

NUMERICAL SIMULATION OF CLASSICAL AND FRACTIONAL EPIDEMIC MODELS

BY

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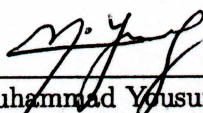
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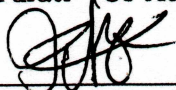
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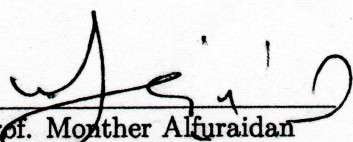

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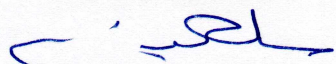

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*I dedicate this thesis to my lovely family (my mother, father, husband,
and children), whose unwavering faith in my aspirations has inspired
my passion for success.*

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DISSERTATION ABSTRACT

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Epidemic models are tools used to understand and predict the spread of infectious diseases. These models consist of systems of nonlinear differential equations. Most of the fractional models cannot be solved analytically; therefore, efficient numerical methods are crucial to solving such problems. Recently, there has been a growing interest in devising effective and reliable numerical approaches to solve time and/or space fractional models.

We shall use fourth-order Runge-Kutta methods for numerical simulation of some classical epidemic models. For the fractional epidemic models, we have developed a class of predictor-corrector type numerical methods. We applied these methods to fractional epidemic models with Caputo fractional derivative and Atangana-Baleanu-Caputo (ABC) fractional derivative.

For the diffusive epidemic models, we have developed a third-order L -stable exponential time differencing scheme using restricted Padé approximation of the matrix exponential functions. We employ a Fourier spectral approach which results in a diagonal representation of the fractional Laplacian. We also use sophisticated mathematical techniques, namely, discrete sine and cosine transforms, to improve the computational efficiency of the methods. An algorithm is devised for straightforward implementation of the method. This algorithm can easily be implemented to space fractional reaction-diffusion systems with homogeneous Dirichlet as well as homogeneous Neumann boundary conditions. The third-order accuracy of the method is proved analytically and demonstrated numerically using a test problem. Linear error analysis of these method is presented, stability regions of the method are computed, and their graphs are plotted.

For numerical simulation of the time-space fractional epidemic models, a computationally efficient and highly accurate predictor-corrector numerical scheme is developed. The fractional Laplacian is discretized using the matrix transfer technique based on a fourth-order compact finite difference method. The predictor-corrector time stepping scheme developed uses the fractional Euler method as a predictor and the fractional Adams method as a corrector. We apply the well-known Richardson extrapolation to further enhance the accuracy of the scheme. Utilizing the discrete Fourier transform, a computationally efficient algorithm is developed for straightforward implementation of the scheme to solve time-space fractional reaction-diffusion systems. The dynamics of the solutions due to the memory effect and/or anomalous diffusion is demonstrated through numerical experiments.

The findings of this thesis shall contribute to better understanding the dynamics of

epidemic diseases and provide some valuable tools for epidemiologists and for people in the public health sectors. Moreover, this study contributes to the development of the field of computational epidemiology by providing methods for better prediction and management of the disease outbreak.

