

SUBLATTICE-DEPENDENT NONLINEARITY IN THE TOPOLOGICAL SSH MODEL

BY

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Dedication

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THESIS ABSTRACT

NAME: Ahmed Alharthy

TITLE OF STUDY: Sublattice-Dependent Nonlinearity in the Topological SSH Model

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Recently, new phases of matter known as topological insulators have been discovered, where a material is an insulator in the bulk and a conductor at the boundary. Moreover, this interesting property is topologically protected, meaning it is robust under perturbation which makes the conducting states on the boundaries (surfaces) exhibit special scattering-resistant behavior. Such phases of matter are described mathematically by topological models. A one dimensional topological model which has been studied extensively is the Su–Schrieffer–Heeger (SSH) model. While the notion of band topology is inherently linked to linear systems, recently there has been growing interest in the interplay between nonlinearities, topological phases and edge states. From this perspective comes our consideration of a sublattice-dependent onsite nonlinearity in the topological SSH model with real lattice and

reciprocal lattice studies. Mathematically speaking, the problem in the real lattice space is solved under open boundary conditions, where in the reciprocal lattice space it is solved under periodic boundary conditions, with different signs' variation of the sublattice-nonlinearity strengths, where g_A denotes the nonlinearity strength on sublattice A , and g_B is the nonlinearity strength on sublattice B . In the real lattice space study, the problem was solved numerically using the Self-Consistent Field Iterative Method. In the case where g_A and g_B are both positive, it was found that in the topologically nontrivial regime ($J_1 < J_2$) and at sufficiently high nonlinearities, the energy of the left (right) edge state depends only on g_A (g_B) and is constant with respect to g_B (g_A). Moreover, the existence of only one edge state was found in the model for a particular set of values of a sublattice-dependent nonlinearity strength. In addition, in the topologically trivial and nontrivial (including when $J_1 = J_2$) regimes and at sufficiently high nonlinearities we noted the presence of a touching point between two energy bands. This touching point was found to only shift (and not disappear) under perturbation, which could indicate a topological origin analogous to gapless topological phases observed in Weyl semimetal. In the case where g_A and g_B have opposite signs, the persistence of bulk's delocalized solutions was observed even at extremely high nonlinearities' strengths. In the reciprocal lattice space study, we assumed Bloch's state solutions and used semi-analytical approach to obtain energy bands, applied dynamical stability analysis, and studied nonlinear/conventional Zak phases. Furthermore, a general expression of the nonlinear Zak phase, which holds for any well-behaved two-level real Hamiltonian with C^1 state-dependent (through $\Sigma = |\psi_2(t)|^2 - |\psi_1(t)|^2$) onsite non-

linearities, was derived. In the case where g_A and g_B are positive and at sufficiently high nonlinearity strengths, energy gap-closing accompanied with discontinuity in the nonlinear Zak phase were observed, marking a nonlinearity-induced topological phase transition.

ملخص الرسالة

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عنوان الدراسة: اللاخطية المعتمدة على تحت-الشبكة في نموذج SSH الطوبولوجي

التخصص: قسم الفيزياء

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تم مؤخرًا اكتشاف أطوار جديدة من المادة تُعرف باسم العوازل الطوبولوجية، حيث يكون للمادة خصائص عازلة في الحجم وموصلة على الحدود. وعلاوة على ذلك، فإن هذه الخاصية المثيرة للاهتمام محمية طوبولوجيًا، مما يعني أنها مقاومة للتشويشات، وهو ما يجعل الحالات الموصلة على الحدود (السطوح) تُظهر سلوكًا خاصًا مقاومًا للتبعثر. يتم وصف هذه الأطوار من المادة رياضيًا باستخدام نماذج طوبولوجية. أحد النماذج الطوبولوجية أحادية البعد التي تمت دراستها بشكل مكثف هو نموذج Su-Schrieffer-Heeger (SSH). وعلى الرغم من أن مفهوم طوبولوجيا الحزمة مرتبط بطبيعته بالأنظمة الخطية، إلا أن اهتمامًا متزايدًا قد ظهر مؤخرًا حول التفاعل بين اللاخطية، الأطوار الطوبولوجية، والحالات الحدية. من هذا المنطلق، تأتي دراستنا للاخطية المعتمدة على تحت-الشبكة في نموذج SSH الطوبولوجي، مع دراسة كل من الفضاء الحقيقي والفضاء المتبادل للشبكة البلورية. تم حل المشكلة في فضاء الشبكة الحقيقي باستخدام شروط حدودية مفتوحة، بينما تم حلها في الفضاء المتبادل باستخدام شروط حدودية دورية، مع تباين في إشارات شدة اللاخطية الخاصة بكل تحت-شبكة، حيث يُرمز إلى شدة اللاخطية في تحت-الشبكة A بالرمز g_A ، وفي تحت-الشبكة B بالرمز g_B . في دراسة الفضاء الحقيقي، تم حل المشكلة عددًا باستخدام طريقة الحقل الذاتي المتسقة التكرارية. خلال عملية التكرار، تم اختيار جميع حالات الطاقة الذاتية من النموذج الخطي المقابل كنقاط انطلاق، وتم جمع ورسم فقط الحلول غير الخطية التي تتقارب بسرعة كافية. في الحالة التي يكون فيها كل من g_A و g_B موجبين، تم التوصل إلى أنه في النظام الطوبولوجي غير البسيط ($J_1 < J_2$) وعند لاختطيات عالية بما فيه الكفاية، فإن طاقة الحالة الحدية اليسرى

(أو اليمنى) تعتمد فقط على g_A (أو g_B) وتبقى ثابتة بالنسبة لـ g_B (أو g_A). علاوة على ذلك، تم العثور على وجود حالة حدية واحدة فقط في النموذج لمجموعة معينة من قيم الاخطية المعتمدة على تحت-الشبكة. بالإضافة إلى ذلك، سواء في النظام الطوبولوجي البسيط أو غير البسيط (بما في ذلك عندما $J_1 = J_2$) وعند لاخطيات عالية كفاية، لاحظنا وجود نقطة تماس بين حزمتين طاقيتين. هذه النقطة لم تختفِ تحت التشويش، بل فقط تغير موقعها، مما قد يشير إلى أصل طوبولوجي مشابه للأطوار الطوبولوجية غير الفارغة التي نلاحظ في أشباه الفلزات من نوع Weyl. أما في الحالة التي يكون فيها لـ g_A و g_B إشارات متعاكسة، فقد لوحظ استمرار وجود حلول غير متموضعة في الحجم حتى عند لاخطيات عالية جداً. في دراسة الفضاء المتبادل للشبكة، افترضنا حلولاً من نوع حالات Bloch، واستخدمنا منهجاً شبه تحليلي للحصول على حزم الطاقة، وقمنا بتحليل الاستقرار الديناميكي، ودرسنا أطوار Zak الاخطية والتقليدية. علاوة على ذلك، تم اشتقاق تعبير عام لطور Zak الاخطي ينطبق على أي هاميلتوني لاخطي معتمد على الحالة. في الحالة التي يكون فيها كل من g_A و g_B موجبين وعند لاخطيات عالية كفاية، لوحظ انغلاق في فجوة الطاقة مصحوب بانقطاع في طور Zak الاخطي، مما يدل على انتقال طوبولوجي في الطور ناتج عن الاخطية.

CHAPTER 1

INTRODUCTION

It is generally taught early in school that matter commonly exists in three different phases, solid, gas, and liquid. As one progresses further, a fourth new phase is introduced, which is plasma. Moreover, we have learned that the solid phase can be further classified based on its transport properties into different phases, insulating and conducting phases.

Quantum mechanically, insulators differ from conductors in having an energy gap in their band structure, separating valence from conduction bands. This energy gap forms an energy barrier that prevents electrons from hopping from the valence band to the conduction band.

Recently, new phases of matter known as topological insulators have been discovered, where a material is an insulator in the bulk and a conductor at the boundary. Moreover, this interesting property is topologically protected, meaning it is robust under perturbation, making the edge states (conducting states) on the boundaries (surfaces) exhibit special scattering-resistant behavior. Such phases of matter are described mathematically by what are called "topological

models". In this context, a topologically trivial phase is the normal insulating phase (without edge states), whereas the topologically nontrivial phase is the one which supports edge states. A one dimensional topological model which has been studied extensively is the Su–Schrieffer–Heeger (SSH) model.

The model has been realized and applied in different areas of physics such as electronic [1], [2], photonic [3]–[7], plasmonic [8], polaritonic [9] as well as mechanical systems [10]. This is due to its simplicity and its symmetry-protected topological nature which makes it support both trivial as well as nontrivial topological phases. Indeed, edge states in the model were found to be robust under different forms of disorder as long as chiral symmetry is maintained [11].

The SSH model in its linear form is a noninteracting model, real-life systems on the other hand, are interacting in nature, so considering interaction in the SSH is of essence from both a theoretical and practical point of view. Studying many-body problems is known to be a challenge due to mathematical and computational complications. A common approach often used to include interactions is by introducing nonlinear terms to the model, reducing the problem to an effective nonlinear one-body problem.

Formally speaking, the notion of band topology is inherently linked to linear systems, however, recently there has been growing interest in the interplay between nonlinearities, topological phases and edge states [12]–[15]. It is from this perspective that for this thesis we decided to investigate a sublattice-dependent onsite nonlinearity in the topological SSH model with real lattice and reciprocal lattice studies.

1.1 Su–Schrieffer–Heeger (SSH) model

The Su-Schrieffer-Heeger (SSH) model [1] is a nearest-neighbor tight-binding model on a one dimensional lattice, with staggered hopping amplitudes. The lattice is divided into N unit cells with each unit cell containing two sublattices, A and B . The model is given as a single particle Hamiltonian of the form:

$$\hat{H}_{SSH} = J_1 \sum_{m=1}^N (\Psi_{B,m} \Psi_{A,m}^+ + h.c.) + J_2 \sum_{m=1}^{N-1} (\Psi_{A,m+1} \Psi_{B,m}^+ + h.c.) \quad (1.1)$$

where J_1 is the energy required to hop within the unit cell, and J_2 is the energy required to hop between adjacent unit cells, $\Psi_{q,m}$ is all zero column vector except one component being equal to one, this component correspond to the site on the chain at unit cell number m , sublattice q , with $m \in \{1, 2, 3, \dots, N\}, q \in \{A, B\}$, and $h.c.$ denotes Hermitian-Conjugate.

The SSH model is known to support both trivial (for $J_1 > J_2$) as well as nontrivial (for $J_1 < J_2$) topological phases [1], [16]. Adding an extra staggered onsite potential term breaks chiral symmetry, and we get what is known as Rice-Mele model [17] with the Hamiltonian:

$$\hat{H}_{RM} = \hat{H}_{SSH} + \sum_{m=1}^N v (\Psi_{A,m} \Psi_{A,m}^+ - \Psi_{B,m} \Psi_{B,m}^+) \quad (1.2)$$

where v is the onsite potential strength. After chiral symmetry-breaking, edge states lose topological protection and their energies scatter under disorder.

1.2 Global Onsite Nonlinearity in the model

In [18], a global onsite nonlinearity (fixed nonlinearity strength on all sites throughout the lattice) was introduced to the Rice-Mele Hamiltonian (1.2) in the form given below:

$$\hat{H} = \hat{H}_{RM} + \sum_{m=1}^N [g |\psi_{A,m}|^2 \Psi_{A,m} \Psi_{A,m}^\dagger + g |\psi_{B,m}|^2 \Psi_{B,m} \Psi_{B,m}^\dagger] \quad (1.3)$$

where $\Psi_{q,m}$ is all zero column vector except one component being equal to one, this component correspond to the site on the chain at unit cell number m , sublattice q , with $m \in \{1, 2, 3, \dots, N\}$, $q \in \{A, B\}$, $\psi_{q,m}$ represents the wavefunction's amplitude at site with unit cell number m , sublattice q and g denotes the global onsite nonlinearity strength.

The system was studied in real lattice space (under open boundary conditions) as well as reciprocal lattice space (under periodic boundary conditions). In the real lattice space study, in-gap localized stationary state solutions were observed to form effective edges induced by the nonlinearity within the bulk of the lattice. That is, due to the Hamiltonian dependence on the state, a wave function strongly localized at one site increases the potential energy there, effectively creating a potential barrier, which can be a large on-site potential for strong nonlinearity strength, thus effectively behaving like a physical edge. Indeed, this explanation was further supported after comparing the spatial profiles of such localized solutions with eigenstates of the corresponding linear model with a certain impurity

potential added. This idea explains the restoration of degeneracy of edge solitons localized at the edges.

In the reciprocal lattice space study, on the other hand, a proposed characteristic quantity for the nonlinear system was introduced and named therein, “Nonlinear Zak phase” which accounts for the geometric contributions in dynamical phases due to the nonlinearity. For sufficiently high nonlinearity, nonlinear Zak phases of all nonlinear energy bands were found to sum up to a quantized quantity (0 modulo 2π), which was not found to be the case for corresponding conventional Zak phases. Moreover, it was shown that for sufficiently high nonlinearity, the nonlinear Zak phase for each nonlinear energy band, although not quantized, is closer than conventional Zak phase to either 0 or π .

These findings suggest that despite symmetry-breaking caused by the staggered onsite potential term, nonlinearity seems to recover some topological order already lost by symmetry-breaking.

1.3 Motivation

The investigation in [18] had been done for fixed nonlinearity strength on all sites on the lattice (global nonlinearity strength). The encouraging results stimulated our interest in considering the more general case where the nonlinearity strength is sublattice-dependent. Our investigation of the resulting model includes both real lattice space, as well as reciprocal lattice space studies.

1.4 Outline of thesis

This thesis is divided into four chapters. In the second chapter, we introduce the theoretical framework needed in our study. In particular, we present the numerical method used to obtain stationary solutions in the real lattice study. In addition, we derive a general expression for the nonlinear Zak phase, then consider analytical tools needed to apply a dynamical stability analysis. In the third chapter, we present the main findings and observations in real lattice and reciprocal lattice spaces. Finally, in the fourth chapter, we summarize the main results in our thesis and suggests newly developed tools for future studies of nonlinear systems.

CHAPTER 2

METHODS

Our study in this thesis considers sublattice-dependent onsite nonlinearity in the SSH model. The problem is considered in real lattice space under Open Boundary Conditions (OBCs) as well as in reciprocal lattice space where Bloch's ansatz is assumed and solved under Periodic Boundary Condition (PBC). In the real lattice space case, the problem is solved numerically using the Self-Consistent Iterative Method, whereas in the reciprocal lattice space study the problem simplifies and is solved by a self-consistent analysis.

In this chapter, we start by presenting the numerical method used to obtain stationary solutions in real lattice space, then we show our self-consistent analysis in reciprocal lattice space leading to a general expression of the nonlinear Zak phase (a proposed geometric characteristic of the nonlinear model). Finally, we derive the analytical tools needed to apply a dynamical stability analysis on nonlinear energy bands in the reciprocal lattice space.

2.1 Numerical method: Self-Consistent Field Iterative Method

To solve for stationary solutions in real lattice space we use a numerical method known as the Self-Consistent Field Iterative Method. The iteration for a state-dependent nonlinear Hamiltonian H_{OBC} from state Ψ_n to state Ψ_{n+1} , where n labels the iteration step, goes as follows:

1. Evaluate $H_n = H_{OBC}(\Psi_n)$
2. Find the eigenstates Φ_i of H_n , $i = 1, 2, \dots, 2N$.
3. Select a new state Ψ_{n+1} to be the eigenstate Φ_j that has the maximum inner product with the previous Ψ_n state, i.e. Φ_j that satisfy $|\Psi_n^\dagger \Phi_j| > 1 - s$ where s is the tolerance threshold, and is set to $s = 10^{-10}$.

