Spectral Representation of the Love Wave Operator

by

Abdul Sami'E Muhammad Abu Safiya

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICS

June, 1981

INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

U·M·I

University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800/521-0600

Order Number 1355701

Spectral representation of the Love wave operator

Abu Safiya, Abdul Sami'e Muhammad, M.S.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1981

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

SPECTRAL REPRESENTATION OF THE LOVE WAVE OPERATOR

BY

ABDUL SAMI'E MUHAMMAD ABU SAFIYA

The Library
University of Petroleum & Minerals
Dahran, Saudi Arabia

A THESIS PRESENTED TO THE

COLLEGE OF GRADUATE STUDIES

UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

MASTER OF SCIENCE IN MATHEMATICS

JUNE, 1981

UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

This thesis, written by

ABDUL SAMIE MUHAMMAD ABU-SAFIYA

under the direction of his Thesis Committee, and approved by all its members, has been presented to and accepted by the Dean, College of Graduate Studies, in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE

Special
A
1
A292
c.2

Dean, College of Graduate Studies

Date 8/2-1/1/cc

M. P. h June 7/81

Department Chairman

THESIS COMMITTEE

Mumtaz H. Kazi

Chairman

Member

Member

بسم الله الرحمن الرحيم

ACKNOWLEDGEMENTS

I am very much indebted to my advisor and thesis supervisor, Prof. M.H. Kazi for his expert guidance during the preparation of this manuscript.

I thank Dr. David Willmer and Dr. Adnan Niazy for their great assistance and constant encouragement.

I am also indebted to the University of Petroleum and Minerals for its financial support during the time I spent in Dhahran.

Thanks are also due to Mr. Saadat A. Siddiqui who was responsible for the typing of my thesis manuscript carrying out this onerous task with skill and understanding.

Finally, I would like to thank my parents and brother Sai'd for the moral support and encouragement that they have provided during my years at UPM. Without their help I would not have been able to complete my thesis research.

TABLE OF CONTENTS

ABSTRACT

In many Geophysical problems of diffraction of Love waves at a laterally discontinuous change in elevation or in material properties of layered structures, we need to express the displacements on either side of the discontinuity, in terms of a complete set of functions, proper or improper, associated with the Love wave operator in order to be able to apply such powerful tools as integral equations and variational principles. Kazi (1976) presented a method for obtaining the spectral representation of the two-dimensional Love wave operator, associated with the propagation of monochromatic SH waves in a laterally uniform layered strip or half-space. Kazi (1976) found such a representation for a two-layer model of an infinite strip, overlying another infinite strip or a half-space, with constant rigidity and density within each layer.

In this thesis we determine the spectral representation of the two dimensional Love wave operator associated with the propagation of monochromatic SH waves in three-layer models of two infinite strips, overlying another infinite strip or a half-space, with constant rigidity and density within each layer. This representation will be of considerable use in tackling Love wave diffraction problems in horizontally discontinuous structures involving three layers.

CHAPTER 1

GENERAL INTRODUCTION.

STURM-LIOUVILLE SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

1.1 General Introduction

One of the principal problems in mathematical geophysics is to investigate the effects of velocity changes, and especially of discontinuities or boundaries (such as continental margins) on the propagation of seismic surface waves (Love and Rayleigh waves) which are characterized by the property that they are restricted to the neighborhood of the surface of the earth, a region which is not easily investigated by body-wave studies. Surface wave studies are, therefore, important in determining the velocity structure in the crust.

In order to construct a theoretical analysis of surface wave reflection, transmission and diffraction at continent/ocean margins with large impedence contrasts or other lateral discontinuities, it is necessary to formulate the problem mathematically and to obtain exact or approximate solutions. However, the mathematical structure of the problems is complicated by the number of parameters involved in the description of the discontinuities and only idealized models have been used so far. Even then the mathematical treatment of these problems is formidable.

In recent years, various problems associated with transmission, reflection and diffraction of these waves through horizontally discontinuous structures have been investigated by a number of authors who have used a variety of analytical and numerical techniques (Sato (1961), Mal and Knopoff (1965), Alsop (1966), Babich and Molotkov (1966), Wolf (1967), Knopoff and Hudson (1964), Boore (1970), Wolf (1970), Lysmer and Drake (1971), Gregersen and Alsop (1974, 1976), Drake and Bolt (1980), Suteau and Martel (1980). However, none of these techniques take into account body-wave conversion which is considerable at certain frequencies.

In a series of papers (Kazi (1978a,b), Kazi (1979), Niazy and Kazi (1980)) the authors use a method, based upon an integral equation formulation together with the application of Schwinger-Levine variational principle to investigate the two-dimensional problems of the propagation of plane, harmonic, monochromatic Love waves, incident normally (from either side) upon the plane of discontinuity in laterally discontinuous structures involving step-wise change in surface topography or change in material properties. Diffraction of Love waves is described by means of a scattering matrix and approximate expressions for its elements are sought through the variational principle. Reflection and transmission coefficients are then obtained through a transmission matrix related to the scattering matrix. The method has the advantage over all other methods used so far, in as much as it takes into account the body-wave contributions. However, the method pre-supposes the existence of a complete set of proper or improper eigenfunctions, in terms of which the

displacements on either side of the discontinuity may be expressed. In order to accomplish this Kazi,(1976) gave a method of finding the spectral representation of the two-dimensional Love wave operator, associated with the propagation of monochromatic SH waves (horizontal component of transverse waves) in a laterally-uniform layered strip or half-space. However, specific spectral representations were found only for two-layer models of an infinite strip overlying another infinite strip or a half-space with constant rigidity and density within each layer.

In this thesis we follow the same procedure as in Kazi (1976) to determine the spectral representation of the Love wave operator associated with monochromatic SH-waves in three-layer models comprising two infinite strips overlying another strip or a half-space, with constant rigidity and density within each layer. This representation will find its usage in tackling Love wave diffraction problems associated with three layers over a half-space by the method described in the preceding paragraph.

Chapter 2 deals with the construction of Green's function from which eigenfunctions (proper as well as improper) can be generated and Chapter 3 is devoted to the determination of the spectral representation of the Love wave operators for the problems mentioned above.

The work done in Chapters 2 and 3 is original to the best of the author's knowledge.

1.2 Regular Sturm-Liouville Systems with Discontinuous Coefficients

Consider the differential equations

$$L(y) - \lambda py = -(r(z)y'(z))' + q(z)y - \lambda p(z)y = 0 \qquad (1.2.1)$$

defined over an interval (a,b), where

$$L = -\frac{d}{dz}[r(z)\frac{d}{dz}] + q(z)$$

and (') denotes differentiation with respect to $z;\lambda$ is a parameter, real or complex.

Let

$$a = z_0 < z_1 < z_2 < \ldots < z_{N+1} = b$$

and $p(z) \equiv p_j(z)$, $q(z) \equiv q_j(z)$, $r(z) \equiv r_j(z)$ and $y \equiv y_j(z)$, when $z \in I_j = \{z: z_j < z < z_{j+1}\}$, $j = 0,1,2,\ldots,N$.

We assume that in each subinterval I_j, the functions $p_j(z)$, $q_j(z)$ and $r_j(z)$ are continuous and single-valued and that their limits when z tends to z_j or z_{j+1} exist and are finite. Moreover, $r(z)$ does not vanish at any point of a subinterval and preserves the same sign in all the subintervals.

We suppose that at each interface or point of discontinuity z_k, $k = 1,2,\ldots N$, the y_j satisfy the conditions

$$y_k(z_k) = d_k y_{k-1}(z_k) + e_k y'_{k-1}(z_k) \tag{1.2.2a}$$

$$y'_k(z_k) = b_k y_{k-1}(z_k) + c_k y'_{k-1}(z_k), \tag{1.2.2b}$$

where $d_k c_k - b_k e_k \neq 0$ and at the end-points $y(z)$ satisfies the Sturmian conditions:

$$B_a(y) \equiv \alpha_1 y(a) + \alpha_2 y'(a) = 0 \tag{1.2.3a}$$

$$B_b(y) \equiv \beta_1 y(b) + \beta_2 y'(b) = 0 \tag{1.2.3b}$$

where

$$|\alpha_1| + |\alpha_2| \neq 0 \quad \text{and} \quad |\beta_1| + |\beta_2| \neq 0$$

The system (1.2.1) — (1.2.3) is the regular, self-adjoint Sturm-Liouville (SL - system in brief) with N points of discontinuity.

The usual results of ordinary Sturm-Liouville theory for regular systems hold for regular, S-L systems with discontinuous coefficients and interface conditions after necessary modifications.

We state here the most important results regarding these systems; the proofs of most of these results can be found in Sangren (1953).

I. The eigenvalues λ_n of the system are real, if the following conditions are satisfied:

1) $r(z)$ preserves the same sign and does not vanish throughout its domain of definition.

2) $p_i(z)$ is positive or identically zero in the subinterval I_i , $i = 0,1,\ldots N$ and for at least one subinterval $p_i(z) > 0$,

3) $c_j d_j - b_j e_j > 0$, $j = 1,2,\ldots N$.

II. The eigenfunctions form a complete orthonormal set. The modified orthonormality condition is given by:

$$< \phi^m(z), \phi^n(z) > = \int_a^b A(z)p(z)\phi^m(z)\overline{\phi^n}(z)dz = \delta_{mn} \qquad (1.2.4)$$

where $\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$; $\phi^m(z)$ and $\phi^n(z)$ are the normalized eigenfunctions belonging to the disinct eigenvalues λ_m and λ_n , respectively. $\overline{\phi^n}(z)$ denotes the complex conjugate of $\phi^n(z)$, and $A(z)$ is a piecewise constant function defined by

$$A(z) = A_i \ , \quad z \in I_i \ , \ i = 0,1,2,\ldots,N \qquad (1.2.5)$$

where

$$A_i = \frac{r_{i-1}(z_i)}{r_i(z_i)}(c_i d_i - b_i e_i)^{-1} A_{i-1}, \ i = 1,2,\ldots N \qquad (1.2.6)$$

and A_0 is a non-zero constant which may be chosen arbitrarily. Conditions (1.2.6) ensure the self-adjointness of the system.

III. The spectrum of eigenvalues is discrete.

IV. The system (1.2.1) — (1.2.3) has an infinite number of eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$ forming a monotone increasing sequence $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. Moreover, the eigenfunction $\phi^n(z)$ corresponding to λ_n has exactly n zeros on (a,b).

V. (Expansion Theorem)

Given an arbitrary continuous and piecewise differentiable function $f(z)$ which vanishes at the end points of the interval when $\phi^n(z)$ vanishes, then the series $\sum_{n=0}^{\infty} C_n\phi^n(z)$ where $C_n = \int_a^b f(z)p(z)\phi^n(z)dz$ converges uniformly and absolutely and has the sum $f(z)$.

Properties of Green's Function for Regular S-L Systems with Discontinuous Coefficients

If $\lambda = 0$ is not eigenvalue of the S-L system (1.2.1) — (1.2.3) then there exists a unique function $G(z,\xi;\lambda)$ satisfying the following conditions:

(G_1) $G_j(z,\xi;\lambda)$, the restriction of the function $G(z,\xi;\lambda)$ to the interval $z_j \leq z \leq z_{j+1}$, $j = 0,1,\ldots, N$, is continuous at each point of the interval.

(G_2) Each $G_j(z,\xi;\lambda)$ possesses a continuous first order derivative for $j \neq k$, where k is such that $\xi \in I_k$: G_k possesses a continuous first-order derivative at each

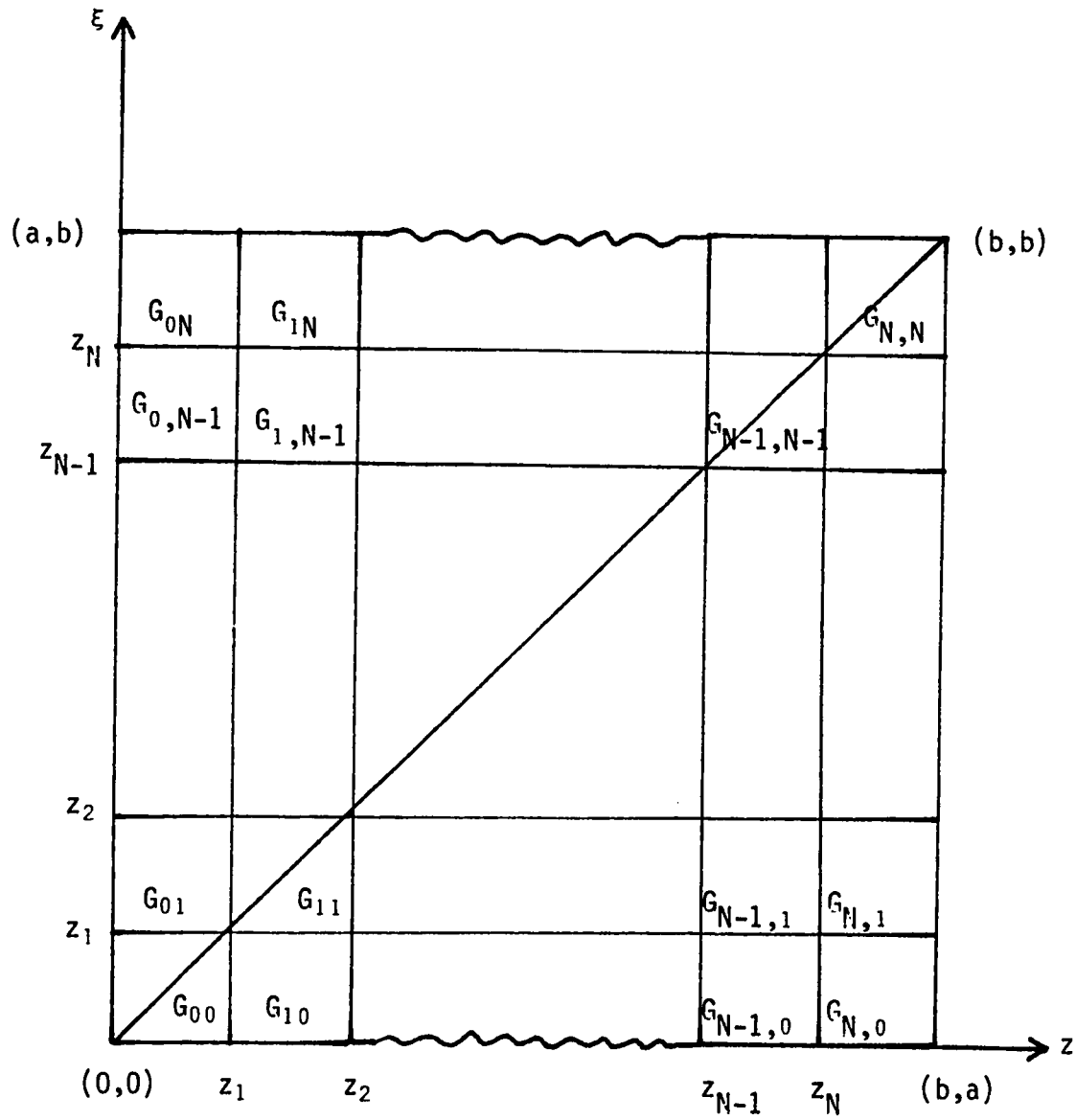

Fig. 1. The character of the Green's function

point of the interval I_k except at the point $z = \xi$, where it has a jump discontinuity, given by:

$$G_k'(\xi^+,\xi;\lambda) - G_k'(\xi^-,\xi;\lambda) = -\frac{1}{r(\xi)}$$

(G_3) $L(G_j) = 0$, $z \in I_j$, $j \neq k$; $L(G_k) = 0$, $z \neq \xi$.

(G_4) $G(z,\xi;\lambda)$ satisfies the interface conditions (1.2.2) and the end point condition, (1.2.3), (see Fig. 1).

The Green's function is unique and symmetric. We shall construct the Green's function for the Love wave operator associated with the three layered finite depth problem in Chapter 2 and the spectral representation will then be determined by integrating Green's function in Chapter 3.

1.3 Singular Sturm-Liouville Systems with Discontinuous Coefficients. Weyl's Theorem

If in the system (1.2.1) — (1.2.3), a or b is infinite or if on the finite interval $r(z)$ vanishes or $q(z)$ or $p(z)$ become infinite at the end-points, then the Sturm-Liouville system is called a self-adjoint singular S-L system with discontinuous coefficients and interface conditions.

Let us consider the Green's function $G(z,\xi;\lambda)$ which is the solution of the system:

$$\left.\begin{array}{l} L(G) - \lambda p(z)G = \delta(z-\xi) \\ B_a(G) = 0 \\ \text{interface conditions (1.2.2)} \end{array}\right\} \qquad (1.3.1)$$

together with a condition at infinity.

It is the condition at infinity which presents certain difficulties for the construction of a unique Green's function to the singular problem.

In order to investigate this difficulty we look at the following theorem due to Weyl.

<u>Weyl's Theorem</u>

<u>Statement</u> : Let a be a regular point and b a singular point and consider the equation

$$(-pu')' + qu - \lambda su = 0 \qquad a < z < b \qquad (1.3.2)$$

where p, q, s are continuous in the interval $a < z < b$.

1. If for some particular value of λ, every solution is of finite s-norm over (a,b), then for any other value of λ, every solution is again of finite s-norm over (a,b) [The s-norm of a function is here defined by $\|u\|_s = [\int_a^b s|u|^2 dz]^{\frac{1}{2}}$. The above condition

is thus expressed as $\int_a^b s|u|^2 dz < \infty$.].

2. For every λ with $I(\lambda) \neq 0$, there exists at least one solution of finite s-norm over (a,b).

According to the theorem we can classify the behavior at the singular point b as follows: Either all solutions are of finite s-norm over (a,b) for all λ (the so-called <u>limit-circle</u> case at b), or for $I(\lambda) \neq 0$ there is exactly one solution (up to a multiplicative constant) of finite s-norm over (a,b) (we then say that we have the <u>limit-point</u> case at b). The terms limit-circle and limit-point will be discussed in more detail shortly.

We shall now extend Weyl's theorem to singular S-L systems with discontinuous coefficients and interface conditions. Firstly, we give the following definition:

A function u is said to be of finite p-norm over (a,b) if u satisfies the following condition

$$||u||_p^2 \equiv \langle u, u\rangle_p^{\frac{1}{2}} = \int_a^b A(z)p(z)|u(z)|^2 dz < \infty ,$$

where A(z) is the weight function given by (1.2.5).

Let $\phi(z,\lambda)$ and $\psi(z,\lambda)$ be two linearly independent solutions of

$$Lu - \lambda p(z)\, u = 0 \quad (\lambda \text{ is a complex number}) \qquad (1.3.3)$$

satisfying the interface conditions (1.2.2) and the following initial conditions at the point a :

$$\left.\begin{aligned} \phi(a,\lambda) &= -\alpha_2 \quad , \quad \phi'(a,\lambda) = \frac{\alpha_1}{r(a)} \\ \psi(a,\lambda) &= \alpha_1 \quad , \quad \psi'(a,\lambda) = \frac{\alpha_2}{r(a)} \end{aligned}\right\} \tag{1.3.4}$$

where α_1 and α_2 are real and $|\alpha_1| + |\alpha_2| \neq 0$
[We observe that ϕ and ψ are indeed independent, since $r(a)W(\phi, \psi; a) = -\alpha_1^2 - \alpha_2^2 \neq 0$ where $W(\phi, \psi; a)$ is the Wronskian of ϕ and ψ (Sangern 1953)].

Each solution of (1.3.2) (except the multiples of ψ) can be expressed as a multiple of $u = \phi + m\psi$, where m is a complex number.

