

ADVANCING ROBUST REGRESSION WITH A
HYBRID M-ESTIMATOR FOR IMPROVED
OUTLIER HANDLING

BY

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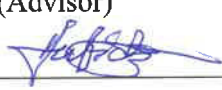
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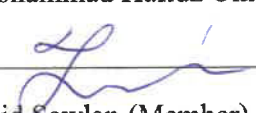
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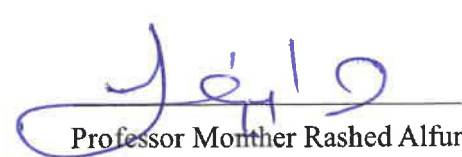
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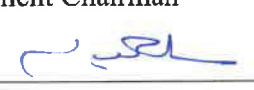
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To my parents, family and siblings

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LIST OF ABBREVIATIONS

AARL	:	Average of the Average Run Length
ARE	:	Asymptotic Relative Efficiency
ARL	:	Average Run Length
CV-MSE	:	Cross-Validated Mean Squared Error
EWMA	:	Exponential Weighted Moving Average
IC	:	In-Control
IRLS	:	Iteratively Reweighted Least Squares
LAV	:	Least Absolute Value
L-BFGS-B	:	Limited-memory Broyden-Fletcher-Goldfarb-Shanno with Box Constraints
LOOCV	:	Leave-One-Out Cross-Validation
LSE	:	Least Squares Estimation
MAD	:	Median Absolute Deviation
MDARL	:	Median Average Run Length
MSE	:	Mean Square Error
OC	:	Out-of-Control

Proposed CV : Proposed Estimator after Cross-Validation

SDARL : Standard Deviation Average Run Length

SEs : Standard Errors

SPC : Statistical Process Control

SSE : Sum of Squared Errors

ABSTRACT

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This study addresses the limitations of least squares estimation (LSE) in the presence of outliers by proposing a robust regression framework that merges the robustness of Huber's M-estimator with the efficiency of Tukey's Bisquare estimator. Conventional LSE methods often yield biased and inefficient parameter estimates when data is contaminated, a common issue in real-world modeling. The proposed method introduces two tuning constants, k_1 and k_2 , which define a trapezoidal influence function with three residual regions: fully weighted for small residuals, decaying weights for moderate residuals, and full rejection of extreme outliers. This design enhances adaptability to both mild and severe outliers. Optimization of the estimator is achieved through a Two-Stage L-BFGS-B algorithm guided by adaptive cross-validation, minimizing the cross-validated mean squared error (CV-MSE) and improving estimation precision and generalizability. Monte Carlo simulations under varying contamination levels were used to evaluate the performance of the estimators in terms of mean, bias squared, MSE, standard error, and asymptotic relative efficiency (ARE). Compared with LSE, Huber, and Bisquare estimators in both simple and multiple linear regression, the proposed estimator consistently outperformed LSE and Huber while matching the efficiency of Bisquare. The

study extends the application to Phase I linear profile monitoring using robust estimators for parameters such as intercept, slope, and error variance. Control chart simulations show Huber-based EWMA charts maintain more stable in-control (IC) values under contamination, while LSE performs better in ideal conditions. The study concludes that the hybrid M-estimator can serve as a valuable statistical tool for researchers and practitioners in fields such as engineering, manufacturing, healthcare, and environmental science, where data quality and modeling precision are crucial.

ملخص الرسالة

الاسم الكامل: خميس لوباتيلة

عنوان الرسالة: الانحدار الخطي المتقدم مع مقدر M لتحسين التعامل مع القيم المتطرفة

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تاريخ الحصول على الدرجة: مايو ٢٠٢٥

تتناول هذه الدراسة قيود تقدير المربعات الصغرى (LSE) في ظل وجود قيم متطرفة، وذلك باقتراح إطار انحدار متين يجمع بين متانة مقدر هوبر M وكفاءة مقدر توكي بيسكوير. غالبًا ما تُنتج طرق تقدير المربعات الصغرى التقليدية تقديرات متحيزة وغير فعالة للمعلمات عند تلوث البيانات، وهي مشكلة شائعة في النمذجة الواقعية. تُقدم الطريقة المقترحة ثابتي ضبط، k_1 و k_2 ، يُحددان دالة تأثير شبه منحرف بثلاث مناطق متبقية: مرجحة بالكامل للقيم المتطرفة الصغيرة، ومتضائلة الأوزان للقيم المتطرفة المتوسطة، ورفض كامل للقيم المتطرفة المتطرفة. يُعزز هذا التصميم القدرة على التكيف مع القيم المتطرفة الصغيرة والكبيرة. يتم تحقيق تحسين المقدر من خلال خوارزمية L-BFGS-B ثنائية المرحلتين والموجهة بالتحقق المتبادل التكيفي، مما يقلل من خطأ متوسط التربيع المتحقق المتبادل (CV-MSE) ويحسن دقة التقدير وقابليته للتعميم. تم استخدام محاكاة مونت كارلو في ظل مستويات تلوث متفاوتة لتقييم أداء المقدرين من حيث المتوسط، ومربع التحيز، و MSE ، والخطأ المعياري، والكفاءة النسبية التقريبية (ARE). بالمقارنة مع مقدرات LSE و Huber و Bisquare في كل من الانحدار الخطي البسيط والمتعدد، تفوق المقدر المقترح باستمرار على LSE و Huber مع مطابقة كفاءة Bisquare. توسع الدراسة التطبيق ليشمل مراقبة المعالم الخطية للمرحلة الأولى باستخدام مقدرين فعالين لمعلمات مثل القاطع، والميل، وتباين الخطأ. تُظهر محاكاة مخططات التحكم أن مخططات EWMA القائمة على Huber تحافظ على قيم أكثر استقرارًا للتحكم (IC) في ظل التلوث، بينما يعمل LSE بشكل أفضل في الظروف المثالية. وخلصت الدراسة إلى أن مقدر M الهجين يمكن أن يكون بمثابة أداة إحصائية

قيمة للباحثين والممارسين في مجالات مثل الهندسة والتصنيع والرعاية الصحية والعلوم البيئية، حيث تكون جودة البيانات ودقة النمذجة أمرًا بالغ الأهمية.

CHAPTER 1

INTRODUCTION

This chapter provides an overview of regression analysis, its significance in statistical modeling, and its widespread applications across various fields. Additionally, this chapter outlines the motivation behind robust regression, defines the research objectives and the need for an improved estimator. Finally, it introduces linear profile monitoring, a statistical quality control technique used to assess the stability of regression relationships over time and the organization of the contents in the thesis.

1.1. Overview of regression analysis

Regression analysis is fundamental to statistical modeling, employed to understand and foresee the relationship between a dependent variable and one or more independent variables [1], [2], [3], and [4]. Simple linear regression is a regression model that uses only one predictor and one response variable. Multiple linear regression expands on this concept by using several independent variables to better explain the variability in the dependent variable [5] and [6]. Regression has been applied in various fields, including engineering, natural sciences, economics, management, biology, life sciences, and social sciences, to facilitate decision-making and enhance research works [2] and [5]. Researchers and policymakers can test hypotheses, find trends, and create data-driven strategies with the aid of regression models. Additionally, they make it possible to account for confounding variables, which raises estimate accuracy and lowers bias. Regression models are helpful,

but when they are misused or misinterpreted, they can produce false results, which highlights the significance of accurate model specification and validation [7].

Despite its wide range of applications, employing the traditional least squares estimation (LSE) method is very sensitive to outliers, which can significantly skew parameter estimates [3]. To overcome these limitations, robust regression techniques have been designed to reduce or entirely disregard the influence of outliers, ensuring accurate and reliable parameter estimates [3], [8], [9], [10], [11], [12], and [13]. Unlike LSE, robust regression methods do not depend heavily on assumptions such as normally distributed errors or homoscedasticity [5], [14], and [15]. This flexibility has made robust regression a critical tool in handling contaminated datasets in fields such as signal processing [16], real estate [17], manufacturing processes [5], education [6], telecommunications [18], and agriculture [2], [19]. Thus, the presence of outliers significantly necessitates the development of robust regression methods such as the well-known M-estimators.

A class of robust estimation known as M-estimators was initially proposed by Huber in 1964 [3] and [20]. It extends the idea of M-estimation of location, which strengthens the mean concept [16], [21]. M-estimators generalize maximum likelihood estimation in order to achieve robustness in parameter estimation by reducing the impact of residuals [19] and [22]. In 1973, Huber refined the concept of M-estimation for estimation in regression settings [20], [23], [24], [1], [25], and [13]. M-estimators minimize the impact of outliers in regression settings by replacing a least squares method with a slower-growing influence function [20] and [22]. These estimators are scale-invariant, ensuring consistency and

efficiency even under moderate levels of data contamination [14], [22], and [25]. M-estimators achieves the maximum breakdown point with a minimum of $\frac{1}{n}$ outliers in the data and the presence of leverage points may restrict its efficacy [17], [26]. How many outliers can be included in the data before affecting the estimations is known as the breakdown point [3], [14], [23]. Moreover, some of these methods simultaneously estimate the scale parameter and the regression coefficient, thereby addressing the problem of optimizing efficiency under a constrained influence function [24]. The Huber and Tukey Bisquare are renowned M-estimators that have gotten increased attention in recent research [1], [19], and [15].

1.2. Linear profile Monitoring

In statistical process control (SPC), monitoring linear profiles is a quite new approach which involves determining whether the quality of a process or product exhibits a linear relationship over time [27] and [28]. This linear relationship, referred to as a profile, represents the relationship between a response variable and one or more independent variables, which can be employed to represent the quality of a process or product in real-world contexts [29], [30], [31] and [32]. In addition to simple linear profiles, there are also multiple linear profiles, polynomial profiles, multivariate linear profiles, nonlinear profiles, and spline profiles. Therefore, the determination of whether the profiles are linear, nonlinear, or generalized relies on their relationship between response variable and the predictor variables [31] and [33]. Also, the monitoring of a process modeled by a profile is conducted in two phases. In practice:

- **Phase I:** According to [28], [34], and [35], historical data are analyzed to compute estimates for intercept, slope, and error variance. OLS-based approaches use traditional least squares, whereas the Huber M-estimator provides more stable estimates by down-weighting outliers. In some cases, the insights obtained from Phase I may suffice for overseeing ongoing processes in Phase II, except for validating the sustainability of the process adjustments [36].
- **Phase II:** The control charts monitor the process by tracking deviations in the estimated intercept, slope, and mean square error (MSE). Shifts in these parameters are detected when the estimates fall outside pre-determined control limits, which are derived based on the in-control distributions of the estimators [28]. Most of the research on control charting pertains to phase II methodologies and their effectiveness [8]. For example, a bivariate T^2 chart assesses regression coefficient stability, while a univariate Shewhart chart tracks variability around the mean line. In addition, univariate methods like EWMA charts track intercept, slope, and error term variation [28] and [36]. However, other methods, like the T^2 control chart, may not identify some parameter variations, limiting their practical use in this phase [36].

1.3. Motivation of the Study

Compared to Bisquare, which offers robust resistance to extreme outliers but struggles with smaller residuals, the Huber M-estimator works well against moderate outliers but is severely affected by extreme residuals. Both approaches fail miserably when it comes to striking a balance between efficiency and robustness. To overcome this drawback, the

proposed hybrid M-estimator provides a more practical and reliable estimation framework that balances robustness and efficiency when handling the influence of outliers in a range of proportions and magnitudes and hence advances the application of robust regression in disciplines requiring high-precision modeling, such as finance, healthcare, and environmental studies. The study also underscores the considerable benefits of utilizing robust estimators, particularly Huber's M-estimators as alternative methods for classical least squares-based control charts when it fails due to presence of outliers and enhancing accuracy of Phase I analysis.

1.4. Thesis Objectives

The primary objective of this study is to develop and evaluate a robust regression method that enhances parameter estimation accuracy in the presence of outliers by striking a balance between efficiency of Bisquare and the robustness of Huber's M-estimators. Specifically, the study aims to address the following objectives: analyze the limitations of traditional Least Squares Estimation (LSE), explore the functional properties of existing robust M-estimators, propose a hybrid M-estimator that balances robustness and efficiency, implementing adaptive cross-validation for optimizing robust regression parameters, and assessing its applicability in both simple and multiple linear regression models. Additionally, the study investigates the use of robust estimators in linear profile monitoring to improve detection and control of process variations, ensuring more reliable statistical modeling in real-world applications. Each of these objectives has been systematically implemented as follows:

(i) To analyze the limitations of LSE in regression modeling.

This objective is covered in Chapter Two (Literature Review), where it discusses the Least Squares Estimation (LSE) method, its assumptions, and its vulnerability to outliers. It highlights how LSE performs poorly when data contamination occurs, leading to biased parameter estimates and increased variance. Various studies are reviewed to indicate real-world cases where LSE fails, providing a foundation for the need for robust regression techniques.

(ii) To explore the functional properties of existing robust M-estimators.

This objective is addressed in Chapters Two and Three, where different robust M-estimators, including Huber and Tukey's Bisquare, are explored in detail. The thesis explains the weighting functions of these estimators and how they mitigate the impact of outliers while maintaining efficiency. The functional properties, such as breakdown points, efficiency, and robustness against different types of contamination, are analyzed through theoretical discussions and empirical studies.

(iii) To propose and evaluate a hybrid M-estimator for simple and multiple linear regression across various proportions and magnitudes of outliers.

Chapters Three and Four are dedicated to developing and testing the proposed hybrid M-estimator. A new estimator that combines the robustness of Huber's estimator with the efficiency of Tukey's Bisquare is introduced. Monte Carlo simulations are used to compare the performance of the proposed estimator against LSE, Huber, and Bisquare estimators in

both simple and multiple linear regression models. Performance is evaluated using mean, standard errors, bias, mean squared error (MSE), and asymptotic relative efficiency (ARE) measures.

(iv) To implement an adaptive cross-validation for optimizing robust regression parameters for the proposed estimator.

This objective is addressed in Chapter Three, where a Two-Stage L-BFGS-B optimization is used to dynamically refine parameter estimates, minimizing cross-validated mean squared error (CV-MSE). The initial stage optimizes the parameters k_1 and k_2 within predefined bounds, while the adaptive refinement stage iteratively narrows the search space to enhance accuracy. This approach ensures efficient parameter estimation, improving model robustness against data contamination.

(v) To assess the application of robust estimators in linear profile monitoring.

This objective is implemented in Chapters Five, where Huber's M-estimator is applied to Phase I monitoring of linear profiles. This chapter discusses how classical regression-based control charts can be improved using robust estimators, minimizing false alarms caused by extreme observations. Real-world datasets, such as traffic volume and temperature monitoring, are analyzed to demonstrate the practical effectiveness of robust regression in maintaining stable profile monitoring.

1.5. Organization of the thesis

This thesis is structured in a way that provides a comprehensive analysis of robust regression methods. Chapter Two presents a literature review on regression estimation techniques and the development of robust M-estimators, highlighting their advantages over traditional methods. Chapter Three delves into the application of robust M-estimators in simple linear regression, explaining their mathematical formulation and implementation. Chapter Four expands on this by exploring how robust M-estimators can be applied to multiple linear regression, addressing challenges related to high-dimensional data and outliers. Chapter Five focuses on the use of robust Phase I estimators for monitoring linear profiles, demonstrating their effectiveness in detecting deviations in regression relationships over time. Finally, Chapter Six concludes with a summary of the key findings of the study as well as the recommendations for future studies aimed at enhancing robust regression methodologies.

CHAPTER 2

LITERATURE REVIEW

This chapter reviews existing regression methods, highlighting the limitations of LSE and the advantages of robust regression. Various M-estimators are discussed, including Huber and Tukey's Bisquare, which provide robustness against outliers by replacing the loss function of least squares estimation with a less rapidly growing objective function. We also discussed Cross-validation techniques and the Limited-memory Broyden-Fletcher-Goldfarb-Shanno with Box Constraints or Bound (L-BFGS-B) optimization algorithm for parameter selection and estimation stability.

2.1. Least Squares Estimation (LSE) and Its Limitations

The Least squares estimation was independently formulated by Gauss in Germany in 1795 and Legendre in France in 1805. Rousseeuw and Leroy (1987), as mentioned by [5], stated that LSE is the widely used method in regression analysis and is preferred across many disciplines due to its straightforward mathematical framework. It establishes the line of best fit, which minimizes the sum of squared deviations between predicted and observed values, known as model residuals [2]. This method was first applied to data from geodetic and astronomical sources [5]. LSE assumes model errors are normally distributed with a constant variance, which is rarely true in practice [2], [3], [5], [6], and [17]. As statistical computing advances, most data are analyzed without pre-processing, resulting in

unreported outliers. Consequently, the performance of the LSE is entirely compromised due to the presence of the outliers [6], [37], and [18].

2.2. The Impact of Outliers and the Need for Robust Estimation

Outliers are data points that obviously differ from other observations and are abandoned from the prevailing pattern [6], [14]. [16] suggested that outliers can be characterized by position or by influence in the regression line. Leverage points are observations that deviate significantly from the majority and have an impact on the explanatory variable. Atypical observations that deviate greatly from the fitted line are referred to as problematic leverage points. Influential observations attract the regression line towards the response variable and affect the model parameters [16], [38], [17]. Both forms of outliers may exist in the datasets. Even a few outliers can affect LSE estimations, accuracy, and predictions [39], [21], and [12]. Therefore, outliers reduce the validity of source information or the data generation process [40], [5], and [14]. Due to these reasons, LSE cannot be used on non-normal error distributions or contaminated datasets. Other approaches, such as robust regression, have been suggested as alternatives [19], [18], and [41].

2.3. Robust Regression and M-Estimators

Robust regression has gained popularity because of computer advances that simplify complex iterative algorithms and eliminate the need for early outlier detection [26]. M-estimators, S-estimators, MM-estimators, least trimmed square (LTS), least median square (LMS), R regression estimator, GR regression estimator, HBR, and regression estimator

are some of the robust regression techniques that have been studied to reduce outliers in estimates of the model [23], [42], [24], [15], [3], and [2]. However, M-estimators, especially Bisquare and Huber's, are gaining popularity due to their computational efficiency and robustness.

Famous monotone M-estimators like the Huber estimator balance robustness and efficiency [20], [24], and [23]. It behaves like LSE for small residuals, and for large residuals, it matches the least absolute value (LAV) in a linear increase that minimizes the influence of outliers and improves robustness. When errors are normally distributed, it maximizes efficiency [14], [22]. The Huber estimator works effectively on small to moderately contaminated datasets due to its dual behavior. The Bisquare (or bi-weight) estimator is a notable M-estimator within the category of redescending M-estimators, distinguished by its strong resistance against extreme outliers [38], [18], and [39]. Redescending M-estimators are robust estimators with an influence function that increases for small residuals but decreases to zero for big residuals while preserving efficiency. This characteristic minimizes the impact of extreme outliers on estimation. Examples include Tukey's bi-weight, Andrews-sine, Hampel, and the Qadir Beta estimator [18], [39], and [43]. The Bisquare technique is beneficial in applications where extreme data can skew parameter estimations.

Many robust estimators, especially redescending ones, have non-convex objective functions, as [1] and [44] noted. Additionally, non-convexity adds more local minima or saddle points to the optimization problem, making it more difficult to estimate parameters

and decreasing the effectiveness of gradient descent and quasi-Newton methods. Thus, global optimization approaches are necessary to ensure that the optimization process converges to the global minimum rather than staying in suboptimal solutions.

2.4. Optimization Strategies for Robust Estimation

A global optimization (GO) strategy for robust estimators [45] produces global rather than local optima. Traditional iterative reweighted least squares (IRLS) methods are compared to GO for redescending estimators such as Geman–McClure, Welsch, and Tukey. According to the study, GO excels in sequential robust estimates for M-split problems and geomatics applications. Redefining estimators like Tukey's biweight to avoid piecewise definitions simplifies GO implementation. These contributions improve robust estimation methods and their practical applicability in complex datasets with many outliers. Here, genetic algorithms and simulated annealing are extremely useful. For instance, the probabilistic "simulated annealing" method allows intermittent uphill advances to avoid local minima while exploring the solution space [46].

However, L-BFGS [47] strategically improves BFGS corrections to boost computing efficiency. Though effective in box-constrained functions, this strategy is best for large-scale continuous optimization. According to [48], the L-BFGS-B algorithm has substantially improved large-scale bound-constrained optimization. It is scalable, user-friendly, and computationally cheap per iteration, making it useful for many applications.

2.5. Optimizing Bias-Variance Trade-off using Cross-Validation

In machine learning, K-fold cross-validation has attracted scholarly attention, particularly in high-stakes domains like healthcare. [49] describes k-fold cross-validation, which partitions datasets into k equal-sized folds. This approach makes it simple to systematically balance bias and variation. K-fold cross-validation has more bias but less variance than the leave-one-out cross-validation (LOOCV). In a similar study, [50] introduces cross-validation techniques and claims that k-fold yields less biased out-of-sample performance estimates than holdout validation. They use the MIMIC-III dataset to emphasize the need for cross-validation best practices to ensure that models can reliably predict outcomes for unseen test samples. They also highlight practical factors and common model evaluation errors. Further contributions from [51] expand on the bias-variance trade-off in choosing the optimal fold number. Empirical evidence suggests that k values between 5 and 10 produce a more consistent prediction error estimate, which is essential for model selection. Cross-validation results are also affected by training set size; therefore, practitioners should avoid overestimating variation when utilizing smaller testing sets.

The J-K-fold cross-validation technique [52] was a breakthrough in reducing the variability of standard k-fold techniques. This method uses several independent k-fold Cross-validation to reduce performance estimate uncertainty, which might mask model efficacy. Reduce variability rather than bias for effective model tuning, especially when underestimating prediction error could be harmful. Cross-validation in cardiovascular imaging performance estimations was expanded by showing that train or test sample

methods with random seed selection significantly affect the area under the curve (AUC) [53]. For accurate predictive models and comparisons, medical applications with small sample sizes require strong validation methods.

2.6. Robust Improvement of Phase I Estimates

Early methodologies for linear profile monitoring often depend on the assumption of normality in the error distribution and employed classical least squares estimation (LSE) or maximum likelihood estimation (MLE) to obtain estimates of the profile parameters. However, according to studies by [34], [54], [55], [56], and [57] revealed that the statistical characteristics of commonly used charts, which work best under normality assumptions, are probably greatly impacted by the fact that the basic process distribution in many applications is not normal. This is true even when it comes to profile monitoring. This argument is further supported by [36] and [35], who explained that deviation from normality can jeopardize the statistical validity of all phase I methodologies due to the possibility for a significant increase in the false alarm rate. Additionally, numerous studies including [35], [58], [59], and [60] have demonstrated the importance of using robust estimators, like M-estimators, in both contaminated and uncontaminated data in order to improve estimates and control chart performance in phase II. The importance of the robust estimators is particularly apparent in phase I, where the control limits for further monitoring process in phase II are established using historical data [28]. The study by [59] and [60] show that M-estimators perform better than least squares in terms of unbiasedness, efficiency, and accuracy in error diagnosis, particularly when working with datasets that have high rates of contamination. These studies highlight how using robust estimation

techniques can result in distinguished improvements in the accuracy and reliability of Phase I analysis.

2.7. Enhancing Phase II Linear Profile Monitoring

[28] introduced a methodology for monitoring linear profiles in Phase II of a process by constructing separate control charts for the intercept, slope, and variability. Their approach relies on the assumption that residuals are normally distributed, thereby justifying the use of classical least square method and related control limits. The least squares method of estimation (LSE) estimates the parameters based on the assumption that the error term in the regression model is normally distributed. Although the approach demonstrated to be successful in in-control settings, it can be susceptible to model deviations, especially when there are outliers in the data and can significantly impact their statistical effectiveness [29], [35], [36], [57], and [58]. Alternative estimation methods are needed to estimate profile parameters that work better when the process is under control and when contamination is present [35] and [58].

The existing studies have laid a strong foundation for robust regression methods, but gaps remain in understanding the trade-offs between robustness and efficiency, optimizing computational techniques, and applying robust estimators to specialized domains. Therefore, this study enhances computational optimization techniques to attain a compromise between robustness and efficiency. By bridging these gaps, the study contributes to the theoretical development of a more reliable, efficient, and adaptable robust regression model capable of handling real-world data complexities

CHAPTER 3

COMPARATIVE ANALYSIS OF ESTIMATORS IN SIMPLE LINEAR REGRESSION

This chapter introduces the concept of classical least squares, robust M-estimators and their implementation using the Iteratively Reweighted Least Squares (IRLS) algorithm. Furthermore, it presents mathematical formulation of a hybrid robust M-estimator along with a comparison of its graphic properties to Huber's and Bisquare. A Monte Carlo simulation is conducted to generate predictor values from a uniform distribution and response values based on a linear model with normally distributed noise. Outliers are introduced using a Chi-square distribution, and their impact on estimation accuracy is examined and compared across all estimators.

3.1. Conventional Least Squares Estimation (LSE)

According to [16], assuming we collect n random observations $y_1, y_2, \dots, y_n$, with the corresponding fixed values $x_1, x_2, \dots, x_n$, we can estimate the regression coefficients β_0 , β_1 , and σ^2 . The anticipated regression model can be straightforward defined as

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \dots, n \quad (3.1)$$

ϵ_i denotes a normally distributed error term with mean 0 and variance σ^2 . ϵ_i and ϵ_j are uncorrelated terms with zero covariance (i.e., $cov(\epsilon_i, \epsilon_j) = 0$ for all $i, j; i \neq j$), where

$i = 1, \dots, n$. Further generalization of the model for multiple predictor variables is given as follows:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_{k-1} x_{i,k-1} + \epsilon_i \quad (3.2)$$

In least squares estimation, we want to find the estimates $\hat{\beta}_0, \dots, \hat{\beta}_n$ that minimize the mean square error using the following objective function:

$$\hat{\beta} = \arg \min_{\beta} \sum_{i=1}^n (y_i - x_i^T \beta)^2 \quad (3.3)$$

where β is a vector of regression coefficients, and $\hat{\beta}$ denotes the estimated coefficients. As stated by [8], when the design matrix X is a full rank matrix in equation (3.3), then the unique solution is given by

$$\hat{\beta}_{LS} = (X^T X)^{-1} X^T Y \quad (3.4)$$

3.2. M-estimators

According to [14], M-estimators are constructed by applying the maximum likelihood function to the density function $\frac{1}{\sigma} f\left(\frac{e}{\sigma}\right)$ where σ scaling parameter and $e = y_i - x_i' \beta$. The density of y_i' s is represented as $\frac{1}{\sigma} f\left(\frac{y_i - x_i' \beta}{\sigma}\right)$, and the likelihood function for β and σ is specified as follows:

$$L(\beta, \sigma) = \frac{1}{\sigma^n} \prod_{i=1}^n f\left(\frac{y_i - x_i' \beta}{\sigma}\right) \quad (3.5)$$

After manipulating equation (3.5), we get the objective function which minimizes a less rapidly growing function than a square:

$$\sum_{i=1}^n \rho(r_i) = \sum_{i=1}^n \rho\left(\frac{y_i - x_i' \beta}{\sigma}\right) \quad (3.6)$$

Hence, r_i represents scaled residuals. To get the estimating equations, set the partial derivatives of β from equation (3.6) to zero, as shown below

$$\sum_{i=1}^n \psi(y_i - x_i' \beta) x_i = 0 \quad (3.7)$$

where $\psi(e) = \rho'(e)$ denotes the influence function. To solve equation (3.7), assign the weight function $W(e)$ as follows.

$$W(e) = \begin{cases} \frac{\psi(e)}{e}, & e \neq 0 \\ \psi'(0), & e = 0. \end{cases} \quad (3.8)$$

The weights for scaled residuals are expressed as $w_i = W(r_i)$. This solves (3.7) above as follows:

$$\sum_{i=1}^n \psi(y_i - x_i' \beta) x_i = 0 \quad (3.9)$$

Equation (3.9) provides weighted least squares (WLS) estimates if the weight, w_i , is known. The optimal solution is attained through the application of the Iterative reweighted least squares (IRLS) algorithm [39] and Newton-Raphson method [3].

3.3. Iterative Reweighted Least Squares (IRLS) Algorithm

The IRLS algorithm solves the regression problem by repeatedly using the residuals to alter the weights for each observation until the best parameters are obtained [3], [22], [61], [62] and [19]. Implementing the IRLS algorithm involves the following steps:

- (i) Set iteration variable $j = 0$ and select initial estimate value $\beta^{(0)}$. Calculate the initial residuals, $e_i^{(0)} = y_i - x_i' \beta^{(0)}$, then estimate the scale and weight of the residuals ($s^{(0)}$ and $w_i^{(0)}$), where;

$$s^{(0)} = \frac{1}{0.6745} \text{median}\left(\left|e_i^{(0)} - \text{median}(e_i^{(0)})\right|\right) \quad (3.10)$$

For normally distributed independent observations, the constant 0.6745 provides an approximate unbiased estimate of σ , which is expressed $\hat{\sigma}$ or s depending on sample variance.

- (ii) Use equation (3.11) to calculate scaled residuals e_i .

$$r_i = \frac{e_i}{s} \quad (3.11)$$

(iii) Apply current weights to solve estimating equations by minimizing $\sum_i^n w_i^{(j-1)} r_i^2$ to obtain $\beta^{(j)}$ at each iteration j . The WLS solution is given as:

$$\beta_{IRLS}^{(j)} = (x'Wx)^{-1}x'Wy \quad (3.12)$$

where, $x_{(n \times k+1)}$ denotes a design matrix with x_i' as its i -th observation or row, and

$W_{(n \times n)}$ is defined as $diag(w_i^{(j-1)})$.

(iv) Repeat steps (ii) and (iii) until estimates converge, that is when, $\beta^{(j)} - \beta^{(j-1)} \approx 0$.

The literature [3],[14], [1], [26],[22], and [19], highlights that the objective function $\rho(\cdot)$ satisfies a number of significant constraints. First, for all e , it is non-negative ($\rho(e) \geq 0$). Secondly, it is symmetrical ($\rho(e) = \rho(-e)$). Furthermore, $\rho(\cdot)$ is zero at the origin, (i.e., $\rho(0) = 0$). Lastly, for any two values e_i and e_j where $e_i > e_j$ it holds that $\rho(e_i) \geq \rho(e_j)$, indicating a non-decreasing nature with respect to the magnitude of e . Multiple choices for $\rho(\cdot)$ have been proposed, if $\rho(e) = \frac{1}{2}e^2$, the normal equation solution is acknowledged as least squares [8], [13]. Conversely, $\rho(e) = |e|$ is a least absolute deviation (LAV), as defined below:

$$\hat{\beta}_{LAV} = \arg \min_{\beta} \sum_{i=1}^n |y_i - x_i^T \beta| \quad (3.13)$$

3.4. Huber M-estimator

The Huber objective function grows more gradually than LSE as residuals deviate from zero, balancing robustness and efficiency through the tuning parameter k . A commonly used value, $k = 1.345\sigma$, achieves approximately 95% efficiency relative to LSE under normality by fully weighting small residuals while downweighting larger ones to enhance outlier resistance [22], [63]. The objective, influence, and weight functions are defined as follows.

Objective function:

$$\rho_H(e) = \begin{cases} \frac{e^2}{2} & , |e| \leq k \\ k|e| - \frac{k^2}{2} & , |e| > k \end{cases} \quad (3.14)$$

Influence function:

$$\psi_H(e) = \begin{cases} e & , |e| \leq k \\ k \text{sign}(e) & , |e| > k \end{cases} \quad (3.15)$$

and weight function:

$$w_H(e) = \begin{cases} 1 & , |e| \leq k \\ \frac{k}{|e|} & , |e| > k \end{cases} \quad (3.16)$$

3.5. Tukey's bi-weight M-estimator

The Tukey Bisquare M-estimator is highly robust to large residuals, as its influence function drops to zero for extreme outliers, effectively eliminating their impact. Unlike Huber's estimator, which still assigns partial weight to large residuals, the Bisquare estimator rapidly reduces weights as residuals increase, providing stronger resistance to severe outliers. The objective, influence, and weight functions are given by:

Objective function

$$\rho_B(e) = \begin{cases} \frac{k^2}{6} \left(1 - \left[1 - \left(\frac{e}{k} \right)^2 \right]^3 \right) & , |e| \leq k \\ \frac{k^2}{6} & , |e| > k \end{cases} \quad (3.17)$$

Influence function

$$\psi_B(e) = \begin{cases} e \left(\left[1 - \left(\frac{e}{k} \right)^2 \right]^2 \right) & , |e| \leq k \\ 0 & , |e| > k \end{cases} \quad (3.18)$$

and weight function:

$$w_B(e) = \begin{cases} \left[1 - \left(\frac{e}{k} \right)^2 \right]^2 & , |e| \leq k \\ 0 & , |e| > k \end{cases} \quad (3.19)$$

In the Tukey bi-weight method, the center and tail behaviors of the estimator are determined by the tuning constant $k = 4.685\sigma$, which guarantees 95% efficiency relative to LSE in the presence of a normal distribution.

3.6. The Proposed Hybrid M-estimator

The Proposed estimator is a class of redescending M-estimators that incorporates two key tuning parameters, one coming from Huber's M-estimator and the other from Tukey's bi-weight M-estimator. Redescending M-estimators are unique since they can entirely remove the impact of severe outliers. Examples of redescending M-estimators include Tukey's bi-weight, Andrews-sine, Hampel, and the Qadir Beta estimator [39]. This hybrid estimator seeks to enhance the robustness of Huber's method against smaller outliers while improving the efficiency of Tukey's bi-weight method for extreme outliers, hence advancing outlier handling. For small residuals up to the Huber tuning constant (k_1), the estimator applies uniform weights, comparable to conventional least squares estimation, hence improving stability for smaller deviations. Residuals that fall between the Huber and Tukey bi-weight thresholds are subsequently downweighted in accordance with the Bisquare principle, enhancing efficiency for moderate residuals. For extremely large residuals that exceed the Tukey bi-weight threshold (k_2), the estimator assigns a weight of zero, thereby removing these outliers to improve robustness.

3.6.1. Formulation of the Proposed M-estimator

The design of the Proposed estimator started with the objective of creating a weighting function that integrates the robustness of Huber's approach with the efficiency of Tukey biweight. The new weighting function was formulated using geometric translation techniques to ensure that the plotted weights would show a trapezoidal structure. The weighting function, $W_P(e)$, is a piecewise function consisting of two tuning constants, k_1

and k_2 , which ensures that the weighting is carried out in three unique regions based on the domain of $|e|$. The three zones form a trapezoidal weighting structure that emphasizes values in the "**core region**," reduces emphasis in the "**transition zone**", and ignores data in the "**rejection region**". Moreover, the flexibility provided by this weighting scheme allows the function to be applied in various settings, making it an excellent option for research necessitating controlled weights and exclusions. The weighting function, $w_p(e)$, is mathematically defined in accordance with the previously given properties as follows:

$$w_p(e) = \begin{cases} 1 & , \quad |e| \leq k_1 \\ \left[1 - \left(\frac{|e| - k_1}{k_2 - k_1}\right)^2\right]^2 & , \quad k_1 < |e| \leq k_2 \\ 0 & , \quad |e| > k_2 \end{cases} \quad (3.20)$$

where e denotes residuals, k_1 and k_2 are positive constants with, $k_1 < k_2$, and $k_1 = 1.345$ and $k_2 = 4.685$.

The Proposed M-estimator's influence function $\psi_p(e)$ is a derivative of the objective function or the product of residuals and weight function. It helps identify influential data points that may impact parameter estimates, and ensures this impact is minimized accordingly. Consequently, the influence function is expressed as

$$\psi_p(e) = \begin{cases} e & , \quad |e| \leq k_1 \\ e \left[1 - \left(\frac{|e| - k_1}{k_2 - k_1}\right)^2\right]^2 & , \quad k_1 < |e| \leq k_2 \\ 0 & , \quad |e| > k_2 \end{cases} \quad (3.21)$$

3.6.2. Derivation of the Objective Function of the Proposed M-estimator

The objective function, $\rho_P(e)$ is derived by calculating the antiderivative of the influence function. The main region of this objective function is assumed directly from Huber's M-estimator, while the outlying region is determined by the factored quantity of the objective function following the integration of the transition region with respect to residuals. To determine the transition zone for the allocation of gradual weights, we examined the integral of the function for the specified transition region as shown in **Appendix B**. Therefore, the objective function of the proposed model is given by:

$$\rho_P(e) = \begin{cases} \frac{e^2}{2} & , |e| \leq k_1 \\ \frac{k_2^6}{6(k_2 - k_1)^4} \left[\frac{3}{k_2^4} (k_2^2 - 4k_1k_2 + 4k_1^2)e^2 + 3 \left(2\frac{k_1^2}{k_2^6} - \frac{1}{k_2^4} + 2\frac{k_1}{k_2^5} \right) e^4 \right] & , k_1 < |e| \leq k_2 \\ \frac{k_2^6}{6(k_2 - k_1)^4} & , |e| > k_2 \end{cases} \quad (3.22)$$

3.7. Functional Properties of M-estimators

Figure 3.1 below illustrates the functional characteristics of M-estimators, depicting their responses to different magnitudes of outliers. The illustration emphasizes the robustness of the Huber to small residuals and the efficiency of Bisquare, which demonstrates superior resistance to extreme outliers. The Proposed M-estimator minimize the effect of outliers by downweighting large residuals, hence improving robustness. This comparison demonstrates the distinctive methodology employed by M-estimators in handling outliers.

Subplot (h) demonstrates a more linear shape in the central part of the influence function

of the proposed estimator, enabling efficient and consistent estimation for inliers, while effectively regulating larger residuals, thereby providing the flexibility to handle both contaminated and non-contaminated data. This functional analysis emphasizes the need to choose estimators that correspond with data characteristics in the presence of outliers.

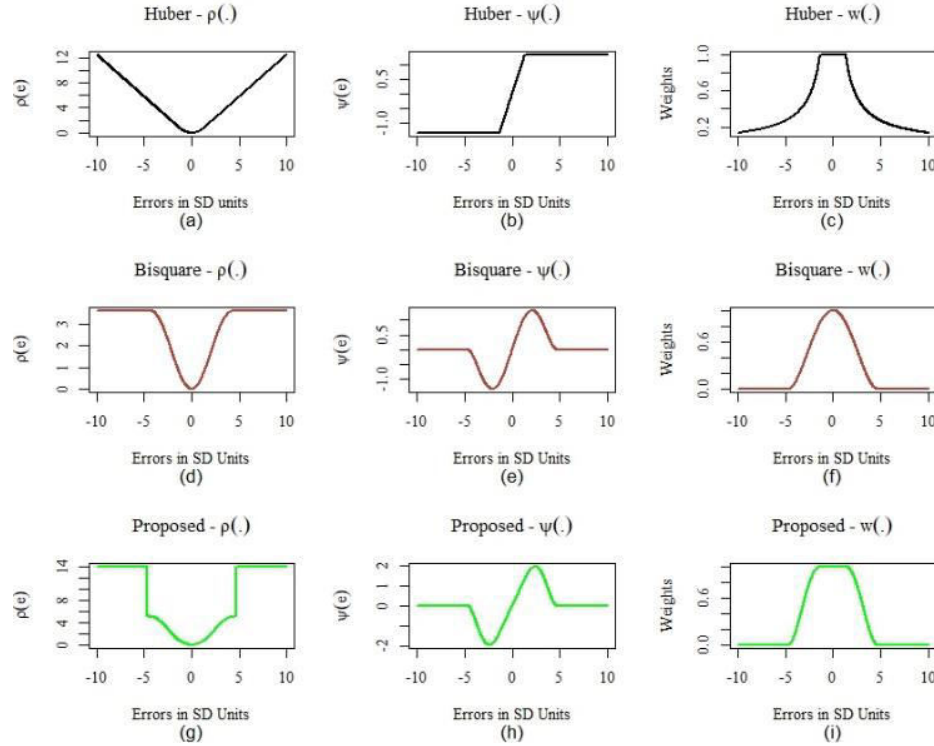


Figure 3.1 Functional properties for M-estimators.

3.8. Adaptive Cross-validation using a Two-Stage L-BFGS-B

Optimization

Adaptive Cross-Validation is a sophisticated optimization technique which facilitates the finding of accurate estimates that minimize mean squared error (MSE) across various contamination scenarios using cross-validation technique. In contrast to conventional cross-validation techniques that employ manual tuning or fixed parameter grids, adaptive

cross-validation makes use of a Two-Stage Limited-memory Broyden-Fletcher-Goldfarb-Shanno algorithm with Box constraints (L-BFGS-B) to dynamically modify parameter values within predetermined bounds. L-BFGS-B is ideal for large-scale optimization problems with bound variable constraints, gradient information, and a smooth objective function. This strategy improves the effectiveness and accuracy of parameter search. The algorithm is also suitable and efficient for solving unbounded constraints problems [48] and for more details about the L-BFGS-B algorithm see [64].

1. **Initial optimization stage:** The initial set of optimized parameters for k_1 and k_2 is obtained in this step by minimizing the cross-validated mean squared error (CV-MSE) within predefined parameter bounds (k_1 between 1 and 1.345, and k_2 between 1.346 and 4.685).
2. **Adaptive refinement stage:** Following the initial optimization, an adaptive cross-validation is used to further refine the parameter estimates. Every time the L-BFGS-B algorithm updates the previously optimized values by focusing the search in a more limited region ($k_1 \pm 0.05$ and $k_2 \pm 0.05$), to improve accuracy. Over the course of ten cycles, the estimates are gradually improved until convergence is reached. Finally, the optimized k_1 and k_2 are used to fit the proposed model, ensuring accurate and robust data representation. The flowchart for Adaptive Cross-Validation via Two-Stage L-BFGS-B optimization technique is presented in **Appendix C**.

3.8.1. Algorithm: Optimizing Parameters Using Cross-Validation and L-BFGS-B

Algorithm 1: Adaptive Cross-Validation using L-BFGS-B

Input: Design Matrix $X \in \mathbb{R}^{(n \times p)}$, response vector $y \in \mathbb{R}$, Initial bounds $[k_{1,lower}, k_{1,upper}]$, $[k_{2,lower}, k_{2,upper}]$, number of folds (n_{folds}) and Maximum iterations ($n_{iterations}$).

Output: Optimized parameters k_1^* and k_2^* , fitted regression model using k_1^* and k_2^* .

Step 1: Initialize Parameters

Set $k_1^0 = 1$, $k_2^0 = 1.346$, $k_{1,lower} = 1$, $k_{1,upper} = 1.345$, and $k_{2,lower} = 1.346$, $k_{2,upper} = 4.685$.

Step 2: Cross-Validation Function

For given k_1 and k_2 split the data into n_{folds} folds: $D = \{D_{train}, D_{test}\}$

Train the proposed model using D_{train} : $\hat{y} = \ell(X; k_1, k_2)$

Find MSE based on D_{test} : $MSE_{fold} = n_{test}^{-1} \sum_{i \in test} (y_i - \hat{y}_i)^2$

Average Cross-validated MSE: $MSE_{CV} = n_{folds}^{-1} \sum_{k=1}^{n_{folds}} (y_i - \hat{y}_i)^2$

Step 3: Optimization Using L-BFGS-B to obtain initial k_1^* and k_2^*

$$\text{Min}_{k_1, k_2} MSE_{CV}(k_1, k_2) \text{ Subject to } k_{1,lower} \leq k_1 \leq k_{1,upper}, k_{2,lower} \leq k_2 \leq k_{2,upper}$$

Step 4: Adaptive Cross-Validation

for $t = 1, 2, \dots, n_{iterations}$ do:

Update parameter bounds: $k_{1,lower} = k_1^* - 0.05$, $k_{1,upper} = k_1^* + 0.05$

$$k_{2,lower} = k_2^* - 0.05, k_{2,upper} = k_2^* + 0.05$$

Re-optimize using updated bounds to refine k_1^* and k_2^*

If: k_1^* and k_2^* are optimal (i.e., $MSE_{CV} < \text{Least MSE for OLS and Existing } M - \text{Estimators}$)

break the loop

Else: Proceed to the next iteration

end for

Step 5: Final Model Fitting

Fit Model: $\hat{y} = \ell(X; k_1^*, k_2^*)$ and obtain final MSE $\hat{\sigma}^2 = n_{iter}^{-1} \sum_{i=1}^n (y_i - \hat{y}_i)^2$

3.9. Simulation Design and Data Generation Process for Simple Linear Regression Model

The Monte Carlo simulation with one million (1,000,000) replications served as the basis for the simulation study for the simple models. This was done to guarantee the accuracy and reliability of estimation results under various contamination scenarios. Additionally, the simulation was conducted using R statistical software and the "MASS" package to perform robust regression. The following seven steps were employed to generate the predictor and response variables and fit the models in each estimator for the specified number of iterations:

- (i) A set of $n = 100$ predictor values (x_i) is generated uniformly from the interval $[0, 10]$

$$x_i \sim U(0,10), i = 1, 2, \dots, n$$

- (ii) The response variable (y_i) is generated using a linear regression model with random normal noise with parameters $\beta_0 = 12$, $\beta_1 = 4.3$ and $\sigma^2 = 1$

$$y_i = 12 + 4.3x_i + \epsilon_i, \epsilon_i \sim N(0, \sigma^2)$$

- (iii) Outliers are introduced with a probability α , representing the proportion of data points that will be contaminated such that

$$O = \{i: u_i < \alpha, u_i \sim U(0,1)\}$$

where $|O| = \lfloor n \cdot \alpha \rfloor$ denotes the number of outlier points.

- (iv) The outliers (m_i) are sampled from a chi-square distribution with $df = 1$ to ensure moderate randomness in outlier size such that

$$m_i \sim \chi_1^2, i \in O$$

(v) For the identified outlier indices $i \in O$, both the predictor (x) and response (y) variables are modified:

- The value of x_i is perturbed by adding the scaled outlier magnitude $(\lambda \cdot m_i)$

$$x_i \leftarrow x_i + \lambda \cdot m_i, i \in O$$

where λ denotes a scaling factor controlling the size of outliers.

- The value of y_i is adjusted by subtracting the scaled outlier magnitude $(\lambda \cdot m_i)$

$$y_i \leftarrow y_i - \lambda \cdot m_i, i \in O$$

(vi) The dataset with n pairs of x and y values is generated, with some points ($i \in O$) modified to include outliers. The dataset is structured as:

$$D = \{(x_i, y_i): i = 1, 2, \dots, n\}$$

(vii) For each generated data, the regression models were fitted to obtain estimates of the coefficients $\hat{\beta}_0$, $\hat{\beta}_1$, and the mean squared error ($\hat{\sigma}^2$).

Both regression models are fitted sequentially using traditional least square estimation (LSE), Huber, Bisquare, and the proposed M-estimator. Each iteration, model coefficients and MSE are calculated till the maximum number of iterations is attained. The parameter λ regulates perturbation magnitude (0, 1, 2, 3, 4, and 5), while the proportion of outliers (α) is set at 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, and 0.10.

3.10. Performance Evaluation Metrics

Regression estimators are assessed using the methodological approach by [24], [25], [62], and [23]. It emphasizes important performance indicators such as asymptotic relative efficiency (ARE), bias squared and mean squared error (MSE). The expected values and standard errors of the parameter estimates are also included as additional evaluation criteria. For each regression estimator, the MSE is systematically broken down into two components: the bias squared and the variance of the estimator. The asymptotic relative efficiency (ARE) of the estimators is the ratio of the MSE of M-estimators to the conventional least squares MSE. Robust estimators outperform the LSE when ARE value is less than one, whereas an ARE value greater than one indicates that the LSE provides more accurate estimates with less variation. If an estimator is inefficient, ARE may approach zero. The squared bias, MSE, and ARE for each estimator were calculated using the following formulas:

$$ARE(LSE, M - Estimator) = \frac{MSE(\hat{\beta}_{M-Estimator})}{MSE(\hat{\beta}_{LSE})} \quad (3.23)$$

$$Bias^2(\hat{\beta}) = (E(\hat{\beta}) - \beta)^2 \quad (3.24)$$

$$MSE(\hat{\beta}) = Bias^2(\hat{\beta}) + Var(\hat{\beta}) \quad (3.25)$$

3.11. Results and Discussion

3.11.1. Simulation Results for Simple Linear Regression before Tuning Parameter Optimization

This section presents the results and discussion of the proposed robust regression approach alongside Huber, Bisquare, and LSE in simple linear regression. The analysis of the proposed estimator begins with the tuning constants for Huber and Bisquare M-estimators as baseline parameters, balancing robustness and efficiency. To enhance accuracy, an adaptive algorithm iteratively refines these parameters, optimizing performance by reducing estimation errors while maintaining robustness. The evaluation is based on key metrics such as bias square, mean squared error (MSE), and asymptotic relative efficiency (ARE), along with standard errors and expected values to compare the refined and initial estimates. **Table A1-A3** in **Appendix A** show the results of the simple linear regression model based on the existing parameters from Huber and Bisquare M-estimators $k_1 = 1.345$ and $k_2 = 4.685$.

[A]: A Comparative Analysis of Estimators based on Mean, Bias Squared, and Standard Errors (SEs)

Intercept (β_0)

For uncontaminated data ($\alpha = 0.00$), all estimators provide accurate estimates close to the true parameter, demonstrating reliability in clean datasets. However, as contamination increases, the LSE becomes highly sensitive to outliers, leading to significant deviations.

At $\alpha = 0.05$ and $\lambda = 5$, its mean estimate rises to 17.156, and at $\alpha = 0.10$ and $\lambda = 5$, it further deteriorates to 20.624. In contrast, the robust estimators, particularly Bisquare and the proposed method, maintain stability, with the proposed estimator producing a mean estimate of 12.035 under extreme contamination. Though slightly outperformed by Bisquare in some cases, the proposed estimator consistently exhibits strong robustness.

Similarly, squared bias for LSE increases sharply with contamination, reaching 74.373 at $\alpha = 0.10$ and $\lambda = 5$, underscoring its vulnerability to outliers. Conversely, Huber, Bisquare, and the proposed estimator maintain significantly lower bias values, with the proposed method achieving the smallest deviation (0.001) at the highest contamination level. While Bisquare occasionally demonstrates marginally lower bias, altogether outperform LSE and Huber in preserving estimation accuracy.

Standard errors (SEs) follow a similar pattern. Under uncontaminated conditions, all estimators yield comparable SEs (0.202 to 0.208). However, LSE becomes increasingly unstable as contamination rises, with its SE increase suddenly to 5.372 at $\alpha = 0.10$ and $\lambda = 5$. In contrast, the robust estimators maintain considerably lower SEs, all remaining below 1.099. The proposed estimator strikes a favorable balance, consistently producing lower SE values than Huber while preserving the efficiency of Bisquare.

Overall, in the presence of outliers, robust estimators substantially outperform LSE by maintaining accuracy, minimizing bias, and ensuring stable variability. While Bisquare demonstrates slightly better performance, the proposed estimator remains highly resistant to contamination, reinforcing the importance of using robust estimation methods for

regression models applied to real-world or contaminated datasets. **Figure 3.2** below exhibits the performance of both estimators in terms of bias.

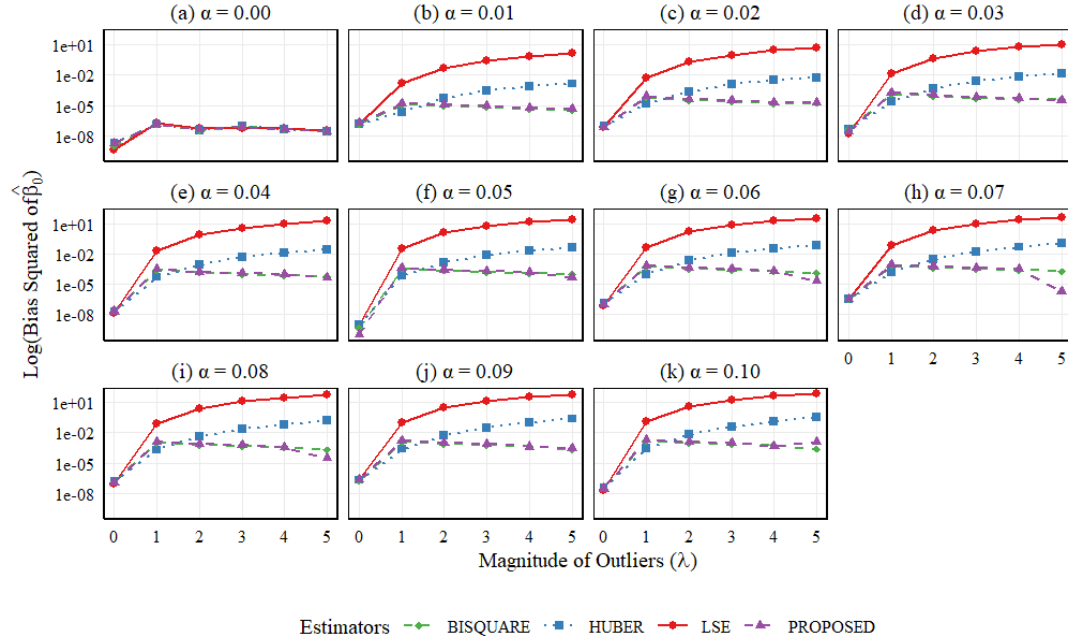


Figure 3.2 The bias square for the intercept estimates for simple linear regression before optimization of parameters.

Slope (β_1)

Across all estimators, mean estimates remain accurate in uncontaminated data ($\alpha=0.00$), but LSE quickly deteriorates as outliers increase. While the robust estimators, Huber (4.268), Bisquare (4.300), and the proposed (4.299), maintain much closer estimates, the mean estimate for LSE diverges from the true value of 4.300 at $\alpha=0.03$ and $\lambda=5$, falling to 3.520. Under extreme contamination ($\alpha=0.10$, $\lambda=5$), LSE falls drastically to 2.299,

highlighting its sensitivity to outliers, while Bisquare (4.297) and the proposed estimator (4.285) exhibit strong resistance.

The squared bias further reinforces these findings. The bias of LSE inflates with increasing contamination, reaching 1.436 at $\alpha=0.05$, $\lambda=5$, and worsening to 4.006 at $\alpha=0.10$, $\lambda=5$. In contrast, Bisquare and the proposed estimator maintain near-zero bias, demonstrating their superior robustness. Although Huber remains relatively stable, it exhibits slightly higher bias (0.025 at $\alpha=0.10$, $\lambda=5$) compared to Bisquare and the proposed estimator.

While all estimators have similar Standard errors (SE) values of approximately 0.035 in clean data, the variability of LSE escalates drastically as contamination rises, reaching 1.107 at $\alpha=0.07$, $\lambda=5$, and 1.097 at $\alpha=0.10$, $\lambda=5$. Meanwhile, Bisquare (0.085), Huber (0.275) and the proposed estimator (0.233) sustain significantly lower SEs, striking a balance between robustness and efficiency.

In general, the proposed estimator shows good resistance to contamination, preserving low bias and controlled variability. Although Bisquare performs marginally better in extreme situations, the proposed estimator is still a good substitute that efficiently reduces the impact of outliers. The performance of both estimators in terms of bias is shown in **Figure 3.3** below.

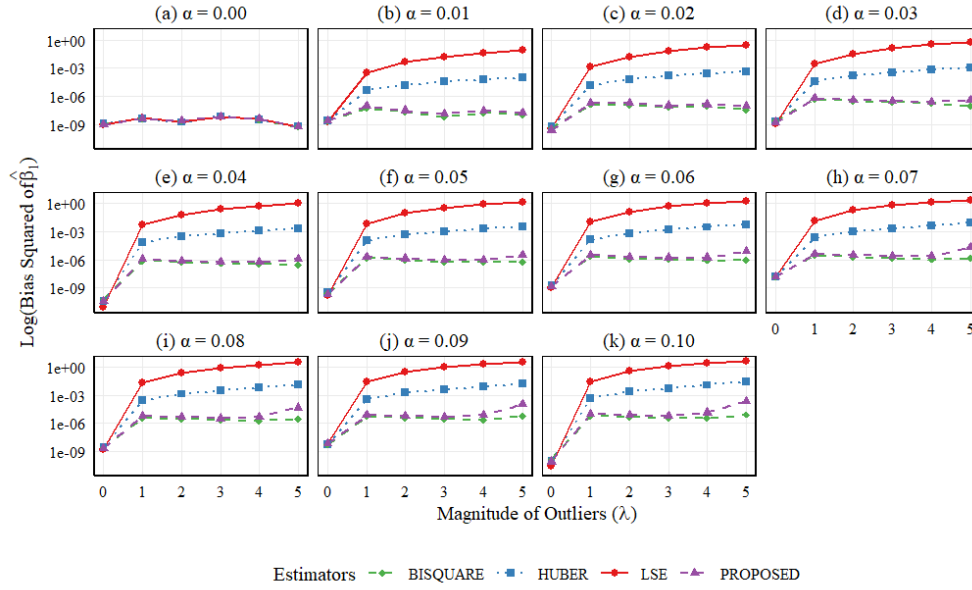


Figure 3.3 The bias square for the slope estimates simple linear regression before optimization of parameters.

Mean square error (σ^2)

Across all contamination levels, the proposed estimator consistently balances accuracy, bias reduction, and stability in standard error, making it a robust alternative to LSE. In uncontaminated data ($\alpha=0.00$), all estimators perform similarly, with LSE achieving an MSE of 1.000 and robust estimators closely matching it. However, as contamination increases, LSE becomes highly unstable, with MSE increasing to 89.952 at $\alpha = 0.10$, $\lambda = 5$, while the proposed estimator maintains a significantly lower MSE (2.042), outperforming Huber (2.722) and closely matching Bisquare (1.308).

Bias square trends emphasize this pattern, while all estimators remain unbiased in clean data, bias for LSE explodes under contamination, reaching 7912.526 at $\alpha = 0.10$, $\lambda = 5$,

whereas the proposed estimator maintains a much lower bias (1.085), performing better than Huber (2.965).

Similarly, standard error (SE) for LSE reveals extreme variability, with SE peaking at 49.453 under severe contamination ($\alpha = 0.10$, $\lambda = 5$). However, robust estimators demonstrate superior robustness with Huber (12.318), the proposed estimator (14.870), and Bisquare (5.028). Therefore, the proposed estimator successfully reduces the effect of outliers, providing competitive trade-off between robustness and efficiency against Bisquare and Huber. **Figure 3.4** below illustrates how well both estimators performed in terms of bias.

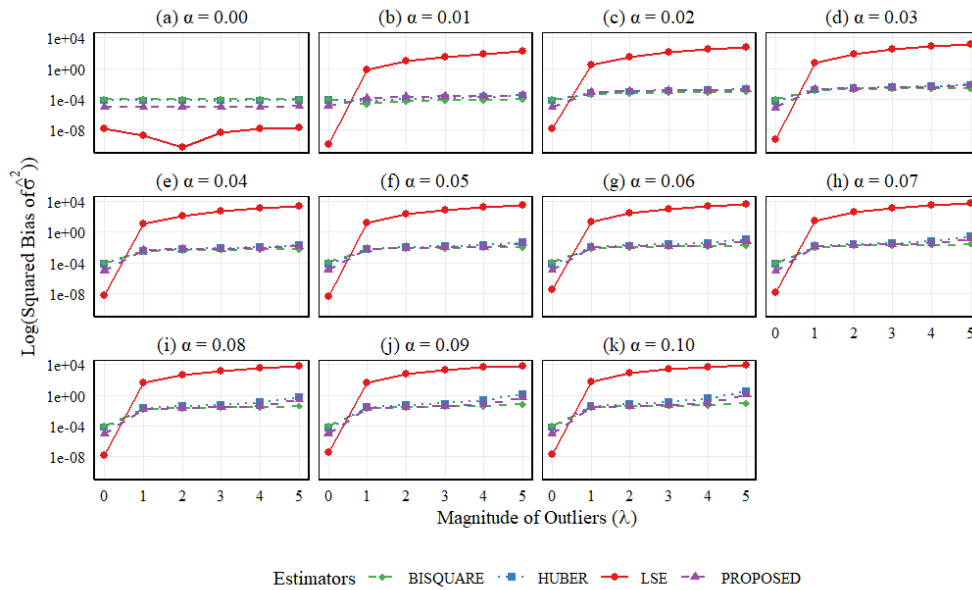


Figure 3.4 The bias square for the mean square error (MSE) estimates for simple linear regression before optimization of parameters.

[B]: A Comparative Analysis of Estimators based on the Mean Squared Error (MSE) and Asymptotic Relative Efficiency (ARE)

Intercept (β_0)

In uncontaminated data ($\alpha = 0.00$), all estimators perform similarly in mean square error (MSE), with the proposed estimator (0.042) slightly outperforming Huber and Bisquare both having MSE of 0.043. However, as contamination increases, LSE deteriorates significantly, with its MSE severely growing to 103.232 at $\alpha = 0.10$, $\lambda = 5$, while robust estimators maintain much lower values Bisquare (0.174), Proposed (1.208), and Huber (1.916). The proposed estimator consistently balances robustness and efficiency, staying closer to Bisquare and significantly outperforming LSE.

Asymptotic relative efficiency (ARE) follows a similar trend. In clean data, robust estimators have ARE values slightly above 1, but as contamination rises, they decline to verify their efficiency. At $\alpha = 0.10$, $\lambda = 5$, ARE drops to 0.02 for Huber and 0.002 for Bisquare, while the proposed estimator maintains 0.012, demonstrating superior trade-off between robustness and efficiency. Overall, the proposed estimator proves to be a strong alternative to LSE, consistently maintaining reliability in contaminated data while balancing efficiency and robustness. **Figure 3.5** and **Figure 3.6** below present a clear trend for MSE and ARE for the intercept coefficient.

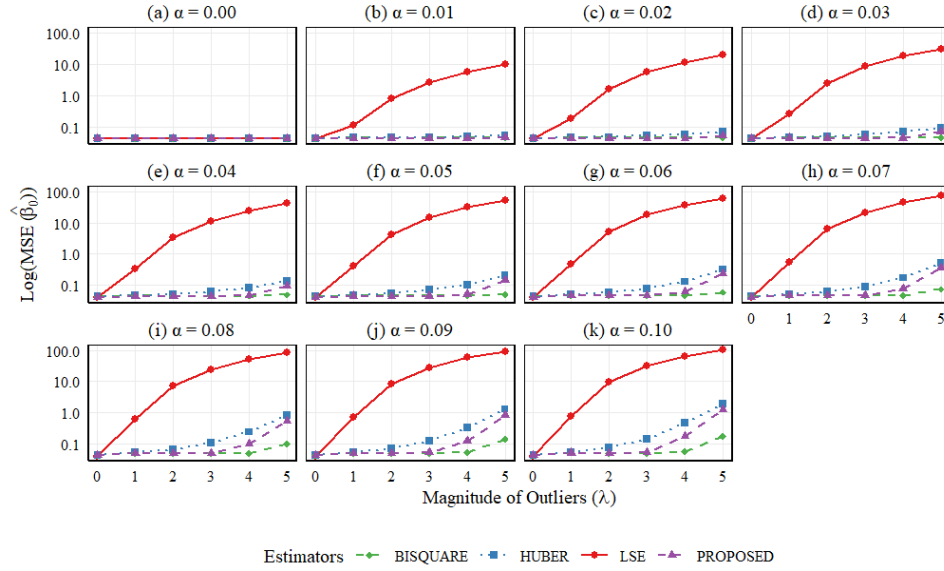


Figure 3.5 The mean square error (MSE) for the intercept estimates for simple linear regression before optimization of parameters.