At the beginning of each iteration process, an energy eigenstate from the corresponding linear model is selected as an initial trial state. In our study, all energy eigenstates from the corresponding linear model were selected as initial trial states. However, only nonlinear stationary solutions that converged sufficiently quickly were collected and plotted later. This implementation of the numerical method while highlights dynamically stable solutions (solutions which are converged to from different trial states), leaves solutions that are not converging quickly enough not considered.

Recall that the Inverse Participation Ratio (IPR):

$$IPR(\Psi) = \frac{1}{\sum_{s=1}^{2N} |\psi_s|^4}$$

where s is the lattice site's index, and for a lattice with N unit cells, s runs from 1 to $2N$, and ψ_s is the wavefunction amplitude at site s . Note that the IPR for a localized solution is around 1, and for a delocalized solution $\rightarrow \infty$. In this thesis, we use the term "soliton" loosely, to refer to any localized state that is not directly related to an edge state of the model in the linear limit. In the plotting process, the IPR and the normalization condition were used to classify solutions based on their localization properties. More specifically, the four following conditions (tests) were implemented in order, as else-if statements:

1. If

$$0.15 < |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

is satisfied, meaning the wavefunction's probability at the four leftmost sites makes significant contribution, then the solution is a left edge (state/soliton) and is plotted in cyan.

2. If

$$0.15 < |\psi_{2N-3}|^2 + |\psi_{2N-2}|^2 + |\psi_{2N-1}|^2 + |\psi_{2N}|^2$$

is satisfied, meaning the wavefunction's probability at the four rightmost sites makes significant contribution, then the solution is a right edge (state/soliton) and is plotted in red.

3. If

$$0.85 < IPR(\Psi) < 5$$

is satisfied, then the solution is a soliton bulk state and is plotted in black.

4. If none of the above conditions are satisfied, then the solution is a delocalized bulk state and is plotted in gray.

The numerical values in the above inequalities were set after studying wave-function's profiles for representative solutions with corresponding quantities.

2.2 General Expression of Nonlinear Zak Phase

A Hamiltonian of state-dependent nonlinearity of the form Eq.(1.3) is known to embed geometric phase contributions in its dynamical phases due to nonlinearity [18]. This fact may indicate that the conventional Zak phase $i\Phi + \frac{d\Phi}{dt}$ may not provide the complete geometric description of the adiabatic process in the nonlinear system. In [18], a proposed (geometric) characteristic quantity for the nonlinear system was introduced and named therein, “Nonlinear Zak phase” which accounts for the geometric contributions in dynamical phases due to the nonlinearity.

Indeed, it was observed in [18] that the nonlinear Zak phases can be significantly different from the conventional Zak phases. Moreover, it was found that for sufficiently high nonlinearity, nonlinear Zak phases of all nonlinear energy bands were found to sum up to a quantized quantity (0 modulo 2π), which was not found to be the case for corresponding conventional Zak phases. In addition, it was shown that for sufficiently high nonlinearity, the nonlinear Zak phase for each nonlinear energy band, although not quantized, is closer than conventional Zak phase to either 0 or π . The fact that only nonlinear Zak phases may recover quantization is a remarkable observation, which suggests that geometric characterization of nonlinear lattice systems has unique features absent in linear systems.

In what follows, we derive a general expression of the nonlinear Zak phase, which holds for any well-behaved two-level real Hamiltonian with C^1 state-dependent

(through $\Sigma = |\psi_2(t)|^2 - |\psi_1(t)|^2$) onsite nonlinearities.

Consider a time-dependent nonlinear Schrodinger equation:

$$i \frac{\partial \Psi(t)}{\partial t} = H(\Sigma, k) \Psi(t) \quad (2.1)$$

with $\hbar = 1$, $\Psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$, $\Sigma = |\psi_2(t)|^2 - |\psi_1(t)|^2$ and the Hamiltonian has the form:

$$H(\Sigma, k) = h_x(k)\sigma_x + h_y(k)\sigma_y + h_z(\Sigma)\sigma_z + \frac{S(\Sigma)}{2}I \quad (2.2)$$

where k is the quasimomentum. $h_x(k), h_y(k)$ ⁰¹ are well behaved real functions of k , $h_z(\Sigma)$, $S(\Sigma)$ are real C^1 functions of Σ . On the other hand, σ_x , σ_y , σ_z are the 2×2 Pauli matrices, and I is the 2×2 identity matrix. Note that Σ is a state-dependent real variable that might have implicit dependence on k .

Considering $\Psi(t) = e^{if(t)}\Phi$ where $f(t)$ is a real scalar function, and $\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$.

This form of $\Psi(t)$ was considered to separate explicit time-dependence in one term, namely $e^{if(t)}$, note however that Φ have implicit time-dependence through an adiabatic parameter. Plugging $\Psi(t)$ into the nonlinear Schrodinger equation Eq. (2.1), we get:

$$\frac{df}{dt}\Phi = i \frac{d\Phi}{dt} - H\Phi \quad (2.3)$$

⁰¹ $h_x(k), h_y(k)$ are to be fixed later to be the coupling terms in the SSH model

Multiplying Eq.(2.1) from the left hand side by $\Psi(t)^\dagger$, we get:

$$\frac{df}{dt} = i\Phi^\dagger \frac{d\Phi}{dt} - \Phi^\dagger H \Phi \quad (2.4)$$

Now, we perturbatively expand the following in terms of an adiabatic parameter

ϵ :

$$\begin{aligned} \frac{df}{dt} &= \alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2 + \dots \\ \Phi &= \Phi^{(0)} + \epsilon\Phi^{(1)} + \epsilon^2\Phi^{(2)} + \dots \end{aligned} \quad (2.5)$$

$$H = H^{(0)} + \epsilon H^{(1)} + \epsilon^2 H^{(2)} + \dots$$

where $H^{(0)} = H(\Sigma, k)|_{\epsilon=0} = H(\Sigma^{(0)}, k)$ and $H^{(1)} = \frac{dH}{d\epsilon}|_{\epsilon=0} = \frac{d\Sigma}{d\epsilon}|_{\epsilon=0} \left[\frac{dh_z}{d\Sigma} \Big|_{\Sigma=\Sigma^{(0)}} \sigma_z + \frac{1}{2} \frac{dS}{d\Sigma} \Big|_{\Sigma=\Sigma^{(0)}} I \right]$, with $\Sigma^{(0)}$ denoting Σ evaluated at $\epsilon = 0$.

Next, we impose a condition (which states that the system is initially prepared to be in a definite energy state Φ_E):

$$H^{(0)}\Phi_E = E\Phi_E \quad (2.6)$$

with $\Phi_E = \Phi^{(0)}$. Without loss of generality, we may parametrize an initial state $\Phi^{(0)}$ (using Bloch sphere representation) as:

$$\Phi^{(0)} = \begin{pmatrix} \varphi_1^{(0)} \\ \varphi_2^{(0)} \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \quad (2.7)$$

Where $\theta \in [0, \pi)$ is the polar angle, and $\phi \in [0, 2\pi)$ is the azimuthal angle, with $e^{i\phi}$ being [16]

$$e^{i\phi} = \frac{h_x(k) + ih_y(k)}{\sqrt{h_x^2 + h_y^2}} \quad (2.8)$$

On the other hand, the normalization condition at the beginning of the adiabatic process (when the system is at $\Phi^{(0)}$) means that we need to have:

$$|\varphi_1^{(0)}|^2 + |\varphi_2^{(0)}|^2 = 1 \quad (2.9)$$

we get after substituting Φ from Eq.(2.5) into:

$$|\psi_1(t)|^2 + |\psi_2(t)|^2 = |\varphi_1|^2 + |\varphi_2|^2 = 1$$

and equating terms up to ϵ^1 , we obtain the condition:

$$\begin{aligned} \Re\left(\varphi_2^{(0)*} \varphi_2^{(1)}\right) &= -\Re\left(\varphi_1^{(0)*} \varphi_1^{(1)}\right) \\ \Rightarrow \Re\left(e^{-i\phi} \varphi_2^{(1)}\right) &= -\cot\left(\frac{\theta}{2}\right) \Re\left(\varphi_1^{(1)}\right) \end{aligned} \quad (2.10)$$

After substituting Φ from Eq.(2.5) in Σ , and making use of Eqs.(2.7, 2.10), Σ has the form (up to ϵ^1 terms):

$$\Sigma = |\psi_2(t)|^2 - |\psi_1(t)|^2 = |\varphi_2|^2 - |\varphi_1|^2 = |\varphi_2^{(0)}|^2 - |\varphi_1^{(0)}|^2 - 4\epsilon \Re\left(\varphi_1^{(0)*} \varphi_1^{(1)}\right) + \dots \quad (2.11)$$

with $\Sigma^{(0)} = |\varphi_2^{(0)}|^2 - |\varphi_1^{(0)}|^2 = -\cos\theta$. Going back to Eq.(2.4), and expanding it up to ϵ^1 , we get the two conditions:

$$\alpha_0 = -E \quad (2.12)$$

$$\epsilon\alpha_1 = i\Phi^{(0)+}\frac{d\Phi^{(0)}}{dt} - \epsilon\Phi^{(0)+}H^{(1)}\Phi^{(0)} \quad (2.13)$$

where the eigenvalue equation $H^{(0)}\Phi^{(0)} = E\Phi^{(0)}$ was also used.

Moreover, after substituting expanded $\frac{df}{dt}$, Φ and H in Eq.(2.3) and considering only the ϵ^1 term of the first component's in the equation Eq.(2.3), we get:

$$\begin{aligned} \epsilon\alpha_1\varphi_1^{(0)} &= i\frac{d\varphi_1^{(0)}}{dt} - \epsilon\left[\alpha_0\varphi_1^{(1)} + H_{11}^{(0)}\varphi_1^{(1)} + H_{12}^{(0)}\varphi_2^{(1)} + H_{11}^{(1)}\varphi_1^{(0)}\right] \\ \epsilon\alpha_1\varphi_1^{(0)} &= i\frac{d\varphi_1^{(0)}}{dt} - \epsilon\left[\alpha_0\varphi_1^{(1)} + \left(h_z(\Sigma)|_{\epsilon=0} + \frac{S(\Sigma)|_{\epsilon=0}}{2}\right)\varphi_1^{(1)} \right. \\ &\quad \left. + (h_x(k) - ih_y(k))\varphi_2^{(1)} + \frac{d\Sigma}{d\epsilon}|_{\epsilon=0}\left(\frac{dh_z}{d\Sigma}|_{\Sigma=\Sigma^{(0)}} + \frac{1}{2}\frac{dS}{d\Sigma}|_{\Sigma=\Sigma^{(0)}}\right)\varphi_1^{(0)}\right] \end{aligned} \quad (2.14)$$

Taking the real component of both sides of Eq.(2.14), we get:

$$\begin{aligned} \epsilon\alpha_1\varphi_1^{(0)} &= i\frac{d\varphi_1^{(0)}}{dt} - \epsilon\left[\alpha_0\Re(\varphi_1^{(1)}) + \left(h_z(\Sigma)|_{\epsilon=0} + \frac{S(\Sigma)|_{\epsilon=0}}{2}\right)\Re(\varphi_1^{(1)}) \right. \\ &\quad \left. + \Re\left[(h_x(k) - ih_y(k))\varphi_2^{(1)}\right] + \frac{d\Sigma}{d\epsilon}|_{\epsilon=0}\left(\frac{dh_z}{d\Sigma}|_{\Sigma=\Sigma^{(0)}} + \frac{1}{2}\frac{dS}{d\Sigma}|_{\Sigma=\Sigma^{(0)}}\right)\varphi_1^{(0)}\right] \end{aligned} \quad (2.15)$$

Making use of Eqs.(2.8, 2.10), we replace $\Re\left[(h_x(k) - ih_y(k))\varphi_2^{(1)}\right]$ in Eq.(2.15) with $-\sqrt{h_x^2 + h_y^2} \cot\left(\frac{\theta}{2}\right)\Re(\varphi_1^{(1)})$. After using Eq.(2.12), then evaluating

$\frac{d\Sigma}{d\epsilon}\big|_{\epsilon=0} = -4 \varphi_1^{(0)} \Re(\varphi_1^{(1)})$ and recalling that $\varphi_1^{(0)}$ is real in our case, we get:

$$\begin{aligned} \epsilon \alpha_1 \varphi_1^{(0)} &= i \frac{d\varphi_1^{(0)}}{dt} - \epsilon \Re(\varphi_1^{(1)}) \left\{ -E + h_z(\Sigma) \big|_{\epsilon=0} \right. \\ &\quad \left. + \frac{S(\Sigma) \big|_{\epsilon=0}}{2} - \sqrt{h_x^2 + h_y^2} \cot \frac{\theta}{2} - 4 \left(\varphi_1^{(0)} \right)^2 \left[\frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} + \frac{1}{2} \frac{dS}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} \right] \right\} \end{aligned}$$

Substituting by $\varphi_1^{(0)} = \cos \frac{\theta}{2}$ then solving for $\epsilon \Re(\varphi_1^{(1)})$, we get:

$$\epsilon \Re(\varphi_1^{(1)}) = \frac{i \frac{d\varphi_1^{(0)}}{dt} - \epsilon \alpha_1 \cos \frac{\theta}{2}}{\left\{ -E + h_z(\Sigma) \big|_{\epsilon=0} + \frac{1}{2} S(\Sigma) \big|_{\epsilon=0} - \sqrt{h_x^2 + h_y^2} \cot \frac{\theta}{2} - 4 \cos^2 \left(\frac{\theta}{2} \right) \left[\frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} + \frac{1}{2} \frac{dS}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} \right] \right\}} \quad (2.16)$$

Recall that when using the explicit form of $H^{(1)}$, Eq.(2.13) takes the form:

$$\epsilon \alpha_1 = i \Phi^{(0)} + \frac{d\Phi^{(0)}}{dt} - 4 \cos \frac{\theta}{2} \left(\frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} \Sigma^{(0)} - \frac{1}{2} \frac{dS}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} \right) \epsilon \Re(\varphi_1^{(1)}) \quad (2.17)$$

By substituting $\epsilon \Re(\varphi_1^{(1)})$ from Eq.(2.16) into Eq.(2.17) then solving for $\epsilon \alpha_1$, we

find:

$$\epsilon \alpha_1 = \frac{i \Phi^{(0)} + \frac{d\Phi^{(0)}}{dt} + 4i \cos \frac{\theta}{2} \frac{d\varphi_1^{(0)}}{dt} \left(\frac{\frac{1}{2} \frac{dS}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} - \Sigma^{(0)} \frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}}}{-E + h_z(\Sigma) \big|_{\epsilon=0} + \frac{1}{2} S(\Sigma) \big|_{\epsilon=0} - \sqrt{h_x^2 + h_y^2} \cot \frac{\theta}{2} - 4 \cos^2 \left(\frac{\theta}{2} \right) \left[\frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} + \frac{1}{2} \frac{dS}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} \right]} \right)}{1 + 4 \cos^2 \left(\frac{\theta}{2} \right) \left(\frac{\frac{1}{2} \frac{dS}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} - \Sigma^{(0)} \frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}}}{-E + h_z(\Sigma) \big|_{\epsilon=0} + \frac{1}{2} S(\Sigma) \big|_{\epsilon=0} - \sqrt{h_x^2 + h_y^2} \cot \frac{\theta}{2} - 4 \cos^2 \left(\frac{\theta}{2} \right) \left[\frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} + \frac{1}{2} \frac{dS}{d\Sigma} \big|_{\Sigma=\Sigma^{(0)}} \right]} \right)} \quad (2.18)$$

Which is a general expression for the nonlinear Zak phase. Please note that the expression holds for any real-valued well-behaved functions $h_x(k), h_y(k)$, and for any real C^1 functions $h_z(\Sigma)$, and $S(\Sigma)$. Note also that the fraction within () vanish for linear Hamiltonians and the expression reduces to the conventional Zak phase $i \Phi^{(0)} + \frac{d\Phi^{(0)}}{dt}$ as expected. Furthermore, using the expression in Eq.(2.18) in

the limit $g_A = g_B$, we managed to successfully reproduce the Zak phases plots published in [18].