Further, we impose the following boundary condition at a point $b_0 < \infty$:

$$\beta_1 u(b_0,\lambda) + \beta_2 r(b_0)u'(b_0,\lambda) = 0 \tag{1.3.5}$$

where β_1 and β_2 are arbitrary real numbers and $|\beta_1| + |\beta_2| \neq 0$.
We are interested in investigating the limit as $b_0 \to \infty$(the boundary condition (1.3.5) cannot be imposed directly at ∞, because there is no guarantee that u or u' is finite at ∞). A function u satisfies the condition (1.3.5) for some real β_1, β_2 if and only if

$$r(b_0)W(u,\bar{u};b_0) = 0 \tag{1.3.6}$$

If and only if $u = \phi + m\psi$ then

$$m = -\frac{h\,\phi(b_0,\lambda) + r(b_0)\phi'(b_0,\lambda)}{h\,\psi(b_0,\lambda) + r(b_0)\psi'(b_0,\lambda)}, \quad \text{where} \quad h = \frac{\beta_1}{\beta_2} \tag{1.3.7}$$

and

$$r(b_0)\,W(\phi,\bar{\phi};b_0) + m\bar{m}\,W(\psi,\bar{\psi};b_0) + m\,W(\psi,\bar{\phi};b_0) + \bar{m}.W(\phi,\bar{\psi};b_0) = 0 \tag{1.3.8}$$

where bar denotes the complex conjugate.

Now let h run through all real numbers. Then, m describes a circle in the complex plane with

$$\text{centre} \quad S = -\frac{W(\phi,\bar{\psi};b_0)}{W(\psi,\bar{\psi};b_0)} \tag{1.3.9}$$

and

$$\text{radius} \quad R = \frac{(\alpha_1^2 + \alpha_2^2)^{\frac{1}{2}}}{2|I(\lambda)|\,[\int_a^{b_0} A(z)p(z)|\psi|^2 dz]} \tag{1.3.10}$$

as being as $I(\lambda) \neq 0$.

As b_0 increases with λ fixed, the radius R decreases and each circle is contained in those for smaller value of b_0. Hence in the limit as $b_0 \to \infty$, the circles approach a limit-circle $C_\infty(\lambda)$ or a limit-point $m_\infty(\lambda)$. It can be shown that if $m = m_\infty(\lambda)$ or m lies on $C_\infty(\lambda)$, m is the interior to all circles and therefore the solution $u = m\psi + \phi$ is square-integrable over the interval (a,∞), relative to

the weight function $A(z)p(z)$. It follows from (1.3.10) that in the limit-point case there is just one solution $u = \phi(z,\lambda) + m_\infty(\lambda)\psi(z,\lambda)$ which satisfies the conditions:

$$\|u\| = \int_a^\infty A(z)p(z)|u(z)|^2 dz < \infty \tag{1.3.11}$$

and in the limit-circle case the independent solutions ψ and $\phi + m\psi$ and therefore every solution of (1.3.3) satisfies the condition (1.3.11). Although this has been proved only for $I(\lambda) \neq 0$, it can be shown (Stakgold(1967) section 4.4), that if for some value of λ every solution of (1.3.3) satisfies the condition (1.3.11), then so does every solution for any other value of λ. We conclude therefore, that in the limit-circle case all solution are square-integrable, relative to the weight function $A(z)p(z)$, over the interval (a,∞).

The Green's function $G(z,\xi;\lambda)$ is characterized as the only solution of the system (1.3.1) which satisfies:

$$\int_a^\infty A(z)p(z)|G|^2 dz < \infty,$$

when $I(\lambda) \neq 0$ and the singularity at infinity is of the limit-point type. For real values of λ, we apply the principle of analytic continuation and define the Green's function as the limit of G, regarded as a function of complex λ, as λ approaches the real axis. As λ may approach the real axis from either side, we may ensure uniqueness by

supposing, for example, that $I(\sqrt{\lambda}) > 0$ (See Friedman (1956, pp.230-231)).

We shall use these ideas to construct the Green's function for the Love wave operator associated with two layers over a half-space in Chapter 2 and the spectral representation will then be determined by integrating this Green's function in Chapter 3.

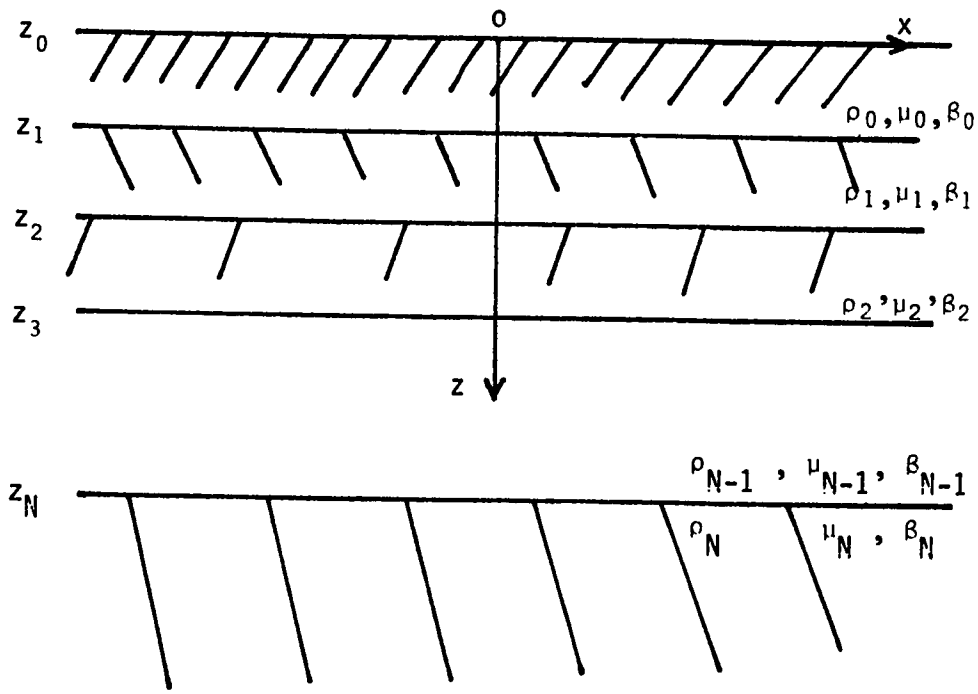

Fig. 2. Geometry of the problem for N layers

CHAPTER 2

GREEN'S FUNCTION FOR THE LOVE WAVE OPERATOR

In this chapter we construct Green's functions for Love wave operators associated with structures consisting of two uniform layers, overlying another uniform layer of finite depth or a uniform half space. We shall use these Green's functions to obtain the spectral representation of the Love wave operators in Chapter 3.

2.1 Equations of Motion

We consider the two-dimensional motion of a laterally homogeneous layered structure in a general way. The motion will consist of waves propagating along the x-axis shown in fig. 2. The medium to be considered is made up to $N+1$ parallel, isotropic and solid layers. It may be finite in depth, with $N+1$ layers contained between two parallel plane surfaces, or may be semi-infinite in depth with N layers overlying a half-space. For the finite depth problem we suppose the upper and lower plane surfaces to be stress-free.

In the case of a layered half-space, we assume that the upper plane surface is stress free and the displacements as functions of depth are square integrable, relative to the rigidity function over the semi-infinite interval.

The density ρ and the rigidity μ of the material in each layer may vary continuously with depth but are uniform in any direction parallel to the boundaries. The density ρ and the rigidity μ may, however, be discontinuous across the N plane interfaces.

We choose the axes in such a way that the xy-plane coincides with the upper free surface, the positive z-axis is directed into the medium and all displacements are uniform in the y-direction. The various layers and the interfaces are numbered from the free surface as shown in Fig. 2.

The upper free plane surface has the equation $z = z_0 = 0$. The lower bounding plane surface, in the case of finite depth, has the equation $z = z_{N+1} = H$, and the N interfaces have the equations $z = z_i$, $i = 1, 2, \ldots, N$.

We shall consider horizontally polarized shear waves only, which means that there are no displacements in the x and z directions, and the motion is in the y-direction only. Let $v(x,z,t)$ be the y-component of displacement. It must satisfy the Love wave differential equation

$$\rho(z) \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \left(\mu(z) \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z} \left(\mu(z) \frac{\partial v}{\partial z}\right) \tag{2.1.1}$$

$$\left.\begin{aligned} \text{where} \quad & \mu(z) = \mu_i(z) , \\ \text{and} \quad & \rho(z) = \rho_i(z) , \end{aligned}\right\} \tag{2.1.2}$$

and $z \in I_i = \{z: z_i < z < z_{i+1}\}$, $i = 0, 1, \ldots, N$.

(In the case of a layered half-space, we take $I_N = \{z: z_N < z\}$)

We assume that the functions $\mu_i(z)$, $\partial\mu_i/\partial z$ and $\rho_i(z)$ are continuous, and μ_i and ρ_i are positive in the subintervals I_i , $i = 0, 1, \ldots , N$, and that these limits, when $z \longrightarrow z_i$ or z_{i+1} , are finite.

In order to obtain a general representation, we first of all examine harmonic waves travelling in the positive x-direction with positive real frequency ω and wave number k.

$$v(x,z,t) = V(z) \exp[\, i(\omega t - kx)] \tag{2.1.3}$$

(We shall assume ω to be fixed and choose k to satisfy the propagation conditions.)

Since $v(x,z)$, the y-component of displacement satisfies equation (2.1.1), we have

$$L(V) = \frac{d}{dz}\left(\mu(z)\,\frac{dV}{dz}\right) + (\omega^2\rho(z) - k^2\mu(z))V = 0 \tag{2.1.4}$$

L being the Love wave operator and

$$V(z) \equiv V_i(z) \qquad , \quad z \in I_i \quad , \quad i = 0, 1, \ldots , N$$

The interface conditions of welded contact between the layers are given by the equations:

$$V_k(z_k) = V_{k-1}(z_k) \quad , \quad k = 1, 2, \ldots, N \tag{2.1.5}$$

and

$$\mu_k(z_k)V_k'(z_k) = \mu_{k-1}(z_k)V_{k-1}'(z_k) \tag{2.1.6}$$

where (') denotes differentiation with respect to z.

The end-conditions for the finite depth problem will be chosen as

$$V_0'(0) = 0 \tag{2.1.7a}$$

and

$$V_N'(H) = 0 \tag{2.1.7b}$$

For the layered half-space problem we shall assume, in addition to the end-condition $V_0'(0) = 0$, the following condition:

$$\int_0^\infty \mu(z)|V(z)|^2 dz < \infty \tag{2.1.8}$$

The system (2.1.4) —— (2.1.7) is a regular Sturm - Liouville system (§ 1.2) and the system (2.1.4) —— (2.1.6), (2.1.7a)(2.1.8) is a singular S-L system (§ 1.3) with N points of discontinuity and corresponding interface conditions, with

$$p_i = - r_i = \mu_i(z) \ , \quad i = 0,1,2,\ldots,N$$

$$q_i = \rho_i\omega^2 \ , \quad \lambda = k^2 \ , \quad d_k = 1 \ , \quad e_k = 0 \ , \quad b_k = 0$$

and $$c_k = \mu_{k-1}(z_k)/\mu_k(z_k) \ , \quad k = 1,2,\ldots,N.$$

Hence

$$A_i = \frac{r_{i-1}(z_i)}{r_i(z_i)} (c_i d_i - b_i e_i)^{-1} A_{i-1}$$

$$= A_{i-1} \ , \quad i = 1,2,\ldots,N.$$

We choose $A_0 = 1$, thus giving $A_i = 1$, $i = 1,2,\ldots,N$.

The conditions 1), 2) and 3) given in I(§ 1.2) are all satisfied. Hence the results I, II and III are valid and the orthonormality condition of eigenfunctions becomes :

$$< \phi^m(z) \ , \ \phi^n(z) > = \int_{-H_1}^{H_3} \mu(z)\phi^m(z)\overline{\phi^n}(z)dz$$

$$= \delta_{mn}$$

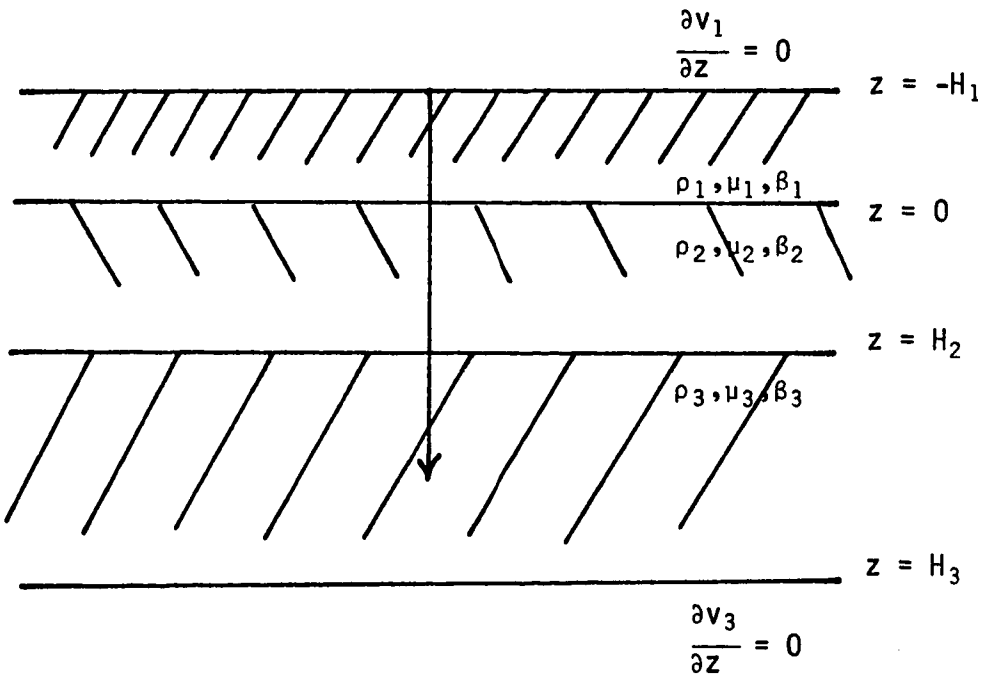

Fig. 3. Geometry of three layered problem

2.2 Green's Function for the Love wave operator for three layered finite depth problem

In this section we shall construct the Green's function for the Love wave operator for the three-layered finite depth problem. We consider an infinite strip consisting of a layer of depth $H_3 - H_2$, rigidity μ_3, shear velocity β_3 and density ρ_3, overlaid by two infinite strips, consisting of a layer of depth $H_2(<H_3)$, density ρ_2, rigidity μ_2 $(<\mu_3)$ and shear velocity $\beta_2(<\beta_3)$, and another layer of depth H_1 $(<H_2)$, density ρ_1, rigidity μ_1 $(<\mu_2)$ and shear velocity β_1 $(<\beta_2)$ (see fig. 3). We suppose the density and the rigidity of each layer to be constant and the top and the bottom plane surfaces to be stress-free.

Let $v(x,z,t) = V(z) \exp[i(\omega t - kx)]$ be the y-compoent of displacement, where

$$
\begin{aligned}
V(z) &\equiv V_1(z) && -H_1 < z < 0 \\
&\equiv V_2(z) && 0 < z < H_2 \\
&\equiv V_3(z) && H_2 < z < H_3
\end{aligned}
$$

Then $V_1(z)$, $V_2(z)$ and $V_3(z)$ satisfy the following equations (see equation (2.1.4))

$$\frac{d^2V_1}{dz^2} + \sigma_1^2 V_1 = 0 \quad \sigma_1^2 = (\frac{\omega^2}{\beta_1^2} - \lambda), \ \lambda = k^2, \ \beta_1^2 = \frac{\mu_1}{\rho_1} \qquad -H_1 < z < 0 \tag{2.2.1}$$

$$\frac{d^2V_2}{dz^2} + \sigma_2^2 V_2 = 0 \quad \sigma_2^2 = (\frac{\omega^2}{\beta_2^2} - \lambda), \ \beta_2^2 = \frac{\mu_2}{\rho_2}, \quad 0 < z < H_2 \tag{2.2.2}$$

$$\frac{d^2V_3}{dz^2} - \sigma_3^2 V_3 = 0 \quad \sigma_3^2 = (\lambda - \frac{\omega^2}{\beta_3^2}), \ \beta_3^2 = \frac{\mu_3}{\rho_3}, \quad H_2 < z < H_3 \tag{2.2.3}$$

with the interface conditions

$$V_1(0) = V_2(0) \tag{2.2.4a}$$

$$\mu_1 V_1'(0) = \mu_2 V_2'(0) \tag{2.2.4b}$$

and

$$V_2(H_2) = V_3(H_2) \tag{2.2.5a}$$

$$\mu_2 V_2'(H_2) = \mu_3 V_3'(H_2) \tag{2.2.5b}$$

and the boundary conditions

$$V_1'(-H_1) = 0 \tag{2.2.6a}$$

$$V_3'(H_3) = 0 \tag{2.2.6b}$$

Now we find the Green's function $G(z, \xi, \lambda)$ (see Chapter 1 Section (1.2) for properties) belonging to the system (2.2.1) —— (2.2.6).

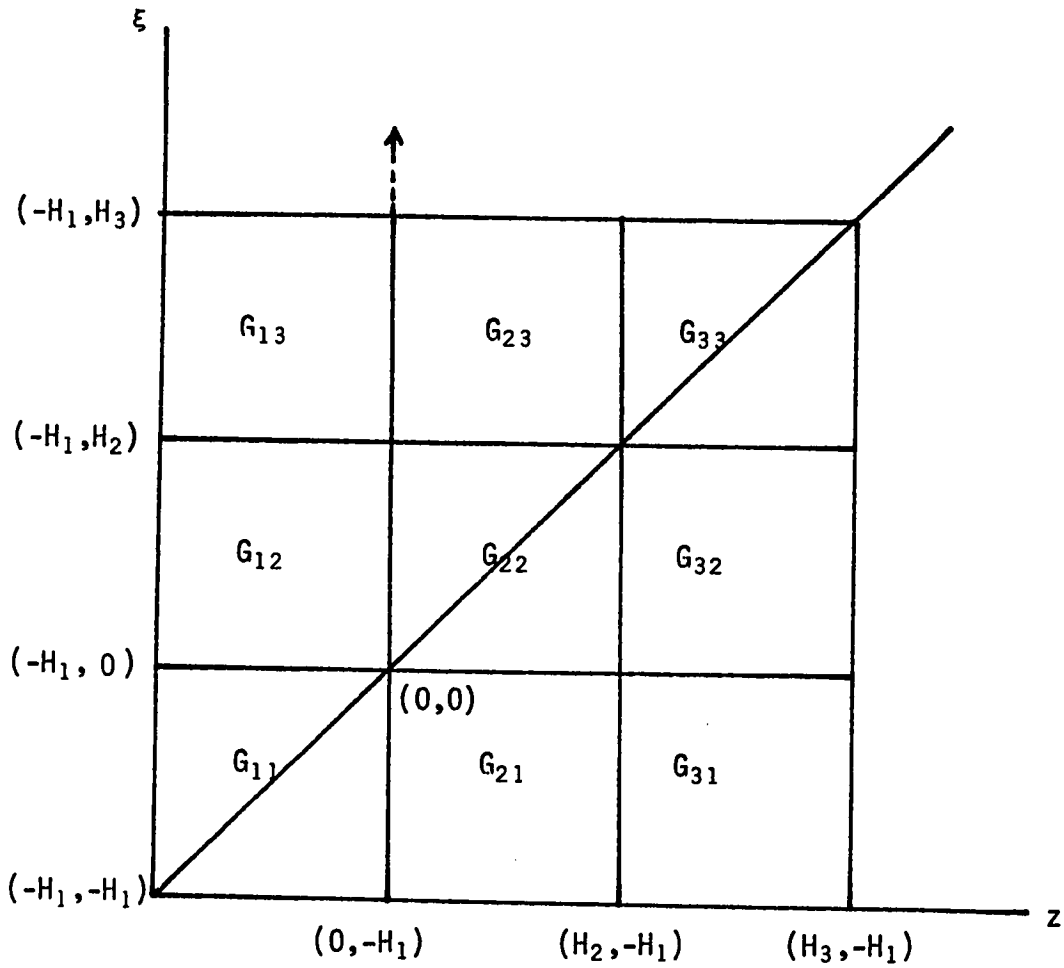

Fig. 4. The character of Green's Function for the three layered problem.

