New efficient frequency domain algorithm for H_∞ approximation with applications to controller reduction

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Abstract: New frequency domain computational schemes for the weighted and unweighted H_∞ norm system approximation problems are introduced. The schemes are applicable in both continuous and discrete-time cases. The new algorithm is used to obtain reduced order controllers for a well known control problem.

1 Introduction

The optimal H_∞ norm approximation problem, which is known as complex rational Chebychev approximation problem among mathematicians, is defined as follows.

Definition 1: Given a transfer function $G(z) \in RH_\infty$ with degree n, find a transfer function $G_r(z)$ with degree r such that $\|G - G_r\|_\infty$ is minimised, where $\|G\|_\infty := \sup_\omega |G(e^{j\omega})|$. In the weighted case, one has $\|W(G - G_r)\|_\infty$ for a given weight $W(z)$.

It is known that there exists an optimal solution to this problem, and that the solution is not necessarily unique (see [1] and references therein). There is no known general algorithm to obtain the optimal solution. The characterisation of the solution in the frequency domain is also non-trivial. Only some sufficient conditions are available.

A state-space characterisation of the optimal solution for the H_∞ model reduction problem was given in [2]. However, this characterisation does not lead to a computable algorithm. Suboptimal implementations of the algorithm are reported in [3, 4].

Computation of the optimal solution in general is an open problem. In [5, 6], a computational algorithm is proposed to obtain local best approximations. In [7], a new H_∞ norm approximation technique was introduced. The technique is based on a series of identification steps, where the excitation is being updated.

For the H_∞ norm model reduction problem there are several well known suboptimal computational techniques, such as balanced model reduction [8], and Hankel norm model reduction [9]. For the weighted case the situation is more difficult. The most commonly used technique is weighted balanced model reduction [10]. There are known H_∞ norm bounds for balanced and Hankel norm approximation schemes that give an idea about the level of optimality of the solution for the model reduction problem.

IEE Proceedings online no. 20010298

DOI: 10.1049/ip-cta:20010298

Paper first received 24th October 1997 and in revised form 30th June 1998

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For the weighted case there is no known norm bound for the weighted balanced approximation; as a matter of fact, the approximant is not even guaranteed to be stable in some cases. Recently, Zhou [11] introduced a solution for the weighted Hankel norm approximation problem, which has good H_∞ norm error bounds for certain types of weights.

In this paper we develop frequency domain generalisations of the scheme introduced in [7]. The scheme is applicable to both continuous and discrete-time problems without resorting to the bilinear transformation.

2 Preliminaries

2.1 Characterisation problem

In this Section, known results on characterisation of the optimal solution for the H_∞ norm approximation problem are summarised. An alternative way of stating the H_∞ norm approximation problem is as follows [12]: Find the best $G_r(z)$ so that the error function $E(e^{j\omega}) = G(e^{j\omega}) - G_r(e^{j\omega})$ is contained inside a disc around 0 having minimal radius.

If $|E(e^{j\omega})| = \text{constant}\ \forall \omega \in [0, 2\pi]$, the error curve is said to be circular. *A priori* not much is known about the problem, except that the error curve should be enclosed inside a circle with minimum radius and it must touch the circle on at least $r+2$ points. The experience shows that for most cases the optimal error curve has winding number $2r+1$ and is near circular. The following theorem relates near circularity to near optimality [12].

Theorem 1: Suppose the error curve for a $G_r(z)$ has winding number at least $2r+1$ about the origin. Then

$$\min_\omega |G(e^{j\omega}) - G_r(e^{j\omega})| \le |G(e^{j\omega}) - G_r^*(e^{j\omega})| \le \|G - G_r\|_\infty$$

where $G_r^*(z)$ is an optimal approximant.

An immediate corollary of this theorem is that if for such a $G_r(z)$ the upper and lower bounds are equal, i.e., the error is perfectly circular, then $G_r(z)$ is optimal.