2.3 Theory of Dynamical Stability Analysis

In this section, we derive the analytical tools needed to apply a dynamical stability analysis on nonlinear energy bands in the reciprocal lattice space study.

Consider a time-dependent Schrodinger equation and its complex conjugate equation:

$$\begin{aligned} i \frac{\partial \Psi}{\partial t} &= H(\Sigma, k) \Psi \\ -i \frac{\partial \Psi^*}{\partial t} &= [H(\Sigma, k) \Psi]^* \end{aligned} \quad (2.19)$$

where $\hbar = 1$, $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, $\Sigma = |\psi_2|^2 - |\psi_1|^2$, k is the quasimomentum, and the Hamiltonian $H = H(\Sigma, k)$ is assumed to be hermitian(i.e. $H^* = H$).

Suppose the system is initially prepared to be in a stationary state $\Psi = \Psi^{(0)}$, such that:

$$i \frac{\partial \Psi}{\partial t} = H^{(0)} \Psi^{(0)} = E \Psi^{(0)} \quad (2.20)$$

Where $H^{(0)} = H(\Sigma^{(0)}, k)$, $\Psi^{(0)} = \begin{pmatrix} \psi_1^{(0)} \\ \psi_2^{(0)} \end{pmatrix}$ and $\Sigma^{(0)} = |\psi_2^{(0)}|^2 - |\psi_1^{(0)}|^2$.

Suppose at later time, the system evolves to be in a state Ψ' :

$$\Psi = \Psi^{(0)} \longrightarrow \Psi' = \Psi^{(0)} + \delta \Psi, \quad \Psi'^* = \Psi^{(0)*} + (\delta \Psi)^* \quad (2.21)$$

Where $\delta \Psi = \begin{pmatrix} \delta \psi_1 \\ \delta \psi_2 \end{pmatrix}$ is a small perturbation to $\Psi^{(0)}$.

It follows that:

$$\Sigma = |\psi_2|^2 - |\psi_1|^2 \longrightarrow \Sigma' = \Sigma^{(0)} + \psi_2^{(0)*} \delta\psi_2 + \psi_2^{(0)} \delta\psi_2^* - \psi_1^{(0)*} \delta\psi_1 - \psi_1^{(0)} \delta\psi_1^* + \delta\psi_2^* \delta\psi_2 - \delta\psi_1^* \delta\psi_1 \quad (2.22)$$

By rewriting the equations in Eq.(2.19) for Ψ' , one finds:

$$\begin{aligned} i \frac{\partial \Psi'}{\partial t} &= H'(\Sigma', k) \Psi' \\ -i \frac{\partial \Psi'^*}{\partial t} &= [H'(\Sigma', k) \Psi']^* \end{aligned} \quad (2.23)$$

By considering $\delta\psi_1$, $\delta\psi_2$, $\delta\psi_1^*$ and $\delta\psi_2^*$ as independent degrees of freedom and solving the system of equations in (2.23) for each independently, one get up to the first order in perturbation a differential equation of the form:

$$i \partial_t \begin{pmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \delta\psi_1^* \\ \delta\psi_2^* \end{pmatrix} = \mathcal{L} \begin{pmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \delta\psi_1^* \\ \delta\psi_2^* \end{pmatrix} \quad (2.24)$$

where $\mathcal{L}$ is a 4×4 matrix, that is generally non-hermitian, and depends on the Hamiltonian in question.

A solution of equations of Eq.(2.24) is known to depend on the eigenvalues $\{\lambda_n, \forall n\}$ of $\mathcal{L}$ through $e^{-i \lambda_n t}$. Recall that $\mathcal{L}$ is generally non-hermitian, therefore its eigenvalues are generally complex. Accordingly, for the state's perturbation to not grow to infinity as time increases, the eigenvalues have to fulfill the following

condition:

$$\Im(\lambda_n) = 0, \forall n \quad (2.25)$$

in other words, all eigenvalues have to be real. If a state's perturbation doesn't grow to infinity as time increases (i.e. the condition on eigenvalues is satisfied), the state $\Psi = \Psi^{(0)}$ is said to be dynamically stable, otherwise it is dynamically unstable.

In our numerical application of the stability analysis to a given state, we evaluate and study the quantity:

$$\max [|\Im(\lambda_n)|], \forall n \quad (2.26)$$

Equivalently to the condition in Eq.(2.25), if $\max [|\Im(\lambda_n)|] = 0, \forall n$, the state is dynamically stable, otherwise it is dynamically unstable.

CHAPTER 3

RESULTS AND DISCUSSION

3.1 Real Lattice Space Study

In the real lattice space study we consider N dimerized nonlinear SSH lattice, with a state-dependent Hamiltonian in the form:

$$\hat{H} = \hat{H}_{SSH} + \sum_{m=1}^N [v (\Psi_{A,m} \Psi_{A,m}^+ - \Psi_{B,m} \Psi_{B,m}^+) + g_A |\psi_{A,m}|^2 \Psi_{A,m} \Psi_{A,m}^+ + g_B |\psi_{B,m}|^2 \Psi_{B,m} \Psi_{B,m}^+] \quad (3.1)$$

where $\Psi_{q,m}$ is all zero column vector except one component being equal to one, this component corresponds to the site on the chain at unit cell number m , sublattice q , with $m \in \{1, 2, 3, \dots, N\}$, $q \in \{A, B\}$, $\psi_{q,m}$ represents the wavefunction's amplitude at site with unit cell number m , sublattice q , and g_A denotes the nonlinearity strength on sublattice A, and g_B is the nonlinearity strength on sublattice B.

For the Hamiltonian (3.1), a corresponding set of nonlinear Schrodinger equa-

tions has the form:

$$\begin{aligned} i \frac{d\psi_{A,m}}{dt} &= J_1 \psi_{B,m} + J_2 \psi_{B,m-1} + v \psi_{A,m} + g_A |\psi_{A,m}|^2 \psi_{A,m} \\ i \frac{d\psi_{B,m}}{dt} &= J_1 \psi_{A,m} + J_2 \psi_{A,m+1} - v \psi_{B,m} + g_B |\psi_{B,m}|^2 \psi_{B,m} \end{aligned} \quad (3.2)$$

where $m = 1, 2, 3, \dots, N$, and N is an even number that is large enough for the thermodynamical limit $N \rightarrow \infty$ to apply. In particular, we seek stationary solutions of the nonlinear time-independent Schrodinger equation, which reads in compact form:

$$\hat{H}(\Psi)\Psi = E\Psi \quad (3.3)$$

where $\Psi = \begin{pmatrix} \psi_{A,1} \\ \psi_{B,1} \\ \vdots \\ \psi_{A,N} \\ \psi_{B,N} \end{pmatrix}$, and H is the Hamiltonian in (3.1). Component-wise, Eq. (3.3), reads:

$$\begin{aligned} J_1 \psi_{B,m} + J_2 \psi_{B,m-1} + v \psi_{A,m} + g_A |\psi_{A,m}|^2 \psi_{A,m} &= E \psi_{A,m} \\ J_1 \psi_{A,m} + J_2 \psi_{A,m+1} - v \psi_{B,m} + g_B |\psi_{B,m}|^2 \psi_{B,m} &= E \psi_{B,m} \end{aligned} \quad (3.4)$$

The state-dependent nonlinear eigenvalue problem is solved assuming Open Boundary Conditions (OBC), which are given below:

$$\psi_{B,0} = \psi_{A,N+1} = 0 \quad (3.5)$$

This choice of OBC is natural when studying the SSH lattice, as it ensures that the lattice starts at sublattice A and ends at sublattice B , given that each unit cell contain two sublattices A and B .

A numerical method known as the Self-Consistent Field (SCF) iterative method is used to solve the problem, which is explained in Sec. (2.1). Note that the Hamiltonian (3.1) has 5 parameters (J_1 , J_2 , v , g_A , g_B). The iteration process in this numerical method starts by selecting an initial trial state, which was selected in our case from corresponding linear Hamiltonian, i.e. the Hamiltonian with the following parameters (J_1 , J_2 , v , 0 , 0). All energy eigenstates from the corresponding linear Hamiltonian were eventually selected as initial trial states, and only sufficiently quickly converging solutions were picked and plotted as shown below. Our study in the real lattice space considered different sign variations of the nonlinearity strengths g_A and g_B . In what follows, we show the main results observed.

In the case when g_A , and g_B are both positive, it was found that in the topologically nontrivial regime ($J_1 < J_2$) and at sufficiently high nonlinearities, the energy of the left (right) edge state depends only on g_A (g_B) and is constant with respect to g_B (g_A) (see Figs.(3.1, 3.4, 3.5) below). Moreover, the existence of only one edge state was found in the model for a particular set of values of a sublattice-dependent nonlinearity strength. For instance, note the disappearance of the left edge state in the region around $g_A = 3$ in Fig.(3.1), where the nonexistence of the left edge state was also verified after checking the wave profiles of all solutions near $g_A = 3$. These observations could potentially have significant roles in Quantum

state transfer applications.

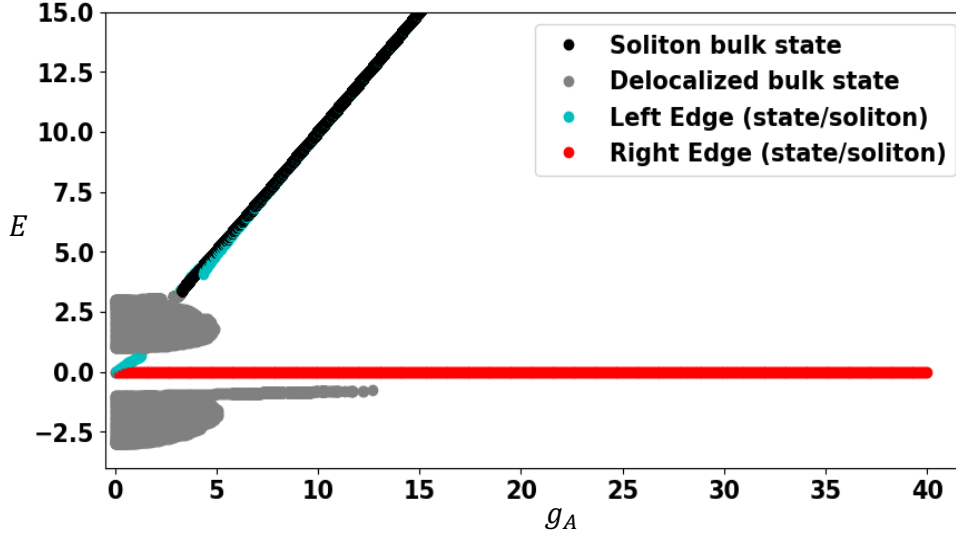
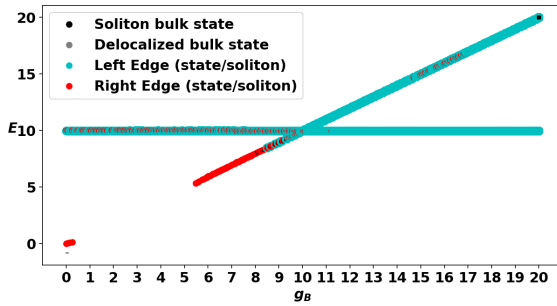
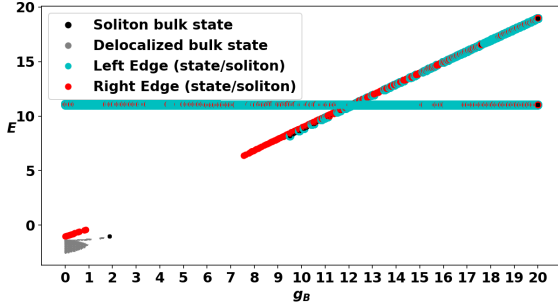


Figure 3.1: Energy vs parameter g_A for $J_1 = 1$, $J_2 = 2$, $v = 0.0001$, $g_B = 0$

In addition, for any values of J_1 and J_2 , and at sufficiently high nonlinearities, we note the presence of a touching point between the two energy bands as shown in Fig.(3.2a). This touching point was found to only shift (and not disappear) under v perturbation (see Fig.(3.2b)), which could indicate a topological origin analogous to gapless topological phases observed in Weyl semimetal for instance. Such a touching point was not observed in the corresponding case with global nonlinearity ($g_A = g_B$).



(a) Energy vs parameter g_B for $v = 0.0001$



(b) Energy vs parameter g_B for $v = 1$

Figure 3.2: Plots of Energy vs parameter g_B for $J_1 = 1$, $J_2 = 2$, $g_A = 10$

Moreover, in the case when g_A , and g_B have a sign difference, it was found that delocalized bulk solutions persist even at high nonlinearity strengths as shown in Fig.(3.3). An anticipation of such result can be reached by recalling that solving Eq.(3.4) iteratively for g_A , and g_B with sign difference is basically solving a coupled Gross-Pitaevskii type equations, one with attractive interaction and the other repulsive. Furthermore, we noted that the two delocalized bands shift as g_A changes such that their center is in the vicinity of $|\frac{g_A}{g_B}| \simeq 1$.

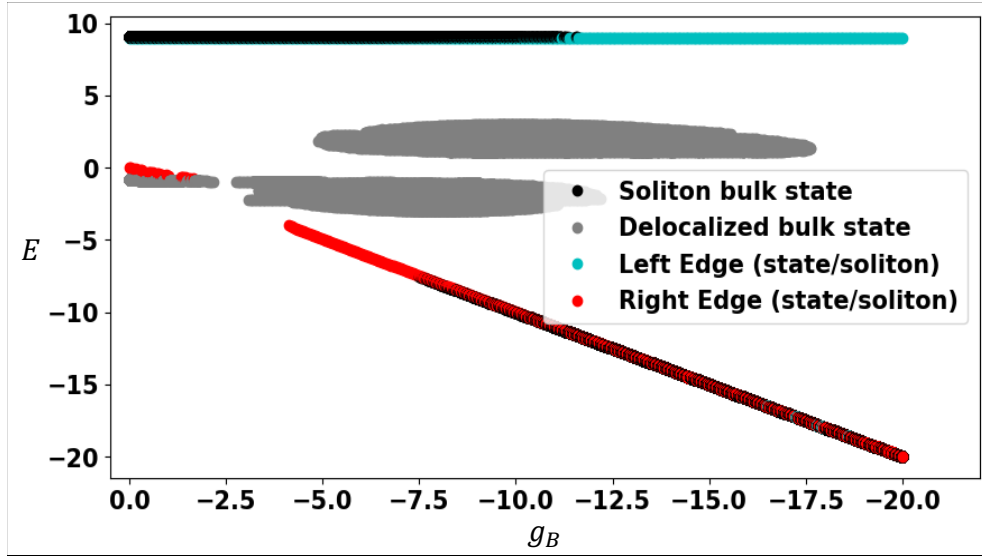


Figure 3.3: Energy vs parameter g_B for $J_1 = 1$, $J_2 = 2$, $v = 0.0001$, $g_A = 9$

An interesting observation in real lattice space is the appearance of a robust state which we named "Wave-Packet" state (reason behind the name is stated below). This state is characterized by its energy being the highest energy in the lower energy band where its value is around -1, as shown in Fig.(3.4) in gray, it persists with almost constant energy just below the right edge state (in red) as g_A increases. It is worth noting, however, that the robustness of the "Wave-

"Packet" state seems to depend on the model's parameters. Note, for example, how the robustness of the "Wave-Packet" state varies from energy plot in Fig.(3.1) compared to Fig.(3.4) just by setting $v = 0.0001$ instead of $v = 0.5$. Furthermore, in contrast to the energy plot in Fig.(3.4), when energy was plotted against g_B (not g_A as Fig.(3.4)), the "Wave-Packet" state did not persist at high nonlinearity strength as can be seen in Fig.(3.5).

