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A finite difference approximation for a class of singular boundary value problems

Abu-Zaid, Issam Taher Mohammad, Ph.D.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1992

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A FINITE DIFFERENCE APPROXIMATION FOR A CLASS OF SINGULAR BOUNDARY VALUE PROBLEMS

BY

Issam Taher Mohammad Abu-Zaid

A Dissertation Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICAL SCIENCES

August, 1992

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This dissertation, written by **Issam Taher Mohammad Abu-Zaid** under the direction of his Dissertation Advisor and approved by his Dissertation Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICAL SCIENCES.**

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540C

To My Parents

540C

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TABLE OF CONTENTS

540C

540C

LIST OF TABLES

TABLE *PAGE*

LIST OF FIGURES

DISSERTATION ABSTRACT

NAME OF STUDENT : ISSAM TAHER MOHAMMAD ABU-ZAID

TITLE OF STUDY : A Finite Difference Approximation for a Class of Singular Boundary Value Problems

MAJOR FIELD : Mathematical Sciences

DATE OF DEGREE : August, 1992

540C

Numerical solutions of singular boundary value problems have drawn considerable interest in the last two decades. This work is a contribution in that direction.

The objective of this dissertation is to extend certain results in the literature to a wider class of singular self-adjoint boundary value problems with minimal constraints on the data of the problem.

A finite difference method is used to approximate the solutions, eigenvalues and eigenvectors of the associated differential operators. Order h^2 convergence is achieved in these approximations.

Numerical examples are given to demonstrate the $O(h^2)$ convergence obtained theoretically.

DOCTOR OF PHILOSOPHY DEGREE

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Dhahran, Saudi Arabia

August, 1992

خلاصة رسالة

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540C

لقد حازت المسائل الفردية ذات القيم الحدّية على اهتمام الباحثين خلال العقدين الماضيين حيث عالجها كثيرٌ منهم بالتحليل العددي . ان هذا الجهد هو مساهمة في هذا الأتجاه .

هدف هذا البحث هو توسيع نطاق بعض النتائج التي تحققت حتى الآن في معالجة المسائل التفاضلية الفردية ذات القيم الحديّة وذاتيّة القرين وذلك مع المحافظة على حد ادنى من التقييد على معطيات المسألة .

في هذه الأطروحة استعملنا طريقة الفروقات المحدودة لتقريب الحلول لهذه المسائل وكذلك لتقريب قيم الأرقام والمتجهات الخفيّة المتعلقة بها.

لقد بيّنا نظرياً وكذلك بالتحليل العددي على مجموعة من الأمثلة ان الحلول التقريبية في هذا البحث تؤول الى الحلول الحقيقية بتناسبٍ طرديّ مع مربع المسافة الدقيقة المستعملة في تقسيم فترة المسألة .

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آب ١٩٩٢

INTRODUCTION

Consider a formally self-adjoint differential expression

$$l(y) = (-1)^n \left(p_0 y^{(n)}\right)^{(n)} + (-1)^{n-1}\left(p_1 y^{(n-1)}\right)^{(n-1)} + \cdots + p_n y$$

in the interval $a < x < b$, where each coefficient $p_{n-k}(x)$ is real and has a derivative up to the kth order inclusive. The expression $l(y)$ is said to be regular if the interval (a,b) is finite and the functions $1/p_0(x), p_1(x), \cdots, p_n(x)$ are absolutely integrable in the whole interval (a,b); otherwise $l(y)$ is called singular.

In this dissertation, we will consider the second order formally self-adjoint singular differential operator $l(y)$ defined by

$$l(y) = (p(x)y')' - p(x)q(x)y, \quad 0 < x < 1,$$

where $p^{-1}(x) \in L^1_{loc}(0,1]$ but $p^{-1}(x) \notin L^1[0,\delta]$ for any $\delta > 0$. We will use the available results on the existence and uniqueness of a solution for the self-adjoint singular boundary value problem

$$\begin{aligned} l(y) &= p(x)f(x), \quad 0 < x < 1, \\ \lim_{x \to 0^+} p(x)y' &= 0 \\ y(1) &= 0 \end{aligned} \tag{0.1}$$

and then use finite difference methods to provide $O(h^2)$ numerical approximation to its solution and approximate its eigenvalues and eigenfunctions in appropriate function spaces.

540C

Problems of this type arise when Poisson operator $-\Delta u = f$ is considered on a domain in $\Re^n$ with spherical symmetry. If the data depends only on the radial coordinate, then using polar coordinates reduces the problem to a one dimensional problem of the type above. Moreover, second order differential operators which are self-adjoint and non-positive are also used in statistics in connection with strings in the study of the relation between the past and the future of a real, one-dimensional Gausian process [8].

There is a significant amount of literature concerning the numerical solution of singular two point boundary value problems. Methods used by authors in the last two decades include finite differences (e.g. Chawla [4], Jamet [12] and Nassif [17]), collocation (e.g. Doedel and Reddien [7], Russell and Shampine [20] and Ritz-Galerkin methods (e.g. Ciarlet, Natterer and Varga [6], Erricsson and Thome´e [10], Jespersen [13]). Chapter 1 of this dissertation is devoted to the literature review of numerical solutions of second order singular boundary value problems.

Chapter 2 includes three sections. Section 2.1 provides a review of the existing theory on self-adjoint extensions of differential operators as illustrated by Naimark [16]. Section 2.2 includes a review of the general theory on the convergence of eigenvalues and eigenvectors that is assembled and discussed by Chatelin [3]. Section 2.3 is a brief discussion on irreducible matrices, partial ordering and M-matrices. This material is based on Ortega and Rheinboldt [18]. This chapter contains the background that is needed for our work in Chapters 3 and 4.

540C

Most published work in this subject treat the special case in which $p(x) = x^\alpha, \alpha > 0$. An exception is Jamet [12] who considered a more general function $p(x)$ and obtained $O(h^{1-\alpha})$ numerical approximation using standard finite difference methods. Chawla [4] used another difference scheme with $p(x) = x^\alpha, \alpha \geq 1$ and obtained $O(h^2)$ convergence. In Chapter 3, we will apply Chawla's difference scheme to the above problem with a wider class of functions $p(x)$ and with less restrictions on the data of the problem. We will prove that the difference scheme used provides $O(h^2)$ convergence.

In this dissertation we let $L^2(0,1)$ denote the Hilbert space of all complex valued functions which are square integrable on (0,1). If $y \in L^2(0,1)$ then $\|y\|_2 = \left(\int_0^1 |y|^2 dt\right)^{1/2}$. We also let $C[0,1]$ denote the normed space of continuous functions on [0,1] with the infinity norm $\|\cdot\|_\infty$. If $y \in C[0,1]$, then $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$.

Under certain conditions set on the coefficient functions $p(x)$ and $q(x)$, the differential expression $l(y)$ induces differential operators in the spaces

$L^2(0,1)$ and $C[0,1]$ which have discrete eigenvalues. In Chapter 4 we present an analysis of such operators, showing compactness of their inverses and therefore discreteness of their spectrum. The finite difference scheme introduced in Chapter 3 will be used to approximate the eigenvalues and eigenfunctions for these operators.

In Chapter 5, we illustrate the theoretical results obtained in Chapter 3 through numerical examples where $p(x)$ is one of the following functions

$$\sin\left(\frac{\pi}{2}x\right), \quad \sqrt{x}\left(\sqrt{x}+1\right)\mathrm{Ln}\left(\sqrt{x}+1\right), \quad \left(2x-x^2\right)^{3/2}.$$

The eigenvalue problem treated in Chapter 4 will be illustrated numerically in Chapter 5 with $p(x)=\sin\left(\frac{\pi}{2}x\right)$.

540C

CHAPTER 1

LITERATURE REVIEW

540C

In this chapter we give an overview of the literature on the numerical analysis of second order singular boundary value problems. We will consider numerical treatments of problems which are similar to the problem considered in this dissertation. In each of the cases discussed below, we will state the boundary value problem, the numerical method used in approximating its solution and the error estimates established for the obtained numerical solution.

In his study of numerical methods for generalized axially symmetric potentials in a rectangle, Parter [19] was led to the differential operator

$$Ly = y'' + \frac{\alpha}{x} y' - qy, \quad |\alpha| < 1, \quad q > 0, \quad 0 < x < 1 \tag{1.1}$$

He used the following finite difference operator to approximate L

$$\left(L_h^{(1)} Y\right)_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{\alpha}{2ih^2}\left(y_{i+1} - y_{i-1}\right) - q_i y_i.$$

Upon writing (1.1) as

$$Ly = \frac{1}{x^\alpha}\left(x^\alpha y'\right)' - qy$$

Parter also used the following finite difference operator to approximate L

$$\left(L_h^{(2)}Y\right)_i = \frac{\left((i+\frac{1}{2})h\right)^{\alpha}(y_{i+1}-y_i)-\left((i-\frac{1}{2})h\right)^{\alpha}(y_i-y_{i-1})}{h^2(ih)^{\alpha}} - q_i y_i.$$

Contrary to our definition of singular problems, some authors, e.g. Parter [19] and Jamet [12], consider problems like (1.1) with $|\alpha|<1$ as singular. For completeness of presentation, we include reference to their work in this chapter.

Jamet [12] considered convergence of finite difference approximations to two special singular boundary-value problems. Depending on the nature of the singularity, he considered a two-point or a one-point boundary-value problem. He discussed results regarding existence, uniqueness and convergence of solutions. He also provided results on error analysis for certain special cases. In particular, Jamet considered the following ordinary differential equation

(1.2) $$Ly = y'' + f(x)y' - g(x)y = h(x), \quad x \in (0,1).$$

Or written differently,

(1.3) $$Ly = \frac{1}{p(x)}(p(x)y')' - g(x)y = h(x),$$

540 C

where

(1.4) $$p(x)=\exp\left(-\int_x^1 f(t)\,dt\right).$$

He assumed that

(1.5) $$\begin{aligned}&f(x)\in C(0,1]\\&f(x)\to\infty \text{ as } x\to 0\\&g(x),\quad h(x)\in C[0,1]\\&g(x)\ge 0.\end{aligned}$$

In the case in which $f(x)$ satisfies the condition

(1.6) $$\begin{aligned}&f(x)<\frac{\alpha}{x}, \text{ for } x \text{ small enough, and}\\&0<\alpha<1,\end{aligned}$$

he considered the following two-point boundary value problem:

(1.7) $$\begin{aligned}&Ly(x)=h(x),\quad 0<x<1\\&y(0)=a\\&y(1)=b\\&y\in C^2(0,1)\cap C[0,1].\end{aligned}$$

For $\alpha=1$ and $f(x)$ satisfying the condition

(1.8) $$f(x)>\frac{1}{x}-c \text{ for } x \text{ small enough and } c\ge 0$$

he considered the problem

$$(1.9)\qquad \begin{aligned} &Ly(x)=h(x),\quad 0<x<1\\ &y(1)=b\\ &y\in C^2(0,1)\cap C(0,1]\cap B(0,1), \end{aligned}$$

where $B(0,1)$ is the set of bounded functions on $(0,1)$.

With a uniform partition $0<x_1<x_2<\cdots<x_{N-1}<1$, $h=1/N$, Jamet considered the following finite difference approximation of the operator L.

$$(1.10)\qquad \left(L_h^{(1)}Y\right)_i=\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+f_i\frac{y_{i+1}-y_{i-1}}{2h}-g_iy_i$$

$$(1.11)\qquad \left(L_h^{(2)}Y\right)_i=\frac{1}{p_i}\frac{1}{h}\left(P_{i+\frac{1}{2}}\frac{y_{i+1}-y_i}{h}-P_{i-\frac{1}{2}}\frac{y_i-y_{i-1}}{h}\right)-g_iy_i$$

where $y_i=y(ih)$ and $Y=(y_0,\cdots,y_N)$.

Jamet then considered the difference scheme

$$(1.12)\qquad \begin{aligned} &(L_hY)_i=h_i,\quad 1\le i\le N-1\\ &Y_0=a\\ &Y_N=b \end{aligned}$$

where the difference operator L_h is either (1.10) or (1.11) if condition (1.6) is satisfied or the operator (1.11) if condition (1.6) is not satisfied. He showed that problem (1.7) satisfying condition (1.6) has a unique solution $y(x)$ and that problem (1.12) has a unique solution Y which converges uniformly to $y(x)$ as $h\to 0$.

540C

He further showed that problem (1.9) satisfying condition (1.8) has a unique solution $y(x)$ and that problem (1.12) has a unique solution Y which converges uniformly to $y(x)$ as $h \to 0$ on any compact subinterval $[\beta, 1]$, $0 < \beta < 1$.

On the other hand, Jamet discussed uniform error estimates for the two-point boundary value problem

$$
\begin{aligned}
& Ly = y'' + \frac{\alpha}{x} y' - qy = 0, \quad q > 0, \quad 0 < \alpha < 1 \\
& y(0) = 1 \\
& y(1) = 0
\end{aligned}
\tag{1.13}
$$

using the finite difference scheme

$$
\begin{aligned}
(L_h Y)_i &= 0, \quad 1 \le i \le N-1 \\
y_0 &= 1 \\
y_N &= 0
\end{aligned}
\tag{1.14}
$$

where the operator L_h is given by

$$
(L_h Y)_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \frac{\alpha}{ih} \frac{y_{i+1} - y_{i-1}}{2h} - qy_i. \tag{1.15}
$$

He showed that if $y(x)$ is a solution for problem (1.13) and $Y = (y_0, \cdots, y_N)$ is a solution for problem (1.14), then

$$
\|Y - y(x)\|_{L_\infty} < ch^{1-\alpha} \tag{1.16}
$$

where c is a constant. Finally, Jamet considered the boundary value

540C

problem

$$
\text{(1.17)}\qquad
\begin{aligned}
&Ly = y'' + f(x)y' - qy = h(x), \quad 0 < x < 1\\
&q \text{ is a nonnegative constant}\\
&y'(0) = 0\\
&y(1) = b\\
&f(x) \text{ is positive, decreasing and belongs to } C(0,1)\\
&h(x) \in C[0,1]
\end{aligned}
$$

together with the finite difference scheme

$$
\text{(1.18)}\qquad
\begin{aligned}
(L_h Y)_i &= h_i, \quad 1 \le i \le N-1\\
y_0 - y_1 &= 0\\
y_N &= b
\end{aligned}
$$

where L_h is the difference operator (1.10). Jamet showed that (1.18) has, for each h, a unique solution which converges uniformly in the interval (0,1) to a function $y(x)$ which satisfies (1.17). In particular if $f(x)$ satisfies (1.8), then the limit function $y(x)$ is the unique solution for problem (1.9).

Ciarlet, Natterer and Varga [6] considered the following two-point boundary value problem

$$\text{(1.19)}\qquad Ly = \left(p(x)y'\right)' = f(x, y(x)), \quad 0 < x < 1$$

$$\text{(1.20)}\qquad y(0) = y(1) = 0$$

where they assumed that the function $p(x)$ satisfies

$$(1.21)\qquad \begin{array}{ll} \text{(i)} & p(x) > 0 \text{ in } (0,1) \\ \text{(ii)} & p \in C^1(0,1), \text{ and} \\ \text{(iii)} & \dfrac{1}{p} \in L^1[0,1]. \end{array}$$

Ciarlet, et al, [6] used the Rayleigh - Ritz - Galerkin method with problem (1.19) - (1.20) to approximate its solution in a weighted Sobolev space S. The space S is defined as the linear space of all real valued functions $y \in C^0[0,1]$ satisfying the boundary condition (1.20), such that y is absolutely continuous on $[0,1]$, and such that

$$(1.22)\qquad \sqrt{p(x)}\,y'(x) \in L^2[0,1]$$

It is assumed that the real valued function $f(x,y)$ given in (1.19) is continuous in $[0,1] \times \Re$, and continuously differentiable with respect to y for all $0 \le x \le 1$, and all real y. It is also assumed that there exists a constant γ such that

$$(1.23)\qquad f_y(x,y) = \frac{\partial f(x,y)}{\partial y} \ge \gamma > -\Lambda$$

for all $0 \le x \le 1$ and all real y, where Λ is a positive constant that is determined by the fraction $p(x)$ and the space S. The approximating finite dimensional subspace that Ciarlet, et al, used in the application of Rayleigh - Ritz - Galerkin method is defined as follows:

540C

Let $\Pi : 0 = x_0 < x_1 < x_2 < \cdots < x_{N+1} = 1$ be any partition Π of $[0,1]$. Define the space S^{Π} as the subspace of S whose functions y satisfy

(1.24) $$(p(x)y'(x))' = 0 \quad x_i < x < x_{i+1} \text{ for all } 0 \le i \le N.$$

The basis for S^{Π} is defined in terms of the function $h_i(x) = r(x) - r(x_i)$ where

$$r(x) = \int_0^x \frac{1}{p(\tau)} d\tau.$$

The basis functions $w_i(x)$, $1 \le i \le N$ are defined as follows:

(1.25) $$w_i(x) = \begin{cases} 0, & 0 \le x \le x_{i-1}, \\ h_{i-1}(x)/h_{i-1}(x_i) & x_{i-1} \le x \le x_i, \\ 1 - [h_i(x)/h_i(x_{i+1})] & x_i \le x \le x_{i+1}, \\ 0 & x_{i+1} \le x \le 1. \end{cases}$$

Ciarlet, et al, proved that if ϕ is the solution of (1.19) - (1.20) and $\tilde{y}$ is the Galerkin solution over the subspace S^{Π}, then there exists a constant c, independent of the partition Π, such that

(1.26) $$\|\tilde{y} - \phi\|_{L^{\infty}[0,1]} \le c\ell(\Pi)$$

where

(1.27) $$\ell(\Pi) = \max_{0 \le i \le N} \left\{ (x_{i+1} - x_i) \int_{x_i}^{x_{i+1}} \frac{1}{p(\tau)} d\tau \right\}.$$

540C

In the special case of a uniform partition Π^h with mesh size $h = 1/(N+1)$ and with $p(x) = x^\sigma$, $0 \le \sigma < 1$, then,

(1.28) $$\|\tilde{y} - \phi\|_{L^\infty[0,1]} \le c\ell\left(\Pi^h\right) = c_1 h^{2-\sigma}.$$

This result in (1.28) is an improvement of Jamet's result who used finite differences with a linear problem and obtained convergence of order $h^{1-\sigma}$. In fact, Jamet's results could be obtained via the above Galerkin method if the approximating subspace S is chosen to be the space of all continuous piecewise linear functions with a uniform partition Π^h of mesh size h. In this case, the analog of (1.28) when $p(x) = x^\sigma$, $0 \le \sigma < 1$ will be

(1.29) $$\|\tilde{y} - \phi\|_{L^\infty[0,1]} \le c_2 h^{1-\sigma}.$$

It is worth noting here that the computation of Jamet's solution and Ciarlet's Galerkin appromation $\tilde{y}$ in the linear case are comparable in the sense that they both require the solution of a triadiagonal matrix system.