ملخص الرسالة

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النماذج الوبائية هي أدوات تُستخدم لفهم الأمراض المعدية والتنبؤ بانتشارها. تتكون هذه النماذج من أنظمة من المعادلات التفاضلية غير الخطية. ومعظم النماذج الكسرية لا يمكن حلها تحليلياً؛ ولذلك، تُعد الطرق العددية الفعالة ضرورية لحل مثل هذه المشاكل. وفي الآونة الأخيرة، كان هناك اهتمام متزايد بابتكار مناهج عددية فعالة وموثوقة لحل النماذج الكسرية الزمانية و/أو المكانية. سنستخدم طرق رانج-كوتا من الدرجة الرابعة للمحاكاة العددية لبعض النماذج الوبائية الكلاسيكية. بالنسبة للنماذج الوبائية الكسرية، قمنا بتطوير فئة من الطرق العددية من نوع مصحح التنبؤ. وطبقنا هذه الطرق على نماذج الأوبئة الكسرية ذات المشتق الكسري الكابوتو والمشتق الكسري أتانجانا-باليانو-كابوتو (ABC). بالنسبة للنماذج الوبائية الانتشارية، قمنا بتطوير مخطط التفاضل الزمني الأسّي المستقر من الدرجة الثالثة باستخدام تقريب Padé المقيد للدوال الأسية المصفوفية. نستخدم نهج فورييه الطيفي الذي ينتج عنه تمثيل قطري لدوال لابلاسيان الكسرية. نستخدم أيضاً تقنيات رياضية متطورة، وهي تحويلات الجيب وجيب التمام المتقطعة لتحسين الكفاءة الحسابية للطرق. تم ابتكار خوارزمية للتنفيذ المباشر للطريقة. يمكن تطبيق هذه الخوارزمية بسهولة على أنظمة الانتشار التفاعلي الجزئي الفضائي مع شروط حدود ديريشليت المتجانسة وكذلك شروط حدود نيومان المتجانسة. تم إثبات دقة الدرجة الثالثة من دقة الطريقة تحليلياً وتوضيحها عددياً باستخدام مشكلة اختبار. يتم عرض تحليل الخطأ الخطي لهذه الطريقة، ويتم حساب مناطق استقرار الطريقة، ويتم رسم الرسوم البيانية الخاصة بها. من أجل المحاكاة العددية لنماذج الأوبئة الكسرية الزمانية والمكانية للوباء تم تطوير مخطط عددي مصحح تنبؤي عددي عالي الكفاءة الحسابية والدقة. يتم تقسيم اللابلاسيان الكسري باستخدام تقنية نقل المصفوفة استناداً إلى طريقة فرق محدود مضغوط من الدرجة الرابعة. يستخدم

مخطط التنبؤ-المصحح الزمني الذي تم تطويره طريقة أويلر الكسرية كمتنبئ وطريقة آدمز الكسرية كمصحح. نطبق استقراء ريتشاردسون المعروف لتعزيز دقة المخطط. بالاستفادة من تحويل فورييه المتقطع، تم تطوير خوارزمية فعالة حسابيًا للتنفيذ المباشر للمخطط لحل أنظمة التفاعل والانتشار الجزئي الزمني والمكاني. يتم توضيح ديناميكيات الحلول الناتجة عن تأثير الذاكرة و/أو الانتشار الشاذ من خلال التجارب العددية. ستسهم نتائج هذه الأطروحة في فهم ديناميكيات الأمراض الوبائية بشكل أفضل، وتوفير بعض الأدوات القيمة لعلماء الأوبئة وللعاملين في قطاعات الصحة العامة. وعلاوة على ذلك، تساهم هذه الدراسة في تطوير مجال علم الأوبئة الحسابي من خلال توفير طرق للتنبؤ بتفشي الأمراض وإدارتها بشكل أفضل.

CHAPTER 1

INTRODUCTION

1.1 Background of the Models

Epidemiological mathematical models continue to play pivotal role in understanding, capturing, and predicting the spread of infectious diseases. These models use several mathematical techniques to examine parameters such as rates of infection, periods of recuperation, and population interactions, allowing academics and public health officials to predict the spread of epidemics and develop effective remedies. Epidemic models leverage simulations and mathematical equations to provide vital insights that influence decision making during epidemics and eventually assist in the management and prevention of transmissible diseases. Numerous studies have been conducted to investigate the dynamics of COVID-19 transmission since the virus began to pose a hazard to human health on a worldwide scale. The two basic types of epidemic models are as follows:

- *SIR* (Susceptible-Infected-Recovered) model: In this model the individuals in

a population are divided into three states: susceptible (S), infected (I), and recovered (R). This model describes the disease's transitions between these states as well as how it spreads throughout the population.

- *SEIR* (Susceptible-Exposed-Infected-Recovered) model: An extra compartment of "Exposed" is added to the *SIR* model.

The *SIR* model is often used to analyze infectious illness transmission. This model consists of three non-linear ordinary differential equations (ODEs), expressed as follows:

$$\begin{aligned}\frac{dS(t)}{dt} &= -\beta \frac{S(t)I(t)}{N}, \\ \frac{dI(t)}{dt} &= \beta \frac{S(t)I(t)}{N} - \gamma I(t), \\ \frac{dR(t)}{dt} &= \gamma I(t),\end{aligned}\tag{1.1}$$

where N is the total population size divided into susceptible, infected, and recovered individuals during time (t), that is, $N = S(t) + I(t) + R(t)$. In this basic *SIR* model, a susceptible (S) individual acquires an illness when he/she becomes infected (I) and subsequently recovers (R) to become immune. This *SIR* model has two parameters: the rate of infection for susceptible hosts β and the rate of recovery for infected people γ . Researchers have modified and improved this *SIR* model in order to come up with a more realistic framework for comprehending the dynamics of disease transmission. In particular, by considering the latent period to obtain a more accurate representation of transmission for infectious diseases. During the latent period (E), infections are