Definition 2: Given $G(z) \in RH_\infty$ with a minimal realisation

$$G(z) = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right],$$

define the controllability gramian, $\mathbf{L_c}$, and the observability gramian, $\mathbf{L_o}$, to be the solution of the following Lyapunov equations:

$$\mathbf{L_c} - \mathbf{A}\mathbf{L_c}\mathbf{A}^T = \mathbf{B}\mathbf{B}^T, \qquad \mathbf{L_o} - \mathbf{A}^T\mathbf{L_o}\mathbf{A} = \mathbf{C}^T\mathbf{C}$$

then $\sigma_i = [\lambda_i(\mathbf{L_c}\mathbf{L_o})]^{1/2}$ (in the decreasing order) are called the Hankel singular values (HSV) of the system $G(z)$.

The following lemma is well known.

Lemma 1: Given an nth order transfer function $G(z)$, let $G_r(z)$ be an rth order transfer function. We then have $\sigma_{r+1} \leq \|G - G_r\|_\infty$, where σ_{r+1} is the $(r+1)$th HSV of $G(z)$.

Remark 1: Lemma 1 provides a lower bound for the H_∞ norm approximation. Unlike Hankel norm model reduction, the lower bound is not necessarily achievable in the H_∞ norm approximation problem.

2.2 l_2 and l_∞ approximation, and Lawson's algorithm

Lawson, in his PhD thesis in 1961 [13], considered computation of l_∞ approximations by means of weighted l_p approximations with $p < \infty$, where the functions to be approximated are defined on a finite point set.

In [14], Lawson's algorithm was generalised to obtain the solution of the H_∞ one-block problem via the weighted H_2 optimisation problem. In optimal l_∞ regression, the aim is to approximate the values $f(x_i) = f_i$, $i = 1, 2, \ldots, m$ on the set $X = \{x_i \mid i = 1, 2, \ldots, m\}$ by the approximating function

$$L(\mathbf{a}, x) = \sum_{i=1}^{n} a_i \phi_i(x)$$

such that $\max_i |f(x_i) - L(\mathbf{a}, x_i)|$ is minimised, where $\{\phi_i(x)\}$ is a Chebychev set. The following is the definition of Lawson's algorithm [15].

Definition 3: Consider the sequence of weights defined by

$$u^{l+1}(x) = \frac{u^l(x)|f(x) - L(\mathbf{a}^l, x)|}{\sum_{x \in X} u^l(x)|f(x) - L(\mathbf{a}^l, x)|}$$

where $u^l(x_i) > 0$ and $L(\mathbf{a}^l, x)$ is the optimal l_2 approximation to $f(x)$ with weights $u^l(x)$. If $u^l(x_i) = 0$ at some x_i for which the error exceeds that of the non-zero weights, then $u^l(x_i)$ is set to a non-zero value and the algorithm is restarted.

Theorem 2 [15]: In the limit, Lawson's algorithm converges to the solution of the l_∞ regression problem.

In [16], Lawson's algorithm was applied to the case of the complex linear Chebychev approximation via discretising the objective function on the unit circle (therefore converting it into a finite point set problem).

In this paper we generalise Lawson's algorithm to solve the H_∞ approximation problem from a sequence of least squares identification problems.

2.3 Controller reduction for uncertain systems

Consider the feedback system given in Fig. 1. A sufficient condition for $K_r(s)$ to be a stabilising controller is given in the following lemma [17].

Lemma 2: Given a stabilising controller $K(s)$ for the plant $G(s)$. If $K_r(s)$ has the same number of unstable poles as $K(s)$ and

$$\left\| \frac{G}{1 + KG}(K - K_r) \right\|_\infty \leq 1 \qquad (1)$$

then $K_r(s)$ is also a stabilising controller.

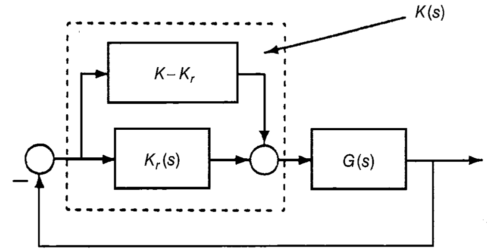

Fig. 1 *Controller reduction*

In controller reduction we would like to find a reduced order controller that not only stabilizes the closed loop system, but also delivers good performance. Recently, several results in this regard have been reported [18–22]. For example, in [21], it is shown that the controller reduction problem for uncertain systems with guaranteed closed loop performance is equivalent to a weighted H_∞ norm system approximation problem. In this Section we give a summary of the controller reduction scheme for uncertain systems presented in [21]. The system under consideration is shown in Fig. 2, where Δ is a block diagonal matrix with complex and/or real blocks. The generic controller reduction problem is defined below.

