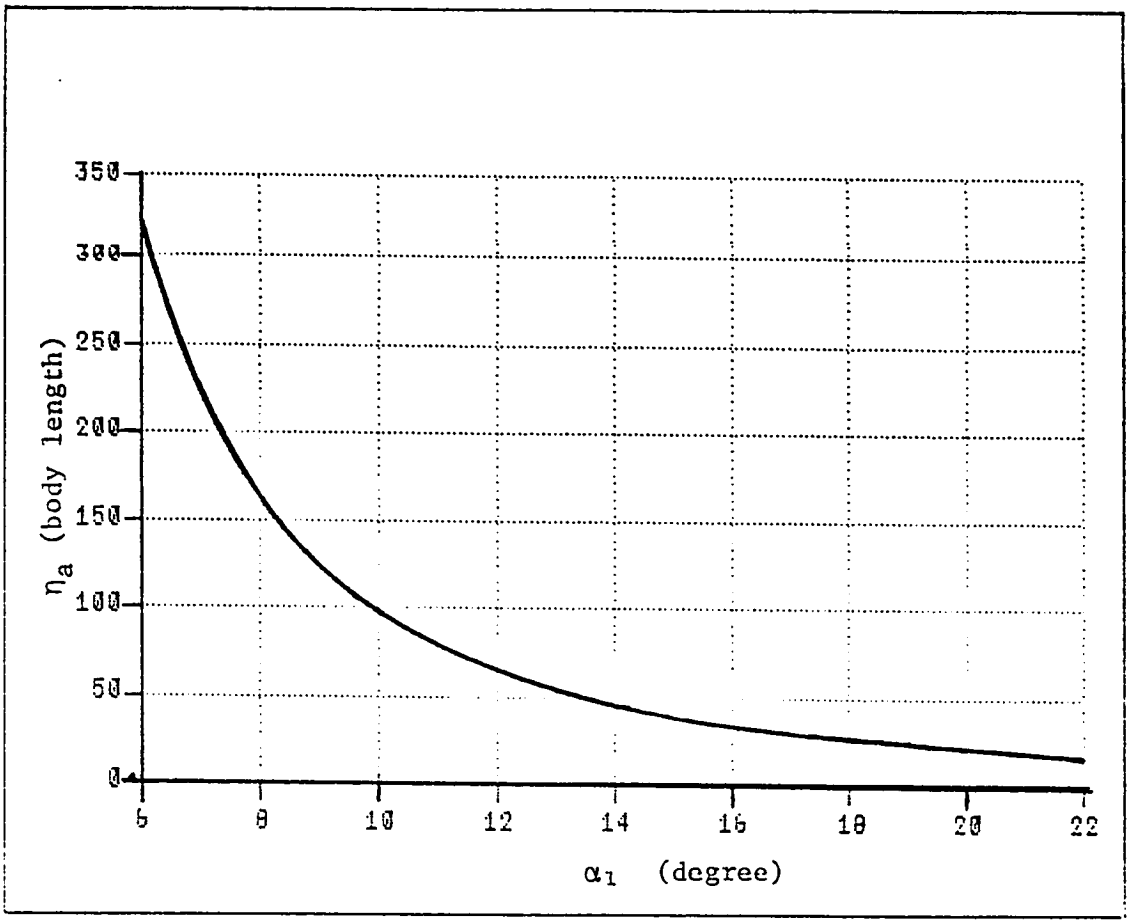

Fig. 4.8. The limit of the conical region (η_a) for a delta wing with straight leading edges as it varies with (α_1). ($M = \sqrt{2}$, $\eta = 100$)

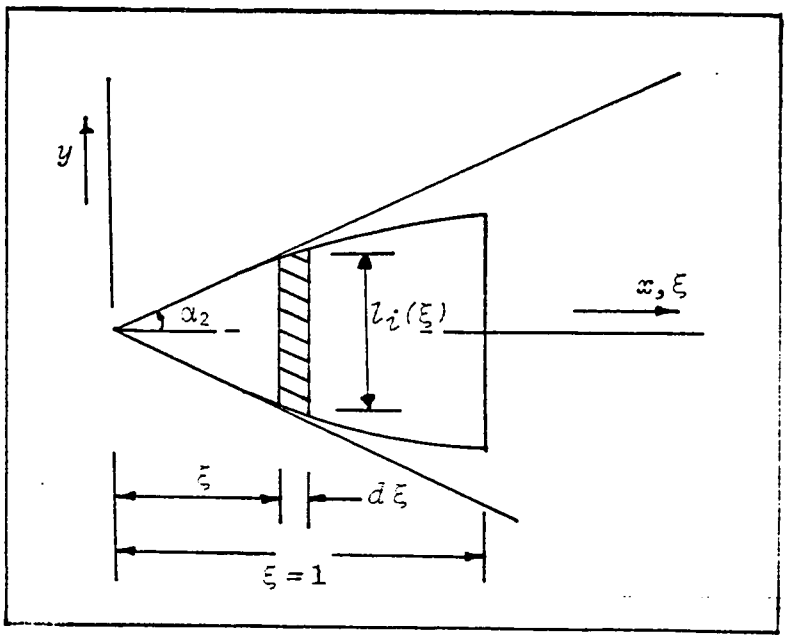

Fig. 4.9. Delta wing with curved leading edges.

where

$$K_1 = CL \cdot \tan\alpha_2 \cdot \sqrt{M_o^2 - 1}$$

$$K_2 = CL \cdot 2C \cdot \sqrt{M_o^2 - 1}$$

$$CL = 2\ \Delta P / \rho U^2.$$

Generally, the Whitham F-Function is given by:

$$F(\xi^*) = \frac{1}{2\pi} \int_0^{\xi^*} \frac{S''(\xi)}{\sqrt{\xi^* - \xi}}\, d\xi \qquad (4.40)$$

where ξ^* is an arbitrary argument of the F-Function.

Substituting equation (4.39) in equation (4.40) yields:

$$F(\xi^*) = \frac{1}{2\pi} \int_0^{\xi^*} \frac{K_1 - K_2\xi}{\sqrt{\xi^* - \xi}}\, d\xi$$

Carrying on the integration, one can finally get:

$$F(\xi^*) = \frac{1}{\pi} \left\{ K_1 (\xi^*)^{1/2} - \frac{2K_2}{3} (\xi^*)^{3/2} \right\}$$

or

$$F(\xi) = \frac{1}{\pi} \left\{ K_1 (\xi)^{1/2} - \frac{2K_2}{3} (\xi)^{3/2} \right\} \qquad (4.41)$$

The equivalent body of revolution for the curved wing will be a cone-cylinder configuration (see Fig. 4.10).

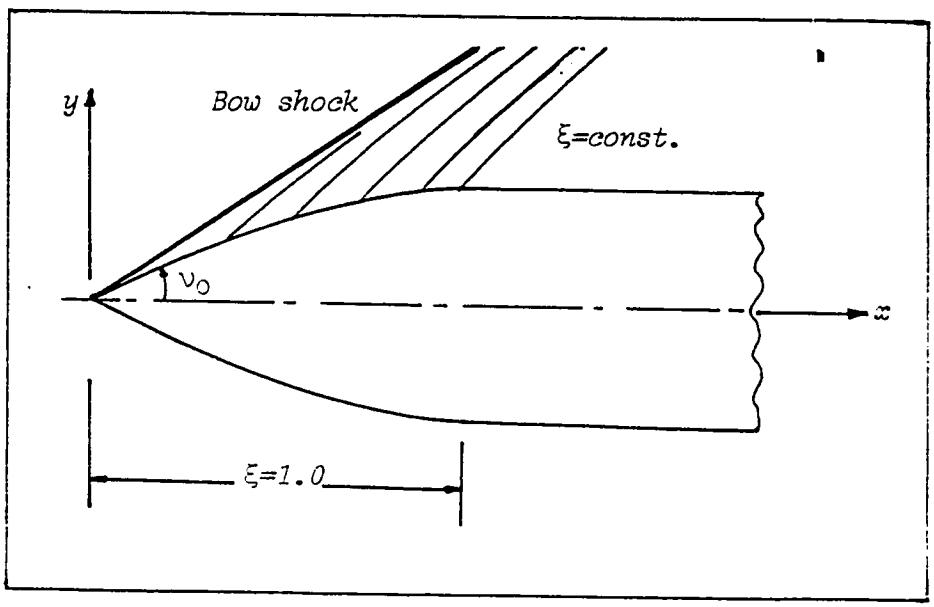

Fig. 4.10. Equivalent body of revolution for the wing with curved leading edges.

4.2.2 : THE RELATION BETWEEN (ν_0) AND (α_2)

Again we have to find a relation between the half-apex angle of the wing and that of the equivalent body of revolution.

Substituting equation (4.36) in equation (4.37) and integrating the resultant equation yields:

$$S(\xi) = CL \cdot \sqrt{M_0^2 - 1} \left(\frac{\tan\alpha_2}{2} \xi^2 - \frac{C}{3} \xi^3 \right) \tag{4.42}$$

At the tip of the body $\xi << 1$, and $\xi^2 >> \xi^3$, therefore, at the tip of the body a cone can be assumed, whose cross-sectional area is given as:

$$\begin{aligned} A = \pi r^2 &= \pi(\xi \tan\nu_0)^2 \\ &= \pi \tan^2\nu_0 \, \xi^2 \end{aligned} \tag{4.43}$$

Neglecting higher order terms in equation (4.42) and equating the resultant equation with equation (4.43) one can finally get:

$$\tan\nu_0 = \left(\frac{CL}{2\pi} \cdot \tan\alpha_2 \cdot \sqrt{M_0^2 - 1} \right)^{\frac{1}{2}} \tag{4.44}$$

4.2.3 : DETERMINATION OF THE ARBITRARY CONSTANT (C)

Consider the cross-sectional area distribution of the equivalent body of revolution:

$$S(\xi) = \pi \, r^2 = CL \cdot \sqrt{M_0^2 - 1} \left(\frac{\tan\alpha_2}{2} \xi^2 - \frac{C}{3} \xi^3 \right)$$

From equation (4.44) one can get:

$$CL \cdot \sqrt{M_O^2 - 1} = \frac{2\pi \tan^2\nu_O}{\tan\alpha_2} \tag{4.45}$$

Substituting equation (4.45) in equation (4.44) one can get:

$$r^2 = \frac{2\tan^2\nu_O}{\tan\alpha_2}\left(\frac{\tan\alpha_2}{2}\xi^2 - \frac{C}{3}\xi^3\right) \tag{4.46}$$

The condition which is applied here to avoid any expansion is that no corner is needed at $\xi = 1$ of the equivalent body of revolution; in order to avoid any axisymmetric Prandtl-Meyer expansions (i.e., the slope is zero at $\xi = 1.0$). One should have:

$$\left.\frac{dr}{d\xi}\right|_{\xi=1} = 0 \tag{4.47}$$

Applying equation (4.47) to equation (4.46), one can get:

$$C = \tan\alpha_2 \tag{4.48}$$

4.2.4 : THE VALUE OF NEUTRAL MACH LINE (ξ_n)

Recall equation (4.41), the Whitham F-Function has a root when the value between brackets is zero, i.e., $F(\xi_n)=0$:

$$K_1(\xi_n)^{1/2} - \frac{2K_2}{3}\xi_n^{3/2} = 0$$

dividing through by $\sqrt{\xi_n}$ ($\xi_n \neq 0$) yields:

$$K_1 - \frac{2K_2}{3}\xi_n = 0 \tag{4.49}$$

for $C = \tan\alpha_2$, one obtains that $K_2 = 2K_1$.

