



## TIP DEFLECTION ESTIMATION OF A ROTATING CANTILEVERED BEAM

A. El-Sinawi<sup>1</sup>, and M. N. Hamdan<sup>2</sup>

*Assistant Professor, Department of Mechanical Engineering, KFUPM*

*Associate Professor, Department of Mechanical Engineering, KFUPM*

*KFUPM Box 1887, Dhahran 31261, Saudi Arabia. Email: elsinawi@kfupm.edu.sa*

### ABSTRACT

*To eliminate the need for sensor placement on rotating flexible beams such as turbine blades, helicopter rotors, and similar applications, a new approach has been developed based on the linear quadratic estimator (LQE) technique for estimating the vibration of any point on the span of a rotating flexible beam mounted on a compliant hub (plant) in the presence of process and measurements noise. A nonlinear model of the plant is utilized by this study to mimic the actual plant behavior. The corresponding plant dynamics of the LQE are in the form of a reduced order linear model constructed from the eigenvalues and eigenfunctions of a finite element dynamic model of the plant formulated in the state space. A virtual hub deflection (that mimics the actual measurement of the vertical hub deflection needed by the estimation process) is generated by the non-linear model of the plant. The LQE reconstructs the states of the plant, including transverse deflection of the beam at any point, from the measurements of the vertical deflection of the hub, assuming that it is the most accessible state for measurement. Estimated beam tip deflection obtained by the proposed technique is then compared to the tip deflection generated by the non-linear model and the results show good agreement.*

**Keywords:** *Optimal Control, Estimation, Finite Elements, Flexible Structures.*

## 1. INTRODUCTION

Rotating flexible beams similar to Figure 1, model turbine blades, helicopter rotors, robot arms, and similar systems. Extensive studies on the vibration of rotating flexible beams have been dedicated to the development of theoretical models that describe the characteristics of such vibration in a pure mathematical fashion [Zhu et al., 1997], [Hu et al., 1994], [Zheng et al., 2000], [Mitchell et al., 1988], [Swaminadham et al., 1994], [Bazoune et al., 2001], [Kane et al., 1987]. However, few studies have discussed the method by which the vibration of the rotating beams could be monitored and/or controlled. Khulief [Khulief, 2000] has proposed the control of a rotating beam mounted on a rigid hub using linear quadratic regulator, which required the placement of the sensor and actuator on the rotating beam. Elliott *et al.* [Elliot et al., 2001], Yousefi *et al.* [Yousefi et al., 2000], Zimmerman *et al.* [Zimmerman et al., 1989], and Mallory *et al.* [Mallory et al., 2000] all have also suggested placing sensors at various locations along the span of the beam for monitoring and control purposes. However, their research has focused on either non-rotating beams or large space structures.

Real life applications that require on-line monitoring and/or control of rotating beam may face many problems regarding sensor(s) placement on the beam such as high speeds, extreme temperatures, and high centrifugal forces.

Another issue regarding many of the research studies dealing with the monitoring and/or control of vibration of flexible rotating beams is that, they are limited to deterministic models of the system. In reality, however, components of structural and mechanical systems often exhibit considerable stochastic variations in their properties. Thus, the characteristics of a structure corresponding to these properties show some stochastic variations [El-Sinawi et al. 2001]. This makes it necessary to take account of the uncertainties of system parameters if highly reliable models and/or control schemes are to be utilized.

To eliminate the need for a sensor placement on rotating flexible beam such as turbine blade, helicopter rotors, and like applications, a new approach based on the linear quadratic estimator (LQE) technique for estimating the vibration of any point on the span of a rotating flexible beam mounted on a compliant hub and undergoing large planar deformation (plant) and subject to process and measurements noise, has been developed. A nonlinear model of the plant is utilized to mimic the actual plant behavior [El-Sinawi et al., 2001]. The corresponding plant dynamics of the LQE are in the form of a reduced order linear model constructed from the eigenvalues and eigenfunctions of a finite element dynamic model of the plant and formulated in the state space [Bazoune et al., 2001], [El-Sinawi et al., 2001], [El-Sinawi, 1999], [El-Sinawi et al., 1999]. The LQE reconstructs the states of the plant, including transverse deflection of the beam, from the virtual measurements of the vertical deflection of the hub, which is assumed to be the most accessible state for measurement.