Generic controller reduction problem: Let $K(s)$ be a robustly stabilising controller for the uncertain set of plants with the nominal plant $G(s)$. Find an rth order robustly stabilising controller $K_r(s)$ that has the same number of unstable poles as $K(s)$.

Fig. 3 shows equivalent block diagram representations of the feedback system with a reduced order controller. The controller reduction problem can be restated as: find K_r as defined in the generic controller problem such that the feedback system is stable for all allowable Δ in the set. There is no known direct method to find such a $K_r(s)$. The system in Fig. 3 is shown [21] to be equivalent to the μ-analysis problem shown in Fig. 4 with $\sigma_{max}(\Delta_K) = \sigma_{max}(K - K_r)$. The frequency-dependent upper bound $\nu(\omega)$ is obtained such that $\sigma_{max}(K - K_r) < \nu(\omega)$ for all frequencies.

Computation of the upper bound can be done using the μ tools software [23]. One has to perform a few bisection steps on the size of Δ_K for each frequency to determine the maximum size of Δ_K before instability occurs. The optimal controller reduction problem can be cast as a standard L_∞ norm approximation problem:

$$\min_{K_r} \|W(K - K_r)\|_\infty \qquad (2)$$

where $W(s)$ is such that $|W(j\omega)| \geq 1/\nu(\omega)$. Finally, note that the derived condition is sufficient only. If the weighted error is less than one then specifications are guaranteed. In

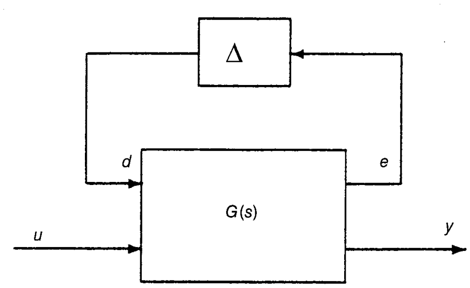

Fig. 2 *Uncertain set of plants*

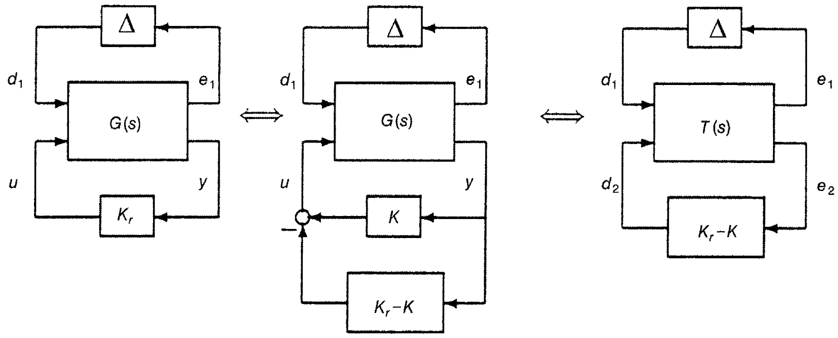

Fig. 3 *Controller reduction approach*

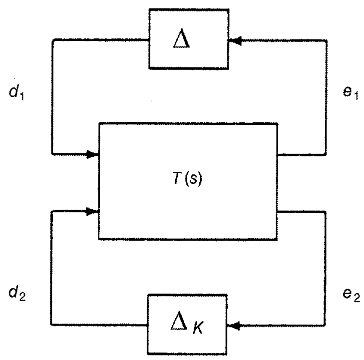

Fig. 4 *'Sufficient' μ-analysis problem*

the following we give a summary of the controller reduction procedure.

Controller reduction procedure for uncertain systems:

Step 1: Determine the upper bound for $\sigma_{max}(\Delta_K(j\omega)) \leq v(\omega)$ such that the system in Fig. 4 is robustly stable.

Step 2: Given r, the desired controller order, find a $K_r(s)$ with the same number of unstable poles as $K(s)$, such that

$$\sigma_{max}(K(j\omega) - K_r(j\omega)) \leq v(\omega), \qquad \forall\omega \tag{3}$$

is satisfied.

If such $K_r(s)$ could not be obtained, increase r and repeat step 2.