Substituting this value in equation (4.49) one can obtain that the neutral Mach line (ξ_n) occurs at:

$$\xi_n = \frac{3}{4} \tag{4.50}$$

For the considered case the neutral Mach line hits the shock at infinity, hence, ξ should be less than or equal to ξ_n, (i.e., $\xi \leqslant \xi_n$) in the shock calculations, but η goes from $0 \to \infty$, i.e., $0 < \eta < \infty$.

4.2.5 : CALCULATIONS OF THE SHOCK WAVE GEOMETRY AND STRENGTH

(A) Shock Wave Geometry:

To calculate the shock wave geometry, let's consider the general expression of the shock geometry (Oswatitsch 46):

$$\sqrt{\eta} = \frac{\tan\alpha_o}{S_o \cdot \delta} \cdot \frac{1}{F^2(\xi)} \int_0^{\xi} F(\xi)d\xi \tag{4.51}$$

Recalling equation (4.41), the integral on the right hand side of equation (4.51) becomes:

$$\int_0^{\xi} F(\xi)d\xi = \int_0^{\xi} \frac{1}{\pi}\left(K_1\xi^{1/2} - \frac{2K_3}{3}(\xi)^{3/2}\right)d\xi$$

$$= \frac{2}{3\pi}\left(K_1\xi^{3/2} - \frac{2K_2}{5}\xi^{5/2}\right) \tag{4.52}$$

and

$$F^2(\xi) = \frac{1}{\pi^2}\left(K_1^2\xi + \frac{4}{9}K_2^2\xi^3 - \frac{4}{3}K_1K_2\xi^2\right) \tag{4.53}$$

Substituting equations (4.52) and (4.53) in equation (4.50) yields:

$$\sqrt{\eta} = \frac{2\,\pi\,\tan\alpha_0}{3 \cdot S_0 \cdot \delta} \cdot \frac{\left(K_1 \xi^{3/2} - \frac{2K_2}{5}\,\xi^{5/2}\right)}{\left(K_1^2 \xi + \frac{4}{9}\,K_2^2 \xi^3 - \frac{4}{3}\,K_1 K_2 \xi^2\right)}$$

or

$$\sqrt{\eta} = \frac{2\,\pi\,\tan\alpha_0}{3 \cdot S_0 \cdot \delta} \cdot \frac{\left(K_1 \xi^{1/2} - \frac{2K_2}{5}\,\xi^{3/2}\right)}{\left(K_1^2 + \frac{4}{9}\,K_2^2 \xi^2 - \frac{4}{3}\,K_1 K_2 \xi\right)} \tag{4.54}$$

This equation gives the relationship $\xi = \xi(\eta)$ along the shock which is needed in the calculation of the pressure jump across the shock. This equation cannot be solved (i.e., finding $\xi = \xi(\eta)$ explicitly) by direct analytical method but it can be solved by numerical iteration.

In a more compact form equation (4.54) can be written as:

$$\sqrt{\eta} = K_3 \cdot \frac{\left(K_1 \xi^{1/2} - \frac{2K_2}{5}\,\xi^{3/2}\right)}{\left(K_1^2 + \frac{4}{9}\,K_2^2 \xi^2 - \frac{4}{3}\,K_1 K_2 \xi\right)}$$

where

$$K_3 = \frac{2\,\pi\,\tan\alpha_0}{3 \cdot S_0 \cdot \delta}$$

from which one can get:

$$\sqrt{\eta}\left(K_1^2 + \frac{4}{9}K_2^2\xi^2 - \frac{4}{3}K_1K_2\xi\right) = K_3\left(K_1\xi^{1/2} - \frac{2K_2}{5}\xi^{3/2}\right)$$

or

$$\sqrt{\eta}\left(K_1^2 + \frac{4}{9}K_2^2\xi^2 - \frac{4}{3}K_1K_2\xi\right) - K_3\left(K_1\xi^{1/2} - \frac{2K_2}{5}\xi^{3/2}\right) = 0 \qquad (4.55)$$

for a given distance η, the only unknown is ξ which can be found by solving the above equation by iteration method, the general shape of this equation will be:

$$A_1 + B_1\xi^2 - C_1\xi - D_1\xi^{1/2} + E_1\xi^{3/2} = 0 \qquad (4.56)$$

where

$$A_1 = \sqrt{\eta}\ K_1^2$$

$$B_1 = \sqrt{\eta} \cdot \frac{4}{9}K_2^2$$

$$C_1 = \frac{4}{3}K_1K_2\ \sqrt{\eta}$$

$$D_1 = K_3K_1$$

$$E_1 = \frac{2}{5}K_1K_2$$

Equation (4.56) is solved by using Newton iteration method. Starting with an initial value of about ($.9\ \xi_n$) and bearing in mind that ξ_{max} is .75 only ($\xi_n = .75$).

(B) <u>Shock Wave Strength</u>:

To calculate the pressure jump across the shock, recall the general formula of the shock strength:

$$\frac{\Delta P}{P_o} = \gamma\ M_o^2\ \tan\alpha_o \cdot \delta \cdot \Lambda \qquad (4.15)$$

and

$$\Lambda \tan\alpha_o = \frac{1}{\sqrt{\eta}} F(\xi) \tag{4.17}$$

Substituting equation (4.41) in equation (4.17) yields:

$$\Lambda \tan\alpha_o = \frac{1}{\sqrt{\eta}} \cdot \frac{1}{\pi}\left(K_1\xi^{\frac{1}{2}} - \frac{2K_2}{3}\xi^{3/2}\right) \tag{4.57}$$

Substituting equation (4.57) in equation (4.15) yields:

$$\frac{\Delta P}{P_o} = \gamma M_o^2 \cdot \delta \cdot \frac{1}{\pi\sqrt{\eta}}\left(K_1\xi^{\frac{1}{2}} - \frac{2K_2}{3}\xi^{3/2}\right) \tag{4.58}$$

but for the considered profile, it is already found that $\underline{K_2 = 2K_1}$.

Substituting this value in equation (4.58) yields:

$$\frac{\Delta P}{P_o} = \gamma M_o^2 \cdot \delta \cdot \frac{K_1}{\pi\sqrt{\eta}}\left(\xi^{\frac{1}{2}} - \frac{4}{3}\xi^{3/2}\right) \tag{4.59}$$

This equation [equation (4.59)] with the aid of equation (4.55) gives the pressure jump across the shock wave.

The shock strength for the wing with curved leading edges vs. the body length is shown in Fig. 4.11 for a Mach number equal to 2.0.

As can be seen from equation (4.59) the shock strength varies also with the flow Mach number, and the half-apex angle of the wing.

In Fig. 4.12 the variation of the shock strength with the Mach number for a constant distance of 100 body length

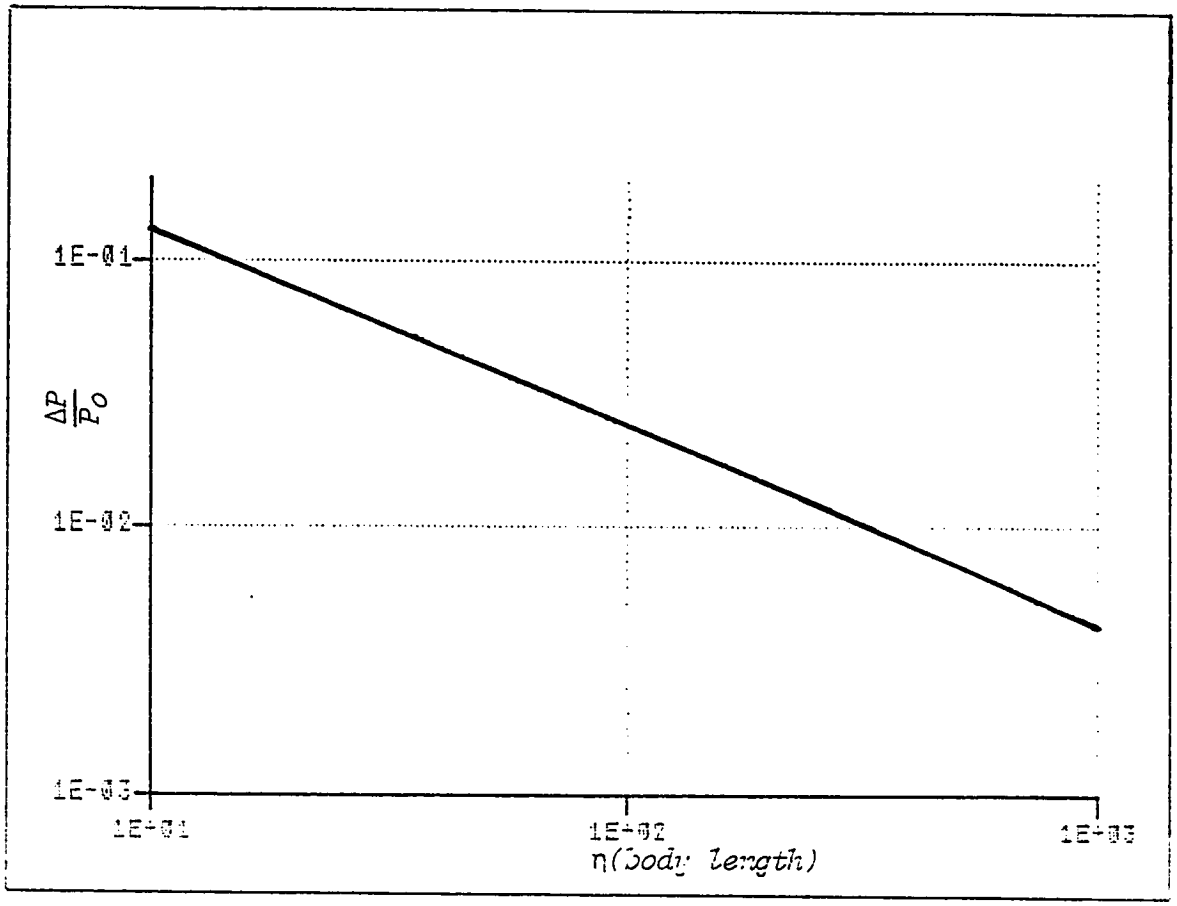

Fig. 4.11. Bow shock strength of a delta wing with curved leading edges as it varies with the distance (η). ($M = 2.0$, $\alpha_2 = 37.8^{\circ}$)