## 2. FINITE ELEMENT MODEL

To implement the proposed scheme on a distributed parameter system such as the beam in Figure 1, the structure is discretized into finite elements forming an  $n$ -dimensional discrete spring-mass-damper system whose dynamics is described by second order matrix differential equation of

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = u(t) \quad (1)$$

where  $M$ ,  $K$  and  $C$  are the mass, stiffness, and damping coefficient square, symmetric matrices with their dimensions equal to the number of degrees of freedom  $n$  [Gawronski et al., 1991].  $x(t)$  and  $u(t)$  are the displacement and force vectors, respectively. For systems with classical damping, i.e., the systems with damping matrix  $C$  proportional to the mass  $M$  and stiffness  $K$  matrices,  $M$ ,  $K$  and  $C$  can be diagonalized using normalized orthonormal eigenvectors as the columns of the transformation matrix [El-Sinawi, 1999], resulting

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = Q_i u(t), \quad i = 1, \dots, n \quad (2)$$

where  $\eta_i$ ,  $\omega_i$ , and  $\zeta_i$  represent the transformed coordinates, natural frequency, and damping ratio of the structure's  $i$ -th mode of vibration. When the input is point force (as in the case of actuators),  $Q_i$  are the vector components of the  $i$ -th eigenfunction evaluated at the force input location.

$$\dot{z} = \begin{bmatrix} 0 & I \\ -\Omega^2 & -2\zeta\Omega \end{bmatrix} z + \begin{bmatrix} 0 \\ Q \end{bmatrix} u \quad (3)$$

$$y = \begin{bmatrix} W & 0 \end{bmatrix} z + Du \quad (4)$$

The formulation presented by equations (3) and (4) is the basis for state-space modeling of flexible structures, having point force(s) as the input(s) and point displacement(s) as the measured output(s).

where

state vector:

$$z(t) = \begin{Bmatrix} \eta(t) \\ \dot{\eta}(t) \end{Bmatrix}$$

number of modes:	$N_m$
number of inputs:	$N_u$
number of outputs:	$N_y$
modal displacement:	$\eta(t) = \{\eta_1(t), \eta_2(t), \dots, \eta_{N_m}(t)\}^T$
modal velocity:	$\dot{\eta}(t) = \{\dot{\eta}_1(t), \dot{\eta}_2(t), \dots, \dot{\eta}_{N_m}(t)\}^T$
input:	$u(t) = \{u_1(t), u_2(t), \dots, u_{N_u}(t)\}^T$
spatial coordinates	$r_i$
output:	$y(t) = \{x(r_1, t), x(r_2, t), \dots, x(r_{N_y}, t)\}^T$
natural frequency:	$\Omega = \text{diag} \{ \{\omega_1, \omega_2, \dots, \omega_{N_m}\} \}$
modal damping:	$\zeta = \text{diag} \{ \{\zeta_1, \zeta_2, \dots, \zeta_{N_m}\} \}$
eigenfunction $i$ at location $j$	$\psi_{ij}$

input matrix

$$Q = \begin{bmatrix} \psi_{1,1}, & \dots, & \psi_{1,N_u} \\ \vdots & \ddots & \vdots \\ \psi_{N_m,1}, & \dots, & \psi_{N_m,N_u} \end{bmatrix}$$

output matrix

$$W = \begin{bmatrix} \psi_{1,1}, & \dots, & \psi_{N_m,1} \\ \vdots & \ddots & \vdots \\ \psi_{1,N_y}, & \dots, & \psi_{N_m,N_y} \end{bmatrix}$$

$A$ ,  $B$ ,  $C$ , and  $D$  matrices shown in equations (3) and (4), are the dynamic, input, output, and direct input matrices, respectively. These matrices describe the state-space model of the flexible structure, which are functions of the system (natural frequency, damping ratio, and mode shapes (eigenfunctions), i.e., if we assume  $\theta = f([\omega_i, \zeta_i, \text{ and } \psi_i]_{i=1, \dots, n})$  then the resulting state-space model format is;

$$\dot{z} = A(\theta)z + B(\theta)u \tag{5}$$

$$y = C(\theta)z + D(\theta)u \tag{6}$$

The information needed to construct the  $A$ ,  $B$ ,  $C$ , and  $D$  matrices of equations (5) and (6), i.e., (mode shapes and natural frequencies,) can be easily obtained by performing finite element analysis on a solid model of the plant using one of the readily available FEA commercial software packages. A portion of the finite element model used to generate the state-space model of the plant is shown in Figure 2

### 3. OPTIMAL STATE ESTIMATOR (OBSERVER) DESIGN

The first step in designing the proposed optimal estimation technique is the design of the Kalman gains [Kalman, 1960], [Kalman et al., 1961]. The plant used for this purpose is the one represented by equations (5) and (6).