Jespersen [13] also used Ritz - Galerkin methods with linear and non-linear singular boundary-value problems. He considered the linear problem

(1.30) $$-y''(x) - \frac{\alpha}{x} y'(x) = f(x), \quad x \in (0,1), \quad \alpha \ge 1$$

(1.31) $$y'(0) = 0, \quad y(1) = 0$$

This equation arises, with $\alpha = n-1$, when a change of variables is employed in a radially symmetric Poisson's equation in $\Re^n$. After writing (1.30) in the form

(1.32) $$-(x^\alpha y')' = x^\alpha f, \quad x \in (0,1).$$

Jespersen seeks to approximate a Galerkin solution y for problem (1.31) - (1.32) in the space $\overset{o}{H}{}^1_\alpha(0,1) \equiv H^1_\alpha(0,1) \cap \{y \in C(0,1]: y(1) = 0\}$

where

$$H^1_\alpha(0,1) \equiv \left\{ y \in L_2(0,1): \int_0^1 x^\alpha (y^2 + (y')^2) dx < \infty \right\}.$$

The approximating finite dimensional subspace that Jespersen used in the application of the Ritz - Galerkin method is defined as follows:

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be any partition of the interval $I = [0,1]$. $I_i = (x_{i-1}, x_i)$, $h_i = x_{i+1} - x_i$, $h = \max_i h_i$. For J an interval, let $\Pi_k(J)$ denote the set of polynomials of order k (degree $< k$) on J. Let $0 \le \nu \le k-1$ be integers. Define

$$S^h = S^h_{k,\nu} = \left\{ y \in H^\nu(I): y\big|_{I_i} \in \Pi_k(I_i), \quad 1 \le i \le N \right\},$$

where $H^\nu(I)$ is the space of all function $y \in C^{\nu-1}(I)$ such that $y^{(\nu-1)}$ is absolutely continuous and $y^{(\nu)} \in L_2(I)$. The approximating subspace in which a Galerkin solution y_h is determined is

540C

$$\overset{o}{S}_h = \{y \in S^h : y(1) = 0\}.$$

Jespersen indicates the existence of a unique solution $y \in \overset{o}{H}{}^1_\alpha(I)$ for problem (1.31) - (1.32) and a unique Galerkin solution $y_h \in \overset{o}{S}_h$. With the assumption of a quisi-uniform partition of I, i.e. there are constants $M \geq 0$ and $\gamma \geq 1$ independent of h such that

$$x_j / x_i \leq M(j/i)^\gamma \text{ for } 1 \leq i \leq j \leq N.$$

Jespersen establishes the following error estimate

(1.33) $$\|y - y_h\|_{L_2} \leq Ch\|y' - y_h'\|_{L_2}.$$

He further proves that if $y \in H^j(I)$ where $1 \leq j \leq k$,

then

(1.34) $$\|y - y_h\|_{L_2} \leq Ch^j\|y^{(j)}\|_{L_2}.$$

Through more refined analysis, Jespersen derives the following uniform error estimate

$$\|y_h - y\|_{L_\infty} \leq C(\ln\frac{1}{h})^{\bar{j}}\, h^j\|y^{(j)}\|_{L_\infty},$$

where $\bar{j} = 1$ if $j = 2$ and $\bar{j} = 0$ if $j > 2$.

540 C

Further, Jespersen considered the following nonlinear problem

(1.35) $$-\left(x^{\alpha}y'\right)' = x^{\alpha}f(x,y), \quad x \in (0,1)$$

(1.36) $$y'(0) = 0, \quad y(1) = 0$$

He assumed that $f(x,y)$ is smooth as a function of x and differentiable as a function of y, and

(1.37) $$\left|f_y(x,y)\right| \le K$$

and

(1.38) $$f_y(x,y) \le (1-c)\lambda_0^2$$

where $c > 0$ and λ_0^2 is the smallest positive eigenvalue of the problem

(1.39) $$-(x^{\alpha}y')' = \lambda_0^2 x^{\alpha} y, \quad x \in (0,1)$$

(1.40) $$y'(0) = 0, \quad y(1) = 0.$$

Finally, Jespersen provided the following error estimate for the nonlinear problem (1.35) - (1.36) provided that $y \in H^k(I)$ and $f_{yy}(x,y) \le K$.

(1.41) $$\|y - y_h\|_{L_2} + h\|y' - y_h'\|_{L_2} \le Ch^k\|u^{(k)}\|_{L_2}$$

Eriksson and Thome´e [10] also used Galerkin methods to approximate solutions of singular boundary value problems. They considered Galerkin piecewise polynomial approximation methods for the following singular two-point boundary value problem:

(1.42) $Ly(x) = -y''(x) - \frac{\alpha}{x} y'(x) + y'(x) + q(x)y(x) = f(x), \quad x \in I = (0,1),$

(1.43) $$y'(0) = 0, \quad y(1) = 0.$$

Eriksson, et al, considered approximating a solution y for (1.42) - (1.43) in the space H^1 of all $y \in C(0,1]$ which vanish at $x = 1$ and for which $x^{\alpha/2} y' \in L_2[0,1]$. The approximating finite dimensional subspace S_h that Eriksson, et al, used in the application of the Galerkin method consists of continuous functions which vanish at $x = 1$ and which reduce to polynomials of degree at most $k-1$ on each subinterval $I_i = (x_i, x_{i+1})$ of the partition of I defined by $0 = x_0 < x_1 < \cdots < x_N = 1$. Using variational methods, Eriksson, et al, proved the following weighted norm error estimate which was established earlier by Jespersen [13]

(1.44) $$\|x^{\alpha/2}(y_h - y)\|_{L_2} \le Ch^j \|x^{\alpha/2} y^{(j)}\|_{L_2}$$

where $1 \le j \le k$, y is the unique solution of (1.42) - (1.43) in H^1 and y_h is the unique Galerkin solution in S_h.

Eriksson, et al, also proved (1.44) in the uniform norm for the case $\alpha > 1$, i.e. for $\alpha > 1$, the following uniform norm error estimate holds

(1.45) $$\|x^{\alpha/2}(y_h - y)\|_{L_\infty} \le Ch^j \|x^{\alpha/2} y^{(j)}\|_{L_\infty}.$$

540C

Doedel and Reddien [7] used finite difference methods with a class of singular two-point boundary value problems. They considered the following classes of problems that were also considered by Jamet [12].

$$Ly = y'' + \frac{\alpha}{x}y' - q(x)y = f(x), \quad 0 < x < 1, \qquad (1.46)$$
$$q(x), \quad f(x) \in C[0,1].$$

According to the value of α they considered the two-point problem (1.46) with $0 < \alpha < 1$ and the conditions

$$y(0) = a, \quad y(1) = b \qquad (1.47)$$
$$y \in C^2(0,1] \cap C[0,1],$$

or the one point problem (1.46) with $\alpha \geq 1$ and the conditions

$$y(1) = b \qquad (1.48)$$
$$y \in C^2(0,1] \cap C[0,1].$$

The finite difference scheme they used may be described as follows:
Consider a partition $0 < x_1 < x_2 \cdots < x_N = 1$ with $h_j = x_j - x_{j-1}$, and $h = \max_j h_j$. Let y_j be an approximation for $y(x_j)$, $j = 0,1,\cdots,N$. The finite difference approximation at the point x_j is given by

$$L_h y_j \equiv \sum_{i=-1}^{1} d_{j,i} y_{j+i} = \sum_{i=1}^{m} e_{j,i} \; f(z_{j,i}). \qquad (1.49)$$

The points $z_{j,i}$ need not coincide with the mesh points. m is a positive

540C

integer that could vary with j or may be kept fixed. The coefficients $d_{j,i}$ and $e_{j,i}$ in (1.49) may be determined as follows:

Let $\{\phi_{j,l}\}_{\ell=1}^{m+2}$ be a basis for a certain function space on the interval $[x_{j-1}, x_{j+1}]$, $j=1,2,\cdots,N-1$ where the basis functions $\phi_{j,l}$ are chosen consecutively from the list

$$1, x^{1-\alpha}, \quad x^{2-\alpha}, \quad x^2, \quad x^{3-\alpha}, \quad x^3, \quad \cdots.$$

The criteria for their selection is that their span can approximate the behavior of the solution well. Since the basis functions depend on j, a different choice may be made near 0. The coefficients $d_{j,i}$ and $e_{j,i}$ are determined from the following $m+3$ equations

$$\begin{aligned} &\sum_{i=-1}^{1} d_{j,i}\, \phi_{j,\ell}(x_{j+i}) = \sum_{i=1}^{m} e_{j,i}\, L\phi_{j,\ell}(z_{j,i}), \quad 1 \le \ell \le m+2 \\ &\sum_{i=1}^{m} e_{j,i} = 1 \end{aligned} \tag{1.50}$$

The finite difference scheme is given by (1.49) with the appropriate boundary conditions. For problem (1.47) the boundary conditions are $y_0 = 0$ and $y_N = b$. For problem (1.48) the continuous boundary conditions $y'(0) = 0$ and $y(1) = b$ are used which yield the discrete approximations

540C

$$y_N = b \text{ and } \sum_{j=0}^{n} \beta_j y_j = 0.$$

They proved that the finite difference scheme (1.49) is equivalent to the following collocation scheme:

Find $\{p_j(x)\}_{j=1}^{N-1}$ where

$$p_j(x) = \sum_{\ell=1}^{m+2} \alpha_{j,\ell} \phi_{j,\ell}(x),$$

such that there exist numbers $v_i, \quad i = 0, \cdots, N$, so that

$$\text{(1.51)} \quad \begin{array}{ll} (1) & p_j(x_{j+i}) = v_{j+i}, \qquad -1 \le i \le 1 \\ (2) & Lp_j(z_{j,i}) = f(z_{j,i}), \;\; 1 \le i \le m, \;\; 1 \le j \le N-1 \\ (3) & p_j\text{'s satisfy the appropriate boundary conditions.} \end{array}$$

For boundary conditions (1.47), the collocation method uses boundary conditions $p_1(0) = a$ and $p_{N-1}(1) = b$ while for boundary conditions (1.48), the collocation method uses the boundary conditions

$$\sum_{j=0}^{n} \beta_j v_j = 0 \text{ and } p_{N-1}(1) = b.$$

Doedel and Reddien under certain conditions provide stability and convergence results for the two cases $0 < \alpha < 1$ and $\alpha \ge 1$ as outlined below. For the case $0 < \alpha < 1$. Let

$$L_0 y \equiv (x^\alpha y')' \text{ and } X = L_0^{-1} C[0,1]$$

where the inversion is with respect to the boundary conditions (1.47). For $y \in X$, define the norm

(1.52) $$\|y\|_X = \|y\|_{L^\infty} + \|x^\alpha y'\|_{L^\infty} + \|(x^\alpha y')'\|_{L^\infty}.$$

The results for the case $0 < \alpha < 1$, may be summarized as follows:

(1) If the homogeneous problem associated with (1.46) - (1.47) has only the trivial solution, then (1.46) - (1.47) has a unique solution y and

(1.53) $$\|y\|_X \le c \ \|f\|_{L^\infty}.$$

(2) Under certain conditions [7] and for h sufficiently small, (1.51) has a unique solution $P_h = \{P_j\}$ and

(1.54) $$\|P_h\|_X \le c \ \|f\|_{L^\infty}.$$

(3) If the solution y of (1.46) - (1.47) satisfies the following expansion

(1.55) $$(x^\alpha y')' = A(x) + x^\alpha B(x) \text{ where } A(x), B(x) \in C^k[0,1]$$

and under the conditions of (2) above, then the following error estimate holds

(1.56) $$\|y - p_k\|_{L_\infty} \le c\, h^k.$$

540C

Stability results for the case $\alpha \ge 1$ may be summarized as follows:

(1) If the boundary value problem

(1.57) $$y'' + \frac{\alpha y'}{x} - q(x)y = f \quad 0 < x < 1, \quad \alpha \ge 1$$

where $q(x) \in C[0,1]$ has only the trivial solution when $f = 0$, then there exists a unique solution y for every $f \in C[0,1]$ and

(1.58) $$\|x^{-1}y'\|_{L_\infty} + \|y''\|_{L_\infty} \le c\|f\|_{L_\infty}$$

(2) Under certain conditions [7] and for h small enough, there exists a unique solution p_k for (1.51) and

(1.59) $$\|p_k\|_X \le c\|f\|_{L_\infty}$$

where

$$\|p_k\|_X = \|p_k\|_{L_\infty} + \|x^{-1}p_k'\|_{L_\infty} + \|p_k''\|_{L_\infty}.$$

Chawla, et al, [4] considered the following class of singular nonlinear two-point boundary value problems

(1.60) $$y'' + \frac{\alpha}{x}y' + f(x,y) = 0, \quad x \in (0,1]$$

(1.61) $$y'(0^+) = 0, \quad y(1) = a$$

where $\alpha \geq 1$, a is a constant, $f(x,y)$ is continuous for $(x,y) \in \{[0,1] \times \Re\}$, $\dfrac{\partial f}{\partial y}$ exists and is continuous and $\dfrac{\partial f}{\partial y} \leq 0$.

They used the following finite difference scheme to approximate the solution of (1.60) - (1.61). For $N \geq 2$, consider a uniform mesh over [0,1] and let

$$h = 1/N, \quad x_k = kh, \quad y_k = y(x_k) \text{ and } f_k = f(x_k, y_k) \text{ for } k = 0, 1, \cdots, N.$$

Chawla, et al, [4] established the following identity for $k \geq 2$:

(1.62) $$\frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \frac{I_k^+}{J_k} + \frac{I_k^-}{J_{k-1}},$$

where

(1.63) $$\begin{cases} I_k^+ = \dfrac{1}{\alpha - 1} \displaystyle\int_{x_k}^{x_{k+1}} (x_{k+1}^{1-\alpha} - t^{1-\alpha}) t^\alpha f(t)\, dt \\ I_k^- = \dfrac{1}{\alpha - 1} \displaystyle\int_{x_{k-1}}^{x_k} (t^{1-\alpha} - x_{k-1}^{1-\alpha}) t^\alpha f(t)\, dt, \quad \text{and} \end{cases}$$

(1.64) $$J_k = (x_{k+1}^{1-\alpha} - x_k^{1-\alpha}) / (1 - \alpha).$$

For $k = 1$, the following identify is used:

(1.65) $$\frac{y_2 - y_1}{J_1} = -\int_0^{x_1} t^\alpha f(t) dt - \frac{1}{(1-\alpha) J_1} \int_{x_1}^{x_2} (x_2^{1-\alpha} - t^{1-\alpha}) t^\alpha f(t)\ dt$$

where

(1.66) $$J_1 = \left(x_2^{1-\alpha} - x_1^{1-\alpha}\right)/(1-\alpha).$$

540C

The finite difference scheme (1.62) & (1.65) is used to obtain order (h^2) approximation for the solution y of (1.60) - (1.61) with the following assumptions

(1.67) $$|f'| \le C_1 \text{ and } x|f''| \le C_2 \text{ for } 0 < x \le 1$$

for suitable positive constants C_1 and C_2. Taylor series expansion of $f(t)$ about $x = x_k$ is used in the above scheme. An approximation for the value of y at $x = 0$ is computed from the following equation

(1.68) $$y_0 = y_1 + \frac{h^2}{2(\alpha+1)} f_1 - \frac{\alpha+4}{6(\alpha+1)(\alpha+2)} h^3 f'(\xi_0), \quad 0 < \xi_0 < x_1.$$

In another work [5], Chawla, et al, used the same finite difference scheme (1.62) & (1.65) with problem (1.60) - (1.61) and obtained order (h^4) convergence by replacing $f(t)$ by the cubic polynomial which interpolates to f at x_{k-1}, x_k, x_{k+1} and to f' at x_k. The following additional restrictions on f were made

(1.69) $$|f'''| \le C_3 \text{ and } x|f^{(4)}| \le C_4, \quad 0 \le x \le 1.$$

More recently, Fink, et al, [11] and Baxley [2] considered singular nonlinear boundary value problems that include the following problem:

$$y'' + \frac{n-1}{x} y' + f(x,y) = 0, \quad x \in (0,1) \tag{1.70}$$

$$y'(0) = 0, \quad y(1) = 0. \tag{1.71}$$

In particular, the case when $f(x,y)$ is singular at $y = 0$ is included in their analysis. The special case $f(x,y) = a(x)y^{-p}$, $p > 0$ and $a(x)$ is continuous is also included in the results. In both papers [11] and [2], general existence and uniqueness results for positive solutions are given. In particular, Fink, et at, [11], prove that if

(1) f:$[0,1) \times (0,\infty) \to (0,\infty)$ is continuous, and

(2) $f(x,y)$ is strictly decreasing in y for $x \in (0,1)$, and integrable over $[0,1]$ for each fixed $y > 0$,

then problem (1.70)-(1.71) has exactly one positive solution belonging to $C^1[0,1) \cap C^2(0,1)$. The proof of the above result also provides a tool for approximating the solution.

Nassif [17] considered eigenvalue - eigenfunction finite difference approximations for regular and singular Sturm - Liouville problems. He obtained error estimates for eigenvalue - eigenfunction approximations by using two difference schemes: Numerov scheme to solve the Schrödinger singular equation and the central difference formula for Sturm - Liouville

540 C

problems. In particular, Nassif considered the Schrödinger operator

(1.72) $$Ly = -y'' + q(x)y, \quad 0 < x < \infty$$

with boundary conditions

(1.73) $$B[y] = cy'(0) + dy(0) = 0$$

(1.74) $$y(x) \text{ bounded on } (0,\infty).$$

Let $x_i = ih, 0 \le i \le N, x_0 = 0, X = x_N = Nh, \lim_{h\to 0} X = \lim_{h\to 0} N = \infty.$

The Numerov difference scheme determines a vector $Y = \{Y_i\}_{0\le i\le N}$ and approximate eigenvalue $\lambda_h \in \Re$ such that

(1.75) $$\left(-Y_{i-1} + 2Y_i - Y_{i+1}\right)/h^2 + \left(q_{i-1}Y_{i-1} + 10q_iY_i + q_{i+1}Y_{i+1}\right)/12$$
$$= \lambda_h\left(Y_{i-1} + 10Y_i + Y_{i+1}\right)/12$$

(1.76) $$B_h[Y] = 0 \text{ and } Y_N = 0$$

where B_h is the difference approximation to B. Nassif proved that under the following assumptions

(1) $q \in C^{\infty}(0,\infty) \cap C[0,\infty]$, $q \to 0$ as $x \to \infty$

(2) $\alpha = \inf_{x\in\Re_+} q(x) < 0$, $M = \sup_{x\in\Re_+} |q(x)|$,

(3) q is Lipschitz continuous on $(0,\infty)$

(4) $q', q'', q''', q^{(4)}$ are bounded on $(0,\infty)$

(5) $c = 0$ and $d = 1$ in (1.73),

then for every isolated eigenvalue λ_k, $k \ge 1$, with multiplicity 1 of the operator L in (1.72) - (1.74), the Numerov scheme gives a sequence of operators $L_h : \Re^{N-1} \to \Re^{N-1}$, and a sequence of isolated eigenvalues $\lambda_{k,h}$ of L_h, with the same multiplicity as λ_k, such that for some choice of $\{h, X(h)\}$ the following error estimate holds

(1.77) $$\left|\lambda_k - \lambda_{k,h}\right| \le ch^4.$$

Further,

(1.78) $$\delta\left(r_h E_k,\ F_{k,h}\right) \le c\, h^4$$

where E_k and $F_{k,h}$ are the invariant subspaces corresponding to λ_k and $\lambda_{k,h}$ respectively and

$$r_h : L^2(0,\infty) \to \Re^{N-1}$$

such that

$$r_h f = \left\{f(x_i)\right\},\ 1 \le i \le N-1.$$

$\delta(X,Y)$ is the aperture or gap between the subspaces X and Y. He used the L_2 norm in his definition of the gap between subspaces. Nassif also considered the Regular Sturm - Liouville problem.

(1.79) $$-(p(x)y')' + q(x)y = \lambda f(x)y, \quad a < x < b$$

(1.80) $$y(a) = y(b) = 0$$

where p, q and s are continuous and positive on $[a, b]$. Further, he required sufficient regularity on p, q and f so that $y \in C^4(a, b)$; that is

(1.81) $$p \in C^3(a,b), q, f \in C^2(a,b).$$

With a partition $x = a + ih$, $0 \le i \le N$ where $h = (b-a)/N$, and a discretization of (1.79) - (1.80) by using the central difference formula

$$\delta_{h/2}\, y(x) = \frac{y(x+h/2) - y(x-h/2)}{h}$$

he obtained the finite difference scheme

(1.82) $$-\delta_{h/2}\,(p_i\, \delta_{h/2}\, Y_i) + q_i\, Y_i = \lambda_h f_i\, Y_i, \quad 0 < i < N$$

(1.83) $$Y_0 = Y_N = 0.$$

The operators L and L_h are defined in $L^2(a,b)$ and $\Re^{N-1}$ respectively by

$$Ly = \frac{-(-p(x)y')' + q(x)y}{f(x)}$$

$$(L_h Y)_i = \frac{-\delta_{h/2}\left(p_i\, \delta_{h/2}\, Y_i\right) + q_i Y}{f_i}, 0 < i < N$$

where $Y \in \Re^{N-1}$.