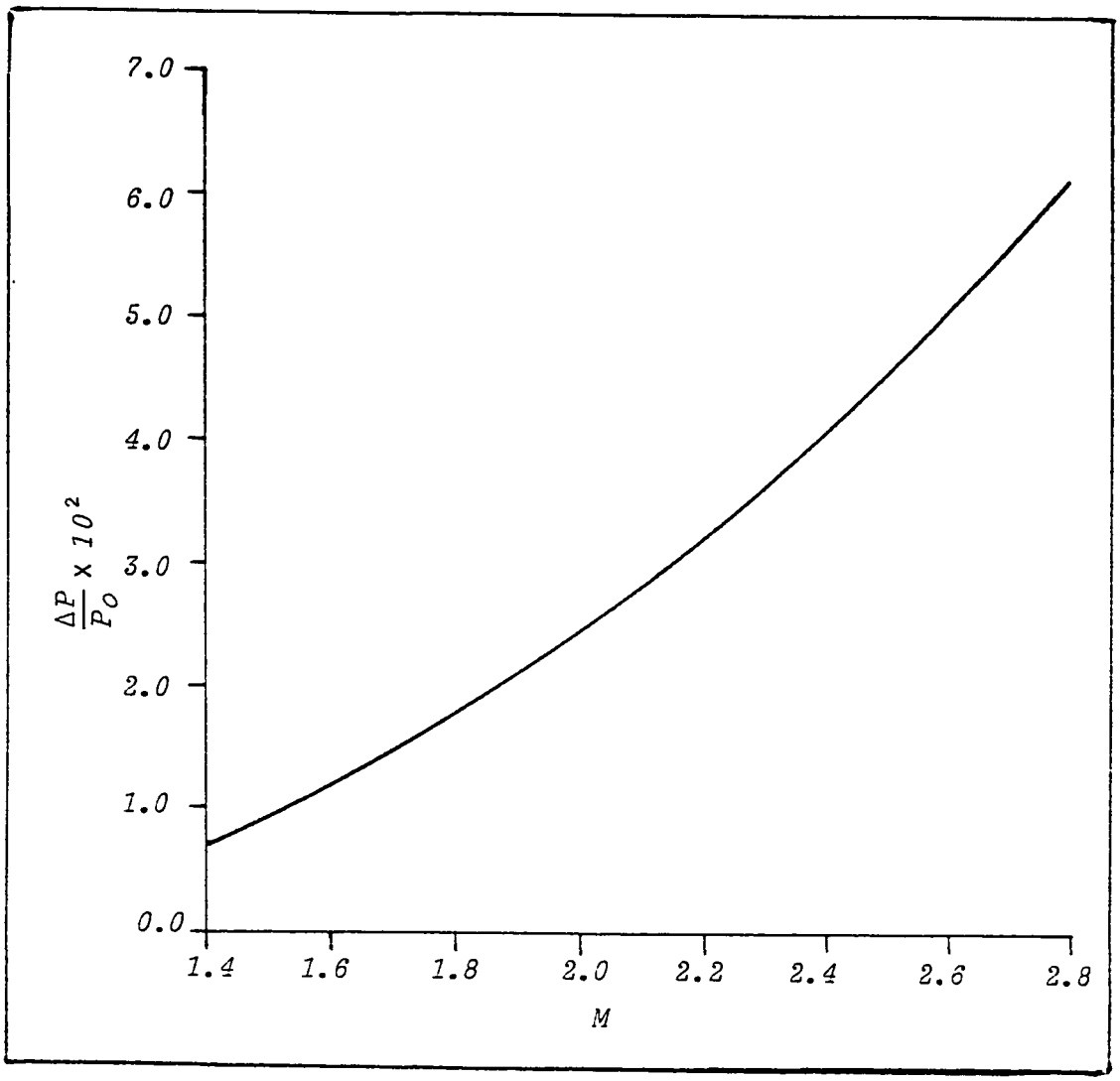

Fig. 4.12. Shock strength of a delta wing with curved leading edges as it varies with the Mach number. ($\eta = 100$, $\alpha_2 = 37.8^o$)

and a half-apex angle of 37.8° is shown.

In Fig. 4.13 the variation of the shock strength with the half-apex angle of the curved wing is plotted for a constant Mach number of $\sqrt{2}$, and at a distance of 100 body length.

To illustrate the effect of the pressure jump across the bow shock wave on an observer in the vertical plane; whether it is annoying or not, let's consider the following practical example:

(a) A supersonic aircraft with delta wings (having straight leading edges) of length 20 m and half-apex angle of 15° is flying at an altitude of 8060 m and $M = \sqrt{2}$:

From the given data:

$$\eta = \frac{8060}{20} = 403 \text{ (body length)}$$

From equation (4.35):

$$\frac{\Delta P}{P_o} = .0030834$$

The pressure in the undisturbed flow field (P_o) at the given altitude is found to be 3.536×10^4 Pa.

Hence;

$$\Delta P = 109 \text{ Pa}$$

It is known that if the pressure jump is greater than 100 Pa it is annoying. Therefore 109 Pa is an annoying pressure jump.

But if the given altitude is increased to 1060 m the pressure jump is found to be 68 Pa, which is not annoying.

(b) A supersonic aircraft with curved wings of length 20 m and half-apex angle of 37.8^{o} is flying at an altitude of 8000 m and $M = \sqrt{2}$:

Again:

$$\eta = 400 \text{ (body length)}$$

From equation (4.59):

$$\frac{\Delta P}{P_o} = .0026277$$

The pressure in the undisturbed flow field (P_o) at the given altitude is found to be 3.565×10^4 Pa.

Hence;

$$\Delta P = 93.7$$

This pressure jump is not annoying. But if the aircraft is flying at an altitude, say, 7000 m, the pressure jump will be 119 Pa which is an annoying pressure jump.

4.3 The Relation Between (α_1) and (α_2) for the Same Lift

For the two configurations considered (the delta wing and the one with curved leading edges) to have the same total lift from both configurations, the total area should be the same (assuming that $\Delta P'$ is the same for both configurations).

(a) The Delta Wing: (see Fig. 4.1)

$$A_D = \int_0^1 l_i(\xi)\,d\xi$$

$$= \int_0^1 2\xi \tan\alpha_1 \, d\xi$$

$$= \tan\alpha_1 \cdot (1)^2$$

from which:

$$A_D = \tan\alpha_1 \tag{4.60}$$

where A_D is the delta wing area.

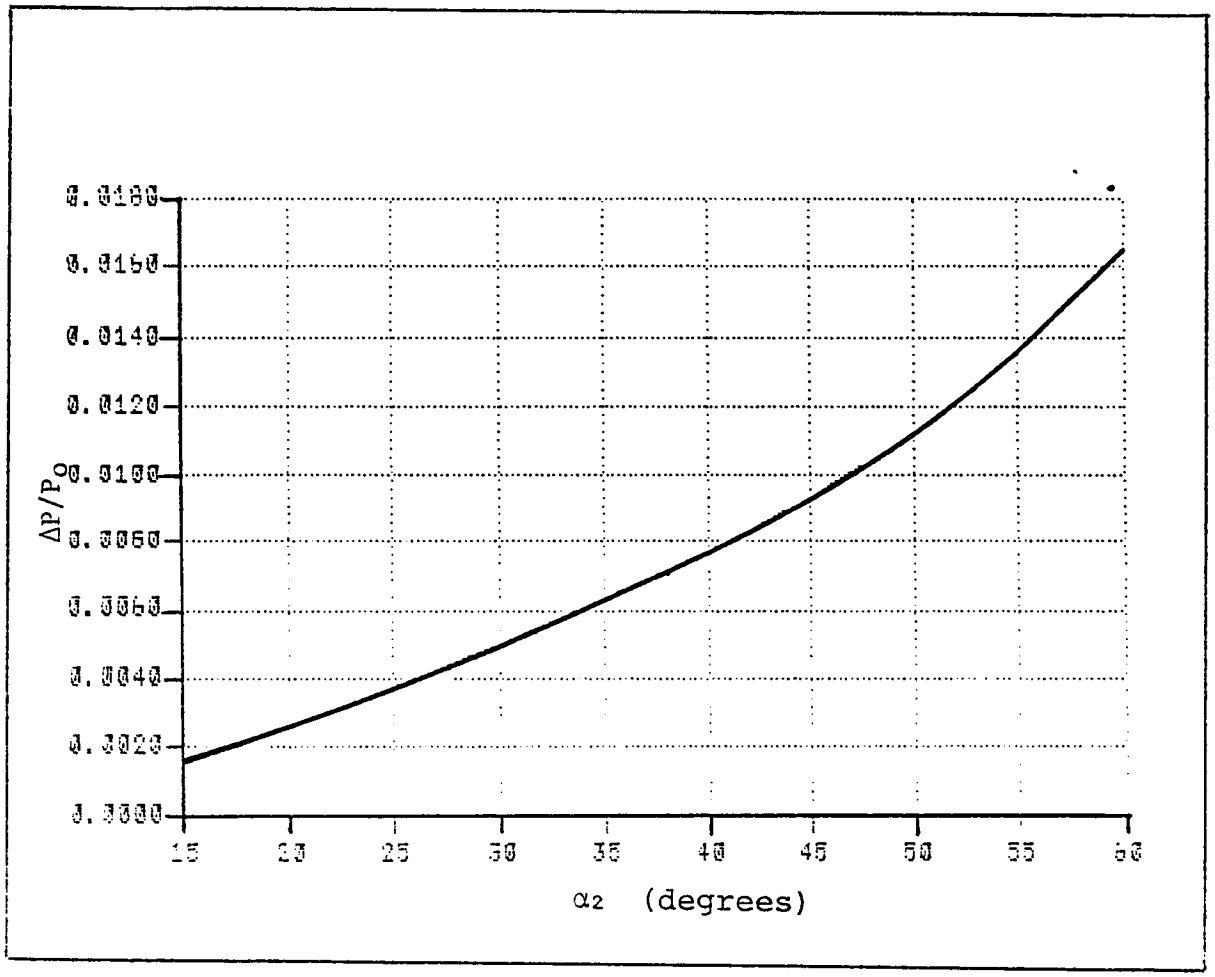

Fig. 4.13. The shock wave strength of a delta wing with curved leading edges as it varies with (α_2). ($M = \sqrt{2}$, $\eta = 100$)

(b) <u>The Delta Wing with Curved Leading Edges</u>: (see Fig.4.2)

$$A_C = \int_0^1 l_i(\xi)d\xi$$

$$= \int_0^1 2(\xi \tan\alpha_2 - C\xi^2)d\xi$$

but we have already found the $C = \tan\alpha_2$,

hence;

$$A_C = 2 \tan\alpha_2 \int_0^1 (\xi - \xi^2)d\xi$$

$$= 2 \tan\alpha_2 \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$A_D = \frac{\tan\alpha_2}{3} \qquad (4.61)$$

where A_C is the area of the delta wing with curved leading edges.

