Some Properties of Bicyclic Extensions

by

Sayed Omar

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

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This thesis written by Sayed Omar under the direction of his Thesis advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfilment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICS.

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December, 1998

Abstract

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Title:	Some Properties of Bicyclic Extensions
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Warne in [21] replaced "**group**" by "**monoid**" and termed the resulting construction a bicyclic extension. We give explicit proof of an important property of this bicyclic extension semigroup in Theorem 31.

Abundant semigroup theory was initiated by Fountain [6]. Asibong-Ibe [1] extended Reilly's [10] results to the class of abundant semigroups. We characterize the starred Green's relations on Bicyclic extensions and show as main result that in case of a cancellative monoid, the bicyclic extension is an E^*-bisimple semigroup, by analogy with Warne [13] and Asibong-Ibe [1].

This work forms an important first step towards obtaining a unified structure theory (of some class of E^*-bisimple semigroups) that will yield the results of Asibong-Ibe [1] and Warne [13] as special cases.

FOR THE DEGREE OF MASTER OF SCIENCE

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
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في هذه الرسالة ندرس بعض من الخصائص الأساسية للإمتدادات الثنائية الدائرية (Bicyclic Extensions) التي قدمها وارن عام ١٩٩٧ م ثم نعطي البرهان لأحد نظريات وارن الهامة والتي ذكرها بدون برهان .

بعد ذلك قدمنا الصفات التي تميز علاقات جرين على الإمتدادات الدائرية الثنائية.

وفي الجزء الأخيرمن الرسالة أثبتنا انه في حالة المونويد القابل للاختصار يكون الامتداد الدائري الثنائي هو نصف زمرة من النوع E*- bisimple و هذه نظرية هامة في هذا المجال حيث أن النتائج التي توصل لها ايزبونج عام ١٩٨٥ م و وارن عام ١٩٩٧ م تعتبر حالات خاصة من هذه النظرية.

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Table of Contents

Abstract ii

Arabic Abstract iii

Acknowledgement iv

Table of Contents v

0.1 Introduction 1

0.2 List of Symbols 3

1 Elementary Concepts 5

1.1 Basic Definitions 5

2 Bicyclic Extension of a Monoid 16

2.1 Translational Hull 16

3 Green's Relations on Bicyclic Extensions 41

3.1 Bruck-Reilly Extension 58

3.2 The Starred or Generalized Green's Relations 59

3.3 Generalized Bruck-Reilly Extension 63

4 Starred Green's Relations on Bicyclic Extensions 66

4.1 $\mathcal{L}^*$ and $\mathcal{R}^*$ Relations on Bicyclic Extensions 66

4.2 Some Properties of Bicyclic Extensions and Cancellative Monoids 75

4.3 E^*-Bisimple Semigroup 77

Appendix A 80

Appendix B 81

References 82

0.1 Introduction:

Let T be a monoid with group of units U. Let θ be a homomorphism of T into U (θ° denoting the identity automorphism of T). Let P and K be disjoint sets and γ be a homomorphism of T into G_K . the full transformation group on K. Let $S = ((N^0 \times \{0\}) \times (T \times P)) \cup ((N^0 \times N) \times (T \times K))$ where N $[N^0]$ is the set of natural numbers [non-negative integers] under the multiplication $((n,k),(g,p))((r,s),(h,q)) = ((n+r-t,k+s-t),(g\theta^{r-t}h\theta^{k-t},x))$ where $t = min(k,r)$ and $x = q$ or $p(h\theta^{k-r-1}\gamma)$ according to whether $k \leq r$ or $k > r$. Warne introduced bicyclic extensions S in [21] and also characterized bicyclic extensions of finite chains of groups in [21]. This result generalized results of Warne [15, 16], Reilly [10], Koçin [8], Munn [9], Warne [20]. The translational hull of a bicyclic extension of a group has been studied by Warne [20].

The bicyclic semigroup B ($B = N^0 \times N^0$) under the multiplication $((m,n)(p,q)) = (m+p-t, n+q-t)$ where $t = min(n,p)$ has played a central role in the theory of simple semigroups (Any simple semigroup which is not completely simple is locally bicyclic [3, 7]. (See, for example, [3], [14], [17], [18], [19]).

In **chapter 1** of this thesis, we introduce basic elementary concepts and propositions on semigroups used throughout this thesis. We introduce the basic terminology and notation; this is short preparation of general nature which is needed for specialized subject treated in this thesis. These definitions are written from Clifford & Preston [4] and Howie [7].

To construct more general structure theory of simple semigroup, Warne in [21] replaced "group" by "monoid" and termed the resulting construction a bicyclic extension. In **chapter 2** of this thesis, we give explicit proof of an important property of this bicyclic extension semigroup in Theorem 31.

Green's equivalence relations are fundamental equivalence relations. definable in any semigroup, were first introduced and studied by J. A. Green (1951). In **chapter 3**, we give examples, lemma and theorems of Green's equivalence property related to Bicyclic Extensions. In [21] a lemma is stated without proof and we give a proof (Theorem 43).

Abundant semigroup theory was initiated by Fountain [6]. Asibong-Ibe [1] extended Reilly's [10] structure theorem to the class of abundant semigroups. In, **chapter 4**, we characterize the Green's relations on Bicyclic extensions and show that as main result that in case of a cancellative monoid. the bicyclic extension is an $E^* - bisimple$ semigroup.

0.2 List of symbols:

$\mid A \mid$	means the cardinal number of the set A.
$A \times B$	means the set of all ordered pairs (a, b) with a in A and b in B.
$a \vee b$	the join of a and b : least upper bound of $\{a, b\}$.
$a \wedge b$	the meet of a and b : greatest lower bound of $\{a, b\}$.
$BR(T, \theta)$	the Bruck-Reilly extension.
$BR^*(T, \theta)$	the generalized Bruck-Reilly extension.
$\mathcal{D}$	the Green's relation: $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ $(= \mathcal{R} \circ \mathcal{L})$: $\mathcal{L} \vee \mathcal{R}$: the smallest equivalence relation that containing both $\mathcal{L}$ and $\mathcal{R}$.
$\mathcal{D}^*$	the join of $\mathcal{L}^*$ and $\mathcal{R}^*$ equivalences.
$\Box$	denotes the empty set.
$\mathcal{H}$	the Green's relation: $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.
$\mathcal{H}^*$	the generalized Green's relation: $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$.
i_S	the diagonal relation of S : $i_S = \{(a, a) : a \in S\}$.
$\mathcal{J}$	$a\mathcal{J}b$ $(a, b$ in $S)$ to mean $S^1aS^1 = S^1bS^1$ where $S^1aS^1 = SaS \cup aS \cup Sa \cup \{a\}$ and $S^1bS^1 = SbS \cup bS \cup Sb \cup \{b\}$.
$\mathcal{J}^*$	the starred Green's relation defined by equality of principal $*$-Ideals.
λ_a	denotes the inner left translation $x \rightarrow ax$ of a semigroup S, where a is fixed element of S.
L_a	the $\mathcal{L}$-class containing a. (= The set of all elements of S which are $\mathcal{L}$-equivalent to a).

L_a^*	the $\mathcal{L}^*$–class containing $a \in S$.
$\mathcal{L}$	the green's relation: $\mathcal{L} = \{(a,b) \in S \times S : S^1a = S^1b\}$.
$\mathcal{L}^*$	$(a,b) \in \mathcal{L}_S^* \Longleftrightarrow (a,b) \in \mathcal{L}_T$ for some over-semigroup T of S (i.e. $S \subset T$).
ρ_a	denotes the inner right translation $x \to xa$ of a semigroup S, where a is fixed of S.
$\rho \circ \sigma$	the composition of ρ and σ.
R_a	the $\mathcal{R}$–class containing a.
$\mathcal{R}$	the Green's relation: $\mathcal{R} = \{(a,b) \in S \times S : aS^1 = bS^1\}$.
$\mathcal{R}^*$	$(a,b) \in \mathcal{R}_S^* \Longleftrightarrow (a,b) \in \mathcal{R}_T$ for some over-semigroup T of S.
R_a^*	the $\mathcal{R}^*$–class containing $a \in S$.
S^0	the semigroups S with adjoined zero 0.
S^1	the semigroups S with adjoined identity 1.

Chapter 1

Elementary Concepts

In this introductory chapter, we introduce basic elementary concepts and propositions on semigroups used throughout this thesis. We introduce the basic terminology and notation; this is short preparation of general nature which is needed for specialized subject treated in this thesis. These definitions are written from Clifford & Preston [4] and Howie [7].

1.1 Basic Definitions

Definition 1 : *If X is a non-empty set, then a subset ρ of $X \times X$ is called a* ***binary relation*** *on X. The empty subset $\Box$ of $X \times X$ will be included among the binary relations. Other special binary relations worthy of mention are the universal relation $X \times X$ and the equality relation (or "diagonal" of $X \times X$), defined by $(a, b) \in \iota$ if and only if $a = b$. $\iota_X = \{(x, x) : x \in X\}$.*

If $(a, b) \in \rho$, where a and b are elements of X, we may also write $a\rho b$. If ρ and

σ are relations on X, their composition $\rho \circ \sigma$ is defined as follows: $(a, b) \in \rho \circ \sigma$ if there exists x in X such that $(a, x) \in \rho$ and $(x, b) \in \sigma$. i.e. $\rho \circ \sigma = \{(a, b) \in X \times X : (\exists x \in X)$ such that $(a, x) \in \rho$ and $(x, b) \in \sigma\}$.

The converse ρ^{-1} of a relation ρ is defined by $(a, b) \in \rho^{-1}$ if and only if $(b, a) \in \rho$.

Definition 2 : *A* ***function*** *(or a* ***mapping****) f from a set X to a set Y (written $f : X \longrightarrow Y$) assigns to each $a \in X$ exactly one element $b \in Y$; b is called the value of the function at a or the image of a. X is the domain of the function and Y is the codomain or range. Two functions are equal if they have the same domain and range and have the same value for each element of their common domain. A mapping $f : X \longrightarrow Y$ is said to be* **onto** *(or* **upon** *or* **surjective** *) if every element of Y is the image under f of at least one element of X. A function is said one – to – one (or* 1 – 1 *or injective) if distinct element of X are mapped by f into distinct element of Y. A function f is said to be* **bijective** *(or bijection or one-to-one correspondence) if it is both one – to – one and onto.*

Definition 3 : *A* **binary operation** *($\star$) on a set S is a mapping of $S \times S$ (the set of all ordered pairs of elements of S) into S (written $\star : S \times S \longrightarrow S$) which to each ordered pair $(a, b) \in S \times S$ associates a unique element of S. The image in S of the element (a, b) of $S \times S$ will be denoted by $a \star b$. Other symbols which we may use to denote binary operations are $\cdot$, $+$ and $\circ$. Frequently we shall omit the $\star$, writing ab for $a \star b$.*

Definition 4 : *A* **groupoid** *is a system $(S, \cdot)$ consisting of a non-empty set S with a binary operation "$\cdot$" on S. We shall usually write S instead of $(S, \cdot)$ when there is* *no danger of ambiguity.*

A **partial binary operation** *on a set S is a mapping of a non-empty subset of $S \times S$ into S. By a* **partial groupoid** *we shall mean a system $(S, \cdot)$ consisting of a non-empty set S together with a partial binary operation $(\cdot)$ on S.*

Definition 5 *: A binary operation $(\cdot)$ on a set S is called* **associative** *if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c in S.*

Definition 6 *: A* **semigroup** *is a groupoid $(S, \cdot)$ such that the operation $(\cdot)$ is associative. We frequently use the expression "S is a semigroup with respect to $(\cdot)$ ", to mean that $(\cdot)$ is an associative binary operation on S.*

Definition 7 *: An element e of a groupoid S is called a* ***left [right] identity*** *element of S if $ea = a$ [$ae = a$] for all a in S. An element e of S is called a* ***two-sided identity*** *(or simply* ***identity****) element of S if it is both a left and a right identity element of S. We note that if S contains a left identity e and a right identity f, then $e = f$; for $ef = f$ since e is a left identity, and $ef = e$ since f is a right identity. As a consequence of this, we see that exactly one of the following statements must hold for a groupoid S.*

(1): S has no left and no right identity;

(2): S has one or more left identity elements, but no right identity elements;

(3): S has one or more right identity elements, but no left identity element;

(4): S has a unique two-sided identity element, and no other right or left identity element.

Definition 8 *: An element z of a groupoid S is called a* **left [right] zero** *element if $za = z$ $[az = z]$ for every a in S. An element z of S is called a* **zero** *element of S if it is both a left and a right zero element of S. There can be at most one such element, since if*

$$(\forall x \in S) \qquad x0' = 0'x = 0',$$

then

$$\begin{aligned} 0' &= 00' \text{(since } 0' \text{ is a zero)} \\ &= 0 \quad \text{(since } 0 \text{ is a zero)} \end{aligned}$$

So, the foregoing tetrachotomy holds if we replace the word "identity" by "zero".

Let S be a semigroup, and 1 be a symbol not representing any element of S. We may adjoin an identity element 1 to S, extending the given binary operation in S to one in $S \cup 1$ by defining $11 = 1$ and $1a = a1 = a$ for every a in S. It is quickly verified that $S \cup 1$ is a semigroup with identity element 1. Similarly one may adjoin a zero element 0 to S by defining $00 = 0a = a0 = 0$ for all a in S. A semigroup with identity is called a **monoid**.

We shall use the notation S^1 and S^0 with the following meaning:

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identiy element} \\ S \cup \{1\} & \text{if } S \text{ has no identity element} \end{cases}$$

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element and } |S| > 1 \\ S \cup \{0\} & \text{if } S \text{ has no zero element} \end{cases}$$

An element e of a groupoid S is called **idempotent** *if $ee = e$ $(e^2 = e)$.*

If every element of a semigroup S is idempotent, we shall say that S itself is idempo-

tent, or S is a band. Note that if S is a semigroup with a single element e ($S = \{e\}$) then e is the zero and the identity element of S.

Example 9 *: Let $S = [0, 1/2)$ (= set of all real numbers x such that $0 \leq x < 1/2$). Let the binary relation $\star$ be defined as follows: for any $x, y \in S$, $x \star y = min\{x, y\}$. Now, we check the associativity: Let $x, y, z \in S$. Then*

$$x \star (y \star z) = x \star (min\{y, z\}) = min\{x, y, z\} = min\{min\{x, y\}, z\} = (x \star y) \star z$$

Hence S is a semigroup. Since, $x \star 0 = 0 = 0 \star x$, so, 0 is the zero element of $S = S^0$. We may adjoin an identity element 1 to S, such that $1 \star x = x = x \star 1$ $\forall x \in S$, so, 1 is the identity of $S^1 = S \cup \{1\}$.

Definition 10 *: An element a of a groupoid S is said to be* ***left [right] cancellable*** *if, for any x and y in S, $ax = ay$ [$xa = ya$] implies $x = y$. A groupoid S is called* ***left [right] cancellative*** *if every element of S is left [right] cancellable. We say that S is* ***cancellative*** *(or is a cancellation groupoid) if it is both left and right cancellative.*

Remark 1 :

(I): If e is an idempotent element of a left cancellative semigroup S, then e is a left identity element of S. (For, $e(ex) = ex \Rightarrow ex = x$).

(II): A cancellative semigroup can contain at most one idempotent element, namely an identity element. (Suppose that e_1 and e_2 are idempotent of S. Then, $e_1e_2 = e_1(e_1e_2) \Rightarrow e_2 = e_1e_2$ (by left cancellation). However $e_1e_2 = (e_1e_2)e_2 \Rightarrow e_1 = e_1e_2$(by right cancellation). So $e_1 = e_2$.).

(III): If S is a cancellative semigroup, so is S^1.

(IV): If S is left zero semigroup with $|S| > 1$, then S is right cancellative. but S^1 is not. For, let $a \neq 1$. $1a = 1$ (since S is left zero), then it implies that $1a = a1 = 1.1$ and therefore S can not be right cancellative, otherwise $a = 1$. but we have $a \neq 1$.

Example 11 : *We show that, with respect to matrix multiplication, the set*

$$S = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} : a, b \in R,\ a, b > 0 \right\}$$

is a cancellative semigroup without identity.

Solution: For any $a, b, c, d > 0$, we have

$$\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ d & 1 \end{bmatrix} = \begin{bmatrix} ac & 0 \\ bc + d & 1 \end{bmatrix} \in S \text{ (since } ac > 0 \text{ and } bc + d > 0)$$

therefore S is a semigroup. To show that S is cancellative. let $A_1 A = A_2 A$ where

$$A = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} a_1 & 0 \\ b_1 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} a_2 & 0 \\ b_2 & 1 \end{bmatrix},\ a, b, a_1, b_1, a_2, b_2 > 0.$$

So,

$$\begin{bmatrix} a_1 & 0 \\ b_1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} a_2 & 0 \\ b_2 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix},$$

then it follows that

$$\begin{bmatrix} a_1 a & 0 \\ b_1 a + b & 1 \end{bmatrix} = \begin{bmatrix} a_2 a & 0 \\ b_2 a + b & 1 \end{bmatrix} \Longrightarrow a_1 a = a_2 a \text{ and } b_1 a + b = b_2 a + b$$

$\Longrightarrow a_1 = a_2$ and $b_1 = b_2 \Longrightarrow A_1 = A_2$. Similarly, we can show that $AA_1 = AA_2$ implies $A_1 = A_2$. Therefore S is cancellative. Obviously the identity $1_S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$.

Definition 12 : *A relation ρ is said to be* (1) : ***reflexive*** *if* $\iota \subset \rho$. (2) : ***symmetric*** *if* $\rho \subseteq \rho^{-1}$ *(and hence* $\rho = \rho^{-1}$*), and* (3) : ***transitive*** *if* $\rho \circ \rho \subseteq \rho$. *A relation ρ on a set X is called an* ***equivalence relation*** *on X if it is reflexive, symmetric, and transitive.*

Definition 13 : *By a* **transformation** *of a set X we shall mean a single valued mapping of X into itself. We denote the image of an element $x \in X$ under a transformation or mapping α by $x\alpha$ (rather than $\alpha(x)$ or αx). By product (or* **iterate** *or* **composition***) of two transformations α and β of X we mean the transformation $\alpha\beta$ defined by $x(\alpha\beta) = (x\alpha)\beta$ for all x in X (that is , α followed by β).*

Example 14 : *The set $\mathcal{F}_X$ of all transformation of X is a semigroup with respect to iteration. The associative law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds since, $\forall\ x \in X$,*

$$x((\alpha\beta)\gamma) = (x(\alpha\beta))\gamma = ((x\alpha)\beta)\gamma = (x\alpha)(\beta\gamma) = x(\alpha(\beta\gamma)).$$

We call $\mathcal{F}_X$ the full transformation semigroup on X.

Definition 15 : *Let S and S' be groupoids. A mapping ϕ of S into S' is called a* ***homomorphism*** *if $(ab)\phi = (a\phi)(b\phi)$ for all a, b in S. The range $S\phi$ of ϕ, i.e., the set of all element $a\phi$ of S' with a in S, is a subgroupoid of S'. We say that S is* ***homomorphic*** *with $S\phi$, and that $S\phi$ is a homomorphic image of S; we write* $\mathbf{S} \sim \mathbf{S\phi}$. *If S is a semigroup, so is $S\phi$. A one-to-one homomorphism ϕ of S into S' is called an* ***isomorphism*** *of S into S'. The groupoids S and $S\phi$ are then said to be* ***isomorphic****, and we write* $\mathbf{S} \cong \mathbf{S\phi}$. *A homomorphism of S into itself is called an* ***endomorphism****, and an isomorphism of S upon itself is called an* ***automorphism****.*

Definition 16 : *If A and B are subsets of a groupoid S, then by the set* **product** *AB of A and B we shall mean the set of all elements ab of S with $a \in A$ and $b \in B$. When dealing with singleton sets we shall use the notational simplications that are customary in algebra, writing (for example) Ab rather than $A\{b\}$.*

If a is an element of a semigroup S without identity *then Sa need not contain a. We will use the following standard notations:*

$$S^1a = Sa \cup \{a\}$$

$$aS^1 = aS \cup \{a\}$$

$$S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$$

Notice that S^1a, aS^1 and S^1aS^1 are all subsets of S—they do not contain the identity element 1.

Definition 17 : *A non-empty subset A ($A \neq \Box$) of semigroup S is a* **left ideal** *of S if $s \in S$, $a \in A$ imply $sa \in A$ ($A \neq \Box$, $A \subseteq S \Longrightarrow SA \subseteq A$). A nonempty subset A of S is a* **right ideal** *of a groupoid S if $s \in S$, $a \in A$ imply $as \in A$ ($A \neq \Box$, $A \subseteq S \Longrightarrow AS \subseteq A$). A is an* **ideal** *(two sided ideal) of S if it is both a left and a right ideal of S. The* ***left ideal of S generated*** *by A is $A \cup SA = S^1A$, the* ***right ideal of S generated*** *by A is $A \cup AS = AS^1$ and the* ***(two sided) ideal of S generated*** *by A is $A \cup SA \cup AS \cup SAS = S^1AS^1$. If, in particular, A consists of a single element a, then we call $L(a) = S^1a$, $R(a) = aS^1$, and $J(a) = S^1aS^1$ the* ***principal left, principal right, principal two-sided ideal of*** *S, respectively,* ***generated*** *by a.*

Definition 18 : *A binary relation w on a set X (that is, a subset w of $X \times X$) is called a* ***(partial) order*** *if*

(1): $(x,x) \in w$ for all x in X—that is, w is reflexive;

(2): $(x,y) \in w$ and $(y,x) \in w$ imply $x = y$—that is, w is antisymmetric;

(3): $(x,y) \in w$ and $(y,z) \in w$ imply $(x,z) \in w$ (x,y,z in X)—that is, w is transitive. In other words, a partial ordering is a reflexive, antisymmetric, and transitive relation.

