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Error Bounds in Approximating Time-Delay Systems

Samir H. Al-Amer and Fouad M. AL-Sunni*
Systems Engineering Department
King Fahd University of Petroleum and Mineral.
Dhahran, Saudi Arabia
Email: samir@ccse.kfupm.edu.sa alsunni@ccse.kfupm.edu.sa

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Abstract

In this paper, bounds on approximating time-delay systems are proposed. Bounds on the infinity norm of the weighted error are obtained when the approximating function is a general rational function, all-pass function, Pade' and Laguerre approximations. An example is presented in illustration.

1. Introduction

A transfer function of the form

$$G(s) = P(s)e^{-sd} \tag{1.1}$$

may be used to represent many dynamical systems. In (1.1), $P(s)$ represents the dynamics of the system, and e^{-sd} represents the input delay.

In many situations it is desirable to approximate the infinite-dimensional time-delay systems by finite-dimensional rational transfer functions. Several techniques are available for approximating time-delay systems. Padé approximation has been widely used to approximate e^{-sd}(See, for example, [1]). Formulas for Padé approximation of any desirable order is available together with error bounds in the L_2 and L_∞ sense [1]. Hankel approximation of $P(s)e^{-sd}$ with stable and strictly proper $P(s)$ was obtained in [3]. In this case, better approximation was obtained but with a much larger computation burden. Methods based on truncation of Fourier-Laguerre series were developed in [4] and [7]. The methods are extended to a larger class of time-delay systems and are computationally efficient. However, the resulted finite-dimensional approximation may be of considerably high order and further model reduction is needed. Yoon and Lee [9], obtained rational approximation of e^{-sd} and $P(s)e^{-sd}$ based on truncated Blascke product together with L_2 and L_∞-norm bounds.

The error bounds are valuable in assessing the quality of the approximation. They can be used as a guide in the selection of the order of the approximating function. Several bounds are available [1], [9], [10]. However, most of the bounds are conservative.

In this paper, bounds on the infinity-norm of the weighted approximation error are derived for different types of approximating functions: the general rational functions, the all-pass functions, the Padé and Laguerre approximations.

In the following section, the statement of the problem is presented. Some preliminary results are provided in Section 3. The main results are reported in section 4. An illustrative example is presented in Section 5 and our conclusion is given in Section 6.

2. Problem Statement

Our objective is to obtain bounds on the error introduced in approximating time-delay systems. The system under study is assumed to be a single-input single

*Corresponding Author

output system described by (1.1) where $P(s)$ is assumed to be rational, stable and strictly proper. The approximate of $G(s)$ is given by $P(s)G_r(s)$ where $G_r(s)$ is an r th order rational approximation of e^{-sd} obtained such that

$$\epsilon = \left\| \left(G_r(sd) - e^{-sd}\right) W(s) \right\|_\infty \tag{2.1}$$

is reasonably small. The weighting function $W(s)$ is assumed to have the following form

$$W(s) = M\left(\frac{1}{1+\tau s}\right)^k \tag{2.2}$$

where $k \geq 1$. The parameters k, M and τ are selected such that

$$|P(j\omega)| \leq \frac{M}{|1+j\omega\tau|^k} \qquad \forall\omega$$

Re-naming the variables and re-arranging, one can easily show that the error, ϵ, given by (2.1) is equivalent to

$$\epsilon = \left\| \left(G_r(s) - e^{-s}\right) W(\frac{s}{d}) \right\|_\infty \tag{2.3}$$

The approximating function G_r is assumed to be in one of the following sets:

1. General r th order transfer functions of the form

$$G_r(s) = \frac{\sum_{i=0}^{r} b_i\, s^i}{1 + \sum_{i=1}^{r} a_i\, s^i} \tag{2.4}$$

2. All-pass r th order transfer functions of the form

$$G_r(s) = \frac{1 + \sum_{i=1}^{r} a_i(-s)^i}{1 + \sum_{i=1}^{r} a_i s^i} \tag{2.5}$$

3. An r th order Padé Approximations with a transfer function[1]

$$G_r(s) = \frac{1 + \sum_{i=1}^{r} a_i(-s)^i}{1 + \sum_{i=1}^{r} a_i s^i} \tag{2.6}$$

where a_i are given by

$$a_i = \frac{(2r-i)!r!}{(2r)!i!(r-i)!}$$

4. An r th order Laguerre approximation ($r = 2n$) [7]

$$G_r(s) = \left[\frac{1 - \frac{s}{2n}}{1 + \frac{s}{2n}}\right]^n \tag{2.7}$$

Bounds on the error when the approximating function belongs to one of the above classes will be derived.

