

GENERALIZED EXPONENTIAL TIME DIFFERENCING SCHEMES FOR FRACTIONAL OSCILLATION MODELS

BY

ALJOWHARA HASAN A. HONAIN

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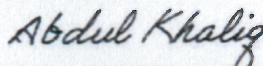
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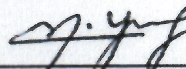
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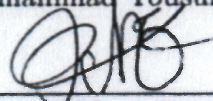
Prof. Khaled M. Furati (Adviser)



Prof. Abdul Q. M. Khaliq (Co-Adviser)



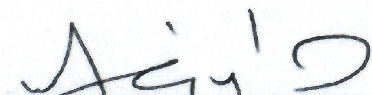
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Prof. Nasr Aldeen Tatar (Member)



Dr. Rachid Ait Haddou (Member)



Prof. Monther Alfuraidean
Department Chairman



Prof. Suliman Al-Homidan
Dean of Graduate Studies

Date



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Dedication

To Allah, All-Knowing, All-Wise, The Most Generous, without Whom this could not have been accomplished. To my esteemed advisor, Prof. Khaled Furati, for your constant guidance and support throughout this journey. To my loving family, whose constant encouragement and belief in me fueled my determination to complete this thesis. To my beloved husband, Hatim, for his constant support, encouragement, and companionship during late-night brainstorming sessions and countless hours of work, which kept me going. And to myself, for enduring the stress and late nights involved in this work.

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DISSERTATION ABSTRACT

NAME: Aljowhara Hasan A. Honain

TITLE OF STUDY: Generalized Exponential Time Differencing Schemes for
Fractional Oscillation Models

MAJOR FIELD: Mathematics

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Due to the fact that real processes are typically or most likely fractional, using the concept of fractional calculus may be a step closer to the real world. Fractional calculus has been utilized to mathematically describe actual issues that have arisen in various fields of science and engineering. Fractional models proved to provide accurate description of various phenomena and observed experimental data.

Oscillations occur in various processes and are of great significance to understand, analyze and simulate real phenomena. Fractional oscillatory type equations provide an effective tool to model some anomalous oscillatory behaviors. The solution of fractional oscillation equations exhibits oscillatory behavior. It is a challenging task to construct efficient numerical methods for fractional oscillation equations that preserve accurately the oscillatory behavior of the solution.

This dissertation aims to develop efficient numerical methods tailored to a class of fractional oscillation problems. The proposed methods are rooted in exponential time differencing and supplemented by advanced numerical techniques to enhance accuracy and computational efficiency. Special emphasis is placed on the implementation of these methods using rational approximations of the oscillatory Mittag-Leffler function. Comprehensive numerical experiments will be conducted to rigorously evaluate the methods convergence rates, stability, and ability to capture the intricate dynamics of fractional oscillatory systems.

ملخص الرسالة

الاسم:

الجوهرة حسن علي حُتَيْن

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نظرًا لأن العمليات الحقيقية عادةً ما تكون، أو من المرجح أن تكون، ذات طبيعة كسرية، فإن استخدام مفهوم حساب التفاضل والتكامل الكسري قد يكون خطوة أقرب لفهم العالم الحقيقي. لقد تم استخدام حساب التفاضل والتكامل الكسري لوصف المشكلات الفعلية التي ظهرت في مجالات مختلفة من العلوم والهندسة. وأثبتت النماذج الكسرية أنها تقدم وصفًا دقيقًا لمجموعة متنوعة من الظواهر والبيانات التجريبية الملحوظة.

تحدث التذبذبات في العديد من العمليات وهي ذات أهمية كبيرة لفهم وتحليل ومحاكاة الظواهر الواقعية. توفر المعادلات الكسرية من النوع التذبذبي أداة فعالة لنمذجة بعض السلوكيات التذبذبية الشاذة. وتظهر حلول معادلات التذبذب الكسري سلوكًا تذبذبيًا. يُعد بناء طرق عددية فعالة لمعالجة هذه المعادلات مع الحفاظ بدقة على السلوك التذبذبي للحل مهمةً مليئةً بالتحديات.

تهدف هذه الأطروحة إلى تطوير طرق عددية فعالة مخصصة لفئة من مشكلات التذبذب الكسري. تستند الطرق المقترحة إلى استخدام الفروقات الزمنية الأسية وتُدعم بتقنيات عددية متقدمة لتعزيز الدقة والكفاءة الحسابية. يركز البحث بشكل خاص على تنفيذ هذه الطرق باستخدام تقريبات نسبية لدالة ميتاغ-ليفنر التذبذبية. سيتم إجراء تجارب عددية شاملة لتقييم معدلات تقارب الطرق واستقرارها وقدرتها على التقاط الديناميكيات المعقدة للأنظمة التذبذبية الكسرية بدقة.

CHAPTER 1

INTRODUCTION

This chapter provides an overview, starting with a brief discussion of the motivation behind the problem addressed in this thesis. We present a clear problem statement, followed by an outline of our research methodology, and a summary of our main accomplishments.

1.1 Motivation

In many real-world processes, observed behavior often diverges from classical models, making traditional integer-order calculus inadequate for accurately capturing complex, real-world phenomena. Fractional calculus, by contrast, provides a closer representation of these processes, incorporating both hereditary and memory effects that are crucial for modeling a wide range of phenomena in science and engineering [3]. This approach has led to the development of fractional models that can better reflect experimental data and provide more accurate descriptions of various physical and engineered systems.

Oscillatory processes, which occur across many natural and artificial systems, are particularly significant for understanding and simulating real-world dynamics. However, anomalous oscillatory behaviors, which do not conform to integer-order models, present a unique challenge. Fractional oscillatory equations offer an effective framework for capturing these irregular oscillations. The solutions of fractional oscillation equations exhibit characteristic oscillatory patterns, which makes it essential to develop efficient numerical methods that accurately preserve this behavior over time.

The solution of fractional differential equations (FDEs) typically lacks exact closed-form expressions, making numerical methods indispensable. Currently, the numerical treatment of integer-order differential equations is well established; however, the field of fractional differential equations is still developing and poses unique challenges [4]. Developing efficient, reliable numerical methods for these equations is crucial to further advancing the field.

Exponential time differencing (ETD) is one promising approach for developing numerical schemes for FDEs, including fractional oscillation equations. This method leverages the exact solution of the homogeneous part of the equation while approximating the integral of the inhomogeneous part, making it particularly suitable for capturing complex dynamics in fractional systems [5]. To further enhance computational efficiency, rational approximations of the Mittag-Leffler function, which appears in the solution of many fractional models, will be employed in our implementations.

Many existing numerical methods for FDEs, such as finite difference schemes, iterative methods, and integral methods, are effective but can have limitations. Explicit difference methods, for instance, are straightforward but may lack stability,

while implicit methods, though more stable, can be computationally intensive [6]. For fractional oscillation problems, methods like the Adams-Bashforth-Moulton scheme, finite-difference approaches, and collocation methods have been used, yet they often fail to preserve the oscillatory nature of the solution over longer intervals.

In this dissertation, we aim to develop accurate and efficient numerical methods for fractional oscillation problems, including cases with both linear and nonlinear forcing terms. Our approach involves constructing suitable approximations for the oscillatory Mittag-Leffler functions to ensure computational efficiency. We will also extend these schemes to semi-discretized partial differential equations (PDEs) to explore their applicability in more complex models.

The outcomes of this research are anticipated to enhance the understanding of fractional oscillation models and expand their application potential. By providing a set of reliable numerical methods, this work will support future studies in the modeling and simulation of fractional systems, impacting diverse scientific fields and contributing to the broader development of fractional calculus methodologies.

1.2 Problem Description

In this dissertation, motivated by the need to accurately simulate anomalous oscillatory behaviors, we consider the following class of initial boundary value problems for fractional oscillation equations. These equations are essential for capturing the complexities observed in real-world systems where classical models fall short.

We define the initial value problem as follows:

$$\begin{aligned}
{}^cD^\alpha u + Au &= f, \quad t > 0, \quad \alpha \in (1, 2) \\
u(0) &= u_0, \\
u'(0) &= u_1,
\end{aligned} \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (1, 2)$ defined by

$${}^cD^\alpha f(t) := \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} f''(\tau) d\tau, & 1 < \alpha < 2, \\ \frac{d^2 f(t)}{dt^2}, & \alpha = 2. \end{cases}$$

In this context, the coefficient A is an $m \times m$ matrix that represents relevant physical parameters and characteristics of the oscillatory system, while $u = [u_1, u_2, \dots, u_m]^T$ is the solution vector, which describes the displacement of the oscillating system. The vectors $u_0 = [u_{01}, u_{02}, \dots, u_{0m}]^T$ and the initial conditions are $u_1 = [u_{11}, u_{12}, \dots, u_{1m}]^T$, with the superscript T denoting the transpose operation.

The function $f = [f_1, f_2, \dots, f_m]^T$ represents the external force acting on the system and can be either a linear external force $f(t)$ or a nonlinear external force $f(u, t)$.

In the special case where $\alpha = 2$, we recover the well-known ordinary differential equation for oscillators, which is foundational in both classical and quantum mechanics. This highlights the importance of fractional calculus in extending our understanding of dynamical systems that exhibit non-integer order behaviors.

to be referred later.

1.3 Research Achievements

The following objectives were achieved in this thesis:

- Development of rational approximations for Mittag-Leffler functions, including a new rational approximation for α in $(1, 2)$.
- Design and implementation of GETD schemes for fractional oscillation equations with linear external forces.
- Development of GETD schemes for fractional oscillation problems with nonlinear external forces.
- Extension of these GETD schemes to systems of fractional differential equations (FDEs).
- Comprehensive study of the applicability of the developed GETD schemes to semi-discretized partial differential equations (PDEs).

1.4 Research Methodology

In this work, we use the following tools, methods, and approaches:

- Approximations of Mittag-Leffler functions.
- Laplace transform and its inverse to the semi-discrete system, to obtain a mild solution of the system as an integral equation
- Polynomial interpolation techniques for integral approximations.

- Exponential time differencing methods.
- Various Discrete inequalities together with bounds on Mittag-Leffler functions to prove stability and convergence results
- Graded mesh techniques to achieve optimal convergence results.
- Matrix factorization methods to efficiently calculate Mittag-Leffler functions of matrices.
- Rational approximations of Mittag-Leffler functions and their partial fractions for efficient implementation of the methods
- Analysis of the rate of convergence and stability requirements for the schemes.

1.5 Scholarly Outcomes

In this section, we present our scholarly contributions to the field of fractional calculus, focusing on numerical methods for fractional oscillation models and Mittag-Leffler functions. Our research aims to advance understanding and computational techniques in these complex areas, as reflected in the following publications.

1. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2024). Generalized Exponential Time Differencing for Fractional Oscillation Models. *Journal of Computational and Applied Mathematics*, ISSN 0377-0427.
2. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2024). Rational approximations for the oscillatory two-parameter Mittag-Leffler function.

Fractal and Fractional, 8(6), 319.

3. Honain, A. H., & Furati, K. M. (2024). Exponential Time Differencing Scheme for Fractional Plasma Oscillations. *IFAC-PapersOnLine*, 58(12), 169-173.
4. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2023). Rational approximation for oscillatory Mittag-Leffler function. In *2023 International Conference on Fractional Differentiation and Its Applications (ICFDA)*, pp. 1–5. IEEE.
5. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2024). Generalized Exponential Time Differencing Scheme for Non-linear Fractional Oscillation Models. *Prepared for submission*.

1.6 Dissertation Outline

This thesis is structured as follows. In Chapter 2, we provide overview on fractional calculus, general definitions of differential and integral operators, and present some preliminary results. Chapter 3 focuses on rational approximations for the oscillatory two-parameter Mittag–Leffler function, where we illustrate their accuracy and efficiency. The construction of generalized exponential time differencing (GETD) schemes is discussed in Chapter 4, specifically for linear fractional oscillation models, along with convergence and stability analyses. Chapter 5 extends these methods to nonlinear fractional oscillation models. Finally, in Chapter 6, we explore various applications of the proposed approaches to fractional oscillation models.

CHAPTER 2

PRELIMINARIES

This Preliminaries Chapter establishes the foundational concepts and mathematical tools necessary for the development and analysis of fractional differential equations in this thesis. Fractional calculus, which generalizes classical differentiation and integration to arbitrary (non-integer) orders, plays a central role in the study of complex systems exhibiting memory and hereditary properties. This chapter introduces key definitions, properties, and functions essential for understanding fractional calculus and its applications.

We begin with the Gamma Function and Beta Function, which form the basis of fractional integral and derivative formulations. The chapter then delves into the Riemann-Liouville and Caputo fractional derivatives, presenting their formal definitions, integral representations, and key properties. These derivatives are fundamental in modeling dynamic systems with fractional-order behavior, particularly in physics, engineering, and biology.

The chapter also explores the Mittag-Leffler function, a generalization of the exponential function, which frequently appears in solutions of fractional differential equa-

tions. We provide its one-parameter and two-parameter formulations, along with their properties, recurrence relations, and asymptotic behaviors. The Mittag-Leffler function's versatility in capturing complex dynamics, especially in oscillatory and history-dependent systems, underscores its importance.

Finally, the chapter concludes with a short discussion on oscillation, a key concept in dynamic systems analysis. Definitions and classifications of oscillatory and periodic functions are provided, alongside examples and properties of oscillatory behavior. These concepts are essential for analyzing solutions of fractional oscillation models, which are a focus of this thesis.

By laying out these foundational concepts, this chapter provides the mathematical framework for the subsequent chapters, enabling a rigorous exploration of fractional calculus and its applications in solving nonlinear fractional differential equations.

2.1 Fractional Calculus

Fractional calculus generalizes the concept of integer-order differentiation and integration to non-integer (fractional) orders. This section introduces fundamental definitions and properties related to fractional derivatives and integrals.

Definition 2.1 (Gamma Function) *The Gamma function, denoted by $\Gamma(z)$, is defined as:*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad z > 0, \quad (2.1)$$

which satisfies the recurrence relation:

$$\Gamma(z + 1) = z\Gamma(z). \quad (2.2)$$

Definition 2.2 (Beta Function) *The Beta function, denoted by $B(x, y)$, is defined as:*

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, y > 0. \quad (2.3)$$

Definition 2.3 (Riemann-Liouville Fractional Integral [7]) *Let $[a, b] \subset \mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a,x}^\alpha$ and $I_{x,b}^\alpha$ of order $\alpha > 0$ are given by*

$$I_{a,x}^\alpha f(x) := \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt, \quad x > a, \quad (2.4)$$

and

$$I_{x,b}^\alpha f(x) := \int_x^b \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt, \quad x < b, \quad (2.5)$$

respectively.

The integrals $I_{a,x}^\alpha$ and $I_{x,b}^\alpha$ are referred to as the left-sided and right-sided Riemann-Liouville fractional integrals, respectively.

Definition 2.4 (Riemann-Liouville Fractional Derivative [7]) *Let $\alpha \in (n-1, n), n \in \mathbb{N}$. The Riemann-Liouville fractional derivatives $D_{a,x}^\alpha$ and $D_{x,b}^\alpha$ of order $\alpha > 0$ are given by*

$$D_{a,x}^\alpha f(x) = \left(\frac{d}{dx}\right)^n I_{a,x}^{n-\alpha} f(x), \quad x > a \quad (2.6)$$

and

$$D_{x,b}^\alpha f(x) = \left(-\frac{d}{dx}\right)^n I_{x,b}^{n-\alpha} f(x), \quad x < b \quad (2.7)$$

respectively.

The derivatives $D_{a,x}^\alpha$ and $D_{x,b}^\alpha$ are referred to as the left-sided and right-sided Riemann-Liouville fractional derivatives, respectively.

Definition 2.5 (Caputo Fractional Derivative [7]) Let $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that $\alpha \in (n-1, n)$. The left and right sided Caputo fractional derivatives ${}^c D_{a,x}^\alpha$ and ${}^c D_{x,b}^\alpha$ of order $\alpha > 0$ are given in terms of the Riemann-Liouville fractional derivatives by

$${}^c D_{a,x}^\alpha f(x) := D_{a,x}^\alpha \left[f(x) - \sum_{\ell=0}^{n-1} \frac{f^{(\ell)}(a)}{\ell!} (x-a)^\ell \right], \quad x > a, \quad (2.8)$$

and

$${}^c D_{x,b}^\alpha f(x) := D_{x,b}^\alpha \left[f(x) - \sum_{\ell=0}^{n-1} \frac{f^{(\ell)}(a)}{\ell!} (x-a)^\ell \right], \quad x < b, \quad (2.9)$$

respectively.

Theorem 2.1 ([7]) Let $\alpha > 0$ and $n = \lfloor \alpha \rfloor + 1$. If $f \in AC^n[a, b]$, then the Caputo fractional derivatives ${}^c D_{a,x}^\alpha f(x)$ and ${}^c D_{x,b}^\alpha f(x)$ exist almost everywhere on $[a, b]$.

If $\alpha \notin \{0, 1, 2, \dots\}$, the fractional derivatives are given by:

$${}^c D_{a,x}^\alpha f(x) = \int_a^x \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(t) dt =: I_{a,x}^{n-\alpha} D^n f(x), \quad x > a, \quad (2.10)$$

and

$${}^c D_{x,b}^\alpha f(x) = (-1)^n \int_x^b \frac{(t-x)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(t) dt =: (-1)^n I_{x,b}^{n-\alpha} D^n f(x), \quad x < b, \quad (2.11)$$

where $D^n = d^n/dx^n$ denotes the n -th derivative.

In particular, for $1 < \alpha < 2$, and $f \in C^2[a, b]$, the fractional derivatives simplify to:

$${}^c D_{a,x}^\alpha f(x) = \int_a^x \frac{(x-t)^{1-\alpha}}{\Gamma(2-\alpha)} f''(t) dt =: I_{a,x}^{2-\alpha} D^2 f(x), \quad x > a, \quad (2.12)$$

and

$${}^c D_{x,b}^\alpha f(x) = \int_x^b \frac{(t-x)^{1-\alpha}}{\Gamma(2-\alpha)} f''(t) dt =: I_{x,b}^{2-\alpha} D^2 f(x), \quad x < b. \quad (2.13)$$

For convenience, when the lower limit of the Caputo derivative equals zero, we adopt the simplified notation:

$${}^c D_{a,x}^\alpha \equiv {}^c D^\alpha.$$

These definitions form the foundation of fractional calculus, which extends classical

calculus to arbitrary orders, enabling the modeling of complex physical and dynamic systems.

2.2 Notes on Mittag-Leffler Functions

Definition 2.6 *The one-parameter Mittag-Leffler function is defined as [8]:*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \operatorname{Re} \alpha > 0, \quad z \in \mathbb{C}. \quad (2.14)$$

which generalizes the exponential function:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n + 1)}. \quad (2.15)$$

Special Cases

Using the definition, the following relations hold [8]:

$$E_1(\pm z) = e^{\pm z}, \quad E_2(-z^2) = \cos z, \quad (2.16)$$

$$E_2(z^2) = \cosh z, \quad E_{\frac{1}{2}}(\pm z^{\frac{1}{2}}) = e^z \operatorname{erfc}(\mp z^{\frac{1}{2}}). \quad (2.17)$$

Here, $\operatorname{erf}(z)$ and $\operatorname{erfc}(z)$ denote the error and complementary error functions, respectively:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} du, \quad \operatorname{erfc}(z) = 1 - \operatorname{erf}(z). \quad (2.18)$$

Definition 2.7 *The two-parameter Mittag-Leffler function is defined as [8]:*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \operatorname{Re} \alpha > 0, \beta \in \mathbb{C}, \quad z \in \mathbb{C}. \quad (2.19)$$

which generalizes the one-parameter Mittag-Leffler function:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \operatorname{Re} \alpha > 0, \quad z \in \mathbb{C}. \quad (2.20)$$

In particular, $E_{\alpha,1}(z) = E_{\alpha}(z)$. For $\alpha, \beta \in \mathbb{C}$, the condition $\operatorname{Re}(\alpha) > 0$ must hold to ensure convergence.

Special Cases

Using the definition, the following relations hold [8]:

$$E_{1,1}(z) = e^z, \quad E_{1,2}(z) = \frac{e^z - 1}{z}, \quad (2.21)$$

$$E_{2,1}(z) = \cosh \sqrt{z}, \quad E_{2,2}(z) = \frac{\sinh \sqrt{z}}{\sqrt{z}}. \quad (2.22)$$

Recurrence Relations

The two-parameter Mittag-Leffler function satisfies the following recurrence relations:

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z E_{\alpha,\beta+\alpha}(z), \quad (2.23)$$

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z). \quad (2.24)$$

Asymptotic Expansion of the Mittag-Leffler Function

Theorem 2.2 [8][*Asymptotic Expansion of $E_{\alpha,\beta}(z)$*] Let $\alpha \in (0, 2)$, $\beta \in \mathbb{C}$, and $\mu \in \mathbb{R}$ such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$. Then, for $\mu \leq |\arg z| \leq \pi$, the following asymptotic expansion holds:

$$E_{\alpha,\beta}(z) = - \sum_{\ell=1}^n \frac{z^{-\ell}}{\Gamma(\beta - \alpha\ell)} + \mathcal{O}(|z|^{-(n+1)}), \quad \text{as } |z| \rightarrow \infty, \quad n \geq 1.$$

2.3 Notes on Oscillation

Oscillation refers to the recurring variation of a system between two states or about a central value over time. It describes a periodic or non-periodic motion observed in various physical and mathematical systems. In the context of fractional oscillations, it is crucial to understand the underlying concepts of oscillatory behavior.

Examples of oscillation include:

- The motion of a pendulum.
- Vibrations of a tuning fork.
- The sine wave, often used to model oscillatory behavior mathematically.
- The decay of a damped oscillator, which may be governed by fractional differential equations.

2.3.1 Periodic and Oscillatory Motion

Oscillatory motion involves a system moving back and forth around an equilibrium position. Periodic motion is a special type of oscillation that repeats at regular intervals.

However, it is important to note:

- Not all oscillatory motions are periodic (e.g., $e^{-x} \sin(x)$).
- Not all periodic motions are oscillatory (e.g., uniform circular motion).

2.3.2 Oscillating Functions

An oscillating function exhibits recurring changes in value over its domain. Periodic functions, a subset of oscillating functions, satisfy the property:

$$f(x + P) = f(x),$$

where $P > 0$ is the fundamental period. For instance:

$$\sin(x) = \sin(x + 2\pi).$$

A function is oscillating if it alternates between values or directions as its independent variable changes. This includes periodic functions and functions that oscillate without repeating.

Definition 2.8 (Periodic Oscillatory Function) *A function is periodic and oscillatory if it repeats its values over a fixed period while alternating about an equilibrium*

value. For example, $\sin(x)$ is both periodic and oscillatory.

Definition 2.9 (Non-Periodic Oscillatory Function) *A function is non-periodic and oscillatory if it alternates values without repeating exactly. For example, $e^{-x} \sin(x)$ oscillates and decays but does not repeat.*

2.3.3 Behavior of Oscillatory Functions

In the study of fractional oscillations, the behavior of oscillatory functions can be categorized as follows:

- **Purely periodic:** Functions like $\sin(x)$ or $\cos(x)$, which repeat their values over a fixed interval.
- **Damped oscillations:** Functions like $e^{-x} \cos(x)$, which oscillate while their amplitude decays to zero over time.
- **Fractional oscillations:** Oscillations governed by fractional differential equations, often exhibiting memory effects or non-integer decay rates.
- **Monotonic approach:** Functions like $\frac{1}{x+1}$, which are not oscillatory but converge monotonically to zero.

2.3.4 Applications of Oscillation in Fractional Models

Fractional oscillatory models are widely used to describe systems where traditional integer-order models fail to capture complex behaviors. These include:

- Viscoelastic materials, where memory effects play a crucial role.

- Damped harmonic oscillators with fractional damping terms.
- Biological systems exhibiting anomalous diffusion or irregular oscillatory patterns.

Such models leverage fractional derivatives to introduce non-local behavior and better describe oscillatory systems with history-dependent properties.

CHAPTER 3

RATIONAL APPROXIMATIONS FOR THE OSCILLATORY TWO-PARAMETER MITTAG-LEFFLER FUNCTION

3.1 Introduction

We are concerned with the approximation of the two parameter Mittag-Leffler function (MLF) $E_{\alpha,\beta}$, defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \operatorname{Re} \alpha > 0, \beta \in \mathbb{C}, \quad z \in \mathbb{C}. \quad (3.1)$$

This entire function generalizes the MLF of one-parameter, $E_{\alpha} = E_{\alpha,1}$, and contains several well-known functions as special cases. In particular, E_1 is the exponential func-

tion, $E_2(-z^2)$ and $zE_{2,2}(-z^2)$ are the cosine and sine functions, respectively, among others. For some surveys on the Mittag-Leffler functions, see for example [7], [8].

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ arises frequently in the solutions of many physical problems described by differential and/or integral equations of fractional order. In the case $\alpha \in (1, 2)$, the MLF appears naturally in the solutions of fractional diffusion-wave equations, fractional differential equation for motion, and fractional plasma equations, see [9], [10], [11], [12], [13].

Computing the MLF $E_{\alpha,\beta}$ is usually challenging (devising and implementing suitable algorithms) and expensive (computation time). Although the series (3.1) converges for all $z \in \mathbb{C}$, it is impractical or ineffective to use it computationally for $|z| \geq 1$ because the series converges very slowly. As a result, various methods have been developed to evaluate the MLF. In [14], an algorithm based on the location of the argument z in the complex plane is developed, whereby for large $|z|$ values the asymptotic series as $|z| \rightarrow \infty$ is used, for $|z| < 1$ the series definition (3.1) is used, and for values in intermediate regions the integral representation is used. In [15], an approach based on numerical inversion of Laplace transform is proposed. However, these approaches are often computationally time consuming, see [16], [17] for some CPU-time comparisons.

Various rational approximations have been sought to approximate $E_{\alpha,\beta}$ efficiently and accurately. Some of them are based solely on the series definition and provide accurate approximations for small argument values, see for example [18], [19]. Some others are based on global Padé approach [20] in which a hybrid of the local series definition and the asymptotic series representation is used. These global approxima-

tions lead to approximants that are accurate over a wide range of arguments, see [16], [17], [21], [22], [23].

The existing rational approximants are effective when the MLF is completely monotone, which is the case when $\alpha \in (0, 1)$ and $\beta \geq \alpha$. However, when $\alpha > 1$, as discussed in [24], [1], [25], for some α - β combinations, $E_{\alpha,\beta}$ is often oscillatory and having multiple real zeros. In these situations, the rational approximants might fail to trace the oscillation profile and match the large zeros.

In this chapter, we consider in the $\alpha\beta$ -plane the strip

$$\Omega = \{(\alpha, \beta) : 1 < \alpha < 2, \beta \geq 1\}. \quad (3.2)$$

The regions where $E_{\alpha,\beta}$ is monotone and where it is oscillatory are classified by studying the zeros of its derivative. Additionally, we introduce new rational approximants that sufficiently capture more of the MLF roots and can trace their oscillations effectively. This is achieved by decomposing $E_{\alpha,\beta}$ with real roots into another weighted Mittag-Leffler function devoid of roots, accompanied by a polynomial. Based on this decomposition, we construct new rational approximants as a sum of the global Padé approximant and a polynomial. Using this approach, we are able to capture sufficiently many roots of $E_{\alpha,\beta}$ by choosing the appropriate degree of the polynomial. Further, we generalize these approximants to matrix MLF and discuss different approaches in which the approximant can be implemented for matrix arguments. Apart from the computation cost, all approaches yield close approximations.

Numerical experiments are presented to illustrate the performance of our approx-

imants. Considering MLF with scalar arguments, we observe the limitation of the existing global Padé approximants in approximating the oscillatory MLFs. It is then demonstrated that the new approximants introduced here are able to improve the situation thereby providing sufficiently accurate approximations. For the matrix MLF, we consider four approaches for computing the rational approximants and compare them to the matrix MLF algorithm developed in [26]. Using the values obtained from this algorithm as a reference, it is seen that our approximants provide close values with significantly less computation time.

The rest of the chapter is organized as follows: a characterization of the oscillatory behavior of the MLF is given in Section 3.2. In Section 3.3, the derooting representation is discussed. Global Padé approximants of MLF with scalar and matrix arguments are discussed in sections 3.4 and 3.5, respectively. Some applications to the evaluation of solutions of fractional evolution equations are discussed in Section 3.6.

3.2 Monotonicity and oscillatory properties

In this section, we characterize the oscillatory behavior of

$$E_{\alpha,\beta}(-t), \quad (\alpha, \beta) \in \Omega, \quad t \geq 0.$$

For the discussion that follows, the following formula for the derivative [7] plays a fundamental role:

$$\frac{d}{dt}E_{\alpha,\beta}(-t) = -\frac{1}{\alpha} [E_{\alpha,\beta+\alpha-1}(-t) - (\beta-1)E_{\alpha,\beta+\alpha}(-t)]. \quad (3.3)$$

Since $E_{\alpha,\beta}(0) = 1/\Gamma(\beta)$, it follow from (3.3) that

$$\frac{d}{dt}E_{\alpha,\beta}(0) = -\frac{1}{\alpha} \left[\frac{1}{\Gamma(\beta+\alpha-1)} - \frac{\beta-1}{\Gamma(\beta+\alpha)} \right] = -\frac{1}{\Gamma(\beta+\alpha)} < 0. \quad (3.4)$$

The function $E_{\alpha,\beta}(-t)$, $t \geq 0$, is known to be completely monotone for $0 < \alpha \leq 1$, $\beta \geq \alpha$. However, this monotonicity is not necessarily preserved when $(\alpha, \beta) \in \Omega$ since either the function or its derivative could have roots. On the other hand, it has been established in [1] that $E_{\alpha,\beta}$ has at most a finite number of roots when $(\alpha, \beta) \in \Omega$. Thus, $E_{\alpha,\beta}(-t)$ is monotone for sufficiently large t . Furthermore, it is shown in [1] that there exists only one zero when α is sufficiently close to 1 while the number of zeros increases as α increases towards 2.

In Figure 3.1, the curve ϕ is the boundary given in [1, Table 1] such that $E_{\alpha,\beta}$ has a finite number of real roots when (α, β) is below it and none above it. However, the function in the region below $\phi(\alpha)$ differs with respect to the number of roots. Typically, the number of roots increases as α goes to 2. Similarly, we constructed the boundary ψ for the derivative. The corresponding data is presented in Table 3.1.

The monotonicity and oscillatory behavior of $E_{\alpha,\beta}$ is characterized by the different regions marked in Figure 3.1 and described in Table 3.2. It follows from (3.4) that

$E_{\alpha,\beta}(-t)$ is monotonically decreasing for (α, β) in the regions (D) and (F). In the regions (B) and (C), although the function has no real roots, it could have a finite number of oscillations due to the roots of the derivative.

Remark 1 *Since $E_{\alpha,\beta}(-t) \rightarrow 0$ as $t \rightarrow \infty$, it is obvious that each root of it is followed by at least one root of its derivative. This is consistent with the inequality*

$$\phi(\alpha) < \psi(\alpha).$$

For the region below the line $\beta = \alpha + 1$, which includes the regions (A), (B) and (D), the following decomposition is presented in [24],

$$E_{\alpha,\beta}(-t) = g_{\alpha,\beta}(-t) + f_{\alpha,\beta}(-t), \quad t > 0, \quad \beta < \alpha + 1, \quad (3.5)$$

where the function $f_{\alpha,\beta}(-t)$ is asymptotically approaching zero as $t \rightarrow \infty$ and $g_{\alpha,\beta}(-t)$ is an oscillatory function.