Traditionally one writes $x \leq y$ rather than $(x,y) \in w$. We shall follow this convention, and also write $x \geqslant y$, $x < y$ and $x > y$ to mean (respectively) $(y,x) \in w$, $(x,y) \in w$ and $x \neq y$, and $(y,x) \in w$ and $x \neq y$.

A partial order having the extra property

(4): $(x,y) \in w$ or $(y,x) \in w$ for every $x,y \in X$

will be called a ***total order***.

An idempotent element f of S is called ***primitive*** *if $f \neq 0$ and if $e \leq f$ implies $e = 0$ or $e = f$*

Proposition 19 : *Let E be the set of idempotents of a semigroup S. If $e, f \in E$, we* ***define*** *$e \leq f$ to mean $ef = fe = e$. Then $\leq$ is a partial ordering of E.*

Proof. Let $e, f, g \in E$.

(1): $e^2 = e$, and hence $e \leq e$.

(2): If $e \leq f$ and $f \leq e$ then $ef = fe = e$ and $fe = ef = f$, whence $e = f$.

(3): If $e \leq f$ and $f \leq g$ then $ef = fe = e$ and $fg = gf = f$, whence $eg = (ef)g = e(fg) = ef = e$ and $ge = g(fe) = (gf)e = fe = e$. Hence $e \leq g$.) If S contains a zero element 0, then $0 \leq e$ for every $e \in E$. We shall call $\leq$ the **natural partial ordering** of E. ■

Definition 20 : *An element a of a semigroup S is called **regular** if $a \in aSa$, that is, if $axa = a$ for some x in S. A semigroup S is called regular if every element of S is regular.*

Remark 2 : *We note that if $axa = a$ then $e = ax$ and $f = xa$ are idempotent elements of S. We also note that if a is a regular element of S, then the principal right ideal $aS^1 = \{a\} \cup aS$ generated by a is just aS, for $a = axa = a(xa) = af$ implies $a \in aS$. Similarly $S^1a = Sa$.*

Definition 21 : *Two elements a and b of a semigroup S are said to be **inverses** of each other if*

$$aba = a \qquad \text{and} \qquad bab = b$$

If an element a of a semigroup S has an inverse in S, then a is evidently regular.

Lemma 22 : *If a is a regular element of a semigroup S, say $axa = a$ with x in S, then a has at least one inverse in S, in particular xax.*

Proof. See [2, Lemma 1.14]. ■

Definition 23 : *An element b of a partially ordered set X is called an **upper bound** (UB) of a subset Y of X if $y \leq b$ for every y in Y. An upper bound b of Y is called a **least upper bound** (LUB) or **join** of Y if $b \leq c$ for every upper bound c of Y. **Lower bound** (LB) and **greatest lower bound** (GLB) or **meet** are defined dually.*

A partially ordered set X is called an **upper semilattice** *[***lower semilattice***] if every two-element subset $\{a,b\}$ of X has a join [meet] in X; it follows that every finite subset of X has a join [meet]. The join [meet] of $\{a,b\}$ will be denoted by $a \vee b$ $[a \wedge b]$. A* **lattice** *is a partially ordered set which is both an upper and a lower semilattice. A lattice X is said to be* **complete** *if every subset of X has a join and a meet.*

Definition 24 *: A groupoid S is called* **left simple** *[***right simple***] if S is the only left ideal [right ideal] of S.*

A semigroup S without zero is called **simple semigroup** *if it has no proper ideals. A semigroup S with zero is called* **0-simple semigroup** *if (i): $\{0\}$ and S are the only ideals and (ii): $S^2 \neq \{0\}$.*

By a **completely 0-simple semigroup** *we mean a 0-simple semigroup containing a primitive idempotent.*

For example any finite 0-simple semigroup is completely 0-simple. For S must contain an idempotent, so that $E \neq \square$. Further more $E \neq 0$, since $E = 0$ would imply that every element of S, and hence S itself, is nilpotent, contrary to $S^2 = S$. It is then clear that the finite partially ordered set $E \setminus 0$ must contain a minimal element, that is, a primitive idempotent.

Definition 25 *: Let S be a semigroup whose set E of idempotents is non-empty. We define a partial ordering $\geqslant$ on E by the rule that $e \geqslant f$ if and only if $ef = f = fe$. If $E = \{e_i : i \in N^0\}$, where N^0 denotes the set of all non-negative integers, and if the elements of E form the* **chain** *(or totally ordered) $e_0 > e_1 > e_2 > \cdots$, then S is called an $\omega -$* **semigroup**.

Chapter 2

Bicyclic Extension of a Monoid

2.1 Translational Hull

To construct more general structure theory of simple semigroup, Warne in [21] replaced "group" by "monoid" and termed the resulting construction a bicyclic extension. In this chapter, we give explicit proof of an important property of this bicyclic extension semigroup in Theorem 31.

Definition 26 : *Let* $S = ((N^0 \times \{0\}) \times (T \times P)) \cup ((N^0 \times N) \times (T \times K))$. *Let* T *be a monoid with group of units* U ($U = \{y \in T \mid xy = yx = 1_T$ *for some* $x \in T$ $\})$. *Let* θ *be a homomorphism of* T *into* U. *Let* θ° *denote the identity automorphism of* T. *Let* P *and* K *be disjoint sets,* N $[N^0]$ *be the set of natural numbers [non-negative integers] and* γ *be a homomorphism of* T *into* G_K, *the full transformation group on* K. *Define the following multiplication on* S :

$((n,k),(g,p))((r,s)\,,(h,q)) = ((n+r-t, k+s-t), (g\theta^{r-t}h\theta^{k-t}, x))$ *where* $t = min(k,r)$

and $x = q$ *or* $p(h\theta^{k-r-1}\gamma)$ *according to whether* $k \leq r$ *or* $k > r$. *We term* $S = (T, P, K, \theta, \gamma)$ *under this multiplication a* ***bicyclic extension*** *of* T.

Example 27 *: Let* $S = ((N^0 \times \{0\}) \times (T \times P)) \cup ((N^0 \times N) \times (T \times K))$ *where* $T = \{o, e, 1_T\}$ *is the monoid under multiplication given in table*

$\cdot$	o	e	1_T
o	o	o	o
e	o	o	e
1_T	o	e	1_T

with group of units $U = \{1_T\}$*;* $P = \{a, b, c\}$*;* $K = \{g, h\}$*; Let* $\theta : T \longrightarrow U$ *be defined such that for* $x \in T$, $x\theta = 1_T$ *and* $\theta^0 : T \longrightarrow T$ *such that* $x\theta^0 = x$ *for any* $x \in T$. *Let* $\gamma : T \longrightarrow G_K$ *be defined such that*

$o\gamma : \{g, h\} \longrightarrow \{g, h\}$	$e\gamma : \{g, h\} \longrightarrow \{g, h\}$	$1_T\gamma : \{g, h\} \longrightarrow \{g, h\}$
$g(o\gamma) = h$	$g(e\gamma) = g$	$g(1_T\gamma) = g$
$h(o\gamma) = g$	$h(e\gamma) = h$	$h(1_T\gamma) = h$

Then let $s_1 = ((0,0), (1_T, a))$, $s_2 = ((2,3), (o, g))$, $s_3 = ((0,1), (e, h)) \in S$ and we consider

$$
\begin{aligned}
s_1 s_2 &= ((0,0), (1_T, a))((2,3), (o, g)) \\
&= ((0 + 2 - 0, 0 + 3 - 0), (1_T\theta^{2-0} o\theta^{0-0}, g)) \\
&= ((2,3), (1_T(\theta^2) o\theta^0, g)) \\
&= ((2,3), (1_T o, g)) \text{ since } 1_T\theta^2 = (1_T\theta)\theta = 1_T\theta = 1_T \\
&= ((2,3), (o, g));
\end{aligned}
$$

$$
\begin{aligned}
s_2 s_3 &= ((2,3),(o,g))((0,1),(e,h)) \\
&= ((2+0-0,3+1-0),(o\theta^{0-0}e\theta^{3-0},g(e\theta^{3-0-0}\gamma))) \\
&= ((2,3),(o1_T,g(e\theta^3\gamma))) \text{ since } 1_T\theta^3 = (1_T\theta)\theta^2 = 1_T\theta^2 = 1_T \\
&= ((2,3),(o,g(e\theta(\theta^2)\gamma))) \\
&= ((2,3),(o,g(1_T(\theta^2)\gamma))) \\
&= ((2,3),(o,g(1_T\gamma))) \\
&= ((2,3),(o,g)); \\
s_1 s_3 s_1 &= ((0,0),(1_T,a))((0,1),(e,h))((0,0),(1_T,a)) \\
&= ((0,1),(1_T\theta^{0-0}e\theta^{0-0},h))((0,0),(1_T,a)) \\
&= ((0,1),(1_Te,h))((0,0),(1_T,a)) \\
&= ((0,1),(e,h))((0,0),(1_T,a)) \\
&= s_3 s_1
\end{aligned}
$$

Definition 28 : *A transformation ρ of a semigroup S is called a* ***right translation*** *of S if $x(y\rho) = (xy)\rho$ for all x, y in S. A transformation $\lambda : S \longrightarrow S$ is called a* ***left translation*** *if $(x\lambda)y = (xy)\lambda$ $\forall x, y \in S$.*

Definition 29 : *Let S be a semigroup. With each element a of S we associate a transformation $\rho_a : S \longrightarrow S$ $[\lambda_a : S \longrightarrow S]$ of S defined by $x\rho_a = xa$ $[x\lambda_a = ax]$ $\forall$ $x \in S$. We call $\rho_a[\lambda_a]$* ***the inner right [left] translation*** *of S.*

Remark 3 : $x\rho_a\rho_b = (x\rho_a)\rho_b = (xa)\rho_b = (xa)b = x(ab) = x\rho_{ab}$ *for all $x \in S$, so for any $a, b \in S$, we have $\rho_a\rho_b = \rho_{ab}$.*

Similarly, $x\lambda_a\lambda_b = (x\lambda_a)\lambda_b = (ax)\lambda_b = b(ax) = (ba)x = x\lambda_{ba}$ *for all* $x \in S$. *so for any* $a, b \in S$, *we have* $\lambda_a\lambda_b = \lambda_{ba}$.

Definition 30 *: A left translation* λ *and a right translation* ρ *are said to be* ***linked*** *if* $x(y\lambda) = (x\rho)y$ *for all* x, y *in* S. *For example, let* $a \in S$, *then the inner translations* λ_a *and* ρ_a *are linked, since* $x(y\lambda_a) = x(ay) = (xa)y = (x\rho_a)y$ *for all* x, y *in* S. *We define the* ***translational hull*** $\bar{S}$ *of a semigroup* S *to be the set of all pairs* (λ, ρ) *of linked left and right translations* λ *and* ρ *of* S. *If* (λ_1, ρ_1) *and* (λ_2, ρ_2) *are in* $\bar{S}$, *then so is* $(\lambda_2\lambda_1, \rho_1\rho_2) \in \bar{S}$, *since, for any* x *and* y *in* S, *we have*

$$\begin{aligned} x(z(\lambda_2\lambda_1) &= x((z\lambda_2)\lambda_1) \text{ for all } x, z \in S \\ &= x(y\lambda_1) \text{ where } z\lambda_2 = y \in S \\ &= (x\rho_1)y \text{ since } \rho_1 \text{and } \lambda_1 \text{ are linked} \\ &= (x\rho_1)(z\lambda_2) \text{where } z\lambda_2 = y \in S \\ &= ((x\rho_1)\rho_2)z \text{ since } \rho_2 \text{ and } \lambda_2 \text{ are linked} \\ &= (x(\rho_1\rho_2))z \end{aligned}$$

We may therefore define a binary operation in $\bar{S}$ *by*

$$(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_2\lambda_1, \rho_1\rho_2) \tag{2.0}$$

Now, we show that $\bar{S}$ with the above binary operation is associative.

Let $(\lambda_1, \rho_1), (\lambda_2, \rho_2), (\lambda_3, \rho_3) \in \bar{S}$

$$\begin{aligned} (\lambda_1, \rho_1)((\lambda_2, \rho_2), (\lambda_3, \rho_3)) &= (\lambda_1, \rho_1)(\lambda_3\lambda_2, \rho_2\rho_3) \\ &= (\lambda_3\lambda_2\lambda_1, \rho_1\rho_2\rho_3) \end{aligned}$$

$$= (\lambda_3(\lambda_2\lambda_1), (\rho_1\rho_2)\rho_3)$$

$$= (\lambda_2\lambda_1, \rho_1\rho_2)(\lambda_3, \rho_3)$$

$$= ((\lambda_1, \rho_1)(\lambda_2, \rho_2))(\lambda_3, \rho_3)$$

Theorem 31 *: Let $S = (T, P, K, \theta, \gamma)$ be a bicyclic extension and let $W = \{(g, \delta) : g \in T$ and $\delta : P \longrightarrow P$ is a non-constant mapping $\}$. Let $\bar{S}$ be the translational hull of S. Then, $\bar{S} \cong S \cup W$ under the multiplication*

$$(g_1, \delta_1) \cdot (g_2, \delta_2) = \begin{cases} (g_1g_2, \delta_1\delta_2) & \textit{if } \delta_1 \circ \delta_2 \textit{ is not constant} \\ ((0,0), (g_1g_2, k)) & \textit{if } (p)\delta_1 \circ \delta_2 = k \textit{ for all } p \in P \end{cases}$$

$$((a, b), (h, q)) \cdot (g, \delta) = \begin{cases} ((a, b), (h(g\theta^b), q(g\theta^{b-1}\gamma))) & \textit{if } b > 0 \\ ((a, 0), (hg, q\delta)) & \textit{if } b = 0 \end{cases}$$

$$(g, \delta) \cdot ((a, b), (h, q)) = ((a, b), ((g\theta^a)h, q))$$

where juxtaposition denotes the multiplication in T and $\circ$ denotes iteration of mappings, and $v_1 \cdot v_2 = v_1v_2$ where $v_1, v_2 \in S$, and also juxtaposition denotes multiplication in S.

Proof. To prove the above theorem, we need to go the following steps and prove each of the following:

Step(i): First, we show that $e_0 = ((0,0), (1_T, p))$ is a left identity of S.

Now

$$\begin{aligned} e_0((m,n),(z,s)) &= ((0,0),(1_T,p))((m,n),(z,s)) \\ &= ((m,n),(1_T\theta^m z\theta^0, s)),\ (t = \min(0,m) = 0) \end{aligned}$$

$$= ((m,n),(1_T z, s)), \text{ since } 1_T\theta^m = 1_T \text{ and } z\theta^0 = z$$

$$= ((m,n),(z,s)), \text{ since } 1_T z = z$$

Therefore, $((0,0),(1_T,p))$ is a left identity of S for any $p \in P$.

Note $\gamma : (T,.) \overset{\text{homo}}{\longrightarrow} (G_K, \circ)$ where G_K the set of all one to one and onto mappings from K to K (G_K is the full transformation group on K). Since $1_T \in T$, so $1_T\gamma \in G_K$. Since γ is a homomorphism, then $(1_T a)\gamma = 1_T\gamma \circ a\gamma$, $\forall\ a \in T$. So, $a\gamma = 1_T\gamma \circ a\gamma$. Also, $(a1_T)\gamma = a\gamma \circ 1_T\gamma$ and, hence $a\gamma = a\gamma \circ 1_T\gamma$. Therefore $1_T\gamma = 1_{G_K}$ is the identity of G_K.

Step(ii): Let ρ be any transformation of S. Define $\phi : P \longrightarrow S$ such that $\phi(p) = ((0,0),(1_T,p))\rho$ for $p \in P$. Then ρ is a right translation if and only if

$$((n,k),(g,q))\rho = \begin{cases} ((n,k),(g,q))\phi(z) \text{ for any } z \in P & \textbf{if } k > 0 \\ ((n,k),(g,q))\phi(q) & \textbf{if } k = 0 \end{cases} \tag{2.1}$$

Proof. $(\Longrightarrow)$ Suppose that ρ is a right translation. We consider cases:

Case 1: $k = 0$. Consider $\phi(q) = ((0,0),(1_T,q))\rho$ for $q \in P$. Then

$$\begin{aligned}
& ((n,0),(g,p))((0,0),(1_T,q)) \\
= \ & ((n+0-0, 0+0-0),((g\theta^{0-0}1\theta^{0-0}),q)) \ \textbf{if } k = 0 \ (k \leqslant r = 0) \\
= \ & ((n,0),(g1_T,q)) \text{ , since } g\theta^0 = g \text{ and } 1_T\theta^0 = 1_T \\
= \ & ((n,0),(g,q)), \text{ since } g, 1_T \in T \text{ and } g1_T = g.
\end{aligned}$$

Therefore,

$$\begin{aligned}
((n,0),(g,q))\rho &= [((n,0),(g,q))((0,0),(1_T,q))]\rho, \textbf{ if } k = 0 \text{ (then } q \in P \text{)} \\
&= ((n,0),(g,q))[((0,0),(1_T,q))\rho], \text{ since } \rho \text{ is a right translation}
\end{aligned}$$

$$= ((n,0),(g,q))\phi(q), \text{ since } \phi(q) = ((0,0),(1_T,q))\rho,\ q \in P.$$

Case 2: $k > 0$. Then, for any $p \in P$,

$$\begin{aligned}
& ((n,k),(g,q))((0,0),(1_T,p)) \\
= \ & ((n+0-0,k+0-0),((g\theta^{0-0}1_T\theta^{k-0}),q(1_T\theta^{k-0-1}\gamma))), \text{ if } k>0 \\
= \ & ((n,k),(g\theta^0 1_T\theta^k, q(1_T\theta^{k-1}\gamma)))\ , \text{ if } k>0 \\
= \ & ((n,k),(g1_T, q(1_T\gamma\ ))), \text{since } g\theta^0 = g, 1_T\theta^k = 1_T \ \&\ 1_T\theta^{k-1} = 1_T \\
= \ & ((n,k),(g,q)), \text{ since } 1_T\gamma \text{ is identity of } G_K.
\end{aligned}$$

Thus,

$$\begin{aligned}
((n,k),(g,q))\rho &= [((n,k),(g,q))((0,0),(1_T,q))]\rho \\
&= ((n,k),(g,q))[((0,0),(1_T,p))\rho], \text{ since } \rho \text{ is a right translation} \\
&= ((n,k),(g,q))\phi(q).
\end{aligned}$$

($\Longleftarrow$) : Let $\rho = \rho_\phi$ be defined as in (2.1). We show that ρ_ϕ is a right translation of S.

Case I: If $k \leqq r$, by multiplication on left side, we have

$$((n,k),(g,p))((r,s),(h,q)) = ((n+r-k,s),(g\theta^{r-k}h\theta^{k-k},q))$$

$$\Longrightarrow [((n,k),(g,p))((r,s),(h,q))]\rho_\phi = ((n+r-k,s),(g\theta^{r-k}h,q))\rho_\phi$$

Now, we consider two subcases $s > 0$ and $s = 0$.

(a): If $s > 0$ and $k \leqq r$. Then

$$\begin{aligned}
& [((n,k),(g,p))((r,s),(h,q)]\rho_\phi \\
= \ & ((n+r-k,s),(g\theta^{r-k}h,q))\rho_\phi \text{ since } k \leqq r
\end{aligned}$$

$$= ((n+r-k, s), (g\theta^{r-k}h, q))\phi(z), \forall\ z \in P \text{ , by (2.1)}$$

$$= [((n,k),(g,p))((r,s),(h,q)]\phi(z), \text{ for all } z \in P, \ \phi(z) \in S$$

$$= [((n,k),(g,p))][((r,s),(h,q))\phi(z)], \text{ for all } z \in P. \text{ by associativity in } S$$

$$= [((n,k),(g,p))][((r,s),(h,q))\rho_\phi] \text{ , by (2.1)}$$

(b): If $s = 0$ and $k \leqq r$.

$$\begin{aligned}
[((n,k),(g,p))((r,0),(h,q)]\rho_\phi &= ((n+r-k,0),(g\theta^{r-k}h,q))\rho_\phi \\
&= ((n+r-k,0),(g\theta^{r-k}h,q))\phi(q), \text{ by (2.1)} \\
&= [((n,k),(g,p))((r,0),(h,q)]\phi(q) \\
&= [((n,k),(g,p))][((r,0),(h,q))\phi(q)], \text{ by associativity in } S \\
&= [((n,k),(g,p))][((r,0),(h,q)]\rho_\phi] \text{ . by (2.1)}
\end{aligned}$$

Case II: If $k > r$, then

$$\begin{aligned}
((n,k),(g,p))((r,s),(h,q)) &= ((n,k+s-r),(g\theta^{r-r}h\theta^{k-r}, p(h\theta^{k-r-1}\gamma))) \\
&= ((n,k+s-r),(g\theta^0 h\theta^{k-r}, p(h\theta^{k-r-1}\gamma))) \\
&= ((n,k+s-r),(g(h\theta^{k-r}), p(h\theta^{k-r-1}\gamma))) \text{ since } g\theta^0 = g
\end{aligned}$$

Now, we consider two subcases:

(a): $k > r$ and $s = 0$

$$\begin{aligned}
&[((n,k),(g,p))((r,0),(h,q))]\rho_\phi \\
&= [((n,k+0-r),(g(h\theta^{k-r}), p(h\theta^{k-r-1}\gamma)))]\rho_\phi \text{ ; } k-r > 0 \\
&= ((n,k-r),(g\theta^0 h\theta^{k-r}, p(h\theta^{k-r-1}\gamma)))\rho_\phi, \ k-r > 0 \\
&= ((n,k-r),(g\theta^0 h\theta^{k-r}, p(h\theta^{k-r-1}\gamma)))\phi(z) \ \forall\ z \in P, \text{ by (2.1)}
\end{aligned}$$

$$= \quad [((n,k),(g,p))((r,0),(h,q))]\phi(z) \text{ for all } z \in P$$

$$= \quad ((n,k),(g,p))[((r,0),(h,q))\phi(z)] \text{ for all } z \in P,$$

$$= \quad ((n,k),(g,p))[((r,0),(h,q))\rho_\phi]$$

(b): $k > r$ and $s > 0$

$$[((n,k),(g,p))((r,s),(h,q))]\rho_\phi$$

$$= \quad [((n,k+s-r),(g\theta^{r-r}h\theta^{k-r}, p(h\theta^{k-r-1}\gamma)))]\rho_\phi\ ,\ k-r>0$$

$$= \quad ((n,k+s-r),(g\theta^0 h\theta^{k-r}, p(h\theta^{k-r-1}\gamma)))\phi(z), \forall\ z \in P, \text{ by (2.1)}$$

$$= \quad [((n,k),(g,p))((r,s),(h,q))]\phi(z) \text{ for all } z \in P, k > r$$

$$= \quad ((n,k),(g,p))[((r,s),(h,q))\phi(z)] \text{ for all } z \in P, \text{by associativity in } S$$

$$= \quad ((n,k),(g,p))[((r,s),(h,q))\rho_\phi] \text{ since } s > 0.$$

Therefore $\rho = \rho_\phi$ is a right translation of S.