When we studied the "Wave-Packet" state's profile in the first case (corresponding to Fig.(3.4)) starting from zero nonlinearity strength ($g_A = 0$), we found that the state becomes more and more localized (as shown in Fig.(3.7)) as the nonlinearity strength g_A increases. This localization, however, is accompanied with oscillatory behavior hence the name "Wave-Packet" state. At non-zero nonlinearity strength ($g_A > 0$) the state's envelope seems to have exponential decay.

However, when we looked at the "Wave-Packet" state's profile in the second case (corresponding to Fig.(3.5)) as the nonlinearity strength g_B increases, it was clear that the nonlinearity strength g_B has stronger effect on the "Wave-Packet" state's localization than g_A . To see that clearly, compare the "Wave-Packet" state's localization in Fig.(3.8) with that in Fig.(3.7) for $g_A = g_B = 2$ for instance.

An explanation of why as the nonlinearity strength increases, the "Wave-Packet" state becomes more and more localized (as shown in Figs.(3.7, 3.8)) can be made by the argument proposed in [18] of nonlinearity-induced effective edges in the bulk. That is, due to the Hamiltonian dependence on the state, a wave function strongly localized at one site increases the potential energy there, effectively creating a potential barrier, which can be a large on-site potential for strong

nonlinearity strength, thus effectively behaving like a physical edge. The location of this effective edge can be thought of to be at the center of the "Wave-Packet" state, where it acts as an effective potential barrier that admits a strongly localized wave function, whose probability density decays seemingly exponentially with the distance from the effective edge.

Finally, the "Wave-Packet" state seems to be numerically stable, as at sufficiently high nonlinearity we got multiple degenerate states with equal energies up to 10^{-6} in dimensionless units. Interestingly, the "Wave-Packet" state exists even in the topologically trivial regime ($J_1 > J_2$) as shown in Fig.(3.6), so despite the nonexistence of edge states one could still have a state with almost constant energy in the topologically trivial regime.

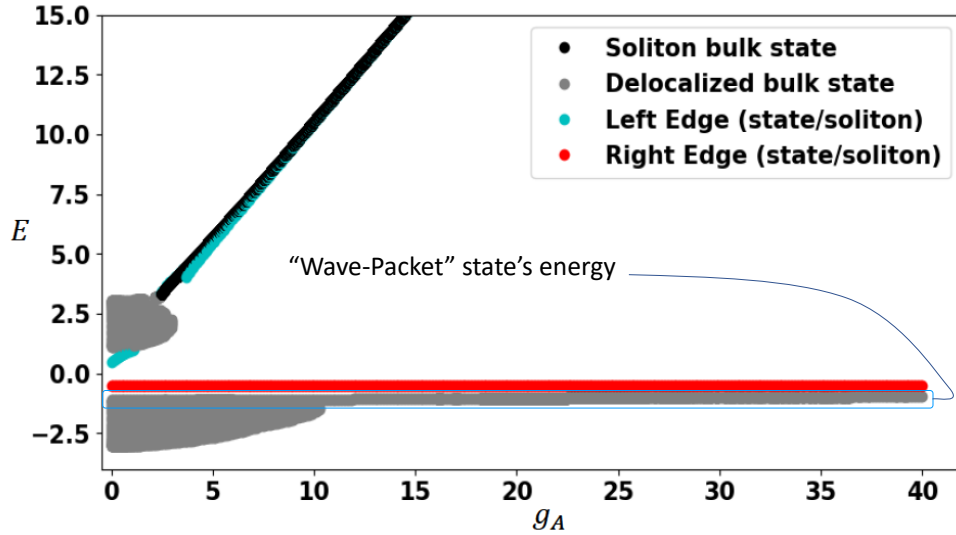


Figure 3.4: Energy vs parameter g_A for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 0$

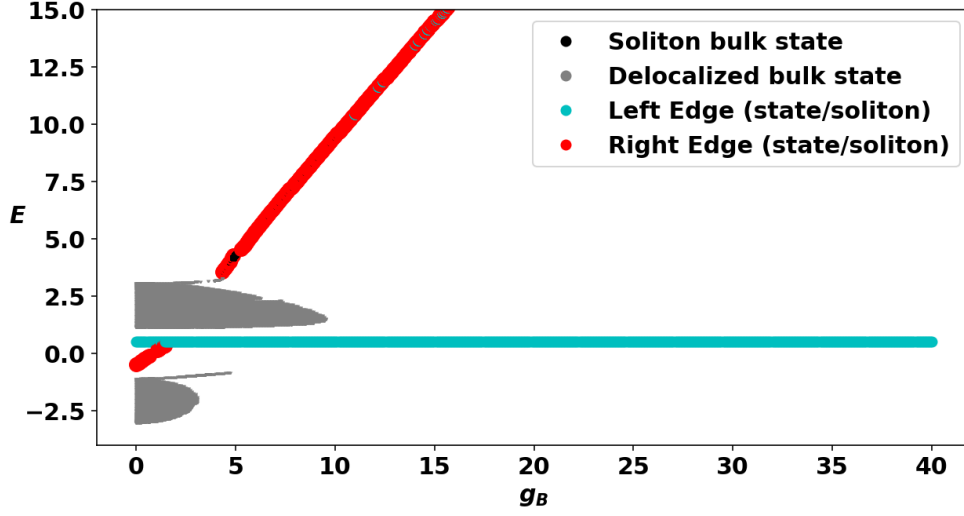


Figure 3.5: Energy vs parameter g_B for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 0$

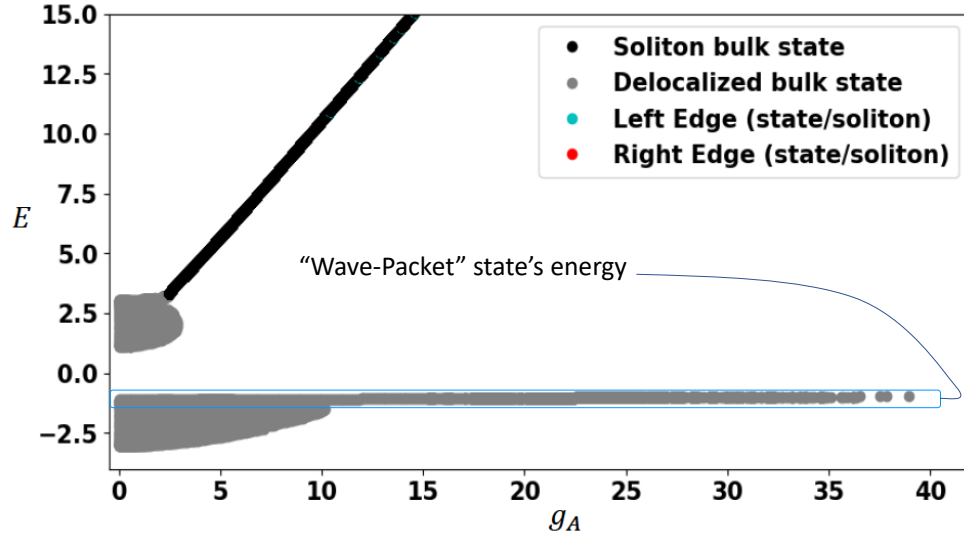


Figure 3.6: Energy vs parameter g_A for $J_1 = 2$, $J_2 = 1$, $v = 0.5$, $g_B = 0$

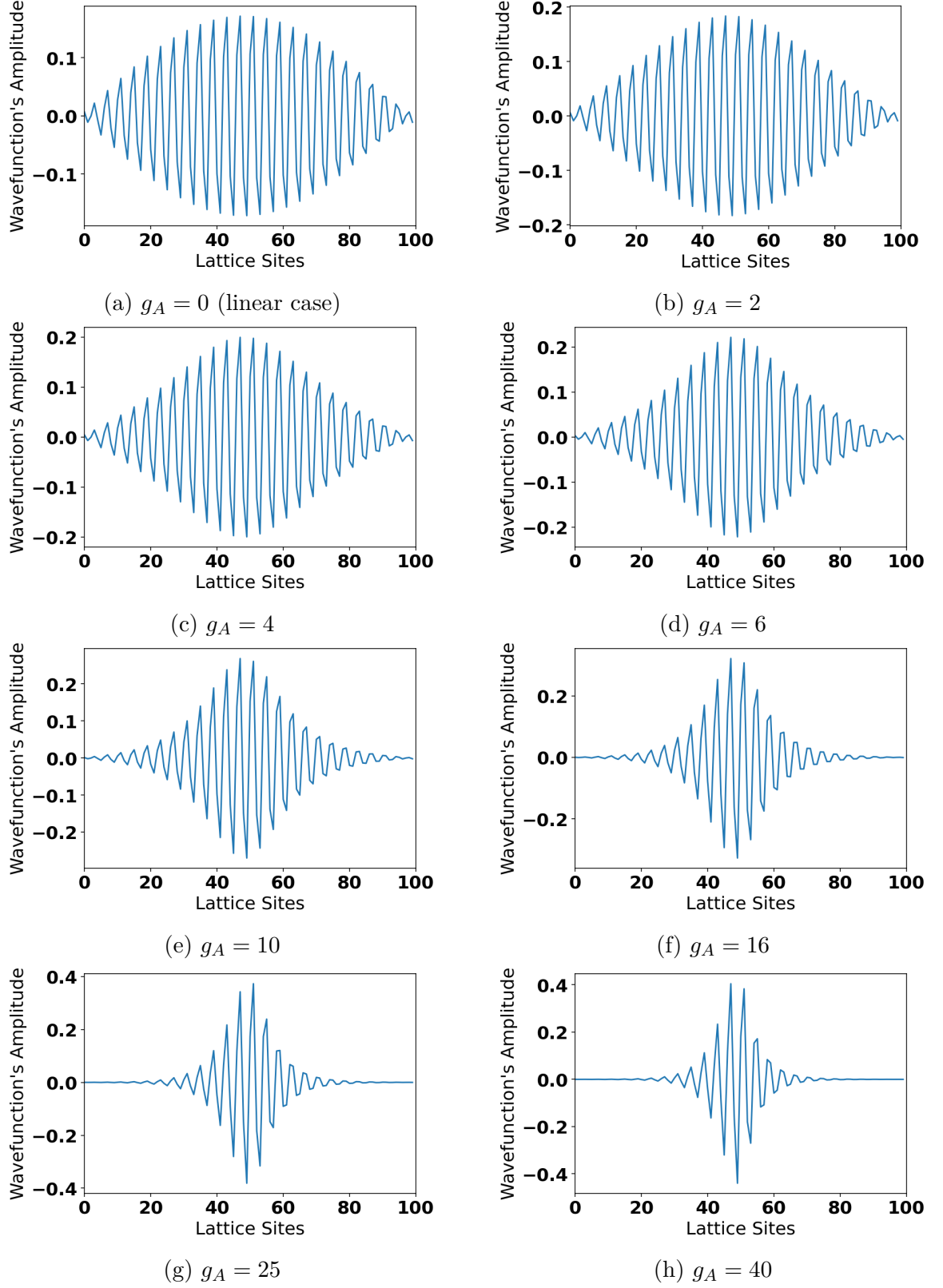


Figure 3.7: Plots of the "Wave-Packet" state's profile, as nonlinearity strength g_A increases, starting from $g_A = 0$ (top-left), to $g_A = 40$ (bottom-right) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 0$

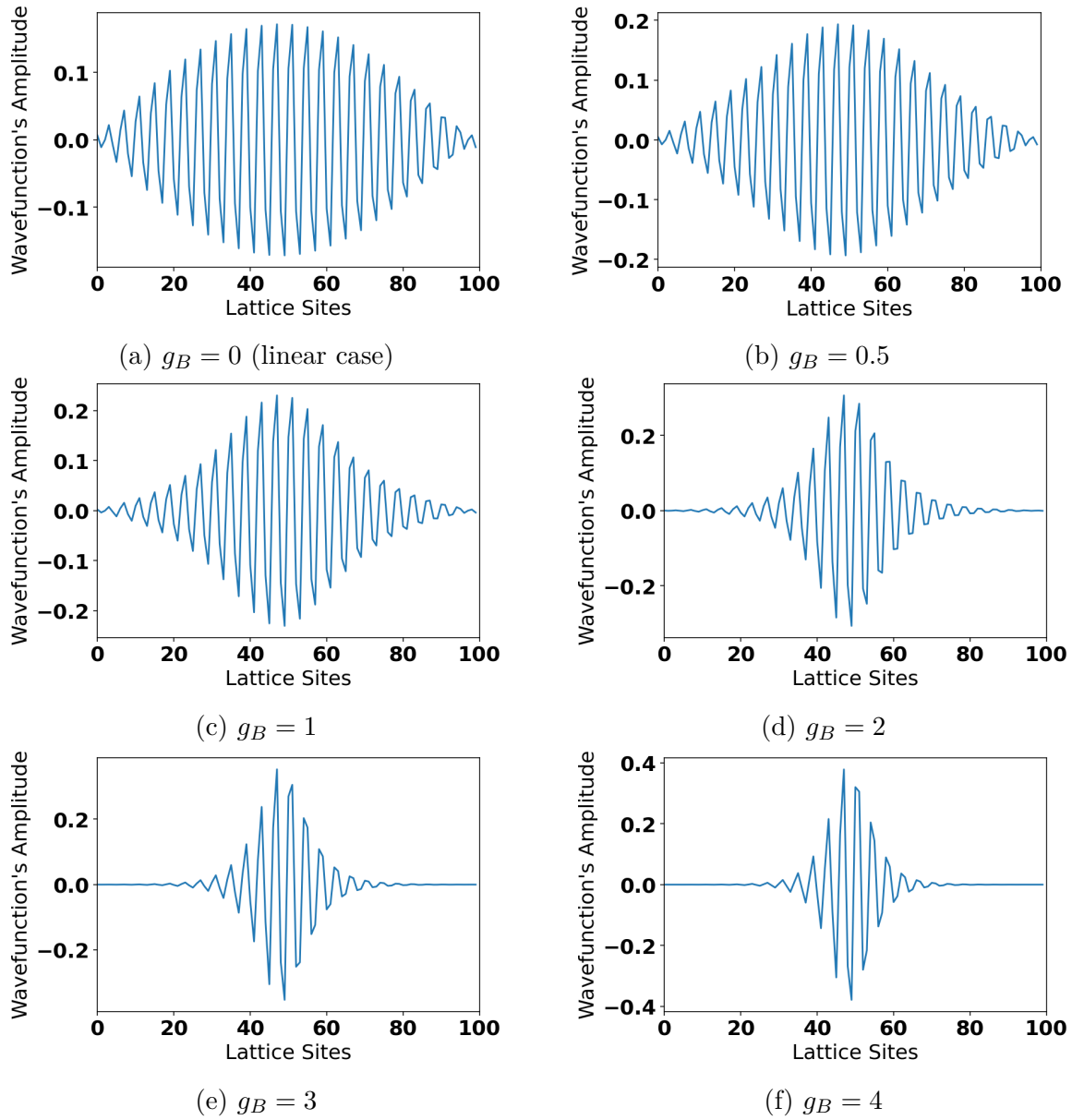


Figure 3.8: Plots of the "Wave-Packet" state's profile, as nonlinearity strength g_B increases, starting from $g_B = 0$ (top-left), to $g_B = 4$ (bottom-right) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 0$