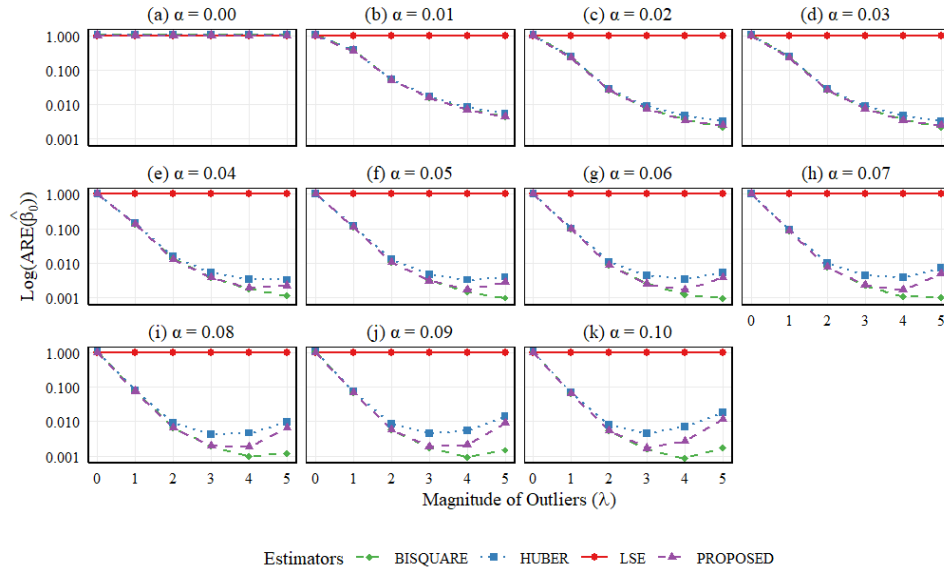


Figure 3.6 The asymptotic relative efficiency (ARE) of the intercept estimates for simple linear regression before optimization of parameters.

Slope (β_1)

All estimators show identical, low MSE values (0.001) in uncontaminated data ($\alpha=0.00$), demonstrating same performance. Immediate vulnerability of LSE is highlighted at ($\alpha=0.01$, $\lambda=1$), its MSE jumps to 0.006, whereas Huber, Bisquare, and the proposed estimator maintain 0.001. As contamination rises to ($\lambda=5$) its MSE spikes to 0.478 while Huber goes to 0.002, Bisquare and the proposed estimator stay at 0.001. LSE further degrades to 5.210 under extreme contamination ($\alpha=0.10$, $\lambda=5$), whereas the proposed estimator (0.055) stays in between Huber and Bisquare, indicating superior compromise between robustness and efficiency.

In uncontaminated data, LSE is the most efficient, while robust estimators exhibit slightly higher ARE values, Huber (1.053), Bisquare (1.059), and the proposed estimator (1.024). As contamination increases ($\alpha=0.01$, $\lambda=5$), the efficiency of robust estimators remains competitive, with ARE values of 0.004 for Huber and 0.003 for both Bisquare and the proposed estimator. Despite optimal performance of LSE in clean data, its efficiency collapses under contamination, whereas the proposed estimator consistently balances robustness and efficiency, closely matching the performance of Bisquare, especially under extreme conditions ($\lambda=4$, $\lambda=5$).

Overall, the proposed estimator offers a strong balance between minimizing error and maintaining efficiency under contamination. While LSE performs best in ideal conditions, its rapid decline in MSE and ARE highlights its unsuitability for contaminated data. The proposed estimator, slightly outperforming Huber and closely matching Bisquare, proves

to be a highly effective alternative for handling outliers without sacrificing efficiency. A distinct trend for MSE and ARE for the slope coefficient is shown in **Figure 3.7** and **Figure 3.8** below.

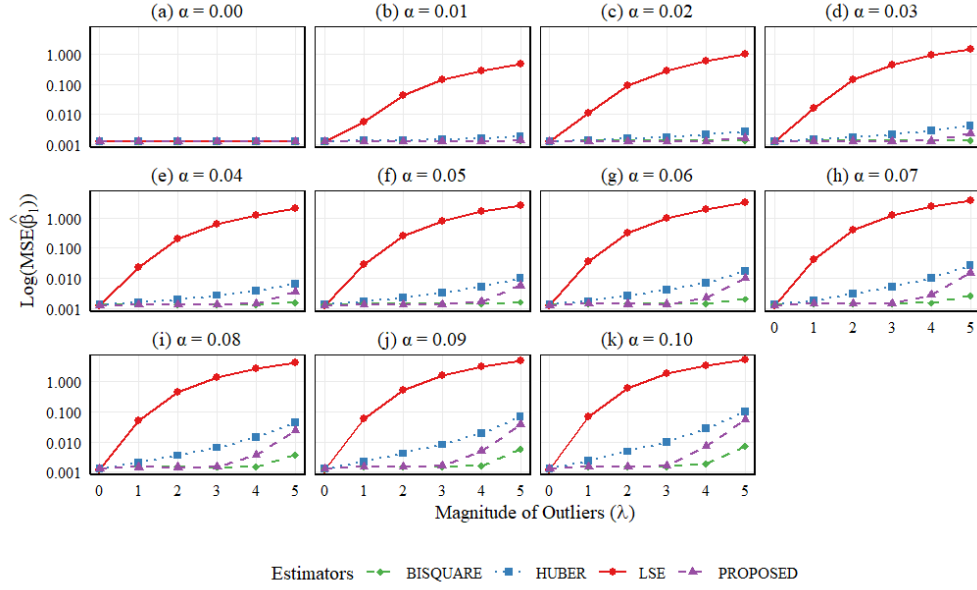


Figure 3.7 The mean square error (MSE) for the slope estimates for simple linear regression before optimization of parameters.

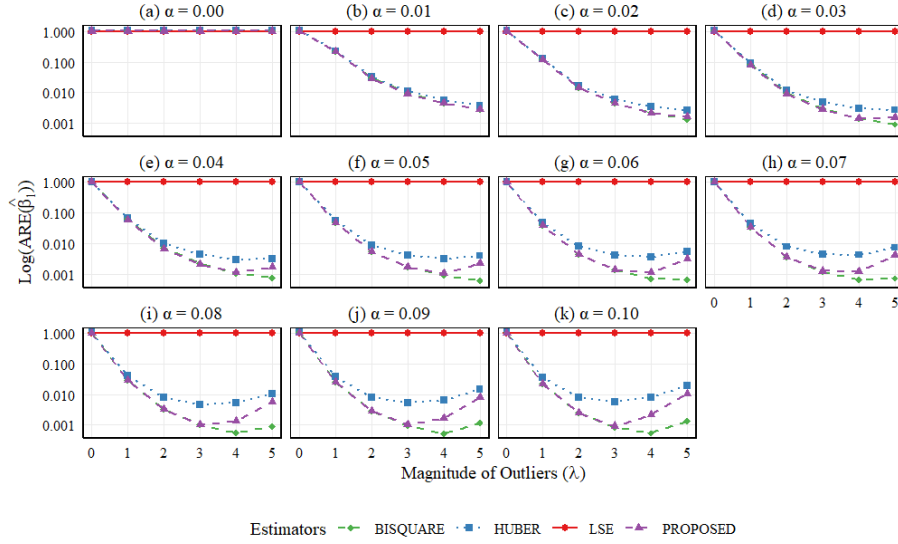


Figure 3.8 The asymptotic relative efficiency (ARE) of the slope estimates for simple linear regression before optimization of parameters.

Mean square error (σ^2)

In uncontaminated data ($\alpha=0.00$), LSE achieves the lowest MSE (0.020), while robust estimators show slightly higher but comparable values 0.054 each, maintaining accuracy despite their robust mechanisms. However, even under minimal contamination ($\alpha=0.01$, $\lambda=1$), robust estimators remain stable at 0.056 while LSE MSE increases to 7.484. At higher contamination levels ($\lambda=5$), the MSE for LSE deteriorates further to 941.402, while Huber and Bisquare maintain lower values (0.231), and the proposed estimator (0.462) though slightly higher but exhibits strong robustness. In extreme contamination ($\alpha = 0.10$, $\lambda = 5$), LSE becomes highly unstable, reaching an MSE of 10,358.151, whereas the proposed estimator (222.196) remains competitive with Huber (154.697) and Bisquare (25.377), the latter demonstrating the strongest resistance to outliers.

On the other hand, ARE reinforces that in clean data, robust estimators show higher ARE values, indicating a compromised efficiency. However, under minor contamination ($\alpha = 0.01, \lambda = 1$), the efficiency of LSE collapses, with ARE for robust estimators dropping to 0.007. As contamination increases ($\lambda = 5$), ARE values approach zero, confirming complete inefficiency of LSE in handling outliers. Under extreme contamination ($\alpha=0.10, \lambda=5$), Bisquare achieves the highest efficiency with (0.002), followed by Huber (0.015) and the proposed estimator (0.021). The proposed estimator balances robustness and efficiency across contamination levels, while Bisquare emerges as the most resilient under extreme outlier influence. In this case, the MSE and ARE pattern for the mean square error of simple linear regression model is shown in **Figure 3.9** and **Figure 3.10**.

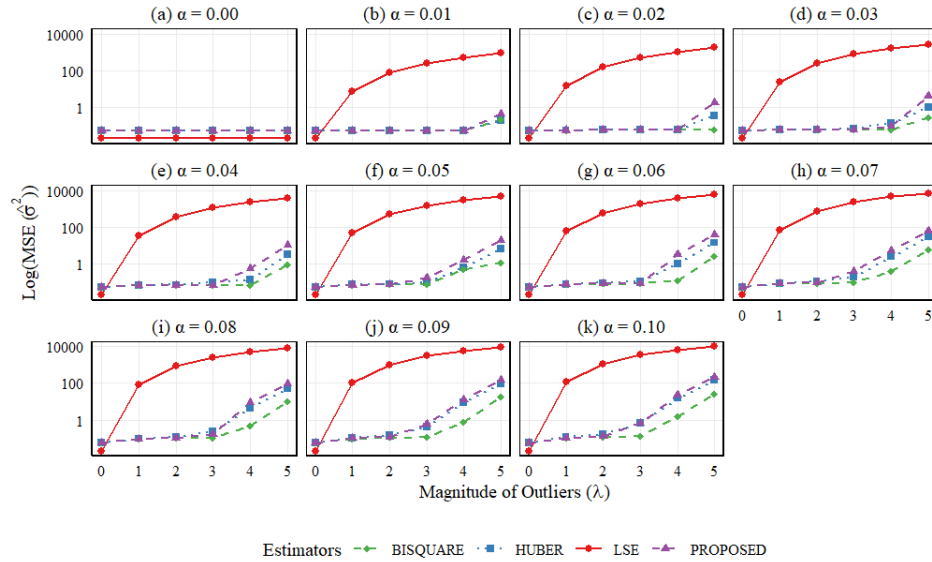


Figure 3.9 The mean square error (MSE) of MSE estimates for simple linear regression before optimization of the parameters for the proposed estimator.

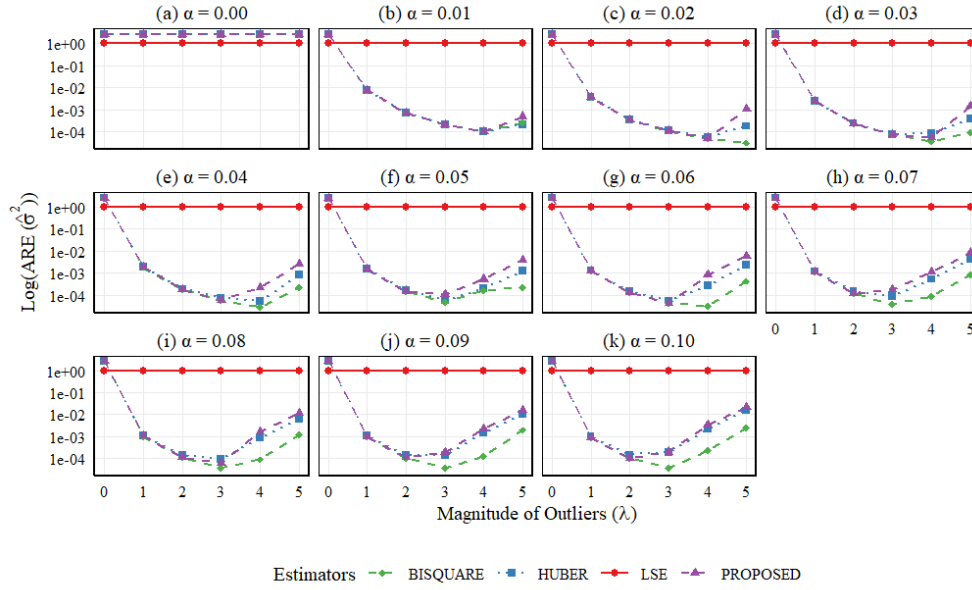


Figure 3.10 The asymptotic relative efficiency (ARE) for the mean square error estimates for simple linear regression before optimization of the parameters for the proposed estimator.

3.11.2. Simulation Results for Simple Linear Regression after Optimization of Parameter Optimization

The findings of a simple linear regression model are presented in this section following parameter optimization using the adaptive cross-validation algorithm using the optimal values $k_1 = 1.020$ and $k_2 = 2.015$. In **Appendix A**, detailed results for the simple linear regression model are shown in **Table A4 - A6**.

[A]: A Comparative Analysis of Estimators based on Mean, Bias Squared, and Standard Errors

Intercept (β_0)

The true intercept value of 12.000 is accurately estimated by all estimators when there are no outliers ($\alpha = 0.00$). However, the performance of LSE sharply deteriorates as contamination increases, reaching 20.624 at $\alpha = 0.10$, $\lambda = 5$. Bisquare remains relatively stable, indicating only insignificant fluctuations, whereas Huber is the most affected robust estimator, with its mean estimate increasing to 12.603. The proposed estimator shows a high degree of resistance to contamination by consistently maintaining mean estimates close to 12.000. In addition, the bias square of LSE increases significantly up to 74.368 at $\alpha = 0.10$, $\lambda=5$.

Again, Huber is slightly impacted by outliers producing a bias square of 0.364 whereas Bisquare and the proposed approach have negligible bias, about 0.000. The robustness of the proposed estimator is demonstrated by its lowest bias square and remains consistent across all contamination levels.

Similarly, standard errors (SEs) for LSE increase sharply from 0.202 to 5.374 as contamination increases. Huber exhibits a moderate increase from 0.207 to 1.257 but remains relatively stable. Bisquare continuously maintains low SE values from 0.207 to 0.429, signifying its resistance to contamination. The proposed estimator is confirmed to be more robust than LSE and to consistently balance efficiency and robustness, stabilizing

at 0.269 for all contamination levels. The performance trend of each estimator in terms of bias square following turning parameter optimization is shown in **Figure 3.11** below.

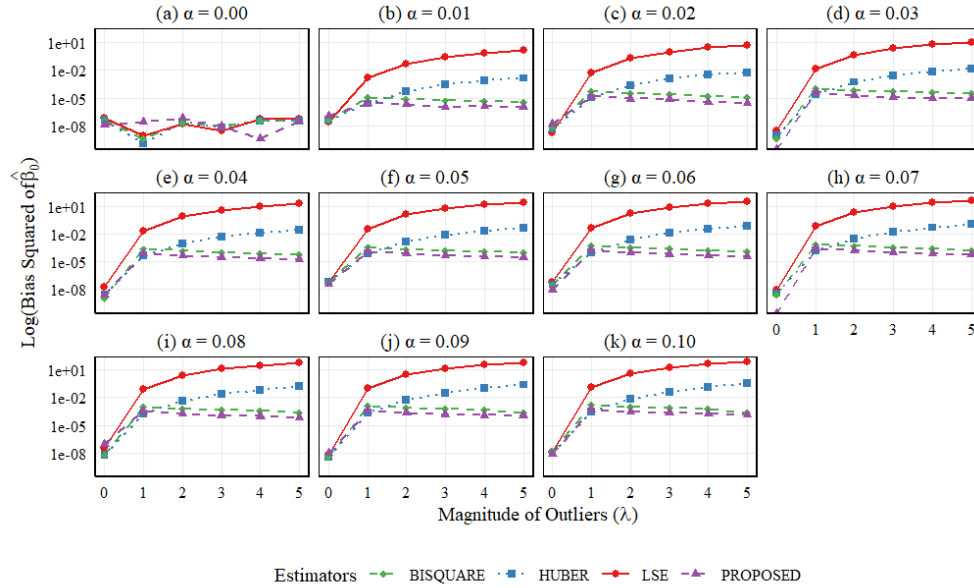


Figure 3.11 The bias square for the intercept estimates for simple linear regression after optimization of the parameters for the proposed estimator.

Slope (β_1)

In uncontaminated data, all estimators accurately estimate 4.300, just like the intercept estimates do. However, the performance of the least squares drastically deteriorates when contamination levels rise ($\alpha = 0.10$, $\lambda = 5$), with its estimate falling to 2.299. Likewise, Huber is the robust estimator that is most impacted by contamination, declining slightly to 4.140. However, both the Bisquare and the proposed estimators remain near 4.300, with

the proposed approach exhibiting the highest stability, thereby highlighting its resilience in parameter estimation.

Similarly, at $\alpha = 0.10$, $\lambda=5$, the bias square of LSE drastically rises from 0.000 to 4.005, highlighting its high vulnerability to contamination. Bisquare and the proposed estimator show resilience by remaining below 0.026 even though Huber establishes slight increases, reaching 0.026. The proposed estimator consistently maintains lower squared bias values than the LSE and other robust estimators, demonstrating its robustness to outliers.

Same trends are observed in standard errors, where the LSE abruptly grows from 0.035 to 1.098 and exhibits notable instability. The proposed estimator continuously maintains low standard error values (0.047), whereas Huber and Bisquare only experience slight increases and remain relatively stable. The proposed estimator is more effective at handling contaminated data because it strikes a balance between robustness and efficiency, generating accurate and trustworthy estimates across a range of contamination levels. The performance pattern of each estimator concerning bias square of the slope coefficient after tuning parameter optimization is illustrated in **Figure 3.12** below.

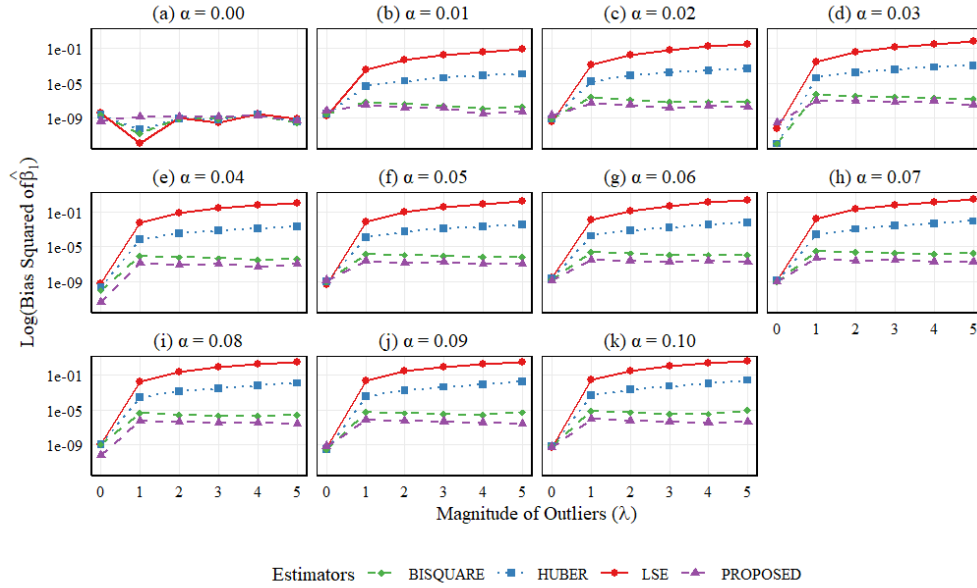


Figure 3.12 The bias square for the slope estimates for simple linear regression after optimization of the parameters for the proposed estimator.

Mean square error (σ^2)

In the presence of contamination, the performance of the LSE is severely reduced. Initially, LSE maintains a mean square error (MSE) estimate of 1.000 under uncontaminated conditions; however, as contamination increases ($\alpha = 0.10$, $\lambda = 5$), its MSE significantly escalates, reaching 89.949. In contrast, the Huber, Bisquare, and the proposed estimators consistently maintain values close to 1.000, with the proposed estimator showing slightly greater variability than Bisquare but remaining within a stable range.

The same pattern is observed in bias squared, with the LSE rapidly increasing to 7911.944 at $\alpha = 0.10$, $\lambda = 5$. By contrast, the bias squared values of the Huber and Bisquare estimators

are substantially lower, at 3.036 and 0.098 respectively. Notably, the proposed estimator maintains its stability at 0.001, demonstrating both its reliability in parameter estimation and its high resistance to contamination.

Furthermore, the standard errors associated with LSE for MSE increased drastically from 0.143 to 49.487 as contamination increased. While Huber and Bisquare estimators indicate moderate increases, they remain within an acceptable range. The proposed estimator maintains a standard error below 0.674, further underscoring its robustness in handling contaminated data and its ability to balance efficiency and robustness. **Figure 3.13** below illustrates the bias square behavior of each estimator for the mean square error of the model after optimizing the tuning parameter.

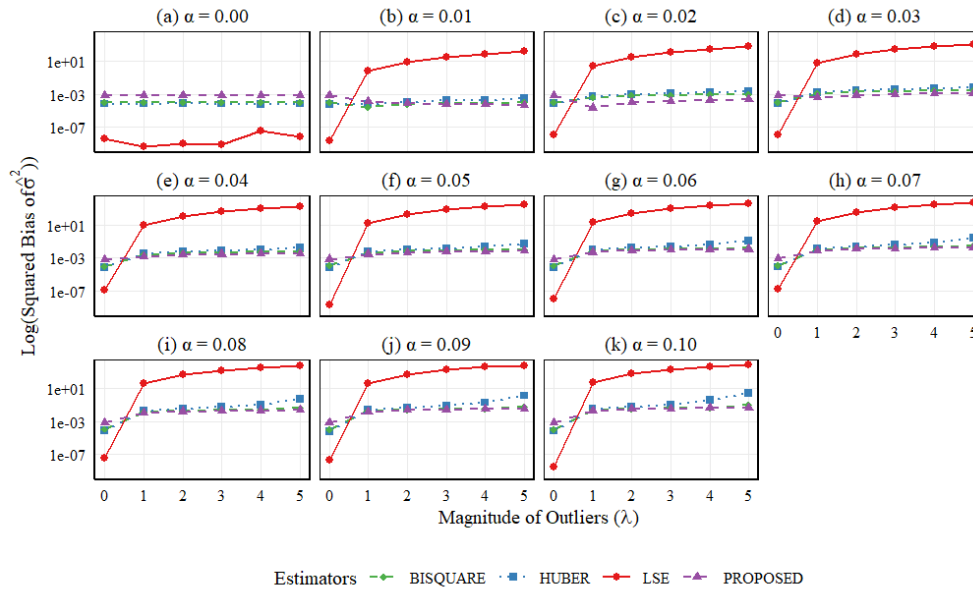


Figure 3.13 The bias square for the mean square error (MSE) estimates for simple linear regression after optimization of the parameters for the proposed estimator.

[B]: A Comparative Analysis of Estimators based on the Mean Squared Error (MSE) and Asymptotic Relative Efficiency (ARE)

Intercept (β_0)

In uncontaminated conditions ($\alpha = 0.00$) all estimators performed equally with Huber and Bisquare reporting somewhat higher values of MSE (0.043) and LSE (0.041). The slightly conservative nature of the proposed estimator in clean data settings is reflected in its higher MSE of 0.073. The extreme sensitivity of LSE to outliers is demonstrated by its increase in MSE to 103.244 as contamination rises at ($\alpha = 0.10, \lambda = 5$). The proposed estimator, Huber, and Bisquare, on the other hand, maintain significantly lower MSE values at 1.944, 0.184, and 0.072, respectively. It is noteworthy that Bisquare exhibits the lowest MSE in cases of extreme contamination, while the proposed estimator maintains relatively stable values, demonstrating its capacity to reduce the effects of contamination without sacrificing efficiency.

However, as demonstrated by the asymptotic relative efficiency (ARE) the robust estimators are less efficient than the LSE in uncontaminated conditions ($\alpha = 0.00$) with Huber (1.054), Bisquare (1.059), and the proposed estimator (1.784). The ARE values for robust estimators, on the other hand, dramatically decrease with increasing contamination, demonstrating their effectiveness. Huber and Bisquare achieve the ARE of 0.019 and 0.002, respectively, at $\alpha = 0.10, \lambda = 5$. The proposed estimator maintains a more stable efficiency with ARE close to 0.000, proving its capacity to strike a balance between efficiency and robustness. This suggests that the efficiency of LSE totally breaks down at

this level of contamination. **Figure 3.14** and **Figure 3.15** below depict the clear pattern of MSE and ARE for the intercept coefficient after optimizing the tuning parameters.

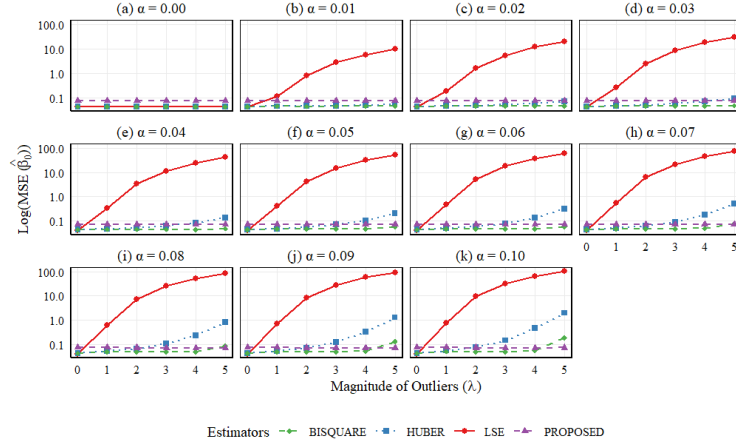


Figure 3.14 The mean square error (MSE) for the intercept estimates for simple linear regression after optimization of the parameters for the proposed estimator.

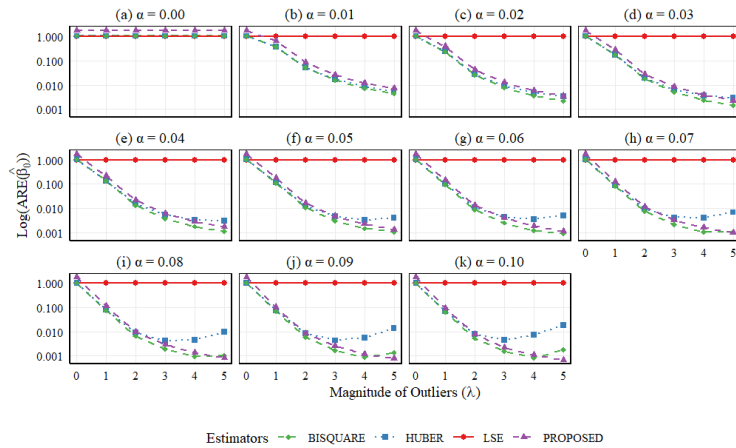


Figure 3.15 The asymptotic relative efficiency (ARE) of the intercept estimates for simple linear regression after optimization of the parameters for the proposed estimator.

Slope (β_1)

For the slope coefficient, all estimators yield nearly the same MSE values under ideal data ($\alpha = 0.00$), with the proposed estimator indicating slightly higher values at 0.002, while LSE, Huber, and Bisquare report values of 0.001. Nonetheless, the MSE for LSE exhibits extreme inefficiency as contamination rises, rising from 0.001 to 5.210. The MSE values for Huber and Bisquare at $\alpha = 0.10$, $\lambda = 5$ are 0.102 and 0.008, respectively, indicating that they are still relatively stable. An MSE of 0.002 is maintained throughout by the proposed estimator, demonstrating its exceptional capability to strike a balance between efficiency and robustness.

Likewise, robust estimators initially exhibit lower efficiency than LSE in uncontaminated settings with ARE values such as Huber (1.053), Bisquare (1.059), and the proposed estimator (1.787). Conversely, ARE values for robust estimators sharply decrease with increasing contamination, which is consistent with the MSE trend. At $\alpha = 0.10$, $\lambda = 5$, Huber, Bisquare, and the proposed estimator attains the highest level of efficiency with ARE equal to 0.020, 0.002, and 0.000, respectively. This highlights that the proposed estimator maintains slightly higher efficiency than Bisquare, making it a better trade-off between robustness and efficiency. A noticeable trend in MSE and ARE is evident in **Figure 3.16** and **Figure 3.17** below.

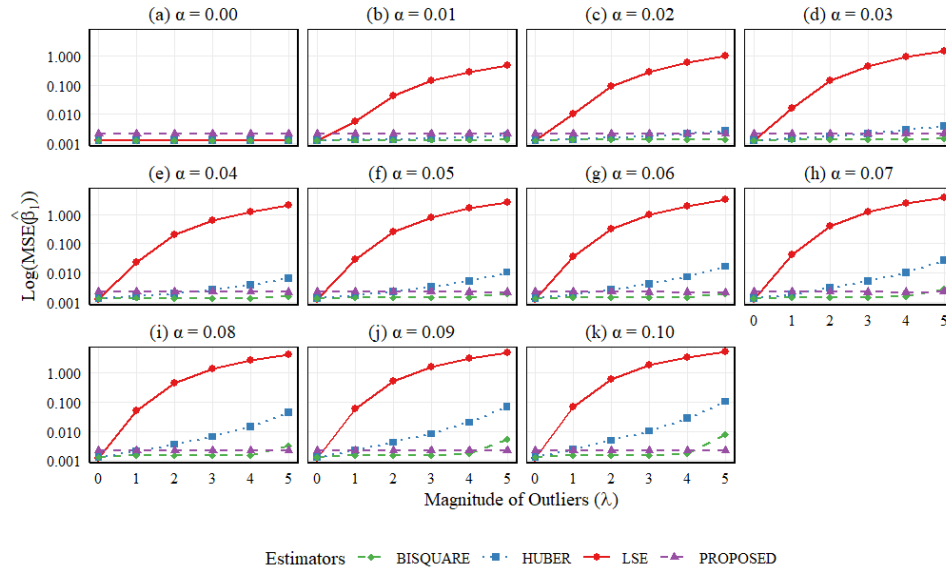


Figure 3.16 The mean square error (MSE) for the slope estimates for simple linear regression after optimization of the parameters for the proposed estimator.

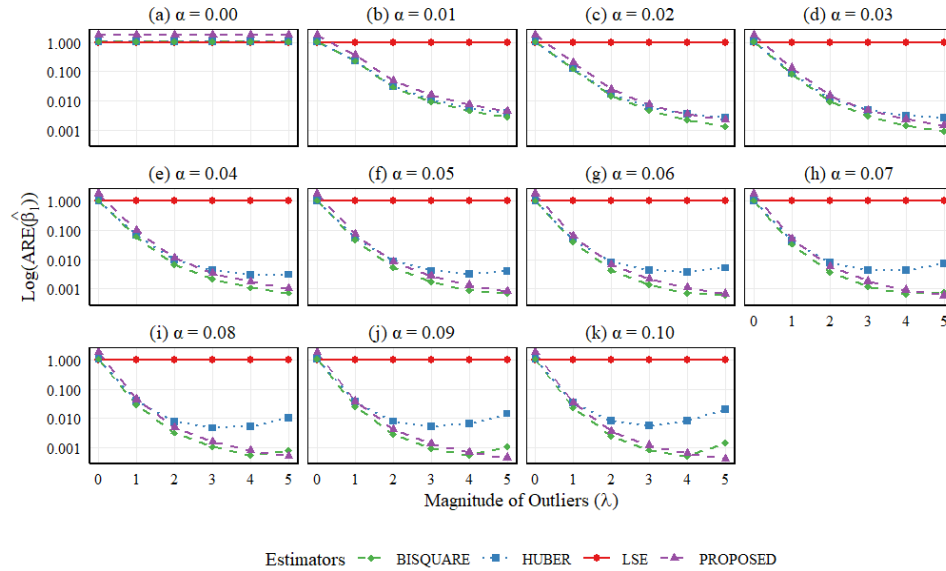


Figure 3.17 The asymptotic relative efficiency (ARE) of the slope estimates for simple linear regression after optimization of the parameters for the proposed estimator.

Mean square error (σ^2)

In terms of mean square error (MSE) estimation, the LSE performs well when there is no contamination. However, as contamination increases, its MSE rises sharply from 0.020 to 10,360.906, indicating a complete breakdown. In contrast, the Huber and Bisquare estimators maintain much lower MSE values, with Huber reaching 157.791 and Bisquare 26.186. The proposed estimator remains stable at 0.497, demonstrating its robustness in variance estimation.

A similar pattern is observed in the asymptotic relative efficiency (ARE) values. Robust estimators tend to be less efficient in uncontaminated data, but as contamination increases, their asymptotic relative efficiency values drop significantly, improving their efficiency. At $\alpha = 0.10$ and $\lambda = 5$, the ARE value for Huber falls to 0.015, Bisquare to 0.003, and the proposed estimator to 0.000. This suggests that under extreme contamination, robust estimators become more efficient compared to their performance in the absence of outliers. However, at higher contamination levels, the proposed estimator maintains relatively higher efficiency than Bisquare. Further details on the trends of ARE and MSE can be found in **Figure 3.18** and **Figure 3.19** below.

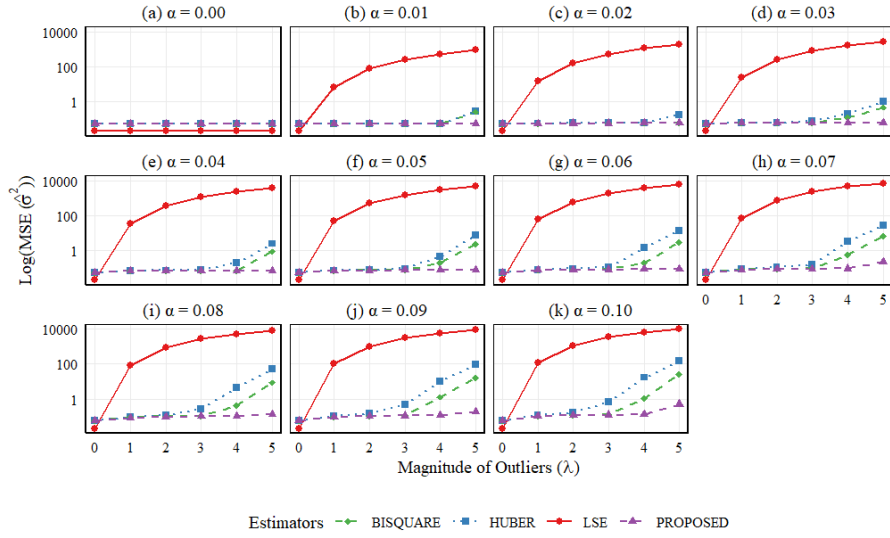


Figure 3.18 The mean square error (MSE) of MSE estimates for simple linear regression after optimization of the parameters for the proposed estimator.

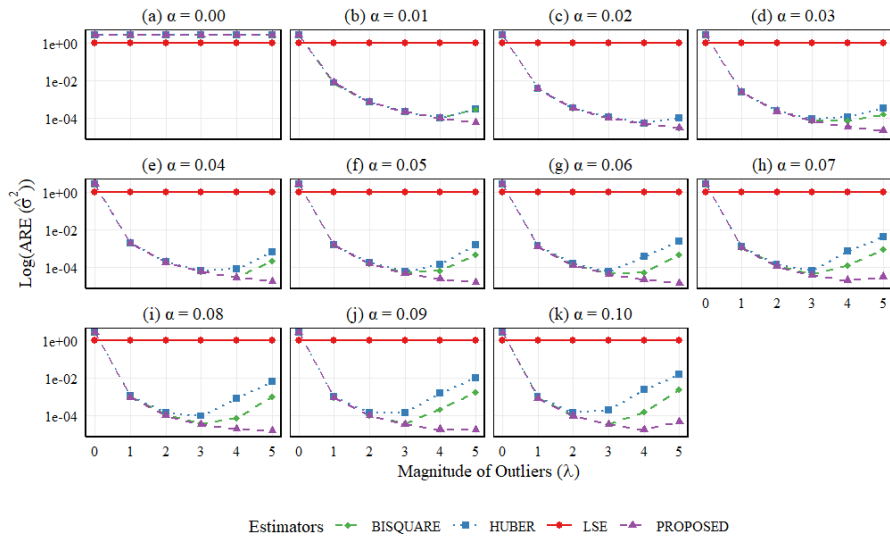


Figure 3.19 The asymptotic relative efficiency for the mean square error (MSE) estimates for simple linear regression after optimization of the parameters for the proposed estimator.

3.12. Application on Real Datasets

The study applied the "Metro Interstate Traffic Volume 2012-2018" data from [65], which contains hourly traffic volume for MN DoT ATR station 301 between Minneapolis and St. Paul, MN. The dataset has 48,204 observations and 9 variables. The response variable traffic volume (Hourly I-94 ATR 301 reported westbound traffic volume) was predicted using four numeric predictor variables: temp (Average temperature measured in kelvin), rain_1h, snow_1h, and clouds_all (Percentage of cloud cover). We randomly selected 150 observations and removed unrealistic weather events to create an ideal scenario. Minimal snowfall was simulated under ideal conditions by adding tiny snowfall values to prevent contamination by extreme or invalid values, as shown in **Figure 3.20** below. Furthermore, this example is implemented in both simple linear regression in Chapter Two and multiple linear regression in Chapter Four.

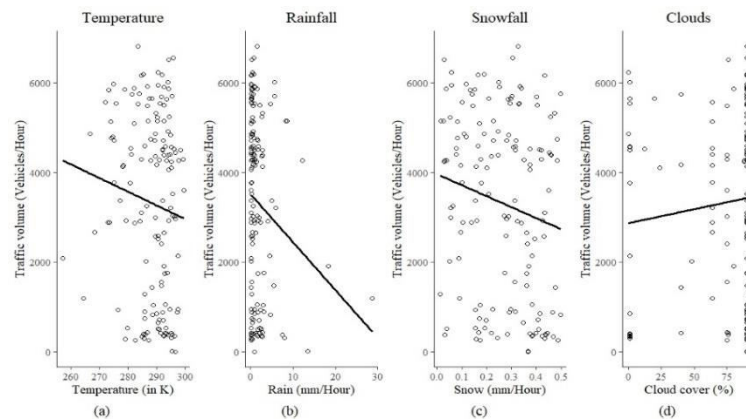


Figure 3.20 The scatter plots for Metro Interstate Traffic Volume against Temperature, Rainfall, Snowfall and Clouds all obtained at Minneapolis-St Paul, MN Traffic Volume for Westbound I-94 from 2012-2018

Figure 3.20 above demonstrates how average temperature, rainfall, snowfall, and cloud cover affect Minneapolis and St. Paul traffic volume. **Subplot (a)** shows that greater temperatures lower traffic volume. This shows that colder seasons increase automobile use while warmer seasons support alternate mobility. In **subplot (b)**, rainfall decreases traffic volume. Snowfall also negatively affects traffic volume in **subplot (c)**, but so less than rainfall. **Subplot (d)** shows that cloud cover slightly decreases traffic volume. We prepared summary statistics to see the behavior of the variables before continuing with the analysis, as shown in **Table 3.1** below.

Table 3.1 The summary statistics for Minneapolis and St. Paul, MN Metro Interstate Traffic Volume in absence of extreme weather conditions (Outliers).

	Temp	Rain	Snow	Clouds	Traffic Volume
Statistics	(K)	(mm/h)	(mm/h)	(%)	(Vehicles/h)
Min.	257.400	0.250	0.012	0.000	3.000
1st Qu.	284.800	0.280	0.149	75.000	1045.000
Median	290.500	0.650	0.278	90.000	3770.000
Mean	288.300	1.890	0.259	72.690	3321.000
3rd Qu.	294.000	2.000	0.379	90.000	5042.000
Max.	299.600	28.700	0.499	92.000	6807.000

According to summary statistics in **Table 3.1** above, when temperatures are reasonable, rainfall is light, and snowfall is not excessive, Minneapolis and St. Paul citizens maintain traffic flow. The typical temperature is 257.4 to 299.6 K, with extra winter to moderate temperature. This suggests that residents are probably accustomed to commuting in such temperatures, and the climate there supports consistent traffic flow. Light rain ranging from

0.25 to 28.7 mm/hour rainfall. Snow falls between 0.012 and 0.499 mm/hour with no extreme events. Partly cloudy and overcast conditions are common in the region, which has 0% to 92% cloud cover. At busy times, there are 6807 vehicles every hour, while at quiet times there are only 3 vehicles.

3.12.1. Illustration in Simple Linear Regression Model

The relationship between traffic volume and temperature in Minneapolis under ideal weather conditions was investigated using a straightforward linear regression model as shown below:

$$\text{Traffic Volume} = \beta_0 + \beta_1 \times \text{Temperature} + \epsilon \quad (3.26)$$

where Traffic volume is a dependent variable, temperature denotes predictor variable measured in Kelvin, intercept (β_0), slope (β_1), and error term (ϵ). As indicated in **Table 3.2** below, the main goal in this example is to ascertain how accurate estimators are at estimating the parameters of the given model.

Table 3.2 Estimation results for the simple linear regression model on Traffic volume, and Temperature variables based on clean and contaminated Metro Interstate Traffic Volume dataset.

Case I – Uncontaminated Scenario							
Estimator	Intercept	SE	Temp	SE	MSE	ARE	Turning Parameters
LSE	12112.230	6281.180	-30.492	21.780	4133916	1.000	—
Huber	12112.230	6281.177	-30.492	21.780	7809163	1.889	$k_1 = 1.345$

Bisquare	12281.200	6736.190	-31.005	23.357	7813324	1.890	$k_2 = 4.685$
Proposed	12112.230	6281.177	-30.492	21.780	7809163	1.889	$k_1 = 1.345, k_2 = 4.685$
Case II – Contaminated Scenario							
LSE	-6740.523	9212.810	36.983	31.880	9859633	1.000	—
Huber	6459.935	6954.242	-10.043	24.061	7483317	0.759	$k_1 = 1.345$
Bisquare	15627.378	7198.094	-42.201	24.905	7478742	0.759	$k_2 = 4.685$
Proposed	14499.482	6543.962	-38.198	22.642	7288673	0.739	$k_1 = 1.345, k_2 = 4.685$
Proposed CV	8962.237	4411.809	-16.048	15.265	5364842	0.544	$k_1 = 0.928, k_2 = 1.411$

In **Table 3.2** above, Case I shows the ideal Metro Interstate Traffic Volume condition, where the LSE estimator has a lowest mean square error (MSE) of 4,133,916, an intercept estimate of 12,112.230, and a slope of -30.492. implying a decline in traffic flow for every unit increase in temperature, while robust estimators are less efficient confirmed by ARE values 1.890 each. Thus, in ideal conditions, the LSE is most efficient.

3.12.2. Introducing outliers into Traffic volume and Temperature variables

As shown in **Figure 3.21** below, temperature and traffic volume variables were perturbed using simple linear regression simulation procedure presented in **Section 3.9**, and traffic volume data was stretched by 8500 vehicles/hour to generate suitable outliers.

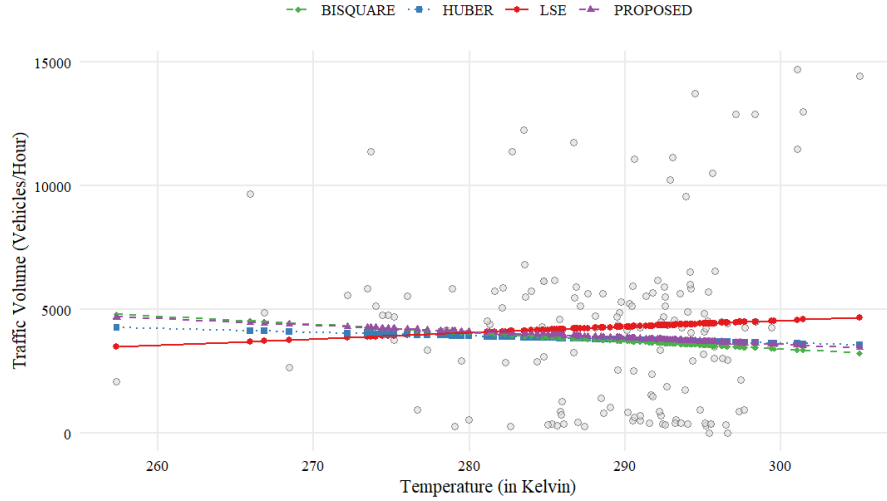


Figure 3.21 The relationship between Traffic volume and Temperature after introducing outliers.

In **Table 3.2** above, Case II indicates that LSE is more influenced by outliers, with an MSE of 9,859,633, showing its sensitivity in contaminated settings. The Huber estimator is more robust with an intercept of 6,459.935 and a slope of -10.043. MSE of 7,483,317 and ARE of 0.759 suggesting significant outlier reduction. Bisquare estimator increases robustness with an intercept of 15,627.378, a higher negative slope of -42.201 and the MSE of 7,478,742. The proposed estimator balances robustness and efficiency with an intercept of 14,499.482, a slope of -38.198 and MSE of 7,288,673. The performance of the proposed estimator is greatly improved using adaptive cross-validation, hence outperforms all other estimators, as evidenced by its 8956.916 intercept, -16.028 slope, MSE of 5,364,536 which is the least among all estimators, reflecting a substantial improvement in efficiency and robustness.

3.12.3. Conclusion

The chapter provides a comprehensive examination of robust regression methodologies, contrasting robust M-estimators like Huber's, Tukey's Bisquare, and a newly proposed hybrid M-estimator with conventional least squares estimation (LSE). Through detailed mathematical formulations and the application of the Iteratively Reweighted Least Squares (IRLS) algorithm, the study demonstrates how robust techniques successfully reduce the impact of outliers. Monte Carlo simulation and comparative analyses reveal that while LSE is highly susceptible to contamination, robust estimators sustain lower bias and variability. Adaptive cross-validation further improves the performance of the proposed hybrid estimator, which outstandingly strikes a good balance between robustness and efficiency even under extreme contamination scenarios. The practical application of the estimator on real-world traffic volume data highlights its potential for strong statistical modeling in a variety of contexts.

CHAPTER 4

COMPARATIVE ANALYSIS OF ESTIMATORS IN MULTIPLE LINEAR REGRESSION

This chapter builds upon the evaluation of the proposed M-estimator by extending its analysis, alongside LSE, Huber, and Bisquare estimators, to the multiple linear regression model. A simulation study is conducted to assess the impact of influential and leverage points on parameter estimation. This analysis aims to compare the effectiveness of robust estimators against the traditional LSE, highlighting their ability to handle data contamination and maintain reliable performance under various conditions.

4.1. Simulation Design and Data Generation Process for Multiple Linear Regression

The research additionally evaluated estimation techniques within a multiple linear regression context, focusing on influential and low-leverage points. The predictor variables were generated from diverse distributions and utilized to generate the response variable through a multiple linear regression model. The framework for the simulation is provided below:

1. Parameter Initialization

- Sample size: $n = 100$
- Parameter values: $\beta_0 = 12, \beta_1 = 4.3, \beta_2 = 2.5, \beta_3 = 0.8, \beta_4 = 0.5$ and $\sigma_0^2 = 1$.

- Proportion of data points subjected to extreme deviations was denoted by (α)
- Scale of deviation or magnitude applied to outlying points was denoted by (λ)

2. Generation of Predictor Variables

- $x_1 = \min(U, 8)$ where $U \sim \text{Uniform}(0,10)$
- $x_2 = \min(E, 15)$ where $E \sim \text{Exponential}(\lambda, 0.5)$
- $x_3 = \min(\chi^2, 10)$ where $\chi^2 \sim \chi^2(2)$
- $x_4 = \min(10 \cdot B, 8)$ where $B \sim \text{Beta}(\text{shape1} = 2, \text{shape2} = 5)$

3. Generation of Response Variable

- $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \epsilon$, where $\epsilon \sim N(0, \sigma_0^2 = 1)$

4. Introduction of Outliers to mimic real-world scenarios with influential observations

(a) Selection of Influential Points

- $I = \{i \in \{1, 2, \dots, n\} : U_i < \alpha\}$, where I denotes corresponding indices for outliers, and $U_i \sim \text{Uniform}(0,1)$ and $\alpha \in [0,1]$ is the probability threshold.

(b) Modification of Dependent Variable

- $y + M_i$ where $M_i = \lambda \cdot E_i, \forall i \in I$ is the magnitude of influence for the i -th influential observation, $E_i \sim \text{Exponential}(\text{rate} = 0.3)$ is an exponential random variable with rate parameter 0.3 and $\lambda > 0$ scales the exponential values to control the magnitude of influence

(c) Adjustment of Predictors:

- For a predictor x_k and its corresponding reduction factor c_k , the general mathematical expression is:

$$x_{k,i} = \max(x_{k,i} - \lambda \cdot c_k, 0), \forall i \in I,$$

where: k indexes the predictors ($k = 1, 2, 3, 4$), c_k is the specific reduction factor for predictor k ($c_1 = 0.5, c_2 = 0.3, c_3 = 0.2, c_4 = 0.1$) and I is the set of influential indices

(d) Data frame Formation

- The adjusted predictors and modified response variables were combined to form the contaminated dataset.

5. Data Storage

- For each iteration out of 1,000,000 iterations, the predictors and response variables were combined into a dataset to fit the least squares and robust regression models as follows:

(a) Least Squares Estimation (LSE):

- The LSE model was fitted using the `lm ()` function in R.
- Estimated coefficients $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$, and $\hat{\beta}_4$ and mean squared error ($\hat{\sigma}^2$) were recorded.

(b) Robust M-estimator: Uses `rlm ()` function with the help of “MASS” package in R

- Coefficients and MSE were also stored for Huber, Bisquare and Proposed estimators.

The estimation results for mean, bias square, standard error (SE), mean square error (MSE), and asymptotic relative efficiency (ARE), are presented in **Tables A7-A12** in **Appendix A**.

4.2. Results and Discussion

4.2.1. Simulation Results for Multiple Linear Regression before Tuning Parameter Optimization

This section presents the simulation results for the multiple linear regression model before optimizing the tuning parameter for the proposed estimator. Initially, the analysis employs the predefined tuning parameters $k_1 = 1.345$ and $k_2 = 4.685$, which are derived from the well-established Huber and Bisquare M-estimators. These values serve as a reference point for assessing the initial performance of the proposed estimator before further refinement. The goal is to evaluate how well the estimator performs under these standard parameters and to establish a basis for subsequent optimization.

[A]: A Comparative Analysis of Estimators based on Mean, Bias Squared, and Standard Errors

Intercept (β_0)

All estimators accurately estimate the true parameter ($\beta_0 = 12$) in uncontaminated data ($\alpha = 0.00$), proving their consistency. As contamination increases, LSE becomes unreliable, with estimates diverging sharply from the actual parameter, showing its sensitivity to outliers. For instance, at $\alpha = 0.10$ and $\lambda = 5$ LSE shoots to 21.852, Huber experiences slight instability due to its weighting function, which reduces but does not eliminate extreme residuals with a mean estimate of 12.902. Bisquare (12.003) and the proposed estimator (12.004) effectively suppresses outliers, keeping estimates close to the true parameter. The proposed estimator balances robustness and efficiency, maintaining accuracy even under increasing contamination.

Bias squared trends highlight the limitations of LSE, with bias rising to 97.062 at $\alpha=0.10$ and $\lambda = 5$, shifting the estimate to 21.852, almost double the true value. Huber remains stable at low contamination but accumulates bias as λ increases, peaking at 0.814. Bisquare and the proposed estimator keeps bias near zero by strongly rejecting outliers. The proposed estimator controls bias effectively, while preserving more information than Bisquare.

Standard errors (SE) further reveal estimator stability. In clean data, all estimators have similar SE close to 0.32. As contamination rises, the SE for LSE exceeds 3 at $\alpha = 0.10$ and

$\lambda = 5$, showing extreme variability. Huber experiences moderate increases, from 0.319 to 0.617. Both Bisquare and the proposed estimator maintain low SE closer to 0.350, with the proposed method striking a balance between efficiency and robustness.

Generally, LSE is highly sensitive to outliers, while Huber offers moderate robustness with some bias, and Bisquare discards too much information. The proposed estimator achieves a balance of bias control, variance stability, and accuracy, making it the preferred choice for robustness and efficiency. **Figure 4.1** below illustrates the bias square trends for all estimators.

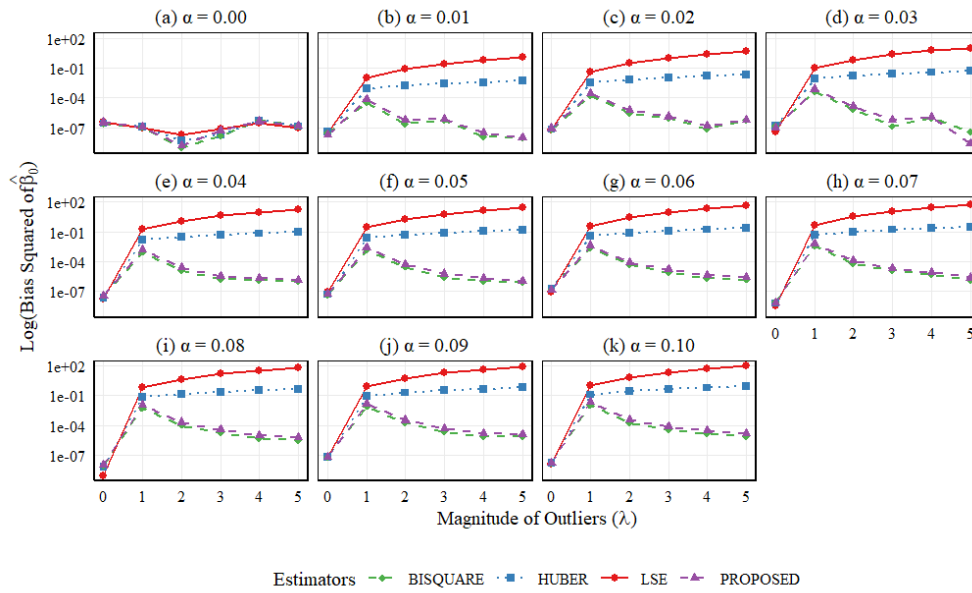


Figure 4.1 The bias square for the intercept estimates for multiple linear regression before optimization of the parameters for the proposed estimator.

Slope Coefficients ($\beta_1, \beta_2, \beta_3$, and β_4)

For slope (β_1) all estimators perform well in clean data context ($\alpha = 0.00$), with LSE maintaining a mean of 4.300 across λ values. However, as contamination increases, LSE becomes highly sensitive to outliers, with its mean estimates dropping by 17% to 3.564 at $\alpha = 0.10, \lambda = 5$. In contrast, Huber and Bisquare estimators remain stable, with means of 4.228 and 4.300, respectively, though Huber is slightly more affected. The proposed estimator demonstrates superior stability, consistently retaining a mean estimate of 4.300 across all contamination levels, highlighting its robustness.

Bias trends further support these findings. While all estimators have a bias square of 0.000 in clean data, the bias square for LSE rises sharply to 0.541 at $\alpha = 0.10, \lambda = 5$, indicating poor accuracy under contamination. Huber exhibits an insignificant increase to 0.005, while Bisquare remains at 0.000, demonstrating greater resistance to extreme contamination. The proposed estimator also maintains a bias square of 0.000, ensuring robust and efficient estimation.

Standard error (SE) trends reveal instability of LSE, with SE rising from 0.040 in clean data to 0.388 at $\alpha = 0.10, \lambda = 5$. Huber shows moderate robustness with an SE of 0.063, while Bisquare and the proposed estimator maintain lower values between 0.040 and 0.042, outperforming both LSE and Huber. The proposed estimator indicates minimal variability and superior stability under contamination.

A comparable pattern is observed for β_2 , β_3 , and β_4 . Under contamination, LSE reveals significant deviation and high variability, whereas Bisquare and the proposed estimator provide the most stable and reliable slope estimates with minimal bias. Huber remains robust but demonstrates slight performance degradation.

In summary, LSE is significantly susceptible to contamination, while Huber provides moderate robustness but remains biased. The Bisquare estimator effectively controls bias; however, it excessively downweight certain observations, potentially reducing efficiency in some cases, whereas the proposed estimator ensures low bias, stable SE, and superior robustness. **Figure 4.2** below illustrates the slope estimate tendencies across contamination levels.

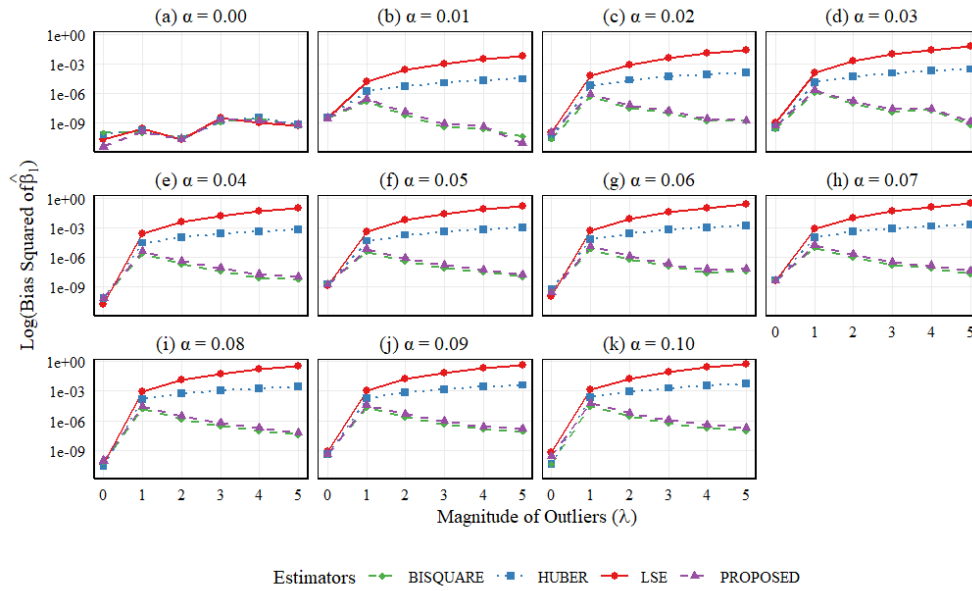


Figure 4.2 The bias square for the slope estimates for multiple linear regression before the optimization of the parameters of the proposed estimator.

Mean square error (σ^2)

Uncontaminated data LSE performs well in terms of mean estimates, bias square, and standard error but immediately collapses under contamination. Initially, all estimators approximate the true variance ($\sigma^2 = 1$), with LSE achieving an exact 1.000, while robust estimators slightly underestimate (Huber = 0.950, Bisquare = 0.945, and Proposed = 0.962). However, with $\alpha = 0.10$ and $\lambda = 5$, LSE inflates drastically to 101.022, while robust methods remain close to 1. Bias square follows the same trend; LSE begins at zero but inflates to 10004.38 under extreme contamination, whereas robust estimators maintain much lower values (Huber = 0.189, Bisquare = 0.048, and Proposed = 0.057). Standard errors further confirm this instability, LSE values for SE increases from 0.145 to 53.813, while Bisquare and the proposed estimator remain more stable (Bisquare = 0.308, Proposed = 0.358). Collectively, Bisquare stood up as the most reliable due to its minimal bias and controlled variance, making it the best choice in contaminated environments. **Figure 4.3** below visually presents these findings.

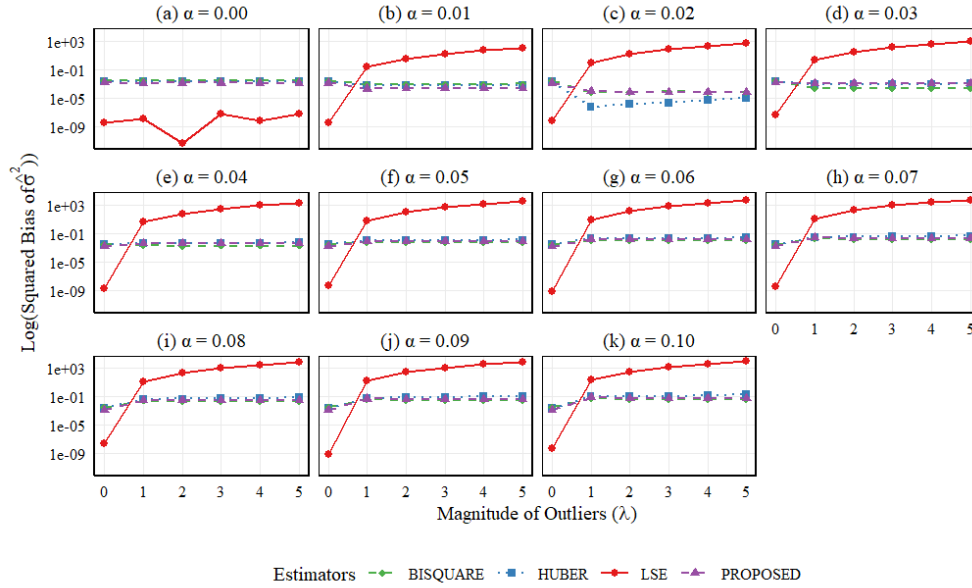


Figure 4.3 The bias square for the mean square error (MSE) estimates for multiple linear regression before optimization of the parameters for the proposed estimator

[B]: A Comparative Analysis of Estimators based on Mean square Errors and Asymptotic relative efficiency for Multiple Regression Parameters

Intercept (β_0)

In uncontaminated settings ($\alpha = 0.00$), all estimators reveal similar MSE values, showing comparable efficiency. However, as contamination increases, the MSE for LSE rises dramatically until 112.746 at $\alpha = 0.10$, $\lambda = 5$, while the proposed estimator remains stable at 0.119 comparable to Bisquare. Robust estimators maintain significantly lower MSE values, with Huber showing moderate resistance and Bisquare excelling in outlier rejection.

The proposed estimator closely follows Bisquare but with slightly enhanced robustness, ensuring stable performance.

Asymptotic relative efficiency (ARE) further highlights the inefficiency of LSE in contaminated settings. While LSE has better efficiency in clean data, the efficiency of robust increases as contamination rises. At moderate contamination ($\alpha = 0.01$ to 0.03 and $\lambda = 1$), the ARE for Huber remains at (0.656 and 0.351) indicating its closer efficiency to LSE, while Bisquare and the proposed estimator in most cases maintain better efficiency. Under severe contamination ($\alpha = 0.10$, $\lambda = 5$), LSE becomes highly inadequate, with Huber's ARE dropping to 0.011 and Bisquare and the proposed estimator reaching 0.001. This suggests LSE would require 100 times additional samples to achieve the same accuracy as the proposed estimator, emphasizing its superior balance of efficiency and robustness. **Figure 4.4** and **Figure 4.5** below illustrate the MSE and ARE trends.

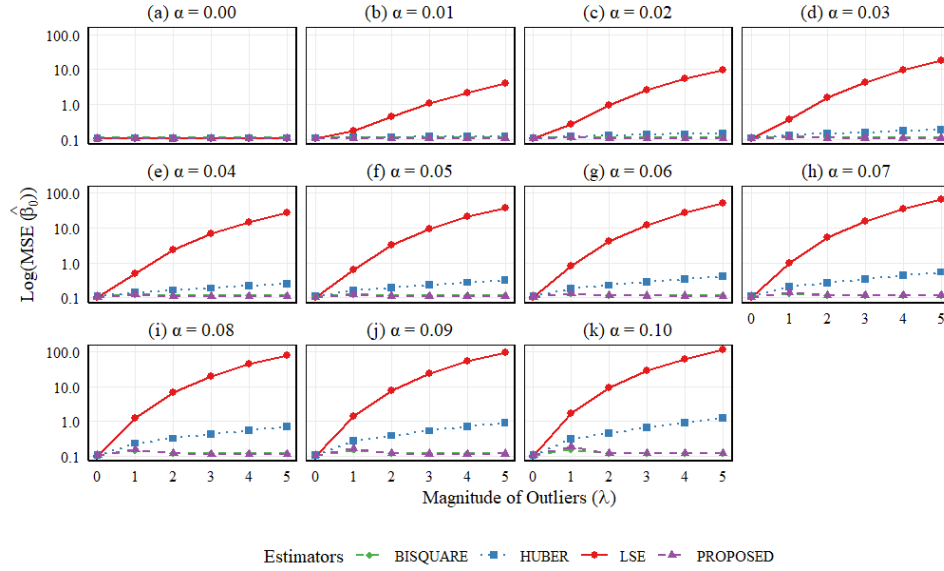


Figure 4.4 The mean square error (MSE) for the intercept estimates for multiple linear regression before optimization of the parameters of the proposed estimator.