When $\beta = 1$ and $1 < \alpha < 2$, the function $f_{\alpha,1}(-t)$ is known to be a completely monotonic function while the function $g_{\alpha,1}(-t)$ is oscillatory with exponentially decreasing amplitude [9]. The properties of E_α , $1 < \alpha < 2$, such as monotonicity, roots, and the oscillatory behavior have been discussed in details in [27].

In the region (B), although $E_{\alpha,\beta}(-t)$ has no roots as it decays to zero, it undergoes oscillations due to the roots of its derivative as demonstrated in Figure 3.2. This behavior is also consistent with the oscillatory nature of $g_{\alpha,\beta}$ there. On the other hand, in the region (D) in which $E_{\alpha,\beta}$ is monotone, the behavior of the oscillatory

Table 3.1: The data for the boundary ϕ [1, Table 1] and the boundary ψ . The function $E_{\alpha,\beta}$ possesses real roots below the curve $\phi(\alpha)$ and has no real roots above it. Conversely, below the curve $\psi(\alpha)$, $E_{\alpha,\beta}$ exhibits non-monotonic behavior, whereas above it, the function demonstrates monotonicity.

α	$\phi(\alpha)$	$\psi(\alpha)$
1.00	1.00000	1.00000
1.05	1.05924	1.06640
1.10	1.12400	1.14204
1.15	1.19325	1.22532
1.20	1.26674	1.31565
1.25	1.34437	1.41277
1.30	1.42608	1.51654
1.35	1.51187	1.62689
1.40	1.60173	1.74374
1.45	1.69565	1.86706
1.50	1.79365	1.99685
1.55	1.89573	2.13306
1.60	2.00191	2.27568
1.65	2.11219	2.42471
1.70	2.22660	2.58014
1.75	2.34513	2.74196
1.80	2.46779	2.91017
1.85	2.59460	3.08477
1.90	2.72557	3.26575
1.95	2.86070	3.45311
2.00	3.00000	3.64686

Table 3.2: Number of roots in the regions of Figure 3.1.

Region	$E_{\alpha,\beta}(-t)$	$dE_{\alpha,\beta}/dt$
A	Finite number of roots	Finite number of roots
B	No real roots	Finite number of roots
C	No real roots	Finite number of roots
D	No real roots	No real roots
F	No real roots	No real roots

function $g_{\alpha,\beta}$ fades away and $f_{\alpha,\beta}(-t)$ dominates.

3.3 Derooting decomposition

A key step to accurately approximate $E_{\alpha,\beta}(-t)$ when $1 < \alpha < 2$ is to capture its oscillatory behavior and roots. However, in region (A) the number of roots in general could exceed the ones of rational approximants. This makes these approximants valid only for small arguments.

A Derooting approach can be used to extend the validity of rational approximants to large intervals. For this purpose, we use the following recursive identity [28]

$$\begin{aligned} E_{\alpha,\beta}(t) &= t^r E_{\alpha,\beta+\alpha r}(t) + P_{\alpha,\beta}^{r-1}(t), \\ \alpha > 0, \quad \beta > 0, \quad t \in \mathbb{R}, \quad r &= 0, 1, 2, \dots, \end{aligned} \tag{3.6}$$

where,

$$P_{\alpha,\beta}^{r-1}(t) := \begin{cases} \sum_{k=0}^{r-1} \frac{t^k}{\Gamma(\alpha k + \beta)}, & r = 1, 2, 3, \dots \\ 0, & r = 0. \end{cases} \tag{3.7}$$

In this identity, $E_{\alpha,\beta}$ is decomposed into another MLF with shifted second parameter and a polynomial of degree $r - 1$. This allows us to replace the parameters (α, β) in region (A) by parameters $(\alpha, \beta + \alpha r)$ above the boundaries $\phi(\alpha)$ or above $\psi(\alpha)$. Consequently, we obtain a function $E_{\alpha,\beta+\alpha r}$ that has no roots or even could be monotone. Therefore, the approximation of $E_{\alpha,\beta}(-t)$, $1 < \alpha < 2$, with a finite number of real roots can be replaced by approximating another one that has no roots or one that is monotone.

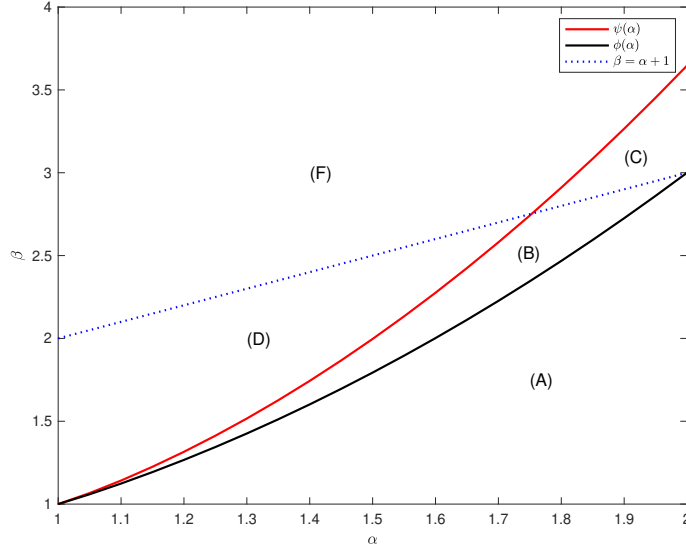


Figure 3.1: α - β phase diagram for $E_{\alpha,\beta}(-t)$ and its derivative, illustrating the function's behavior for five distinct regions. In region (A), $E_{\alpha,\beta}$ exhibits oscillatory behavior with the presence of real roots. In regions (B) and (C), $E_{\alpha,\beta}$ oscillates without roots. In regions (D) and (F), $E_{\alpha,\beta}$ displays monotonic behavior.

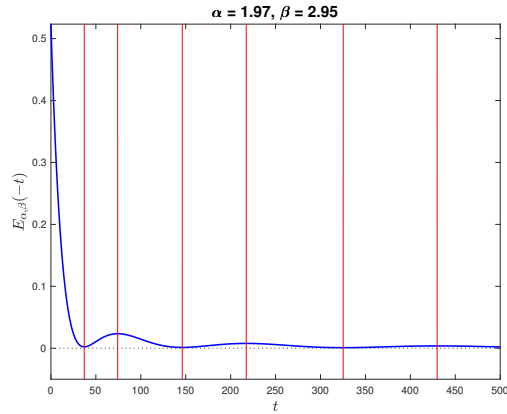


Figure 3.2: Typical oscillations in region (B) about the roots (marked by the red lines) of the derivative.

3.4 Rational approximation

Based on the global rational approximation technique introduced for transcendental functions in [20], a variety of global Padé approximants for $E_{\alpha,\beta}(-t)$, $0 < \alpha < 1$, are developed and implemented [16], [17], [21], [22], [23]. These approximants are still valid for $E_{\alpha,\beta}(-t)$, $1 < \alpha < 2$, over intervals in which the number of its roots does not exceed the number of approximant roots. This renders these approximants inadequate when approximating MLF over an interval in which it has multiple roots, as the case when (α, β) in region (A) of Figure 3.1.

In general, the global Padé approximation $R_{\alpha,\beta}^{m,n}(t)$, where m and n are positive integers, for $E_{\alpha,\beta}(-t)$ takes the form [16], [17]

$$R_{\alpha,\beta}^{m,n}(t) = \frac{1}{\Gamma(\beta - \alpha)} \frac{p_1 + p_2 t \cdots + t^{\nu-1}}{q_0 + q_1 t + \cdots + t^\nu}, \quad v = \frac{m + n - 1}{2}. \quad (3.8)$$

The construction of the global Padé approximation depends on the series definition (3.1) of the MLF as well as its asymptotic expansion given by the following theorem.

Theorem 3.1 [4] *Let $\alpha \in (0, 2)$, $\beta \in \mathbb{C}$, and $\mu \in \mathbb{R}$ such that $\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}$.*

Then for $\mu \leq |\arg z| \leq \pi$,

$$E_{\alpha,\beta}(z) = - \sum_{k=1}^n \frac{(z)^{-k}}{\Gamma(\beta - \alpha k)} + \mathcal{O}(|z|^{-(n+1)}), \quad \text{as } |z| \rightarrow \infty, \quad n \geq 1.$$

It has been shown in [16] that the approximant $R_{\alpha,\beta}^{m,n}$ has the asymptotic error

$$|E_{\alpha,\beta}(-t) - R_{\alpha,\beta}^{m,n}(t)| = \begin{cases} \mathcal{O}\left(t^{\frac{m-n-1}{2}}\right), & m \geq n+1, \text{ as } t \rightarrow 0, \\ \mathcal{O}\left(t^{-(n+1)}\right), & n > 1, \text{ as } t \rightarrow \infty. \end{cases} \quad (3.9)$$

In the rest of this section we consider the global Padé approximants $R_{\alpha,\beta}^{7,2}$ and $R_{\alpha,\beta}^{13,4}$ developed in [16], [17] and examine their performance when $1 < \alpha < 2$. Then we describe how the derooting decomposition leads to approximants with increased number of roots and thus able to track more of MLF roots.

3.4.1 Global Padé approximation

For $\{(\alpha, \beta) : 1 < \alpha < 2, \beta \geq 1, \alpha \neq \beta\}$, we consider the global Padé approximants

$$R_{\alpha,\beta}^{7,2}(t) = \frac{1}{\Gamma(\beta - \alpha)} \frac{p_1 + p_2 t + p_3 t^2 + t^3}{q_0 + q_1 t + q_2 t^2 + q_3 t^3 + t^4}, \quad t \geq 0, \quad (3.10)$$

and

$$R_{\alpha,\beta}^{13,4}(t) = \frac{1}{\Gamma(\beta - \alpha)} \frac{p_1 + p_2 t + \cdots + p_7 t^6 + t^7}{q_0 + q_1 t + q_2 t^2 + \cdots + q_7 t^7 + t^8}, \quad t \geq 0, \quad (3.11)$$

where the coefficients satisfy respectively the systems

$$\begin{pmatrix} 1 & 0 & 0 & a_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 1 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & 0 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & b_0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -a_0 \\ -a_1 \\ -b_1 \end{pmatrix}, \quad (3.12)$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & b_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
p_3 \\
p_4 \\
p_5 \\
p_6 \\
p_7 \\
q_0 \\
q_1 \\
q_2 \\
q_3 \\
q_4 \\
q_5 \\
q_6 \\
q_7
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
-1 \\
-a_0 \\
-a_1 \\
-a_2 \\
-a_3 \\
-b_3 \\
-b_2 \\
-b_1
\end{pmatrix}, \tag{3.13}$$

$$a_j = \frac{(-1)^{j+1}\Gamma(\beta - \alpha)}{\Gamma(\beta + j\alpha)}, \quad b_j = \frac{(-1)^{j+1}\Gamma(\beta - \alpha)}{\Gamma(\beta - (j+1)\alpha)}, \quad j = 0, 1, \dots$$

Remark 2 As mentioned in [17], it is important to highlight that the systems (3.12) and (3.13), for determining the coefficients of the approximants $R_{\alpha,\beta}^{7,2}$ and $R_{\alpha,\beta}^{13,4}$, exhibit relatively high condition numbers. Consequently, solving these systems necessitates careful attention to avoid such accuracy loss due to possible ill conditioning. In the

implementations, we make use of the Matlab function "linsolve" or the correspondence Python function "scipy.linalg.solve", which handle such range of condition numbers.

Computationally, it is observed that $R_{\alpha,\beta}^{13,4}$ has at most three real roots, while $R_{\alpha,\beta}^{7,2}$ has exactly one real root. Consequently, in approximating some of the oscillatory MLFs, each approximant is only able to trace the function up to its largest root. To illustrate the accuracy and limitations of both approximants when $1 < \alpha < 2$, we compare with the Matlab routine `ml` [15] and `ml_matrix` [26] for the scalar and matrix arguments, respectively, as our reference values. Other implementations of the MLF such as Matlab function `mlf` [29] and Mathematica `MittagLefflerE` function can also be used. However, these functions are not available for Matrix arguments which are of primary interest in the current work. As such, we adopt those in the aforementioned references for our comparisons.

In Figure 3.3, two examples from region (A) are shown. In plot (a), the MLF has only one root which is typically the case for $1 < \alpha < 1.4$. As observed, $R_{\alpha,\beta}^{13,4}$ provides a good approximation in this cases. In plot (b), the MLF has many roots which typically the case as α gets closer to 2. As expected, each approximant fails beyond its largest root to capture the oscillations of the MLF .

In Figure 3.4, typical cases from Region (B) and (C) are shown. The corresponding MLFs in both regions is root-free but are oscillatory. As shown, $R_{\alpha,\beta}^{13,4}$ is sufficiently accurate for an extended interval. However, eventually it takes negative values as observed in plot (b).

In Figure 3.5, typical cases from regions (D) and (F) are illustrated. In both

regions the MLF is monotone and globally positive. As seen clearly, $R_{\alpha,\beta}^{13,4}$ is a good approximation for an extended interval.

As general guidelines, we have the following:

- When (α, β) in Regions (A), the existing global Padé approximants are not effective over extended intervals due to the existence of roots.
- When (α, β) in region (B) or (C), more accurate approximants should be used due to the oscillatory behavior of MLF.
- When (α, β) in Region (D) or (F), the approximant $R_{\alpha,\beta}^{13,4}$ is sufficiently accurate since MLF is globally monotone.

3.4.2 Rational approximation for oscillatory MLFs

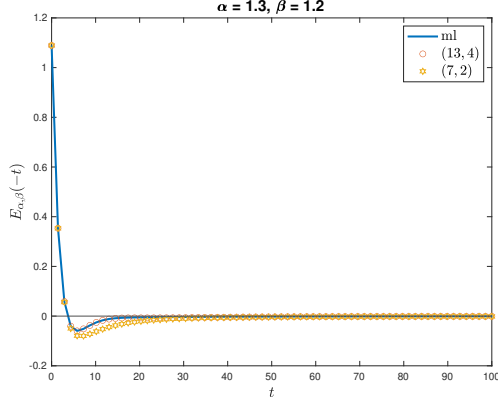
As observed in the above subsection, when (α, β) lies below the boundary $\psi(\alpha)$ in Figure 3.1, then more accurate and multi roots approximants are essential for extended intervals. Next, we propose a class of rational approximants that have the desirable performance.

It follows from (3.6) that we can write

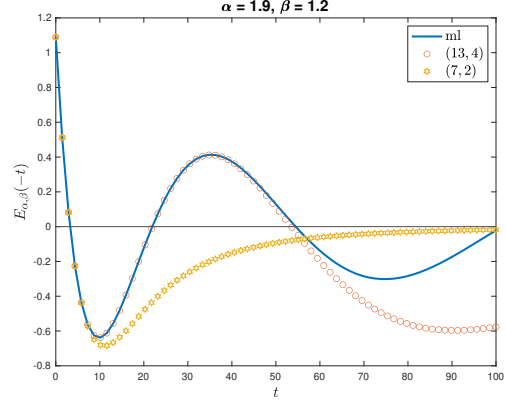
$$E_{\alpha,\beta}(-t) = (-t)^r E_{\alpha,\beta+\alpha r}(-t) + P_{\alpha,\beta}^{r-1}(-t), \quad t > 0. \quad (3.14)$$

Let $R_{\alpha,\beta}^{m,n}$ be the global Padé approximants described in [16]. We introduce the rational approximant

$$E_{\alpha,\beta}(-t) \approx R_{\alpha,\beta}^{m,n,r}(t), \quad t > 0, \quad (3.15)$$

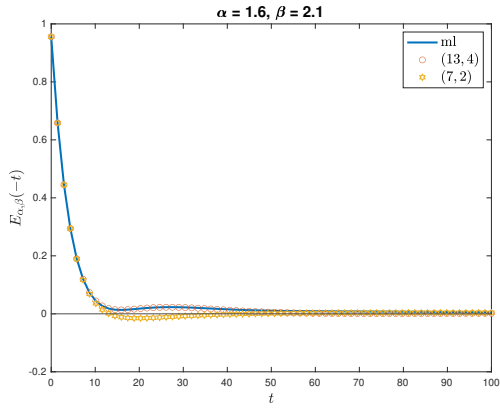


(a) Region(A): one root

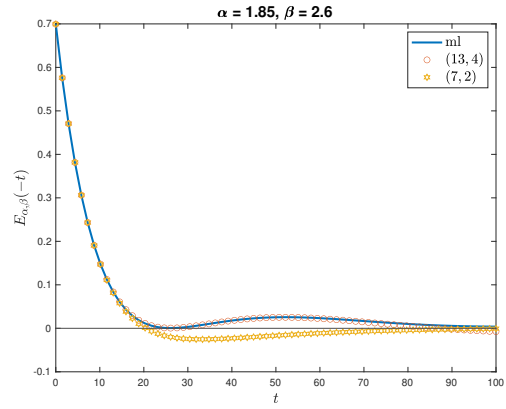


(b) Region(A): many roots

Figure 3.3: Plots of $R_{\alpha,\beta}^{13,4}$ and $R_{\alpha,\beta}^{7,2}$ approximants of $E_{\alpha,\beta}$, (α, β) in region (A).



(a) Region(B): no roots and oscillatory



(b) Region(C): no roots and oscillatory

Figure 3.4: Plots of $R_{\alpha,\beta}^{13,4}$ and $R_{\alpha,\beta}^{7,2}$ approximants of $E_{\alpha,\beta}$, (α, β) in regions (B) and (C).

where,

$$R_{\alpha,\beta}^{m,n,r}(t) := (-t)^r R_{\alpha,\beta+\alpha r}^{m,n}(t) + P_{\alpha,\beta}^{r-1}(-t), \quad r = 0, 1, 2, \dots \quad (3.16)$$

Note that as a special case we have

$$R_{\alpha,\beta}^{m,n,0} = R_{\alpha,\beta}^{m,n}.$$

The integer r could be chosen so that $(\alpha, \beta + \alpha r)$ lies above the boundary $\psi(\alpha)$ in Figure 3.1. As demonstrated in the previous subsection, this shifting of parameters would enhance the performance of $R_{\alpha,\beta+\alpha r}^{m,n}$ since in this case $E_{\alpha,\beta+\alpha r}(-t)$ is root-free and non-oscillatory. Furthermore, r could be chosen large enough so that the approximant $R_{\alpha,\beta}^{m,n,r}$ captures the desired number of roots and oscillations of $E_{\alpha,\beta}$ over an extended interval.

3.4.3 Asymptotic behavior of the approximation error

We define the error

$$e_{\alpha,\beta}^{m,n,r}(t) := E_{\alpha,\beta}(-t) - R_{\alpha,\beta}^{m,n,r}(t).$$

Subtracting the decompositions (3.14) and (3.16), then using the asymptotic error (3.9), we have the asymptotic behavior:

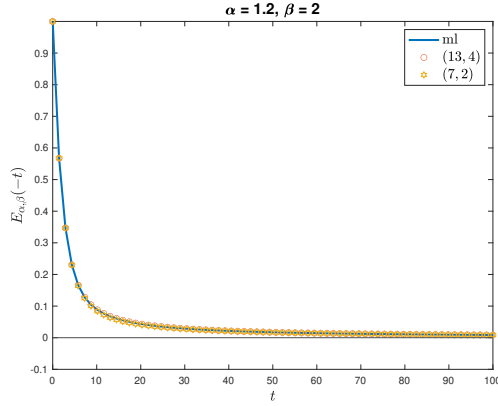
$$\begin{aligned}
|e_{\alpha,\beta}^{m,n,r}(t)| &= |(-t)^r| [E_{\alpha,\beta+\alpha r}(-t) - R_{\alpha,\beta+\alpha r}^{m,n}(t)] \\
&= t^r \begin{cases} \mathcal{O}\left(t^{\frac{m-n-1}{2}}\right), & m \geq n+1, \text{ as } t \rightarrow 0, \\ \mathcal{O}\left(t^{-(n+1)}\right), & n > 1, \text{ as } t \rightarrow \infty, \end{cases} \\
&= \begin{cases} \mathcal{O}\left(t^{\frac{m-n-1}{2}+r}\right), & m \geq n+1, \text{ as } t \rightarrow 0, \\ \mathcal{O}\left(t^{-n-1+r}\right), & n > 1, \text{ as } t \rightarrow \infty. \end{cases} \tag{3.17}
\end{aligned}$$

Note that although the exponent $(-n-1+r)$ in the error term (3.17) as $t \rightarrow \infty$ is positive when $r > n+1$, large values of r increase the number of roots in the modified approximant $R_{\alpha,\beta}^{m,n,r}$, which improves its capability to capture the oscillations of the MLF over an extended interval. Consequently, the actual error starts to grow only after the final root of the approximant.

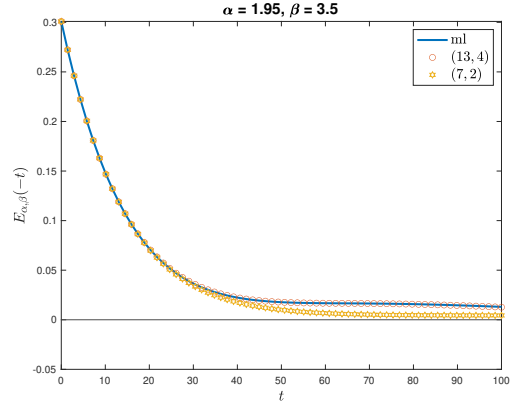
The performance of the generalized approximant $R_{\alpha,\beta}^{13,4,r}$ when (α, β) in regions (A) and (C) is demonstrated in Figures 3.6 and 3.7, respectively, where

$$|e_{\alpha,\beta}^{m,n,r}(t)| := |E_{\alpha,\beta}(-t) - R_{\alpha,\beta}^{m,n,r}(t)|.$$

As can be observed, the interval of approximation can be extended by increasing r . The generalized approximant $R_{\alpha,\beta}^{7,2,r}$ has the same features, although it is not as accurate as $R_{\alpha,\beta}^{14,3,r}$, as shown in Figure 3.8.



(a) Region(D): monotone and positive



(b) Region(F): monotone and positive

Figure 3.5: Plots of $R_{\alpha,\beta}^{13,4}$ and $R_{\alpha,\beta}^{7,2}$ approximants of $E_{\alpha,\beta}$, (α, β) in regions (D) and (F).

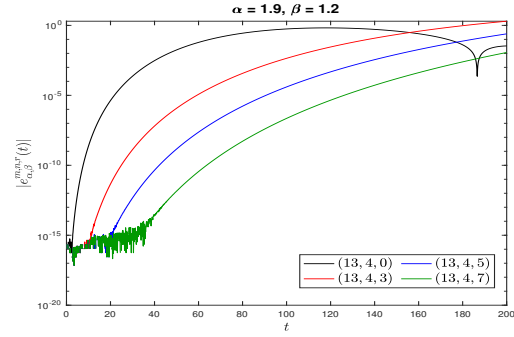
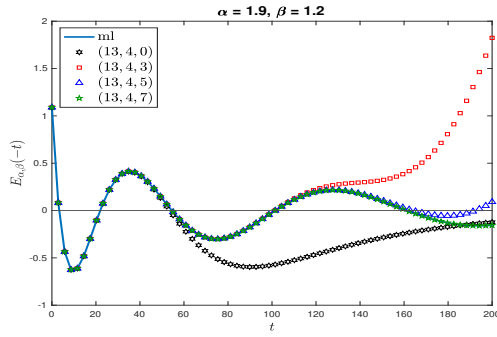


Figure 3.6: Plots of $R_{\alpha,\beta}^{13,4,r}$ for different values of r when (α, β) in region (A).

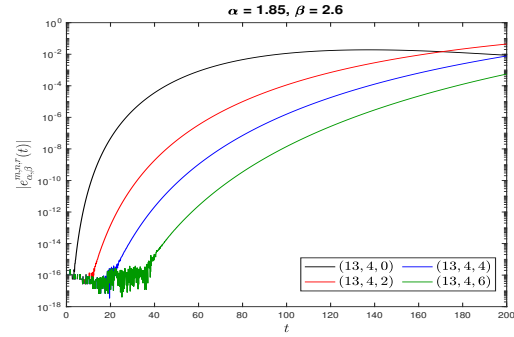
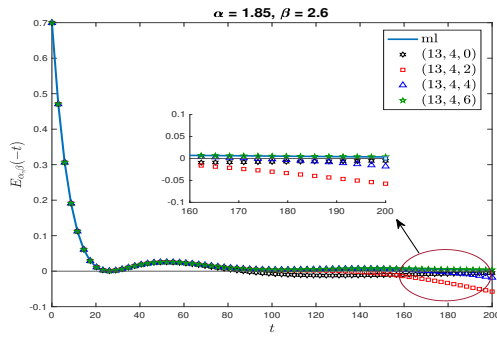


Figure 3.7: Plots of $R_{\alpha,\beta}^{13,4,r}$ for different values of r when (α, β) in region (C).

3.5 Approximation of matrix Mittag–Leffler

In this section we study the approximation of matrix MLF defined by [30]

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \operatorname{Re} \alpha > 0, \beta \in \mathbb{C}, \quad A \in \mathbb{C}^{n \times n}. \quad (3.18)$$

Using the definition of rational functions of matrices, see [31], we define the global Padé approximation of $E_{\alpha,\beta}(A)$ by

$$R_{\alpha,\beta}^{m,n}(A) := \frac{1}{\Gamma(\beta - \alpha)} [q(A)]^{-1} p(A), \quad (3.19)$$

for some appropriate matrix A , where p and q denote the numerator and denominator, respectively, of $\Gamma(\beta - \alpha)R_{\alpha,\beta}^{m,n}$. Furthermore, by extending the decomposition (3.6) to matrix arguments, we define the rational approximant

$$R_{\alpha,\beta}^{m,n,r}(A) := (-A)^r R_{\alpha,\beta+r\alpha}^{m,n}(A) + P_{\alpha,\beta}^{r-1}(-A), \quad r = 0, 1, 2, \dots, \quad (3.20)$$

where $P_{\alpha,\beta}^{r-1}$ is the polynomial given by (4.9).

Several approaches exist for extending a scalar rational function to matrix arguments. For a comprehensive overview, we recommend consulting [31]. Next we discuss some approaches for implementing (3.19) and compare their accuracy and computation time.

1. Linear System approach

A straightforward approach is to evaluate $p(A)$ and $q(A)$ by efficient methods

for the evaluation of matrix polynomials such as Paterson–Stockmeyer method and Horner’s method (nested multiplication) [31]. Additionally, the powers $A^2, A^3, \dots$, can be precomputed and used in the computation of both $p(A)$ and $q(A)$ to minimize the overall cost. It is noteworthy that research in the field of matrix polynomial evaluation is increasingly active. In particular, a new family of methods for evaluating matrix polynomials which is more efficient than the established Paterson–Stockmeyer method is proposed in [31]. This area could be a subject of future research for us, as this section concentrates on introducing general techniques for approximating the MLF matrix using the rational approximation, aiming for a general comparison.

Using this approach, the approximant $R_{\alpha,\beta}^{m,n}(A)$ is obtained by solving the matrix system

$$q(A)R_{\alpha,\beta}^{m,n}(A) = p(A).$$

This requires solving N systems for an $N \times N$ matrix.

2. Partial Fraction approach

Partial fraction decomposition is known to provide an efficient form for evaluating rational functions. For the global Padé approximants, it has been discussed in [16], [17] that these approximants have complex conjugate roots which can add to the efficient implementation. As an example, the approximant $R_{\alpha,\beta}^{13,4}(t)$ which admits the partial fraction decomposition

$$R_{\alpha,\beta}^{13,4}(t) = \sum_{i=1}^4 \left[\frac{c_i}{t - s_i} + \frac{\bar{c}_i}{t - \bar{s}_i} \right], \quad (3.21)$$

where $\{c_1, c_2, c_3, c_4\}$ and $\{s_1, s_2, s_3, s_4\}$ are the non-conjugate residues and poles, respectively, can be written as

$$R_{\alpha,\beta}^{13,4}(t) = 2 \operatorname{Re} \sum_{i=1}^4 \left[\frac{c_i}{t - s_i} \right]. \quad (3.22)$$

So for a matrix argument A , the approximant can be calculated as

$$E_{\alpha,\beta}(-A) \approx R_{\alpha,\beta}^{13,4}(A) := 2 \operatorname{Re} \sum_{j=1}^4 c_j (A - s_j I)^{-1}, \quad (3.23)$$

where I is the identity matrix.

For example, in the implementation using the partial fraction approach, the partial fraction decompositions (3.22) is used to compute matrix-vector products as follows.

For a given square matrix A and a vector v , the matrix-vector product $E_{\alpha,\beta}(-A)v$ is computed using (3.20) as

$$E_{\alpha,\beta}(-A)v \approx R_{\alpha,\beta}^{m,n,r}(A)v := (-A)^r R_{\alpha,\beta+\alpha r}^{m,n}(A)v + P_{\alpha,\beta}^{r-1}(-A)v,$$

where the term

$$R_{\alpha,\beta+\alpha r}^{m,n}(A)v,$$

is computed using the partial fraction decomposition approach. From (3.23), we

use the approximation

$$E_{\alpha,\beta+\alpha r}(-A)v \approx R_{\alpha,\beta+\alpha r}^{m,n}(A)v = 2 \operatorname{Re} \sum_{j=1}^4 c_j (A - s_j \mathbf{I})^{-1} v.$$

By solving the systems

$$(A - s_j \mathbf{I}) x_j = v, \quad \text{for } j = 1, 2, 3, 4, \quad (3.24)$$

we obtain

$$E_{\alpha,\beta+\alpha r}(-A)v \approx 2 \operatorname{Re} \sum_{j=1}^4 c_j x_j,$$

where x_j , are the solutions of the linear systems (3.24). As such, no matrix inversion is required. Moreover, using the partial fraction decomposition of the rational approximations considers an efficient and cost-effective alternative. As mentioned in [17], the poles and residues in the partial fraction decomposition (3.22) rely on α and β . Therefore, their computation can be achieved independently of the argument A . The Python or MATLAB "residue" function could be used for this purpose.

3. Matrix Diagonalization approach

When the matrix argument A is diagonalizable which is a scenario frequently encountered in matrices derived from the semi-discretization of partial differential equations, then the factorization $A = ZDZ^{-1}$ could be considered, where D is the diagonal matrix containing the eigenvalues and the columns of Z are

the corresponding eigenvectors. In this case, the matrix MLF can be computed as [31], [32]

$$E_{\alpha,\beta}(A) = Z E_{\alpha,\beta}(D) Z^{-1} = Z \text{diag}(E_{\alpha,\beta}(\lambda_i)) Z^{-1}.$$

Accordingly, the approximant $R_{\alpha,\beta}^{m,n}$ could be computed as

$$R_{\alpha,\beta}^{m,n}(A) = Z \text{diag}(R_{\alpha,\beta}^{m,n}(\lambda_i)) Z^{-1}.$$

By employing this approach, we only need to calculate the rational approximation for scalar arguments and perform matrix multiplications which considerably reduces the computational time. We note that using the diagonalization approach is advisable only under the condition that the matrix is guaranteed to be well conditioned [31].