3.2 Reciprocal Lattice Space Study

In the reciprocal lattice space study we assume Bloch's state solutions

$$\begin{aligned}\psi_{A,j} &= \Phi_A e^{ikj} \\ \psi_{B,j} &= \Phi_B e^{ikj}\end{aligned}\tag{3.6}$$

under the Periodic Boundary Conditions (PBCs) $\psi_{A,N+1} = \psi_{A,1}$, and $\psi_{B,0} = \psi_{B,N}$. In what follows we denote Φ_A and Φ_B by Φ_1 and Φ_2 respectively, for notational consistency with derivation in Sec.(2.2). Substituting Eq.(3.6) into Eq.(3.2), and considering stationary solutions, we get the nonlinear eigenvalue problem

$$H(\Sigma, k)\Phi = E\Phi\tag{3.7}$$

$k \in [0, 2\pi)$, with the Hamiltonian

$$H(\Sigma, k) = h_x(k)\sigma_x + h_y(k)\sigma_y + h_z(\Sigma)\sigma_z + \frac{S(\Sigma)}{2}I\tag{3.8}$$

where $h_x(k), h_y(k)$ are the coupling terms in the SSH model, namely $h_x(k) = J_1 + J_2 \cos k$, $h_y(k) = J_2 \sin k$, the diagonal terms $h_z(\Sigma)$ and $S(\Sigma)$ are $h_z(\Sigma) = v + \frac{1}{4}[g_A - g_B - \Sigma(g_A + g_B)]$, $S(\Sigma) = \frac{1}{2}[(g_B - g_A)\Sigma + g_A + g_B]$. Moreover, Φ is a pseudo-spinor $\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ and $\Sigma = |\varphi_2|^2 - |\varphi_1|^2$, on the other hand, σ_x , σ_y , σ_z are 2×2 Pauli matrices, and I is the 2×2 identity matrix acting on Φ .

Note that in the reciprocal lattice space study the general Hamiltonian sim-

plifies and becomes block-diagonal for each k as shown in (3.8). Accordingly, the normalization condition in the reciprocal lattice space study is applied for a given Φ only within its corresponding unit cell k in the reciprocal lattice space.

In addition to assuming Bloch's state solutions, this observation highlights another major difference from the real lattice space study, where the normalization condition for a given Ψ was applied to the whole lattice sites. Due to these differences, the two problems are to be considered as separate problems.

To solve Eq.(3.7) with the Hamiltonian Eq.(3.8) for stationary solutions with corresponding energy eigenvalues for a fixed set of parameters (J_1 , J_2 , v , g_A , g_B), we consider the equation:

$$\det [H(\Sigma, k) - E I] = \begin{vmatrix} h_z(\Sigma) + \frac{S(\Sigma)}{2} - E & h_x(k) - ih_y(k) \\ h_x(k) + ih_y(k) & -h_z(\Sigma) + \frac{S(\Sigma)}{2} - E \end{vmatrix} = 0$$

which when solved for E gives the two energy roots:

$$E_{1,2} = \frac{S(\Sigma)}{2} \pm \sqrt{h_x^2 + h_y^2 + h_z^2} \quad (3.9)$$

Without loss of generality, we may parametrize our state Φ (using Bloch sphere representation) as:

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \quad (3.10)$$

Where $\theta \in [0, \pi)$, and $\phi \in [0, 2\pi)$. Moreover, from the Bloch sphere representa-

tion, we get the identity [16]:

$$h_z(\Sigma) = \sqrt{h_x^2 + h_y^2 + h_z^2} \cos \theta \quad (3.11)$$

By squaring both sides of Eq.(3.11), and writing $|\varphi_2|^2 = 1 - |\varphi_1|^2 = 1 - \cos^2 \frac{\theta}{2}$ after using the normalization condition, we have reduced the equation to depend on one unknown, namely in $\cos^2 \frac{\theta}{2}$. For notational convenience we denote $x = \cos^2 \frac{\theta}{2}$, and after some algebra Eq.(3.11) becomes:

$$\begin{aligned} & x^4 [(g_A)^2 + (g_B)^2 + 2 g_A g_B] + x^3 [4v(g_A + g_B) - 4g_A g_B - (g_A)^2 - 3(g_B)^2] \\ & + x^2 [4(J_1^2 + J_2^2 + v^2 + 2 J_1 J_2 \cos k) + 3 (g_B)^2 + 2 g_A g_B - 4v g_A - 8 v g_B] \\ & + x [4v g_B - 4(J_1^2 + J_2^2 + v^2 + 2 J_1 J_2 \cos k) - (g_B)^2] + J_1^2 + J_2^2 + 2 J_1 J_2 \cos k = 0 \end{aligned} \quad (3.12)$$

The quartic polynomial in Eq.(3.12) is solved numerically for a given k value, where $k \in [0, 2\pi)$, and only real roots $x \in \mathbb{R}$ with their values being less than or equal to one and greater than zero $0 < x \leq 1$, are selected. The latter condition follows from the normalization condition. Note that in the x notation, $\Sigma = 1 - 2x$. It's now straight forward to evaluate corresponding energies from Eq.(3.9).

Moreover, note that $\varphi_1 = \sqrt{x}$ and $\varphi_2 = \frac{h_x(k) + ih_y(k)}{\sqrt{h_x^2 + h_y^2}} \sqrt{1 - x}$ where the expression in Eq.(2.8) has been used. Evaluating φ_1 and φ_2 numerically is needed for numerical calculation of the nonlinear Zak phase.

An expression of the nonlinear Zak phase for our Hamiltonian Eq.(3.8) can be obtained by substituting corresponding $h_x(k)$, $h_y(k)$, $h_z(\Sigma)$ and $S(\Sigma)$ into the

general expression of nonlinear Zak phase Eq.(2.18) which was derived in Sec.(2.2).

This leads to:

$$\epsilon\alpha_1 = \frac{i\Phi^+ \frac{d\Phi}{dt} + 4i \cos \frac{\theta}{2} \frac{d\varphi_1}{dt} \left(\frac{\frac{1}{4}[g_B - g_A + \Sigma(g_A + g_B)]}{-E + v + \frac{g_A}{2}(1 - \Sigma) - \sqrt{h_x^2 + h_y^2} \cot \frac{\theta}{2} + 2g_A \cos^2 \frac{\theta}{2}} \right)}{1 + 4 \cos^2 \left(\frac{\theta}{2} \right) \left(\frac{\frac{1}{4}[g_B - g_A + \Sigma(g_A + g_B)]}{-E + v + \frac{g_A}{2}(1 - \Sigma) - \sqrt{h_x^2 + h_y^2} \cot \frac{\theta}{2} + 2g_A \cos^2 \frac{\theta}{2}} \right)} \quad (3.13)$$

It's clear from the nonlinear Zak phase expression in Eq.(3.13) that in the case where $g_A = g_B = 0$ (the linear limit), the fraction within () vanish and the expression reduces to the conventional Zak phase $i\Phi^+ \frac{d\Phi}{dt}$ as expected.

Recall that the expression of the nonlinear Zak phase derived in [18] (for the case when $S(\Sigma) = 0$), reads:

$$\epsilon\alpha_1 = i\Phi^+ \frac{d\Phi}{dt} \left(1 + \frac{\frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma(0)} \cos \theta (1 + \cos \theta)}{E + \frac{dh_z}{d\Sigma} \big|_{\Sigma=\Sigma(0)} \sin^2 \theta} \right) \quad (3.14)$$

Despite algebraic difference in our expression of the nonlinear Zak phase Eq.(2.1) and the expression in Eq.(3.14) used in [18], in studied cases, the two yield the same result in the limit $g_A = g_B$. In addition, in the linear limit both expressions reduce to the conventional Zak phase $i\Phi^+ \frac{d\Phi}{dt}$ as expected.

The difference in the two expressions' algebraic forms might be attributed to how each expression was derived, in [18] the derivation relied on a hidden eigenstate relation of the Hamiltonian $H^{(0)}$ in Eq.(2.6) for the case when $S(\Sigma) = q$ in Eq.(2.2), where q is just a constant. This relation can be obtained by noting that :

$$\Phi^{(0)\perp} = \begin{pmatrix} \sin \frac{\theta}{2} \\ -e^{i\phi} \cos \frac{\theta}{2} \end{pmatrix}$$

is an (hidden) eigenstate of $H^{(0)}$

$$H^{(0)}\Phi^{(0)\perp} = (E - \frac{q}{2})\Phi^{(0)\perp} \quad (3.15)$$

where $H^{(0)} = H(\Sigma^{(0)}, k)$ and $\Sigma^{(0)}$ corresponds to the state $\Phi^{(0)}$ in Eq.(2.7). In our derivation, however, we did not use the relation in Eq.(3.15), instead we used Eq.(2.8) as illustrated in Sec.(2.2). It is known in mathematics that showing algebraic equivalence of two expressions just by algebraic manipulation might not be a trivial task, which turned out to apply in our case. However, the facts that both expressions were derived self-consistently, yielded the same result in studied cases, and in the linear limit approach the same limit, strongly suggest their equivalence. Prior to studying the regime $g_A \neq g_B$, we successfully bench-marked our code by reproducing energy and Zak phases plots published in [18] in the limit $g_A = g_B$.

In what follows, we adapt the following coloring of energy bands in our reciprocal lattice space study. For a given Hamiltonian, the lowest energy band is colored in blue and designated E_1 , whereas the highest energy band is colored red and designated E_4 . In the case where the Hamiltonian has third and fourth bands (due to nonlinearity), the second lowest energy band is colored in magenta and designated E_2 , and the second highest energy band is colored in cyan and

designated E_3 . This coloring association will also follow during the study of Zak phases as well as dynamical stability analysis of nonlinear energy bands.

A method that we propose to easily identify emerging energy bands, as well as energy gap-closing in a nonlinear system is by plotting the energy bands (for all k values) versus the parameter of the problem, from the plot one can visually identify at which parameter value an observation of interest take place. As an illustrative example, we consider the following plot in Fig.(3.9) for the case when both g_A and g_B are positive. Note, how at around $g_A = 2.3$ new energy bands (colored mainly in magenta) emerge between the two energy bands E_1 (in blue) and E_4 (in red). These emerging energy bands turned out to correspond to the loop structure which is depicted in Fig.(3.10,b) below. Moreover, from the same plot in Fig.(3.9), gap-closing can be easily seen at $g_A = 5$, where the energy band E_4 (in red) emerges with the loop structure. The latter gap-closing corresponds to what is most likely a topological phase transition, this identification is further supported by the jump discontinuity in the nonlinear Zak phase of energy band E_4 (in red) at $g_A = 5$ as shown in Fig.(3.11)

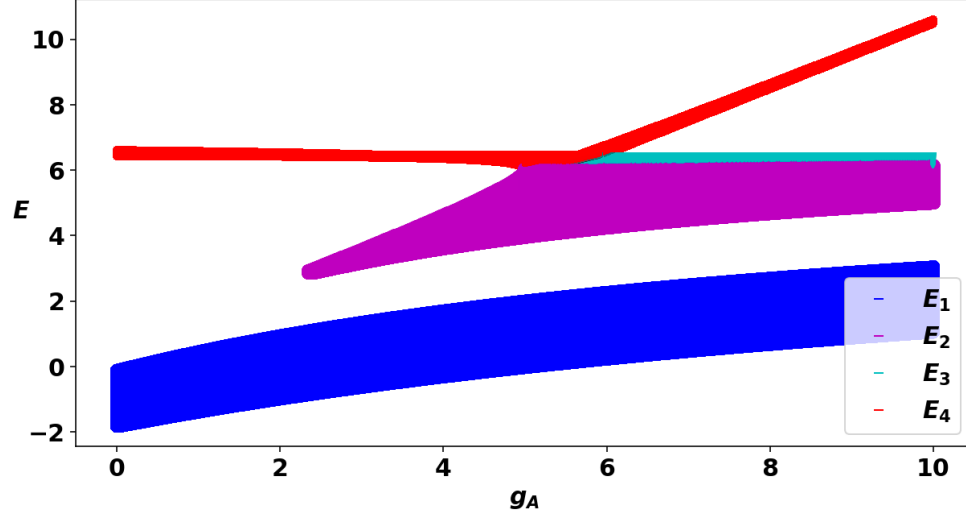


Figure 3.9: Energy vs parameter g_A for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 7$

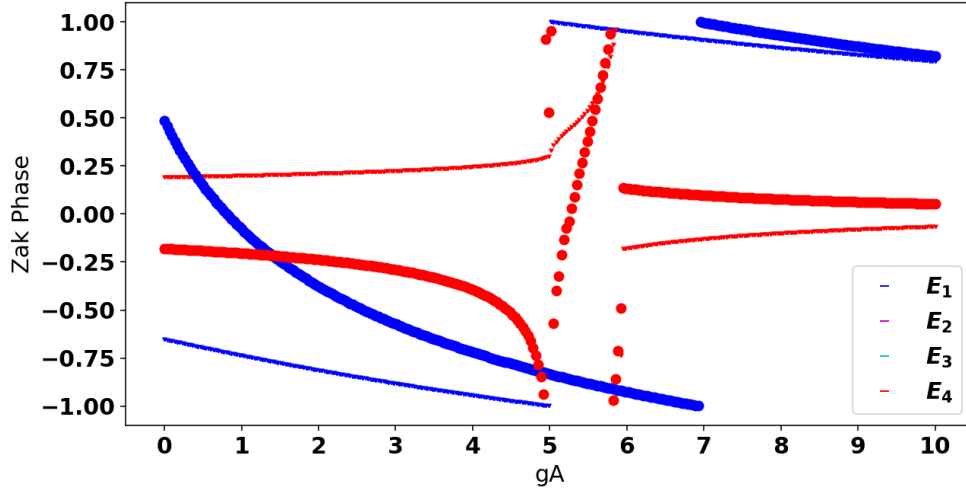


Figure 3.11: Nonlinear Zak phase (in bold), and Conventional Zak phase (in thin) vs parameter g_A for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 7$

Similar energy bands structure to that shown in Fig.(3.9) - although with inverted energy bands - was observed in the case of negative g_A and g_B . However, when g_A and g_B have different signs no gap-closing was observed (only two energy bands existed) as depicted in a representative energy plot Fig.(3.12). Although the red energy bands in Fig.(3.12) may seem flat, they are not actually (see Fig.(3.13)).