Kalman estimator is used to estimate, on the basis of noisy measurements, the values of the state variables of a system subject to stochastic input-output disturbances. The input disturbances are included in the state-space model by adding the noise input vector  $v$  to the exogenous input vector  $u$ . Moreover, to include measurement noise, the vector  $w$  is added to the output of the system. Such noise signals are usually part of the actual mode of the system.

Kalman estimator takes on a particularly simple structure closely resembling the original system [Kalman, 1960], [Kalman et al., 1961]. The state-space model of the Kalman estimator is expressed by the following equation,

$$\dot{\hat{y}} = A\hat{y} + L[\tilde{y} - C\hat{y}] + Bu \quad (7)$$

and,

$$L = S_0 C^T R^{-1} \quad (8)$$

where  $\hat{y}$  is the vector of the state estimates,  $\tilde{y}$  is the vector of measured states from the actual system,  $L$  is the Kalman matrix of gains, and  $S_0$  is the steady state solution of the following matrix *Riccati* differential equation [Kalman, 1960], [Kalman et al., 1961], [Hassibi et al., 1999].

$$\dot{S} = AS + S A^T - SC^T R^{-1} CS + B Q B^T \quad (9)$$

Matrices  $R$  and  $Q$  are the symmetric, nonnegative matrices that minimizes the following performance index  $J$ :

$$J = \int_0^{\infty} (x^T BQB^T x + u^T Ru) dt \tag{10}$$

It is worth mentioning that knowledge of the absolute magnitude of  $R$ , and  $Q$  is not important, rather, only their relative magnitude is important [Kalman, 1960], [Hassibi et al., 1999], [Gawronski et al., 1991].

Now, with the linear time-invariant model of the system available, namely equations (5) and (6), along with Kalman matrix of gains ( $L$ ), optimal estimates of the plant states can be obtained according to equation (7).

It is worth mentioning that the controllability and observability of the estimator are found by obtaining the rank of their corresponding gramians. Full rank gramians indicate a controllable and observable system [El-Sinawi, 1999], [Hassibi et al., 1999], [Gawronski et al., 1991]. The proposed estimation scheme is shown in Figure 3.

#### 4. NON-LINEAR MODEL OF THE BEAM-HUB SYSTEM (PLANT)

A mathematical model for a flexible beam undergoing large planar flexural deformations, continuously rotating under the effect of hub torque and mounted on a compliant hub has been developed by Al-Bedoor *et al* [Al-Bedoor et al., 2001]. Lagrangian dynamics in conjunction with the assumed modes method were utilized to derive, directly, the non-linear equivalent temporal equations of motion. The following four coupled non-linear ordinary differential equations represent the generated non-linear mathematical model:

$$\begin{aligned} (1 + \mu)\ddot{X} + \frac{1}{2}[(1 + 2C + b_3q^2)\sin\theta - 2b_1q\cos\theta]\ddot{\theta} \\ + \frac{1}{2}[(1 + 2C + b_3q^2)\cos\theta + 2b_1q\sin\theta]\dot{\theta}^2 \\ + 2[b_3q\sin\theta - b_1\cos\theta]\dot{q}\dot{\theta} - (b_3\cos\theta)\dot{q}^2 \\ - [b_1\sin\theta + b_3q\cos\theta]\ddot{q} + \beta s_1^2 X = F_x \end{aligned} \tag{11}$$