3. Preliminary Results

In this section several preliminary results are presented. The results presented here will prove useful in deriving the main results of the following section.

Lemma 1. *Let $G_r(s)$ be a transfer function whose frequency response is $G_r(j\omega) = |G_r(j\omega)|\, e^{j\Phi(\omega)}$ then*

$$\left|e^{-j\omega} - G_r(j\omega)\right| =$$

$$\sqrt{1 + |G_r(j\omega)|^2 - 2\,|G_r(j\omega)| \cos(\Phi(\omega)+\omega)} \tag{3.1}$$

$$= \sqrt{\left(|G_r(j\omega)| - \cos(\Phi(\omega)+\omega)\right)^2 + \sin^2(\Phi(\omega)+\omega)} \tag{3.2}$$

and if $G_r(s)$ is an all-pass transfer function then

$$\left|e^{-j\omega} - G_r(j\omega)\right| =$$

$$\sqrt{\left(1 - \cos(\Phi(\omega)+\omega)\right)^2 + \sin^2(\Phi(\omega)+\omega)} \tag{3.3}$$

$$= \sqrt{2 - 2\cos(\Phi(\omega)+\omega)} \tag{3.4}$$

Proof:

Using Euler identity for $e^{-j\omega}$ and $G_r(j\omega)$ and manipulating the expression will result in (3.1). Equation (3.2) is obtained by adding and subtracting $\cos(\Phi(\omega)+\omega)$ and simplifying the resulted expression. The expression of the error for the all-pass case are obtained by direct substitution of $|G_r(j\omega)| = 1$ in (3.1) and (3.2).

The following results can be easily derived from Lemma 1.

Corollary 1. *If a transfer function $G_r(s)$ with a phase angle $\Phi(\omega)$, and if $\Phi(\omega_1) + \omega_1 \geq \frac{\pi}{2}$ then there exists $\omega_0 \leq \omega_1$ such that $\left|e^{-j\omega_0} - G_r(j\omega_0)\right| \geq 1$*

Proof: *If $\Phi(\omega_0) + \omega_0 = \frac{\pi}{2}$, then $\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = \sqrt{\left|G_r(j\omega_0)\right|^2 + 1} \geq 1$.*

Corollary 2. *If $G_r(s)$ is an all-pass transfer function with a phase angle $\Phi(\omega)$, and if $\Phi(\omega_1) + \omega_1 \geq \pi$, then there exists $\omega_0 \leq \omega_1$ such that $\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = 2$.*

Proof: *When $\Phi(\omega_0) + \omega_0 = \pi$ then $\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = \sqrt{2 - 2\cos(\pi)} = 2$.*

The following is an important result from linear control theory.

Fact 1: The phase angle $\Phi(\omega)$ of an r^{th} order transfer function satisfies the following bound for all ω

$$\Phi(\omega) \geq \begin{cases} -r\pi & \text{if } G_r \text{ is non-minimum phase of the forms given in (2.4)-(2.7)} \\ -\frac{r\pi}{2} & \text{if } G_r \text{ is minimum phase of the form given in (2.4)} \end{cases}$$

4. Main Results

In this section, bounds on the weighted approximation error will be derived. We start by presenting the lower bounds.

4.1. Error Bounds.

Theorem 4.1. *Let e^{-sd} be approximated by an r^{th} order transfer function $G_r(s)$ then*

$$\left\| \frac{\left(e^{-sd} - G_r\right) M}{(1+\tau s)^k} \right\|_\infty \geq M\ \gamma \left[1 + \frac{\tau^2}{d^2\beta}\left(r\pi + \frac{\pi}{\alpha}\right)^2\right]^{-\frac{k}{2}}$$

where

1. $\alpha = 2,\ \beta = 1,\ \gamma = 1$ *for non-minimum phase approximation*
2. $\alpha = 1,\ \beta = 1,\ \gamma = 2$ *for all-pass approximation*
3. $\alpha = 1,\ \beta = 4,\ \gamma = 1$ *for minimum phase approximation*

Proof: *For part 1, Fact 1 implies that $\Phi(\omega) \geq -r\pi$, and therefore $\Phi(\omega_1) + \omega_1 \geq \frac{\pi}{2}$ for all $\omega_1 \geq r\pi + \frac{\pi}{2}$. Using Corollary 1, there exists an $\omega_0 \leq r\pi + \frac{\pi}{2}$ such that*

$$\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = 1.$$

Using the fact that $\frac{M}{\left(1+j\omega\frac{\tau}{d}\right)^k}$ is monotonically decreasing with respect to ω, we have

$$\left|\frac{\left(e^{-j\omega_1} - G_r(j\omega_0)\right) M}{\left(1 + j\omega_0\frac{\tau}{d}\right)^k}\right| \geq \frac{M}{\left[\sqrt{1 + \left(r\pi + \frac{\pi}{2}\right)^2 \left(\frac{\tau}{d}\right)^2}\right]^k}$$