Let $G(z, \xi; \lambda) = G_{ij}$

where $i,j = 1, 2, 3$. The subscript i refers to the z-interval, the subscript j refers to the ξ-interval and 1, 2 & 3 refer to the intervals $(-H_1, 0)$, $(0, H_2)$ and (H_2, H_3), respectively (see fig. 4). Then G_{ij} determines the Green's function $G(z, \xi; \lambda)$ completely.

If $-H_1 \leq \xi \leq 0$, then G_{11}, G_{21} and G_{31} satisfy the differential equations

$$\frac{\partial^2 G_{11}}{\partial z^2} + \sigma_1^2 G_{11} = \delta(z - \xi) \tag{2.2.7}$$

$$\frac{\partial^2 G_{21}}{\partial z^2} + \sigma_2^2 G_{21} = 0 \tag{2.2.8}$$

and

$$\frac{\partial^2 G_{31}}{\partial z^2} - \sigma_3^2 G_{31} = 0 \tag{2.2.9}$$

together with the following conditions

$$G_{11}' = 0 \qquad \text{at} \quad z = -H_1 \tag{2.2.10a}$$

(' refers to differentiation with reference to z)

$$G_{11} = G_{21} \qquad \text{at} \quad z = 0 \tag{2.2.10b}$$

$$\mu_1 G_{11}' = \mu_2 G_{21}' \qquad \text{at} \quad z = 0 \tag{2.2.10c}$$

$$G_{21} = G_{31} \qquad \text{at} \quad z = H_2 \tag{2.2.10d}$$

$$\mu_2 G'_{21} = \mu_3 G'_{31} \qquad \text{at} \quad z = H_2 \tag{2.2.10e}$$

$$G'_{31} = 0 \qquad \text{at} \quad z = H_3 \tag{2.2.10f}$$

$$G_{11}(z, \xi+0; \lambda) = G_{11}(z, \xi-0; \lambda) \tag{2.2.10h}$$

$$\lim_{z \to \xi+0} G'_{11} = \lim_{z \to \xi-0} G'_{11} + \frac{1}{\mu_1} \tag{2.2.10k}$$

From (2.2.7) — (2.2.9), (2.2.10a), (2.2.10f) we have

$$G_{11} = A \cos\sigma_1 (z + H_1) \qquad -H_1 < z < \xi$$

$$= B_1 \cos\sigma_1 (z + H_1) + C_1 \sin\sigma_1 (z + H_1) \qquad \xi < z < 0$$

$$G_{21} = B_2 \cos\sigma_2 (z - H_2) + C_2 \sin\sigma_2 (z - H_2) \qquad 0 < z < H_2$$

and

$$G_{31} = D \cosh\sigma_3 (z - H_3) \qquad H_2 < z < H_3$$

Conditions (2.2.10h) and (2.2.10k) imply

$$A = B_1 + C_1 \tan\sigma_1 (\xi + H_1) \tag{2.2.11}$$

and

$$-B_1 \sin\sigma_1 (\xi + H_1) + C_1 \cos\sigma_1 (\xi + H_1)$$

$$= -A \sin\sigma_1 (\xi + H_1) + \frac{1}{\sigma_1 \mu_1} \tag{2.2.12}$$

Substitute the value of A from (2.2.11) in (2.2.12) we have

$$C_1 = \frac{\cos\sigma_1(\xi + H_1)}{\mu_1\sigma_1}$$

whence

$$G_{11} = B_1\cos\sigma_1(z+H_1) + \frac{(\sin\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1))}{\mu_1\sigma_1} \qquad -H_1 < z < \xi$$

$$= B_1\cos\sigma_1(z+H_1) + \frac{(\sin\sigma_1(z+H_1)\cos\sigma_1(\xi+H_1))}{\mu_1\sigma_1} \qquad \xi < z < 0$$

$$G_{21} = B_2\cos\sigma_2(z - H_2) + C_2\sin\sigma_2(z - H_2) \qquad 0 < z < H_2$$

$$G_{31} = D\cosh\sigma_3(z - H_3) \qquad H_2 < z < H_3$$

Since $G_{21} = G_{31}$ and $\mu_2 G'_{21} = \mu_3 G'_{31}$ at $z = H_2$ we have

$$B_2 = D\cosh\sigma_3(H_2 - H_3)$$

and

$$C_2 = D\frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sinh\sigma_3(H_2 - H_3) \qquad (2.2.13)$$

Therefore

$$G_{21} = B_2\left[\cos\sigma_2(z-H_2) + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sin\sigma_2(z-H_2)\tanh\sigma_3(H_2-H_3)\right]$$

Since $G_{11} = G_{21}$ and $\mu_1 G'_{11} = \mu_2 G'_{21}$ at $z = 0$

we have

$$B_1 = \sec\sigma_1 H_1 \left[B_2 \cos\sigma_2 H_2 - C_2 \sin\sigma_2 H_2 - \frac{(\sin\sigma_1 H_1 \cos\sigma_1(\xi + H_1))}{\mu_1 \sigma_1} \right] \tag{2.2.14}$$

and $$- B_1 \mu_1 \sigma_1 \sin\sigma_1 H_1 + \cos\sigma_1 H_1 \cos\sigma_1(\xi + H_1)$$

$$= B_2 \mu_2 \sigma_2 \sin\sigma_2 H_2 + C_2 \mu_2 \sigma_2 \cos\sigma_2 H_2 \tag{2.2.15}$$

Substituting B_1 from (2.2.14) in (2.2.15) we have

$$C_2(\mu_1 \sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 H_2 - \mu_2 \sigma_2 \cos\sigma_2 H_2)$$

$$= B_2(\mu_2 \sigma_2 \sin\sigma_2 H_2 + \mu_1 \sigma_1 \tan\sigma_1 H_1 \cos\sigma_2 H_2)$$

$$- \cos\sigma_1 H_1 \cos\sigma_1(\xi + H_1) - \tan\sigma_1 H_1 \sin\sigma_1 H_1 \cos\sigma_1(\xi + H_1)$$

$$\Rightarrow \quad C_2 = B_2\left(\frac{\mu_2 \sigma_2 \sin\sigma_2 H_2 + \mu_1 \sigma_1 \tan\sigma_1 H_1 \cos\sigma_2 H_2}{\mu_1 \sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 H_2 - \mu_2 \sigma_2 \cos\sigma_2 H_2}\right)$$

$$- \left(\frac{\cos\sigma_1(\xi + H_1)(\cos\sigma_1 H_1 + \tan\sigma_1 H_1 \sin\sigma_1 H_1)}{\mu_1 \sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 H_2 - \mu_2 \sigma_2 \cos\sigma_2 H_2}\right)$$

$$\Rightarrow \quad C_2 = B_2 \gamma_1 - \frac{\cos\sigma_1(\xi + H_1)}{\cos\sigma_1 H_1 (\mu_1 \sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 H_2 - \mu_2 \sigma_2 \cos\sigma_2 H_2)} \tag{2.2.16}$$

where $$\gamma_1 = \frac{\mu_1 \sigma_1 \sin\sigma_2 H_2 + \mu_1 \sigma_1 \tan\sigma_1 H_1 \cos\sigma_2 H_2}{\mu_1 \sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 H_2 - \mu_2 \sigma_2 \cos\sigma_2 H_2} \tag{2.2.17}$$

From equation (2.2.13) we substitute the value of C_2 in (2.2.16) to get

$$B_2 \frac{\mu_3\sigma_3}{\mu_2\sigma_2} \tanh \sigma_3(H_2 - H_3) - B_2\gamma_1$$

$$= - \frac{\cos\sigma_1(\xi + H_1)}{\cos\sigma_1 H_1(\mu_1\sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 H_2 - \mu_2\sigma_2 \cos\sigma_2 H_2)}$$

$$\Rightarrow B_2 = \frac{\cos\sigma_1(\xi + H_1)}{[(\gamma_1 - \frac{\mu_3\sigma_3}{\mu_2\sigma_2} \tanh\sigma_3(H_2-H_3))\cos\sigma_1 H_1 \cos\sigma_2 H_2(\mu_1\sigma_1 \tan\sigma_1 H_1 \tan\sigma_2 H_2 - \mu_2\sigma_2)]}$$

$$\Rightarrow B_2 = \frac{\mu_2\sigma_2 \cos\sigma_1(\xi + H_1)}{[(\mu_2\sigma_2\gamma_1 - \mu_3\sigma_3 \tanh\sigma_3(H_2-H_3))\cos\sigma_1 H_1 \cos\sigma_2 H_2(\mu_1\sigma_1 \tan\sigma_1 H_1 \tan\sigma_2 H_2 - \mu_2\sigma_2)]}$$

$$= \frac{\mu_2\sigma_2 \cos\sigma_1(\xi + H_1)}{\Delta_1 \cos\sigma_1 H_1 \cos\sigma_2 H_2} \tag{2.2.18}$$

where

$$\Delta_1 = \mu_2^2\sigma_2^2 \tan\sigma_2 H_2 + \mu_1\sigma_1\mu_2\sigma_2 \tan\sigma_1 H_1 + \mu_2\sigma_2\mu_3\sigma_3 \tanh\sigma_3(H_2 - H_3)$$

$$- \mu_1\sigma_1\mu_3\sigma_3 \tanh\sigma_3(H_2 - H_3) \tan\sigma_1 H_1 \tan\sigma_2 H_2 \tag{2.2.19}$$

Equation (2.2.13) gives

$$C_2 = \frac{[\mu_3\sigma_3 \cos\sigma_1(\xi + H_1) \tanh\sigma_3(H_2 - H_3)]}{\Delta_1 \cos\sigma_1 H_1 \cos\sigma_2 H_2} \tag{2.2.20}$$

substituting the value of C_2 and B_2 in (2.2.14) , we have

$$B_1 = \sec\sigma_1 H_1 \left[\frac{\mu_2\sigma_2 \cos\sigma_2 H_2 \cos\sigma_1(\xi+H_1)}{\Delta_1 \cos\sigma_1 H_1 \cos\sigma_2 H_2} - \frac{\mu_3\sigma_3 \cos\sigma_1(\xi+H_1)\sin\sigma_2 H_2 \tanh\sigma_3(H_2-H_3)}{\Delta_1 \cos\sigma_1 H_1 \cos\sigma_2 H_2} - \frac{(\sin\sigma_1 H_1 \cos\sigma_1(\xi+H_1)}{\mu_1\sigma_1} \right] \qquad (2.2.21)$$

from (2.2.18) and (2.2.20) — (2.2.21) we conclude that

$$G_{11} = \frac{1}{\Delta_1 \cos^2\sigma_1 H_1} \left[\cos\sigma_1(z + H_1)\cos\sigma_1(\xi + H_1)M_1 \right] + \frac{\cos\sigma_1(z + H_1)\sin\sigma_1\xi}{\mu_1\sigma_1\cos\sigma_1 H_1} \qquad -H_1 < z < \xi$$

$$= \frac{1}{\Delta_1 \cos^2\sigma_1 H_1} \left[\cos\sigma_1(z + H_1)\cos\sigma_1(\xi + H_1)M_1 \right] + \frac{\cos\sigma_1(\xi + H_1)\sin\sigma_1 z}{\mu_1\sigma_1\cos\sigma_1 H_1} \qquad \xi < z < 0$$

$$= \frac{M_1 \cos\sigma_1(z + H_1)\cos\sigma_1(\xi + H_1)}{\Delta_1 \cos^2\sigma_1 H_1} + \frac{1}{\mu_1\sigma_1\cos\sigma_1 H_1} B(z,\xi),$$

where $M_1 = \mu_2\sigma_2 - \mu_3\sigma_3 \tanh\sigma_3(H_2 - H_3)\tan\sigma_2 H_2.$ (2.2.22)

and $B(z,\xi) = \cos\sigma_1(z+H_1)\sin\sigma_1\xi\theta(\xi-z)+\cos\sigma_1(\xi+H_1)\sin\sigma_1 z\theta(z-\xi)$

$$G_{21} = \frac{\cos\sigma_1(\xi + H_1)\ \tilde{A}(z)}{\Delta_1\cos\sigma_1 H_1\cos\sigma_2 H_2}$$

where

$$\tilde{A}(z) = \mu_2\sigma_2\cos\sigma_2(z-H_2) + \mu_3\sigma_3\tanh\sigma_3(H_2-H_3)\sin\sigma_2(z-H_2) \qquad (2.2.23)$$

and $$G_{31} = \frac{\mu_2\sigma_2\cosh\sigma_3(z - H_3)\cos\sigma_1(\xi + H_1)}{\Delta_1\cos\sigma_1 H_1\cos\sigma_2 H_2\cosh\sigma_3(H_2 - H_3)}$$

where $\theta(\xi - z) = 1 \qquad \xi > z$

$\qquad\qquad = 0 \qquad \xi < z$

is the Heaviside unit function.

If $0 \leq \xi \leq H_2$, then G_{12} , G_{22} and G_{32} satisfy the differential equations

$$\frac{\partial^2 G_{12}}{\partial z^2} + \sigma_1^2 G_{12} = 0 \qquad (2.2.24)$$

$$\frac{\partial^2 G_{22}}{\partial z^2} + \sigma_2^2 G_{22} = \delta(z - \xi) \tag{2.2.25}$$

and

$$\frac{\partial^2 G_{32}}{\partial z^2} - \sigma_3^2 G_{32} = 0 \tag{2.2.26}$$

together with the modified form of conditions (2.2.10a) — (2.2.10k).

From (2.2.24) — (2.2.26), (2.2.10a) and (2.2.10f) we have

$$G_{12} = A \cos \sigma_1 (z + H_1) \qquad - H_1 < z < 0$$

$$G_{22} = B_1 \cos \sigma_2 (z - H_2) + C_1 \sin \sigma_2 (z - H_2) \qquad 0 < z < \xi$$

$$= B_2 \cos \sigma_2 (z - H_2) + C_2 \sin \sigma_2 (z - H_2) \qquad \xi < z < H_2$$

and

$$G_{32} = D \cosh \sigma_3 (z - H_3) \qquad H_2 < z < H_3$$

From the modified conditions (2.2.10h) — (2.2.10k) and (2.2.10b) — (2.2.10e) we have from the above forms of G_{12} , G_{22} and G_{32} , the following equations:

$$A \cos \sigma_1 H_1 = B_1 \tag{2.2.27}$$

$$- A\mu_1\sigma_1 \sin\sigma_1 H_1 = \mu_2\sigma_2 C_1 \tag{2.2.28}$$

$$D \cosh\sigma_3(H_2 - H_3) = B_2 \tag{2.2.29}$$

$$D\mu_3\sigma_3 \sinh \sigma_3(H_2 - H_3) = \mu_2\sigma_2 C_2 \tag{2.2.30}$$

$$B_1\cos\sigma_2\xi + C_1\sin\sigma_2\xi = B_2\cos\sigma_2(\xi-H_2) + C_2\sin\sigma_2(\xi-H_2) \tag{2.2.31}$$

$$- B_1\sin\sigma_2\xi + C_1\cos\sigma_2\xi + \frac{1}{\mu_2\sigma_2} = - B_2\sin\sigma_2(\xi-H_2) + C_2\cos\sigma_2(\xi-H_2) \tag{2.2.32}$$

Multiplying (2.2.31) by $\cos\sigma_2(\xi - H_2)$ and subtracting (2.2.32) multiplied by $\sin\sigma_2(\xi - H_2)$ gives

$$B_2 = B_1\cos\sigma_2 H_2 + C_1\sin\sigma_2 H_2 - \frac{\sin\sigma_2(\xi - H_2)}{\mu_2\sigma_2}$$

Likewise

$$C_2 = - B_1\sin\sigma_2 H_2 + C_1\cos\sigma_2 H_2 + \frac{\cos\sigma_2(\xi - H_2)}{\mu_2\sigma_2}$$

Using (2.2.27) — (2.2.30) we therefore have

$$\frac{C_2}{B_2} = \frac{\mu_3\sigma_3}{\mu_2\sigma_2} \tanh \sigma_3(H_2 - H_3)$$

$$= \frac{-(\mu_1\sigma_1\sin\sigma_1 H_1\cos\sigma_2 H_2 + \mu_2\sigma_2\cos\sigma_1 H_1\sin\sigma_2 H_2)A + \cos\sigma_2(\xi-H_2)}{(-\mu_1\sigma_1\sin\sigma_1 H_1\sin\sigma_2 H_2 + \mu_2\sigma_2\cos\sigma_1 H_1\cos\sigma_2 H_2)A - \sin\sigma_2(\xi-H_2)}$$

or $A(-\mu_1\sigma_1\mu_3\sigma_3\tanh\sigma_3(H_2-H_3)\sin\sigma_1H_1\sin\sigma_2H_2+\mu_2\sigma_2\mu_3\sigma_3\tanh\sigma_3(H_2-H_3)$

$\cos\sigma_1H_1\cos\sigma_2H_2+\mu_1\sigma_1\mu_2\sigma_2\sin\sigma_1H_1\cos\sigma_2H_2+\mu_2^2\sigma_2^2\cos\sigma_1H_1\sin\sigma_2H_2)$

$=(\mu_2\sigma_2\cos\sigma_2(\xi-H_2)+\mu_3\sigma_3\tanh\sigma_3(H_2-H_3)\sin\sigma_2(\xi-H_2))$

$\therefore \quad A\cos\sigma_1H_1\cos\sigma_2H_2\Delta_1=\tilde{A}(\xi)$

i.e. $$A=\frac{\tilde{A}(\xi)}{\Delta_1\cos\sigma_1H_1\cos\sigma_2H_2}$$

where $\tilde{A}(\xi)$ and Δ_1 are given respectively by (2.2.23) and (2.2.19).