Remark 3 *The above approaches for computing the MLF matrix are discussed for completeness and general comparisons. However, they are mainly developed to compute matrix-vector products involved in the implementation of numerical methods for solving fractional differential equations. To the best of our knowledge, computing matrix-vector products involving MLF remains a fundamental challenge since the current algorithms for evaluating MLF of scalar or matrix arguments [14], [15], [26] require a considerably high implementation cost. Therefore, developing such rational approximants is essential, as they have been proven to offer comparable accuracy and efficient approximations at a low cost.*

Table 3.3: Maximum absolute error (AE), maximum relative error (RE) and the runtime (in seconds) for computing the matrix MLF of a 100×100 Redheffer matrix, $\beta = 1$.

	$\alpha = 1.5$			$\alpha = 1.9$		
	AE	RE	Runtime	AE	RE	Runtime
Linear System	1.13E-04	7.20E-03	1.90E-03	2.41E-08	3.00E-06	1.10E-03
Partial Fraction	1.13E-04	7.20E-03	3.70E-03	2.41E-08	3.00E-06	2.30E-03
ml_matrix			1.44E-01			9.73E+00

For experimental purposes, we consider the performance of the approximant (3.16) when applied to Redheffer matrix of size 100×100 , in comparison to the reference values by ml_matrix. In Figure 3.9, a color map is showing the component-wise values and relative errors for $R_{\alpha,\beta}^{13,4}$, $(\alpha, \beta) = (1.9, 1)$, when calculated using partial fractions. In Table 3.3, a comparison of the absolute error, relative error, and runtime is provided for the different techniques in computing $R_{\alpha,\beta}^{13,4}$. The absolute and relative error are given by

$$|E_{\alpha,\beta}(-t) - R_{\alpha,\beta}^{m,n}(t)|,$$

and

$$\left| \frac{E_{\alpha,\beta}(-t) - R_{\alpha,\beta}^{m,n}(t)}{E_{\alpha,\beta}(-t)} \right|,$$

respectively.

3.6 Applications and numerical experiments

We present here some applications of the general Padé approximant (3.16) related to fractional oscillation equations where MLFs arise naturally as solutions. Numerical experiments are included to highlight the efficiency and accuracy of these approxi-

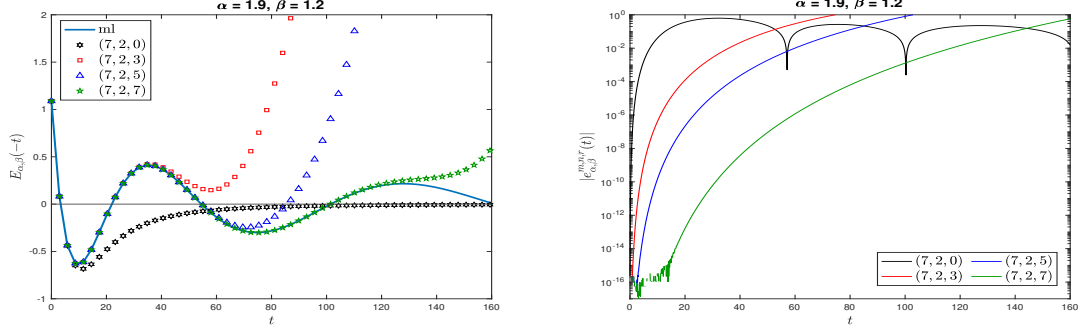


Figure 3.8: Plots of $R_{\alpha,\beta}^{7,2,r}$ for different values of r when (α, β) in region (A).

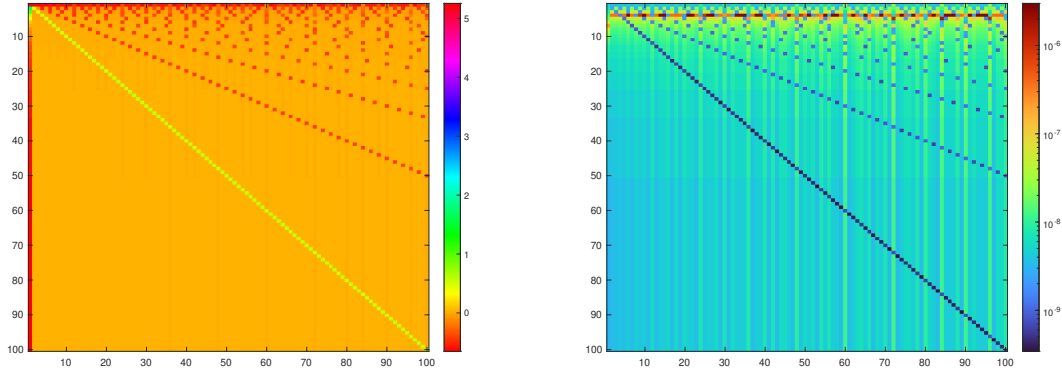


Figure 3.9: MLF matrix approximation $R_{\alpha,\beta}^{13,4}$, $(\alpha, \beta) = (1.9, 1)$, of 100×100 Redheffer matrix using partial fractions (Left) and its relative error (Right).

ments.

3.6.1 Application: Fractional plasma oscillations

The fractional plasma oscillation model, as in [33], is given by

$$\begin{aligned} {}^cD^\alpha u(t) + Au(t) &= f(t), \\ u(0) &= u_0, \\ u'(0) &= u_1, \end{aligned} \tag{3.25}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (1, 2)$ defined by

$${}^cD^\alpha u(t) := \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} u''(\tau) d\tau, & 1 < \alpha < 2, \\ u''(t), & \alpha = 2. \end{cases}$$

The constant A is the fractional electron plasma frequency and $f(t)$ is the electric field. We consider the model with static electric field and with no electric field.

Fractional plasma oscillations model with static electric field

Consider the special case of problem (3.25),

$${}^cD^\alpha u(t) + u(t) = 1, \quad u(0) = 1, \quad u'(0) = -1.$$

Table 3.4: Maximum relative error (RE) and runtime in computing the solution (3.26), over the time mesh $[0, 0.17, 0.34, \dots, 10]$, $\alpha = 1.9$.

	RE	Runtime
$R_{\alpha,\beta}^{13,4}$	3.00E-01	5.69E-04
$R_{\alpha,\beta}^{13,4,2}$	5.95E-03	5.19E-04
$R_{\alpha,\beta}^{13,4,5}$	2.23E-06	5.20E-04
$R_{\alpha,\beta}^{13,4,8}$	3.55E-10	5.52E-04
ml	-	2.38E-02

The exact solution for this initial value problem is

$$u(t) = E_{\alpha}(-t^{\alpha}) - tE_{\alpha,2}(-t^{\alpha}) + t^{\alpha}E_{\alpha,\alpha+1}(-t^{\alpha}). \quad (3.26)$$

In Figure 3.10, a comparison of approximations of (3.26) when $\alpha = 1.2$ is provided. Considering the three terms in (3.26), the pair (α, β) when $\beta = 1$ falls in region (A) of the phase diagram 3.1, where the MLF is oscillatory. While the pairs $(1.2, 2)$ and $(1.2, 2.2)$ fall in region (D) and (F), respectively, in which the MLF has no real roots and no oscillations. A similar comparison when $\alpha = 1.9$ is included in Figure 3.11. In this case, the terms $E_{1.9,1}$ and $E_{1.9,2}$ correspond to (α, β) in parts of Region (A) in which the MLF is highly oscillatory and has many real roots.

To illustrate the computation efficiency, it is seen in Table 3.4 that $R_{\alpha,\beta}^{13,4}$ and $R_{\alpha,\beta}^{13,4,r}$ with $r = 2, 5, 8$ have significantly less computation cost (in time) than the ml function. Moreover, while the modified approximant $R_{\alpha,\beta}^{13,4,8}$ yields more accurate approximations than $R_{\alpha,\beta}^{13,4}$ when compared to the reference values, they have comparable computation time.

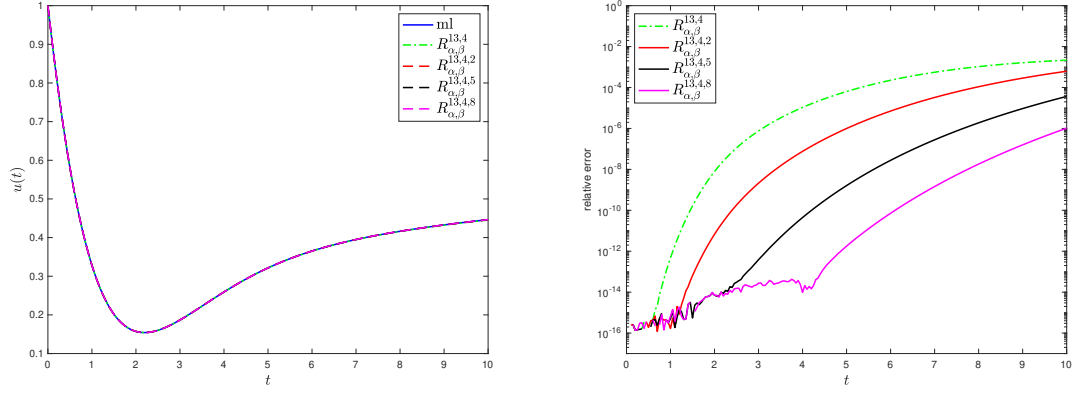


Figure 3.10: Profiles (left) and relative error (right) in approximating the solution (3.26), $\alpha = 1.2$.

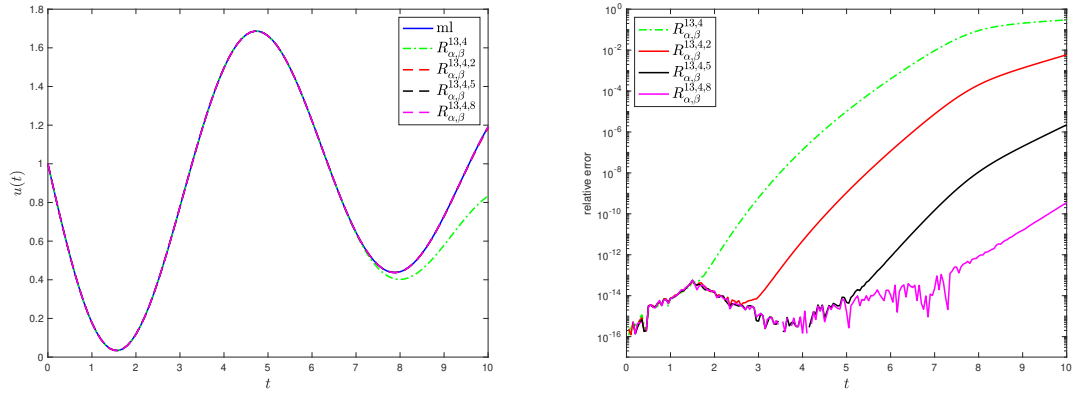


Figure 3.11: Profiles (left) and relative error (right) in approximating the solution (3.26), $\alpha = 1.9$.

Fractional plasma oscillations model with no electric field

Consider the initial values problem

$${}^cD^\alpha u(t) + u(t) = 0, \quad u(0) = 0.2, \quad u'(0) = 0.1.$$

The exact solution of this problem is given by

$$u(t) = 0.2E_\alpha(-t^\alpha) + 0.1tE_{\alpha,2}(-t^\alpha). \quad (3.27)$$

A comparison of approximations of (3.27) when $\alpha = 1.9$ is provided in Figure 3.12. Unlike the static electric field case, it is observed that over an extended time interval, the solution of the fractional oscillation equation has more oscillations around zero. To sufficiently capture these oscillations on the interval $[0, 20]$, the approximant $R_{\alpha,\beta}^{13,4,15}$ can be used. Clearly, as the values of r increase, the number of roots in the rational approximation also increases. As a result, the precision of the approximants improves and their ability to capture a greater number of roots and oscillations across a relatively broader range is enhanced. Consequently, the accuracy of the approximants improves as well as their capability for capturing more roots and oscillations over a relatively extended interval is enhanced.

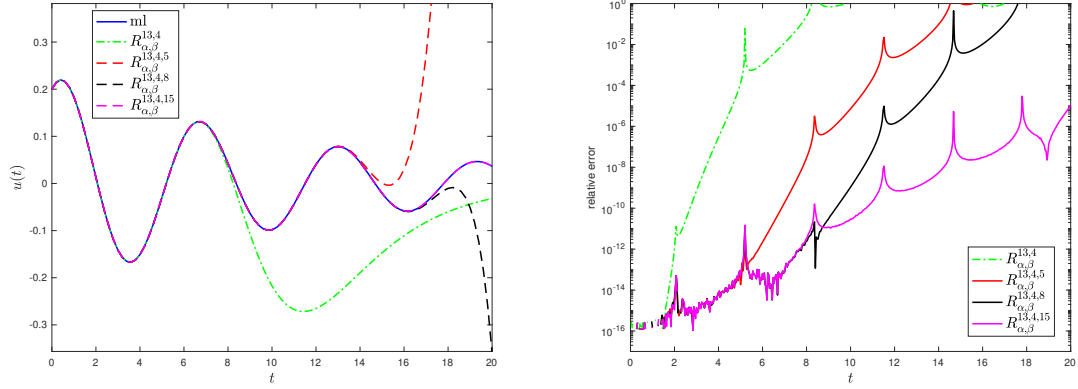


Figure 3.12: Profiles (left) and relative error (right) in approximating the solution (3.27), $\alpha = 1.9$.

3.6.2 Application: Time fractional diffusion-wave equation

Consider the fractional diffusion-wave problem, [34]:

$$\begin{aligned}
 {}^c\partial_t^\alpha u(x, t) &= \partial_x^2 u(x, t), \quad 0 \leq x \leq \pi, \\
 u(x, 0) &= \sin(x), \\
 \partial_t u(x, t)|_{t=0} &= 0, \\
 u(0, t) &= u(\pi, t) = 0, \quad t \in (0, T],
 \end{aligned} \tag{3.28}$$

where ${}^c\partial_t^\alpha$ and ∂_t are the fractional partial time derivative of order $\alpha \in (1, 2)$ and first partial derivative in t , respectively, and ∂_x^2 is the second partial derivative in space.

The exact solution of (3.28) is $u(x, t) = E_\alpha(-t^\alpha) \sin(x)$, however, this problem is intended to illustrate the performance of the approximants (3.16). To this end, applying the second central difference approximation to (3.28) over uniform spatial

mesh with step size $h = \pi/(m+1)$, leads to a system of the form

$$\begin{aligned} {}^cD^\alpha U(t) + AU(t) &= 0, \quad t > 0, \\ U(0) &= U_0, \\ U'(0) &= 0, \end{aligned} \tag{3.29}$$

where the $m \times m$ matrix $A = [a_{ij}]$ is tridiagonal with $a_{i,i} = 2$ and $a_{i,j} = -1$ for $j = i \pm 1$. Further, $U(t) = [u_1(t), u_2(t), \dots, u_m(t)]^T$, $U(0) = [\sin(x_1), \sin(x_2), \dots, \sin(x_m)]^T$, where $u_1(t) = u(x_1, t)$ and $x_i = i \times \pi/m$. The solution of the above system of differential equations is given by $U(t) = E_\alpha(-At^\alpha)U_0$. As an experiment, we take $h = \pi/100$, so the coefficient matrix A is of size 99×99 . In addition, the rational approximation of the matrix MLF is computed by the diagonalization approach.

In Figure 3.13, the solution profile is presented for the case $\alpha = 1.2$. As can be seen, both the global Padé approximant $R_{1.2,1}^{13,4}$ and its generalization $R_{1.2,1}^{13,4,2}$ are comparable. This is expected since the solution has only one root and not much oscillatory behavior is experienced. In Figure 3.14, the case $\alpha = 1.9$ where the MLF is more oscillatory is presented. In this case the rational approximant $R_{1.9,1}^{13,4}$ fails to capture the oscillations for large t and thus yields undesirable approximations. However, the modified $R_{1.9,1}^{13,4,r}$ with an appropriate choice $r = 2$ rectifies this issue and gives more accurate approximate values.

To illustrate the efficiency of these approximants, their runtime is compared with that of `ml_martix` Matlab function. Table 3.5 showcases the CPU time for computing the solution $U(t) = E_\alpha(-At^\alpha)U_0$ on a time mesh $\{t_n = 0.1n, n = 0, 1, \dots, 100\}$. The

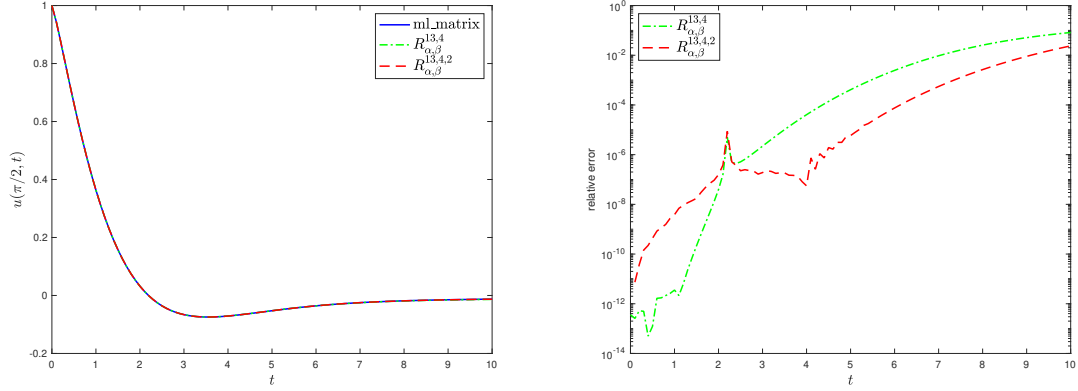


Figure 3.13: Application 3.6.2: approximation of $u(\pi/2, t)$ (left) and relative error (right), $\alpha = 1.2$.

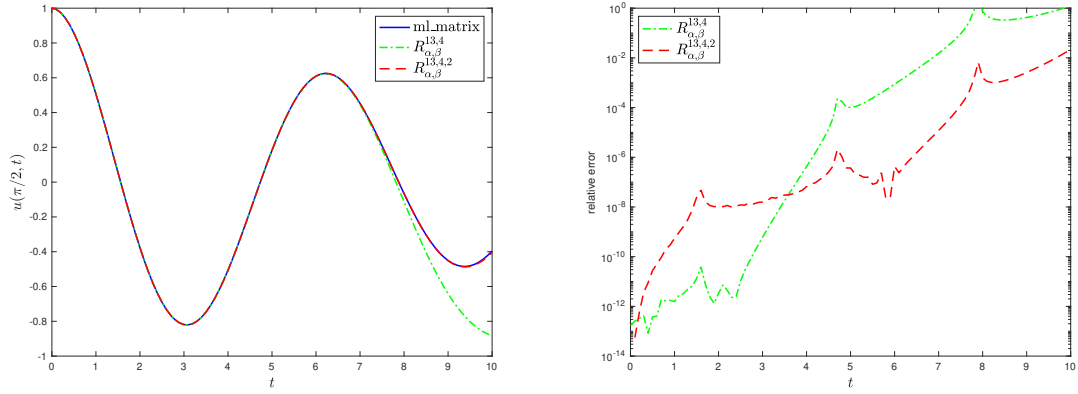


Figure 3.14: Application 3.6.2: approximation of $u(\pi/2, t)$ (left) and relative error (right), $\alpha = 1.9$.

results show that the approximant $R_{1.9,1}^{13,4,2}$ performs about seven times faster than the ml_matrix . Moreover, while the approximant $R_{1.9,1}^{13,4,2}$ on an extended interval is able to achieve higher accuracy than $R_{1.9,1}^{13,4}$, their runtime is comparable.

3.7 Concluding Remarks

In this chapter, a characterization of the oscillatory and monotone behavior of the two-parameter Mittag-Leffler function is given. Furthermore, generalized rational ap-

Table 3.5: Application 3.6.2: runtime in computing the solution of (3.29) over the time mesh $[0, 0.1, 0.2, \dots, 10]$, $\alpha = 1.9$.

	Runtime
$R_{1.9,1}^{13,4}$	8.00E-02
$R_{1.9,1}^{13,4,2}$	9.95E-02
ml_matrix	6.796E-01

proximants over extended intervals are developed.

These approximants are based on decomposing a Mittag-Leffler function that possesses real roots into a combination of a weighted rootless Mittag-Leffler function and a polynomial. The approximants have good tracking capabilities of the roots and oscillations over extended intervals. To show the efficiency of these approximants, numerical experiments were presented alongside various applications. Furthermore, to demonstrate computation efficiency, comparisons with the ml and ml_matrix Matlab functions are included. Such approximants play an effective role in the implementation of numerical methods for fractional oscillation equations.

CHAPTER 4

GENERALIZED EXPONENTIAL TIME DIFFERENCING FOR LINEAR FRACTIONAL OSCILLATION MODELS

4.1 Introduction

We consider the initial value problem for the system of equations

$$\begin{aligned} {}^cD^\alpha \mathbf{u}(t) + \mathbf{A}\mathbf{u}(t) &= \mathbf{f}(t), \quad t > 0, \quad \alpha \in (1, 2), \\ \mathbf{u}(0) &= \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{u}_0^1, \end{aligned} \tag{4.1}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$ defined by [7]

$${}^cD^\alpha \mathbf{u}(t) := \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \mathbf{u}''(\tau) d\tau, & 1 < \alpha < 2, \\ \mathbf{u}''(t), & \alpha = 2. \end{cases}$$

The coefficient $\mathbf{A}$ is an $m \times m$ matrix which is relevant to the physical parameters such as the external oscillation frequency. The vector-valued function $\mathbf{f} = [f_1, f_2, \dots, f_m]^T$ is the external force and $\mathbf{u} = [u_1, u_2, \dots, u_m]^T$ is the solution (displacement) vector. $\mathbf{u}_0 = [u_{01}, u_{02}, \dots, u_{0m}]^T$, $\mathbf{u}_0^1 = [u_{01}^1, u_{02}^1, \dots, u_{0m}^1]^T$ are the initial conditions.

When $\alpha = 2$, the well-known harmonic oscillator equation is recovered. This oscillator is one of the simplest mechanical systems whose motion is governed by a second-order differential equation. A wide mathematical interest has been witnessed to this kind of problems and various numerical methods have been developed for this purpose [35], [36].

As many real-world processes are inherently memory dependent in nature, incorporating the concept of fractional calculus could represent a significant stride towards a more realistic modeling approach. Fractional calculus has been employed to mathematically articulate real-world challenges encountered across diverse scientific domains. Fractional order derivatives are recognized by scientists and engineers as a powerful instrument for describing various complex phenomena. A detailed review of fractional calculus is included in [7], [4], [37].

Various mathematical models describing fractional oscillators and their analytical solutions are explored in the literature. In [11], the integral equation governing the motion of the harmonic oscillator is generalized by considering a fractional order integral. This modification results in the formulation of the equation of motion for a fractional oscillator. The fractional oscillator equation involving a time fractional derivative of Riemann-Liouville type is discussed in [38]. Additionally, the linearly damped oscillator equation with Caputo fractional derivative damping term is explored in [39]. Novel analytical solutions for mass-spring-damper systems described using Liouville–Caputo left generalized fractional derivative are presented in [40]. For comprehensive explanations of fractional oscillator systems and their applications, readers may refer to [11], [33], [41], [42], [10], [43], [13], [44].

Typically, obtaining suitable and computable exact solutions for fractional models is challenging or unfeasible. As a result, it becomes crucial to develop numerical methods tailored to such models. Hence, there is a necessity to create efficient and reliable numerical approaches for addressing models involving fractional differential equations.

In the realm of numerical methods, the Adams-Bashforth-Moulton method and a finite-difference scheme for solving models of linear fractional oscillators are introduced in [2]. In [45], the reproducing kernel Hilbert space method is proposed for obtaining a numerical solution to the fractional-order Duffing-Van der Pol oscillator equation. In [46], a numerical method to solve the fractional-order Duffing–Van der Pol oscillator equation is proposed, based on using collocation points and approximating the solution employing Bernoulli wavelets. In [47], the Jacobi spectral collocation method to

solve the fractional model of Liénard and Duffing equations with the Liouville–Caputo fractional derivative is presented. These equations represent a generalization of the spring–mass system equation and capture the dynamics of an oscillating circuit.

Another class of numerical methods, trigonometric integrators, which are designed to efficiently solve second-order differential equations with oscillatory or periodic behavior, such as those encountered in mechanical oscillators or wave phenomena. These integrators excel in accurately capturing the oscillatory nature of solutions without requiring excessively small time steps. In contrast to exponential integrators, which are tailored for systems with rapidly decaying behaviors (refer to [48] for an extensive survey), trigonometric integrators are designed to tackle systems exhibiting fast oscillations. Further insights into trigonometric and exponential integrators can be found in [49], [50], [51], [52], [53], [54].

Exponential time differencing (ETD) is a technique for developing numerical methods applicable to ordinary differential equations or for time integration of partial differential equations, particularly those exhibiting stiffness. The fundamental principle underlying ETD hinges on the precise solution of the homogeneous part, paired with a suitable approximation of the integral pertaining to the non-homogeneous part. ETD techniques furnish explicit methods that are inherently stable for systems with fast-decaying or oscillating modes, commonly referred to as stiff systems. Notably, these methods impose no stringent constraints on time step sizes, distinguishing them from standard explicit time-stepping approaches. Stiffness, in a broader sense, may exhibit traits of both rapid decay and rapid oscillation [55].

The term generalized exponential time differencing (GETD) denotes the extension

of ETD to fractional differential equations, as introduced in [5]. The application of GETD extends the flexibility and efficiency of ETD techniques to the field of fractional calculus, offering a valuable tool for handling complex dynamics characterized by fractional derivatives.

In this chapter, we consider a general form of the driven fractional oscillation equation. We obtain an exact closed-form of the solution which is in terms of Mittag-Leffler function (MLF) and a convolution integral. We approximate the solution using the exponential time differencing technique. We approximate the integral using a linear interpolation polynomial and we use the generalized rational approximation for the MLF established in [27], [56], [16], [17]. As a result, we develop an efficient second-order numerical scheme for a class of fractional oscillation problems.

Higher orders of convergence can be achieved using, for example, quadratic interpolation. However, we are not aiming for high-order accuracy in this chapter. Unlike many methods in the literature where the order depends on α , the order of our numerical scheme remains independent of α . The strength of our approach is its robustness, consistently achieving a second-order convergence rate across different scenarios, including both oscillatory and non-oscillatory solutions, as well as smooth and non-smooth source term models.

The structure of this manuscript is as follows. In the coming Section 5.2, the description of the numerical scheme is introduced. The stability and convergence analysis for smooth and non-smooth force terms is presented in Section 5.6. Section 4.4 showcases various numerical experiments. Concluding remarks can be found in Section 4.5.

4.2 Derivation and implementation of the GETD scheme

As a motivation, when $\alpha = 2$ in the model problem (4.1), we obtain a second-order system of differential equations. By applying the Laplace transform and its inversion, and utilizing the series expansion of trigonometric matrix functions, the exact solution is expressed in integral form

$$\mathbf{u}(t) = \cos(\mathbf{B}t)\mathbf{u}_0 + \mathbf{B}^{-1} \sin(\mathbf{B}t)\mathbf{u}_0^1 + \mathbf{B}^{-1} \int_0^t \sin(\mathbf{B}(t - \tau))\mathbf{f}(\tau) d\tau, \quad (4.2)$$

where the matrix $\mathbf{B}$ is such that $\mathbf{B}^2 = \mathbf{A}$. Subsequently, trigonometric integrators are obtained by employing suitable approximations of the integral on the right-hand side.

Next, we turn our attention to the derivation and implementation of the GETD scheme for the model problem (4.1).

4.2.1 The GETD Scheme

Using Laplace transform and its inversion, we obtain the mild solution of (4.1),

$$\mathbf{u}(t) = E_\alpha(-\mathbf{A}t^\alpha)\mathbf{u}_0 + tE_{\alpha,2}(-\mathbf{A}t^\alpha)\mathbf{u}_0^1 + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\mathbf{A}(t - \tau)^\alpha)\mathbf{f}(\tau) d\tau, \quad (4.3)$$

where $E_{\alpha,\beta}$ is the two-parameter MLF defined for any square matrix $\mathbf{B}$ by [30]

$$E_{\alpha,\beta}(\mathbf{B}) = \sum_{k=0}^{\infty} \frac{\mathbf{B}^k}{\Gamma(\alpha k + \beta)}, \quad \operatorname{Re}(\alpha) > 0, \quad \beta \in \mathbb{C},$$

and $E_{\alpha} = E_{\alpha,1}$.

To approximate (4.3) over a specific interval $[0, T]$, inspired by the trigonometric integrators presented in [50], [57], [58], we introduce mesh points $0 = t_0 < t_1 < \dots < t_N = T$. Subsequently, let $I_n = [t_n, t_{n+1}]$, $k_n = t_{n+1} - t_n$, $n = 0, 1, \dots, N-1$, and $k = \max_n k_n$. Now, for $i = 1, 2, \dots, m$, let $p_{n,i}(\tau)$ denote the linear interpolation of f_i on the interval I_n , $n = 0, 1, \dots, N-1$. Specifically, letting $f_i^n = f_i(t_n)$, we have

$$p_{n,i}(\tau) = f_i^n + \frac{\tau - t_n}{k_n} (f_i^{n+1} - f_i^n). \quad (4.4)$$

Further, define the vector-valued function

$$\mathbf{P}_n : I_n \rightarrow \mathbb{R}^m, \quad \mathbf{P}_n(\tau) = [p_{n,1}(\tau), p_{n,2}(\tau), \dots, p_{n,m}(\tau)]^{\mathbb{T}}.$$

Setting $\mathbf{u}^0 = \mathbf{u}(0)$, and $\mathbf{u}^n \approx \mathbf{u}(t_n)$, $1 \leq n \leq N$, the numerical scheme for (4.1) is defined by

$$\begin{aligned} \mathbf{u}^n = & E_{\alpha}(-\mathbf{A}t_n^{\alpha})\mathbf{u}^0 + t_n E_{\alpha,2}(-\mathbf{A}t_n^{\alpha})\mathbf{u}_0^1 + \\ & \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\mathbf{A}(t_n - \tau)^{\alpha}) \mathbf{P}_j(\tau) d\tau. \end{aligned} \quad (4.5)$$

By employing integration by parts on (4.5), the numerical scheme amounts to

$$\mathbf{u}^n = E_\alpha(-\mathbf{A}t_n^\alpha) \mathbf{u}^0 + t_n E_{\alpha,2}(-\mathbf{A}t_n^\alpha) \mathbf{u}_0^1 + \mathcal{E}_n^{\alpha,\alpha}(0; \mathbf{A}) \mathbf{f}^0 + \mathcal{F}_n, \quad n = 1, \dots, N, \quad (4.6)$$

where

$$\mathcal{E}_n^{\alpha,\beta}(\tau; \mathbf{A}) = (t_n - \tau)^\beta E_{\alpha,\beta+1}(-\mathbf{A}(t_n - \tau)^\alpha),$$

and

$$\mathcal{F}_n = \sum_{j=0}^{n-1} \mathcal{E}_n^{\alpha,\alpha+1}(t_j; \mathbf{A}) (\Delta \mathbf{f}^{j+1} - \Delta \mathbf{f}^j), \quad (4.7)$$

with $\Delta \mathbf{f}^0 = 0$, $\Delta \mathbf{f}^j = \frac{\mathbf{f}^j - \mathbf{f}^{j-1}}{k_{j-1}}$, $j = 1, 2, \dots, N$.