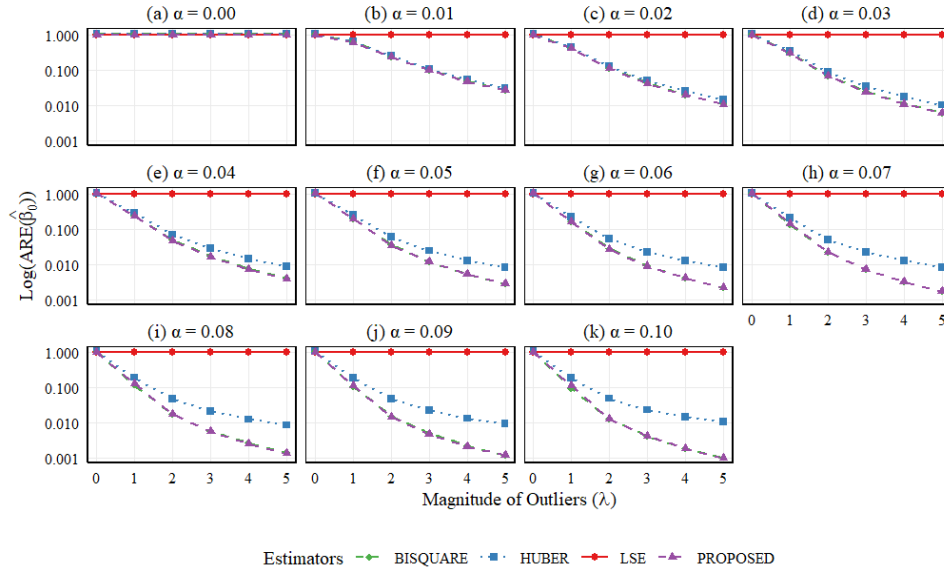


Figure 4.5 The asymptotic relative efficiency (ARE) of the intercept estimates for multiple linear regression before optimization of the proposed estimator.

Slope Coefficients (β_1 , β_2 , β_3 , and β_4)

The Mean squared error (MSE) and Asymptotic relative efficiency (ARE) reveal the strengths and weaknesses of the estimators in both uncontaminated and contaminated environments. In a clean dataset ($\alpha = 0.00$), all four estimators yield similar MSE values (0.002), confirming their efficiency. However, as the proportion of outliers increases, significant differences emerge. The Least squares estimator is particularly vulnerable, with its MSE shoot up to 0.692 at $\alpha = 0.10$ and $\lambda = 5$, indicating a complete failure in estimation accuracy. The Huber estimator shows better robustness but still indicates a gradual increase in MSE, suggesting it cannot fully withstand severe contamination. In contrast, the Bisquare and proposed estimators maintain significantly lower MSE (0.002) across all contamination levels. Bisquare reduces the impact of outliers effectively, and the proposed estimator works similarly but does a slightly better job at staying accurate and reliable throughout.

The ARE metric further underscores the inefficiency of LSE in contaminated datasets. While LSE is optimal in the absence of outliers ($\alpha = 0.00$), its efficiency rapidly deteriorates under contamination, demanding substantially more observations to match the performance of robust estimators. For extreme contamination ($\alpha = 0.10$, $\lambda = 5$), Bisquare and the proposed estimator yield ARE values as low as 0.003, suggesting that LSE would require to utilize an approximate of 333 extra observations in order achieve this efficiency. The remaining slope coefficients, β_2 , β_3 , and β_4 , follow similar patterns in both MS and ARE. Their performance matches with the trends observed for β_1 , where LSE struggles

with contamination, Huber provides moderate resistance, and both the Bisquare and proposed estimators demonstrate superior robustness and efficiency across varying levels of outliers. **Figure 4.6** and **Figure 4.7** below illustrate these trends for MSE and ARE for slope coefficients.

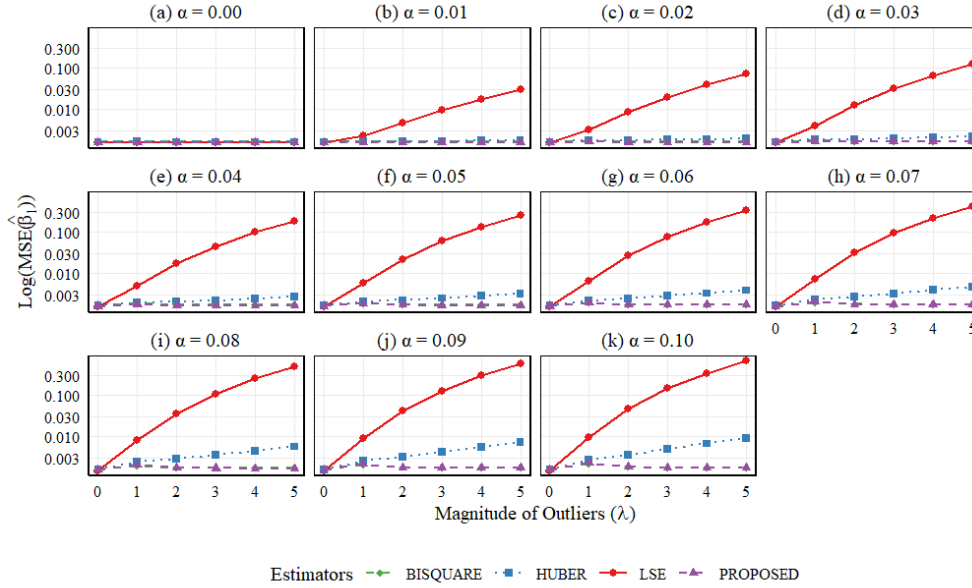


Figure 4.6 The mean square error (MSE) for the slope estimates for multiple linear regression before optimization of the parameters for the proposed estimator.

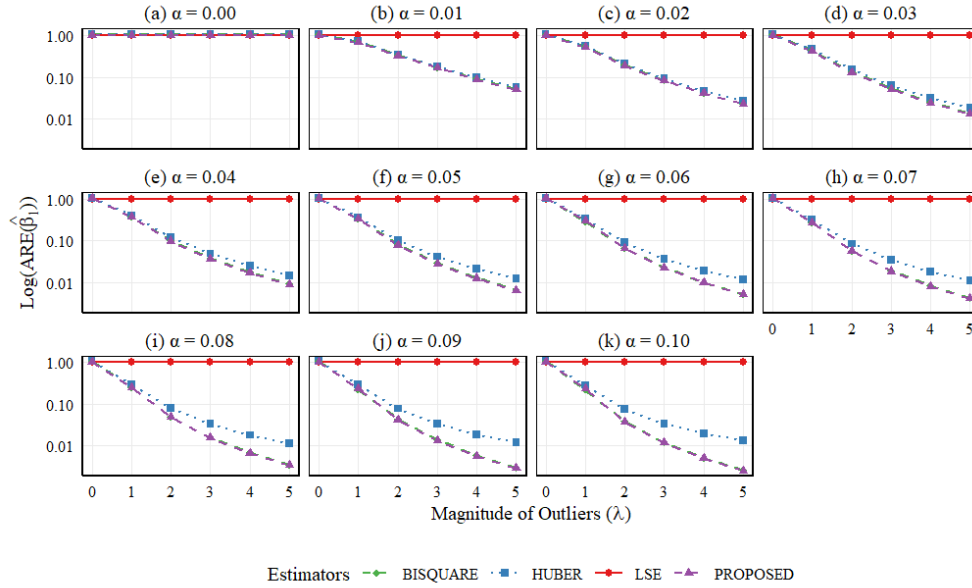


Figure 4.7 The asymptotic relative efficiency (ARE) of the slope estimates for multiple linear regression before optimization of the parameters for the proposed estimator.

Mean square error (σ^2)

In the absence of outliers ($\alpha = 0.00$), all estimators perform comparably, with MSE values between 0.021 and 0.054. However, as contamination increases, LSE becomes highly unsuitable, with its MSE shooting up to 4584.34 at $\alpha = 0.05$ and $\lambda = 5$, while robust estimators remain much lower at 0.091, 0.072, and 0.076 for Huber, Bisquare and the proposed estimator respectively. This indicates that LSE struggles with outliers, while robust methods handle them effectively. Likewise, LSE loses efficiency in uncontaminated data as confirmed by the lowest ARE values for robust estimators. While it performs best in clean data, its efficiency drops to nearly zero at $\alpha = 0.10$ and $\lambda = 5$, meaning it would need infinitely many additional observations to match the efficiency of robust estimators.

Among robust techniques, Bisquare provides the most reliable performance, making it the best choice for handling outliers. **Figure 4.8** and **Figure 4.9** below explicitly display these patterns.

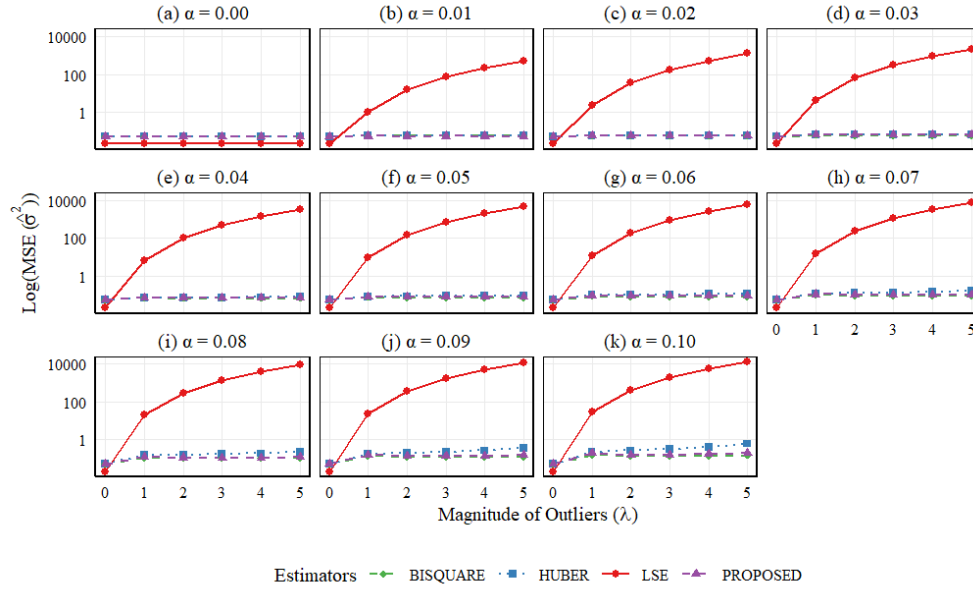


Figure 4.8 The mean square error (MSE) for the MSE estimates for multiple linear regression before optimization of the parameters for the proposed estimator.

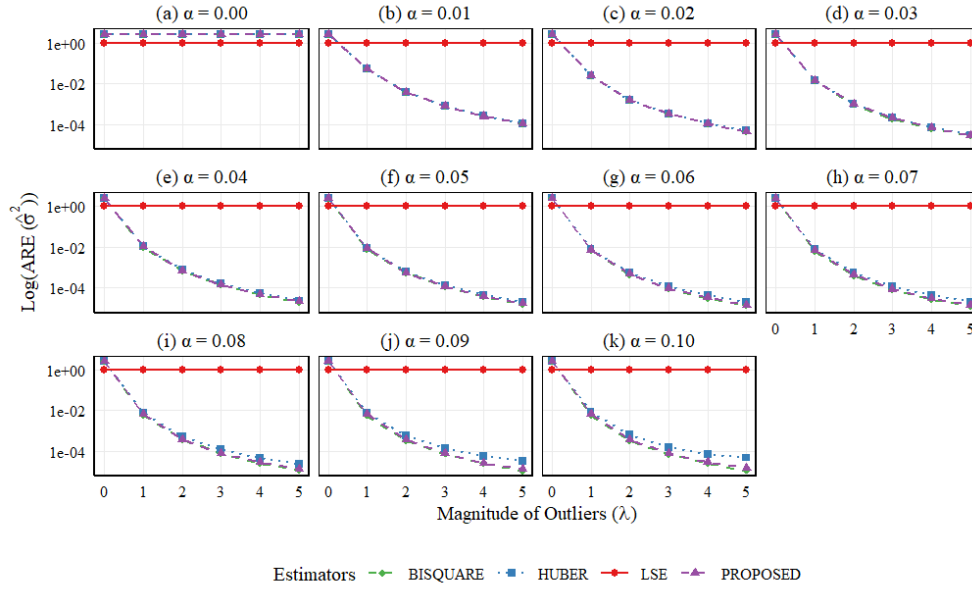


Figure 4.9. The asymptotic relative efficiency (ARE) of the MSE estimates for multiple linear regression before optimization of the parameters for the proposed estimator.

4.2.2. Simulation Results for Multiple Linear Regression after Optimization of the Tuning Parameters.

This section presents the simulation results for the multiple linear regression model after optimizing the tuning parameters for the proposed estimator. The estimator is fine-tuned using adaptive cross-validation, which determines the optimal values $k_1 = 1.095$ and $k_2 = 2.908$. These values enhance the performance estimator by balancing robustness and efficiency in parameter estimation. The primary objective of this analysis is to assess how adaptive cross-validation improves the accuracy and stability of the proposed estimator, especially in handling contaminated data. Additional details can be found in **Tables A13** –

A18 in **Appendix A**. Below is a discussion of the estimates for the intercept, slope, and mean squared error (MSE) in terms of mean, bias square, and standard errors.

[A]: A Comparative Analysis of Estimators based on Mean, Squared Bias, and Standard Errors for Multiple Regression Parameters

Intercept (β_0)

In uncontaminated condition ($\alpha = 0.00$), all estimators accurately estimate the intercept and standard errors remaining within the range of 0.319 (LSE) to 0.359 (proposed estimator). However, as contamination increases, LSE becomes highly unstable, with its mean estimate growing to 21.844 at $\alpha = 0.10$, $\lambda = 5$, accompanied by an alarming bias square of 96.906. This large deviation confirms the extreme sensitivity of LSE to outliers.

On the other hand, Huber and Bisquare estimators show greater stability. The mean estimate for Huber rises moderately to 12.901, with bias square increasing to 0.812, whereas Bisquare remains relatively stable at 12.002, maintaining minimal bias square (0.000). The proposed estimator also yields 12.001 which is much closer to the actual parameter, exhibiting the lowest bias square among all robust estimators, confirming its robustness in handling contaminated data.

The standard errors for LSE increased from 0.319 to 3.950, reflecting significant variability. Huber and Bisquare experience moderate growth, but the proposed estimator achieves the best balance between robustness and efficiency and stabilizes at 0.362, stood

up to be the most reliable estimator under extreme contamination. The bias square trend for intercept estimates ($\hat{\beta}_0$) for each estimator is shown in **Figure 4.10** below.

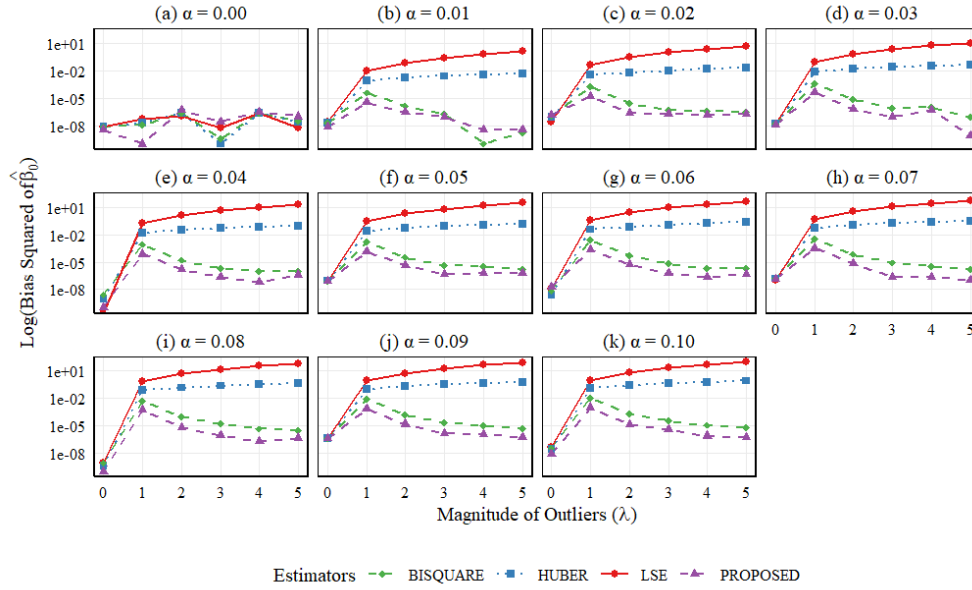


Figure 4.10 The bias square for the intercept estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

Slope Coefficients (β_1 , β_2 , β_3 , and β_4)

Like the intercept, all estimators appropriately estimate the slope coefficients in the uncontaminated scenario ($\alpha = 0.00$), with mean estimates closely matching the true values. However, LSE begins to deteriorate as the contamination increases, deviating significantly across all slopes. For example, at $\alpha = 0.10$, $\lambda = 5$, The estimate for β_1 in LSE drops from 4.300 to 3.565, with bias square increasing from 0.000 to 0.540 and standard errors inflating from 0.039 to 0.387. Huber shows slight sensitivity, with the estimate declining

to 4.228 and bias square rises to 0.005, but standard errors remain lower than LSE at 0.063. Nonetheless, Bisquare and the proposed estimator maintain values closer to the actual parameter (4.300), with negligible bias square (0.000) and stable standard errors (0.044 for the proposed estimator, 0.042 for Bisquare), indicating strong robustness. For β_2 , β_3 , and β_4 , a similar pattern is noted. Under contamination, LSE experiences substantial deviation and increased variability, while robust estimators, especially Bisquare and the proposed estimator, provide the most stable and reliable estimates for slope coefficients with low bias squared, while Huber experiences slight degradation. The bias square trend for slope estimates ($\hat{\beta}_1$) for each estimator is shown in **Figure 4.11** below.

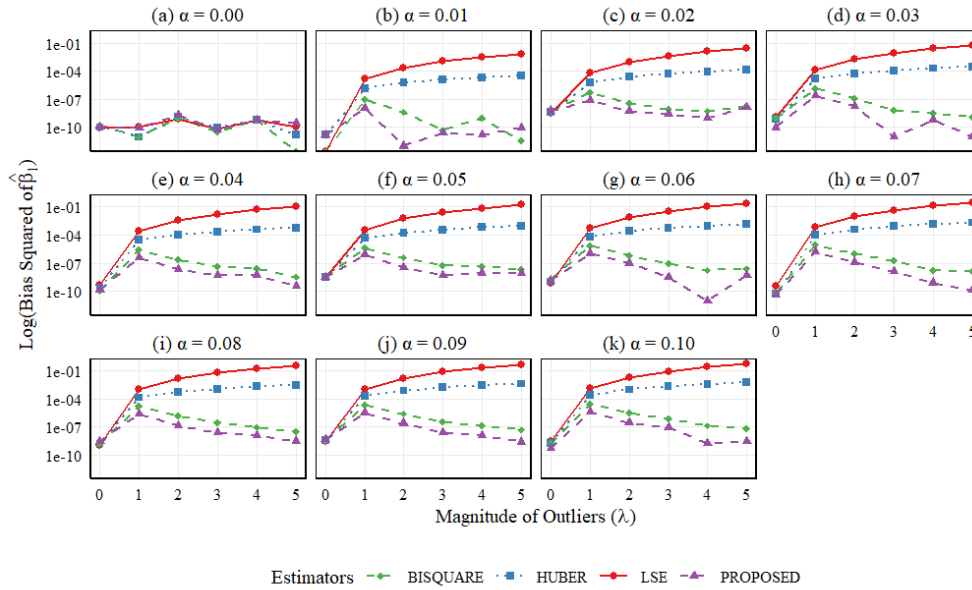


Figure 4.11 The bias square for the slope estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

Mean square error (σ^2)

The mean square error estimation follows a similar pattern, where LSE initially performs well but collapses under contamination. For instance, at $\alpha = 0.10$, $\lambda = 5$, the mean square error (MSE) estimate of LSE increases drastically to 101.028 which is an increase of more than 100 units, confirming its complete breakdown in the presence of outliers. In contrast, among the robust estimators the MSE for Huber estimator remains lower at 1.433, though it experiences noticeable increases. Bisquare (1.218) and the proposed estimator (1.218) remain significantly more stable, supporting their robustness in MSE estimation. The proposed estimator indicates the best overall performance, maintaining stable mean square error estimates across all contamination levels. The bias square trend for the mean square error estimate ($\hat{\sigma}^2$) for each estimator is shown in **Figure 4.12** below.

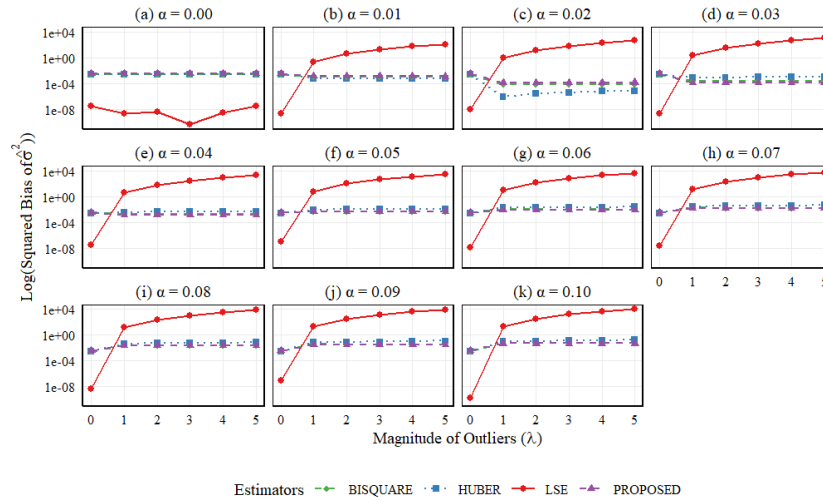


Figure 4.12 The bias square for the mean square error (MSE) estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

**[B]: A Comparative Analysis of Estimators based on Analysis of Mean
Square Errors and Asymptotic Relative Efficiency for Multiple
Regression Parameters**

The MSE and ARE at various contamination levels allow comparisons between LSE, Huber, Bisquare, and the proposed estimators. All estimators perform well under ideal conditions ($\alpha = 0.00$), but LSE has the lowest MSE. As contamination increases, the MSE for LSE declines considerably. Strong contamination resistance allows robust estimators like Bisquare and the proposed estimator to maintain lower MSE values. When outliers are absent, LSE maintains its ARE advantage. When contaminated, the ARE for robust estimators drop significantly, demonstrating their increased robustness against LSE. The proposed estimator consistently maintains a balance between robustness and efficiency, making it a strong substitute for LSE and other robust estimators. Further discussion for each parameter is provided below:

Intercept (β_0)

Under uncontaminated scenarios, LSE yields the lowest MSE of 0.102, followed closely by Huber (0.107), Bisquare (0.109), and the proposed estimator (0.129). However, as contamination increases ($\alpha = 0.10, \lambda = 5$), the MSE for LSE rises sharply to 112.508, confirming its extreme vulnerability to outliers. In contrast, the proposed estimator maintains an MSE of 0.131, demonstrating its significant robustness. The efficiency of LSE in an ideal situation is much better compared to robust estimators. However, as contamination increases, the ARE values of robust estimators drop drastically, confirming

their greater efficiency. For instance, at $\alpha = 0.10$, $\lambda = 5$, Huber (0.011), Bisquare (0.001), and the proposed estimator (0.001) all have ARE values much lower than 1, indicating that while they are less efficient than LSE in absence of atypic observations, they are more stable under contamination. This indicates that LSE completely breaks down under contamination whereas the proposed estimator and Bisquare provide the most stable and reliable estimates. **Figure 4.13** and **Figure 4.14** below provide further trend details in MSE and ARE for the intercept estimates ($\hat{\beta}_0$).

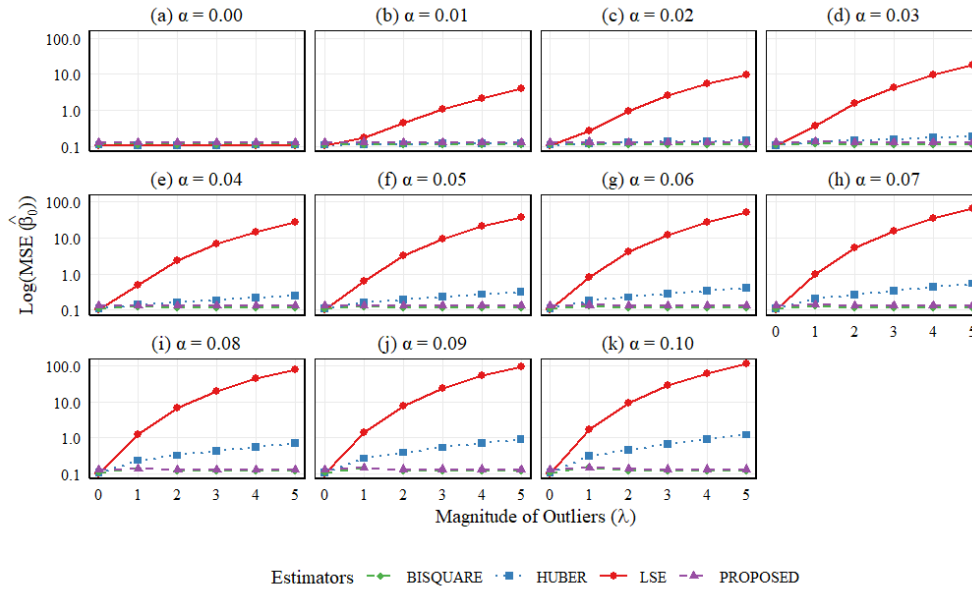


Figure 4.13 The mean square error (MSE) for the intercept estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

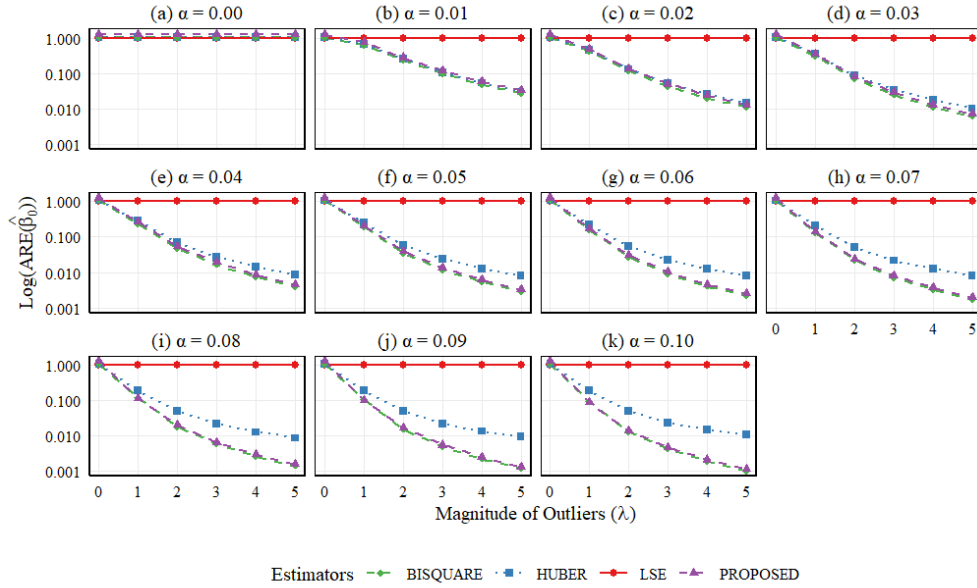


Figure 4.14 The asymptotic relative efficiency (ARE) of the intercept estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

Slope Coefficients (β_1 , β_2 , β_3 , and β_4)

Similar trends are observed across the slope coefficients. Under no contamination, all estimators show comparable performance. However, as contamination increases, the mean square error for LSE increases dramatically, highlighting its inability to handle contaminated data effectively. For example, at $\alpha=0.10$, $\lambda = 5$, all the slope coefficients show comparable patterns. The performance of all estimators is similar when there is no contamination. The MSE for LSE, however, increases as contamination rises, underscoring its inability to manage tainted data efficiently. Take the case of $\alpha=0.10$, $\lambda = 5$, MSE for β_1 for the least square estimator rises from 0.002 to 0.690, while the proposed estimator

maintains a much lower value of 0.004 closer to Bisquare. Also, the ARE for robust estimators drops significantly, with Huber at 0.013, and 0.003 for both Bisquare and the proposed estimator, verifying their effectiveness in reducing contamination impact with higher efficiency. The proposed estimator consistently maintains lower MSE values compared to LSE under contamination, making it a more reliable choice for slope estimation. **Figure 4.15** and **Figure 4.16** below provide additional details on the trends in MSE and ARE for slope estimate ($\hat{\beta}_1$).

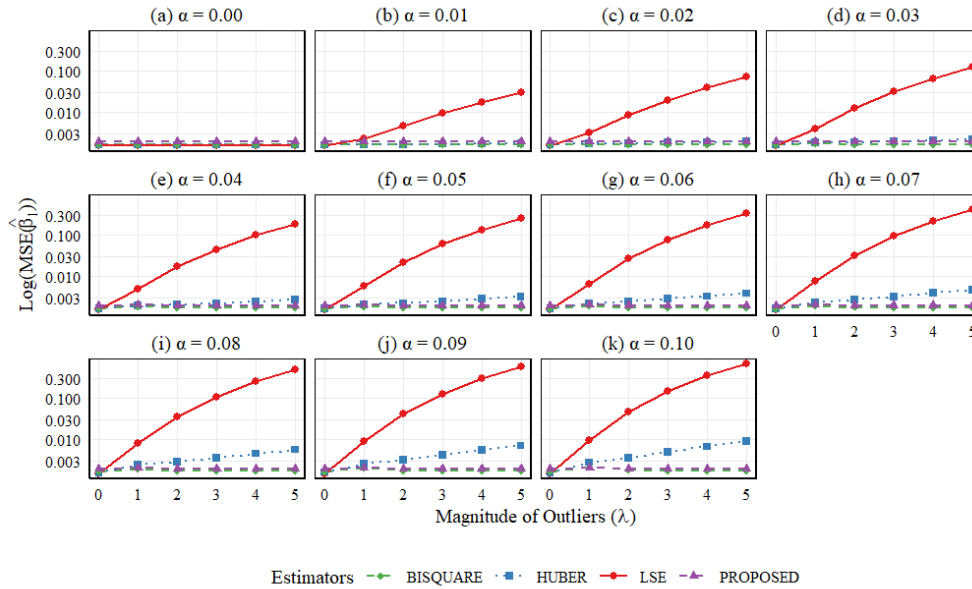


Figure 4.15 The mean square error (MSE) for the slope estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

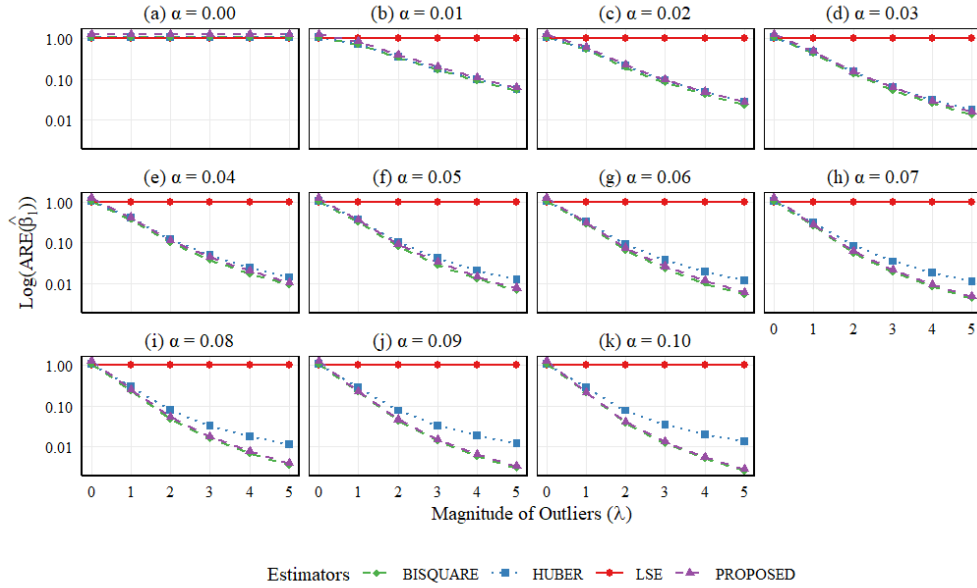


Figure 4.16 The asymptotic relative efficiency (ARE) of the slope estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

Mean square error (σ^2)

The most severe effects of contamination are observed in mean square error estimation. In the absence of contamination, LSE performs well with an MSE of 0.021, comparable to robust estimators. However, as contamination increases, for example at $\alpha = 0.10$, $\lambda = 5$, the MSE for LSE jumps from 0.021 to 12910.303, indicating complete breakdown. The proposed estimator sustains an MSE of only 0.142, demonstrating its superior resistance to contamination. For ARE, LSE outperforms robust estimators when there is no contamination, but as contamination increases, robust estimators become significantly more stable. The proposed estimator, while initially less efficient, maintains a consistently

lower MSE and a superior trade-off between robustness and efficiency as contamination increases, reinforcing its advantage in real-world contaminated datasets. **Figure 4.17** and **Figure 4.18** below provide further details on the patterns of MSE and ARE for the MSE estimates ($\hat{\sigma}^2$).

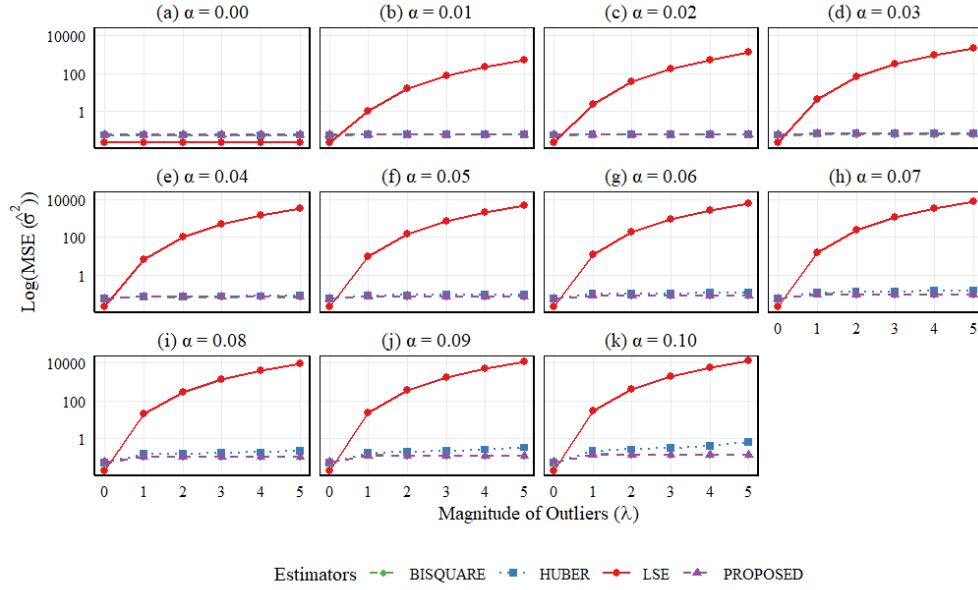


Figure 4.17 The mean square error (MSE) for the MSE estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

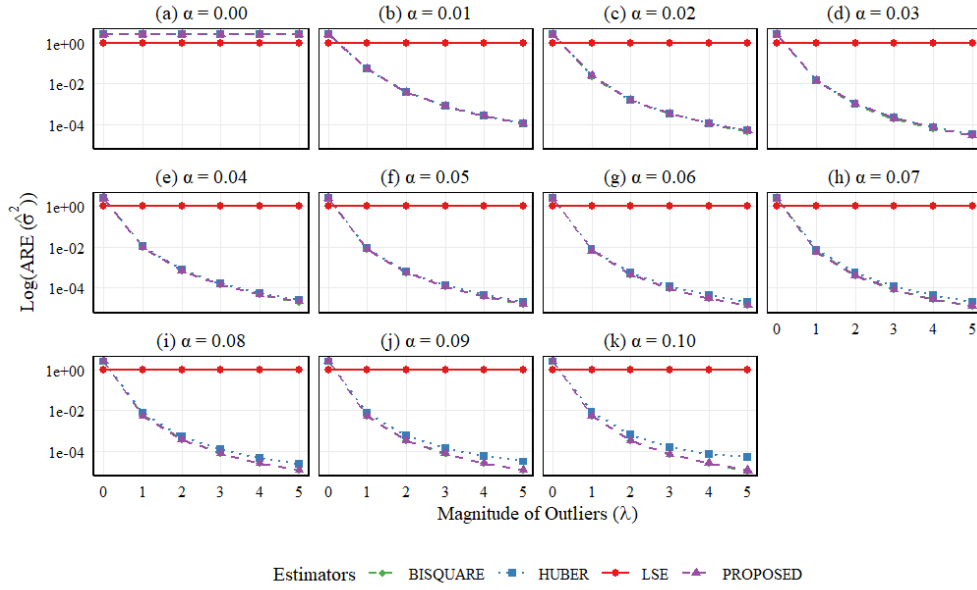


Figure 4.18 The asymptotic relative efficiency (ARE) of the MSE estimates for multiple linear regression after optimization of the parameters for the proposed estimator.

4.3. Practical Example in Multiple Linear Regression

The fit was also extended to multiple regression, assuming no outliers. Outliers are introduced using simulation design from **Section 4.1** and the following model is fitted and the results are presented in **Table 4.1** below.

$$\text{Traffic Volume} = \beta_0 + \beta_1 \times \text{Temp} + \beta_2 \times \text{Rain} + \beta_3 \times \text{Snow} + \beta_4 \times \text{Clouds} + \epsilon \quad (4.1)$$

The LSE estimator is effective under ideal conditions, as demonstrated by Case I in **Table 4.1**, which has an intercept of 7,898.700 and slopes of -14.439 for temperature, -119.524 for rainfall, -2,355.812 for snow, and 5.773 for cloud cover. The Huber estimator yields

MSE of 6,740,005 higher than LSE and ARE of 1.710, indicating its ineffectiveness in the absence of outliers. The Bisquare estimator is more robust compared to other robust estimators with an intercept of 7957.507 and slopes of -14.525 (temperature), -121.889 (rainfall), -2494.065 (snowfall), and 6.128 (cloudsall). Its ARE and MSE are 6,702,165 and 1.700 respectively. In terms of efficiency, the proposed estimator is comparable to Huber. When used properly, it is a flexible choice because it attains accuracy that is near LSE while adding robustness against minute fluctuations.

Table 4.1 Estimation results for the multiple linear regression model based on Contaminated and non-contaminated Metro Interstate Traffic Volume dataset.

Case I – Uncontaminated Scenario								
Estimator	Intercept	Temp	Rainfall	Snowfall	Clouds_all	MSE	ARE	Turning Parameters
LSE	7898.700	-14.439	-119.524	-2355.812	5.773			
	(6493.153)	(22.897)	(48.078)	(1258.932)	(5.285)	3942327	1.000	—
Huber	7981.900	-14.737	-119.994	-2357.662	5.846			
	(6615.285)	(23.328)	(48.982)	(1282.611)	(5.385)	6740005	1.710	$k_1 = 1.345$
Bisquare	7957.507	-14.525	-121.889	-2494.065	6.128			
	(6969.394)	(24.577)	(51.604)	(1351.268)	(5.673)	6702165	1.700	$k_2 = 4.685$
Proposed	7899.871	-14.443	-119.530	-2355.854	5.774			$k_1 = 1.345, k_2 =$
	(6493.016)	(22.897)	(48.077)	(1258.905)	(5.285)	6737939	1.709	4.685
Case II – Contaminated Scenario								
LSE	9066.530	-18.080	212.645	-5859.270	12.169			—
	(8749.300)	(30.818)	(49.916)	(1689.576)	(7.091)	7122800	1.000	
Huber	13840.240	-35.529	253.378	-4436.395	9.548	6382659		
	(8032.594)	(28.294)	(45.827)	(1551.173)	(6.510)		0.896	$k_1 = 1.345$

	17402.270	-48.362	332.296	-4068.439	8.630			
Bisquare	(8378.123)	(29.511)	(47.798)	(1617.898)	(6.790)	6294050	0.884	$k_2 = 4.685$
	18629.600	-52.481	341.757	-4234.898	8.153			$k_1 = 1.345, k_2 =$
Proposed	(7785.527)	(27.424)	(44.417)	(1503.462)	(6.310)	6268450	0.880	4.685
	15303.710	-48.598	263.992	-1322.066	35.390			$k_1 = 0.873, k_2 =$
Proposed CV	(4682.559)	(16.494)	(26.714)	(904.248)	(3.795)	4727358	0.664	= 1.539

Note: Standard errors (SE)

4.4. Introducing outliers into Traffic volume, Temperature, Rainfall, Snowfall and Cloudsall variables

The simulation design for multiple linear regression in **Section 4.1** was used to introduce outliers in response variables (traffic volume), and predictor variables temperature, rainfall, snowfall, and cloudsall. Outliers were generated by adding substantial deviations to the response variable and elevated by 8500 (vehicles/hour) to reflect the effect of extreme values, while adjustments were made to the predictors to balance their influence and ensure realistic outlier behavior as illustrated in **Figure 4.19** below.

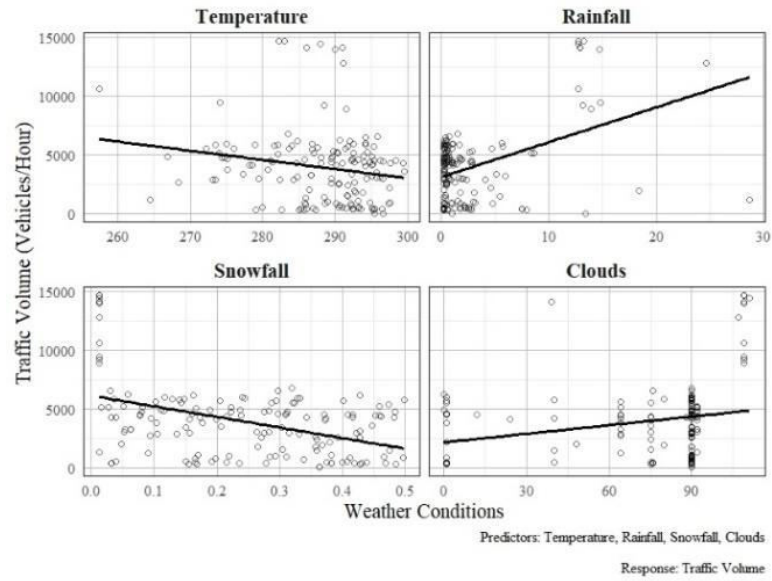


Figure 4.19 Metro Interstate Traffic Volume scatter plots based on tainted data.

Case II in **Table 4.1** presents the findings for contaminated data. In clean settings, the LSE estimator is most efficient, but in contaminated conditions, it struggles. It has the greatest MSE compared to all estimators, showing outlier sensitivity.

The Huber estimator enhances robustness with an improved MSE of 6,382,659 and an ARE of 0.896, demonstrating better stability and reduced variability. The Bisquare estimator performs similarly but introduces aggressive down-weighting to extreme deviations yielding lower MSE of 6,294,050. The proposed estimator exhibits a more consistent pattern in managing outliers, balancing efficiency and robustness with an ARE of 0.880 and an MSE of 6,268,450. Using adaptive cross-validation with the L-BFGS-B optimization algorithm, it produces even better estimates with an MSE of 4,727,358. Using

flexible parameter selection, the proposed estimator effectively reduces the influence outliers making it an excellent tool for modeling traffic dynamics in Minneapolis.

4.5. Conclusion

For modeling Metro Interstate Traffic Volume in Minneapolis under typical weather conditions, this example demonstrates that the LSE estimator is the most accurate and efficient approach. Its sensitivity to outliers limits its use in contaminated situations. The proposed robust estimator is the most efficient and adaptable according to data features, producing lower MSE and ARE values make it ideal for complicated and noisy data. The study underlines the necessity of choosing the appropriate estimator based on data characteristic to accurately and insightfully analyze Minneapolis traffic dynamics across weather contexts.

CHAPTER 5

COMPARATIVE ANALYSIS OF EWMA CONTROL CHARTS USING LEAST SQUARES AND HUBER M-ESTIMATOR

This chapter investigates Exponentially Weighted Moving Average (EWMA) control charts for monitoring linear regression profiles using classical Least Squares Estimation (LSE) and robust Huber M-Estimator methods. It outlines the in-control process model with intercept, slope, and error variance parameters. A Monte Carlo simulation framework is used to generate contaminated data with outliers and obtain the average run length (ARL). Phase I employs LSE and Huber M-estimators to estimate in-control values, while Phase II uses EWMA charts (EWMA1, EWMA2, EWMA3) to detect shifts in these parameters. Performance is evaluated using the Average of the Average Run Length (AARL), Median of the Average Run Length (MDARL), and Standard Deviation of the Average Run Length (SDARL). The chapter also highlights the robustness of Huber-based chart to outliers compared to LSE-based chart, which is more sensitive to small shifts but less stable under contamination. Results, presented in tables and figures, show both methods converge for large shifts, but Huber excels in noisy data. The chapter entirely emphasizes the role of robust estimation in enhancing EWMA chart reliability for industrial applications with data contamination.

5.1. Fundamental Assumptions and Process Model

The methodology assumes an in-control process modeled as:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, i = 1, 2, \dots, n \quad (5.1)$$

where $\epsilon_i \sim N(0, \sigma^2)$ are independent and identically distributed (i.i.d.) errors. The x_i values were fixed and equally spaced from 2 to 40 in increments of 2, resulting in $n = 20$ observations. For centering, we defined $x_i^* = x_i - \bar{x}$, so that the design matrix is centered around zero for simplicity and numerical stability. The in-control (IC) model parameters were assumed to be known and set to $\beta_0 = 3$, $\beta_1 = 2$, and $\sigma_0 = 1$.

5.2. Phase I Parameter Estimation

In Phase I, we simulate $k = 100$ in-control profiles based on the following model:

$$y_i^* = \beta_0^* + \beta_1^* x_i^* + \epsilon_i^*, i = 1, 2, \dots, n \quad (5.2)$$

With a random error term $\epsilon_i^* \sim N(0, \sigma^2)$, $\beta_0^* = \beta_0 + \beta_1 \bar{x}$, and $\beta_1^* = \beta_1$. Additionally, a mild outlier is introduced at the sixteenth observation $y_i^*[16]$ by subtracting a product of $y_i^*[16]$ with magnitudes $\gamma \in [0.00, 0.1]$ from the actual value of $y_i^*[16]$. The contaminated Phase I data are used to estimate reference parameters such as intercept ($\hat{\beta}_0$), slope ($\hat{\beta}_1$), and error variance ($\hat{\sigma}^2$) using both LSE and Huber's regression techniques. These estimates are averaged across all profiles to obtain the in-control parameter estimates as given below:

$$\bar{\beta}_0 = \frac{1}{k} \sum_{j=1}^k \hat{\beta}_{0j}, \bar{\beta}_1 = \frac{1}{k} \sum_{j=1}^k \hat{\beta}_{1j}, \bar{\sigma}^2 = \frac{1}{k} \sum_{j=1}^k \hat{\sigma}_j^2 \quad (5.3)$$

LSE applied closed form of intercept, slope and error variance to obtain the above estimates as shown below:

$$\begin{aligned}
\hat{\beta}_{0k} &= y_k - \hat{\beta}_{1k}\bar{x} \\
\hat{\beta}_{1k} &= \frac{S_{xy(k)}}{S_{xx}} \\
\hat{\sigma}_{jk}^2 &= \frac{SSE}{(n-2)}
\end{aligned} \tag{5.4}$$

where $S_{xx} = \sum_{i=1}^n (x_i^*)^2$ is the sum of squares for the centered predictor and the Huber M-estimators applied an Iteratively Reweighted Least Squares (IRLS), which reduces the influence of outliers and estimate variance using a robust MAD-based scale.

5.3. Process Monitoring in Phase II

For the in-control state, we set $\delta_1 = 0$, $\delta_2 = 0$, and $\delta_3 = 1$. The model under monitoring could experience shifts in intercept, slope, or error variance such that, $\beta_0^s = \beta_0 + \delta_1\sigma_0$, with $\delta_1 \in [0, 2, 0.2]$, $\beta_1^s = \beta_1 + \delta_2\sigma_0$, where $\delta_2 \in [0, 0.25, 0.025]$ and $\sigma_1 = \delta_3\sigma_0$, with $\delta_3 \in [1, 3, 0.2]$.

5.4. Simulation Design and Validation

The simulation procedure follows a two-phase structure. In Phase I, k contaminated samples of size $n = 100$ were generated to construct control limits and estimate baseline parameters also known as in-control profile parameters. In Phase II, 10,000 out-of-control processes are simulated for each processes shift denoted by δ , and run lengths were recorded until a signal occurs or a truncation limit is reached. Validation checks include verifying convergence of robust IRLS algorithm and consistency between MAD-based scale estimates computed in C and output generated using `rlm()` package in R.

5.5. EWMA3 Control Charts Implementation

The EWMA statistics for intercept (EWMA1), slope (EWMA2), and error variance (EWMA3) are updated iteratively using:

$$\begin{cases} EWMA_1^{(t)} = \lambda Z_1^{(t)} + (1 - \lambda)EWMA_1^{(t-1)} \\ EWMA_2^{(t)} = \lambda Z_2^{(t)} + (1 - \lambda)EWMA_2^{(t-1)} \\ EWMA_3^{(t)} = \max(0, \lambda Z_3^{(t)} + (1 - \lambda)EWMA_3^{(t-1)}) \end{cases} \quad (5.5)$$

where Z_1, Z_2 , and Z_3 are the standardized statistics obtained from the estimated coefficients and error variance after the process goes out of control (OC) values after shifting of profile parameters with respect to in control estimates which are given as:

$$\begin{cases} Z_1 = \frac{\hat{\beta}_0^s - \bar{\beta}_0}{\sqrt{\bar{\sigma}^2/n}} \\ Z_2 = \frac{\hat{\beta}_1^s - \bar{\beta}_1}{\sqrt{\bar{\sigma}^2/n}} \\ Z_3 = \frac{\sigma_1^2 - \bar{\sigma}^2}{\sqrt{2\bar{\sigma}^4/(n-2)}} \end{cases} \quad (5.6)$$

The control charts for LSE and robust M-Estimators are configured with a smoothing constant ($\lambda = 0.2$) and control limit coefficients for each estimator when all three EWMA charts are combined, the overall in-control ARL is about 20 as shown in

Table 5.1 below:

Table 5.1 Limit Coefficients for Least Squares and Robust M- Estimators (In-Control $ARL_0 \cong 200$)

Estimator	L_1	L_2	L_3
LSE	3.0814	3.0814	3.7772
Huber M-Estimator	3.2472	3.2472	3.5590

These coefficients were initiated from the known parameter case where the actual limits were 3.021 for L_1 and L_2 and 3.311 for L_3 derived based on the asymptotic variance factor $(\frac{\lambda}{2-\lambda})$ as given in the control limits structure below:

$$\begin{cases} CL_k = L_k \cdot \sqrt{\frac{\lambda}{2-\lambda}}, \\ \text{for } k = 1, 2, 3 \end{cases} \quad (5.7)$$

A signal is triggered for intercept and slope control charts if any statistic exceeds $\pm L_k \cdot \sqrt{\frac{\lambda}{2-\lambda}}$ and $L_k \cdot \sqrt{\frac{\lambda}{2-\lambda}}$ for error variance and the multiplier (L_k) are selected to provide a given in-control ARL value.

5.6. Performance Metrics

The performance of the charts for classical and Huber estimators were evaluated using 10,000 Monte Carlo trials, with the expected ARL (average run length to signal), SDARL (standard deviation of ARL), and MDARL (median of ARL) calculated for each magnitude of outliers γ and process shift δ .

5.6.1. Average of the Average Run Length (AARL)

The AARL generalizes control chart performance, where a high in-control AARL ($AARL_o$) minimizes false alarms, while a low out-of-control AARL ($AARL_1$) ensures quick detection of shifts. For robust methods such as Huber M-estimator AARL quantifies effectiveness across contamination levels.

5.6.2. Median Average Run Length (MDARL)

The MDARL provides a robust performance summary, resilient to skewed ARL distributions. Higher values reflect delayed responses, reflecting the context-dependent interpretation of AARL, whereas lower MDARL indicates rapid shift detection.

5.6.3. Standard Deviation of Average Run Length (SDARL)

SDARL measures performance stability. Low SDARL implies consistent detection across conditions, while high SDARL signals sensitivity to shifts or contamination, reducing reliability.

5.6.4. Sensitivity to Outliers

Huber based control charts are assessed under varying outlier magnitudes to validate stability; comparison with traditional LSE charts highlights their superiority in maintaining detection performance despite contamination.

5.7. Results and Discussions

5.7.1. Classical LSE-Based Control Chart

The **Table 5.2** below summarizes the in-control performance of an Exponentially Weighted Moving Average (EWMA) control chart based on Least Squares (LS) estimation for monitoring linear regression parameters. It displays the Average Run Length ($AARL_0$), Standard Deviation of Run Length ($SDARL_0$), and Median Run Length ($MDARL_0$) under different outlier magnitudes, represented by gamma (γ), ranging from 0.00 to 0.10. These metrics, derived from Monte Carlo simulations, evaluate the behavior of the chart when there is no occurrence of shifts in the process, but with the changing magnitudes of outliers. The EWMA chart tracks standardized LSE estimates to detect deviations, providing insights into its stability as outliers impact the regression model.

Table 5.2 $AARL_0$, $SDARL_0$, and $MDARL_0$ for LSE based Control Chart

γ	$AARL_0$	$SDARL_0$	$MDARL_0$
0.00	199.71	71.36	190.25
0.01	169.27	73.24	156.57
0.02	105.25	52.90	92.87
0.03	62.54	28.03	56.16
0.04	42.70	16.32	39.39
0.05	33.03	10.68	30.77
0.06	27.85	8.27	26.17
0.07	24.53	6.41	23.38
0.08	22.75	5.58	21.70
0.09	21.46	4.84	20.62
0.10	20.58	4.40	19.84

Implications and Limitations

Implications

The results show that the performance of the LSE-based EWMA chart deteriorates as outlier magnitude (γ) increases. At $\gamma = 0.00$, the $AARL_0$ is 199.71, indicating a stable in-control run length of about 200 runs, with an $MDARL_0$ of 190.25 confirming consistent behavior. However, as γ rises to 0.10, $AARL_0$ drops sharply to 20.58, and $MDARL_0$ falls to 19.84, signifying frequent false alarms. This rapid decay implies that even small quantity of outliers, for instance at ($\gamma = 0.01$, , $AARL_0 = 169.27$) disrupt LSE estimates, making the chart overly sensitive. The $SDARL_0$, peaking at 73.24 for $\gamma = 0.01$ and decreasing to 4.40 at $\gamma = 0.10$, shows that larger outliers lead to more predictable but shorter run lengths. These trends highlight the utility of the chart in clean data environments but suggest it struggles in settings with outliers, necessitating careful consideration for the application in process monitoring.

Limitations

The primary limitation of the chart is its lack of robustness to outliers, as evidenced by the steep, $AARL_0$ decline from 199.71 to 20.58 across γ values. The sensitivity of LSE to extreme values causes large residuals, triggering false signals too often, especially at $\gamma \geq 0.05$ $AARL_0$ stays below 33.03. High $SDARL_0$ at low γ (71.36 at $\gamma = 0.00$) indicates unstable performance, making run lengths unpredictable closer to in-control process. Moreover, the right-skewed run length distribution ($MDARL_0 < AARL_0$) implies early signals are common, which may confuse practitioners expecting stable performance. These

issues limit the reliability of the chart in noisy processes, confirming a need for robust alternatives to maintain effective control without excessive false alarms.

5.7.2. Huber M-Estimators-Based Control Chart

Similarly, the in-control performance metrics for Huber M-Estimator-based control chart, evaluated across varying magnitudes of outliers. These results were found using Exponentially Weighted Moving Average (EWMA) control charts, a statistical process control tool designed to detect small shifts in the process by assigning exponentially decreasing weights to past observations. The Huber M-Estimator, known for its robustness to outliers, is integrated into the EWMA framework to enhance the ability of the chart to maintain performance in the presence of data contamination. The metrics $AARL_0$, $SDARL_0$, and $MDARL_0$ provide insights into the behavior of the run length of the chart under in-control conditions as shown in **Table 5.3** below:

Table 5.3 $AARL_0$, $SDARL_0$, and $MDARL_0$ for Huber M-Estimator Control Chart

γ	$AARL_0$	$SDARL_0$	$MDARL_0$
0.00	201.97	130.49	171.24
0.01	185.38	117.50	155.34
0.02	142.10	87.96	119.38
0.03	107.86	63.99	91.70
0.04	91.73	52.66	78.23
0.05	85.67	46.91	73.88
0.06	83.44	45.66	72.24
0.07	82.79	43.98	72.10
0.08	84.02	48.10	71.57
0.09	83.26	45.67	71.93
0.10	83.82	45.80	72.14

Implications and Limitations

Implications

The Huber M-Estimator-based EWMA control chart demonstrates remarkable robustness in maintaining in-control performance, as its $AARL_0$ values remain consistently close to the target of 200 even as outlier magnitude (γ) increases, outperforming the LSE-based EWMA chart significantly. At $\gamma = 0.00$, the Huber chart achieves an $AARL_0$ of 201.97, aligning well with the ideal in-control run length of 200, while the LSE chart starts slightly lower at 199.71. As γ increases to 0.01, the $AARL_0$ only slightly decreases to 185.38, staying closer to 200, whereas the $AARL_0$ for LSE drops more noticeably to 169.27, indicating a quick deviation from the desired in-control value. When $\gamma = 0.10$, the $AARL_0$ for Huber is 83.82, still retaining a reasonable proximity to 200, while the $AARL_0$ for LSE falls to 20.58, a drastic fall that signals frequent false alarms and process issues. Similarly, $SDARL_0$ and $MDARL_0$ at $\gamma = 0.10$ are 45.80 and 72.14, respectively, both providing a slightly better estimates for in-control values compared to $SDARL_0$ of 4.40 and $MDARL_0$ of 19.84 for LSE.

Limitations

While the Huber-based EWMA control chart excels in maintaining in-control values closer to 200, it is not entirely immune to the effects of outliers, as $AARL_0$ does decline from 201.97 at $\gamma = 0.00$ to 83.82 at $\gamma = 0.10$. However, this drop is far less severe compared to the LSE-based control chart which declined from 199.71 to 20.58, indicating that Huber still provides greater stability in profiles monitoring. At $\gamma \geq 0.05$, Huber stabilizes around

83, although below 200 but it is considerably closer to $AARL_0$ compared to LSE which falls below 33.03. Similarly, the $SDARL_0$ and $MDARL_0$ for Huber at $\gamma = 0.10$ falls between 45.80 and 72.14 respectively, although it deviates from 200 but remain much closer than 4.40 and 19.84 for LSE, showing that the in-control performance for Huber-based control chart is less affected by outliers. This suggests that while not entirely unaffected, Huber demonstrates greater stability, less performance degradation, and more reliable in-control detection compared to traditional LSE-based methods and fine-tuning parameters like the Huber constant or EWMA smoothing constant could further enhance its ability to maintain $AARL_0$, $SDARL_0$, and $MDARL_0$ even closer to 200, ensuring prolonged in-control stability in noisy processes. A clear visualization of this trend is presented in **Figure 5.1** to **Figure 5.3** below:

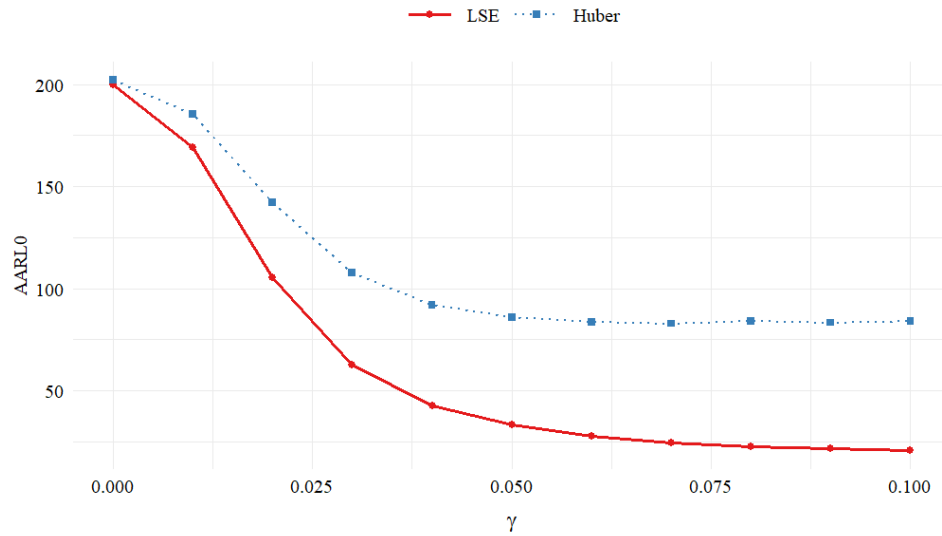


Figure 5.1 $AARL_0$ trend for LSE and Huber M-estimator based control chart using EWMA.

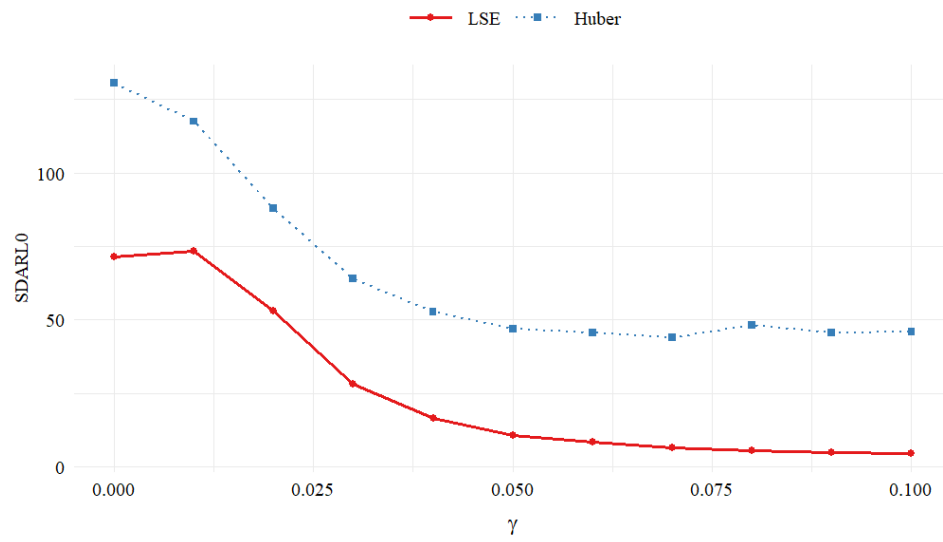


Figure 5.2 SDARL₀ trend for LSE and Huber M-estimator based control chart using EWMA.

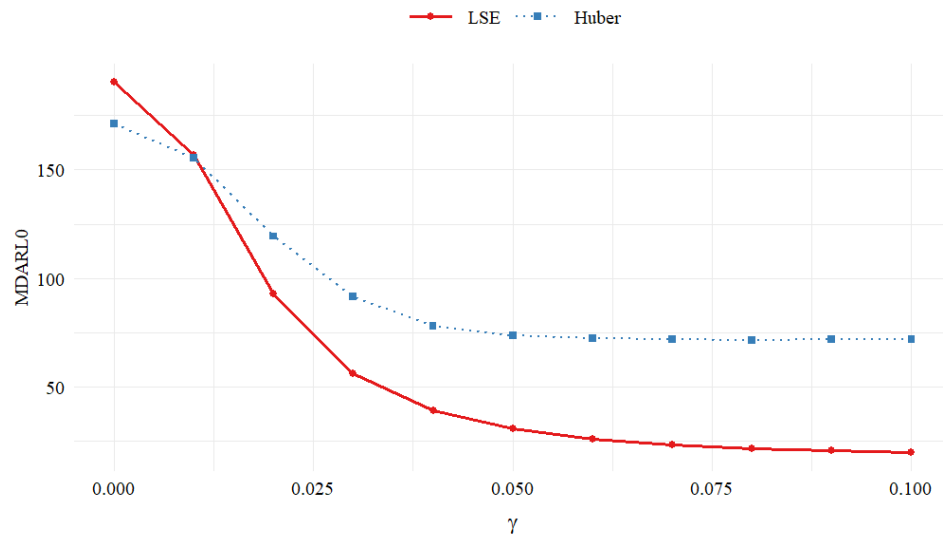


Figure 5.3 MDARL₀ trend for LSE and Huber M-estimator based control chart using EWMA.