Multiplying (2.2.31) by $\cos\sigma_2\xi$ and subtracting (2.2.32) multiplied by $\sin\sigma_2\xi$ gives

$$B_1=B_2\cos\sigma_2H_2-C_2\sin\sigma_2H_2+\frac{\sin\sigma_2\xi}{\mu_2\sigma_2}$$

Likewise $$C_1=B_2\sin\sigma_2H_2+C_2\cos\sigma_2H_2-\frac{\cos\sigma_2\xi}{\mu_2\sigma_2}$$

Using (2.2.27) —— (2.2.30) we, therefore, have

$$\frac{C_1}{B_1}=\frac{-\mu_1\sigma_1}{\mu_2\sigma_2}\tan\sigma_1H_1$$

$$=\frac{(\cosh\sigma_3(H_2-H_3)\sin\sigma_2H_2+\frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sinh\sigma_3(H_2-H_3)\cos\sigma_2H_2)D-\frac{\cos\sigma_2\xi}{\mu_2\sigma_2}}{(\cosh\sigma_3(H_2-H_3)\cos\sigma_2H_2-\frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sinh\sigma_3(H_2-H_3)\sin\sigma_2H_2)D+\frac{\sin\sigma_2\xi}{\mu_2\sigma_2}}$$

or $$D\Big[-\mu_2\sigma_2\mu_1\sigma_1\tan\sigma_1H_1\cosh\sigma_3(H_2-H_3)\cos\sigma_2H_2+$$

$$\mu_1\sigma_1\mu_3\sigma_3\tan\sigma_1H_1\sinh\sigma_3(H_2-H_3)\sin\sigma_2H_2-\mu_2^2\sigma_2^2\cosh\sigma_3(H_2-H_3)$$

$$\sin\sigma_2H_2-\mu_2\sigma_2\mu_3\sigma_3\sinh\sigma_3(H_2-H_3)\cos\sigma_2H_2\Big]$$

$$= -\mu_2\sigma_2\cos\sigma_2\xi+\mu_1\sigma_1\tan\sigma_1H_1\sin\sigma_2\xi$$

$$\therefore \quad -D\cosh\sigma_3(H_2-H_3)\cos\sigma_2H_2\Delta_1 = -\mu_2\sigma_2\cos\sigma_2\xi+\mu_1\sigma_1\tan\sigma_1H_1\sin\sigma_2\xi$$

i.e. $$D = \frac{\mu_2\sigma_2\cos\sigma_2\xi-\mu_1\sigma_1\tan\sigma_1H_1\sin\sigma_2\xi}{\Delta_1\cosh\sigma_3(H_2-H_3)\cos\sigma_2H_2},$$

whence

$$G_{12} = \frac{\tilde{A}(\xi)\cos\sigma_2(z+H_1)}{\Delta_1\cos\sigma_1H_1\cos\sigma_2H_2} \qquad -H_1 < z < 0$$

$$G_{22} = \frac{\tilde{A}(\xi)}{\Delta_1\mu_2\sigma_2\cos\sigma_2H_2}\Big[\mu_2\sigma_2\cos\sigma_2z-\mu_1\sigma_1\tan\sigma_1H_1\sin\sigma_2z\Big] \qquad 0 < z < \xi$$

$$= \frac{\tilde{A}(z)}{\Delta_1\mu_2\sigma_2\cos\sigma_2H_2}\Big[\mu_2\sigma_2\cos\sigma_2\xi-\mu_1\sigma_1\tan\sigma_1H_1\sin\sigma_2\xi\Big] \qquad \xi < z < H_2$$

By using

$$\mu_1\sigma_1\tan\sigma_1H_1 = \frac{\Delta_1-\mu_2^2\sigma_2^2\tan\sigma_2H_2-\mu_2\sigma_2\mu_3\sigma_3\tanh\sigma_3(H_2-H_3)}{\mu_2\sigma_2-\mu_3\sigma_3\tanh\sigma_3(H_2-H_3)\tan\sigma_2H_2} \qquad (2.2.33)$$

we can put G_{22} into the form

$$G_{22} = \frac{\tilde{A}(z)\tilde{A}(\xi)}{M_1\Delta_1\cos^2\sigma_2 H_2} - \frac{\tilde{A}(\xi)\sin\sigma_2 z\theta(\xi-z) + \tilde{A}(z)\sin\sigma_2\xi\theta(z-\xi)}{M_1\mu_2\sigma_2\cos^2\sigma_2 H_2} \qquad 0 < z < H_2$$

Finally

$$G_{32} = \frac{\cosh\sigma_3(z-H_3)}{\Delta_1\cos\sigma_2 H_2\cosh\sigma_3(H_2-H_3)}(\mu_2\sigma_2\cos\sigma_2\xi - \mu_1\sigma_1\tan\sigma_1 H_1\sin\sigma_2\xi) \quad H_2 < z < H_3$$

where $\tilde{A}(z)$, M_1 and Δ_1 are given by (2.2.23), (2.2.22) and (2.2.19) respectively.

If $H_2 \leq \xi \leq H_3$, then G_{13}, G_{23} and G_{33} satisfy the differential equations

$$\frac{\partial^2 G_{13}}{\partial z^2} + \sigma_1^2 G_{13} = 0 \tag{2.2.34}$$

$$\frac{\partial^2 G_{23}}{\partial z^2} + \sigma_2^2 G_{23} = 0 \tag{2.2.35}$$

and

$$\frac{\partial^2 G_{33}}{\partial z^2} - \sigma_3^2 G_{33} = \delta(z - \xi) \tag{2.2.36}$$

together with conditions (2.2.10a) — (2.2.10k) (suitably modified).

G_{13} , G_{23} and G_{33} can be obtained in the same manner as G_{11} , G_{21} and G_{31} , so we list down their expressions without giving detailed calculations.

$$G_{13} = \frac{\mu_2\sigma_2\cos\sigma_1(z+H_1)\cos\sigma_1(z+H_1)}{\Delta_1\cos\sigma_1 H_1\cos\sigma_2 H_2\cosh\sigma_3(H_2-H_3)} \qquad -H_1 < z < 0$$

$$G_{23} = \frac{\mu_2\sigma_2\cosh\sigma_3(\xi-H_3)\tilde{A}(z)}{\Delta_1 M_1\cos\sigma_2 H_2\cosh\sigma_3(H_2-H_3)} - \frac{\cosh\sigma_3(\xi-H_3)}{M_1\cos\sigma_2 H_2\cosh\sigma_3(H_2-H_3)} \qquad 0 < z < H_2$$

and

$$G_{33} = \frac{\mu_2^2\sigma_2^2\cosh\sigma_3(\xi-H_3)\cosh\sigma_3(z-H_3)}{M_1\Delta_1\cos^2\sigma_2 H_2\cosh^2\sigma_3(H_2-H_3)}$$

$$- \frac{\cosh\sigma_3(\xi-H_3)\cosh\sigma_3(z-H_3)(\mu_2\sigma_2\tanh\sigma_3(H_2-H_3) - \mu_3\sigma_3\tan\sigma_2 H_2)}{M_1\mu_3\sigma_3}$$

$$- \frac{1}{\mu_3\sigma_3}\{\cosh\sigma_3(\xi-H_3)\sinh\sigma_3(z-H_3)\theta(\xi-z)$$

$$+ \cosh\sigma_3(z-H_3)\sinh\sigma_3(\xi-H_3)\theta(z-\xi)\} \qquad H_2 < z < H_3$$

Various component of the Green's functions for the problem under consideration are given by:

$$G_{11} = \frac{1}{\Delta_1 \cos^2\sigma_1 H_1}\left[M_1 \cos\sigma_1(z+H_1)\cos\sigma_2(\xi+H_1)\right]$$

$$+ \frac{1}{\mu_1\sigma_1\cos\sigma_1 H_1}\left[\cos\sigma_1(z+H_1)\sin\sigma_1\xi\theta(\xi-z) + \cos\sigma_1(\xi+H_1)\sin\sigma_1 z\theta(z-\xi)\right]$$

$$-H_1 < z < 0 \qquad (2.2.37)$$

$$G_{21} = \frac{\tilde{A}(z)\cos\sigma_1(\xi+H_1)}{\Delta_1\cos\sigma_1 H_1\cos\sigma_2 H_2} \qquad 0 < z < H_2 \qquad (2.2.38)$$

$$G_{31} = \frac{\mu_2\sigma_2\cosh\sigma_3(z-H_3)\cos\sigma_1(\xi+H_1)}{\Delta_1\cos\sigma_1 H_1\cos\sigma_2 H_2\cosh\sigma_3(H_2-H_3)} \qquad H_2 < z < H_3 \qquad (2.2.39)$$

$$G_{12} = \frac{\tilde{A}(\xi)\cos\sigma_1(z+H_1)}{\Delta_1\cos\sigma_1 H_1\cos\sigma_2 H_2} \qquad -H_1 < z < 0 \qquad (2.2.40)$$

$$G_{22} = \frac{\tilde{A}(z)\tilde{A}(\xi)}{M_1\Delta_1\cos^2\sigma_2 H_2} - \frac{(\tilde{A}(\xi)\sin\sigma_2 z\theta(\xi-z) + \tilde{A}(z)\sin\sigma_2\xi\theta(z-\xi))}{M_1\mu_2\sigma_2\cos^2\sigma_2 H_2}$$

$$0 < z < H_2 \qquad (2.2.41)$$

$$G_{32} = \frac{\mu_2\sigma_2\cosh\sigma_3(z-H_3)\tilde{A}(\xi)}{M_1\Delta_1\cos^2\sigma_2 H_2\cosh\sigma_3(H_2-H_3)} - \frac{\cosh\sigma_3(z-H_3)}{M_1\cos\sigma_2 H_2\cosh\sigma_3(H_2-H_3)}$$

$$H_2 < z < H_3 \qquad (2.2.42)$$

$$G_{13} = \frac{\mu_2\sigma_2\cosh\sigma_3(\xi-H_3)\cos\sigma_1(z+H_1)}{\Delta_1\cos\sigma_1H_1\cos\sigma_2H_2\cosh\sigma_3(H_2-H_3)} \qquad -H_1 < z < 0 \qquad (2.2.43)$$

$$G_{23} = \frac{\tilde{A}(z)\,\mu_2\sigma_2\cosh\sigma_3(\xi-H_3)}{M_1\Delta_1\cos^2\sigma_2H_2\cosh\sigma_3(H_2-H_3)} - \frac{\cosh\sigma_3(\xi-H_3)}{M_1\cos\sigma_2H_2\cosh\sigma_3(H_2-H_3)}$$

$$0 < z < H_2 \qquad (2.2.44)$$

$$G_{33} = \frac{\mu_2^2\sigma_2^2\cosh\sigma_3(\xi-H_3)\cosh\sigma_3(z-H_3)}{M_1\Delta_1\cos^2\sigma_2H_2\cosh^2\sigma_3(H_2-H_3)}$$

$$- \frac{\cosh\sigma_3(\xi-H_3)\cosh\sigma_3(z-H_3)(\mu_2\sigma_2\tanh\sigma_3(H_2-H_3) - \mu_3\sigma_3\tan\sigma_2H_2)}{M_1\mu_3\sigma_3}$$

$$- \frac{1}{\mu_3\sigma_3}\{\cosh\sigma_3(\xi-H_3)\sinh\sigma_3(z-H_3)\theta(\xi-z)$$

$$+ \cosh\sigma_3(z-H_3)\sinh\sigma_3(\xi-H_3)\theta(z-\xi)\} \qquad H_2 < z < H_3 \qquad (2.2.45)$$

where Δ_1, M_1 and $\tilde{A}(z)$ are given by (2.2.19), (2.2.22) and (2.2.23) respectively.

We note the symmetry of the Green's function from the fact that

$G_{ij}(z, \xi; \lambda) = G_{ji}(\xi, z; \lambda) \qquad i,j = 1, 2, 3.$

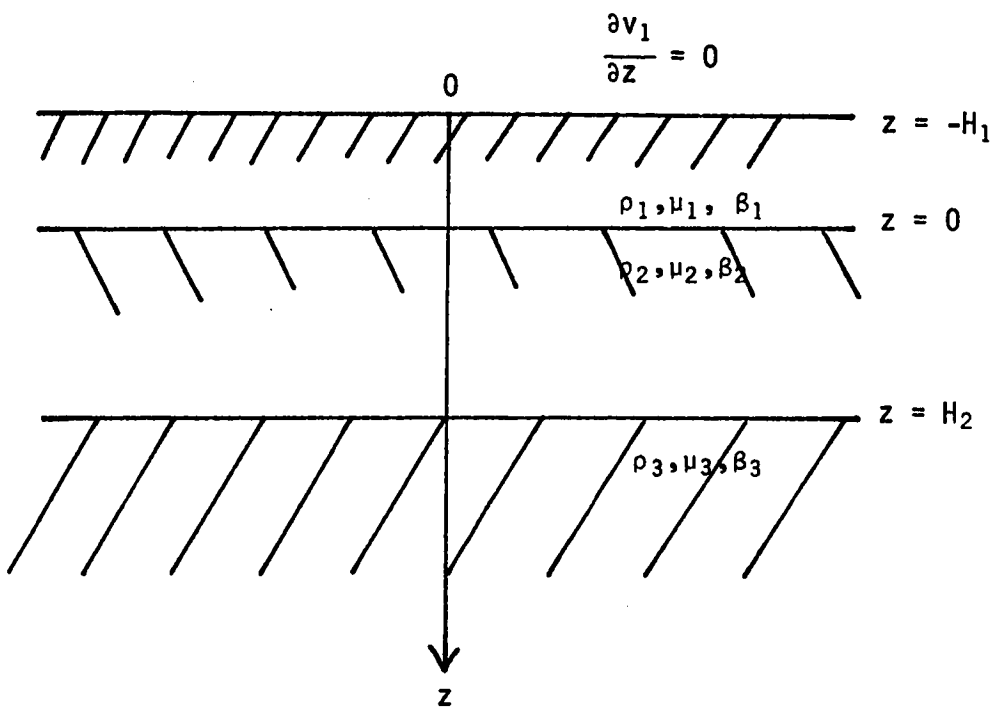

Fig. 5. The geometry of the problem of two layers over a half-space.

2.3 Green's Function for Love wave operator for two layers over a half-space

In this section we shall construct the Green's function for the Love wave operator for two uniform layers over a uniform half-space. We consider a layer of infinite depth, rigidity μ_3, shear velocity β_3 and density ρ_3, overlaid by two infinite strips, consisting of a layer of finite depth H_2, density ρ_2, rigidity $\mu_2(<\mu_3)$ and shear velocity $\beta_2(<\beta_3)$, and another layer of depth $H_1(<H_2)$, density ρ_1, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$(see fig. 5). We suppose the density and the rigidity of each layer to be constant, and the top plane surface to be stress-free. The system of equations is given by (2.2.1) — (2.2.6a) with (2.2.6b) replaced by the following condition:

$$\int_0^\infty \mu(z)\,|V(z)|^2 dz < \infty.$$

Let $G(z, \xi; \lambda) = G_{ij}$

where G_{ij} have the same meaning as in section 2.2, with the interval (H_2, H_3) replaced by (H_2, ∞).

If $-H_1 \leq \xi \leq 0$, then G_{11}, G_{21} and G_{31} satisfy the differential equation

$$\frac{\partial^2 G_{11}}{\partial z^2} + \sigma_1^2 G_{11} = \delta(z-\xi) \tag{2.3.1}$$

$$\frac{\partial^2 G_{21}}{\partial z^2} + \sigma_2^2 G_{21} = 0 \qquad (2.3.2)$$

and

$$\frac{\partial^2 G_{31}}{\partial z^2} - \sigma_3^2 G_{31} = 0 \qquad (2.3.3)$$

together with the following conditions

$$G'_{11} = 0 \quad \text{at} \quad z = -H_1 \qquad (2.3.4a)$$

$$G_{11} = G_{21} \quad \text{at} \quad z = 0 \qquad (2.3.4b)$$

$$\mu_1 G'_{11} = \mu_2 G'_{21} \quad \text{at} \quad z = 0 \qquad (2.3.4c)$$

$$G_{21} = G_{31} \quad \text{at} \quad z = H_2 \qquad (2.3.4d)$$

$$\mu_2 G'_{21} = \mu_3 G'_{31} \quad \text{at} \quad z = H_2 \qquad (2.3.4e)$$

$$G_{11}(z, \xi+0;\lambda) = G_{11}(z, \xi-0;\lambda) \qquad (2.3.4f)$$

$$\lim_{z\to\xi+0} G'_{11} = \lim_{z\to\xi-0} G'_{11} + \frac{1}{\mu_1} \qquad (2.3.4k)$$

and

$$\int_{-H_1}^{\infty} \mu(z)|G(z)|^2 dz < \infty \qquad (2.3.4L)$$

From (2.3.1) — (2.3.3) (2.3.4a), (2.3.4L) we have

$$G_{11} = A\cos\sigma_1(z + H_1) \qquad -H_1 < z < \xi$$

$$= B_1\cos\sigma_1(z + H_1) + C_1\sin\sigma_1(z + H_1) \qquad \xi < z < 0$$

$$G_{21} = B_2\cos\sigma_2(z - H_2) + C_2\sin\sigma_2(z - H_2) \qquad 0 < z < H_2$$

and $$G_{31} = D\, e^{-\sigma_3(z - H_2)} \qquad H_2 < z$$

Using the conditions (2.3.4f) , (2.3.4k) we find that

$$G_{11} = B_1\cos\sigma_1(z+H_1) + \frac{\sin\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{\mu_1\sigma_1} \qquad -H_1 < z < \xi$$

$$= B_1\cos\sigma_1(z+H_1) + \frac{\sin\sigma_1(z+H_1)\cos\sigma_1(\xi+H_1)}{\mu_1\sigma_1} \qquad \xi < z < 0$$

$$G_{21} = B_2\cos\sigma_2(z-H_2) + C_2\sin\sigma_2(z-H_2) \qquad 0 < z < H_2$$

and $$G_{31} = D\, e^{-\sigma_3(z - H_2)} \qquad H_2 < z$$

From $G_{21} = G_{31}$ and $\mu_2 G'_{21} = \mu_3 G'_{31}$ at $z = H_2$
we have

$$D = B_2 \tag{2.3.5}$$

and $$C_2\mu_2\sigma_2 = -\mu_3\sigma_3 D$$

$$\Rightarrow \qquad C_2 = \frac{-\mu_3\sigma_3}{\mu_2\sigma_2} B_2 \tag{2.3.6}$$

From $G_{11} = G_{21}$ and $\mu_1 G'_{11} = \mu_2 G'_{21}$ at $z = 0$

we have

$$B_1\cos\sigma_1 H_1 + (\sin\sigma_1 H_1 \cos\sigma_1(\xi+H_1))/\mu_1\sigma_1$$

$$= D(\cos\sigma_2 H_2 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sin\sigma_2 H_2)$$

$$\Rightarrow \quad B_1 = \sec\sigma_1 H_1 \left[D(\cos\sigma_2 H_2 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sin\sigma_2 H_2) - (\frac{\sin\sigma_1 H_1 \cos\sigma_1(\xi+H_1)}{\mu_1\sigma_1})\right] \tag{2.3.7}$$

and $$-\mu_1\sigma_1 B_1 \sin\sigma_1 H_1 + \cos\sigma_1 H_1 \cos\sigma_1(\xi+H_1)$$

$$= D(\mu_2\sigma_2 \sin\sigma_2 H_2 - \mu_3\sigma_3 \cos\sigma_2 H_2)$$

$$\Rightarrow \quad -\mu_1\sigma_1 \tan\sigma_1 H_1 \left[D\ (\cos\sigma_2 H_2 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sin\sigma_2 H_2) - \frac{\sin\sigma_1 H_1 \cos\sigma_1(\xi+H_1)}{\mu_1\sigma_1}\right]$$

$$+ \cos\sigma_1 H_1 \cos\sigma_1(\xi+H_1) = D(\mu_2\sigma_2 \sin\sigma_2 H_2 - \mu_3\sigma_3 \cos\sigma_2 H_2)$$

$$\Rightarrow \quad D\left[\mu_1\sigma_1 \tan\sigma_1 H_1 \cos\sigma_2 H_2 + \frac{\mu_1\sigma_1\mu_3\sigma_3}{\mu_2\sigma_2}\tan\sigma_1 H_1 \sin\sigma_2 H_2\right.$$

$$\left. + \mu_2\sigma_2 \sin\sigma_2 H_2 - \mu_3\sigma_3 \cos\sigma_2 H_2\right]$$

$$= \tan\sigma_1 H_1 \sin\sigma_1 H_1 \cos\sigma_1(\xi+H_1) + \cos\sigma_1 H_1 \cos\sigma_1(\xi+H_1)$$

$$\Rightarrow \quad D\left[\mu_1\sigma_1\mu_2\sigma_2 \tan\sigma_1 H_1 + \mu_1\sigma_1\mu_3\sigma_3 \tan\sigma_1 H_1 \tan\sigma_2 H_2 + \mu_2^2\sigma_2^2 \tan\sigma_2 H_2 - \mu_2\sigma_2\mu_3\sigma_3\right]$$

$$= \frac{\mu_2\sigma_2 \cos\sigma_1(\xi+H_2)}{\cos\sigma_2 H_2 \cos\sigma_1 H_1}$$

$$\Rightarrow \quad D = \frac{\mu_2\sigma_2 \cos\sigma_1(\xi+H_1)}{\Delta_2 \cos\sigma_2 H_2 \cos\sigma_1 H_1} \tag{2.3.8}$$

where $\Delta_2 = \mu_1\sigma_1\mu_2\sigma_2 \tan\sigma_1 H_1 + \mu_1\sigma_1\mu_3\sigma_3 \tan\sigma_2 H_2 \tan\sigma_1 H_1$

$$-\mu_3\sigma_3\mu_2\sigma_2 + \mu_2^2\sigma_2^2 \tan\sigma_2 H_2 \tag{2.3.9}$$

Also equation (2.3.7) becomes

$$B_1 = \frac{\mu_2\sigma_2 \cos\sigma_1(\xi+H_1)}{\Delta_2 \cos\sigma_2 H_2 \cos^2\sigma_1 H_1}\left(\cos\sigma_2 H_2 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\sin\sigma_2 H_2\right) - \frac{\tan\sigma_1 H_1 \cos\sigma_1(\xi+H_1))}{\mu_1\sigma_1} \tag{2.3.10}$$