3 New H_∞ norm approximation schemes

3.1 Motivation

Consider the H_∞ norm approximation problem

$$\min_{G_r \in H_\infty} \|G - G_r\|_\infty \tag{4}$$

where

$$G_r(z) = \frac{n_r(z)}{d_r(z)} := \frac{b_0 + b_1 z^{-1} + \cdots + b_r z^{-r}}{1 + a_1 z^{-1} + \cdots + a_r z^{-r}} \tag{5}$$

In the FIR case ($d_r(z) = 1$), it is well known that the optimal approximant $G_r(z) = n_r(z) = b_0 + b_1 z^{-1} + \cdots + b_r z^{-n}$ can be computed using Lawson's algorithm [16, 24].

FIR approximation scheme using Lawson's algorithm:

Step 1: Discretise $G(j\omega)$ on N points in $[0, \pi]$, say $G(e^{j\omega_l})$.

Step 2: Compute the limit of the iteration

$$\min_{b_0,\ldots,b_n} \sum_{l=1}^{N} |(G(e^{j\omega_l}) - n_r(e^{j\omega_l}))U^k(e^{j\omega_l})|^2$$

where $U^k(e^{j\omega_l})$ is updated according to

$$U^{k+1}(e^{j\omega_l}) = (G(e^{j\omega_l}) - n_r(e^{j\omega_l}))U^k(e^{j\omega_l})/\alpha,$$

where $$\alpha = \sum_{l=1}^{N} (|U^{k+1}(e^{j\omega_l})|^2)^{1/2}$$

Note that step 2 in the algorithm corresponds to Lawson's algorithm, and it is guaranteed to converge to the optimal solution of the original FIR approximation problem (eqn. 5) if N is large enough.

The main feature of Lawson's algorithm is that it is applicable to H_∞ norm minimisation of an error if the approximant is given as a linear combination of some basis functions. For example, in the FIR case the error is given as a linear combination of $1, z^{-1}, z^{-2}, \ldots, z^{-r}$. However, in the rational approximation problem the approximant is not given as a linear combination of some basis functions. It is given as a ratio of two polynomials of unknown coefficients. Therefore, Lawson's algorithm is not directly applicable to this problem.

Consider the optimisation problem

$$\min_{d_r, n_r} \|Gd_r - n_r\|_\infty \tag{6}$$

instead of eqn. 4, which was first suggested in [25–27] for real rational approximation on a finite set, and is known as Loeb's approach.

Definition 4: We have the following definitions.

$$\text{equation error: } G(z)d_r(z) - n_r(z), \quad \text{error: } G(z) - \frac{n_r(z)}{d_r(z)}$$

Notice that eqn. 6 is linear on the optimisation parameters $b_0, \ldots, b_r, a_1, \ldots, a_r$, and therefore Lawson's algorithm can be used to compute the optimal solution of eqn. 6, which would lead to the following algorithm.

H_∞ approximation using Loeb's approach and Lawson's algorithm:

Step 1: Discretise $G(j\omega)$ on N points in $[0, \pi]$, say $G(e^{j\omega_l})$.

Step 2: Compute the limit of the iteration

$$\min_{b_0,\ldots,b_n,a_r,\ldots,a_r} \sum_{l=1}^{N} |(G(e^{j\omega_l})d_r(e^{j\omega_l}) - n_r(e^{j\omega_l}))U^k(e^{j\omega_l})|^2$$

where $U^k(e^{j\omega_l})$ is updated according to

$$U^{k+1}(e^{j\omega_l}) = \big(G(e^{j\omega_l})d_r(e^{j\omega_l}) - n_r(e^{j\omega_l})\big)U^k(e^{j\omega_l})/\alpha,$$

where $$\alpha = \sum_{l=1}^{N}(|U^{k+1}(e^{j\omega_l})|^2)^{1/2}$$

Unfortunately, most of the time the optimal solution of eqn. 6 does not lead to a good solution for the original problem (eqn. 4). When the actual error, $\|G - G_r\|_\infty$, is computed it is often quite larger than the optimal error. A more serious problem is that the approximant is often not stable. Therefore, this algorithm does not help in our problem. A slight modification of this algorithm turns out to work very well. In the modified algorithm, the weight is being updated using the following formula

$$U^{k+1}(e^{j\omega_l}) = \left(G(e^{j\omega_l}) - \frac{n_r(e^{j\omega_l})}{d_r(e^{j\omega_l})}\right)U^k(e^{j\omega_l})/\alpha,$$

where $$\alpha = \sum_{l=1}^{N}(|U^{k+1}(e^{j\omega_l})|^2)^{1/2}.$$

The optimisation problem in step 2 is linear in the unknown parameters and the weights are updated based on the actual error. In this case the algorithm is not Lawson's algorithm, and therefore convergence to the optimal solution of eqn. 6 is not assured.