5.7.3. AARL Comparisons

In this case the in-control state was set to $\delta_1 = 0$, $\delta_2 = 0$, and $\delta_3 = 1$. The model under monitoring phase (Phase II) could experience three types of shifts. Shifts in intercept, slope, or error variance such that, $\beta_0^s = \beta_0 + \delta_1 \sigma_0$, with $\delta_1 \in [0, 2, 0.2]$, $\beta_1^s = \beta_1 + \delta_2 \sigma_0$, where $\delta_2 \in [0, 0.25, 0.025]$ and $\sigma_1 = \delta_3 \sigma_0$, with $\delta_3 \in [1, 3, 0.2]$. **Table 5.4** below presents the AARL performance of EWMA control charts based on Least Squares Estimation (LSE) and the Huber M-estimator under various magnitudes of intercept shifts in a simple linear regression model, with the in-control AARL approximately equal to 200.

Table 5.4 AARL Comparisons Under Intercept Shifts from β_0 to $\beta_0 + \delta_1 \sigma_0$ (In-Control AARL $\cong 200$)

Chart	δ_1										
	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
EWMA_{LSE}	200.00	14.55	4.56	2.78	2.10	1.76	1.41	1.13	1.02	1.00	1.00
EWMA_{Huber}	200.00	15.64	4.75	2.87	2.15	1.81	1.47	1.17	1.03	1.00	1.00

Under in-control conditions ($\delta_1 = 0$), both LSE and Huber-based charts yield an approximate AARL of 200, confirming suitable stability. When small shifts are introduced (δ_1 between 0.2 and 0.6), the LSE-based chart exhibits slightly lower AARLs (14.55, 4.56, and 2.78) compared to Huber (15.64, 4.75, and 2.87), indicating marginally better responsiveness in detecting minor changes. As the shift magnitude increases (δ_1 from 0.8 to 2.0), both charts show improved detection, but the LSE-based chart continues to outperform Huber by maintaining slightly lower AARLs across all shift values. For instance, at $\delta_1 = 1.2$ and $\delta_1 = 1.4$, LSE return AARLs of 1.41 and 1.13, respectively, compared to Huber's 1.47 and 1.17. Although the performance gap becomes minimal at

higher shift magnitudes, the consistent pattern shows that LSE-based EWMA control charts detect intercept shifts more efficiently. Overall, while the Huber estimator is known for its robustness to data contamination and outliers, in clean data conditions where early detection of change is critical, the LSE-based EWMA control chart demonstrates superior performance in linear profile monitoring of the intercept parameter. This pattern is clearly shown in **Figure 5.4** below.

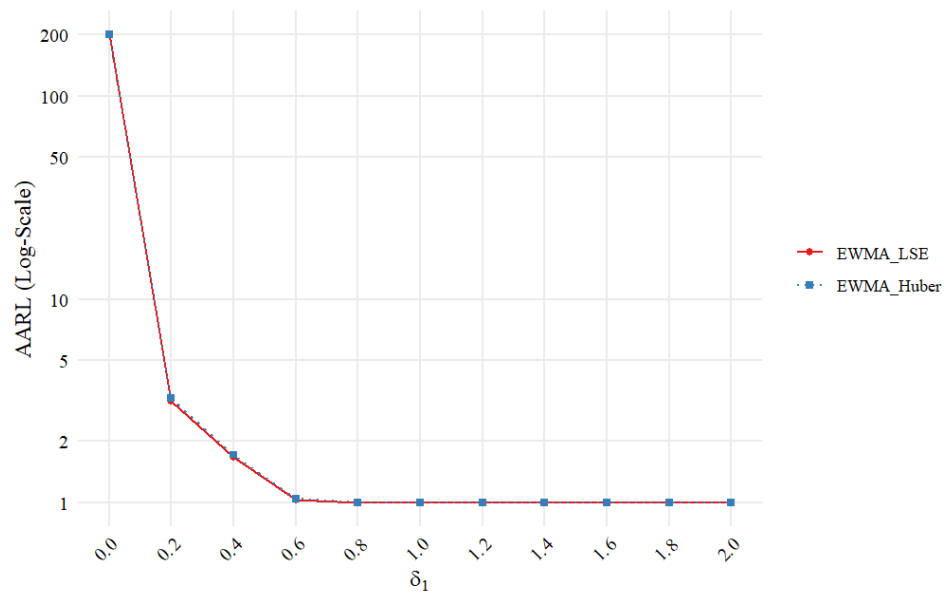


Figure 5.4 AARL Comparisons Under Intercept Shifts δ_1

Table 5.5 AARL Comparisons Under Slope Shifts from β_1 to $\beta_1 + \delta_2\sigma_0$ (In-Control AARL $\cong 200$)

Chart	δ_2										
	0	0.025	0.05	0.075	0.1	0.125	0.15	0.175	0.2	0.225	0.25
EWMA_{LSE}	200.00	3.16	1.67	1.03	1.00	1.00	1.00	1.00	1.00	1.00	1.00
EWMA_{Huber}	200.00	3.26	1.72	1.04	1.00	1.00	1.00	1.00	1.00	1.00	1.00

The findings in **Table 5.5** above presents the AARL comparisons under slope shifts ranging from $\delta_2 = 0$ to $\delta_2 = 0.25$, with an in-control AARL of approximately 200. For ($\delta_2 = 0$), both charts show identical AARL values of 200.00, confirming their expected in-control performance. As the shift magnitude increases to ($\delta_2 = 0.025$), the AARL for EWMA LSE-based chart decreases to 3.16, slightly outperforming EWMA Huber-based chart, which records an AARL of 3.26. This trend continues at ($\delta_2 = 0.05$ to $\delta_2 = 0.075$), with LSE achieving an AARL of 1.67 and 1.03 compared to Huber's 1.72 and 1.04 respectively. These results suggest that EWMA LSE-based chart is slightly more sensitive to smaller slope shifts from $\delta_2 = 0.025$ to $\delta_2 = 0.075$, detecting deviations slightly faster than Huber-based chart. However, for ($\delta_2 \geq 0.1$), both charts stabilize at an AARL of 1.00, indicating equivalent performance in detecting larger shifts, as both reach the minimum AARL value, reflecting rapid detection of significant deviations. The slight advantage of LSE EWMA-based chart for smaller shifts may be attributed to its least squares estimation, which could be more responsive to slight changes in the presence of normal data, whereas Huber's EWMA-based robust estimation may mitigate the impact of outliers but slightly delay detection for small shifts. Generally, while LSE EWMA-based chart demonstrates a superiority in sensitivity for small slope shifts, both charts perform comparably for larger shifts, making them both effective for monitoring processes under varying shift magnitudes, with the choice between them potentially depending on the expected data characteristics and robustness requirements. The **Figure 5.5** below provides a visual representation of this trend.

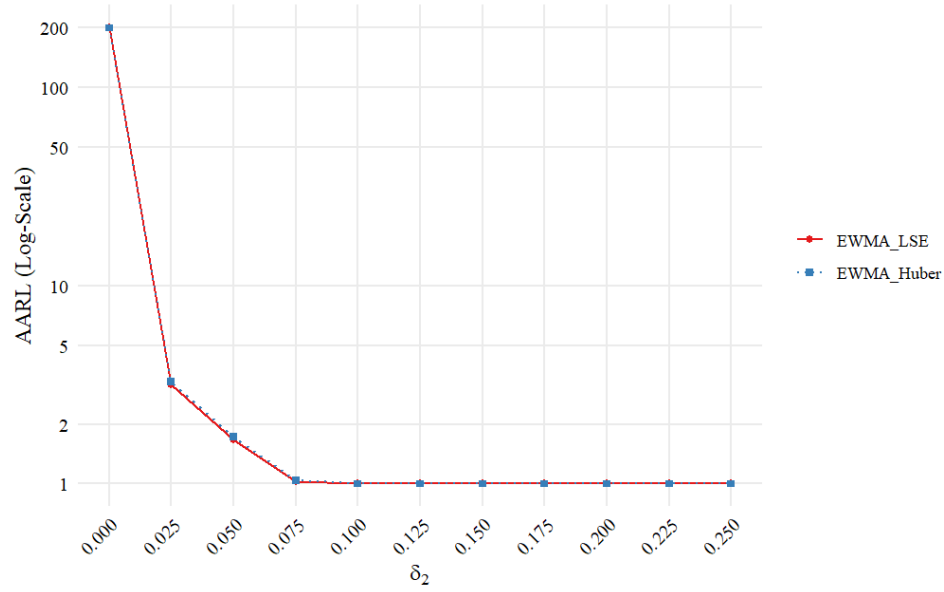


Figure 5.5 AARL Comparisons Under Slope Shifts δ_2

Table 5.6 AARL Comparisons Under Standard Deviation Shifts from σ_0 to $\delta_3\sigma_0$ (In-Control AARL = 200)

Chart	δ_3										
	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
EWMA_{LSE}	200.00	7.82	3.01	1.91	1.44	1.21	1.09	1.04	1.02	1.01	1.00
EWMA_{Huber}	200.00	7.06	2.94	1.90	1.44	1.21	1.09	1.04	1.02	1.01	1.00

The standard deviation shifts are presented in **Table 5.6** above, with (δ_3) ranging from 1 to 3, and an in-control AARL of approximately 200. At $(\delta_3 = 1)$, both charts indicate an AARL of 200.00, confirming their anticipated in-control performance. As the standard deviation shift increases to $(\delta_3 = 1.2)$, Huber's EWMA-based chart outperforms LSE with an AARL of 7.06 compared to 7.82, indicating quicker detection of moderate shift. This trend continues at $(\delta_3 = 1.4)$, where AARL is 2.94 for Huber-based EWMA chart, slightly better than 3.01 for LSE. For $(\delta_3 = 1.6)$, the AARL values are nearly alike, with Huber-

based chart at 1.90 and LSE at 1.91, suggesting comparable performance. From ($\delta_3 = 1.8$) onward, both charts converge, with AARL values of 1.44 at ($\delta_3 = 1.8$), 1.21 at ($\delta_3 = 2$), 1.09 at ($\delta_3 = 2.2$), 1.04 at ($\delta_3 = 2.4$), 1.02 at ($\delta_3 = 2.6$), 1.01 at ($\delta_3 = 2.8$), and 1.00 at ($\delta_3 = 3$). These findings indicate that Huber EWMA-based control chart is more sensitive to smaller scale shifts ($\delta_3 = 1.2$ to 1.4), likely due to its robust estimation approach, which effectively handle variations in standard deviation without being overly influenced by extreme values. In contrast, LSE-based EWMA chart, relying on the least squares estimation, is slightly less responsive to these moderate shifts but matches Huber-based EWMA chart performance for larger shifts from $\delta_3 = 1.8$ and beyond. The convergence of AARL values for larger shifts suggests that both charts are equally effective in detecting significant changes in standard deviation, achieving faster detection as AARL approaches 1.00. Therefore, Huber-based EWMA chart may be preferred in scenarios where smaller scale shifts are of primary concern, particularly in processes prone to outliers, while both charts are equally suitable for detecting larger shifts, with the choice depending on the

specific process characteristics and robustness needs. Additionally, this scenario is shown in detail in the **Figure 5.6** below.

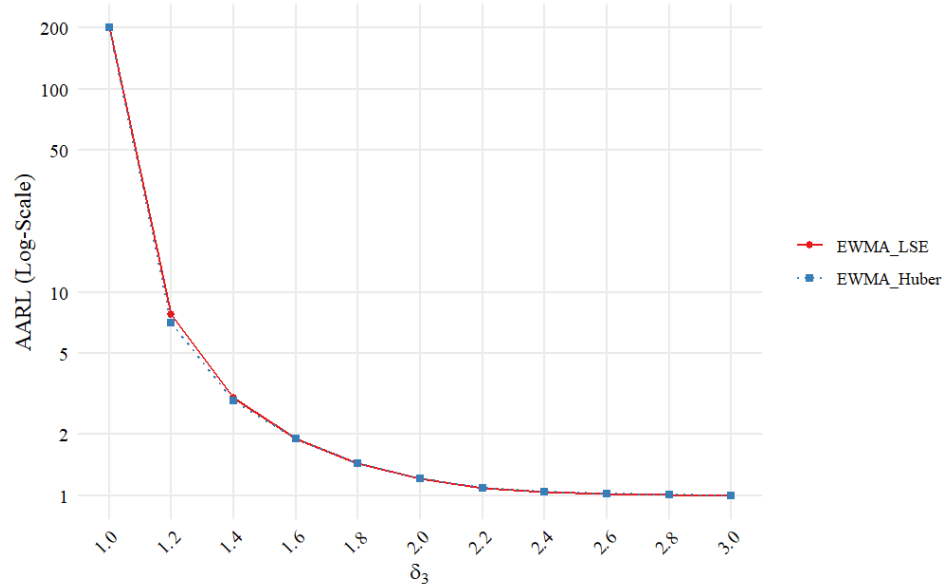


Figure 5.6 AARL Comparisons Under Standard Deviation Shifts δ_3

5.8. Conclusion

The Chapter has provided a detailed evaluation of EWMA control charts for monitoring linear regression profiles, comparing the performance of classical LSE-based charts with robust Huber M-Estimator-based charts under various shift scenarios and outlier conditions. The results demonstrate that both charts achieve the desired in-control AARL of approximately 200 under clean conditions, but their performance diverges significantly in the presence of outliers and process shifts. The LSE-based EWMA chart shows superior sensitivity to small intercept and slope shifts, with lower AARL values for intercept shifts for instance at ($\delta_1 = 0.2$), and for slope shifts for example at ($\delta_2 = 0.025$), making it

preferable in clean data context where early detection is critical. However, its performance deteriorates rapidly with increasing outlier magnitudes, with in-control AARL dropping from 199.71 to 20.58 as outlier magnitude γ increases from 0 to 0.10, indicating a lack of robustness. In contrast, the Huber-based EWMA chart maintains greater stability, with in-control AARL declining more gradually from 201.97 to 83.82 over the same range, and it outperforms LSE for small to moderate standard deviation shifts as can be seen at ($\delta_3 = 1.2$), showing to its robust estimation that mitigates the impact of outliers. For larger shifts across all parameters, both charts converge to AARL values near 1.00, indicating rapid detection. These findings underscore the trade-off between sensitivity and robustness, LSE-based charts are more responsive in ideal conditions, while Huber-based charts offer reliable performance in contaminated data scenarios. The chapter highlights the importance of selecting an appropriate estimation method based on process characteristics, such as expected outlier presence, and suggests that fine-tuning parameters like the Huber constant or EWMA smoothing constant could further enhance robust chart performance. Future research could explore hybrid approaches or alternative robust estimators to balance sensitivity and stability, ensuring effective monitoring in diverse industrial applications.

CHAPTER 6

SUMMARY, CONCLUSIONS, LIMITATIONS AND FUTURE RECOMMENDATIONS

6.1. Summary and Conclusions

In Chapter One, the thesis lays a strong foundation for the significance of regression analysis and the challenges posed by outliers. This introductory section lays out the rationale for employing robust regression techniques, particularly emphasizing the drawbacks of conventional least squares estimation (LSE) in the presence of outliers. Chapter Two entails a comprehensive literature review on existing methodologies such as LSE, Huber's and Tukey's Bisquare M-estimators, and discusses their respective strengths and weaknesses. In addition, reviews on cross-validation, optimization techniques and linear profile monitoring were pointed out. Chapters Three and Four then advance the study by providing a detailed comparative simulation analysis of the proposed hybrid M-estimator against the traditional LSE, Huber, and Bisquare estimators. Chapter Five focuses on the practical application of robust regression to monitor linear profiles in Phase I process control. It demonstrates how robust estimator, particularly the Huber M-estimator, is applied to accurately estimate the in-control parameters. This chapter emphasizes how robust estimation can be used to stabilize process behavior while reducing false alarms, thereby enhancing the usefulness of robust M-estimators in actual quality control and process monitoring scenarios. The simulation results evaluated through measures such as bias, mean squared error (MSE), and asymptotic relative efficiency (ARE) demonstrate that the hybrid estimator achieves a favorable balance between robustness and efficiency,

especially in the presence of outliers. Furthermore, the innovative use of an adaptive cross-validation strategy based on a Two-Stage L-BFGS-B optimization, detailed in Chapter Three, plays a crucial role in dynamically refining the estimator parameters. Overall, the thesis concludes that the proposed hybrid M-estimator significantly enhances the robustness and accuracy of regression analysis in the presence of outliers. Through the integration of Huber's and Tukey's Bisquare M-estimators, the hybrid approach successfully strikes a balance between maintaining parameter estimation efficiency and the requirement for robustness against atypical observations. Extensive Monte Carlo simulation results demonstrate that under various levels of outlier contamination, traditional least squares estimation (LSE) rapidly deteriorates in performance, whereas the robust methods particularly the hybrid estimator yields minimal bias, stable standard errors, and a favorable mean squared errors. Moreover, the adaptive tuning process using a Two-Stage L-BFGS-B optimization algorithm further refines the parameters of the estimator, ensuring high reliability even in challenging contamination scenarios. Overall, the findings underscore the value of robust regression techniques in delivering accurate model estimates where conventional methods fail.

6.2. Limitations of the Study

The study has a few shortcomings that affect how widely its results can be applied. For one, the new hybrid M-estimator, while solid, still struggles with extreme outliers, which can throw off its accuracy in heavily contaminated data. Also, the study relies on Monte Carlo simulations, which are great for testing in a controlled setting but might not fully reflect the messiness of real-world data with complex outlier patterns, like clustered or high

leverage points. It mainly focuses on simple and multiple linear regression, so it is not directly suited for nonlinear or high-dimensional problems common in modern data analysis. The Huber-based EWMA control charts, though robust, lose some stability as outliers get stronger, suggesting there is room to further optimize the tuning parameters for better performance. Finally, the Two-Stage L-BFGS-B optimization method, while effective, can be computationally costly, which might be an issue in situations with limited computing power, so finding simpler optimization approaches could be a next step.

6.3. Future Recommendations

According to the findings and insights discussed throughout the thesis, especially in Chapters Three, Four, and Five. It is strongly recommended that practitioners and researchers adopt robust regression techniques, such as the proposed hybrid M-estimator, simulation studies have demonstrated that it performs better than traditional LSE (Chapters Three and Four) by successfully striking a balance between robustness against outlier contamination and efficiency. Also, the integration of adaptive cross-validation via a Two-Stage L-BFGS-B optimization is highlighted as a viable approach for dynamically tuning parameters of the estimator, and further incorporation of its routines into statistical software could facilitate broader practical application of the proposed estimator.

Moreover, several prospects for further research are noted. Beyond the simple and multiple linear regression that were the focus of the thesis, it would be beneficial to expand the current robust estimation framework to high-dimensional settings and investigate its suitability in nonlinear or generalized linear models. The promising results in Phase I

monitoring of linear profiles (Chapter Five) suggest that future work in real-time process monitoring and quality control could be beneficial, particularly for industries where quick detection of process shifts is critical. Exploring alternative global optimization methods to enhance or complement the adaptive cross-validation procedure and evaluating the estimator under diverse outlier patterns such as clustered or leverage points are also important directions. Collectively, these recommendations and research challenges pave the way for refining robust regression methods, ensuring their efficacy across a wider range of practical applications.

APPENDICES

Appendix A: Simulation results for Simple and Multiple Linear regression

Hint: 1st Row – Mean, Bias² and Standard Errors, 2nd Row – Mean Square Error (MSE) and Asymptotic Relative Efficiency (ARE)

Table A1: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the intercept coefficient (β_0) for simple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)
		(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)
	HUBER	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)
		(0.043, 1.054)	(0.043, 1.053)	(0.043, 1.054)	(0.043, 1.054)	(0.043, 1.054)	(0.043, 1.054)
	BISQUARE	(12.000, 0.000, 0.208)	(12.000, 0.000, 0.208)	(12.000, 0.000, 0.208)	(12.000, 0.000, 0.208)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.208)
0.01	LSE	(12.000, 0.000, 0.204)	(12.000, 0.000, 0.204)	(12.000, 0.000, 0.204)	(12.000, 0.000, 0.204)	(12.000, 0.000, 0.204)	(12.000, 0.000, 0.204)
		(0.042, 1.024)	(0.042, 1.023)	(0.042, 1.024)	(0.042, 1.023)	(0.042, 1.024)	(0.042, 1.024)
	LSE	(12.000, 0.000, 0.202)	(12.039, 0.002, 0.336)	(12.228, 0.052, 0.874)	(12.515, 0.266, 1.572)	(12.850, 0.723, 2.276)	(13.210, 1.463, 2.947)
		(0.041, 1.000)	(0.114, 1.000)	(0.816, 1.000)	(2.736, 1.000)	(5.904, 1.000)	(10.146, 1.000)
	HUBER	(12.000, 0.000, 0.207)	(11.999, 0.000, 0.209)	(12.007, 0.000, 0.211)	(12.017, 0.000, 0.214)	(12.027, 0.001, 0.219)	(12.037, 0.001, 0.228)
0.02	LSE	(12.000, 0.000, 0.208)	(11.997, 0.000, 0.209)	(11.997, 0.000, 0.208)	(11.997, 0.000, 0.209)	(11.998, 0.000, 0.209)	(11.998, 0.000, 0.211)
		(0.043, 1.060)	(0.044, 0.381)	(0.043, 0.053)	(0.044, 0.016)	(0.044, 0.007)	(0.045, 0.004)
	LSE	(12.000, 0.000, 0.204)	(11.996, 0.000, 0.206)	(11.997, 0.000, 0.205)	(11.997, 0.000, 0.205)	(11.998, 0.000, 0.205)	(11.998, 0.000, 0.210)
		(0.042, 1.024)	(0.042, 0.370)	(0.042, 0.052)	(0.042, 0.015)	(0.042, 0.007)	(0.044, 0.004)
	PROPOSED	(12.000, 0.000, 0.202)	(12.075, 0.006, 0.424)	(12.451, 0.204, 1.200)	(13.006, 1.013, 2.139)	(13.649, 2.719, 3.055)	(14.325, 5.407, 3.895)
0.02	LSE	(12.000, 0.000, 0.202)	(12.075, 0.006, 0.424)	(12.451, 0.204, 1.200)	(13.006, 1.013, 2.139)	(13.649, 2.719, 3.055)	(14.325, 5.407, 3.895)
		(0.041, 1.000)	(0.185, 1.000)	(1.642, 1.000)	(5.589, 1.000)	(12.055, 1.000)	(20.577, 1.000)
	LSE	(12.000, 0.000, 0.207)	(11.996, 0.000, 0.211)	(12.015, 0.000, 0.215)	(12.034, 0.001, 0.222)	(12.055, 0.003, 0.232)	(12.076, 0.006, 0.249)
		(0.043, 1.053)	(0.045, 0.241)	(0.047, 0.028)	(0.050, 0.009)	(0.057, 0.005)	(0.068, 0.003)
	BISQUARE	(12.000, 0.000, 0.208)	(11.992, 0.000, 0.210)	(11.994, 0.000, 0.210)	(11.995, 0.000, 0.210)	(11.996, 0.000, 0.210)	(11.996, 0.000, 0.210)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.03		(0.043, 1.059)	(0.044, 0.238)	(0.044, 0.027)	(0.044, 0.008)	(0.044, 0.004)	(0.044, 0.002)
		(12.000, 0.000, 0.204)	(11.991, 0.000, 0.207)	(11.994, 0.000, 0.207)	(11.995, 0.000, 0.207)	(11.996, 0.000, 0.206)	(11.996, 0.000, 0.225)
	PROPOSED	(0.042, 1.023)	(0.043, 0.231)	(0.043, 0.026)	(0.043, 0.008)	(0.043, 0.004)	(0.051, 0.002)
		(12.000, 0.000, 0.202)	(12.113, 0.013, 0.497)	(12.666, 0.443, 1.430)	(13.476, 2.178, 2.526)	(14.396, 5.741, 3.558)	(15.354, 11.249, 4.470)
	LSE	(0.041, 1.000)	(0.260, 1.000)	(2.489, 1.000)	(8.559, 1.000)	(18.402, 1.000)	(31.230, 1.000)
		(12.000, 0.000, 0.207)	(11.995, 0.000, 0.214)	(12.022, 0.000, 0.220)	(12.052, 0.003, 0.230)	(12.084, 0.007, 0.246)	(12.117, 0.014, 0.282)
	HUBER	(0.043, 1.053)	(0.046, 0.176)	(0.049, 0.020)	(0.056, 0.006)	(0.068, 0.004)	(0.093, 0.003)
		(12.000, 0.000, 0.207)	(11.989, 0.000, 0.211)	(11.991, 0.000, 0.211)	(11.993, 0.000, 0.211)	(11.994, 0.000, 0.211)	(11.994, 0.000, 0.213)
	BISQUARE	(0.043, 1.059)	(0.045, 0.172)	(0.045, 0.018)	(0.044, 0.005)	(0.045, 0.002)	(0.045, 0.001)
	PROPOSED	(12.000, 0.000, 0.204)	(11.988, 0.000, 0.208)	(11.990, 0.000, 0.208)	(11.992, 0.000, 0.208)	(11.993, 0.000, 0.208)	(11.994, 0.000, 0.258)
0.04		(12.000, 0.000, 0.201)	(12.149, 0.022, 0.558)	(12.876, 0.767, 1.614)	(13.925, 3.706, 2.819)	(15.098, 9.597, 3.911)	(16.286, 18.373, 4.835)
	LSE	(0.041, 1.000)	(0.334, 1.000)	(3.372, 1.000)	(11.653, 1.000)	(24.896, 1.000)	(41.753, 1.000)
		(12.000, 0.000, 0.207)	(11.993, 0.000, 0.216)	(12.030, 0.001, 0.224)	(12.071, 0.005, 0.239)	(12.114, 0.013, 0.261)	(12.161, 0.026, 0.332)
	HUBER	(0.043, 1.054)	(0.047, 0.140)	(0.051, 0.015)	(0.062, 0.005)	(0.081, 0.003)	(0.136, 0.003)
		(12.000, 0.000, 0.207)	(11.985, 0.000, 0.212)	(11.988, 0.000, 0.212)	(11.990, 0.000, 0.212)	(11.992, 0.000, 0.212)	(11.992, 0.000, 0.220)
	BISQUARE	(0.043, 1.060)	(0.045, 0.136)	(0.045, 0.013)	(0.045, 0.004)	(0.045, 0.002)	(0.049, 0.001)
		(12.000, 0.000, 0.204)	(11.983, 0.000, 0.210)	(11.987, 0.000, 0.209)	(11.989, 0.000, 0.209)	(11.991, 0.000, 0.216)	(11.993, 0.000, 0.304)
	PROPOSED	(0.042, 1.024)	(0.044, 0.133)	(0.044, 0.013)	(0.044, 0.004)	(0.047, 0.002)	(0.093, 0.002)
		(12.000, 0.000, 0.201)	(12.183, 0.034, 0.611)	(13.080, 1.166, 1.770)	(14.349, 5.519, 3.045)	(15.760, 14.136, 4.169)	(17.156, 26.587, 5.075)
	LSE	(0.041, 1.000)	(0.407, 1.000)	(4.300, 1.000)	(14.788, 1.000)	(31.516, 1.000)	(52.348, 1.000)
0.05		(12.000, 0.000, 0.207)	(11.992, 0.000, 0.219)	(12.038, 0.001, 0.229)	(12.090, 0.008, 0.248)	(12.146, 0.021, 0.284)	(12.210, 0.044, 0.401)
	HUBER	(0.043, 1.054)	(0.048, 0.118)	(0.054, 0.013)	(0.070, 0.005)	(0.102, 0.003)	(0.205, 0.004)
		(12.000, 0.000, 0.207)	(11.981, 0.000, 0.214)	(11.985, 0.000, 0.213)	(11.988, 0.000, 0.213)	(11.989, 0.000, 0.217)	(11.990, 0.000, 0.225)
	BISQUARE	(0.043, 1.060)	(0.046, 0.113)	(0.046, 0.011)	(0.046, 0.003)	(0.047, 0.001)	(0.051, 0.001)
		(12.000, 0.000, 0.204)	(11.979, 0.000, 0.211)	(11.983, 0.000, 0.211)	(11.986, 0.000, 0.212)	(11.988, 0.000, 0.227)	(11.994, 0.000, 0.380)
	PROPOSED	(0.042, 1.024)	(0.045, 0.111)	(0.045, 0.010)	(0.045, 0.003)	(0.052, 0.002)	(0.145, 0.003)
		(12.000, 0.000, 0.202)	(12.218, 0.048, 0.660)	(13.274, 1.622, 1.894)	(14.764, 7.642, 3.230)	(16.373, 19.127, 4.355)	(17.956, 35.472, 5.234)
	LSE	(0.041, 1.000)	(0.483, 1.000)	(5.211, 1.000)	(18.075, 1.000)	(38.089, 1.000)	(62.871, 1.000)
		(12.000, 0.000, 0.207)	(11.990, 0.001, 0.221)	(12.046, 0.002, 0.234)	(12.111, 0.012, 0.257)	(12.180, 0.032, 0.312)	(12.264, 0.069, 0.503)
	HUBER	(0.043, 1.053)	(0.049, 0.101)	(0.057, 0.011)	(0.079, 0.004)	(0.130, 0.003)	(0.322, 0.005)
0.06		(12.000, 0.000, 0.208)	(11.977, 0.001, 0.215)	(11.982, 0.000, 0.215)	(11.985, 0.000, 0.215)	(11.986, 0.000, 0.215)	(11.988, 0.000, 0.244)
	BISQUARE	(0.043, 1.059)	(0.047, 0.096)	(0.046, 0.009)	(0.046, 0.003)	(0.046, 0.001)	(0.060, 0.001)
		(12.000, 0.000, 0.204)	(11.975, 0.001, 0.213)	(11.979, 0.000, 0.212)	(11.983, 0.000, 0.212)	(11.985, 0.000, 0.252)	(11.995, 0.000, 0.484)
	PROPOSED	(0.042, 1.023)	(0.046, 0.095)	(0.046, 0.009)	(0.045, 0.003)	(0.064, 0.002)	(0.234, 0.004)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.07	LSE	(12.001, 0.000, 0.202)	(12.251, 0.063, 0.702)	(13.470, 2.160, 2.017)	(15.153, 9.943, 3.367)	(16.957, 24.575, 4.486)	(18.707, 44.981, 5.326)
		(0.041, 1.000)	(0.556, 1.000)	(6.229, 1.000)	(21.277, 1.000)	(44.702, 1.000)	(73.343, 1.000)
	HUBER	(12.001, 0.000, 0.207)	(11.988, 0.001, 0.223)	(12.056, 0.003, 0.240)	(12.132, 0.018, 0.270)	(12.217, 0.047, 0.354)	(12.327, 0.107, 0.639)
		(0.043, 1.053)	(0.050, 0.090)	(0.061, 0.010)	(0.091, 0.004)	(0.172, 0.004)	(0.515, 0.007)
	BISQUARE	(12.001, 0.000, 0.207)	(11.973, 0.001, 0.216)	(11.979, 0.000, 0.216)	(11.982, 0.000, 0.216)	(11.984, 0.000, 0.219)	(11.987, 0.000, 0.272)
0.08	LSE	(12.001, 0.000, 0.204)	(11.969, 0.001, 0.214)	(11.976, 0.001, 0.214)	(11.980, 0.000, 0.218)	(11.983, 0.000, 0.275)	(11.999, 0.000, 0.594)
		(0.042, 1.024)	(0.047, 0.084)	(0.046, 0.007)	(0.048, 0.002)	(0.076, 0.002)	(0.353, 0.005)
	HUBER	(12.000, 0.000, 0.202)	(12.284, 0.080, 0.743)	(13.655, 2.738, 2.111)	(15.518, 12.379, 3.486)	(17.513, 30.396, 4.596)	(19.390, 54.605, 5.371)
		(0.041, 1.000)	(0.632, 1.000)	(7.194, 1.000)	(24.530, 1.000)	(51.520, 1.000)	(83.450, 1.000)
	BISQUARE	(12.000, 0.000, 0.207)	(11.986, 0.001, 0.226)	(12.065, 0.004, 0.245)	(12.155, 0.024, 0.282)	(12.260, 0.067, 0.412)	(12.401, 0.161, 0.805)
0.09	LSE	(12.000, 0.000, 0.208)	(11.968, 0.001, 0.218)	(11.975, 0.001, 0.217)	(11.979, 0.000, 0.217)	(11.982, 0.000, 0.221)	(11.986, 0.000, 0.313)
		(0.043, 1.059)	(0.048, 0.077)	(0.048, 0.007)	(0.048, 0.002)	(0.049, 0.001)	(0.098, 0.001)
	HUBER	(12.000, 0.000, 0.204)	(11.964, 0.001, 0.216)	(11.972, 0.001, 0.216)	(11.976, 0.001, 0.216)	(11.982, 0.000, 0.311)	(12.006, 0.000, 0.729)
		(0.042, 1.023)	(0.048, 0.076)	(0.047, 0.007)	(0.047, 0.002)	(0.097, 0.002)	(0.532, 0.006)
	PROPOSED	(12.001, 0.000, 0.202)	(12.314, 0.099, 0.778)	(13.838, 3.377, 2.201)	(15.882, 15.067, 3.579)	(18.022, 36.264, 4.651)	(20.024, 64.390, 5.380)
0.1	LSE	(12.000, 0.000, 0.207)	(11.984, 0.001, 0.229)	(12.074, 0.006, 0.251)	(12.179, 0.032, 0.300)	(12.305, 0.093, 0.478)	(12.492, 0.242, 1.021)
		(0.043, 1.053)	(0.053, 0.075)	(0.069, 0.008)	(0.122, 0.004)	(0.321, 0.006)	(1.284, 0.014)
	HUBER	(12.000, 0.000, 0.208)	(11.964, 0.001, 0.219)	(11.972, 0.001, 0.219)	(11.976, 0.001, 0.218)	(11.979, 0.000, 0.226)	(11.985, 0.000, 0.371)
		(0.043, 1.059)	(0.049, 0.070)	(0.049, 0.006)	(0.048, 0.002)	(0.052, 0.001)	(0.138, 0.001)
	PROPOSED	(12.000, 0.000, 0.204)	(11.959, 0.002, 0.218)	(11.968, 0.001, 0.217)	(11.973, 0.001, 0.223)	(11.980, 0.000, 0.349)	(12.017, 0.000, 0.908)
0.1	LSE	(12.000, 0.000, 0.204)	(11.959, 0.002, 0.218)	(11.968, 0.001, 0.217)	(11.973, 0.001, 0.223)	(11.980, 0.000, 0.349)	(12.017, 0.000, 0.908)
		(0.042, 1.023)	(0.049, 0.070)	(0.048, 0.006)	(0.051, 0.002)	(0.122, 0.002)	(0.825, 0.009)
	HUBER	(12.000, 0.000, 0.202)	(12.346, 0.120, 0.813)	(14.013, 4.053, 2.279)	(16.230, 17.895, 3.667)	(18.502, 42.281, 4.694)	(20.624, 74.373, 5.372)
		(0.041, 1.000)	(0.781, 1.000)	(9.248, 1.000)	(31.343, 1.000)	(64.316, 1.000)	(103.232, 1.000)
	BISQUARE	(12.000, 0.000, 0.207)	(11.982, 0.002, 0.232)	(12.084, 0.007, 0.258)	(12.204, 0.042, 0.319)	(12.357, 0.127, 0.569)	(12.600, 0.360, 1.247)
0.1	LSE	(12.000, 0.000, 0.207)	(11.982, 0.002, 0.232)	(12.084, 0.007, 0.258)	(12.204, 0.042, 0.319)	(12.357, 0.127, 0.569)	(12.600, 0.360, 1.247)
		(0.043, 1.053)	(0.054, 0.069)	(0.074, 0.008)	(0.144, 0.005)	(0.451, 0.007)	(1.916, 0.019)
	HUBER	(12.000, 0.000, 0.208)	(11.959, 0.002, 0.220)	(11.968, 0.001, 0.220)	(11.973, 0.001, 0.220)	(11.977, 0.001, 0.236)	(11.984, 0.000, 0.417)
		(0.043, 1.059)	(0.050, 0.064)	(0.050, 0.005)	(0.049, 0.002)	(0.056, 0.001)	(0.174, 0.002)
	PROPOSED	(12.000, 0.000, 0.204)	(11.954, 0.002, 0.220)	(11.963, 0.001, 0.219)	(11.969, 0.001, 0.227)	(11.980, 0.000, 0.416)	(12.035, 0.001, 1.099)
0.1	LSE	(12.000, 0.000, 0.204)	(11.954, 0.002, 0.220)	(11.963, 0.001, 0.219)	(11.969, 0.001, 0.227)	(11.980, 0.000, 0.416)	(12.035, 0.001, 1.099)
		(0.042, 1.023)	(0.050, 0.064)	(0.049, 0.005)	(0.052, 0.002)	(0.174, 0.003)	(1.208, 0.012)
	HUBER	(12.000, 0.000, 0.207)	(11.982, 0.002, 0.232)	(12.084, 0.007, 0.258)	(12.204, 0.042, 0.319)	(12.357, 0.127, 0.569)	(12.600, 0.360, 1.247)
		(0.043, 1.053)	(0.054, 0.069)	(0.074, 0.008)	(0.144, 0.005)	(0.451, 0.007)	(1.916, 0.019)
	BISQUARE	(12.000, 0.000, 0.208)	(11.959, 0.002, 0.220)	(11.968, 0.001, 0.220)	(11.973, 0.001, 0.220)	(11.977, 0.001, 0.236)	(11.984, 0.000, 0.417)

Table A2: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_1) for simple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)
		(0.001, 1.000)	(0.001, 1.000)	(0.001, 1.000)	(0.001, 1.000)	(0.001, 1.000)	(0.001, 1.000)
	HUBER	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)
		(0.001, 1.053)	(0.001, 1.053)	(0.001, 1.054)	(0.001, 1.054)	(0.001, 1.054)	(0.001, 1.053)
	BISQUARE	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)
0.01	LSE	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.035)
		(0.001, 1.024)	(0.001, 1.023)	(0.001, 1.024)	(0.001, 1.023)	(0.001, 1.024)	(0.001, 1.024)
	HUBER	(4.300, 0.000, 0.035)	(4.282, 0.000, 0.074)	(4.235, 0.004, 0.197)	(4.169, 0.017, 0.346)	(4.096, 0.042, 0.494)	(4.018, 0.079, 0.631)
		(0.001, 1.000)	(0.006, 1.000)	(0.043, 1.000)	(0.137, 1.000)	(0.286, 1.000)	(0.478, 1.000)
	BISQUARE	(4.300, 0.000, 0.036)	(4.298, 0.000, 0.036)	(4.296, 0.000, 0.037)	(4.294, 0.000, 0.038)	(4.292, 0.000, 0.039)	(4.290, 0.000, 0.041)
0.02	LSE	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.037)
		(0.001, 1.053)	(0.001, 0.229)	(0.001, 0.032)	(0.001, 0.011)	(0.002, 0.006)	(0.002, 0.004)
	HUBER	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.037)
		(0.001, 1.059)	(0.001, 0.225)	(0.001, 0.030)	(0.001, 0.010)	(0.001, 0.005)	(0.001, 0.003)
	PROPOSED	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.037)
0.03	LSE	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.037)
		(0.001, 1.023)	(0.001, 0.218)	(0.001, 0.029)	(0.001, 0.009)	(0.001, 0.004)	(0.001, 0.003)
	HUBER	(4.300, 0.000, 0.035)	(4.264, 0.001, 0.097)	(4.171, 0.017, 0.272)	(4.045, 0.065, 0.471)	(3.904, 0.157, 0.660)	(3.759, 0.293, 0.831)
		(0.001, 1.000)	(0.011, 1.000)	(0.090, 1.000)	(0.287, 1.000)	(0.592, 1.000)	(0.984, 1.000)
	BISQUARE	(4.300, 0.000, 0.036)	(4.296, 0.000, 0.037)	(4.292, 0.000, 0.038)	(4.288, 0.000, 0.040)	(4.283, 0.000, 0.043)	(4.279, 0.000, 0.047)
0.04	LSE	(4.300, 0.000, 0.036)	(4.296, 0.000, 0.037)	(4.292, 0.000, 0.038)	(4.288, 0.000, 0.040)	(4.283, 0.000, 0.043)	(4.279, 0.000, 0.047)
		(0.001, 1.053)	(0.001, 0.129)	(0.002, 0.017)	(0.002, 0.006)	(0.002, 0.004)	(0.003, 0.003)
	HUBER	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)
		(0.001, 1.059)	(0.001, 0.123)	(0.001, 0.015)	(0.001, 0.005)	(0.001, 0.002)	(0.001, 0.001)
	PROPOSED	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.041)
0.05	LSE	(4.300, 0.000, 0.035)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.036)	(4.300, 0.000, 0.041)
		(0.001, 1.023)	(0.001, 0.120)	(0.001, 0.014)	(0.001, 0.004)	(0.001, 0.002)	(0.002, 0.002)
	HUBER	(4.300, 0.000, 0.035)	(4.246, 0.003, 0.116)	(4.109, 0.037, 0.324)	(3.925, 0.140, 0.554)	(3.724, 0.331, 0.765)	(3.520, 0.608, 0.949)
		(0.001, 1.000)	(0.016, 1.000)	(0.141, 1.000)	(0.447, 1.000)	(0.917, 1.000)	(1.509, 1.000)
	BISQUARE	(4.300, 0.000, 0.036)	(4.294, 0.000, 0.038)	(4.288, 0.000, 0.040)	(4.281, 0.000, 0.042)	(4.274, 0.001, 0.047)	(4.268, 0.001, 0.055)
0.06	LSE	(4.300, 0.000, 0.036)	(4.294, 0.000, 0.038)	(4.288, 0.000, 0.040)	(4.281, 0.000, 0.042)	(4.274, 0.001, 0.047)	(4.268, 0.001, 0.055)
		(0.001, 1.053)	(0.001, 0.089)	(0.002, 0.012)	(0.002, 0.005)	(0.003, 0.003)	(0.004, 0.003)
	HUBER	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.300, 0.000, 0.037)	(4.300, 0.000, 0.037)	(4.300, 0.000, 0.037)
		(0.001, 1.059)	(0.001, 0.082)	(0.001, 0.009)	(0.001, 0.003)	(0.001, 0.001)	(0.001, 0.001)
	PROPOSED	(4.300, 0.000, 0.035)	(4.299, 0.000, 0.036)	(4.299, 0.000, 0.036)	(4.299, 0.000, 0.036)	(4.299, 0.000, 0.036)	(4.299, 0.000, 0.048)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.001, 1.023)	(0.001, 0.080)	(0.001, 0.009)	(0.001, 0.003)	(0.001, 0.001)	(0.002, 0.002)
0.04	LSE	(4.300, 0.000, 0.035)	(4.229, 0.005, 0.131)	(4.049, 0.063, 0.365)	(3.811, 0.239, 0.616)	(3.556, 0.553, 0.838)	(3.303, 0.993, 1.022)
		(0.001, 1.000)	(0.022, 1.000)	(0.196, 1.000)	(0.618, 1.000)	(1.255, 1.000)	(2.037, 1.000)
	HUBER	(4.300, 0.000, 0.036)	(4.292, 0.000, 0.038)	(4.283, 0.000, 0.041)	(4.274, 0.001, 0.045)	(4.265, 0.001, 0.051)	(4.255, 0.002, 0.067)
		(0.001, 1.054)	(0.002, 0.069)	(0.002, 0.010)	(0.003, 0.004)	(0.004, 0.003)	(0.006, 0.003)
	BISQUARE	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.039)
		(0.001, 1.060)	(0.001, 0.061)	(0.001, 0.007)	(0.001, 0.002)	(0.001, 0.001)	(0.002, 0.001)
0.05	LSE	(4.300, 0.000, 0.035)	(4.299, 0.000, 0.036)	(4.299, 0.000, 0.036)	(4.299, 0.000, 0.036)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.058)
		(0.001, 1.024)	(0.001, 0.060)	(0.001, 0.007)	(0.001, 0.002)	(0.001, 0.001)	(0.003, 0.002)
	HUBER	(4.300, 0.000, 0.035)	(4.212, 0.008, 0.144)	(3.990, 0.096, 0.399)	(3.703, 0.356, 0.663)	(3.397, 0.815, 0.888)	(3.102, 1.436, 1.067)
		(0.001, 1.000)	(0.028, 1.000)	(0.256, 1.000)	(0.795, 1.000)	(1.604, 1.000)	(2.573, 1.000)
	HUBER	(4.300, 0.000, 0.036)	(4.290, 0.000, 0.039)	(4.279, 0.000, 0.042)	(4.267, 0.001, 0.047)	(4.255, 0.002, 0.056)	(4.242, 0.003, 0.083)
		(0.001, 1.054)	(0.002, 0.057)	(0.002, 0.009)	(0.003, 0.004)	(0.005, 0.003)	(0.010, 0.004)
0.06	BISQUARE	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.040)
		(0.001, 1.060)	(0.001, 0.048)	(0.001, 0.005)	(0.001, 0.002)	(0.001, 0.001)	(0.002, 0.001)
	PROPOSED	(4.300, 0.000, 0.035)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.041)	(4.298, 0.000, 0.075)
		(0.001, 1.024)	(0.001, 0.048)	(0.001, 0.005)	(0.001, 0.002)	(0.002, 0.001)	(0.006, 0.002)
	LSE	(4.300, 0.000, 0.035)	(4.195, 0.011, 0.156)	(3.933, 0.135, 0.426)	(3.598, 0.493, 0.700)	(3.250, 1.103, 0.924)	(2.916, 1.915, 1.094)
		(0.001, 1.000)	(0.035, 1.000)	(0.316, 1.000)	(0.984, 1.000)	(1.956, 1.000)	(3.112, 1.000)
0.07	HUBER	(4.300, 0.000, 0.036)	(4.287, 0.000, 0.040)	(4.274, 0.001, 0.044)	(4.260, 0.002, 0.050)	(4.245, 0.003, 0.064)	(4.228, 0.005, 0.106)
		(0.001, 1.053)	(0.002, 0.049)	(0.003, 0.008)	(0.004, 0.004)	(0.007, 0.004)	(0.017, 0.005)
	BISQUARE	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.045)
		(0.001, 1.060)	(0.001, 0.040)	(0.001, 0.004)	(0.001, 0.001)	(0.001, 0.001)	(0.002, 0.001)
	PROPOSED	(4.300, 0.000, 0.035)	(4.298, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.047)	(4.297, 0.000, 0.099)
		(0.001, 1.023)	(0.001, 0.039)	(0.001, 0.004)	(0.001, 0.001)	(0.002, 0.001)	(0.010, 0.003)
0.08	LSE	(4.300, 0.000, 0.035)	(4.178, 0.015, 0.166)	(3.877, 0.179, 0.452)	(3.498, 0.643, 0.728)	(3.110, 1.417, 0.947)	(2.742, 2.426, 1.107)
		(0.001, 1.000)	(0.042, 1.000)	(0.384, 1.000)	(1.172, 1.000)	(2.314, 1.000)	(3.652, 1.000)
	HUBER	(4.300, 0.000, 0.036)	(4.285, 0.000, 0.040)	(4.269, 0.001, 0.045)	(4.252, 0.002, 0.054)	(4.234, 0.004, 0.074)	(4.211, 0.008, 0.138)
		(0.001, 1.053)	(0.002, 0.044)	(0.003, 0.008)	(0.005, 0.004)	(0.010, 0.004)	(0.027, 0.007)
	BISQUARE	(4.300, 0.000, 0.036)	(4.298, 0.000, 0.038)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.051)
		(0.001, 1.059)	(0.001, 0.033)	(0.001, 0.004)	(0.001, 0.001)	(0.001, 0.001)	(0.003, 0.001)
0.08	PROPOSED	(4.300, 0.000, 0.035)	(4.298, 0.000, 0.037)	(4.298, 0.000, 0.037)	(4.298, 0.000, 0.038)	(4.298, 0.000, 0.052)	(4.296, 0.000, 0.123)
		(0.001, 1.024)	(0.001, 0.033)	(0.001, 0.004)	(0.001, 0.001)	(0.003, 0.001)	(0.015, 0.004)
0.08	LSE	(4.300, 0.000, 0.035)	(4.162, 0.019, 0.175)	(3.823, 0.228, 0.472)	(3.405, 0.802, 0.750)	(2.977, 1.751, 0.966)	(2.584, 2.945, 1.110)
		(0.001, 1.000)	(0.050, 1.000)	(0.451, 1.000)	(1.365, 1.000)	(2.684, 1.000)	(4.177, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.05	HUBER	(4.300, 0.000, 0.036) (0.001, 1.054)	(4.283, 0.000, 0.041) (0.002, 0.040)	(4.264, 0.001, 0.047) (0.004, 0.008)	(4.244, 0.003, 0.057) (0.006, 0.005)	(4.222, 0.006, 0.088) (0.014, 0.005)	(4.191, 0.012, 0.175) (0.042, 0.010)
		(4.300, 0.000, 0.036) (0.001, 1.060)	(4.298, 0.000, 0.038) (0.001, 0.029)	(4.298, 0.000, 0.038) (0.001, 0.003)	(4.299, 0.000, 0.038) (0.001, 0.003)	(4.299, 0.000, 0.039) (0.002, 0.001)	(4.298, 0.000, 0.061) (0.004, 0.001)
	BISQUARE	(4.300, 0.000, 0.035) (0.001, 1.024)	(4.298, 0.000, 0.038) (0.001, 0.029)	(4.298, 0.000, 0.038) (0.001, 0.003)	(4.298, 0.000, 0.038) (0.001, 0.003)	(4.298, 0.000, 0.060) (0.004, 0.001)	(4.293, 0.000, 0.153) (0.023, 0.006)
		(4.300, 0.000, 0.035) (0.001, 1.000)	(4.146, 0.024, 0.184) (0.066, 1.000)	(3.769, 0.281, 0.490) (0.522, 1.000)	(3.312, 0.976, 0.768) (1.565, 1.000)	(2.854, 2.090, 0.973) (3.036, 1.000)	(2.437, 3.470, 1.105) (4.692, 1.000)
	PROPOSED	(4.300, 0.000, 0.036) (0.001, 1.053)	(4.280, 0.000, 0.042) (0.002, 0.037)	(4.259, 0.002, 0.049) (0.004, 0.008)	(4.236, 0.004, 0.062) (0.008, 0.005)	(4.209, 0.008, 0.104) (0.019, 0.006)	(4.168, 0.017, 0.224) (0.068, 0.014)
		(4.300, 0.000, 0.036) (0.001, 1.060)	(4.298, 0.000, 0.038) (0.001, 0.025)	(4.298, 0.000, 0.038) (0.001, 0.003)	(4.298, 0.000, 0.038) (0.001, 0.001)	(4.298, 0.000, 0.040) (0.002, 0.001)	(4.298, 0.000, 0.075) (0.006, 0.001)
	LSE	(4.300, 0.000, 0.035) (0.001, 1.024)	(4.297, 0.000, 0.038) (0.001, 0.025)	(4.298, 0.000, 0.038) (0.001, 0.003)	(4.298, 0.000, 0.040) (0.002, 0.001)	(4.297, 0.000, 0.070) (0.005, 0.002)	(4.290, 0.000, 0.192) (0.037, 0.008)
		(4.300, 0.000, 0.035) (0.001, 1.000)	(4.130, 0.029, 0.192) (0.066, 1.000)	(3.718, 0.339, 0.507) (0.595, 1.000)	(3.223, 1.160, 0.783) (1.773, 1.000)	(2.739, 2.438, 0.977) (3.392, 1.000)	(2.299, 4.006, 1.097) (5.210, 1.000)
	HUBER	(4.300, 0.000, 0.036) (0.001, 1.054)	(4.278, 0.000, 0.043) (0.002, 0.035)	(4.253, 0.002, 0.051) (0.005, 0.008)	(4.227, 0.005, 0.068) (0.010, 0.006)	(4.194, 0.011, 0.126) (0.027, 0.008)	(4.140, 0.025, 0.275) (0.101, 0.019)
		(4.300, 0.000, 0.036) (0.001, 1.060)	(4.297, 0.000, 0.038) (0.001, 0.023)	(4.298, 0.000, 0.038) (0.001, 0.002)	(4.298, 0.000, 0.038) (0.001, 0.001)	(4.298, 0.000, 0.043) (0.002, 0.001)	(4.297, 0.000, 0.085) (0.007, 0.001)
	PROPOSED	(4.300, 0.000, 0.035) (0.001, 1.024)	(4.297, 0.000, 0.038) (0.001, 0.023)	(4.297, 0.000, 0.038) (0.001, 0.002)	(4.298, 0.000, 0.040) (0.002, 0.001)	(4.296, 0.000, 0.085) (0.007, 0.002)	(4.285, 0.000, 0.233) (0.055, 0.010)

Table A3: Estimates for (Mean, Bias2, Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the mean square error (σ^2) for simple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(1.000, 0.000, 0.143) (0.020, 1.000)	(1.000, 0.000, 0.143) (0.020, 1.000)	(1.000, 0.000, 0.143) (0.020, 1.000)	(1.000, 0.000, 0.143) (0.020, 1.000)	(1.000, 0.000, 0.143) (0.020, 1.000)	(1.000, 0.000, 0.143) (0.020, 1.000)
		(0.992, 0.000, 0.232) (0.054, 2.637)	(0.992, 0.000, 0.232) (0.054, 2.637)	(0.992, 0.000, 0.232) (0.054, 2.641)	(0.992, 0.000, 0.232) (0.054, 2.632)	(0.992, 0.000, 0.232) (0.054, 2.639)	(0.992, 0.000, 0.232) (0.054, 2.640)
	HUBER	(0.990, 0.000, 0.231) (0.054, 2.637)	(0.990, 0.000, 0.231) (0.054, 2.637)	(0.990, 0.000, 0.232) (0.054, 2.641)	(0.990, 0.000, 0.231) (0.054, 2.632)	(0.990, 0.000, 0.232) (0.054, 2.639)	(0.990, 0.000, 0.232) (0.054, 2.640)
		(0.990, 0.000, 0.231) (0.054, 2.637)	(0.990, 0.000, 0.231) (0.054, 2.637)	(0.990, 0.000, 0.232) (0.054, 2.641)	(0.990, 0.000, 0.231) (0.054, 2.632)	(0.990, 0.000, 0.232) (0.054, 2.639)	(0.990, 0.000, 0.232) (0.054, 2.640)

		λ						
α	ESTIMATOR	0	1	2	3	4	5	
0.01	PROPOSED	(0.054, 2.630)	(0.054, 2.630)	(0.054, 2.635)	(0.054, 2.626)	(0.054, 2.632)	(0.054, 2.634)	
		(0.997, 0.000, 0.232)	(0.997, 0.000, 0.232)	(0.997, 0.000, 0.232)	(0.997, 0.000, 0.232)	(0.997, 0.000, 0.232)	(0.997, 0.000, 0.232)	
		(0.054, 2.634)	(0.054, 2.634)	(0.054, 2.639)	(0.054, 2.632)	(0.054, 2.637)	(0.054, 2.637)	
	LSE	(1.000, 0.000, 0.143)	(1.809, 0.655, 2.613)	(3.923, 8.542, 8.428)	(6.837, 34.066, 15.205)	(10.102, 82.844, 21.820)	(13.531, 157.013, 28.007)	
		(0.020, 1.000)	(7.484, 1.000)	(79.580, 1.000)	(265.250, 1.000)	(558.937, 1.000)	(941.402, 1.000)	
	HUBER	(0.992, 0.000, 0.232)	(1.008, 0.000, 0.236)	(1.010, 0.000, 0.237)	(1.012, 0.000, 0.238)	(1.014, 0.000, 0.239)	(1.017, 0.000, 0.449)	
		(0.054, 2.630)	(0.056, 0.007)	(0.056, 0.001)	(0.057, 0.000)	(0.057, 0.000)	(0.202, 0.000)	
	BISQUARE	(0.990, 0.000, 0.231)	(1.005, 0.000, 0.236)	(1.007, 0.000, 0.236)	(1.008, 0.000, 0.236)	(1.009, 0.000, 0.237)	(1.010, 0.000, 0.480)	
		(0.054, 2.625)	(0.056, 0.007)	(0.056, 0.001)	(0.056, 0.000)	(0.056, 0.000)	(0.231, 0.000)	
	PROPOSED	(0.997, 0.000, 0.232)	(1.012, 0.000, 0.236)	(1.014, 0.000, 0.237)	(1.015, 0.000, 0.237)	(1.016, 0.000, 0.237)	(1.018, 0.000, 0.680)	
		(0.054, 2.629)	(0.056, 0.007)	(0.056, 0.001)	(0.056, 0.000)	(0.056, 0.000)	(0.462, 0.000)	
	0.02	LSE	(1.000, 0.000, 0.143)	(2.598, 2.554, 3.630)	(6.769, 33.286, 11.681)	(12.396, 129.872, 20.713)	(18.649, 311.502, 29.238)	(25.090, 580.349, 36.965)
(0.020, 1.000)			(15.734, 1.000)	(169.730, 1.000)	(558.908, 1.000)	(1166.334, 1.000)	(1946.759, 1.000)	
HUBER		(0.992, 0.000, 0.232)	(1.024, 0.000, 0.241)	(1.030, 0.001, 0.243)	(1.035, 0.001, 0.245)	(1.039, 0.002, 0.247)	(1.046, 0.002, 0.588)	
		(0.054, 2.635)	(0.059, 0.004)	(0.060, 0.000)	(0.061, 0.000)	(0.063, 0.000)	(0.348, 0.000)	
BISQUARE		(0.990, 0.000, 0.231)	(1.020, 0.000, 0.240)	(1.025, 0.001, 0.241)	(1.028, 0.001, 0.242)	(1.029, 0.001, 0.242)	(1.030, 0.001, 0.242)	
		(0.054, 2.629)	(0.061, 0.002)	(0.059, 0.000)	(0.059, 0.000)	(0.059, 0.000)	(0.060, 0.000)	
PROPOSED		(0.997, 0.000, 0.232)	(1.027, 0.001, 0.240)	(1.032, 0.001, 0.241)	(1.035, 0.001, 0.242)	(1.036, 0.001, 0.243)	(1.043, 0.002, 1.410)	
		(0.054, 2.633)	(0.062, 0.002)	(0.059, 0.000)	(0.060, 0.000)	(0.060, 0.000)	(1.990, 0.001)	
0.03		LSE	(1.000, 0.000, 0.143)	(3.389, 5.707, 4.425)	(9.519, 72.570, 13.952)	(17.707, 279.115, 24.415)	(26.648, 657.818, 33.938)	(35.708, 1204.636, 42.230)
			(0.020, 1.000)	(25.286, 1.000)	(267.238, 1.000)	(875.187, 1.000)	(1809.622, 1.000)	(2987.978, 1.000)
		HUBER	(0.992, 0.000, 0.232)	(1.040, 0.001, 0.246)	(1.051, 0.003, 0.249)	(1.059, 0.003, 0.253)	(1.068, 0.005, 0.377)	(1.084, 0.007, 1.045)
			(0.054, 2.634)	(0.062, 0.002)	(0.064, 0.000)	(0.067, 0.000)	(0.147, 0.000)	(1.098, 0.000)
	BISQUARE	(0.990, 0.000, 0.232)	(1.036, 0.001, 0.244)	(1.043, 0.002, 0.246)	(1.047, 0.002, 0.247)	(1.049, 0.002, 0.248)	(1.052, 0.003, 0.522)	
		(0.054, 2.628)	(0.061, 0.002)	(0.062, 0.000)	(0.063, 0.000)	(0.064, 0.000)	(0.275, 0.000)	
	PROPOSED	(0.997, 0.000, 0.232)	(1.043, 0.002, 0.245)	(1.050, 0.003, 0.246)	(1.054, 0.003, 0.248)	(1.056, 0.003, 0.313)	(1.073, 0.005, 2.087)	
		(0.054, 2.632)	(0.062, 0.002)	(0.063, 0.000)	(0.064, 0.000)	(0.101, 0.000)	(4.362, 0.001)	
	0.04	LSE	(1.000, 0.000, 0.143)	(4.174, 10.074, 5.078)	(12.220, 125.881, 15.776)	(22.786, 474.630, 27.163)	(34.125, 1097.262, 37.181)	(45.360, 1967.795, 45.520)
			(0.020, 1.000)	(35.864, 1.000)	(374.774, 1.000)	(1212.453, 1.000)	(2479.680, 1.000)	(4039.896, 1.000)
		HUBER	(0.992, 0.000, 0.232)	(1.058, 0.003, 0.251)	(1.073, 0.005, 0.256)	(1.086, 0.007, 0.289)	(1.100, 0.010, 0.347)	(1.137, 0.019, 1.800)
			(0.054, 2.632)	(0.066, 0.002)	(0.071, 0.000)	(0.091, 0.000)	(0.130, 0.000)	(3.260, 0.001)
BISQUARE		(0.990, 0.000, 0.232)	(1.051, 0.003, 0.249)	(1.062, 0.004, 0.251)	(1.067, 0.005, 0.253)	(1.071, 0.005, 0.253)	(1.075, 0.006, 0.920)	
		(0.054, 2.626)	(0.064, 0.002)	(0.067, 0.000)	(0.068, 0.000)	(0.069, 0.000)	(0.853, 0.000)	
PROPOSED		(0.997, 0.000, 0.232)	(1.059, 0.003, 0.249)	(1.070, 0.005, 0.252)	(1.075, 0.006, 0.253)	(1.080, 0.006, 0.726)	(1.112, 0.013, 3.226)	
		(0.054, 2.629)	(0.062, 0.002)	(0.068, 0.000)	(0.070, 0.000)	(0.534, 0.000)	(10.418, 0.003)	