Remark 4 *Other varieties of GETD schemes for solving (4.1) can be obtained by considering different approaches to approximating the integral in (4.5). For example, Garrappa and Popolizio [59] discussed collocation-type GETD schemes for initial value problems of order $n-1 < \alpha \leq n$, $n = 1, 2, \dots$, and demonstrated that for a sufficiently smooth source term, $O(k^{d+1})$ convergence rates can be achieved using quadrature rules with degree d precision. In the local polynomial interpolation approach, the integral is evaluated exactly using analytic closed-form formulas. In this approach, higher-degree polynomial interpolants, such as quadratic interpolation with midpoints, can be utilized to achieve a higher order GETD scheme. These higher-order interpolants can also be used with nonuniform meshes to maintain the desired accuracy when f is nonsmooth.*

Our primary focus in this chapter, however, is to develop a GETD scheme for (4.1), aiming to take advantage of its robustness to the inherent oscillatory nature of the problem. To this end, we have considered local linear interpolants in order

to develop a second order accurate scheme, where nonuniform meshes are used to maintain the optimal convergence rate.

4.2.2 MLF Computation

To compute the MLFs $E_{\alpha,\beta}$ with $1 < \alpha < 2$ appearing in the numerical scheme (4.6), we employ rational approximants capable of capturing the roots and oscillations inherent in these functions. These approximants are introduced for the cases $\beta = 1$ in [27], which subsequently been extended and discussed for $\beta \in \mathbb{R}$ in [56].

Consider the following recurrence relation

$$\begin{aligned} E_{\alpha,\beta}(t) &= t^r E_{\alpha,\beta+\alpha r}(t) + P_{\alpha,\beta}^{r-1}(t), \\ \alpha > 0, \quad \beta > 0, \quad t \in \mathbb{R}, \quad r &= 0, 1, 2, \dots, \end{aligned} \tag{4.8}$$

where,

$$P_{\alpha,\beta}^{r-1}(t) := \begin{cases} \sum_{k=0}^{r-1} \frac{t^k}{\Gamma(\alpha k + \beta)}, & r = 1, 2, 3, \dots, \\ 0, & r = 0. \end{cases} \tag{4.9}$$

Based on the relation, we define the following approximant for $E_{\alpha,\beta}(-t)$,

$$R_{\alpha,\beta}^{m,n,r}(t) := (-t)^r R_{\alpha,\beta+\alpha r}^{m,n}(t) + P_{\alpha,\beta}^{r-1}(-t), \quad r = 0, 1, 2, \dots, \tag{4.10}$$

where, $R_{\alpha,\beta+\alpha r}^{m,n}(t)$ is the global Padé approximant described in [16].

For a given square matrix $\mathbf{B}$ and a vector $\mathbf{v}$, the matrix-vector product $E_{\alpha,\beta}(-\mathbf{B})\mathbf{v}$

is computed as

$$E_{\alpha,\beta}(-\mathbf{B})\mathbf{v} \approx R_{\alpha,\beta}^{m,n,r}(\mathbf{B})\mathbf{v} := (-\mathbf{B})^r R_{\alpha,\beta+\alpha r}^{m,n}(\mathbf{B})\mathbf{v} + P_{\alpha,\beta}^{r-1}(-\mathbf{B})\mathbf{v}, \quad (4.11)$$

where the term

$$R_{\alpha,\beta+\alpha r}^{m,n}(\mathbf{B})\mathbf{v} \quad (4.12)$$

is computed using the partial fraction decomposition approach discussed in [17].

4.3 Analysis of stability and convergence

In this section, we investigate the stability and convergence rate of the numerical scheme (4.6). In the upcoming analysis, we use $\|\cdot\|$ to represent a sub-multiplicative matrix norm, such as the infinity norm defined for a $p \times q$ matrix $\mathbf{B} = [b_{ij}]$ by

$$\|\mathbf{B}\| = \max_{1 \leq i \leq p} \sum_{j=1}^q |b_{ij}|.$$

We begin by extending the following theorem to the matrix arguments for later use.

Theorem 4.1 [4] *If $\alpha < 2$, β is an arbitrary real number, μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ and M is a positive constant, then*

$$|E_{\alpha,\beta}(z)| \leq \frac{M}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi. \quad (4.13)$$

Lemma 4.1 *Let $\mathbf{A}$ be an $m \times m$ diagonalizable matrix with positive eigenvalues.*

Then, for $\beta > 0$ with $\alpha \in (1, 2)$, there is a positive constant C , which depends on

$\{\alpha, \beta, \mathbf{A}\}$, such that

$$\|E_{\alpha, \beta}(-\mathbf{A})\| \leq C, \quad t \geq 0. \quad (4.14)$$

More specifically,

$$\|E_{\alpha, \beta}(-\mathbf{A})\| \leq \frac{C}{1 + \min_i \lambda_i},$$

where λ_i , $i = 1, 2, \dots, m$, are the eigenvalues of $\mathbf{A}$.

Proof. Let $\lambda_i > 0$, $i = 1, 2, \dots, m$, be the eigenvalues of $\mathbf{A}$. Since $\mathbf{A}$ is diagonalizable, there exists a non-singular $m \times m$ matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ with $\mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_m)$. Thus, we can write

$$E_{\alpha, \beta}(-\mathbf{A}) = \mathbf{P}E_{\alpha, \beta}(-\mathbf{D})\mathbf{P}^{-1}.$$

From the sub-multiplicativity property of the matrix norm and (6.8), we obtain the bound

$$\|E_{\alpha, \beta}(-\mathbf{A})\| \leq \|\mathbf{P}\| \|E_{\alpha, \beta}(-\mathbf{D})\| \|\mathbf{P}^{-1}\| \leq \max_{1 \leq i \leq m} \frac{M}{1 + \lambda_i},$$

for some $M > 0$. This implies (4.14). ■

Next, we establish the following stability result.

Theorem 4.2 Suppose that $\mathbf{f} \in C[0, T]$ and $\mathbf{A}$ is a diagonalizable matrix with positive eigenvalues. Let $\mathbf{u}^n$ represent the numerical solution of (4.1) defined by (4.5). Then, there exists a positive constant C such that

$$\|\mathbf{u}^n\| \leq C \left[\|\mathbf{u}_0\| + t_n \|\mathbf{u}_0^1\| + \frac{4t_n^\alpha}{\alpha} \max_{0 \leq t \leq t_n} \|\mathbf{f}(t)\| \right], \quad 1 \leq n \leq N. \quad (4.15)$$

Proof. By straightforward calculations, it is evident that

$$\|\mathbf{P}_j(s)\| \leq 4 \max_{0 \leq j \leq n} \|\mathbf{f}^j\| \leq 4 \max_{0 \leq t \leq t_n} \|\mathbf{f}(t)\|, \quad j = 0, \dots, n-1. \quad (4.16)$$

Inserting (6.9) into (4.5) yields

$$\begin{aligned} \|\mathbf{u}^n\| &\leq \|E_\alpha(-\mathbf{A}t_n^\alpha)\| \|\mathbf{u}^0\| + t_n \|E_{\alpha,2}(-\mathbf{A}t_n^\alpha)\| \|\mathbf{u}_0^1\| \\ &\quad + 4 \max_{0 \leq t \leq t_n} \|\mathbf{f}(t)\| \int_0^{t_n} (t_n - \tau)^{\alpha-1} \|E_{\alpha,\alpha}(-\mathbf{A}(t_n - \tau)^\alpha)\| d\tau. \end{aligned} \quad (4.17)$$

Then, it follows from Lemma 4.1 that there is $C > 0$ such that

$$\|\mathbf{u}^n\| \leq C \left[\|\mathbf{u}^0\| + t_n \|\mathbf{u}_0^1\| + 4 \max_{0 \leq t \leq t_n} \|\mathbf{f}(t)\| \int_0^{t_n} (t_n - \tau)^{\alpha-1} d\tau \right]. \quad (4.18)$$

Computing the integral completes the proof. ■

For a force term with sufficient regularity, the convergence result is as follows.

Theorem 4.3 *Let $\mathbf{u}$ denote the solution of (4.1) with a force term $\mathbf{f} \in \mathcal{C}^2[0, T]$. Let $\mathbf{u}^n$ represent the numerical solution defined by (4.5). Then,*

$$\|\mathbf{u}^n - \mathbf{u}(t_n)\| \leq \frac{Ck^2t_n^\alpha}{2!\alpha} \max_{0 \leq t \leq t_n} \|\mathbf{f}^{(2)}(t)\|, \quad n = 1, \dots, N.$$

Proof. From the definition of the exact and numerical solution, it follows that for

$$1 \leq n \leq N,$$

$$\begin{aligned} \mathbf{e}^n &:= \mathbf{u}^n - \mathbf{u}(t_n) \\ &= \sum_{j=0}^{n-1} \left[\int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\mathbf{A}(t_n - \tau)^\alpha) [\mathbf{P}_j(\tau) - \mathbf{f}(\tau)] d\tau \right]. \end{aligned} \quad (4.19)$$

Using the known Lagrange interpolation error formula, we have

$$|p_{j,i}(\tau) - f_i(\tau)| \leq \frac{k^2}{2!} \max_{t \in I_j} |f_i^{(2)}(t)|, \quad \text{for } \tau \in I_j, \quad i = 0, \dots, m. \quad (4.20)$$

Thus,

$$\|\mathbf{P}_j(\tau) - \mathbf{f}(\tau)\| \leq \frac{k^2}{2!} \max_{0 \leq i \leq m} \max_{t \in I_j} |f_i^{(2)}(t)|. \quad (4.21)$$

Plugging this into (4.19) results in,

$$\|\mathbf{e}^n\| \leq \frac{k^2}{2!} \max_{0 \leq t \leq t_n} \|\mathbf{f}^{(2)}(t)\| \int_0^{t_n} (t_n - \tau)^{\alpha-1} \|E_{\alpha,\alpha}(-\mathbf{A}(t_n - \tau)^\alpha)\| d\tau. \quad (4.22)$$

Applying Lemma 4.1 yields

$$\|\mathbf{e}^n\| \leq \frac{Ck^2}{2!} \max_{0 \leq t \leq t_n} \|\mathbf{f}^{(2)}(t)\| \int_0^{t_n} (t_n - \tau)^{\alpha-1} d\tau. \quad (4.23)$$

The proof is completed upon evaluating the integral. ■

Next, we establish that when the force term f exhibits a singularity near the origin such that

$$|f_i^{(\ell)}(t)| \leq \sigma t^{\gamma-\ell}, \quad t > 0, \gamma > 0, \ell \in \{1, 2\}, \quad (4.24)$$

for some $1 \leq i \leq m$, then the convergence rate in Theorem 6.3 remains achievable by utilizing the graded mesh points

$$t_n = (nk)^\delta, \quad n = 0, 1, \dots, N, \quad k = T^{1/\delta}/N, \quad \delta \geq 1. \quad (4.25)$$

As demonstrated in [60], this graded mesh satisfies the inequalities

$$k_{n+1} \leq 2^{\delta-1}k_n, \quad k_n \leq \delta k t_{n+1}^{1-1/\delta}, \quad 0 \leq n \leq N-2, \quad (4.26)$$

and

$$t_{n+1} \leq 2^\delta t_n, \quad 1 \leq n \leq N-1. \quad (4.27)$$

Employing these properties, we establish the following theorem.

Theorem 4.4 *Let $\mathbf{u}$ denote the solution of (4.1) with $\mathbf{f}$ satisfying (4.24). Let $\mathbf{u}^n$ represent the approximate solution defined by (4.5) using the graded mesh (4.25) with mesh exponent δ . Then, there exists $C > 0$, which depends on $\{\alpha, \sigma, \gamma, \delta\}$ but not on k , such that for $n = 1, \dots, N$,*

$$\|\mathbf{u}^n - \mathbf{u}(t_n)\| \leq Ck^2, \quad \text{for } \delta \geq \max\left\{1, \frac{2}{\gamma+1}\right\}.$$

The parameters σ and γ are as in (4.24).

Proof. Using (4.4), we can show that for $s \in I_0$,

$$p_{0,i}(s) - f_i(s) = \frac{s}{k_0} \int_0^{t_1} f'_i(t) dt - \int_0^s f'_i(t) dt. \quad (4.28)$$

Thus,

$$|p_{0,i}(s) - f_i(s)| \leq 2 \int_0^{t_1} |f'_i(t)| dt, \quad \text{for } s \in I_0. \quad (4.29)$$

Therefore, the regularity assumption (4.24) implies that

$$|p_{0,i}(s) - f_i(s)| \leq \frac{2\sigma}{\gamma} t_1^\gamma, \quad \text{for } s \in I_0. \quad (4.30)$$

Thus, for $s \in I_0$, there is $C_1 > 0$ such that

$$\|\mathbf{P}_0(s) - \mathbf{f}(s)\| \leq C_1 t_1^\gamma. \quad (4.31)$$

Similarly, for $s \in I_j$, $j = 1, \dots, N-1$, it follows from (6.11), (4.24), and (4.26) that

$$|p_{j,i}(s) - f_i(s)| \leq C_2 k_j^2 \max_{t \in I_j} |f_i^{(2)}(t)| \leq C_2 k^2 t_{j+1}^{2(1-1/\delta)} t_j^{\gamma-2}. \quad (4.32)$$

for some $C_2 > 0$.

Then, by the mesh property (4.27), we have

$$|p_{j,i}(s) - f_i(s)| \leq C_2 k^2 t_j^{2(1-1/\delta)} t_{j+1}^{\gamma-2}. \quad (4.33)$$

Thus, for $s \in I_j$, there is $C_3 > 0$ such that

$$\|\mathbf{P}_j(s) - \mathbf{f}(s)\| \leq C_3 k^2 t_j^{2(1-1/\delta)} t_{j+1}^{\gamma-2}. \quad (4.34)$$

Using the bounds (4.14), (4.31), and (4.34) in (4.19) there is $C_4 > 0$ such that

$$\begin{aligned}\|\mathbf{e}^n\| &\leq C_4 \left\{ t_1^\gamma \int_0^{t_1} (t_n - \tau)^{\alpha-1} d\tau + k^2 \sum_{j=1}^{n-1} t_j^{2(1-1/\delta)} t_{j+1}^{\gamma-2} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} d\tau \right\} \\ &= C_4 \left\{ t_1^\gamma [t_n^\alpha - (t_n - t_1)^\alpha] + k^2 \int_{t_1}^{t_n} \tau^{\gamma-2/\delta} (t_n - \tau)^{\alpha-1} d\tau \right\}.\end{aligned}\quad (4.35)$$

Therefore, applying the inequality $t_1^\gamma [t_n^\alpha - (t_n - t_1)^\alpha] \leq t_n^{\alpha-1} t_1^{\gamma+1}$, and the identity

$$\int_0^t \tau^{v_1-1} (t - \tau)^{v_2-1} d\tau = \frac{\Gamma(v_1)\Gamma(v_2)}{\Gamma(v_1 + v_2)} t^{v_1+v_2-1}, \quad v_1, v_2 > 0, \quad (4.36)$$

there exists $C > 0$ such that for $n = 1, \dots, N$,

$$\|\mathbf{e}^n\| \leq C \{t_n^{\alpha-1} t_1^{\gamma+1} + k^2 t_n^{\gamma+\alpha-2/\delta}\}. \quad (4.37)$$

Since $t_1 = k^\delta$, we get

$$\|\mathbf{e}^n\| \leq C \{t_n^{\alpha-1} k^{\delta(\gamma+1)} + k^2 t_n^{\gamma+\alpha-2/\delta}\}. \quad (4.38)$$

Employing the assumption $\delta(\gamma + 1) \geq 2$ completes the proof. ■

Remark 5 *In the case where f is nonsmooth and satisfying the regularity property (4.24), it is evident that the Lagrange error term involving the second derivative $f^{(2)}(t)$ is not suitable on the interval $[0, t_1]$ due to the term $t^{\gamma-2}$. However, by using the graded mesh, one can take advantage of the properties (4.26) and (4.27) to achieve a desirable estimate on the first subinterval. From a computational point of view, the approximation error of the numerical solution is expectedly larger near the*

origin, owing to the singularity of the approximated function f around $t_0 = 0$. The objective of the graded mesh, for each fixed N , is to cluster some mesh points near the singularity by taking smaller step sizes near t_0 to minimize the approximation error around this initial singularity. The size of the mesh elements close to t_0 reduces as the mesh exponent δ increases.

4.4 Numerical Experiments

To validate the accuracy of the numerical scheme and assess the effect of a graded mesh in the case of non-smooth force term, we present some relevant experiments. For numerical illustration of the convergence properties, we compute the maximum absolute error as

$$e(N) = \max_{1 \leq n \leq N} \|\mathbf{u}(t_n) - \mathbf{u}^n\|,$$

when the exact solution $\mathbf{u}(t)$ is known. In the absence of the exact solution, we compute the error through successive approximation:

$$e(N) = \max_{1 \leq n \leq N} \|\mathbf{u}_*^{2n} - \mathbf{u}^n\|,$$

where $\mathbf{u}_*^n$ is an implementation of the numerical scheme with twice the number of mesh points used in the $\mathbf{u}^n$ implementation. In both cases, the order of convergence is computed as $\log_2[e(N)/e(2N)]$.

Example 4.5 (*Non-oscillatory solution with smooth and nonsmooth force*)

Consider the following problem

$$\begin{aligned} {}^cD^\alpha u(t) + Au(t) &= \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} + A(t^r + c), \quad t > 0, \\ u(0) &= c, \quad u'(0) = 0. \end{aligned} \tag{4.39}$$

The exact solution is given by

$$u(t) = t^r + c. \tag{4.40}$$

For experimental purposes, we consider $t \in [0, 1]$, $A = 2$ and $c = 5$.

To demonstrate the convergence under a sufficiently smooth force term, we require $f \in \mathcal{C}^2[0, 1]$. To meet this condition, we set $r = 4$ in (4.39) and thus $2 \leq r - \alpha \leq 3$, which ensures that the force term is $\mathcal{C}^2[0, 1]$. As shown in Table 4.1, the expected $O(k^2)$ order of convergence is achieved by the numerical scheme using a uniform mesh, in conformance with our theoretical result.

For nonsmooth force term, choosing $r = 2$ leads to $0 < r - \alpha \leq 1$, resulting in singularity of f at $t = 0$. Additionally, we choose $\alpha \in \{1.2, 1.9\}$, yielding corresponding values of γ in Theorem 4.4 as $\{0.8, 0.1\}$. To attain the optimal $O(k^2)$ rate, we require $\delta = 2/1.8$ for $\alpha = 1.2$ and $\delta = 2/1.1$ for $\alpha = 1.9$. The findings in Table 4.2 substantiate these assertions. It is evident that the optimal convergence rates are achieved for these specific values of mesh exponent δ , while rates deteriorate over a uniform mesh ($\delta = 1$). Consistent with our theoretical results, it is observed that for nonsmooth force conditions, only an $O(k^{1+\gamma})$ rate is achieved.

Figure 4.3 displays the profiles of the absolute approximation error for the non-smooth problem. Notably, the use of a graded mesh enhances accuracy across the

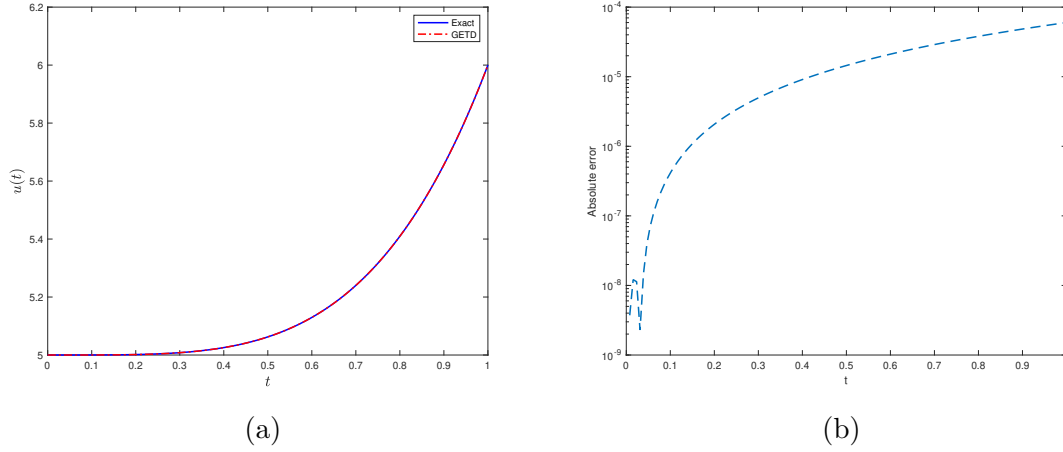


Figure 4.1: Example 4.5, smooth force, ($r = 4$): plots of the exact and approximate solution over uniform mesh, $\alpha = 1.9$ and $N = 128$, (left), along with the absolute error (right).

computation interval. Additionally, Figures 4.1 and 4.2 depict the profiles of the solution and its approximations for smooth and nonsmooth force terms, respectively.

Example 4.6 (*Driven oscillation with a smooth periodic external force*)

In this example we examine the fractional mass-damper equation following the formulation in [40]. In problem (4.1), we introduce a periodic force function with amplitude ϕ and frequency ψ , specifically $f(t) = \phi \cos(\psi t)$. Our model problem then takes the form

$$\begin{aligned}
 {}^c D^\alpha u(t) + Au(t) &= \phi \cos(\psi t), \quad t > 0, \\
 u(0) &= u_0, \quad u'(0) = u_0^1.
 \end{aligned}
 \tag{4.41}$$

In this case, the exact solution is given by

$$\begin{aligned}
 u(t) &= E_\alpha(-At^\alpha) u_0 + t E_{\alpha,2}(-At^\alpha) u_0^1 \\
 &\quad + \phi \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-A(t-\tau)^\alpha) \cos(\psi \tau) d\tau.
 \end{aligned}
 \tag{4.42}$$

Table 4.1: Example 4.5, smooth force, ($r = 4$): maximum absolute errors and convergence rates over uniform mesh.

N	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
16	2.51e-03	1.9960	2.53e-03	1.9730	2.42e-03	1.9264	2.28e-03	1.8244
32	6.25e-04	2.0000	6.19e-04	1.9917	6.17e-04	1.9730	6.00e-04	1.9279
64	1.56e-04	2.0004	1.56e-04	1.9972	1.55e-04	1.9895	1.53e-04	1.9683
128	3.91e-05	2.0004	3.91e-05	1.9991	3.89e-05	1.9958	3.87e-05	1.9856

Table 4.2: Example 4.5, nonsmooth force, ($r = 2$): maximum absolute errors and convergence rates over uniform and graded meshes with $\delta = 2/(3 - \alpha)$.

N	$\alpha = 1.2$				$\alpha = 1.9$			
	$\delta = 1$		$\delta = 2/1.8$		$\delta = 1$		$\delta = 2/1.1$	
16	5.30e-03	1.8184	3.03e-03	2.0117	6.02e-02	1.0919	4.92e-03	1.9285
32	1.52e-03	1.7985	7.71e-04	1.9787	2.82e-02	1.0946	1.27e-03	1.9504
64	4.41e-04	1.7892	1.96e-04	1.9735	1.31e-02	1.0968	3.29e-04	1.9526
128	1.28e-04	1.7870	5.00e-04	1.9713	6.16e-03	1.0980	8.49e-05	1.9541

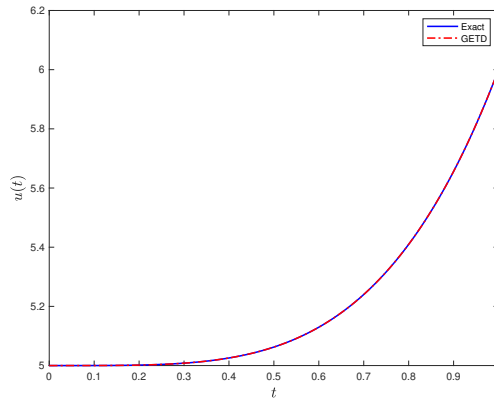
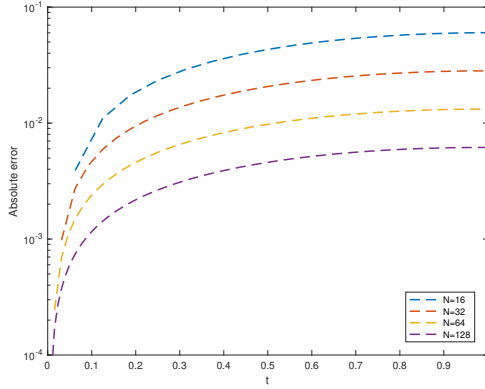
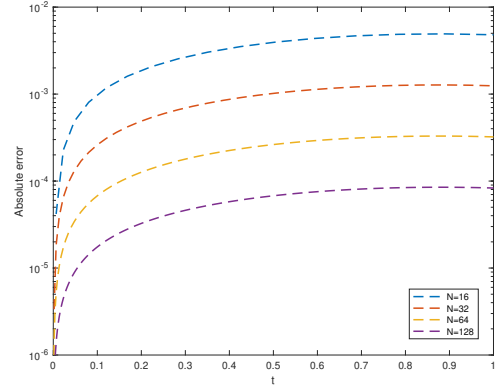


Figure 4.2: Example 4.5, nonsmooth force, ($r = 2$): plots of the exact and approximate solution for $\alpha = 1.9$, $N = 128$.



(a) Uniform mesh: $\delta = 1$



(b) Graded mesh: $\delta = 2/1.1$

Figure 4.3: Example 4.5, nonsmooth force: plots of absolute error for the GETD scheme over uniform and graded meshes for $\alpha = 1.9$, for different values of N .

This expression involves a convolution that may not be easily obtained explicitly in closed form. Therefore, to demonstrate the convergence results, we compute the error using successive approximation. For numerical implementation, we set $A = 1$, $u_0 = u_0^1 = 1$, $\phi = \psi = 1$ and consider $t \in [0, 20]$. The force term here is sufficiently smooth, and Table 4.3 indicates that the expected $O(k^2)$ rates are obtained on a uniform mesh.

Further, Figure 4.4 presents plots of the approximate solutions for varying values of α . Notably, when specifically examining $\alpha = 2$, where the exact solution is available in closed form and serves as a reference, it is evident that as α approaches 2, the profiles of the approximate solutions progressively align with the reference solution.

To demonstrate the stability and applicability of the proposed scheme, we analyze its performance on highly oscillatory problems. As such, we consider (4.41) when $A = 5, 10, 50, 100$ and 1000 and $t \in [0, 10]$. The numerical results are summarized in Table 4.4. Figures 4.5 and 4.6 demonstrate the profiles of the approximate solution

for different values of α compared to the exact solution when $\alpha = 2$. In this specific experiment, we applied our numerical scheme using the Matlab routine `ml` [15].

As it is established by [11], [41], [44], the motion of the fractional oscillator reflects the characteristics exhibited by an ordinary damped oscillator. This is clearly evident in Figures 4.4, 4.5 and 4.6. The damping inherent in the fractional oscillator is an integral aspect of its equation of motion, in contrast to the ordinary damped oscillator where damping is introduced through a damping force. Notably, the damping effects are linked to the order of the fractional derivative. While when $\alpha = 2$ the oscillator is conservative (the total energy is conserved), the fractional oscillator is dissipative.

Table 4.3: Example 4.6, maximum absolute errors and convergence rates

N	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
16	2.35e-02	2.4080	3.89e-02	2.3969	9.51e-02	1.8530	3.45e-01	1.8722
32	5.28e-03	2.9487	9.83e-03	2.1644	2.46e-02	1.9510	8.73e-02	1.9830
64	1.31e-03	2.1526	2.46e-03	1.9831	6.17e-03	1.9963	2.18e-02	2.0006
128	3.29e-04	2.0075	6.16e-04	1.9955	1.54e-03	1.9979	5.44e-03	2.0030

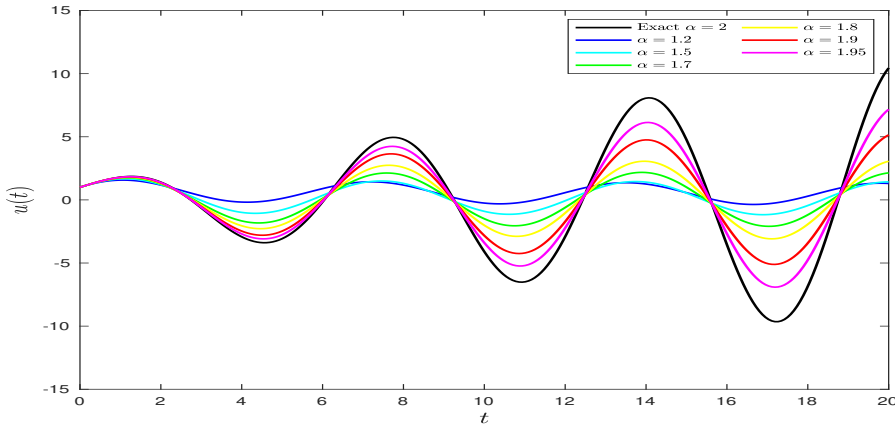


Figure 4.4: Example 4.6, plots of the approximate solution for different values of α compared to the exact solution when $\alpha = 2$, $N = 128$.

Example 4.7 (*Oscillatory solution with nonsmooth forcing term*)

Table 4.4: Example 4.6, maximum absolute errors and convergence rates when $\alpha = 1.9$, $N = 256$ for different oscillation frequencies.