present but do not cause symptoms; individuals may pass on the disease to others through direct or indirect interactions. As shown by Chen et al. [23], the system of non-linear ODE's for the basic *SEIR* model is illustrated as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \frac{-\beta S(t)I(t)}{N}, \\
\frac{dE(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - \sigma E(t), \\
\frac{dI(t)}{dt} &= \sigma E(t) - \gamma I(t), \\
\frac{dR(t)}{dt} &= \gamma I(t),
\end{aligned} \tag{1.2}$$

where σ represents the removal of susceptible individuals from an exposed class.

Furthermore, researchers incorporated birth and mortality rates into the *SEIR* model to obtain more realistic results to real-world circumstances. Jose [19] suggested an *SEIR* model by considering the birth rate Λ , the natural mortality rate μ and the average virus-related fatality rate ϵ . The enhanced *SEIR* model is represented as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \frac{\beta S(t)I(t)}{N} - \mu S(t) \\
\frac{dE(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - (\sigma + \mu)E(t) \\
\frac{dI(t)}{dt} &= \sigma E(t) - (\gamma + \mu + \epsilon)I(t) \\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t)
\end{aligned} \tag{1.3}$$

The classical *SEIR* model is further improved by the addition of diffusion terms, which account for a description of disease transmission over diverse spatial regions.

This expansion takes into consideration population movement, allowing for a better understanding of how illnesses can spread across geographical regions. This inclusion results in a system of non-linear reaction-diffusion equations that describe the spatial as well as temporal evolution of the disease. Al-Showaikh and Twizel in [5] considered the following diffusive epidemic model:

$$\begin{aligned}
\frac{\partial S}{\partial t} &= \mu N - (\mu + \beta I) S + \alpha \frac{\partial^2 S}{\partial x^2}, \\
\frac{\partial E}{\partial t} &= \beta I S - (\mu + \sigma) E + \alpha \frac{\partial^2 E}{\partial x^2}, \\
\frac{\partial I}{\partial t} &= \sigma E - (\mu + \gamma) I + \alpha \frac{\partial^2 I}{\partial x^2},
\end{aligned} \tag{1.4}$$

where α is the diffusion rate. The model is used with the following initial conditions,

$$S(x, 0) = S_0(x), \quad E(x, 0) = E_0(x), \quad I(x, 0) = I_0(x),$$

and the boundary conditions,

$$\frac{\partial S(\pm L, t)}{\partial x} = 0, \quad \frac{\partial E(\pm L, t)}{\partial x} = 0, \quad \frac{\partial I(\pm L, t)}{\partial x} = 0.$$

Furthermore, Emmanuel et al. [48] considered a space fractional diffusive *SIR* model,

$$\begin{aligned}
\frac{\partial S}{\partial t} &= -\beta I S + K_\alpha(-\infty D_x^\alpha S) \\
\frac{\partial I}{\partial t} &= \beta I S - \gamma I + K_\alpha(-\infty D_x^\alpha I) \\
\frac{\partial R}{\partial t} &= \gamma I
\end{aligned} \tag{1.5}$$

where K_α is the diffusion coefficient, $1 < \alpha \leq 2$ and the Riesz/Weyl fractional order operator is defined as,

$${}_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^x \frac{f(y)}{(x-y)^{\alpha-1}} dy \quad (1.6)$$

This fractional diffusion epidemic model, in fact, provides an advanced framework for simulating disease dynamics in complex, real-world scenarios.

Another approach in developing epidemic models consists of incorporating the time fractional derivative to capture the non-locality and the memory effects of the model. The $SEI_A I_S HRD$ model extends the classical $SEIR$ model by involving the Caputo derivative. It is studied by Biala and Khaliq in [13]. The compartments included in this model are susceptible (S), exposed (E), infected-asymptomatic (I_A), infected-symptomatic (I_S), infected-hospitalized (H), and recovered-dead (R). The discussed model is as follows:

$$\begin{aligned} {}_0D_t^\alpha S(t) &= b_1 N - \frac{S(t)}{N} \left(\beta_A I_A(t) + \beta_S I_S(t) \right) - b_1 S(t) \\ {}_0D_t^\alpha E(t) &= \frac{S(t)}{N} \left(\beta_A I_A(t) + \beta_S I_S(t) \right) - (\sigma + \mu_1) E(t) \\ {}_0D_t^\alpha I_A(t) &= \eta \sigma E(t) - (\gamma_1 + \mu_2) I_A(t) \\ {}_0D_t^\alpha I_S(t) &= (1 - \eta) \sigma E(t) - (\gamma_2 + \mu_3) I_S(t) \\ {}_0D_t^\alpha H(t) &= \gamma_1 I_A(t) + \gamma_2 I_S(t) - (\rho + \mu_4) H(t) \\ {}_0D_t^\alpha R(t) &= \rho H(t) - b_1 R(t) \\ {}_0D_t^\alpha D(t) &= \mu_1 E(t) + \mu_2 I_A(t) + \mu_3 I_S(t) + \mu_4 H(t) \end{aligned} \quad (1.7)$$

where ${}_0D_t^\alpha$ with $(0 < \alpha \leq 1)$ is the Caputo fractional derivative that is defined as,

$${}_tD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t f'(x)(t-x)^{-\alpha} dx \quad (1.8)$$

In the time-fractional model (1.7), β_A represents the transmission rate between susceptible individuals and those affected with asymptomatic illness, β_S is the rate of interaction between susceptible and infected individuals experiencing symptoms. The asymptomatic individual carries the germs without exhibiting any infectious symptoms. The latent time for the exposed population is denoted by σ^{-1} and the proportion of the exposed class that shifts to the asymptomatic class after the latent period is represented by η , while the remainder, who remain symptomatic, is denoted as $1-\eta$.

The parameters γ_1, γ_2 represent the hospitalization rates for asymptomatic and symptomatic infectious conditions, respectively. The hospitalized patients recover at a rate ρ , and b_1 denotes the natural birth rate, while μ_1, μ_2, μ_3 , and μ_4 account for the mortality rates for the exposed, asymptomatic infected, symptomatic infected and hospitalized groups, respectively.

The total population is denoted by N which is divided into $S(t)$, $E(t)$, $I_A(t)$, $I_S(t)$, $H(t)$, $R(t)$, and $D(t)$. The data associated with the system is shown as:

$$S(0) = S_0 \geq 0, \quad E(0) = E_0 \geq 0, \quad I_A(0) = I_{A0} \geq 0, \quad I_S(0) = I_{S0} \geq 0$$

$$H(0) = H_0 \geq 0, \quad R(0) = R_0 \geq 0, \quad D(0) = D_0 \geq 0$$

The diagram below explains the movement between the compartments and their transmission rates [13],

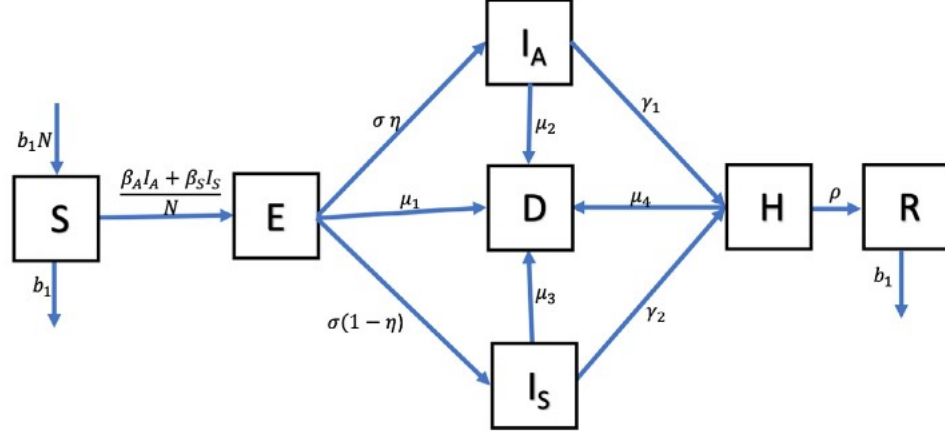


Figure 1.1: The diagram of $SEI_A I_S HRD$ model

Furati et al. [41] proposed a fractional-order memory-dependent model to simulate the spread of COVID-19. The impact of governmental interventions and public perception are incorporated as part of the non-linear, time-varying transmission rate. The proposed model is as follows:

$$\begin{aligned}
 {}_0D_t^\alpha S(t) &= \mu^\alpha - \frac{\beta SI}{N} - \mu^\alpha S, \\
 {}_0D_t^\alpha E(t) &= \frac{\beta SI}{N} - (\sigma^\alpha + \mu^\alpha)E, \\
 {}_0D_t^\alpha I(t) &= \sigma^\alpha E - (\gamma^\alpha + \mu^\alpha)I, \\
 {}_0D_t^\alpha R(t) &= \gamma^\alpha I - \mu^\alpha R, \\
 {}_0D_t^\alpha N(t) &= \mu^\alpha - \mu^\alpha N, \\
 {}_0D_t^\alpha P(t) &= \phi \gamma^\alpha I - \omega^\alpha P.
 \end{aligned} \tag{1.9}$$

The total population N at any given time is distributed among the four compartments.

That is,

$$N(t) = S(t) + E(t) + I(t) + R(t).$$

The variable

where the public perception grows at rate $\phi\gamma$ and decreases at rate ω . The parameter ϕ is the proportion of severe cases while ω^{-1} represents the mean time of public reaction. The mean of the latent period is σ^{-1} while γ is the rate at which contagious individuals recover from the illness.

The nonnegative initial data associated with the above model are

$$S(0) = S_0, E(0) = E_0, I(0) = I_0, N(0) = N_0, P(0) = P_0$$

The diagram below explains the movement between the compartments and their transmission rates.

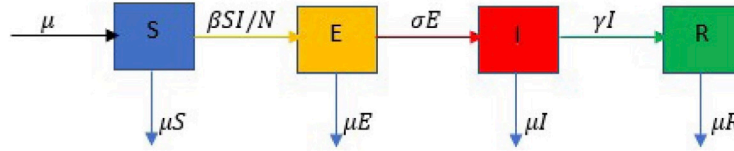


Figure 1.2: Transfer diagram between compartments of the model (1.9).

1.2 Literature Review

The first mathematical epidemic model was established by Bernoulli in 1760, during his investigation on the transmission dynamics of the smallpox virus [40]. In the

early 20th century, Hamer and Soper constructed a formula to qualitatively study the infectiousness of measles [47]. The well known Kermack–McKendrick model, which consists of two-dimensional system of ODEs (ordinary differential equations), was studied by Brauer [15] in 1927. In his study, it is shown that periodic solutions are impossible as S is a monotonically decreasing function of time. Furthermore, there cannot be backward bifurcations or numerous endemic equilibria because I is getting closer to zero. In fact, this mathematical model plays a crucial role in investigating many infectious diseases. The contribution in analyzing epidemic models assists epidemiologists in understanding the transition dynamics and then control and combat the spread of many infectious diseases, such as the eradication of smallpox, HIV/AIDS, Influenza, Cholera, and Malaria [99], [21], [53], [56], [107], [38], and recently the most ongoing disease that has been identified is the COVID-19 pandemic [57], [103], [108], [50]. These research studies concentrated on the development of mathematical models intended to describe the transmission of diseases among humans as well as between humans and animals. The contributions were not only limited by demonstrating the models but also to the establishment of many numerical schemes to solve the epidemic models. Having in mind that these kinds of models are based on non-linear differential equations that are challenging to be solved.

In fact, the attribution of introducing the earlier *SIR* (Susceptible-Infectious-Removed) model is attributed to Ronald Ross in 1908, when he was renowned for unveiling the transmission of malaria by mosquitoes [6]. Whereas Kermack and McKendrick subsequently expanded and intricately examined the *SIR* model to establish

the *SEIR* model as well as Soper [89]. Following that, numerous researchers elaborated on these epidemic models by adding more compartments depending on the dynamics of disease transmission, its treatments, or the methods utilized or implemented for disease prevention. Some of the extended *SIR* models, such as *SEIR* [36], [101], [59], *SITR* [77], [58], [44], *SEQIJR* [92], and the *SIS* model [76], [98], [42], were explored by many researchers. By adding sex, age, and geographic differences, Worden and Porco [100] extended the Cartesian product to the epidemic mathematical structures by adding host heterogeneity.