Step (iii): We show that every left translation of a bicyclic extension S is an inner left translation.

Proof. Let λ be a left translation of S (i.e. $\lambda : S \longrightarrow S$ such that $(rs)\lambda = (r\lambda)s$ for all $r, s \in S$).

Since by step (i), $((n,k),(g,q)) = ((0,0),(1_T,p))((n,k),(g,q))$ for all $p \in P$ and $((n,k),(g,q)) \in S$. Then

$$\begin{aligned} ((n,k),(g,q))\lambda &= [((0,0),(1_T,p))((n,k),(g,q))]\lambda \\ &= [((0,0),(1_T,p))\lambda]((n,k),(g,q)) \text{ since } \lambda \text{ is a left translation} \\ &= t((n,k),(g,q)), \text{where } t = ((0,0),(1_T,p))\lambda \\ &= ((n,k),(g,q))\lambda_t\ \forall((n,k),(g,q)) \in S \end{aligned}$$

Therefore λ is the inner left translation λ_t.

Step (iv): Let ρ_ϕ be a right translation of a bicyclic extension S which is linked to some left translation of S. Then either ρ_ϕ is an inner right translation of S or ϕ is a non-constant mapping such that $\phi(p) = ((0,0),(g_\phi, p\delta_\phi))$ where $\delta_\phi : P \longrightarrow P$ is a non-constant mapping and $g_\phi \in T$ (i.e. g_ϕ depends just on ϕ not on p).

Proof. Let $E_0 = \{((0,0),(1_T,p)) : p \in P\}$ and $f_0 \in E_0$. Suppose ρ_ϕ is linked with λ_t ($t \in S$ and $t-$fixed). Let $e_0 = ((0,0),(1_T,q))$ for some $q \in P$. Then

$$\begin{aligned}(e_0\rho_\phi)f_0 &= e_0(f_0\lambda_t) \text{ since } \rho_\phi \text{ and } \lambda_t \text{ are linked}\\ &= e_0(tf_0) \text{ since by step}(iii)\ \lambda_t \text{ is an inner left translation}\\ &= tf_0 \text{ by step } (i).\end{aligned}$$

Thus

$$(((0,0),(1_T,q))\rho_\phi)f_0 = tf_0 \text{ for all } q \in P \text{ and all } f_0 \in E_0$$

Also since

$$\begin{aligned}((0,0),(1_T,q))\rho_\phi &= ((0,0),(1_T,q))\phi(q) \text{ by Step } (ii)\\ &= \phi(q), \text{by Step } (i),\end{aligned}$$

we deduce that

$$\phi(q)f_0 = tf_0\ \forall q \in P \text{ and } \forall f_0 \in E_0,$$

Hence $\phi(q)f_0 = tf_0$ ($t-$fixed) and $\phi(u)f_0 = tf_0$ ($t-$fixed) for $q, u \in P$. So

$$\phi(q)f_0 = \phi(u)f_0 \text{ for all } f_0 \in E_0. \tag{2.2}$$

Let $\phi(q) = ((a_1, b_1), (x_1, y_1))$ and $\phi(u) = ((a_2, b_2), (x_2, y_2))$. Thus, for all $p \in P$ we have

$$((a_1, b_1), (x_1, y_1))((0,0), (1_T, p)) = ((a_2, b_2), (x_2, y_2))((0,0), (1_T, p))$$
$$\Leftrightarrow \; ((a_1 - l_1, b_1 - l_1), (x_1\theta^{0-l_1} 1_T\theta^{b_1 - l_1}, s_1)) = ((a_2 - l_2, b_2 - l_2), (x_2\theta^{0-l_2} 1_T\theta^{b_2 - l_2}, s_2))$$

where $l_1 = \min(b_1, 0) = 0$, $l_2 = \min(b_2, 0) = 0$,

$$s_1 = \begin{cases} p \;\text{ if } b_1 \leqslant 0 \; (b_1 = 0) \\ y_1(1_T\theta^{b_1 - 0 - 1}\gamma) \;\text{ if } b_1 > 0 \end{cases} \quad \text{and } s_2 = \begin{cases} p \;\text{ if } b_2 \leqslant 0 \; (b_2 = 0) \\ y_2(1_T\theta^{b_2 - 0 - 1}\gamma) \;\text{ if } b_2 > 0 \end{cases}.$$

Hence

$$((a_1, b_1), (x_1\theta^0 1_T\theta^{b_1}, s_1)) = ((a_2, b_2), (x_2\theta^0 1_T\theta^{b_2}, s_2)) \tag{2.3}$$

So $a_1 = a_2$ and $b_1 = b_2$. Now if , $b_1 > 0 (\Longleftrightarrow b_2 > 0)$ then

$$((a_1, b_1), (x_1\theta^0 1_T\theta^{b_1}, y_1(1_T\theta^{b_1 - 0 - 1}\gamma))) = ((a_2, b_2), (x_2\theta^0 1_T\theta^{b_2}, y_2(1_T\theta^{b_2 - 0 - 1}\gamma)))$$
$$\Longleftrightarrow \; ((a_1, b_1), (x_1 1_T, y_1(1_T\gamma))) = ((a_2, b_2), (x_2 1_T, y_2(1_T\gamma)))$$

since $x_1\theta^0 = x_1, 1_T\theta^{b_1} = 1_T, 1_T\theta^{b_1 - 1} = 1_T$ and $x_2\theta^0 = x_2, 1_T\theta^{b_2} = 1_T, 1\theta^{b_2 - 1} = 1_T$.

Thus,

$$((a_1, b_1), (x_1 1_T, y_1(1_{G_K}))) = ((a_2, b_2), (x_2 1_T, y_2(1_{G_K})))$$
$$\Longleftrightarrow \; ((a_1, b_1), (x_1, y_1)) = ((a_2, b_2), (x_2, y_2)),$$

since $x_1 1_T = x_1$, $y_1 \in K$, $1_T\gamma = 1_{G_K}$, the identity mapping of K onto K (i.e. the identity of G_K), $y_1(1_T\gamma) = y_1 1_{G_K} = y_1$ and similarly $x_2 1_T = x_2$, $y_2 \in K$, $y_2(1_T\gamma) = y_2 1_{G_K} = y_2$.

Therefore, if $b_1 > 0$ then $a_1 = a_2$, $b_1 = b_2$, $x_1 = x_2$ and $y_1 = y_2$. So $\phi(q) = \phi(u)$ for any $q, u \in P$, i.e., ϕ is a constant mapping. i.e. $\phi(l) = w$ fixed element of S for all $l \in P$. Thus by Step (ii)

$$\begin{aligned}((n,k),(g,q))\rho_\phi &= ((n,k),(g,q))w, \ (w = \phi(l), \text{ for all } l \in P)\\ &= ((n,k),(g,q))\rho_w \text{ for all } ((n,k),(g,q)) \in S\end{aligned}$$

So ρ_ϕ is an inner right translation, ρ_w.

On the other hand if $b_1 = 0 = b_2$ then, from (2.3), we have

$$((a_1,0),(x_1\theta^0 1_T\theta^0,p)) = ((a_2,0),(x_2\theta^0 1_T\theta^0,p))$$

$$\Longleftrightarrow \quad ((a_1,0),(x_1,p)) = ((a_2,0),(x_2,p))$$

Thus $a_1 = a_2$, $x_1 = x_2$, so, we may write (note: $\phi : P \longrightarrow S$)

$$\begin{aligned}\phi(q) &= ((a_1,0),(x_1,y_1))\\ \phi(u) &= ((a_1,0),(x_1,y_2))\end{aligned} \tag{2.4}$$

for any u, $q \in P$. By definition of ρ_ϕ , we then have

$$\begin{aligned}((0,1),(1_T,p))\rho_\phi &= ((0,1),(1_T,p)\phi(q)\\ &= ((0,1),(1_T,p))\phi(u)\end{aligned}$$

So, $((0,1),(1_T,p))\phi(q) = ((0,1),(1_T,p))\phi(u)$. Since $\phi(q) = ((a_1,0),(x_1,y_1)), q \in P$ and $\phi(u) = ((a_1,0),(x_1,y_2))$, $u \in P$, then

$$((0,1),(1_T,p))((a_1,0),(x_1,y_1)) = ((0,1),(1_T,p))((a_1,0),(x_1,y_2)) \tag{2.5}$$

Now, if $a_1 > 0$ (so, $a_1 \geqslant 1$), then $((a_1 - 1,0),(x_1,y_1)) = ((a_1 - 1,0),(x_1,y_2))$. Thus, $y_1 = y_2$. So, $\phi(u) = ((a_1,0),(x_1,y_1)) = \phi(q)$ for any $u,q \in P$. Thus, ϕ is a constant mapping and, hence ρ_ϕ is an inner right translation.

Suppose $a_1 = 0$. Thus, by (2.5), we have

$$((0,1),(1_T,p))((0,0),(x_1,y_1)) = ((0,1),(1_T,p))((0,0),(x_1,y_2))$$

$$\iff ((0,1),(1_T\theta^{0-0}x_1\theta^{1-0},p(x_1\theta^{1-0-1}\gamma))) = ((0,1),(1_T\theta^{0-0}x_1\theta^{1-0},p(x_1\theta^{1-0-1}\gamma)))$$

$$\iff ((0,1),(x_1\theta,p(x_1\gamma))) = ((0,1),(x_1\theta,p(x_1\gamma)))$$

and, no new information is yielded. However,

$$\phi(q) = ((0,0),(x_1,y_1))$$

$$\phi(u) = ((0,0),(x_1,y_2))$$

If $y_1 = y_2$, $\phi(q) = \phi(u)$ and so ϕ is a constant mapping and hence ρ_ϕ is an inner right translation. If not ($y_1 \neq y_2$) we may write $\phi(p) = ((0,0),(g_\phi,p\delta_\phi))$ where $g_\phi \in T$ (i.e. g_ϕ depends just on ϕ not on p) and $\delta_\phi : P \longrightarrow P$ is a non-constant mapping (the fourth co-ordinate depends on p and ϕ and the third co-ordinate just on ϕ).

In the case ρ_ϕ is not an inner right translation, we may write

$$\phi(p) = ((0,0),(g,p\delta)) = (g,\delta)p \text{ (since } \phi \text{ is here fixed).}$$

So, a right translation ρ of S that is linked with some left translation of S is either an inner right translation or $\rho = \rho_{(g,\delta)}$ where $(g,\delta)(p) = ((0,0),(g,p\delta))$ for $p \in P$ where $\delta : P \to P$ is a non-constant mapping. (Note of course that any right translation is uniquely determined by ϕ).

Step (V): We consider multiplication of any two right translations such that each one is linked to some left translation. In Step (*iv*), we showed that any right translation of S which is linked to some left translation is either an inner right translation ρ_v where $v \in S$ or of the form $\rho = \rho_{(g,\delta)}$ where $(g,\delta)(p) = ((0,0),(g,p\delta))$ for $p \in P$, such that $\delta : P \to P$ is a non-constant mapping.

Case 1: Since $\rho_{(g_1,\delta_1)}$ and $\rho_{(g_2,\delta_2)}$ are not inner right translations, then, by Step (iv) $\rho_{(g_1,\delta_1)}, \rho_{(g_2,\delta_2)} \in W'$ where $W' = \{\rho_{(g,\delta)} : g \in T$ and $\delta : P \longrightarrow P$ is a non-constant mapping$\}$. Let $e_0 = ((0,0),(1_T,p)) \in E_0$.

For all $p \in P$, we show that

$$((0,0),(1_T,p))\rho_{(g_1,\delta_1)} = ((0,0),(g_1,p\delta_1)) \tag{2.6}$$

By Step (ii), we have

$$\begin{aligned} & ((0,0),(1_T,p))\rho_{(g_1,\delta_1)} \\ = \; & ((0,0),(1_T,p))(g_1,\delta_1)(p), \text{by Step } (ii) \\ = \; & ((0,0),(1_T,p))((0,0),(g_1,p\delta_1)), \text{ since } (g_1,\delta_1)(p) = ((0,0),(g_1,p\delta_1)) \\ = \; & ((0,0),(g_1,p\delta_1)), \text{by Step } (i) \end{aligned}$$

So, $e_0\rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)} = ((0,0),(g_1,p\delta_1))\rho_{(g_2,\delta_2)}$

Since $\rho = \rho_{(g,\delta)}$ where $(g,\delta)(p) = ((0,0),(g,p\delta))$ for $p \in P$, such that $\delta : P \rightarrow P$ is a non-constant mapping. Then,

$$\begin{aligned} (g_1,\delta_1)(p) &= ((0,0),(g_1,p\delta_1)) \\ &= e_0\rho_{(g_1,\delta_1)}, \text{ by (2.6)} \end{aligned}$$

$$\begin{aligned} & e_0\rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)} \\ = \; & ((0,0),(g_1,p\delta_1))\rho_{(g_2,\delta_2)}, \text{ by (2.6)} \\ = \; & ((0,0),(g_1,p\delta_1))(g_2,\delta_2)(p\delta_1), \text{ by Step } (ii) \\ = \; & ((0,0),(g_1,p\delta_1))((0,0),(g_2,(p\delta_1)\delta_2)) \\ = \; & ((0,0),(g_1\theta^{0-0}g_2\theta^{0-0},(p\delta_1)\delta_2)), \text{by multiplication in } S \end{aligned}$$

$$= ((0,0),(g_1g_2,p(\delta_1\circ\delta_2)))$$

$$= \begin{cases} ((0,0),(g_1g_2,p(\delta_1\circ\delta_2))) \text{ if } (\delta_1\circ\delta_2) \text{ is not a constant mapping.} \\ ((0,0),(g_1g_2,k_0)) \text{ if } p(\delta_1\circ\delta_2) = k_0 \text{ for } p \in P. \end{cases}$$

$$= \begin{cases} ((0,0),(g_1g_2,p(\delta_1\circ\delta_2))) \text{ if } (\delta_1\circ\delta_2) \text{ is not a constant mapping.} \\ e_0((0,0),(g_1g_2,k_0)) \text{ if } p(\delta_1\circ\delta_2) = k_0 \text{ for } p \in P \text{ (by Step } (i)) \end{cases}$$

$$= \begin{cases} e_0\rho_{(g_1g_2,\delta_1\circ\delta_2)} \text{ if } (\delta_1\circ\delta_2) \text{ is not a constant mapping (by (2.6))} \\ e_0\rho_{((0,0),(g_1g_2,k_0))} \text{ if } k_0 \text{ is a constant (by Step } (iv) \text{ since } \rho_{((0,0),(g_1g_2,k_0))} \text{ is} \\ \text{an inner right translation)} \end{cases}$$

Thus,

$$e_0\rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)} = \begin{cases} e_0\rho_{(g_1g_2,\delta_1\circ\delta_2)} \text{ if } (\delta_1\circ\delta_2) \text{ is non-constant mapping} \\ e_0\rho_{((0,0),(g_1g_2,k_0))} \text{ if } p(\delta_1\circ\delta_2) = k_0 \text{ for } p \in P \end{cases} \quad (2.7)$$

Fact: We show that

$$\forall\, x \in S,\ \exists\, e_0 \in E_0 = \{((0,0),(1_T,p) : p \in P\} \text{ such that } xe_0 = x \quad (2.8)$$

Proof. let $x = ((n,k),(g,q))$.

Case $1_{\mathbf{a}}$: If $k = 0$. Consider $e_0 = ((0,0),(1,q))$ for $q \in P$

$$((n,0),(g,q))((0,0),(1_T,q))$$

$$= ((n+0-0,0+0-0),(g\theta^{0-0}1_T\theta^{0-0}),q)) \text{ if } k = 0\ (k \leqslant r = 0)$$

$$= ((n,0),(g1_T),q))\ , \text{ since } g\theta^0 = g \text{ and } 1_T\theta^0 = 1_T$$

$$= ((n,0),(g,q)), \text{ since } g, 1_T \in T \text{ and } g1_T = g$$

Therefore, $xe_0 = x$.

Case 1_b: If $k > 0$. Consider $e_0 = ((0,0),(1_T,p))$ for $p \in P$. Then.

$$\begin{aligned}
& ((n,k),(g,q))((0,0),(1_T,p)) \\
= \; & ((n+0-0,k+0-0),(g\theta^{0-0}1_T\theta^{k-0}),q(1_T\theta^{k-0-1}\gamma))) \text{ if } k>0 \\
= \; & ((n,k),(g\theta^0 1_T\theta^k, q(1_T\theta^{k-1}\gamma))) \text{ if } k>0 \\
= \; & ((n,k),(g1_T,q(1_T\gamma\ ))), \text{since } g\theta^0 = g, 1_T\theta^k = 1_T \;\&\; 1_T\theta^{k-1} = 1_T \\
= \; & ((n,k),(g,q)), \text{ since } 1_T\gamma \text{ is identity of } G_K.
\end{aligned}$$

Therefore, $xe_0 = x$. Thus, for any $x \in S$, $\exists\, e_0 \in E_0 = \{((0,0),(1,p) : p \in P\}$ such that $xe_0 = x$.

$\forall\, x \in S$, by (2.7), we have

$$x(e_0\rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)}) = \begin{cases} x(e_0\rho_{(g_1g_2,\delta_1{}_\circ\delta_2)}) \text{ if } (\delta_1{}_\circ\delta_2) \text{ is non-constant mapping} \\ x(e_0\rho_{((0,0),(g_1g_2,k_0))}) \text{ if } p(\delta_1{}_\circ\delta_2) = \; k_0 \text{ for } p \in P \end{cases}$$

By definition of right translation, we have

$$(xe_0)\rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)}) = \begin{cases} (xe_0)\rho_{(g_1g_2,\delta_1{}_\circ\delta_2)} \text{ if } (\delta_1{}_\circ\delta_2) \text{ is non-constant mapping} \\ (xe_0)\rho_{((0,0),(g_1g_2,k_0))} \text{ if } p(\delta_1{}_\circ\delta_2) = \; k_0 \text{ for } p \in P \end{cases}$$

Using (2.8), $\forall\, x \in S$,

$$x\rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)} = \begin{cases} x\rho_{(g_1g_2,\delta_1{}_\circ\delta_2)} \text{ if } (\delta_1{}_\circ\delta_2) \text{ is non-constant mapping} \\ x\rho_{((0,0),(g_1g_2,k_0))} \text{ if } p(\delta_1{}_\circ\delta_2) = \; k_0 \text{ for } p \in P \end{cases}$$

Thus

$$\rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)} = \begin{cases} \rho_{(g_1g_2,\delta_1\circ\delta_2)} \text{ if } \delta_1{}_\circ\delta_2 \text{ is not a constant mapping} \\ \rho((0,0),(g_1g_2,k)) \quad \text{if } p(\delta_1{}_\circ\delta_2) = k \text{ for all } p \in P \end{cases} \tag{2.9}$$

Case 2: Suppose that $\rho_{v_1=((a_1,b_1),(h_1,q_1))}$ is an inner right translation and $\rho_{(g_2,\delta_2)}$ is not an inner right translation. Then $\rho = \rho_{(g_2,\delta_2)}$ where $(g_2,\delta_2)(p) = ((0,0),(g_2,p\delta_2))$ for

$p \in P$, such that $\delta : P \to P$ is a non-constant mapping, by step (iv). Therefore, for any $p \in P$, we have

$$[((0,0),(1_T,p)\rho_{v_1}]\rho_{(g_2,\delta_2)}$$

$$= [((0,0),(1_T,p)v_1]\rho_{(g_2,\delta_2)} \text{ since } \rho_{v_1} \text{is an inner right translation}$$

$$= [((0,0),(1_T,p)((a_1,b_1),(h_1,q_1))]\rho_{(g_2,\delta_2)} \text{ since } \rho_{(g_2,\delta_2)} \text{ a right translation}$$

$$= ((a_1,b_1),(h_1,q_1))\rho_{(g_2,\delta_2)}, \text{ by Step } (i)$$

$$= \begin{cases} ((a_1,b_1),(h_1,q_1))(g_2,\delta_2)(q_1), (\text{where } z = q_1 \text{ in Step } (ii)) \text{ if } b_1 > 0 \\ ((a_1,0),(h_1,q_1))(g_2,\delta_2)(q_1) \text{ if } b_1 = 0 \end{cases} \quad \text{by Step } (ii)$$

$$= \begin{cases} ((a_1,b_1),(h_1,q_1))((0,0),(g_2,q_1\delta_2) \text{ if } b_1 > 0 \\ ((a_1,0),(h_1,q_1))((0,0),(g_2,q_1\delta_2) \text{ if } b_1 = 0 \end{cases}$$

$$= \begin{cases} ((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma))) \text{ if } b_1 > 0 \\ ((a_1,0),(h_1g_2,q_1\delta_2)) \text{ if } b_1 = 0 \end{cases} \quad \text{by multiplication in } S$$

$$= \begin{cases} e_0((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma))) \text{ if } b_1 > 0 \\ e_0((a_1,0),(h_1g_2,q_1\delta_2)) \text{ if } b_1 = 0 \end{cases} , \; e_0 = ((0,0),(1,p)), \text{ by step } (i)$$

$$= \begin{cases} e_0\rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))} \text{ if } b_1 > 0 \\ e_0\rho_{((a_1,0),(h_1g_2,q_1\delta_2))} \text{ if } b_1 = 0 \end{cases} \quad \text{where } \rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))} \text{ and}$$

$\rho_{((a_1,0),(h_1g_2,q_1\delta_2))}$ are inner right translation.