To prove Part 2, Corollary 2 is used to show that there exist $\omega_0 \leq r\pi + \pi$ such that

$$\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = 2,$$

and the rest follows the proof of Part 1. For the third part, Fact 1 and Corollary 1 are used to show that there exist $\omega_0 \leq \frac{r\pi}{2} + \frac{\pi}{2}$ such that

$$\left|e^{-j\omega_0} - G_r(j\omega_0)\right| = 1.$$

The remaining part of the proof follows that of Part 1.

An upper bound may be obtained by replacing the original weight in (3) by

$$M\left|\frac{1}{\tau s}\right|^k$$

Note that

$$M\left|\frac{1}{1 + j\omega\tau}\right|^k \leq M\left|\frac{1}{j\omega\tau}\right|^k$$

We now present the following corollary.

Corollary 3. *For all pass systems,*

$$\left\|\left(e^{-sd} - G_r(sd)\right)\frac{M}{(1+\tau s)^k}\right\|_\infty$$

$$\leq \max_\omega \frac{\sqrt{2 - 2\cos(\omega + \Phi(\omega))}M}{(\omega)^k}\left(\frac{d}{\tau}\right)^k$$

Proof:

$$\left|\left(e^{-j\omega} - G_r(j\omega)\right) \frac{M}{\left(1 + j\frac{\tau}{d}\omega\right)^k}\right|$$
$$= \sqrt{2 - 2\cos(\omega + \Phi(\omega))}M \left|\frac{1}{1 + j\omega\frac{\tau}{d}}\right|^k$$
$$\leq \sqrt{2 - 2\cos(\omega + \Phi(\omega))}M \left|\frac{1}{j\omega\frac{\tau}{d}}\right|^k$$

Now, taking the maximum of the right hand side gives the above result.

4.2. Error Bounds for Padé and Laguerre Approximations

The structure of the Padé and Laguerre approximations are well known and therefore we expect that tighter bounds may be obtained. Consider an r^{th} order all-pass transfer function $G_r(s)$ with a phase angle $\Phi(\omega)$. The following inequality is valid for all θ,

$$\left\|\frac{\left(e^{-sd} - G_r(sd)\right) M}{(1 + \tau s)^k}\right\|_\infty \geq M \sqrt{\frac{2 - 2\cos(\theta)}{\left(1 + (\omega_r^*)^2 \left(\frac{\tau}{d}\right)^2\right)^k}} \tag{4.1}$$

where ω_r^* is the smallest frequency that satisfies

$$\omega_r^* + \Phi(\omega_r^*) = \theta$$

Note that ω_r^* depends on the family of the approximating function and on the selected order. Determining the largest lower bound may not be easy. However in the following subsection we give an approximate behavior of the right hand side of (4.1).

Lemma 2. *If $G_r(s)$ is an r^{th} order Laguerre approximation of e^{-s}, then*

$$\left|\left(e^{-j\omega d} - G_r(j\omega)\right) \frac{M}{(1 + j\omega\tau)^k}\right| \leq M \ \frac{\min(2, \frac{\omega^3}{12r^2})}{(1 + \omega^2\tau^2)^{k/2}}$$

Proof: *The phase angle of $G_r(s)$ is*

$$\Phi(\omega) = -2r \arctan(\frac{\omega}{2r})$$

The Taylor series expansion of $\Phi(\omega) + \omega$ is given by

$$\Phi(\omega) + \omega = 2r \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)} \left(\frac{\omega}{2r}\right)^{2k+1}$$

Note than if $\frac{\omega}{2r} < 1$ then the alternating series converges and

$$|\Phi(\omega) + \omega| \leq \frac{\omega^3}{12r^2}$$

Using the standard inequality

$$\left|1 - e^{j\theta}\right| \leq |\theta|$$

gives us the following bound

$$\left|e^{-j\omega} - G_r(j\omega)\right| \leq \frac{\omega^3}{12r^2} \qquad if \quad \omega < 2r$$

4.3. Approximation of the weighted Error

Since the formulas of the Padé and Laguerre approximations are known, one can easily obtain tighter upper and lower bounds using curve fitting algorithms. In this section, curve fitting will be used to find approximate behavior of (4.1), and approximations of the infinity-norm of the weighted approximation error for both Padé and Laguerre.

From experimenting with many examples, it is observed that the largest bound in (4.1) is obtained when $\theta \approx \pi$. Fixing $\theta = \pi$, the corresponding ω_r^* can be calculated. These values are given in Table 1.