Nassif showed that under the assumption (1.81), and for every isolated eigenvalue λ_k of the operator L with associated invariant subspace E_k the finite difference scheme (1.82) - (1.83) gives a sequence of operators

$L_h : \Re^{N-1} \to \Re^{N-1}$ and a sequence of isolated eigenvalues $\lambda_{k,h}$ with corresponding eigenspaces $E_{k,h} \subset \Re^{N-1}$ such that for all h the following error estimates hold

$$\left|\lambda_k - \lambda_{k,h}\right| \leq ch^2$$

and

$$\delta_h\left(r_h E_k,\ F_{k,h}\right) \leq ch^2$$

where δ_h and r_h are as defined in Nassif's first problem.

CHAPTER 2

BACKGROUND

This chapter includes the necessary background from the literature that is needed in subsequent chapters. It contains three sections. Section 1 includes existing results on self-adjoint extensions of a closed symmetric operator in a Hilbert space. This material is based on the treatment provided by Naimark [15] and [16]. Section 2 includes a review of the necessary results on the spectral approximation of a closed linear operator and is based on the treatment discussed by Chatelin [3]. Section 3 is a brief discussion that includes results relating to irreducible matrices, partial ordering of matrices and non-negative matrices. This section is based on the treatment provided by Ortega and Rheinboldt [18].

2.1 SELF-ADJOINT EXTENSIONS OF CLOSED SYMMETRIC OPERATORS IN A HILBERT SPACE

2.1.1 Projection and Isometric Operators

Let H be a separable Hilbert space and M a closed subspace in H. Then any vector x can be expressed uniquely as $x = x_1 + x_2$ where $x_1 \in M$ and $x_2 \perp M$. An *orthogonal projection* P on H is defined by

$$P(x) = x_1.$$

A linear operator U is called an *isometry* if $(U(x),U(y))=(x,y)$ for all $x,y \in$ Domain U where $(.,.)$ denotes the inner product in H.

540C

2.1.2 The Direct Sum of Hilbert Spaces and the Graph of an Operator

Let $H_1,H_2,\cdots,H_n$ be Hilbert spaces. Let H be the set of all vectors $(x_1,x_2,\cdots,x_n)$ such that $x_1 \in H_1, x_2 \in H_2,\cdots,x_n \in H_n$. We define scalar multiplication and addition in H by

$$\alpha(x_1,x_2,\cdots,x_n)=(\alpha x_1,\alpha x_2,\cdots,\alpha x_n)$$

$$(x_1,x_2,\cdots,x_n)+(x_1',x_2',\cdots,x_n')=(x_1+x_1',x_2+x_2',\cdots,x_n+x_n').$$

We also define inner product in H by

$$((x_1,x_2,\cdots,x_n),(x_1',x_2',\cdots,x_n'))=(x_1,x_1')+(x_2,x_2')+\cdots+(x_n,x_n').$$

With these operations, H becomes a Hilbert space. It is called the direct sum of $H_1,H_2,\cdots,H_n$ and is denoted by

$$H_1 \oplus H_2 \oplus \cdots \oplus H_n.$$

Let L be an operator in a Hilbert space H. The set of all ordered pairs (x,Lx), $x \in$ Domain L in the direct sum $H \oplus H$ is called the *graph* of the operator L and is denoted by G_L. It is easily verified that an operator L is linear if and only if its graph is a subspace of $H \oplus H$.

2.1.3 Closed Operators; Closure of an Operator

An operator L is said to be *closed* if its graph G_L is closed in $H \oplus H$, i.e. an operator is closed if and only if the following condition holds:

If

$$x_n \in \text{ Domain } L,\ x_n \to x \text{ and } Lx_n \to y$$

then

$$x \in \text{ Domain } L \text{ and } Lx = y.$$

If the closure of the graph G_L of an operator L is the graph of a certain operator, then we say the operator L admits a closure $\bar{L}$. Such an operator $\bar{L}$, if it exists, is the minimal closed extension of the operator L. An operator L admits a closure if and only if

$$x_n \in \text{ Domain } L,\ x_n \to 0 \text{ and } Lx_n \to y, \text{ then } y = 0.$$

A vector x will be in the domain of $\bar{L}$ if there exists a sequence of vectors $x_n \in$ Domain L such that $x_n \to x$ and Lx_n converges as $n \to \infty$. $\bar{L}x$ is defined as $\lim\limits_{n\to\infty} Lx_n$.

2.1.4 The Adjoint of an Operator; Hermitian, Symmetric and Self-Adjoint Operators

Let L be a linear operator whose domain is dense in H. The *adjoint* L^* of L is an operator in H whose domain is the set of all $y \in H$ such that

$$(Lx, y) = (x, z)$$

for some $z \in H$ and for all $x \in$ Domain L. We define

$$L^* y = z \text{ for all } y \in \text{ Domain } L^*.$$

The vector z in the above equation is unique for a given y, for else, assume

$$(Lx, y) = (x, z')$$

then $(x, z - z') = 0 \quad \forall x \in$ Domain L. This means that domain L is not dense in H which is a contradiction.

Let L be a linear operator whose domain is dense in H, then L^* is a closed linear operator. To see this, we consider the operator U in $H \oplus H$ defined by

$$U(x, y) = (iy, -ix).$$

Let $A = U(G_L)$, the image of the graph of L under U. Define $B \subseteq H \oplus H$ by

$$B = H \oplus H - A,$$

then B is the graph of L^* in $H \oplus H$. To see this, let $(y, z) \in B$, then

$$((iLx, -ix), \quad (y, z)) = 0 \quad \forall x \in \text{ Domain } L.$$

The above equation is equivalent to

$$(Lx, y) = (x, z),$$

therefore, $y \in$ Domain L^* and $z = L^*y$. Since B is an orthogonal complement in a Hilbert space, then it is closed and hence L^* is closed.

An operator L is called *Hermitian* if $\forall x, y \in$ Domain L, we have

$$(Lx, y) = (x, Ly).$$

A *symmetric* operator is a Hermitian operator whose domain is dense in H. We conclude that an operator L whose domain is dense in H is symmetric if and only if

$$L \subset L^*.$$

We further conclude that a symmetric operator always admits a closure.

An operator L is called *self-adjoint* if its domain is dense in H and $L = L^*$. Therefore a self-adjoint operator is necessarily closed.

2.1.5 Spectrum of a Self-Adjoint Operator

Let L be any operator in a Hilbert space H. A complex number λ is called a *regular point* of the operator L if $R_\lambda = (L - \lambda I)^{-1}$ exists, is bounded and has domain the whole space H. The operator R_λ is called the *resolvent* of L with respect to λ. All points other than the regular points belong to the *spectrum*. In particular the eigenvalues belong to the spectrum. The set of all eigenvalues is called the discrete (or point) spectrum of the operator. The spectrum and resolvent of an operator will be discussed in more detail in Section 2.2.

For a self-adjoint operator, two eigenvectors that belong to two distinct eigenvalues are orthogonal. Therefore, any set of eigenvectors that correspond to distinct eigenvalues must be a finite or countable orthogonal system in H. We conclude that the discrete spectrum of a self-adjoint operator is a finite or countable set of real numbers. For a self-adjoint operator, any non-real number is a regular point.

2.1.6 Compact Operators

An operator L in a complete normed space A is said to be *compact* if it maps a bounded set into a relatively compact set. Following are some properties of compact operators that will be of use to us later in this work.

1. Every compact operator is bounded.
2. If L is compact and M is bounded and defined everywhere in A, then the operators LM and ML are compact.
3. If an operator L is compact, then its adjoint L^* is also compact.
4. If $\{L_n\}$ is a sequence of compact operators such that

$$\|L - L_n\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

then L is compact.

5. The non-zero points of the spectrum of a Hermitian, compact operator are all real eigenvalues with finite multiplicities and can only accumulate at $\lambda = 0$.

A complex valued function $U(x,\tau)$, $a < x, \tau < b$, of two real variables that satisfies the following condition

$$\int_a^b \int_a^b |U(x,\tau)|^2 dx\, d\tau < \infty$$

is called a *Hilbert-Schmidt* kernel. The integral operator in the Hilbert space $L^2(a,b)$ defined by

$$g(x) = \int_a^b K(x,\tau) f(\tau) d\tau, \quad f(x) \in L^2(a,b),$$

is a compact operator. The integral operators with Hilbert-Schmidt kernel make an important class of compact operators.

2.1.7 Extension of a Symmetric Operator

The main objective of the remaining subsections is to discuss symmetric extensions of a given symmetric operator. In particular, the problem of constructing self-adjoint extensions for a given symmetric operator is the

540C

one that interests us. Let M be a symmetric extension of the symmetric operator L, then $L \subset M$. It can be easily shown that $M^* \subset L^*$. Since M is a symmetric operator we must have $M \subset M^*$ and hence

$$L \subset M \subset M^* \subset L^*.$$

We conclude that every symmetric extension of the operator L, is a restriction of the adjoint operator L^*. Self-adjoint extensions will be obtained in subsection 2.1.11.

2.1.8 Deficiency Spaces of a Symmetric Operator

Let L, be a symmetric operator and λ an arbitrary non-real number. Let R_λ and $R_{\bar{\lambda}}$ denote the ranges of the operators $(L - \lambda I)$ and $(L - \bar{\lambda} I)$ respectively. Further, let

$$N_\lambda = H - R_\lambda, \text{ and}$$
$$N_{\bar{\lambda}} = H - R_{\bar{\lambda}}.$$

Since N_λ and $N_{\bar{\lambda}}$ are orthogonal complements in H for the subspaces R_λ and $R_{\bar{\lambda}}$, they are subspaces by themselves (in fact closed subspaces). N_λ and $N_{\bar{\lambda}}$ will be called the *deficiency spaces* of the operator L, with respect to λ. Deficiency spaces will prove to be instrumental in constructing self-adjoint extensions for a given symmetric operator.

Lemma 2.1

The deficiency spaces N_λ and $N_{\bar{\lambda}}$ *are the eigenspaces of the operator* L^* *which belong to the eigenvalues* $\bar{\lambda}$ and λ *respectively.*

Proof

Let $x \in N_\lambda$, and $y \in$ Domain L, then

$$(Ly - \lambda y, x) = 0.$$

Hence,

$$(Ly, x) = (y, \bar{\lambda} x).$$

Therefore, $x \in$ Domain L^* and $L^* x = \bar{\lambda} x$. Conversely, assume $L^* x = \bar{\lambda} x$, then for each $y \in$ Domain L we have

$$(Ly, x) = (y, \bar{\lambda} x) \Rightarrow (Ly - \lambda y, x) = 0 \Rightarrow x \in N_\lambda.$$

2.1.9 The Domain of the Adjoint Operator

The subspaces $H_1, H_2, \cdots, H_n$ are called *linearly independent* if any set of n vectors $\{x_i\}_{i=1}^n$, $x_i \in H_i$ is linearly independent in H. If

$$x \in H_1 \oplus H_2 \oplus \cdots \oplus H_n$$

then x has a unique representation

$$x = x_1 + x_2 + \cdots + x_n, \quad x_i \in H_i, \quad i = 1, 2, \cdots, n.$$

540C

In what follows we will find out that the domain of the adjoint of a closed symmetric operator can be expressed as the direct sum of the domain of the operator and two of its deficiency spaces.

540C

Theorem 2.1

Let L be a closed symmetric operator, then the subspaces Domain L, N_λ and $N_{\bar{\lambda}}$ *are linearly independent and*

$$\text{Domain } L^* = \text{Domain } L \oplus N_\lambda \oplus N_{\bar{\lambda}}. \tag{2.1}$$

Proof

First we show linear independence. Let $x_1 \in$ Domain L, $x_2 \in N_\lambda$ and $x_3 \in N_{\bar{\lambda}}$ such that

$$x_1 + x_2 + x_3 = 0, \tag{2.2}$$

then

$$\begin{aligned} (L^* - \lambda I)(x_1 + x_2 + x_3) &= 0 \\ (L - \lambda I)x_1 + (L^* - \lambda I)x_2 + 0 &= 0 \\ (L - \lambda I)x_1 + (\bar{\lambda} - \lambda)x_2 &= 0. \end{aligned}$$

Since R_λ and N_λ are linearly independent subspaces we conclude that

$$(L - \lambda I)x_1 = 0 \text{ and } (\bar{\lambda} - \lambda)x_2 = 0.$$

Since L is a symmetric operator and $\lambda \neq \bar{\lambda}$ we must have $x_1 = 0$ and

$x_2 = 0$. Substituting in (2.2) we obtain $x_3 = 0$.

To prove (2.1) we note that Domain L, N_λ and $N_{\bar{\lambda}}$ are subspaces of Domain L^* and hence

$$\text{Domain } L \oplus N_\lambda \oplus N_{\bar{\lambda}} \subseteq \text{Domain } L^*.$$

To prove the converse of the above statement, let $x \in$ Domain L^*, we need to show that x can be represented as

$$x = x_1 + x_2 + x_3 \text{ where } x_1 \in \text{ Domain } L, x_2 \in N_\lambda, x_3 \in N_{\bar{\lambda}}.$$

Since the operator L is closed, R_λ is a closed subspace of H ([16] Theorem 3, page 28). Therefore, H can be written as the direct sum of R_λ and N_λ. Hence the vector $(L^* - \lambda I)x$ can be written as

(2.3) $$(L^* - \lambda I)x = y_1 + y_2 \text{ where } y_1 \in R_\lambda \text{ and } y_2 \in N_\lambda.$$

(2.3) can be rewritten as

$$(L^* - \lambda I)x = (L - \lambda I)x_1 + (\bar{\lambda} - \lambda)x_2 \text{ where } x_1 \in \text{ Domain } L \text{ and } x_2 \in N_\lambda,$$

hence

$$(L^* - \lambda I)x = (L^* - \lambda I)x_1 + (L^* - \lambda I)x_2,$$

and

$$(L^* - \lambda I)(x - x_1 - x_2) = 0.$$

Therefore,

$$x - x_1 - x_2 = x_3 \in N_{\bar{\lambda}},$$

and

$$x = x_1 + x_2 + x_3 \text{ where } x_1 \in \text{ Domain } L,\ x_2 \in N_\lambda \text{ and } x_3 \in N_{\bar{\lambda}}.$$

The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.1

A closed symmetric operator L is self-adjoint if and only if

$$N_\lambda = N_{\bar{\lambda}} = \{0\} \textit{ for each } \lambda.$$

Theorem 2.2

Let L be a symmetric operator and λ any complex number with $\operatorname{Im}(\lambda) > 0$. *Then*

$$\dim N_\lambda = \dim N_i \text{ and } \dim N_{\bar{\lambda}} = \dim N_{-i}.$$

2.1.10 Deficiency Indices

Let $m = \dim N_\lambda$ and $n = \dim N_{\bar{\lambda}}$, $\operatorname{Im}(\lambda) > 0$. The numbers m and n are called the *deficiency indices* of the operator L. We note that the deficiency indices are well defined in view of Theorem 2.2 above. Corollary 2.1 could therefore be restated as follows:

A closed symmetric operator L is self-adjoint if and only if its deficiency *indices are* $m = 0$ and $n = 0$.

540C

The following useful result is also discussed in [16], page 33.

Theorem 2.3

Let L *be a closed symmetric operator and* M *a bounded, self-adjoint operator whose domain is the whole space* H. *The operators* L *and* $L+M$ *have the same deficiency indices.*

2.1.11 Construction of a Symmetric Extension for a Given Symmetric Operator

In this subsection we give the main result which provides the mechanism for constructing a closed symmetric extension of a given symmetric operator.

Theorem 2.4

Let L *be a closed symmetric operator with deficiency spaces* N_{λ} *and* $N_{\bar{\lambda}}$ *where* $\operatorname{Im}\lambda > 0$. *The operator* L *has a closed symmetric extension* L' *if and only if there exists an isometric operator* S *whose domain* D *is a closed subspace of* $N_{\bar{\lambda}}$ *and whose range* R *is a closed subspace of* N_{λ}. *The operator* L' *has domain all vectors* x' *of the form*

(2.4) $$x' = x + x_1 - Sx_1, \quad x \in \text{ Domain } L, \; x_1 \in D,$$

and is defined by

(2.5) $$L'(x') = Lx + \lambda x_1 - \bar{\lambda} S x_1.$$

540C

The following theorem can easily be concluded from Theorem 2.4.

Theorem 2.5

An operator L has a self-adjoint extension if and only if its deficiency indices are equal.

2.1.12 Singular Self-Adjoint Differential Expressions

Any formally self-adjoint differential expression with real, sufficiently differentiable functions in the interval $a < x < b$ is necessarily of even order and can be expressed in the form

(2.6) $$\ell(y) = (-1)^n \left(p_0 y^{(n)}\right)^{(n)} + (-1)^{n-1} \left((p_1 y^{(n-1)}\right)^{(n-1)} + \cdots + p_n y$$

([15], page 8). The expression $\ell(y)$ is said to be *regular* if the interval (a,b) is finite and the functions

(2.7) $$1/p_0(x),\ p_1(x),\ \cdots,\ p_n(x)$$

are absolutely integrable in the whole interval (a,b). Otherwise $\ell(y)$ is said to be *singular*. The end-point $x = a$ is regular if $a > -\infty$ and the functions (2.7) are absolutely integrable in every interval $[a,\delta]$ where $\delta < b$.

Otherwise the end-point a is called singular. Therefore, the end-point a is singular if $a = -\infty$ or if at least one of the functions in (2.7) is not absolutely integrable in an interval $[a, \delta]$ for some $\delta < b$. Regularity and singularity of the right end-point $x = b$ is defined similarly.

In chapters 3 and 4 we will consider a certain class of second order singular differential operators on an interval $[a,b]$ where the end-point a is singular and the end-point b is regular. We will provide numerical approximation for the solution of a certain class of boundary value problems involving these operators. We will also provide numerical approximations for the eigenvalues and eigenvectors for these operators.

2.2 SPECTRAL APPROXIMATION OF A CLOSED LINEAR OPERATOR

In this section, we will be concerned with approximation of the eigenelements of a closed linear operator. We will discuss sufficient conditions under which we obtain convergence of numerical approximations of eigenvalues and eigenvectors. Throughout this section we let X denote a complex Banach space and H a complex Hilbert space.

2.2.1 Basic Definitions of the Resolvent Set and Spectrum of an Operator

Let L be a closed operator in X, and z any complex number. We observe the following definitions:

the *resolvent* set $\rho(L)$ is the set of all complex numbers z such that $(L-z)^{-1}$ exists, has domain X, and is bounded. We let $R(L,z)=(L-z)^{-1}$;

the *spectrum* $\sigma(L)$ of L is the complementary set of $\rho(L)$ in the complex plane. $\sigma(L)$ is divided into three mutually exclusive parts:

point spectrum $P\sigma(L)$ is the set of all $z\in\sigma(L)$ such that $L-z$ has no inverse;

continuous spectrum $C\,\sigma(L)$ is the set of all $z\in\sigma(L)$ such that $(L-z)$ has an unbounded inverses with domain dense in X; and

residual spectrum $R\sigma(L)$ is the set of all $z\in\sigma(L)$ such that $(L-z)$ has an inverse (bounded or unbounded) with domain not dense in X.

A nonzero vector $x \in$ Domain L such that $Lx = \lambda x$ is called an eigenvector of L corresponding to the eigenvalue λ. Therefore, the point spectrum of L consists of all the eigenvalues λ of L. The kernel of $(L - \lambda I)$ is the eigenspace of L corresponding to λ. Its dimension g is called the *geometric multiplicity* of λ.

The resolvent operator $R(L,z)$, $z \in \rho(L)$ satisfies the following two identities which are needed later.

The first resolvent equation (Hilbert)

$$\begin{aligned} R(L,z_1) - R(L,z_2) &= (z_1 - z_2)R(L,z_1)R(L,z_2) \\ &= (z_1 - z_2)R(L,z_2)R(L,z_1). \end{aligned} \tag{2.8}$$

Proof

$$\begin{aligned} (z_1 - z_2)R(L,z_1)R(L,z_2) &= R(L,z_1)((L - z_2) - (L - z_1))R(L,z_2) \\ &= R(L,z_1) - R(L,z_2). \end{aligned}$$

The other part of (2.8) may be proved similarly.