3.2 New algorithms

Consider a system $G(z)$. For a given input $U(z)$ one has $Y(z) = G(z)U(z)$. On the other hand, if one computes the output of $G_r(z)$, the approximate system, one gets $Y_r(z) = G_r(z)U(z)$. Then the output error is

$$Y(z) - Y_r(z) = (G(z) - G_r(z))U(z) \qquad (7)$$

For a given $U(z)$, consider the optimal approximation problem

$$\min_{G_r(z) \in H_\infty} \|(G - G_r)U\|_2 \qquad (8)$$

This is a weighted H_2 approximation with weight $U(z)$. Naturally, the solution $G_r(z)$ will depend on the weight $U(z)$.

On the other hand, let's say the optimal approximant $G_r(z)$ is given; then there exist a $U^*(z)$ such that

$$\|G - G_r\|_\infty = \sup_{\|U\|_2=1} \|(G - G_r)U\|_2 = \|(G - G_r)U^*\|_2 \qquad (9)$$

Remark 2: It is well known that the $U^*(z)$, corresponding to the worst case input, has the property that it is necessarily zero except for the frequencies corresponding to $|G(e^{j\omega_l}) - G_r(e^{j\omega_l})| = \|G - G_r\|_\infty$, unless $G(z) - G_r(z)$ is allpass.

The above observations suggest that there might be an input signal $U^*(z)$ for which eqn. 8 will lead to the optimal solution of the H_∞ norm approximation problem. There are two points to be noted for the above approach. First, does there exist such a $U^*(j\omega)$, and if it exists, how do you find it? Second, even if $U^*(j\omega)$ is known, solving the optimisation problem in eqn. 8 is probably no easier than the original problem since it is also a nonlinear minimisation problem.

Assume $G_r(z)$ is the optimal approximant and $|G(e^{j\omega_l}) - G_r(e^{j\omega_l})| = \|G - G_r\|_\infty$, for $\omega_1, \ldots, \omega_N$. Then, for every $U^*(j\omega)$ achieving the minimum we have

$$\|(G - G_r)U^*\|_2 = \|G - G_r\|_\infty$$

Now instead of the output error minimisation problem, consider the equation error minimisation problem:

$$\|(d_rG - n_r)U\|_2 = \|(G - G_r)d_rU\|_\infty \qquad (10)$$

This criterion is similar to the output error criterion except that $d_r(j\omega)$ is multiplying $U(j\omega)$. As was mentioned before, the optimal excitation $U^*(j\omega)$ is zero except for a few frequencies. Therefore, the effect of $d_r(z)$ is just scaling the excitation in those few points. Since the optimal error is equal in all the nonzero frequencies of the optimal excitation, the magnitude being different does not affect its property of achieving the H_∞ norm error. The new scheme does not suffer from nonlinearity of the parameters and our experience shows that it works better than the output error scheme.

The preceding discussion leads us to suggest the following algorithm, which is based on a sequence of weighted equation error minimisation steps. The update rule is the modified version of the equation error based method as explained before.

The approach developed above is summarised in the following algorithms.

Algorithm 1: H_∞ model reduction algorithm

Step 1: Take $U^0(z) = 1$.

Step 2: Solve for $G_r^k(z)$ from

$$\min_{n_r(z), d_r(z)} \|(d_r(z)G(z) - n_r(z))U^k(z)\|_2 \qquad (11)$$

Step 3: Update the weight

$$U^{k+1}(z) = (G(z) - G_r^k(z))U^k(z)/\alpha, \quad \text{where } \alpha := \|U^{k+1}\|_2$$

and go to step 2.

No proof of convergence is available. The algorithm is terminated after a fixed number of iterations (usually between 10 and 30), and G_r is taken as the one that has the least $\|G - G_r\|_\infty$.

The weighted case is treated similarly with the following algorithm.