From (2.3.5) — (2.3.9) we have

$$G_{11} = \frac{\mu_2\sigma_2 \cos\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{\Delta_2 \cos^2\sigma_1 H_1}\left(1 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\tan\sigma_2 H_2\right)$$

$$-\left[\frac{\tan\sigma_1 H_1 \cos\sigma_1(z+H_1)\cos\sigma_1(\xi+H_1) - \sin\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{\mu_1\sigma_1}\right]$$

$$-H_1 < z < \xi$$

$$= \frac{\mu_2\sigma_2\cos\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{\Delta_2\cos^2\sigma_1 H_1}\left(1 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\tan\sigma_2 H_2\right)$$

$$- \left[\frac{\tan\sigma_1 H_1\cos\sigma_1(z+H_1)\cos\sigma_1(\xi+H_1) - \sin\sigma_1(z+H_1)\cos\sigma_1(\xi+H_1)}{\mu_1\sigma_1}\right] \qquad \xi < z < 0$$

Hence

$$G_{11} = \frac{\mu_2\sigma_2\cos\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{\Delta_2\cos^2\sigma_1 H_1}\left(1 + \frac{\mu_3\sigma_3}{\mu_2\sigma_2}\tan\sigma_2 H_2\right) + \frac{1}{\mu_1\sigma_1\cos\sigma_1 H_1}$$

$$\left[\cos\sigma_1(z+H_1)\sin\sigma_1\xi\,\theta(\xi-z) + \cos\sigma_1(\xi+H_1)\sin\sigma_1 z\,\theta(z-\xi)\right]$$

$$G_{21} = \frac{\cos\sigma_1(\xi+H_1)}{\Delta_2\cos\sigma_2 H_2\cos\sigma_1 H_1}\left(\mu_2\sigma_2\cos\sigma_2(z-H_2) - \mu_3\sigma_3\sin\sigma_2(z-H_2)\right) \qquad 0 < z < H_2$$

and

$$G_{31} = \frac{\mu_2\sigma_2\cos\sigma_1(\xi+H_1)}{\Delta_2\cos\sigma_2 H_2\cos\sigma_1 H_1}\, e^{-\sigma_3(z-H_2)} \qquad H_2 < z$$

If $0 \leq \xi \leq H_2$, then G_{12} , G_{22} and G_{32} satisfy the differential equations

$$\frac{\partial^2 G_{12}}{\partial z^2} + \sigma_1^2 G_{12} = 0 \qquad (2.3.11)$$

$$\frac{\partial^2 G_{22}}{\partial z^2} + \sigma_2^2 G_{22} = \delta(z - \xi) \tag{2.3.12}$$

and

$$\frac{\partial^2 G_{32}}{\partial z^2} - \sigma_3^2 G_{32} = 0 \tag{2.3.13}$$

together with the conditions (2.3.4a) — (2.3.4L) , after suitable modifications for the present case.

From (2.3.11) — (2.2.13) , (2.3.4a) and (2.3.4L) we have

$$G_{12} = A \cos\sigma_1(z + H_1) \qquad - H_1 < z < 0$$

$$G_{22} = B_1 \cos\sigma_2(z - H_2) + C_1 \sin\sigma_2(z - H_2) \qquad 0 < z < \xi$$

$$= B_2 \cos\sigma_2(z - H_2) + C_2 \sin\sigma_2(z - H_2) \qquad \xi < z < H_2$$

and

$$G_{32} = D\, e^{-\sigma_3(z - H_2)} \qquad H_2 < z$$

From the modified conditions (2.3.4b) — (2.3.4k) we have

$$A \cos\sigma_1 H_1 = B_1 \tag{2.3.14}$$

$$- A\, \mu_1\sigma_1 \sin\sigma_1 H_1 = \mu_2\sigma_2 C_1 \tag{2.3.15}$$

$$D = B_2 \tag{2.3.16}$$

$$\mu_2\sigma_2 C_2 = -\ \mu_3\sigma_3 D \tag{2.3.17}$$

$$B_1\cos\sigma_2\xi + C_1\sin\sigma_2\xi = B_2\cos\sigma_2(\xi-H_2) + C_2\sin\sigma_2(\xi-H_2) \tag{2.3.18}$$

$$-B_1\sin\sigma_2\xi + C_1\cos\sigma_2\xi = -B_2\sin\sigma_2(\xi-H_2) + C_2\cos\sigma_2(\xi-H_2) \tag{2.3.19}$$

Using the method described in § (2.2) in dealing with the equations (2.2.27) — (2.2.32) we find from (2.3.14) — (2.3.19) that

$$\frac{C_2}{B_2} = \frac{-\ \mu_3\sigma_3}{\mu_2\sigma_2}$$

$$= \frac{-(\mu_1\sigma_1\sin\sigma_1H_1\cos\sigma_2H_2 + \mu_2\sigma_2\cos\sigma_1H_1\sin\sigma_2H_2)A + \cos\sigma_2(\xi-H_2)}{(-\ \mu_1\sigma_1\sin\sigma_1H_1\sin\sigma_2H_2 + \mu_2\sigma_2\cos\sigma_1H_1\cos\sigma_2H_2)A - \sin\sigma_2(\xi-H_2)}$$

or $A(\mu_3\sigma_3\mu_1\sigma_1\sin\sigma_1H_1\sin\sigma_2H_2 - \mu_2\sigma_2\mu_3\sigma_3\cos\sigma_1H_1\cos\sigma_2H_2$

$$+ \mu_1\sigma_1\mu_2\sigma_2\sin\sigma_1H_1\cos\sigma_2H_2 + \mu_2^2\sigma_2^2\cos\sigma_1H_1\sin\sigma_2H_2)$$

$$= \mu_2\sigma_2\cos\sigma_2(\xi-H_2) - \mu_3\sigma_3\sin\sigma_2(\xi-H_2)$$

$$\therefore\quad A\cos\sigma_1H_1\cos\sigma_2H_2\Delta_2 = \mu_2\sigma_2\cos\sigma_2(\xi-H_2) - \mu_3\sigma_3\sin\sigma_2(\xi-H_2)$$

i.e. $$A = \frac{B(\xi)}{\Delta_2 \cos\sigma_1 H_1 \cos\sigma_2 H_2}$$

where $$B(\xi) = \mu_2\sigma_2\cos\sigma_2(\xi - H_2) - \mu_3\sigma_3\sin\sigma_2(\xi - H_2) \qquad (2.3.20)$$

and Δ_2 is given by (2.3.9)

Also $$\frac{C_1}{B_1} = \frac{-\mu_1\sigma_1}{\mu_2\sigma_2}\tan\sigma_1 H_1$$

$$= \frac{(\mu_2\sigma_2\sin\sigma_2 H_2 - \mu_3\sigma_3\cos\sigma_2 H_2)D - \cos\sigma_2\xi}{(\mu_2\sigma_2\cos\sigma_2 H_2 + \mu_3\sigma_3\sin\sigma_2 H_2)D + \sin\sigma_2\xi}$$

or $$D(-\mu_1\sigma_1\mu_2\sigma_2\cos\sigma_2 H_2\tan\sigma_1 H_1 - \mu_1\sigma_1\mu_3\sigma_3\sin\sigma_2 H_2\tan\sigma_1 H_1 - \mu_2^2\sigma_2^2\sin\sigma_2 H_2$$

$$+ \mu_2\sigma_2\mu_3\sigma_3\cos\sigma_2 H_2) = \mu_1\sigma_1\sin\sigma_1\xi\tan\sigma_1 H_1 - \mu_2\sigma_2\cos\sigma_2\xi$$

$$\therefore \qquad D = \frac{\mu_2\sigma_2\cos\sigma_2\xi - \mu_1\sigma_1\sin\sigma_2\xi\tan\sigma_1 H_1}{\Delta_2\cos\sigma_2 H_2}$$

where

$$G_{12} = \frac{\cos\sigma_1(z+H_1)B(\xi)}{\Delta_2\cos\sigma_1 H_1\cos\sigma_2 H_2} \qquad -H_1 < z < 0$$

$$G_{22} = \frac{B(z)(\mu_2\sigma_2\cos\sigma_2\xi - \mu_1\sigma_1\tan\sigma_1 H_1\sin\sigma_2\xi)}{\mu_2\sigma_2\Delta_2\cos\sigma_2 H_2} \qquad \xi < z$$

$$= \frac{B(\xi)(\mu_2\sigma_2\cos\sigma_2 z - \mu_1\sigma_1\tan\sigma_1 H_1\sin\sigma_2 z)}{\mu_2\sigma_2\Delta_2\cos\sigma_2 H_2} \qquad z < \xi$$

By using

$$\mu_1\sigma_1\tan\sigma_1 H_1 = \frac{\Delta_2 - \mu_2^2\sigma_2^2\tan\sigma_2 H_2 + \mu_3\sigma_3\mu_2\sigma_2}{\mu_2\sigma_2 + \mu_3\sigma_3\tan\sigma_2 H_2} \tag{2.3.21}$$

we can put G_{22} into the form

$$G_{22} = \frac{B(\xi)B(z)}{M_2\Delta_2\cos^2\sigma_2 H_2}$$

$$- \frac{(B(\xi)\sin\sigma_2 z\theta(\xi-z) + B(z)\sin\sigma_2\xi\theta(z-\xi))}{M_2\mu_2\sigma_2\cos\sigma_2 H_2} \qquad 0 < z < H_2$$

Finally

$$G_{32} = \frac{1}{\Delta_2\cos\sigma_2 H_2}(\mu_2\sigma_2\cos\sigma_2\xi - \mu_1\sigma_1\tan\sigma_1 H_1\sin\sigma_2\xi)e^{-\sigma_3(z-H_2)} \qquad H_2 < z$$

where $$M_2 = \mu_2\sigma_2 + \mu_3\sigma_3\tan\sigma_2 H_2 \tag{2.3.22}$$

and Δ_2 , $B(z)$ are defined by (2.3.9) and (2.3.20) respectively.

If $H_2 \leq \xi < \infty$, then G_{13} , G_{23} and G_{33} satisfy the differential equations

$$\frac{\partial^2 G_{13}}{\partial z^2} + \sigma_1^2 G_{13} = 0 \tag{2.3.23}$$

$$\frac{\partial^2 G_{23}}{\partial z^2} + \sigma_1^2 G_{23} = 0 \qquad (2.3.24)$$

and

$$\frac{\partial^2 G_{33}}{\partial z^2} - \sigma_3^2 G_{33} = \delta(z - \xi) \qquad (2.3.25)$$

together with conditions (2.3.4a) — (2.3.4L) (suitably modified).

From (2.3.23) — (2.3.25), (2.3.4a), (2.3.4L) we have

$$G_{13} = A \cos\sigma_1(z + H_1) \qquad -H_1 < z < 0$$

$$G_{23} = B_1 \cos\sigma_2(z - H_2) + C_1 \sin\sigma_2(z - H_2) \qquad 0 < z < H_2$$

and

$$G_{33} = D_1 e^{\sigma_3(z-H_2)} + D_2 e^{-\sigma_3(z-H_2)} \qquad H_2 < z < \xi$$

$$= E\, e^{-\sigma_3(z-H_2)} \qquad H_2 < \xi < z$$

Since $G_{13} = G_{23}$ and $\mu_1 G'_{13} = \mu_2 G'_{23}$ at $z = 0$

we have $A \cos\sigma_1 H_1 = B_1 \cos\sigma_2 H_2 - C_1 \sin\sigma_2 H_2$

=> $A = \sec\sigma_1 H_1 (B_1 \cos\sigma_2 H_2 - C_1 \sin\sigma_2 H_2)$

and $-A\mu_1\sigma_1 \sin\sigma_1 H_1 = B_1 \mu_2 \sigma_2 \sin\sigma_2 H_2 + C_1 \mu_2 \sigma_2 \cos\sigma_2 H_2$

$$\Rightarrow \quad -\mu_1\sigma_1\tan\sigma_1H_1(B_1\cos\sigma_2H_2 - C_1\sin\sigma_2H_2)$$

$$= B_1\mu_2\sigma_2\sin\sigma_2H_2 + C_1\mu_2\sigma_2\cos\sigma_2H_2$$

$$\Rightarrow \quad C_1(\mu_1\sigma_1\tan\sigma_1H_1\sin\sigma_2H_2 - \mu_2\sigma_2\cos\sigma_2H_2)$$

$$= B_1(\mu_2\sigma_2\sin\sigma_2H_2 + \mu_1\sigma_1\tan\sigma_1H_1\cos\sigma_2H_2)$$

$$\Rightarrow \quad C_1 = B_1\left(\frac{\mu_2\sigma_2\sin\sigma_2H_2 + \mu_1\sigma_1\tan\sigma_1H_1\cos\sigma_2H_2}{\mu_1\sigma_1\tan\sigma_1H_1\sin\sigma_2H_2 - \mu_2\sigma_2\cos\sigma_2H_2}\right)$$

$$= B_1\gamma_1$$

where
$$\gamma_1 = \left(\frac{\mu_2\sigma_2\tan\sigma_2H_2 + \mu_1\sigma_1\tan\sigma_1H_1}{\mu_1\sigma_1\tan\sigma_1H_1\tan\sigma_2H_2 - \mu_2\sigma_2}\right) \qquad (2.3.26)$$

$G_{23} = G_{33}$ and $\mu_2G'_{23} = \mu_3G'_{33}$ at $z = H_2$

so we have

$$B_1 = D_1 + D_2$$

and
$$C_1\mu_2\sigma_2 = \mu_3\sigma_3D_1 - \mu_3\sigma_3D_2$$

$$\Rightarrow \quad \mu_2\sigma_2\gamma_1D_1 + \mu_2\sigma_2\gamma_1D_2 = \mu_3\sigma_3D_1 - \mu_3\sigma_3D_2$$

$$\Rightarrow \quad D_2(\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1) = (\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1)D_1$$

$$\Rightarrow \qquad D_1 = \left(\frac{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)D_2$$

$$\Rightarrow \qquad B_1 = D_2\left(1 + \frac{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)$$

$$= D_2\left(\frac{2\mu_3\sigma_3}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)$$

$$C_1 = D_2\left(\frac{2\mu_3\sigma_3\gamma_1}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)$$

and $$A = \frac{D_2}{\cos\sigma_1 H_1}\left(\frac{2\mu_3\sigma_3}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)(\cos\sigma_2 H_2 - \gamma_1\sin\sigma_2 H_2)$$

$$= \frac{-D_2\mu_2\sigma_2}{\cos\sigma_1 H_1}\left(\frac{2\mu_3\sigma_3}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)$$

G_{33} is continuous at $z = \xi$ and $\lim\limits_{z \to \xi+0} G'_{33} = \lim\limits_{z \to \xi-0} G'_{33} + \frac{1}{\mu_3}$

and so

$$E\, e^{-\sigma_3(\xi-H_2)} = D_1 e^{\sigma_3(\xi-H_2)} + D_2 e^{-\sigma_3(\xi-H_2)}$$

$$\Rightarrow \quad E = D_2 + D_1 e^{2\sigma_3(\xi-H_2)}$$

and $$- E\, e^{-\sigma_3(\xi-H_2)} = D_1 e^{\sigma_3(\xi-H_2)} - D_2 e^{-\sigma_3(\xi-H_2)} + \frac{1}{\mu_3\sigma_3}$$

$$\Rightarrow \quad - D_2 e^{-\sigma_3(\xi-H_2)} - D_1 e^{\sigma_3(\xi-H_2)} = D_1 e^{\sigma_3(\xi-H_2)} - D_2 e^{-\sigma_3(\xi-H_2)} + \frac{1}{\mu_3\sigma_3}$$

$$\Rightarrow \quad D_1 = \frac{- e^{-\sigma_3(\xi-H_2)}}{2\mu_3\sigma_3}$$

and $$E = D_2 - \frac{e^{\sigma_3(\xi-H_2)}}{2\mu_3\sigma_3}$$

Moreover, we have

$$D_2 = \left(\frac{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}\right) \frac{- e^{-\sigma_3(\xi-H_2)}}{2\mu_3\sigma_3}$$

$$B_1 = \left(\frac{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}\right)\left(\frac{- e^{-\sigma_3(\xi-H_2)}}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)$$

$$= \frac{- e^{-\sigma_3(\xi-H_2)}}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}$$

$$A = \left(\frac{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}\right)\left(\frac{2\mu_3\sigma_3}{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}\right)$$

$$(\cos\sigma_2 H_2 - \gamma_1 \sin\sigma_2 H_2) \frac{- e^{-\sigma_3(\xi-H_2)}}{2\sigma_3\mu_3\cos\sigma_1 H_1}$$

$$= \frac{-\,e^{-\sigma_3(\xi-H_2)}(\cos\sigma_2 H_2 - \gamma_1 \sin\sigma_2 H_2)}{\cos\sigma_1 H_1(\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1)}$$

$$= \frac{\mu_2\sigma_2(\mu_1\sigma_1 \tan\sigma_1 H_1 \tan\sigma_2 H_2 - \mu_2\sigma_2)e^{-\sigma_3(\xi-H_2)}}{\Delta_2 \cos\sigma_1 H_1(\mu_1\sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 H_2 - \mu_2\sigma_2 \cos\sigma_2 H_2)}$$

$$= \frac{\mu_2\sigma_2 e^{-\sigma_3(\xi-H_2)}}{\Delta_2 \cos\sigma_1 H_1 \cos\sigma_2 H_2}$$

and

$$E = -\left(\frac{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}\right)\frac{e^{-\sigma_3(\xi-H_2)}}{2\mu_3\sigma_3} - \frac{e^{\sigma_3(\xi-H_2)}}{2\mu_3\sigma_3}$$

Therefore, we have

$$G_{13} = \frac{\mu_2\sigma_2 \cos\sigma_1(z+H_1)}{\Delta_2 \cos\sigma_1 H_1 \cos\sigma_2 H_2}\, e^{-\sigma_3(\xi-H_2)}$$

$$G_{23} = B_1(\cos\sigma_2(z-H_2) + \gamma_1 \sin\sigma_2(z-H_2))$$

$$= \frac{-(\cos\sigma_2(z-H_2) + \gamma_1 \sin\sigma_2(z-H_2)}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}\, e^{-\sigma_3(\xi-H_2)}$$

$$= \frac{e^{-\sigma_3(\xi-H_2)}}{\Delta_2 \cos\sigma_2 H_2}(\mu_2\sigma_2 \cos\sigma_2 z - \mu_1\sigma_1 \tan\sigma_1 H_1 \sin\sigma_2 z)$$

and

$$G_{33} = \frac{-e^{\sigma_3(\xi-H_2)}e^{\sigma_3(z-H_2)}}{2\mu_3\sigma_3} - \frac{e^{-\sigma_3(\xi-H_2)}e^{-\sigma_3(z-H_2)}}{2\mu_3\sigma_3}\left(\frac{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}\right)$$

$$z < \xi$$

$$= \frac{-e^{\sigma_3(z-H_2)}e^{\sigma_3(\xi-H_2)}}{2\mu_3\sigma_3} - \frac{e^{-\sigma_3(z-H_2)}e^{-\sigma_3(\xi-H_2)}}{2\mu_3\sigma_3}\left(\frac{\mu_3\sigma_3 - \mu_2\sigma_2\gamma_1}{\mu_3\sigma_3 + \mu_2\sigma_2\gamma_1}\right)$$

$$\xi < z$$

$$= \frac{\mu_2^2\sigma_2^2 e^{-\sigma_3(\xi-H_2)}e^{-\sigma_3(z-H_2)}}{M_2\Delta_2\cos^2\sigma_2H_2} - \frac{e^{-\sigma_3(\xi-H_2)}e^{-\sigma_3(z-H_2)}(\mu_3\sigma_3\tan\sigma_2H_2-\mu_2\sigma_2)}{2\mu_3\sigma_3M_2}$$

$$-\left[\frac{e^{-\sigma_3(\xi-z)}}{2\mu_3\sigma_3}\,\theta(\xi-z) + \frac{e^{-\sigma_3(z-\xi)}}{2\mu_3\sigma_3}\,\theta(z-\xi)\right]$$

where M_2 is given by (2.3.20).