As stated above, to have the same lift from both wings both wing areas should be the same:

$$A_C = A_D \qquad (4.62)$$

Substituting equations (4.60) and (4.61) in equation (4.62) yields:

$$\frac{\tan\alpha_2}{3} = \tan\alpha_1$$

from which

$$\tan\alpha_2 = 3 \tan\alpha_1 \qquad (4.63)$$

For example, if we choose a half-apex angle of a delta wing to be 15.0° the equivalent curved leading edges wing will have a half-apex angle of 38.79°.

4.4 THE SEPARATION ANGLE

One might ask what is the maximum apex-angle which is allowed for both wings to avoid separation?

To answer this question we introduce the following equation given by Oswatitch [46]:

$$\beta_s = \frac{\gamma + 1}{4} \cdot \frac{\tan^2\nu_o}{(.86)^2} \tag{4.64}$$

where

$$\beta_s = \frac{M_o - 1}{M_o + 1}$$

where ν_o is the half-apex angle of the equivalent body of revolution given by equation (4.14):

$$\tan^2\nu_o = \left(\tan\alpha_1 \cdot \frac{CL}{2\pi} \cdot \sqrt{M_o^2 - 1} \right) \tag{4.14}$$

for example, if one considers $M_o = \sqrt{2}$, CL = 1.0, $\gamma = 1.4$, then:

$$\tan^2\nu_o = \frac{\tan\alpha_1}{2\pi} \tag{4.65}$$

$$\beta_s = .17157.$$

Substituting this value in equation (4.64) one can get that the maximum allowable (ν_0) to avoid separation is 24.697°.

Substituting this value of ν_0 in equation (4.65) above, the maximum value of α_1 to avoid separation is 53.04°.

The maximum angle (α_2) to avoid seperation can be obtained from equation (4.44) as 53.04°.

4.5 THE SHOCK DRAG

If a body moves in a perfect (frictionless) fluid and shock waves are generated, a drag force called either the shock drag or the wave drag is produced.

In supersonic flow there are basically three different mechanisms whereby drag is created. The first two are essentially the same as in subsonic flow, namely, through the action of viscosity in the boundary layer and through the release of vorticity that accompanies production of lift. The determination of the skin friction drag involves calculation of the boundary layer in a manner similar to what is considered in incompressible flow.

The vortex drag arises from the momentum, and hence kinetic energy, left in the fluid as a lifting vehicle travels through it. Since the vorticity remains essentially stationary with the fluid there is no fundamental difference

between subsonic and supersonic speeds.

In the supersonic case, however, the lift will induce an additional drag component, namely wave drag or shock drag. The shock drag is an aerodynamic phenomenon unique to supersonic flow and is associated with the energy radiated away from the vehicle in the form of pressure waves in much the same way as a fast-moving ship causes waves on the water surface. Bearing in mind that the shock waves are longitudinal waves while ship waves are transversal dispersive ones.

For a planar wing one can distinguish between shock wave drag due to thickness and that due to lift. The sum of the shock drag and vortex drag is often called the pressure drag, since it is manifested by the pressure times the chordwise slope of wing or body surface.

According to Lighthill, in the isentropic approximation the power of the shock drag is radiated as acoustic power into the asymptotic long-range field.

The power of the shock drag is given by Lighthill as:

$$L_A = \frac{1}{2} \int_0^{\xi_n} \int_0^{\pi} \rho_o \, a_o^3 \, M_o^3 \, F^2(\xi) d\theta_n \, d\xi \qquad (4.66)$$

From which the shock drag is expressible as:

$$D_S = \frac{1}{2} \int_0^{\xi_n} \int_0^{\pi} \rho_o \, a_o^2 \, M_o^2 \, F^2(\xi) d\theta_n \, d\xi \qquad (4.67)$$

4.5.1 : THE SHOCK DRAG FOR THE DELTA WING WITH STRAIGHT LEADING EDGES

As mentioned before the flow over this wing is composed of a conical region and an expansion region. Each region has its own F-function. Hence, the F-function mentioned in equation (4.67) is composed of the conical F-function and the expansion F-function:

$$F(\xi) = F_c(\xi) + F_e(\xi) \tag{4.68}$$

where

$F_c(\xi)$ - is the Whitham F-function in the conical region,

$$F_c(\xi) = \frac{2}{\pi}\sqrt{\xi} \cdot \frac{\cos\theta_n}{N_1}$$

$F_e(\xi)$ - is the Whitham F-function in the expansion region,

$$F_e(\xi) = \frac{2}{\pi}\left\{1 - \sqrt{\frac{1+T_s}{2T_s}}\,(\xi - 1)\right\}\frac{\cos\theta_n}{N_1}$$

θ_n - is angle of the observer inclination from the vertical plane,

$$\left(\cos\theta_n = \frac{z}{\sqrt{y^2 + z^2}}\right)$$

for the vertical plane, $\cos\theta_n = 1$

$N_1 = 1 - \beta_o^2 \tan^2\alpha_1 \sin^2\theta_n$

Substituting equations (4.69) and (4.68) in equation (4.67) yields:

$$D_S = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{\rho_o a_o^2 M_o^2}{(1 - \beta_o^2\tan^2\alpha_1\sin^2\theta_n)^2} \cdot \left\{\int_o^1 F_c^2(\xi)d\xi + \int_1^{\xi_n} F_e^2(\xi)d\xi\right\} d\xi d\theta_n$$

$$= \frac{\rho_o a_o^2 M_o^2}{2}\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_n}{(1 - E^2\sin^2\theta_n)^2}\, d\theta_n \cdot \left\{\int_o^1 \frac{4}{\pi^2}\,\xi d\xi + \frac{4}{\pi^2}\int_1^{\xi_n}[1 - B(\xi-1)^2]d\xi\right\} \tag{4.69}$$

where

$$E = \beta_o \tan\alpha_1$$

$$B = \sqrt{\frac{1 + T_s}{2T_s}}$$

Consider

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_n}{(1 - E^2\sin^2\theta_n)^2}\, d\theta_n = \int_{-\pi/2}^{\pi/2} \frac{(1 + \cos 2\theta_n)/2}{\left[1 - E^2\left(\frac{1 - \cos 2\theta_n}{2}\right)\right]^2}\, d\theta_n$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \frac{1 + \cos\gamma_n}{\left[\left(1 - \frac{E^2}{2}\right) + \frac{E^2}{2}\cos\gamma_n\right]^2}\, d\gamma_n$$

where

$$\gamma_n = 2\theta_n$$

Evaluating the above integral one can finally get:

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_n}{(1 - E^2\sin^2\theta_n)^2}\, d\theta_n = \frac{(C-K)}{2(C^2-K^2)}\left\{\frac{\pi}{\sqrt{C^2-K^2}}\right\} \tag{4.70}$$

where

$$C = 1 - \frac{E^2}{2}$$

$$K = \frac{E^2}{2}$$

Simplifying more, equation (4.70) can be rewritten as:

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_n}{(1-E^2\sin^2\theta_n)^2}\, d\theta_n = \frac{\pi}{2\sqrt{1 - \beta_o^2 \tan^2\alpha_1}} \tag{4.71}$$

Equation (4.71) is a constant, let this constant be equated to H, i.e.,

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_n}{(1 - E^2\sin^2\theta_n)^2}\, d\theta_n = H \tag{4.72}$$

Consider the other integrals in equation (4.69):

$$\int_0^1 \frac{4}{\pi^2}\, \xi d\xi = \frac{4}{\pi^2} \cdot \frac{1}{2} \tag{4.73}$$

$$\int_1^{\xi_n} \frac{4}{\pi^2}\, [1-B(\xi-1)]^2 d\xi = \frac{4}{\pi^2} \int_1^{\xi_n} [(1+B) - B\xi]^2 d\xi$$

$$= \frac{4}{\pi^2} \left\{ (1+B)^2\xi - 2B(1+B)\, \frac{\xi^2}{2} + B^2\frac{\xi^3}{3} \right\} \Bigg|_1^{\xi_n}$$

$$= \frac{4}{\pi^2} \left[\left\{ (1+B)^2\xi_n - B(1+B)\xi_n^2 + \frac{B^2\xi_n^3}{3} \right\} \right.$$

$$\left. - \left\{ (1+B)^2 - B(1+B) + \frac{B^2}{3} \right\} \right]$$

Noticing that $\xi_n = 1 + \sqrt{\frac{2T_s}{1 + T_s}}$, $B = \sqrt{\frac{1 + T_s}{2T_s}}$, yields:

$$\xi_n = \frac{B + 1}{B}$$

Substituting this value in the above equation yields:

$$\int_1^{\xi_n} \frac{4}{\pi^2}\, [1 - B(\xi-1)]^2 d\xi = \frac{4}{\pi^2} \cdot \frac{1}{3B} \tag{4.74}$$

Substituting equations (4.74), (4.73) and (4.72) into equation (4.69) yields:

$$D_S = \frac{\rho_o\ a_o^2\ M_o^2}{2} \cdot H \cdot \left\{\frac{4}{\pi^2} \cdot \frac{1}{2} + \frac{4}{\pi^2} \cdot \frac{1}{3B}\right\}$$

$$= \frac{\rho_o\ a_o^2\ M_o^2 \cdot 4H}{2\pi^2}\left(\frac{1}{2} + \frac{1}{3B}\right)$$

$$= \frac{\rho_o\ a_o^2\ M_o^2 \cdot H}{3\pi^2}\left(3 + 2\sqrt{\frac{2T_s}{1 + T_s}}\right) \qquad (4.75)$$

where

H is a constant given by equation (4.71), and

$T_s = \tan\nu_o \cot\alpha_o$.

Equation (4.75) gives the total shock drag of the shock wave generated on a delta wing with α_1 as a half-apex angle. The variation of this shock drag with (M) and (α_1) at an altitude of 2000 m is shown in Figs.(4.14) and (4.15) respectively.

For the lateral direction of the delta wing an assumption has been made in calculating the shock drag that the two expected expansion fans are considered to collapse into one expansion fan emanating from the trailing edge.

4.5.2 : THE SHOCK DRAG FOR THE WING WITH CURVED LEADING EDGES

For the wing with curved leading edges, the solution of the shock wave is uniformly valid. The Whitham F-function of the curved wing in any plane is:

$$F(\xi) = \frac{\cos\theta_n\ K_1}{\pi \cdot N_2}\left\{\xi^{1/2} - \frac{4}{3}\xi^{3/2}\right\} \qquad (4.76)$$

where $K_1 = CL\ \tan\alpha_2 \cdot \beta_o$

$\cos\theta_n = 1.0$ for the plane normal to wing

$N_2 = 1-\beta_o^2\ \tan^2\alpha_2\ \sin^2\theta_n$.