$$\begin{aligned} (1 + \mu)\ddot{Y} + \frac{1}{2}[(1 + 2C - b_3q^2)\cos\theta - 2b_1q\sin\theta]\ddot{\theta} \\ - \frac{1}{2}[(1 + 2C - b_3q^2)\sin\theta + 2b_1q\cos\theta]\dot{\theta}^2 \\ - 2[b_3q\cos\theta + b_1\sin\theta]\dot{q}\dot{\theta} - (b_3\sin\theta)\dot{q}^2 \\ + [b_1\cos\theta - b_3q\sin\theta]\ddot{q} + \beta s_2^2 Y = F_y \end{aligned} \tag{12}$$

$$\begin{aligned}
& [b_o + b_{12}q_i^2]\ddot{\theta} + 2b_{12}q\dot{q}\dot{\theta} + \frac{1}{2}[(1+2C + b_3q^2)\sin\theta - 2b_1q\cos\theta]\ddot{X} \\
& + \frac{1}{2}[(1+2C - b_3q^2)\cos\theta - 2b_1q\sin\theta]\ddot{Y} \\
& + \frac{1}{2}[b_{14} + b_7q^2]\ddot{q} + b_7q\dot{q}^2 = \frac{T}{m_B l^2}
\end{aligned} \tag{13}$$

$$\begin{aligned}
& (b_2 + b_8q^2)\ddot{q} + \frac{1}{2}(b_{14} + b_7q^2)\ddot{\theta} + b_8q\dot{q}^2 \\
& - (b_1\sin\theta + b_3q\cos\theta)\ddot{X} \\
& + (b_1\cos\theta - b_3q\sin\theta)\ddot{Y} \\
& - (2b_{13}q^3 + b_{12}q)\dot{\theta}^2 + \beta^2 b_9q + 2\beta^2 b_1q^3 = 0
\end{aligned} \tag{14}$$

where  $X$ , and  $Y$  are the hub horizontal and vertical deflections, respectively.  $\theta$  is the rigid body rotation and  $q$  as the beam  $i$ -th modal degree of freedom. The base flexibility is defined as a ratio to the beam flexibility using the parameters  $s_1 = K_x l^3 / EI$  and  $s_2 = K_y l^3 / EI$ . For more information on the non-linear model see [Al-Bedoor et al. 2001].

## 5. NUMERICAL SIMULATION AND RESULTS

The proposed technique is used to estimate the tip deflection of a beam mounted on a compliant hub and rotating at a constant angular speed of 2400 rpm and subject to external torque. Dimensions and material properties of the beam hub system used in the non-linear model are given in Table 1. This model, as we have mentioned before, is assumed to be a representation of the actual model. Of course this model is needed to generate what is assumed to be the measurable states of the actual system, i.e., the vertical hub deflection as shown in Figure 3.

States of the system are generated by numerically integrating equations (11) through (14) using Matlab®.

The reduced order linear elastodynamic model of the beam-hub system, i.e., (plant) needed for the construction of the Linear Quadratic Estimator (LQE) and the corresponding Kalman gains matrix ( $L$ ) is generated from the eigenvalues and eigenfunctions of the finite element model according to equations (5) and (6). The finite element model generates the natural frequencies and mode shapes from the dynamic solution of a solid model of the plant discretized by 162- 8 node shell elements each has a thickness equal to the beam-hub depth.

The linear state-space model of the plant is then constructed from the first four modes of vibration predicted by the FEA method. The natural frequencies of the first four modes are listed in Table 2. Figure 4 shows the first mode of vibration of the beam-hub system as predicted by finite element analysis.

Modal damping of 1% has been added to the system (assuming natural material damping). Thus the damping matrix  $C$  is constructed based on the initial assumption that  $C$  is proportional to the mass and stiffness matrices namely,  $M$  and  $K$ .

**Table 1: Simulation data for the beam-hub system**

Item	Value
<i>Beam Length</i>	<i>3 m</i>
<i>Beam's mass/unit length</i>	<i>6.44 kg/m</i>
<i>Beam's flexural rigidity</i>	<i>3614</i>
<i>Hub radius</i>	<i>0.2 m</i>
<i>Hub mass</i>	<i>50 Kg</i>
<i>Hub stiffness</i>	$K_x = K_y = 1e6 N/m$
<i>Torque Applied</i>	<i>40 N.m @ 5 Hz square wave</i> <i>1 N.m @ 20 Hz sine wave</i>

**Table 2: Modal frequencies generated by FEA**

Mode	Modal Frequency (Hz)
<i>1<sup>st</sup> mode</i>	<i>6.1183</i>
<i>2<sup>nd</sup> mode</i>	<i>17.191</i>
<i>3<sup>rd</sup> mode</i>	<i>34.745</i>
<i>4<sup>th</sup> mode</i>	<i>61.704</i>

With the matrices  $A$ ,  $B$ , and  $C$  are now available from the finite element model, the kalman matrix of gains ( $L$ ) is obtained by solving equations (8) through (10), and the Linear Quadratic Estimator is implemented to estimate the tip deflection of the beam for the first four modes of vibration.