The second resolvent equation (Hilbert)

$$\begin{aligned} R(L_1,z) - R(L_2,z) &= R(L_1,z)(L_2 - L_1)R(L_2,z) \\ &= R(L_2,z)(L_2 - L_1)R(L_1,z). \end{aligned} \tag{2.9}$$

540C

540 C

Proof

$$\begin{aligned} R(L_1,z)(L_2-L_1)R(L_2,z) &= R(L_1,z)(L_2-z-(L_1-z))R(L_2,z) \\ &= R(L_1,z)-R(L_2,z). \end{aligned}$$

The other part of (2.9) may be proved similarly.

2.2.2 Separation of the Spectrum

Let L be a closed operator in X, with domain domain L. A subspace M of X is called *invariant* under L if

$$L[(\text{Domain } L)\cap M]\subseteq M.$$

Let L_M denote the restriction of L to the subspace Domain $L\cap M$. Let $X=M\oplus N$ where M and N are closed subspaces and each is invariant under L. Let P denote the projection on M along N. The operator L is said to be completely reduced by the invariant subspaces M and N if $P(\text{Domain } L)\subset \text{Domain } L$.

The following theorem includes results that will be used extensively in subsequent chapters.

Theorem 2.6

Let $\sigma(L)=\sigma_1\cup\sigma_2$ where σ_1 is bounded and separated from σ_2 in such a *way that a closed Jordan curve Γ can be drawn in $\rho(L)$ around σ_1 leaving σ_2 in its exterior. Then*

1. *The operator*

$$P = -\frac{1}{2\pi i}\int_{\Gamma} R(z)dz \tag{2.10}$$

is a projection on X.

2. *Let $M = PX$ and $N = (1-P)X$, then $X = M \oplus N$ and L is completely reduced by M and N.*

3. *Let L_M and L_N be defined as before, the spectra of L_M and L_N are σ_1 and σ_2 respectively and L_M is bounded.*

2.2.3 Isolated Eigenvalues

In the sequel we will be interested in the cases when the spectrum of a closed operator has an isolated point λ. The spectrum $\sigma(L)$ can then be written as $\sigma(L) = \{\lambda\} \cup \sigma_1$, where $\sigma_1 = \sigma - \{\lambda\}$. If Γ is any closed Jordan curve enclosing λ but no other points in $\sigma(L)$, then the projection

$$P = -\frac{1}{2\pi i}\int_{\Gamma} R(z)dz$$

depends on λ only, because λ is the only singularity of $R(z)$ inside Γ.

The projection P is called the *spectral projection* associated with λ. For $x \in X$, Px is defined by

$$Px = -\frac{1}{2\pi i}\int_{\Gamma} R(z)x\,dz.$$

The subspace $M = PX$ is called the *invariant subspace* associated with λ. We assume that the dimension of $M = m < \infty$, then $\{\lambda\}$ is the spectrum of the finite rank operator L_M of *algebraic multiplicity m*. Therefore, λ is an eigenvalue of L of algebraic multiplicity m. We let $g =$ dimension $E =$ Kernel $(L-\lambda)$, i.e., we assume λ has *geometric multiplicity* g. We will also assume that λ has *ascent* ℓ. Let L_n, $n = 1,2,\cdots$ be a sequence of closed operators such that Domain $L_n =$ Domain L. If Γ lies in $\rho(L_n)$, then we may define the resolvent $R_n(z) = (L_n - z)^{-1}$ for any $z \in \Gamma$. We may also define the projection

$$P_n = -\frac{1}{2\pi i}\int_{\Gamma} R_n(z)\,dz.$$

Let Δ denote the domain enclosed by Γ. The set $\sigma(L_n) \cap \Delta$ is the spectrum of L_N inside Δ. We also define the invariant subspace $M_n = P_n X$ of L_n that is associated with $\sigma(L_n) \cap \Delta$.

The sequence $\{L_n\}$, $n = 1,2,\cdots$, is said to be an *approximation* of L if

(2.11) $\quad L_n x \to Lx$ as $n \to \infty$, for all $x \in$ Domain L.

540C

i.e. L_n converges to L pointwise, and we write $L_n \xrightarrow{p} L$.

540C

The remainder of this section will be concerned with the analysis of the convergence of $\sigma(L_n)\cap\Delta$ to $\{\lambda\}$ and the convergence of the invariant subspaces M_n to M.

2.2.4 Convergence of the Spectrum $\sigma(L_n)\cap\Delta$

Let E and E_n, $n\in N$, be subsets of the complex plane.

Definition

$E=\overline{\lim}\ E_n$ *if for any* $x\in E$, $\exists$ *an infinite subset* $N_1\subseteq N$, *where* N *is the set of natural numbers, such that for any* $n\in N_1$ *and* $x_n\in E_n$, *we have* $|x_n-x|\to 0$.

Definition

$E=\underline{\lim}\ E_n$ *if for any* $x\in E$, $\exists$ $x_n\in E_n$ *such that* $|x_n-x|\to 0$, $n\in N$.

Definition

If $\overline{\lim}\,E_n=\underline{\lim}\,E_n=E$, then we define $E=\lim\ E_n$.

Let, as before, λ be an isolated eigenvalue of L, Γ a closed Jordan curve in $\rho(L)$ isolating λ, Δ be the domain enclosed by Γ and $\sigma(L)\cap\Delta=\{\lambda\}$.

Definition

The spectrum of L_n in Δ *converges to* λ *if*

(2.12) $$\lim_{n\to\infty}\left(\sigma(L_n)\cap\Delta\right)=\{\lambda\}.$$

We note that condition (2.12) demonstrates the continuity of $\sigma(L)$ when L is perturbed by $L-L_n$. i.e. $L_n=L-(L-L_n)$. This is equivalent to the union of the lower semicontinuity

$$\{\lambda\}\subset\ \underline{\lim}\left(\sigma(L_n)\cap\Delta\right)$$

and the upper semicontinuity

$$\{\lambda\}\supset\overline{\lim}\ \left(\sigma(L_n)\cap\Delta\right).$$

The lower semicontinuity is equivalent to the condition that given $\varepsilon>0$, there is $\lambda_n\in\sigma(L_n)\cap\Delta$ such that $|\lambda-\lambda_n|<\varepsilon$. The upper semicontinuity is equivalent to the condition that no point other than λ could be the limit of any subsequence of points in $\sigma(L_n)\cap\Delta$.

We note have that condition (2.11) is not sufficient to imply convergence of the spectrum (2.12). To see examples that support this statement, the reader may refer to [3], page 230.

540C

2.2.5 Sufficient Conditions for the Convergence of $\sigma(L_n)\cap\Delta$

540C

Definition

Let $z \in \rho(L)$, we define stable convergence at z, $L_n - z \xrightarrow{s} L - z$ if the following conditions are satisfied

1. $L_n - z \xrightarrow{p} L - z$

2. $\exists N(z)$ *such that* $\forall n > N(z)$, $z \in \rho(L_n)$ *and* $\|R_n(z)\| \le M(z)$ *where* $M(z)$ *is a positive constant depending possibly on* z.

Lemma 2.2

The following are equivalent for all $z \in \rho(L)$

(i) $L_n - z \xrightarrow{s} L - z$, and

(ii) $R_n(z) \xrightarrow{p} R(z)$ and $L_n \xrightarrow{p} L$.

Proof

By the Second Resolvent Equation (2.9), we have

$$R_n(z) - R(z) = R_n(z)(L - L_n)R(z).$$

Since $R_n(z)$ and $R(z)$ are bounded, it is easy to see that (i) and (ii) are equivalent.

Lemma 2.3

If $L_n - z \xrightarrow{s} L - z$, *then the function* $z \to R_n(z)$ *is continuous in* $\rho(L)$. *This continuity is uniform in* n for n *large enough.*

Proof

If $z_0 \in \rho(L)$, then

(2.13) $$L_n - z = L_n - z_0 + z_0 - z = (L_n - z_0)\left[1 - (z - z_0)\, R_n(z_0)\right].$$

For z such that $|z - z_0|\,\|R_n(z_0)\| < 1$, we conclude from (2.13) that

(2.14) $$R_n(z) = \left(\sum_{i=0}^{\infty}\left[(z - z_o)\, R_n(z_0)\right]^i\right) R_n(z_0).$$

Since $L_n - z \xrightarrow{s} L - z$, we have $\|R_n(z_0)\| < M(z_0)$ for $n > N(z_0)$. Given ε, $0 < \varepsilon < 1$, then for $n > N(z_0)$ and any z such that $|z - z_0| < \dfrac{\varepsilon}{M(z_0)}$,

we may conclude from (2.14) that

$$\|R_n(z) - R_n(z_0)\| \le \|R_n(z_0)\| \sum_{i=1}^{\infty} \varepsilon^i \le M(z_0)\frac{\varepsilon}{1-\varepsilon}.$$

Hence the function $z \to R_n(z)$ is continuous for $z \in \rho(L)$, uniformly in n for n large enough.

540C

Lemma 2.4

If $L_n - z \xrightarrow{s} L - z$ for all $z \in K$, K is a compact subset of $\rho(L)$, then

$$\sup_{z \in K,\, n > N(K)} \|R_n(z)\| \leq M(K). \tag{2.15}$$

Proof

Since K is compact, it may be covered by a finite family of sets of diameter less than ε. Then for any $z \in K$, $\exists\, z_0$ such that $|z - z_0| < \varepsilon$ and hence

$$\|R_n(z)\| \leq \|R_n(z) - R_n(z_0)\| + \|R_n(z_0)\|.$$

Since K is compact, the numbers z_0 can be chosen to make a finite set. Following the outline of the proof for Lemma 2.3, we may conclude (2.15)

Lemma 2.5

Let $L_n - z \xrightarrow{s} L - z$ for all $z \in \Gamma$, where Γ is a closed Jordan curve isolating λ, where λ is an eigenvalue of L, then for $n > N(\Gamma)$,

$$\Gamma \subset \rho(L_n).$$

Proof

Since Γ is compact, the result follows directly from Lemma 2.4.

Lemma 2.6

Let $L_n - z \xrightarrow{s} L - z$ for all $z \in \Gamma$, then $P_n x \to Px$ for all $x \in X$.

540C

Proof

By the Second Resolvent Equation (2.9) we have

$$R_n(z)-R(z)=R_n(z)(L-L_n)R(z).$$

Integrating both sides on Γ, we get

$$(2.16)\qquad (P_n-P)x=-\frac{1}{2\pi i}\int_{\Gamma}R_n(z)(L-L_n)R(z)x\,dz.$$

Since $(L-L_n)R(z)$ is bounded, then for a fixed $x\in X$, the function $z\to(L-L_n)R(z)x$ is continuous for $z\in\rho(L)$. Therefore for any $x\in X$, and $n>N(\Gamma)$ we obtain from (2.16)

$$\|(P-P_n)x\|\le\frac{\text{meas}\,\Gamma}{2\pi}M(\Gamma)\max_{z\in\Gamma}\|(L-L_n)R(z)x\|.$$

Lemma 2.7

Let $L_n-z\xrightarrow{s}L-z$ *for all* $z\in\Gamma$, *then* $\dim P_nX\ge\dim PX$ *for* n *large enough.*

Proof

From Lemma 2.6 we conclude that $P_nx\to Px$ for all $x\in X$. Since $\dim PX=m$, let $\{x_i\}, i=1,\cdots,m$ be a basis for $M=PX$. Further, let $\{x_i^*\}, i=1,\cdots,m$ be the adjoint basis in M^* such that $(x_i,x_j^*)=\delta_{ij}, i,j=1,\cdots,m$. Since $P_nx\to Px$ $\forall x\in X$, we must have $(P_nx_i,x_j^*)\to\delta_{ij}$. Therefore for n large enough, the vectors $P_nx_i, i=1,\cdots,m$

are linearly independent and have $\dim P_n X \geq \dim PX$.

540C

We are now ready to state the main result on the convergence of the spectrum $\sigma(L_n) \cap \Delta$.

Theorem 2.7

Let $L_n - z \xrightarrow{s} L - z$ *for all* $z \in \Delta - \{\lambda\}$, *then*

(2.17) $$\lim_{n\to\infty}(\sigma(L_n) \cap \Delta) = \{\lambda\}.$$

Proof

We prove (2.17) by showing lower semicontinuity and upper semi-continuity. Let $\varepsilon > 0$. Let $\Delta_\varepsilon = \Delta - \{z : |z - \lambda| < \varepsilon\}$ and $\Gamma_\varepsilon = \{z : |z - \lambda| = \varepsilon\}$. Δ_ε is a compact subset of $\rho(L)$ and Γ_ε is a circle isolating λ and lies also in $\rho(L)$.

We define

$$P_\varepsilon = \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} R(z)\,dz$$

and

$$P_{n\varepsilon} = \frac{-1}{2\pi i} \int_{\Gamma\varepsilon} R_n(z)\,dz.$$

It is easy to see that $P_\varepsilon = P$. By Lemma 2.7 we have $\dim P_{n\varepsilon} X \geq \dim P_\varepsilon X = m$ for $n > N(\Gamma_\varepsilon)$. This means that $\sigma(L_n) \cap \Delta$ is not empty for n large enough and that for any $\varepsilon > 0, \exists$ a point $\lambda_n \in \sigma(L_n) \cap \Delta$ such that $|\lambda - \lambda_n| < \varepsilon$. This is the lower semicontinuity of $\sigma(L)$. Now, by Lemma 2.4, we have

$$\sup_{z \in \Delta_\varepsilon, n > N(\Delta_\varepsilon)} \|R_n(z)\| \leq M(\Delta_\varepsilon)$$

and therefore, $\Delta_\varepsilon \subset \rho(L_n)$ for $n > N(\Delta_\varepsilon)$. This means that no point other than λ can be the limit of any subsequence of points in $\sigma(L_n \cap \Delta)$. i.e.

$$\{\lambda\} \supset \overline{\lim}(\sigma(L_n) \cap \Delta).$$

Therefore,

$$\lim_{n \to \infty}(\sigma(L_n) \cap \Delta) = \{\lambda\}.$$

The condition $L_n - z \xrightarrow{s} L - z$ is a sufficient condition but not necessary for the convergence of the $\sigma(L_n) \cap \Delta$. Moreover, convergence of the eigenvalues as complex numbers is not sufficient for the convergence of the associated eigenvectors. To ensure convergence of the eigenvectors we need convergence of the eigenvalues and preservation of the algebraic multiplicities as we shall see below. The reader is referred to [3], pages 233-234, for examples that illustrate the above statements.

2.2.6 Convergence of the Eigenvalues and Preservation of the Multiplicities

In this subsection we establish a sufficient condition for the convergence of the eigenvalues and the associated eigenvectors.

Definition

Let λ be an isolated eigenvalue of L with finite algebraic multiplicity, we say that $\{L_n\}_N$ is a strongly stable approximation of L, $L_n - z \xrightarrow{ss} L - z$, in Δ if the following conditions are satisfied

1) $L_n - z \xrightarrow{s} L - z$ *in* $\Delta - \{\lambda\}$,

2) $\dim P_n X = \dim PX = m$ *for* n *large enough where* P *and* P_n *are the spectral projections associated with* $\{\lambda\}$ *and* $\sigma(L_n) \cap \Delta$ *respectively. Strongly stable convergence on* Γ *is defined similarly but condition 1) is satisfied for* $z \in \Gamma$.

Lemma 2.8

Let $L_n - z \xrightarrow{ss} L - z$ in Δ, then for n large enough $\sigma(L_n) \cap \Delta$ consists of m eigenvalues, counting their multiplicities.

Proof

Since $L_n - z \xrightarrow{s} L - z$ in $\Delta, \Gamma \in \rho(L_n)$ for n large enough by Lemma 5. Therefore, by Theorem 2.5, $\sigma(L_n) \cap \Delta$ is the spectrum of $P_n L_n P_n$ when restricted to $M_n = P_n X$. Since $\dim M_n = m$, this operator is of rank m and hence has m eigenvalues counting their multiplicities.

2.2.7 Gap Convergence of the Invariant Subspaces M_n and M and Convergence of the Eigenvectors

A natural measure of the accuracy of approximation of the eigenvectors is the *gap (or aperture)* of the subspaces M_n and M. We start by the following definition.

Definition

Let M and N be closed subspaces of X. Let

$$\delta(M,N) = \sup_{\substack{x \in M \\ \|x\|=1}} \operatorname{dist}(x,N).$$

The gap between the subspaces M and N, $\theta(M,N)$ is defined by

$$\theta(M,N) = \max\left(\delta(M,N), \delta(N,M)\right).$$

We note that $\theta(M,N) \leq 1$.

The following results will lead us to the conclusion that the strong stability in Δ of an approximation L_n will guarantee the gap convergence of the invariant subspaces M_n and M. We will also establish for any sequence of eigenvalues $\{\lambda_n\}_N$ that converges to λ, the convergence of a subsequence of the associated normalized eigenvectors.

Theorem 2.8

If $L_n - z \xrightarrow{ss} L - z$ *on* Γ, *then* $\theta(M_n, M) \to 0$ *as* $n \to \infty$.

Theorem 2.9

Let $L_n - z \xrightarrow{ss} L - z$ *in* Δ, *then for any sequence of eigenvalues* $\lambda_n \in M_n$ *converging to* $\lambda \in M$ *and for any sequence of associated normalized eigenvectors* $\{x_n\}_{N'}$, *there exists a subsequence* $\{x_n\}_{N_1 \subset N}$ *converging to an eigenvector* x *associated with* λ *and*

$$\dim Ker\,(L_n - \lambda_n) \leq \dim Ker\,(L - \lambda).$$

2.2.8 Theoretical Error Bounds

Let

$$\gamma_n = \|(L - L_n)P\|.$$

γ_n is well defined since $(L - L_n)P$ is bounded by the closed graph theorem. Moreover since P has finite rank $\gamma_n \to 0$ as $n \to \infty$.

Theorem 2.10

Let $L_n z \xrightarrow{ss} L - z$ *in* Δ, *then for* n *large enough we have* $\theta(M, M_n)$ *and* $\lambda - \hat{\lambda}_n$ *are at least of order* γ_n *where* $\hat{\lambda}_n$ *is the arithmetic mean of the* m *eigenvalues* λ_n *in* $\sigma(L_n) \cap \Delta$ *counting their multiplicities.*

540C

The following theorem provides a sufficient condition for strongly stable convergence when T and T_n are compact ([3], page 351].

Theorem 2.11

Let λ be an isolated eigenvalue of the compact operator L with finite algebraic multiplicity. Let L_n be a sequence of compact operators that converges uniformly to L, then $L_n - z \xrightarrow{ss} L - z$ in Δ, where Δ is as defined in 2.2.3.

2.3 IRREDUCIBLE MATRICES, PARTIAL ORDERING OF MATRICES AND NON-NEGATIVE MATRICES

This brief section includes results on irreducible matrices, partial ordering and non-negative matrices. These results will be needed later in Chapter 3 to establish error estimates of numerical solutions for a certain class of singular boundary value problems.

2.3.1 Irreducible Matrices

Definition

An $n \times n$ real or complex matrix $H = (h_{ij})$ is reducible if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix},$$

where H_{11} and H_{22} are square matrices. H is called irreducible if it is not reducible.

Lemma 2.9

A matrix H is reducible if and only if there is a nonempty subset $J \subset \{1,\cdots,n\}$ such that

$$h_{ij} = 0 \text{ for all } i \in J,\, j \notin J.$$

Lemma 2.10

A matrix H is irreducible if and only if for any two indices $1 \le i < j \le n$, there exists a sequence of nonzero elements of H of the form

$$h_{i,i_1}, h_{i_1,i_2}, \cdots, h_{i_m j}.$$

As an application of Lemma 2.10 we consider the matrix

$$\begin{bmatrix} h_{11} & h_{12} & & O \\ h_{21} & & \ddots & \\ & \ddots & \ddots & h_{n-1,n} \\ O & & h_{n,n-1} & h_{nn} \end{bmatrix}$$

where $|h_{ii}| > 0$, $|h_{i,i+1}| > 0$ and $|h_{i-1,i}| > 0$.