Various components of the Green's function for the half-space problem are given by

$$G_{11} = \frac{M_2\cos\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{\Delta_2\cos^2\sigma_1H_1}$$

$$+ \frac{1}{\mu_1\sigma_1\cos\sigma_1H_1}\,(\cos\sigma_1(z+H_1)\sin\sigma_1\xi\,\theta(\xi-z)$$

$$+ \cos\sigma_1(\xi+H_1)\sin\sigma_1z\,\theta(z-\xi)) \qquad -H_1 < z < 0 \qquad (2.3.27)$$

$$G_{21} = \frac{B(z)\cos\sigma_1(\xi+H_1)}{\Delta_2\cos\sigma_2H_2\cos\sigma_1H_1} \qquad 0 < z < H_2 \qquad (2.3.28)$$

$$G_{31} = \frac{\mu_2\sigma_2\cos\sigma_1(\xi+H_1)e^{-\sigma_3(z-H_2)}}{\Delta_2\cos\sigma_2H_2\cos\sigma_1H_1} \qquad H_2 < z \qquad (2.3.29)$$

$$G_{12} = \frac{\cos\sigma_1(z+H_1)B(\xi)}{\Delta_2\cos\sigma_1H_1\cos\sigma_2H_2} \qquad -H_1 < z < 0 \qquad (2.3.30)$$

$$G_{22} = \frac{B(z)B(\xi)}{M_2\Delta_2\cos^2\sigma_2H_2} - \frac{(B(\xi)\sin\sigma_2z\theta(\xi-z) + B(z)\sin\sigma_2\xi\theta(z-\xi))}{\mu_2\sigma_2M_2\cos\sigma_2H_2}$$

$$0 < z < H_2 \qquad (2.3.31)$$

$$G_{32} = \frac{B(\xi)\mu_2\sigma_2e^{-\sigma_3(z-H_2)}}{M_2\Delta_2\cos^2\sigma_2H_2} - \frac{e^{-\sigma_3(z-H_2)}}{M_2\cos\sigma_2H_2} \qquad H_2 < z \qquad (2.3.32)$$

$$G_{13} = \frac{\mu_2\sigma_2\cos\sigma_1(z+H_1)}{\Delta_2\cos\sigma_1H_1\cos\sigma_2H_2}\, e^{-\sigma_3(\xi-H_2)} \qquad -H_1 < z < 0 \qquad (2.3.33)$$

$$G_{23} = \frac{B(z)\mu_2\sigma_2e^{-\sigma_2(\xi-H_2)}}{M_2\Delta_2\cos^2\sigma_2H_2} - \frac{e^{-\sigma_3(\xi-H_2)}}{M_2\cos\sigma_2H_2} \qquad 0 < z < H_2 \qquad (2.3.34)$$

$$G_{33} = \frac{\mu_2^2\sigma_2^2 e^{-\sigma_3(\xi-H_2)} e^{-\sigma_3(z-H_2)}}{M_2\Delta_2\cos^2\sigma_2 H_2}$$

$$- \frac{e^{-\sigma_3(\xi-H_2)} e^{-\sigma_3(z-H_2)} (\mu_3\sigma_3\tan\sigma_2 H_2 - \mu_2\sigma_2)}{2\mu_3\sigma_3 M_2}$$

$$- \left[\frac{e^{-\sigma_3(\xi-z)}}{2\mu_3\sigma_3}\ \theta(\xi-z) + \frac{e^{-\sigma_3(z-\xi)}}{2\mu_3\sigma_3}\ \theta(z-\xi) \right]$$

$$H_2 < z \qquad (2.3.35)$$

where Δ_2 , $B(z)$ and M_2 are given by (2.3.9), (2.3.20) and (2.3.22) respectively. We note that $G_{ij}(z, \xi; \lambda) = G_{ij}(\xi, z ;\lambda)$ $i,j = 1, 2, 3$ i.e. the Green's function is symmetric.

We remark that the Green's function for the finite depth problem as $H_3 \to \infty$ assumes the same expression as the Green's function for the half-space problem. However, the Green's function for the half-space problem has been derived under more general conditions.

CHAPTER 3

SPECTRAL REPRESENTATION OF THE LOVE WAVE OPERATOR

In this chapter, we shall find the spectral representation of the operators associated with the propagation of Love waves in linearly elastic structures consisting of two infinite homogeneous strips overlying another infinite homogeneous layer of finite depth or a uniform half-space, by using the corresponding Green's functions obtained in Chapter 2.

3.1 The Formula for the Spectral Representation for the Regular Case

Consider the inhomogeneous problem,

$$Ly - \lambda p(z)y = f(z) \tag{3.1.1}$$

together with equations (1.2.2) and (1.2.3)

Then
$$y(z) = \int_a^b G(z, \xi; \lambda) f(\xi) d\xi \tag{3.1.2}$$

is the unique solution of (3.1.1), where $G(z, \xi; \lambda)$ is the Green's function for the system (1.2.1) — (1.2.3).

Moreover, by the Expansion Theorem (§ 1.2) we can set up the series expansion of $y(z)$ in terms of the complete set of normalized eigenfunctions $\{\phi^n(z)\}$ of the associated homogeneous system.

Let $$y(z) = \sum_n \alpha_n \phi^n(z) \tag{3.1.3}$$

Since the eigenfunctions form a complete orthonormal set and the modified orthonormality condition is given by

$$< \phi^m , \phi^n > \equiv \int_a^b A(z)p(z)\phi^m(z)\overline{\phi^n}(z)dz = \delta_{mn}$$

$$\text{where} \quad \delta_{mn} = 0 \quad \text{if} \quad m \neq n$$
$$= 1 \quad \text{if} \quad m = n$$

and $A(z)$ defined by (1.2.5) is a piecewise constant function, then

$$< y(z), \phi^k(z) > = < \sum_n \alpha_n \phi^n(z) , \phi^k(z) >$$

$$= \sum_n \alpha_n < \phi^n , \phi^k >$$

$$= \alpha_k$$

whence $$y(z) = \sum_n < y , \phi^n > \phi^n(z) \tag{3.1.4}$$

If $y = \delta(z-\xi)$, then we get

$$< \delta(z-\xi),\ \phi^n > = \int_a^b A(z)p(z)\overline{\phi^n}(z)\delta(z-\xi)dz$$

$$= A(\xi)p(\xi)\overline{\phi^n}(\xi) \tag{3.1.5}$$

Equations (3.1.4) and (3.1.5) yield the following representation of the delta function:

$$\delta(z-\xi) = \sum_n < \delta(z-\xi),\ \phi^n > \overline{\phi^n}(z)$$

$$= \sum_n A(\xi)p(\xi)\overline{\phi^n}(\xi)\phi^n(z)$$

$$= A(\xi)p(\xi) \sum_n \overline{\phi^n}(\xi)\phi^n(z) \tag{3.1.6}$$

In order to find the bilinear series for the Green's function, we multiply equation (3.1.1) by $A(z)\overline{\phi^n}(z)$ and integrate with respect to z between the limits a to b. We get

$$\int_a^b A(z)\overline{\phi^n}(z)\ L(y)dz - \lambda\int_a^b p(z)A(z)\overline{\phi^n}(z)y(z)dz$$

$$= \int_a^b A(z)\overline{\phi^n}(z)f(z)dz \tag{3.1.7}$$

Now $\displaystyle\int_a^b p(z)A(z)\overline{\phi^n}(z)y(z)dz = < y,\ \phi^n >$

and since L is a self adjoint operator,

$$\int_a^b L(y)\ A(z)\ \overline{\phi^n}(z)dz = \int_a^b y(z)\ A(z)\ L(\overline{\phi^n}(z))\,dz$$

$$= \lambda_n \int_a^b y(z)\ A(z)\ p(z)\ \overline{\phi^n}(z)\,dz$$

$$= \lambda_n < y(z),\ \phi^n(z) > \qquad (3.1.8)$$

where λ_n is the eigenvalue corresponding to the eigenfunction $\phi^n(z)$.

Thus equation (3.1.7) becomes

$$(\lambda_n-\lambda) < y(z),\ \phi^n(z) > = \alpha_n(\lambda_n-\lambda)$$

$$= \int_a^b A(z)\ f(z)\ \overline{\phi^n}(z)dz \qquad (3.1.9)$$

If $\lambda \neq \lambda_n$, then (3.1.9) implies

$$\alpha_n = \frac{1}{(\lambda_n - \lambda)} \int_a^b A(z)\ f(z)\ \overline{\phi^n}(z)dz \qquad (3.1.10)$$

From (3.1.3) and (3.1.10) we obtain

$$y(z) = \sum_n \frac{\phi^n(z)}{\lambda_n - \lambda} \int_a^b A(z)\ f(z)\ \overline{\phi^n}(z)dz \qquad (3.1.11)$$

which is the unique solution of (3.1.1).

In the case $f(z) = \delta(z-\xi)$ $y \equiv G(z, \xi; \lambda)$ and so

$$G(z, \xi ; \lambda) = \sum_n \frac{A(\xi)\phi^n(z)\overline{\phi^n}(\xi)}{\lambda_n - \lambda} \tag{3.1.12}$$

which is the bilinear series for $G(z, \xi ; \lambda)$. It can be shown (Stakgold 1967) that Green's function $G(z, \xi ; \lambda)$ is a meromorphic function of λ with simple poles located at the points $\lambda_1, \lambda_2, \ldots, \lambda_n$ on the real axis as its only singularities. The spectrum of eigenvalues is, therefore, discrete in this case.

Integrating (3.1.12) around a large circle $|\lambda| = R$ in the complex λ - plane we get

$$\lim_{R\to\infty} \int_{|\lambda|=R} G(z, \xi; \lambda)d\lambda = -2\pi i \sum A(\xi)\phi^n(z)\overline{\phi^n}(\xi)$$

where R does not take the values $|\lambda_n|$.

So we have by using (3.1.6)

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{|\lambda|=R} G(z, \xi ; \lambda) = -\frac{\delta(z - \xi)}{p(\xi)} \tag{3.1.13}$$

For the system (2.1.4) — (2.1.8) the formula (3.1.13) becomes

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{|\lambda|=R} G(z, \xi ; \lambda)d\lambda = -\sum \phi^n(z)\overline{\phi^n}(\xi)$$

$$= \frac{-\delta(z - \xi)}{\mu(\xi)} \tag{3.1.14}$$

The equation (3.1.14) enables us to find the spectrum and the corresponding eigenfunctions from the knowledge of Green's function. The set of poles of the integrand, all lying on the real axis in the complex λ - plane, constitutes the spectrum of eigenvalues. The corresponding normalized eigenfunction may then be identified from the sum of the residues. We shall illustrate the method by means of a three-layer model, with piecewise constant distribution of the rigidity and the density functions in the next section.

3.2 Spectral Representation of the Love Wave Operator for the regular case

In this section we find the spectral representation of the Love wave operator for an infinite strip consisting of a layer of depth H_3-H_2, rigidity μ_3, shear velocity β_3 and density ρ_3, overlaid by two infinite strips, consisting of a layer of depth $H_2(<H_3)$, density ρ_2, rigidity $\mu_2(<\mu_3)$ and shear velocity $\beta_2(<\beta_3)$ and another layer of depth $H_1(<H_2)$, density ρ_1, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$ (see Figure 2).

We shall use the formula (3.1.14)

$$\lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} G(z,\xi;\lambda)d\lambda = -\sum \phi^n(z)\overline{\phi^n}(\xi) = \frac{-\delta(z-\xi)}{\mu(\xi)}$$

to find the spectrum and the corresponding set of eigenfunctions $\{\phi^n(z)\}$.

Consider

$$I_{11} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{11}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{1}{\Delta_1 \cos^2\sigma_1 H_1} \left[M_1 \cos\sigma_1(z+H_1)\cos\sigma_2(\xi+H_1)\right]$$

$$+ \frac{1}{\mu_1\sigma_1\cos\sigma_1 H_1} \left[\cos\sigma_1(z+H_1)\sin\sigma_1\xi\theta(\xi-z) + \cos\sigma_1(\xi+H_1)\sin\sigma_1 z\theta(z-\xi)\right]d\lambda \tag{3.2.1}$$

where M_1 is given by (2.2.22).

We note that the integrand is an even function of σ_1, σ_2 and σ_3 and therefore the points $\lambda = \frac{\omega^2}{\beta_1^2}$, $\frac{\omega^2}{\beta_2^2}$ and $\frac{\omega^2}{\beta_3^2}$ are not branch points of the integrand. The only singularities of G_{11} are simple and real poles, which are the roots of $\Delta_1 = 0$ (See (2.2.19)). This observation is valid for all the integrands we shall encounter in this section and will not be repeated subsequently.

On evaluating (3.2.1) as the sum of the residues at the poles, we get:

$$I_{11} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{M_1 \cos\sigma_1(z+H_1)\cos\sigma_1(\xi+H_1)}{\Delta_1 \cos^2\sigma_1 H_1} d\lambda$$

$$= -\sum_{n=1}^{\infty} \frac{\cos\sigma_1^{(n)}(z+H_1)\cos\sigma_1^{(n)}(\xi+H_1)(M_1)_{\lambda=\lambda_n}}{\cos^2\sigma_1^{(n)} H_1 \frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_1^n(z)\phi_1^n(\xi) \tag{3.2.2}$$

where

$$\phi_1(z) = \frac{\cos\sigma_1(z+H_1)}{\cos\sigma_1 H_1} \left(\frac{M_1}{\frac{\partial}{\partial\lambda}[-\Delta_1]} \right)^{\frac{1}{2}} \tag{3.2.3}$$

$$\sigma_1^{(n)} = \left(\frac{\omega^2}{\beta_1^2} - \lambda_n \right)^{\frac{1}{2}}, \quad \sigma_2^{(n)} = \left(\frac{\omega^2}{\beta_2^2} - \lambda_n \right)^{\frac{1}{2}}$$

and

$$\sigma_3^{(n)} = \left(\lambda_n - \frac{\omega^2}{\beta_3^2} \right)^{\frac{1}{2}} \tag{3.2.4}$$

The superscript n on ϕ_1 (and on all subsequent ϕ's) refers to evaluation at the pole $\lambda = \lambda_n$ of Δ_1.

Consider

$$I_{21} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{21}(z,\xi;\lambda)\,d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{\tilde{A}(z)\cos\sigma_1(\xi+H_1)}{\Delta_1 \cos\sigma_1 H_1 \cos\sigma_2 H_2}\,d\lambda \tag{3.2.5}$$

where $\tilde{A}(z)$ and Δ_1 are given by (2.2.23) and (2.2.19) respectively.

By evaluating the integrand (3.2.5) as the sum of the residues at the poles, we get

$$I_{21} = - \sum_{n=1}^{\infty} \frac{\cos\sigma_1^{(n)} (\xi+H_1) (\tilde{A}(z))_{\lambda=\lambda_n}}{\cos\sigma_1^{(n)} H_1 \cos\sigma_2^{(n)} H_2 \frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_2^n(z)\phi_1^n(\xi) \tag{3.2.6}$$

where
$$\phi_2(z) = \frac{\tilde{A}(z)}{\cos\sigma_2 H_2} \left(\frac{1}{M_1 \frac{\partial}{\partial\lambda} [-\Delta_1]} \right)^{\frac{1}{2}} \tag{3.2.7}$$

Since $G_{21}(z,\xi;\lambda) = G_{12}(\xi,z;\lambda)$ we have

$$I_{12} = - \sum_{n=1}^{\infty} \frac{\cos\sigma_1^{(n)} (z+H_1)(\tilde{A}(\xi))_{\lambda=\lambda_n}}{\cos\sigma_1^{(n)} H_1 \cos\sigma_2^{(n)} H_2 \frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_1^n(z)\phi_2^n(\xi) \tag{3.2.8}$$

Consider

$$I_{31} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{31}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{\mu_2\sigma_2 \cosh\sigma_3(z-H_3)\cos\sigma_1(\xi+H_1)}{\Delta_1 \cos\sigma_1 H_1 \cos\sigma_2 H_2 \cosh\sigma_3(H_2-H_3)} d\lambda \tag{3.2.9}$$

Evaluating (3.2.9) as the sum of the residues at the poles, we get

$$I_{31} = -\sum_{n=1}^{\infty} \frac{\mu_2\sigma_2^{(n)}\cosh\sigma_3^{(n)}(z-H_3)\cos\sigma_1^{(n)}(\xi+H_1)}{\cos\sigma_1^{(n)}H_1\cos\sigma_2^{(n)}H_2\cosh\sigma_3^{(n)}(H_2-H_3)\frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_3^n(z)\phi_1^n(\xi) \tag{3.2.10}$$

$$I_{13} = -\sum_{n=1}^{\infty} \frac{\mu_2\sigma_2^{(n)}\cosh\sigma_3^{(n)}(\xi-H_3)\cos\sigma_1^{(n)}(z+H_1)}{\cos\sigma_1^{(n)}H_1\cos\sigma_2^{(n)}H_2\cosh\sigma_3^{(n)}(H_2-H_3)\frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_1^n(z)\phi_3^n(\xi) \tag{3.2.11}$$

where $$\phi_3(z) = \frac{\mu_2\sigma_2\cosh\sigma_3(z-H_3)}{\cos\sigma_2H_2\cosh\sigma_3(H_2-H_3)}\left(\frac{1}{M_1\frac{\partial}{\partial\lambda}[-\Delta_1]}\right)^{\frac{1}{2}} \tag{3.2.12}$$

Consider

$$I_{22} = \lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R} G_{22}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R}\left[\frac{\tilde{A}(z)\tilde{A}(\xi)}{M_1\Delta_1\cos^2\sigma_2H_2}\right.$$

$$\left. - \frac{(\tilde{A}(\xi)\ \sin\sigma_2z\theta(\xi-z) + \tilde{A}(z)\ \sin\sigma_2\xi\theta(z-\xi))}{\mu_2\sigma_2M_1\cos^2\sigma_2H_2}\right]d\lambda \tag{3.2.13}$$

Evaluating (3.2.13) as the sum of the residues at the poles, we get

$$I_{22} = - \sum_{n=1}^{\infty} \frac{(\tilde{A}(z)\ \tilde{A}(\xi))_{\lambda=\lambda_n}}{\cos^2\sigma_2^{(n)} H_2 (M_1)_{\lambda=\lambda_n} \frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_2^n(z)\phi_2^n(\xi) \qquad (3.2.14)$$

where $\phi_2(z)$ is given by (3.2.7).