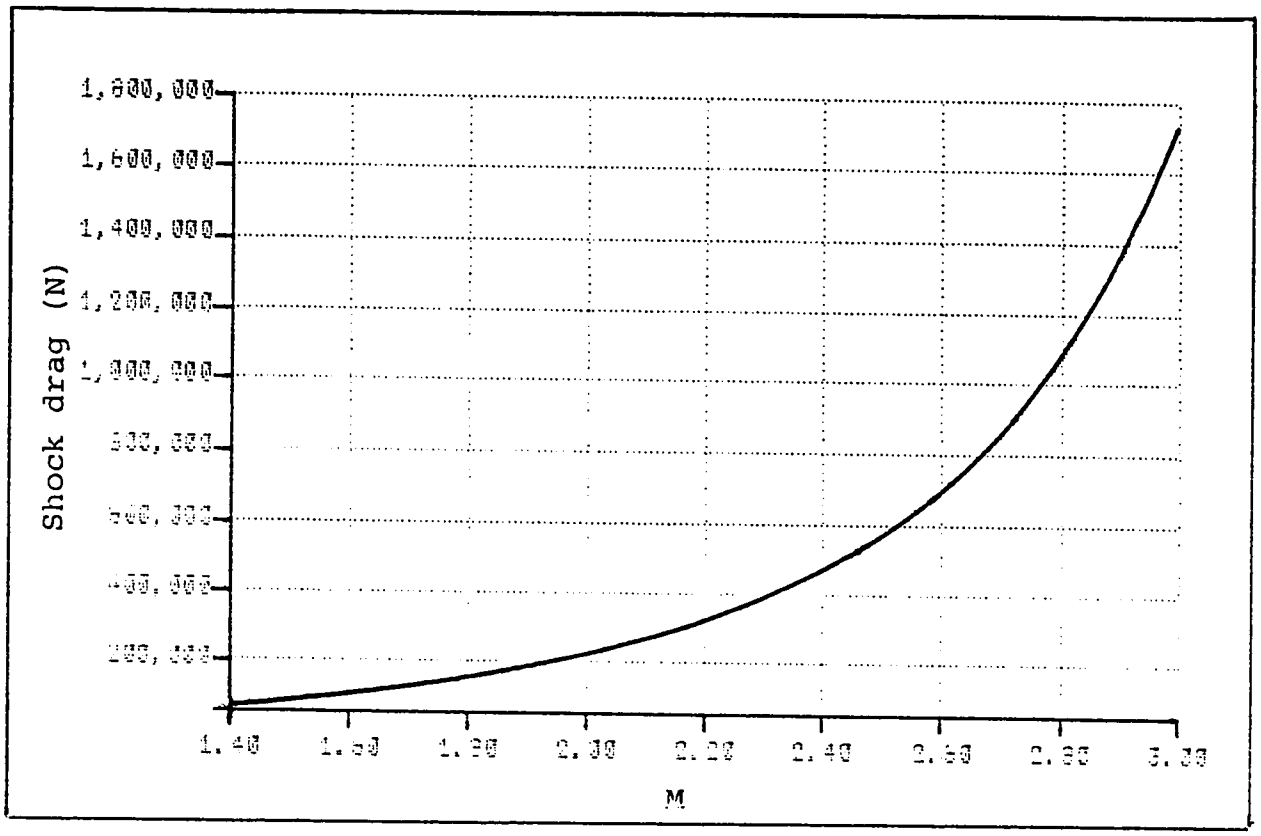

Fig. 4.14. Variation of the shock drag of a delta wing with the Mach number. ($\alpha_1 = 15^o$, altitude = 2000 m)

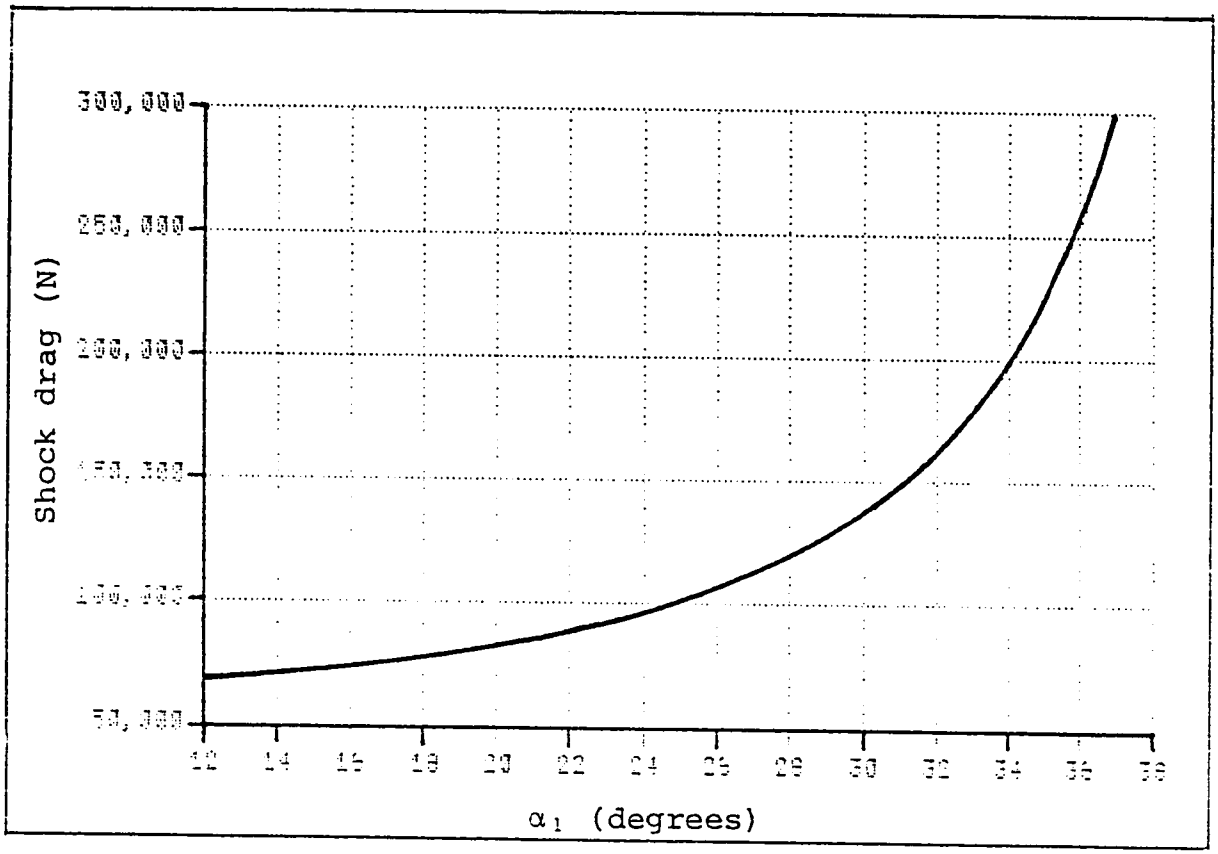

Fig. 4.15. Variation of the shock drag of a delta wing with the half-apex angle (α_1). ($M = \sqrt{2}$, altitude = 2000 m)

The total shock drag for the curved wing is also given by equation (4.67):

$$D_S = \frac{1}{2} \int_{o}^{\xi_n} \int_{-\pi/2}^{\pi/2} \rho_o a_o^2 M_o^2 \, F^2(\xi) d\xi \, d\gamma \qquad (4.67)$$

Substituting equation (4.76) into equation (4.67) yields:

$$D_S = \frac{\rho_o a_o^2 M_o^2 \cdot K_1^2}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_n}{(1-\beta_o^2 \tan^2\alpha_2 \sin^2\theta_n)^2} \, d\theta_n \cdot \int_{o}^{\xi_n} \left(\xi^{1/2} - \frac{4}{3} \xi^{3/2} \right)^2 d\xi \qquad (4.77)$$

With the help of equation (4.71):

$$\int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta_n}{(1-\beta_o^2 \tan^2\alpha_2 \sin^2\theta_n)^2} \, d\theta_n = \frac{\pi}{2\sqrt{1-\beta_o^2 \tan^2\alpha_2}}$$

$$= H_1 \qquad (4.78)$$

Consider

$$\int_{o}^{\xi_n} \left(\xi^{1/2} - \frac{4}{3} \xi^{3/2} \right)^2 d\xi = \frac{\xi_n^2}{2} + \frac{4}{9} \xi_n^4 - \frac{8}{9} \xi_n^3$$

but it is already found that $\xi_n = \frac{3}{4}$

Hence;

$$\int_{o}^{\xi_n} \left(\xi^{1/2} - \frac{4}{3} \xi^{3/2} \right)^2 d\xi = \frac{(.75)^2}{2} + \frac{4}{9} (.75)^4 - \frac{8}{9} (.75)^3$$

$$= \frac{3}{64} \qquad (4.79)$$

Substituting equations (4.79) and (4.78) into equation (4.77) yields:

$$D_s = \frac{\rho_o \, a_o^2 \, M_o^2 \, K_1^2}{2\pi^2} \cdot H_1 \cdot \frac{3}{64}$$

$$= \frac{3}{128\pi^2} \cdot \rho_o \, a_o^2 \, M_o^2 \, K_1^2 \cdot H_1 \qquad (4.80)$$

where H_1 is a constant given by equation (4.78).

As a result the shock drag for both configurations considered is directly proportional to the flow Mach number, lift coefficient, the half-apex angle of the wing, the density of fluid and the sound velocity.

The variation of the shock drag of the curved wing with the (M) and (α_2) at an altitude of (2000 m) is shown in Figs. (4.16) and (4.17) respectively.