The matrix above is irreducible because given any $1 \le i < j \le n$, a sequence of nonzero elements of H may be provided as follows:

$$h_{i,i+1},\ h_{i+1,i+2},\ \cdots,\ h_{j-1,j}.$$

Definition

An $n \times n$ real matrix $H = (h_{ij})$ is diagonally dominant if

$$\sum_{j=1, i \neq j}^{n} |h_{ij}| \leq |h_{ii}|, \quad \forall i = 1, \cdots, n.$$

H is strictly diagonally dominant if the above inequality is strict for all $i = 1, \cdots, n$, and is irreducibly diagonally dominant if it is irreducible, diagonally dominant and the above inequality is strict for at least one value of $i = 1, \cdots, n$.

Theorem 2.12

If an $n \times n$ matrix is strictly or irreducibly diagonally dominant, then it is invertible.

2.3.2 Partial ordering of Matrices and Non-negative Matrices

Later on in this work, we will need to compare matrices entry by entry. We may do this by defining a *partial ordering*, on the space of $m \times n$ matrices as follows.

Let $A_{m \times n}$ and $B_{m \times n}$ be real matrices, then

$$A \leq B \text{ if } a_{ij} \leq b_{ij} \quad \forall i = 1, \cdots, m, \ \forall j = 1, \cdots, n.$$

The above ordering relation $\leq$ is reflexive, symmetric and transitive. Moreover, it satisfies the following two properties

$$\text{if } A \leq B, \text{ then } \alpha A \leq \alpha B \quad \forall \alpha \geq 0,$$

and

$$\text{if } A \leq B, \text{ then } A + C \leq B + C,$$

where $A, B,$ and C are any $m \times n$ matrices.

If $A \geq 0$, then it is called non-negative.

Definition

An $n \times n$ real matrix H is called an $M-$matrix if it is invertible, $H^{-1} \geq 0$ and $h_{ij} \leq 0$ for all $i, j = 1, \cdots, n, \quad i \neq j$.

Lemma 2.11

If H is an irreducibly diagonally dominant real $m \times n$ matrix such that $a_{ij} \leq 0, i \neq j$ and $a_{ii} > 0 \quad \forall i = 1, \cdots, n$, then it is an $M-$matrix.

Lemma 2.12

Let H be an $M-$matrix and A an $n \times n$ nonnegative diagonal matrix, then $H + A$ is an $M-$matrix and

$$(H + A)^{-1} \leq H^{-1}.$$

CHAPTER 3

A FINITE DIFFERENCE METHOD FOR APPROXIMATING THE SOLUTION

3.1 INTRODUCTION

In this chapter we state the specific assumptions on the boundary value problem (0.1) and introduce the finite difference scheme that will be used to approximate its solution. This chapter is organized as follows. In Section 3.1 we define the problem and set the conditions on its coefficient functions. Section 3.2 includes the preliminaries in which we relate the problem to the work done in [9] and prove that the spectrum of the differential operator involved lies in the negative part of the real axis. In Section 3.3 we prove the existence and uniqueness of the solution to our problem and that this solution is continuously differentiable on $[0,1]$. Section 3.4 provides the finite difference scheme that is used to approximate the solution to the problem. Convergence of the finite difference scheme and the rate of convergence is treated in Section 3.5.

Consider the self-adjoint singular boundary value problem

$$(3.1) \qquad \frac{1}{p(x)}(p(x)y')' - q(x)y = f(x) \quad x \in (0,1)$$

(3.2) $$\lim_{x\to 0^+} p(x)y' = 0$$

(3.3) $$y(1) = 0$$

(3.4) $$p(x) \geq 0, p(0) = 0$$

(3.5) $$p^{-1}(x) \in L^1_{loc}(0,1]$$

(3.6) $$q(x), f(x) \in C[0,1] \text{ and } |p'(x)| \leq M_1 < \infty$$

(3.7) $$q(x) \geq 0,$$

(3.8) $$\int_0^1 \left(\int_x^1 \frac{1}{p(\tau)} d\tau \right)^2 dx < \infty \text{ and } p(x) \text{ is}$$
increasing in a neighborhood of 0.

We will discuss a finite difference approximation to the solution of the above problem. In [4], Chawla considers the special case with $p(x) = x^\alpha$, $\alpha \geq 1$, however, $f(x)$ in the right hand side of (3.1) is non-linear. In this chapter, we extend the work in [4] by applying Chawla's finite difference scheme to a self-adjoint boundary value problem. We prove $O(h^2)$ convergence under less differentiability conditions on $q(x)$ and $f(x)$.

3.2 PRELIMINARIES

In this section we establish the self-adjointness of the problem (3.1) - (3.3) in $L^2(0,1)$.

Lemma 3.1

If $|p'(x)| \le M_1$, $M_1 > 0$ *and* $p(x) \ge 0$ $\forall x \in [0,1]$, *then*

(1) $p(x) > 0$ for $x \in (0,1]$.

(2) $\int_0^1 \frac{1}{p(x)} dx = \infty$.

(3) $\int_0^1 \int_x^1 \frac{1}{p(\tau)} d\tau \, dx < \infty$.

Proof

(1) Assume $p(x_0) = 0$ for some $x_0 \in (0,1)$, then

(3.9) $|p(x)| = |x - x_0| |p'(\xi)| \le |x - x_0| M_1$ (where $x_0 \le \xi \le 1$), therefore

(3.10) $$\int_{x_0}^1 \frac{1}{|p(x)|} dx \ge \frac{1}{M_1} \int_{x_0}^1 \frac{1}{|x - x_0|} dx = \infty.$$

This contradicts condition (3.5) above.

The case $x_0 = 1$ can be treated similarly.

(2) This is easily seen in view of (3.4) and (1) above.

(3) This is a direct application of the Cauchy-Schwartz inequality.

Let $L^2(0,1)$ denote the Hilbert space of all complex-valued measurable functions y which are square integrable on $(0,1)$. Inner product for $y, z \in L^2(0,1)$ is defined by $(y,z) = \int_0^1 y(x)\overline{z(x)} dx$. Let ℓ be the formally self-adjoint differential expression defined by $\ell(y) = (py')'$. Let $x = 0$ be a singular point for ℓ and $x = 1$ be a regular point. We also define the so called maximal and minimal operators L_M and L_0 by

(3.11) $$D(L_M) = \{y \in L^2(0,1) : \ell(y) \in L^2(0,1)\}$$

(3.12) $$L_M y = \ell(y)$$

(3.13) $$D(L_0) = \{y \in D(L_M) : y(1) = y'(1) = 0,\ [y,z]_{0^+} = 0 \text{ for all } z \in D(L_M)\}$$

(3.14) $$L_0 y = \ell(y)$$

where for $y, z \in D(L_M)$ we have

(3.15) $$[y,z]_t = p(t)y(t)\overline{z'(t)} - p(t)y'(t)\overline{z(t)}.$$

It is indicated in [9] that $D(L_0)$ is dense in $L^2(0,1)$ and that L_0 is a closed symmetric operator with $L_0^* = L_M$ where L_0^* denotes the adjoint of L_0. Any self-adjoint extension L of L_0 is such that $L_0 < L < L_M$. It is also shown in [9] that under our assumptions on $p(x)$ the operator L_0 has deficiency indices (2,2) and that the operator $L: L^2(0,1) \to L^2(0,1)$ defined by

(3.16) $$D(L) = \left\{y \in D(L_M) : \lim_{x \to 0^+} p(x)y'(x) = 0,\ y(1) = 0\right\}$$

(3.17) $$Ly = \ell(y) = (py')'$$

is a self-adjoint extension of L_0.

In the remainder of this chapter, c will denote a generic constant.

540C

Lemma 3.2

If $y \in D(L)$, then

(3.18) $$(Ly,y) \le -\frac{1}{c}\|y\|_2^2$$

and hence the spectrum of L lies in the negative part of the real axis.

Proof

(3.19) $$(Ly,y) = \int_0^1 (py')'\bar{y}dx = -\int_0^1 p|y'|^2 dx.$$

Moreover,

(3.20) $$y(x) = -\int_x^1 y'(t)dt = -\int_x^1 \frac{\sqrt{p}}{\sqrt{p}} y' dt.$$

So

(3.21) $$|y(x)|^2 \le \int_x^1 \frac{1}{p} dt \int_x^1 p|y'|^2 dt \le \int_x^1 \frac{1}{p} dt \int_0^1 p|y'|^2 dt,$$

and therefore,

$$\|y\|_2^2 = \int_0^1 |y(x)|^2 dx \le \left(\int_0^1\int_x^1 \frac{1}{p} dt \, \mathrm{dx}\right)\int_0^1 p|y'|^2 dt$$
$$\le -c(Ly,y)$$

where $c = \int_0^1\int_x^1 \frac{1}{p} dt\, dx$. Therefore

(3.22) $$(Ly,y) \le -\frac{1}{c}\|y\|_2^2.$$

540C

3.3 EXISTENCE, UNIQUENESS AND REGULARITY OF THE SOLUTION

In this section we establish existence and uniqueness of the solution to our problem and that this solution is continuously differentiable on [0,1].

Theorem 3.1

The equation

$$Ly - pqy = pf \tag{3.23}$$

where $q(x)$ and $p(x)$ satisfy conditions (3.7) and (3.8) respectively, has a unique solution which belongs to $C^1[0,1]$.

Proof

From (3.22), we have

$$(Ly,y) \le -\frac{1}{c}(y,y) \quad \forall y \in D(L).$$

Since

$$(pqy,y) \ge (y,y) \ \inf pq, \tag{3.24}$$

we have

$$\begin{aligned} ((pq-L)y,y) &\ge \left(\inf pq + \frac{1}{c}\right)(y,y) \\ &= c(y,y) \ \text{where} \ c > 0. \end{aligned} \tag{3.25}$$

540C

Therefore, the operator $(pq-L)$ is coercive (i.e. $((pq-L)y,y)\geq c\|y\|_2 \forall y\in D(L)$ and $c>0$) and hence onto (see [1], corollary on page 48). This proves existence of a solution. To show uniqueness, let

$$(L-pq)y_1 = pf \text{ and } (L-pq)y_2 = pf,\ \ y_1,y_2\in D(L),$$

then

$$(L-pq)(y_1-y_2)=0,$$

and

$$(pq-L)(y_1-y_2)=0.$$

But

$$((pq-L)(y_1-y_2),(y_1-y_2))\geq c\|y_1-y_2\|_2^2. \tag{3.26}$$

So $0\geq c\|y_1-y_2\|_2^2$ and hence $y_1=y_2$.

It remains to show that $y\in C^1[0,1]$. Let y be a solution of

$$(py')'-pqy=pf$$

then

$$y'=\frac{1}{p}\int_0^x pqy\,dt+\frac{1}{p}\int_0^x pf\,dt. \tag{3.27}$$

Therefore, $y\in C^1(0,1]$ and

$$|y'|\leq\frac{1}{p}\left|\int_0^x pqy\,dt\right|+\frac{1}{p}\left|\int_0^x pf\,dt\right|.$$

Now,

$$\lim_{x\to0^+}\frac{1}{p^2}\left|\int_0^x pqy\,\mathrm{dt}\right|^2 \le \lim_{x\to0^+}\frac{1}{p^2}\int_0^x p^2q^2\,dt\int_0^x |y|^2\,dt \tag{3.28}$$

$$\le c\lim_{x\to0^+}\frac{1}{p^2}\int_0^x p^2dt\|y\|_2^2$$

$$= 0 \text{ by assumption (3.8).} \tag{3.29}$$

A similar argument may be used to prove that

$$\lim_{x\to0^+}\frac{1}{p}\int_0^x pf\,dt = 0. \tag{3.30}$$

Therefore $\lim_{x\to0^+} y'(x) = 0$ and hence $y \in C^1[0,1]$.

3.4 THE FINITE DIFFERENCE SCHEME

Rewrite (3.23) in the form

$$(py')' = p(qy + f) \tag{3.31}$$

and integrating and applying conditions (3.2) and (3.3) we get

$$y = -\int_x^1 \frac{1}{p(\tau)}\int_0^\tau p(t)(q(t)y + f(t))dt\,d\tau. \tag{3.32}$$

540C

Interchanging the order of integration in (3.32) we get

$$(3.33)\qquad \begin{aligned} y = &-\int_0^x\left(\int_x^1 \frac{1}{p(\tau)}d\tau\right)p(t)(q(t)y+f(t))\,dt \\ &-\int_x^1\left(\int_t^1 \frac{1}{p(\tau)}d\tau\right)p(t)(q(t)y+f(t))\,dt. \end{aligned}$$

Written differently, (3.33) becomes

$$(3.34)\qquad y = -\int_0^1 K(x,t)p(t)(q(t)y+f(t))\,dt$$

where $K(x,t)$ is the kernel and is given by

$$(3.35)\qquad K(x,t) = \begin{cases} \displaystyle\int_x^1 \frac{1}{p(\tau)}d\tau & t \le x \\ \displaystyle\int_t^1 \frac{1}{p(\tau)}d\tau & t > x. \end{cases}$$

Now for $N \ge 2$ we consider a uniform mesh over the interval $[0,1]$. Let $x_k = kh,\ k = 0,\cdots,N,\ h = 1/N$. Let $y_k = y(x_k)$. For ease of notation, let $g(t) = q(t)y + f(t)$. Using (3.34) and (3.35) we obtain for $x = x_k$ and $x = x_{k+1},\ k \ge 1$

$$(3.36)\qquad \begin{aligned} y_{k+1} - y_k = &\int_{x_k}^{x_{k+1}} \frac{1}{p(\tau)}d\tau \int_0^{x_k} p(t)g(t)\,dt \\ &+\int_{x_k}^{x_{k+1}}\left(\int_t^{x_{k+1}} \frac{1}{p(\tau)}d\tau\right)p(t)g(t)\,\mathrm{dt}. \end{aligned}$$

540C

Similarly, for $k \geq 2$ we have

$$
(3.37) \qquad \begin{aligned} y_k - y_{k-1} &= \int_{x_{k-1}}^{x_k} \frac{1}{p(\tau)}\, d\tau \int_0^{x_k} p(t)g(t)\, dt \\ &- \int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} \frac{1}{p(\tau)}\, d\tau \right) p(t)g(t)\, dt. \end{aligned}
$$

In order to simplify notation, we let

$$
(3.38) \qquad \psi_k(t) = \int_t^{x_{k+1}} \frac{1}{p(\tau)}\, d\tau, \quad k \geq 1.
$$

Eliminating $\int_0^{x_k} p(t)g(t)\, dt$ from (3.36) and (3.37) we obtain for $k \geq 2$

$$
(3.39) \qquad \begin{aligned} \Delta y_k &\equiv -\frac{y_{k+1}}{\psi_k(x_k)} + \left(\frac{1}{\psi_k(x_k)} + \frac{1}{\psi_{k-1}(x_{k-1})} \right) y_k - \frac{y_{k-1}}{\psi_{k-1}(x_{k-1})} \\ &= -\int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)g(t)\, dt \end{aligned}
$$

where

$$
(3.40) \qquad U_k(t) = \begin{cases} 1 - \psi_{k-1}(t) / \psi_{k-1}(x_{k-1}) & \text{if } x_{k-1} \leq t \leq x_k \\ \psi_k(t) / \psi_k(x_k) & \text{if } x_k \leq t \leq x_{k+1}. \end{cases}
$$

At this point we will assume that the functions $q(x)$ and $f(x)$ are differentiable on $[0,1]$ and that $|f'(x)| < \infty$ and $|f'(x)| < \infty$ $\forall x \in [0,1]$. This condition will enable us to approximate integrals involving q and f by functional values without sacrificing $O(h^2)$ convergence. However, we will lift these additional restrictions on q and f later in this chapter and maintain $O(h^2)$ convergence.

Now, expanding $g(t)$ in Taylor series in the interval $[x_{k-1}, x_{k+1}]$ around x_k we may rewrite (3.39) as follows:

(3.41) $$\Delta y_k = -\int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)\{g_k + (t-x_k)g'(\xi_k)\}\, dt.$$
$$x_{k-1} \le \xi_k \le x_{k+1}.$$

Therefore,

(3.42) $$\Delta y_k = -g_k \int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)\, dt - \int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)(t-x_k)g'(\xi_k)\, dt,$$

or,

(3.43) $$\Delta y_k = -g_k A_k - B_k \text{ for } k \ge 2$$

where

$$A_k = \int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)\, dt \text{ and } B_k = \int_{x_{k-1}}^{x_{k+1}} p(t)U_k(t)(t-x_k)g'(\xi_k)\, dt.$$

We note that $A_k > 0$ for $k \ge 2$.

The discretization at $k=1$ may be obtained from (3.36) as follows:

(3.44) $$y_2 - y_1 = \psi_1(x_1) \int_0^{x_1} p(t)g(t)\, dt + \int_{x_1}^{x_2} \psi_1(t)p(t)g(t)\, dt,$$

or

(3.45) $$\frac{y_2}{\psi_1(x_1)} - \frac{y_1}{\psi_1(x_1)} = \int_0^{x_2} p(t)U_1(t)g(t)\, dt$$

where

$$U_1(t) = \begin{cases} 1 & \text{if } \ 0 \le t \le x_1 \\ \psi_1(t)/\,\psi_1(x_1) & \text{if } \ x_1 \le t \le x_2. \end{cases} \tag{3.46}$$

By expanding $g(t)$ in the interval $[0,x_2]$ about $t = x_1$, we may rewrite (3.45) as

$$\frac{y_2}{\psi_1(x_1)} - \frac{y_1}{\psi_1(x_1)} = \int_0^{x_2} p(t)U_1(t)\ \{g_1 + (t - x_1)g'(\xi_1)\}\ dt, \tag{3.47}$$

$$0 \le \xi_1 \le x_2$$

or

$$\frac{y_1}{\psi_1(x_1)} - \frac{y_2}{\psi_1(x_1)} = -g_1 A_1 - B_1 \tag{3.48}$$

where

$$A_1 = \int_0^{x_2} p(t)U_1(t)\ dt \ \text{ and } \ B_1 = \int_0^{x_2} p(t)U_1(t)(t - x_1)g'(\xi_1)\ dt. \tag{3.49}$$

We note that $A_1 > 0$. The discretization (3.43) for $k \ge 2$ and (3.48) for $k = 1$ may be expressed in matrix notation by the equation

$$HY + AY = B(h) + F \tag{3.50}$$

where $H=(h_{ij})$ is a tridiagonal matrix whose elements are

$$h_{11}=\frac{1}{\psi_1(x_1)},\quad h_{ii}=\frac{1}{\psi_{i-1}(x_{i-1})}+\frac{1}{\psi_i(x_i)},\quad i=2,\cdots,N-1$$
$$h_{i,i-1}=-\frac{1}{\psi_{i-1}(x_{i-1})},\quad i=2(1)N-1$$
$$h_{i,i+1}=-\frac{1}{\psi_i(x_i)},\quad i=1(1)N-2.$$

And

$$Y=(y_1,\cdots,y_{N-1})^T$$

$A=(a_{ij})$ denotes the diagonal matrix with

$$a_{ii}=A_iq_i,\quad i=1,\cdots,N-1$$
$$B(h)=(-B_1,\cdots,-B_{N-1})^T$$
$$F=(-A_1f_1,\cdots,-A_{N-1}f_{N-1})^T.$$

The method that we consider here determines an approximation $\tilde{Y}$ for Y by solving the $(N-1)$ by $(N-1)$ linear system

(3.51) $$H\tilde{Y}+A\tilde{Y}=F.$$

To approximate y_0 we proceed as follows:

$$y'=\frac{1}{p(x)}\int_0^x p(t)(q(t)y+f(t))\,dt.$$

Integrating both sides from $x=0$ to $x=x_1$ we get

$$y_1 - y_0 = \int_0^{x_1}\int_0^{x}\frac{p(t)}{p(x)}g(t)\,dt\,dx = \int_0^{x_1}\int_t^{x_1}\frac{1}{p(x)}dxp(t)g(t)\,dt.$$

Expanding $g(t)$ in the interval $[0,x_1]$ about $t=x_1$, we get

$$y_1 - y_0 = \int_0^{x_1} p(t)\int_t^{x_1}\frac{1}{p(x)}dx\,dtg_1 + \int_0^{x_1} p(t)\int_t^{x_1}\frac{1}{p(x)}dx(t-x_1)g'(\xi_0)\,dt$$

where $0 \le \xi_0 \le x_1$.