Consider

$$I_{32} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{32}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \left[\frac{\mu_2\sigma_2 \cosh\sigma_3(z-H_3)\tilde{A}(\xi)}{M_1\Delta_1 \cos^2\sigma_2 H_2 \cosh\sigma_3(H_2-H_3)} \right.$$

$$\left. - \frac{\cosh\sigma_3(z-H_3)}{M_1 \cos\sigma_2 H_2 \cosh\sigma_3(H_2-H_3)} \right] d\lambda \qquad (3.2.15)$$

Evaluating (3.2.15) as the sum of the resides at the poles, we get

$$I_{32} = - \sum_{n=1}^{\infty} \frac{\mu_2\sigma_2^{(n)} \cosh\sigma_3^{(n)}(z-H_3)(\tilde{A}(\xi))_{\lambda=\lambda_n}}{\cos^2\sigma_2^{(n)} H_2 \cosh\sigma_3^{(n)}(H_2-H_3)(M_1)_{\lambda=\lambda_n} \frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_3^n(z)\phi_2^n(\xi) \tag{3.2.16}$$

$$I_{23} = - \sum_{n=1}^{\infty} \frac{\mu_2\sigma_2^{(n)}\cosh\sigma_3^{(n)}(\xi-H_3)(\tilde{A}(z))_{\lambda=\lambda_n}}{\cos^2\sigma_2^{(n)}H_2\cosh\sigma_3^{(n)}(H_2-H_3)(M_1)_{\lambda=\lambda_n}\frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_2{}^n(z)\phi_3^n(\xi) \tag{3.2.17}$$

where $\phi_2(z)$ and $\phi_3(\xi)$ are given by (3.2.7) and (3.2.12) respectively.

Consider

$$I_{33} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{33}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \left[\frac{\mu_2^2\sigma_2^2\cosh\sigma_3(\xi-H_3)\cosh\sigma_3(z-H_3)}{M_1\Delta_1\cos^2\sigma_2 H_2\cosh^2\sigma_3(H_2-H_3)} \right.$$

$$- \frac{\cosh\sigma_3(\xi-H_3)\cosh\sigma_3(z-H_3)(\mu_2\sigma_2\tanh\sigma_3(H_2-H_3)-\mu_3\sigma_3\tan\sigma_2 H_2)}{M_1\mu_3\sigma_3}$$

$$- \frac{1}{\mu_3\sigma_3}\{\cosh\sigma_3(\xi-H_3)\sinh\sigma_3(z-H_3)\theta(\xi-z)$$

$$\left. + \cosh\sigma_3(z-H_3)\sinh\sigma_3(\xi-H_3)\theta(z-\xi)\}\right] d\lambda \tag{3.2.18}$$

Evaluating (3.2.18) as the sum of the residues at the poles, we get

$$I_{33} = - \sum \frac{\mu_2^2(\sigma_2^{(n)})^2 \cosh\sigma_3^{(n)}(\xi-H_3)\cosh\sigma_3^{(n)}(z-H_3)}{\cos^2\sigma_2^{(n)} H_2 \cosh^2\sigma_3^{(n)}(H_2-H_3)(M_1)_{\lambda=\lambda_n} \frac{\partial}{\partial\lambda}[\Delta_1]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{\infty} \phi_3^n(z)\phi_3^n(\xi) \tag{3.2.19}$$

The final results are

$$I_{ij} = \sum_{n=1}^{\infty} \phi_i^n(z)\phi_j^n(\xi) \qquad i,j = 1,2,3. \tag{3.2.20}$$

where $\phi_1(z)$, $\phi_2(z)$ and $\phi_3(z)$ are given by (3.2.3), (3.2.7) and (3.2.12) respectively.

From (3.1.14) and (3.2.20) we obtain the following representation of the delta function

$$\delta(z-\xi) = \sum_{n=1}^{\infty} \mu(\xi)\phi^n(z)\overline{\phi^n}(\xi) \tag{3.2.21}$$

where

$$\left.\begin{aligned} \phi^n(z) &= \phi_1^n(z) & -H_1 \leqslant z \leqslant 0 \\ &= \phi_2^n(z) & 0 \leqslant z \leqslant H_2 \\ &= \phi_3^n(z) & H_2 \leqslant z \leqslant H_3 \end{aligned}\right\} \tag{3.2.22}$$

and
$$\left.\begin{aligned} \mu(z) &= \mu_1 & -H_1 \leq z \leq 0 \\ &= \mu_2 & 0 \leq z \leq H_2 \\ &= \mu_3 & H_2 \leq z \leq H_3 \end{aligned}\right\} \qquad (3.2.23)$$

The representation of the delta function as a sum of eigenfunctions as in (3.2.21) can be used to obtain a similar representation for any function $f(z)$ as follows.

If $f(z)$ is any function of finite μ-norm, then multiplying (3.2.21) by $f(\xi)$, and integrating with respect to ξ over the interval $(-H_1, H_3)$, we have

$$\int_{-H_1}^{H_3} f(\xi)\delta(z-\xi)d\xi = \sum_{n=1}^{\infty} \int_{-H_1}^{H_3} \mu(\xi)f(\xi)\phi^n(z)\phi^n(\xi)d\xi$$

$$= \sum_{n=1}^{\infty} \phi^n(z) \int_{-H_1}^{H_3} \mu(\xi)f(\xi)\phi^n(\xi)d\xi$$

Thus
$$f(z) = \sum_{n=1}^{\infty} f_n\phi^n(z)$$

where
$$f_n = \langle f, \phi^n \rangle = \int_{-H_1}^{H_3} \mu(\xi)f(\xi)\phi^n(\xi)d\xi.$$

In particular, when $f(z) = \phi^m(z)$ we get the following orthonormality relations:

$$\langle \phi^m, \phi^n \rangle = \int_{-H_1}^{H_3} \mu(z)\phi^m(z)\phi^n(z)dz = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

3.3 The Formula for Spectral Representation for the Singular Case

We remarked in section (3.2) that the spectrum of the Love wave operator when the layered structure is of finite depth is discrete. However, the spectrum of the Love wave operator for the layered half-space problem is the disjoint union of the point-spectrum (giving rise to proper eigenfunctions), and a continuous spectrum (which yields improper eigenfunctions), because the Green's function in this case has a branch point singularity (see Friedman 1956 p.p. 214 and Stakgold 1967 (§ 4.4)). We shall see that in addition to poles the contour integral of the Green's functions over the complex λ domain reduces to a sum of residues at the poles plus the integral along the branch cut.

We follow Stakgold 1967 (§ 4.4) in assuming the validity of the formula

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{|\lambda|=R} G(z,\xi;\lambda)d\lambda = - \frac{\delta(z-\xi)}{p(\xi)} \tag{3.3.1}$$

in the singular as well as the regular case.

The continuous spectrum will be the set of point on the branch-cut along a portion of the real axis and the discrete spectrum will be the set of poles located on the real axis. We can obtain the representation of the delta function in the following form:

$$\delta(z-\xi) = \sum A(\xi)p(\xi)\phi^n(z)\overline{\phi^n}(\xi) + \int A(\xi)p(\xi)\psi(z,\lambda)\overline{\psi(\xi,\lambda)}d \tag{3.3.2}$$

where $\phi^n(z)$ are the normalized eigenfunctions belonging to the discrete spectrum and $\psi(z,\lambda)$ are the normalized improper eigenfunctions belonging to the continuous spectrum.

The normalization of the eigenfunctions $\{\phi^n(z)\}$ may be achieved by the condition

$$< \phi^m(z), \phi^n(z) > = \int_a^\infty A(z)p(z)\phi^m(z)\overline{\phi^n}(z)dz = \delta_{mn} \qquad (3.3.3)$$

In order to obtain the normalized conditions for the improper eigenfunctions $\{\psi(z,\lambda)\}$, we multiply (3.3.2) by $\psi(\xi,\lambda')$ and integrate from a to ∞ to get

$$\psi(z,\lambda') = \sum\phi^n(z) < \psi(\xi,\lambda'), \phi^n(\xi) >$$

$$+\int\psi(z,\lambda) < \psi(\xi,\lambda'), \psi(\xi,\lambda) > d\lambda \qquad (3.3.4)$$

For (3.3.4) to be valid, we must have

$$< \psi(z,\lambda'), \phi^n(z) > = \int_a^\infty A(z)\psi(z,\lambda')\overline{\phi^n}(z)dz = 0 \qquad (3.3.5)$$

and $$< \psi(z,\lambda'), \psi(z,\lambda) > = \int_a^\infty A(z)p(z)\psi(z,\lambda')\overline{\psi(z,\lambda)}dz = \delta(\lambda-\lambda') \qquad (3.3.6)$$

Hence the improper eigenfunctions must be normalized according to (3.3.5) and (3.3.6).

For the system (2.1.4) — (2.1.6), (2.1.8) , the formulae (3.3.2) — (3.3.3) and (3.3.5) — (3.3.6) become:

$$\delta(z-\xi) = \sum \mu(\xi)\phi^n(z)\,\overline{\phi^n(\xi)} + \int \mu(\xi)\psi(z,\lambda)\overline{\psi(\xi,\lambda)}d\lambda \tag{3.3.7}$$

$$< \phi^m, \phi^n > = \int_0^\infty \mu(z)\phi^m(z)\overline{\phi^n(z)}dz = \delta_{mn} \tag{3.3.8}$$

$$< \psi(z,\lambda'), \phi^m(z) > = \int_0^\infty \mu(z)\psi(z,\lambda')\overline{\phi^m(z)}dz = 0 \tag{3.3.9}$$

and

$$< \psi(z,\lambda'), \psi(z,\lambda) > = \int_0^\infty \mu(z)\psi(z,\lambda')\overline{\psi(z,\lambda)}dz = \delta(\lambda-\lambda') \tag{3.3.10}$$

From (3.3.1) and (3.3.7) we have the formula

$$\lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G(z,\xi;\lambda)d\lambda = \sum \phi^n(z)\overline{\phi^n(\xi)} + \int \psi(z,\lambda)\overline{\psi(\xi,\lambda)}d\lambda \tag{3.3.11}$$

Any function $V(z)$ which is square - integrable relative to the weight function $\mu(z)$, can be expressed as

$$V(z) = \sum_n < V, \phi^n > \phi^n(z) + \int < V, \psi > \psi(z,\lambda)d\lambda$$

We shall find the spectral representation of the Love wave operator for two layers over a half space in the next section.

3.4 Spectral Representation of the Love Wave Operator for two Infinite Layers of Finite Thickness over a Uniform half-space

In this section we find the spectral representation of the Love wave operator for a uniform half-space of rigidity μ_3, shear velocity β_3 and density ρ_3, overlaid by two infinite strips consisting of a layer of finite depth H_2, density ρ_2, rigidity $\mu_2(<\mu_3)$ and shear velocity $\beta_2(<\beta_3)$ in and another layer of depth $H_1(<H_2)$, density ρ_1, rigidity $\mu_1(<\mu_2)$ and shear velocity $\beta_1(<\beta_2)$ (see fig. 4).

In order to find the discrete and continuous spectrum along with the corresponding eigenfunctions $\{\phi^n(z)\}$ and improper eigenfunctions $\{\psi(z,\lambda)\}$ we use formulae (3.3.1) and (3.3.2).

First we consider

$$I_{11}' = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{11}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \left[\frac{\cos\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)M_2}{\Delta_2\cos^2\sigma_1 H_1} + \frac{1}{\mu_1\sigma_1\cos\sigma_1 H_1}\right.$$

$$\left.\{\cos\sigma_1(z+H_1)\sin\sigma_1\xi\theta(\xi-z)+\cos\sigma_1(\xi+H_1)\sin\sigma_1 z\theta(z-\xi)\}\right]d\lambda \tag{3.4.1}$$

where M_2 is given by (2.3.22) and Δ_2 by (2.3.9).

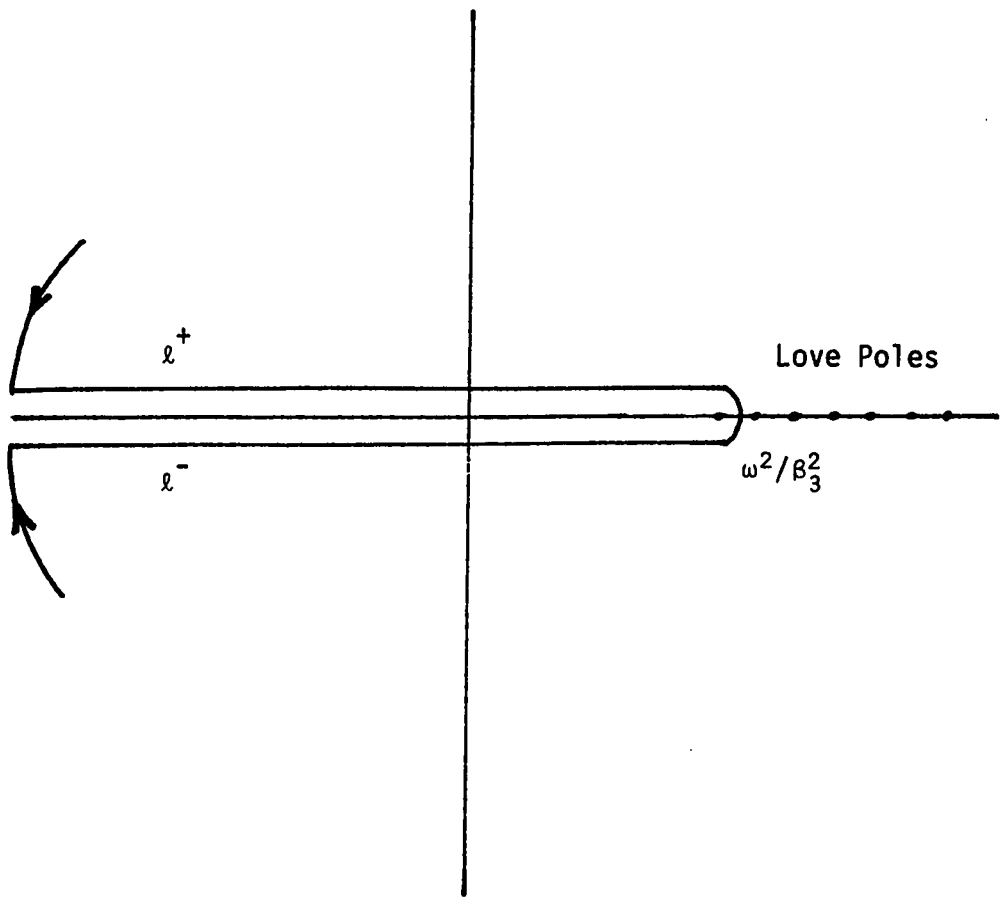

Fig. 6. The Contour of Integration

We note that $\lambda = \omega^2/\beta_3^2$ is the only branch-point singularity of the integrand of (3.4.1) and the poles are the roots of $\Delta_2 = 0$, which is the dispersion equation for Love wave propogation in two layers over a half space (see Ewing et al. 1957 p.p.229).

The poles are all simple, finite in number and are located in the interval $(\omega^2/\beta_3^2 , \omega^2/\beta_1^2]$. The set of these poles constitutes the discrete spectrum. The continuous spectrum is related to the integral over the branch-lines ℓ^+ , ℓ^- in the complex λ - plane, and the path of integration is shown in figure 6.

We assume that $Re(\sigma_3) > 0$ for $I(\lambda) \neq 0$. This means that on the branch-line ℓ^+ ,$\sigma_3 = i\, s_3$ and on ℓ^- ,$\sigma_3 = -i\, s_3$,

where $s_3 = (\omega^2/\beta_3^2 - \lambda)^{\frac{1}{2}}$ is real and positive for $\lambda < \omega^2/\beta_3^2$.

Let $$\gamma_1 = \frac{M_2}{\Delta_2} = \frac{\mu_2\sigma_2 + \mu_3\sigma_3 \tan\sigma_2 H_2}{\Delta_2}$$

Then $$\gamma_1^+ - \gamma_1^- = 2i\, I(\gamma_1^+)$$

$$= \frac{2i\mu_2^2\sigma_2^2\mu_3 s_3 \sec^2\sigma_2 H_2}{p^2 + q^2}$$

where $$p = \mu_1\sigma_1\mu_2\sigma_2 \tan\sigma_1 H_1 + \mu_2^2\sigma_2^2 \tan\sigma_2 H_2 \qquad (3.4.2)$$

& $$q = \mu_1\sigma_1\mu_3 s_3 \tan\sigma_2 H_2 \tan\sigma_1 H_1 - \mu_2\sigma_2\mu_3 s_3 \qquad (3.4.3)$$

The contribution to I'_{11} from ℓ^+ and ℓ^- is

$$\frac{-1}{2\pi i}\left(\int_{\ell^+} G_{11}d\lambda - \int_{\ell^-} G_{11}d\lambda\right) = \frac{-1}{2\pi i}\int_{-\infty}^{\omega^2/\beta_3^2}(G_{11}{}^+ - G_{11}{}^-)d\lambda$$

$$= \frac{-1}{2\pi i}\int_{-\infty}^{\omega^2/\beta_3^2}\frac{(\gamma_1{}^+ - \gamma_1{}^-)\cos\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{\cos^2\sigma_1 H_1}d\lambda$$

where the superscripts + and - refer to the values at the branches ℓ^+ and ℓ^- respectively, and so the branch-line contribution to the integral (3.4.1) is given by

$$\frac{-\mu_3\mu_2^2}{\pi}\int_{-\infty}^{\omega^2/\beta_3^2}\frac{s_3\sigma_2^2\cos\sigma_1(\xi+H_1)\cos\sigma_1(z+H_1)}{(p^2+q^2)\cos^2\sigma_2 H_2\cos^2\sigma_1 H_1}d\lambda$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2}\psi_1(z,\lambda)\psi_1(\xi,\lambda)d\lambda$$

where $$\psi_1(z,\lambda) = \frac{\mu_2\sigma_2\mu_3 s_3\cos\sigma_1(z+H_1)\cos\theta}{p\sqrt{\pi\ \mu_3 s_3}\ \cos\sigma_1 H_1\cos\sigma_2 H_2} \tag{3.4.4}$$

and $$\theta = \tan^{-1}\frac{q}{p} \tag{3.4.5}$$

p and q are given by (3.4.2) and (3.4.3) respectively.

The sum of the residues at the finite number of the poles which lie in the interval $(\omega^2/\beta_3^2 , \omega^2/\beta_1^2]$ is given by

$$- \sum_{n=1}^{N} \frac{\cos\sigma_1^{(n)}(\xi+H_1)\cos\sigma_1^{(n)}(z+H_1)(M_2)_{\lambda=\lambda_n}}{\cos^2\sigma_1^{(n)}H_1 \frac{\partial}{\partial\lambda}[\Delta_2]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{N} \phi_1^n(\xi)\phi_1^n(z)$$

where
$$\phi_1(z) = \frac{\cos\sigma_1(z+H_1)}{\cos\sigma_1 H_1} \left(\frac{M_2}{\frac{\partial}{\partial\lambda}[-\Delta_2]} \right)^{\frac{1}{2}} \tag{3.4.6}$$

Consider

$$I_{21}' = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{21}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{B(z)\cos\sigma_1(\xi+H_1)}{\Delta_2\cos\sigma_2 H_2\cos\sigma_1 H_1} d\lambda \tag{3.4.7}$$

where $B(z)$ is given by (2.3.20).

Again $\lambda = \omega^2/\beta_3^2$ is the only branch-point singularity of the integrand of (3.4.7) and the poles are the roots of $\Delta_2 = 0$.