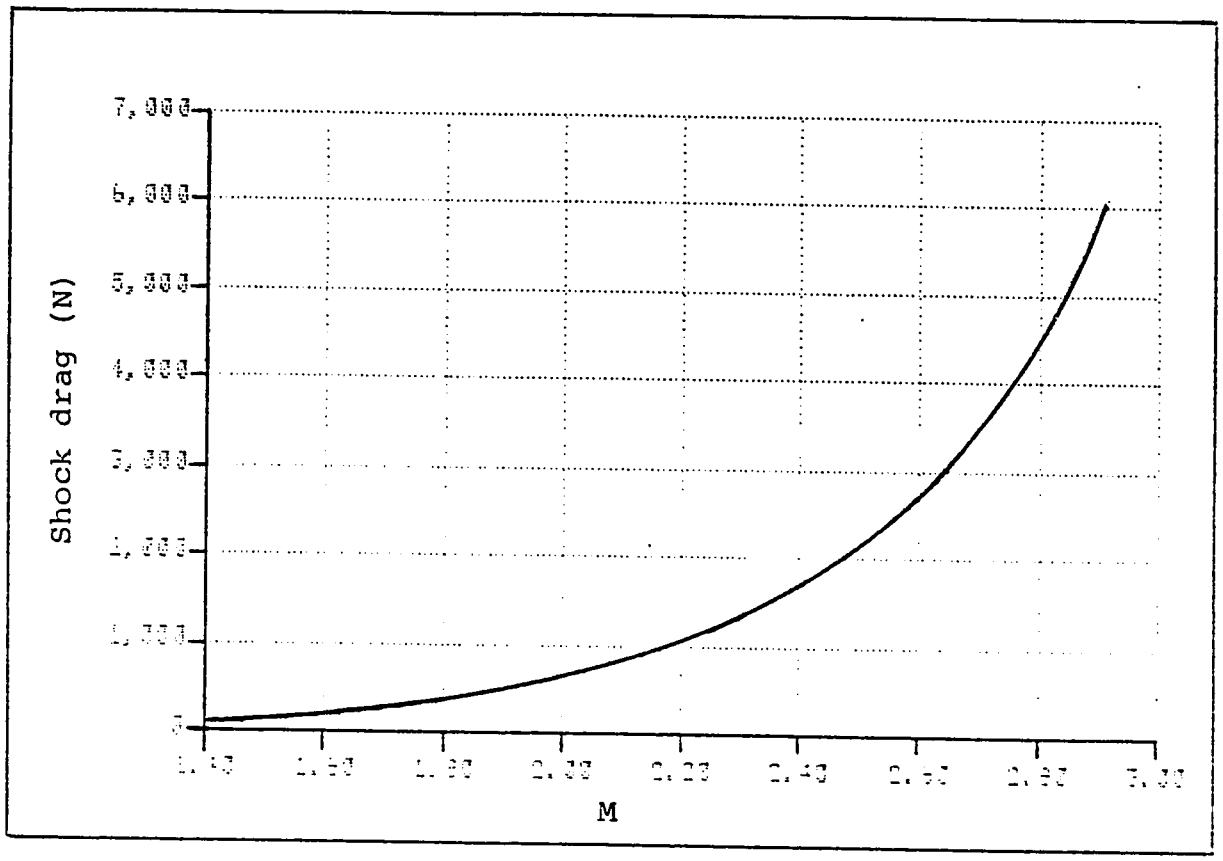

Fig. 4.16. Variation of the shock drag of a curved wing with the Mach number (M). (α_2 = 18°, altitude = 2000 m)

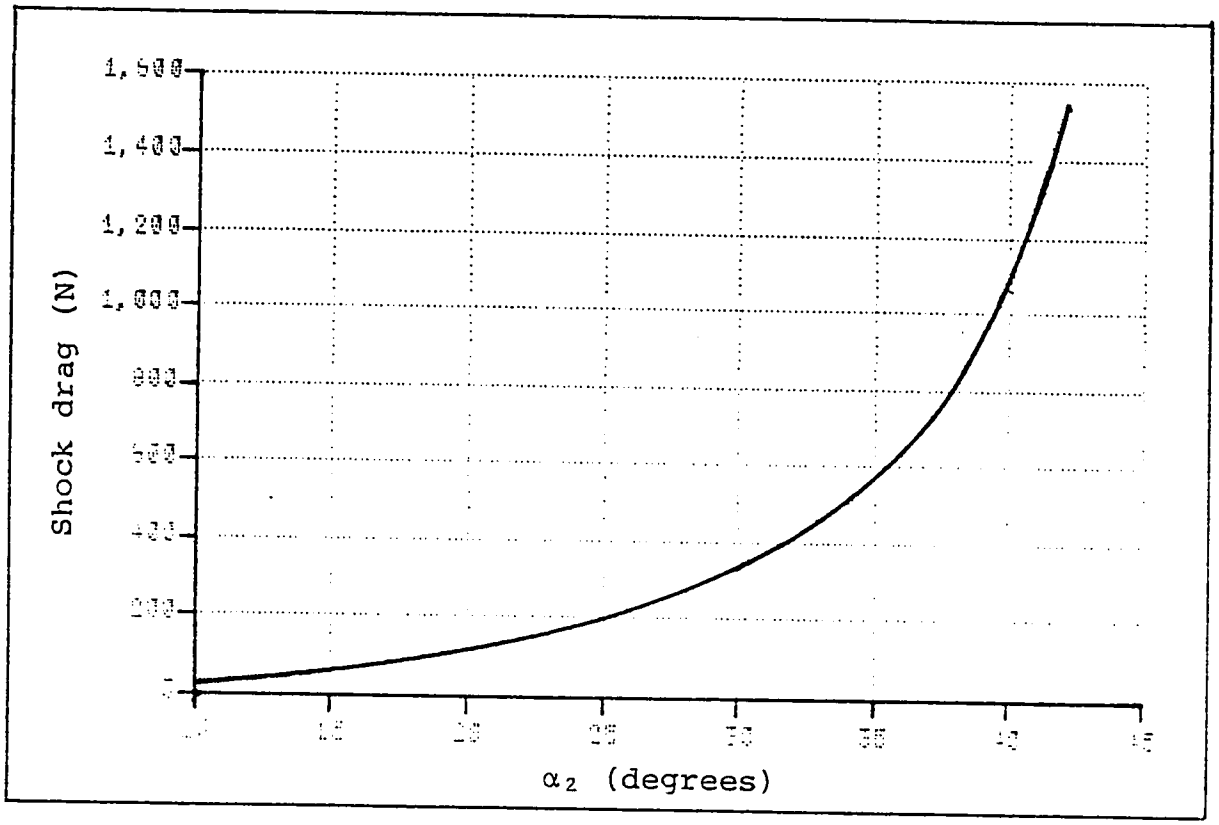

Fig. 4.17. Variation of the shock drag of a curved wing with the half-apex angle (α_2). ($M = \sqrt{2}$, altitude = 2000 m)

CHAPTER 5

THE PERTURBED COORDINATES FOR TWO-DIMENSIONAL STEADY SUPERSONIC FLOWS UP TO FIRST ORDER

In this chapter the position of the characteristics is corrected upto first order using the Analytical Method of Characteristics.

In a steady supersonic parallel flow in the x-direction with a Mach angle equal to (α_0); two sets of characteristics are formed; namely ξ = constant and η = constant (see Fig. 5.1). The two sets of characteristics can be dealt with by Oswatitsch Analytical Method of Characteristics while Whitham method can deal with one set of characteristics only.

The slope of the characteristics (Fig. 5.2) is given as:

$$\left.\begin{aligned} \left.\frac{dx}{dy}\right|_{\xi = \text{const.}} &= \cot(\nu + \alpha) \\ \left.\frac{dx}{dy}\right|_{\eta = \text{const.}} &= \cot(\nu - \alpha) \end{aligned}\right\} \qquad (5.1)$$

where α is the local Mach angle,

ν is the angle of the streamline (Mach line) relative to some constant direction (x-direction)

Equation (5.1) can be rewritten as:

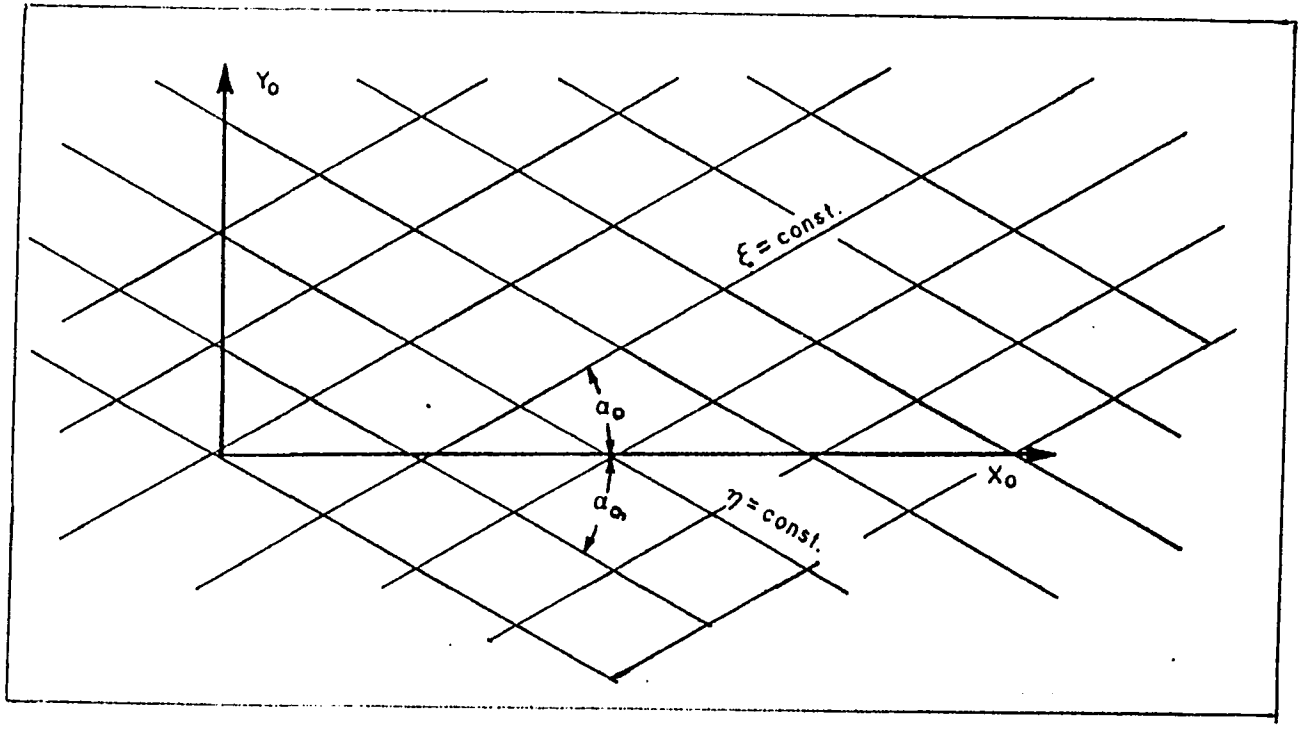

Fig. 5.1. System of Mach lines in undisturbed parallel flow.