Since $y \in C^1[0,1]$ by Theorem 3.1 and using condition (3.6), there exists $M_2 > 0$ such that $g'(x) \le M_2$ for $x \in [0,1]$. By applying Taylor's series and Holder's inequality to the second term in the above equation we get

$$\begin{aligned}\left|\int_0^{x_1} p(t)\int_t^{x_1}\frac{1}{p(x)}dx(t-x_1)g'(\xi_0)dt\right| &\le \int_0^{x_1} tp'(\tau_0)\int_t^{x_1}\frac{1}{p(x)}dx(t-x_1)dtM_2,\ \ 0\le\tau_0\le x_1\\ &\le h^2\int_0^{x_1}\int_t^{x_1}\frac{1}{p(x)}dxdtM_2 \sup_{t\in[0,1]} p'(t)\\ &\le h^2c_1M_2M_1\end{aligned}$$

where $c_1 = \int_0^{x_1}\int_t^{x_1}\frac{1}{p(x)}dxdt$ and $c_1 < \infty$ by (3.8). Therefore, the equation

$$\tilde{y}_0 = \tilde{y}_1 - \left\{\int_0^{x_1} p(t)\int_t^{x_1}\frac{1}{p(x)}dxdt\right\}g_1 \tag{3.52}$$

may be used to compute an $O(h^2)$ approximation $\tilde{y}_0$ for y_0.

540C

3.5 CONVERGENCE OF THE FINITE DIFFERENCE SCHEME AND RATE OF CONVERGENCE

Theorem 3.2

The finite difference scheme represented by (3.51) is convergent of $O(h^2)$.

Proof

Let $E=(e_1,\cdots,e_{N-1})^T=Y-\tilde{Y}$. From (3.50) and (3.51) we get the error equation

(3.53) $$(H+A)E=B(h).$$

It can be easily checked that H is irreducible and irreducibly diagonally dominant. Moreover, since the diagonal elements of H are positive and the off diagonal elements are negative, H is an M-matrix by Lemma 2.11, i.e., H is invertible and $H^{-1}\geq 0$. Now, since A is a nonnegative diagonal matrix and H is an M-matrix, then $H+A$ is also an M-matrix and $(H+A)^{-1}\leq H^{-1}$ by Lemma 2.12. Now from (3.53) we have

(3.54) $$\|E\|\leq\left\|H^{-1}|B(h)|\right\|$$

in the uniform norm, where

$$|B(h)|=\left(|-B_1|,\cdots,|-B_{N-1}|\right)^T.$$

Since H is symmetric and tridiagonal, it can be checked that $H^{-1}=\left(h_{ij}^{-1}\right)$ is given by

$$h_{ij}^{-1} = \begin{cases} \int_{x_j}^{1} \frac{1}{p(\tau)} d\tau, & i \le j \\ \int_{x_i}^{1} \frac{1}{p(\tau)} d\tau, & i \ge j. \end{cases} \tag{3.55}$$

Now, we obtain bounds for the local truncation error.

For, $k \geq 2$, we have

$$B_k = \int_{x_{k-1}}^{x_{k+1}} p(t) U_k(t)(t - x_k) g'(\xi_k) dt,$$

and

$$|B_k| \le |D_k| M_2$$

where

$$\begin{aligned} D_k &= \int_{x_{k-1}}^{x_{k+1}} p(t) U_k(t)(t - x_k) dt \\ &= \int_{x_{k-1}}^{x_k} p(t) \left[1 - \frac{\psi_{k-1}(t)}{\psi_{k-1}(x_{k-1})} \right] (t - x_k) dt \\ &\quad + \int_{x_k}^{x_{k+1}} p(t) \frac{\psi_k(t)}{\psi_k(x_k)} (t - x_k) dt. \end{aligned} \tag{3.56}$$

Using integration by parts in (3.56) we obtain

$$
\begin{aligned}
D_k &= -\int_{x_{k-1}}^{x_k} \frac{d}{dt}\left\{p(t)\left[1-\frac{\psi_{k-1}(t)}{\psi_{k-1}(x_{k-1})}\right]\right\}\frac{(t-x_k)^2}{2}dt \\
&\quad -\int_{x_k}^{x_{k+1}} \frac{d}{dt}\left\{p(t)\frac{\psi_k(t)}{\psi_k(x_k)}\right\}\frac{(t-x_k)^2}{2}dt \\
&= -\int_{x_{k-1}}^{x_k} p'(t)\left[1-\frac{\psi_{k-1}(t)}{\psi_{k-1}(x_{k-1})}\right]\frac{(t-x_k)^2}{2}dt \\
&\quad -\int_{x_{k-1}}^{x_k} \frac{1}{\psi_{k-1}(x_{k-1})}\frac{(t-x_k)^2}{2}dt \\
&\quad -\int_{x_k}^{x_{k+1}} p'(t)\left[\frac{\psi_k(t)}{\psi_k(x_k)}\right]\frac{(t-x_k)^2}{2}dt + \int_{x_k}^{x_{k+1}} \frac{1}{\psi_k(x_k)}\frac{(t-x_k)^2}{2}dt.
\end{aligned} \tag{3.57}
$$

Therefore,

$$
\begin{aligned}
|D_k| \le &\left| -\int_{x_{k-1}}^{x_k} p'(t)\left[1-\frac{\psi_{k-1}(t)}{\psi_{k-1}(x_{k-1})}\right]\frac{(t-x_k)^2}{2}dt \right. \\
&\left. -\int_{x_k}^{x_{k+1}} p'(t)\left[\frac{\psi_k(t)}{\psi_k(x_k)}\right]\frac{(t-x_k)^2}{2}dt \right| \\
&+\left|\int_{x_k}^{x_{k+1}} \frac{1}{\psi_k(x_k)}\frac{(t-x_k)^2}{2}dt - \int_{x_{k-1}}^{x_k} \frac{1}{\psi_{k-1}(x_{k-1})}\frac{(t-x_k)^2}{2}dt\right|.
\end{aligned} \tag{3.58}
$$

The first term in (3.58) is bounded by $\frac{h^3}{6}M_1$ because

$$0 \le \frac{\psi_k(t)}{\psi_k(x_k)} \le 1 \ \ \forall k \ge 2 \ \text{ and } \ |p'(t)| \le M_1.$$

The second term is equal to

$$\begin{aligned}
\left|\frac{h^3}{6}\left(\frac{1}{\psi_k(x_k)}-\frac{1}{\psi_{k-1}(x_{k-1})}\right)\right| &= \left|\frac{h^3}{6}\left(\frac{\psi_{k-1}(x_{k-1})-\psi_k(x_k)}{\psi_k(x_k)\psi_{k-1}(x_{k-1})}\right)\right| \\
&= \left|\frac{h^3}{6}\left(\frac{\frac{1}{p(\xi_1)}h-\frac{1}{p(\xi_2)}h}{\frac{1}{p(\xi_2)}h\frac{1}{p(\xi_1)}h}\right)\right|, \quad x_{k-1}\le \xi_1,\xi_2\le x_{k+1} \\
&= \left|\frac{h^2}{6}(p(\xi_2)-p(\xi_1))\right| \\
&\le \left|\frac{h^2}{6}(\xi_2-\xi_1)M_1\right| \\
&\le \frac{1}{3}h^3M_1.
\end{aligned}$$

Therefore, for $k\ge 2$, we have

$$|B_k|\le \frac{1}{2}h^3M_1M_2. \tag{3.59}$$

Now, for $k=1$ we have from (3.49)

$$B_1 = \int_0^{x_2} p(t)U_1(t)(t-x_1)g'(\xi_1)dt$$

and

$$|B_1|\le |D_1|M_2$$

where

$$\text{(3.60)}\quad \begin{aligned} D_1 &= \int_0^{x_2} p(t)U_1(t)(t-x_1)dt \\ &= \int_0^{x_1} p(t)(t-x_1)dt + \int_{x_1}^{x_2} p(t)\frac{\psi_1(t)}{\psi_1(x_1)}(t-x_1)dt. \end{aligned}$$

Using integration by parts in (3.60) and the fact that $p(0)=0$, we obtain

$$\begin{aligned} D_1 = &-\int_0^{x_1}\frac{1}{2}p'(t)(t-x_1)^2dt - \int_{x_1}^{x_2}\frac{1}{2}p'(t)\frac{\psi_1(t)}{\psi_1(x_1)}(t-x_1)^2dt \\ &+\int_{x_1}^{x_2}\frac{1}{2}\frac{1}{\psi_1(x_1)}(t-x_1)^2dt \end{aligned}$$

and we have

$$\left|\int_0^{x_1}\frac{1}{2}p'(t)(t-x_1)^2dt\right| \le \frac{1}{6}M_1h^3$$

$$\left|\int_{x_1}^{x_2}\frac{1}{2}p'(t)\frac{\psi_1(t)}{\psi_1(x_1)}(t-x_1)^2dt\right| \le \frac{1}{6}M_1h^3 \quad \left(\text{note that } \frac{\psi_1(t)}{\psi_1(x_1)} \le 1, x_1 \le t \le x_2\right)$$

$$\begin{aligned} \int_{x_1}^{x_2}\frac{1}{2}\frac{1}{\psi_1(x_1)}(t-x_1)^2dt &\le \frac{1}{6}\frac{1}{\int_{x_1}^{x_2}\frac{1}{p(\tau)}d\tau}h^3 \\ &= \frac{1}{6}\frac{p(\alpha_1)}{h}h^3, \quad x_1 \le \alpha_1 \le x_2 \\ &\le \frac{1}{6}\frac{hp'(\beta_1)}{h}h^3, \quad 0 \le \beta_1 \le \alpha_1 \\ &\le \frac{1}{6}h^3M_1. \end{aligned}$$

We then conclude

$$|D_1(h)| \le \frac{1}{2}h^3M_1$$

and

(3.61) $$|B_1(h)| \le \frac{1}{2} h^3 M_1 M_2.$$

Therefore,

(3.62) $$|B_k(h)| \le \frac{1}{2} M_1 M_2 h^3 \text{ for all } k = 1(1)N-1.$$

Now, from (3.55) we see that the first element in each column of the matrix H is the greatest. Therefore, using (3.54), (3.55) and (3.62) we obtain

$$\begin{aligned}
\|E\|_\infty &\le \sum_{j=1}^{N-1} \int_{x_j}^{1} \frac{1}{p(\tau)} d\tau \left(\frac{1}{2} M_1 M_2 h^3\right) \\
&= \frac{1}{2} M_1 M_2 h^2 \sum_{j=1}^{N-1} h \int_{x_j}^{1} \frac{1}{p(\tau)} d\tau \\
&\le \frac{1}{2} M_1 M_2 h^2 \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} \int_{x_j}^{1} \frac{1}{p(\tau)} d\tau\, dt \\
&\le \frac{1}{2} M_1 M_2 h^2 \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} \int_{t}^{1} \frac{1}{p(\tau)} d\tau\, dt \\
&\le \frac{1}{2} M_1 M_2 h^2 \int_0^1 \int_t^1 \frac{1}{p(\tau)} d\tau\, dt \\
&\le \frac{1}{2} M_1 M_2 c h^2
\end{aligned}$$

where $c = \int_0^1 \int_t^1 \frac{1}{p(\tau)} d\tau\, dt$. This proves that the above finite difference method is $O(h^2)$ - convergent.

The finite difference scheme developed in Section 3.4 could also be applied to problems (3.1) - (3.3) if the functions $q(x)$ and $f(x)$ belong to $C[0,1]$. To see this we expand $y(t)$ in Equation (3.39) in Taylor series in the interval $[x_{k-1}, x_{k+1}]$ around x_k to obtain

$$\Delta y_k = -A_k' y_k - B_k' - C_k \quad \text{for} \quad k \geq 2 \tag{3.43$'$}$$

where

$$A_k' = \int_{x_{k-1}}^{x_{k+1}} p(t) U_k(t) q(t) dt,$$
$$B_k' = \int_{x_{k-1}}^{x_{k+1}} p(t) U_k(t) q(t)(t - x_k)\, y'(\xi_k) dt, \ x_{k-1} \leq \xi_k \leq x_{k+1},$$
$$C_k = \int_{x_{k-1}}^{x_{k+1}} p(t) U_k(t) f(t) dt,$$

and $U_k(t)$ is given by (3.40).

Similarly, the discretization for $k = 1$ can be shown to be given by

$$\frac{y_1}{\psi_1(x_1)} - \frac{y_2}{\psi_1(x_1)} = -A_1' y_1 - B_1' - C_1 \tag{3.48$'$}$$

where

$$A_1' = \int_0^{x_2} p(t) U_1(t) q(t) dt$$
$$B_1' = \int_0^{x_2} p(t) U_1(t) q(t)(t - x_1)\, y'(\xi_1) dt, \quad 0 \leq \xi_1 \leq x_2$$
$$C_1 = \int_0^{x_2} p(t) U_1(t) f(t) dt.$$

540C

In matrix notation, (3.43)′ and (3.48)′ be expressed as

$$HY + A'Y = B'(h) + C$$

where H and Y are the same as in (3.50) and

$A' = (a'_{ij})$ denotes the diagonal matrix with
$a'_{ii} = A'_i, \quad i = 1, \cdots, N-1,$
$B'(h) = (-B'_1, \cdots, -B'_{N-1})^T$
$C = (-C_1, \cdots, -C_{N-1})^T.$

As in (3.51), the method we consider determines an approximation $\tilde{Y}$ for Y by solving the $(N-1)$ by $(N-1)$ linear system

$$\text{(3.51)}' \qquad H\tilde{Y} + A\tilde{Y} = C.$$

To approximate y_0 we proceed in a fashion similar to what we did in Section 3.4 to obtain

$$\text{(3.52)}' \quad \tilde{y}_0 = \tilde{y}_1 - \int_0^{x_1} p(t)q(t)\int_t^{x_1} \frac{1}{p(x)}dx\,dt\;\tilde{y}_1 + \int_0^{x_1} p(t)f(t)\int_t^{x_1} \frac{1}{p(x)}dx\,dt.$$

Following the outline of the proof of Theorem 3.2 we can show that the difference scheme represented by (3.51)′ is convergent of $O(h^2)$ in the uniform norm.

540C

CHAPTER 4

NUMERICAL APPROXIMATION OF EIGENVALUES AND EIGENVECTORS FOR SINGULAR BOUNDARY VALUE PROBLEMS

4.1. INTRODUCTION

In Chapter 3 we discussed approximations for the solution of (3.1) - (3.3) from a numerical analysis point of view. There was no need to formally define discrete operators and discuss their convergence. In this chapter, however, and in order to make use of the existing theory on convergence of eigenelements we approach the problem from an operator theory point of view.

In this chapter we study the approximation of the eigenpairs of the singular two point boundary value problem

$$(4.1) \qquad \frac{1}{p(x)}\left(p(x)y'(x)\right)' - q(x)y(x) = \lambda y(x), \quad x \in (0,1),$$

with boundary conditions (3.2) and (3.3).

Under certain conditions set on the coefficient functions $p(x)$ and $q(x)$, (4.1) induces differential operators in the spaces $L^2(0,1)$ and $C[0,1]$ which

540C

have discrete eigenvalues. In this chapter we present an analysis of such operators showing compactness of their inverses and therefore discreteness of their spectrum. The finite difference method introduced in Chapter 3 will be used to approximate the eigenpairs of the problem. It will be shown that the method is $O(h^2)$ under minimal assumptions on the smoothness of the coefficient functions. The error analysis will be carried out in both the energy norm and the uniform norm.

Eigenvalue problems of this type were considered by Nassif [17] and Mills [14] where the analysis was done in the energy norm for general operators and then the eigenvalue problem was given as a special case. This work is more general in the sense that the analysis is done for a class of singular boundary value problems rather than a specific problem.

The content of this chapter is as follows. In Section 2 we state the problem and the assumptions on the coefficient functions. The operators involved are then defined and the discreteness of the spectrum is proved. In section 3 we study the convergence of the approximate operators arising from the difference scheme to the exact operators. Finally the convergence of the eigenpairs is studied in section 4.

4.2 THE EIGENVALUE PROBLEM

We consider here the eigenvalue problem

540C

$$\text{(4.2)} \qquad \frac{1}{p(x)}\left(p(x)y'(x)\right)' - q(x)y(x) = \lambda y(x) \quad x \in (0,1),$$

with boundary conditions

$$\text{(4.3)} \qquad \lim_{x\to 0^+}\left(p(x)y'\right) = 0,$$

$$\text{(4.4)} \qquad y(1) = 0,$$

where we make the following assumptions:

(A) $p(0)=0$, $p^{-1}(x) \in L^1_{loc}(0,1]$, $p(x) \ge 0$ and $\sup_{x\in[0,1]}|p'(x)| = M < \infty$.

(B) $q(x) \ge 0$ and $q(x) \in C[0,1]$.

(C) $N = \int_0^1\left(\int_x^1 \frac{1}{p(\tau)}d\tau\right)^2 dx < \infty$, p is increasing in a neighborhood $[0,\delta]$ of 0 and $\frac{p'(x)}{p(x)} = 0\left(\frac{1}{x}\right)$.

We will show in this section that, under assumptions (A) - (C), problem (4.2) - (4.4) has a discrete set of eigenvalues. It should be noted that, from the theory of differential equations, each eigenvalue can have at most two distinct eigenfunctions. To carry out the analysis we introduce the following operators. The operator L defined by

$$\text{(4.5)} \qquad \begin{gathered} D(L) = \left\{ y \in L^2(0,1) : Ly \in L^2(0,1),\ \lim_{x\to 0^+} p(x)y' = y(1) = 0 \right\}, \\ Ly = (py')'. \end{gathered}$$

540C

The operator K is defined by

$$D(K)=D(L) \tag{4.6}$$
$$Ky=(py')'-pqy.$$

Then (4.2) can be written as

$$\frac{1}{p}Ky=\lambda y,$$

i.e. problem (4.2) - (4.4) reduces to one of investigating the spectrum of the operator $\frac{1}{p}K$.

Before we proceed any further, we need the following Lemma.

Lemma 4.1

Under assumption (A), $p(x)$ is bounded away from zero on any compact subinterval $[\delta,1]$ of $(0,1]$, i.e. $\exists c>0$ such that $p(x)\geq c \ \forall x\in[\delta,1]$.

Proof

By Lemma 3.1, we have $p(x)>0$ for $x\in(0,1]$. Now assume that $p(x)$ is not bounded from below on some interval $[\delta,1]$, $\delta>0$. Then $\exists$ a sequence $\{x_i\}_N\subset[\delta,1]$ such that $p(x_i)\to 0$ as $i\to\infty$. But then there exists a subsequence $\{x_i\}_{N_1}$ such that $x_i\to x_0\in[\delta,1]$. Thus $p(x_0)=0$ which is not possible by Lemma 3.1.

Theorem 4.1

Under assumptions (A) - (C), the operator

540C

$$K : L^2(0,1) \to L^2(0,1)$$

is self adjoint and has a compact inverse.

Proof

In [9] it was shown by investigating the deficiency indices of the so called minimal operator for L that L is self adjoint with compact resolvent. Since K is obtained from L by addition of a bounded self adjoint operator; namely multiplication by $-pq$; which does not affect the deficiency indices, it follows from Theorem 2.3 that K is self adjoint with compact resolvent. It thus remains to show that 0 is in the resolvent set of K. For this it suffices to show that 0 is in the domain of regularity of K since self adjoint operators do not have residual spectrum. Note first that for $y \in D(L)$ we have $py' \in AC[0,1]$, the space of absolutely continuous functions on $[0,1]$. Then by Lemma 4.1 we deduce that y' is continuous on any compact subinterval of $[0,1]$. Thus for any $x \in (0,1]$ we have

$$y(x) = \int_x^1 y' dt$$

and, recalling condition (C),

$$\begin{aligned}(y,y) &= \int_0^1 y^2 dx = \int_0^1 \left(\int_x^1 y' dt \right)^2 dx \le \int_0^1 \left(\int_x^1 \frac{1}{p} dt \right) \int_x^1 p y'^2 dt \, dx \\ &\le \int_0^1 \left(\int_x^1 \frac{1}{p} dt \right) dx \int_0^1 p y'^2 dt \le c(py', y') \\ &= -c(Ly,y) = -c(Ky,y) - c(pqy,y) \le -c(Ky,y).\end{aligned}$$

540C

Therefore, $|(Ky,y)| \geq \frac{1}{c}\|y\|_2^2$. By the Cauchy-Schwartz inequality we have $\|Ky\|_2 \geq \frac{1}{c}\|y\|_2$.