Thus $\forall\, x \in S$ and $\forall\, p \in P$

$$x[((0,0),(1,p))\rho_{((a_1,b_1),(h_1,q_1))}]\rho_{(g_2,\delta_2)} = \begin{cases} x[e_0\rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))}] \text{ if } b_1 > 0 \\ x[e_0\rho_{((a_1,0),(h_1g_2,q_1\delta_2))}] \text{ if } b_1 = 0 \end{cases}$$

By right translation property, we have

$$[xe_0\rho_{((a_1,b_1),(h_1,q_1))}]\rho_{(g_2,\delta_2)} = \begin{cases} [xe_0]\rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))} \text{ if } b_1 > 0 \\ [xe_0]\rho_{((a_1,0),(h_1g_2,q_1\delta_2))} \text{ if } b_1 = 0 \end{cases}$$

Using (2.8) $\exists\, e_0 \in E_0 = \{((0,0),(1,p)) : p \in P\}$ such that $xe_0 = x$. So

$$[x\rho_{((a_1,b_1),(h_1,q_1))}]\rho_{(g_2,\delta_2)} = \begin{cases} x\rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))} \text{ if } b_1 > 0 \\ x\rho_{((a_1,0),(h_1g_2,q_1\delta_2))} \text{ if } b_1 = 0 \end{cases}$$

$$\rho_{((a_1,b_1),(h_1,q_1))}\rho_{(g_2,\delta_2)} = \begin{cases} \rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))} \text{ if } b_1 > 0 \\ \rho_{((a_1,0),(h_1g_2,q_1\delta_2))} \text{ if } b_1 = 0 \end{cases} \tag{2.10}$$

Case 3: Since that $\rho_{(g_1,\delta_1)}$ is not an inner right translation and $\rho_{v_2=((a_2,b_2),(h_2,q_2))}$ is an inner right translation, then, by Step (iv) $\rho = \rho_{(g_1,\delta_1)}$ where $(g_1,\delta_1)(p) = ((0,0),(g_1,p\delta_1))$ for $p \in P$, such that $\delta : P \to P$ is a non-constant mapping.

Let $e_0 \in E_0 = \{((0,0),(1_T,p)) : p \in P\}$, then,

$$\begin{aligned}
& e_0\rho_{(g_1,\delta_1)}\rho_{((a_2,b_2),(h_2,q_2))} \\
= \; & ((0,0),(g_1,p\delta_1))\rho_{((a_2,b_2),(h_2,q_2))} \text{ by (2.6)} \\
= \; & ((0,0),(g_1,p\delta_1))((a_2,b_2),(h_2,q_2)) \text{ since } \rho_{((a_2,b_2),(h_2,q_2))} \text{ is an inner right translation} \\
= \; & ((a_2,b_2),((g_1\theta^{a_2})h_2,q_2)) \\
= \; & e_0((a_2,b_2),((g_1\theta^{a_2})h_2,q_2)) \text{ by Step } (i) \\
= \; & e_0\rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))}, \text{ where } \rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))} \text{ is a right translation}
\end{aligned}$$

Thus

$$x[e_0\rho_{(g_1,\delta_1)}\rho_{((a_2,b_2),(h_2,q_2))}] = x[e_0\rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))}]$$

By definition of right translation

$$[xe_0]\rho_{(g_1,\delta_1)}\rho_{((a_2,b_2),(h_2,q_2))} = [xe_0]\rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))}$$

Using (2.8), $\forall\, x \in S$, $\exists\, e_0 \in E_0 = \{((0,0),(1_T,p)) : p \in P\}$ such that $xe_0 = x$. So, $\forall$ $x \in S$,

$$x\rho_{(g_1,\delta_1)}\rho_{((a_2,b_2),(h_2,q_2))} = x\rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))}$$

$$\Longrightarrow \rho_{(g_1,\delta_1)}\rho_{((a_2,b_2),(h_2,q_2))} = \rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))} \tag{2.11}$$

Case 4: If both ρ_{v_1} and ρ_{v_2} are inner right translations, we show that

$$\rho_{v_1}\rho_{v_2} = \rho_{v_1v_2} \text{ for } v_1, v_2 \in S. \tag{2.12}$$

Proof. Let $x, v_1, v_2 \in S$,

$$\begin{aligned} x(\rho_{v_1}\rho_{v_2}) &= (x\rho_{v_1})\rho_{v_2} \\ &= (xv_1)\rho_{v_2} \text{ since } \rho_{v_1} \text{is an inner right translation} \\ &= (xv_1)v_2 \text{ since } \rho_{v_2} \text{ is an inner right translation} \\ &= x(v_1v_2) \text{ by associativity in } S, \text{ for any } x, v_1, v_2 \in S \\ &= x\rho_{v_1v_2} \end{aligned}$$

Step (vi): Let S be a bicyclic extension and let λ_v, λ_s be two inner left translation of S, $v, s \in S$. Then $\lambda_v = \lambda_s \Leftrightarrow v((0,0),(1_T,p) = s((0,0),(1_T,p))$ for all $p \in P$.

Proof. ($\Rightarrow$) Suppose that $\lambda_v = \lambda_s$. Then

$$((n,k),(g,p))\lambda_v = ((n,k),(g,p))\lambda_s \text{ for all } ((n,k),(g,p)) \in S.$$

Since $((0,0),(1_T,p)) \in S$, so,

$$((0,0),(1_T,p))\lambda_v = ((0,0),(1_T,p))\lambda_s \text{ for all } p \in P$$

Therefore

$$v((0,0),(1_T,p)) = s((0,0),(1_T,p)) \text{ for all } p \in P$$

$(\Longleftarrow)$: Suppose $v((0,0),(1_T,p)) = s((0,0),(1_T,p))$, for any $v, s \in S$ and all $p \in P$. Then, for any $((n,k),(g,p)) \in S$,

$$v((0,0),(1_T,p))((n,k),(g,p)) = s((0,0),(1_T,p))((n,k),(g,p)) \text{ for any } ((n,k),(g,p)) \in S$$

then, by Step (i)

$$v((n,k),(g,p)) = s((n,k),(g,p)) \text{ for any } ((n,k),(g,p)) \in S$$

$$\Longleftrightarrow \quad ((n,k),(g,p))\lambda_v = ((n,k),(g,p))\lambda_s \text{ for any } ((n,k),(g,p)) \in S$$

Hence, $\lambda_v = \lambda_s$.

Step (vii): We will show that $\varphi : \bar{S} \longrightarrow S \cup W$ ($W = \{(g,\delta) : g \in T$ and $\delta : P \longrightarrow P$, a non-constant map$\}$) defined by $\varphi(\lambda_v, \rho_v) = v$ and $\varphi(\lambda_s, \rho_{(g,\delta)}) = (g,\delta)$ defines an isomorphism from $\bar{S}$ onto $S \cup W$ under the multiplication given in the statement of the theorem.

First we need to show that φ is well defined. So, we need to show following:

(i): $(\lambda_{v_1}, \rho_{v_1}) = (\lambda_{v_2}, \rho_{v_2}) \Longrightarrow \varphi(\lambda_{v_1}, \rho_{v_1}) = \varphi(\lambda_{v_2}, \rho_{v_2})$

(ii): $(\lambda_{s_1}, \rho_{(g_1,\delta_1)}) = (\lambda_{s_2}, \rho_{(g_2,\delta_2)}) \Longrightarrow \varphi(\lambda_{s_1}, \rho_{(g_1,\delta_1)}) = \varphi(\lambda_{s_2}, \rho_{(g_2,\delta_2)})$.

(iii): $(\lambda_v, \rho_s) \in \bar{S} \Longrightarrow \varphi(\lambda_v, \rho_s) \in S \cup W$.

Let $(\lambda_{v_1}, \rho_{v_1}) = (\lambda_{v_2}, \rho_{v_2})$ then

$$\begin{aligned}
& \lambda_{v_1} = \lambda_{v_2} \text{ and } \rho_{v_1} = \rho_{v_2} \\
\Longrightarrow \; & ((0,0),(1_T,p))\rho_{v_1} = ((0,0),(1_T,p))\rho_{v_2} \text{ for } p \in P \\
\Longrightarrow \; & ((0,0),(1_T,p))v_1 = ((0,0),(1_T,p))v_2 \text{ for } p \in P \\
\Longrightarrow \; & v_1 = v_2 \text{ by Step } (i).
\end{aligned}$$

Thus $\varphi(\lambda_{v_1}, \rho_{v_1}) = \varphi(\lambda_{v_2}, \rho_{v_2})$.

Let $(\lambda_{s_1}, \rho_{(g_1,\delta_1)}) = (\lambda_{s_2}, \rho_{(g_2,\delta_2)})$. then $\lambda_{s_1} = \lambda_{s_2}$ and $\rho_{(g_1,\delta_1)} = \rho_{(g_2,\delta_2)}$, so

$$\begin{aligned}
& ((0,0),(1_T,p))\rho_{(g_1,\delta_1)} = ((0,0),(1_T,p))\rho_{(g_2,\delta_2)} \\
\Longrightarrow \; & ((0,0),(g_1,p\delta_1)) = ((0,0),(g_2,p\delta_2)) \text{ for any } p \in P \\
\Longrightarrow \; & g_1 = g_2 \text{ and } p\delta_1 = p\delta_2 \text{ for any } p \in P \\
\Longrightarrow \; & g_1 = g_2 \text{ and } \delta_1 = \delta_2 \\
\Longrightarrow \; & (g_1,\delta_1) = (g_2,\delta_2)
\end{aligned}$$

Therefore $\varphi(\lambda_{s_1}, \rho_{(g_1,\delta_1)}) = \varphi(\lambda_{s_2}, \rho_{(g_2,\delta_2)})$.

Suppose $(\lambda_v, \rho_s) \in \bar{S}$. Then λ_v and ρ_s are linked. So, for all $x, y \in S$, we have

$$\begin{aligned}
& x(y\lambda_v) = (x\rho_s)y \\
\Longrightarrow \; & x(vy) = (xs)y \\
\Longrightarrow \; & x(vy) = x(sy) \\
\Longrightarrow \; & x(y\lambda_v) = x(y\lambda_s).
\end{aligned}$$

Thus $\lambda_v = \lambda_s$ and

$$\varphi(\lambda_v, \rho_s) = \varphi(\lambda_s, \rho_s) = s \in S \cup W$$

Therefore φ is well defined.

Next we show that φ is a homomorphism.

(a):

$$\varphi((\lambda_{s_1}, \rho_{(g_1,\delta_1)})(\lambda_{s_2}, \rho_{(g_2,\delta_2)})) = \varphi(\lambda_{s_2}\lambda_{s_1}, \rho_{(g_1,\delta_1)}\rho_{(g_2,\delta_2)}) \text{ by (2.0)}$$

$$= \begin{cases} \varphi(\lambda_{s_2}\lambda_{s_1}, \rho_{(g_1g_2,\delta_1\circ\delta_2)}) \text{ if } \delta_1 \circ \delta_2 \text{ is non-constant} \\ \varphi(\lambda_{s_1s_2}, \rho_{((0,0),(g_1g_2,k))}) \text{ if } (p)\delta_1 \circ \delta_2 = k \ \forall p \in P \end{cases} \text{ by (2.9)}$$

$$= \begin{cases} (g_1g_2, \delta_1 \circ \delta_2) \text{ if } \delta_1 \circ \delta_2 \text{ is non-constant} \\ ((0,0), (g_1g_2, k) \text{ if } (p)\delta_1 \circ \delta_2 = k \ \forall p \in P \end{cases} \text{ by the definition of } \varphi$$

$$= (g_1, \delta_1)(g_2, \delta_2)$$

$$= \varphi(\lambda_{s_1}, \rho_{(g_1,\delta_1)})\varphi(\lambda_{s_2}, \rho_{(g_2,\delta_2)}) \text{ by the definition of } \varphi$$

(b): $s_1 = ((a_1, b_1), (h_1, q_1)), s_2 = ((a_2, b_2), (h_2, q_2))$

$$\varphi((\lambda_{s_1}, \rho_{((a_1,b_1),(h_1,q_1))})(\lambda_{((a_2,b_2),(h_2,q_2))}, \rho_{(g_2,\delta_2)}))$$

$$= \varphi(\lambda_{((a_2,b_2),(h_2,q_2))}\lambda_{s_1}, \rho_{((a_1,b_1),(h_1,q_1))}\rho_{(g_2,\delta_2)}) \text{ by (2.0)}$$

$$= \begin{cases} \varphi(\lambda_{s_1((a_2,b_2),(h_2,q_2))}, \rho_{((a_1,b_1),(h_1(g_2\theta^{b_1}),q_1(g_2\theta^{b_1-1}\gamma)))} \text{ if } b_1 > 0 \\ \varphi(\lambda_{s_1((a_2,b_2),(h_2,q_2))}, \rho_{((a_1,0),(h_1g_2,q_1\delta_2))} \text{ if } b_1 = 0 \end{cases} . \text{ by (2.10)}$$

$$= \begin{cases} ((a_1, b_1), (h_1(g_2\theta^{b_1}), q_1(g_2\theta^{b_1-1}\gamma))) \text{ if } b_1 > 0 \\ ((a_1, 0), (h_1g_2, q_1\delta_2)) \text{ if } b_1 = 0 \end{cases} \text{ by the definition of } \varphi$$

$$= ((a_1, b_1), (h_1, q_1))(g_2, \delta_2) \text{ by the definition of product in } S \cup W$$

(c): $s_1 = ((a_1, b_1), (h_1, q_1)), s_2 = ((a_2, b_2), (h_2, q_2))$

$$\varphi((\lambda_{s_1}, \rho_{(g_1,\delta_1)})(\lambda_{((a_1,b_1),(h_1,q_1))}, \rho_{((a_2,b_2),(h_2,q_2))}))$$

$$= \varphi(\lambda_{((a_1,b_1),(h_1,q_1))}\lambda_{s_1}, \rho_{(g_1,\delta_1)}\rho_{((a_2,b_2),(h_2,q_2))}) \text{ by (2.0)}$$

$$= \varphi(\lambda_{s_1((a_1,b_1),(h_1,q_1))}, \rho_{((a_2,b_2),((g_1\theta^{a_2})h_2,q_2))}) \text{ by (2.11)}$$

$$= ((a_2, b_2), ((g_1\theta^{a_2})h_2, q_2)) \text{ by definition of } \varphi$$

$$= (g_1, \delta_1)((a_2, b_2), (h_2, q_2)) \text{ by definition in } S \cup W$$

$$= \varphi(\lambda_{s_1}, \rho_{(g_1,\delta_1)})\varphi(\lambda_{((a_1,b_1),(h_1,q_1))}, \rho_{((a_2,b_2),(h_2,q_2))}) \text{ by definition of } \varphi.$$

$$= \varphi(\lambda_{s_1}, \rho_{((a_1,b_1),(h_1,q_1))})\varphi(\lambda_{((a_2,b_2),(h_2,q_2))}, \rho_{(g_2,\delta_2)}) \text{ by the definition of } \varphi$$

(d):

$$\begin{aligned}
\varphi((\lambda_{v_1}, \rho_{v_1})(\lambda_{v_2}, \rho_{v_2})) &= \varphi(\lambda_{v_2}\lambda_{v_1}, \rho_{v_1}\rho_{v_2}) \text{ by (2.0)} \\
&= \varphi(\lambda_{v_2 v_1}, \rho_{v_1 v_2}) \text{ by (2.12)} \\
&= v_1 v_2 \text{ by definition of } \varphi \\
&= \varphi(\lambda_{v_1}, \rho_{v_1})\varphi(\lambda_{v_2}, \rho_{v_2}) \text{ by the definition of } \varphi.
\end{aligned}$$

We show φ maps $\bar{S}$ onto $S \cup W$. If $v \in S, \varphi(\lambda_s, \rho_v) = v$. Suppose, $(g, \delta) \in W$. Let $s = ((0,0),(g,k))$. Then,

$$\begin{aligned}
(g,\delta)(q)((0,0),(1_T,p)) &= ((0,0),(g,q\delta))((0,0),(1_T,p)) \\
&= ((0,0),(g,p)) \text{ by multiplication in } S \\
&= ((0,0),(g,k))((0,0),(1_T,p)) \text{ by multiplication in } S \\
&= s((0,0),(1_T,p)) \text{ for all } p \in P, \text{ since } s = ((0,0),(g,k)).
\end{aligned}$$

If $y \in S$, then

$$(g,\delta)(q)((0,0),(1_T,p))y = s((0,0),(1_T,p))y \text{ for all } p \in P, \text{by above equalilty.}$$

$$(g,\delta)(q)y = sy \text{ by Step } (i)$$

So, $\forall\, x \in S$, $x((g,\delta)(q)y) = x(sy)$, then $(x(g,\delta)(q))y = x(sy)$,by associativity in S. Hence, $(x\rho_{(g,\delta)})y = x(y\lambda_s)$ for $x \in S$. Thus, $(\lambda_s, \rho_{(g,\delta)}) \in \bar{S}$ and $\varphi(\lambda_s, \rho_{(g,\delta)}) = (g,\delta)$.

Next, we show that φ is one-to-one. Suppose $\varphi(\lambda_{s_1}, \rho_{(g_1,\delta_1)}) = \varphi(\lambda_{s_2}, \rho_{(g_2,\delta_2)})$. Then

$$(g_1, \delta_1) = (g_2, \delta_2) \text{ by the definition of } \varphi$$

$$\Longrightarrow \quad \rho_{(g_1,\delta_1)} = \rho_{(g_2,\delta_2)}$$

Thus

$$\forall q \in P, \ ((0,0),(1_T,q))\rho_{(g_1,\delta_1)} = ((0,0),(1_T,q))\rho_{(g_2,\delta_2)}$$

However, by (2.6), we have

$$\forall q \in P, \ ((0,0),(g_1,q\delta_1)) = ((0,0),(g_2,q\delta_2))$$

So, $g_1 = g_2$ and $\delta_1 = \delta_2$. Then,

$$\forall q \in P, \ (g_1,\delta_1)(q)f_0 = (g_2,\delta_2)(q)f_0, \text{ where } f_0 \in E_0 = \{((0,0),(1_T,p)) : p \in P\}$$

From the left hand side, we have

$$\begin{aligned}
(g_1,\delta_1)(q)f_0 &= ((0,0),(g_1,q\delta_1)((0,0),(1_T,p)) \\
&= ((0,0),(g_1,p)) \\
&= ((0,0),(g_1,k))((0,0),(1_T,p)) \\
&= s_1 f_0, \text{ where } s_1 = ((0,0),(g_1,k))
\end{aligned}$$

From the right hand side, we have

$$\begin{aligned}
(g_2,\delta_2)(q)f_0 &= ((0,0),(g_2,q\delta_2))((0,0),(1_T,p)) \\
&= ((0,0),(g_2,p)) \\
&= ((0,0),(g_2,k))((0,0),(1_T,p)) \\
&= s_2 f_0, \text{ where } s_2 = ((0,0),(g_2,k)).
\end{aligned}$$

So, $s_1 f_0 = s_2 f_0$ for all $p \in P$ and $f_0 \in E_0$. Then we have

$$
\begin{aligned}
& (s_1 f_0)x = (s_2 f_0)x \text{ for any } x \in S \\
\Longrightarrow\ & s_1(f_0 x) = s_2(f_0 x) \text{ for any } x \in S \\
\Longrightarrow\ & (f_0 x)\lambda_{s_1} = (f_0 x)\lambda_{s_2} \text{ for any } x \in S \\
\Longrightarrow\ & x\lambda_{s_1} = x\lambda_{s_2} \text{ for any } x \in S \text{ (since } f_0 x = x) \\
\Longrightarrow\ & \lambda_{s_1} = \lambda_{s_2}
\end{aligned}
$$

Thus, $(\lambda_{s_1}, \rho_{(g_1,\delta_1)}) = (\lambda_{s_2}, \rho_{(g_2,\delta_2)})$. Clearly, $\varphi(\lambda_v, \rho_v) = \varphi(\lambda_t, \rho_t)$ implies $v = t$, so $(\lambda_v, \rho_v) = (\lambda_t, \rho_t)$.