Table 1: The smallest frequencies at which the un-weighted error is 2, $M = k = 1$

r	ω_r^*
1	5.595
2	7.917
3	10.175
4	12.393
5	14.585
6	16.757
7	18.193
8	21.057
9	23.191
10	25.317

The frequency ω_r^* can be accurately represented by a second order polynomial

$$\omega_r^* = -0.0047r^2 + 2.2297r + 3.4928$$

A lower bound on the approximation error is given by

$$\left\| \left(e^{-sd} - G_r\right) \frac{M}{(1+\tau s)^k} \right\|_\infty \geq \frac{2}{\sqrt{\left(1 + (-0.0047r^2 + 2.2297r + 3.4928)^2 \left(\frac{\tau}{d}\right)^2\right)^k}} \quad (4.2)$$

Similarly, a lower bound on Laguerre approximation can be obtained by finding ω_r^*.

Table 2: The smallest frequencies at which the un-weighted error is 2.

r	ω_r^*
1	5.597
2	7.455
3	9.056
4	10.499
5	11.834
6	13.086
7	14.272
8	15.405
9	16.493
10	17.542

A good approximation of ω_r^* is given by

$$\omega_r^* = (6.6717r + 7.0805)^{2/3}$$

which give rise to the following bound

$$\left\| \left(e^{-sd} - G_r\right) \frac{M}{(1+\tau s)^k} \right\|_\infty \geq \frac{2}{\sqrt{\left(1 + (6.6717r + 7.0805)^{4/3} \left(\frac{\tau}{d}\right)^2\right)^k}}$$

The $\infty-$ norm of the weighted error for $W(s) = \frac{1}{(1+\tau s)^2}$ and $G_r(sd)$ can be generated for a range of values of τ, d and r. However a plot of the exact error for the range $r \in [1, 20]$ and $\frac{\tau}{d} \in [0, 5]$ suggests that the behavior can be described as follows. For the rth order approximation, the weighted infinity-norm error may be fitted as

$$\left\| \frac{(G_r(sd) - e^{-sd})}{(1+\tau s)^2} \right\|_\infty \approx \frac{1}{\left|(a_2r^2 + a_1r + a_0)\frac{\tau}{d} + (b_2r^2 + b_1r + b_0)\right|^2} \quad (4.3)$$

where

$$a_2 = 0.0062, a_1 = -0.1010,\ a_0 = 0.4848,$$
$$b_2 = -0.0197, b_1 = 1.8359, b_0 = 1.0637$$

for the rth order Pade approximation and

$$a_2 = 0.0041, a_1 = -0.0729,\ a_0 = 0.4764,$$
$$b_2 = -0.0351, b_1 = 1.2894, b_0 = 1.6376$$

for the rth order Laguerre approximation. The deviation from the true infinity-norm of the above formulas is less than 0.01 over the range $\frac{\tau}{d} \leq 5$ and $r < 20$.

5. An Illustrative Example

It is required to use Padé approximation to obtain rational approximations of $\frac{e^{-s}}{(1+s)^2}$. The rth order Padé approximation $G_r(s)$ is obtained so that the error

$$\left\| \left(e^{-s} - G_r(s)\right) \frac{1}{(1+s)^2} \right\|_\infty$$

is reasonably small. Lower bounds for Padé' approximation of order 1 to 10 are obtained and are given in Table 3. The actual error is given in column 2. The lower bounds given by Theorem 1 are listed in column 3. Column 4 shows the bounds given by (4.2). The Last column represents the approximate values of the weighted infinity-norm error provided by (4.3).

Table 3: Actual error and different error bounds

Order r	Actual Error	Lower Bound 1	Lower Bound 2	predicted Error
1	0.0989	0.0862	0.0594	0.0935
2	0.0403	0.0319	0.0313	0.0406
3	0.0225	0.0164	0.0193	0.0227
4	0.0146	0.0100	0.0131	0.0146
5	0.0103	0.0067	0.0094	0.0102
6	0.0076	0.0048	0.0071	0.0076
7	0.0059	0.0036	0.0056	0.0059
8	0.0047	0.0028	0.0045	0.0047
9	0.0039	0.0022	0.0037	0.0039
10	0.0032	0.0018	0.0031	0.0033

6. Conclusions

In this paper, bounds on approximation of time-delay systems are obtained. Lower bounds on the error when the approximating function is an r^{th} order minimal phase, non-minimal phase, all pass, Padé and Laguerre transfer functions are given. Fitting formulas for the weighted infinity-norm error of Padé and Laguerre approximations are also obtained.

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