A	Maximum Absolute Error	Convergence Rate
5	1.12e - 04	2.0251
10	2.21e - 04	1.9818
50	3.98e - 03	1.9065
100	1.98e - 03	1.8985
1000	1.78e - 03	1.8229

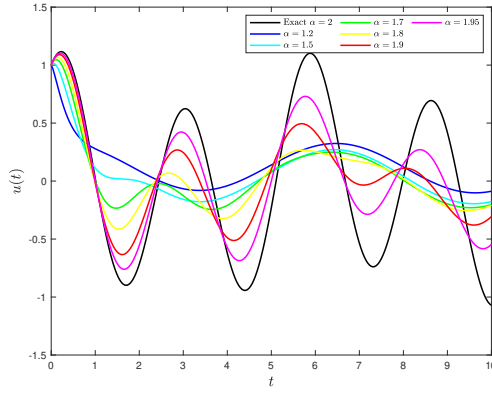
In order to draw a comparison with the results reported in [2], we analyze the following problem involving a nonsmooth forcing term:

$$\begin{aligned}
{}^c D^\alpha u(t) + Au(t) &= \frac{\mu t^c}{\Gamma(1+c)}, & t > 0, \quad \mu > 0, \quad 0 < c < 1, \\
u(0) &= u_0, & u'(0) &= u_0^1.
\end{aligned} \tag{4.43}$$

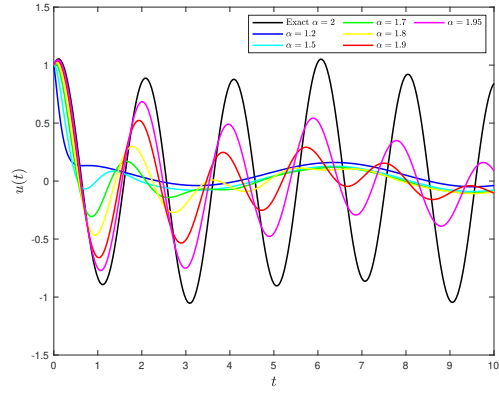
The exact solution to (4.43) is given by:

$$u(t) = E_{\alpha,1}(-At^\alpha)u_0 + tE_{\alpha,2}(-At^\alpha)u_0^1 + \mu t^{\alpha+c}E_{\alpha,\alpha+c+1}(-At^\alpha), \tag{4.44}$$

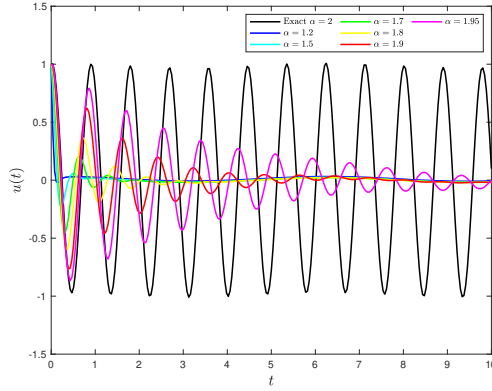
In this example, we set $c = 0.3$, $\mu = 0.5$, $A = 2^\alpha$, $u_0 = u_0^1 = 1$, $\alpha = 1.8$ and $t \in [0, 1]$, as in [2]. Table 4.5 presents the maximum absolute errors for various values of N , comparing the results obtained using both the GETD and the Adams-Bashforth-Moulton methods. The results indicate that the GETD method achieves significantly lower absolute errors across the mesh refinements compared to the ABM method, showcasing its effectiveness in handling nonsmooth forcing terms in fractional differential equations. Figure 4.7 compares GETD and ABM schemes for a nonsmooth forcing term. The left plot shows that the GETD approximation closely follows the exact solution, while ABM deviates, especially in the early and mid-range of t values.



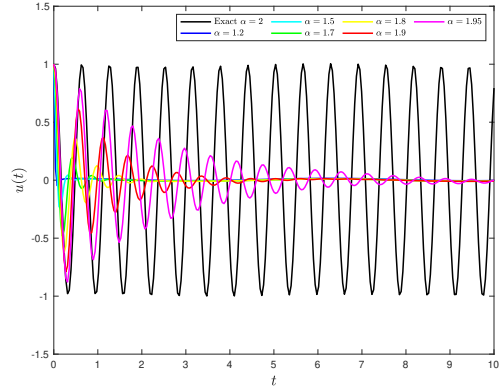
(a) $A = 5$



(b) $A = 10$



(c) $A = 50$



(d) $A = 100$

Figure 4.5: Example 4.6, plots of the approximate solution for different values of α compared to the exact solution when $\alpha = 2$, $N = 256$ for different oscillation frequencies.

The right plot, displaying the absolute error, confirms that GETD maintains lower error levels across the interval, demonstrating superior accuracy and stability over ABM for this problem.

For highly oscillatory problems, as illustrated in Figure 4.8, where the oscillation frequency $A = 100^\alpha$, $\alpha = 1.8$, and the number of steps $N = 80$, the comparison between the GETD and ABM schemes highlights significant differences in stability. The ABM scheme exhibits severe instability, characterized by rapidly increasing os-

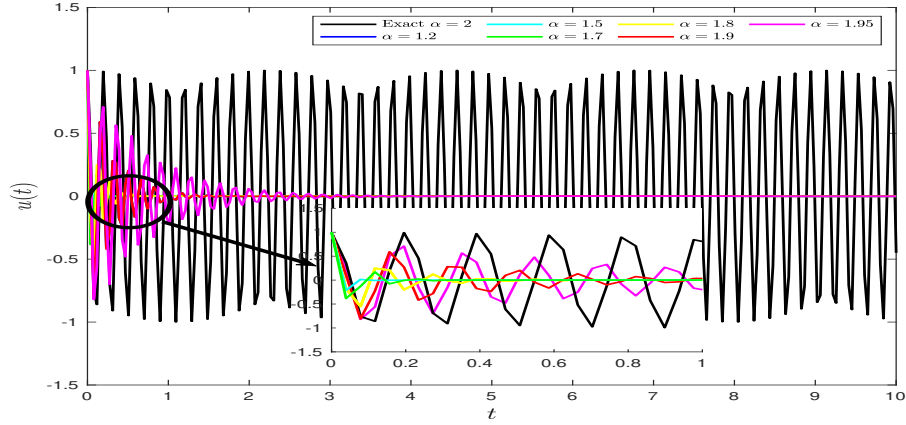


Figure 4.6: Example 4.6, plots of the approximate solution for different values of α compared to the exact solution when $\alpha = 2$, $N = 256$, $A = 1000$.

Table 4.5: Example 4.7, nonsmooth force: maximum absolute errors for $\alpha = 1.8$ using GETD and ABM.

N	GETD	ABM
16	9.76e − 04	3.93e − 02
32	2.56e − 04	4.32e − 02
64	6.70e − 05	4.46e − 02
128	1.75e − 05	4.51e − 02

cillations and divergence from the exact solution, as shown in the left plot of Figure 4.8. This instability is expected due to the explicit nature of the ABM scheme, which is inherently sensitive to highly oscillatory problems and fails to maintain stability at larger step sizes. In contrast, the GETD scheme demonstrates remarkable stability and accuracy. The numerical solution closely tracks the exact solution throughout the entire time interval, effectively handling the highly oscillatory nature of the problem without exhibiting the numerical blow-up observed in the ABM scheme. These results reinforce that explicit methods like ABM are unsuitable for highly oscillatory cases, whereas the stability and robustness of the GETD scheme make it a more reliable choice, particularly for fractional problems with $\alpha \in (1, 2)$.

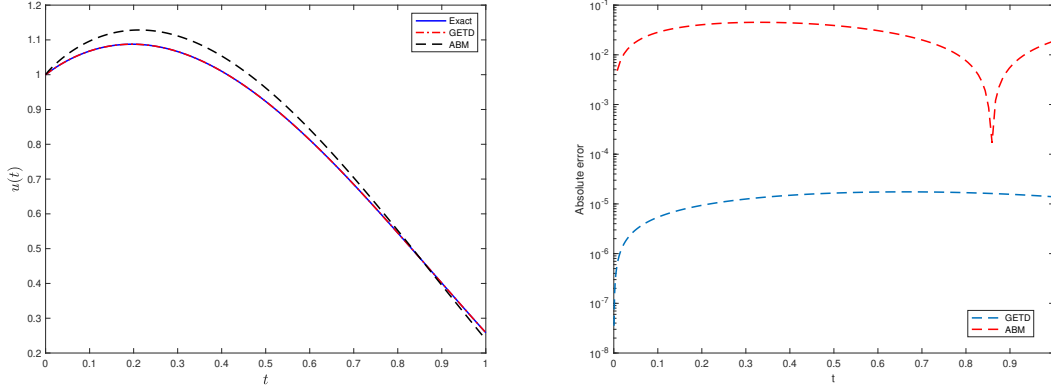


Figure 4.7: Example 4.7, nonsmooth force: plots of the exact and approximate solution using GETD and ABM schemes [2], $N = 128$, $\alpha = 1.8$. The left plot shows the solution profiles, while the right plot shows the absolute error for both methods.

In Figure 4.9, we present the profile of the solution $u(t)$ for the highly oscillatory case using the ABM scheme. It is observed that while the profile obtained from ABM with a step size $k = 1/80$ exhibits instability in the form of oscillations, as shown in the left plot of Figure 4.8, the outcome from ABM does not suffer such instability when a much smaller step size $k = 1/640$ is used. However, by significantly reducing the step size, the solution profile from ABM with $k = 1/640$ agrees with that of the GETD scheme with $k = 1/80$, as shown in the earlier comparisons.

In the context of efficient computation, with $T = 1$, using $k = 1/80$ for the GETD scheme requires only $N = 80$ mesh elements, whereas stabilizing the ABM scheme requires $k = 1/640$, corresponding to $N = 640$ mesh elements. This highlights the efficiency of the GETD scheme, which achieves stability and accuracy with larger step sizes, effectively reducing the computational cost compared to the ABM scheme. The ability of the GETD scheme to maintain stability with fewer mesh elements underscores its superiority for highly oscillatory problems.

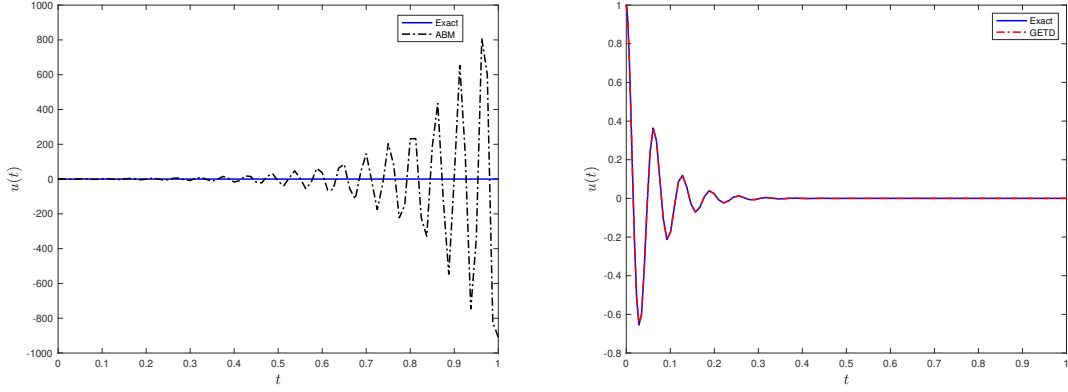


Figure 4.8: Example 4.7, nonsmooth force: plots of the exact and approximate solution using GETD and ABM schemes [2], $A = 100^\alpha$, $\alpha = 1.8$, and $N = 80$. The left plot shows ABM instability, while the right plot demonstrates GETD stability.

For further numerical implementation, we select $\alpha \in \{1.2, 1.9\}$ in (4.43). To achieve optimal convergence at $O(k^2)$, Theorem 4.4 suggests using a graded mesh parameter $\delta = 2/1.3$. In [2], a first-order accuracy was obtained using ABM method for $t \in [0, 1]$. As shown in Table 5.8, the expected $O(k^2)$ convergence rate is achieved on a graded mesh, while a uniform mesh with $\delta = 1$ results in an $O(k^{1.3})$ rate, consistent with our theoretical results. Profiles of the exact and approximate solutions for $\alpha \in \{1.2, 1.9\}$ over $t \in [0, 4]$ are presented in Figure 4.10.

Example 4.8 (*Non oscillatory behavior of solution with non-smooth force*)

Here we explore the model problem (4.1) featuring a smooth and rapidly decaying solution to highlight certain stability characteristics of the scheme. Specifically, we select the solution $u(t) = e^{-t}$ and appropriately determine the initial data $u_0 = 1$, $u_0^1 = -1$, and the force term as

$$f(t) = t^{2-\alpha} E_{1,3-\alpha}(-t) + Ae^{-t}.$$

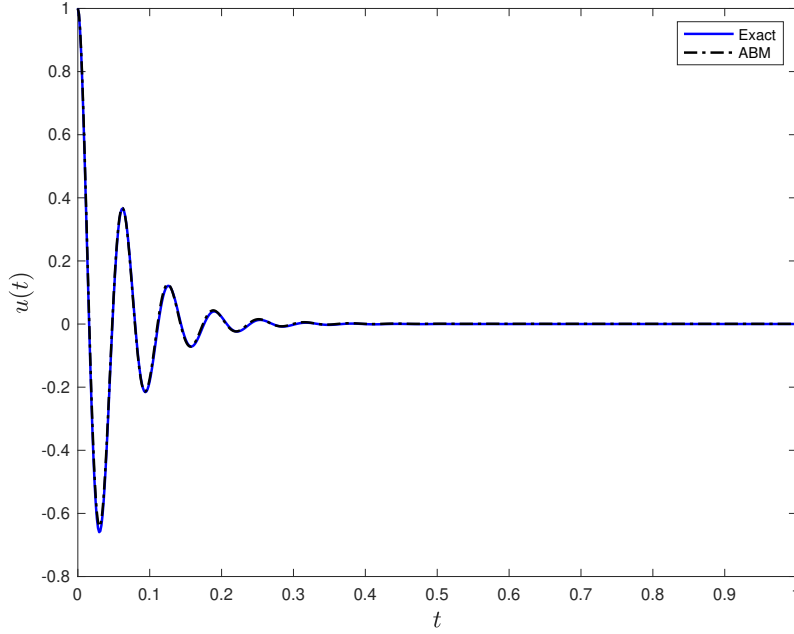
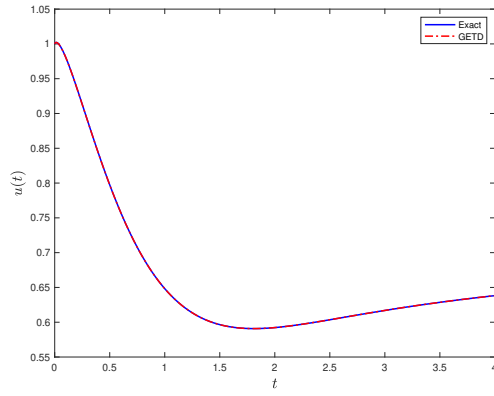


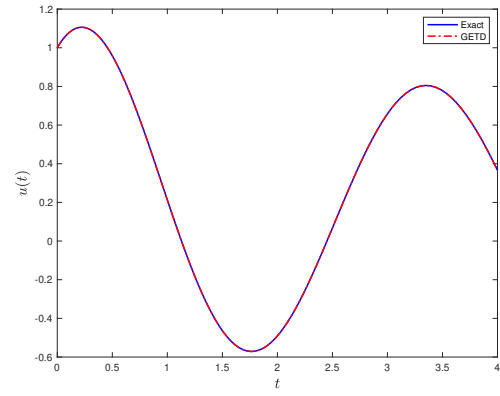
Figure 4.9: Example 4.7, nonsmooth force: plots of the exact and approximate solution using the ABM scheme [2], $A = 100^\alpha$, $\alpha = 1.8$, and $N = 640$. This illustrates the stabilization of ABM for a much smaller step size.

It is clear that the force term is nonsmooth, with $0 < \gamma = 2 - \alpha < 1$ in Theorem 4.4. The severity of the singularity of f depends on the value of α and worsens as α approaches 2 (excluding 2). According to aforementioned theorem, achieving an optimal $O(k^2)$ convergence rate requires $\delta = 2/(3 - \alpha)$.

For testing purposes, we set $A = 1$, consider $t \in [0, 1]$, and select $\alpha \in \{1.2, 1.9\}$. Figure 4.11 presents plots of the exact and approximate solution for $\alpha = 1.9$. The numerical results in Table 4.7 substantiate our theoretical findings. Furthermore, examining the plots of absolute error in Figure 4.12, it becomes apparent that the impact of graded meshes extends beyond achieving optimal rates of convergence. As anticipated, by using a graded mesh to manage the error in the vicinity of the singularity (at $t = 0$), the resulting growth over the interval $[0, 1]$ is diminished compared to that



(a) $\alpha = 1.2$



(b) $\alpha = 1.9$

Figure 4.10: Example 4.7, nonsmooth source: plots of the exact and approximate solution , $N = 128$.

of observed with uniform mesh.

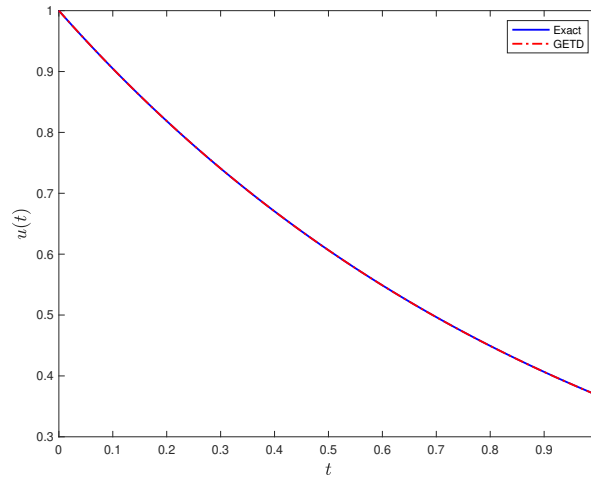
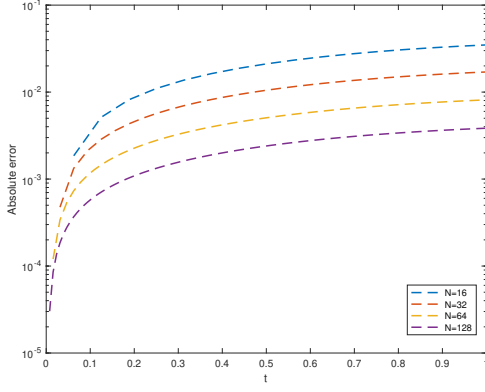
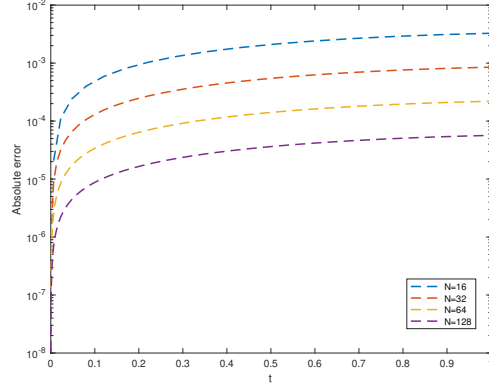


Figure 4.11: Example 4.8, nonsmooth force: plots of the exact and approximate solution when $\alpha = 1.9$, $N = 128$.

Example 4.9 (*Triple Oscillator System*)



(a) Uniform mesh: $\delta = 1$



(b) Graded mesh: $\delta = 2/1.1$

Figure 4.12: Example 4.8, nonsmooth force: plots of absolute error for the GETD scheme over uniform and graded meshes for $\alpha = 1.9$, for different values of N .

Consider the system of three interconnected oscillators described by (4.1) with

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 5 \end{pmatrix}, \quad \mathbf{u}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{u}_0^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{f}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (t^\mu + 3), \quad \mu > 0. \quad (4.45)$$

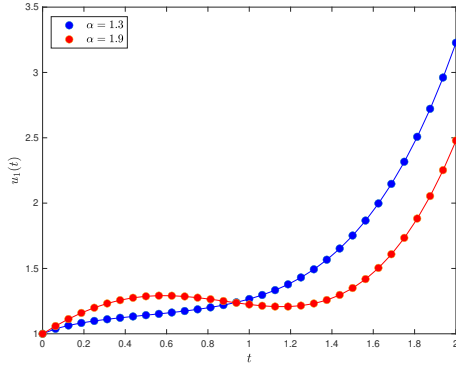
The exact solution of this system is given by

$$\begin{aligned} \mathbf{u}(t) = & E_{\alpha}(-\mathbf{A}t^{\alpha}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + tE_{\alpha,2}(-\mathbf{A}t^{\alpha}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ & + t^{\alpha}E_{\alpha,\alpha+1}(-\mathbf{A}t^{\alpha}) \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \Gamma(\mu+1)t^{\mu+\alpha}E_{\alpha,\mu+\alpha+1}(-\mathbf{A}t^{\alpha}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned} \quad (4.46)$$

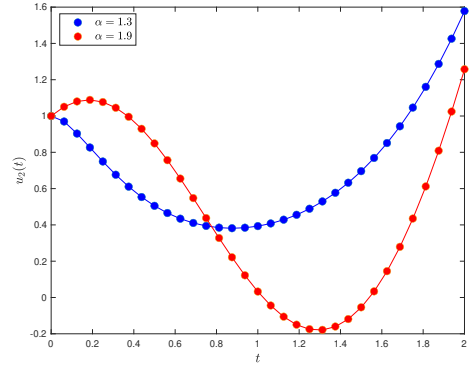
For implementation purposes, we take $t \in [0, 2]$. In the case of a sufficiently smooth force term, we choose $3 < \mu = 5 - \alpha < 4$ so that each $f_i \in \mathcal{C}^3$. As shown in Table 4.8, the expected $O(k^2)$ rate is achieved on a uniform mesh for various values of α . In fact, the first example illustrates that having $\mathbf{f} \in \mathcal{C}^2$ is sufficient for the optimal convergence rate.

For a non-smooth forcing term, we set $\mu = 2 - \alpha$, which implies that $\mathbf{f}$ has a singularity near the origin. Our analysis suggests that using a non-uniform mesh with exponent δ , the scheme is of $O(k^2)$ when $\delta \geq \max\left\{1, \frac{2}{\gamma+1}\right\}$. The results in Table 4.9 affirm these assertions. It is seen that while the optimal convergence cannot be obtained over a uniform mesh, due to the singularity of the force, it is attained over a non-uniform mesh with $\delta = \frac{2}{\gamma+1}$. Plots of the absolute error in Figure 4.14 illustrate the influence of the graded mesh for achieving the order of convergence for the non-smooth problem.

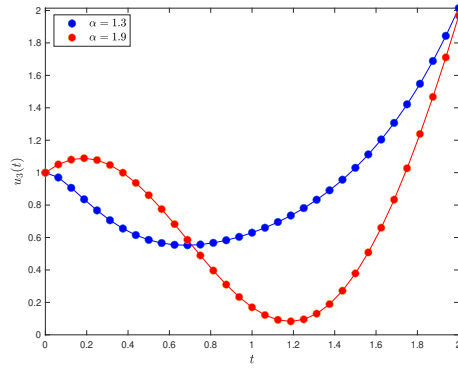
Further, Figure 4.15 shows the norm of the errors at each time step. Subfigure (a)



(a) $u_1(t)$



(b) $u_2(t)$



(c) $u_3(t)$

Figure 4.13: example 4.9, smooth force: plots of the exact and approximate solutions, $\alpha = 1.3, 1.9$ and $N = 32$. The solid lines represent the exact solutions

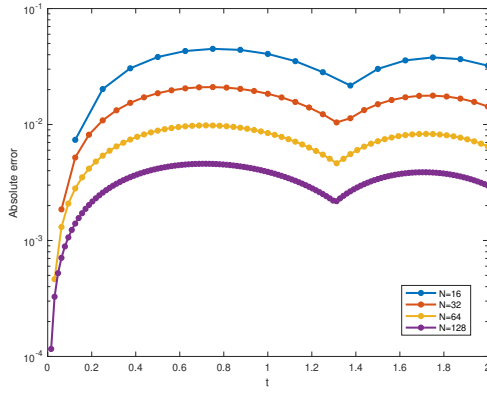
shows that the numerical scheme accuracy is enhanced when using the mesh exponent $\delta = 2/(\gamma + 1) = 1.82$. Subfigure (b), on the other hand, shows that further increasing δ beyond this value may not be necessary, as the resulting approximations exhibit comparable accuracy.

Example 4.10 (*Fractional diffusion-wave equation*)

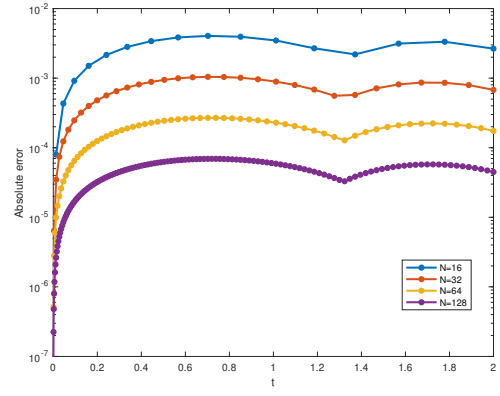
To illustrate the effectiveness of the GETD scheme as a time-integrator for diffusion-wave equations and its applicability in solving large systems, we con-

Table 4.6: Example 4.7, nonsmooth source: maximum absolute errors and convergence rates over uniform and graded meshes.

N	$\alpha = 1.2$				$\alpha = 1.9$			
	$\delta = 1$		$\delta = 2/1.3$		$\delta = 1$		$\delta = 2/1.3$	
16	4.10e-02	1.2565	6.65e-03	1.7780	3.43e-02	1.2841	5.48e-03	1.9494
32	1.79e-02	1.1934	1.76e-03	1.9212	1.44e-02	1.2506	1.47e-03	1.9002
64	7.39e-03	1.2794	4.62e-04	1.9284	5.91e-03	1.2875	3.85e-04	1.9321
128	3.02e-03	1.2913	1.21e-04	1.9303	2.41e-03	1.2955	1.01e-04	1.9328

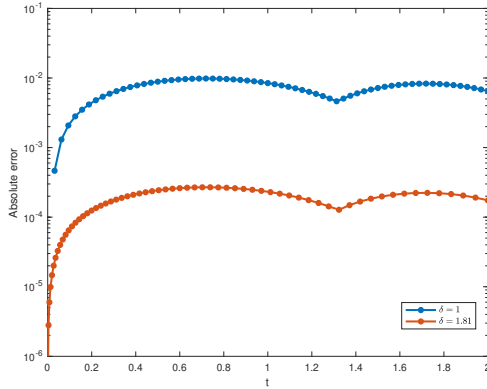


(a) Uniform mesh: $\delta = 1$

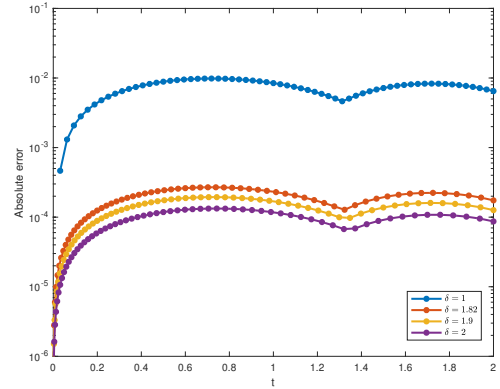


(b) Graded mesh: $\delta = 2/1.1$

Figure 4.14: Example 4.9, nonsmooth force: plots of absolute error for the GETD scheme over uniform and graded meshes for $\alpha = 1.9$, for different N 's.



(a)



(b)

Figure 4.15: Example 4.9, nonsmooth force: the impact of the graded mesh on the absolute error with $\alpha = 1.9$, for $N = 64$.

Table 4.7: Example 4.8, nonsmooth force: maximum absolute errors and convergence rates over uniform and graded meshes.

N	$\alpha = 1.2$				$\alpha = 1.9$			
	$\delta = 1$		$\delta = 2/1.8$		$\delta = 1$		$\delta = 2/1.1$	
16	1.58e-03	1.5274	2.89e-04	1.7936	1.70e-02	1.0329	8.50e-04	1.9435
32	5.19e-04	1.6040	8.07e-05	1.8408	4.88e-02	1.0657	2.20e-04	1.9521
64	1.65e-04	1.6520	2.20e-05	1.8721	8.14e-03	1.0823	5.67e-05	1.9556
128	5.14e-05	1.6844	5.92e-06	1.8943	3.84e-03	1.0908	1.46e-05	1.9574

Table 4.8: example 4.9, smooth force: maximum absolute errors and convergence rates.

N	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
16	9.65e-03	2.0119	7.79e-03	2.0149	6.39e-03	2.0138	5.36e-03	2.0128
32	2.40e-03	2.0064	1.94e-03	2.0054	1.59e-03	2.0040	1.34e-03	2.0040
64	5.99e-04	2.0032	4.84e-04	2.0019	3.98e-04	2.0011	3.34e-04	2.0010
128	1.49e-04	2.0029	1.21e-04	2.0010	9.94e-05	2.0003	8.34e-05	2.0002

sider the following problem from [61]:

$$\begin{aligned}
& {}^c\partial_t^\alpha u(x, t) = \partial_x^2 u(x, t) + \exp(x) [\Gamma(1 + \alpha) - t^\alpha], \quad 0 < x < 1, \quad 0 < t \leq 1, \\
& u(x, 0) = 0, \quad \partial_t u(x, 0) = 0, \quad 0 \leq x \leq 1, \\
& u(0, t) = t^\alpha, \quad u(1, t) = \exp(1)t^\alpha, \quad 0 < t \leq 1,
\end{aligned} \tag{4.47}$$

where ${}^c\partial_t^\alpha$ is the Caputo fractional partial time derivative of order $\alpha \in (1, 2)$, and ∂_x^2

Table 4.9: Example 4.9, nonsmooth force: maximum absolute errors and convergence rates over uniform and graded meshes.

N	$\alpha = 1.3$				$\alpha = 1.9$			
	$\delta = 1$		$\delta = 2/1.7$		$\delta = 1$		$\delta = 2/1.1$	
16	8.82e-03	1.4792	3.88e-03	1.8993	4.49e-02	1.0655	4.05e-03	1.9251
32	2.75e-03	1.6821	1.01e-03	1.9414	2.10e-02	1.0949	1.04e-03	1.9551
64	8.59e-04	1.6772	2.60e-04	1.9538	9.83e-03	1.0978	2.69e-04	1.9567
128	2.67e-04	1.6855	6.71e-05	1.9572	4.59e-03	1.0993	6.92e-05	1.9571

is the second order partial derivative in space. The exact solution of this problem is

$$u(x, t) = \exp(x)t^\alpha.$$

Implementing the second-order central difference approximation to (4.47) over a uniform spatial mesh with step size $h = 1/(m + 1)$ results in a system of the form (4.1), where the matrix $\mathbf{A} = [a_{ij}]$ is tridiagonal with $a_{i,i} = 2$ and $a_{i,j} = -1$ for $j = i \pm 1$. The initial data and force term in (4.1) can be written as $\mathbf{u}_0 = \mathbf{u}_0^1 = \mathbf{0}_m$, and

$$\mathbf{f}(t) = \begin{bmatrix} g(h, t) + t^\alpha \\ g(2h, t) \\ \vdots \\ g((m-1)h, t) \\ g(mh, t) + t^\alpha \exp(1) \end{bmatrix},$$

where $\mathbf{0}_m$ is an $m \times 1$ zero vector and $g(x, t) = \exp(x) [\Gamma(1 + \alpha) - t^\alpha]$.

To illustrate the convergence rates in time, we select the spatial step-size h such that the time error is dominant, taking into account that the spatial error is $O(h^2)$. For this, we choose $h = 1/512$ and, as such, $\mathbf{A}$ is of size 511×511 . Table 4.10 shows the outcomes of our numerical experiment for $\alpha = 1.2$. It is seen that by choosing the appropriate graded mesh exponent $\delta = 2/1.2$, we are able to achieve the optimal convergence rate.

Plots of the error profiles are presented in Figure 4.16, showing the effect of the graded mesh. The time $t = 1/32$ is the first time step on the uniform mesh and the

fifth time step on the graded mesh. It is seen in the figure on the left that using the non-uniform time step to refine the mesh improves the absolute error.

Table 4.10: Example 4.10, maximum absolute errors and convergence rates, $\alpha = 1.2$.