Moreover $(\lambda_s, \rho_{(g,\delta)}) \neq (\lambda_v, \rho_v)$. Suppose that $(\lambda_s, \rho_{(g,\delta)}) = (\lambda_v, \rho_v)$, then $\rho_{(g,\delta)} = \rho_v$ a contradiction, since ρ_v is an inner right translation but $\rho_{(g,\delta)}$ is not because δ is not a constant mapping.

Therefore φ is an isomorphism . Thus $\bar{S} \cong S \cup W$. ■

Chapter 3

Green's Relations on Bicyclic Extensions

Green's equivalence relations are fundamental equivalence relations, definable in any semigroup, were first introduced and studied by J. A. Green (1951). In this chapter we give examples, lemma and theorems on characterizations of Green's equivalences related to Bicyclic Extensions.

Definition 32 : *If a is an element of a semigroup S, the smallest left ideal containing a is $Sa \cup \{a\}$, ($A = Sa \cup \{a\} \Rightarrow SA \subseteq A$) which we may conveniently write as S^1a, and which we shall call the* ***principal left ideal generated by*** *a. An equivalence relation $\mathcal{L}$ on S is then defined by the rule that $a\mathcal{L}b$ if and only if a and b generate the same principal left ideal, i.e. if and only if $S^1a = S^1b$. In other words, $\mathcal{L}$ is the subset of $S \times S$ consisting of all pairs (a, b) such that $a \cup Sa = b \cup Sb$. We say that a and b are* ***$\mathcal{L}-$equivalent.*** *By L_a we mean the set of all elements of S which are*

*$\mathcal{L}$–equivalent to a (a in S), that is, L_a is the equivalence class mod $\mathcal{L}$ containing a; we call $L_a = \{x \mid x\mathcal{L}a\}$ the $\mathcal{L}$-***class** *containing a. Clearly $\mathcal{L}$ is an equivalence relation such that if $a\mathcal{L}b$ then $ac\mathcal{L}bc$ for all $c \in S$, that is $\mathcal{L}$ is* ***a right congruence.***

Dually we define $a\mathcal{R}b$ to mean $aS^1 = bS^1 (a, b \in S)$, and note that $\mathcal{R}$ is a ***left congruence*** *on S. By R_a we mean the equivalence class of S mod $\mathcal{R}$ or the $\mathcal{R}$-***class**, *containing a. We say that a and b are $\mathcal{R}$–****equivalent.*** *$a\mathcal{R}b$ if and only if a and b generate the same principal right ideal, i.e. if and only if $aS^1 = bS^1, (a, b \in S)$.*

An alternative characterization, making the "mutual divisibility" aspect of these equivalences more explicit, is given in the following lemma.

Lemma 33 : *Let a, b be element of a semigroup S. Then $a\mathcal{L}b$ if and only if there exist x and y in S^1 such that $xa = b$, $yb = a$. Also, $a\mathcal{R}b$ if and only if there exist u and v in S^1 such that $au = b$, $bv = a$.*

Proof. Let $a, b \in S$.

(a) : ($\Longleftarrow$) Let $a = bx$, $b = ay$ for some $x, y \in S^1$. We show that $aS^1 = bS^1$. Let $u \in aS^1$. Then $u = av$ for some $v \in S^1$. So, $u = av = (bx)v = b(xv) \in bS$ (since $a = bx$) . Therefore $aS^1 \subseteq bS^1$. Now, let $p \in bS^1$. Then, $p = bq$ for some $q \in S^1$. Then, $p = (ay)q$ (since $b = ay$) . So $p = a(yq) \in aS$. Therefore $aS^1 \supseteq bS^1$. Thus $aS^1 = bS^1$, so, $a\mathcal{L}b$.

(b) : ($\Longrightarrow$) Let $a\mathcal{L}b$. then, $aS^1 = bS^1$. So, $a = bx$, $b = ay$ for some $x, y \in S^1$. A similar argument establishes that $a\mathcal{R}b \Longleftrightarrow \exists\ u\ ,\ v \in S^1$ such that $au = b$, $bv = a$. ∎

Definition 34 : *The intersection of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{H}$, i.e. $\mathcal{R} \cap \mathcal{L} = \mathcal{H}$. We denote the $\mathcal{H}$-class by H_a. Clearly $H_a = R_a \cap L_a$.*

The join $\mathcal{L} \vee \mathcal{R}$ (the smallest equivalence relation that containing both $\mathcal{L}$ and $\mathcal{R}$) is also of great importance, and is denoted by $\mathcal{D}$, i.e. $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. The $\mathcal{D}$-class containing a of S will be denoted by D_a.

Lemma 35 : *If a is a regular element of a semigroup S, then every element of $\mathcal{D}_a$ is regular.*

Proof. *Since a is regular, then $a = axa$ for some $x \in S$. We show that every element in L_a and R_a is regular. Let $b \in L_a$, then there exist $u, v \in S^1$such that $ua = b, vb = a$, so*

$$b = ua = uaxa = bxa = b(xv)b$$

then b is regular. Similarly, we cam show that every element of R_a is regular. Thus D_a is regular. See [7, Proposition 3.1]. ■

Remark 4 : *So, by the above lemma, if, D is a $\mathcal{D}-$class then either every element of D is regular or no element of D is regular. Since idempotents are regular $(eee = e)$ it follows that a $\mathcal{D}-$class containing an idempotent is regular. Conversely, we can show that a regular $\mathcal{D}-$class must contain at least one idempotent.*

Lemma 36 : *In a regular $\mathcal{D}$ − class each $\mathcal{L}$-class and each $\mathcal{R}$ − class contains at least one idempotent*

Proof. See [7, Proposition 3.2]. ■

Lemma 37 : *The relations $\mathcal{L}$ and $\mathcal{R}$ commute ($\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$) and so the relation $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ is the smallest equivalence relation $\mathcal{L} \vee \mathcal{R}$ containing both $\mathcal{L}$ and $\mathcal{R}$.*

Proof. See [4, Lemma 2.1]. ■

Remark 5 : *Note that if R is an $\mathcal{R}$-class of a semigroup S, and L is and $\mathcal{L}$-class of S, then $a\mathcal{D}b$ if and only if $R \cap L \neq \Box$ (empty set). For, let $a \in R$ and $b \in L$. Then $a\mathcal{D}b \Leftrightarrow \exists\, c \in S$ such that $a\mathcal{R}c$ and $c\mathcal{L}b$. But this condition on c is equivalent to $c \in R$ and $c \in L$, that is $c \in R \cap L \neq \Box$ (empty set).*

Definition 38 : *We define $a\mathcal{J}b$ (a, b in S) to mean $S^1aS^1 = S^1bS^1$ ($SaS \cup aS \cup Sa \cup \{a\} = SbS \cup bS \cup Sb \cup \{b\}$), that is, a and b are $\mathcal{J}-$**equivalent** if and only if they generate the same two-sided principal ideal. $a\mathcal{J}b$ if and only if there exist x, y, u, v in S^1 for which $xay = b$, $ubv = a$. Since $\mathcal{L} \subseteq \mathcal{D}$, $\mathcal{L} \subseteq \mathcal{J}$, $\mathcal{R} \subseteq \mathcal{D}$ and $\mathcal{R} \subseteq \mathcal{J}$. so, $\mathcal{D} \subseteq \mathcal{J}$; in general, $\mathcal{D} \neq \mathcal{J}$.*

We will explain these definitions with the following examples.

Example 39 : *Let C be a cancellative semigroup and suppose that C has no identity.*

We show that there cannot be any pair of elements e, a in C for which $ea = a$ or for which $ae = e$, and deduce that $\mathcal{L} = \mathcal{R} = \mathcal{D} = i_C$.

Solution:

Suppose by way of contradiction that $ea = a$ for some e, a in C. Then we have

$$ea = 1a \text{ (} 1 \notin C \text{ is an identity adjoined to } C.)$$

$$\Longrightarrow \quad e = 1.$$

Now, we show that $\mathcal{L} = \mathcal{R}$. Let $a, b \in \mathcal{L}$. Suppose that $(a, b) \in \mathcal{L}$ then there exist c, d in C^1 such that

$$ca = b \text{ and } db = a$$

from which it follows that

$$b = ca = (cd)b$$

which is a contradiction by above (i), unless $a = b$ and $c = d = 1$.

Similarly we can show that $(a, b) \in \mathcal{R}$ implies $a = b$. Therefore $\mathcal{L} = \mathcal{R} = \mathcal{D} = i_C$.

In Example 11, we have $\mathcal{L} = \mathcal{R} = \mathcal{D} = i_S$, $\mathcal{J} = S \times S$.

Proposition 40 : *In a commutative semigroup S (for any $a, b \in S$, $ab = ba$), we have*

$$\mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = i_S$$

Proof. First we show that $\mathcal{L} = \mathcal{R}$. So suppose that $(a. b) \in \mathcal{L}$

$$\begin{aligned} & (a, b) \in \mathcal{L} \\ \iff & \exists\ c, d \in S^1 \text{such that } ca = b \text{ and } db = a \\ \iff & \exists\ c, d \in S^1 \text{such that } ac = b \text{ and } bd = a \text{ (by commutativity)} \\ \iff & (a, b) \in \mathcal{R} \end{aligned}$$

Thus $\mathcal{L} \subseteq \mathcal{R}$.

Similarly, we may show that $\mathcal{R} \subseteq \mathcal{L}$ and hence $\mathcal{L} = \mathcal{R}$.

Therefor $\mathcal{L} = \mathcal{R} = \mathcal{H} = \mathcal{D}$.

Next, suppose that $(a, b) \in \mathcal{J}$. Then there exist $c_1, d_1, c_2, d_2 \in S$ such that

$$c_1 a d_1 = b \text{ and } c_2 b d_2 = a.$$

However, by commutativity, this is equivalent to

$$(c_1 d_1)a = b \text{ and } (c_2 d_2)b = a.$$

Then $(a,b) \in \mathcal{L} = \mathcal{R}$.

Thus $\mathcal{J} = \mathcal{D}$. ■

Example 41 : *We consider the set* $S_3^- = \{0, a, e_1, e_2, e_3, 1\}$ *where*

$$0 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix},$$

$$e_2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}, e_3 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

and we define our binary operation to be the usual composite.

The multiplication of all elements of S_3^- *is as shown in the table below:*

$\bullet$	0	a	e_1	e_2	e_3	1
0	0	0	0	0	0	0
a	0	0	0	a	a	a
e_1	0	a	e_1	0	a	e_1
e_2	0	0	0	e_2	e_2	e_2
e_3	0	0	0	e_3	e_3	e_3
1	0	a	e_1	e_2	e_3	1

So, we have

$\mathcal{R}$*-classes*: $\{0\}, \{a\}, \{e_1\}, \{e_2\}, \{e_3\}, \{1\}$.

$\mathcal{L}$*-classes:* $\{0\}$, $\{a\}$, $\{e_1\}$, $\{e_2, e_3\}, \{1\}$.

$\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \{(0,0), (a,a), (e_1,e_1), (e_2,e_2), (e_3,e_3), (1,1)\} = \mathcal{R}$.

Thus, $i = \mathcal{H} = \mathcal{R} \subseteq \mathcal{L} = \mathcal{D} = \mathcal{J} = w$-universal relation.

Remark 6 : *It is convenient to visualize a* $\mathcal{D}$*-class* D *of a semigroup* S *as what Clifford and Preston (1961) have called an eggbox picture in which each row represents* *an $\mathcal{R}$-class in D, each column an $\mathcal{L}$-class in D and each cell an $\mathcal{H}$-class. If $a, b \in D$ are such that $a\mathcal{R}b$ (so that a and b lie in the same row of the eggbox), then the egg-box for the above example is as follows:*

D	L_0	L_a	L_{e_1}	$L_{e_2} = L_{e_3}$	L_1
R_0	0				
R_a		a			
R_{e_1}			e_1		
R_{e_2}				e_2	
R_{e_3}				e_3	
R_1					1

So, $D_0 = \{0\}$, $D_a = \{a\}$, $D_{e_1} = \{e_1\}$, $D_{e_2} = D_{e_3} = \{e_2, e_3\}$, $D_1 = \{1\}$.

Example 42 : *We obtain the Green's equivalences on the following semigroup*

L_n – a left-zero semigroup with n-elements

Solution: Let $L_n = \{l_1, l_2, l_3, ..., l_n\}$. We adjoin the identity element $L_n^1 = L_n \cup \{1\} = \{l_1, l_2, l_3, ..., l_n, 1\}$, and, $l_i * l_j = l_i$ for all i , $j \in \{1, 2, ..., n\}$.

$\mathcal{R}$-classes: $\{l_1\}, \{l_2\}, \{l_3\}, ..., \{l_n\}, \{1\}$.

$\mathcal{L}$-classes: $\{l_1, l_2, l_3, ..., l_n\}, \{1\}$.

$\mathcal{J}$-classes: $\{l_1, l_2, l_3, ..., l_n\}, \{1\}$.

$\mathcal{D}$-classes: $\{l_1, l_2, l_3, ..., l_n\}, \{1\}$.

Thus, $i = \mathcal{H} = \mathcal{R}$, $\mathcal{L} = \mathcal{D} = \mathcal{J} = w-$universal relation (i.e. $w = S \times S$)

Despite the ease with which we characterized the Green's relations on these special semigroups, the problem of characterizing the Green's relations is far from trivial in general.

In [21] the following theorem is stated without proof and we now give a proof:

Theorem 43 *: [21, Lemma 2.2]. Let $S = (T, K, P, \theta, \gamma)$. Then,*

(a) $((n, k), (g, p))\mathcal{R}((r, s), (h, q))$ if and only if $n = r$ and $g\mathcal{R}h$ *(in T)*.

(b) $((n, k), (g, p))\mathcal{L}((r, s), (h, q))$ if and only if $k = s$, $p = q$, and $g\mathcal{L}h$ (in T).

(c) $((n, k), (g, p))\mathcal{D}((r, s), (h, q))$ if and only if $g\mathcal{D}h$ (in T).

(d) $E(S) = \{((0, 0), (f, p)) : p \in P,\ f \in E(T)\} \cup \{((n, n), (g, q)) : n \in N,\ q \in K,\ g \in E(T)\}$, (where $E(S)$ is the set of idempotent of S and $E(T)$ is the set of idempotent of T).

(e) Let e, $f \in E(T)$. Then $((k, k), (e, p)) < ((r, r), (f, q))$ if and only if $k > r$ or $k = r$, $p = q$, and $e < f$.

Proof. (a): $(\Longrightarrow)$ Suppose that $((n, k), (g, p))\mathcal{R}((r, s), (h, q))$.

Then $\exists$ $((r_1, s_1), (h_1, q_1)), ((r_2, s_2), (h_2, q_2)) \in S^1$ such that

$$((n, k), (g, p)) = ((r, s), (h, q))((r_1, s_1), (h_1, q_1)) \tag{3.1}$$

and

$$((r, s), (h, q)) = ((n, k), (g, p))((r_2, s_2), (h_2, q_2)) \tag{3.2}$$

Case (i): $s \leq r_1$ and $k \leq r_2$

$$\begin{aligned} ((n, k), (g, p)) &= ((r, s), (h, q))((r_1, s_1), (h_1, q_1)) \\ &= ((r + r_1 - s, s + s_1 - s), (h\theta^{r_1 - s}h_1\theta^{s-s}, q_1)) \end{aligned}$$

$$= ((r + r_1 - s, s_1), (h\theta^{r_1 - s} h_1 \theta^0, q_1))$$

$$\Longleftrightarrow n = r + r_1 - s, k = s_1, \ g = h\theta^{r_1-s} h_1 \theta^0 \text{ and } p = q_1 \text{ where } s \leq r_1 \tag{3.3}$$

And, also by (3.2), we have

$$((r, s), (h, q)) = ((n, k), (g, p))((r_2, s_2), (h_2, q_2))$$

$$\Longleftrightarrow \quad ((r, s), (h, q)) = ((n + r_2 - k, k + s_2 - k), (g\theta^{r_2-k} h_2 \theta^{k-k}, q_2)) \text{ where } k \leq r_2$$

$$\Longleftrightarrow \quad ((r, s), (h, q)) = ((n + r_2 - k, s_2), (g\theta^{r_2-k} h_2 \theta^0, q_2)) \text{ where } k \leq r_2$$

$$\Longleftrightarrow \quad r = n + r_2 - k, s = s_2, h = g\theta^{r_2-k} h_2 \theta^0 \text{ and } q = q_2 \text{ where } k \leq r_2$$

$$\Longleftrightarrow \quad n = r + k - r_2, s = s_2, h = g\theta^{r_2-k} h_2 \theta^0 \text{ and } q = q_2 \text{ where } k \leq r_2 \tag{3.4}$$

Considering (3.3) and (3.4) we have,

$$n = r + r_1 - s, \ k = s_1, \ g = h\theta^{r_1-s} h_1 \theta^{s-s} \text{ and } p = q_1 \text{ where } s \leq r_1;$$

$$n = r + k - r_2, \ s = s_2, \ h = g\theta^{r_2-k} h_2 \theta^0 \text{ and } q = q_2 \text{ where } k \leq r_2.$$

So, $r + r_1 - s = r + k - r_2 \Longrightarrow r_1 - s = k - r_2$. Since $r_1 - s \geqslant 0$ and $k - r_2 \leq 0$, then $r_1 - s = 0$ and $k - r_2 = 0$, so, $r_1 = s$ and $k = r_2$. Then, we have

$$\begin{cases} n = r + s - s, k = s_1, g = h\theta^{s-s} h_1 \theta^{s-s} \text{ and } p = q_1; \\ n = r + k - k, \ s = s_2, h = g\theta^{k-k} h_2 \theta^0 \text{ and } q = q_2 \end{cases}$$

$$\Longleftrightarrow n = r \text{ and } g = hh_1 \text{ and } h = gh_2 \Longleftrightarrow n = r \text{ and } g\mathcal{R}h$$

Case (ii): $s > r_1$ and $k \leq r_2$

$$\begin{aligned} ((n, k), (g, p)) &= ((r, s), (h, q))((r_1, s_1), (h_1, q_1)) \\ &= ((r + r_1 - r_1, s + s_1 - r_1), (h\theta^{r_1 - r_1} h_1 \theta^{s - r_1}, q(h_1 \theta^{s - r_1 - 1} \gamma))) \\ &= ((r, s + s_1 - r_1), (h\theta^0 h_1 \theta^{s - r_1}, q(h_1 \theta^{s - r_1 - 1} \gamma))) \end{aligned}$$

$$\iff n = r,\ k = s + s_1 - r_1,\ g = h\theta^0 h_1 \theta^{s-r_1} \text{ and } p = q(h_1\theta^{s-r_1-1}\gamma) \text{ where } s > r_1 \tag{3.5}$$

By (3.2), we have

$$\begin{aligned} &((r,s),(h,q)) = ((n,k),(g,p))((r_2,s_2),(h_2,q_2)) \\ \iff\ & n = r + k - r_2,\ s = s_2,\ h = g\theta^{r_2-k}h_2\theta^0 \text{ and } q = q_2 \text{ where } k \leq r_2 \end{aligned} \tag{3.6}$$

Considering (3.5) and (3.6), we have

$$n = r,\ n = r + k - r_2,\ k = s + s_1 - r_1,\ s = s_2$$

and

$$g = h\theta^0 h_1 \theta^{s-r_1},\ h = g\theta^{r_2-k}h_2\theta^0, \text{ where } s > r_1 \text{ and } k \leq r_2$$

Then, $n = r$, $k - r_2 = 0$ $(k = r_2)$, $g = h(h_1\theta^{s-r_1})$ and $h = g\theta^0 h_2$. So, we have,

$n = r,\ g = h(h_1\theta^{s-r_1})$ and $h = gh_2$, since θ^0 is the identity automorphism of T.

Thus, $n = r$ and $g\mathcal{R}h$.

Case (iii): $k > r_2$ and $s > r_1$

$$\begin{aligned} ((r,s),(h,q)) &= ((n,k),(g,p))((r_2,s_2),(h_2,q_2)) \\ &= ((n + r_2 - r_2, k + s_2 - r_2),(g\theta^{r_2-r_2}h_2\theta^{k-r_2}, p(h_2\theta^{k-r_2-1}\gamma))) \\ &= ((n, k + s_2 - r_2),(g\theta^0 h_2\theta^{k-r_2}, p(h_2\theta^{k-r_2-1}\gamma))) \end{aligned}$$

$$\iff r = n,\ s = k + s_2 - r_2,\ h = g\theta^0 h_2 \theta^{k-r_2},\ q = p(h_2\theta^{k-r_2-1}\gamma) \tag{3.7}$$

By (3.2), we have,

$$((n,k),(g,p)) = ((r,s),(h,q))((r_1,s_1),(h_1,q_1))$$

$$\Longleftrightarrow \quad n=r, k=s+s_1-r_1, g=h\theta^0 h_1\theta^{s-r_1} \text{and } p=q(h_1\theta^{s-r_1-1}\gamma) \text{ where } s>r_1.$$

Considering the last two conditions, where $k>r_2$, $s>r_1$, we have,

$$r=n,\ s=k+s_2-r_2,\ k=s+s_1-r_1$$

and

$$h=g\theta^0 h_2\theta^{k-r_2},\ g=h\theta^0 h_1\theta^{s-r_1}$$

Therefore $r=n$, $h=g(h_2\theta^{k-r_2})$ and $g=h(h_1\theta^{s-r_1})$, that is, $n=r$ and $g\mathcal{R}h$, since $h_2\theta^{k-r_2}, h_1\theta^{s-r_1}\in T$.