Let $\gamma_2 = \dfrac{B(z)}{\Delta_2}$

Then $\gamma_2^+ - \gamma_2^- = 2i\ I(\gamma_2^+)$

$$= \frac{2i\mu_2\sigma_2\mu_3 s_3 C(z)}{(p^2+q^2)\cos\sigma_2 H_2}$$

where $C(z) = (\mu_2\sigma_2\cos\sigma_2 z - \mu_1\sigma_1\sin\sigma_2 z\tan\sigma_1 H_1)$ (3.4.8)

and the branch-line contribution to the integral is given by

$$\frac{-\mu_2\mu_3}{\pi}\int_{-\infty}^{\omega^2/\beta_3^2} \frac{\sigma_2 s_3\cos\sigma_1(\xi+H_1)C(z)}{(p^2+q^2)\cos^2\sigma_2 H_2\cos\sigma_1 H_1}\,d\lambda$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2} \psi_2(z,\lambda)\psi_1(\xi,\lambda)\,d\lambda$$

where $\psi_1(\xi,\lambda)$ is given by (3.4.4)

and $\psi_2(z,\lambda) = \dfrac{\mu_3 s_3 C(z)\cos\theta}{p(\cos\sigma_2 H_2)\sqrt{\pi\mu_3 s_3}}$ (3.4.9)

where θ and $C(z)$ are given by (3.4.5) and (3.4.8) respectively.

Contribution from the poles is given by

$$-\sum_{n=1}^{N} \frac{\cos\sigma_1^{(n)}(\xi+H_1)(B(z))_{\lambda=\lambda_n}}{\cos\sigma_2^{(n)}H_2\cos\sigma_1^{(n)}H_1\ \frac{\partial}{\partial\lambda}[\Delta_2]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{N} \phi_2^n(z)\phi_1^n(\xi)$$

where $\phi_1(\xi)$ is given by (3.4.6) and

$$\phi_2(z) = \frac{B(z)}{\cos\sigma_2 H_2} \left(\frac{1}{M_2 \frac{\partial}{\partial\lambda} [-\Delta_2]} \right)^{\frac{1}{2}} \tag{3.4.10}$$

Consider

$$I'_{31} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{31}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \frac{\mu_2\sigma_2\cos\sigma_1(\xi+H_1)e^{-\sigma_3(z-H_2)}}{\Delta_2\cos\sigma_2 H_2\cos\sigma_1 H_1} d\lambda \tag{3.4.11}$$

$\lambda = \omega^2/\beta_3^2$ is the only branch-point singularity of the integrand of (3.4.11) and the poles are the roots of $\Delta_2 = 0$.

Let $$\gamma_3 = \frac{e^{-\sigma_3(z-H_2)}}{\Delta_2}$$

Then $$\gamma_3^+ - \gamma_3^- = 2i\ I(\gamma_3^+)$$

$$= \frac{-\ 2i\ D(z)}{p^2 + q^2}$$

where $$D(z) = p \sin s_3(z-H_2) + q \cos s_3(z-H_2) \tag{3.4.12}$$

Therefore $$G_{31}^{+} - G_{31}^{-} = \frac{-2i\mu_2\sigma_2\cos\theta\cos\sigma_1(\xi+H_1)\sin(\theta+s_3(z-H_2))}{p\cos\sigma_2H_2\cos\sigma_1H_1} \tag{3.4.13}$$

and so the branch-line contribution to the integral (3.4.11) is given by

$$\frac{+\mu_2}{\pi}\int_{-\infty}^{\omega^2/\beta_3^2}\frac{\sigma_2\cos\sigma_1(\xi+H_1)\cos\theta\sin(\theta+s_3(z-H_2))}{p\cos\sigma_2H_2\cos\sigma_1H_1}\,d\lambda$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2}\psi_1(\xi,\lambda)\psi_3(z,\lambda)\,d\lambda$$

where $$\psi_3(z,\lambda) = \frac{-\sin(\theta+s_3(z-H_2))}{\sqrt{\pi\mu_3 s_3}} \tag{3.4.14}$$

Contribution from the poles is given by

$$-\sum_{n=1}^{N}\frac{\mu_2\sigma_2^{(n)}\cos\sigma_1^{(n)}(\xi+H_1)\;e^{-\sigma_3^{(n)}(z-H_2)}}{\cos\sigma_2^{(n)}H_2\cos\sigma_1^{(n)}H_1\;\frac{\partial}{\partial\lambda}[\Delta_2]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{N}\phi_3^n(z)\phi_1^n(\xi)$$

where $$\phi_3(z) = \frac{\mu_2\sigma_2 e^{-\sigma_3(z-H_2)}}{\cos\sigma_2H_2}\left(\frac{1}{M_2\;\frac{\partial}{\partial\lambda}[-\Delta_2]}\right)^{\frac{1}{2}} \tag{3.4.15}$$

Consider

$$I'_{22} = \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} G_{22}(z,\xi;\lambda)d\lambda$$

$$= \lim_{R\to\infty} \frac{-1}{2\pi i} \oint_{|\lambda|=R} \left[\frac{B(z)B(\xi)}{M_2\Delta_2\cos^2\sigma_2 H_2}\right.$$

$$\left. - \frac{B(\xi)\sin\sigma_2 z\theta(\xi-z) + B(z)\sin\sigma_2\xi\theta(z-\xi))}{\mu_2\sigma_2 M_2\cos\sigma_2 H_2}\right] d\lambda \qquad (3.4.16)$$

$\lambda = \omega^2/\beta_3^2$ is the only branch-point singularity of the integrand of (3.4.16) and the poles are the roots of $\Delta_2 = 0$.

Let $\gamma_4 = \dfrac{B(z)B(\xi)}{\Delta_2 M_2}$

Then $\gamma_4^+ - \gamma_4^- = 2i\ I(\gamma_4^+)$

$$= \frac{2i\mu_3 s_3 C(z)C(\xi)}{p^2 + q^2}$$

where $C(z)$ is given by (3.4.8)

and so the branch-line contribution to the integral (3.4.16) is

given by

$$\frac{-1}{\pi}\int_{-\infty}^{\omega^2/\beta_2^2}\frac{\mu_3 s_3 C(z)C(\xi)}{(p^2+q^2)\cos^2\sigma_2 H_2}\,d\lambda$$

$$=-\int_{-\infty}^{\omega^2/\beta_2^2}\psi_2(\xi,\lambda)\psi_2(z,\lambda)\,d\lambda$$

where $\psi_2(z,\lambda)$ is given by (3.4.9).

Contribution from the poles is given by

$$-\sum_{n=1}^{N}\frac{[B(z)B(\xi)]_{\lambda=\lambda_n}}{\cos^2\sigma_2^{(n)}H_2(M_2)_{\lambda=\lambda_n}\ \frac{\partial}{\partial\lambda}[\Delta_2]_{\lambda=\lambda_n}}$$

$$=\sum_{i=1}^{N}\phi_2^n(\xi)\phi_2^n(z)$$

Consider

$$I_{32}' = \lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R}G_{32}(z,\xi;\lambda)\,d\lambda$$

$$= \lim_{R\to\infty}\frac{-1}{2\pi i}\oint_{|\lambda|=R}\left[\frac{\mu_2\sigma_2 B(\xi)e^{-\sigma_3(z-H_2)}}{\Delta_2 M_2\cos^2\sigma_2 H_2}-\frac{e^{-\sigma_3(z-H_2)}}{M_2\cos\sigma_2 H_2}\right]d\lambda \tag{3.4.17}$$

$\lambda = \omega^2/\beta_3^2$ is the only branch-point singularity of the integrand of (3.4.17) and the poles are the roots of $\Delta_2 = 0$.

Let $$\gamma_5 = \frac{B(\xi)e^{-\sigma_3(z-H_2)}}{M_2\Delta_2}$$

Then $$\gamma_5^+ - \gamma_5^- = 2i\ I(\gamma_5^+)$$

$$= \frac{-2iD(z)}{p^2 + q^2}$$

where $D(z)$ is given by (3.4.12).

and so $$G_{32}^+ - G_{32}^- = \frac{-2iC(\xi)\cos\theta\sin(\theta+s_3(z-H_2))}{p\ \cos\sigma_2 H_2}$$

Therefore, the branch-line contribution to the integral (3.4.17) is given by

$$\frac{1}{\pi}\int_{-\infty}^{\omega^2/\beta_3^2} \frac{C(\xi)\cos\theta\sin(\theta+s_3(z-H_2))}{p\ \cos\sigma_2 H_2}$$

$$= -\int_{-\infty}^{\omega^2/\beta_3^2} \psi_3(z,\lambda)\psi_2(\xi,\lambda)d\lambda$$

Contribution from poles is given by

$$-\sum_{n=1}^{N} \frac{\mu_2\sigma_2^{(n)}\ e^{-\sigma_3^{(n)}(z-H_2)}\ (B(\xi))_{\lambda=\lambda_n}}{\cos\sigma_2^{(n)}H_2(M_2)_{\lambda=\lambda_n}\ \frac{\partial}{\partial\lambda}[\Delta_2]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{N} \phi_3^n(z)\phi_2^n(\xi)$$

Finally we consider

$$I'_{33} = \lim_{R\to\infty} \oint_{|\lambda|=R} G_{33}(z,\xi;\lambda)\,d\lambda$$

$$= \lim_{R\to\infty} \oint_{|\lambda|=R} \left[\frac{\mu_2^2\sigma_2^2 e^{-\sigma_3(\xi-H_2)} e^{-\sigma_3(z-H_2)}}{\Delta_2 M_2 \cos^2\sigma_2 H_2} \right.$$

$$- \frac{e^{-\sigma_3(\xi-H_2)} e^{-\sigma_3(z-H_2)} (\mu_3\sigma_3 \tan\sigma_2 H_2 - \mu_2\sigma_2)}{2\mu_3\sigma_3 M_2}$$

$$\left. - \left(\frac{e^{-\sigma_3(\xi-z)}}{2\mu_3\sigma_3}\theta(\xi-z) + \frac{e^{-\sigma_3(z-\xi)}}{2\mu_3\sigma_3}\theta(z-\xi) \right) \right] d\lambda \qquad (3.4.18)$$

$\lambda = \omega^2/\beta_3^2$ is the only branch point singularity and the poles are the roots of $\Delta_2 = 0$.

Let $$\gamma_6 = \frac{e^{-\sigma\ (\xi-H_2)} e^{-\sigma_3(z-H_2)}}{\Delta_2 M_2}$$

Then $$\gamma_6^+ - \gamma_6^- = 2i\ I(\gamma_6^+)$$

Also $$G_{33}^+ - G_{33}^- = 2i\ I(G_{33}^+)$$

$$= \frac{2i}{\mu_3 s_3} \left[\cos s_3(z-\xi) - \cos[s_3(z+\xi-2H_2) + 2\theta] \right]$$

$$= \frac{2i}{\mu_3 s_3} \sin(\theta + s_3(\xi-H_2)) \sin(\theta + s_3(z-H_2))$$

Therefore, the branch-line contribution to the integral (3.4.18) is given by

$$\frac{-1}{\pi} \int_{-\infty}^{\omega^2/\beta_3^2} \psi_3(\xi,\lambda)\psi_3(z,\lambda)d\lambda$$

Contribution from the poles is given by

$$- \sum_{n=1}^{N} \frac{\mu_2^2(\sigma_2^n)^2 \exp[-\sigma_3^{(n)}(\xi-H_2)]\exp[-\sigma_3^{(n)}(z-H_2)]}{\cos^2\sigma_2^{(n)}H_2(M_2)_{\lambda=\lambda_n} \frac{\partial}{\partial\lambda}[\Delta_2]_{\lambda=\lambda_n}}$$

$$= \sum_{n=1}^{N} \phi_3^n(\xi)\phi_3^n(z)$$

The final results are

$$I'_{ij} = \sum_{n=1}^{N} \phi_i^n(\xi)\phi_j^n(z) - \int_{-\infty}^{\omega^2/\beta_3^2} \psi_i(z,\lambda)\psi_j(\xi,\lambda)d\lambda \qquad (3.4.19)$$

$$i,j = 1,2,3$$

where $\phi_1(z)$, $\phi_2(z)$ and $\phi_3(z)$ are given by (3.4.6), (3.4.10) and (3.4.15) respectively, and $\psi_1(z,\lambda)$, $\psi_2(z,\lambda)$ and $\psi_3(z,\lambda)$ are given by (3.4.4), (3.4.9) and (3.4.14) respectively.

From (3.3.1), (3.3.2) and (3.4.19) we obtain the following representation of the delta function

$$\delta(z-\xi) = \sum_{n=1}^{N} \mu(\xi)\phi^n(z)\phi^n(\xi) - \int_{-\infty}^{\omega^2/\beta_3^2} \mu(\xi)\psi(z,\lambda)\psi(\xi,\lambda)d\lambda \qquad (3.4.20)$$

where

$$\left.\begin{aligned} \phi^n(z) &= \phi_1^n(z) & -H_1 \leq z \leq 0 \\ &= \phi_2^n(z) & 0 \leq z \leq H_2 \\ &= \phi_3^n(z) & H_2 \leq z \leq H_3 \end{aligned}\right\} \qquad (3.4.21)$$

and $\mu(z)$ is given by (3.2.27) are the normalized eigenfunctions,

and

$$\left.\begin{aligned} \psi(z,\lambda) &= \psi_1(z,\lambda) & -H_1 \leq z \leq 0 \\ &= \psi_2(z,\lambda) & 0 \leq z \leq H_2 \\ &= \psi_3(z,\lambda) & H_2 \leq z \leq H_3 \end{aligned}\right\} \qquad (3.4.22)$$

where $\psi_1(z,\lambda)$, $\psi_2(z,\lambda)$ and $\psi_3(z,\lambda)$ are given by (3.4.4), (3.4.9) and (3.4.14) respectively are the normalized improper eigenfunctions.

If $f(z)$ is of finite μ-norm over the interval $(-H_1,\infty)$, then the representation of $f(z)$ in terms of eigenfunctions $\{\phi^n(z)\}$ and improper eigenfunctions $\{\psi(z,\lambda)\}$ can be obtained on multiplying

(3.4.20) by $f(\xi)$ and integrating with respect to ξ from $-H_1$ to ∞, we get

$$\int_{-H_1}^{\infty} f(\xi)\delta(z-\xi)d\xi = \sum_{n=1}^{N} \phi^n(z) \int_{-H_1}^{\infty} \mu(\xi)f(\xi)\phi^n(\xi)d\xi$$

$$- \int_{-\infty}^{\omega^2/\beta_3^2} \psi(\lambda,z)dz \int_{-H_1}^{\infty} \mu(\xi)\psi(\xi,\lambda)f(\xi)d\xi$$

i.e. $$f(z) = \sum_{n=1}^{N} f_n\phi^n(z) - \int_{-\infty}^{\omega^2/\beta_3^2} f_\lambda\psi(\lambda,z)dz \qquad (3.4.23)$$

where $$f_n = \langle f,\phi^n \rangle \quad \text{and} \quad f_\lambda = \langle f,\psi(\xi,\lambda)\rangle \qquad (3.4.24)$$

In particular, if $f(z) = \phi^m(z)$ or $\psi(z,\lambda')$, then (3.4.23), (3.4.24) yield the following orthonormality relations

$$\int_{-H_1}^{\infty} \mu(z)\phi^m(z)\phi^n(z)dz = \delta_{mn} = \langle \phi^m, \phi^n \rangle \quad 1 \leq m,n \leq N$$

$$\int_{-H_1}^{\infty} \mu(z)\psi(z,\lambda)\psi(z,\lambda')dz = \delta(\lambda-\lambda') = \langle \psi(z,\lambda) , \psi(z,\lambda') \rangle$$

$$-\infty \leq \lambda , \lambda' < \omega^2/\beta_3^2$$

and $$\int_{-H_1}^{\infty} \mu(z)\phi^m(z)\psi(z,\lambda)dz = 0 = \langle \phi^m , \psi \rangle \quad 1 \leq m \leq N$$

$$-\infty \leq \lambda < \omega^2/\beta_3^2$$

3.5 CONCLUSION

We have obtained the spectral representation of the two-dimensional Love wave operator associated with monochromatic SH-waves in three-layered models comprising two infinite strips overlying another strip or a half-space. The spectral representation enables us to tackle a class of problems associated with the transmission and reflection of Love waves at a horizontally discontinuous change in elevation or in material properties of three layered structures, using the method based on an integral equation formulation together with the application of Schwinger-Levine variational principle as in Kazi (1978a,b), Kazi (1979) and Niazy and Kazi (1980).

Our future line of research will be to investigate the above-mentioned problems.

REFERENCES

[1] Alsop, L.E.: Transmission and reflection of Love waves at a vertical discontinuity, J. Geophys. Res. 71, 3969-3984., 1966.

[2] Babich, V.M. and Molotkov, I.A.: Propagation of Love waves in an elastic half-space, inhomogeneous in the directions of two axes. Bull. (IZV) Acad. Sci. USSR. Earth Physics, 6, 34-38,1966.

[3] Boore, D.M.: Love waves in non-uniform waveguides: Finite difference calculation, J. Geophys. Res., 75, 1512-1527, 1970.

[4] Drake, L.A. and Bolt, B.A.: Love waves normally incident at a continental boundary. Bull. Seism. Soc. Am., 70, 1103-1125, 1980.

[5] Ewing, W.M., Press, F. and Jardetsky, W.S.: Elastic waves in Layered Media. McGraw-Hill, 1957.

[6] Friedman, B.: Principles and techniques of applied mathematics, Wiley, New York, 1956.

[7] Gregersen, S. and Alsop, L.E.: Amplitudes of horizontally reflected Love waves, Bull. Seis. Soc. Am., 64, 535-553, 1974.

[8] Gregersen, S. and Alsop, L.E.: Mode Conversion of Love waves at a continental margin, Bull. Seism. Soc. Am., 66, 1855-1872, 1976.

[9] Kazi, M.H.: Spectral representation of the Love wave operator, Geophys. J. Roy. Ast. Soc. 47, 225-249, 1976.

[10] Kazi, M.H.: The Love wave scattering matrix for a continental margin (Theoretical), Geophys. J. Roy. Ast. Soc. 52, 25-44, 1978a.

[11] Kazi, M.H.: The Love wave scattering matrix for a continental margin (Numerical), Geophys. J. Roy. Ast. Soc. 53, 227-243, 1978b.

[12] Kazi, M.H.: Transmission, Reflection and Diffraction of Love waves in an infinite strip with a surface step, J. Phys. A; Math. Gen., Vol. 12, 1441-1455, 1979.

[13] Knopoff, L. and Hudson, J.A.: Transmission of Love waves past a continental margin, J. Geophys. Res., 69, 1649-1653, 1964.

[14] Lysmer, J. and Drake, L.A.: The propagation of Love waves across non-horizontally layered structures, Bull. Seism. Soc. Am., 61, 1233-1251, 1971.

[15] Mal, A.K. and Knopoff, L.: Transmission of Rayleigh waves past a step change in elevation, Bull. Seism. Soc. Am., 55, 319-334, 1965.

[16] Niazy, A. and Kazi, M.H.: On the Love wave scattering problem for welded layered quarter-spaces with applications,Bull. Seism. Soc. Am., Vol. 70, No.6, pp. 2071-2095, 1980.

[17] Sangren, W.C.: ORNL 1566, Oakridge National Laboratory, USA, 1953.

[18] Sato, R.: Love waves in case the surface layer is variable in thickness, J. of Physics of the Earth,9, 19-36, 1961.

[19] Stakgold, I.: Boundary value problems of mathematical physics, Volume I, Macmillan, 1967.

[20] Suteau, A. and Martel, L.: Surface wave structure with locally irregular boundaries, Bull. Seism. Soc. Am., 70, 791-808, 1980.

[21] Wolf, B.: Propagation of Love waves in surface layers of varying thickness, Pure Appl. Geophys., 67, 76-82, 1967.

[22] Wolf, B.: Propagation of Love waves in layers with irregular boundaries, Pure Appl. Geophys., 78, 48-57, 1970.