Algorithm 2: Weighted H_∞ model reduction algorithm

Step 1: Take $U^0(z) = 1$.

Step 2: Solve for $G_r^k(z)$ from

$$\min_{n_r(z), d_r(z)} \|(d_r(z)G(z) - n_r(z))W(z)U^k(z)\|_2 \qquad (12)$$

Step 3: Update the weight

$$U^{k+1}(z) = \big(G(z) - G_r^k(z)\big)W(z)U^k(z)/\alpha,$$

where

$$\alpha := \|U^{k+1}\|_2$$

and go to step 2.

The stopping rule is similar to that for algorithm 1.

Notice that if $W(z) = 1$, algorithm 2 is the same as algorithm 1. Therefore, it is enough to develop algorithm 2.

3.3 Implementation of frequency domain algorithms

The algorithm is implemented by discretising the objective function on the unit circle. In order to have accurate results we need to take large number of points on the unit circle.

Implementation of algorithm 2:

Step 1: Take $U^0(z) = 1$.

Step 2: Sample $G(e^{j\omega})$ on N (a large number) points.

Step 3: Solve for $G_r^k(z)$ from

$$\min_{n_r(z),d_r(z)} \sum_{l=1}^{N} |(d_r(e^{j\omega_l})G(e^{j\omega_l}) - n_r(e^{j\omega_l}))W(e^{j\omega_l})U^k(e^{j\omega_l})|^2$$

Step 4: Update the weight

$$U^{k+1}(e^{j\omega_l}) = (G(e^{j\omega_l}) - G_r^k(e^{j\omega_l}))W(e^{j\omega_l})U^k(e^{j\omega_l})/\alpha,$$

where $$\alpha := \sum_{l=1}^{N} (|U^{k+1}(e^{j\omega_l})|^2)^{1/2}$$

and go to step 3.

Remark 3: It is important to note that the amount of computation required in each iteration can be reduced significantly by using the following observations:

(i) The optimal excitation has only at most $2r+2$ nonzero frequency components.
(ii) The location of peaks changes only slightly after initial iterations.

Using these two points one can concentrate on only some regions of the frequency axis, rather than concentrating on the whole axis, by taking N equally spaced frequencies. Initially one can take equally spaced frequencies, and later on focus on some regions of the frequency axis, and in the limit converge to points.

Remark 4: The frequency domain algorithms are directly applicable to the continuous time case without any modifications to the algorithm. One would only need to determine the frequency range of importance from the beginning since the real line is not a compact region. One can take $\omega \in [0, \omega_u) + \infty$ as the region for a large enough ω_u.

Generalisations into a two-dimensional system approximation are similarly possible.

Remark 5: The frequency domain algorithms presented in this paper can be used for frequency domain H_∞ norm identification, where the only difference is that one is given $G(e^{j\omega_l})$ instead of $G(z)$, which naturally fits in our approach.

4 Examples

In this Section several examples are given to illustrate the presented approximation algorithms.

4.1 H_∞ norm approximation examples

Consider the following 6th order discrete-time system.

$$G(z) = \frac{0.04z^6+0.72z^5+5.4z^4+21.6z^3+48.6z^2+58.32z+29.16}{z^6+1.2z^4+0.36z^2}$$

The approximations are also calculated using the Hankel norm model reduction technique with 'D' term [9], and using the new technique. The results are summarized in Table 1 where ALG1 stands for approximations using Algorithm 1, HA stands for Hankel norm approximation [9], and HSV stands for the $(r+1)$th Hankel singular value of the system (which is a lower bound for the optimal error by lemma 1).

Table 1: H_∞ norm error for model reduction scheme and Hankel approximation

Degree of approximant	1	2	3	4
Calculated norm-ALG1	208.3918	101.6728	93.3071	31.5316
Calculated norm-HA	351.7961	166.3895	109.8171	32.5014
Lower bound (HSV)	193.0026	89.5016	84.2125	29.7294

4.2 Weighted H_∞ approximation

We now consider example 2 reported in [11], where several well known weighted approximation schemes are compared. In this example, $G(s)$ and $W(s)$ are given as

$$G(s) = \frac{s^2+0.2s+1.01}{s^2+0.2s+4.04}\,\frac{s^2+0.2s+9.01}{s^2+0.2s+16.02},$$
$$W(s) = \frac{(s-1)^2}{s^2-0.2s+1}$$