Case (iv): $s\le r_1$ and $k>r_2$.

By (3.1), we have

$$((n,k),(g,p)) = ((r,s),(h,q))((r_1,s_1),(h_1,q_1))$$

$$\Leftrightarrow \quad n=r+r_1-s,\ k=s_1,\ g=h\theta^{r_1-s}h_1\theta^0 \text{ where } s\le r_1$$

By (3.2), we have

$$((r,s),(h,q)) = ((n,k),(g,p))((r_2,s_2),(h_2,q_2))$$

$$\Leftrightarrow \quad r=n,\ s=k+s_2-r_2,\ h=g\theta^0 h_2\theta^{k-r_2} \text{ where } k>r_2$$

Considering the last two conditions, where $s\le r_1, k>r_2$, we have,

$$r=n,\ r_1-s=0,\ g=h\theta^{r_1-s}h_1 \text{ and } h=g(h_2\theta^{k-r_2})$$

Then $r=n$, $g=h\theta^0 h_1$ and $h=g(h_2\theta^{k-r_2})$. So $n=r$, $g=hh_1$ and $h=g(h_2\theta^{k-r_2})$.

Therefore, $n=r$ and $g\mathcal{R}h$.

($\Longleftarrow$) Conversely, let $((n,k),(g,p)),((r,s),(h,q)) \in S$, and suppose that $n = r$ and $g\mathcal{R}h$. We need to show that $((n,k),(g,p))\mathcal{R}((r,s),(h,q))$. That is, we need to find two elements $((r_1,s_1),(h_1,q_1)),((r_2,s_2),(h_2,q_2)) \in S^1$, such that (3.1) and (3.2) are satisfied. Let $s_1 = r_2 = k$, $r_1 = s_2 = s$, $q_1 = p$, $q_2 = q$. Since $g\mathcal{R}h$, then exist $h_1, h_2 \in T$ such that $g = hh_1$ and $h = gh_2$. Therefore $g = h\theta^{r_1-s}h_1\theta^0$, $h = g\theta^{r_2-k}h_2\theta^0$. Thus,

$$\begin{aligned}
&((r,s),(h,q))((r_1,s_1),(h_1,q_1)) \\
= \ & ((r+r_1-s, s+s_1-s),(h\theta^{r_1-s}h_1\theta^{s-s},q_1)) \text{ since } r_1 = s \\
= \ & ((n,k),(g,p)) \text{ since } s_1 = k,\ q_1 = p,\ g = h\theta^{r_1-s}h_1\theta^0,\ h = g\theta^{r_2-k}h_2\theta^0
\end{aligned}$$

and

$$\begin{aligned}
&((n,k),(g,p))((r_2,s_2),(h_2,q_2)) \\
= \ & ((n+r_2-k, k+s_2-k),(g\theta^{r_2-k}h_2\theta^{k-k},q_2)) \text{ since } k = r_2 \\
= \ & ((n,s),(h,q)) \text{ since } s_2 = s,\ r_2 = k,\ h = g\theta^{r_2-k}h_2\theta^{k-k},\ q = q_2.
\end{aligned}$$

(b): ($\Longrightarrow$) Suppose that $((n,k),(g,p))\mathcal{L}((r,s),(h,q))$.

Then $\exists$ $((r_1,s_1),(h_1,q_1)),((r_2,s_2),(h_2,q_2)) \in S^1$ such that

$$((n,k),(g,p)) = ((r_1,s_1),(h_1,q_1))((r,s),(h,q)) \tag{3.8}$$

and

$$((r,s),(h,q)) = ((r_2,s_2),(h_2,q_2))((n,k),(g,p)) \tag{3.9}$$

we consider the case $s_1 \leq r$ and $s_2 \leq n$. Thus

$$((n,k),(g,p)) = ((r_1,s_1),(h_1,q_1))((r,s),(h,q))$$
$$= ((r_1+r-s_1, s_1+s-s_1),(h_1\theta^{r-s_1}h\theta^{s_1-s_1},q))$$
$$= ((r_1+r-s_1,s),(h_1\theta^{r-s_1}h\theta^0,q))$$

$$\Longrightarrow n = r_1+r-s_1,\ k=s,\ g = h_1\theta^{r-s_1}h\theta^0 \text{ and } p=q \text{ where } s_1 \leq r$$

$$\Longleftrightarrow n = r_1+r-s_1,\ k=s,\ g = h_1\theta^{r-s_1}h \text{ and } p=q \text{ where } s_1 \leq r \qquad (3.10)$$

And, also by (3.9), we have

$$\begin{aligned} & ((r,s),(h,q)) \\ = \quad & ((r_2,s_2),(h_2,q_2))((n,k),(g,p)) \\ = \quad & ((r_2+n-s_2, s_2+k-s_2),(h_2\theta^{n-s_2}g\theta^{s_2-s_2},p)) \\ \Longleftrightarrow \quad & r = r_2+n-s_2,\ s=k,\ h = h_2\theta^{n-s_2}g \text{ and } q=p \text{ where } s_2 \leq n \quad (3.11) \end{aligned}$$

Considering (3.10) and (3.11) we have,

$$n = r_1+r-s_1,\ k=s,\ g = h_1\theta^{r-s_1}h\theta^0 \text{ and } p=q \text{ where } s_1 \leq r$$

$$r = r_2+n-s_2, s=k,\ h = h_2\theta^{n-s_2}g\theta^0 \text{ and } q=p \text{ where } s_2 \leq n$$

Then, we have

$k=s,\ g=(h_1\theta^{r-s_1})h,\ h=(h_2\theta^{n-s_2})g$ and $p=q$ where $s_1 \leq r,\ s_2 \leq n$. Therefore

$k=s, p=q$ and $g\mathcal{L}h$.

The other cases are similar.

($\Longleftarrow$) Conversely, let $((n,k),(g,p)),((r,s),(h,q)) \in S$, and suppose that $k=s$, $p=q$ and $g\mathcal{L}h$ (in T). We need to show that $((n,s),(g,p))\mathcal{L}((r,k),(h,p))$ by finding two elements

$$((r_1, s_1), (h_1, q_1)), ((r_2, s_2), (h_2, q_2)) \in S^1,$$

such that (3.8) and (3.9) are satisfied. Let $s_1 = n = r_2$ and $r_1 - r = s_2$. Since $g\mathcal{L}h$, then there exist $h_1, h_2 \in T$ such that $g = h_2h$ and $h = h_1g$. Therefore,

$$\begin{aligned} & ((r_1, s_1), (h_1, q_1))((n, k), (g, p)) \\ = \;& ((r_1 + n - n, s_1 + k - s_1), (h_1\theta^0 g\theta^{s_1 - r}, p)) \\ = \;& ((r, s), (h, q)) \text{ since } r_1 = n,\ k = s,\ h = h_1g \text{ and } p = q. \end{aligned}$$

and

$$\begin{aligned} & ((r_2, s_2), (h_2, q_2))((r, s), (h, q)) \\ = \;& ((r_2 + r - r, s_2 + s - s), (h_2\theta^0 h\theta^{s_2 - r}, q)) \\ = \;& ((n, k), (g, p)) \text{ since } r_2 = n,\ k = s,\ g = h_2h \text{ and } q = p. \end{aligned}$$

(c): ($\Longrightarrow$) Let $((n, k), (g, p)), ((r, s), (h, q)) \in S$. Then

$$\begin{aligned} & ((n, k), (g, p))\mathcal{D}((r, s), (h, q)) \\ \Longrightarrow \;& ((n, k), (g, p))\mathcal{L}((n_1, k_1), (g_1, p_1))\mathcal{R}((r, s), (h, q)) \end{aligned}$$

for some $((n_1, k_1), (g_1, p_1)) \in S^1$. By parts (a) and (b) above it follows that $k = k_1, p = p_1$, $g\mathcal{L}g_1$ (in T) and $n_1 = r$, $g_1\mathcal{R}h$ (in T)

$\Longrightarrow g\mathcal{D}h$ (in T).

($\Longleftarrow$) : Let $((n, k), (g, p)), ((r, s), (h, q)) \in S$ and suppose that $g\mathcal{D}h$ (in T). Then there exist g_1 in T^1 such that $g\mathcal{L}g_1\mathcal{R}h$. However, by parts (a) and (b) we have

$$\begin{aligned} & ((n, k), (g, p))\mathcal{L}((r, k), (g_1, p))\mathcal{R}((r, s), (h, q)). \\ \Longrightarrow \;& ((n, k), (g, p))\mathcal{D}((r, s), (h, q)). \end{aligned}$$

(d): We show that $E(S) = \{((0,0),(f,p)) : p \in P, f \in E(T)\} \cup \{((n,n),(g,q)) : n \in N, q \in K, g \in E(T)\}$.

$(\Longrightarrow)$: We show that $E(S) \subseteq \{((0,0),(f,p)) : p \in P, f \in E(T)\} \cup \{((n,n),(g,q)) : n \in N, q \in K, g \in E(T)\}$.

Let $((n,k),(g,p)) \in E(S)$. Then

$$\begin{aligned}((n,k),(g,p)) &= ((n,k),(g,p))((n,k),(g,p)) \\ &= \begin{cases} ((n+n-k, k+k-k),(g\theta^{n-k}g\theta^{k-k}, p)) \text{ where } k \le n \\ ((n+n-n, k+k-n),(g\theta^{n-n}g\theta^{k-n}, p(g\theta^{k-n-1}\gamma))) \text{ where } k > n \end{cases} \\ &= \begin{cases} ((2n-k, k),(g\theta^{n-k}g, p)) \text{ where } k \le n \\ ((n, 2k-n),(g(g\theta^{k-n}), p(g\theta^{k-n-1}\gamma))) \text{ where } k > n \end{cases}\end{aligned}$$

which implies that

$$\begin{cases} n = 2n-k \text{ and } g = g\theta^{n-k}g \quad \text{when } k \le n; \text{ or} \\ k = 2k-n,\ g = g(g\theta^{k-n}) \text{ and } p = p(g\theta^{k-n-1}\gamma))) \quad \text{when } k > n \end{cases}$$

$$\Longrightarrow \begin{cases} n = k \text{ and } g = gg \quad \text{when } k \le n; \text{ or} \\ k = n \text{ but } k > n,\ g = gg \text{ and } p = p(g\theta^{0-1}\gamma))) \text{ (impossible).} \end{cases}$$

$$\Longrightarrow\ n = k \text{ and } g = gg = g^2$$

So, we have $((n,k),(g,p)) = \begin{cases} ((0,0),(g,p)), \text{ where } p \in P,\ g = gg \textbf{ if } n = 0 \\ ((n,n),(g,p)), \text{ where } p \in K,\ g = gg, n = k = p \textbf{ if } n \ne 0 \end{cases}$

Thus, $((n,k),(g,p))^2 = ((n,k),(g,p)) \in E_0 \cup E_n$ where $E_0 = \{((0,0),(f,p)) : p \in P, f \in E(T)\}$, $E_n = \{((n,n),(g,q)) : n \in N, q \in K, g \in E(T)\}$.

$(\Longleftarrow)$: We show that $E(S) \supseteq \{((0,0),(f,p)) : p \in P, f \in E(T)\} \cup \{((n,n),(g,q) : n \in N, q \in K, g \in E(T)\}$.

Notice that

$$\begin{aligned}((0,0),(g,p))((0,0),(g,p)) &= ((0,0),(g\theta^{0-0}g\theta^{0-0},p)) = ((0,0),(gg,p)) \\ &= ((0,0),(g,p)) \text{ since } g \in E(T)\end{aligned}$$

Therefore $((0,0),(g,p)) \in E(S)$

Similarly,

$$\begin{aligned}((n,n),(g,q))((n,n),(g,q)) &= ((n+n-n,n+n-n),(g\theta^{n-n}g^{n-n},q) \\ &= ((n,n),(gg,q)) \\ &= ((n,n),(g,q)) \text{ since } g \in E(T).\end{aligned}$$

So, $((n,k),(g,p)) \in E(S)$. Hence the proof is complete.

(e): Let e, $f \in E(T)$. We need to show that

$$((k,k),(e,p)) < ((r,r),(f,q)) \Longleftrightarrow k > r \text{ or } k = r, p = q, \text{and } e < f.$$

Let $((k,k),(e,p)) < ((r,r),(f,q))$. Then

$$((k,k),(e,p))((r,r),(f,q)) = ((r,r),(f,q))((k,k),(e,p)) = ((k,k),(e,p))$$

$$\Longleftrightarrow \begin{cases} ((k,k),(e,p)) = ((r,r),(f,q))((k,k),(e,p)) \text{ and} \\ ((k,k),(e,p)) = ((k,k),(e,p))((r,r),(f,q)) \end{cases}$$

However,

$$\begin{aligned}((k,k),(e,p)) &= ((r,r),(f,q))((k,k),(e,p)) \\ &= \begin{cases} ((r+k-r,r+k-r),(f\theta^{k-r}e\theta^{r-r},p)) \text{ if } r \leqq k \\ ((r+k-k,r+k-k),(f\theta^{k-k}e\theta^{r-r},q(e\theta^{r-k-1}\gamma))), \text{ if } r > k \end{cases}\end{aligned}$$

$$= \begin{cases} ((k,k),(f\theta^{k-r}e\theta^0,p)) \text{ if } r \leqq k \\ ((r,r),(f\theta^0 e\theta^0, q(e\theta^{r-k-1}\gamma))), \text{ if } r > k \end{cases}$$

$$\iff \begin{cases} \text{If } r \leqq k, \text{ then } e = f\theta^{k-r}e\theta^0 \text{ and } p = p \\ \text{If } r > k, \text{ then } e = fe \text{ and } p = q(e\theta^{r-k-1}\gamma) \end{cases}$$

Also,

$$\begin{aligned} ((k,k),(e,p)) &= ((k,k),(e,p))((r,r),(f,q)) \\ &= \begin{cases} ((k+r-k, k+r-k),(e\theta^{r-k}f\theta^{k-k}.q)) \text{ if } k \leqq r \\ ((k+r-r, k+r-r),(e\theta^{r-r}f\theta^{k-r}.p(f\theta^{k-r-1}\gamma))), \text{ if } k > r \end{cases} \\ &= \begin{cases} ((r,r),(e\theta^{r-k}f\theta^0.q)) \text{ if } k \leqq r \\ ((k,k),(e\theta^0 f\theta^{k-r}.p(f\theta^{k-r-1}\gamma))), \text{ if } k > r \end{cases} \\ &\iff \begin{cases} \text{If } k \leqq r, \text{ then } k = r \text{ and } e = e\theta^{r-k}f\theta^0. \text{ and } p = q \\ \text{If } k > r, \text{ then } e = e(f\theta^{k-r}) \text{ and } p = p(f\theta^{k-r-1}\gamma) \end{cases} \end{aligned}$$

Thus, we deduce that

$k > r$ or $k = r$, $e = fe = ef$ ($e < f$), and $p = q$. ■

It follow that we may represent the $\mathcal{L}$–and $\mathcal{R}$–classes of a bicyclic extension $S = (T, K, P, \theta, \gamma)$ as follows:

$$L_{(k_1,g,q_1)} = \{((n,k_1),(g,q_1)) : n \in N^0\};$$

$$R_{(n_2,g)} = \{((n_2,k),(g,q)) : \text{either } k = 0, q \in P \text{ or } k \in N, q \in K\};$$

$$D_g = \{((n,k_1),(g,q_1)) : n \in N^0 \text{ and (either } k = 0, q \in P \text{ or } k \in N, q \in K)\},$$

where $g \in T$.

An immediate consequence of this is

Corollary 44 : *The bicyclic extension $S = (T, K, P, \theta, \gamma)$ is regular if and only if T is regular.*

Proof. This follows from the observed fact that every $\mathcal{L}$-class [$\mathcal{R}$-class] of S contains an idempotent if and only if every $\mathcal{L}$ -class [$\mathcal{R}$-class] of T contains an idempotent.

■

Definition 45 : *A semigroup S is called* ***$\mathcal{D}$-simple*** *or* ***bisimple*** *if it consists of a single $\mathcal{D}$-class.*

Definition 46 : *A regular semigroup S in which the idempotents form a subsemigroup is called an* ***orthodox semigroup.*** *In other words the regular semigroup S is orthodox if and only if the set E of idempotents of S is closed under multiplication.*

3.1 Bruck-Reilly Extension

The bicyclic semigroup B ($B = N^0 \times N^0$) under the multiplication $((m, n)(p, q)) = (m + p - t, n + q - t)$ where $t = min(n, p)$ has played a central role in the theory of simple semigroups (Any simple semigroup which is not completely simple is locally bicyclic [3, 7]. (See, for example, [3], [14], [17], [18], [19]). Reilly in [10] shows B is $\omega-$semigroup. Reilly's theorem [10, Theorem 2.2] shows that how we can generate bisimple $\omega-$semigroups from any group and any endomorphism of that group.

Theorem 47 : *[10, Theorem 2.2 & Theorem 3.5] Let G be a group and let θ be endomorphism of G. Let $S = BR(G, \theta) = \{((m, n); g) \in B \times G : (m, n) \in N^0 \times N^0$*

and $g \in G\}$. *Define multiplication on* S *as follows:*

$$(m_1, n_1); g_1)(m_2, n_2); g_2) = ((m_1 + n_1 - t, m_2 + n_2 - t); g_1\theta^{m_2-t} g_2\theta^{n_1-t})),$$

where $t = min(n_1, m_2)$ *and we take* θ^0 *to the identity automorphism of* G. *Then* S *is a bisimple* $\omega-$*semigroup.*

Conversely, every bisimple $\omega-$*semigroup is isomorphic to* $BR(G, \theta)$ *for some* G.

Definition 48 : *The collection* $E(R)$ *or* $\mathcal{R}$ *classes of* E *may be partially ordered by the following rule. If* R_1, $R_2 \in E(R)$, $R_1 < R_2$ *if and only if* $e < f$ *for all* $e \in R_1$ *and* $f \in R_2$. *(*$e \leq f$ *if and only if* $ef = fe = e$*). If E(R), under this order, is order isomorphic to* N^0, *the non-negative integers, under the reverse of the usual order.* E *is* **called a naturally** *ordered band.*

A bisimple semigroup whose idempotents form a naturally ordered band is termed an **E − bisimple semigroup.**

A semigroup with set of E *of idempotents is called L-**unipotent** [R-**unipotent**] if it is regular and if* $\forall e, f \in E$ *we have* $efe = ef$ *[*$efe = fe$*].*

A semigroup with set of E *of idempotents is called* L^**-**unipotent*** *[*R^**-**unipotent**] if it is abundant and if* $\forall e, f \in E$ *we have* $efe = ef$ $[efe = fe]$.

3.2 The Starred or Generalized Green's Relations

In this section, we introduce the starred green's relations as in [6] and we give some examples about these relations.

Definition 49 : *On a semigroup* S *the relation* $\mathcal{L}^*$ *[*$\mathcal{R}^*$*] is defined by the rule that*

$(a,b) \in \mathcal{L}^*$ [$\mathcal{R}^*$] if and only if the elements a, b are related by the Green's relation $\mathcal{L}$ [$\mathcal{R}$] in some over-semigroup T of S, (i.e. $S \subset T$).

In other words: Let S be a semigroup and let $a, b \in S$.

$$(a,b) \in \mathcal{L}_S^* \iff (a,b) \in \mathcal{L}_T \text{ for some over-semigroup } T \text{ of } S, \ (S \subset T).$$

We write $a\mathcal{L}_S^*b$. a, b are said to be **$\mathcal{L}^*$-equivalent.**

Definition 50 : *Dually, $\mathcal{R}_S^*$ can be defined as*

$$(a,b) \in \mathcal{R}_S^* \iff (a,b) \in \mathcal{R}_T$$

*for some over-semigroup T of S, ($S \subset T$). Then a, b are said to be **$\mathcal{R}^*$-equivalent.** We write $a\mathcal{R}_S^*b$.*

Moreover, $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$, $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$ (Note that in general $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$).

*We introduce $*-$**ideals** to obtain the starred analogue of the Green's relation $\mathcal{J}$. The $\mathcal{L}^*-$*class *containing the element a is denoted by L_a^*. The corresponding notation is used for the other relations. We define a **left** $*-$**ideals** [**right** $*-$**ideals**] of a semigroup S to be a left [right] ideal I of S for which $L_a^* \subseteq I$ [$R_a^* \subseteq I$] for all elements a of I. A subset I of S is a $*-$**ideals** if it is both a left $*-$ideal and a right $*-$ideal. The principal $*-$**ideals** J*(a) generated by the element a of S is the intersection of all $*-$**ideals** of S to which a belongs. The relation $\mathcal{J}^*$ is defined by the rule that $a\mathcal{J}^*b$ if and only if $J^*(a) = J^*(b)$.*

Obviously, on any semigroup S we have $\mathcal{L} \subseteq \mathcal{L}^*$, for we may take S as an over-semigroup of itself. Similarly, we have $\mathcal{R} \subseteq \mathcal{R}^*, \mathcal{H} \subseteq \mathcal{H}^*$, $\mathcal{D} \subseteq \mathcal{D}^*$.