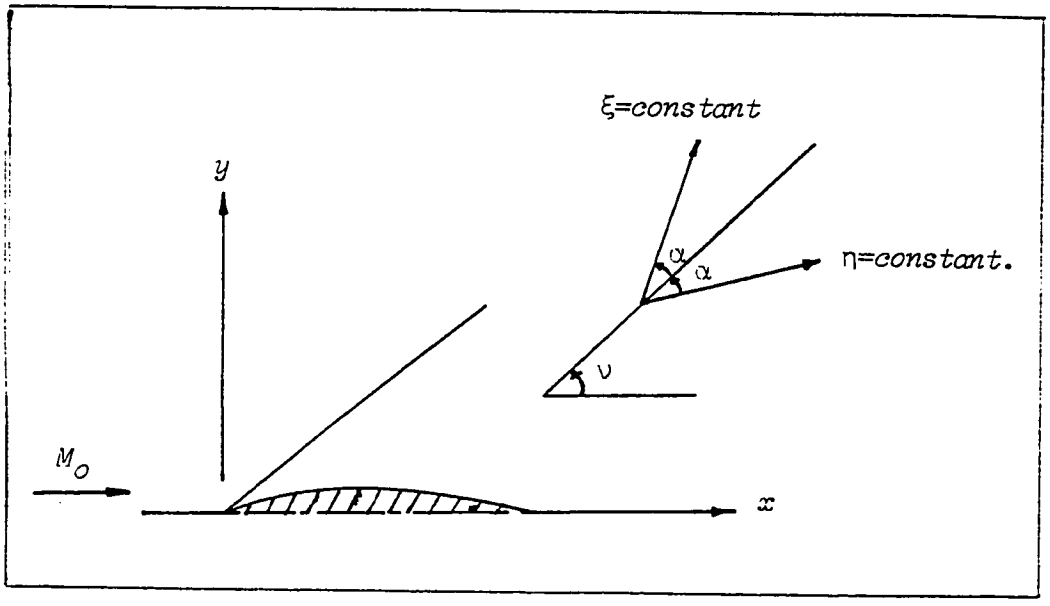

Fig. 5.2. Slope of the characteristics.

$$\left.\begin{aligned} dx|_{\xi} &= \cot(\nu + \alpha)dy|_{\xi} \\ dx|_{\eta} &= \cot(\nu - \alpha)dy|_{\eta} \end{aligned}\right\} \tag{5.2}$$

but $x = x(\xi,\eta)$ hence;

$$dx = \frac{\partial x}{\partial \xi}\, d\xi + \frac{\partial x}{\partial \eta}\, d\eta \tag{5.3}$$

from which:

$$\left.\begin{aligned} dx|_{\xi} &= \frac{\partial x}{\partial \eta}\, d\eta \\ dx|_{\eta} &= \frac{\partial x}{\partial \xi}\, d\xi \end{aligned}\right\} \tag{5.4}$$

Similarly, one can find that:

$$\left.\begin{aligned} dy|_{\xi} &= \frac{\partial y}{\partial \eta}\, d\eta \\ dy|_{\eta} &= \frac{\partial y}{\partial \xi}\, d\xi \end{aligned}\right\} \tag{5.5}$$

Substituting equation (5.4) and equation (5.5) into equation (5.2) yields:

$$\left.\begin{aligned} x_{\xi} - \cot(\nu - \alpha)y_{\xi} &= 0 \\ x_{\eta} - \cot(\nu + \alpha)y_{\eta} &= 0 \end{aligned}\right\} \tag{5.6}$$

where $x_{\xi} = \frac{\partial x}{\partial \xi}$, $y_{\xi} = \frac{\partial y}{\partial \xi}$, etc.

For the unperturbed flow which is parallel to x-direction:

$$\nu_o = 0 \ , \quad \alpha = \alpha_o \tag{5.7}$$

Substituting equation (5.7) into equation (5.6) yields:

$$\left.\begin{aligned} x_\xi + \cot\alpha_o\, y_\xi &= 0 \\ x_\eta - \cot\alpha_o\, y_\eta &= 0 \end{aligned}\right\} \qquad (5.8)$$

Expanding the physical coordinates x, y in a power series:

$$\left.\begin{aligned} x(\xi,\eta) &= x_o(\xi,\eta) + x_1(\xi,\eta) + x_2(\xi,\eta) + \ldots\ldots \\ y(\xi,\eta) &= y_o(\xi,\eta) + y_1(\xi,\eta) + y_2(\xi,\eta) + \ldots\ldots \end{aligned}\right\} \qquad (5.9)$$

where x_o and y_o are the coordinates of the characteristic space (Fig. 5.1). x_1 and y_1 are the first order corrections of the coordinates x_o and y_o. x_2 and y_2 are the second order corrections of the coordinates x_o and y_o.

As a first order approximation, equation (5.9) reads:

$$\left.\begin{aligned} x(\xi,\eta) &= x_o(\xi,\eta) \\ y(\xi,\eta) &= y_o(\xi,\eta) \end{aligned}\right\} \qquad (5.10)$$

Substituting equation (5.10) in equation (5.8) yields:

$$\left.\begin{aligned} x_{o\xi} + \cot\alpha_o\, y_{o\xi} &= 0 \\ x_{o\eta} - \cot\alpha_o\, y_{o\eta} &= 0 \end{aligned}\right\} \qquad (5.11)$$

Integrating equation (5.11), the following relation can be found:

$$\left.\begin{aligned} \xi &= x_o - y_o \cot\alpha_o \\ \eta &= x_o + y_o \cot\alpha_o \end{aligned}\right\} \qquad (5.12)$$

The inverse relations can be found from equation (5.12) as:

$$\left.\begin{aligned} x_o &= \frac{1}{2}(\xi + \eta) \\ y_o &= \frac{1}{2}\tan\alpha_o(-\xi + \eta) \end{aligned}\right\} \qquad (5.13)$$

From equation (5.13) one can also get that:

$$\left.\begin{aligned} x_{o_\xi} &= x_{o_\eta} = \frac{1}{2} \\ -y_{o_\xi} &= y_{o_\eta} = \frac{1}{2}\tan\alpha_o \end{aligned}\right\} \qquad (5.14)$$

The first law of thermodynamics for a steady-state, steady-flow process can be written as:

$$q_o^* + g_o^* + \frac{q_o^2}{2} + h_o = g_2^* + \frac{q_2^2}{2} + h_2 + w_2^* \qquad (5.15)$$

where

q^* is the heat transfer per unit mass
g^* is the potential energy
h is the specific enthalpy
q is the flow velocity
w^* is the work per unit mass
and the subscript (o) refers to the undisturbed flow.

For two-dimensional flow over a certain body, equation (5.15) can be written as:

$$h_o = \frac{q_2^2}{2} + h_2 \qquad (5.16)$$

where $q_o^* - w_2^* = g_1^* - g_2^* = q_1 = 0.$

For ideal gases the enthalpy is a function of temperature only. The enthalpy is given by:

$$h(T) = \int C_p \, dT \tag{5.17}$$

where T is the temperature

C_p is the specific heat at constant pressure.

From equation (5.17) one can get that:

$$\left.\begin{aligned} h_o(T) &= C_{p_o} T_o \\ h_2(T) &= C_{p_2} T_2 \end{aligned}\right\} \tag{5.18}$$

Substituting equation (5.18) in equation (5.16) yields:

$$q_2^2 = 2(C_{p_o}T_o - C_{p_2}T_o) \tag{5.19}$$

where q_2 will be a maximum if $h_2 = 0$, hence;

$$q_{max} = \sqrt{2 \, C_{p_o} \, T_o} \tag{5.20}$$

where q_{max} is the maximum velocity which can be obtained in the disturbed flow field.

The sound velocity (a) for a perfect gas is expressible as:

$$\begin{aligned} a^2 &= \frac{\gamma p}{\rho} \\ &= C_p(\gamma - 1)T \end{aligned} \tag{5.21}$$

where γ is the ratio of specific heats.

Substituting equation (5.21) in equation (5.19) yields:

$$\frac{q_2^2}{2} + \frac{a_2^2}{\gamma - 1} = C_{p_o} T_o \tag{5.22}$$

Substituting equation (5.20) into equation (5.22) yields:

$$\frac{q_2^2}{2} + \frac{a_2^2}{\gamma - 1} = \frac{q_{max}^2}{2} \tag{5.23}$$

which can be rewritten as:

$$a^2 = \frac{\gamma - 1}{2} (q_{max}^2 - q^2) \tag{5.24}$$

From the definition of the Mach angle (α) one has:

$$\begin{aligned} \sin\alpha &= \frac{1}{M} \\ &= \frac{a}{q} \end{aligned} \tag{5.25}$$

Now substituting equation (5.24) in equation (5.25) yields:

$$\sin\alpha = \frac{\sqrt{\gamma - 1}}{\sqrt{2}\ q} \cdot \sqrt{q_{max}^2 - q^2} \tag{5.26}$$

Differentiating equation (5.26) and making the suitable arrangements one can finally get:

$$d\alpha = -\tan\alpha \left(1 + \frac{\gamma - 1}{2} M^2 \right) \frac{dq}{q} \tag{5.27}$$

In the case of small perturbation theory up to first order one has:

$$\left.\begin{aligned} \alpha &= \alpha_o + \alpha_1 \\ q &= q_o + q_1 \\ \nu &= \nu_1 = \frac{V}{U_o} \\ U &= U_o + u \\ V &= v \end{aligned}\right\} \tag{5.28}$$

where U and V are the velocity components in the x and y-directions respectively

u and v are the velocity perturbation in the x and y-directions respectively.

From equation (5.28):

$$d\alpha = \alpha_1 \tag{5.29}$$

Substituting equation (5.29) in equation (5.27) yields:

$$\alpha_1 = -\tan\alpha_o \left(1 + \frac{\gamma - 1}{2} M_o^2\right) \frac{dq}{q} \tag{5.30}$$

The flow velocity (q) is expressible as:

$$q^2 = U^2 + V^2$$

$$= [U_o + (U - U_o)]^2 + V^2$$

from which:

$$q = U_o \left[1 + 2\left(\frac{U}{U_o} - 1\right) + \left(\frac{U}{U_o} - 1\right)^2 + \left(\frac{V}{U_o}\right)^2\right]^{\frac{1}{2}} \tag{5.31}$$

Upon expanding the square root into a Tayler series and neglecting higher order terms one can get:

$$q \cong U_o \left[1 + \left(\frac{U}{U_o} - 1\right) + \frac{1}{2}\left(\frac{U}{U_o} - 1\right)^2 + \frac{1}{2}\left(\frac{V}{U_o}\right)^2\right]$$

From which it can be found that:

$$\frac{dq}{q} \cong \left(\frac{U}{U_o} - 1\right) + \frac{1}{2}\left(\frac{V}{U_o}\right) \tag{5.32}$$

From equation (5.30), equation (5.28) and equation (5.32) one can write (after neglecting higher order terms):

$$\nu_1 \pm \alpha_1 = \frac{v}{U_o} \pm \tan\alpha_o \left(1 + \frac{\gamma - 1}{2} M_o^2 \right) \left(\frac{U}{U_o} - 1 \right) \tag{5.33}$$

Consider:

$$\cot(\nu \pm \alpha) = \cot[\nu_1 \pm (\alpha_o + \alpha_1)]$$
$$= \cot[(\nu_1 \pm \alpha_1) \pm \alpha_o]$$

Expanding the right hand side of this equation yields:

$$\cot(\nu \pm \alpha) = \pm\cot\alpha_o - \frac{1}{\sin^2\alpha_o} \cdot (\nu_1 \pm \alpha_1) + \ldots\ldots \tag{5.34}$$