		λ					
α	ESTIMATOR	0	1	2	3	4	5
0.05	LSE	(1.000, 0.000, 0.143)	(4.935, 15.480, 5.606)	(14.854, 191.946, 17.283)	(27.610, 708.093, 29.249)	(41.170, 1613.638, 39.495)	(54.300, 2840.886, 47.582)
		(0.020, 1.000)	(46.908, 1.000)	(490.632, 1.000)	(1563.592, 1.000)	(3173.503, 1.000)	(5104.952, 1.000)
	HUBER	(0.992, 0.000, 0.232)	(1.076, 0.005, 0.256)	(1.098, 0.010, 0.264)	(1.116, 0.013, 0.283)	(1.140, 0.020, 0.791)	(1.211, 0.044, 2.538)
		(0.054, 2.627)	(0.071, 0.002)	(0.079, 0.000)	(0.094, 0.000)	(0.646, 0.000)	(6.486, 0.001)
	BISQUARE	(0.990, 0.000, 0.231)	(1.069, 0.005, 0.253)	(1.082, 0.007, 0.257)	(1.088, 0.008, 0.258)	(1.093, 0.009, 0.696)	(1.098, 0.010, 1.043)
		(0.054, 2.621)	(0.069, 0.001)	(0.073, 0.000)	(0.075, 0.000)	(0.493, 0.000)	(1.097, 0.000)
0.06	PROPOSED	(0.997, 0.000, 0.232)	(1.076, 0.006, 0.254)	(1.090, 0.008, 0.257)	(1.096, 0.009, 0.392)	(1.104, 0.011, 1.288)	(1.168, 0.028, 4.444)
		(0.054, 2.625)	(0.070, 0.001)	(0.074, 0.000)	(0.163, 0.000)	(1.670, 0.001)	(19.775, 0.004)
	LSE	(1.000, 0.000, 0.143)	(5.688, 21.974, 6.082)	(17.370, 267.961, 18.476)	(32.304, 979.962, 30.965)	(47.742, 2184.777, 41.115)	(62.558, 3789.429, 48.904)
		(0.020, 1.000)	(58.964, 1.000)	(609.315, 1.000)	(1938.816, 1.000)	(3875.242, 1.000)	(6180.993, 1.000)
	HUBER	(0.992, 0.000, 0.232)	(1.096, 0.007, 0.262)	(1.123, 0.015, 0.273)	(1.148, 0.022, 0.291)	(1.188, 0.035, 0.995)	(1.324, 0.105, 3.766)
		(0.054, 2.639)	(0.078, 0.001)	(0.090, 0.000)	(0.107, 0.000)	(1.026, 0.000)	(14.288, 0.002)
0.07	BISQUARE	(0.990, 0.000, 0.232)	(1.086, 0.007, 0.258)	(1.102, 0.010, 0.262)	(1.110, 0.012, 0.265)	(1.115, 0.013, 0.330)	(1.127, 0.016, 1.617)
		(0.054, 2.633)	(0.074, 0.001)	(0.079, 0.000)	(0.082, 0.000)	(0.122, 0.000)	(2.630, 0.000)
	PROPOSED	(0.997, 0.000, 0.232)	(1.094, 0.009, 0.259)	(1.109, 0.012, 0.263)	(1.118, 0.014, 0.265)	(1.135, 0.018, 1.822)	(1.252, 0.064, 6.099)
		(0.054, 2.637)	(0.076, 0.001)	(0.081, 0.000)	(0.084, 0.000)	(3.339, 0.001)	(37.265, 0.006)
	LSE	(1.000, 0.000, 0.143)	(6.433, 29.514, 6.492)	(19.887, 356.712, 19.633)	(36.705, 1274.830, 32.191)	(53.973, 2806.105, 42.235)	(70.278, 4799.505, 49.589)
		(0.020, 1.000)	(71.663, 1.000)	(742.165, 1.000)	(2311.080, 1.000)	(4589.864, 1.000)	(7258.602, 1.000)
0.08	HUBER	(0.992, 0.000, 0.232)	(1.116, 0.011, 0.268)	(1.152, 0.023, 0.284)	(1.186, 0.035, 0.401)	(1.251, 0.063, 1.531)	(1.502, 0.252, 5.483)
		(0.054, 2.644)	(0.085, 0.001)	(0.103, 0.000)	(0.196, 0.000)	(2.408, 0.001)	(30.312, 0.004)
	BISQUARE	(0.990, 0.000, 0.232)	(1.103, 0.011, 0.262)	(1.123, 0.015, 0.268)	(1.132, 0.018, 0.271)	(1.139, 0.019, 0.610)	(1.162, 0.026, 2.456)
		(0.054, 2.638)	(0.079, 0.001)	(0.087, 0.000)	(0.091, 0.000)	(0.392, 0.000)	(6.059, 0.001)
	PROPOSED	(0.997, 0.000, 0.232)	(1.111, 0.012, 0.264)	(1.131, 0.017, 0.269)	(1.142, 0.020, 0.613)	(1.167, 0.028, 2.316)	(1.360, 0.129, 7.785)
		(0.054, 2.642)	(0.082, 0.001)	(0.090, 0.000)	(0.396, 0.000)	(5.390, 0.001)	(60.737, 0.008)
0.09	LSE	(1.000, 0.000, 0.143)	(7.168, 38.046, 6.901)	(22.288, 453.191, 20.502)	(40.911, 1592.870, 33.227)	(59.895, 3468.610, 43.130)	(77.297, 5821.199, 49.795)
		(0.020, 1.000)	(85.671, 1.000)	(873.531, 1.000)	(2696.871, 1.000)	(5328.828, 1.000)	(8300.728, 1.000)
	HUBER	(0.992, 0.000, 0.232)	(1.137, 0.015, 0.275)	(1.182, 0.033, 0.296)	(1.227, 0.051, 0.435)	(1.333, 0.111, 2.118)	(1.759, 0.576, 7.282)
		(0.054, 2.638)	(0.094, 0.001)	(0.120, 0.000)	(0.241, 0.000)	(4.595, 0.001)	(53.607, 0.006)
	BISQUARE	(0.990, 0.000, 0.231)	(1.121, 0.015, 0.268)	(1.144, 0.021, 0.274)	(1.155, 0.024, 0.277)	(1.163, 0.027, 0.663)	(1.202, 0.041, 3.217)
		(0.054, 2.631)	(0.086, 0.001)	(0.096, 0.000)	(0.101, 0.000)	(0.466, 0.000)	(10.388, 0.001)
0.10	PROPOSED	(0.997, 0.000, 0.232)	(1.129, 0.017, 0.269)	(1.153, 0.023, 0.275)	(1.164, 0.027, 0.365)	(1.204, 0.042, 2.954)	(1.511, 0.262, 9.666)
		(0.054, 2.636)	(0.089, 0.001)	(0.099, 0.000)	(0.160, 0.000)	(8.769, 0.002)	(93.692, 0.011)
	LSE	(1.000, 0.000, 0.143)	(7.882, 47.360, 7.237)	(24.667, 560.118, 21.338)	(45.007, 1936.608, 34.058)	(65.316, 4136.601, 43.485)	(83.807, 6856.918, 49.711)
0.11	HUBER	(0.992, 0.000, 0.232)	(1.159, 0.019, 0.282)	(1.214, 0.046, 0.310)	(1.276, 0.076, 0.600)	(1.442, 0.195, 2.896)	(2.170, 1.369, 9.770)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.1		(0.054, 2.639)	(0.105, 0.001)	(0.142, 0.000)	(0.436, 0.000)	(8.584, 0.001)	(96.812, 0.010)
		(0.990, 0.000, 0.232)	(1.139, 0.019, 0.273)	(1.166, 0.028, 0.281)	(1.179, 0.032, 0.285)	(1.190, 0.036, 0.835)	(1.255, 0.065, 4.224)
	BISQUARE	(0.054, 2.633)	(0.094, 0.001)	(0.106, 0.000)	(0.113, 0.000)	(0.733, 0.000)	(17.903, 0.002)
		(0.997, 0.000, 0.232)	(1.148, 0.022, 0.274)	(1.175, 0.031, 0.282)	(1.189, 0.036, 0.741)	(1.249, 0.062, 3.631)	(1.738, 0.544, 12.161)
	PROPOSED	(0.054, 2.638)	(0.097, 0.001)	(0.110, 0.000)	(0.585, 0.000)	(13.248, 0.002)	(148.445, 0.016)
		(1.000, 0.000, 0.143)	(8.613, 57.952, 7.578)	(26.960, 673.936, 22.040)	(48.969, 2301.031, 34.774)	(70.473, 4826.463, 43.794)	(89.952, 7912.526, 49.453)
	LSE	(0.020, 1.000)	(115.377, 1.000)	(1159.688, 1.000)	(3510.230, 1.000)	(6744.339, 1.000)	(10358.151, 1.000)
		(0.992, 0.000, 0.232)	(1.182, 0.025, 0.290)	(1.250, 0.062, 0.328)	(1.333, 0.111, 0.726)	(1.593, 0.351, 3.834)	(2.722, 2.965, 12.318)
	HUBER	(0.054, 2.635)	(0.117, 0.001)	(0.170, 0.000)	(0.638, 0.000)	(15.053, 0.002)	(154.697, 0.015)
		(0.990, 0.000, 0.231)	(1.158, 0.025, 0.278)	(1.189, 0.036, 0.287)	(1.204, 0.042, 0.292)	(1.219, 0.048, 1.231)	(1.308, 0.095, 5.028)
0.1	BISQUARE	(0.054, 2.628)	(0.102, 0.001)	(0.118, 0.000)	(0.127, 0.000)	(1.562, 0.000)	(25.377, 0.002)
		(0.997, 0.000, 0.232)	(1.167, 0.028, 0.280)	(1.198, 0.039, 0.289)	(1.215, 0.046, 0.808)	(1.314, 0.098, 4.843)	(2.042, 1.085, 14.870)
	PROPOSED	(0.054, 2.633)	(0.106, 0.001)	(0.123, 0.000)	(0.699, 0.000)	(23.551, 0.003)	(222.196, 0.021)

Table A4: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the intercept coefficient (β_0) for simple linear regression model after optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0		(12.000, 0.000, 0.202)	(12.000, 0.000, 0.201)	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)	(12.000, 0.000, 0.202)
	LSE	(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)	(0.041, 1.000)
		(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)
	HUBER	(0.043, 1.054)	(0.043, 1.054)	(0.043, 1.053)	(0.043, 1.053)	(0.043, 1.053)	(0.043, 1.053)
		(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.207)	(12.000, 0.000, 0.208)	(12.000, 0.000, 0.208)	(12.000, 0.000, 0.208)
	BISQUARE	(0.043, 1.060)	(0.043, 1.060)	(0.043, 1.059)	(0.043, 1.059)	(0.043, 1.059)	(0.043, 1.059)
		(12.000, 0.000, 0.269)	(12.000, 0.000, 0.269)	(12.000, 0.000, 0.269)	(12.000, 0.000, 0.269)	(12.000, 0.000, 0.269)	(12.000, 0.000, 0.269)
	PROPOSED	(0.073, 1.785)	(0.073, 1.789)	(0.072, 1.783)	(0.073, 1.783)	(0.073, 1.780)	(0.072, 1.782)
		(12.000, 0.000, 0.202)	(12.038, 0.001, 0.333)	(12.230, 0.053, 0.879)	(12.519, 0.270, 1.583)	(12.849, 0.721, 2.273)	(13.211, 1.467, 2.943)
	LSE	(0.041, 1.000)	(0.112, 1.000)	(0.826, 1.000)	(2.774, 1.000)	(5.887, 1.000)	(10.130, 1.000)
0.01		(12.000, 0.000, 0.207)	(11.998, 0.000, 0.209)	(12.007, 0.000, 0.211)	(12.017, 0.000, 0.214)	(12.027, 0.001, 0.219)	(12.037, 0.001, 0.228)
	HUBER	(0.043, 1.054)	(0.044, 0.389)	(0.045, 0.054)	(0.046, 0.017)	(0.049, 0.008)	(0.053, 0.005)
		(12.000, 0.000, 0.208)	(11.997, 0.000, 0.209)	(11.997, 0.000, 0.209)	(11.998, 0.000, 0.209)	(11.998, 0.000, 0.209)	(11.998, 0.000, 0.211)
	BISQUARE	(0.043, 1.060)	(0.044, 0.387)	(0.043, 0.053)	(0.044, 0.016)	(0.044, 0.007)	(0.044, 0.004)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.02	PROPOSED	(12.000, 0.000, 0.269) (0.072, 1.782)	(11.998, 0.000, 0.269) (0.073, 0.645)	(11.999, 0.000, 0.269) (0.072, 0.088)	(11.999, 0.000, 0.269) (0.072, 0.026)	(11.999, 0.000, 0.269) (0.072, 0.012)	(11.999, 0.000, 0.269) (0.072, 0.007)
		(12.000, 0.000, 0.202) (0.041, 1.000)	(12.076, 0.006, 0.422) (0.184, 1.000)	(12.451, 0.203, 1.199) (1.642, 1.000)	(13.005, 1.010, 2.134) (5.565, 1.000)	(13.652, 2.729, 3.064) (12.115, 1.000)	(14.317, 5.366, 3.883) (20.446, 1.000)
	LSE	(12.000, 0.000, 0.207) (0.043, 1.053)	(11.997, 0.000, 0.211) (0.045, 0.242)	(12.015, 0.000, 0.215) (0.046, 0.028)	(12.034, 0.001, 0.222) (0.050, 0.009)	(12.055, 0.003, 0.232) (0.057, 0.005)	(12.076, 0.006, 0.249) (0.068, 0.003)
	HUBER	(12.000, 0.000, 0.207) (0.043, 1.059)	(11.993, 0.000, 0.210) (0.044, 0.239)	(11.995, 0.000, 0.209) (0.044, 0.027)	(11.995, 0.000, 0.210) (0.044, 0.008)	(11.996, 0.000, 0.210) (0.044, 0.004)	(11.996, 0.000, 0.210) (0.044, 0.002)
	BISQUARE	(12.000, 0.000, 0.269) (0.072, 1.784)	(11.996, 0.000, 0.269) (0.072, 0.393)	(11.997, 0.000, 0.268) (0.072, 0.044)	(11.997, 0.000, 0.269) (0.072, 0.013)	(11.998, 0.000, 0.268) (0.072, 0.006)	(11.998, 0.000, 0.268) (0.072, 0.004)
	PROPOSED	(12.000, 0.000, 0.202) (0.041, 1.000)	(12.113, 0.013, 0.498) (0.260, 1.000)	(12.668, 0.447, 1.433) (2.501, 1.000)	(13.474, 2.172, 2.526) (8.552, 1.000)	(14.395, 5.738, 3.560) (18.410, 1.000)	(15.344, 11.180, 4.462) (31.087, 1.000)
0.03	LSE	(12.000, 0.000, 0.207) (0.043, 1.054)	(11.995, 0.000, 0.214) (0.046, 0.176)	(12.022, 0.001, 0.220) (0.049, 0.019)	(12.052, 0.003, 0.230) (0.056, 0.007)	(12.084, 0.007, 0.248) (0.068, 0.004)	(12.117, 0.014, 0.279) (0.092, 0.003)
	HUBER	(12.000, 0.000, 0.208) (0.043, 1.060)	(11.989, 0.000, 0.211) (0.045, 0.172)	(11.992, 0.000, 0.211) (0.045, 0.018)	(11.993, 0.000, 0.211) (0.045, 0.005)	(11.994, 0.000, 0.212) (0.045, 0.002)	(11.994, 0.000, 0.215) (0.046, 0.001)
	BISQUARE	(12.000, 0.000, 0.269) (0.073, 1.786)	(11.994, 0.000, 0.269) (0.073, 0.279)	(11.996, 0.000, 0.268) (0.072, 0.029)	(11.996, 0.000, 0.268) (0.072, 0.008)	(11.997, 0.000, 0.268) (0.072, 0.004)	(11.997, 0.000, 0.267) (0.072, 0.002)
	PROPOSED	(12.000, 0.000, 0.202) (0.041, 1.000)	(12.148, 0.022, 0.558) (0.334, 1.000)	(12.874, 0.765, 1.612) (3.362, 1.000)	(13.926, 3.710, 2.821) (11.668, 1.000)	(15.097, 9.593, 3.913) (24.906, 1.000)	(16.293, 18.429, 4.835) (41.804, 1.000)
	LSE	(12.000, 0.000, 0.207) (0.043, 1.053)	(11.993, 0.000, 0.216) (0.047, 0.140)	(12.030, 0.001, 0.224) (0.051, 0.015)	(12.071, 0.005, 0.238) (0.062, 0.005)	(12.114, 0.013, 0.263) (0.082, 0.003)	(12.161, 0.026, 0.324) (0.131, 0.003)
	HUBER	(12.000, 0.000, 0.207) (0.043, 1.059)	(11.985, 0.000, 0.212) (0.045, 0.136)	(11.988, 0.000, 0.212) (0.045, 0.013)	(11.990, 0.000, 0.212) (0.045, 0.004)	(11.991, 0.000, 0.212) (0.045, 0.002)	(11.992, 0.000, 0.220) (0.048, 0.001)
0.04	BISQUARE	(12.000, 0.000, 0.269) (0.072, 1.783)	(11.992, 0.000, 0.269) (0.072, 0.217)	(11.994, 0.000, 0.268) (0.072, 0.021)	(11.995, 0.000, 0.268) (0.072, 0.006)	(11.995, 0.000, 0.267) (0.072, 0.003)	(11.996, 0.000, 0.267) (0.071, 0.002)
	PROPOSED	(12.000, 0.000, 0.202) (0.041, 1.000)	(12.184, 0.034, 0.612) (0.409, 1.000)	(13.080, 1.166, 1.770) (4.299, 1.000)	(14.353, 5.536, 3.051) (14.847, 1.000)	(15.754, 14.095, 4.163) (31.427, 1.000)	(17.160, 26.626, 5.081) (52.448, 1.000)
	LSE	(12.000, 0.000, 0.207) (0.043, 1.053)	(11.992, 0.000, 0.218) (0.048, 0.117)	(12.038, 0.001, 0.229) (0.054, 0.013)	(12.090, 0.008, 0.248) (0.070, 0.005)	(12.146, 0.021, 0.283) (0.102, 0.003)	(12.209, 0.044, 0.405) (0.208, 0.004)
	HUBER	(12.000, 0.000, 0.207) (0.043, 1.060)	(11.981, 0.000, 0.213) (0.046, 0.112)	(11.985, 0.000, 0.214) (0.046, 0.011)	(11.988, 0.000, 0.213) (0.046, 0.003)	(11.989, 0.000, 0.214) (0.046, 0.001)	(11.991, 0.000, 0.237) (0.056, 0.001)
	BISQUARE	(12.000, 0.000, 0.269) (0.072, 1.784)	(11.989, 0.000, 0.269) (0.072, 0.177)	(11.992, 0.000, 0.268) (0.072, 0.017)	(11.994, 0.000, 0.267) (0.071, 0.005)	(11.994, 0.000, 0.267) (0.071, 0.002)	(11.995, 0.000, 0.266) (0.071, 0.001)
	PROPOSED	(12.000, 0.000, 0.202) (0.041, 1.000)	(12.217, 0.047, 0.660) (0.409, 1.000)	(13.275, 1.626, 1.901) (4.299, 1.000)	(14.759, 7.613, 3.228) (14.847, 1.000)	(16.377, 19.160, 4.353) (31.427, 1.000)	(17.960, 35.526, 5.234) (52.448, 1.000)
0.06	LSE	(12.000, 0.000, 0.207) (0.043, 1.053)	(11.992, 0.000, 0.218) (0.048, 0.117)	(12.038, 0.001, 0.229) (0.054, 0.013)	(12.090, 0.008, 0.248) (0.070, 0.005)	(12.146, 0.021, 0.283) (0.102, 0.003)	(12.209, 0.044, 0.405) (0.208, 0.004)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.041, 1.000)	(0.483, 1.000)	(5.239, 1.000)	(18.036, 1.000)	(38.108, 1.000)	(62.925, 1.000)
0.05	HUBER	(12.000, 0.000, 0.207)	(11.990, 0.001, 0.221)	(12.047, 0.002, 0.234)	(12.111, 0.012, 0.258)	(12.181, 0.033, 0.315)	(12.263, 0.069, 0.496)
		(0.043, 1.053)	(0.049, 0.101)	(0.057, 0.011)	(0.079, 0.004)	(0.132, 0.003)	(0.315, 0.005)
	BISQUARE	(12.000, 0.000, 0.207)	(11.977, 0.001, 0.215)	(11.982, 0.000, 0.215)	(11.985, 0.000, 0.215)	(11.987, 0.000, 0.214)	(11.989, 0.000, 0.243)
		(0.043, 1.059)	(0.047, 0.097)	(0.046, 0.009)	(0.046, 0.003)	(0.046, 0.001)	(0.059, 0.001)
	PROPOSED	(12.000, 0.000, 0.269)	(11.987, 0.000, 0.269)	(11.990, 0.000, 0.268)	(11.992, 0.000, 0.267)	(11.994, 0.000, 0.266)	(11.994, 0.000, 0.265)
		(0.072, 1.781)	(0.073, 0.150)	(0.072, 0.014)	(0.071, 0.004)	(0.071, 0.002)	(0.070, 0.001)
	LSE	(12.000, 0.000, 0.202)	(12.250, 0.063, 0.702)	(13.474, 2.172, 2.019)	(15.161, 9.991, 3.379)	(16.962, 24.622, 4.489)	(18.698, 44.863, 5.322)
		(0.041, 1.000)	(0.556, 1.000)	(6.248, 1.000)	(21.412, 1.000)	(44.770, 1.000)	(73.185, 1.000)
	HUBER	(12.000, 0.000, 0.207)	(11.988, 0.001, 0.224)	(12.056, 0.003, 0.239)	(12.133, 0.018, 0.270)	(12.218, 0.048, 0.360)	(12.326, 0.106, 0.632)
		(0.043, 1.053)	(0.050, 0.090)	(0.060, 0.010)	(0.090, 0.004)	(0.177, 0.004)	(0.505, 0.007)
0.07	BISQUARE	(12.000, 0.000, 0.207)	(11.973, 0.001, 0.216)	(11.979, 0.000, 0.216)	(11.982, 0.000, 0.216)	(11.984, 0.000, 0.220)	(11.987, 0.000, 0.278)
		(0.043, 1.059)	(0.048, 0.085)	(0.047, 0.008)	(0.047, 0.002)	(0.049, 0.001)	(0.077, 0.001)
	PROPOSED	(12.000, 0.000, 0.269)	(11.985, 0.000, 0.269)	(11.988, 0.000, 0.268)	(11.990, 0.000, 0.267)	(11.992, 0.000, 0.266)	(11.992, 0.000, 0.267)
		(0.072, 1.784)	(0.073, 0.131)	(0.072, 0.011)	(0.071, 0.003)	(0.071, 0.002)	(0.071, 0.001)
	LSE	(12.000, 0.000, 0.202)	(12.283, 0.080, 0.742)	(13.650, 2.724, 2.108)	(15.532, 12.472, 3.495)	(17.512, 30.378, 4.588)	(19.391, 54.631, 5.374)
		(0.041, 1.000)	(0.630, 1.000)	(7.168, 1.000)	(24.690, 1.000)	(51.427, 1.000)	(83.510, 1.000)
	HUBER	(12.000, 0.000, 0.207)	(11.986, 0.001, 0.227)	(12.065, 0.004, 0.245)	(12.155, 0.024, 0.283)	(12.259, 0.067, 0.406)	(12.401, 0.161, 0.803)
		(0.043, 1.054)	(0.052, 0.082)	(0.064, 0.009)	(0.104, 0.004)	(0.232, 0.005)	(0.806, 0.010)
	BISQUARE	(12.000, 0.000, 0.208)	(11.968, 0.001, 0.218)	(11.975, 0.001, 0.217)	(11.979, 0.000, 0.217)	(11.982, 0.000, 0.220)	(11.986, 0.000, 0.296)
		(0.043, 1.060)	(0.048, 0.077)	(0.048, 0.007)	(0.048, 0.002)	(0.049, 0.001)	(0.088, 0.001)
0.08	PROPOSED	(12.000, 0.000, 0.270)	(11.982, 0.000, 0.270)	(11.987, 0.000, 0.267)	(11.989, 0.000, 0.267)	(11.990, 0.000, 0.265)	(11.992, 0.000, 0.266)
		(0.073, 1.784)	(0.073, 0.116)	(0.072, 0.010)	(0.071, 0.003)	(0.071, 0.001)	(0.071, 0.001)
	LSE	(12.000, 0.000, 0.202)	(12.315, 0.099, 0.778)	(13.836, 3.370, 2.201)	(15.884, 15.082, 3.588)	(18.019, 36.234, 4.650)	(20.029, 64.465, 5.385)
		(0.041, 1.000)	(0.705, 1.000)	(8.212, 1.000)	(27.954, 1.000)	(57.856, 1.000)	(93.463, 1.000)
	HUBER	(12.000, 0.000, 0.207)	(11.985, 0.001, 0.229)	(12.074, 0.006, 0.251)	(12.179, 0.032, 0.301)	(12.305, 0.093, 0.487)	(12.492, 0.242, 1.017)
		(0.043, 1.053)	(0.053, 0.075)	(0.069, 0.008)	(0.123, 0.004)	(0.331, 0.006)	(1.276, 0.014)
	BISQUARE	(12.000, 0.000, 0.207)	(11.964, 0.001, 0.219)	(11.972, 0.001, 0.219)	(11.976, 0.001, 0.218)	(11.979, 0.000, 0.229)	(11.985, 0.000, 0.360)
		(0.043, 1.059)	(0.049, 0.070)	(0.049, 0.006)	(0.048, 0.002)	(0.053, 0.001)	(0.130, 0.001)
	PROPOSED	(12.000, 0.000, 0.269)	(11.980, 0.000, 0.269)	(11.985, 0.000, 0.267)	(11.988, 0.000, 0.266)	(11.989, 0.000, 0.265)	(11.990, 0.000, 0.266)
		(0.072, 1.781)	(0.073, 0.103)	(0.072, 0.009)	(0.071, 0.003)	(0.070, 0.001)	(0.071, 0.001)
0.1	LSE	(12.000, 0.000, 0.202)	(12.346, 0.120, 0.813)	(14.009, 4.036, 2.270)	(16.228, 17.872, 3.669)	(18.513, 42.413, 4.697)	(20.624, 74.368, 5.374)
		(0.041, 1.000)	(0.780, 1.000)	(9.191, 1.000)	(31.335, 1.000)	(64.479, 1.000)	(103.244, 1.000)
	HUBER	(12.000, 0.000, 0.207)	(11.983, 0.002, 0.232)	(12.085, 0.007, 0.257)	(12.204, 0.042, 0.321)	(12.358, 0.128, 0.582)	(12.603, 0.364, 1.257)
		(0.043, 1.054)	(0.054, 0.069)	(0.073, 0.008)	(0.145, 0.005)	(0.467, 0.007)	(1.944, 0.019)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.002, 1.787)	(0.002, 0.204)	(0.002, 0.024)	(0.002, 0.008)	(0.002, 0.004)	(0.002, 0.002)
0.03	LSE	(4.300, 0.000, 0.035)	(4.246, 0.003, 0.116)	(4.108, 0.037, 0.324)	(3.926, 0.140, 0.554)	(3.725, 0.331, 0.765)	(3.522, 0.605, 0.948)
		(0.001, 1.000)	(0.016, 1.000)	(0.142, 1.000)	(0.447, 1.000)	(0.917, 1.000)	(1.503, 1.000)
	HUBER	(4.300, 0.000, 0.036)	(4.294, 0.000, 0.038)	(4.287, 0.000, 0.040)	(4.281, 0.000, 0.042)	(4.274, 0.001, 0.047)	(4.268, 0.001, 0.054)
		(0.001, 1.054)	(0.001, 0.089)	(0.002, 0.012)	(0.002, 0.005)	(0.003, 0.003)	(0.004, 0.003)
	BISQUARE	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.300, 0.000, 0.037)	(4.300, 0.000, 0.037)	(4.300, 0.000, 0.038)
		(0.001, 1.060)	(0.001, 0.083)	(0.001, 0.009)	(0.001, 0.003)	(0.001, 0.001)	(0.001, 0.001)
0.04	LSE	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)
		(0.002, 1.786)	(0.002, 0.134)	(0.002, 0.015)	(0.002, 0.005)	(0.002, 0.002)	(0.002, 0.001)
	HUBER	(4.300, 0.000, 0.035)	(4.229, 0.005, 0.131)	(4.049, 0.063, 0.364)	(3.811, 0.239, 0.616)	(3.556, 0.553, 0.838)	(3.302, 0.996, 1.021)
		(0.001, 1.000)	(0.022, 1.000)	(0.196, 1.000)	(0.619, 1.000)	(1.255, 1.000)	(2.040, 1.000)
	HUBER	(4.300, 0.000, 0.036)	(4.292, 0.000, 0.038)	(4.283, 0.000, 0.041)	(4.274, 0.001, 0.045)	(4.265, 0.001, 0.051)	(4.255, 0.002, 0.065)
		(0.001, 1.053)	(0.002, 0.069)	(0.002, 0.010)	(0.003, 0.004)	(0.004, 0.003)	(0.006, 0.003)
0.05	BISQUARE	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.039)
		(0.001, 1.059)	(0.001, 0.061)	(0.001, 0.007)	(0.001, 0.002)	(0.001, 0.001)	(0.001, 0.001)
	PROPOSED	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)
		(0.002, 1.786)	(0.002, 0.098)	(0.002, 0.011)	(0.002, 0.003)	(0.002, 0.002)	(0.002, 0.001)
	LSE	(4.300, 0.000, 0.035)	(4.212, 0.008, 0.144)	(3.990, 0.096, 0.399)	(3.702, 0.357, 0.664)	(3.398, 0.813, 0.888)	(3.101, 1.438, 1.068)
		(0.001, 1.000)	(0.029, 1.000)	(0.256, 1.000)	(0.798, 1.000)	(1.601, 1.000)	(2.578, 1.000)
0.06	HUBER	(4.300, 0.000, 0.036)	(4.290, 0.000, 0.039)	(4.279, 0.000, 0.042)	(4.267, 0.001, 0.047)	(4.256, 0.002, 0.056)	(4.242, 0.003, 0.084)
		(0.001, 1.053)	(0.002, 0.057)	(0.002, 0.009)	(0.003, 0.004)	(0.005, 0.003)	(0.010, 0.004)
	BISQUARE	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.043)
		(0.001, 1.060)	(0.001, 0.048)	(0.001, 0.005)	(0.001, 0.002)	(0.001, 0.001)	(0.002, 0.001)
	PROPOSED	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)
		(0.002, 1.786)	(0.002, 0.077)	(0.002, 0.008)	(0.002, 0.003)	(0.002, 0.001)	(0.002, 0.001)
0.07	LSE	(4.300, 0.000, 0.035)	(4.195, 0.011, 0.156)	(3.933, 0.135, 0.427)	(3.599, 0.492, 0.700)	(3.249, 1.105, 0.924)	(2.915, 1.917, 1.094)
		(0.001, 1.000)	(0.035, 1.000)	(0.317, 1.000)	(0.982, 1.000)	(1.958, 1.000)	(3.114, 1.000)
	HUBER	(4.300, 0.000, 0.036)	(4.287, 0.000, 0.040)	(4.274, 0.001, 0.044)	(4.260, 0.002, 0.050)	(4.245, 0.003, 0.064)	(4.228, 0.005, 0.105)
		(0.001, 1.053)	(0.002, 0.049)	(0.003, 0.008)	(0.004, 0.004)	(0.007, 0.004)	(0.016, 0.005)
	BISQUARE	(4.300, 0.000, 0.036)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.037)	(4.299, 0.000, 0.044)
		(0.001, 1.059)	(0.001, 0.040)	(0.001, 0.004)	(0.001, 0.001)	(0.001, 0.001)	(0.002, 0.001)
0.07	PROPOSED	(4.300, 0.000, 0.047)	(4.299, 0.000, 0.047)	(4.300, 0.000, 0.047)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)
		(0.002, 1.785)	(0.002, 0.062)	(0.002, 0.007)	(0.002, 0.002)	(0.002, 0.001)	(0.002, 0.001)
0.07	LSE	(4.300, 0.000, 0.035)	(4.179, 0.015, 0.166)	(3.876, 0.180, 0.452)	(3.497, 0.645, 0.730)	(3.109, 1.419, 0.948)	(2.744, 2.421, 1.106)
		(0.001, 1.000)	(0.042, 1.000)	(0.385, 1.000)	(1.178, 1.000)	(2.318, 1.000)	(3.645, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.05		(4.300, 0.000, 0.036)	(4.285, 0.000, 0.040)	(4.269, 0.001, 0.045)	(4.252, 0.002, 0.054)	(4.234, 0.004, 0.075)	(4.211, 0.008, 0.136)
	HUBER	(0.001, 1.053)	(0.002, 0.044)	(0.003, 0.008)	(0.005, 0.004)	(0.010, 0.004)	(0.026, 0.007)
		(4.300, 0.000, 0.036)	(4.298, 0.000, 0.038)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.039)	(4.299, 0.000, 0.052)
	BISQUARE	(0.001, 1.059)	(0.001, 0.034)	(0.001, 0.004)	(0.001, 0.001)	(0.001, 0.001)	(0.003, 0.001)
		(4.300, 0.000, 0.047)	(4.299, 0.000, 0.047)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)	(4.300, 0.000, 0.046)
	PROPOSED	(0.002, 1.787)	(0.002, 0.052)	(0.002, 0.006)	(0.002, 0.002)	(0.002, 0.001)	(0.002, 0.001)
		(4.300, 0.000, 0.035)	(4.162, 0.019, 0.175)	(3.824, 0.227, 0.471)	(3.402, 0.806, 0.752)	(2.977, 1.750, 0.964)	(2.584, 2.944, 1.110)
	LSE	(0.001, 1.000)	(0.050, 1.000)	(0.449, 1.000)	(1.372, 1.000)	(2.680, 1.000)	(4.177, 1.000)
		(4.300, 0.000, 0.036)	(4.283, 0.000, 0.041)	(4.264, 0.001, 0.047)	(4.244, 0.003, 0.058)	(4.222, 0.006, 0.087)	(4.191, 0.012, 0.174)
	HUBER	(0.001, 1.054)	(0.002, 0.040)	(0.004, 0.008)	(0.006, 0.005)	(0.014, 0.005)	(0.042, 0.010)
0.08		(4.300, 0.000, 0.036)	(4.298, 0.000, 0.038)	(4.298, 0.000, 0.038)	(4.299, 0.000, 0.038)	(4.299, 0.000, 0.039)	(4.298, 0.000, 0.057)
	BISQUARE	(0.001, 1.060)	(0.001, 0.029)	(0.001, 0.003)	(0.001, 0.001)	(0.001, 0.001)	(0.003, 0.001)
		(4.300, 0.000, 0.047)	(4.299, 0.000, 0.047)	(4.299, 0.000, 0.047)	(4.299, 0.000, 0.046)	(4.299, 0.000, 0.046)	(4.300, 0.000, 0.046)
	PROPOSED	(0.002, 1.788)	(0.002, 0.044)	(0.002, 0.005)	(0.002, 0.002)	(0.002, 0.001)	(0.002, 0.001)
		(4.300, 0.000, 0.035)	(4.146, 0.024, 0.184)	(3.770, 0.281, 0.490)	(3.312, 0.977, 0.769)	(2.855, 2.089, 0.973)	(2.436, 3.474, 1.107)
	LSE	(0.001, 1.000)	(0.058, 1.000)	(0.522, 1.000)	(1.568, 1.000)	(3.035, 1.000)	(4.698, 1.000)
		(4.300, 0.000, 0.036)	(4.280, 0.000, 0.042)	(4.259, 0.002, 0.049)	(4.236, 0.004, 0.063)	(4.208, 0.008, 0.107)	(4.168, 0.017, 0.223)
	HUBER	(0.001, 1.053)	(0.002, 0.037)	(0.004, 0.008)	(0.008, 0.005)	(0.020, 0.007)	(0.067, 0.014)
		(4.300, 0.000, 0.036)	(4.298, 0.000, 0.038)	(4.298, 0.000, 0.038)	(4.298, 0.000, 0.038)	(4.298, 0.000, 0.041)	(4.298, 0.000, 0.072)
	BISQUARE	(0.001, 1.059)	(0.001, 0.025)	(0.001, 0.003)	(0.001, 0.001)	(0.002, 0.001)	(0.005, 0.001)
0.09		(4.300, 0.000, 0.047)	(4.299, 0.000, 0.047)	(4.299, 0.000, 0.046)	(4.299, 0.000, 0.046)	(4.299, 0.000, 0.046)	(4.300, 0.000, 0.046)
	PROPOSED	(0.002, 1.787)	(0.002, 0.038)	(0.002, 0.004)	(0.002, 0.001)	(0.002, 0.001)	(0.002, 0.000)
		(4.300, 0.000, 0.035)	(4.130, 0.029, 0.192)	(3.719, 0.338, 0.505)	(3.224, 1.158, 0.784)	(2.737, 2.444, 0.977)	(2.299, 4.005, 1.098)
	LSE	(0.001, 1.000)	(0.066, 1.000)	(0.593, 1.000)	(1.772, 1.000)	(3.399, 1.000)	(5.210, 1.000)
		(4.300, 0.000, 0.036)	(4.278, 0.000, 0.043)	(4.253, 0.002, 0.051)	(4.227, 0.005, 0.068)	(4.193, 0.011, 0.129)	(4.140, 0.026, 0.277)
	HUBER	(0.001, 1.054)	(0.002, 0.035)	(0.005, 0.008)	(0.010, 0.006)	(0.028, 0.008)	(0.102, 0.020)
		(4.300, 0.000, 0.036)	(4.297, 0.000, 0.038)	(4.298, 0.000, 0.038)	(4.298, 0.000, 0.038)	(4.298, 0.000, 0.041)	(4.297, 0.000, 0.088)
	BISQUARE	(0.001, 1.060)	(0.001, 0.023)	(0.001, 0.002)	(0.001, 0.001)	(0.002, 0.001)	(0.008, 0.001)
		(4.300, 0.000, 0.047)	(4.299, 0.000, 0.047)	(4.299, 0.000, 0.046)	(4.299, 0.000, 0.046)	(4.299, 0.000, 0.046)	(4.299, 0.000, 0.047)
	PROPOSED	(0.002, 1.786)	(0.002, 0.033)	(0.002, 0.004)	(0.002, 0.001)	(0.002, 0.001)	(0.002, 0.000)

Table A6: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the mean square error (σ^2) for simple linear regression model after optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(1.000, 0.000, 0.143)	(1.000, 0.000, 0.143)	(1.000, 0.000, 0.143)	(1.000, 0.000, 0.143)	(1.000, 0.000, 0.143)	(1.000, 0.000, 0.143)
		(0.020, 1.000)	(0.020, 1.000)	(0.020, 1.000)	(0.020, 1.000)	(0.020, 1.000)	(0.020, 1.000)
	HUBER	(0.992, 0.000, 0.232)	(0.992, 0.000, 0.232)	(0.992, 0.000, 0.232)	(0.992, 0.000, 0.232)	(0.992, 0.000, 0.232)	(0.992, 0.000, 0.232)
		(0.054, 2.635)	(0.054, 2.644)	(0.054, 2.632)	(0.054, 2.626)	(0.054, 2.629)	(0.054, 2.638)
	BISQUARE	(0.990, 0.000, 0.231)	(0.990, 0.000, 0.232)	(0.990, 0.000, 0.231)	(0.990, 0.000, 0.231)	(0.990, 0.000, 0.231)	(0.990, 0.000, 0.231)
		(0.054, 2.629)	(0.054, 2.637)	(0.054, 2.625)	(0.054, 2.620)	(0.054, 2.624)	(0.054, 2.632)
0.01	LSE	(0.974, 0.001, 0.233)	(0.974, 0.001, 0.233)	(0.974, 0.001, 0.233)	(0.974, 0.001, 0.232)	(0.975, 0.001, 0.232)	(0.975, 0.001, 0.233)
		(0.055, 2.686)	(0.055, 2.688)	(0.055, 2.681)	(0.055, 2.676)	(0.055, 2.674)	(0.055, 2.683)
	HUBER	(1.000, 0.000, 0.143)	(1.801, 0.642, 2.583)	(3.933, 8.600, 8.467)	(6.876, 34.533, 15.296)	(10.087, 82.579, 21.778)	(13.556, 157.663, 28.021)
		(0.020, 1.000)	(7.313, 1.000)	(80.286, 1.000)	(268.494, 1.000)	(556.841, 1.000)	(942.854, 1.000)
	BISQUARE	(0.992, 0.000, 0.232)	(1.007, 0.000, 0.236)	(1.010, 0.000, 0.237)	(1.013, 0.000, 0.237)	(1.014, 0.000, 0.239)	(1.017, 0.000, 0.529)
		(0.054, 2.635)	(0.056, 0.008)	(0.056, 0.001)	(0.057, 0.000)	(0.057, 0.000)	(0.280, 0.000)
0.02	LSE	(0.990, 0.000, 0.231)	(1.005, 0.000, 0.236)	(1.007, 0.000, 0.236)	(1.009, 0.000, 0.236)	(1.009, 0.000, 0.237)	(1.010, 0.000, 0.521)
		(0.054, 2.628)	(0.056, 0.008)	(0.056, 0.001)	(0.056, 0.000)	(0.056, 0.000)	(0.272, 0.000)
	HUBER	(0.975, 0.001, 0.233)	(0.989, 0.000, 0.237)	(0.991, 0.000, 0.237)	(0.993, 0.000, 0.238)	(0.993, 0.000, 0.238)	(0.993, 0.000, 0.238)
		(0.055, 2.682)	(0.056, 0.008)	(0.056, 0.001)	(0.057, 0.000)	(0.057, 0.000)	(0.057, 0.000)
	BISQUARE	(1.000, 0.000, 0.143)	(2.601, 2.563, 3.630)	(6.763, 33.211, 11.671)	(12.389, 129.717, 20.667)	(18.690, 312.940, 29.320)	(25.007, 576.333, 36.851)
		(0.020, 1.000)	(15.738, 1.000)	(169.412, 1.000)	(556.826, 1.000)	(1172.577, 1.000)	(1934.317, 1.000)
0.03	LSE	(0.992, 0.000, 0.232)	(1.024, 0.000, 0.241)	(1.030, 0.001, 0.243)	(1.035, 0.001, 0.244)	(1.039, 0.002, 0.247)	(1.046, 0.002, 0.431)
		(0.054, 2.628)	(0.059, 0.004)	(0.060, 0.000)	(0.061, 0.000)	(0.063, 0.000)	(0.188, 0.000)
	BISQUARE	(0.990, 0.000, 0.231)	(1.021, 0.000, 0.240)	(1.025, 0.001, 0.241)	(1.027, 0.001, 0.242)	(1.029, 0.001, 0.242)	(1.030, 0.001, 0.242)
		(0.054, 2.621)	(0.058, 0.004)	(0.059, 0.000)	(0.059, 0.000)	(0.059, 0.000)	(0.060, 0.000)
	HUBER	(0.974, 0.001, 0.232)	(1.005, 0.000, 0.241)	(1.009, 0.000, 0.242)	(1.012, 0.000, 0.243)	(1.013, 0.000, 0.243)	(1.014, 0.000, 0.244)
		(0.055, 2.679)	(0.058, 0.004)	(0.059, 0.000)	(0.059, 0.000)	(0.059, 0.000)	(0.060, 0.000)
0.04	LSE	(1.000, 0.000, 0.143)	(3.390, 5.714, 4.428)	(9.544, 72.996, 13.977)	(17.676, 278.081, 24.387)	(26.634, 657.105, 33.958)	(35.611, 1197.913, 42.159)
		(0.020, 1.000)	(25.325, 1.000)	(268.365, 1.000)	(872.809, 1.000)	(1810.233, 1.000)	(2975.331, 1.000)
	HUBER	(0.992, 0.000, 0.232)	(1.040, 0.001, 0.245)	(1.051, 0.003, 0.249)	(1.059, 0.003, 0.269)	(1.068, 0.005, 0.455)	(1.083, 0.007, 1.009)
		(0.054, 2.631)	(0.062, 0.002)	(0.065, 0.000)	(0.076, 0.000)	(0.211, 0.000)	(1.024, 0.000)
	BISQUARE	(0.990, 0.000, 0.231)	(1.036, 0.001, 0.244)	(1.044, 0.002, 0.246)	(1.047, 0.002, 0.247)	(1.050, 0.002, 0.360)	(1.052, 0.003, 0.698)
		(0.054, 2.625)	(0.061, 0.002)	(0.062, 0.000)	(0.063, 0.000)	(0.132, 0.000)	(0.490, 0.000)
0.05	PROPOSED	(0.974, 0.001, 0.232)	(1.020, 0.000, 0.245)	(1.028, 0.001, 0.247)	(1.031, 0.001, 0.249)	(1.034, 0.001, 0.249)	(1.035, 0.001, 0.249)

		λ					
α	ESTIMATOR	0	1	2	3	4	5
		(0.055, 2.682)	(0.060, 0.002)	(0.062, 0.000)	(0.063, 0.000)	(0.063, 0.000)	(0.063, 0.000)
0.04		(1.000, 0.000, 0.143)	(4.163, 10.006, 5.061)	(12.214, 125.753, 15.762)	(22.818, 476.013, 27.197)	(34.120, 1096.966, 37.224)	(45.409, 1972.130, 45.515)
	LSE	(0.020, 1.000)	(35.617, 1.000)	(374.179, 1.000)	(1215.692, 1.000)	(2482.611, 1.000)	(4043.754, 1.000)
		(0.992, 0.000, 0.232)	(1.058, 0.003, 0.250)	(1.074, 0.005, 0.256)	(1.086, 0.007, 0.262)	(1.101, 0.010, 0.424)	(1.133, 0.018, 1.548)
	HUBER	(0.054, 2.635)	(0.066, 0.002)	(0.071, 0.000)	(0.076, 0.000)	(0.190, 0.000)	(2.413, 0.001)
		(0.990, 0.000, 0.232)	(1.052, 0.003, 0.248)	(1.063, 0.004, 0.251)	(1.068, 0.005, 0.253)	(1.070, 0.005, 0.253)	(1.075, 0.006, 0.915)
	BISQUARE	(0.054, 2.629)	(0.064, 0.002)	(0.067, 0.000)	(0.068, 0.000)	(0.069, 0.000)	(0.843, 0.000)
0.05		(0.975, 0.001, 0.233)	(1.036, 0.001, 0.249)	(1.047, 0.002, 0.252)	(1.052, 0.003, 0.254)	(1.055, 0.003, 0.255)	(1.057, 0.003, 0.255)
	PROPOSED	(0.055, 2.685)	(0.063, 0.002)	(0.066, 0.000)	(0.067, 0.000)	(0.068, 0.000)	(0.068, 0.000)
		(1.000, 0.000, 0.143)	(4.937, 15.498, 5.617)	(14.846, 191.723, 17.276)	(27.639, 709.627, 29.302)	(41.153, 1612.301, 39.485)	(54.340, 2845.178, 47.654)
	LSE	(0.020, 1.000)	(47.047, 1.000)	(490.190, 1.000)	(1568.213, 1.000)	(3171.378, 1.000)	(5116.060, 1.000)
		(0.992, 0.000, 0.232)	(1.076, 0.005, 0.255)	(1.098, 0.010, 0.264)	(1.115, 0.013, 0.277)	(1.140, 0.019, 0.635)	(1.212, 0.045, 2.695)
	HUBER	(0.054, 2.633)	(0.071, 0.002)	(0.079, 0.000)	(0.090, 0.000)	(0.423, 0.000)	(7.306, 0.001)
0.06		(0.990, 0.000, 0.232)	(1.068, 0.005, 0.252)	(1.082, 0.007, 0.256)	(1.088, 0.008, 0.258)	(1.092, 0.009, 0.427)	(1.102, 0.010, 1.495)
	BISQUARE	(0.054, 2.627)	(0.068, 0.001)	(0.072, 0.000)	(0.074, 0.000)	(0.191, 0.000)	(2.245, 0.000)
		(0.974, 0.001, 0.233)	(1.052, 0.003, 0.253)	(1.066, 0.004, 0.257)	(1.072, 0.005, 0.259)	(1.076, 0.006, 0.261)	(1.080, 0.006, 0.261)
	PROPOSED	(0.055, 2.678)	(0.067, 0.001)	(0.071, 0.000)	(0.072, 0.000)	(0.074, 0.000)	(0.075, 0.000)
		(1.000, 0.000, 0.143)	(5.685, 21.945, 6.082)	(17.387, 268.531, 18.541)	(32.251, 976.608, 30.933)	(47.786, 2188.934, 41.128)	(62.570, 3790.891, 48.883)
	LSE	(0.020, 1.000)	(58.936, 1.000)	(612.287, 1.000)	(1933.472, 1.000)	(3880.474, 1.000)	(6180.398, 1.000)
0.07		(0.992, 0.000, 0.232)	(1.096, 0.007, 0.262)	(1.123, 0.015, 0.273)	(1.148, 0.022, 0.297)	(1.190, 0.036, 1.159)	(1.318, 0.101, 3.732)
	HUBER	(0.054, 2.634)	(0.078, 0.001)	(0.090, 0.000)	(0.110, 0.000)	(1.380, 0.000)	(14.026, 0.002)
		(0.990, 0.000, 0.231)	(1.085, 0.007, 0.258)	(1.102, 0.010, 0.262)	(1.110, 0.012, 0.264)	(1.115, 0.013, 0.417)	(1.127, 0.016, 1.664)
	BISQUARE	(0.054, 2.628)	(0.074, 0.001)	(0.079, 0.000)	(0.082, 0.000)	(0.187, 0.000)	(2.785, 0.000)
		(0.974, 0.001, 0.233)	(1.069, 0.005, 0.259)	(1.086, 0.007, 0.263)	(1.094, 0.009, 0.266)	(1.099, 0.010, 0.267)	(1.103, 0.011, 0.268)
	PROPOSED	(0.055, 2.683)	(0.072, 0.001)	(0.077, 0.000)	(0.079, 0.000)	(0.081, 0.000)	(0.082, 0.000)
0.08		(1.000, 0.000, 0.143)	(6.422, 29.403, 6.504)	(19.929, 358.319, 19.629)	(36.803, 1281.823, 32.316)	(54.015, 2810.576, 42.278)	(70.181, 4785.947, 49.535)
	LSE	(0.020, 1.000)	(71.700, 1.000)	(743.611, 1.000)	(2326.127, 1.000)	(4598.036, 1.000)	(7239.709, 1.000)
		(0.991, 0.000, 0.232)	(1.116, 0.011, 0.268)	(1.152, 0.023, 0.284)	(1.186, 0.035, 0.342)	(1.254, 0.064, 1.754)	(1.491, 0.241, 5.297)
	HUBER	(0.054, 2.635)	(0.085, 0.001)	(0.104, 0.000)	(0.152, 0.000)	(3.140, 0.001)	(28.300, 0.004)
		(0.989, 0.000, 0.231)	(1.103, 0.011, 0.263)	(1.123, 0.015, 0.268)	(1.133, 0.018, 0.271)	(1.139, 0.019, 0.732)	(1.163, 0.026, 2.504)
	BISQUARE	(0.054, 2.628)	(0.080, 0.001)	(0.087, 0.000)	(0.091, 0.000)	(0.556, 0.000)	(6.294, 0.001)
0.08		(0.974, 0.001, 0.233)	(1.086, 0.007, 0.263)	(1.106, 0.011, 0.269)	(1.116, 0.014, 0.272)	(1.122, 0.015, 0.273)	(1.127, 0.016, 0.451)
	PROPOSED	(0.055, 2.683)	(0.077, 0.001)	(0.084, 0.000)	(0.088, 0.000)	(0.090, 0.000)	(0.219, 0.000)
0.08		(1.000, 0.000, 0.143)	(7.160, 37.947, 6.895)	(22.244, 451.296, 20.448)	(41.000, 1599.970, 33.312)	(59.856, 3464.048, 43.016)	(77.288, 5819.848, 49.813)
	LSE	(0.020, 1.000)	(85.482, 1.000)	(869.422, 1.000)	(2709.684, 1.000)	(5314.388, 1.000)	(8301.152, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.09	HUBER	(0.991, 0.000, 0.232)	(1.137, 0.015, 0.275)	(1.181, 0.033, 0.296)	(1.227, 0.051, 0.459)	(1.332, 0.110, 2.032)	(1.757, 0.572, 7.216)
		(0.054, 2.636)	(0.094, 0.001)	(0.120, 0.000)	(0.262, 0.000)	(4.240, 0.001)	(52.642, 0.006)
	BISQUARE	(0.989, 0.000, 0.232)	(1.121, 0.015, 0.268)	(1.144, 0.021, 0.274)	(1.155, 0.024, 0.277)	(1.163, 0.027, 0.613)	(1.195, 0.038, 2.905)
		(0.054, 2.630)	(0.086, 0.001)	(0.096, 0.000)	(0.101, 0.000)	(0.402, 0.000)	(8.475, 0.001)
	PROPOSED	(0.974, 0.001, 0.233)	(1.104, 0.011, 0.268)	(1.128, 0.016, 0.275)	(1.139, 0.019, 0.278)	(1.146, 0.021, 0.281)	(1.151, 0.023, 0.329)
		(0.055, 2.683)	(0.083, 0.001)	(0.092, 0.000)	(0.097, 0.000)	(0.100, 0.000)	(0.131, 0.000)
	LSE	(1.000, 0.000, 0.143)	(7.892, 47.500, 7.241)	(24.654, 559.492, 21.331)	(45.001, 1936.048, 34.091)	(65.292, 4133.423, 43.504)	(83.838, 6862.199, 49.768)
		(0.020, 1.000)	(99.937, 1.000)	(1014.489, 1.000)	(3098.215, 1.000)	(6026.002, 1.000)	(9339.050, 1.000)
	HUBER	(0.992, 0.000, 0.232)	(1.159, 0.019, 0.282)	(1.214, 0.046, 0.310)	(1.276, 0.076, 0.613)	(1.449, 0.201, 3.086)	(2.167, 1.361, 9.772)
		(0.054, 2.636)	(0.105, 0.001)	(0.141, 0.000)	(0.452, 0.000)	(9.725, 0.002)	(96.848, 0.010)
0.1	BISQUARE	(0.990, 0.000, 0.232)	(1.139, 0.019, 0.273)	(1.166, 0.028, 0.281)	(1.179, 0.032, 0.284)	(1.191, 0.037, 1.109)	(1.250, 0.063, 4.090)
		(0.054, 2.630)	(0.094, 0.001)	(0.106, 0.000)	(0.113, 0.000)	(1.266, 0.000)	(16.787, 0.002)
	PROPOSED	(0.975, 0.001, 0.233)	(1.122, 0.015, 0.273)	(1.149, 0.022, 0.281)	(1.163, 0.026, 0.285)	(1.171, 0.029, 0.288)	(1.177, 0.031, 0.377)
		(0.055, 2.682)	(0.089, 0.001)	(0.101, 0.000)	(0.108, 0.000)	(0.112, 0.000)	(0.173, 0.000)
	LSE	(1.000, 0.000, 0.143)	(8.610, 57.909, 7.580)	(26.919, 671.786, 21.988)	(48.940, 2298.198, 34.814)	(70.549, 4837.127, 43.795)	(89.949, 7911.944, 49.487)
		(0.020, 1.000)	(115.358, 1.000)	(1155.268, 1.000)	(3510.230, 1.000)	(6755.124, 1.000)	(10360.906, 1.000)
	HUBER	(0.992, 0.000, 0.232)	(1.182, 0.025, 0.290)	(1.250, 0.062, 0.327)	(1.332, 0.110, 0.761)	(1.606, 0.367, 4.076)	(2.742, 3.036, 12.440)
		(0.054, 2.638)	(0.117, 0.001)	(0.170, 0.000)	(0.690, 0.000)	(16.978, 0.003)	(157.791, 0.015)
	BISQUARE	(0.990, 0.000, 0.231)	(1.158, 0.025, 0.278)	(1.189, 0.036, 0.287)	(1.204, 0.041, 0.292)	(1.217, 0.047, 1.035)	(1.312, 0.098, 5.108)
		(0.054, 2.631)	(0.102, 0.001)	(0.118, 0.000)	(0.127, 0.000)	(1.119, 0.000)	(26.186, 0.003)
0.1	PROPOSED	(0.974, 0.001, 0.233)	(1.140, 0.020, 0.278)	(1.172, 0.030, 0.287)	(1.187, 0.035, 0.292)	(1.196, 0.039, 0.295)	(1.204, 0.042, 0.674)
		(0.055, 2.685)	(0.097, 0.001)	(0.112, 0.000)	(0.121, 0.000)	(0.125, 0.000)	(0.497, 0.000)