Corollary 4.1

The operator $\frac{1}{p}K$ has a discrete spectrum.

Proof

Since multiplication by p is a continuous operator, and K^{-1} is compact, then $K^{-1}p$ is compact and hence has a discrete spectrum and so does $\frac{1}{p}K$.

We turn next to the study of the operator $\frac{1}{p}K$ on the space $C[0,1]$. In this case we restrict the domain of the operator to those functions in $C[0,1]$ which are mapped into $C[0,1]$. We can state the following theorem.

Theorem 4.2

The operator $K^{-1}p{:}C[0,1] \to C[0,1]$ is compact. Consequently $\frac{1}{p}K$ has a discrete spectrum.

Proof

Consider first the operator $\frac{1}{p}L$. We can write $L^{-1}p$ as an integral operator with kernel

$$K(x,t)=\begin{cases} p(t)\int_x^1 \frac{1}{p(\tau)}d\tau & t\le x \\ p(t)\int_t^1 \frac{1}{p(\tau)}d\tau & t>x. \end{cases}$$

We will show that K is uniformly bounded and that $\int_0^1 |K_x(x,t)|dt$ is uniformly bounded in x. It then follows from Arzela-Ascoli's theorem that $L^{-1}p$ is compact.

Now $K(x,t)\le p(t)\int_t^1 \frac{1}{p(\tau)}d\tau = p(t)\left\{\int_t^\delta \frac{1}{p(\tau)}d\tau + \int_\delta^1 \frac{1}{p(\tau)}d\tau\right\}$. With the second part of the last parenthesis bounded by Lemma 4.1. As for the first part we have

$$p(t)\int_t^\delta \frac{1}{p(\tau)}d\tau \le p(t)\frac{1}{p(t)}\int_t^\delta d\tau = (\delta - t).$$

Therefore, the right hand side of the above inequality is bounded on $[0,\delta]$. Thus $K(x,t)$ is uniformly bounded on $[0,1]$. Also since

$$\int_0^1 |K_x(x,t)|dt = \int_0^x \frac{p(t)}{p(x)}dt = \frac{\int_0^\delta p(t)dt + \int_\delta^x p(t)dt}{p(x)},$$

a similar argument shows that $\int_0^1 |K_x(x,t)|dt$ is uniformly bounded in x. Thus $L^{-1}p$ is compact. Now the operator $K^{-1}p = \left(1-L^{-1}pq\right)^{-1}L^{-1}p$ will be compact once we show that $\left(1-L^{-1}pq\right)^{-1}$ is bounded. Since the multiplication by q is a continuous operator (see assumption (B)), the operator $L^{-1}pq$ is compact. Hence it suffices to show that 1 is not an eigenvalue for $L^{-1}pq$. So let $L^{-1}pqy = y$, then $(L-pq)y = 0$. Therefore,

$$0 = ((L-pq)y, y) = -\int_0^1 py'^2 dx - \int_0^1 pqy^2 dx$$

and thus $py'^2 = 0$. Thus $y' \equiv 0$ (see Lemma 4.1). Since $y \in D(L)$ we must also have $y(1) = 0$. Therefore $y \equiv 0$. This completes the proof of the theorem.

The following Lemma provides some "reverse" inequalities, i.e. bounds on the norms of y', y'' for solutions of the problem $\frac{1}{p}Ky = f$ in terms of the data f of the problem. These estimates will be needed later in this chapter.

Lemma 4.2

Let $y \in D(K)$ solve $\frac{1}{p}Ky = f$, where $f \in C[0,1]$, then

$$\|y^{(m)}\|_s \le c\|f\|_s, \quad m = 0,1,2; \quad s = 2, \infty.$$

Proof

For $m = 0$, the result follows from theorems 4.1 and 4.2 since $K^{-1}p$ is compact in both the energy and the uniform norms.

For $m = 1$, we write

(4.7) $$y'(x) = \frac{1}{p(x)}\int_0^x p(t)q(t)y(t)dt + \frac{1}{p(x)}\int_0^x p(t)f(t)dt$$

then

$$
(4.8)\qquad \begin{aligned}
|y'(x)| &\le \frac{1}{p(x)}\int_0^x p(t)dt\left(\|q\|_\infty \|y\|_\infty + \|f\|_\infty\right) \\
&\le c\frac{1}{p(x)}\int_0^x p(t)dt\|f\|_\infty \quad \text{(by using the first part of this proof).}
\end{aligned}
$$

Since $p(x)$ is increasing in the neighborhood of zero, then using Lemma 4.1 we can show that $\frac{1}{p}\int_0^x pdt \to 0$ as $x \to 0^+$ and that it is bounded on $[0,1]$. Hence we get

$$\|y'\|_\infty \le c\|f\|_\infty .$$

Returning to (4.7) and using the Cauchy-Schwartz inequality we have the following estimates

$$
\begin{aligned}
|y'(x)|^2 &\le c\left\{\left(\frac{1}{p}\int_0^x pqy\,dt\right)^2 + \left(\frac{1}{p}\int_0^x pf\,dt\right)^2\right\} \\
&\le c\left(\frac{1}{p^2}\int_0^x p^2 dt\right)\left(\|y\|_2^2 + \|f\|_2^2\right) \\
&\le c\left(\frac{1}{p^2}\int_0^x p^2 dt\right)\|f\|_2^2 .
\end{aligned}
$$

We can again show that $\frac{1}{p^2}\int_0^x p^2 dt$ is a bounded function on $[0,1]$. Hence we get

$$\|y'\|_2 \le c\|f\|_2 .$$

For $m=2$, write

$$y'' = f + qy - \frac{p'}{p}y'.$$

Then

$$\|y''\|_s \le c\left(\|f\|_s + \|y\|_s + \left\|\frac{p'}{p}y'\right\|_s\right) \le c\left(\|f\|_\infty + \left\|\frac{p'}{p}y'\right\|_\infty\right), \quad s = 2, \infty.$$

Hence we need to estimate $\left\|\frac{p'}{p}y'\right\|_s$ in terms of $\|f\|_s$.

For $s=\infty$, we have

$$\frac{p'}{p}y' = \frac{p'}{p^2}\int_0^x pqy\,dt + \frac{p'}{p^2}\int_0^x pf\,dt.$$

Then

$$\left|\frac{p'}{p}y'(x)\right| \le c\left(\frac{|p'|}{p^2}\int_0^x p\,dt\right)\left(\|y\|_\infty + \|f\|_\infty\right).$$

Using condition (C) and Lemma 4.1 we can show that $\frac{|p'|}{p^2}\int_0^x pdt$ is bounded in $[0,1]$.

Thus

$$\left\|\frac{p'}{p}y'\right\|_\infty \le c\|f\|_\infty,$$

540C

and therefore,

$$\|y''\|_\infty \le c\|f\|_\infty .$$

For $s=2$, write

$$\frac{p'}{p}y' = \frac{p'}{p^2}\int_0^x pg\,dt$$

where $g = f + qy$.

Then, for δ small enough,

$$\left\|\frac{p'}{p^2}\int_0^x pq\,dt\right\|_2^2 = \int_0^1\left(\frac{p'}{p^2}\right)^2\left(\int_0^x pg\,dt\right)^2 dx$$
$$= \int_0^\delta\left(\frac{p'}{p^2}\right)^2\left(\int_0^x pg\,dt\right)^2 dx + \int_\delta^1\left(\frac{p'}{p^2}\right)^2\left(\int_0^x pg\,dt\right)^2 dx.$$

Since $p(x)$ is bounded away from zero on $[\delta,1]$ and $p'(x)$ is bounded, the second summand is bounded by $c\|g\|_2^2$. Using assumption (C) we estimate the first summand as follows

$$\int_0^\delta\left(\frac{p'}{p^2}\right)^2\left(\int_0^x pg\,dt\right)^2 dx \le \int_0^\delta\left(\frac{p'}{p^2}\right)^2\left(\int_0^x p|g|dt\right)^2 dx$$
$$\le c\int_0^\delta\left(\frac{1}{x}\int_0^x|g|dt\right)^2 dx.$$

Recalling Hardy's Lemma we have

$$\int_0^\delta\left(\frac{1}{x}\int_0^x|g|dt\right)^2 dx \le \left\|\frac{1}{x}\int_0^x|g|dt\right\|_2^2 \le 2\|g\|_2^2.$$

We have thus established that

$$\left\|\frac{p'}{p}y'\right\|_2^2 \le c\|f+qy\|_2^2.$$

This implies that

$$\|y''\|_2 \le c\|f\|_2.$$

4.3 APPROXIMATION OF THE OPERATOR $K^{-1}p$ AND ORDER OF CONVERGENCE

In this section we investigate the convergence properties of the difference scheme (3.51)′ and (3.52)′. We prove in Theorem 4.3 below the uniform convergence of the approximate operators to the operator $K^{-1}p$. This in turn will give us optimal order of convergence of the eigenvalues and eigenfunctions which will be studied in Section 4.4.

Since it follows from Theorem 4.1 that $D(K)\subset \dot{C}[0,1]\subset L^2(0,1)$ where $\dot{C}[0,1]$ is the space of continuous functions that vanish at $x=1$, we may start by defining the subspaces $H_h=\{\vartheta_h\in\dot{C}[0,1]\colon \vartheta_h$ is linear on $(x_{i-1},x_i), i=1,\cdots,N\}$ and the projections $I_h\colon\dot{C}[0,1]\to H_h$ which interpolate at the mesh points $x_0,\cdots,x_N$ by peicewise linear polynomials. Furthermore, we define the operators J_h on $L^2(0,1)\,(C[0,1])$ into H_h as follows. For a given $f\in L^2(0,1)\,(C[0,1])$ we assign the function $\vartheta_h\in H_h$ whose values at

540C

the mesh points are the components of the solution vector $\tilde{Y}$ of the matrix equation (3.51)′ for $k \geq 1$ and $= \tilde{y}_0$ given by (3.52)′ at $k=0$. (The solution exists since H is an M−matrix, i.e. H is invertible and $H^{-1} \geq 0$. Also $H+A$ is an M−matrix and $(H+A)^{-1} \leq H^{-1}$.) We now have the following theorem.

Theorem 4.3

The operators J_h converge uniformly to $K^{-1}p$ as $h \to 0$ in the energy norm and the uniform norm. The order of convergence is h^2.

Proof

Let $f \in L^2(0,1)\,(C[0,1])$. Let $y = K^{-1}pf$ and $y_h = J_h f$.

Then for $s \in \{2,\infty\}$, $\|K^{-1}pf - J_h f\|_s = \|y - y_h\|_s \leq \|y - I_h y\|_s + \|I_h y - y_h\|_s$.

Using the standard approximation properties of $L^2(0,1)$ and $C[0,1]$ spaces we get

$$\|y - I_h y\|_s \leq ch^2 \|y''\|_s .$$

Lemma 4.2 now yields

$$\|y - I_h y\|_s \leq ch^2 \|f\|_s . \tag{4.9}$$

The theorem will then be proved once we show a similar estimate for $\|I_h y - y_h\|_s$. Since H_h is isomorphic to $\Re^N$ and since all norms on $\Re^N$ are equivalent we can write

540C

(4.10) $$\|I_h y - y_h\|_{\infty} \le c\left(|y_0 - \tilde{y}_0| + \|Y - \tilde{Y}\|_{\infty}\right)$$

where $y_0 = y(0)$, $\tilde{y}_0$ is given by (3.52)′, $Y = (y_1, \cdots, y_N)^T$ and $\tilde{Y}$ is given by (3.51)′.

Now from Section 3.5 we have

(4.11) $$\|Y - \tilde{Y}\|_{\infty} \le ch^2 \|y''\|_{\infty}.$$

Also, using (3.52)′ we get

$$|y_0 - \tilde{y}_0| \le |y_1 - \tilde{y}_1| + \left|\int_0^{x_1} p(t)q(t)\psi_0(t)dt\right| |y_1 - \tilde{y}_1| + \left|\int_0^{x_1} p(t)q(t)(t - x_1)\psi_0(t)dt\right| \|y'\|_{\infty}.$$

From (4.11) we have

(4.12) $$|y_1 - \tilde{y}_1| \le ch^2 \|y''\|_{\infty}.$$

On the other hand

$$\left|\int_0^{x_1} p(t)q(t)\,\psi_0(t)dt\right| \le \|q\|_{\infty} \int_0^{x_1} p(t) \int_t^{x_1} \frac{1}{p(\tau)} d\tau dt \le c \text{ (by condition C).}$$

Finally we note that since $p(t)\int_t^{x_1} \frac{1}{p(\tau)} d\tau$ vanishes at 0 and is bounded on $[0,h]$ we have

$$\lim_{h \to 0^+} \left| \frac{1}{h^2} \int_0^{x_1} p(t)(t - x_1) \int_t^{x_1} \frac{1}{p(\tau)} d\tau dt \right|$$
$$\le \lim_{h \to 0^+} \frac{1}{h} \int_0^h p(t) \int_t^h \frac{1}{p(\tau)} d\tau dt \le c.$$

540C

This means that $\int_0^{x_1} p(t)(t-x_1)\ \psi_0(t)dt$ is $O(h^2)$ as $h \to 0$. This together with (4.12) yields

(4.13) $$|y_0 - \tilde{y}_0| \le ch^2 \|y'\|_\infty .$$

From (4.10), (4.11) and (4.13), we get

(4.14) $$\|I_h y - y_h\|_\infty \le ch^2 \|y'\|_\infty .$$

But from Lemma 4.2, we obtain

(4.15) $$\|y'\|_\infty \le c\|f\|_\infty$$

(4.14) and (4.15) establish the required estimate.

4.4 APPROXIMATION OF EIGENELEMENTS

We now consider the problem of approximating the eigenvalues and eigenfunctions of the problem (4.2) - (4.4), or, equivalently, approximating the eigenvalues and eigenfunctions of the operator $K^{-1}p$ by the scheme (3.51)′ and (3.52)′. By the analysis of Section 4.2, the operator $K^{-1}p$ has real isolated eigenvalues with ascent 1 and multiplicity at most 2. In the sequel X denotes either $L^2(0,1)$ or $C[0,1]$, λ_k denotes an eigenvalue of $K^{-1}p$ and Γ a closed Jordan curve around λ_k such that the domain Δ enclosed by Γ contains no other eigenvalues of $K^{-1}p$. P_k will denote the spectral projection associated with λ_k i.e.,

540C

$$P_k = -\frac{1}{2\pi i}\int_\Gamma R(z)dz$$

where $R(z)$ is the resolvent of the operator $K^{-1}p$. The invariant space associated with $K^{-1}p$ and λ_k will be denoted by M_k;

$$M_k = P_k X.$$

With J_h defined as in Section 4.3 we may define the resolvent $R_h(z) = (J_h - z)^{-1}$ for each $z \in \Gamma$ provided that Γ lies in the resolvent set $\rho(J_h)$ of the operator J_h. The spectral projection

$$P_h^{(k)} = -\frac{1}{2\pi i}\int_\Gamma R_h(z)dz$$

projects on to the invariant subspace associated with J_h and its eigenvalues inside Γ. The latter set is denoted by $\sigma(J_h) \cap \Delta$. The invariant space $P_h^{(k)}X$ associated with J_h and $\sigma(J_h) \cap \Delta$ will be denoted by $M_h^{(k)}$.

With the foregoing definitions at hand we can now use Theorem 4.3 and the results in Section 2.2 to state the rate of approximation of the eigenvalues and eigenfunctions of the problem (4.2) - (4.4) in the following theorem.

Theorem 4.4

Let λ_k be an eigenvalue of the operator $K^{-1}p$ isolated by the Jordan curve Γ. Then for h small enough, Γ will enclose at most two eigenvalues of J_h and

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$$\left|\lambda_k - \overline{\lambda}_h^{(k)}\right| \le ch^2,$$
$$\theta\left(M_k, M_h^{(k)}\right) \le ch^2,$$

where $\overline{\lambda}_h^{(k)}$ *denotes the average of the two eigenvalues of* J_h *inside* Γ.

CHAPTER 5

NUMERICAL RESULTS

In this chapter we provide numerical examples which verify $O(h^2)$ convergence of the finite difference scheme employed in Chapters 3 and 4. The computer application program "Mathematica" was used to execute the algorithms that were used with the numerical examples.

5.1 NUMERICAL APPROXIMATION OF THE SOLUTION OF THE SINGULAR BOUNDARY VALUE PROBLEM

Three examples were used with three different functions $p(x)$. Uniform mesh was used with $N = 16, 32$ and 64. Following are the examples and the numerical results obtained.

Example 1

$$\begin{gathered} \left(\left(\sin\frac{\pi}{2}x\right)y'\right)' - \frac{\pi^2}{2}\left(\sin\frac{\pi}{2}x\right)y = -\frac{\pi^2}{2}\sin\pi x, \quad 0 \le x \le 1 \\ \lim_{x\to 0^+}\left(\sin\frac{\pi}{2}x\right)y' = 0, \\ y(1) = 0. \end{gathered} \tag{5.1}$$

It is easily checked that $y(x) = \cos\frac{\pi}{2}x$ is an exact solution for the above problem. Table I below shows $O(h^2)$ convergence of the finite difference

540C

used. In this table, the vectors Y and $\tilde{Y}$ are the exact and approximate solutions as defined by (3.50) and (3.51) respectively. Figure 5.1 shows the graphs for Y and $\tilde{Y}$ in the same plane for N=32 and Figure 5.2 shows the graph for the error $Y - \tilde{Y}$.

TABLE I: Numerical Results for Example 1

N	$\|Y-\tilde{Y}\|_\infty$	h^2
16	7.8 (-4)*	3.9 (-3)
32	2.0 (-4)	9.8 (-4)
64	5.0 (-5)	2.4 (-4)

*The numbers in parenthesis are powers of 10, e.g. 7.8(-4)=7.8 $\times$ 10^{-4}.