The second and third order weighted approximations are computed. The approximate systems are

$$G_2(s) = \frac{0.7854s^2+2.1795s+3.0315}{s^2+0.2994s+16.6218},$$
$$G_3(s) = \frac{3.4840s^3+6.2187s^2+58.5105s+0.6177}{s^3+9.1493s^2+18.0468s+144.9743}$$

From Table 2 it is seen that the new weighted approximation algorithm gives better results for the examples studied where LA stands for the weighted Hankel approximation proposed in Latham and Anderson [28], AI and AII stand for the two algorithms proposed in [11], Enns stands for the weighted balanced approximation introduced in [10], and ALG. 2 stands for the new weighted approximation algorithm, algorithm 2, of this paper.

4.3 Controller reduction

Consider the four disk control system given in [18, page 517]. The system is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{b}_2 u$$
$$\mathbf{z} = \begin{bmatrix} \mathbf{H} \\ \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \mathbf{c}_2\mathbf{x} + \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}\mathbf{w}$$

Table 2: H_∞ error for model reduction error for example 2

Degree of approximant	2	3
Lower bound (HSV)	2.7037	2.5267
LA	20.08	11.94
AI	4.827	8.20
AII	4.822	3.946
Enns	5.128	4.993
ALG. 2	4.6284	3.8447

where

$$\mathbf{A}=\begin{bmatrix} -0.161 & -6.004 & -0.58215 & -9.9835 & -0.40727 & -0.3982 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{b}_2=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{H}=10^{-3}\times[0\ 0\ 0\ 0\ 0.55\ 11\ 1.32\ 18]$$

$$\mathbf{B}_1=[\mathbf{b}_2\ \mathbf{0}]$$

$$\mathbf{c}_2=[0\ 0\ 6.4432\times10^{-3}\ 2.3196\times10^{-3}\ 7.1252\times10^{-2}\ 1.0002\ 0.10455\ 0.99551]$$

The optimal $\|T_{zw}\|_\infty = 1.1272$. An eighth order suboptimal controller is designed using $\gamma = 1.2$. The full order controller is

$$K(s)=\frac{-0.8179s^7-0.1574s^6-4.9150s^5-0.6311s^4-8.1819s^3-0.5926s^2-3.2694s-0.1057}{s^8+1.9376s^7+7.6359s^6+11.5915s^5+18.4317s^4+19.9626s^3+15.7459s^2+9.1034s+3.8220}$$

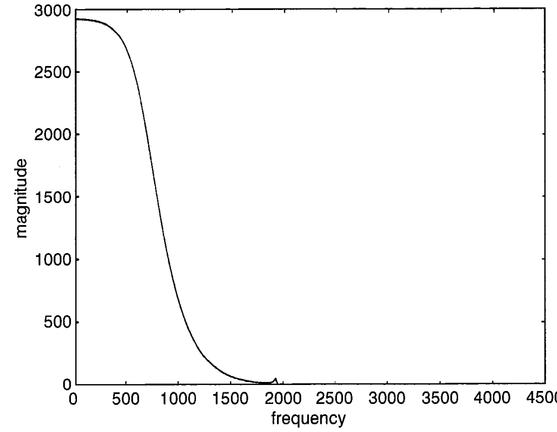

Fig. 5 *Weighting function obtained in controller reduction*

The algorithm presented in Section 2.3 is used to obtain the weight $W(\omega)$ for the controller reduction problem. The weight is shown in Fig. 5. Algorithm 2 is then used to solve the resulted weighted H_∞ approximation problem. The result is summarised in Table 3.

From Table 3, it is clear that controllers with order 4 or higher are needed to achieve the required performance level. Note also that the obtained controllers are proper. If one restricts the controllers to be strictly proper, algorithm 2 can be used with the increase of the weighted error. When reduced order controllers of lower orders are required one needs to relax the performance level.