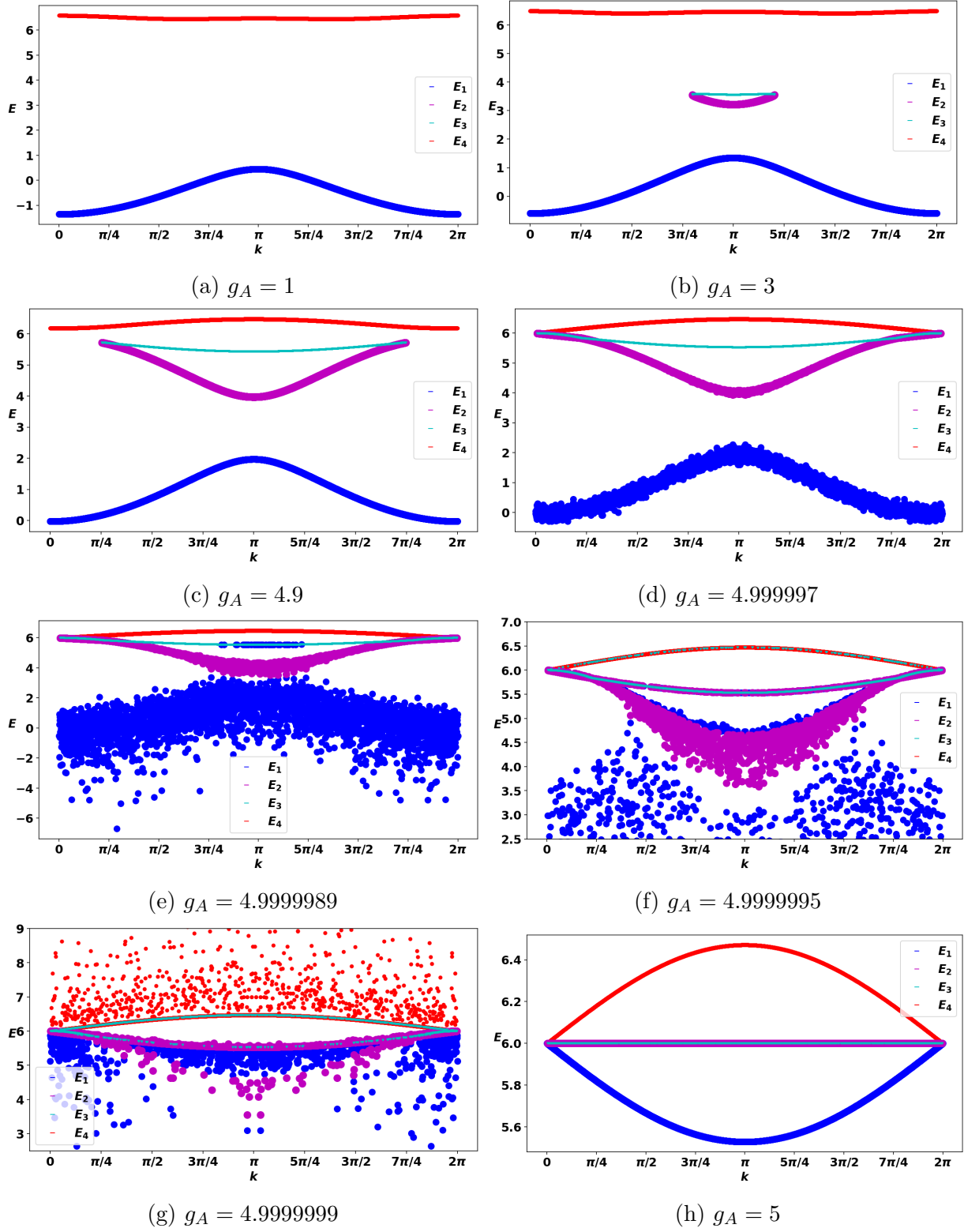


Figure 3.10: Plots of energy bands (E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 7$, as nonlinearity strength g_A increases (see Fig.(3.9)).

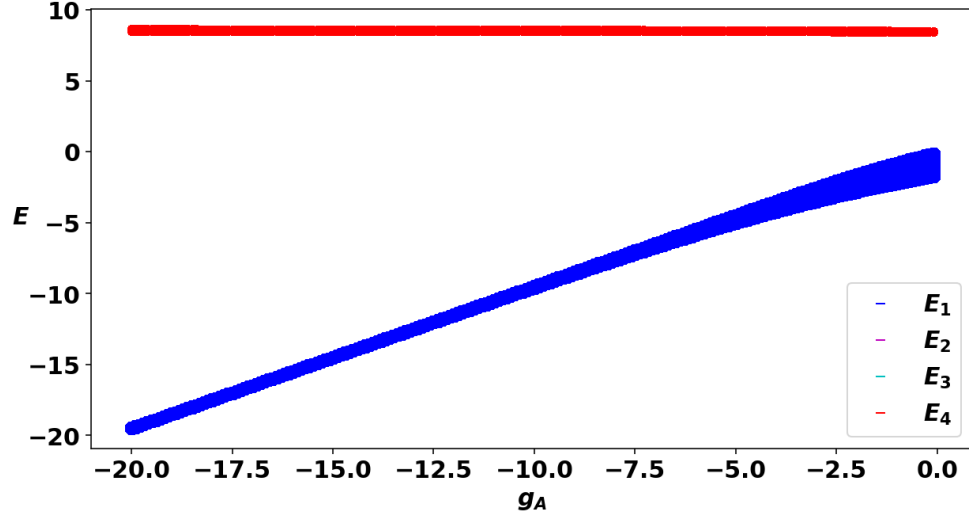


Figure 3.12: Energy vs parameter g_A for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 9$

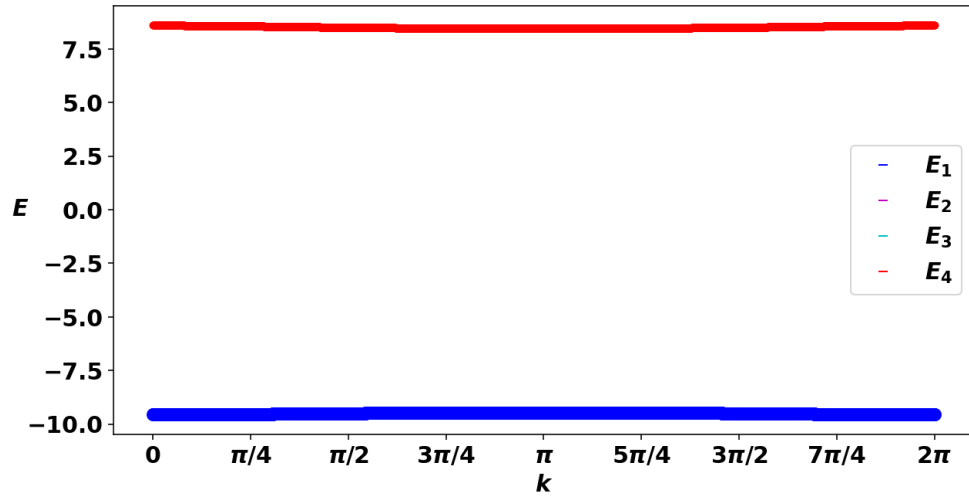


Figure 3.13: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = -10$, $g_B = 9$

When Zak phases were computed for the energy bands in Fig.(3.12), we see from the plot in Fig.(3.14) that although the conventional Zak phase is not generally quantized, at sufficiently high nonlinearity strength the nonlinear Zak phase approaches quantized value.

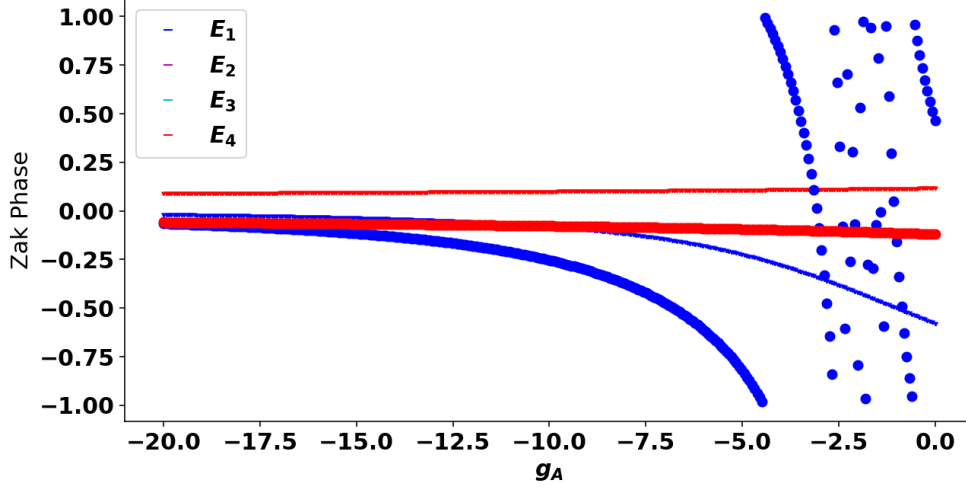


Figure 3.14: Nonlinear Zak phase (in bold), and Conventional Zak phase (in thin) vs parameter g_A for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 9$

To study the dynamical stability of a state $\Psi^{(0)} = \begin{pmatrix} \psi_1^{(0)} \\ \psi_2^{(0)} \end{pmatrix}$ for the Hamiltonian (3.8), we need to evaluate the corresponding $\mathcal{L}$ matrix that was defined in Eq.(2.24), then find its eigenvalues. Following the steps detailed in Sec.(2.3), we get:

$$i \partial_t \begin{pmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \delta\psi_1^* \\ \delta\psi_2^* \end{pmatrix} = \mathcal{L} \begin{pmatrix} \delta\psi_1 \\ \delta\psi_2 \\ \delta\psi_1^* \\ \delta\psi_2^* \end{pmatrix}$$

where

$$\mathcal{L} = \begin{pmatrix} v + 2g_A|\psi_1^{(0)}|^2 - E & h_x(k) - ih_y(k) & g_A\left(\psi_1^{(0)}\right)^2 & 0 \\ h_x(k) + ih_y(k) & -v + 2g_B|\psi_2^{(0)}|^2 - E & 0 & g_B\left(\psi_2^{(0)}\right)^2 \\ -g_A\left(\psi_1^{(0)*}\right)^2 & 0 & -\left(v + 2g_A|\psi_1^{(0)}|^2 - E\right) & -[h_x(k) + ih_y(k)] \\ 0 & -g_B\left(\psi_2^{(0)*}\right)^2 & -[h_x(k) - ih_y(k)] & -\left(-v + 2g_B|\psi_2^{(0)}|^2 - E\right) \end{pmatrix} \quad (3.16)$$

To test the dynamical stability of the state, we find the eigenvalues of $\mathcal{L}$ from the characteristic polynomial:

$$\det[\mathcal{L} - \lambda I] = 0 \quad (3.17)$$

which is solved numerically, then we evaluate the quantity

$$\max[|\Im(\lambda_n)|], \quad \forall n \quad (3.18)$$

If $\max[|\Im(\lambda_n)|] = 0$, $\forall n$, the state is dynamically stable, otherwise it is dynamically unstable.

Next, we show plots for a dynamical stability analysis for representative energy bands for different nonlinearity strength g_A . In particular, we consider energy bands near the topological phase transition shown in Fig.(3.10). Specifically, we begin by fixing the parameters ($J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_B = 7$) while varying g_A . Starting with energy band where $g_A = 1$

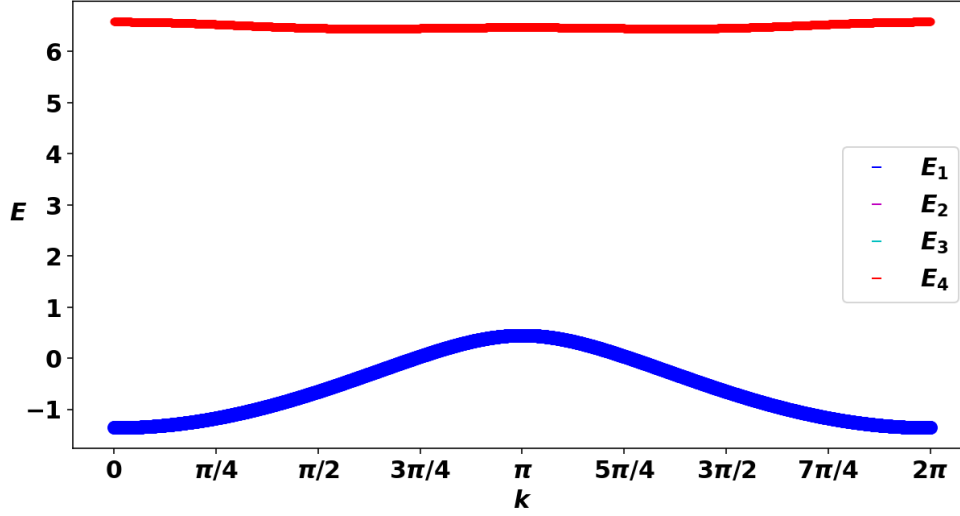


Figure 3.15: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 1$, $g_B = 7$

The stability analysis of the energy band in Fig.(3.15):

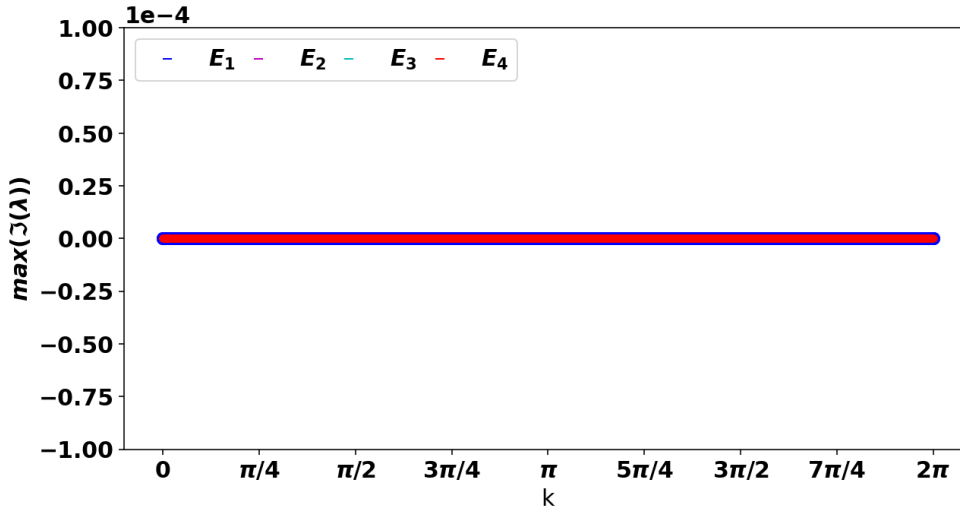


Figure 3.16: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 1$, $g_B = 7$

which shows in Fig.(3.15) that both energy bands E_1 (in blue) and E_4 (in red) are dynamically stable. As g_A increases further to $g_A = 3$, the energy band has the form Fig.(3.17)