Table A7: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the intercept coefficient (β_0) for multiple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(12.001, 0.000, 0.318)	(12.000, 0.000, 0.320)	(12.000, 0.000, 0.319)	(12.000, 0.000, 0.319)	(11.999, 0.000, 0.319)	(12.000, 0.000, 0.319)
		(0.101, 1.000)	(0.102, 1.000)	(0.102, 1.000)	(0.102, 1.000)	(0.102, 1.000)	(0.102, 1.000)
	HUBER	(12.001, 0.000, 0.327)	(12.000, 0.000, 0.328)	(12.000, 0.000, 0.327)	(12.000, 0.000, 0.328)	(11.999, 0.000, 0.328)	(12.000, 0.000, 0.328)
		(0.107, 1.054)	(0.107, 1.055)	(0.107, 1.055)	(0.108, 1.055)	(0.107, 1.056)	(0.108, 1.055)
0	BISQUARE	(12.001, 0.000, 0.329)	(12.000, 0.000, 0.330)	(12.000, 0.000, 0.329)	(12.000, 0.000, 0.330)	(11.999, 0.000, 0.329)	(12.000, 0.000, 0.330)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.108, 1.065)	(0.108, 1.066)	(0.108, 1.065)	(0.109, 1.066)	(0.108, 1.066)	(0.109, 1.066)
0.01	PROPOSED	(12.001, 0.000, 0.323)	(12.000, 0.000, 0.324)	(12.000, 0.000, 0.323)	(12.000, 0.000, 0.324)	(11.999, 0.000, 0.323)	(12.000, 0.000, 0.323)
		(0.104, 1.026)	(0.104, 1.026)	(0.104, 1.026)	(0.105, 1.026)	(0.104, 1.027)	(0.105, 1.026)
	LSE	(12.000, 0.000, 0.319)	(12.105, 0.011, 0.402)	(12.281, 0.079, 0.611)	(12.522, 0.272, 0.895)	(12.812, 0.659, 1.235)	(13.144, 1.309, 1.617)
		(0.102, 1.000)	(0.172, 1.000)	(0.453, 1.000)	(1.077, 1.000)	(2.186, 1.000)	(3.908, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.029, 0.000, 0.335)	(12.040, 0.002, 0.336)	(12.051, 0.003, 0.337)	(12.061, 0.004, 0.339)	(12.070, 0.005, 0.341)
		(0.107, 1.054)	(0.113, 0.656)	(0.114, 0.253)	(0.117, 0.108)	(0.119, 0.054)	(0.121, 0.031)
	BISQUARE	(12.000, 0.000, 0.329)	(12.006, 0.000, 0.333)	(12.001, 0.000, 0.331)	(12.001, 0.000, 0.331)	(12.000, 0.000, 0.331)	(12.000, 0.000, 0.331)
		(0.108, 1.065)	(0.111, 0.644)	(0.109, 0.242)	(0.110, 0.102)	(0.110, 0.050)	(0.109, 0.028)
	PROPOSED	(12.000, 0.000, 0.323)	(12.008, 0.000, 0.328)	(12.001, 0.000, 0.325)	(12.001, 0.000, 0.325)	(12.000, 0.000, 0.325)	(12.000, 0.000, 0.325)
		(0.104, 1.026)	(0.107, 0.624)	(0.106, 0.233)	(0.106, 0.098)	(0.106, 0.048)	(0.105, 0.027)
0.02	LSE	(12.000, 0.000, 0.319)	(12.209, 0.044, 0.469)	(12.560, 0.313, 0.796)	(13.030, 1.061, 1.207)	(13.599, 2.557, 1.679)	(14.251, 5.067, 2.200)
		(0.102, 1.000)	(0.264, 1.000)	(0.947, 1.000)	(2.518, 1.000)	(5.375, 1.000)	(9.908, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.058, 0.000, 0.343)	(12.082, 0.007, 0.345)	(12.104, 0.011, 0.348)	(12.124, 0.015, 0.352)	(12.144, 0.021, 0.355)
		(0.107, 1.056)	(0.121, 0.459)	(0.126, 0.133)	(0.132, 0.052)	(0.139, 0.026)	(0.147, 0.015)
	BISQUARE	(12.000, 0.000, 0.329)	(12.013, 0.000, 0.337)	(12.002, 0.000, 0.332)	(12.001, 0.000, 0.332)	(12.000, 0.000, 0.332)	(12.001, 0.000, 0.332)
		(0.109, 1.066)	(0.114, 0.432)	(0.110, 0.117)	(0.110, 0.044)	(0.110, 0.021)	(0.110, 0.011)
	PROPOSED	(12.000, 0.000, 0.323)	(12.017, 0.000, 0.333)	(12.002, 0.000, 0.327)	(12.001, 0.000, 0.326)	(12.000, 0.000, 0.327)	(12.001, 0.000, 0.326)
		(0.104, 1.027)	(0.111, 0.423)	(0.107, 0.113)	(0.106, 0.042)	(0.107, 0.020)	(0.106, 0.011)
	LSE	(12.000, 0.000, 0.319)	(12.311, 0.097, 0.524)	(12.834, 0.695, 0.939)	(13.529, 2.339, 1.438)	(14.366, 5.600, 2.001)	(15.318, 11.010, 2.612)
		(0.102, 1.000)	(0.372, 1.000)	(1.578, 1.000)	(4.406, 1.000)	(9.603, 1.000)	(17.832, 1.000)
0.03	HUBER	(12.000, 0.000, 0.328)	(12.089, 0.000, 0.350)	(12.126, 0.016, 0.355)	(12.158, 0.025, 0.360)	(12.190, 0.036, 0.365)	(12.219, 0.048, 0.371)
		(0.107, 1.055)	(0.131, 0.351)	(0.142, 0.090)	(0.154, 0.035)	(0.169, 0.018)	(0.186, 0.010)
	BISQUARE	(12.000, 0.000, 0.329)	(12.021, 0.000, 0.340)	(12.003, 0.000, 0.334)	(12.000, 0.000, 0.334)	(12.001, 0.000, 0.333)	(12.000, 0.000, 0.333)
		(0.108, 1.065)	(0.116, 0.312)	(0.112, 0.071)	(0.111, 0.025)	(0.111, 0.012)	(0.111, 0.006)
	PROPOSED	(12.000, 0.000, 0.323)	(12.028, 0.001, 0.338)	(12.004, 0.000, 0.329)	(12.001, 0.000, 0.328)	(12.001, 0.000, 0.328)	(12.000, 0.000, 0.328)
		(0.104, 1.026)	(0.115, 0.309)	(0.108, 0.069)	(0.108, 0.024)	(0.107, 0.011)	(0.107, 0.006)
	LSE	(12.000, 0.000, 0.319)	(12.414, 0.171, 0.574)	(13.102, 1.215, 1.056)	(14.018, 4.072, 1.623)	(15.113, 9.694, 2.255)	(16.344, 18.874, 2.924)
		(0.102, 1.000)	(0.500, 1.000)	(2.331, 1.000)	(6.707, 1.000)	(14.780, 1.000)	(27.426, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.121, 0.001, 0.359)	(12.170, 0.029, 0.365)	(12.216, 0.047, 0.372)	(12.260, 0.067, 0.381)	(12.300, 0.090, 0.390)
		(0.107, 1.055)	(0.144, 0.287)	(0.163, 0.070)	(0.185, 0.028)	(0.213, 0.014)	(0.242, 0.009)
0.04	BISQUARE	(12.000, 0.000, 0.329)	(12.029, 0.001, 0.345)	(12.004, 0.000, 0.336)	(12.001, 0.000, 0.335)	(12.001, 0.000, 0.335)	(12.001, 0.000, 0.335)
		(0.109, 1.065)	(0.120, 0.240)	(0.113, 0.048)	(0.112, 0.017)	(0.112, 0.008)	(0.112, 0.004)
	PROPOSED	(12.000, 0.000, 0.323)	(12.039, 0.002, 0.344)	(12.005, 0.000, 0.331)	(12.002, 0.000, 0.330)	(12.001, 0.000, 0.330)	(12.001, 0.000, 0.330)
		(0.105, 1.026)	(0.120, 0.240)	(0.110, 0.047)	(0.109, 0.016)	(0.109, 0.007)	(0.109, 0.004)
	LSE	(12.000, 0.000, 0.319)	(12.414, 0.171, 0.574)	(13.102, 1.215, 1.056)	(14.018, 4.072, 1.623)	(15.113, 9.694, 2.255)	(16.344, 18.874, 2.924)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.05	LSE	(12.000, 0.000, 0.319)	(12.514, 0.264, 0.618)	(13.367, 1.868, 1.157)	(14.494, 6.222, 1.777)	(15.833, 14.694, 2.459)	(17.350, 28.625, 3.185)
		(0.102, 1.000)	(0.646, 1.000)	(3.206, 1.000)	(9.379, 1.000)	(20.741, 1.000)	(38.767, 1.000)
	HUBER	(12.000, 0.000, 0.327)	(12.154, 0.001, 0.368)	(12.218, 0.048, 0.377)	(12.276, 0.076, 0.387)	(12.331, 0.110, 0.398)	(12.384, 0.147, 0.410)
		(0.107, 1.054)	(0.159, 0.246)	(0.190, 0.059)	(0.226, 0.024)	(0.268, 0.013)	(0.316, 0.008)
	BISQUARE	(12.000, 0.000, 0.329)	(12.038, 0.001, 0.349)	(12.005, 0.000, 0.338)	(12.002, 0.000, 0.337)	(12.001, 0.000, 0.337)	(12.001, 0.000, 0.336)
0.06	LSE	(12.000, 0.000, 0.323)	(12.052, 0.003, 0.350)	(12.007, 0.000, 0.333)	(12.002, 0.000, 0.332)	(12.002, 0.000, 0.332)	(12.001, 0.000, 0.331)
		(0.104, 1.026)	(0.125, 0.194)	(0.111, 0.035)	(0.110, 0.012)	(0.110, 0.005)	(0.110, 0.003)
	HUBER	(12.000, 0.000, 0.319)	(12.613, 0.376, 0.659)	(13.629, 2.653, 1.248)	(14.964, 8.783, 1.916)	(16.542, 20.628, 2.635)	(18.298, 39.671, 3.392)
		(0.102, 1.000)	(0.810, 1.000)	(4.211, 1.000)	(12.456, 1.000)	(27.569, 1.000)	(51.179, 1.000)
	BISQUARE	(12.000, 0.000, 0.328)	(12.188, 0.002, 0.378)	(12.268, 0.072, 0.390)	(12.340, 0.116, 0.404)	(12.408, 0.166, 0.418)	(12.472, 0.223, 0.435)
0.07	LSE	(12.000, 0.000, 0.328)	(12.188, 0.002, 0.378)	(12.268, 0.072, 0.390)	(12.340, 0.116, 0.404)	(12.408, 0.166, 0.418)	(12.472, 0.223, 0.435)
		(0.108, 1.055)	(0.179, 0.220)	(0.224, 0.053)	(0.279, 0.022)	(0.341, 0.012)	(0.412, 0.008)
	HUBER	(12.000, 0.000, 0.329)	(12.048, 0.002, 0.355)	(12.007, 0.000, 0.340)	(12.003, 0.000, 0.339)	(12.001, 0.000, 0.338)	(12.001, 0.000, 0.338)
		(0.109, 1.066)	(0.128, 0.158)	(0.116, 0.027)	(0.115, 0.009)	(0.114, 0.004)	(0.114, 0.002)
	PROPOSED	(12.000, 0.000, 0.323)	(12.065, 0.004, 0.358)	(12.009, 0.000, 0.336)	(12.004, 0.000, 0.334)	(12.002, 0.000, 0.333)	(12.002, 0.000, 0.333)
0.08	LSE	(12.000, 0.000, 0.323)	(12.065, 0.004, 0.358)	(12.009, 0.000, 0.336)	(12.004, 0.000, 0.334)	(12.002, 0.000, 0.333)	(12.002, 0.000, 0.333)
		(0.105, 1.026)	(0.133, 0.164)	(0.113, 0.027)	(0.112, 0.009)	(0.111, 0.004)	(0.111, 0.002)
	HUBER	(12.000, 0.000, 0.319)	(12.713, 0.508, 0.696)	(13.882, 3.542, 1.326)	(15.420, 11.696, 2.031)	(17.230, 27.358, 2.783)	(19.232, 52.303, 3.572)
		(0.102, 1.000)	(0.992, 1.000)	(5.299, 1.000)	(15.822, 1.000)	(35.103, 1.000)	(65.061, 1.000)
	BISQUARE	(12.000, 0.000, 0.328)	(12.225, 0.004, 0.389)	(12.319, 0.102, 0.404)	(12.407, 0.166, 0.422)	(12.489, 0.239, 0.442)	(12.566, 0.321, 0.464)
0.09	LSE	(12.000, 0.000, 0.328)	(12.225, 0.004, 0.389)	(12.319, 0.102, 0.404)	(12.407, 0.166, 0.422)	(12.489, 0.239, 0.442)	(12.566, 0.321, 0.464)
		(0.108, 1.055)	(0.202, 0.204)	(0.265, 0.050)	(0.344, 0.022)	(0.435, 0.012)	(0.536, 0.008)
	HUBER	(12.000, 0.000, 0.330)	(12.060, 0.004, 0.360)	(12.008, 0.000, 0.342)	(12.003, 0.000, 0.340)	(12.002, 0.000, 0.340)	(12.001, 0.000, 0.340)
		(0.109, 1.066)	(0.133, 0.135)	(0.117, 0.022)	(0.116, 0.007)	(0.116, 0.003)	(0.115, 0.002)
	PROPOSED	(12.000, 0.000, 0.323)	(12.081, 0.007, 0.367)	(12.011, 0.000, 0.338)	(12.005, 0.000, 0.336)	(12.003, 0.000, 0.335)	(12.002, 0.000, 0.335)
0.10	LSE	(12.000, 0.000, 0.323)	(12.081, 0.007, 0.367)	(12.011, 0.000, 0.338)	(12.005, 0.000, 0.336)	(12.003, 0.000, 0.335)	(12.002, 0.000, 0.335)
		(0.105, 1.026)	(0.141, 0.142)	(0.115, 0.022)	(0.113, 0.007)	(0.112, 0.003)	(0.112, 0.002)
	HUBER	(12.000, 0.000, 0.319)	(12.811, 0.657, 0.729)	(14.137, 4.566, 1.396)	(15.865, 14.937, 2.134)	(17.893, 34.730, 2.915)	(20.134, 66.160, 3.715)
		(0.102, 1.000)	(1.189, 1.000)	(6.515, 1.000)	(19.491, 1.000)	(43.225, 1.000)	(79.958, 1.000)
	BISQUARE	(12.000, 0.000, 0.327)	(12.263, 0.005, 0.402)	(12.374, 0.140, 0.421)	(12.478, 0.228, 0.443)	(12.575, 0.330, 0.470)	(12.669, 0.447, 0.501)
0.11	LSE	(12.000, 0.000, 0.327)	(12.263, 0.005, 0.402)	(12.374, 0.140, 0.421)	(12.478, 0.228, 0.443)	(12.575, 0.330, 0.470)	(12.669, 0.447, 0.501)
		(0.107, 1.055)	(0.231, 0.194)	(0.317, 0.049)	(0.425, 0.022)	(0.551, 0.013)	(0.698, 0.009)
	HUBER	(12.000, 0.000, 0.329)	(12.074, 0.005, 0.367)	(12.010, 0.000, 0.344)	(12.004, 0.000, 0.342)	(12.002, 0.000, 0.342)	(12.002, 0.000, 0.341)
		(0.108, 1.065)	(0.140, 0.118)	(0.119, 0.018)	(0.117, 0.006)	(0.117, 0.003)	(0.117, 0.001)
	PROPOSED	(12.000, 0.000, 0.323)	(12.100, 0.010, 0.377)	(12.014, 0.000, 0.341)	(12.006, 0.000, 0.338)	(12.003, 0.000, 0.338)	(12.002, 0.000, 0.338)
0.12	LSE	(12.000, 0.000, 0.323)	(12.100, 0.010, 0.377)	(12.014, 0.000, 0.341)	(12.006, 0.000, 0.338)	(12.003, 0.000, 0.338)	(12.002, 0.000, 0.338)
		(0.104, 1.026)	(0.152, 0.128)	(0.117, 0.018)	(0.114, 0.006)	(0.114, 0.003)	(0.114, 0.001)
	HUBER	(12.000, 0.000, 0.319)	(12.908, 0.824, 0.761)	(14.385, 5.686, 1.463)	(16.299, 18.483, 2.227)	(18.541, 42.783, 3.025)	(21.006, 81.100, 3.850)
0.13	LSE	(12.000, 0.000, 0.319)	(12.908, 0.824, 0.761)	(14.385, 5.686, 1.463)	(16.299, 18.483, 2.227)	(18.541, 42.783, 3.025)	(21.006, 81.100, 3.850)
		(0.102, 1.000)	(1.404, 1.000)	(7.826, 1.000)	(23.442, 1.000)	(51.935, 1.000)	(95.921, 1.000)
0.14	HUBER	(12.000, 0.000, 0.328)	(12.303, 0.008, 0.415)	(12.433, 0.188, 0.439)	(12.552, 0.305, 0.469)	(12.668, 0.446, 0.505)	(12.780, 0.609, 0.551)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.1		(0.107, 1.054)	(0.264, 0.188)	(0.381, 0.049)	(0.525, 0.022)	(0.701, 0.013)	(0.912, 0.010)
		(12.000, 0.000, 0.329)	(12.088, 0.008, 0.375)	(12.013, 0.000, 0.346)	(12.005, 0.000, 0.344)	(12.003, 0.000, 0.343)	(12.003, 0.000, 0.343)
	BISQUARE	(0.108, 1.065)	(0.148, 0.105)	(0.120, 0.015)	(0.118, 0.005)	(0.118, 0.002)	(0.118, 0.001)
		(12.000, 0.000, 0.323)	(12.120, 0.014, 0.389)	(12.018, 0.000, 0.344)	(12.007, 0.000, 0.340)	(12.004, 0.000, 0.339)	(12.003, 0.000, 0.341)
	PROPOSED	(0.104, 1.026)	(0.165, 0.118)	(0.119, 0.015)	(0.116, 0.005)	(0.115, 0.002)	(0.116, 0.001)
		(12.000, 0.000, 0.319)	(13.005, 1.009, 0.789)	(14.625, 6.893, 1.519)	(16.731, 22.386, 2.311)	(19.174, 51.466, 3.125)	(21.852, 97.062, 3.960)
	LSE	(0.102, 1.000)	(1.632, 1.000)	(9.201, 1.000)	(27.726, 1.000)	(61.231, 1.000)	(112.746, 1.000)
		(12.000, 0.000, 0.328)	(12.346, 0.011, 0.429)	(12.494, 0.244, 0.461)	(12.634, 0.402, 0.501)	(12.770, 0.593, 0.551)	(12.902, 0.814, 0.617)
	HUBER	(0.108, 1.055)	(0.303, 0.186)	(0.457, 0.050)	(0.653, 0.024)	(0.896, 0.015)	(1.195, 0.011)
		(12.000, 0.000, 0.329)	(12.104, 0.011, 0.383)	(12.014, 0.000, 0.349)	(12.006, 0.000, 0.346)	(12.004, 0.000, 0.345)	(12.003, 0.000, 0.345)
0.1	BISQUARE	(0.109, 1.065)	(0.158, 0.097)	(0.122, 0.013)	(0.120, 0.004)	(0.119, 0.002)	(0.119, 0.001)
		(12.000, 0.000, 0.323)	(12.143, 0.021, 0.402)	(12.020, 0.000, 0.348)	(12.008, 0.000, 0.343)	(12.005, 0.000, 0.343)	(12.004, 0.000, 0.344)
	PROPOSED	(0.105, 1.026)	(0.182, 0.112)	(0.121, 0.013)	(0.118, 0.004)	(0.118, 0.002)	(0.119, 0.001)

Table A8: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_1) for multiple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0		(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)
	LSE	(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)
		(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
	HUBER	(0.002, 1.054)	(0.002, 1.055)	(0.002, 1.056)	(0.002, 1.055)	(0.002, 1.055)	(0.002, 1.055)
		(4.300, 0.000, 0.040)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
	BISQUARE	(0.002, 1.064)	(0.002, 1.066)	(0.002, 1.066)	(0.002, 1.065)	(0.002, 1.065)	(0.002, 1.065)
		(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
	PROPOSED	(0.002, 1.025)	(0.002, 1.026)	(0.002, 1.026)	(0.002, 1.026)	(0.002, 1.026)	(0.002, 1.026)
		(4.300, 0.000, 0.039)	(4.296, 0.000, 0.048)	(4.284, 0.000, 0.068)	(4.266, 0.001, 0.092)	(4.242, 0.003, 0.120)	(4.213, 0.008, 0.151)
	LSE	(0.002, 1.000)	(0.002, 1.000)	(0.005, 1.000)	(0.010, 1.000)	(0.018, 1.000)	(0.030, 1.000)
0.01		(4.300, 0.000, 0.040)	(4.299, 0.000, 0.041)	(4.298, 0.000, 0.041)	(4.296, 0.000, 0.041)	(4.295, 0.000, 0.041)	(4.294, 0.000, 0.041)
	HUBER	(0.002, 1.055)	(0.002, 0.726)	(0.002, 0.351)	(0.002, 0.176)	(0.002, 0.096)	(0.002, 0.057)
		(4.300, 0.000, 0.040)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
	BISQUARE	(0.002, 1.066)	(0.002, 0.719)	(0.002, 0.342)	(0.002, 0.171)	(0.002, 0.093)	(0.002, 0.055)

		λ					
α	ESTIMATOR	0	1	2	3	4	5
0.02	PROPOSED	(4.300, 0.000, 0.040)	(4.299, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
		(0.002, 1.026)	(0.002, 0.697)	(0.002, 0.330)	(0.002, 0.165)	(0.002, 0.089)	(0.002, 0.053)
	LSE	(4.300, 0.000, 0.039)	(4.292, 0.000, 0.055)	(4.269, 0.001, 0.087)	(4.233, 0.004, 0.123)	(4.186, 0.013, 0.163)	(4.129, 0.029, 0.205)
		(0.002, 1.000)	(0.003, 1.000)	(0.008, 1.000)	(0.020, 1.000)	(0.040, 1.000)	(0.071, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.297, 0.000, 0.042)	(4.295, 0.000, 0.042)	(4.293, 0.000, 0.042)	(4.291, 0.000, 0.042)	(4.289, 0.000, 0.042)
		(0.002, 1.056)	(0.002, 0.562)	(0.002, 0.210)	(0.002, 0.093)	(0.002, 0.047)	(0.002, 0.027)
	BISQUARE	(4.300, 0.000, 0.040)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.066)	(0.002, 0.544)	(0.002, 0.197)	(0.002, 0.085)	(0.002, 0.042)	(0.002, 0.023)
	PROPOSED	(4.300, 0.000, 0.040)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
		(0.002, 1.027)	(0.002, 0.531)	(0.002, 0.190)	(0.002, 0.082)	(0.002, 0.041)	(0.002, 0.023)
0.03	LSE	(4.300, 0.000, 0.039)	(4.288, 0.000, 0.062)	(4.254, 0.002, 0.102)	(4.201, 0.010, 0.147)	(4.131, 0.028, 0.195)	(4.048, 0.063, 0.245)
		(0.002, 1.000)	(0.004, 1.000)	(0.013, 1.000)	(0.031, 1.000)	(0.066, 1.000)	(0.123, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.296, 0.000, 0.043)	(4.292, 0.000, 0.043)	(4.289, 0.000, 0.043)	(4.286, 0.000, 0.043)	(4.283, 0.000, 0.044)
		(0.002, 1.055)	(0.002, 0.464)	(0.002, 0.151)	(0.002, 0.063)	(0.002, 0.032)	(0.002, 0.018)
	BISQUARE	(4.300, 0.000, 0.040)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.066)	(0.002, 0.438)	(0.002, 0.134)	(0.002, 0.054)	(0.002, 0.025)	(0.002, 0.014)
	PROPOSED	(4.300, 0.000, 0.040)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
		(0.002, 1.026)	(0.002, 0.431)	(0.002, 0.130)	(0.002, 0.052)	(0.002, 0.024)	(0.002, 0.013)
	LSE	(4.300, 0.000, 0.039)	(4.284, 0.000, 0.067)	(4.239, 0.004, 0.115)	(4.169, 0.017, 0.166)	(4.079, 0.049, 0.220)	(3.971, 0.108, 0.276)
		(0.002, 1.000)	(0.005, 1.000)	(0.017, 1.000)	(0.045, 1.000)	(0.097, 1.000)	(0.184, 1.000)
0.04	HUBER	(4.300, 0.000, 0.040)	(4.295, 0.000, 0.044)	(4.290, 0.000, 0.044)	(4.285, 0.000, 0.044)	(4.280, 0.000, 0.045)	(4.276, 0.001, 0.045)
		(0.002, 1.055)	(0.002, 0.403)	(0.002, 0.121)	(0.002, 0.049)	(0.002, 0.024)	(0.003, 0.014)
	BISQUARE	(4.300, 0.000, 0.040)	(4.298, 0.000, 0.042)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.065)	(0.002, 0.371)	(0.002, 0.101)	(0.002, 0.038)	(0.002, 0.017)	(0.002, 0.009)
	PROPOSED	(4.300, 0.000, 0.040)	(4.298, 0.000, 0.042)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.040)
		(0.002, 1.026)	(0.002, 0.368)	(0.002, 0.098)	(0.002, 0.037)	(0.002, 0.017)	(0.002, 0.009)
	LSE	(4.300, 0.000, 0.039)	(4.281, 0.000, 0.072)	(4.225, 0.006, 0.126)	(4.140, 0.026, 0.182)	(4.029, 0.074, 0.241)	(3.896, 0.163, 0.302)
		(0.002, 1.000)	(0.006, 1.000)	(0.022, 1.000)	(0.059, 1.000)	(0.132, 1.000)	(0.254, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.293, 0.000, 0.045)	(4.287, 0.000, 0.045)	(4.281, 0.000, 0.045)	(4.275, 0.001, 0.046)	(4.269, 0.001, 0.047)
		(0.002, 1.055)	(0.002, 0.363)	(0.002, 0.102)	(0.002, 0.041)	(0.003, 0.021)	(0.003, 0.012)
0.05	BISQUARE	(4.300, 0.000, 0.041)	(4.298, 0.000, 0.043)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.065)	(0.002, 0.326)	(0.002, 0.080)	(0.002, 0.029)	(0.002, 0.013)	(0.002, 0.007)
	PROPOSED	(4.300, 0.000, 0.040)	(4.297, 0.000, 0.043)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.026)	0.002, 0.326)	(0.002, 0.078)	(0.002, 0.028)	(0.002, 0.013)	(0.002, 0.006)
	LSE	(4.300, 0.000, 0.039)	(4.277, 0.001, 0.077)	(4.211, 0.008, 0.136)	(4.110, 0.036, 0.198)	(3.979, 0.103, 0.260)	(3.825, 0.225, 0.324)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.07		(0.002, 1.000)	(0.006, 1.000)	(0.026, 1.000)	(0.075, 1.000)	(0.171, 1.000)	(0.330, 1.000)
	HUBER	(4.300, 0.000, 0.040) (0.002, 1.055)	(4.292, 0.000, 0.046) (0.002, 0.333)	(4.284, 0.000, 0.046) (0.002, 0.091)	(4.276, 0.001, 0.047) (0.003, 0.037)	(4.269, 0.001, 0.048) (0.003, 0.019)	(4.262, 0.001, 0.049) (0.004, 0.011)
	BISQUARE	(4.300, 0.000, 0.041) (0.002, 1.065)	(4.297, 0.000, 0.043) (0.002, 0.290)	(4.299, 0.000, 0.042) (0.002, 0.066)	(4.300, 0.000, 0.042) (0.002, 0.023)	(4.300, 0.000, 0.042) (0.002, 0.010)	(4.300, 0.000, 0.042) (0.002, 0.005)
	PROPOSED	(4.300, 0.000, 0.040) (0.002, 1.026)	(4.297, 0.000, 0.043) (0.002, 0.293)	(4.299, 0.000, 0.041) (0.002, 0.065)	(4.300, 0.000, 0.041) (0.002, 0.022)	(4.300, 0.000, 0.041) (0.002, 0.010)	(4.300, 0.000, 0.041) (0.002, 0.005)
	LSE	(4.300, 0.000, 0.039) (0.002, 1.000)	(4.273, 0.001, 0.081) (0.007, 1.000)	(4.198, 0.010, 0.145) (0.031, 1.000)	(4.082, 0.048, 0.211) (0.092, 1.000)	(3.932, 0.136, 0.276) (0.212, 1.000)	(3.757, 0.295, 0.343) (0.413, 1.000)
	HUBER	(4.300, 0.000, 0.040) (0.002, 1.056)	(4.290, 0.000, 0.047) (0.002, 0.312)	(4.281, 0.000, 0.048) (0.003, 0.084)	(4.272, 0.001, 0.048) (0.003, 0.034)	(4.263, 0.001, 0.050) (0.004, 0.018)	(4.255, 0.002, 0.051) (0.005, 0.011)
	BISQUARE	(4.300, 0.000, 0.040) (0.002, 1.066)	(4.297, 0.000, 0.044) (0.002, 0.264)	(4.299, 0.000, 0.042) (0.002, 0.056)	(4.300, 0.000, 0.042) (0.002, 0.019)	(4.300, 0.000, 0.042) (0.002, 0.008)	(4.300, 0.000, 0.042) (0.002, 0.004)
	PROPOSED	(4.300, 0.000, 0.040) (0.002, 1.027)	(4.296, 0.000, 0.044) (0.002, 0.270)	(4.299, 0.000, 0.042) (0.002, 0.055)	(4.299, 0.000, 0.041) (0.002, 0.018)	(4.300, 0.000, 0.041) (0.002, 0.008)	(4.300, 0.000, 0.041) (0.002, 0.004)
	LSE	(4.300, 0.000, 0.039) (0.002, 1.000)	(4.270, 0.001, 0.086) (0.008, 1.000)	(4.185, 0.013, 0.153) (0.037, 1.000)	(4.054, 0.060, 0.222) (0.110, 1.000)	(3.886, 0.171, 0.291) (0.256, 1.000)	(3.689, 0.373, 0.359) (0.502, 1.000)
	HUBER	(4.300, 0.000, 0.040) (0.002, 1.055)	(4.289, 0.000, 0.048) (0.002, 0.298)	(4.277, 0.001, 0.049) (0.003, 0.080)	(4.267, 0.001, 0.050) (0.004, 0.033)	(4.256, 0.002, 0.052) (0.005, 0.018)	(4.247, 0.003, 0.054) (0.006, 0.011)
0.08	BISQUARE	(4.300, 0.000, 0.040) (0.002, 1.065)	(4.296, 0.000, 0.045) (0.002, 0.243)	(4.299, 0.000, 0.042) (0.002, 0.049)	(4.299, 0.000, 0.042) (0.002, 0.016)	(4.300, 0.000, 0.042) (0.002, 0.007)	(4.300, 0.000, 0.042) (0.002, 0.004)
	PROPOSED	(4.300, 0.000, 0.040) (0.002, 1.026)	(4.295, 0.000, 0.045) (0.002, 0.253)	(4.298, 0.000, 0.042) (0.002, 0.048)	(4.299, 0.000, 0.042) (0.002, 0.016)	(4.300, 0.000, 0.041) (0.002, 0.007)	(4.300, 0.000, 0.041) (0.002, 0.003)
	LSE	(4.300, 0.000, 0.039) (0.002, 1.000)	(4.266, 0.001, 0.089) (0.009, 1.000)	(4.173, 0.016, 0.161) (0.042, 1.000)	(4.028, 0.074, 0.233) (0.128, 1.000)	(3.842, 0.210, 0.304) (0.302, 1.000)	(3.626, 0.455, 0.374) (0.595, 1.000)
	HUBER	(4.300, 0.000, 0.040) (0.002, 1.055)	(4.287, 0.000, 0.049) (0.003, 0.287)	(4.274, 0.001, 0.051) (0.003, 0.077)	(4.261, 0.002, 0.052) (0.004, 0.033)	(4.249, 0.003, 0.055) (0.006, 0.018)	(4.238, 0.004, 0.058) (0.007, 0.012)
	BISQUARE	(4.300, 0.000, 0.040) (0.002, 1.065)	(4.295, 0.000, 0.045) (0.002, 0.227)	(4.298, 0.000, 0.043) (0.002, 0.043)	(4.299, 0.000, 0.042) (0.002, 0.014)	(4.300, 0.000, 0.042) (0.002, 0.006)	(4.300, 0.000, 0.042) (0.002, 0.003)
0.09	PROPOSED	(4.300, 0.000, 0.040) (0.002, 1.026)	(4.294, 0.000, 0.046) (0.002, 0.240)	(4.298, 0.000, 0.042) (0.002, 0.042)	(4.299, 0.000, 0.042) (0.002, 0.014)	(4.300, 0.000, 0.042) (0.002, 0.006)	(4.300, 0.000, 0.042) (0.002, 0.003)
	LSE	(4.300, 0.000, 0.039) (0.002, 1.000)	(4.263, 0.001, 0.093) (0.010, 1.000)	(4.160, 0.020, 0.168) (0.048, 1.000)	(4.002, 0.089, 0.243) (0.148, 1.000)	(3.799, 0.251, 0.316) (0.350, 1.000)	(3.564, 0.541, 0.388) (0.692, 1.000)
	HUBER	(4.300, 0.000, 0.040) (0.002, 1.055)	(4.285, 0.000, 0.051) (0.003, 0.283)	(4.270, 0.001, 0.053) (0.004, 0.077)	(4.255, 0.002, 0.055) (0.005, 0.034)	(4.241, 0.003, 0.058) (0.007, 0.019)	(4.228, 0.005, 0.063) (0.009, 0.013)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(4.300, 0.000, 0.040)	(4.295, 0.000, 0.046)	(4.298, 0.000, 0.043)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)
	BISQUARE	(0.002, 1.065)	(0.002, 0.217)	(0.002, 0.038)	(0.002, 0.012)	(0.002, 0.005)	(0.002, 0.003)
		(4.300, 0.000, 0.040)	(4.293, 0.000, 0.048)	(4.298, 0.000, 0.043)	(4.299, 0.000, 0.042)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.042)
	PROPOSED	(0.002, 1.026)	(0.002, 0.234)	(0.002, 0.038)	(0.002, 0.012)	(0.002, 0.005)	(0.002, 0.003)

Table A9: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_2) for multiple linear regression model before optimization of the proposed estimator parameters.

		λ					
α	ESTIMATOR	0	1	2	3	4	5
0		(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)
	LSE	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)
		(2.500, 0.000, 0.054)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.055)
	HUBER	(0.003, 1.052)	(0.003, 1.052)	(0.003, 1.052)	(0.003, 1.051)	(0.003, 1.052)	(0.003, 1.051)
		(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
	BISQUARE	(0.003, 1.065)	(0.003, 1.065)	(0.003, 1.065)	(0.003, 1.064)	(0.003, 1.065)	(0.003, 1.064)
0.01		(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)
	PROPOSED	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.025)
		(2.500, 0.000, 0.053)	(2.496, 0.000, 0.065)	(2.484, 0.000, 0.092)	(2.467, 0.001, 0.124)	(2.445, 0.003, 0.156)	(2.422, 0.006, 0.190)
	LSE	(0.003, 1.000)	(0.004, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.028, 1.000)	(0.042, 1.000)
		(2.500, 0.000, 0.054)	(2.499, 0.000, 0.056)	(2.498, 0.000, 0.056)	(2.497, 0.000, 0.056)	(2.496, 0.000, 0.056)	(2.495, 0.000, 0.056)
	HUBER	(0.003, 1.052)	(0.003, 0.723)	(0.003, 1.052)	(0.003, 1.052)	(0.003, 0.113)	(0.003, 0.074)
0.02		(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
	BISQUARE	(0.003, 1.065)	(0.003, 0.717)	(0.003, 1.065)	(0.003, 1.065)	(0.003, 0.110)	(0.003, 0.072)
		(2.500, 0.000, 0.054)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)
	PROPOSED	(0.003, 1.026)	(0.003, 0.695)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 0.106)	(0.003, 0.070)
		(2.500, 0.000, 0.053)	(2.491, 0.000, 0.075)	(2.468, 0.001, 0.119)	(2.434, 0.004, 0.166)	(2.393, 0.011, 0.214)	(2.346, 0.024, 0.260)
	LSE	(0.003, 1.000)	(0.006, 1.000)	(0.015, 1.000)	(0.032, 1.000)	(0.057, 1.000)	(0.091, 1.000)
0.02		(2.500, 0.000, 0.054)	(2.498, 0.000, 0.057)	(2.495, 0.000, 0.057)	(2.493, 0.000, 0.057)	(2.492, 0.000, 0.057)	(2.490, 0.000, 0.057)
	HUBER	(0.003, 1.052)	(0.003, 0.560)	(0.003, 0.215)	(0.003, 0.103)	(0.003, 0.058)	(0.003, 0.036)
		(2.500, 0.000, 0.055)	(2.499, 0.000, 0.056)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
	BISQUARE	(0.003, 1.066)	(0.003, 0.543)	(0.003, 0.203)	(0.003, 0.096)	(0.003, 0.054)	(0.003, 0.033)
	PROPOSED	(2.500, 0.000, 0.054)	(2.499, 0.000, 0.055)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.003, 1.027)	(0.003, 0.530)	(0.003, 0.196)	(0.003, 0.093)	(0.003, 0.052)	(0.003, 0.032)
0.03	LSE	(2.500, 0.000, 0.053)	(2.487, 0.000, 0.084)	(2.453, 0.002, 0.140)	(2.403, 0.009, 0.199)	(2.342, 0.025, 0.257)	(2.468, 0.001, 0.118)
		(0.003, 1.000)	(0.007, 1.000)	(0.022, 1.000)	(0.049, 1.000)	(0.091, 1.000)	(0.015, 1.000)
	HUBER	(2.500, 0.000, 0.054)	(2.497, 0.000, 0.058)	(2.493, 0.000, 0.058)	(2.490, 0.000, 0.058)	(2.487, 0.000, 0.058)	(2.495, 0.000, 0.057)
		(0.003, 1.052)	(0.003, 0.466)	(0.003, 0.158)	(0.003, 0.071)	(0.004, 0.039)	(0.003, 0.216)
	BISQUARE	(2.500, 0.000, 0.055)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.055)
		(0.003, 1.065)	(0.003, 0.442)	(0.003, 0.142)	(0.003, 0.063)	(0.003, 0.034)	(0.003, 0.203)
0.04	LSE	(2.500, 0.000, 0.054)	(2.499, 0.000, 0.056)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.054)
		(0.003, 1.026)	(0.003, 0.435)	(0.003, 0.138)	(0.003, 0.061)	(0.003, 0.033)	(0.003, 0.196)
	HUBER	(2.500, 0.000, 0.053)	(2.483, 0.000, 0.092)	(2.438, 0.004, 0.158)	(2.373, 0.016, 0.226)	(2.293, 0.043, 0.294)	(2.205, 0.087, 0.358)
		(0.003, 1.000)	(0.009, 1.000)	(0.029, 1.000)	(0.067, 1.000)	(0.129, 1.000)	(0.215, 1.000)
	BISQUARE	(2.500, 0.000, 0.054)	(2.495, 0.000, 0.059)	(2.491, 0.000, 0.060)	(2.486, 0.000, 0.060)	(2.483, 0.000, 0.060)	(2.480, 0.000, 0.060)
		(0.003, 1.052)	(0.004, 0.407)	(0.004, 0.127)	(0.004, 0.127)	(0.004, 0.030)	(0.004, 0.019)
0.05	LSE	(2.500, 0.000, 0.055)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
		(0.003, 1.066)	(0.003, 0.377)	(0.003, 0.377)	(0.003, 0.109)	(0.003, 0.024)	(0.003, 0.015)
	HUBER	(2.500, 0.000, 0.054)	(2.498, 0.000, 0.057)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
		(0.003, 1.026)	(0.003, 0.375)	(0.003, 0.375)	(0.003, 0.106)	(0.003, 0.023)	(0.003, 0.014)
	BISQUARE	(2.500, 0.000, 0.053)	(2.479, 0.000, 0.099)	(2.424, 0.006, 0.174)	(2.343, 0.025, 0.250)	(2.246, 0.065, 0.324)	(2.137, 0.132, 0.396)
		(0.003, 1.000)	(0.010, 1.000)	(0.036, 1.000)	(0.087, 1.000)	(0.170, 1.000)	(0.288, 1.000)
0.06	LSE	(2.500, 0.000, 0.054)	(2.494, 0.000, 0.061)	(2.488, 0.000, 0.061)	(2.483, 0.000, 0.061)	(2.478, 0.000, 0.061)	(2.474, 0.001, 0.061)
		(0.003, 1.053)	(0.004, 0.365)	(0.004, 0.108)	(0.004, 0.046)	(0.004, 0.025)	(0.004, 0.025)
	HUBER	(2.500, 0.000, 0.055)	(2.498, 0.000, 0.058)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
		(0.003, 1.066)	(0.003, 0.330)	(0.003, 0.088)	(0.003, 0.036)	(0.003, 0.018)	(0.003, 0.011)
	BISQUARE	(2.500, 0.000, 0.054)	(2.498, 0.000, 0.058)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
		(0.003, 1.027)	(0.003, 0.331)	(0.003, 0.085)	(0.003, 0.035)	(0.003, 0.018)	(0.003, 0.011)
0.07	LSE	(2.500, 0.000, 0.053)	(2.475, 0.001, 0.106)	(2.409, 0.008, 0.188)	(2.314, 0.035, 0.272)	(2.199, 0.091, 0.352)	(2.073, 0.182, 0.430)
		(0.003, 1.000)	(0.012, 1.000)	(0.044, 1.000)	(0.108, 1.000)	(0.215, 1.000)	(0.367, 1.000)
	HUBER	(2.500, 0.000, 0.054)	(2.493, 0.000, 0.062)	(2.485, 0.000, 0.063)	(2.479, 0.000, 0.063)	(2.473, 0.001, 0.063)	(2.468, 0.001, 0.063)
		(0.003, 1.052)	(0.004, 0.335)	(0.004, 0.095)	(0.004, 0.041)	(0.005, 0.022)	(0.005, 0.014)
	BISQUARE	(2.500, 0.000, 0.055)	(2.498, 0.000, 0.059)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
		(0.003, 1.066)	(0.003, 0.295)	(0.003, 0.073)	(0.003, 0.029)	(0.003, 0.015)	(0.003, 0.009)
0.07	LSE	(2.500, 0.000, 0.054)	(2.497, 0.000, 0.059)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.056)
		(0.003, 1.026)	(0.004, 0.299)	(0.003, 0.071)	(0.003, 0.028)	(0.003, 0.014)	(0.003, 0.008)
0.07	LSE	(2.500, 0.000, 0.053)	(2.471, 0.001, 0.112)	(2.396, 0.011, 0.201)	(2.286, 0.046, 0.291)	(2.155, 0.119, 0.378)	(2.011, 0.239, 0.462)
		(0.003, 1.000)	(0.013, 1.000)	(0.051, 1.000)	(0.131, 1.000)	(0.262, 1.000)	(0.452, 1.000)

		λ					
α	ESTIMATOR	0	1	2	3	4	5
		(2.500, 0.000, 0.054)	(2.491, 0.000, 0.064)	(2.482, 0.000, 0.064)	(2.474, 0.001, 0.065)	(2.468, 0.001, 0.065)	(2.462, 0.001, 0.066)
	HUBER	(0.003, 1.052)	(0.004, 0.314)	(0.004, 0.087)	(0.005, 0.037)	(0.005, 0.020)	(0.006, 0.013)
	BISQUARE	(2.500, 0.000, 0.055)	(2.498, 0.000, 0.060)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.057)
		(0.003, 1.065)	(0.004, 0.270)	(0.003, 0.063)	(0.003, 0.025)	(0.003, 0.012)	(0.003, 0.007)
	PROPOSED	(2.500, 0.000, 0.054)	(2.497, 0.000, 0.061)	(2.499, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
		(0.003, 1.026)	(0.004, 0.277)	(0.003, 0.061)	(0.003, 0.024)	(0.003, 0.012)	(0.003, 0.007)
0.08	LSE	(2.500, 0.000, 0.053)	(2.468, 0.001, 0.117)	(2.382, 0.014, 0.213)	(2.259, 0.058, 0.309)	(2.111, 0.151, 0.400)	(1.950, 0.302, 0.489)
		(0.003, 1.000)	(0.015, 1.000)	(0.059, 1.000)	(0.153, 1.000)	(0.312, 1.000)	(0.541, 1.000)
	HUBER	(2.500, 0.000, 0.055)	(2.490, 0.000, 0.066)	(2.479, 0.000, 0.066)	(2.470, 0.001, 0.067)	(2.462, 0.001, 0.067)	(2.455, 0.002, 0.068)
		(0.003, 1.052)	(0.004, 0.300)	(0.005, 0.081)	(0.005, 0.035)	(0.006, 0.019)	(0.007, 0.012)
	BISQUARE	(2.500, 0.000, 0.055)	(2.497, 0.000, 0.061)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
		(0.003, 1.065)	(0.004, 0.251)	(0.003, 0.055)	(0.003, 0.021)	(0.003, 0.010)	(0.003, 0.006)
	PROPOSED	(2.500, 0.000, 0.054)	(2.496, 0.000, 0.062)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
		(0.003, 1.026)	(0.004, 0.262)	(0.003, 0.054)	(0.003, 0.021)	(0.003, 0.010)	(0.003, 0.006)
0.09	LSE	(2.500, 0.000, 0.053)	(2.464, 0.001, 0.122)	(2.369, 0.017, 0.224)	(2.232, 0.072, 0.325)	(2.070, 0.185, 0.422)	(1.893, 0.369, 0.515)
		(0.003, 1.000)	(0.016, 1.000)	(0.067, 1.000)	(0.177, 1.000)	(0.363, 1.000)	(0.634, 1.000)
	HUBER	(2.500, 0.000, 0.054)	(2.488, 0.000, 0.068)	(2.476, 0.001, 0.068)	(2.465, 0.001, 0.069)	(2.456, 0.002, 0.070)	(2.447, 0.003, 0.072)
		(0.003, 1.051)	(0.005, 0.289)	(0.005, 0.078)	(0.006, 0.034)	(0.007, 0.019)	(0.008, 0.013)
	BISQUARE	(2.500, 0.000, 0.055)	(2.496, 0.000, 0.062)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
		(0.003, 1.064)	(0.004, 0.236)	(0.003, 0.049)	(0.003, 0.018)	(0.003, 0.009)	(0.003, 0.005)
	PROPOSED	(2.500, 0.000, 0.054)	(2.495, 0.000, 0.064)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
		(0.003, 1.026)	(0.004, 0.250)	(0.003, 0.048)	(0.003, 0.018)	(0.003, 0.009)	(0.003, 0.005)
0.1	LSE	(2.500, 0.000, 0.053)	(2.461, 0.002, 0.127)	(2.357, 0.021, 0.234)	(2.207, 0.086, 0.341)	(2.030, 0.221, 0.442)	(1.836, 0.441, 0.538)
		(0.003, 1.000)	(0.018, 1.000)	(0.075, 1.000)	(0.202, 1.000)	(0.417, 1.000)	(0.730, 1.000)
	HUBER	(2.500, 0.000, 0.054)	(2.487, 0.000, 0.070)	(2.473, 0.001, 0.071)	(2.460, 0.002, 0.072)	(2.449, 0.003, 0.074)	(2.439, 0.004, 0.076)
		(0.003, 1.052)	(0.005, 0.283)	(0.006, 0.076)	(0.007, 0.033)	(0.008, 0.019)	(0.010, 0.013)
	BISQUARE	(2.500, 0.000, 0.055)	(2.496, 0.000, 0.063)	(2.499, 0.000, 0.058)	(2.500, 0.000, 0.058)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
		(0.003, 1.065)	(0.004, 0.225)	(0.003, 0.044)	(0.003, 0.016)	(0.003, 0.008)	(0.003, 0.005)
	PROPOSED	(2.500, 0.000, 0.054)	(2.494, 0.000, 0.065)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
		(0.003, 1.026)	(0.004, 0.243)	(0.003, 0.044)	(0.003, 0.016)	(0.003, 0.008)	(0.003, 0.004)

Table A10: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_3) for multiple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)
		(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
		(0.003, 1.052)	(0.003, 1.053)	(0.003, 1.053)	(0.003, 1.052)	(0.003, 1.053)	(0.003, 1.052)
	BISQUARE	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
0.01	LSE	(0.003, 1.065)	(0.003, 1.066)	(0.003, 1.065)	(0.003, 1.064)	(0.003, 1.065)	(0.003, 1.064)
		(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)
	HUBER	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)
		(0.800, 0.000, 0.054)	(0.797, 0.000, 0.067)	(0.788, 0.000, 0.094)	(0.774, 0.001, 0.126)	(0.758, 0.002, 0.160)	(0.738, 0.004, 0.193)
	BISQUARE	(0.003, 1.000)	(0.004, 1.000)	(0.009, 1.000)	(0.017, 1.000)	(0.027, 1.000)	(0.041, 1.000)
0.02	LSE	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.057)	(0.798, 0.000, 0.057)	(0.798, 0.000, 0.057)	(0.797, 0.000, 0.057)	(0.796, 0.000, 0.057)
		(0.003, 1.052)	(0.003, 0.726)	(0.003, 0.365)	(0.003, 0.195)	(0.003, 0.118)	(0.003, 0.079)
	HUBER	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
		(0.003, 1.065)	(0.003, 0.720)	(0.003, 0.358)	(0.003, 0.191)	(0.003, 0.115)	(0.003, 0.077)
	PROPOSED	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.055)
0.03	LSE	(0.003, 1.026)	(0.003, 0.698)	(0.003, 0.346)	(0.003, 0.184)	(0.003, 0.111)	(0.003, 0.074)
		(0.800, 0.000, 0.054)	(0.793, 0.000, 0.077)	(0.776, 0.001, 0.121)	(0.749, 0.003, 0.169)	(0.716, 0.007, 0.217)	(0.678, 0.015, 0.265)
	HUBER	(0.003, 1.000)	(0.006, 1.000)	(0.015, 1.000)	(0.031, 1.000)	(0.054, 1.000)	(0.085, 1.000)
		(0.800, 0.000, 0.056)	(0.798, 0.000, 0.058)	(0.797, 0.000, 0.058)	(0.795, 0.000, 0.058)	(0.794, 0.000, 0.058)	(0.792, 0.000, 0.058)
	BISQUARE	(0.003, 1.053)	(0.003, 0.565)	(0.003, 0.222)	(0.003, 0.109)	(0.003, 0.063)	(0.003, 0.040)
0.04	LSE	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
		(0.003, 1.065)	(0.003, 0.548)	(0.003, 0.210)	(0.003, 0.103)	(0.003, 0.059)	(0.003, 0.038)
	HUBER	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.055)	(0.800, 0.000, 0.056)
		(0.003, 1.026)	(0.003, 0.534)	(0.003, 0.203)	(0.003, 0.099)	(0.003, 0.057)	(0.003, 0.036)
	PROPOSED	(0.800, 0.000, 0.054)	(0.790, 0.000, 0.086)	(0.764, 0.001, 0.142)	(0.725, 0.006, 0.202)	(0.676, 0.015, 0.262)	(0.620, 0.032, 0.319)
0.05	LSE	(0.003, 1.000)	(0.007, 1.000)	(0.022, 1.000)	(0.046, 1.000)	(0.084, 1.000)	(0.134, 1.000)
		(0.800, 0.000, 0.056)	(0.798, 0.000, 0.059)	(0.795, 0.000, 0.059)	(0.793, 0.000, 0.059)	(0.790, 0.000, 0.059)	(0.788, 0.000, 0.059)
	HUBER	(0.003, 1.054)	(0.004, 0.471)	(0.004, 0.165)	(0.004, 0.077)	(0.004, 0.043)	(0.004, 0.027)
		(0.800, 0.000, 0.056)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
	PROPOSED	(0.003, 1.066)	(0.003, 0.447)	(0.003, 0.150)	(0.003, 0.069)	(0.003, 0.038)	(0.003, 0.024)
0.06	LSE	(0.800, 0.000, 0.055)	(0.799, 0.000, 0.057)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
		(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)	(0.003, 1.026)
	HUBER	(0.800, 0.000, 0.054)	(0.797, 0.000, 0.067)	(0.788, 0.000, 0.094)	(0.774, 0.001, 0.126)	(0.758, 0.002, 0.160)	(0.738, 0.004, 0.193)
		(0.003, 1.000)	(0.004, 1.000)	(0.009, 1.000)	(0.017, 1.000)	(0.027, 1.000)	(0.041, 1.000)
	BISQUARE	(0.003, 1.052)	(0.003, 1.053)	(0.003, 1.053)	(0.003, 1.052)	(0.003, 1.053)	(0.003, 1.052)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.003, 1.026)	(0.003, 0.439)	(0.003, 0.145)	(0.003, 0.067)	(0.003, 0.037)	(0.003, 0.023)
0.04	LSE	(0.800, 0.000, 0.054)	(0.787, 0.000, 0.094)	(0.753, 0.002, 0.161)	(0.702, 0.010, 0.231)	(0.637, 0.026, 0.298)	(0.565, 0.055, 0.364)
		(0.003, 1.000)	(0.009, 1.000)	(0.028, 1.000)	(0.063, 1.000)	(0.115, 1.000)	(0.187, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.797, 0.000, 0.061)	(0.793, 0.000, 0.061)	(0.790, 0.000, 0.061)	(0.787, 0.000, 0.061)	(0.784, 0.000, 0.061)
		(0.003, 1.052)	(0.004, 0.411)	(0.004, 0.133)	(0.004, 0.060)	(0.004, 0.034)	(0.004, 0.021)
	BISQUARE	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.059)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
		(0.003, 1.065)	(0.003, 0.381)	(0.003, 0.116)	(0.003, 0.052)	(0.003, 0.028)	(0.003, 0.017)
0.05	LSE	(0.800, 0.000, 0.055)	(0.799, 0.000, 0.058)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
		(0.003, 1.026)	(0.003, 0.378)	(0.003, 0.113)	(0.003, 0.050)	(0.003, 0.027)	(0.003, 0.017)
	HUBER	(0.800, 0.000, 0.054)	(0.785, 0.000, 0.101)	(0.742, 0.003, 0.177)	(0.679, 0.015, 0.254)	(0.601, 0.040, 0.331)	(0.511, 0.084, 0.402)
		(0.003, 1.000)	(0.010, 1.000)	(0.035, 1.000)	(0.079, 1.000)	(0.149, 1.000)	(0.245, 1.000)
	BISQUARE	(0.800, 0.000, 0.056)	(0.796, 0.000, 0.062)	(0.791, 0.000, 0.062)	(0.787, 0.000, 0.062)	(0.783, 0.000, 0.062)	(0.780, 0.000, 0.063)
		(0.003, 1.053)	(0.004, 0.369)	(0.004, 0.114)	(0.004, 0.051)	(0.004, 0.028)	(0.004, 0.018)
0.06	LSE	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.059)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
		(0.003, 1.066)	(0.004, 0.335)	(0.003, 0.095)	(0.003, 0.041)	(0.003, 0.022)	(0.003, 0.013)
	HUBER	(0.800, 0.000, 0.055)	(0.799, 0.000, 0.059)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
		(0.003, 1.026)	(0.004, 0.336)	(0.003, 0.092)	(0.003, 0.040)	(0.003, 0.021)	(0.003, 0.013)
	BISQUARE	(0.800, 0.000, 0.054)	(0.782, 0.000, 0.108)	(0.731, 0.005, 0.191)	(0.657, 0.020, 0.276)	(0.565, 0.055, 0.358)	(0.460, 0.115, 0.436)
		(0.003, 1.000)	(0.012, 1.000)	(0.041, 1.000)	(0.097, 1.000)	(0.184, 1.000)	(0.305, 1.000)
0.07	LSE	(0.800, 0.000, 0.056)	(0.795, 0.000, 0.064)	(0.789, 0.000, 0.064)	(0.784, 0.000, 0.064)	(0.780, 0.000, 0.064)	(0.775, 0.001, 0.064)
		(0.003, 1.053)	(0.004, 0.341)	(0.004, 0.101)	(0.004, 0.045)	(0.005, 0.025)	(0.005, 0.016)
	HUBER	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.060)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.058)
		(0.003, 1.065)	(0.004, 0.302)	(0.003, 0.081)	(0.003, 0.034)	(0.003, 0.018)	(0.003, 0.011)
	BISQUARE	(0.800, 0.000, 0.055)	(0.798, 0.000, 0.060)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
		(0.003, 1.026)	(0.004, 0.307)	(0.003, 0.079)	(0.003, 0.033)	(0.003, 0.017)	(0.003, 0.011)
0.08	LSE	(0.800, 0.000, 0.054)	(0.778, 0.000, 0.114)	(0.721, 0.006, 0.205)	(0.635, 0.027, 0.295)	(0.530, 0.073, 0.383)	(0.411, 0.152, 0.467)
		(0.003, 1.000)	(0.013, 1.000)	(0.048, 1.000)	(0.114, 1.000)	(0.220, 1.000)	(0.369, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.794, 0.000, 0.065)	(0.787, 0.000, 0.065)	(0.781, 0.000, 0.066)	(0.775, 0.001, 0.066)	(0.770, 0.001, 0.067)
		(0.003, 1.053)	(0.004, 0.319)	(0.004, 0.092)	(0.005, 0.041)	(0.005, 0.023)	(0.005, 0.014)
	BISQUARE	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.061)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
		(0.003, 1.066)	(0.004, 0.276)	(0.003, 0.070)	(0.003, 0.029)	(0.003, 0.015)	(0.003, 0.009)
0.08	LSE	(0.800, 0.000, 0.055)	(0.798, 0.000, 0.062)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
		(0.003, 1.026)	(0.004, 0.283)	(0.003, 0.068)	(0.003, 0.028)	(0.003, 0.015)	(0.003, 0.009)
0.08	LSE	(0.800, 0.000, 0.054)	(0.776, 0.001, 0.120)	(0.710, 0.008, 0.217)	(0.614, 0.035, 0.314)	(0.497, 0.092, 0.407)	(0.363, 0.191, 0.495)
		(0.003, 1.000)	(0.015, 1.000)	(0.055, 1.000)	(0.133, 1.000)	(0.258, 1.000)	(0.435, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.800, 0.000, 0.056)	(0.793, 0.000, 0.067)	(0.785, 0.000, 0.067)	(0.778, 0.000, 0.068)	(0.771, 0.001, 0.068)	(0.765, 0.001, 0.069)
	HUBER	(0.003, 1.053)	(0.005, 0.305)	(0.005, 0.087)	(0.005, 0.038)	(0.005, 0.021)	(0.006, 0.014)
	BISQUARE	(0.800, 0.000, 0.056)	(0.798, 0.000, 0.062)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
		(0.003, 1.065)	(0.004, 0.258)	(0.003, 0.062)	(0.003, 0.025)	(0.003, 0.013)	(0.003, 0.008)
	PROPOSED	(0.800, 0.000, 0.055)	(0.798, 0.000, 0.063)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
0.09		(0.800, 0.000, 0.054)	(0.773, 0.001, 0.125)	(0.700, 0.010, 0.228)	(0.594, 0.042, 0.330)	(0.464, 0.113, 0.428)	(0.318, 0.232, 0.520)
	LSE	(0.003, 1.000)	(0.016, 1.000)	(0.062, 1.000)	(0.151, 1.000)	(0.296, 1.000)	(0.503, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.792, 0.000, 0.069)	(0.783, 0.000, 0.069)	(0.775, 0.001, 0.070)	(0.767, 0.001, 0.071)	(0.759, 0.002, 0.072)
		(0.003, 1.054)	(0.005, 0.295)	(0.005, 0.083)	(0.006, 0.037)	(0.006, 0.021)	(0.007, 0.014)
	BISQUARE	(0.800, 0.000, 0.056)	(0.798, 0.000, 0.063)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
0.1		(0.003, 1.066)	(0.004, 0.244)	(0.003, 0.056)	(0.003, 0.023)	(0.003, 0.012)	(0.003, 0.007)
	PROPOSED	(0.800, 0.000, 0.055)	(0.797, 0.000, 0.065)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
		(0.003, 1.026)	(0.004, 0.257)	(0.003, 0.055)	(0.003, 0.022)	(0.003, 0.011)	(0.003, 0.007)
	LSE	(0.800, 0.000, 0.054)	(0.770, 0.001, 0.130)	(0.691, 0.012, 0.238)	(0.575, 0.051, 0.345)	(0.433, 0.135, 0.448)	(0.275, 0.276, 0.544)
		(0.003, 1.000)	(0.018, 1.000)	(0.068, 1.000)	(0.170, 1.000)	(0.336, 1.000)	(0.572, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.791, 0.000, 0.071)	(0.781, 0.000, 0.072)	(0.771, 0.001, 0.072)	(0.762, 0.001, 0.074)	(0.753, 0.002, 0.076)
		(0.003, 1.054)	(0.005, 0.289)	(0.006, 0.081)	(0.006, 0.036)	(0.007, 0.021)	(0.008, 0.014)
	BISQUARE	(0.800, 0.000, 0.056)	(0.797, 0.000, 0.064)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.059)
		(0.003, 1.066)	(0.004, 0.232)	(0.003, 0.051)	(0.003, 0.020)	(0.003, 0.010)	(0.003, 0.006)
	PROPOSED	(0.800, 0.000, 0.055)	(0.797, 0.000, 0.066)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
		(0.003, 1.027)	(0.004, 0.249)	(0.003, 0.050)	(0.003, 0.020)	(0.003, 0.010)	(0.003, 0.006)

Table A11: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_4) for multiple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0		(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)
	LSE	(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)
	HUBER	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)
		(0.004, 1.055)	(0.004, 1.055)	(0.004, 1.055)	(0.004, 1.054)	(0.004, 1.054)	(0.004, 1.056)
	BISQUARE	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.01		(0.004, 1.065)	(0.004, 1.066)	(0.004, 1.065)	(0.004, 1.065)	(0.004, 1.065)	(0.004, 1.067)
		(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)
	PROPOSED	(0.004, 1.026)	(0.004, 1.026)	(0.004, 1.026)	(0.004, 1.026)	(0.004, 1.026)	(0.004, 1.027)
		(0.500, 0.000, 0.065)	(0.498, 0.000, 0.079)	(0.491, 0.000, 0.112)	(0.479, 0.000, 0.151)	(0.463, 0.001, 0.192)	(0.444, 0.003, 0.236)
	LSE	(0.004, 1.000)	(0.006, 1.000)	(0.013, 1.000)	(0.023, 1.000)	(0.038, 1.000)	(0.059, 1.000)
		(0.500, 0.000, 0.066)	(0.499, 0.000, 0.068)	(0.499, 0.000, 0.068)	(0.498, 0.000, 0.068)	(0.497, 0.000, 0.068)	(0.497, 0.000, 0.068)
	HUBER	(0.004, 1.054)	(0.005, 0.725)	(0.005, 0.362)	(0.005, 0.197)	(0.005, 0.119)	(0.005, 0.078)
		(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)
	BISQUARE	(0.004, 1.065)	(0.005, 0.718)	(0.004, 0.354)	(0.004, 0.193)	(0.004, 0.117)	(0.004, 0.076)
	PROPOSED	(0.004, 1.026)	(0.004, 0.696)	(0.004, 0.342)	(0.004, 0.186)	(0.004, 0.112)	(0.004, 0.074)
0.02		(0.500, 0.000, 0.065)	(0.495, 0.000, 0.092)	(0.481, 0.000, 0.144)	(0.459, 0.002, 0.203)	(0.428, 0.005, 0.263)	(0.390, 0.012, 0.324)
	LSE	(0.004, 1.000)	(0.008, 1.000)	(0.021, 1.000)	(0.043, 1.000)	(0.074, 1.000)	(0.117, 1.000)
		(0.500, 0.000, 0.066)	(0.499, 0.000, 0.069)	(0.497, 0.000, 0.069)	(0.496, 0.000, 0.069)	(0.495, 0.000, 0.069)	(0.493, 0.000, 0.069)
	HUBER	(0.004, 1.055)	(0.005, 0.564)	(0.005, 0.227)	(0.005, 0.112)	(0.005, 0.065)	(0.005, 0.041)
		(0.500, 0.000, 0.067)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)
	BISQUARE	(0.004, 1.066)	(0.005, 0.547)	(0.005, 0.215)	(0.005, 0.106)	(0.005, 0.061)	(0.005, 0.039)
		(0.500, 0.000, 0.065)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)
	PROPOSED	(0.004, 1.027)	(0.005, 0.534)	(0.004, 0.208)	(0.004, 0.102)	(0.004, 0.059)	(0.004, 0.037)
		(0.500, 0.000, 0.065)	(0.493, 0.000, 0.102)	(0.472, 0.001, 0.170)	(0.438, 0.004, 0.242)	(0.394, 0.011, 0.316)	(0.339, 0.026, 0.390)
	LSE	(0.004, 1.000)	(0.011, 1.000)	(0.030, 1.000)	(0.063, 1.000)	(0.111, 1.000)	(0.178, 1.000)
0.03		(0.500, 0.000, 0.066)	(0.498, 0.000, 0.070)	(0.496, 0.000, 0.070)	(0.494, 0.000, 0.071)	(0.492, 0.000, 0.071)	(0.490, 0.000, 0.071)
	HUBER	(0.004, 1.054)	(0.005, 0.471)	(0.005, 0.168)	(0.005, 0.080)	(0.005, 0.046)	(0.005, 0.029)
		(0.500, 0.000, 0.067)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)
	BISQUARE	(0.004, 1.065)	(0.005, 0.448)	(0.005, 0.153)	(0.005, 0.073)	(0.005, 0.041)	(0.005, 0.026)
		(0.500, 0.000, 0.065)	(0.499, 0.000, 0.068)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)
	PROPOSED	(0.004, 1.026)	(0.005, 0.440)	(0.004, 0.148)	(0.004, 0.070)	(0.004, 0.040)	(0.004, 0.025)
		(0.500, 0.000, 0.065)	(0.490, 0.000, 0.112)	(0.463, 0.001, 0.192)	(0.419, 0.007, 0.275)	(0.360, 0.019, 0.360)	(0.290, 0.044, 0.444)
	LSE	(0.004, 1.000)	(0.013, 1.000)	(0.038, 1.000)	(0.082, 1.000)	(0.149, 1.000)	(0.241, 1.000)
		(0.500, 0.000, 0.066)	(0.497, 0.000, 0.072)	(0.494, 0.000, 0.072)	(0.492, 0.000, 0.072)	(0.489, 0.000, 0.072)	(0.486, 0.000, 0.073)
	HUBER	(0.004, 1.054)	(0.005, 0.412)	(0.005, 0.137)	(0.005, 0.064)	(0.005, 0.036)	(0.005, 0.023)
0.04		(0.500, 0.000, 0.067)	(0.499, 0.000, 0.069)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)
	BISQUARE	(0.004, 1.065)	(0.005, 0.383)	(0.005, 0.121)	(0.005, 0.056)	(0.005, 0.031)	(0.005, 0.019)
		(0.500, 0.000, 0.065)	(0.499, 0.000, 0.069)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)
	PROPOSED	(0.004, 1.026)	(0.005, 0.380)	(0.004, 0.117)	(0.004, 0.054)	(0.004, 0.030)	(0.004, 0.018)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.05	LSE	(0.500, 0.000, 0.065) (0.004, 1.000)	(0.488, 0.000, 0.120) (0.015, 1.000)	(0.454, 0.002, 0.210) (0.046, 1.000)	(0.400, 0.010, 0.304) (0.102, 1.000)	(0.329, 0.029, 0.397) (0.187, 1.000)	(0.242, 0.067, 0.490) (0.307, 1.000)
		(0.500, 0.000, 0.066) (0.004, 1.054)	(0.496, 0.000, 0.073) (0.005, 0.371)	(0.493, 0.000, 0.074) (0.005, 0.119)	(0.489, 0.000, 0.074) (0.006, 0.055)	(0.485, 0.000, 0.074) (0.006, 0.031)	(0.482, 0.000, 0.075) (0.006, 0.019)
	HUBER	(0.500, 0.000, 0.067) (0.004, 1.064)	(0.499, 0.000, 0.070) (0.005, 0.338)	(0.500, 0.000, 0.068) (0.005, 0.100)	(0.500, 0.000, 0.068) (0.005, 0.045)	(0.500, 0.000, 0.068) (0.005, 0.025)	(0.500, 0.000, 0.068) (0.005, 0.015)
	BISQUARE	(0.500, 0.000, 0.065) (0.004, 1.025)	(0.499, 0.000, 0.070) (0.005, 0.338)	(0.500, 0.000, 0.067) (0.005, 0.097)	(0.500, 0.000, 0.067) (0.004, 0.044)	(0.500, 0.000, 0.067) (0.004, 0.024)	(0.500, 0.000, 0.067) (0.004, 0.015)
	PROPOSED	(0.500, 0.000, 0.065) (0.004, 1.000)	(0.486, 0.000, 0.128) (0.017, 1.000)	(0.446, 0.003, 0.228) (0.055, 1.000)	(0.382, 0.014, 0.330) (0.123, 1.000)	(0.299, 0.041, 0.431) (0.227, 1.000)	(0.198, 0.091, 0.530) (0.372, 1.000)
0.06	LSE	(0.500, 0.000, 0.066) (0.004, 1.055)	(0.496, 0.000, 0.075) (0.006, 0.342)	(0.491, 0.000, 0.076) (0.006, 0.106)	(0.487, 0.000, 0.076) (0.006, 0.048)	(0.482, 0.000, 0.076) (0.006, 0.027)	(0.478, 0.001, 0.077) (0.006, 0.017)
		(0.500, 0.000, 0.067) (0.004, 1.066)	(0.499, 0.000, 0.071) (0.005, 0.304)	(0.500, 0.000, 0.069) (0.005, 0.086)	(0.500, 0.000, 0.068) (0.005, 0.038)	(0.500, 0.000, 0.068) (0.005, 0.021)	(0.500, 0.000, 0.068) (0.005, 0.013)
	HUBER	(0.500, 0.000, 0.065) (0.004, 1.026)	(0.499, 0.000, 0.072) (0.005, 0.308)	(0.500, 0.000, 0.068) (0.005, 0.083)	(0.500, 0.000, 0.067) (0.005, 0.037)	(0.500, 0.000, 0.067) (0.005, 0.020)	(0.500, 0.000, 0.067) (0.005, 0.012)
	BISQUARE	(0.500, 0.000, 0.065) (0.004, 1.000)	(0.484, 0.000, 0.136) (0.019, 1.000)	(0.438, 0.004, 0.243) (0.063, 1.000)	(0.365, 0.018, 0.353) (0.143, 1.000)	(0.269, 0.053, 0.462) (0.267, 1.000)	(0.153, 0.120, 0.567) (0.442, 1.000)
	PROPOSED	(0.500, 0.000, 0.066) (0.004, 1.055)	(0.495, 0.000, 0.077) (0.006, 0.321)	(0.490, 0.000, 0.078) (0.006, 0.097)	(0.484, 0.000, 0.078) (0.006, 0.044)	(0.479, 0.000, 0.079) (0.007, 0.025)	(0.473, 0.001, 0.079) (0.007, 0.016)
0.07	LSE	(0.500, 0.000, 0.067) (0.004, 1.065)	(0.499, 0.000, 0.072) (0.005, 0.279)	(0.500, 0.000, 0.069) (0.005, 0.075)	(0.500, 0.000, 0.069) (0.005, 0.033)	(0.500, 0.000, 0.069) (0.005, 0.018)	(0.500, 0.000, 0.069) (0.005, 0.011)
		(0.500, 0.000, 0.065) (0.004, 1.026)	(0.498, 0.000, 0.073) (0.005, 0.285)	(0.500, 0.000, 0.068) (0.005, 0.073)	(0.500, 0.000, 0.068) (0.005, 0.032)	(0.500, 0.000, 0.068) (0.005, 0.017)	(0.500, 0.000, 0.068) (0.005, 0.010)
	HUBER	(0.500, 0.000, 0.065) (0.004, 1.000)	(0.482, 0.000, 0.142) (0.021, 1.000)	(0.430, 0.005, 0.258) (0.071, 1.000)	(0.347, 0.023, 0.375) (0.164, 1.000)	(0.240, 0.067, 0.489) (0.307, 1.000)	(0.113, 0.150, 0.600) (0.509, 1.000)
	BISQUARE	(0.500, 0.000, 0.066) (0.004, 1.055)	(0.494, 0.000, 0.079) (0.006, 0.307)	(0.488, 0.000, 0.080) (0.007, 0.091)	(0.481, 0.000, 0.080) (0.007, 0.042)	(0.475, 0.001, 0.081) (0.007, 0.024)	(0.468, 0.001, 0.082) (0.008, 0.015)
	PROPOSED	(0.500, 0.000, 0.067) (0.004, 1.066)	(0.499, 0.000, 0.073) (0.005, 0.261)	(0.500, 0.000, 0.069) (0.005, 0.067)	(0.500, 0.000, 0.069) (0.005, 0.029)	(0.500, 0.000, 0.069) (0.005, 0.016)	(0.500, 0.000, 0.069) (0.005, 0.009)
0.08	LSE	(0.500, 0.000, 0.065) (0.004, 1.000)	(0.480, 0.000, 0.149) (0.022, 1.000)	(0.422, 0.006, 0.271) (0.080, 1.000)	(0.331, 0.029, 0.394) (0.184, 1.000)	(0.213, 0.082, 0.514) (0.346, 1.000)	(0.072, 0.183, 0.630) (0.580, 1.000)
		(0.500, 0.000, 0.066) (0.004, 1.055)	(0.493, 0.000, 0.081) (0.006, 0.307)	(0.486, 0.000, 0.082) (0.007, 0.091)	(0.478, 0.000, 0.083) (0.007, 0.042)	(0.471, 0.001, 0.084) (0.007, 0.024)	(0.463, 0.001, 0.086) (0.008, 0.015)
	HUBER	(0.500, 0.000, 0.067) (0.004, 1.066)	(0.499, 0.000, 0.073) (0.005, 0.261)	(0.500, 0.000, 0.069) (0.005, 0.067)	(0.500, 0.000, 0.069) (0.005, 0.029)	(0.500, 0.000, 0.069) (0.005, 0.016)	(0.500, 0.000, 0.069) (0.005, 0.009)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.1		(0.004, 1.054)	(0.007, 0.296)	(0.007, 0.087)	(0.007, 0.040)	(0.008, 0.023)	(0.009, 0.015)
		(0.500, 0.000, 0.067)	(0.498, 0.000, 0.074)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)
	BISQUARE	(0.004, 1.065)	(0.006, 0.246)	(0.005, 0.061)	(0.005, 0.026)	(0.005, 0.014)	(0.005, 0.008)
		(0.500, 0.000, 0.065)	(0.498, 0.000, 0.076)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)
	PROPOSED	(0.004, 1.026)	(0.006, 0.259)	(0.005, 0.060)	(0.005, 0.026)	(0.005, 0.013)	(0.005, 0.008)
		(0.500, 0.000, 0.065)	(0.478, 0.001, 0.154)	(0.415, 0.007, 0.284)	(0.314, 0.035, 0.412)	(0.186, 0.098, 0.538)	(0.033, 0.218, 0.658)
	LSE	(0.004, 1.000)	(0.024, 1.000)	(0.088, 1.000)	(0.205, 1.000)	(0.388, 1.000)	(0.650, 1.000)
		(0.500, 0.000, 0.066)	(0.492, 0.000, 0.084)	(0.484, 0.000, 0.085)	(0.475, 0.001, 0.086)	(0.466, 0.001, 0.088)	(0.457, 0.002, 0.090)
	HUBER	(0.004, 1.055)	(0.007, 0.290)	(0.007, 0.085)	(0.008, 0.039)	(0.009, 0.023)	(0.010, 0.015)
		(0.500, 0.000, 0.067)	(0.498, 0.000, 0.076)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.070)
	BISQUARE	(0.004, 1.066)	(0.006, 0.235)	(0.005, 0.056)	(0.005, 0.024)	(0.005, 0.013)	(0.005, 0.007)
	PROPOSED	(0.500, 0.000, 0.065)	(0.497, 0.000, 0.078)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)