On a regular semigroup S, the two relations coincide (correspondingly). In other words, if S is a regular semigroup and $a, b \in S$ then $a\mathcal{L}b \iff a\mathcal{L}^*b$ ($a\mathcal{R}b \iff a\mathcal{R}^*b$, $a\mathcal{J}b \iff a\mathcal{J}^*b$).

Lemma 51 : : *Let S be a semigroup and let $a, b \in S$. Then*

1. $(a, b) \in \mathcal{L}^* \iff$ for all x, y in S^1, $ax = ay$ if and only if $bx = by$.

2. $(a, b) \in \mathcal{R}^* \iff$ for all x, y in S^1, $xa = ya$ if and only if $xb = yb$.

Example 52 : *R_n- a right-zero semigroup with n-elements. Let $R_n = \{r_1, r_2, r_3, ..., r_n\}$. We adjoin the identity element 1. $R_n^1 = R_n \cup \{1\} = \{r_1, r_2, r_3, ..., r_n, 1\}$, and, $r_i \cdot r_j = r_j$ for all $i, j \in \{1, 2, ..., n\}$.*

$\cdot$	r_1	r_2	r_3	$\cdots$	r_n	1
r_1	r_1	r_2	r_3	$\cdots$	r_n	r_1
r_2	r_1	r_2	r_3	$\cdots$	r_n	r_2
r_3	r_1	r_2	r_3	$\cdots$	r_n	r_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
r_n	r_1	r_2	r_3	$\cdots$	r_n	r_n
1	r_1	r_2	r_3	$\cdots$	r_n	1

$\mathcal{R}^*$-class: $\{r_1, r_2, r_3, ..., r_n\}, \{1\}$; $\mathcal{L}^*$-class: $\{r_1\}, \{r_2\}, \{r_3\}, ..., \{r_n\}, \{1\}$.

Thus i=$\mathcal{H}^* = \mathcal{L}^*$ and $\mathcal{R}^*$=$\mathcal{D}^*$=$\mathcal{J}^* = w$.

Example 53 : *Let $N_m = \{0, n_1, n_2, n_3, ..., n_m\}$ such that $x \cdot y = 0$ for all $x, y \in N_m$. We adjoin the identity element 1.*

$\cdot$	0	n_1	n_2	$\cdots$	n_m	1
0	0	0	0	$\cdots$	0	0
n_1	0	0	0	$\cdots$	0	n_1
n_2	0	0	0	$\cdots$	0	n_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n_m	0	0	0	$\cdots$	0	n_m
1	0	n_1	n_2	$\cdots$	n_m	1

Thus,

$\mathcal{L}^*$ – classes : $\{0\}, \{n_1, n_2, ..., n_m\}$;

$\mathcal{R}^*$ – classes : $\{0\}, \{n_1, n_2, ..., n_m\}$;

$\mathcal{D}^*$ – classes : $\{0\}, \{n_1, n_2, ..., n_m\}$.

So $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^* = \mathcal{D}^* = \mathcal{J}^*$.

Recall the semigroup S_3^- in Example 41-it has

$\mathcal{L}^*$ – classes : $\{0\}, \{a, e_2, e_3\}, \{e_1\}, \{1\}$;

$\mathcal{R}^*$ – classes : $\{0\}, \{a, e_1\}, \{e_2\}, \{e_3\}, \{1\}$;

$\mathcal{D}^*$ – classes : $\{0\}, \{a, e_1, e_2, e_3\}, \{1\}$

.$\mathcal{H}^* = i_{S_3^-}$

Definition 54 : *A semigroup S is called* ***left abundant [right abundant]*** *if every $\mathcal{L}^*$-class [$\mathcal{R}^*$-class] contains an idempotent.*

Definition 55 : *A semigroup S in which every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class contains an idempotent is called* **abundant**.

The theory of abundant semigroups initiated by Fountain [6] parallels the theory of regular semigroups in many respects, however, there are some essential differences as well, see Appendix A. A semigroup may be left abundant but not right abundant for example consider $S_3^- \backslash \{e_1\}$−this is left abundant and it is not right abundant. Also, consider $S_3^- \backslash \{e_2\}$−this is right abundant and it is not left abundant.

The class of abundant semigroups includes in particular the class of cancellative monoids (see section 4.1 for its definition), and any full subsemigroup of a regular semigroup is abundant. (By a *full subsemigroup* of a semigroup U we mean simply one which contains all the idempotents of U.)

Definition 56 : *If S is semigroup, then S is* $*-$ **bisimple** *if and only if it has a single* $\mathcal{D}^*$-class.

3.3 Generalized Bruck-Reilly Extension

Consider a monoid M with H_1^* as the $\mathcal{H}^*$-class which contains the identity element of M. The set $S = N^0 \times M \times N^0$ ($N^0 = \{0, 1, 2, ...\}$) with operation defined by

$$(m, x, n)(p, y, q) = (m - n + t, x\theta^{t-n} y\theta^{t-p}, q - p + t)$$

where $t = \max(n, p)$ and θ is an endomorphism of M with images in H_1^* is a semigroup with an identity $(0, 1, 0)$.

If H_1^* is a subgroup of M then $S = N^0 \times H_1^* \times N^0$ is called the well known Bruck-Reilly extension of M. $S = N^0 \times M \times N^0$ is the generalized Bruck-Reilly extension of M determined by θ, and will be denoted by $S = BR^*(M, \theta)$, [see 1 and 10].

Asibong-Ibe [1] immediately deduced that $BR^*(M, \theta)$ is regular if and only if M is regular and this motivated him to consider the starred Green's relations on $BR^*(M, \theta)$

where he obtained the following:

Theorem 57 *: [1, Theorem 2.1] Let $S = BR^*(M,\theta)$ be a generalized Bruck-Reilly extension of a monoid M determined by θ. Suppose that (m,x,n) and (p,y,q) are in S. Then,*

1. $(m,x,n)\mathcal{L}^*_S(p,y,q)$ if and only if $n = q$ and $x\mathcal{L}^*_M y$.

2. $(m,x,n)\mathcal{R}^*_S(p,y,q)$ if and only if $m = p$ and $x\mathcal{R}^*_M y$.

Using this and other results Asibong-Ibe generalized Reilly's [10] work on Bisimple inverse $\omega-$semigroup to what he termed $*-$ *Bisimple* type A $\omega-$semigroups along the lines of Fountain's work, [6].

Definition 58 *A semigroup S whose idempotents commute and in which each $\mathcal{L}^*$-class [$\mathcal{R}^*$-class] contains an idempotent is called a* ***right adequate [left adequate]*** *semigroup. A semigroup which is both left and right adequate will be called* ***adequate semigroup.***

Definition 59 *The following notation will be used. An $\mathcal{L}^* -$ class containing an element $a \in S$ will be denoted by L^*_a. Similarly R^*_a is an $\mathcal{R}^* -$ class with an element $a \in S$. a^* and a^+will be used to denote typical idempotent elements in L^*_a and R^*_a respectively. A right [left] adequate semigroup S is called a* ***right*** *[**left**]* ***type*** *A semigroup if $ea = a(ea)^*$ [$ae = (ae)^+a$] for all elements a in S and all idempotents e in S. An adequate semigroup S is* ***type*** *A if it is both right and left type A.*

Definition 60 : *Let S be a type A semigroup, and let $a, b, c \in S$. The relation $\tilde{D}$ is defined on S by*

$$a\tilde{D}b \text{ if and only if } a^{*}\mathcal{D}\, b^{*}, a^{+}\mathcal{D}\, b^{+} \text{ for some } a^{*}, b^{*}, a^{+} \text{ and } b^{+}.$$

$\tilde{D}$ is an equivalence relation and satisfies the inclusion $\mathcal{D} \subseteq \tilde{D} \subseteq \mathcal{D}^{*}$ on a type A semigroup S. (See [1])

Theorem 61 : [1, *Theorem* 3.4] *Let S be a $*-$bisimple type A $w-$semigroup such that $\mathcal{D}^{*} = \tilde{D}$. Then $S \equiv BR^{*}(M, \theta)$.*

Chapter 4

Starred Green's Relations on Bicyclic Extensions

Abundant semigroup theory was initiated by Fountain [6]. Asibong-Ibe [1] extended Reilly's [10] results to the class of abundant semigroups. We characterize the starred Green's relations on Bicyclic extensions and show as main result that in case of a cancellative monoid, the bicyclic extension is an E^*-*bisimple* semigroup, by analogy with Warne [13] and Asibong-Ibe [1].

4.1 $\mathcal{L}^*$ and $\mathcal{R}^*$ Relations on Bicyclic Extensions

Lemma 62 : *Let $S = (T, P, K, \theta, \gamma)$. Then,*

(a): $((n,k),(g,p))\mathcal{L}^*((r,s),(h,q))$ if and only if $k = s$, $p = q$, and $g\mathcal{L}^*_T h$ (in T).

(b): $((n,k),(g,p))\mathcal{R}^*((r,s),(h,q))$ if and only if $n = r$ and $g\mathcal{R}^*_T h$ (in T).

Proof. (a): $(\Longrightarrow)$ Let $a = ((n,k),(g,p))$, $b = ((r,s),(h,q))$ be elements in S and suppose that $a\mathcal{L}^*_S b$. First notice that $((k,k),(1_T,p))$ is a right identity for a, that is,

$$\begin{aligned}
& ((n,k),(g,p))((k,k),(1_T,p)) = ((n,k),(g,p)) \\
\Longleftrightarrow\ & ((r,s),(h,q))((k,k),(1_T,p)) = ((r,s),(h,q)), \text{(since } a\mathcal{L}^*_S b) \\
\Longleftrightarrow\ & \begin{cases} ((r+k-s, s+k-s),(h\theta^{k-s}1_T\theta^{s-s}, p)) = ((r,s),(h,q)) \text{ if } s \le k \\ ((r+k-k, s+k-k),(h\theta^{k-k}1_T\theta^{s-k}, q(1_T\theta^{s-k-1}\gamma)) = ((r,s),(h,q)) \text{ if } s > k \end{cases} \\
\Longleftrightarrow\ & \begin{cases} ((r+k-s, k),(h\theta^{k-s}, p)) = ((r,s),(h,q)) \text{ if } s \le k \\ ((r,s),(h,q)) = ((r,s),(h,q)) \text{ if } s > k. \end{cases}
\end{aligned}$$

So,

$$k = s,\ p = q,\ \text{or } s > k \tag{4.1}$$

Similarly, by noting that $((s,s),(1_T,q))$ is a right identity for b we obtain

$$\begin{aligned}
& ((r,s),(h,q))((s,s),(1_T,q)) = ((r,s),(h,q)) \\
\Longleftrightarrow\ & ((n,k),(g,p))((s,s),(1_T,q)) = ((n,k),(g,p)), \text{(since } a\mathcal{L}^*_S b) \\
\Longleftrightarrow\ & \begin{cases} ((n+s-s, k+s-s),(g\theta^{s-s}1_T\theta^{k-s}, q)) = ((n,k),(g,p)) \text{ if } s \le k \\ ((n+s-k, k+s-k),(g\theta^{s-k}1_T\theta^{k-k}, p(1_T\theta^{s-k-1}\gamma)) = ((n,k),(g,p)) \text{ if } s > k \end{cases} \\
\Longleftrightarrow\ & \begin{cases} ((n,k),(g,q)) = ((n,k),(g,p)) \text{ if } s \le k \\ ((n+s-k, s),(g\theta^{s-k}, p)) = ((n,k),(g,p)) \text{ if } s > k. \end{cases}
\end{aligned}$$

So,

$$k = s,\ p = q \text{ or } k > s \tag{4.2}$$

Now from (4.1) and (4.2) we deduce that

$$k = s \text{ and } p = q \tag{4.3}$$

Moreover, let x, y be in T. Then for all g in T we have

$$gx = gy$$

$$\iff ((n,k),(gx,p)) = ((n,k),(gy,p))$$

$$\iff ((n,k),(g,p))((k,k),(x,p)) = ((n,k),(g,p))((k,k),(y,p))$$

$$\iff ((r,k),(h,p))((k,k),(x,p)) = ((r,k),(h,q))((k,k),(y,p))$$

(by (4.3) and the fact that $a\mathcal{L}^*_S b$)

$$\iff ((r,k),(hx,p)) = ((r,k),(hy,q))$$

$$\iff hx = hy.$$

Thus $g\mathcal{L}^*_T h$ as required.

$(\Longleftarrow)$: Suppose that $k = s, p = q$ and $g\mathcal{L}^*_T h$ and let $((n_1,k_1),(g_1,p_1)), ((r_1,s_1),(h_1,q_1))$ in S be such that

$$((n,k),(g,p))((n_1,k_1),(g_1,p_1)) = ((n.k),(g,p))((r_1,s_1),(h_1,q_1))$$

$$\Leftrightarrow \begin{cases} ((n+n_1-k,k_1),(g\theta^{n_1-k}g_1,p_1)) \\ = ((n+r_1-k,s_1),(g\theta^{r_1-k}h_1,q_1)) \text{ if } k \le n_1, k \le r_1 \\ ((n+n_1-k,k_1),(g\theta^{n_1-k}g_1,p_1)) \\ = ((n,k+s_1-r_1),(g.h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma))) \text{ if } k \le n_1, k > r_1 \\ ((n,k+k_1-n_1),(g.g_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma))) \\ = ((n+r_1-k,s_1),(g\theta^{r_1-k}h_1,q_1)) \text{ if } k > n_1, k \le r_1 \\ ((n,k+k_1-n_1),(g.g_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma))) \\ = ((n,k+s_1-r_1),(g.h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma))) \text{ if } k > n_1, k > r_1 \end{cases} \tag{4.4}$$

Now equating the first coordinates in each (of the four) case yields:

Case 1. $n + n_1 - k = n + r_1 - k \Rightarrow n_1 = r_1$. Thus,

a): if $n_1 = r_1 = k$, we have by equating third coordinates

$$gg_1 = gh_1$$

$$\iff hg_1 = hh_1 \text{ (since } g\mathcal{L}^*_T h);$$

$$\iff h\theta^{n_1-k}g_1 = h\theta^{n_1-k}h_1 \text{ (since } n_1 = r_1 = k)$$

b): if $n_1 = r_1 > k$ then $g_1\theta^{n_1-k}, h_1\theta^{n_1-k} \in U$ (-the group of units of T) and so by equating third coordinates, we have

$$g\theta^{n_1-k}g_1 = g\theta^{r_1-k}h_1$$

$$\iff g_1 = h_1$$

$$\implies h\theta^{n_1-k}g_1 = h\theta^{n_1-k}h_1$$

So, if $k \leq n_1, k \leq r_1$ then from (4.4) we have

$$((n + n_1 - k, k_1), (g\theta^{n_1-k}g_1, p_1)) = ((n + r_1 - k, s_1), (g\theta^{r_1-k}h_1, q_1))$$

$$\iff$$

$$((n + n_1 - k, k_1), (h\theta^{n_1-k}g_1, p_1)) = ((n + r_1 - k, s_1), (h\theta^{r_1-k}h_1, q_1))$$

$$\iff$$

$$((r+n_1-k, k+k_1-k), (h\theta^{n_1-k}g_1\theta^{k-k}, p_1)) = ((r+r_1-k, k+s_1-k), (h\theta^{r_1-k}h_1\theta^{k-k}, q_1))$$

$$\iff$$

$$((r, k), (h, p)).((n_1, k_1), (g_1, p_1)) = ((r, k), (h, p))((r_1, s_1), (h_1, q_1))$$

Case 2. $n + n_1 - k = n \Rightarrow n_1 = k$. Thus,

$n_1 = k > r_1$ then by equating third coordinates, we have

$$g\theta^{n_1-k}g_1 = g.h_1\theta^{k-r_1}$$

$$\iff gg_1 = g.h_1\theta^{k-r_1}$$

$$\iff hg_1 = h.h_1\theta^{k-r_1} \text{ (since } g\mathcal{L}^*_T h)$$

$$\iff h\theta^{n_1-k}g_1 = h.h_1\theta^{k-r_1} \text{ (since } n_1 = k)$$

So, if $k \le n_1, k > r_1$ then from (4.4) we have

$((n+n_1-k,k_1),(g\theta^{n_1-k}g_1,p_1)) = ((n,k+s_1-r_1),(g.h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma)))$

$\Longleftrightarrow$

$((n+n_1-k,k_1),(h\theta^{n_1-k}g_1\theta^{k-k},p_1)) = ((n,k+s_1-r_1),(h\theta^{r_1-r_1}h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma))$

$\Longleftrightarrow$

$((r+n_1-k,k+k_1-k),(h\theta^{n_1-k}g_1\theta^{k-k},p_1)) = ((r,k+s_1-r_1),(h\theta^0h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma))$

$\Longleftrightarrow$

$((r,k),(h,p)).((n_1,k_1),(g_1,p_1)) = ((r,k),(h,p))((r_1,s_1),(h_1,q_1))$.

Case 3. Since $k > n_1, k \le r_1$ then by equating the first coordinates in this case yields:

$n = n+r_1-k \Rightarrow r_1 = k$. So, $r_1 = k > n_1$. And by equating third coordinates, we have

$$\begin{aligned} & g.g_1\theta^{k-n_1} = gh_1 \\ \Longleftrightarrow\ & h.g_1\theta^{k-n_1} = hh_1 \text{ (since } g\mathcal{L}^*_T h) \\ \Longleftrightarrow\ & h.g_1\theta^{k-n_1} = h\theta^{r_1-k}h_1\theta^{k-k} \end{aligned}$$

So, if $k > n_1, k \le r_1$ then from (4.4) we have

$((n,k+k_1-n_1),(g.g_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma))) = ((n+r_1-k,s_1),(g\theta^{r_1-k}h_1,q_1))$

$\Longleftrightarrow$

$((n,k+k_1-n_1),(h.g_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma))) = ((n+r_1-k,s_1),(h\theta^{r_1-k}h_1\theta^{k-k},q_1))$

$\Longleftrightarrow$

$((r+n_1-n_1,k+k_1-n_1),(h\theta^{n_1-n_1}g_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma)) = ((r+r_1-k,k+r_1-k),(h\theta^{r_1-k}h_1\theta^{k-k},q_1))$

$\Longleftrightarrow$

$((r,k),(h,p)).((n_1,k_1),(g_1,p_1)) = ((r,k),(h,p))((r_1,s_1),(h_1,q_1)).$

Case 4. Since $k > n_1$ and $k > r_1$ then by equating third coordinates. we have

$g.g_1\theta^{k-n_1} = g.h_1\theta^{k-r_1} \iff h.g_1\theta^{k-n_1} = h.h_1\theta^{k-r_1}$ (since $g\mathcal{L}^*_T h$);

So, if $k > n_1$, $k > r_1$ then from (4.4) we have

$$((n,k+k_1-n_1),(g.g_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma))) = ((n,k+s_1-r_1),(g.h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma)))$$

$\iff$

$$((n,k+k_1-n_1),(h.g_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma))) = ((n,k+s_1-r_1),(h.h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma)))$$

$\iff$

$$((r,k+k_1-n_1),(hg_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma)) = ((r,k+s_1-r_1),(h.h_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma)))$$

$\iff$

$$((r,k+k_1-n_1),(hg_1\theta^{k-n_1},p(g_1\theta^{k-n_1-1}\gamma)) = ((r,k+s_1-r_1),(hh_1\theta^{k-r_1},p(h_1\theta^{k-r_1-1}\gamma)))$$

$\iff$

$((r,k),(h,p)).((n_1,k_1),(g_1,p_1)) = ((r,k),(h,p))((r_1,s_1),(h_1,q_1)).$

Therefore we deduce in all cases that

$$\begin{aligned} &((n,k),(g,p))((n_1,k_1),(g_1,p_1)) = ((n,k),(g,p))((r_1,s_1),(h_1,q_1)) \\ \iff\ &((r,k),(h,p)).((n_1,k_1),(g_1,p_1)) = ((r,k),(h,p))((r_1,s_1),(h_1,q_1)) \end{aligned}$$

and so $((n,k),(g,p))\mathcal{L}^*_S((r,k),(h,p))$ as required.

Proof. (b): ($\Longrightarrow$)Let $a = ((n,k),(g,p))$, $b = ((r,s),(h,q))$ be elements in S and suppose that $a\mathcal{R}^*_S b$.

First notice that $((n,n),(1_T,p))$ is a left identity for a, that is,

$((n,n),(1_T,p))((n,k),(g,p)) = ((n+n-n,n+k-n),(1_T\theta^{n-n}g\theta^{n-n},p)) = ((n,k),(g,p)).$

Thus,

$$((n,n),(1_T,p))((n,k),(g,p)) = ((n,k),(g,p))$$

$$\iff ((n,n),(1_T,p))((r,s),(h,q)) = ((r,s),(h,q)), \text{ (since } a\mathcal{L}^*_S b)$$

$$\iff \begin{cases} ((n+r-n, n+s-n), (1_T\theta^{r-n}h\theta^{n-n}, q)) = ((r,s),(h,q)) \text{ if } n \leq r \\ ((n+r-r, n+s-r), (1_T\theta^{r-r}h\theta^{n-r}, p(h\theta^{n-r-1}\gamma))) = ((r,s),(h,q)) \text{ if } n > r \end{cases}$$

$$\iff \begin{cases} ((r,s),(h,q)) = ((r,s),(h,q)) \text{ if } n \leq r \\ ((n, n+s-r), (h, p(h\theta^{n-r-1}\gamma))) = ((r,s),(h,q)) \text{ if } n > r. \end{cases}$$

So, $n \leq r$.