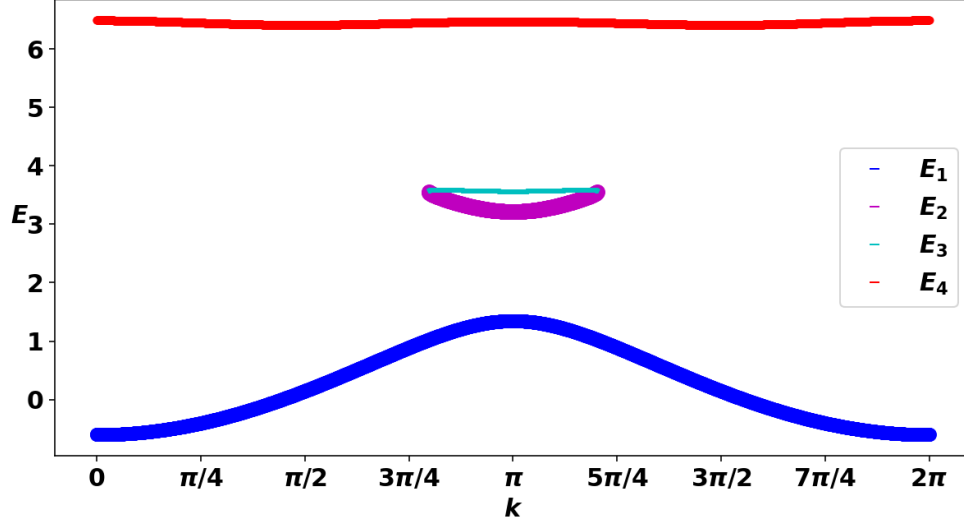


Figure 3.17: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 3$, $g_B = 7$

A loop structure emerges between energy values 3 and 4 and the stability of the bands in Fig.(3.17) shows stable outermost energy bands (E_1 and E_4) and dynamically unstable loop structure (E_2 and E_3) as shown in Fig.(3.18)

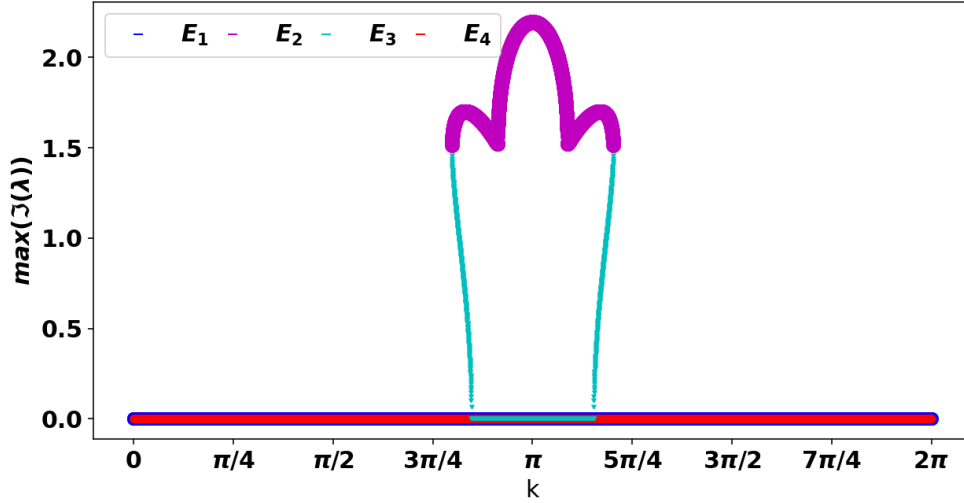


Figure 3.18: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 3$, $g_B = 7$

As $g_A = 4.9$, we have energy bands of the form Fig.(3.19), with their dynamical

ical stability analysis shown in Fig.(3.20). Note how energy bands are becoming increasingly unstable as g_A and g_B become comparable in strength.

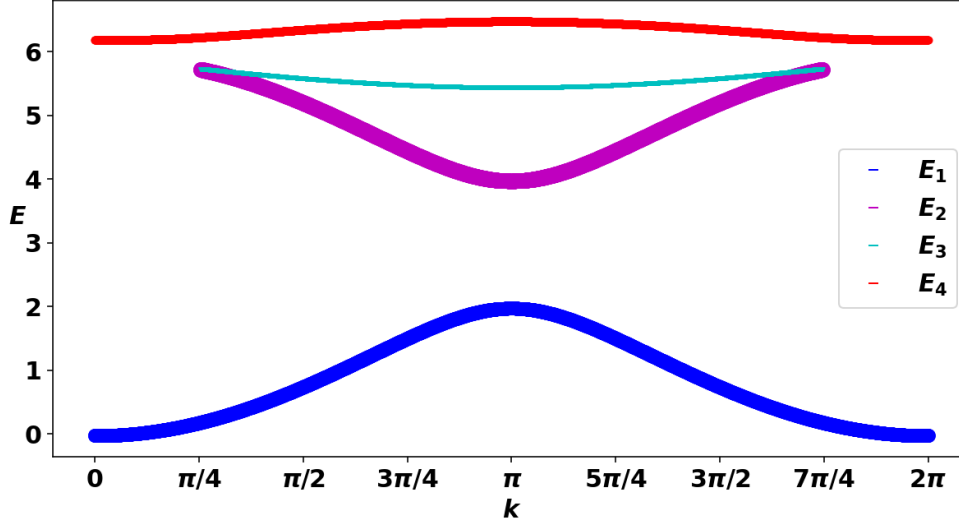


Figure 3.19: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 4.9$, $g_B = 7$

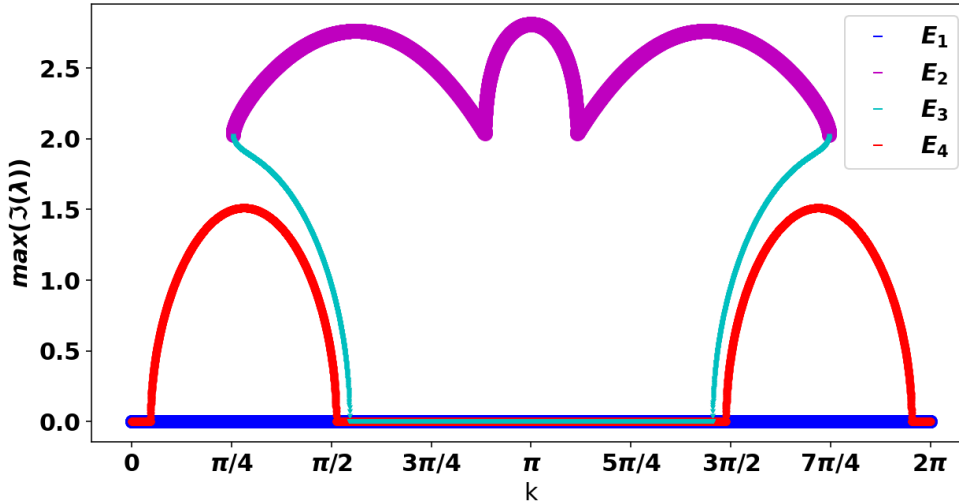


Figure 3.20: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 4.9$, $g_B = 7$

As we get closer to the topological phase transition point at $g_A = 5$, the lowest energy band E_1 starts to deconstruct (lose its typical structure) Fig.(3.21), where the dynamical stability analysis of the energy bands is as shown in Fig.(3.22).

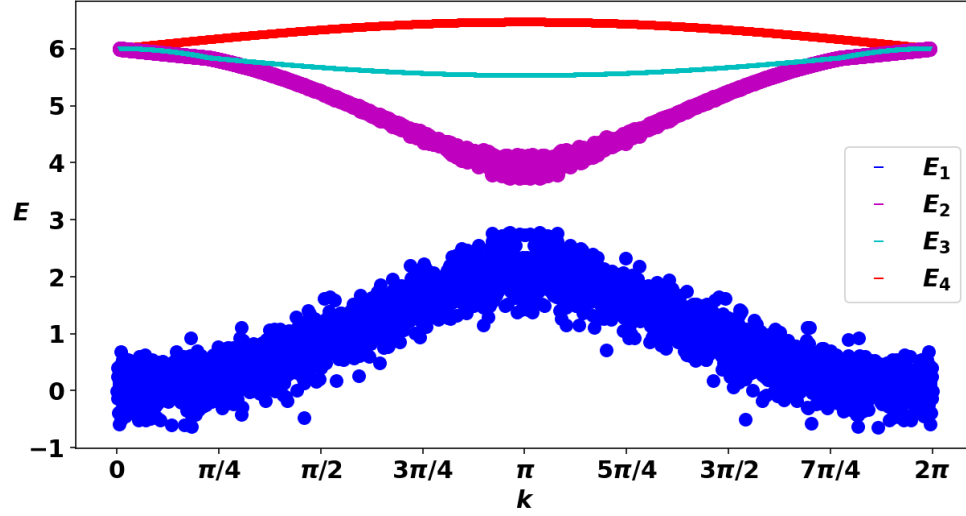


Figure 3.21: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 4.999998$, $g_B = 7$

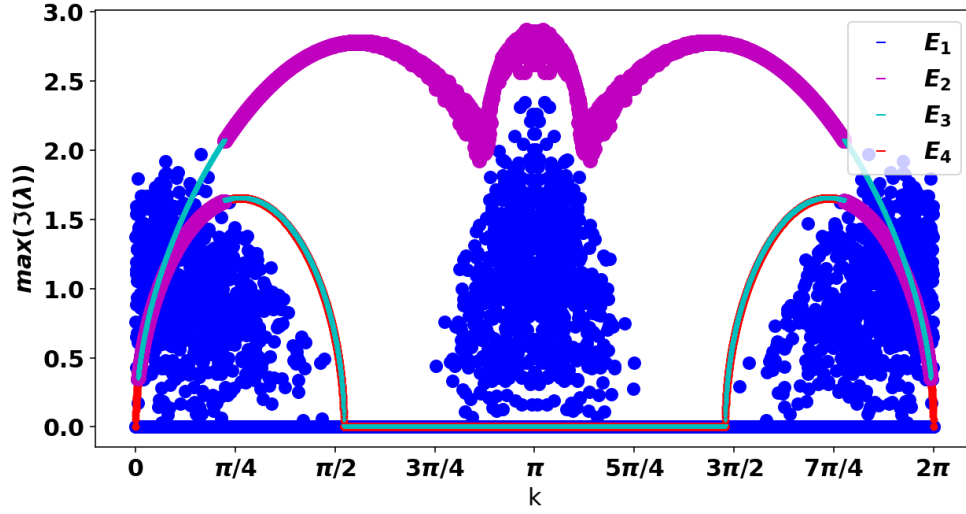


Figure 3.22: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 4.999998$, $g_B = 7$

Eventually, energy bands as in Fig.(3.23) become more dynamically unstable
Fig.(3.24)

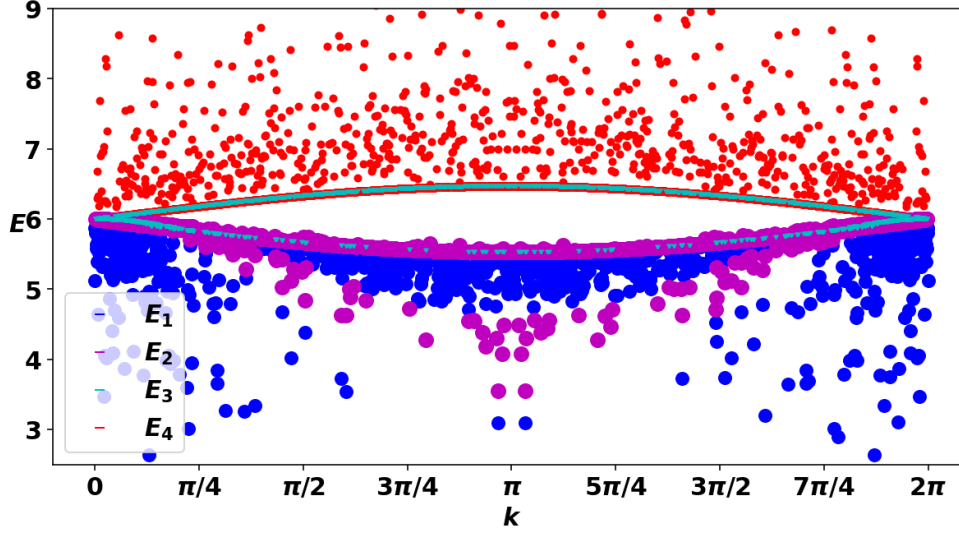


Figure 3.23: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 4.9999999$, $g_B = 7$

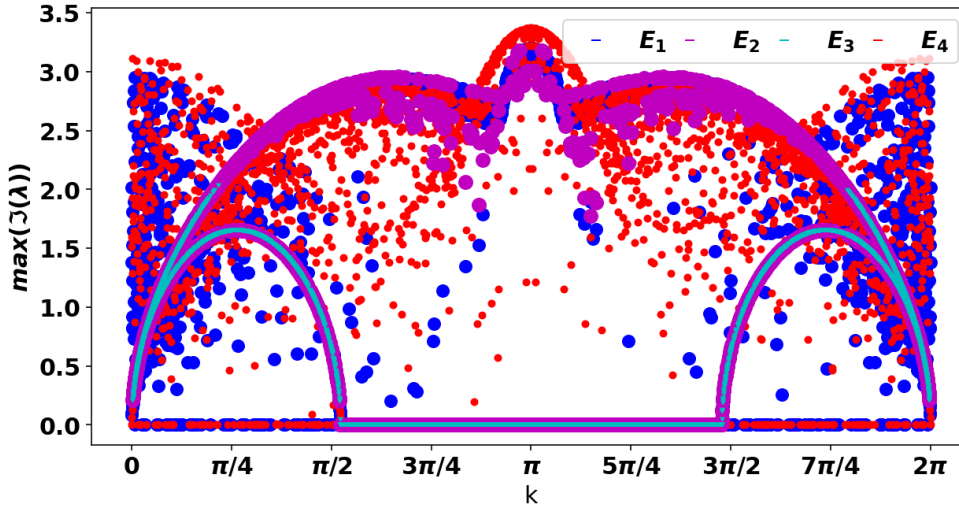


Figure 3.24: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 4.9999999$, $g_B = 7$

Until the topological phase transition point is reached at exactly $g_A = 5$, and all bands join to form a 3 bands loop structure Fig.(3.25) with its dynamical stability analysis shown in Fig.(3.26).