N	$\delta = 1$		$\delta = 2/1.2$	
16	2.82E-03	1.1801	5.48E-04	1.9288
32	1.25E-03	1.1811	1.37E-04	1.9966
64	5.45E-04	1.1913	3.49E-05	1.9770
128	2.38E-04	1.1960		

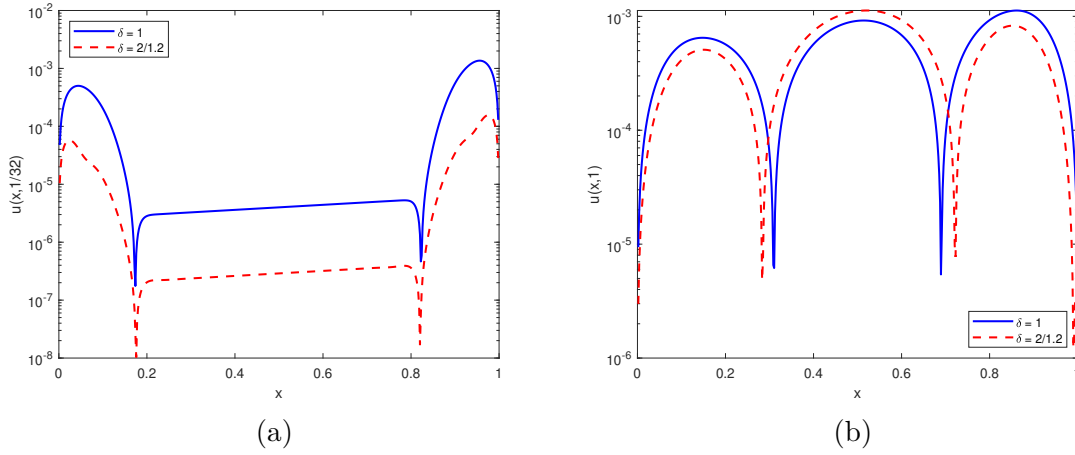


Figure 4.16: Example 4.10, plots of the absolute error: $t = 1/32$, (left), final time step, $T = 1$, (right), illustrating the effect of the graded mesh.

4.5 Concluding Remarks

This chapter studies the accuracy and effectiveness of a second order generalized exponential time differencing numerical method for the solution of fractional evolution equations of oscillatory characteristics. Theoretical analysis establishes the stability and convergence of the numerical solution. These findings are further validated through numerical verification. The proposed scheme is efficiently implemented by

employing rational approximations for the computation of the Mittag-Leffler function.

CHAPTER 5

GENERALIZED EXPONENTIAL TIME DIFFERENCING SCHEME FOR NON-LINEAR FRACTIONAL OSCILLATION MODELS

5.1 Introduction

Fractional order problems are becoming increasingly common in the simulation of real-life models in many fields of engineering, finance, physics, and rheology. These problems involve fractional derivatives that capture the memory effects of previous states in time, making them difficult to solve both analytically and numerically. These char-

acteristics make fractional derivatives useful in modeling complex phenomena such as viscoelasticity, fluid dynamics and biological processes. Consequently, fractional differential equations (FDEs) have attracted considerable interest and ongoing research on efficient numerical methods for computing their solutions.

Recently, various numerical methods have been developed for solving nonlinear fractional differential equations. For instance, the variational iteration method (VIM) and the homotopy analysis method (HAM) in [62] were modified to handle nonlinear FDEs. Notable approaches include the reproducing kernel Hilbert space method for the fractional Duffing–Van der Pol oscillator [45] and the use of variable-order fractional derivatives in the dissipative term of the Duffing oscillator [63]. Additionally, spectral collocation methods such as the Bernoulli wavelets method [46] and the Jacobi spectral method [47] have been applied to fractional models. These methods extend classical approaches to include the effects of fractional-order dynamics, with a focus on nonlinear fractional differential equations.

Another class of numerical schemes devoted to solving second order differential equations with oscillatory behavior commonly found in mechanical oscillators or wave phenomena are trigonometric integrators. These integrators are particularly efficient in capturing the oscillatory nature of solutions without requiring excessively small time steps. In contrast to exponential integrators, which are suited for systems with rapidly decaying behaviors [48], trigonometric integrators are appropriate for systems exhibiting fast oscillations. For further details on trigonometric and exponential integrators, comprehensive surveys can be found in [49], [50], [51], [52], [53], [54].

Exponential time differencing (ETD) is a widely used numerical technique for

solving ordinary and partial differential equations, particularly those involving stiff systems [64], [55], [48], [65]. It is part of a broader class of exponential integrators developed to address stiffness by leveraging the exact integration of the linear (stiff) part using the matrix exponential, while approximating the convolution integral of the source term. ETD methods have demonstrated superior stability compared to implicit-explicit schemes, making them particularly effective for systems with fast-decaying or oscillating modes [53], [66].

In [5], exponential time differencing (ETD) method is generalized to fractional differential equations and the given method is called the generalized exponential time differencing (GETD) method. GETD offers a flexible and efficient framework for solving fractional differential equations, particularly those involving fast-decaying or oscillating modes, by effectively addressing the dynamic complexities of fractional calculus.

In this chapter, we consider the following nonlinear fractional oscillation model:

$$\begin{aligned} {}^c D^\alpha u(t) + \lambda u(t) &= f(t, u(t)), \quad t > 0, \quad \alpha \in (1, 2), \\ u(0) &= u_0^0, \\ u'(0) &= u_0^1, \end{aligned} \tag{5.1}$$

where λ is a constant, and ${}^c D^\alpha$ denotes the Caputo fractional derivative, defined as [7]:

$${}^c D^\alpha u(t) := \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} u''(\tau) d\tau, & 1 < \alpha < 2, \\ u''(t), & \alpha = 2. \end{cases}$$

To ensure the existence of a unique solution to (5.1), we assume that f satisfies the Lipschitz condition with respect to the second variable. Specifically, for a suitable region $\mathcal{D} \subset \mathbb{R}$, the nonlinear source term $f : [0, T] \times \mathcal{D} \rightarrow \mathbb{R}$ satisfies:

$$|f(t, u_1) - f(t, u_2)| \leq L|u_1 - u_2|, \quad \text{for some } L > 0, \quad \text{for all } u_1, u_2 \in \mathcal{D}.$$

Problems of the form (5.1) play critical roles in fields such as electrical circuits, biological systems, and engineering applications. Nonlinear fractional oscillator equations are fundamental for modeling a variety of complex, nonlinear dynamical behaviors in these areas, making them a key area of study in applied mathematics.

In [67], a new method incorporated by the ultraspherical wavelet operational matrix of general order integration and block-pulse functions is adopted to investigate the solution of fractional oscillator type equations. In [68], an optimization method to estimate the solutions of nonlinear oscillation equations is developed. In [69], based on the operational matrix of fractional-order integration, a method for the numerical solution of fractional strongly nonlinear Duffing oscillators with cubic-quintic-heptic nonlinear restoring force is introduced. In [70], an efficient technique for solving fractional nonlinear oscillators, based on the fractional complex transform and global residue harmonic balance method, is proposed.

The main goal of this chapter is to estimate the solution of (5.1) using the exponential time differencing technique, as obtaining an exact solution is challenging due to the complexity of the nonlinear fractional oscillator. In this chapter, we develop multistep GETD schemes for solving (5.1). The proposed schemes are designed to

efficiently solve such equations on graded meshes to achieve uniform optimal convergence rates. We employ partial fraction decompositions of rational approximations for the generalized Mittag-Leffler function, developed in [27], [56], to enhance the computational efficiency of the method.

The structure of this chapter is as follows. Section 5.2 introduces the description of the numerical scheme. Construction of the multistep GETD schemes is discussed in Section 5.3. The stability and convergence analysis for smooth and non-smooth force terms is presented in Section 5.4. Section 5.5 showcases various numerical experiments. Concluding remarks can be found in Section 5.6.

5.2 Derivation of the GETD Scheme

In this section, we derive the Generalized Exponential Time Differencing (GETD) scheme for solving the nonlinear fractional oscillation problem (5.1). The formulation of the scheme is based on the mild solution of the problem, where the homogeneous part is integrated exactly using the Laplace transform and its inversion. The mild solution is expressed as

$$u(t) = E_{\alpha}(-\lambda t^{\alpha}) u_0^0 + t E_{\alpha,2}(-\lambda t^{\alpha}) u_0^1 + \int_0^t \mathcal{E}^{\alpha,\alpha-1}(t-\tau; \lambda) f(\tau, u(\tau)) d\tau, \quad (5.2)$$

where

$$\mathcal{E}^{\alpha,\beta}(s; \lambda) = s^{\beta} E_{\alpha,\beta+1}(-\lambda s^{\alpha}), \quad (5.3)$$

and $E_{\alpha,\beta}$ is the two-parameter Mittag-Leffler function defined as

$$E_{\alpha,\beta}(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(\alpha k + \beta)}, \quad \operatorname{Re}(\alpha) > 0, \quad \beta \in \mathbb{C}.$$

Here, $E_{\alpha}(s) = E_{\alpha,1}(s)$ denotes the one-parameter Mittag-Leffler function.

To numerically approximate the solution $u(t)$ over the interval $[0, T]$, we consider the mesh points

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T,$$

and express the solution at each point t_n , for $1 \leq n \leq N$, as follows

$$\begin{aligned} u(t_n) = & E_{\alpha}(-\lambda t_n^{\alpha}) u_0^0 + t_n E_{\alpha,2}(-\lambda t_n^{\alpha}) u_0^1 \\ & + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathcal{E}^{\alpha,\alpha-1}(t_n - \tau; \lambda) f(\tau, u(\tau)) d\tau. \end{aligned} \tag{5.4}$$

A key step in deriving the GETD scheme is the approximation of the integral in (5.4). This can be particularly delicate, considering that the solution $u(t) \in C^{\alpha}(0, T)$, a space of functions that are continuous with $u'(t)$ being Hölder continuous of order $\alpha - 1$, which reflects the fractional regularity induced by the Caputo derivative. As such, achieving higher-order schemes can be challenging if the regularity property is not adequately taken into consideration [71], [72]. To mitigate this issue, graded meshes have been used successfully in the numerical approximation of integral equations with weakly singular kernels, integrodifferential equations, and fractional differential equations (FDEs) [73], [74], [75], [76]. For this work, we employ a graded

mesh of the form

$$S_N = \{t_n = (nk)^\delta\}_{n=0}^N, \quad \delta \geq 1, \quad \text{with} \quad k = \frac{T^{\frac{1}{\delta}}}{N}, \quad (5.5)$$

where δ is the mesh grading parameter. The mesh sizes $k_n = t_{n+1} - t_n$ and nodes t_n follow these properties [27]

$$k_{n+1} \leq 2^{\delta-1} k_n, \quad k_n \leq \delta k t_{n+1}^{1-\frac{1}{\delta}}, \quad 0 \leq n \leq N-2, \quad (5.6)$$

$$t_{n+1} \leq 2^\delta t_n, \quad 1 \leq n \leq N-1. \quad (5.7)$$

Setting $u^0 = u(0)$ and $u^n \approx u(t_n)$ for $1 \leq n \leq N$, the numerical scheme for the problem (5.1) is given by

$$\begin{aligned} u^n &= E_\alpha(-\lambda t_n^\alpha) u^0 + t_n E_{\alpha,2}(-\lambda t_n^\alpha) u_0^1 \\ &\quad + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \mathcal{E}^{\alpha,\alpha-1}(t_n - \tau; \lambda) \mathcal{P}_{j,r-1}[f; \tau] d\tau. \end{aligned} \quad (5.8)$$

Here, $\mathcal{P}_{j,r-1}[f; \tau]$, $r \in \{1, 2\}$, represents the interpolating polynomial of degree $r-1$ for the function f on the subinterval $I_j = [t_j, t_{j+1}]$. For simplicity, we let $f^n = f(t_n, u^n)$, and

$$\nabla f^j = \frac{f^j - f^{j-1}}{k_{j-1}}, \quad \text{for } j \geq 1,$$

with $\nabla f^0 = 0$.

Next we proceed to develop the multistep GETD schemes.

5.3 Multistep GETD Schemes

In this section, we develop the one-step (GETD-1) and two-step (GETD-2) schemes based on the Generalized Exponential Time Differencing method, applied to graded meshes. These schemes are designed to efficiently handle the nonlinear fractional oscillation problem by utilizing appropriate interpolating polynomials for the source term $f(t, u(t))$.

We begin by describing the construction of the one-step and two-step schemes. These schemes are derived directly from the numerical form (5.8), utilizing graded meshes to improve accuracy near singularities at $t = 0$.

5.3.1 GETD-1 Scheme

The one-step scheme (GETD-1) is constructed by approximating the source term $f(t, u(t))$ using a piecewise constant interpolant, $\mathcal{P}_{j,0}[f; \tau] = f^j$, over each subinterval $[t_j, t_{j+1}]$. Substituting this into the general numerical scheme (5.8), the solution at each mesh point t_n becomes

$$u^n = E_\alpha(-\lambda t_n^\alpha) u^0 + t_n E_{\alpha,2}(-\lambda t_n^\alpha) u_0^1 + \sum_{j=0}^{n-1} f^j \int_{t_j}^{t_{j+1}} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau; \lambda) d\tau.$$

Evaluating the integral results in the GETD-1 scheme

$$u^n = \mathcal{H}(t_n) + \sum_{j=1}^{n-1} k_{j-1} \mathcal{E}^{\alpha, \alpha}(t_n - t_j; \lambda) \nabla f^j, \quad n \geq 1, \quad (5.9)$$

where

$$\mathcal{H}(t_n) := E_\alpha(-\lambda t_n^\alpha) u_0^0 + t_n E_{\alpha,2}(-\lambda t_n^\alpha) u_0^1 + \mathcal{E}^{\alpha,\alpha}(t_n; \lambda) f^0.$$

5.3.2 GETD-2 Scheme

The two-step scheme (GETD-2) improves upon GETD-1 by using a linear interpolating polynomial for the source term. Specifically, we approximate $f(t, u(t))$ on each subinterval $[t_j, t_{j+1}]$ with the first-degree polynomial

$$\mathcal{P}_{j,1}[f; s] = f^j + (s - t_j) \nabla f^{j+1}, \quad j = 0, 1, \dots, n-1. \quad (5.10)$$

To ensure an explicit scheme, $\mathcal{P}_{n-2,1}[f; \tau]$ is used rather than $\mathcal{P}_{n-1,1}[f; \tau]$ on the subinterval I_{n-1} for $n \geq 2$. Consequently, the solution for u^1 is computed using the GETD-1 scheme. Substituting the linear polynomials into (5.8) and integrating by parts, the GETD-2 scheme takes the form

$$u^n = \mathcal{H}(t_n) + \sum_{j=0}^{n-2} \mathcal{E}^{\alpha,\alpha+1}(t_n - t_j; \lambda) (\nabla f^{j+1} - \nabla f^j), \quad n \geq 2. \quad (5.11)$$

5.4 Convergence and Stability Analysis

In this section, we analyze the stability and convergence of the GETD schemes introduced in this chapter. Our analysis relies on several important inequalities that provide the foundation for our theoretical results.

Theorem 5.1 [4] *Let $0 < \alpha < 2$, β be an arbitrary real number, and μ be such that*

$\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Let M be a positive constant. Then, for any complex number z , the following bound holds

$$|E_{\alpha,\beta}(z)| \leq \frac{M}{1+|z|}, \quad \mu \leq |\arg(z)| \leq \pi. \quad (5.12)$$

Using Theorem 5.1, we derive that for any real number $x \geq 0$, the Mittag-Leffler function satisfies the bound

$$|E_{\alpha,\beta}(-x)| \leq C,$$

where C is a positive constant, which depends on $\{\alpha, \beta, x\}$.

Using Theorem 5.1 and the definition of $\mathcal{E}^{\alpha,\alpha-1}$, we can obtain the following estimate

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \mathcal{E}^{\alpha,\alpha-1}(t_n - \tau; \lambda) d\tau &\leq \int_{t_j}^{t_{j+1}} C (t_n - \tau)^{\alpha-1} d\tau \\ &\leq C k_j (t_n - t_j)^{\alpha-1} =: b_{n,j}^{(\alpha)}, \quad n \geq 1. \end{aligned} \quad (5.13)$$

Next, we state a version Grönwall inequality over non uniform mesh [77].

Lemma 5.1 *Let $\alpha \in (1, 2)$, and $g_0 > 0$. Assume the non-negative sequence $\{\psi_n\}_{n=0}^N$ satisfies the recursive relation*

$$\psi_0 \leq g_0, \quad \psi_n \leq \sum_{j=0}^{n-1} b_{n,j}^{(\alpha)} \psi_j + g_0,$$

for $n = 1, 2, \dots, N$. Then, there exists a constant C such that

$$\psi_n \leq Cg_0, \quad n = 1, 2, \dots, N.$$

5.4.1 Stability Analysis

In this section, we analyze the stability of the GETD schemes. Stability, in this context, refers to the behavior of the numerical solution in the presence of small perturbations in the initial conditions as in Definition 4.3 below.

Definition 5.1 [77] *Suppose that u^n and $\tilde{u}^n$ for $n = 1, 2, \dots, N$ are two numerical solutions of the GETD scheme corresponding to different initial values $u_0^{(i)}$ and $\tilde{u}_0^{(i)}$ for $i = 0, 1$. If there exists a constant C , independent of the graded stepsizes k_j for $j = 0, 1, \dots, n$ and n , such that*

$$|u^n - \tilde{u}^n| \leq C \sum_{i=0}^1 \left| u_0^{(i)} - \tilde{u}_0^{(i)} \right|,$$

then the GETD scheme is said to be stable.

Theorem 5.2 *Let u^n for $n = 1, 2, \dots, N$ be the solution of the GETD schemes (5.9) and (5.11) with initial values $u_0^{(i)}$ for $i = 0, 1$. Suppose that the forcing term $f(t, u)$ in the initial value problem (5.1) satisfies the Lipschitz condition with respect to u . Then both the GETD-1 and GETD-2 schemes are stable.*

Proof. We provide the proof for the stability of the GETD-2 scheme. Let $\tilde{u}^n$ denote the perturbed solution corresponding to the perturbed initial data $\tilde{u}_0^{(i)}$ for $i = 0, 1$.

Let $\tilde{f}(\tilde{u}, t)$ be the source term of the perturbed problem. Now define $\tilde{f}^j = f(\tilde{u}^j, t_j)$ and the linear interpolation as polynomial

$$P_{j,1}[\tilde{f}; \tau] = \tilde{f}^j + \frac{\tau - t_j}{k_j} (\tilde{f}^{j+1} - \tilde{f}^j).$$

From the definition of the numerical scheme (5.8), we obtain

$$\begin{aligned} |u^n - \tilde{u}^n| &\leq \sum_{i=0}^1 t_n^i E_{\alpha, i+1}(-\lambda t_n^\alpha) \left| u_0^{(i)} - \tilde{u}_0^{(i)} \right| \\ &\quad + \sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau; \lambda) \left| P_{j,1}[f; \tau] - P_{j,1}[\tilde{f}; \tau] \right| d\tau \\ &\quad + \int_{t_{n-1}}^{t_n} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau; \lambda) \left| P_{n-2,1}[f; \tau] - P_{n-2,1}[\tilde{f}; \tau] \right| d\tau. \end{aligned}$$

Applying the Lipschitz condition for f , which implies that

$$\left| f(u^j, t_j) - f(\tilde{u}^j, t_j) \right| \leq L \left| u^j - \tilde{u}^j \right|,$$

we get

$$\begin{aligned} |u^n - \tilde{u}^n| &\leq C \sum_{i=0}^1 \left| u_0^{(i)} - \tilde{u}_0^{(i)} \right| \\ &\quad + \sum_{j=0}^{n-2} \int_{t_j}^{t_{j+1}} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau; \lambda) (2 \left| u^j - \tilde{u}^j \right| + \left| u^{j+1} - \tilde{u}^{j+1} \right|) d\tau \\ &\quad + \int_{t_{n-1}}^{t_n} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau; \lambda) (2 \left| u^{n-2} - \tilde{u}^{n-2} \right| + \left| u^{n-1} - \tilde{u}^{n-1} \right|) d\tau. \end{aligned}$$

Using the inequality (5.13), and noting that $b_{n,j-1}^{(\alpha)} \leq b_{n,j}^{(\alpha)} \leq 2b_{n,j-1}^{(\alpha)}$ for $1 \leq j \leq$

$n - 1$, we derive

$$|u^n - \tilde{u}^n| \leq C \sum_{i=0}^1 \left| u_0^{(i)} - \tilde{u}_0^{(i)} \right| + 2 \sum_{j=0}^{n-2} b_{n,j}^{(\alpha)} |u^j - \tilde{u}^j| + \sum_{j=0}^{n-2} b_{n,j}^{(\alpha)} |u^{j+1} - \tilde{u}^{j+1}| + 2b_{n,n-1}^{(\alpha)} |u^{n-2} - \tilde{u}^{n-2}|.$$

Thus, we have

$$|u^n - \tilde{u}^n| \leq C \sum_{i=0}^1 \left| u_0^{(i)} - \tilde{u}_0^{(i)} \right| + 8 \sum_{j=0}^{n-1} b_{n,n-j}^{(\alpha)} |u^j - \tilde{u}^{j-1}|,$$

.

Finally, by applying Grönwall's inequality, Lemma 5.1, we conclude

$$|u^n - \tilde{u}^n| \leq C \sum_{i=0}^1 \left| u_0^{(i)} - \tilde{u}_0^{(i)} \right|.$$

I

5.4.2 Convergence Analysis

To begin, we propose regularity assumptions for the solution u and the source term f of equation (5.1). Based on well-known results, such as those in [78], if f is sufficiently smooth with respect to u , the solution u can be decomposed into regular and singular components. By differentiating composite functions, it is clear that any regularity in u propagates to $f(u(t), t)$ as a function of t .

Thus, based on these observations, we consider the following natural assumptions:

Condition $\mathcal{C}_1$:

$$|u^{(\ell)}(t)| \leq Ct^{\alpha-\ell}, \quad \ell = 1, 2.$$

Condition $\mathcal{C}_2$: For $r = 1, 2$, the source term $f(t, \cdot) \in C^r(\mathcal{D})$, i.e., f has continuous derivatives of order r with respect to u .

Condition $\mathcal{C}_3$: We permit $f(t, \cdot)$ to exhibit some weak singularity in t . As such,

$$\left| \frac{\partial^\ell f}{\partial t^\ell} \right| \leq Ct^{\sigma-\ell}, \quad \ell = 1, 2, \quad 0 \leq \sigma \leq 1.$$

Condition $\mathcal{C}_4$: Combining $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, we can have a parameter $0 < \gamma < 1$ such that the solution u and the source term f satisfy the following bound

$$|u^{(\ell)}(t)| + \left| \frac{d^\ell f(t, u)}{dt^\ell} \right| \leq Ct^{\gamma-\ell}, \quad \ell = 1, 2. \quad (5.14)$$

For the error analysis, we use the interpolation polynomial $S_{j,r-1}$ as a comparison function. This polynomial approximates the source term $f(u(s), s)$ over the interval I_j for $j = 0, 1, \dots, n-1$. Specifically, the polynomial $S_{j,r-1}$ is defined as $P_{j,r-1}[f; s]$, where f_j replaces f^j to locally approximate f within each subinterval.

Next, we derive the following error estimates for the interpolation of the source term f .

Lemma 5.2 *The interpolation of the source term f in the GETD scheme satisfies the following error bounds*

$$\int_0^{t_1} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |S_{0,r-1}[f; \tau] - f(\tau, u(\tau))|, d\tau \leq C t_n^{\alpha-1} k^{\delta(\gamma+\alpha)}, \quad n \geq 1, \quad (5.15)$$

and for $n \geq 2$,

$$\sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |S_{j,r-1}[f; \tau] - f(\tau, u(\tau))| d\tau \leq C k^r t_n^{\alpha+\gamma-\frac{r}{\delta}}. \quad (5.16)$$

Proof. Let $g(\tau) := f(\tau, u(\tau))$. First, we prove inequality (5.15) for $r = 2$. Using the regularity assumption $\mathcal{C}_4$, we know that

$$|g'(t)| \leq t^{\gamma-1}.$$

For $\tau \in I_0$, we estimate

$$\begin{aligned} |S_{0,1}[f; \tau] - f(\tau, u(\tau))| &= \left| g(0) + \frac{\tau}{t_1} (g(t_1) - g(0)) - g(\tau) \right| \\ &\leq \int_0^\tau |g'(s)| ds + \frac{\tau}{t_1} \int_0^{t_1} |g'(s)| ds \\ &\leq 2 \int_0^{t_1} |g'(s)| ds \leq 2 \frac{t_1^\gamma}{\gamma} = C t_1^\gamma. \end{aligned} \quad (5.17)$$

Applying this bound, and using (5.13), we get

$$\begin{aligned}
\int_0^{t_1} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |S_{0,1}[f; \tau] - f(\tau, u(\tau))| d\tau &\leq Ct_1^\gamma \int_0^{t_1} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| d\tau \\
&\leq Ct_1^\gamma \int_0^{t_1} (t_n - \tau)^{\alpha-1} d\tau \\
&\leq Ct_1^\gamma t_n^{\alpha-1} t_1 \\
&\leq Ct_1^{\gamma+1} t_n^{\alpha-1} = Ct_n^{\alpha-1} k^{\delta(\gamma+\alpha)}.
\end{aligned} \tag{5.18}$$

Since $t_1 = k^\delta$, the bound in equation (5.15) holds for $r = 2$.

Next, we prove inequality (5.16). By using the Lagrange interpolation error formula and the regularity condition $\mathcal{C}_2$, along with the mesh properties (5.6), for $j \geq 1$, we have:

$$\begin{aligned}
|S_{j,r-1}[g; \tau] - g(\tau)| &\leq Ck_j^r \max_{t \in I_j} |g^{(r)}(t)| \\
&\leq Ck_j^r \max_{t \in I_j} t^{\gamma-r} \\
&\leq Ck^r t_{j+1}^{r(1-1/\delta)} t_j^{\gamma-r},
\end{aligned} \tag{5.19}$$

for $r = 1, 2$ and $j = 1, 2, \dots, N$. Then, using the graded mesh property (5.7) and applying the estimate (5.13), we obtain

$$\begin{aligned}
\sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |S_{j,r-1}[f; \tau] - f(\tau, u)| d\tau &\leq Ck^r \int_{t_1}^{t_n} (t_n - \tau)^{\alpha-1} \tau^{\gamma-\frac{r}{\delta}} d\tau \\
&\leq Ck^r t_n^{\alpha+\gamma-\frac{r}{\delta}}.
\end{aligned} \tag{5.20}$$

In the next theorem, we show the convergence result of the GETD-1 and GETD-2 schemes.

Theorem 5.3 *Let u^n be the solution of the GETD- r schemes defined by (5.9) and (5.11) for $r = 1, 2$, respectively, over the graded mesh (5.5). Let u be the exact solution of (5.1), and suppose that the regularity assumption (5.14) is satisfied with $\ell = r$ for GETD- r . Then, for $\delta \geq 1$, we have the following error estimate*

$$\max_{n=1}^N |u^n - u(t_n)| \leq C k^{\min\{\delta(\gamma+\alpha), r\}}, \quad r = 1, 2.$$

Proof. The proof is presented for the case $r = 2$, as the case for $r = 1$ follows similarly with analogous arguments. We break the proof into several detailed steps for clarity.

For the first time step $n = 1$, the solution u^1 is computed using the GETD-1 scheme. By applying the Lipschitz continuity of f and using the corresponding error estimate from (5.15), we can bound the error at this step. The error $e^1 = u^1 - u(t_1)$ can be written as

$$|e^1| = |u^1 - u(t_1)| = \left| \int_0^{t_1} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau) [P_{0,0}[f; \tau] - f(\tau, u)] d\tau \right|.$$

By triangle inequality, we get

$$|e^1| \leq \int_0^{t_1} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| [|P_{0,0}[f; \tau] - S_{0,0}[f; \tau]| + |S_{0,0}[f; \tau] - f(\tau, u)|] d\tau.$$

By the properties of the interpolating polynomials and the estimate (5.15), the

error in this step can be bounded by

$$|e^1| \leq Ck^{\delta(\gamma+\alpha)}. \quad (5.21)$$

For time steps $n \geq 2$, we decompose the global error $e^n = u^n - u(t_n)$ as a sum of local errors over each previous subinterval

$$e^n = \sum_{j=1}^{n-1} E_j, \quad n \geq 2,$$

where each E_j is given by

$$E_j = \int_{t_j}^{t_{j+1}} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau) [P_{j,1}[f; \tau] - f(\tau, u)] d\tau, \quad 1 \leq j \leq n-2,$$

and for the last interval

$$E_{n-1} = \int_{t_{n-1}}^{t_n} \mathcal{E}^{\alpha, \alpha-1}(t_n - \tau) [P_{n-2,1}[f; \tau] - f(\tau, u)] d\tau.$$

For each subinterval $1 \leq j \leq n-2$, the error E_j is split as

$$|E_j| \leq \int_{t_j}^{t_{j+1}} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| [|P_{j,1}[f; \tau] - S_{j,1}[f; \tau]| + |S_{j,1}[f; \tau] - f(\tau, u)|] d\tau.$$

By applying the Lipschitz continuity of f , we can bound the first term as follows

$$|P_{j,1}[f; \tau] - S_{j,1}[f; \tau]| \leq \frac{t_{j+1} - \tau}{k_j} |f^j - f_j| + \frac{\tau - t_j}{k_j} |f_{j+1} - f^{j+1}| \leq 2L (|e^j| + |e^{j+1}|). \quad (5.22)$$

Then, applying (5.13), we get

$$\int_{t_j}^{t_{j+1}} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |P_{j,1}[f; \tau] - S_{j,1}[f; \tau]| d\tau \leq CLb_{n,j}^{(\alpha)} [|e^j| + |e^{j+1}|],$$

The second term, using (5.16), is bounded such as

$$\int_{t_j}^{t_{j+1}} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |S_{j,1}[f; \tau] - f(\tau, u)| d\tau \leq Ck^2 t_n^{\alpha + \gamma - \frac{2}{\delta}}.$$

Now, combining all the estimates for E_j (for $1 \leq j \leq n-2$), we obtain the following bound

$$\sum_{j=1}^{n-2} |E_j| \leq CL \sum_{j=0}^{n-2} b_{n,j}^{(\alpha)} [|e^j| + |e^{j+1}|] + Ck^2 t_n^{\alpha + \gamma - \frac{2}{\delta}}. \quad (5.23)$$

For the last subinterval E_{n-1} , we apply similar arguments as in the previous step.