Similarly by noting that $((r,r),(1_T,p))$ is a left identity for b we deduce that $r \leq n$. Hence $n = r$. Moreover, let x, y be in T. Then for all g in T we have

$$xg = yg$$

$$\Leftrightarrow ((n,k),(xg,p)) = ((n,k),(yg,p))$$

$$\Leftrightarrow ((n+n-n, n+k-n), (x\theta^{n-n}g\theta^{n-n}, p)) = ((n+n-n, n+k-n), (y\theta^{n-n}g\theta^{n-n}, p))$$

$$\Leftrightarrow ((n,n),(x,p))((n,k),(g,p)) = ((n,n),(y,p))((n,k),(g,p))$$

$$\Leftrightarrow ((n,n),(x,p))((r,s),(h,q)) = ((n,n),(y,p))((r,s),(h,q)) \text{ (by the fact that } a\mathcal{R}^*_S b)$$

$$\Leftrightarrow ((n,n),(x,p))((n,s),(h,q)) = ((n,n),(y,p))((n,s),(h,q)) \text{ (since } n = r)$$

$$\Leftrightarrow ((n,n),(x,p))((n,s),(h,q)) = ((n,n),(y,p))((n,s),(h,q)) \text{ (since } n = r)$$

$$\Leftrightarrow ((n+n-n, n+s-n), (x\theta^{n-n}h\theta^{n-n}, p)) = ((n+n-n, n+s-n), (y\theta^{n-n}h\theta^{n-n}, p))$$

$$\Leftrightarrow ((n,s),(xh,p)) = ((n,s),(yh,p))$$

$$\Leftrightarrow xh = yh$$

Thus $g\mathcal{R}^*_T h$ as required.

($\Longleftarrow$): Suppose that $n = r$ and $g\mathcal{R}_T^* h$ and let $((n_1, k_1), (g_1, p_1)), ((r_1, s_1), (h_1, q_1))$ be in S such that

$$((n_1, k_1), (g_1, p_1))((r, k), (g, p)) = ((r_1, s_1), (h_1, q_1))((r, k), (g, p))$$

$$\Longleftrightarrow \begin{cases} ((n_1 + r - k_1, k_1 + k - k_1), (g_1\theta^{r-k_1} g\theta^{k_1-k_1}, p)) \\ = ((r_1 + r - s_1, s_1 + k - s_1), (h_1\theta^{r-s_1} g\theta^{s_1-s_1}, p)) \text{ if } k_1 \leq r,\ s_1 \leq r \\ ((n_1 + r - k_1, k_1 + k - k_1), (g_1\theta^{r-k_1} g\theta^{k_1-k_1}, p)) \\ = ((r_1 + r - r, s_1 + k - r), (h_1\theta^{r-r} g\theta^{s_1-r}, q_1(g\theta^{s_1-r-1}\gamma))) \text{ if } k_1 \leq r,\ s_1 > r \\ ((n_1 + r - r, k_1 + k - r), (g_1\theta^{r-r} g\theta^{k_1-r}, p_1(g\theta^{k_1-r-1}\gamma))) \\ = ((r_1 + r - s_1, s_1 + k - s_1), (h_1\theta^{r-s_1} g\theta^{s_1-s_1}, p)) \text{ if } k_1 > r,\ s_1 \leq r \\ ((n_1 + r - r, k_1 + k - r), (g_1\theta^{r-r} g\theta^{k_1-r}, p_1(g\theta^{k_1-r-1}\gamma))) \\ = ((r_1 + r - r, s_1 + k - r), (h_1\theta^{r-r} g\theta^{s_1-r}, q_1(g\theta^{s_1-r-1}\gamma))) \text{ if } k_1 > r,\ s_1 > r \end{cases} \tag{4.5}$$

Now, equating the second and third coordinates in each (of the four) case yields:

Case 1: $n_1 + r - k_1 = r_1 + r - s_1$, if $k_1 \leq r$, $s_1 \leq r$.

$$\begin{aligned} & g_1\theta^{r-k_1} g\theta^{k_1-k_1} = h_1\theta^{r-s_1} g\theta^{s_1-s_1} \\ \Longleftrightarrow\ & g_1\theta^{r-k_1} g = h_1\theta^{r-s_1} g \\ \Longleftrightarrow\ & g_1\theta^{r-k_1} h = h_1\theta^{r-s_1} h \text{ since } g\mathcal{R}_T^* h \end{aligned}$$

Then from (4.5) we have

$$\begin{aligned} & ((n_1 + r - k_1, k), (g_1\theta^{r-k_1} g\theta^{k_1-k_1}, p)) = ((r_1 + r - s_1, k), (h_1\theta^{r-s_1} g\theta^{s_1-s_1}, p)) \\ \Leftrightarrow\ & ((n_1 + r - k_1, k), (g_1\theta^{r-k_1} g, p)) = ((r_1 + r - s_1, k), (h_1\theta^{r-s_1} g, p)) \\ \Leftrightarrow\ & ((n_1 + r - k_1, s), (g_1\theta^{r-k_1} h, q)) = ((r_1 + r - s_1, s), (h_1\theta^{r-s_1} h, q)) \\ \Leftrightarrow\ & ((n_1 + r - k_1, k_1 + s - k_1), (g_1\theta^{r-k_1} h, q)) = ((r_1 + r - s_1, s_1 + s - s_1), (h_1\theta^{r-s_1} h, q)) \\ \Leftrightarrow\ & ((n_1, k_1), (g_1, p_1))((r, s), (h, q)) = ((r_1, s_1), (h_1, q_1))((r, s), (h, q)) \end{aligned}$$

Case 2: If $k_1 \leq r$, $s_1 > r$, then from (4.5) by equating the second and third coordinates, we have

$$k = s_1 + k - r \Longrightarrow s_1 = r,$$

but in this case $s_1 > r$. This case is impossible.

Case 3: $k_1 > r$, $s_1 \leq r$, then from (4.5) by equating the second coordinates, we have

$$k_1 + k - r = s_1 + k - s_1 \Longrightarrow k_1 = r$$

but in this case $k_1 > r$. This case is also impossible.

Case 4: $k_1 > r$, $s_1 > r$, then from (4.5) by equating the second and third coordinates, we have

$$k_1 + k - r = s_1 + k - r \Leftrightarrow k_1 = s_1$$

and so

$$\begin{aligned}
& g_1 g \theta^{k_1 - r} = h_1 g \theta^{s_1 - r} \\
\Longleftrightarrow\ & g_1 g \theta^{k_1 - r} = h_1 g \theta^{k_1 - r} \text{ since } k_1 = s_1 \\
\Longleftrightarrow\ & g_1 g \theta^{k_1 - r} = h_1 g \theta^{k_1 - r} \\
\Longleftrightarrow\ & g_1 = h_1 \text{ (since } g\theta^{k_1 - r} \in U) \\
\Longrightarrow\ & g_1 g = h_1 g \\
\Longleftrightarrow\ & g_1 h = h_1 h \text{ since } g \mathcal{R}_T^* h \\
\Longleftrightarrow\ & g_1 \theta^{r-r} h \theta^{k_1 - r} = h_1 \theta^{r-r} h \theta^{s_1 - r} \text{ since } k_1 = s_1
\end{aligned}$$

Then from (4.5) we have

$$((n_1, k_1 + k - r), (g_1 g\theta^{k_1-r}, p_1(g\theta^{k_1-r-1}\gamma))) = ((r_1, s_1 + k - r), (h_1 g\theta^{s_1-r}, q_1(g\theta^{s_1-r-1}\gamma)))$$

$$\Longleftrightarrow$$

$$((n_1, k_1 + k - r), (g_1\theta^{r-r} h\theta^{k_1-r}, p_1(h\theta^{k_1-r-1}\gamma))) = ((r_1 + r - s_1, k), (h_1\theta^{n-s_1} h\theta^{s_1-s_1}, p))$$

$$\Longleftrightarrow$$

$$((n_1, k_1 + s - r), (g_1\theta^0 h\theta^{k_1-r}, p_1(h\theta^{k_1-r-1}\gamma))) = ((r_1, s_1 + s - r), (h_1\theta^0 h\theta^{s_1-r}, q_1(h\theta^{s_1-r-1}\gamma)))$$

$$\Longleftrightarrow$$

$$((n_1, k_1 + s - r), (g_1 h\theta^{k_1-r}, p_1(h\theta^{k_1-r-1}))) = ((r, s_1 + s - r), (h_1 h\theta^{s_1-r}, q_1(h\theta^{s_1-r-1})))$$

$$\Longleftrightarrow$$

$((n_1, k_1), (g_1, p_1))((r, s), (h, q)) = ((r_1, s_1), (h_1, q_1))((r, s), (h, q))$.
Therefore we deduce in all cases that

$$((n_1, k_1), (g_1, p_1))((n, k), (g, p)) = ((r_1, s_1), (h_1, q_1))((n, k), (g, p))$$

$$\Longleftrightarrow \quad ((n_1, k_1), (g_1, p_1))((r, s), (h, q)) = ((r_1, s_1), (h_1, q_1))((r, s), (h, q))$$

and so $((n, k), (g, p))\mathcal{R}^*_S((r, k), (h, p))$ as required.

4.2 Some Properties of Bicyclic Extensions and Cancellative Monoids

It is well-known and indeed easy to show that if T is a cancellative monoid then $\mathcal{H}^* = T \times T$ and hence we deduce from Lemma 62 above:

Lemma 63 : *Let $S = (T, P, K, \theta, \gamma)$, where T is a cancellative monoid. Then S is an abundant semigroup and*

(a) $((n, k), (g, p))\mathcal{L}^*_S((r, s), (h, q))$ *if and only if* $k = s$, $p = q$;

(b) $((n,k),(g,p))\mathcal{R}^*_S((r,s),(h,q))$ *if and only if* $n = r$ *:*

(c) $\mathcal{D}^*$ *is the universal relation in* S*.*

Proof. We only need to show that S is an abundant semigroup and $\mathcal{D}^*$ is the universal relation in S.

Let L^*_a be an $\mathcal{L}^*$-class of S where $a = ((n,k),(g,p)) \in S$. Then,

$$((n,k),(g,p))\mathcal{L}^*_S((k,k),(1_T,p)) \text{ by (a)}$$

So, $((k,k),(1,p)) \in L^*_a$. Also,

$$\begin{aligned} ((k,k),(1_T,p))((k,k),(1_T,p)) &= ((k+k-k,k+k-k),(1_T\theta^{r-r}1_T\theta^{r-r},p)) \\ &= ((k,k),(1_T,p)) \end{aligned}$$

Since L^*_a is an arbitrary $\mathcal{L}^*$class of S , therefore each $\mathcal{L}^*$-class has an idempotent. Now, we show that each $\mathcal{R}^*$-class of S has an idempotent. Let R^*_a be an $\mathcal{R}^*$-class of S where $a = ((n,k),(g,p)) \in S$. Then,

$$((n,k),(g,p))\mathcal{R}^*_S((n,n),(1_T,p)) \text{ by (b)}$$

So, $((n,n),(1,p)) \in R^*_a$. Also,

$$\begin{aligned} ((n,n),(1_T,p))((n,n),(1_T,p)) &= ((n+n-n,n+n-n),(1_T\theta^{n-n}1\theta^{r-r},p)) \\ &= ((n,n),(1_T,p)) \end{aligned}$$

Thus S is an abundant.

(c): We show that $\mathcal{D}^* = S \times S$ (i.e. $\mathcal{D}^*$ is the universal relation in S). Since $\mathcal{D}^* \subseteq S \times S$, so we need to show that $S \times S \subseteq \mathcal{D}^*$. Consider $((n,k),(g,p))$ and $((r,s),(h,q)) \in S$. Then, by (a) and (b) we have $((n,k),(g,p))\mathcal{L}^*_S((r,k),(h,p))$ and $((r,k),(h,p))\mathcal{R}^*_S((r,s),(h,q))$. Thus, $((n,k),(g,p))\mathcal{D}^*((r,s),(h,q))$ ■

Remark 7 : *Note that the idempotents of $S = (T, P, K, \theta, \gamma)$ where T is a cancellative monoid, form a naturally ordered band since for any group G, we have*

$$E(S(T,,P,K,\theta,\gamma)) = E(S(G,,P,K,\theta,\gamma)).$$

4.3 E*-Bisimple Semigroup

Recall definition (45 and 48) : A semigroup S is called $\mathcal{D}$-**simple** or **bisimple** if it consists of a single $\mathcal{D}$-class. Let E be an idempotent semigroup (a band). The collection $E(R)$ or $\mathcal{R}$ classes of E may be partially ordered by the following rule. If R_1, $R_2 \in E(R)$, $R_1 < R_2$ if and only if $e < f$ for all $e \in R_1$ and $f \in R_2$. ($e \leq f$ if and only if $ef = fe = e$). If E(R), under this order, is order isomorphic to N^0, the non-negative integers, under the reverse of the usual order, E is **called a naturally** ordered band.

Definition 64 : *A bisimple semigroup whose idempotents form a naturally ordered band is termed an* **E − bisimple semigroup**.

*A semigroup with set of E of idempotents is called L-**unipotent** [R-**unipotent**] if it is regular and if $\forall e, f \in E$ we have $efe = ef$ [$efe = fe$].*

*Every L-unipotent is orthodox, and also, every R-unipotent is orthodox). A semigroup with set of E of idempotents is called L^*-**unipotent** [R^*-**unipotent**] if it is abundant and if $\forall e, f \in E$ we have $efe = fe$ [$efe = ef$]. .*

*By analogy with orthodox semigroups (see Definition 46) a semigroup is called a **quasi-adequate** semigroup if it is abundant and its idempotents form a subsemigroup. So, alternatively an $\mathcal{L}^* -$ unipotent semigroup is a **quasi-adequate** semigroup with* *a unique idempotent in each $\mathcal{L}^*$-class.*

Recall Definition (56): A semigroup S is called $*-$ **bisimple** *if it consists of a single $\mathcal{D}^*$-class.*

Now we define what we shall call E*-bisimple semigroups as a class of abundant semigroups, by analogy with E-bisimple regular semigroups.

Definition 65 *: An abundant* $*-$ **bisimple** *semigroup whose idempotents form a naturally ordered band will be called an* **E*−bisimple** *semigroup.*

Then we have the main results of this chapter.

Theorem 66 *: Let $S = (T, P, K, \theta, \gamma)$ be as defined in Definition (26) where T is a Cancellative Monoid. Then S is an L^*-unipotent E^*-bisimple semigroup.*

Proof. $E(S) = \{(0,0),(e,p) : p \in P\} \cup \{((n,n),(1_T,q)) : q \in K\}$ where 1_T is the identity of T. We need to show that S is L^*-unipotent. Let $e = ((n,n),(1_T,p)), f = ((m,m),(1_T,k)) \in E(S)$.

Case **(1)**: Let $n \leq m$. Then

$$\begin{aligned} efe &= ((n,n),(1_T,p))((m,m),(1_T,k))((n,n),(1_T,p)) \\ &= ((n+m-n,n+m-n),(1\theta^{m-n}1\theta^{n-n},k))((n,n),(1,p)) \\ &= ((m,m),(1_T,k)((n,n),(1_T,p)) \\ &= fe \end{aligned}$$

Case **(2)**: Let $n > m$. Then

$$fe = ((m,m),(1_T,k))((n,n),(1_T,p))$$

$$= ((m+n-m, m+n-m), (1_T.p))$$
$$= ((n,n), (1_T, p)))$$
$$= e$$

So, $efe = e(fe) = e^2 = e = fe$.

Conclusion 67 *: This work forms an important first step towards obtaining a unified structure theory (of some class of E^*-bisimple semigroup) that will yield the results of Asibong-Ibe* [1] *and Warne* [13] *as special cases.*

[Appendix A]

Regular Semigroup:
Every $\mathcal{L}$-class contains an idempotent
Every $\mathcal{R}$-class contains an idempotent
Every $\mathcal{D}$-class contains an idempotent

Abundant:
Every $\mathcal{L}^*$-class contains an idempotent (left abundant)
Every $\mathcal{R}^*$-class contains an idempotent (right abundant)
LEFT +RIGHT =ABUNDANT

$\mathcal{H}$-class that contains an idempotent is a SUBGROUP
$\mathcal{H}^*$-class that contains an idempotent is a CANCELLATIVE MONOID

[Appendix B]

REGULAR	ABUNDANT
ORTHODOX	QUASI-ADEQUATE
L-UNIPOTENT, R-UNIPOTENT	L^*-UNIPOTENT, R^*-UNIPOTENT
INVERSE	TYPE A

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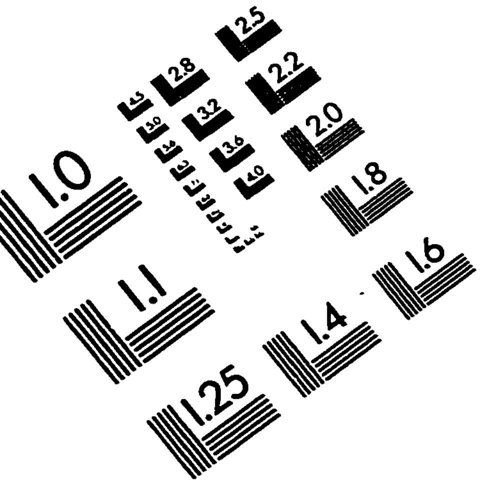
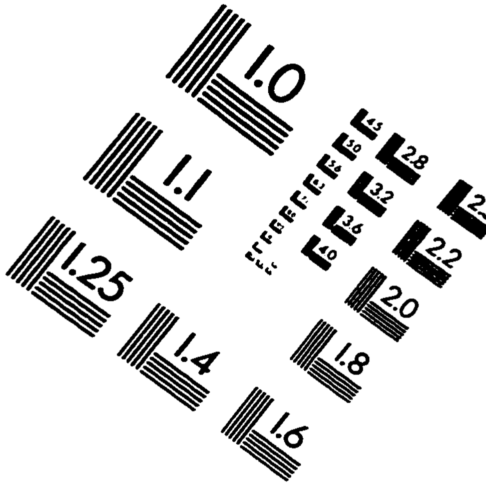
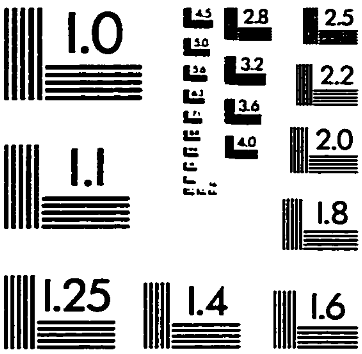
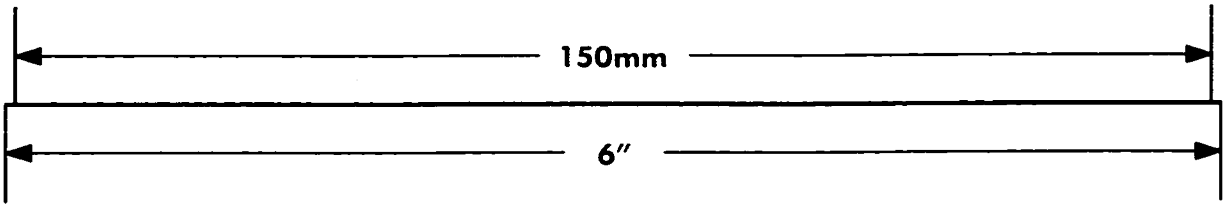
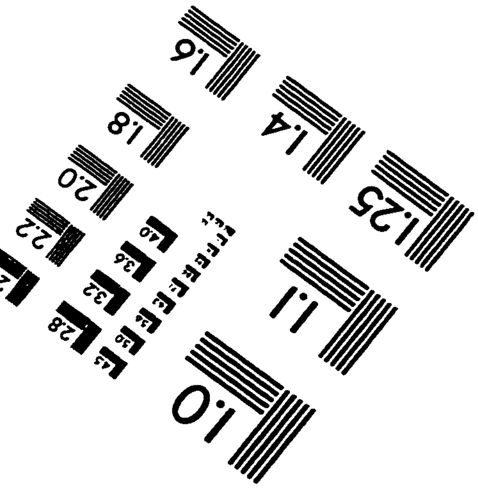
IMAGE EVALUATION TEST TARGET (QA-3)

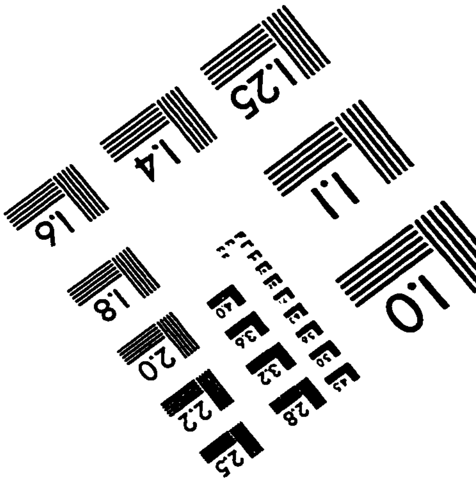












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