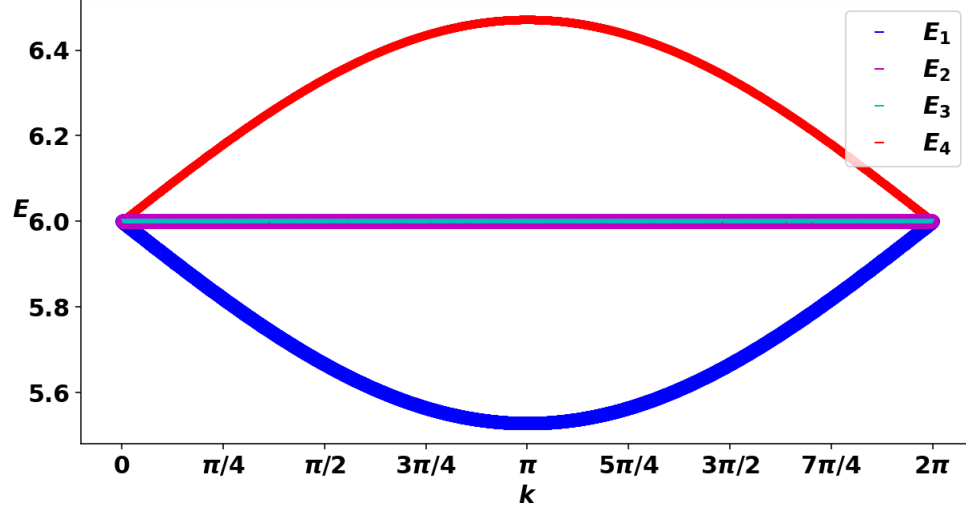


Figure 3.25: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 5$, $g_B = 7$

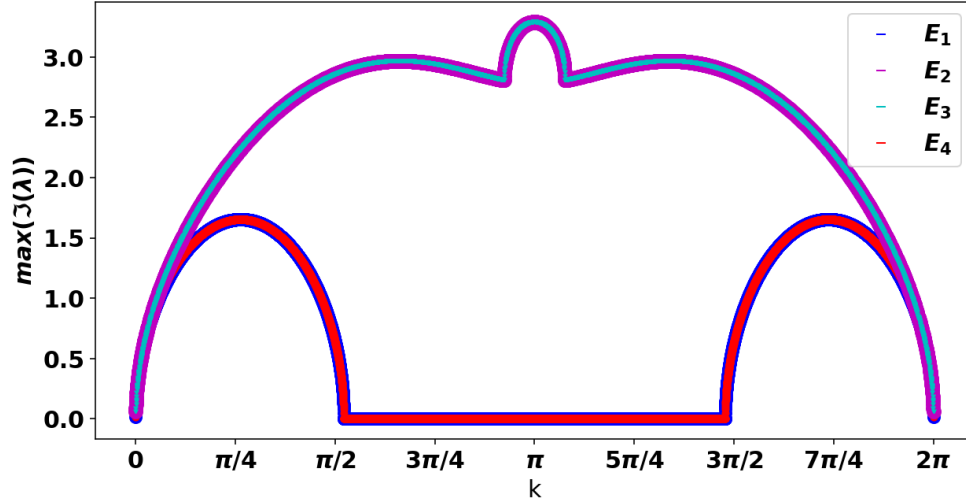


Figure 3.26: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = 5$, $g_B = 7$

Our study showed that energy bands are practically symmetric around $g_A = 5$, meaning that for $g_A < 5 - \delta$, $g_A > 5 + \delta$, where $\delta > 0$, the energy bands structures are identical, and the same thing goes for their dynamical stability.

On the other hand, in the case where g_A and g_B differ in sign, a representative energy bands (when g_A and g_B have big difference in magnitude) looks like

Fig.(3.27) and its dynamical stability is as shown in Fig.(3.28).

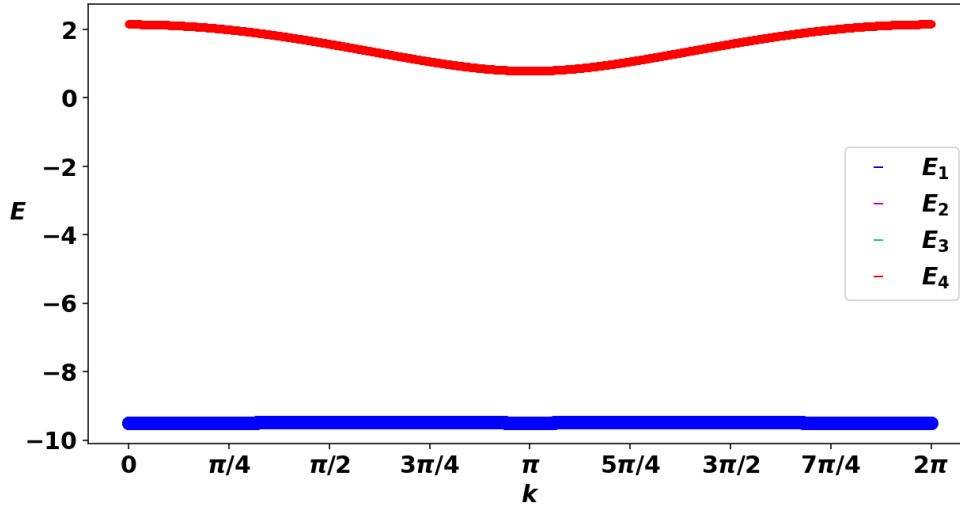


Figure 3.27: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = -10$, $g_B = 1$

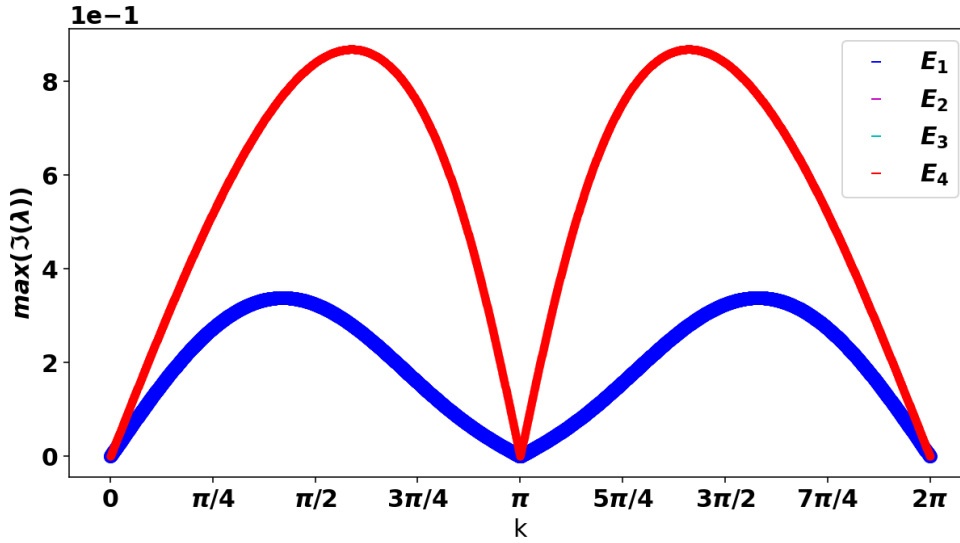


Figure 3.28: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = -10$, $g_B = 1$

Whereas when g_A and g_B have small difference in magnitude, energy bands look like Fig.(3.29)

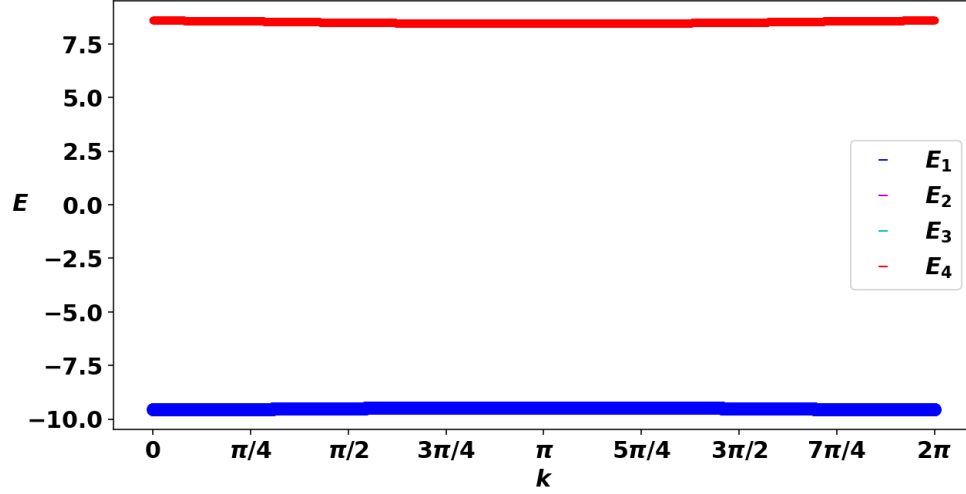


Figure 3.29: Energy(E) vs quasimomentum (k) for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = -10$, $g_B = 9$

where its stability analysis shows Fig.(3.30)

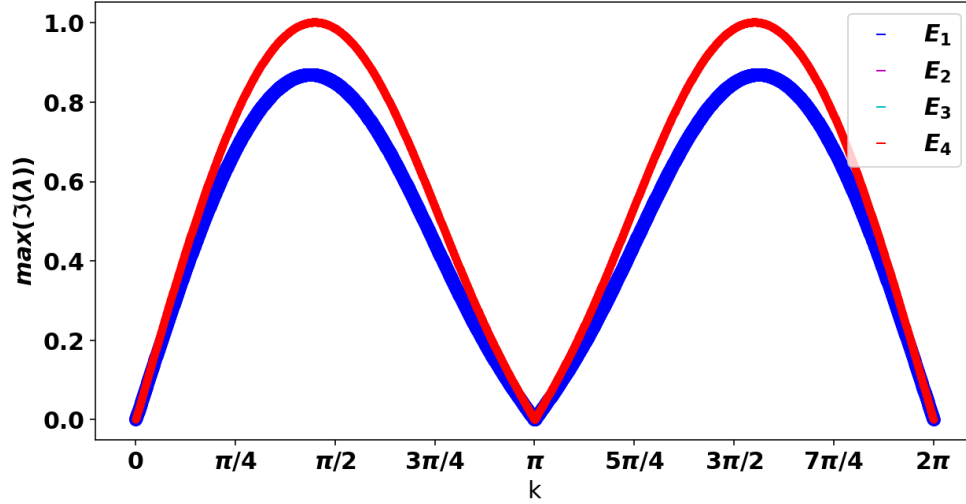


Figure 3.30: Dynamical stability analysis of energy bands for $J_1 = 1$, $J_2 = 2$, $v = 0.5$, $g_A = -10$, $g_B = 9$

Generally speaking, our stability analysis showed that when the two nonlinearity strengths g_A and g_B are comparable in magnitude, the energy bands become increasingly dynamically unstable. This observation was found to hold when g_A and g_B have the same sign, as well as when they have different signs.

CHAPTER 4

CONCLUDING REMARKS

In this thesis, we have considered a sublattice-dependent onsite nonlinearity in the topological SSH model and we performed a real space and reciprocal lattice analysis. The study includes real lattice space (with open boundary conditions) as well as reciprocal lattice space (with periodic boundary conditions). Different signs' variation of the sublattice-nonlinearity strengths were considered, where g_A denotes the nonlinearity strength on sublattice A , and g_B is the nonlinearity strength on sublattice B .

In the real space lattice study, the problem was solved numerically using the Self-Consistent Field Iterative Method. For the iteration process, all energy eigenstates from the corresponding linear model were selected as initial trial states, and only sufficiently quickly converging nonlinear solutions were collected and plotted. In the case where g_A and g_B were both positive, it was found that in the topologically nontrivial regime ($J_1 < J_2$) and at sufficiently high nonlinearities, the energy of the left (right) edge state depends only on g_A (g_B) and is constant with respect to g_B (g_A). Moreover, the existence of only one edge state was found

in the model for a particular set of values of a sublattice-dependent nonlinearity strength. In addition, in the topologically trivial and nontrivial (including when $J_1 = J_2$) regimes and at sufficiently high nonlinearities we noted the presence of a touching point between two energy bands. This touching point was found to only shift (and not disappear) under perturbation, which could indicate a topological origin analogous to gapless topological phases observed in Weyl semimetal. Such a touching point was not observed in the corresponding global nonlinearity ($g_A = g_B$) case. In the case where g_A and g_B have opposite signs, the persistence of the bulk's delocalized solutions was observed even at extremely high nonlinearities' strengths.

In the reciprocal lattice space study, we assumed Bloch's state solutions and used a semi-analytical approach to obtain energy bands, applied a dynamical stability analysis, and studied nonlinear and conventional Zak phases. Furthermore, a general expression of the nonlinear Zak phase, which holds for any well-behaved two-level real Hamiltonian with C^1 state-dependent (through $\Sigma = |\psi_2(t)|^2 - |\psi_1(t)|^2$) onsite nonlinearities, was derived. In the case where g_A and g_B are positive and at sufficiently high nonlinearity strengths, energy gap-closing accompanied with discontinuity in the nonlinear Zak phase were observed, indicating a nonlinearity-induced topological phase transition. Finally, our stability analysis of nonlinear energy bands showed that when the two nonlinearity strengths g_A and g_B are comparable in magnitude, the energy bands become increasingly dynamically unstable. This observation was found to hold for both cases, when g_A and g_B have the same sign as well as when they have different signs.

For future study of nonlinear systems, we proposed a method to easily identify emerging energy bands, as well as energy gap-closing in nonlinear systems. The method is based on plotting the energy bands (for all k values) versus the parameter of the problem, from the plot one can visually identify at which parameter value an observation of interest takes place, in addition, the method was applied and an illustrative example was provided. Moreover, studying the discontinuity of the nonlinear Zak phase Eq.(2.18) could potentially lead to a mathematically necessary condition for the occurrence of topological phase transitions in the nonlinear model under consideration.

REFERENCES

- [1] S. Su and Heeger., “Solitons in polyacetylene,” *Physical Review Letters*, vol. 42, no. 1698, 1979.
- [2] S. S. Heeger, Kivelson, *Solitons in conducting polymers*, 1988, vol. 3, no. 60.
- [3] H. I. Malkova N., *Observation of optical Shockleylike surface states in photonic superlattices*, 2009, vol. 11, no. 34.
- [4] S. H., *Topologically protected midgap states in complex photonic lattices*, 2013, vol. 11, no. 38.
- [5] P. A. Slobozhanyuk A.P., *Subwavelength topological edge states in optically resonant dielectric structures*, 2015, vol. 12, no. 114.
- [6] B. M. Poli C., *Selective enhancement of topologically induced interface states in a dielectric resonator chain*, 2015, no. 6.
- [7] A. I. BlancoRedondo A., *Topological Optical Waveguiding in Silicon and the Transition between Topological and Trivial Defect States*, 2016, vol. 16, no. 116.

- [8] S. K. Poddubny, Miroschnichenko, “Topological majorana states in zigzag chains of plasmonic nanoparticles.” *ACS Photonics*, vol. 2, no. 1, p. 101–105, 2014.
- [9] G. V. StJean P., “Lasing in topological edge states of a onedimensional lattice.” *Nature Photonics*, vol. 10, no. 11, p. 651–656, 2017.
- [10] L. T. Kane C.L., “Topological boundary modes in isostatic lattices.” *Nature Physics*, no. 10, p. 39–45, 2014.
- [11] M. Scollon and M. P. Kennett, “Persistence of chirality in the su-schrieffer-heeger model in the presence of on-site disorder,” *Physical Review B*, vol. 101, no. 144204, 2020.
- [12] L. I. C. K. Smirnova, logo, “Nonlinear topological photonics featured,” *Applied Physics Reviews*, no. 021306, 2020.
- [13] S. Gorlach, “Nonlinear topological states in the su-schrieffer-heeger model,” *NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS*, no. 8, 2017.
- [14] M. Ezawa, “Nonlinearity-induced transition in the nonlinear su-schrieffer-heeger model and a nonlinear higher-order topological system,” *Physical Review B*, no. 235420, 2021.
- [15] G. S. Y. . S. Sone, Ezawa, “Transition from the topological to the chaotic in the nonlinear su-schrieffer-heeger model,” *Nature Communications*, no. 422, 2025.

- [16] P. Asboth, Oroszlany, *A Short Course on Topological Insulators*, 2016.
- [17] M. J. Rice and E. J. Mele, “Elementary excitations of a linearly conjugated diatomic polymer,” *Physical Review Letters*, vol. 49, no. 1455, 1982.
- [18] L. Tuloup, Bomantara and Gong., “Nonlinearity induced topological physics in momentum space and real space,” *Physical Review B*, vol. 102, no. 115411, 2020.

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