Simulation of the proposed technique is carried out using a 5 Hz square wave torque with amplitude of 40 N.m added to it  $\pm 10$  N.m random torque as shown in Figure 4. The purpose of adding a random torque profile is to simulate process noise. Both excitations are shown in Figures 5 and six, respectively.

A random signal was added to the virtual measurements of the vertical hub deflection (output of the non-linear model) to simulate measurements noise. Estimates of the transverse beam tip deflection are obtained by plotting the corresponding state estimate generated by the LQE.

Figures 5, 6, 7 and 8 show the estimated beam tip deflection compared to the assumed tip deflection (tip deflection predicted by the non-linear model) for the first four modes of vibration when the plant is excited by the 5 Hz square wave torque.

## 6. CONCLUSIONS

A new approach for estimating beam tip deflection via measurements of hub deflection using a Linear Quadratic Estimator for a rotating beam mounted on a compliant hub is presented. A readily available non-linear model is used to mimic the actual vibration of the beam-hub system. Approximate reduced order linear model of the beam-hub system is obtained using finite element analysis.

A linear quadratic estimator is formulated based on the reduced order linear model dynamics and the optimal Kalman matrix of gains is determined from the steady state solution of a matrix Riccati differential equation.

With a fully functional estimator available, measurements of the vertical hub deflection is compared to the estimated hub deflection and the difference (error) is multiplied by the corresponding Kalman gain and fed back to the estimator as a second input. The output of the estimator is a vector of two states, namely, hub deflection and beam tip deflection.

Estimated beam tip deflection is plotted and compared to the tip deflection produced by the non-linear model; see Figures 7 through 14. Simulation results show that the proposed approach is capable of producing fairly accurate estimates of the tip deflection for most modes. However, over estimates of the tip deflection for some modes have persisted especially for the second mode (Figures 8 and 12). The authors have not yet been able to justify such discrepancy.

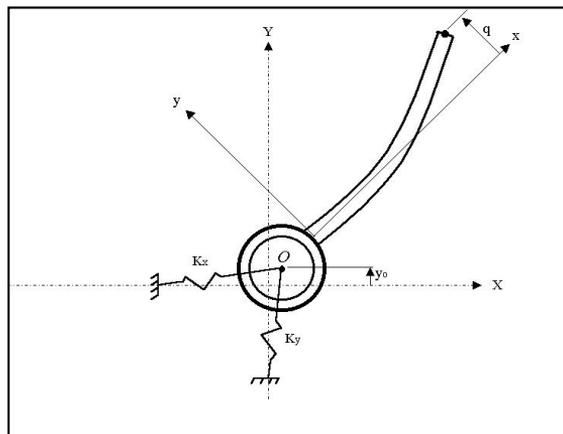
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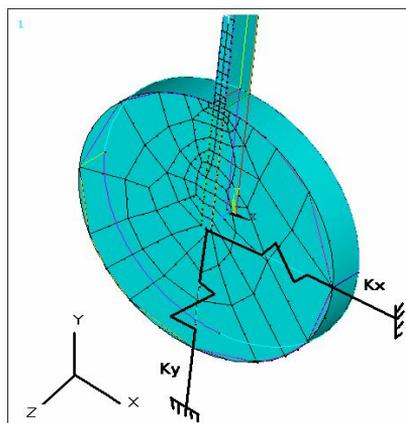
## REFERENCES