Substituting equation (5.33) in equation (5.34) yields:

$$\cot(\nu \pm \alpha) = \pm\cot\alpha_o - M^2_o \cdot \delta(v_1 \pm A \cot\alpha_o \cdot u_1) \tag{5.35}$$

where

$$\delta u_1 = \frac{U}{U_o} - 1$$

$$\delta v_1 = \frac{v}{U_o}$$

and $$A(M_o) = \left(1 + \frac{\gamma - 1}{2} M_o^2 \right) \tan^2\alpha_o$$

Substituting equation (5.35) and equation (5.9) in equation (5.6) yields:

$$x_{o\eta} + x_{1\eta} + \ldots = (y_{o\eta} + y_{1\eta} + \ldots)[\cot\alpha_o - M_o^2\,\delta\,(v_1 - A\cot\alpha_o u_1) + \ldots]$$
$$x_{o\xi} + x_{1\xi} + \ldots = (y_{o\xi} + y_{1\xi} + \ldots)[-\cot\alpha_o - M_o^2\,\delta\,(v_1 + A\cot\alpha_o u_1) + \ldots]$$

(5.36)

Substituting equation (5.14) in equation (5.36) and neglecting higher order terms, equation (5.36) is reduced to:

$$\left.\begin{aligned} x_{1\eta} - \cot\alpha_o\, y_{1\eta} &= -y_{o\eta}\, M_o^2 \cdot \delta(v_1 - A\cot\alpha_o u_1) \\ x_{1\xi} + \cot\alpha_o\, y_{1\xi} &= -y_{o\xi}\, M_o^2 \cdot \delta(v_1 + A\cot\alpha_o u_1) \end{aligned}\right\} \qquad (5.37)$$

From equation (5.14):

$$-y_{o\xi} = +y_{o\eta} = \frac{1}{2}\tan\alpha_o$$

Substituting these values in equation (5.37), and integrating the resultant equation yields: (see Fig. 5.3)

$$\left.\begin{aligned} x_1 - \cot\alpha_o y_1 &= \frac{-1}{2}\tan\alpha_o \cdot M_o^2 \cdot \delta \int_{\eta_u}^{\eta} \left\{v_1(\xi,\bar{\eta}) - A\cot\alpha_o \cdot u_1(\xi,\bar{\eta})\right\} d\eta + K_1(\xi) \\ x_1 + \cot\alpha_o \cdot y_1 &= \frac{1}{2}\tan\alpha_o \cdot M_o^2 \cdot \delta \int_{\xi_u}^{\xi} \left\{v_1(\bar{\xi},\eta) + A\cot\alpha_o \cdot u_1(\bar{\xi},\eta)\right\} d\bar{\xi} + K_2(\eta) \end{aligned}\right\} \qquad (5.38)$$

The first order velocity perturbations u_1 and v_1 can be found from the conservation laws:

$$\left.\begin{aligned} -\cot\alpha_o \frac{\partial u_1}{\partial x_o} + \frac{\partial v_1}{\partial y_o} &= 0 \qquad \text{(Continuity)} \\ \frac{\partial u_1}{\partial y_o} - \frac{\partial v_1}{\partial x_o} &= 0 \qquad \text{(Irrotationality)} \end{aligned}\right\} \qquad (5.39)$$

As can be seen from equation (5.38), the first order velocity perturbations are integrated along the characteristics to give the first order perturbations of the coordinates to correct the position of the characteristics.

In the undisturbed upstream flow the perturbations must vanish in order to match the boundary conditions, from which:

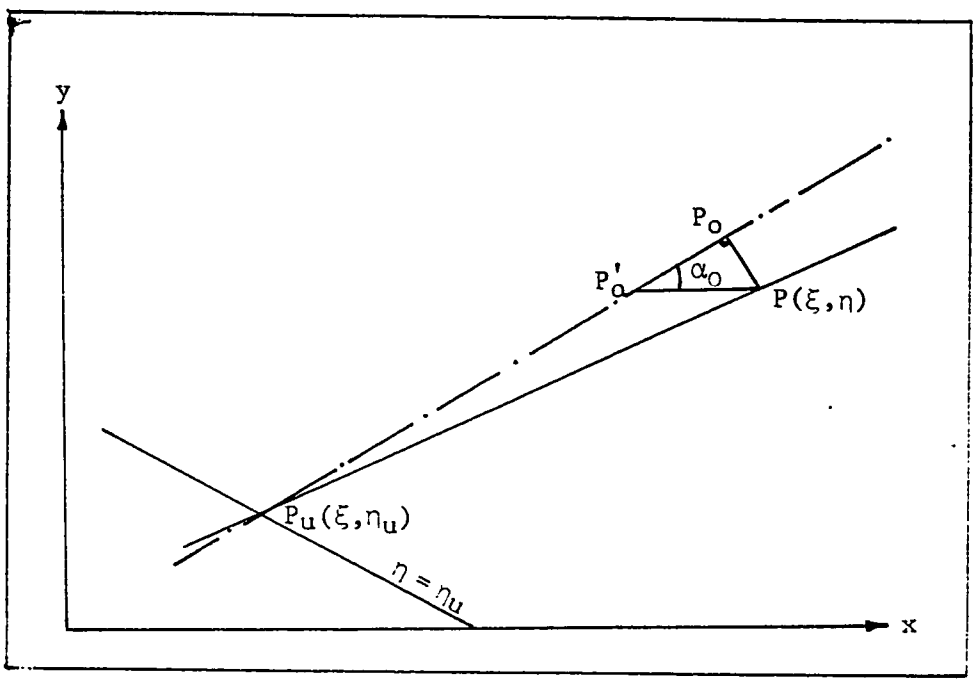

Fig. 5.3. Displacement of point P from its position P_o or P'_o on the Mach line (- - -).

$$K_2(\eta) = 0 \tag{5.40}$$

In accordance with Ackeret's theory it follows that:

$$u_1(\xi)\cot\alpha_o = -v_1(\xi) \tag{5.41}$$

Ackeret's theory assures that the velocity perturbations depends only on the local flow angle and not on the shape of the profile before the perturbation.

The solution of equation (5.38) can be given as:

$$\left.\begin{aligned} x_1 - y_1 \cot\alpha_o &= -S_1 v_1(\xi)(\eta - \eta_u) + K_1(\xi) \\ x_1 + y_1 \cot\alpha_o &= -D \int_0^{\xi} v_1(\bar{\xi})\, d\bar{\xi} \end{aligned}\right\} \tag{5.42}$$

where

$$S_1 = \frac{\gamma + 1}{4} \cdot \frac{M_o^4}{(M_o^2 - 1)^{3/2}}$$

and $\quad D = S_1 - M_o^2 \tan\alpha_o .$

K_1 may be chosen from the condition that either x_1 or y_1 is zero on the x_o-axis. In general it is small and may be neglected, so that:

$$\left.\begin{aligned} x - y \cot\alpha_o &= \xi - S_1 v_1(\xi)(\eta - \xi) \\ x + y \cot\alpha_o &= \eta \end{aligned}\right\} \tag{5.43}$$

Compared to the acoustic approximation there is the

accumulative term $S_1 v_1(\xi)(\eta - \xi)$ which becomes large for large values of η, i.e., at large distance from the profile. For the Mach line ξ = constant one finally obtains the first order slope as:

$$\left.\frac{dy}{dx}\right|_{\xi} = \tan\alpha_o[1 + 2\ S_1 v_1(\xi)] = \tan(\nu + \alpha) \qquad (5.44)$$

The Mach lines with ξ = constant associated with a profile are shown in Fig. 5.4. The disturbances caused by the profile propagate on Mach lines which still are straight lines but due to the slope as given by equation (5.44) are divergent. Weak front and rear shocks may be found as a bisectrix of the Mach line inclination ahead and behind the shock front from the fact that the perturbation potential is continuous across the shock front.

The Ackeret's theory yields Mach lines which are parallel to those of undisturbed flow. Furthermore in Ackeret's theory the perturbations are restricted to the flow field between the dashed lines (Fig. 5.4). There is also no damping of shock waves. The Ackeret's theory is useful near the body but fails far from the body where the AMC gives the correct results.

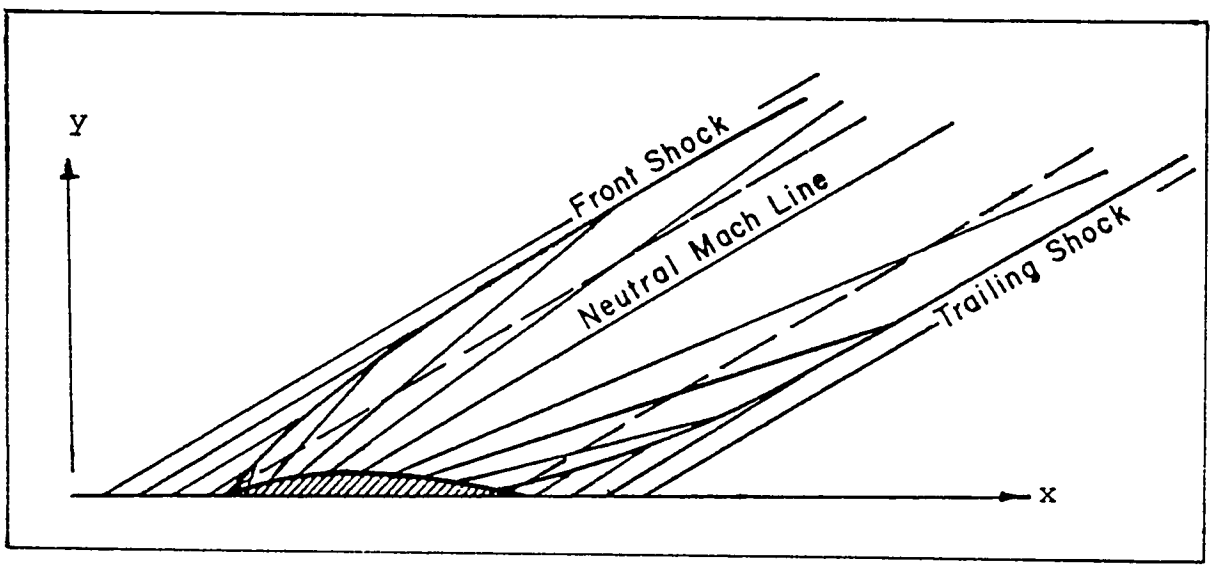

Fig. 5.4. Two-dimensional steady supersonic flow around slender profile.

CHAPTER 6

SUMMARY AND CONCLUDING REMARKS

In this thesis an acceleration potential is presented to account for the effect of different force distributions on the generation and decay of the shock waves formed in a supersonic far flow field.

The acceleration potential solution of the shock strength enables us to calculate the shock strength not only in the plane directly under the wing but also in other inclined planes near the shock.

Introducing the acceleration potential solution as an initial solution into the Analytical Method of Characteristics enables us to calculate the decay of the shock in the plane normal to the body in the far flow field near the shock wave.

The equivalent body of revolution which can be represented by a source distribution on its axis of symmetry can be used throughout the supersonic far flow field.

The shock wave strength of a delta wing is constant in the conical region since no expansion wave runs into the shock.

The size of the conical region varies with the Mach number and the apex angle of the wing for constant density and

specific heats. The shock wave does not terminate at a finite distance for the delta wing whether it has straight or curved leading edges.

The position of the characteristics is corrected up to first order by integrating the first order velocity perturbations along the characteristics.

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