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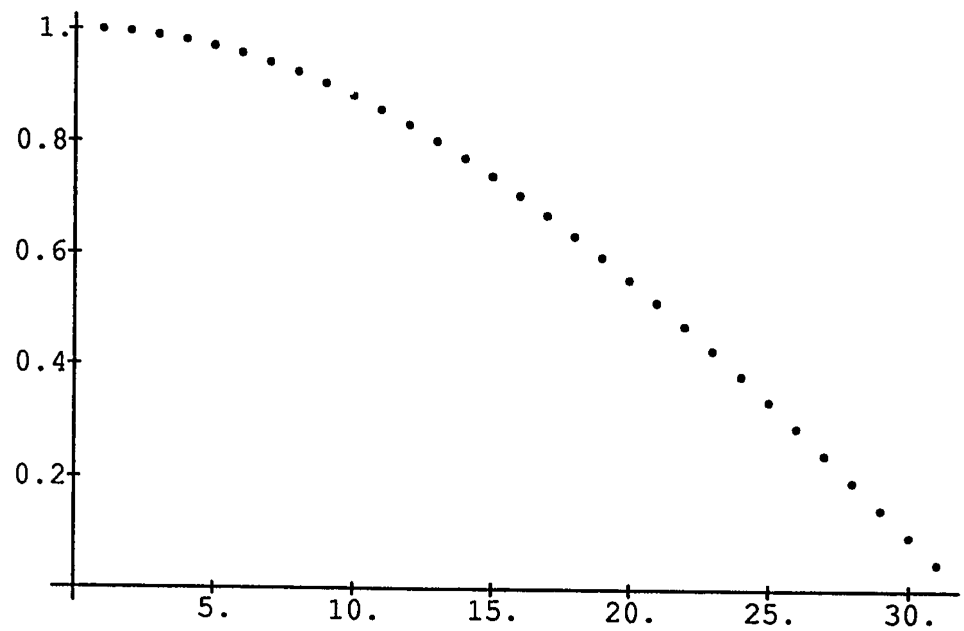

Mesh points

Fig. 5.1. Graph for the exact and approximate solutions of Example 1

540C

Mesh Points

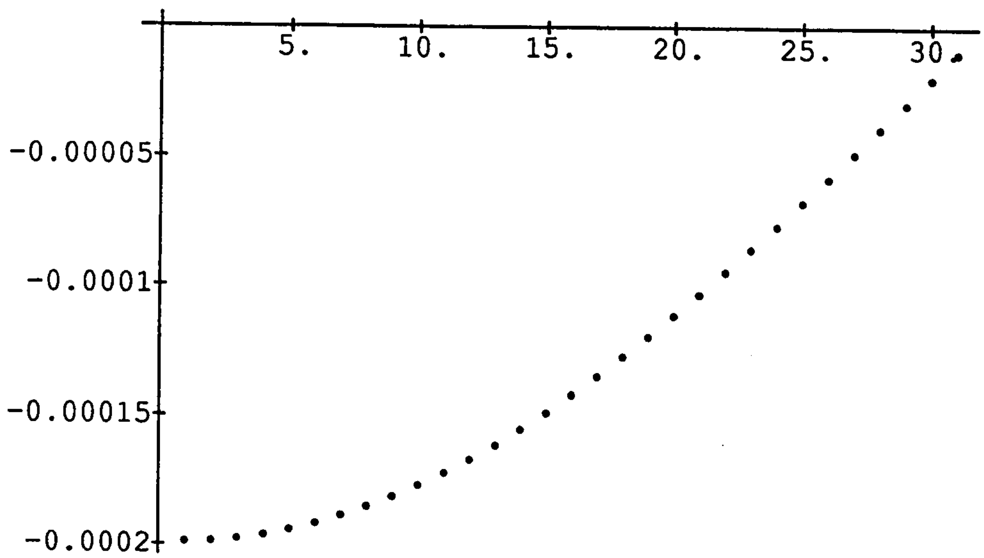

Fig. 5.2 Errors for Example 1

Example 2

(5.2)

$$\left((2x-x^2)^{3/2}y'\right)'-(2x-x^2)^{3/2}y=(2-6x-x^2+x^3)(2x-x^2)^{3/2}$$
$$+(6-15x+9x^2)x^{3/2}(2-x)^{1/2},\quad 0\le x\le 1$$
$$\lim_{x\to 0^+}(2x-x^2)^{3/2}y'=0,$$
$$y(1)=0.$$

The exact solution for (5.2) is $y(x)=x^2(1-x)$. Numerical results showing $O(h^2)$ convergence are given in TABLE II. Figure 5.3 shows the graphs for Y and $\tilde{Y}$ in the same plane for N=32 and Figure 5.4 shows the error $Y-\tilde{Y}$.

Example 3

$$(p(x)y')'-q(x)p(x)y=p(x)f(x),\quad 0\le x\le 1$$
(5.3)
$$\lim_{x\to 0^+}p(x)y'=0,$$
$$y(1)=0,$$

where

$$p(x)=\sqrt{x}\left(\sqrt{x}+1\right)\ln\left(\sqrt{x}+1\right)$$
$$q(x)=1$$
$$f(x)=3-\frac{15}{2}x-x^2+x^3+\frac{\sqrt{x}(2-3x)}{2\left(\sqrt{x}+1\right)}\left(1+\frac{1}{\ln\left(\sqrt{x}+1\right)}\right).$$

The exact solution for (5.3) is $y(x)=x^2(1-x)$.

540C

TABLE II: Numerical Results for Example 2

N	$\|Y-\tilde{Y}\|_\infty$	h^2
16	2.2 (-3)	3.9 (-3)
32	6.1 (-4)	9.8 (-4)
64	1.6 (-4)	2.4 (-4)

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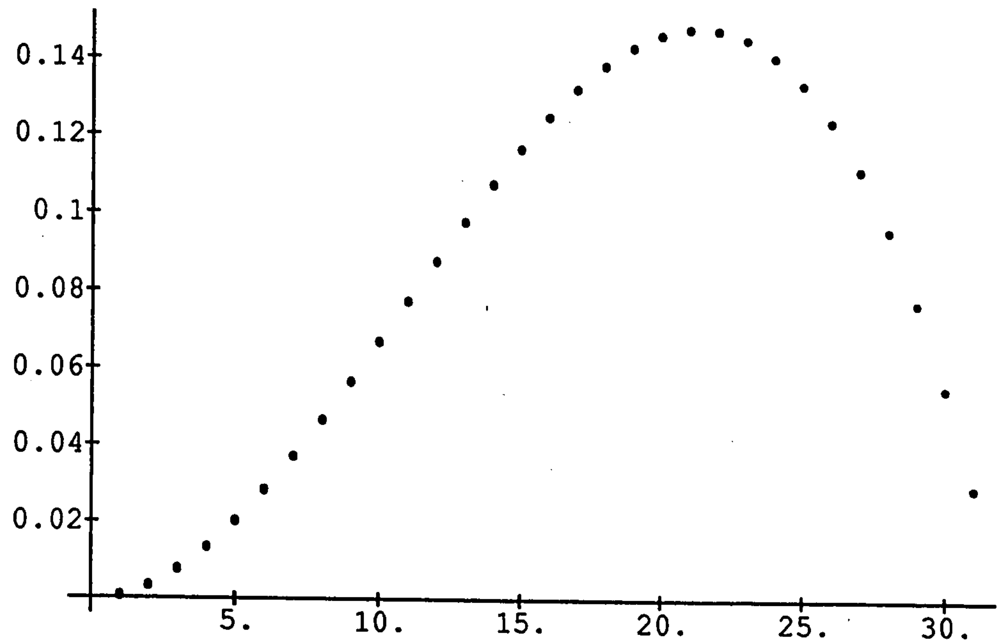


Mesh Points

Fig. 5.2. Graph for the exact and approximate solutions of Example 2

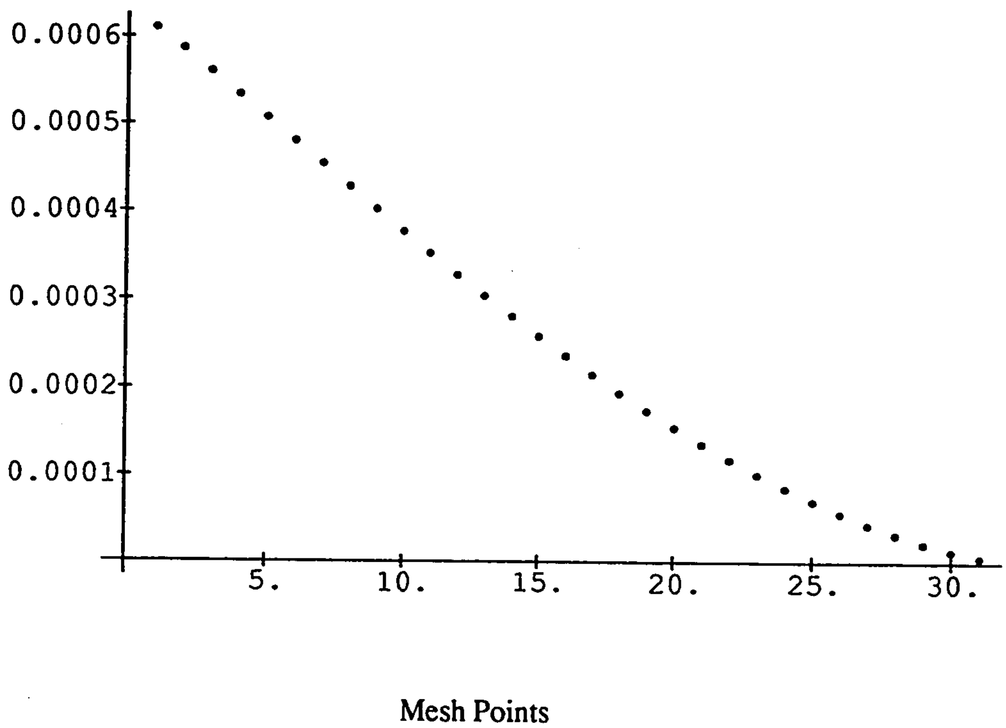
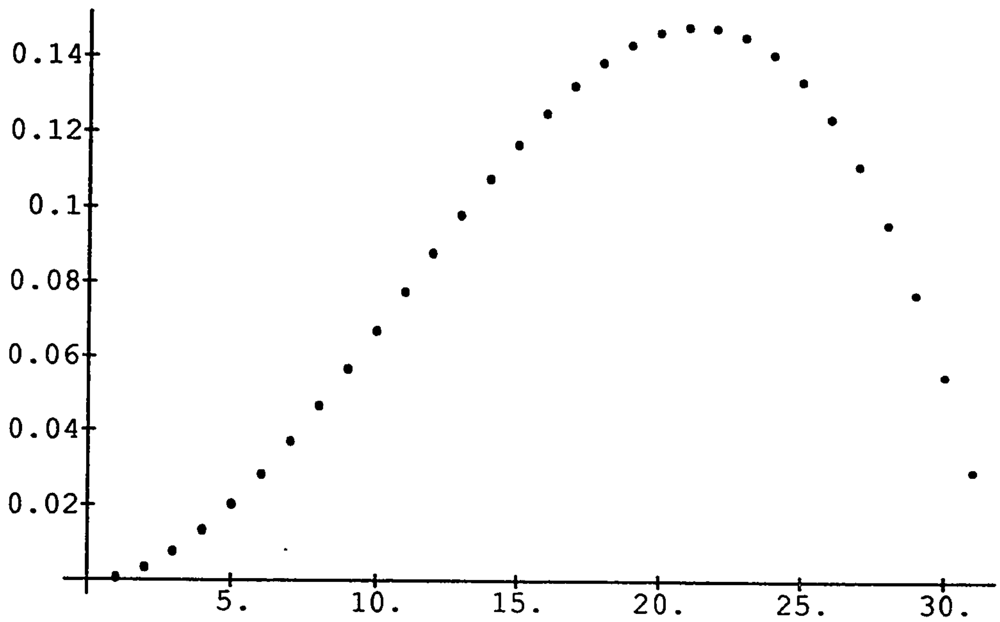


Fig. 5.4. Errors for Example 2

Numerical results for Example 3 are given in TABLE III. Figure 5.5 shows the graphs for Y and $\tilde{Y}$ in the same plane for N=32 and Figure 5.6 shows the error $Y - \tilde{Y}$.

TABLE III: Numerical Results for Example 3

N	$\|Y-\tilde{Y}\|_\infty$	h^2
16	2.1 (-3)	3.9 (-3)
32	5.8 (-4)	9.8 (-4)
64	1.5 (-4)	2.4 (-4)



Mesh Points

Fig. 5.3. Graph for the exact and approximate solution of Example 3

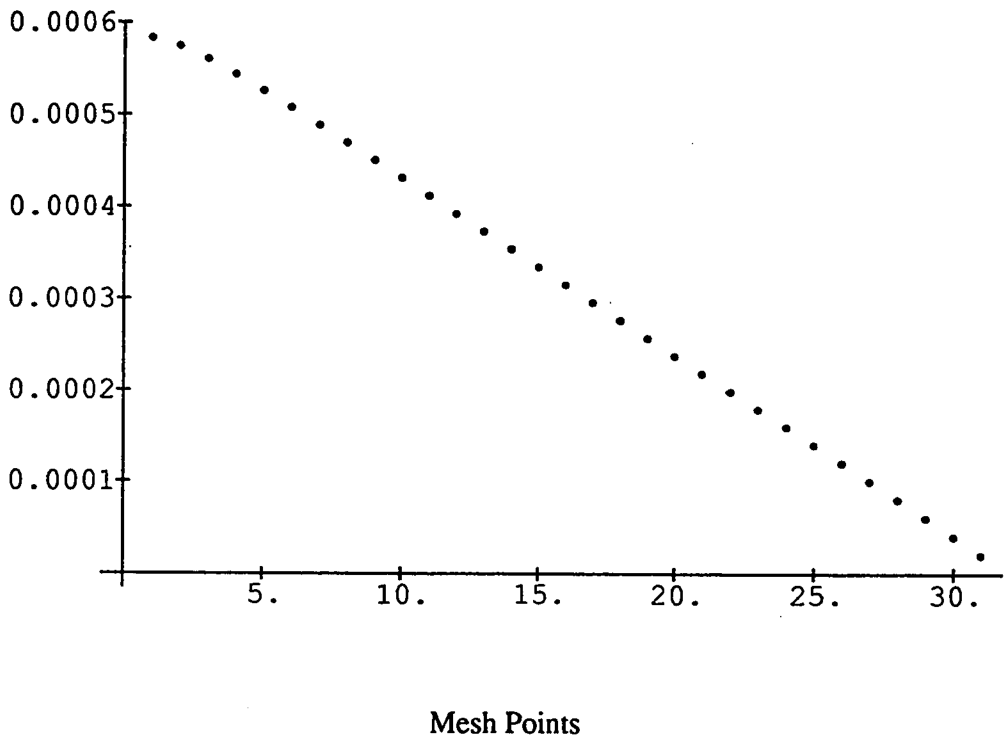


Fig. 5.6. Errors for Example 3

5.2 APPROXIMATION OF THE EIGENELEMENTS

Example 4

Consider the eigenvalue value problem

$$\left(\left(\sin\frac{\pi}{2}x\right)y'\right)' - \frac{\pi^2}{2}\left(\sin\frac{\pi}{2}x\right)y = \lambda\left(\sin\frac{\pi}{2}x\right)y, \quad 0 \le x \le 1$$

$$(5.1) \qquad \lim_{x\to 0^+}\left(\sin\frac{\pi}{2}x\right)y' = 0,$$

$$y(1) = 0.$$

It can easily be verified that $\lambda = -\pi^2$ is an isolated eigenvalue for the above problem and that $y = \cos\frac{\pi}{2}x$ is the associated eigenfunction.

Let $\Delta\lambda$ denote $\left|\pi^2 - \lambda_N\right|$ where λ_N is the numerically obtained eigenvalue with a mesh size N. Let $\theta(A,B)$ denote the gap between A, the eigenspace associated with $\lambda = \pi^2$, and B, the eigenspace associated with λ_N. Table IV shows $O\left(h^2\right)$ convergence for both the eigenvalues and eigenfunctions.

We list below the computer program that was run by Mathematica to approximate the eigenelements of Example 4.

TABLE IV: Numerical Results for Example 4

N	$\Delta\lambda$	$\theta(A,B)$	h^2
16	7.9 (-3)	1.6 (-5)	3.9 (-3)
32	2.0 (-3)	3.9 (-6)	9.8 (-4)
64	4.9 (-4)	5.6 (-6)	2.4 (-4)

APPENDIX

Sample Mathematica Program Used to Approximate the Solution for Example 3

```
Q[x_] := 2Log[Log[x^.5+1]]
P[x_]: = x^.5(x^.5+1)Log[x^.5+1]
F[x_]: = 3-7.5x-x^2+x^3+.5(2-3x)/(x^.5+1)(x^.5+x^.5/Log[x^.5+1]
Y[x_]: = x^2(1-x)
y=N [Table[Y[2^(-5)i], {i,31}], 10]
Do[A[i] := NIntegrate[P[x](Q[x]-Q[2^(-5)(i-1)])/(Q[2^(-5)i]-
Q[2^(-5)(i-1)]),{x, 2^(-5)(i-1), 2^(-5)i}]+
NIntegrate[P[x](-Q[x]+Q[2^(-5)(i+1)])/(-Q[2^(-5)i]+Q[2^(-5)(i+1)]),
{x, 2^(-5)(i), 2^(-5)(i+1)}], {i, 2, 31, 1}]
A[1] := NIntegrate[P[x],{x, 0, 2^(-5)}]+ NIntegrate[P[x](-Q[x]+
Q[2^(-5)(2)])/(-Q[2^(-5)]+Q[2^(-5)(2)]),{x, 2^(-5), 2^(-5)(2)}]
A[1]
Table[A[i], {i, 2, 31}]
v= N [Table[-A[i](F[2^(-5)i]), {i, 1, 31, 1}], 10]
h[1]=1/(Q[2(2^(-5))]-Q[2^(-5)])
Do[h[i] := 1/(Q[2^(-5))i]-Q[2^(-5)(i-1)])+1/(-Q[(2^(-5))i]+
Q[2^(-5)(i+1)]), {i, 2, 31, 1}]
```

```
Do[hl[i] := -1/(Q[(2^(-5))i]-Q[2^(-5)(i-1)]), {i, 2, 31, 1}]

Do[hu[i] := -1/(-Q[(2^(-5))i]+Q[2^(-5)(i+1)]), {i, 1, 30, 1}]

Table[hu[i], {i, 1, 30}]

Table[hl[i], {i, 2, 31}]

m = N [ Table[Switch[i-j, -1,hu[i], 0,h[i]+A[i], 1,hl[i], _,0], {i, 31},
{j, 31}]]

LinearSolve[m, v]

y-%

yo: = N [.000362658-NIntegrate[P[x](Q[2^(-5)]-Q[x])(.000362658+F[x]),
{x, 0, 2^(-5)}]]

yo
```

Sample Mathematica Program Used to Approximate the Solution for Example 4

```
Y[x_] := N [Cos[x(Pi/2)]]
Q[x_] := N [ (2/Pi)Log[Abs[Csc[x(Pi/2)]-Cot[x(Pi/2)]]]]
P[x_] := N [Sin[x(Pi/2)]]
y = N [Table[Y[2^(-4)i], {i, 15}]]
Do[A[i] := (Pi^2/2)(NIntegrate[P[x](Q[x]-Q[2^(-4)(i-1)])/(Q[2^(-4)i]-
Q[2^(-4)(i-1)]),{x, 2^(-4)(i-1), 2^(-4)i}]+ NIntegrate[P[x](-Q[x]+
Q[2^(-4)(i+1)])/(-Q[2^(-4)i]+Q[2^(-4)(i+1)]),{x, 2^(-4)(i), 2^(-4)(i+1)}]),
{i, 2, 15, 1}]
A[1] := (Pi^2/2)(NIntegrate[P[x],{x, 0 2^(-4)}]+ NIntegrate[P[x](-Q[x]+
Q[2^(-4)(2)])/(-Q[2^(-4)]+Q[2^(-4)(2)]),{x, 2^(-4), 2^(-4)(2)}])
A[1]
Table[A[i], {i, 2, 15}]
h[1]=1/(Q[2(2^(-4))]-Q[2^(-4)])
Do[h[i] := 1/(Q[(2^(-4))i]-Q[2^(-4)(i-1)])+1/(-Q[(2^(-4))i]+Q[2^(-4)
(i+1)]), {i, 2, 15,1}]
Do[hl[i] := -1/(Q[2^(-4))i]-Q[2^(-4)(i-1)]), {i, 2, 15, 1}]
Do[hu[i] := -1/(-Q[2^(-4))i]+Q[2^(-4)(i+1)]), {i, 1, 14, 1}]
Table[hu[i], {i, 1, 14}]
Table[hl[i], {i, 2, 15}]
```

```
m = N [ Table[Switch[i-j, -1,-hu[i]/(A[i]/(Pi^2/2)), 0,-h[i]/(A[i]/(Pi^2/2))-
Pi^2/2, 1,-hl[i]/(A[i]/(Pi^2/2)),_,0], {i, 15}, {j, 15}]]
Eigenvalues[m]
Eigenvectors[m]
me = N [ Table[Switch[i-j, -1,-hu[i]/(A[i]/(Pi^2/2)), 0,9.86175-
h[i]/(A[i]/(Pi^2/2), 1,-hl[i]/(A[i]/(Pi^2/2)),_,0], {i, 15}, {j, 15}]]
z=N [Table[0, {i, 15}]]
LinearSolve[me, z]
(-.363385/.995185)y
```

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