As such the error E_{n-1} is split as

$$|E_{n-1}| \leq \int_{t_{n-1}}^{t_n} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| [|P_{n-2,1}[f; \tau] - S_{n-2,1}[f; \tau]| + |S_{n-2,1}[f; \tau] - f(\tau, u)|] d\tau. \quad (5.24)$$

Decomposing and applying (5.22) and (5.13), we can bound the first term as follows

$$\int_{t_{n-1}}^{t_n} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |P_{n-2,1}[f; \tau] - S_{n-2,1}[f; \tau]| d\tau \leq CLb_{n,n-1}^{(\alpha)} [|e^{n-2}| + |e^{n-1}|]. \quad (5.25)$$

To estimate the second term, we observe that

$$S_{n-2,1}[g; \tau] = S_{n-1,1}[g; \tau] - (\tau - t_{n-1}) \left[\frac{g(t_n) - g(t_{n-1})}{k_{n-1}} + \frac{g(t_{n-2}) - g(t_{n-1})}{k_{n-2}} \right].$$

The last term in the bracket equals

$$\frac{1}{k_{n-1}} \int_{t_{n-1}}^{t_n} g''(\xi) (t_n - \xi) d\xi + \frac{1}{k_{n-2}} \int_{t_{n-2}}^{t_{n-1}} g''(\xi) (\xi - t_{n-2}) d\xi.$$

Using this, the inequality $|g''(\xi)| \leq \xi^{\gamma-2}$ (from the regularity condition **C1**), and the mesh properties (5.6) and (5.7), we arrive at the following bound

$$\begin{aligned} |S_{n-2,1}[g; \tau] - g(\tau)| &\leq |S_{n-1,1}[g, \tau] - g(\tau)| + Ck^2 t_{n-1}^{\gamma-\frac{2}{\delta}} \\ &\leq |S_{n-1,1}[g, \tau] - g(\tau)| + Ck^2 t_n^{\gamma-\frac{2}{\delta}}. \end{aligned} \tag{5.26}$$

Thus, following the approach in the proof of (5.16) in Lemma 5.2, we reach

$$\int_{t_{n-1}}^{t_n} |\mathcal{E}^{\alpha, \alpha-1}(t_n - \tau)| |S_{n-2,1}[f; \tau] - f(\tau, u)| d\tau \leq Ck^2 t_n^{\alpha+\gamma-\frac{2}{\delta}}.$$

Inserting this along with (5.25) into (5.24), we achieve the estimate

$$|E_{n-1}| \leq CLb_{n,n-1}^{(\alpha)} [|e^{n-2}| + |e^{n-1}|] + Ck^2 t_n^{\alpha+\gamma-\frac{2}{\delta}}, \quad \text{for } n \geq 2. \tag{5.27}$$

Therefore, by combining equations (5.21), (5.23), and (5.27), and utilizing the fact that $b_{n,j}^{(\alpha)} \leq b_{n,j-1}^{(\alpha)}$ for $1 \leq j \leq n-1$, we arrive at the following bound

$$\begin{aligned}
|e^n| &\leq C \sum_{j=0}^{n-2} b_{n,j}^{(\alpha)} [|e^j| + |e^{j+1}|] + Ck^2 t_n^{\alpha+\gamma-\frac{2}{\delta}} + Ck^{\delta(\alpha+\gamma)} \\
&\leq Ck^{\delta(\alpha+\gamma)} + Ck^2 t_n^{\alpha+\gamma-\frac{2}{\delta}} + C \sum_{j=1}^{n-1} b_{n,j-1}^{(\alpha)} |e^j|, \quad \text{for } n \geq 1.
\end{aligned}$$

Thus, the following estimate is obtained

$$|e^n| \leq Ck^{\delta(\gamma+\alpha)} + Ck^2 t_n^{\alpha+\gamma-\frac{2}{\delta}} + C \sum_{j=1}^{n-1} b_{n,j-1}^{(\alpha)} |e^j|, \quad \text{for } n \geq 1.$$

To complete the proof, we apply Grönwall's inequality as stated in Lemma 5.1. I

5.5 Numerical Experiments

To validate the accuracy of the GETD schemes and assess the effect of a graded mesh in the case of non-smooth force term, we present some relevant experiments. For numerical illustration of the convergence properties, we compute the maximum absolute error as

$$e(N) = \max_{1 \leq n \leq N} \|u(t_n) - u^n\|,$$

when the exact solution $u(t)$ is known.

In the absence of the exact solution, a reference solution is computed using a very refined grid with $N = 1024$. The order of convergence is then determined by comparing the errors for successive grid sizes, using the relationship $\log_2[e(N)/e(2N)]$.

Example 5.4 *Fractional Duffing oscillator*

In this example, we consider the fractional Duffing oscillator with a Caputo fractional derivative, which is commonly used to model systems with memory effects, such as viscoelastic materials and other complex oscillatory systems. The fractional Duffing oscillator can be generally written as:

$${}^cD^\alpha u(t) + \lambda u(t) = \beta u^3(t) + g(t). \quad (5.28)$$

where λ is the linear coefficient, β is the nonlinearity coefficient, $g(t)$ is a driving force or external input, and $u(t)$ is the displacement.

Equation (5.28) describes the behavior of a system where the displacement $u(t)$ is influenced by both linear and nonlinear restoring forces, along with fractional dynamics that capture the system's memory effects.

To demonstrate the theoretical convergence results, Theorem 5.3, we consider the following cases.

Case 1 (Known exact solution)

Choose

$$g(t) = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} + \lambda(t^r + c) - \beta(t^r + c)^3.$$

Then the exact solution is $u(t) = t^r + c$.

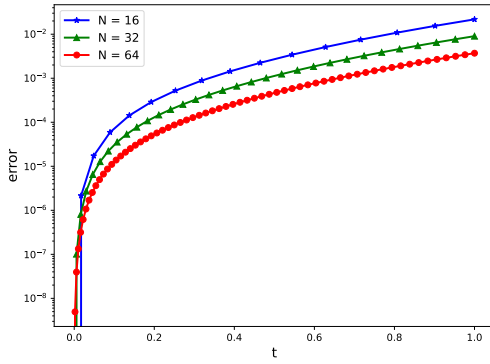
For this case, we choose the following parameters

$$r = 2, \quad \alpha = 1.7, \quad \beta = 0.1, \quad \lambda = 1, \quad c = 0.5.$$

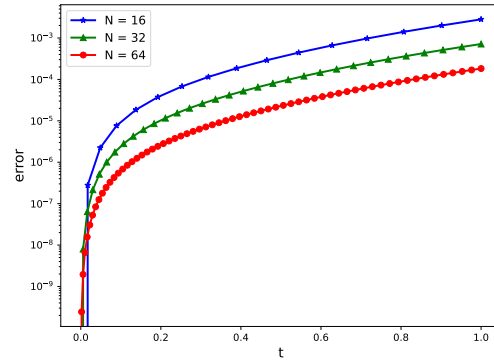
This choice of $g(t)$ ensures that the exact solution $u(t) = t^2 + c$ satisfies (5.28).

Table 5.1: Example 5.4, Case 1 (known exact solution): maximum absolute errors and convergence rates for $\alpha = 1.7$.

N	GETD-1		GETD-2			
	$\delta = 1$		$\delta = 1$	$\delta = 1.5$		
16	7.88e-02		1.45e-02	1.88e-03		
32	3.86e-02	1.0317	6.02e-03	1.2712	4.77e-04	1.9779
64	1.88e-02	1.0331	2.47e-03	1.2826	1.22e-04	1.9620
128	9.22e-03	1.0311	1.01e-03	1.2895	3.14e-05	1.9611
256	4.52e-03	1.0276	4.13e-04	1.2937	8.06e-06	1.9650



(a)

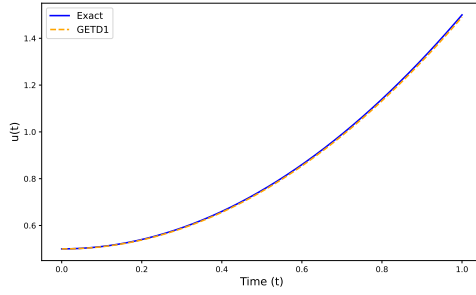


(b)

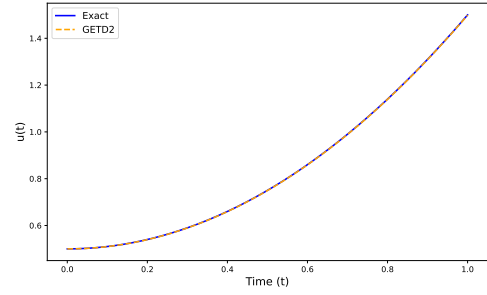
Figure 5.1: Example 5.4, Case 1 (known exact solution): plots of the absolute error, $\alpha = 1.7$ for GETD-2, $\delta = 1$ (left), $\delta = 1.5$ (right).

Table 5.2: Example 5.4, Case 2 (unknown exact solution): maximum absolute errors and convergence rates for $\alpha = 1.8$.

N	GETD-1		GETD-2	
16	2.14e − 03		1.86e − 03	
32	1.08e − 03	0.9835	4.83e − 04	1.9420
64	5.30e − 04	1.0276	1.23e − 04	1.9719
128	2.49e − 04	1.0897	3.11e − 05	1.9868
256	1.07e − 04	1.2175	7.79e − 06	1.9961



(a)



(b)

Figure 5.2: Example 5.4, Case 1 (known exact solution): plots of exact and approximate solutions, $\alpha = 1.7$ for GETD-1 (left), GETD-2 (right).

Case 2 (Unknown exact solution)

In this case, we consider a scenario where the exact solution is unknown. Setting $g(t) = 0$, the governing equation representing the fractional Duffing oscillator has the form

$${}^c D^\alpha u(t) + \lambda u(t) = \beta u^3(t),$$

with the initial conditions $u(0) = 1$ and $u'(0) = 0$

For this case, we choose the following parameters

$$\alpha = 1.8, \quad \beta = 0.3, \quad \lambda = 0.6.$$

Since the exact solution is unknown, a reference solution is computed on a fine grid with $N = 1024$, and the error is calculated by comparing the solutions on coarser grids. The results presented in Table 5.1 and Figure 5.1 demonstrate the performance of the GETD schemes for the fractional Duffing oscillator with the exact solution $u(t) = t^r + c$. The driving term $g(t)$ ensures that the exact solution satisfies the regularity conditions discussed in Section 5.4, though with some limitations for the second derivative. While the solution $u(t)$ satisfies the weak singularity conditions in $g(t)$, its second derivative $u''(t)$ does not fully conform to $\mathcal{C}_1$ (see Section 5.4) due to the lack of decay near $t = 0$, which introduces challenges for optimal convergence on uniform meshes, as seen for GETD-2 with $\delta = 1$. Despite these challenges, the graded mesh parameter $\delta = 1.5$ significantly improves the accuracy of GETD-2, achieving an optimal $O(k^2)$ -rate, consistent with the theoretical convergence results in Theorem 5.3, highlighting the importance of graded meshes for handling solutions with weak singularities near $t = 0$. For GETD-1, which is limited to first-order accuracy, uniform meshes were sufficient, as the first derivative of $u(t)$ satisfies $\mathcal{C}_1$. These observations underscore the robustness of the GETD schemes and validate the interplay between mesh grading and convergence for fractional differential equations with oscillatory and nonlinear dynamics. Profiles of the exact and GETD solutions are given in Figure 5.2.

For Case 2, the fractional Duffing oscillator is considered without an external forcing term ($g(t) = 0$), where the governing equation is nonlinear and the exact solution is unknown. The reference solution is computed on a fine grid with $N = 1024$. The results are summarized in Table 5.2, which highlights the convergence behavior of the GETD-1 and GETD-2 schemes. The GETD-1 scheme exhibits first-

order convergence as indicated by the convergence rate of approximately 1.03 in the table. This result aligns with the theoretical expectations in Theorem 5.3. The GETD-2 scheme demonstrates significantly improved performance, achieving a second-order convergence ($O(k^2)$) for all tested grid sizes. The observed convergence rates approach 2, confirming the robustness of the GETD-2 scheme in handling nonlinearities and fractional dynamics. The absence of an external forcing term in this case simplifies the problem but amplifies the nonlinear effects due to the cubic term $\beta u^3(t)$. Despite these challenges, the GETD-2 scheme maintains its stability and accuracy across different grid sizes. Overall, the results in Table 5.2 validate the theoretical findings of the chapter and highlight the suitability of GETD-2 for nonlinear fractional oscillatory systems, particularly in achieving accurate and efficient solutions.

Example 5.5 *Nonlinear fractional oscillation equation with smooth and nonsmooth forcing term*

Consider the following problem

$$\begin{aligned} {}^c D^\alpha u(t) + \lambda u(t) &= \rho u^2 + g(t), \quad t > 0, \\ u(0) &= c, \quad u'(0) = 0. \end{aligned} \tag{5.29}$$

To demonstrate the theoretical convergence results, we consider the following cases.

Case 1 (Smooth forcing term)

For smooth forcing term, we choose

$$g(t) = t^r.$$

Table 5.3: Example 5.5, Case 1 (smooth forcing term):: maximum absolute errors and convergence rates over uniform mesh.

N	$\alpha = 1.4$		$\alpha = 1.7$	
16	$1.76e - 03$		$1.24e - 03$	
32	$5.09e - 04$	1.7928	$3.34e - 04$	1.8992
64	$1.35e - 04$	1.9140	$8.60e - 05$	1.9595
128	$3.44e - 05$	1.9712	$2.16e - 05$	1.9897
265	$8.51e - 06$	2.0191	$5.32e - 06$	2.0239

Table 5.4: Example 5.5, Case 2 (nonsmooth forcing term): maximum absolute errors and convergence rates over uniform and graded meshes.

N	$\alpha = 1.4$				$\alpha = 1.7$			
	$\delta = 1$		$\delta = 1.5$		$\delta = 1$		$\delta = 1.5$	
16	$5.21e - 03$		$1.98e - 03$		$3.97e - 03$		$1.20e - 03$	
32	$2.00e - 03$	1.3798	$6.75e - 04$	1.5521	$1.51e - 03$	1.3927	$3.37e - 04$	1.8289
64	$7.54e - 04$	1.4096	$1.93e - 04$	1.8040	$5.58e - 04$	1.4385	$8.79e - 05$	1.9406
128	$2.72e - 04$	1.4690	$5.10e - 05$	1.9232	$1.98e - 04$	1.4923	$2.22e - 05$	1.9856
256	$9.11e - 05$	1.5791	$1.29e - 05$	1.9758	$6.56e - 05$	1.5948	$5.62e - 06$	1.9825

To test the GETD2 scheme, we choose $r = 2$ so that $g \in C^2[0, T]$. In addition, we take $\lambda = 2$, $c = 5$ and $\rho = 0.1$.

Case 2 (nonsmooth forcing term)

For nonsmooth forcing term, we choose $g(t)$ as in Case 1. To test the numerical schemes, we considered $r = 0.5$ so clearly $g(t)$ will be nonsmooth. In addition, we let $\lambda = 2$, $c = 5$ and $\rho = 0.01$.

The results in Table 5.3 validate that the GETD-2 scheme performs optimally for smooth forcing terms like $g(t) = t^r$ with $r = 2$. The regularity assumptions $(\mathcal{C}_1\text{--}\mathcal{C}_4)$ are satisfied, ensuring the solution and its derivatives are smooth. This allows the scheme to achieve optimal $O(k^2)$ -convergence on a uniform mesh without requiring additional grading. The results confirm the theoretical results and demonstrate the reliability of the GETD-2 scheme for problems with smooth dynamics.

Table 5.5: Example 5.6, maximum absolute errors and convergence rates for $\alpha = 1.5$ and $\alpha = 1.9$.

N	GETD-1				GETD-2			
	$\alpha = 1.5$		$\alpha = 1.9$		$\alpha = 1.5$		$\alpha = 1.9$	
16	8.79e-03		6.10e-03		7.24e-04		5.86e-03	
32	4.13e-03	1.0911	1.77e-03	1.7891	1.90e-04	1.9328	1.62e-03	1.8562
64	1.95e-03	1.0819	4.87e-04	1.8576	4.81e-05	1.9775	4.27e-04	1.9227
128	8.98e-04	1.1177	2.02e-04	1.2726	1.21e-05	1.9959	1.06e-04	2.0032
256	3.83e-04	1.2313	8.64e-05	1.2236	2.99e-06	2.0129	2.68e-05	1.9912

In addition, the results in Table 5.4 emphasize the importance of a graded mesh ($\delta = 1.5$) for nonsmooth forcing terms like $g(t) = t^r$ with $r = 0.5$. The weak singularity in $g(t)$ near $t = 0$ violates the regularity assumptions $(\mathcal{C}_1\text{--}\mathcal{C}_4)$, causing suboptimal convergence on a uniform mesh. By employing a graded mesh, the GETD-2 scheme mitigates the singularity and achieves an optimal $O(k^2)$ -convergence, underscoring the role of mesh grading in addressing nonsmooth dynamics effectively.

Example 5.6 *Nonlinear fractional oscillation with trigonometric forcing*

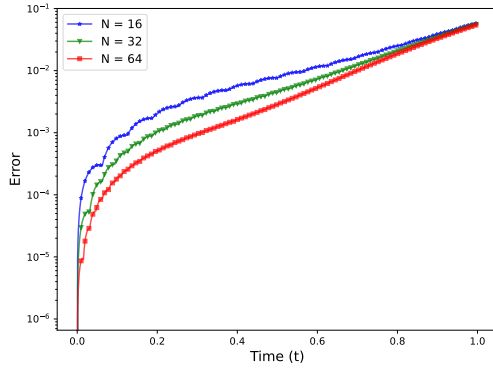
We consider the nonlinear differential equation [62],

$${}^C D^\alpha u(t) + \lambda u(t) - u^3(t) = \sin(t) \cos^2(t), \quad 0 < t \leq 1, \quad (5.30)$$

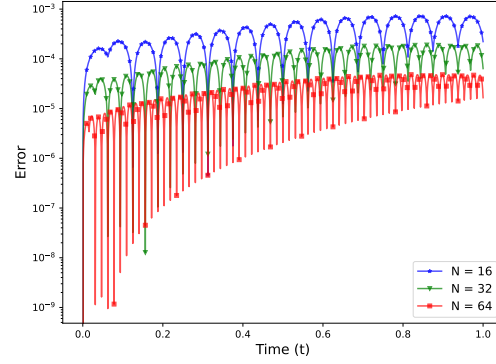
where $1 < \alpha < 2$, subject to the initial conditions

$$u(0) = 0, \quad u'(0) = 1$$

The results in Table 5.5 and Figure 5.3 demonstrate the convergence behavior of the GETD-1 and GETD-2 schemes for solving the nonlinear fractional oscillation



(a)



(b)

Figure 5.3: Example 5.6, plots of the absolute error for different N 's, $\alpha = 1.5$, GETD1 (left), GETD2 (right).

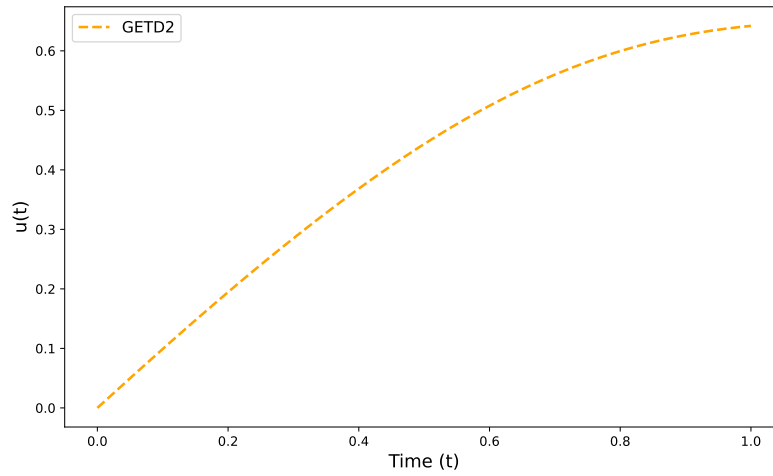


Figure 5.4: Example 5.6, plot of GETD2 solution for $N = 128$, $\alpha = 1.5$.

equation (5.32) with the given initial conditions. The forcing term $\sin(t)\cos^2(t)$ is smooth, ensuring that the regularity assumptions $(\mathcal{C}_1\text{--}\mathcal{C}_4)$ are satisfied. As expected, the GETD-2 scheme achieves an optimal $O(k^2)$ -convergence rate, as evidenced by the observed rates approaching 2 for both $\alpha = 1.5$ and $\alpha = 1.9$. In contrast, the GETD-1 scheme demonstrates first-order convergence, consistent with its design for solving problems with smooth dynamics.

The error plots in Figure 5.3 further illustrate the superior accuracy of GETD-2 compared to GETD-1. The absolute errors for GETD-2 are significantly smaller than those for GETD-1 across all tested grid sizes, highlighting the advantage of higher-order schemes for solving nonlinear fractional differential equations. Profile of the GETD-2 solution is given in Figure 5.4. These results validate the robustness of the GETD schemes, particularly GETD-2, in capturing the nonlinearities and memory effects inherent in fractional oscillatory systems.

Example 5.7 *Nonlinear fractional oscillation equation with exponential and nonsmooth forcing terms*

Consider the fractional nonlinear differential equation [79]:

$${}^c\mathcal{D}^\alpha u(t) - t^2 u e^u + \lambda u(t) = g(t), \quad t > 0, \quad (5.31)$$

where ${}^c\mathcal{D}^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$, and the initial conditions are

$$u(0) = u_0^0, \quad u'(0) = u_0^1.$$

Table 5.6: Example 5.5, Case 1 (Smooth forcing term): maximum absolute errors and convergence rates over uniform mesh.

N	$\alpha = 1.5$		$\alpha = 1.8$	
16	8.17e − 03		1.54e − 02	
32	2.35e − 03	1.7992	3.99e − 03	1.9436
64	6.26e − 04	1.9075	1.02e − 03	1.9703
128	1.60e − 04	1.9647	2.58e − 04	1.9826
256	3.98e − 05	2.0102	6.52e − 05	1.9846

Case 1 (Smooth forcing term)

$$g(t) = e^{2t} \sin(t).$$

For this case, the forcing term is smooth, and we choose the parameters:

$$\alpha = 1.8, \quad \lambda = 1.5, \quad u_0^0 = 1, \quad u_0^1 = 0.$$

Case 2 (Nonsmooth forcing term)

$$g(t) = t^{0.3}, \quad t > 0.$$

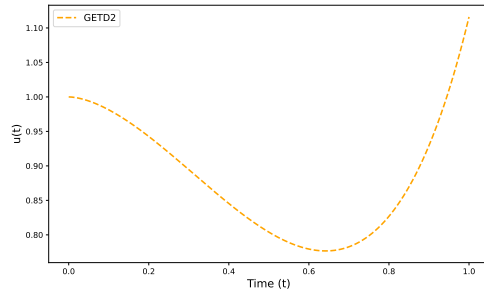
For this case, the forcing term exhibits a weak singularity near $t = 0$, characterized by the fractional power $t^{0.3}$, making it ideal for testing the performance of the schemes with graded meshes. The parameters are the same as in Case 1:

$$\alpha = 1.8, \quad \lambda = 1.5, \quad u_0^0 = 1, \quad u_0^1 = 0.$$

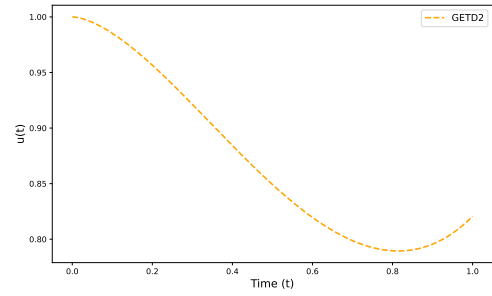
The results in Tables 5.6 and 5.7 illustrate the convergence behavior of the GETD-

Table 5.7: Example 5.6, Case 2 (Nonsmooth forcing term): maximum absolute errors and convergence rates for $\alpha = 1.5$ and $\alpha = 1.8$.

N	$\alpha = 1.5$				$\alpha = 1.8$			
	$\delta = 1$		$\delta = 2$		$\delta = 1$		$\delta = 2$	
16	1.19e-02		1.16e-02		2.66e-02		6.89e-02	
32	4.51e-03	1.3948	3.69e-03	1.6498	7.97e-03	1.7406	1.95e-02	1.8195
64	1.70e-03	1.4073	1.04e-03	1.8237	2.45e-03	1.7037	5.17e-03	1.9165
128	6.35e-04	1.4214	2.75e-04	1.9222	7.74e-04	1.6608	1.33e-03	1.9588
256	2.25e-04	1.4977	7.01e-05	1.9735	2.44e-04	1.6645	3.37e-04	1.9803



(a)



(b)

Figure 5.5: Example 5.7, plot of GETD2 solution $N = 128$, $\alpha = 1.8$, Case 1 (left), Case 2 (right).

2 scheme for solving the nonlinear fractional differential equation (5.31) with smooth and nonsmooth forcing terms. For Case 1, the smooth forcing term $g(t) = e^{2t} \sin(t)$ ensures that the solution satisfies regularity assumptions, leading to optimal $O(k^2)$ convergence rates for $\alpha = 1.5$ and $\alpha = 1.8$, as demonstrated by the observed rates approaching 2. In Case 2, the nonsmooth forcing term $g(t) = t^{0.3}$ introduces a weak singularity, challenging the numerical scheme's ability to maintain accuracy. Nonetheless, the results show that the GETD-2 scheme performs robustly, achieving consistent convergence rates even in the presence of singularities, with graded meshes effectively addressing the reduced regularity near $t = 0$.

The solution profiles in Figure 5.5 illustrate the performance of the GETD-2 scheme for both cases. The approximate solutions align well with the expected behavior, showcasing the capability of the scheme to capture the dynamics of the fractional nonlinear equation accurately.

Example 5.8 *Nonlinear fractional oscillation system with periodic forcing*

Consider the nonlinear fractional differential equation:

$${}^c D^\alpha u(t) + Au(t) = B \cos(u), \quad 0 < t \leq 1, \quad (5.32)$$

where $1 < \alpha < 2$, and the matrices A and B are given by:

$$A = \begin{bmatrix} 10 & 100 & 50 \\ 0 & 0.1 & 25 \\ 20 & 5 & 150 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

Table 5.8: Example 5.8, maximum absolute errors and convergence rates over uniform mesh.

N	$\alpha = 1.7$		$\alpha = 1.9$	
16	4.81e-01		4.54e-01	
32	1.32e-01	1.8646	1.21e-01	1.9106
64	3.14e-02	2.0712	3.09e-02	1.9673
128	7.91e-03	1.9889	7.78e-03	1.9879
256	1.97e-03	2.0020	1.96e-03	1.9893

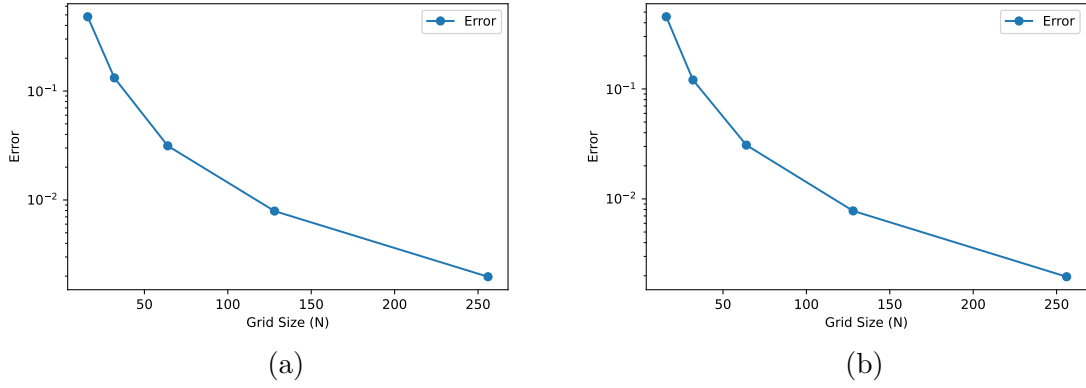


Figure 5.6: Example 5.8, plots of the maximum absolute errors for all N , $\alpha = 1.7$ (left), $\alpha = 1.9$ (right).

subject to the initial conditions

$$u(0) = [1, 1, 1]^T, \quad u'(0) = [0, 0, 0]^T.$$

In this example, we consider a nonlinear fractional oscillation system with periodic forcing, where the matrix A models the interactions between the components, and the forcing term $B \cos(u)$ drives the system with periodic oscillations. Since the exact solution is unknown, a reference solution is computed on a fine grid with $N = 1024$, and the error is analyzed by comparing solutions on coarser grids. The numerical results in Table 5.8 show that the GETD-2 scheme achieves second-order convergence

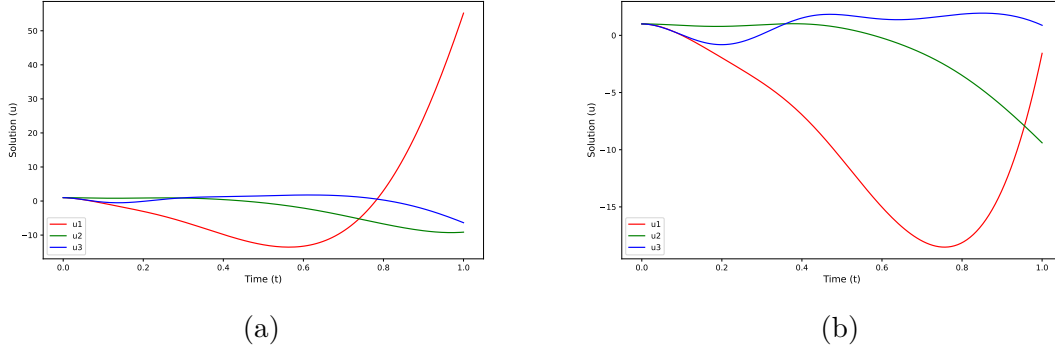


Figure 5.7: Example 5.8, plots of the approximate solutions using GETD2 for $N = 128$, $\alpha = 1.7$ (left), $\alpha = 1.9$ (right).

for both $\alpha = 1.7$ and $\alpha = 1.9$, with the maximum absolute errors for various grid sizes presented in Figure 5.6. The approximate solutions for $N = 128$ are shown in Figure 5.7. These findings validate the robustness of the GETD schemes for solving nonlinear fractional oscillation systems with periodic forcing.

5.6 Conclusion

This chapter presents Generalized Exponential Time Differencing (GETD) schemes for solving nonlinear fractional oscillation equations with Caputo derivative of order $\alpha \in (1, 2)$. The efficiency and accuracy of the GETD schemes in handling both smooth and nonsmooth forcing terms are validated through numerical experiments, including applications to the fractional Duffing oscillator model. For nonsmooth problems, graded meshes significantly improves accuracy, particularly near singularities. The results validate the theoretical convergence rates and highlight the robustness of GETD in solving complex nonlinear fractional differential equations. Future work could involve extending the GETD schemes to other types of fractional differential

equations.

CHAPTER 6

SOME APPLICATIONS TO FRACTIONAL OSCILLATION MODELS

6.1 Introduction

Fractional calculus has gained widespread acceptance for modeling systems exhibiting memory and hereditary properties, which are typical in many real-world processes. Unlike classical integerorder models, fractional models offer a more accurate description of complex phenomena, including anomalous diffusion, viscoelastic behavior, and biological oscillations. In particular, fractional oscillation models, which describe systems with oscillatory behavior subject to memory effects, have found applications in a variety of fields, including mechanical systems, biology, and material science.

This chapter focuses on some applications of fractional oscillation models, demon-

strating the efficacy of the numerical methods developed in previous chapters for solving these models. The aim is to showcase the applicability of the generalized exponential time differencing (GETD) schemes, specifically GETD1 and GETD2, in capturing the oscillatory behavior of fractional systems and providing accurate solutions with appropriate error analysis.

A first order generalized exponential time differencing scheme for fractional plasma oscillations will be developed, analyzed and tested for various examples.

6.2 Modeling Fractional Oscillations

Fractional oscillatory models are often governed by differential equations involving fractional derivatives, typically in the Caputo sense. These models describe the evolution of systems in which the current state depends not only on the present but also on the past states, encapsulating the memory effects inherent in many physical processes.