Table A12: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the overall mean square error (σ^2) for multiple linear regression model before optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0		(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)
	LSE	(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)
		(0.950, 0.003, 0.227)	(0.950, 0.003, 0.227)	(0.950, 0.002, 0.227)	(0.950, 0.002, 0.227)	(0.950, 0.002, 0.227)	(0.950, 0.003, 0.227)
	HUBER	(0.054, 2.567)	(0.054, 2.573)	(0.054, 2.570)	(0.054, 2.577)	(0.054, 2.563)	(0.054, 2.562)
		(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)
	BISQUARE	(0.054, 2.575)	(0.054, 2.581)	(0.054, 2.578)	(0.054, 2.577)	(0.054, 2.572)	(0.054, 2.570)
		(0.962, 0.001, 0.227)	(0.962, 0.001, 0.227)	(0.962, 0.001, 0.227)	(0.962, 0.001, 0.227)	(0.962, 0.001, 0.227)	(0.962, 0.001, 0.227)
	PROPOSED	(0.053, 2.516)	(0.053, 2.525)	(0.053, 2.520)	(0.053, 2.521)	(0.053, 2.515)	(0.053, 2.512)
		(1.000, 0.000, 0.145)	(1.508, 0.258, 0.869)	(2.978, 3.912, 3.339)	(5.335, 18.789, 7.366)	(8.510, 56.406, 12.880)	(12.409, 130.157, 19.768)
	LSE	(0.021, 1.000)	(1.013, 1.000)	(15.036, 1.000)	(73.244, 1.000)	(222.713, 1.000)	(516.775, 1.000)
0.01		(0.950, 0.003, 0.227)	(0.975, 0.001, 0.234)	(0.975, 0.001, 0.234)	(0.975, 0.001, 0.234)	(0.975, 0.001, 0.234)	(0.975, 0.001, 0.234)
	HUBER	(0.054, 2.565)	(0.055, 0.055)	(0.055, 0.004)	(0.055, 0.001)	(0.055, 0.000)	(0.055, 0.000)
		(0.945, 0.003, 0.226)	(0.968, 0.001, 0.232)	(0.968, 0.001, 0.232)	(0.968, 0.001, 0.232)	(0.967, 0.001, 0.232)	(0.968, 0.001, 0.232)
	BISQUARE	(0.054, 2.573)	(0.055, 0.054)	(0.055, 0.004)	(0.055, 0.001)	(0.055, 0.000)	(0.055, 0.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.02	PROPOSED	(0.962, 0.001, 0.227)	(0.985, 0.000, 0.234)	(0.985, 0.000, 0.233)	(0.985, 0.000, 0.233)	(0.985, 0.000, 0.233)	(0.985, 0.000, 0.233)
		(0.053, 2.516)	(0.055, 0.054)	(0.055, 0.004)	(0.055, 0.001)	(0.055, 0.000)	(0.055, 0.000)
	LSE	(1.000, 0.000, 0.145)	(2.008, 1.016, 1.205)	(4.919, 15.356, 4.654)	(9.564, 73.343, 10.267)	(15.814, 219.458, 17.925)	(23.459, 504.399, 27.388)
		(0.021, 1.000)	(2.468, 1.000)	(37.019, 1.000)	(178.744, 1.000)	(540.769, 1.000)	(1254.524, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.001, 0.000, 0.242)	(1.001, 0.000, 0.242)	(1.001, 0.000, 0.242)	(1.002, 0.000, 0.243)	(1.003, 0.000, 0.243)
		(0.054, 2.560)	(0.059, 0.024)	(0.059, 0.002)	(0.059, 0.000)	(0.059, 0.000)	(0.059, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(0.992, 0.000, 0.239)	(0.991, 0.000, 0.239)	(0.991, 0.000, 0.239)	(0.991, 0.000, 0.239)	(0.991, 0.000, 0.239)
		(0.054, 2.569)	(0.057, 0.023)	(0.057, 0.002)	(0.057, 0.000)	(0.057, 0.000)	(0.057, 0.000)
	PROPOSED	(0.962, 0.001, 0.227)	(1.009, 0.000, 0.241)	(1.008, 0.000, 0.240)	(1.008, 0.000, 0.240)	(1.009, 0.000, 0.240)	(1.009, 0.000, 0.240)
		(0.053, 2.513)	(0.058, 0.024)	(0.058, 0.002)	(0.057, 0.000)	(0.058, 0.000)	(0.058, 0.000)
0.03	LSE	(1.000, 0.000, 0.145)	(2.500, 2.249, 1.460)	(6.828, 33.969, 5.646)	(13.711, 161.581, 12.393)	(22.901, 479.653, 21.609)	(34.238, 1104.797, 33.056)
		(0.021, 1.000)	(4.381, 1.000)	(65.846, 1.000)	(315.164, 1.000)	(946.588, 1.000)	(2197.526, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.029, 0.000, 0.250)	(1.031, 0.001, 0.251)	(1.032, 0.001, 0.252)	(1.032, 0.001, 0.253)	(1.035, 0.001, 0.254)
		(0.054, 2.573)	(0.063, 0.014)	(0.064, 0.001)	(0.064, 0.000)	(0.065, 0.000)	(0.066, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.016, 0.000, 0.246)	(1.016, 0.000, 0.246)	(1.016, 0.000, 0.246)	(1.016, 0.000, 0.246)	(1.016, 0.000, 0.246)
		(0.054, 2.581)	(0.061, 0.014)	(0.061, 0.001)	(0.061, 0.000)	(0.061, 0.000)	(0.061, 0.000)
	PROPOSED	(0.962, 0.001, 0.227)	(1.035, 0.001, 0.248)	(1.034, 0.001, 0.247)	(1.033, 0.001, 0.247)	(1.033, 0.001, 0.247)	(1.034, 0.001, 0.247)
		(0.053, 2.526)	(0.063, 0.014)	(0.062, 0.001)	(0.062, 0.000)	(0.062, 0.000)	(0.062, 0.000)
	LSE	(1.000, 0.000, 0.145)	(2.985, 3.939, 1.664)	(8.695, 59.220, 6.435)	(17.766, 281.097, 14.113)	(29.837, 831.576, 24.557)	(44.556, 1897.101, 37.420)
		(0.021, 1.000)	(6.707, 1.000)	(100.623, 1.000)	(480.285, 1.000)	(1434.598, 1.000)	(3297.371, 1.000)
0.04	HUBER	(0.950, 0.003, 0.227)	(1.060, 0.002, 0.261)	(1.062, 0.004, 0.261)	(1.064, 0.004, 0.263)	(1.066, 0.004, 0.264)	(1.069, 0.005, 0.266)
		(0.054, 2.561)	(0.072, 0.011)	(0.072, 0.001)	(0.073, 0.000)	(0.074, 0.000)	(0.075, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.043, 0.002, 0.254)	(1.041, 0.002, 0.253)	(1.041, 0.002, 0.253)	(1.041, 0.002, 0.253)	(1.042, 0.002, 0.253)
		(0.054, 2.571)	(0.067, 0.010)	(0.065, 0.001)	(0.066, 0.000)	(0.066, 0.000)	(0.066, 0.000)
	PROPOSED	(0.962, 0.001, 0.227)	(1.062, 0.004, 0.257)	(1.059, 0.003, 0.254)	(1.059, 0.004, 0.254)	(1.059, 0.003, 0.254)	(1.059, 0.004, 0.254)
		(0.053, 2.515)	(0.067, 0.010)	(0.068, 0.001)	(0.068, 0.000)	(0.068, 0.000)	(0.068, 0.000)
	LSE	(1.000, 0.000, 0.145)	(3.454, 6.023, 1.835)	(10.520, 90.637, 7.070)	(21.734, 429.880, 15.612)	(36.632, 1269.663, 27.055)	(54.756, 2889.760, 41.165)
		(0.021, 1.000)	(9.392, 1.000)	(140.619, 1.000)	(673.625, 1.000)	(2001.639, 1.000)	(4584.343, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.094, 0.005, 0.272)	(1.097, 0.009, 0.273)	(1.100, 0.010, 0.276)	(1.104, 0.011, 0.279)	(1.109, 0.012, 0.282)
		(0.054, 2.558)	(0.083, 0.009)	(0.086, 0.000)	(0.088, 0.000)	(0.088, 0.000)	(0.091, 0.000)
0.05	BISQUARE	(0.945, 0.003, 0.226)	(1.070, 0.005, 0.263)	(1.068, 0.005, 0.261)	(1.068, 0.005, 0.260)	(1.068, 0.005, 0.260)	(1.068, 0.005, 0.260)
		(0.054, 2.566)	(0.074, 0.008)	(0.072, 0.000)	(0.072, 0.000)	(0.072, 0.000)	(0.072, 0.000)
	PROPOSED	(0.962, 0.001, 0.227)	(1.091, 0.008, 0.266)	(1.086, 0.007, 0.262)	(1.086, 0.007, 0.261)	(1.086, 0.007, 0.261)	(1.086, 0.007, 0.261)
		(0.053, 2.510)	(0.079, 0.008)	(0.076, 0.000)	(0.076, 0.000)	(0.076, 0.000)	(0.076, 0.000)
	LSE	(1.000, 0.000, 0.145)	(3.925, 8.554, 1.995)	(12.330, 128.371, 7.692)	(25.650, 607.645, 16.847)	(43.210, 1781.673, 29.199)	(64.411, 4020.976, 44.360)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.021, 1.000)	(12.535, 1.000)	(187.537, 1.000)	(891.467, 1.000)	(2634.233, 1.000)	(5988.829, 1.000)
0.07	HUBER	(0.950, 0.003, 0.227)	(1.130, 0.010, 0.284)	(1.135, 0.018, 0.288)	(1.140, 0.020, 0.291)	(1.146, 0.021, 0.296)	(1.153, 0.024, 0.302)
		(0.054, 2.566)	(0.098, 0.008)	(0.101, 0.001)	(0.104, 0.000)	(0.109, 0.000)	(0.115, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.098, 0.010, 0.272)	(1.095, 0.009, 0.268)	(1.095, 0.009, 0.268)	(1.095, 0.009, 0.268)	(1.095, 0.009, 0.268)
		(0.054, 2.574)	(0.083, 0.007)	(0.081, 0.000)	(0.081, 0.000)	(0.081, 0.000)	(0.081, 0.000)
	PROPOSED	(0.962, 0.001, 0.227)	(1.120, 0.015, 0.276)	(1.114, 0.013, 0.269)	(1.114, 0.013, 0.269)	(1.113, 0.013, 0.269)	(1.113, 0.013, 0.269)
		(0.053, 2.517)	(0.091, 0.007)	(0.086, 0.000)	(0.085, 0.000)	(0.085, 0.000)	(0.085, 0.000)
	LSE	(1.000, 0.000, 0.145)	(4.383, 11.442, 2.122)	(14.080, 171.089, 8.198)	(29.417, 807.515, 17.947)	(49.625, 2364.342, 31.040)	(74.025, 5332.610, 47.070)
		(0.021, 1.000)	(15.946, 1.000)	(238.299, 1.000)	(1129.597, 1.000)	(3327.827, 1.000)	(7548.159, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.169, 0.017, 0.299)	(1.176, 0.031, 0.305)	(1.184, 0.034, 0.311)	(1.194, 0.038, 0.319)	(1.206, 0.042, 0.333)
		(0.054, 2.559)	(0.118, 0.007)	(0.124, 0.001)	(0.131, 0.000)	(0.140, 0.000)	(0.153, 0.000)
0.08	BISQUARE	(0.945, 0.003, 0.226)	(1.129, 0.017, 0.281)	(1.124, 0.015, 0.277)	(1.124, 0.015, 0.277)	(1.124, 0.015, 0.277)	(1.124, 0.015, 0.276)
		(0.054, 2.567)	(0.096, 0.006)	(0.092, 0.000)	(0.092, 0.000)	(0.092, 0.000)	(0.092, 0.000)
	PROPOSED	(0.962, 0.001, 0.227)	(1.152, 0.023, 0.288)	(1.142, 0.020, 0.278)	(1.142, 0.020, 0.278)	(1.143, 0.020, 0.278)	(1.143, 0.020, 0.283)
		(0.053, 2.511)	(0.106, 0.007)	(0.098, 0.000)	(0.097, 0.000)	(0.097, 0.000)	(0.100, 0.000)
	LSE	(1.000, 0.000, 0.145)	(4.831, 14.673, 2.247)	(15.825, 219.792, 8.681)	(33.127, 1032.125, 18.981)	(55.854, 3008.994, 32.675)	(83.250, 6765.010, 49.582)
		(0.021, 1.000)	(19.722, 1.000)	(295.147, 1.000)	(1392.397, 1.000)	(4076.637, 1.000)	(9223.365, 1.000)
	HUBER	(0.950, 0.002, 0.227)	(1.213, 0.026, 0.317)	(1.224, 0.050, 0.326)	(1.235, 0.055, 0.335)	(1.249, 0.062, 0.350)	(1.267, 0.071, 0.384)
		(0.054, 2.565)	(0.146, 0.007)	(0.156, 0.001)	(0.168, 0.000)	(0.184, 0.000)	(0.219, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.161, 0.026, 0.293)	(1.155, 0.024, 0.286)	(1.154, 0.024, 0.285)	(1.154, 0.024, 0.285)	(1.154, 0.024, 0.285)
		(0.054, 2.573)	(0.112, 0.006)	(0.106, 0.000)	(0.105, 0.000)	(0.105, 0.000)	(0.105, 0.000)
0.09	PROPOSED	(0.962, 0.001, 0.227)	(1.187, 0.035, 0.302)	(1.173, 0.030, 0.287)	(1.173, 0.030, 0.286)	(1.173, 0.030, 0.286)	(1.173, 0.030, 0.308)
		(0.053, 2.518)	(0.126, 0.006)	(0.113, 0.000)	(0.112, 0.000)	(0.112, 0.000)	(0.124, 0.000)
	LSE	(1.000, 0.000, 0.145)	(5.273, 18.257, 2.352)	(17.509, 272.537, 9.075)	(36.737, 1277.163, 19.838)	(61.974, 3717.776, 34.222)	(92.251, 8326.714, 51.833)
		(0.021, 1.000)	(23.790, 1.000)	(354.898, 1.000)	(1670.706, 1.000)	(4888.951, 1.000)	(11013.401, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.262, 0.038, 0.338)	(1.276, 0.076, 0.352)	(1.292, 0.085, 0.370)	(1.314, 0.099, 0.404)	(1.341, 0.117, 0.485)
		(0.054, 2.566)	(0.183, 0.008)	(0.200, 0.001)	(0.222, 0.000)	(0.262, 0.000)	(0.352, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.194, 0.038, 0.305)	(1.186, 0.034, 0.295)	(1.185, 0.034, 0.295)	(1.186, 0.034, 0.295)	(1.185, 0.034, 0.294)
		(0.054, 2.575)	(0.131, 0.005)	(0.121, 0.000)	(0.121, 0.000)	(0.122, 0.000)	(0.121, 0.000)
	PROPOSED	(0.962, 0.001, 0.227)	(1.224, 0.050, 0.318)	(1.205, 0.042, 0.298)	(1.204, 0.042, 0.297)	(1.204, 0.042, 0.296)	(1.204, 0.042, 0.326)
		(0.053, 2.520)	(0.152, 0.006)	(0.131, 0.000)	(0.130, 0.000)	(0.130, 0.000)	(0.148, 0.000)
0.1	LSE	(1.000, 0.000, 0.145)	(5.703, 22.120, 2.452)	(19.155, 329.601, 9.443)	(40.270, 1542.160, 20.617)	(67.970, 4484.982, 35.579)	(101.022, 10004.380, 53.813)
		(0.021, 1.000)	(28.131, 1.000)	(418.765, 1.000)	(1967.233, 1.000)	(5750.814, 1.000)	(12900.250, 1.000)
	HUBER	(0.950, 0.002, 0.227)	(1.315, 0.053, 0.364)	(1.336, 0.113, 0.386)	(1.359, 0.129, 0.420)	(1.392, 0.153, 0.502)	(1.434, 0.189, 0.642)
		(0.054, 2.561)	(0.232, 0.008)	(0.262, 0.001)	(0.305, 0.000)	(0.405, 0.000)	(0.600, 0.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.945, 0.003, 0.226)	(1.230, 0.053, 0.320)	(1.219, 0.048, 0.305)	(1.218, 0.048, 0.304)	(1.218, 0.048, 0.305)	(1.219, 0.048, 0.308)
	BISQUARE	(0.054, 2.570)	(0.155, 0.006)	(0.141, 0.000)	(0.140, 0.000)	(0.141, 0.000)	(0.143, 0.000)
		(0.962, 0.001, 0.227)	(1.265, 0.070, 0.340)	(1.238, 0.057, 0.308)	(1.238, 0.056, 0.314)	(1.238, 0.056, 0.331)	(1.238, 0.057, 0.358)
	PROPOSED	(0.053, 2.513)	(0.185, 0.007)	(0.152, 0.000)	(0.155, 0.000)	(0.166, 0.000)	(0.185, 0.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(12.000, 0.000, 0.319)	(12.000, 0.000, 0.319)	(12.000, 0.000, 0.319)	(12.000, 0.000, 0.319)	(12.000, 0.000, 0.319)	(12.000, 0.000, 0.319)
		(0.102, 1.000)	(0.102, 1.000)	(0.102, 1.000)	(0.102, 1.000)	(0.102, 1.000)	(0.102, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.000, 0.000, 0.327)	(12.000, 0.000, 0.328)	(12.000, 0.000, 0.328)	(12.000, 0.000, 0.328)	(12.000, 0.000, 0.328)
		(0.107, 1.055)	(0.107, 1.054)	(0.107, 1.055)	(0.107, 1.055)	(0.107, 1.055)	(0.108, 1.055)
	BISQUARE	(12.000, 0.000, 0.329)	(12.000, 0.000, 0.329)	(12.000, 0.000, 0.329)	(12.000, 0.000, 0.329)	(12.000, 0.000, 0.329)	(12.000, 0.000, 0.330)
		(0.109, 1.065)	(0.108, 1.065)	(0.108, 1.065)	(0.108, 1.065)	(0.109, 1.066)	(0.109, 1.066)
0.01	PROPOSED	(12.000, 0.000, 0.359)	(12.000, 0.000, 0.358)	(12.001, 0.000, 0.359)	(12.000, 0.000, 0.359)	(11.999, 0.000, 0.359)	(12.000, 0.000, 0.359)
		(0.129, 1.265)	(0.128, 1.265)	(0.129, 1.265)	(0.129, 1.264)	(0.129, 1.265)	(0.129, 1.265)
	LSE	(12.000, 0.000, 0.319)	(12.105, 0.011, 0.402)	(12.283, 0.080, 0.611)	(12.522, 0.273, 0.898)	(12.811, 0.658, 1.234)	(13.147, 1.315, 1.618)
		(0.102, 1.000)	(0.173, 1.000)	(0.454, 1.000)	(1.079, 1.000)	(2.181, 1.000)	(3.932, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.029, 0.000, 0.335)	(12.041, 0.002, 0.336)	(12.051, 0.003, 0.337)	(12.061, 0.004, 0.339)	(12.070, 0.005, 0.341)
		(0.108, 1.055)	(0.113, 0.657)	(0.114, 0.252)	(0.116, 0.108)	(0.119, 0.054)	(0.121, 0.031)
0.02	BISQUARE	(12.000, 0.000, 0.330)	(12.006, 0.000, 0.333)	(12.001, 0.000, 0.331)	(12.000, 0.000, 0.331)	(12.000, 0.000, 0.331)	(12.000, 0.000, 0.331)
		(0.109, 1.065)	(0.111, 0.644)	(0.109, 0.241)	(0.109, 0.101)	(0.109, 0.050)	(0.109, 0.028)
	PROPOSED	(12.000, 0.000, 0.359)	(12.002, 0.000, 0.360)	(12.001, 0.000, 0.359)	(12.000, 0.000, 0.359)	(12.000, 0.000, 0.359)	(12.000, 0.000, 0.359)
		(0.129, 1.264)	(0.130, 0.752)	(0.129, 0.284)	(0.129, 0.119)	(0.129, 0.059)	(0.129, 0.033)
	LSE	(12.000, 0.000, 0.319)	(12.209, 0.044, 0.468)	(12.560, 0.314, 0.795)	(13.031, 1.063, 1.208)	(13.602, 2.566, 1.677)	(14.249, 5.058, 2.200)
		(0.102, 1.000)	(0.263, 1.000)	(0.946, 1.000)	(2.521, 1.000)	(5.377, 1.000)	(9.897, 1.000)
0.05	HUBER	(12.000, 0.000, 0.328)	(12.059, 0.000, 0.342)	(12.082, 0.007, 0.345)	(12.104, 0.011, 0.348)	(12.125, 0.016, 0.351)	(12.144, 0.021, 0.356)
		(0.107, 1.055)	(0.121, 0.459)	(0.126, 0.133)	(0.132, 0.052)	(0.139, 0.026)	(0.147, 0.015)
	BISQUARE	(12.000, 0.000, 0.329)	(12.013, 0.000, 0.336)	(12.002, 0.000, 0.332)	(12.001, 0.000, 0.332)	(12.001, 0.000, 0.332)	(12.001, 0.000, 0.332)
		(0.108, 1.065)	(0.113, 0.431)	(0.110, 0.117)	(0.110, 0.044)	(0.110, 0.020)	(0.110, 0.011)
PROPOSED	(12.000, 0.000, 0.359)	(12.004, 0.000, 0.361)	(12.000, 0.000, 0.359)	(12.000, 0.000, 0.359)	(12.000, 0.000, 0.359)	(12.000, 0.000, 0.359)	

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.129, 1.265)	(0.130, 0.495)	(0.129, 0.136)	(0.129, 0.051)	(0.129, 0.024)	(0.129, 0.013)
0.03	LSE	(12.000, 0.000, 0.319)	(12.312, 0.097, 0.525)	(12.833, 0.694, 0.937)	(13.529, 2.337, 1.439)	(14.369, 5.613, 2.000)	(15.317, 11.000, 2.611)
		(0.102, 1.000)	(0.373, 1.000)	(1.573, 1.000)	(4.406, 1.000)	(9.612, 1.000)	(17.815, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.089, 0.000, 0.350)	(12.126, 0.016, 0.355)	(12.159, 0.025, 0.360)	(12.191, 0.036, 0.365)	(12.219, 0.048, 0.371)
		(0.107, 1.054)	(0.131, 0.351)	(0.142, 0.090)	(0.154, 0.035)	(0.170, 0.018)	(0.186, 0.010)
	BISQUARE	(12.000, 0.000, 0.329)	(12.021, 0.000, 0.340)	(12.003, 0.000, 0.334)	(12.001, 0.000, 0.334)	(12.001, 0.000, 0.334)	(12.000, 0.000, 0.333)
		(0.108, 1.065)	(0.116, 0.311)	(0.112, 0.071)	(0.111, 0.025)	(0.111, 0.012)	(0.111, 0.006)
0.04	LSE	(12.000, 0.000, 0.359)	(12.007, 0.000, 0.362)	(12.001, 0.000, 0.360)	(12.000, 0.000, 0.359)	(12.001, 0.000, 0.359)	(12.000, 0.000, 0.359)
		(0.129, 1.264)	(0.131, 0.351)	(0.130, 0.082)	(0.129, 0.029)	(0.129, 0.013)	(0.129, 0.007)
	HUBER	(12.000, 0.000, 0.319)	(12.413, 0.171, 0.574)	(13.102, 1.215, 1.057)	(14.016, 4.064, 1.624)	(15.114, 9.695, 2.254)	(16.347, 18.898, 2.928)
		(0.102, 1.000)	(0.500, 1.000)	(2.333, 1.000)	(6.701, 1.000)	(14.777, 1.000)	(27.472, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.121, 0.001, 0.359)	(12.171, 0.029, 0.365)	(12.216, 0.047, 0.373)	(12.260, 0.067, 0.381)	(12.300, 0.090, 0.389)
		(0.107, 1.055)	(0.143, 0.287)	(0.163, 0.070)	(0.186, 0.028)	(0.212, 0.014)	(0.241, 0.009)
0.05	BISQUARE	(12.000, 0.000, 0.329)	(12.029, 0.001, 0.344)	(12.004, 0.000, 0.336)	(12.001, 0.000, 0.335)	(12.001, 0.000, 0.335)	(12.001, 0.000, 0.335)
		(0.108, 1.066)	(0.119, 0.239)	(0.113, 0.048)	(0.112, 0.017)	(0.112, 0.008)	(0.112, 0.004)
	PROPOSED	(12.000, 0.000, 0.359)	(12.009, 0.000, 0.363)	(12.001, 0.000, 0.360)	(12.000, 0.000, 0.360)	(12.000, 0.000, 0.359)	(12.001, 0.000, 0.359)
		(0.129, 1.265)	(0.132, 0.264)	(0.129, 0.055)	(0.129, 0.019)	(0.129, 0.009)	(0.129, 0.005)
	LSE	(12.000, 0.000, 0.319)	(12.513, 0.264, 0.618)	(13.369, 1.874, 1.158)	(14.496, 6.229, 1.783)	(15.839, 14.736, 2.461)	(17.344, 28.560, 3.181)
		(0.102, 1.000)	(0.646, 1.000)	(3.215, 1.000)	(9.408, 1.000)	(20.794, 1.000)	(38.681, 1.000)
0.06	HUBER	(12.000, 0.000, 0.328)	(12.154, 0.001, 0.368)	(12.219, 0.048, 0.377)	(12.277, 0.077, 0.387)	(12.332, 0.110, 0.398)	(12.384, 0.147, 0.410)
		(0.107, 1.054)	(0.159, 0.247)	(0.190, 0.059)	(0.226, 0.024)	(0.269, 0.013)	(0.316, 0.008)
	BISQUARE	(12.000, 0.000, 0.329)	(12.039, 0.001, 0.349)	(12.006, 0.000, 0.337)	(12.002, 0.000, 0.337)	(12.002, 0.000, 0.337)	(12.001, 0.000, 0.336)
		(0.108, 1.065)	(0.124, 0.191)	(0.114, 0.035)	(0.113, 0.012)	(0.113, 0.005)	(0.113, 0.003)
	PROPOSED	(12.000, 0.000, 0.359)	(12.012, 0.000, 0.365)	(12.002, 0.000, 0.360)	(12.001, 0.000, 0.360)	(12.001, 0.000, 0.360)	(12.001, 0.000, 0.360)
		(0.129, 1.264)	(0.134, 0.207)	(0.129, 0.040)	(0.129, 0.014)	(0.130, 0.006)	(0.129, 0.003)
0.07	LSE	(12.000, 0.000, 0.319)	(12.614, 0.377, 0.659)	(13.626, 2.643, 1.246)	(14.960, 8.761, 1.913)	(16.541, 20.621, 2.634)	(18.305, 39.754, 3.394)
		(0.102, 1.000)	(0.811, 1.000)	(4.195, 1.000)	(12.420, 1.000)	(27.557, 1.000)	(51.274, 1.000)
	HUBER	(12.000, 0.000, 0.328)	(12.189, 0.002, 0.379)	(12.267, 0.072, 0.390)	(12.340, 0.115, 0.403)	(12.408, 0.166, 0.418)	(12.473, 0.223, 0.435)
		(0.107, 1.056)	(0.179, 0.221)	(0.224, 0.053)	(0.278, 0.022)	(0.341, 0.012)	(0.413, 0.008)
	BISQUARE	(12.000, 0.000, 0.329)	(12.049, 0.002, 0.355)	(12.007, 0.000, 0.340)	(12.003, 0.000, 0.338)	(12.001, 0.000, 0.338)	(12.001, 0.000, 0.338)
		(0.108, 1.066)	(0.129, 0.159)	(0.116, 0.028)	(0.115, 0.009)	(0.114, 0.004)	(0.114, 0.002)
0.07	PROPOSED	(12.000, 0.000, 0.359)	(12.015, 0.000, 0.367)	(12.002, 0.000, 0.361)	(12.001, 0.000, 0.360)	(12.000, 0.000, 0.360)	(12.001, 0.000, 0.360)
		(0.129, 1.266)	(0.135, 0.167)	(0.130, 0.031)	(0.130, 0.010)	(0.130, 0.005)	(0.130, 0.003)
0.07	LSE	(12.000, 0.000, 0.319)	(12.712, 0.507, 0.696)	(13.884, 3.549, 1.325)	(15.417, 11.679, 2.028)	(17.224, 27.293, 2.783)	(19.236, 52.361, 3.573)
		(0.102, 1.000)	(0.991, 1.000)	(5.306, 1.000)	(15.792, 1.000)	(35.039, 1.000)	(65.125, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.07		(12.000, 0.000, 0.328) (0.107, 1.056)	(12.225, 0.004, 0.390) (0.202, 0.204)	(12.320, 0.102, 0.404) (0.266, 0.050)	(12.406, 0.165, 0.422) (0.343, 0.022)	(12.488, 0.239, 0.442) (0.434, 0.012)	(12.567, 0.321, 0.465) (0.537, 0.008)
	HUBER						
		(12.000, 0.000, 0.330) (0.109, 1.066)	(12.060, 0.004, 0.361) (0.134, 0.135)	(12.008, 0.000, 0.341) (0.117, 0.022)	(12.003, 0.000, 0.340) (0.116, 0.007)	(12.002, 0.000, 0.340) (0.115, 0.003)	(12.001, 0.000, 0.340) (0.116, 0.002)
	BISQUARE						
		(12.000, 0.000, 0.359) (0.129, 1.267)	(12.018, 0.000, 0.369) (0.136, 0.138)	(12.003, 0.000, 0.361) (0.130, 0.025)	(12.000, 0.000, 0.361) (0.130, 0.008)	(12.000, 0.000, 0.361) (0.130, 0.004)	(12.000, 0.000, 0.361) (0.130, 0.002)
	PROPOSED						
		(12.000, 0.000, 0.319) (0.102, 1.000)	(12.811, 0.657, 0.729) (1.188, 1.000)	(14.136, 4.565, 1.400) (6.525, 1.000)	(15.867, 14.954, 2.134) (19.507, 1.000)	(17.896, 34.757, 2.913) (43.245, 1.000)	(20.135, 66.180, 3.716) (79.991, 1.000)
	LSE						
		(12.000, 0.000, 0.328) (0.108, 1.055)	(12.263, 0.005, 0.401) (0.230, 0.194)	(12.374, 0.140, 0.421) (0.318, 0.049)	(12.477, 0.228, 0.444) (0.425, 0.022)	(12.575, 0.330, 0.469) (0.550, 0.013)	(12.669, 0.447, 0.500) (0.697, 0.009)
	HUBER						
0.08		(12.000, 0.000, 0.330) (0.109, 1.066)	(12.073, 0.005, 0.368) (0.140, 0.118)	(12.010, 0.000, 0.344) (0.118, 0.018)	(12.004, 0.000, 0.342) (0.117, 0.006)	(12.002, 0.000, 0.342) (0.117, 0.003)	(12.002, 0.000, 0.342) (0.117, 0.001)
	BISQUARE						
		(12.000, 0.000, 0.359) (0.129, 1.266)	(12.022, 0.000, 0.372) (0.139, 0.117)	(12.003, 0.000, 0.361) (0.131, 0.020)	(12.001, 0.000, 0.361) (0.131, 0.007)	(12.000, 0.000, 0.361) (0.130, 0.003)	(12.001, 0.000, 0.361) (0.131, 0.002)
	PROPOSED						
		(11.999, 0.000, 0.319) (0.102, 1.000)	(12.907, 0.822, 0.761) (1.401, 1.000)	(14.383, 5.680, 1.463) (7.821, 1.000)	(16.302, 18.510, 2.228) (23.474, 1.000)	(18.544, 42.828, 3.028) (51.995, 1.000)	(21.001, 81.014, 3.846) (95.803, 1.000)
	LSE						
		(11.999, 0.000, 0.328) (0.107, 1.055)	(12.302, 0.008, 0.415) (0.264, 0.188)	(12.433, 0.187, 0.440) (0.381, 0.049)	(12.553, 0.306, 0.469) (0.526, 0.022)	(12.669, 0.447, 0.505) (0.701, 0.013)	(12.779, 0.607, 0.549) (0.908, 0.009)
	HUBER						
		(11.999, 0.000, 0.329) (0.108, 1.065)	(12.087, 0.008, 0.375) (0.148, 0.106)	(12.013, 0.000, 0.346) (0.120, 0.015)	(12.005, 0.000, 0.344) (0.118, 0.005)	(12.003, 0.000, 0.344) (0.118, 0.002)	(12.002, 0.000, 0.343) (0.118, 0.001)
	BISQUARE						
0.09		(11.999, 0.000, 0.359) (0.129, 1.266)	(12.026, 0.001, 0.375) (0.141, 0.101)	(12.004, 0.000, 0.362) (0.131, 0.017)	(12.001, 0.000, 0.362) (0.131, 0.006)	(12.001, 0.000, 0.362) (0.131, 0.003)	(12.001, 0.000, 0.362) (0.131, 0.001)
	PROPOSED						
		(12.000, 0.000, 0.319) (0.102, 1.000)	(13.004, 1.008, 0.789) (1.631, 1.000)	(14.627, 6.899, 1.520) (9.209, 1.000)	(16.728, 22.359, 2.308) (27.685, 1.000)	(19.172, 51.433, 3.126) (61.202, 1.000)	(21.844, 96.906, 3.950) (112.508, 1.000)
	LSE						
		(12.000, 0.000, 0.328) (0.107, 1.055)	(12.345, 0.011, 0.429) (0.303, 0.186)	(12.494, 0.244, 0.462) (0.458, 0.050)	(12.635, 0.403, 0.500) (0.653, 0.024)	(12.769, 0.592, 0.552) (0.896, 0.015)	(12.901, 0.812, 0.616) (1.192, 0.011)
	HUBER						
		(12.000, 0.000, 0.329) (0.108, 1.065)	(12.103, 0.011, 0.384) (0.158, 0.097)	(12.014, 0.000, 0.349) (0.122, 0.013)	(12.006, 0.000, 0.346) (0.119, 0.004)	(12.003, 0.000, 0.346) (0.119, 0.002)	(12.002, 0.000, 0.345) (0.119, 0.001)
	BISQUARE						
		(12.000, 0.000, 0.359) (0.129, 1.265)	(12.031, 0.001, 0.377) (0.143, 0.088)	(12.004, 0.000, 0.364) (0.132, 0.014)	(12.002, 0.000, 0.362) (0.131, 0.005)	(12.001, 0.000, 0.362) (0.131, 0.002)	(12.001, 0.000, 0.362) (0.131, 0.001)
	PROPOSED						

Table A14: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_1) for multiple linear regression model after optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)	(4.300, 0.000, 0.039)
		(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)	(0.002, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
		(0.002, 1.056)	(0.002, 1.056)	(0.002, 1.055)	(0.002, 1.055)	(0.002, 1.055)	(0.002, 1.054)
	BISQUARE	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.040)
0.01	LSE	(0.002, 1.066)	(0.002, 1.066)	(0.002, 1.065)	(0.002, 1.065)	(0.002, 1.065)	(0.002, 1.065)
		(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
	HUBER	(0.002, 1.264)	(0.002, 1.264)	(0.002, 1.263)	(0.002, 1.262)	(0.002, 1.262)	(0.002, 1.263)
		(4.300, 0.000, 0.039)	(4.296, 0.000, 0.048)	(4.284, 0.000, 0.068)	(4.266, 0.001, 0.092)	(4.242, 0.003, 0.120)	(4.213, 0.008, 0.151)
	BISQUARE	(0.002, 1.000)	(0.002, 1.000)	(0.005, 1.000)	(0.010, 1.000)	(0.018, 1.000)	(0.030, 1.000)
0.02	LSE	(4.300, 0.000, 0.040)	(4.299, 0.000, 0.041)	(4.298, 0.000, 0.041)	(4.296, 0.000, 0.041)	(4.295, 0.000, 0.041)	(4.294, 0.000, 0.041)
		(0.002, 1.055)	(0.002, 0.726)	(0.002, 0.350)	(0.002, 0.176)	(0.002, 0.096)	(0.002, 0.057)
	HUBER	(4.300, 0.000, 0.040)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.065)	(0.002, 0.718)	(0.002, 0.341)	(0.002, 0.170)	(0.002, 0.093)	(0.002, 0.054)
	PROPOSED	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
0.03	LSE	(0.002, 1.262)	(0.002, 0.841)	(0.002, 0.402)	(0.002, 0.201)	(0.002, 0.109)	(0.002, 0.064)
		(4.300, 0.000, 0.039)	(4.292, 0.000, 0.055)	(4.269, 0.001, 0.087)	(4.233, 0.004, 0.123)	(4.186, 0.013, 0.163)	(4.129, 0.029, 0.205)
	HUBER	(0.002, 1.000)	(0.003, 1.000)	(0.009, 1.000)	(0.020, 1.000)	(0.040, 1.000)	(0.071, 1.000)
		(4.300, 0.000, 0.040)	(4.297, 0.000, 0.042)	(4.295, 0.000, 0.042)	(4.293, 0.000, 0.042)	(4.291, 0.000, 0.042)	(4.288, 0.000, 0.042)
	BISQUARE	(0.002, 1.055)	(0.002, 0.562)	(0.002, 0.210)	(0.002, 0.092)	(0.002, 0.047)	(0.002, 0.027)
0.04	LSE	(4.300, 0.000, 0.040)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.065)	(0.002, 0.544)	(0.002, 0.196)	(0.002, 0.084)	(0.002, 0.042)	(0.002, 0.023)
	HUBER	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.264)	(0.002, 0.628)	(0.002, 0.229)	(0.002, 0.098)	(0.002, 0.049)	(0.002, 0.027)
	PROPOSED	(4.300, 0.000, 0.039)	(4.288, 0.000, 0.062)	(4.254, 0.002, 0.102)	(4.201, 0.010, 0.147)	(4.131, 0.028, 0.195)	(4.049, 0.063, 0.244)
0.05	LSE	(0.002, 1.000)	(0.004, 1.000)	(0.013, 1.000)	(0.031, 1.000)	(0.066, 1.000)	(0.123, 1.000)
		(4.300, 0.000, 0.040)	(4.296, 0.000, 0.043)	(4.292, 0.000, 0.043)	(4.289, 0.000, 0.043)	(4.286, 0.000, 0.043)	(4.283, 0.000, 0.044)
	HUBER	(0.002, 1.055)	(0.002, 0.465)	(0.002, 0.152)	(0.002, 0.063)	(0.002, 0.032)	(0.002, 0.018)
		(4.300, 0.000, 0.040)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
	PROPOSED	(0.002, 1.066)	(0.002, 0.439)	(0.002, 0.135)	(0.002, 0.054)	(0.002, 0.025)	(0.002, 0.014)
0.06	LSE	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.264)	(0.002, 0.628)	(0.002, 0.229)	(0.002, 0.098)	(0.002, 0.049)	(0.002, 0.027)
	HUBER	(4.300, 0.000, 0.039)	(4.292, 0.000, 0.055)	(4.269, 0.001, 0.087)	(4.233, 0.004, 0.123)	(4.186, 0.013, 0.163)	(4.129, 0.029, 0.205)
		(0.002, 1.000)	(0.003, 1.000)	(0.009, 1.000)	(0.020, 1.000)	(0.040, 1.000)	(0.071, 1.000)
	BISQUARE	(0.002, 1.055)	(0.002, 0.562)	(0.002, 0.210)	(0.002, 0.092)	(0.002, 0.047)	(0.002, 0.027)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.002, 1.265)	(0.002, 0.498)	(0.002, 0.156)	(0.002, 0.062)	(0.002, 0.029)	(0.002, 0.016)
0.04	LSE	(4.300, 0.000, 0.039)	(4.284, 0.000, 0.067)	(4.240, 0.004, 0.115)	(4.170, 0.017, 0.166)	(4.079, 0.049, 0.220)	(3.971, 0.108, 0.276)
		(0.002, 1.000)	(0.005, 1.000)	(0.017, 1.000)	(0.045, 1.000)	(0.097, 1.000)	(0.185, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.295, 0.000, 0.044)	(4.290, 0.000, 0.044)	(4.285, 0.000, 0.044)	(4.280, 0.000, 0.045)	(4.276, 0.001, 0.045)
		(0.002, 1.055)	(0.002, 0.405)	(0.002, 0.121)	(0.002, 0.049)	(0.002, 0.024)	(0.003, 0.014)
	BISQUARE	(4.300, 0.000, 0.040)	(4.298, 0.000, 0.042)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.065)	(0.002, 0.373)	(0.002, 0.101)	(0.002, 0.038)	(0.002, 0.017)	(0.002, 0.009)
0.05	LSE	(4.300, 0.000, 0.044)	(4.299, 0.000, 0.045)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.263)	(0.002, 0.417)	(0.002, 0.116)	(0.002, 0.044)	(0.002, 0.020)	(0.002, 0.011)
	HUBER	(4.300, 0.000, 0.039)	(4.281, 0.000, 0.072)	(4.225, 0.006, 0.126)	(4.140, 0.026, 0.183)	(4.028, 0.074, 0.242)	(3.896, 0.163, 0.302)
		(0.002, 1.000)	(0.006, 1.000)	(0.022, 1.000)	(0.059, 1.000)	(0.132, 1.000)	(0.254, 1.000)
	BISQUARE	(4.300, 0.000, 0.040)	(4.293, 0.000, 0.045)	(4.287, 0.000, 0.045)	(4.281, 0.000, 0.046)	(4.275, 0.001, 0.046)	(4.269, 0.001, 0.047)
		(0.002, 1.055)	(0.002, 0.361)	(0.002, 0.102)	(0.002, 0.041)	(0.003, 0.021)	(0.003, 0.012)
0.06	LSE	(4.300, 0.000, 0.040)	(4.298, 0.000, 0.043)	(4.299, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)	(4.300, 0.000, 0.041)
		(0.002, 1.066)	(0.002, 0.324)	(0.002, 0.080)	(0.002, 0.029)	(0.002, 0.013)	(0.002, 0.007)
	HUBER	(4.300, 0.000, 0.044)	(4.299, 0.000, 0.045)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.265)	(0.002, 0.356)	(0.002, 0.090)	(0.002, 0.033)	(0.002, 0.015)	(0.002, 0.008)
	BISQUARE	(4.300, 0.000, 0.039)	(4.277, 0.001, 0.077)	(4.212, 0.008, 0.136)	(4.110, 0.036, 0.197)	(3.979, 0.103, 0.260)	(3.825, 0.226, 0.324)
		(0.002, 1.000)	(0.006, 1.000)	(0.026, 1.000)	(0.075, 1.000)	(0.170, 1.000)	(0.331, 1.000)
0.07	LSE	(4.300, 0.000, 0.040)	(4.292, 0.000, 0.046)	(4.284, 0.000, 0.046)	(4.276, 0.001, 0.047)	(4.269, 0.001, 0.048)	(4.262, 0.001, 0.049)
		(0.002, 1.055)	(0.002, 0.333)	(0.002, 0.091)	(0.003, 0.037)	(0.003, 0.019)	(0.004, 0.011)
	HUBER	(4.300, 0.000, 0.040)	(4.297, 0.000, 0.043)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)
		(0.002, 1.065)	(0.002, 0.289)	(0.002, 0.066)	(0.002, 0.023)	(0.002, 0.010)	(0.002, 0.005)
	BISQUARE	(4.300, 0.000, 0.044)	(4.299, 0.000, 0.045)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.264)	(0.002, 0.312)	(0.002, 0.074)	(0.002, 0.026)	(0.002, 0.011)	(0.002, 0.006)
0.08	LSE	(4.300, 0.000, 0.039)	(4.273, 0.001, 0.081)	(4.198, 0.010, 0.145)	(4.082, 0.048, 0.210)	(3.932, 0.135, 0.276)	(3.756, 0.296, 0.343)
		(0.002, 1.000)	(0.007, 1.000)	(0.031, 1.000)	(0.092, 1.000)	(0.212, 1.000)	(0.414, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.290, 0.000, 0.047)	(4.281, 0.000, 0.048)	(4.272, 0.001, 0.049)	(4.263, 0.001, 0.050)	(4.255, 0.002, 0.051)
		(0.002, 1.056)	(0.002, 0.312)	(0.003, 0.084)	(0.003, 0.034)	(0.004, 0.018)	(0.005, 0.011)
	BISQUARE	(4.300, 0.000, 0.040)	(4.297, 0.000, 0.044)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)
		(0.002, 1.066)	(0.002, 0.264)	(0.002, 0.056)	(0.002, 0.019)	(0.002, 0.008)	(0.002, 0.004)
0.08	LSE	(4.300, 0.000, 0.044)	(4.299, 0.000, 0.045)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.265)	(0.002, 0.279)	(0.002, 0.062)	(0.002, 0.021)	(0.002, 0.009)	(0.002, 0.005)
0.08	LSE	(4.300, 0.000, 0.039)	(4.270, 0.001, 0.085)	(4.185, 0.013, 0.154)	(4.054, 0.061, 0.222)	(3.886, 0.171, 0.291)	(3.689, 0.373, 0.360)
		(0.002, 1.000)	(0.008, 1.000)	(0.037, 1.000)	(0.110, 1.000)	(0.256, 1.000)	(0.502, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.09	HUBER	(4.300, 0.000, 0.040)	(4.289, 0.000, 0.048)	(4.277, 0.001, 0.049)	(4.267, 0.001, 0.050)	(4.256, 0.002, 0.052)	(4.247, 0.003, 0.054)
		(0.002, 1.055)	(0.002, 0.297)	(0.003, 0.079)	(0.004, 0.033)	(0.005, 0.018)	(0.006, 0.011)
	BISQUARE	(4.300, 0.000, 0.040)	(4.296, 0.000, 0.045)	(4.299, 0.000, 0.042)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)
		(0.002, 1.065)	(0.002, 0.243)	(0.002, 0.048)	(0.002, 0.016)	(0.002, 0.007)	(0.002, 0.003)
	PROPOSED	(4.300, 0.000, 0.044)	(4.298, 0.000, 0.045)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.262)	(0.002, 0.252)	(0.002, 0.054)	(0.002, 0.018)	(0.002, 0.008)	(0.002, 0.004)
	LSE	(4.300, 0.000, 0.039)	(4.267, 0.001, 0.089)	(4.172, 0.016, 0.161)	(4.027, 0.074, 0.233)	(3.842, 0.210, 0.304)	(3.626, 0.455, 0.374)
		(0.002, 1.000)	(0.009, 1.000)	(0.042, 1.000)	(0.129, 1.000)	(0.302, 1.000)	(0.595, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.287, 0.000, 0.049)	(4.274, 0.001, 0.051)	(4.261, 0.002, 0.052)	(4.249, 0.003, 0.055)	(4.238, 0.004, 0.058)
		(0.002, 1.056)	(0.003, 0.289)	(0.003, 0.077)	(0.004, 0.033)	(0.006, 0.018)	(0.007, 0.012)
0.1	BISQUARE	(4.300, 0.000, 0.041)	(4.296, 0.000, 0.045)	(4.299, 0.000, 0.043)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)
		(0.002, 1.066)	(0.002, 0.229)	(0.002, 0.043)	(0.002, 0.014)	(0.002, 0.006)	(0.002, 0.003)
	PROPOSED	(4.300, 0.000, 0.044)	(4.298, 0.000, 0.046)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.264)	(0.002, 0.232)	(0.002, 0.047)	(0.002, 0.015)	(0.002, 0.007)	(0.002, 0.003)
	LSE	(4.300, 0.000, 0.039)	(4.263, 0.001, 0.093)	(4.160, 0.020, 0.168)	(4.001, 0.089, 0.243)	(3.799, 0.251, 0.316)	(3.565, 0.540, 0.387)
		(0.002, 1.000)	(0.010, 1.000)	(0.048, 1.000)	(0.148, 1.000)	(0.350, 1.000)	(0.690, 1.000)
	HUBER	(4.300, 0.000, 0.040)	(4.285, 0.000, 0.051)	(4.270, 0.001, 0.053)	(4.255, 0.002, 0.055)	(4.242, 0.003, 0.058)	(4.228, 0.005, 0.063)
		(0.002, 1.055)	(0.003, 0.283)	(0.004, 0.077)	(0.005, 0.034)	(0.007, 0.019)	(0.009, 0.013)
	BISQUARE	(4.300, 0.000, 0.040)	(4.295, 0.000, 0.046)	(4.298, 0.000, 0.043)	(4.299, 0.000, 0.042)	(4.300, 0.000, 0.042)	(4.300, 0.000, 0.042)
		(0.002, 1.065)	(0.002, 0.217)	(0.002, 0.038)	(0.002, 0.012)	(0.002, 0.005)	(0.002, 0.003)
0.1	PROPOSED	(4.300, 0.000, 0.044)	(4.298, 0.000, 0.046)	(4.299, 0.000, 0.045)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)	(4.300, 0.000, 0.044)
		(0.002, 1.263)	(0.002, 0.214)	(0.002, 0.042)	(0.002, 0.013)	(0.002, 0.006)	(0.002, 0.003)

Table A15: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_2) for multiple linear regression model after optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)	(2.500, 0.000, 0.053)
		(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)
	HUBER	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.054)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
		(0.003, 1.052)	(0.003, 1.052)	(0.003, 1.053)	(0.003, 1.052)	(0.003, 1.053)	(0.003, 1.053)
0	BISQUARE	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.01		(0.003, 1.065)	(0.003, 1.065)	(0.003, 1.066)	(0.003, 1.065)	(0.003, 1.067)	(0.003, 1.066)
		(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
	PROPOSED	(0.004, 1.280)	(0.004, 1.280)	(0.004, 1.282)	(0.004, 1.281)	(0.004, 1.283)	(0.004, 1.285)
		(2.500, 0.000, 0.053)	(2.496, 0.000, 0.065)	(2.484, 0.000, 0.092)	(2.467, 0.001, 0.124)	(2.446, 0.003, 0.157)	(2.422, 0.006, 0.189)
	LSE	(0.003, 1.000)	(0.004, 1.000)	(0.009, 1.000)	(0.017, 1.000)	(0.027, 1.000)	(0.042, 1.000)
		(2.500, 0.000, 0.054)	(2.499, 0.000, 0.056)	(2.498, 0.000, 0.056)	(2.497, 0.000, 0.056)	(2.496, 0.000, 0.056)	(2.495, 0.000, 0.056)
	HUBER	(0.003, 1.053)	(0.003, 0.722)	(0.003, 0.354)	(0.003, 0.188)	(0.003, 0.113)	(0.003, 0.074)
		(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
	BISQUARE	(0.003, 1.066)	(0.003, 0.716)	(0.003, 0.346)	(0.003, 0.184)	(0.003, 0.110)	(0.003, 0.072)
	PROPOSED	(0.004, 1.284)	(0.004, 0.850)	(0.004, 0.412)	(0.004, 0.219)	(0.004, 0.132)	(0.004, 0.086)
0.02		(2.500, 0.000, 0.053)	(2.491, 0.000, 0.075)	(2.468, 0.001, 0.118)	(2.435, 0.004, 0.166)	(2.393, 0.011, 0.214)	(2.347, 0.023, 0.261)
	LSE	(0.003, 1.000)	(0.006, 1.000)	(0.015, 1.000)	(0.032, 1.000)	(0.057, 1.000)	(0.092, 1.000)
		(2.500, 0.000, 0.054)	(2.498, 0.000, 0.057)	(2.495, 0.000, 0.057)	(2.494, 0.000, 0.057)	(2.492, 0.000, 0.057)	(2.490, 0.000, 0.057)
	HUBER	(0.003, 1.053)	(0.003, 0.562)	(0.003, 0.216)	(0.003, 0.103)	(0.003, 0.058)	(0.003, 0.036)
		(2.500, 0.000, 0.055)	(2.499, 0.000, 0.056)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)	(2.500, 0.000, 0.055)
	BISQUARE	(0.003, 1.066)	(0.003, 0.544)	(0.003, 0.204)	(0.003, 0.096)	(0.003, 0.054)	(0.003, 0.033)
		(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
	PROPOSED	(0.004, 1.283)	(0.004, 0.636)	(0.004, 0.241)	(0.004, 0.114)	(0.004, 0.063)	(0.004, 0.040)
		(2.500, 0.000, 0.053)	(2.487, 0.000, 0.084)	(2.453, 0.002, 0.140)	(2.403, 0.009, 0.199)	(2.342, 0.025, 0.257)	(2.274, 0.051, 0.315)
	LSE	(0.003, 1.000)	(0.007, 1.000)	(0.022, 1.000)	(0.049, 1.000)	(0.091, 1.000)	(0.150, 1.000)
0.03		(2.500, 0.000, 0.054)	(2.497, 0.000, 0.058)	(2.493, 0.000, 0.058)	(2.490, 0.000, 0.058)	(2.487, 0.000, 0.058)	(2.485, 0.000, 0.058)
	HUBER	(0.003, 1.052)	(0.003, 0.466)	(0.003, 0.158)	(0.003, 0.071)	(0.004, 0.039)	(0.004, 0.024)
		(2.500, 0.000, 0.055)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
	BISQUARE	(0.003, 1.066)	(0.003, 0.441)	(0.003, 0.143)	(0.003, 0.063)	(0.003, 0.034)	(0.003, 0.021)
		(2.500, 0.000, 0.060)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
	PROPOSED	(0.004, 1.282)	(0.004, 0.508)	(0.004, 0.167)	(0.004, 0.074)	(0.004, 0.040)	(0.004, 0.024)
		(2.500, 0.000, 0.053)	(2.483, 0.000, 0.092)	(2.438, 0.004, 0.158)	(2.372, 0.016, 0.226)	(2.293, 0.043, 0.293)	(2.205, 0.087, 0.358)
	LSE	(0.003, 1.000)	(0.009, 1.000)	(0.029, 1.000)	(0.067, 1.000)	(0.129, 1.000)	(0.215, 1.000)
		(2.500, 0.000, 0.055)	(2.495, 0.000, 0.059)	(2.490, 0.000, 0.060)	(2.486, 0.000, 0.060)	(2.483, 0.000, 0.060)	(2.480, 0.000, 0.060)
	HUBER	(0.003, 1.053)	(0.004, 0.406)	(0.004, 0.126)	(0.004, 0.055)	(0.004, 0.030)	(0.004, 0.019)
0.04		(2.500, 0.000, 0.055)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
	BISQUARE	(0.003, 1.066)	(0.003, 0.375)	(0.003, 0.108)	(0.003, 0.046)	(0.003, 0.024)	(0.003, 0.014)
		(2.500, 0.000, 0.060)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
	PROPOSED	(0.004, 1.284)	(0.004, 0.424)	(0.004, 0.126)	(0.004, 0.054)	(0.004, 0.028)	(0.004, 0.017)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.05	LSE	(2.500, 0.000, 0.053)	(2.479, 0.000, 0.099)	(2.423, 0.006, 0.174)	(2.343, 0.025, 0.250)	(2.245, 0.065, 0.325)	(2.137, 0.132, 0.396)
		(0.003, 1.000)	(0.010, 1.000)	(0.036, 1.000)	(0.087, 1.000)	(0.171, 1.000)	(0.289, 1.000)
	HUBER	(2.500, 0.000, 0.054)	(2.494, 0.000, 0.061)	(2.488, 0.000, 0.061)	(2.483, 0.000, 0.061)	(2.478, 0.000, 0.061)	(2.474, 0.001, 0.062)
		(0.003, 1.052)	(0.004, 0.365)	(0.004, 0.108)	(0.004, 0.046)	(0.004, 0.025)	(0.004, 0.015)
	BISQUARE	(2.500, 0.000, 0.055)	(2.498, 0.000, 0.058)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
0.06	LSE	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
		(0.004, 1.281)	(0.004, 0.366)	(0.004, 0.101)	(0.004, 0.042)	(0.004, 0.021)	(0.004, 0.013)
	HUBER	(2.500, 0.000, 0.053)	(2.475, 0.001, 0.106)	(2.409, 0.008, 0.188)	(2.314, 0.035, 0.272)	(2.199, 0.090, 0.353)	(2.073, 0.182, 0.431)
		(0.003, 1.000)	(0.012, 1.000)	(0.043, 1.000)	(0.108, 1.000)	(0.215, 1.000)	(0.368, 1.000)
	BISQUARE	(2.500, 0.000, 0.054)	(2.493, 0.000, 0.062)	(2.485, 0.000, 0.063)	(2.479, 0.000, 0.063)	(2.473, 0.001, 0.063)	(2.468, 0.001, 0.063)
0.07	LSE	(2.500, 0.000, 0.055)	(2.498, 0.000, 0.059)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)	(2.500, 0.000, 0.056)
		(0.003, 1.066)	(0.003, 0.295)	(0.003, 0.073)	(0.003, 0.029)	(0.003, 0.015)	(0.003, 0.009)
	HUBER	(2.500, 0.000, 0.060)	(2.499, 0.000, 0.062)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
		(0.004, 1.282)	(0.004, 0.321)	(0.004, 0.084)	(0.004, 0.034)	(0.004, 0.017)	(0.004, 0.010)
	PROPOSED	(2.500, 0.000, 0.053)	(2.472, 0.001, 0.111)	(2.396, 0.011, 0.201)	(2.286, 0.046, 0.290)	(2.155, 0.119, 0.379)	(2.010, 0.240, 0.460)
0.08	LSE	(2.500, 0.000, 0.055)	(2.491, 0.000, 0.064)	(2.482, 0.000, 0.064)	(2.475, 0.001, 0.065)	(2.468, 0.001, 0.065)	(2.462, 0.001, 0.066)
		(0.003, 1.052)	(0.004, 0.315)	(0.004, 0.087)	(0.005, 0.037)	(0.005, 0.020)	(0.006, 0.013)
	HUBER	(2.500, 0.000, 0.055)	(2.498, 0.000, 0.060)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
		(0.003, 1.065)	(0.004, 0.271)	(0.003, 0.063)	(0.003, 0.025)	(0.003, 0.012)	(0.003, 0.007)
	PROPOSED	(2.500, 0.000, 0.060)	(2.499, 0.000, 0.062)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
0.09	LSE	(0.004, 1.281)	(0.004, 0.289)	(0.004, 0.071)	(0.004, 0.028)	(0.004, 0.014)	(0.004, 0.008)
		(2.500, 0.000, 0.053)	(2.468, 0.001, 0.117)	(2.382, 0.014, 0.213)	(2.259, 0.058, 0.308)	(2.112, 0.151, 0.401)	(1.950, 0.302, 0.490)
	HUBER	(0.003, 1.000)	(0.015, 1.000)	(0.059, 1.000)	(0.153, 1.000)	(0.312, 1.000)	(0.542, 1.000)
		(2.500, 0.000, 0.054)	(2.490, 0.000, 0.066)	(2.479, 0.000, 0.066)	(2.470, 0.001, 0.067)	(2.462, 0.001, 0.067)	(2.455, 0.002, 0.068)
	BISQUARE	(0.003, 1.053)	(0.004, 0.300)	(0.005, 0.081)	(0.005, 0.035)	(0.006, 0.019)	(0.007, 0.012)
0.10	LSE	(2.500, 0.000, 0.055)	(2.497, 0.000, 0.061)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
		(0.003, 1.066)	(0.004, 0.251)	(0.003, 0.055)	(0.003, 0.021)	(0.003, 0.010)	(0.003, 0.006)
	HUBER	(2.500, 0.000, 0.060)	(2.499, 0.000, 0.062)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.060)	(2.500, 0.000, 0.060)
		(0.004, 1.283)	(0.004, 0.263)	(0.004, 0.062)	(0.004, 0.024)	(0.004, 0.012)	(0.004, 0.007)
	PROPOSED	(2.500, 0.000, 0.053)	(2.464, 0.001, 0.122)	(2.369, 0.017, 0.224)	(2.233, 0.071, 0.325)	(2.069, 0.186, 0.423)	(1.893, 0.368, 0.514)
0.11	LSE	(0.003, 1.000)	(0.016, 1.000)	(0.067, 1.000)	(0.177, 1.000)	(0.364, 1.000)	(0.632, 1.000)
	HUBER	(2.500, 0.000, 0.055)	(2.488, 0.000, 0.068)	(2.476, 0.001, 0.068)	(2.465, 0.001, 0.069)	(2.456, 0.002, 0.070)	(2.447, 0.003, 0.072)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.1		(0.003, 1.052)	(0.005, 0.290)	(0.005, 0.078)	(0.006, 0.034)	(0.007, 0.019)	(0.008, 0.013)
		(2.500, 0.000, 0.055)	(2.496, 0.000, 0.062)	(2.499, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.057)
	BISQUARE	(0.003, 1.065)	(0.004, 0.237)	(0.003, 0.049)	(0.003, 0.018)	(0.003, 0.009)	(0.003, 0.005)
		(2.500, 0.000, 0.060)	(2.499, 0.000, 0.063)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.061)
	PROPOSED	(0.004, 1.280)	(0.004, 0.242)	(0.004, 0.055)	(0.004, 0.021)	(0.004, 0.010)	(0.004, 0.006)
		(2.500, 0.000, 0.053)	(2.461, 0.002, 0.128)	(2.356, 0.021, 0.234)	(2.207, 0.086, 0.340)	(2.030, 0.220, 0.443)	(1.839, 0.438, 0.538)
	LSE	(0.003, 1.000)	(0.018, 1.000)	(0.075, 1.000)	(0.202, 1.000)	(0.416, 1.000)	(0.727, 1.000)
		(2.500, 0.000, 0.054)	(2.487, 0.000, 0.070)	(2.473, 0.001, 0.071)	(2.460, 0.002, 0.072)	(2.449, 0.003, 0.073)	(2.439, 0.004, 0.076)
	HUBER	(0.003, 1.053)	(0.005, 0.283)	(0.006, 0.076)	(0.007, 0.033)	(0.008, 0.019)	(0.010, 0.013)
		(2.500, 0.000, 0.055)	(2.496, 0.000, 0.063)	(2.499, 0.000, 0.058)	(2.500, 0.000, 0.057)	(2.500, 0.000, 0.058)	(2.500, 0.000, 0.057)
0.01		(0.003, 1.066)	(0.004, 0.224)	(0.003, 0.044)	(0.003, 0.016)	(0.003, 0.008)	(0.003, 0.005)
		(2.500, 0.000, 0.060)	(2.499, 0.000, 0.063)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.061)	(2.500, 0.000, 0.061)
	PROPOSED	(0.004, 1.283)	(0.004, 0.224)	(0.004, 0.049)	(0.004, 0.018)	(0.004, 0.009)	(0.004, 0.005)