1. Al-Bedoor, A. El-Sinawi, And M.N. Hamdan, 2002, "Non-linear dynamic model of an inextensible rotating flexible beam," *Journal of sound and vibration*, 251(5), 767-781.
2. Bazoune, A. ,Y. A. Khulief, N. G. Stephen, And M. A. Mohiuddin, 2001, "Dynamic response of spinning tapered Timoshenko beams using modal reduction," *Finite Element in Analysis and Design* 37, 199-219.
3. Blass, M. J., 1978, "Feedback control of flexible systems," *IEEE Transactions on Automatic Control*, AC-23, 673-679.
4. El-Sinawi, A. And A. R. Kashani, 1999, "Active Control of Engine Mounts," *Society of Automotive Engineers*, paper no. 1999-01-1845.
5. El-Sinawi, A. And A. R. Kashani, 1999, "Active vibration isolation of a plate" *proceedings of the International Mechanical Engineering Conference and Exposition, Anaheim, CA*.
6. El-Sinawi, A. , 1999, *Vibration Suppression in Machining Using a Kalman Estimator-based Controller*. Ph. D. Dissertation, University of Dayton, Dayton, OH.
7. El-Sinawi, A. , B. Al-Bedoor And M.N. Hamdan, July 2001, " Non-linear dynamic model of an inextensible rotating flexible beam supported on flexible base" *Proceedings of the ASME PVP conference in Atlanta, Georgia*.
8. Gawronski, W. And T. Williams, 1991, "Modal reduction for flexible space structures," *Journal of Guidance*, 4(10), 68-76.
9. Hassibi, B., A. H. Sayed, And T. Kailath, 1999, *Indefinite-quadratic estimation and control*. SIAM Publications.
10. Hu, F. L. And A. G. Ulsoy, 1994, "Dynamic modeling of constrained flexible robot arm for controller design," *Journal of dynamic systems meas. And control* 116, 56-65.
11. Kalman, R. E., And R. S. Bucy, 1961, New results in linear filtering and prediction theory," *Journal of Basic Engineering*, 2, 95-101.
12. Kalman, R. E., 1960, "A new approach to linear filtering and prediction problems," *Journal of Basic Engineering*, 2, 35-45.
13. Kane, T. R., R. R. Rayan, And A. K. Banerjee, 1987, "Dynamics of a cantilever beam attached to a moving base," *Journal of Guidance* 10( 2), 139-150.
14. Khulief, Y. A., 2000, "Vibration suppression in rotating beams using active modal control," *Journal of sound and vibration*, 242(2), 681-699.
15. Lee And, Y. S., S. J. Elliott, 2001, "Active position control of flexible smart beam using internal model control," *Journal of Sound and Vibration*, 242(5), 767-791.
16. Lin, S.-C ,2000, "Sensitivity of the dynamic behavior of random rotating Timoshenko beams to system parameter change," *ImechE*. 214 Part G, 247-259.
17. Mallory, W. And D. W. Miller, 2000, "Decentralized state estimation for flexible space structures," *Journal of Guidance, Control, and Dynamics*, 23(4), 665-672.
18. Mitchell, T. P. And J. C. Bruch Jr., 1988, "Free vibration of a flexible arm attached to a compliant finite hub," *Journal of Vibration, Acoustics, Stress, and Reliability Design* 110, 18-120.

19. Swaminadham, M. And M. Chinta, 1994, "Vibration of a turbo machine blade with flexible root," *Journal of Sound and Vibration* 174(2), 284-288.
20. Yousefi-Koma, A., And G. Vukovich, 2000, "Vibration suppression of flexible beams with bonded piezo transducers using wave-absorbing controllers," *Journal of Guidance, Control, and Dynamics*, 23(2), 347-354.
21. Zheng, T. And N. Hasebe, 2000, "Nonlinear dynamic behavior of a complex rotor-bearing system," *Journal of Applied mechanics* 57, 485-495.
22. Zhu, W. D. And C. D. Mote Jr., 1997, "Dynamic modeling and optimal control of rotating Euler-Bernolli beams," *Journal of dynamic systems meas. and control* 119, 802-808.
23. Zimmerman, D. C. And H. H. Cudney 1989, *Journal of Vibration, Acoustics, Stress, and Reliability in Design*, 111, 283-289. Practical implementation issues for active control of large flexible structures.