The general form of a fractional oscillation equation is:

$${}^cD^\alpha u(t) + \lambda u(t) = f(t, u(t)), \quad t > 0, \quad \alpha \in (1, 2)$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order α , $u(t)$ represents the oscillatory quantity of interest, and λ is a constant. The non-local nature of the fractional derivative implies that the behavior of the system at time t depends on all previous times, making the solution more complex compared to traditional models with integer-order derivatives.

6.3 Applications to fractional oscillation models

6.3.1 Biological Systems

Fractional models have also been successfully applied to biological systems [37], [80], [81], where they provide a more accurate representation of processes that exhibit memory, such as neuronal oscillations, circadian rhythms, and other physiological oscillations. For example, the dynamics of ion channels in neurons can be modeled using fractional oscillation equations.

A simple model of biological oscillations with fractional derivatives can be written as:

$${}^cD^\alpha u(t) + \lambda u(t) = f(t, u(t))$$

where $u(t)$ might represent the membrane potential or other state variable in a biological system, and $f(t, u(t))$ is a nonlinear function describing the interaction between different components of the system.

6.3.2 Fractional Electromagnetic Waves

Electromagnetic wave propagation in fractional media has been extensively studied for its ability to model complex wave behaviors in dielectric and metamaterial systems [33], [35], [82]. The fractional-order inductance characteristics in waveguides and plasma media can be described by fractional partial differential equations. The fractional derivative terms in these models allow for the inclusion of memory and nonlocal

effects, which are crucial in describing phenomena such as dispersion and attenuation in real materials.

The model for fractional electromagnetic waves in dielectric media is typically governed by the equation:

$${}^cD^\alpha E(t) + \lambda E(t) = f(t)$$

where ${}^cD^\alpha$ represents the fractional derivative, $E(t)$ is the electric field, and λ is the damping factor. For this application, we compare the GETD methods with the fractional ABM method, which is commonly used in solving such fractional wave equations.

6.3.3 Rogue Waves in the Nonlinear Schrödinger Equation

The study of rogue waves in nonlinear systems, especially in the context of the nonlinear Schrödinger equation (NLSE) [83], [84], has attracted significant attention due to their relevance in oceanography, optics, and plasma physics. The NLSE with fractional derivatives provides a natural framework for modeling the appearance of rogue waves and their dynamics under fractional time evolution.

The fractional NLSE can be written as:

$${}^cD^\alpha \psi(t) + i\lambda \nabla^2 \psi(t) + \mu |\psi(t)|^2 \psi(t) = 0$$

where ${}^cD^\alpha$ is the fractional derivative, $\psi(t)$ is the wave function, and λ, μ are parameters that control dispersion and nonlinearity.

6.3.4 Fractional Tajimi-Kanai Model for Earthquake Ground Motion

The Tajimi-Kanai model is widely used to simulate earthquake ground motion [85], incorporating both elastic and inelastic effects in the structural response. By introducing fractional derivatives, we can capture memory effects that better represent real seismic activity, which may exhibit nonlocal and history-dependent characteristics.

The fractional version of the Tajimi-Kanai model is given by:

$${}^cD^\alpha u(t) + \lambda u(t) = f(t)$$

where $u(t)$ is the ground motion displacement, and $f(t)$ is the external force driving the system.

6.3.5 Viscoelastic Fractional Models

Fractional calculus has been widely applied in viscoelasticity to model materials that exhibit both viscous and elastic behaviors [81], [86], [87], [88]. In particular, fractional viscoelastic models describe materials with memory effects, such as polymers and biological tissues. These models often use fractional derivatives to capture the time-dependent behavior of the materials under stress.

The general form of the fractional viscoelastic equation is:

$${}^cD^\alpha \sigma(t) + \lambda \sigma(t) = \varepsilon(t)$$

where $\sigma(t)$ is the stress, $\varepsilon(t)$ is the strain, and λ is the relaxation parameter.

In the next section, we consider an application of fractional calculus to plasma oscillations, discussed in detail. A first-order Generalized Exponential Time Differencing (GETD) scheme will be developed, analyzed, and tested on various examples to illustrate its effectiveness in modeling fractional plasma oscillations, which are governed by memory effects and complex dynamics.

6.4 Exponential Time Differencing Scheme for Fractional Plasma Oscillations

Plasma represents one of the four basic states of matter, along with the solid, liquid, and gas states. Plasma oscillations represent a fundamental concept in plasma physics, and their understanding is crucial in various applications. Fractional order Lorentz models for plasma oscillations offer a more flexible framework for describing the complex dynamics and memory effects that are often observed in plasma systems. In this section, we consider a fractional-order model for plasma oscillations. The model involves Caputo fractional derivative of order between one and two. An exponential time differencing method is developed to approximate the motion of electrons driven by zero, static, and oscillating electric fields.

6.4.1 Overview

Plasma is composed of charged molecules (ions) and freely moving electrons, forming an ionized gas. A plasma oscillation, also known as a Langmuir wave or electron

plasma wave, is a collective oscillation or wave-like behavior of charged particles, primarily electrons, in a plasma. This phenomenon is a fundamental concept in plasma physics and is characterized by the collective motion of charged particles within the plasma, [89].

Plasma oscillations are important in various areas of plasma physics and astrophysics. They can be excited by external sources, such as radio waves, and are relevant in plasma diagnostics, particle acceleration, and the study of space plasmas. Understanding plasma oscillations is crucial in fields like controlled nuclear fusion, where plasmas are used to contain and heat the fusion fuel, and in the study of astrophysical phenomena involving plasmas, like the behavior of stars and the interstellar medium.

Determining the displacement of electrons in plasmas has broad implications in science and technology. It aids in our understanding of natural and controlled plasmas, facilitates the development of clean energy sources, supports space exploration, and contributes to advancements in various fields.

The fractional oscillator equation with fractional time derivatives of Riemann-Liouville type is discussed in [38]. In [39], the linearly damped oscillator equation with a Caputo fractional derivative damping term is considered. Based on the stability switch, the stability of linear fractional oscillator with Caputo derivative is investigated in [90]. In [91], a linear fractional damped oscillator equation with Caputo derivative is solved using Elzaki transformed method. Novel analytical solutions of the mass-spring-damper systems described using Liouville–Caputo left generalized fractional derivative are presented in [40]. Readers might turn to [10], [11], [44], [13], [42], [41], [33], [92], [43] for thorough description of fractional oscillator systems and their

applications.

Exact solutions for fractional models are in general unavailable or impractical to compute. Accordingly, it is essential to develop effective and reliable numerical methods for approximate solutions of these models.

Exponential time differencing is a technique for developing numerical methods for ordinary differential equations or for time integration of partial differential equations. This technique relies on the exact solution of the homogeneous part combined with an appropriate approximation of the integral of the nonhomogeneous part. This leads to explicit methods that are inherently stable for systems with fast decaying or oscillating modes (stiff systems), with no time step size constraints. In general, stiffness may have features of both rapid decay and rapid oscillation, [55].

In this section, we consider a general form of the fractional oscillation equation; that could represent fractional plasma oscillations; with time-dependent external force. Using Laplace transform, we obtain an exact closed-form of the solution in terms of Mittag-Leffler function (MLF) and a convolution integral. Then the solution is approximated using exponential time differencing. For efficient implementation, generalized rational approximations for the MLF constructed in [27], [16], [17] are used. This yields an efficient first order numerical scheme.

The section is structured as follows. In Section 6.4.2, the fractional plasma oscillations model is introduced. In Section 6.4.3, the construction and analysis of the numerical scheme are presented. Various numerical experiments are demonstrated in Section 6.4.5. Concluding remarks are included in Section 6.4.6.

6.4.2 Plasma oscillation model

In the classical Lorentz oscillator model, the electron is modeled as a driven damped harmonic oscillator. The equation of motion takes the form

$$m\ddot{y} + \gamma\dot{y} + ky = qE(t), \quad (6.1)$$

where y is the displacement of the electron from its equilibrium position, m is the electron mass, $\ddot{y}$ is the acceleration, $\gamma\dot{y}$ is the damping force, ky is the restoring force (binding electron and nucleus), and E is the electric field force.

Fractional order Lorentz models for plasma oscillations introduce a new dimension to our understanding of plasma dynamics. They offer a more flexible framework for describing the complex dynamics and memory effects that are often observed in plasma systems.

In [33], the following model is proposed,

$$\frac{1}{\sigma^{2-\alpha}} {}^c D^\alpha y + \frac{4\pi n e^2}{m_e} y = E(t), \quad 1 < \alpha \leq 2, \quad (6.2)$$

where ${}^c D^\alpha$ is the Caputo fractional derivative defined by

$${}^c D^\alpha u(t) := \begin{cases} \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \ddot{u}(\tau) d\tau, & 1 < \alpha < 2, \\ \ddot{u}(t), & \alpha = 2. \end{cases}$$

The parameter n is the electron number density, e is the elementary charge, m_e is the electron mass, and σ is an auxiliary parameter with dimension of time introduced to preserve the dimensionality of the fractional differential equation.

6.4.3 Numerical scheme

Based on (6.2), we consider the following initial value problem

$$\begin{aligned} {}^c D^\alpha u(t) + \omega u(t) &= f(t), \quad t > 0, \quad 1 < \alpha < 2, \\ u(0) &= u_0, \quad u'(0) = u_1, \end{aligned} \tag{6.3}$$

where ω is a positive constant which is relevant to the physical parameters such as the external oscillation frequency. The function f is the external force, u is the solution (displacement) and u_0, u_1 are the given initial data.

Next, we present the derivation of a numerical scheme to solve (6.3) and investigate its stability and convergence rates.

Derivation of the scheme

Construction of the exponential time differencing scheme relies on obtaining the mild solution in a form in which the homogeneous part is integrated exactly. Using Laplace transform and its inversion, we obtain

$$\begin{aligned} u(t) &= u_0 E_\alpha(-\omega t^\alpha) + u_1 t E_{\alpha,2}(-\omega t^\alpha) \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-\tau)^\alpha) f(\tau) d\tau, \end{aligned} \tag{6.4}$$

where $E_{\alpha,\beta}$ is the two-parameter MLF, [8],

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{C}, \quad z \in \mathbb{C},$$

and $E_{\alpha} := E_{\alpha,1}$.

To approximate (6.4) on some interval $[0, T]$, we consider the mesh points $0 = t_0 < t_1 < \dots < t_N = T$. Let $I_j = [t_j, t_{j+1}]$, $j = 0, 1, \dots, N-1$, $k_j = t_{j+1} - t_j$, and $k = \max_j k_j$. We approximate the function f with a piece-wise constant polynomial with $f|_{I_j} \approx f(t_{j+1})$.

Setting $u^0 = u(0)$, $f(t_{j+1}) = f^{j+1}$ and $u^n \approx u(t_n)$, $1 \leq n \leq N$, the numerical scheme for (6.3) is defined by

$$\begin{aligned} u^n &= u^0 E_{\alpha}(-\omega t_n^{\alpha}) + u_1 t_n E_{\alpha,2}(-\omega t_n^{\alpha}) + \\ &\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t_n - \tau)^{\alpha}) f^{j+1} d\tau. \end{aligned} \tag{6.5}$$

Integrating (6.5), the numerical scheme amounts to

$$\begin{aligned} u^n &= u^0 E_{\alpha}(-\omega t_n^{\alpha}) + u_1 t_n E_{\alpha,2}(-\omega t_n^{\alpha}) \\ &+ \mathcal{E}_n^{\alpha,\alpha}(0; \omega) f^1 + \mathcal{G}_n, \quad n = 1, \dots, N, \end{aligned} \tag{6.6}$$

where

$$\mathcal{E}_n^{\alpha,\beta}(\tau; \omega) = (t_n - \tau)^{\beta} E_{\alpha,\beta+1}(-\omega(t_n - \tau)^{\alpha}),$$

and

$$\mathcal{G}_n = \sum_{j=1}^{n-1} \mathcal{E}_n^{\alpha, \alpha}(t_j; \omega) (f^{j+1} - f^j). \quad (6.7)$$

6.4.4 Analysis of the accuracy and stability

In this section, we investigate the stability and convergence rates of the numerical scheme (6.6).

We begin by the following theorem for later use.

Theorem 6.1 ([4]) *If $\alpha < 2$, β is an arbitrary real number, and μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$. Then there is $M > 0$ such that*

$$|E_{\alpha, \beta}(z)| \leq \frac{M}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi. \quad (6.8)$$

Next, we demonstrate the following stability result.

Theorem 6.2 *Suppose that $f \in C[0, T]$. Let u^n be the numerical solution of (6.3) as specified by (6.5). Then, there is a generic positive constant C such that*

$$|u^n| \leq C \left[|u^0| + |u_1|t_n + \frac{t_n^\alpha}{\alpha} \max_{0 \leq t \leq t_n} |f(t)| \right], \quad 1 \leq n \leq N.$$

Proof. clearly we have

$$|f^{j+1}| \leq \max_{0 \leq t \leq t_n} |f(t)|, \quad j = 0, \dots, n-1. \quad (6.9)$$

Inserting (6.9) into (6.5) yields

$$\begin{aligned} |u^n| &\leq |u^0| |E_\alpha(-\omega t_n^\alpha)| + |u_1| t_n |E_{\alpha,2}(-\omega t_n^\alpha)| \\ &+ \max_{0 \leq t \leq t_n} |f(t)| \int_0^{t_n} (t_n - \tau)^{\alpha-1} |E_{\alpha,\alpha}(-\omega(t_n - \tau)^\alpha)| d\tau. \end{aligned}$$

Then, it follows from Theorem 6.1 that there is $C > 0$ such that

$$|u^n| \leq C \left[|u^0| + |u_1| t_n + \max_{0 \leq t \leq t_n} |f(t)| \int_0^{t_n} (t_n - \tau)^{\alpha-1} d\tau \right].$$

The proof is completed upon evaluating the integral. ■

For a source term with sufficient regularity, the following convergence result is established.

Theorem 6.3 *Let u denote the solution of (6.3) with a source term $f \in \mathcal{C}^1[0, T]$. Let u^n denote the numerical solution specified by (6.5). Then*

$$|u^n - u(t_n)| \leq \frac{C k t_n^\alpha}{\alpha} \max_{0 \leq t \leq t_n} |f'(t)|, \quad n = 1, 2, \dots, N.$$

Proof. From the definition of the exact and numerical solution, it follows that

$$\begin{aligned} e^n &:= u^n - u(t_n) \\ &= \sum_{j=0}^{n-1} \left[\int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t_n - \tau)^\alpha) (f(t_{j+1}) - f(\tau)) d\tau \right]. \end{aligned} \tag{6.10}$$

Using the known Lagrange interpolation error formula, we have

$$|f(t_{j+1}) - f(\tau)| \leq k \max_{t \in I_j} |f'(t)|, \quad \text{for } \tau \in I_j. \quad (6.11)$$

Plugging this into (4.19) and applying Theorem 6.1, yields

$$|e^n| \leq Ck \max_{0 \leq t \leq t_n} |f'(t)| \int_0^{t_n} (t_n - \tau)^{\alpha-1} d\tau.$$

Evaluating the integral completes the proof. ■

6.4.5 Numerical experiments

In this subsection we illustrate the accuracy and efficiency of the proposed scheme. For numerical illustration of convergence properties, we compute the maximum absolute error as

$$e(N) := \max_{1 \leq n \leq N} |u(t_n) - u^n|,$$

if the exact solution $u(t)$ is known. Otherwise, we compute the error based on successive approximation

$$e(N) := \max_{1 \leq n \leq N} |u_*^{2n} - u^n|,$$

where u_*^n is an implementation of the numerical scheme using two-times the number of mesh points used in the implementation for u^n . In both cases, the order of convergence p is computed as

$$p = \log_2[e(N)/e(2N)].$$

To compute the Mittag-Leffler functions that appear in the numerical scheme (6.6), we use rational approximants with good tracking capabilities of the roots and oscillations of the functions for $\alpha \in (1, 2)$. Such approximants presented in [27] for $\beta = 1$ have been further investigated for $\beta \in \mathbb{R}$ in [56].

Example 6.4 (*Smooth forcing term*)

We consider problem (6.3) with a sufficiently smooth forcing term f . We take $f(t) = t$ to test our numerical scheme and verify the convergence and stability results given in Theorems 6.2 and 6.3. Our model problem then takes the form

$$\begin{aligned} {}^cD^\alpha u(t) + \omega u(t) &= t, \quad t > 0, \\ u(0) &= u_0, \quad u'(0) = u_1. \end{aligned} \tag{6.12}$$

The exact solution for this problem is

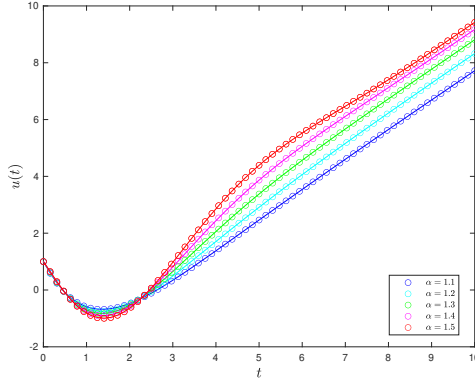
$$\begin{aligned} u(t) &= u_0 E_\alpha(-\omega t^\alpha) + u_1 t E_{\alpha,2}(-\omega t^\alpha) \\ &\quad + t^{\alpha+1} E_{\alpha,\alpha+2}(-\omega t^\alpha). \end{aligned} \tag{6.13}$$

For numerical implementation, we take $\omega = 1$, $u_0 = 1$, $u_1 = -2$ and $t \in [0, 10]$. As shown in Table 6.1, the expected order $O(k)$ is achieved by the numerical scheme, in conformance with our theoretical result. Further, plots of the approximate solution for different values of α are presented in Figure 6.1 compared to the exact solution which is obtained using the “ml” Matlab function by [15].

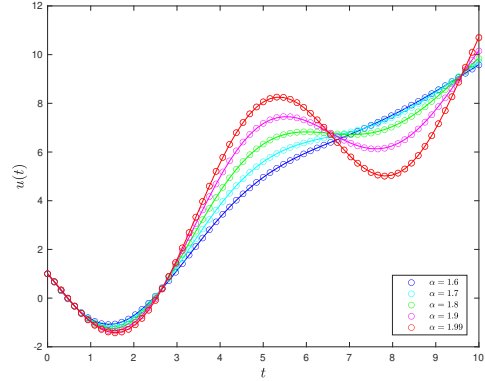
Fractional oscillator contains simultaneously the oscillatory motion and the relaxation Figure 6.1.

Table 6.1: example 6.4: maximum absolute error and convergence rate.

N	$\alpha = 1.2$		$\alpha = 1.9$	
8	3.96e-02		2.97e-02	
16	1.98e-02	0.9986	1.52e-02	0.9738
32	9.93e-03	0.9986	7.64e-03	0.9871
64	4.97e-03	0.9990	3.84e-03	0.9936
128	2.48e-03	0.9993	1.92e-03	0.9968



(a) $\alpha = 1.1, 1.2, 1.3, 1.4$ and 1.5



(b) $\alpha = 1.6, 1.7, 1.8, 1.9$ and 1.99

Figure 6.1: Example 6.4, plots of the exact and approximate solution for different values of α using $N = 64$. The solid lines represent the exact solutions.

Example 6.5 (*Zero and static electric fields*)

Let $f(t) = 0$ and $u_1 = 0$ in (6.3). Then the exact solution is given by

$$u(t) = u_0 E_\alpha(-\omega t^\alpha). \quad (6.14)$$

In particular, when $\alpha = 2$, the solution is the periodic cosine function:

$$u(t) = u_0 \cos(\sqrt{\omega}t).$$

In Figure 6.2, profiles of the approximated solution for different values of α are presented, along with the exact solution when $\alpha = 2$.

As established by [11], [44], [41], the motion of the fractional oscillator reflects the characteristics exhibited by a damped harmonic oscillator. This is clearly evident in Figure 6.2. The damping inherent in the fractional oscillator is an integral aspect of its equation of motion, in contrast to the ordinary damped harmonic oscillator, where damping is introduced through an external force. Notably, the damping effects are linked to the order of the fractional derivative. While when $\alpha = 2$, the oscillator is conservative (the total energy is conserved), the fractional oscillator is dissipative.

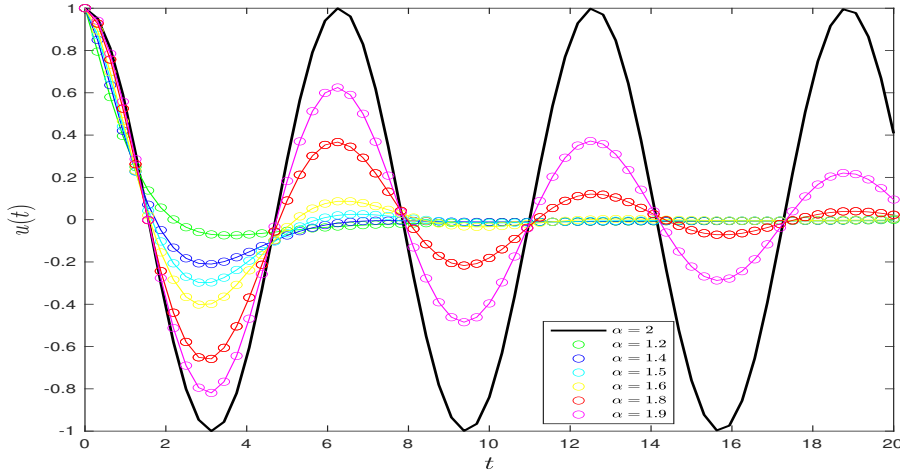


Figure 6.2: Example 6.5, zero-electric field, plots of the exact and approximate solutions for different values of α along with the exact solution for $\alpha = 2$. The solid lines represent the exact solutions, $N = 128$.

Now we consider fractional plasma oscillations driven by a static electric field. We take $f(t) = 1, u_0 = 1, u_1 = -1$ in (6.3), then the exact solution is

$$u(t) = E_{\alpha}(-\omega t^{\alpha}) - tE_{\alpha,2}(-\omega t^{\alpha}) + t^{\alpha}E_{\alpha,\alpha+1}(-\omega t^{\alpha}).$$

The profiles of the approximated solution for different values of α are presented in Figure 6.3. As can be observed, the electron's motion exhibits the characteristics of a

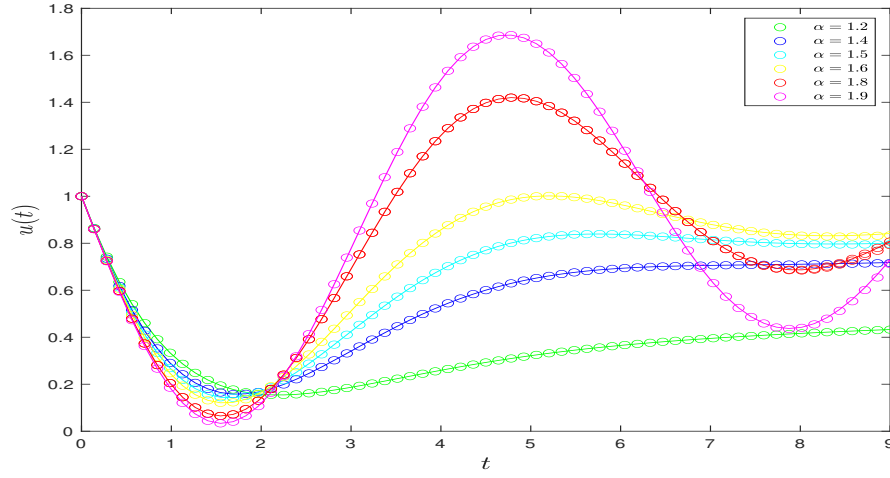


Figure 6.3: Example 6.5, case 2, plots of the exact and approximate solution for different values of α . The solid lines represent the exact solutions, $N = 128$.

damped harmonic oscillator.

In this example the approximate solution is computed using the generalized rational approximants introduced in [27] and the exact solution is obtained using the “ml” Matlab function by [15].

Example 6.6 (*Oscillating electric field*)

We consider a periodic forcing function with amplitude ϕ and frequency ψ , i.e, $f(t) = \phi \sin(\psi t)$. Then problem (6.3) takes the form

$$\begin{aligned} {}^c D^\alpha u(t) + \omega u(t) &= \phi \sin(\psi t), \quad t > 0, \\ u(0) &= u_0, \quad u'(0) = u_1. \end{aligned} \tag{6.15}$$

Table 6.2: example 6.6: maximum absolute error and convergence rate.

N	$\alpha = 1.2$		$\alpha = 1.9$	
8	1.69e-02		1.35e-02	
16	8.39e-03	1.0136	6.88e-03	0.9716
32	4.17e-03	1.0064	3.47e-03	0.9865
64	2.08e-03	1.0029	1.74e-03	0.9935
128	1.04e-03	1.0014	8.74e-04	0.9967

In this case, the exact solution is

$$\begin{aligned}
 u(t) = & u_0 E_\alpha(-\omega t^\alpha) + u_1 t E_{\alpha,2}(-\omega t^\alpha) \\
 & + \phi \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-\tau)^\alpha) \sin(\psi\tau) d\tau.
 \end{aligned} \tag{6.16}$$

Thus, the solution involves a convolution which may not be easily obtained explicitly (in closed form).

For numerical implementation, we take $\omega = 1, u_0 = 1, u_1 = 0, \phi = \psi = 1$ and $t \in [0, 20]$. The source term here is sufficiently smooth and as shown in Table 6.2, the rate of convergence is $O(k)$.

Plots of the approximate solution for different values of α are presented in Figure 6.4 along with the exact solution

$$u(t) = \frac{1}{2}[\sin(t) + (2-t)\cos(t)]$$

when $\alpha = 2$. It is seen that as $\alpha \rightarrow 2$, the profiles of the approximate solutions approach that of $\alpha = 2$.

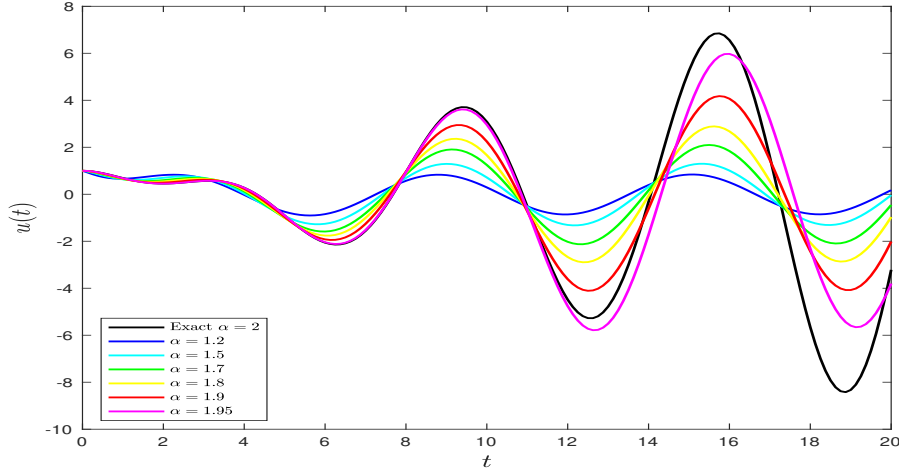


Figure 6.4: Example 6.6, plots of the approximate solution for different values of α compared to the exact solution for $\alpha = 2$, $N = 128$.

6.4.6 Concluding remarks

In this section, the fractional plasma oscillation and its applications are discussed. Moreover, a numerical method based on exponential time differencing technique has been studied for solving fractional plasma oscillation equations. The stability and convergence of the numerical solution are proved theoretically and verified numerically. Efficient implementation of the proposed scheme is carried-out using rational approximations to compute the oscillatory Mittag-Leffler function.

6.5 Conclusion

In this chapter, we have explored the application of fractional oscillation models to a wide range of problems, including electromagnetic wave propagation, rogue waves, earthquake simulation, viscoelasticity, and the fractional Duffing oscillator. The GETD methods have proven to be highly effective in solving fractional plasma os-

cillation models, capturing the oscillatory behavior and handling the complexities introduced by fractional derivatives.

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VITAE

Aljowhara Hasan A. Honain

Email: *aljowhara.honain@gmail.com*

Phone: +966 55 868 9466

Address: Alagrabiya District, Khobar, Saudi Arabia, Zip Code: 34441

Nationality: Saudi

EDUCATION

- Ph.D. in Mathematics

King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia

2024

- M.Sc. in Mathematics

Middle Tennessee State University, Murfreesboro, TN, USA

2016

- B.Sc. in Mathematics

King Abdul-Aziz University, Jeddah, Saudi Arabia

2010

RESEARCH PAPERS

1. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2024). Generalized Exponential Time Differencing for Fractional Oscillation Models. *Journal of Computational and Applied Mathematics*, ISSN 0377-0427.
2. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2024). Rational approximations for the oscillatory two-parameter Mittag-Leffler function. *Fractal and Fractional*, 8(6), 319.
3. Honain, A. H., & Furati, K. M. (2024). Exponential Time Differencing Scheme for Fractional Plasma Oscillations. *IFAC-PapersOnLine*, 58(12), 169-173.
4. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2023). Rational approximation for oscillatory Mittag-Leffler function. In *2023 International Conference on Fractional Differentiation and Its Applications (ICFDA)*, pp. 1–5. IEEE.
5. Honain, A. H., Furati, K. M., Sarumi, I. O., & Khaliq, A. Q. M. (2024). Generalized Exponential Time Differencing Scheme for Non-linear Fractional Oscillation Models. *Prepared for submission*.

TEACHING ACTIVITIES

- Recitation Instructor, King Fahd University of Petroleum and Minerals (KFUPM)

June 2023 - August 2023

- Teaching Assistant, KFUPM
January 2023 - May 2023
- Math Lecturer, Motlow State Community College, Smyrna, TN, USA
January 2019 - December 2019
- Math Teacher, ICM Academy, Murfreesboro, TN, USA
January 2018 - June 2019
- Math and ESL Teacher, Read to Succeed Organization of Murfreesboro, TN, USA
2017 - 2018

PRESENTATIONS

- “Rational Approximation for Oscillatory Mittag-Leffler Function”
March 14-16, 2023, International Conference on Fractional Differentiation and Its Applications (ICFDA’23), Ajman, UAE.
- “Exponential Time Differencing Scheme for Fractional Plasma Oscillations”
July 09-12, 2024, International Conference on Fractional Differentiation and Its Applications (ICFDA’24), Bordeaux, France.
- “Numerical Approximations of Fractional Derivatives”
December 12, 2021, Female Research Day, KFUPM Institute for Knowledge Exchange.

WORKSHOPS AND CONFERENCES ATTENDED

- International Conference on Fractional Differentiation and Its Applications
(ICFDA'24)
July 09-12, 2024, Bordeaux, France
- WebAssign Webinar Series
July 11-13, 2023
- International Conference on Fractional Differentiation and Its Applications
(ICFDA'23)
March 14-16, 2023, Ajman University, UAE
- The 6th Annual MTSU Literacy Research Conference
2018, Murfreesboro, TN, USA

AWARDS

- Grünwald-Letnikov Award: Best Student Paper (Theory)
First Place for “Rational Approximation for Oscillatory Mittag-Leffler Function”
March 2023, ICFDA'23