Table A16: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_3) for multiple linear regression model after optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0		(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)	(0.800, 0.000, 0.054)
	LSE	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)	(0.003, 1.000)
		(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
	HUBER	(0.003, 1.054)	(0.003, 1.053)	(0.003, 1.053)	(0.003, 1.054)	(0.003, 1.052)	(0.003, 1.053)
		(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
	BISQUARE	(0.003, 1.067)	(0.003, 1.065)	(0.003, 1.065)	(0.003, 1.066)	(0.003, 1.065)	(0.003, 1.066)
		(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)
	PROPOSED	(0.004, 1.281)	(0.004, 1.278)	(0.004, 1.280)	(0.004, 1.281)	(0.004, 1.276)	(0.004, 1.280)
		(0.800, 0.000, 0.054)	(0.797, 0.000, 0.067)	(0.788, 0.000, 0.094)	(0.774, 0.001, 0.126)	(0.757, 0.002, 0.159)	(0.738, 0.004, 0.193)
	LSE	(0.003, 1.000)	(0.004, 1.000)	(0.009, 1.000)	(0.017, 1.000)	(0.027, 1.000)	(0.041, 1.000)
0.01		(0.800, 0.000, 0.056)	(0.799, 0.000, 0.057)	(0.798, 0.000, 0.057)	(0.798, 0.000, 0.057)	(0.797, 0.000, 0.057)	(0.796, 0.000, 0.057)
	HUBER	(0.003, 1.053)	(0.003, 0.724)	(0.003, 0.360)	(0.003, 0.194)	(0.003, 0.119)	(0.003, 0.079)
		(0.800, 0.000, 0.056)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.056)
	BISQUARE	(0.003, 1.065)	(0.004, 1.278)	(0.003, 0.352)	(0.003, 0.190)	(0.003, 0.116)	(0.003, 0.077)

		λ					
α	ESTIMATOR	0	1	2	3	4	5
0.02	PROPOSED	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)
		(0.004, 1.278)	(0.004, 0.849)	(0.004, 0.420)	(0.004, 0.227)	(0.004, 0.138)	(0.004, 0.091)
	LSE	(0.800, 0.000, 0.054)	(0.794, 0.000, 0.077)	(0.776, 0.001, 0.121)	(0.749, 0.003, 0.169)	(0.716, 0.007, 0.217)	(0.678, 0.015, 0.265)
		(0.003, 1.000)	(0.006, 1.000)	(0.015, 1.000)	(0.031, 1.000)	(0.054, 1.000)	(0.085, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.798, 0.000, 0.058)	(0.797, 0.000, 0.058)	(0.795, 0.000, 0.058)	(0.794, 0.000, 0.058)	(0.792, 0.000, 0.058)
		(0.003, 1.053)	(0.003, 0.563)	(0.003, 0.222)	(0.003, 0.108)	(0.003, 0.063)	(0.003, 0.040)
	BISQUARE	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.056)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.056)
		(0.003, 1.066)	(0.003, 0.547)	(0.003, 0.209)	(0.003, 0.102)	(0.003, 0.059)	(0.003, 0.037)
	PROPOSED	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)
		(0.004, 1.280)	(0.004, 0.638)	(0.004, 0.248)	(0.004, 0.121)	(0.004, 0.070)	(0.004, 0.044)
0.03	LSE	(0.800, 0.000, 0.054)	(0.790, 0.000, 0.086)	(0.764, 0.001, 0.142)	(0.726, 0.006, 0.202)	(0.676, 0.015, 0.262)	(0.620, 0.032, 0.319)
		(0.003, 1.000)	(0.007, 1.000)	(0.021, 1.000)	(0.046, 1.000)	(0.084, 1.000)	(0.134, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.798, 0.000, 0.059)	(0.795, 0.000, 0.059)	(0.793, 0.000, 0.059)	(0.790, 0.000, 0.059)	(0.788, 0.000, 0.059)
		(0.003, 1.053)	(0.004, 0.470)	(0.004, 0.165)	(0.004, 0.077)	(0.004, 0.043)	(0.004, 0.027)
	BISQUARE	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.058)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
		(0.003, 1.065)	(0.003, 0.446)	(0.003, 0.150)	(0.003, 0.069)	(0.003, 0.038)	(0.003, 0.024)
	PROPOSED	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.061)
		(0.004, 1.280)	(0.004, 0.512)	(0.004, 0.176)	(0.004, 0.081)	(0.004, 0.045)	(0.004, 0.028)
	LSE	(0.800, 0.000, 0.054)	(0.787, 0.000, 0.094)	(0.753, 0.002, 0.161)	(0.702, 0.010, 0.230)	(0.637, 0.026, 0.298)	(0.565, 0.055, 0.363)
		(0.003, 1.000)	(0.009, 1.000)	(0.028, 1.000)	(0.063, 1.000)	(0.115, 1.000)	(0.187, 1.000)
0.04	HUBER	(0.800, 0.000, 0.056)	(0.797, 0.000, 0.061)	(0.793, 0.000, 0.061)	(0.790, 0.000, 0.061)	(0.787, 0.000, 0.061)	(0.784, 0.000, 0.061)
		(0.003, 1.052)	(0.004, 0.412)	(0.004, 0.133)	(0.004, 0.061)	(0.004, 0.034)	(0.004, 0.021)
	BISQUARE	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.058)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
		(0.003, 1.064)	(0.003, 0.382)	(0.003, 0.116)	(0.003, 0.052)	(0.003, 0.028)	(0.003, 0.017)
	PROPOSED	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.061)
		(0.004, 1.277)	(0.004, 0.432)	(0.004, 0.135)	(0.004, 0.060)	(0.004, 0.033)	(0.004, 0.020)
	LSE	(0.800, 0.000, 0.054)	(0.785, 0.000, 0.101)	(0.742, 0.003, 0.176)	(0.678, 0.015, 0.255)	(0.601, 0.040, 0.331)	(0.511, 0.083, 0.402)
		(0.003, 1.000)	(0.010, 1.000)	(0.034, 1.000)	(0.080, 1.000)	(0.149, 1.000)	(0.245, 1.000)
	HUBER	(0.800, 0.000, 0.056)	(0.796, 0.000, 0.062)	(0.791, 0.000, 0.062)	(0.787, 0.000, 0.062)	(0.783, 0.000, 0.062)	(0.780, 0.000, 0.062)
		(0.003, 1.054)	(0.004, 0.370)	(0.004, 0.114)	(0.004, 0.051)	(0.004, 0.028)	(0.004, 0.018)
0.05	BISQUARE	(0.800, 0.000, 0.056)	(0.799, 0.000, 0.059)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
		(0.003, 1.066)	(0.004, 0.336)	(0.003, 0.095)	(0.003, 0.041)	(0.003, 0.022)	(0.003, 0.013)
	PROPOSED	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)	(0.800, 0.000, 0.061)
		(0.004, 1.281)	(0.004, 0.373)	(0.004, 0.110)	(0.004, 0.048)	(0.004, 0.025)	(0.004, 0.015)
	LSE	(0.800, 0.000, 0.054)	(0.782, 0.000, 0.108)	(0.731, 0.005, 0.191)	(0.656, 0.021, 0.275)	(0.565, 0.055, 0.358)	(0.460, 0.116, 0.436)

		λ					
α	ESTIMATOR	0	1	2	3	4	5
0.05		(0.003, 1.000)	(0.012, 1.000)	(0.041, 1.000)	(0.096, 1.000)	(0.184, 1.000)	(0.306, 1.000)
		(0.800, 0.000, 0.056)	(0.795, 0.000, 0.064)	(0.789, 0.000, 0.064)	(0.784, 0.000, 0.064)	(0.780, 0.000, 0.064)	(0.775, 0.001, 0.064)
	HUBER	(0.003, 1.053)	(0.004, 0.341)	(0.004, 0.101)	(0.004, 0.045)	(0.005, 0.025)	(0.005, 0.016)
		(0.800, 0.000, 0.056)	(0.799, 0.000, 0.060)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
	BISQUARE	(0.003, 1.065)	(0.004, 0.302)	(0.003, 0.081)	(0.003, 0.034)	(0.003, 0.018)	(0.003, 0.011)
		(0.800, 0.000, 0.061)	(0.800, 0.000, 0.063)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)
	PROPOSED	(0.004, 1.280)	(0.004, 0.329)	(0.004, 0.092)	(0.004, 0.039)	(0.004, 0.021)	(0.004, 0.012)
		(0.800, 0.000, 0.054)	(0.778, 0.000, 0.114)	(0.721, 0.006, 0.205)	(0.635, 0.027, 0.295)	(0.531, 0.073, 0.384)	(0.411, 0.151, 0.467)
	LSE	(0.003, 1.000)	(0.013, 1.000)	(0.048, 1.000)	(0.114, 1.000)	(0.220, 1.000)	(0.369, 1.000)
		(0.800, 0.000, 0.056)	(0.794, 0.000, 0.065)	(0.787, 0.000, 0.066)	(0.781, 0.000, 0.066)	(0.776, 0.001, 0.066)	(0.770, 0.001, 0.066)
0.07	HUBER	(0.003, 1.053)	(0.004, 0.320)	(0.004, 0.093)	(0.005, 0.041)	(0.005, 0.023)	(0.005, 0.014)
		(0.800, 0.000, 0.056)	(0.799, 0.000, 0.061)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
	BISQUARE	(0.003, 1.066)	(0.004, 0.277)	(0.003, 0.070)	(0.003, 0.029)	(0.003, 0.015)	(0.003, 0.009)
		(0.800, 0.000, 0.061)	(0.800, 0.000, 0.063)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)
	PROPOSED	(0.004, 1.282)	(0.004, 0.296)	(0.004, 0.079)	(0.004, 0.033)	(0.004, 0.017)	(0.004, 0.010)
		(0.800, 0.000, 0.054)	(0.776, 0.001, 0.119)	(0.711, 0.008, 0.216)	(0.614, 0.034, 0.314)	(0.496, 0.092, 0.406)	(0.364, 0.190, 0.495)
	LSE	(0.003, 1.000)	(0.015, 1.000)	(0.055, 1.000)	(0.133, 1.000)	(0.257, 1.000)	(0.435, 1.000)
		(0.800, 0.000, 0.056)	(0.793, 0.000, 0.067)	(0.785, 0.000, 0.067)	(0.778, 0.000, 0.068)	(0.771, 0.001, 0.068)	(0.765, 0.001, 0.069)
	HUBER	(0.003, 1.053)	(0.005, 0.305)	(0.005, 0.087)	(0.005, 0.038)	(0.005, 0.021)	(0.006, 0.014)
		(0.800, 0.000, 0.056)	(0.798, 0.000, 0.062)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
0.08	BISQUARE	(0.003, 1.065)	(0.004, 0.258)	(0.003, 0.062)	(0.003, 0.025)	(0.003, 0.013)	(0.003, 0.008)
		(0.800, 0.000, 0.061)	(0.800, 0.000, 0.063)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)
	PROPOSED	(0.004, 1.280)	(0.004, 0.271)	(0.004, 0.070)	(0.004, 0.029)	(0.004, 0.015)	(0.004, 0.009)
		(0.800, 0.000, 0.054)	(0.773, 0.001, 0.125)	(0.700, 0.010, 0.228)	(0.594, 0.042, 0.331)	(0.463, 0.113, 0.428)	(0.319, 0.232, 0.521)
	LSE	(0.003, 1.000)	(0.016, 1.000)	(0.062, 1.000)	(0.152, 1.000)	(0.297, 1.000)	(0.503, 1.000)
		(0.800, 0.000, 0.056)	(0.792, 0.000, 0.069)	(0.783, 0.000, 0.069)	(0.774, 0.001, 0.070)	(0.767, 0.001, 0.071)	(0.759, 0.002, 0.072)
	HUBER	(0.003, 1.052)	(0.005, 0.294)	(0.005, 0.083)	(0.006, 0.037)	(0.006, 0.021)	(0.007, 0.014)
		(0.800, 0.000, 0.056)	(0.798, 0.000, 0.063)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
	BISQUARE	(0.003, 1.064)	(0.004, 0.243)	(0.003, 0.056)	(0.003, 0.023)	(0.003, 0.011)	(0.003, 0.007)
		(0.800, 0.000, 0.061)	(0.800, 0.000, 0.064)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)
0.09	PROPOSED	(0.004, 1.277)	(0.004, 0.249)	(0.004, 0.062)	(0.004, 0.025)	(0.004, 0.013)	(0.004, 0.008)
		(0.800, 0.000, 0.054)	(0.770, 0.001, 0.130)	(0.691, 0.012, 0.238)	(0.574, 0.051, 0.345)	(0.433, 0.135, 0.448)	(0.274, 0.277, 0.544)
	LSE	(0.003, 1.000)	(0.018, 1.000)	(0.069, 1.000)	(0.170, 1.000)	(0.336, 1.000)	(0.572, 1.000)
		(0.800, 0.000, 0.056)	(0.791, 0.000, 0.071)	(0.780, 0.000, 0.072)	(0.771, 0.001, 0.072)	(0.762, 0.001, 0.074)	(0.753, 0.002, 0.076)
	HUBER	(0.003, 1.052)	(0.005, 0.289)	(0.006, 0.081)	(0.006, 0.036)	(0.007, 0.021)	(0.008, 0.014)
		(0.800, 0.000, 0.056)	(0.799, 0.000, 0.060)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.057)	(0.800, 0.000, 0.057)
	BISQUARE	(0.003, 1.065)	(0.004, 0.302)	(0.003, 0.081)	(0.003, 0.034)	(0.003, 0.018)	(0.003, 0.011)
		(0.800, 0.000, 0.061)	(0.800, 0.000, 0.063)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)
	PROPOSED	(0.004, 1.280)	(0.004, 0.329)	(0.004, 0.092)	(0.004, 0.039)	(0.004, 0.021)	(0.004, 0.012)
		(0.800, 0.000, 0.054)	(0.778, 0.000, 0.114)	(0.721, 0.006, 0.205)	(0.635, 0.027, 0.295)	(0.531, 0.073, 0.384)	(0.411, 0.151, 0.467)
0.1	LSE	(0.003, 1.000)	(0.013, 1.000)	(0.048, 1.000)	(0.114, 1.000)	(0.220, 1.000)	(0.369, 1.000)
		(0.800, 0.000, 0.056)	(0.794, 0.000, 0.065)	(0.787, 0.000, 0.066)	(0.781, 0.000, 0.066)	(0.776, 0.001, 0.066)	(0.770, 0.001, 0.066)
	HUBER	(0.003, 1.053)	(0.004, 0.320)	(0.004, 0.093)	(0.005, 0.041)	(0.005, 0.023)	(0.005, 0.014)
		(0.800, 0.000, 0.056)	(0.799, 0.000, 0.061)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)	(0.800, 0.000, 0.058)
	BISQUARE	(0.003, 1.066)	(0.004, 0.277)	(0.003, 0.070)	(0.003, 0.029)	(0.003, 0.015)	(0.003, 0.009)
		(0.800, 0.000, 0.061)	(0.800, 0.000, 0.063)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)
	PROPOSED	(0.004, 1.282)	(0.004, 0.296)	(0.004, 0.079)	(0.004, 0.033)	(0.004, 0.017)	(0.004, 0.010)
		(0.800, 0.000, 0.054)	(0.776, 0.001, 0.119)	(0.711, 0.008, 0.216)	(0.614, 0.034, 0.314)	(0.496, 0.092, 0.406)	(0.364, 0.190, 0.495)
	LSE	(0.003, 1.000)	(0.015, 1.000)	(0.055, 1.000)	(0.133, 1.000)	(0.257, 1.000)	(0.435, 1.000)
		(0.800, 0.000, 0.056)	(0.793, 0.000, 0.067)	(0.785, 0.000, 0.067)	(0.778, 0.000, 0.068)	(0.771, 0.001, 0.068)	(0.765, 0.001, 0.069)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.800, 0.000, 0.056)	(0.797, 0.000, 0.064)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.059)	(0.800, 0.000, 0.059)
	BISQUARE	(0.003, 1.065)	(0.004, 0.233)	(0.003, 0.051)	(0.003, 0.020)	(0.003, 0.010)	(0.003, 0.006)
		(0.800, 0.000, 0.061)	(0.799, 0.000, 0.064)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)	(0.800, 0.000, 0.062)
	PROPOSED	(0.004, 1.278)	(0.004, 0.233)	(0.004, 0.056)	(0.004, 0.022)	(0.004, 0.011)	(0.004, 0.007)

Table A17: Estimates for (Mean, Bias², Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of the slope coefficient (β_4) for multiple linear regression model after optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)	(0.500, 0.000, 0.065)
		(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)	(0.004, 1.000)
	HUBER	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)	(0.500, 0.000, 0.066)
		(0.004, 1.054)	(0.004, 1.055)	(0.004, 1.055)	(0.004, 1.054)	(0.004, 1.055)	(0.004, 1.055)
	BISQUARE	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)
		(0.004, 1.065)	(0.004, 1.066)	(0.004, 1.065)	(0.004, 1.065)	(0.004, 1.066)	(0.004, 1.065)
0.01	LSE	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
		(0.005, 1.268)	(0.005, 1.268)	(0.005, 1.268)	(0.005, 1.267)	(0.005, 1.269)	(0.005, 1.268)
	HUBER	(0.500, 0.000, 0.065)	(0.497, 0.000, 0.079)	(0.490, 0.000, 0.112)	(0.479, 0.000, 0.151)	(0.463, 0.001, 0.192)	(0.444, 0.003, 0.235)
		(0.004, 1.000)	(0.006, 1.000)	(0.013, 1.000)	(0.023, 1.000)	(0.038, 1.000)	(0.059, 1.000)
	BISQUARE	(0.500, 0.000, 0.066)	(0.499, 0.000, 0.068)	(0.499, 0.000, 0.068)	(0.498, 0.000, 0.068)	(0.497, 0.000, 0.068)	(0.497, 0.000, 0.068)
		(0.004, 1.055)	(0.005, 0.726)	(0.005, 0.363)	(0.005, 0.196)	(0.005, 0.120)	(0.005, 0.078)
0.02	LSE	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)
		(0.004, 1.066)	(0.005, 0.719)	(0.004, 0.356)	(0.004, 0.192)	(0.004, 0.117)	(0.004, 0.076)
	HUBER	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
		(0.005, 1.268)	(0.005, 0.844)	(0.005, 0.420)	(0.005, 0.227)	(0.005, 0.138)	(0.005, 0.090)
	BISQUARE	(0.500, 0.000, 0.065)	(0.495, 0.000, 0.092)	(0.481, 0.000, 0.144)	(0.458, 0.002, 0.202)	(0.428, 0.005, 0.263)	(0.391, 0.012, 0.323)
		(0.004, 1.000)	(0.008, 1.000)	(0.021, 1.000)	(0.043, 1.000)	(0.074, 1.000)	(0.116, 1.000)
	HUBER	(0.500, 0.000, 0.066)	(0.499, 0.000, 0.069)	(0.497, 0.000, 0.069)	(0.496, 0.000, 0.069)	(0.495, 0.000, 0.069)	(0.493, 0.000, 0.069)
		(0.004, 1.054)	(0.005, 0.565)	(0.005, 0.226)	(0.005, 0.112)	(0.005, 0.065)	(0.005, 0.042)
	BISQUARE	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)
		(0.004, 1.064)	(0.005, 0.548)	(0.005, 0.214)	(0.005, 0.106)	(0.005, 0.061)	(0.005, 0.039)
	PROPOSED	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
		(0.005, 1.268)	(0.005, 0.844)	(0.005, 0.420)	(0.005, 0.227)	(0.005, 0.138)	(0.005, 0.090)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.005, 1.267)	(0.005, 0.633)	(0.005, 0.251)	(0.005, 0.124)	(0.005, 0.071)	(0.005, 0.045)
0.03	LSE	(0.500, 0.000, 0.065)	(0.493, 0.000, 0.102)	(0.472, 0.001, 0.170)	(0.438, 0.004, 0.242)	(0.393, 0.011, 0.316)	(0.339, 0.026, 0.389)
		(0.004, 1.000)	(0.011, 1.000)	(0.030, 1.000)	(0.062, 1.000)	(0.111, 1.000)	(0.177, 1.000)
	HUBER	(0.500, 0.000, 0.066)	(0.498, 0.000, 0.070)	(0.496, 0.000, 0.071)	(0.494, 0.000, 0.071)	(0.491, 0.000, 0.071)	(0.489, 0.000, 0.071)
		(0.004, 1.055)	(0.005, 0.471)	(0.005, 0.169)	(0.005, 0.080)	(0.005, 0.046)	(0.005, 0.029)
	BISQUARE	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.067)	(0.500, 0.000, 0.068)
		(0.004, 1.065)	(0.005, 0.447)	(0.005, 0.154)	(0.005, 0.073)	(0.005, 0.041)	(0.005, 0.026)
0.04	LSE	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
		(0.005, 1.268)	(0.005, 0.509)	(0.005, 0.179)	(0.005, 0.085)	(0.005, 0.048)	(0.005, 0.030)
	HUBER	(0.500, 0.000, 0.065)	(0.490, 0.000, 0.112)	(0.463, 0.001, 0.192)	(0.419, 0.006, 0.275)	(0.361, 0.019, 0.360)	(0.290, 0.044, 0.443)
		(0.004, 1.000)	(0.013, 1.000)	(0.038, 1.000)	(0.082, 1.000)	(0.149, 1.000)	(0.241, 1.000)
	BISQUARE	(0.500, 0.000, 0.066)	(0.497, 0.000, 0.072)	(0.494, 0.000, 0.072)	(0.492, 0.000, 0.072)	(0.489, 0.000, 0.072)	(0.486, 0.000, 0.073)
		(0.004, 1.055)	(0.005, 0.412)	(0.005, 0.137)	(0.005, 0.064)	(0.005, 0.036)	(0.005, 0.023)
0.05	LSE	(0.500, 0.000, 0.067)	(0.499, 0.000, 0.069)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)
		(0.004, 1.065)	(0.005, 0.384)	(0.005, 0.121)	(0.005, 0.056)	(0.005, 0.031)	(0.005, 0.019)
	HUBER	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.074)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
		(0.005, 1.267)	(0.005, 0.430)	(0.005, 0.140)	(0.005, 0.065)	(0.005, 0.036)	(0.005, 0.022)
	BISQUARE	(0.500, 0.000, 0.065)	(0.488, 0.000, 0.120)	(0.454, 0.002, 0.211)	(0.400, 0.010, 0.305)	(0.329, 0.029, 0.397)	(0.243, 0.066, 0.489)
		(0.004, 1.000)	(0.015, 1.000)	(0.046, 1.000)	(0.103, 1.000)	(0.187, 1.000)	(0.306, 1.000)
0.06	LSE	(0.500, 0.000, 0.066)	(0.497, 0.000, 0.074)	(0.493, 0.000, 0.074)	(0.489, 0.000, 0.074)	(0.485, 0.000, 0.074)	(0.482, 0.000, 0.075)
		(0.004, 1.053)	(0.005, 0.371)	(0.005, 0.118)	(0.006, 0.054)	(0.006, 0.031)	(0.006, 0.019)
	HUBER	(0.500, 0.000, 0.067)	(0.499, 0.000, 0.070)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)
		(0.004, 1.064)	(0.005, 0.338)	(0.005, 0.100)	(0.005, 0.045)	(0.005, 0.025)	(0.005, 0.015)
	BISQUARE	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.074)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
		(0.005, 1.265)	(0.005, 0.372)	(0.005, 0.114)	(0.005, 0.052)	(0.005, 0.028)	(0.005, 0.017)
0.07	LSE	(0.500, 0.000, 0.065)	(0.486, 0.000, 0.128)	(0.446, 0.003, 0.228)	(0.382, 0.014, 0.330)	(0.298, 0.041, 0.431)	(0.197, 0.092, 0.530)
		(0.004, 1.000)	(0.017, 1.000)	(0.055, 1.000)	(0.123, 1.000)	(0.227, 1.000)	(0.373, 1.000)
	HUBER	(0.500, 0.000, 0.066)	(0.496, 0.000, 0.075)	(0.491, 0.000, 0.076)	(0.487, 0.000, 0.076)	(0.482, 0.000, 0.076)	(0.477, 0.001, 0.077)
		(0.004, 1.055)	(0.006, 0.342)	(0.006, 0.106)	(0.006, 0.048)	(0.006, 0.027)	(0.006, 0.017)
	BISQUARE	(0.500, 0.000, 0.067)	(0.499, 0.000, 0.071)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)	(0.500, 0.000, 0.068)
		(0.004, 1.066)	(0.005, 0.305)	(0.005, 0.086)	(0.005, 0.038)	(0.005, 0.021)	(0.005, 0.013)
0.07	LSE	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.074)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
		(0.005, 1.268)	(0.005, 0.330)	(0.005, 0.097)	(0.005, 0.043)	(0.005, 0.023)	(0.005, 0.014)
0.07	LSE	(0.500, 0.000, 0.065)	(0.484, 0.000, 0.135)	(0.438, 0.004, 0.244)	(0.365, 0.018, 0.353)	(0.268, 0.054, 0.461)	(0.154, 0.120, 0.566)
		(0.004, 1.000)	(0.019, 1.000)	(0.063, 1.000)	(0.143, 1.000)	(0.266, 1.000)	(0.441, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.500, 0.000, 0.066)	(0.495, 0.000, 0.077)	(0.490, 0.000, 0.078)	(0.484, 0.000, 0.078)	(0.478, 0.000, 0.079)	(0.473, 0.001, 0.079)
	HUBER	(0.004, 1.054)	(0.006, 0.322)	(0.006, 0.097)	(0.006, 0.044)	(0.007, 0.025)	(0.007, 0.016)
		(0.500, 0.000, 0.067)	(0.499, 0.000, 0.072)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)
	BISQUARE	(0.004, 1.065)	(0.005, 0.280)	(0.005, 0.075)	(0.005, 0.033)	(0.005, 0.018)	(0.005, 0.011)
	PROPOSED	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.074)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
0.08		(0.500, 0.000, 0.065)	(0.482, 0.000, 0.142)	(0.430, 0.005, 0.258)	(0.347, 0.023, 0.375)	(0.240, 0.068, 0.488)	(0.113, 0.150, 0.599)
	LSE	(0.004, 1.000)	(0.021, 1.000)	(0.071, 1.000)	(0.164, 1.000)	(0.306, 1.000)	(0.509, 1.000)
		(0.500, 0.000, 0.066)	(0.494, 0.000, 0.079)	(0.488, 0.000, 0.080)	(0.481, 0.000, 0.080)	(0.475, 0.001, 0.081)	(0.468, 0.001, 0.082)
	HUBER	(0.004, 1.054)	(0.006, 0.307)	0.007, 0.091)	(0.007, 0.042)	(0.007, 0.024)	(0.008, 0.015)
		(0.500, 0.000, 0.067)	(0.499, 0.000, 0.073)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)
0.09	BISQUARE	(0.004, 1.065)	(0.005, 0.261)	(0.005, 0.067)	(0.005, 0.029)	(0.005, 0.016)	(0.005, 0.009)
		(0.500, 0.000, 0.073)	(0.500, 0.000, 0.075)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
	PROPOSED	(0.005, 1.268)	(0.006, 0.273)	(0.005, 0.075)	(0.005, 0.033)	0.005, 0.017)	(0.005, 0.011)
		(0.500, 0.000, 0.065)	(0.480, 0.000, 0.148)	(0.422, 0.006, 0.272)	(0.331, 0.029, 0.394)	(0.213, 0.082, 0.514)	(0.073, 0.182, 0.630)
	LSE	(0.004, 1.000)	(0.022, 1.000)	(0.080, 1.000)	(0.184, 1.000)	(0.347, 1.000)	(0.580, 1.000)
0.1		(0.500, 0.000, 0.066)	(0.493, 0.000, 0.081)	(0.486, 0.000, 0.082)	(0.478, 0.000, 0.083)	(0.471, 0.001, 0.084)	(0.463, 0.001, 0.086)
	HUBER	(0.004, 1.054)	(0.007, 0.297)	(0.007, 0.087)	(0.007, 0.040)	(0.008, 0.023)	(0.009, 0.015)
		(0.500, 0.000, 0.067)	(0.498, 0.000, 0.074)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)	(0.500, 0.000, 0.069)
	BISQUARE	(0.004, 1.065)	(0.006, 0.248)	(0.005, 0.061)	(0.005, 0.026)	(0.005, 0.014)	(0.005, 0.008)
	PROPOSED	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.075)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
0.1		(0.005, 1.267)	(0.006, 0.254)	(0.005, 0.067)	(0.005, 0.029)	(0.005, 0.015)	(0.005, 0.009)
		(0.500, 0.000, 0.065)	(0.478, 0.001, 0.154)	(0.415, 0.007, 0.284)	(0.316, 0.034, 0.413)	(0.186, 0.098, 0.537)	(0.033, 0.218, 0.658)
	LSE	(0.004, 1.000)	(0.024, 1.000)	(0.088, 1.000)	(0.205, 1.000)	(0.387, 1.000)	(0.650, 1.000)
		(0.500, 0.000, 0.066)	(0.492, 0.000, 0.084)	(0.484, 0.000, 0.085)	(0.475, 0.001, 0.086)	(0.466, 0.001, 0.088)	(0.457, 0.002, 0.090)
	HUBER	(0.004, 1.054)	(0.007, 0.291)	(0.007, 0.085)	(0.008, 0.039)	(0.009, 0.023)	(0.010, 0.015)
0.1		(0.500, 0.000, 0.067)	(0.498, 0.000, 0.076)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.070)	(0.500, 0.000, 0.070)
	BISQUARE	(0.004, 1.065)	(0.006, 0.237)	(0.005, 0.056)	(0.005, 0.024)	(0.005, 0.013)	(0.005, 0.007)
	PROPOSED	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.076)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)	(0.500, 0.000, 0.073)
0.1		(0.005, 1.267)	(0.006, 0.237)	(0.005, 0.061)	(0.005, 0.026)	(0.005, 0.014)	(0.005, 0.008)

Table A18: Estimates for (Mean, Bias2, Standard error (SE), Mean square error (MSE), and Asymptotic relative efficiency (ARE)) of of the overall mean square error (σ^2) for multiple linear regression model after optimization of the proposed estimator parameters.

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0	LSE	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)	(1.000, 0.000, 0.145)
		(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)	(0.021, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(0.950, 0.003, 0.227)	(0.950, 0.003, 0.227)	(0.950, 0.002, 0.227)	(0.950, 0.003, 0.227)	(0.950, 0.003, 0.227)
		(0.054, 2.570)	(0.054, 2.571)	(0.054, 2.563)	(0.054, 2.571)	(0.054, 2.557)	(0.054, 2.576)
	BISQUARE	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.226)	(0.945, 0.003, 0.227)
		(0.054, 2.579)	(0.054, 2.580)	(0.054, 2.571)	(0.054, 2.579)	(0.054, 2.565)	(0.054, 2.584)
0.01	PROPOSED	(0.941, 0.004, 0.229)	(0.941, 0.004, 0.229)	(0.941, 0.003, 0.229)	(0.941, 0.003, 0.229)	(0.941, 0.003, 0.229)	(0.941, 0.004, 0.229)
		(0.056, 2.667)	(0.056, 2.667)	(0.056, 2.653)	(0.056, 2.663)	(0.056, 2.652)	(0.056, 2.670)
	LSE	(1.000, 0.000, 0.145)	(1.508, 0.258, 0.869)	(2.979, 3.918, 3.337)	(5.335, 18.790, 7.350)	(8.496, 56.196, 12.860)	(12.434, 130.736, 19.767)
		(0.021, 1.000)	(1.013, 1.000)	(15.050, 1.000)	(72.810, 1.000)	(221.566, 1.000)	(521.464, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(0.975, 0.001, 0.234)	(0.974, 0.001, 0.234)	(0.975, 0.001, 0.234)	(0.975, 0.001, 0.234)	(0.975, 0.001, 0.234)
		(0.054, 2.565)	(0.055, 0.055)	(0.055, 0.004)	(0.055, 0.001)	(0.055, 0.000)	(0.056, 0.000)
0.02	BISQUARE	(0.945, 0.003, 0.226)	(0.968, 0.001, 0.233)	(0.967, 0.001, 0.232)	(0.968, 0.001, 0.232)	(0.967, 0.001, 0.232)	(0.968, 0.001, 0.233)
		(0.054, 2.573)	(0.055, 0.054)	(0.055, 0.004)	(0.055, 0.001)	(0.055, 0.000)	(0.055, 0.000)
	PROPOSED	(0.941, 0.004, 0.229)	(0.964, 0.001, 0.236)	(0.963, 0.001, 0.235)	(0.964, 0.001, 0.235)	(0.963, 0.001, 0.235)	(0.964, 0.001, 0.235)
		(0.056, 2.658)	(0.057, 0.056)	(0.057, 0.004)	(0.057, 0.001)	(0.057, 0.000)	(0.057, 0.000)
	LSE	(1.000, 0.000, 0.145)	(2.008, 1.017, 1.209)	(4.921, 15.371, 4.655)	(9.582, 73.655, 10.298)	(15.808, 219.287, 17.849)	(23.467, 504.778, 27.407)
		(0.021, 1.000)	(2.479, 1.000)	(37.038, 1.000)	(179.694, 1.000)	(537.867, 1.000)	(1255.941, 1.000)
0.03	HUBER	(0.950, 0.003, 0.227)	(1.001, 0.000, 0.242)	(1.002, 0.000, 0.242)	(1.002, 0.000, 0.243)	(1.003, 0.000, 0.243)	(1.003, 0.000, 0.243)
		(0.054, 2.568)	(0.058, 0.024)	(0.059, 0.002)	(0.059, 0.000)	(0.059, 0.000)	(0.059, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(0.992, 0.000, 0.239)	(0.992, 0.000, 0.239)	(0.991, 0.000, 0.239)	(0.992, 0.000, 0.239)	(0.991, 0.000, 0.239)
		(0.054, 2.577)	(0.057, 0.023)	(0.057, 0.002)	(0.057, 0.000)	(0.057, 0.000)	(0.057, 0.000)
	PROPOSED	(0.941, 0.004, 0.229)	(0.988, 0.000, 0.242)	(0.988, 0.000, 0.242)	(0.988, 0.000, 0.242)	(0.988, 0.000, 0.242)	(0.987, 0.000, 0.242)
		(0.056, 2.662)	(0.059, 0.024)	(0.059, 0.002)	(0.059, 0.000)	(0.059, 0.000)	(0.059, 0.000)
0.04	LSE	(1.000, 0.000, 0.145)	(2.499, 2.247, 1.457)	(6.821, 33.886, 5.636)	(13.713, 161.611, 12.397)	(22.953, 481.943, 21.621)	(34.175, 1100.586, 32.925)
		(0.021, 1.000)	(4.369, 1.000)	(65.648, 1.000)	(315.300, 1.000)	(949.418, 1.000)	(2184.633, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.029, 0.000, 0.250)	(1.030, 0.001, 0.251)	(1.032, 0.001, 0.252)	(1.033, 0.001, 0.252)	(1.034, 0.001, 0.253)
		(0.054, 2.565)	(0.064, 0.015)	(0.064, 0.001)	(0.064, 0.000)	(0.065, 0.000)	(0.065, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.016, 0.000, 0.246)	(1.016, 0.000, 0.246)	(1.016, 0.000, 0.246)	(1.015, 0.000, 0.245)	(1.016, 0.000, 0.246)
		(0.054, 2.573)	(0.061, 0.014)	(0.061, 0.001)	(0.061, 0.000)	(0.061, 0.000)	(0.061, 0.000)
0.05	PROPOSED	(0.941, 0.004, 0.229)	(1.012, 0.000, 0.249)	(1.012, 0.000, 0.249)	(1.012, 0.000, 0.249)	(1.012, 0.000, 0.249)	(1.012, 0.000, 0.249)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
		(0.056, 2.657)	(0.062, 0.014)	(0.062, 0.001)	(0.062, 0.000)	(0.062, 0.000)	(0.062, 0.000)
0.04	LSE	(1.000, 0.000, 0.145)	(2.981, 3.925, 1.661)	(8.696, 59.222, 6.417)	(17.763, 281.008, 14.093)	(29.853, 832.478, 24.508)	(44.550, 1896.611, 37.329)
		(0.021, 1.000)	(6.683, 1.000)	(100.399, 1.000)	(479.618, 1.000)	(1433.134, 1.000)	(3290.041, 1.000)
	HUBER	(0.950, 0.002, 0.227)	(1.060, 0.002, 0.260)	(1.062, 0.004, 0.261)	(1.064, 0.004, 0.263)	(1.067, 0.004, 0.265)	(1.069, 0.005, 0.266)
		(0.054, 2.574)	(0.071, 0.011)	(0.072, 0.001)	(0.073, 0.000)	(0.074, 0.000)	(0.075, 0.000)
	BISQUARE	(0.945, 0.003, 0.227)	(1.042, 0.002, 0.254)	(1.041, 0.002, 0.253)	(1.041, 0.002, 0.253)	(1.041, 0.002, 0.253)	(1.041, 0.002, 0.253)
		(0.054, 2.582)	(0.067, 0.010)	(0.066, 0.001)	(0.066, 0.000)	(0.066, 0.000)	(0.066, 0.000)
0.05	LSE	(0.941, 0.003, 0.229)	(1.038, 0.001, 0.256)	(1.038, 0.001, 0.256)	(1.038, 0.001, 0.256)	(1.038, 0.001, 0.257)	(1.038, 0.001, 0.256)
		(0.056, 2.666)	(0.067, 0.010)	(0.067, 0.001)	(0.067, 0.000)	(0.067, 0.000)	(0.067, 0.000)
	HUBER	(1.000, 0.000, 0.145)	(3.456, 6.032, 1.838)	(10.535, 90.918, 7.104)	(21.761, 431.036, 15.597)	(36.645, 1270.590, 27.122)	(54.716, 2885.433, 41.105)
		(0.021, 1.000)	(9.412, 1.000)	(141.386, 1.000)	(674.311, 1.000)	(2006.192, 1.000)	(4575.037, 1.000)
	BISQUARE	(0.950, 0.002, 0.227)	(1.093, 0.005, 0.271)	(1.097, 0.009, 0.274)	(1.100, 0.010, 0.276)	(1.104, 0.011, 0.278)	(1.109, 0.012, 0.282)
		(0.054, 2.563)	(0.082, 0.009)	(0.084, 0.001)	(0.086, 0.000)	(0.088, 0.000)	(0.091, 0.000)
0.06	LSE	(0.945, 0.003, 0.226)	(1.070, 0.005, 0.262)	(1.068, 0.005, 0.261)	(1.068, 0.005, 0.260)	(1.067, 0.005, 0.260)	(1.068, 0.005, 0.260)
		(0.054, 2.572)	(0.074, 0.008)	(0.072, 0.001)	(0.072, 0.000)	(0.072, 0.000)	(0.072, 0.000)
	HUBER	(0.941, 0.003, 0.229)	(1.065, 0.004, 0.264)	(1.065, 0.004, 0.264)	(1.065, 0.004, 0.264)	(1.065, 0.004, 0.263)	(1.065, 0.004, 0.264)
		(0.056, 2.658)	(0.074, 0.008)	(0.074, 0.001)	(0.074, 0.000)	(0.073, 0.000)	(0.074, 0.000)
	BISQUARE	(1.000, 0.000, 0.145)	(3.922, 8.540, 1.988)	(12.312, 127.960, 7.687)	(25.602, 605.246, 16.838)	(43.169, 1778.234, 29.148)	(64.476, 4029.236, 44.351)
		(0.021, 1.000)	(12.493, 1.000)	(187.043, 1.000)	(888.779, 1.000)	(2627.814, 1.000)	(5996.251, 1.000)
0.07	LSE	(0.950, 0.003, 0.227)	(1.130, 0.010, 0.284)	(1.135, 0.018, 0.288)	(1.140, 0.020, 0.292)	(1.146, 0.021, 0.296)	(1.154, 0.024, 0.303)
		(0.054, 2.561)	(0.098, 0.008)	(0.101, 0.001)	(0.105, 0.000)	(0.109, 0.000)	(0.115, 0.000)
	HUBER	(0.945, 0.003, 0.226)	(1.098, 0.010, 0.271)	(1.096, 0.009, 0.268)	(1.095, 0.009, 0.269)	(1.096, 0.009, 0.268)	(1.095, 0.009, 0.268)
		(0.054, 2.571)	(0.083, 0.007)	(0.081, 0.000)	(0.081, 0.000)	(0.081, 0.000)	(0.081, 0.000)
	BISQUARE	(0.940, 0.004, 0.229)	(1.093, 0.009, 0.272)	(1.093, 0.009, 0.272)	(1.093, 0.009, 0.272)	(1.093, 0.009, 0.272)	(1.093, 0.009, 0.271)
		(0.056, 2.657)	(0.083, 0.007)	(0.082, 0.000)	(0.083, 0.000)	(0.082, 0.000)	(0.082, 0.000)
0.08	LSE	(1.000, 0.000, 0.145)	(4.384, 11.454, 2.128)	(14.094, 171.447, 8.196)	(29.398, 806.458, 17.935)	(49.605, 2362.433, 31.055)	(74.087, 5341.651, 47.192)
		(0.021, 1.000)	(15.980, 1.000)	(238.625, 1.000)	(1128.112, 1.000)	(3326.838, 1.000)	(7568.737, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.169, 0.016, 0.299)	(1.177, 0.031, 0.305)	(1.184, 0.034, 0.311)	(1.194, 0.038, 0.319)	(1.206, 0.042, 0.330)
		(0.054, 2.559)	(0.118, 0.007)	(0.124, 0.001)	(0.131, 0.000)	(0.139, 0.000)	(0.151, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.128, 0.016, 0.282)	(1.125, 0.016, 0.277)	(1.124, 0.015, 0.277)	(1.124, 0.015, 0.277)	(1.124, 0.015, 0.276)
		(0.054, 2.567)	(0.096, 0.006)	(0.092, 0.000)	(0.092, 0.000)	(0.092, 0.000)	(0.092, 0.000)
0.08	LSE	(0.941, 0.004, 0.229)	(1.122, 0.015, 0.281)	(1.122, 0.015, 0.280)	(1.122, 0.015, 0.280)	(1.122, 0.015, 0.280)	(1.122, 0.015, 0.280)
		(0.056, 2.651)	(0.094, 0.006)	(0.094, 0.000)	(0.093, 0.000)	(0.093, 0.000)	(0.093, 0.000)
0.08	LSE	(1.000, 0.000, 0.145)	(4.833, 14.691, 2.245)	(15.806, 219.232, 8.665)	(33.109, 1030.977, 18.921)	(55.871, 3010.865, 32.712)	(83.330, 6778.207, 49.606)
		(0.021, 1.000)	(19.730, 1.000)	(294.306, 1.000)	(1388.988, 1.000)	(4080.913, 1.000)	(9239.006, 1.000)

α	ESTIMATOR	λ					
		0	1	2	3	4	5
0.09	HUBER	(0.950, 0.003, 0.227)	(1.213, 0.026, 0.317)	(1.223, 0.050, 0.325)	(1.235, 0.055, 0.335)	(1.249, 0.062, 0.350)	(1.267, 0.071, 0.377)
		(0.054, 2.565)	(0.146, 0.007)	(0.155, 0.001)	(0.168, 0.000)	(0.184, 0.000)	(0.214, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.160, 0.026, 0.293)	(1.153, 0.024, 0.286)	(1.154, 0.024, 0.285)	(1.154, 0.024, 0.285)	(1.154, 0.024, 0.285)
		(0.054, 2.573)	(0.111, 0.006)	(0.105, 0.000)	(0.105, 0.000)	(0.105, 0.000)	(0.105, 0.000)
	PROPOSED	(0.941, 0.004, 0.229)	(1.152, 0.023, 0.290)	(1.152, 0.023, 0.289)	(1.153, 0.023, 0.289)	(1.152, 0.023, 0.289)	(1.152, 0.023, 0.288)
		(0.056, 2.658)	(0.107, 0.005)	(0.106, 0.000)	(0.107, 0.000)	(0.107, 0.000)	(0.106, 0.000)
	LSE	(1.000, 0.000, 0.145)	(5.270, 18.231, 2.349)	(17.511, 272.627, 9.089)	(36.727, 1276.431, 19.814)	(62.011, 3722.288, 34.284)	(92.288, 8333.510, 51.874)
		(0.021, 1.000)	(23.748, 1.000)	(355.232, 1.000)	(1669.015, 1.000)	(4897.647, 1.000)	(11024.370, 1.000)
	HUBER	(0.949, 0.003, 0.227)	(1.261, 0.038, 0.338)	(1.276, 0.076, 0.352)	(1.292, 0.086, 0.372)	(1.314, 0.099, 0.402)	(1.341, 0.116, 0.465)
		(0.054, 2.568)	(0.182, 0.008)	(0.200, 0.001)	(0.224, 0.000)	(0.260, 0.000)	(0.333, 0.000)
0.1	BISQUARE	(0.944, 0.003, 0.226)	(1.194, 0.038, 0.305)	(1.186, 0.034, 0.295)	(1.185, 0.034, 0.295)	(1.186, 0.034, 0.295)	(1.185, 0.034, 0.295)
		(0.054, 2.577)	(0.130, 0.005)	(0.122, 0.000)	(0.121, 0.000)	(0.121, 0.000)	(0.121, 0.000)
	PROPOSED	(0.940, 0.004, 0.229)	(1.184, 0.034, 0.299)	(1.184, 0.034, 0.298)	(1.184, 0.034, 0.298)	(1.184, 0.034, 0.298)	(1.184, 0.034, 0.298)
		(0.056, 2.663)	(0.123, 0.005)	(0.123, 0.000)	(0.123, 0.000)	(0.123, 0.000)	(0.123, 0.000)
	LSE	(1.000, 0.000, 0.145)	(5.706, 22.147, 2.453)	(19.164, 329.948, 9.456)	(40.268, 1541.955, 20.641)	(67.888, 4474.070, 35.503)	(101.028, 10005.621, 53.895)
		(0.021, 1.000)	(28.162, 1.000)	(419.367, 1.000)	(1968.019, 1.000)	(5734.513, 1.000)	(12910.303, 1.000)
	HUBER	(0.950, 0.003, 0.227)	(1.315, 0.053, 0.363)	(1.336, 0.113, 0.386)	(1.359, 0.129, 0.418)	(1.392, 0.153, 0.495)	(1.433, 0.188, 0.652)
		(0.054, 2.569)	(0.231, 0.008)	(0.262, 0.001)	(0.304, 0.000)	(0.399, 0.000)	(0.612, 0.000)
	BISQUARE	(0.945, 0.003, 0.226)	(1.231, 0.053, 0.319)	(1.218, 0.048, 0.305)	(1.218, 0.048, 0.304)	(1.219, 0.048, 0.304)	(1.218, 0.048, 0.304)
		(0.054, 2.577)	(0.155, 0.006)	(0.141, 0.000)	(0.140, 0.000)	(0.141, 0.000)	(0.140, 0.000)
PROPOSED	(0.941, 0.004, 0.229)	(1.218, 0.048, 0.310)	(1.217, 0.047, 0.308)	(1.217, 0.047, 0.308)	(1.218, 0.047, 0.308)	(1.218, 0.047, 0.308)	
	(0.056, 2.665)	(0.144, 0.005)	(0.142, 0.000)	(0.142, 0.000)	(0.142, 0.000)	(0.142, 0.000)	

Appendix B: Derivation of the Objective Function for the Proposed M-estimator

$$\rho_p(e) = \int_0^e \psi(x) dx \quad (1)$$

where $w(x) = \frac{\psi(x)}{x}$, $w(x)$ represents weighting function, $\psi(x)$ denotes the influence function, and x denotes residuals. Consequently,

$$\rho_p(e) = \int_0^e x \left[1 - \left(\frac{|x| - k_1}{k_2 - k_1} \right)^2 \right]^2 dx$$

$$\rho_p(e) = \frac{1}{(k_2 - k_1)^4} \int_0^e x [(k_2 - k_1)^2 - (|x| - k_1)^2]^2 dx$$

$$\rho_p(e) = \frac{1}{(k_2 - k_1)^4} \int_0^e x [k_2^2 - 2k_1k_2 + k_1^2 - (x^2 - 2k_1|x| + k_1^2)]^2 dx$$

$$\rho_p(e) = \frac{1}{(k_2 - k_1)^4} \int_0^e x [k_2^2 - 2k_1k_2 - x^2 + 2k_1|x|]^2 dx$$

$$\rho_p(e) = \frac{1}{(k_2 - k_1)^4} \int_0^e x [k_2^2 - 2k_1k_2 - (x^2 - 2k_1|x|)]^2 dx$$

we will now expand the integrand,

$$\begin{aligned}
[k_2^2 - 2k_1k_2 - (x^2 - 2k_1|x|)]^2 &= (k_2^2 - 2k_1k_2)^2 - 2[(k_2^2 - 2k_1k_2)(x^2 - 2k_1|x|)] + (x^2 - 2k_1|x|)^2 \\
&= k_2^4 - 4k_1k_2^3 + 4k_1^2k_2^2 - 2(k_2^2x^2 - 2k_1k_2^2|x| - 2k_1k_2x^2 + 4k_1^2k_2|x|) + x^4 - 4k_1x^2|x| + 4k_1^2x^2 \\
&= k_2^4 - 4k_1k_2^3 + 4k_1^2k_2^2 - 2k_2^2x^2 + 4k_1k_2^2|x| + 4k_1k_2x^2 - 8k_1^2k_2|x| + x^4 - 4k_1x^2|x| + 4k_1^2x^2 \\
&= k_2^4 - 4k_1k_2^3 + 4k_1^2k_2^2 + (4k_1^2 - 2k_2^2 + 4k_1k_2)x^2 + 4k_1k_2^2|x| - 8k_1^2k_2|x| - 4k_1x^2|x| + x^4 \\
&= k_2^4 - 4k_1k_2^3 + 4k_1^2k_2^2 + 2(2k_1^2 - k_2^2 + 2k_1k_2)x^2 + 4k_1k_2(k_2 - 2k_1)|x| - 4k_1x^2|x| + x^4
\end{aligned}$$

Thus,

$$\rho_p(e) = \frac{1}{(k_2 - k_1)^4} \int_0^e x(k_2^4 - 4k_1k_2^3 + 4k_1^2k_2^2 + 2(2k_1^2 - k_2^2 + 2k_1k_2)x^2 + 4k_1k_2(k_2 - 2k_1)|x| - 4k_1x^2|x| + x^4)dx \quad (2)$$

Lemma B1: Representation of Integral of the Absolute Value Function and its Properties

Let $T: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $T(x) = |x|$ and let $W: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $W(z, x) = z \cdot \text{sign}(x)$. Then, for any $z, x \in \mathbb{R}$, the following affirmations hold:

(i) Gradient Representation:

$$\text{sign}(x) = \nabla T(x), \text{ for } x \neq 0,$$

(ii) Equality at Identity

$$W(x, x) = T(x),$$

(iii) Inequality relation

$$W(z, x) \geq -T(z)$$

Proof:

Case I: Gradient representation

By definition, $T(x) = |x|$. The derivative of $|x|$ with respect to x is:

$$\nabla T(x) = \frac{d}{dx}|x| = \text{sign}(x), \text{ for } x \neq 0,$$

Case II: Equality at Identity

Substitute $z = x$ into $W(z, x)$:

$$W(z, x) = z \cdot \text{sign}(x) = |x| = T(x)$$

Case III: Inequality relation

For any $z, x \in \mathbb{R}$,

$$W(z, x) = z \cdot \text{sign}(x) \geq -|z| = -T(z),$$

since $z \cdot \text{sign}(x) \geq -|z|$ by the Cauchy-Schwarz inequality.

Integral Representation

Using $|x| = x \cdot \text{sign}(x)$, the integral of $|x|$ becomes:

$$\int |x| dx = \int x \cdot \text{sign}(x) dx$$

Consider,

$$\int x^3 |x| dx = \int x^4 \cdot \text{sign}(x) dx$$

By the substitution $u = |x|$ and $du = \frac{x}{|x|} dx$, the integral simplifies accordingly.

$$\int x^3|x|dx = \int u^4 du = \frac{u^5}{5} + C = \frac{x^4|x|}{5} + C = \frac{x^5}{5} \cdot \text{sign}(x) + C$$

Likewise, for

$$\int x|x|dx = \int x^2 \cdot \frac{x}{|x|} dx = \int u^2 du = \frac{u^3}{3} + C = \frac{x^2|x|}{3} + C = \frac{x^3}{3} \cdot \text{sign}(x) + C$$

This concludes the proof. This lemma formalizes the relationship between the absolute value function, its derivative (the sign function), and integral properties involving these components.

Finally, we get

$$\rho_p(e) = \frac{1}{(k_2 - k_1)^4} \int_0^e x(k_2^4 - 4k_1k_2^3 + 4k_1^2k_2^2) + 2(2k_1^2 - k_2^2 + 2k_1k_2)x^3 + 4k_1k_2(k_2 - 2k_1)x|x| - 4k_1x^3|x| + x^5)dx$$

$$\rho_p(e) = \frac{1}{(k_2 - k_1)^4} \left[\frac{k_2^2}{2}(k_2^2 - 4k_1k_2 + 4k_1^2)e^2 + \frac{1}{2}(2k_1^2 - k_2^2 + 2k_1k_2)e^4 + \frac{4}{3}k_1k_2(k_2 - 2k_1)e^2|e| - \frac{4}{5}k_1e^4|e| + \frac{e^6}{6} \right]$$

$$\rho_p(e) = \frac{k_2^6}{6(k_2 - k_1)^4} \left[\frac{3}{k_2^4}(k_2^2 - 4k_1k_2 + 4k_1^2)e^2 + 3 \left(2\frac{k_1^2}{k_2^6} - \frac{1}{k_2^4} + 2\frac{k_1}{k_2^5} \right) e^4 + \frac{8k_1}{k_2^5}(k_2 - 2k_1)e^2|e| - \frac{24}{5}\frac{k_1}{k_2^6}e^4|e| + \frac{e^6}{k_2^6} \right] \quad (3)$$

This concludes that the objective function of the proposed M-estimator is a non-convex which can be written as:

$$\rho_P(e) = \begin{cases} \frac{e^2}{2} & , |e| \leq k_1 \\ \frac{k_2^6}{6(k_2 - k_1)^4} \left[\frac{3}{k_2^4} (k_2^2 - 4k_1k_2 + 4k_1^2)e^2 + 3 \left(2\frac{k_1^2}{k_2^6} - \frac{1}{k_2^4} + 2\frac{k_1}{k_2^5} \right) e^4 \right. \\ \quad \left. + \frac{8k_1}{k_2^5} (k_2 - 2k_1)e^2|e| - \frac{24}{5} \frac{k_1}{k_2^6} e^4|e| + \frac{e^6}{k_2^6} \right] & , k_1 < |e| \leq k_2 \\ \frac{k_2^6}{6(k_2 - k_1)^4} & , |e| > k_2 \end{cases}$$

To sum up, the Bisquare M-estimator is a special case for the proposed M-estimator when $k_1 = 0$ such that, in $\rho_P(e)$ above we substitute $k_1 = 0$ and let $k_2 = k$ under constraint $k_1 < |e| \leq k_2$ we obtain

$$\rho_P(e) = \frac{k^6}{6k^4} \left[\frac{3}{k^4} (k^2)e^2 + 3 \left(-\frac{1}{k^4} \right) e^4 + \frac{e^6}{k^6} \right]$$

$$\rho_P(e) = \frac{k^2}{6} \left[\frac{3e^2}{k^2} - \frac{3e^4}{k^4} + \frac{e^6}{k^6} \right]$$

Since the expansion of $\left[1 - \left(\frac{e}{k} \right)^2 \right]^3 = 1 - \frac{3e^2}{k^2} + \frac{3e^4}{k^4} - \frac{e^6}{k^6}$, to obtain $\frac{3e^2}{k^2} - \frac{3e^4}{k^4} + \frac{e^6}{k^6}$ we have to take $1 - \left[1 - \frac{3e^2}{k^2} + \frac{3e^4}{k^4} - \frac{e^6}{k^6} \right]$ which is

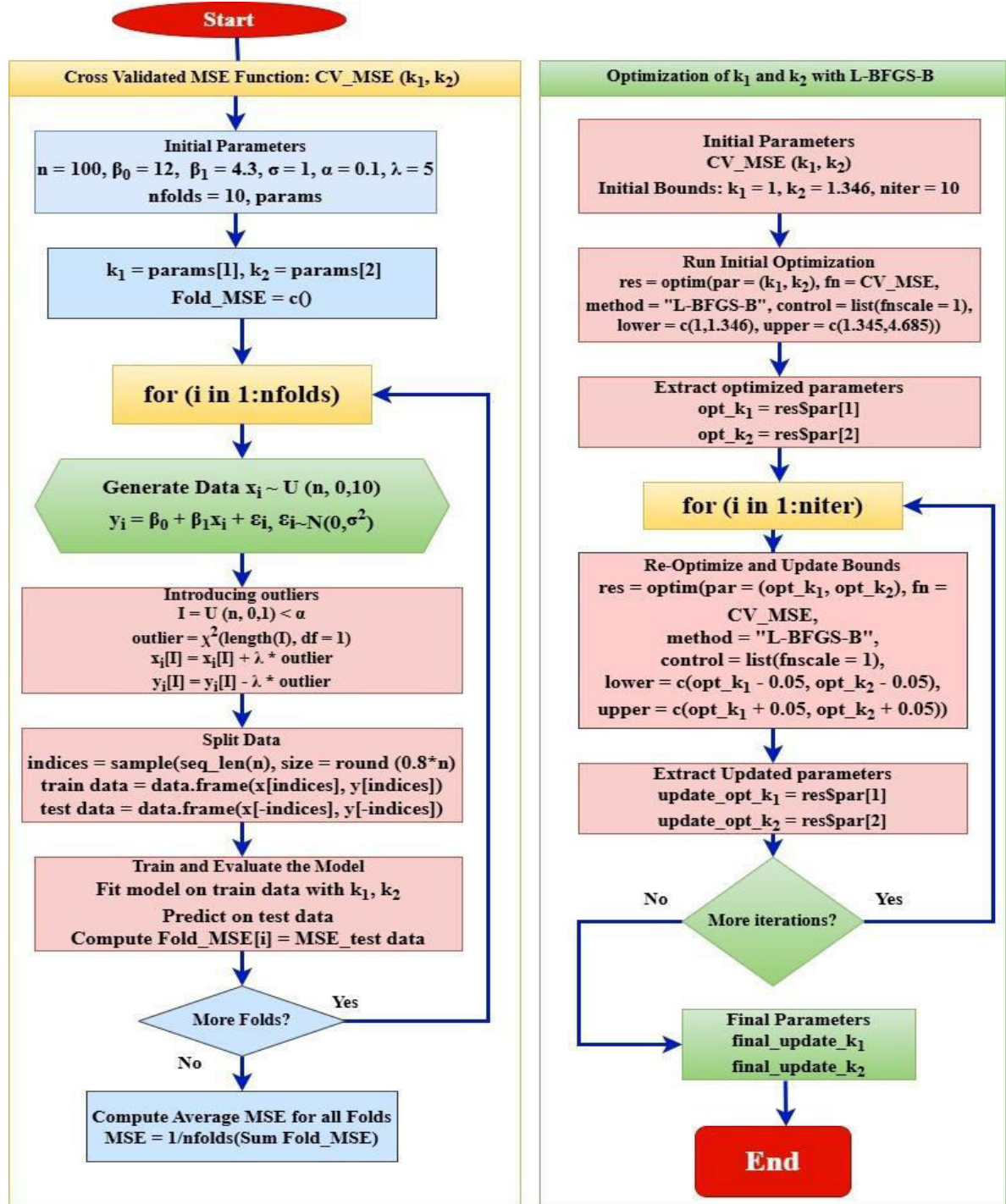
equivalent to writing $1 - \left[1 - \left(\frac{e}{k} \right)^2 \right]^3$

Hence, $\rho_P(e)$ reduces to Bisquare M-estimator which is given by

$$\rho_B(e) = \frac{k^2}{6} \left[1 - \left[1 - \left(\frac{e}{k} \right)^2 \right]^3 \right] \quad (4)$$

Appendix C: Adaptive Cross-Validation using a Two Stage L-BFGS-B

Optimization Technique Flowchart



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Address : P. O. Box 2624, Dar es Salaam - Tanzania

Nationality : Tanzanian

Email : lubbattillah@gmail.com

Date of Birth : 16th October 1986

Academic Background : BA Statistics, 2010

: Postgraduate Diploma in Scientific Computing, 2014

Work Experience:

**2011 – To Date : Statistician at the National Examinations Council of Tanzania
(NECTA).**

Employer Website : <https://www.necta.go.tz/>

Data Analyst Intern : QUEETS-SITT Project

QUEETS-SITT Project by Helvetas Swiss Inter Cooperation in collaboration with Tanzania Teachers Union (TTU) to provide comprehensive capacity building to selected primary schools in all districts of Arusha, Kilimanjaro and Manyara to enhance performance in Mathematics, English and Education for Sustainable Development. (<https://www.helvetas.org/en/tanzania/what-we-do/how-we-work/our-projects/Africa/Tanzania/tanzania-teacher-training>).

Training and Workshops:

- 1. Workshop on “Public Integrity Restoration Initiative (PIRI)” held at University of Dodoma, Tanzania.**
- 2. Training programme on “Official Statistics and Related Methodology” held at National Statistical System Training Academy (NSSTA), Knowledge Park II, Greater Noida, India.**
- 3. Statistical Training Diploma, at the Indian Statistical Education Centre (ISEC) operated by the Indian Statistical Institute Kolkata (ISI), under the auspices of Ministry of Statistics and Programme Implementation, Government of India.**
- 4. Workshop organized by Malawi National Examinations Board (MANEB) to host Southern Africa Association for Educational Assessment (SAEAA) Research Forum held at Crossroads Hotel, Lilongwe Malawi to Formulate of SAEAA Research Forum Strategic Plan 2019/2023 and discussing how research should facilitate Examinations Councils to enhance assessment quality across the Southern Sahara region.**
- 5. Workshop on Strengthening the Quality of Student Learning Data in Tanzania held at Mbezi Wani Dar es Salaam Tanzania. The workshop was financed through the FCDO’s Shule Bora Program and was delivered jointly by the World Bank’s Development Impact Development (DIME) and Cambridge Education.**

Research Experiences:

- 1. Research on “Causes of Primary Schools Pupils Dropouts”, A case of Mbeya Region.**
- 2. Standard Two National Assessment on Reading, Arithmetic and Writing in Tanzania.**
- 3. National Examinations and Assessments Item Response Analysis.**

Career Development and Certifications:

- 1. Use SurveyMonkey to Create a Survey and Analyze Results, an online non-credit project authorized by Coursera Project Network offered through Coursera, April 2025**
- 2. Building Web Applications in PHP course authorized by the University of Michigan through Coursera issued November 2019.**
- 3. Databases and SQL for Data Science course authorized by IBM and offered through Coursera issued on Friday, 09 August 2019 4:07 AM GMT.**
- 4. SQL for Data Science course authorized by University of California, Davis through Coursera issued on Wednesday, July 3, 2019, 4:04 AM GMT.**
- 5. Applied Data Science I: Scientific Computing & Python (with honors) course offered by World Quant University issued in September 2021.**
- 6. Introduction to Data Science an online non-credit course offered by Simplilearn issued in September 2021.**
- 7. Introduction to SQL course offered by Simplilearn issued in October 2021.**