**Figure 1: Deflected rotating flexible beam mounted on a compliant hub**



**Figure 2: Finite element model of the plant**

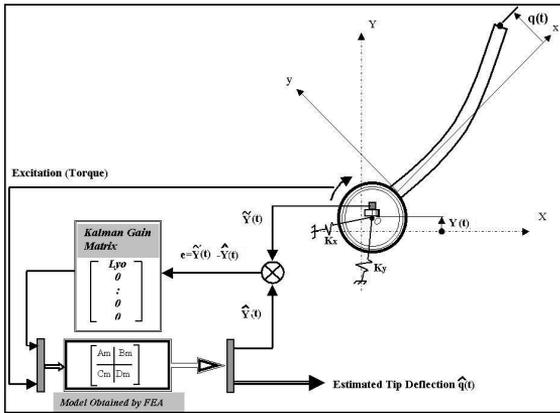


Figure 3: Schematic of the proposed LQE approach for estimating beam tip deflection

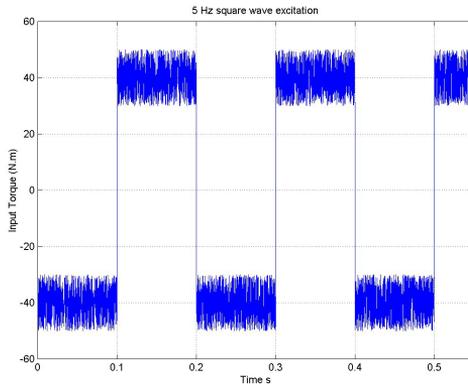


Figure 4: 5 Hz square wave torque profile with added random torque

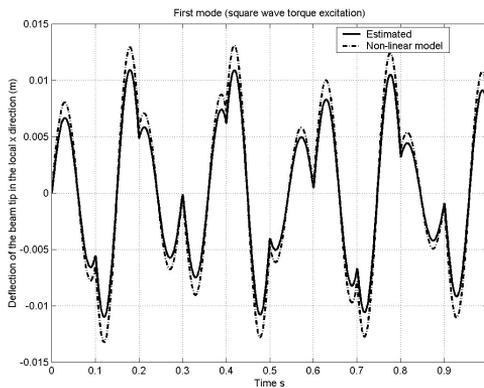


Figure 5: Estimated vs. non-linear beam tip deflection (first mode)

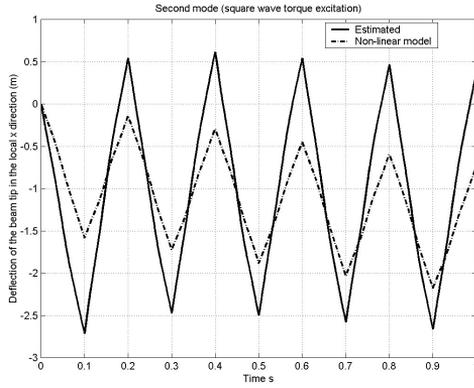


Figure 6: Estimated vs. non-linear beam tip deflection (second mode)

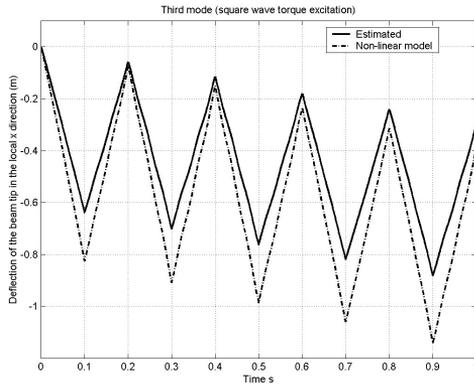


Figure 7: Estimated vs. non-linear beam tip deflection (Third mode)

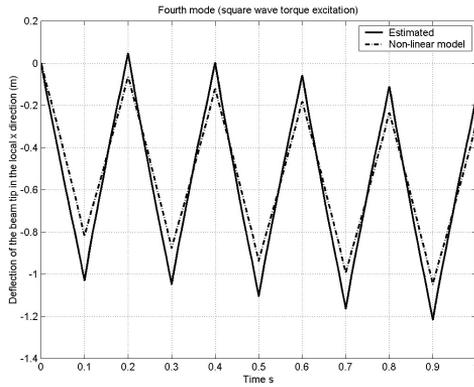


Figure 8: Estimated vs. non-linear beam tip deflection (Fourth mode)