4.3.1 Example 4

In the previous example, at least a fourth order controller is needed to satisfy the specifications. Stabilising controllers of order less than 4 can be obtained with the price of increasing the performance level γ. In this example two approaches of relaxing γ are considered. In the first approach, the controller is designed assuming $\gamma_1 = 1.2$, but in controller reduction γ_1 is relaxed to $\gamma_2 = 1.6$. The results are shown in Table 4. In the second approach, the controller is designed assuming $\gamma_1 = 1.6$, and reduced order

Table 3: Reduced order controllers and corresponding performance level

r	$\|W(K-K_r)\|_\infty$	$\|T_{zw}\|_\infty$	K_r
1	12.9408	73.286	$\frac{-0.2785s-0.0085}{s+0.3156}$
2	11.2638	1.9681	$\frac{-0.1637s^2-0.0975s-0.0030}{s^2+0.4426s+0.1109}$
3	1.42145	1.8801	$\frac{-0.368s^3-0.0188s^2-0.2145s-0.0066}{s^3+0.5146s^2+0.6570s+0.2436}$
4	0.85247	1.1991	$\frac{-0.1987s^4-0.3072s^3-0.1368s^2-0.1771s-0.0057}{s^4+0.8700s^3+1.0805s^2+0.5920s+0.2062}$
5	0.80708	1.1970	$\frac{-0.3725s^5-0.1908s^4-1.3130s^3-0.1675s^2-0.6386s-0.0201}{s^5+1.5405s^4+3.5694s^3+2.4087s^2+1.9525s+0.7314}$
6	0.33393	1.1971	$\frac{-0.0915s^6-0.5034s^5-0.2927s^4-1.3204s^3-0.1967s^2-0.6004s-0.0195}{s^6+1.1854s^5+3.4540s^4+3.2926s^3+3.0921s^2+1.7452s+0.7020}$
7	0.23492	1.1957	$\frac{-0.2608s^7-0.4390s^6-2.4097s^5-1.3426s^4-4.8059s^3-0.8106s^2-2.0312s-0.0649}{s^7+3.0319s^6+7.1582s^5+10.9129s^4+13.4268s^3+10.1391s^2+6.2039s+2.3471}$

Table 4: Reduced order controllers and corresponding performance level ($\gamma_1 = 1.2$, $\gamma_2 = 1.6$)

r	$\|W(K-K_r)\|_\infty$	$\|T_{zw}\|_\infty$	K_r
1	3.3037	1.5070	$\frac{-0.1244s - 0.0049}{s + 0.2554}$
2	2.9844	4.5333	$\frac{-0.0645s^2 - 0.0118s - 0.0021}{s^2 + 0.1552s + 0.1023}$
3	1.0367	1.6853	$\frac{-0.3023s^3 - 0.0101s^2 - 0.1803s - 0.0021}{s^3 + 0.3033s^2 + 0.6772s + 0.1606}$

Table 5: Reduced order controllers and corresponding performance level ($\gamma_1 = \gamma_2 = 1.6$)

r	$\|W(K-K_r)\|_\infty$	$\|T_{zw}\|_\infty$	K_r
1	0.9214	1.5080	$\frac{-0.0992s - 0.0029}{s + 0.1795}$
2	0.8442	1.4733	$\frac{-0.0664s^2 - 0.0431s - 0.0015}{s^2 + 0.3736s + 0.0867}$
3	0.3755	1.4784	$\frac{-0.1047s^3 - 0.0087s^2 - 0.0610s - 0.0019}{s^3 + 0.3558s^2 + 0.4984s + 0.1147}$

controllers satisfying the same performance level are obtained. The results are summarised in Table 5.

One is led to conclude from these examples and others that if one wants to relax the performance level, then it is better to do it before designing the suboptimal controller, and then apply the reduction technique.

5 Discussion

In this paper a frequency domain approach has been developed for the weighted (and unweighted) H_∞ norm approximation problem. The new frequency domain approach has been demonstrated to work well in several meaningful examples. The method was used to obtain reduced order controllers with guaranteed H_∞ performance bounds. The frequency domain technique is applicable to the continuous time case as well. It also naturally generalises onto the H_∞ norm approximation of two-dimensional systems.

Computational complexity of the technique is on the order of the least squares identification technique, which is very efficient and applicable to very high-order systems, as well as filter design via approximating ideal filter characteristics. A preliminary version of this work was presented in [29].

6 Acknowledgments

We acknowledge KFUPM for its support of this research. D. Kavranoğlu would like to thank Keith Glover for the numerous stimulating discussions on the material of the paper during his sabbatical leave at the California Institute of Technology.

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