Several numerical techniques are used to solve the *SEIR* (Susceptible-Exposed-Infectious-Removed) model, which is displayed as ordinary differential equations (ODEs). One of these applied methods is the intuitive Euler's method. Even though this method is a straightforward strategy, its accuracy is low [12]. Thus, researchers explored other methods with higher accuracy than the forward or backward Euler method, such as the explicit [24], [54], [68], [67], the fourth-order Hermite method [66], [88], and the Runge-Kutta methods [81], [49]. Researchers also used fractional-order derivatives to describe epidemic models. Rajagopal et al. [79] analyzed the spread of COVID-19 using the *SEIR* epidemic model involving Caputo fractional derivative. They calculated and explored the equilibrium points and the feasibility region, as well as the stability of the equilibrium points. The fractional Euler method with Caputo fractional derivatives was used and results were compared to those obtained using integer order derivatives. Bearing in mind that the fractional derivative is more accurate than the integer order derivative because of its non-local character, the dynamics of a fractional $S_b I_b S_h I_h H_h F_m P_c C_c$ model for the COVID-19 epidemic model

was investigated in [11]. In this model, the disease is transferred from either a bat or a human with an infected family member or patient to either a clinic or care center. The numerical simulation shows that the changes occurring in time, such as increasing caution or vaccination uptake, depend on the fractional order. Also, fractional order can capture the non-local nature of COVID-19 transmission. The fractional Caputo *SIR* model is used to study the behavior of COVID-19 in Pakistan (see [2]). Numerical approach is obtained using Ulam-Hyers method and its generalized versions.

Due to the fact that numerical solutions for various models might not fit properly with real data, Pan et al. [73] used the Gorenflo-Mainardi-Moretti-Paradisi scheme to obtain the numerical solution of the Ebola epidemic model for a fractional order *SEIR* model. The approximate solution of the fractional *SIR* model using the homotopy analysis method (HAM) is obtained in [1]. It is shown that convergence control parameter contributes significantly in determining the convergence of the solutions.

The epidemic models were further extended by replacing the classical diffusion terms with fractional diffusion terms. Bock and Jayathunga [14] proposed a two-dimensional fractional *SIRUV* epidemic model with fractional diffusion terms. The model was numerically solved by using the Crank-Nicolson method and the Alternating-Direction-Implicit scheme. The acquired results illustrate anomalous diffusive behavior as compared to the results obtained by classical diffusion. By directly simulating air traffic transport using real-world data and adapting a stochastic fractional diffusion reaction equation, Gustafon et al. [45] studied the spatial fractional diffusion SIS model to simulate the spread of epidemics on the air travel network.

1.3 Objectives of the Study

We propose in this thesis to discuss mainly the following objectives:

1. Apply some existing numerical schemes to numerically solve basic and extended epidemic models under different scenarios.
2. Develop and implement high-order numerical schemes to solve classical and fractional diffusive epidemic models.
3. Study the time-fractional epidemic models, systems (1.7) and (1.9), solve them numerically.
4. Study space-time fractional epidemic models and develop some efficient, stable, and reliable numerical schemes to numerically solve them.

1.4 Methodology

- Solve the classical-order basic epidemic model using the fourth-order Runge-Kutta method.
- Solve the fractional epidemic model using the forward Euler method, the predictor-corrector fractional trapezoidal rule, Simpson's rule and a fourth order rule.
- Employ integral approximation techniques.
- Use a third-order exponential time differencing (ETD) scheme to solve spatial-fractional diffusive epidemic models.

- Use spectral approach for the space fractional Laplacian operator.
- Use matrix transfer technique for space-fractional Laplacian discretization.

CHAPTER 2

CLASSICAL EPIDEMIC MODEL

In general, it is unfeasible to solve analytically non-linear ordinary differential equations using standard methods. Therefore, researchers attempt to numerically solve classical non-linear epidemic models while trying to observe the transmission of infectious illnesses over time. They have developed many algorithms and numerical schemes to simulate epidemic models in order to obtain more accurate, reliable, and efficient results. In this chapter, we consider the classical SIR , $SEIR$ epidemic models as systems of non-linear first-order differential equations in order to explore the COVID-19 pandemic. We also provide a numerical simulation of the classical $SEIR$ model using the high-order Runge-Kutta method (RK4) that efficiently solves the system with high accuracy.

2.1 Classical Epidemic Models

Recently, COVID-19 has been classified as a viral respiratory infectious disease caused by the SARS-CoV-2 virus [60]. This pandemic poses globally significant challenges to

public health. Indeed, since it was identified in Wuhan, China, in December 2019, according to the World Health Organization, over 770 million confirmed cases of COVID-19 have been reported, including around 7,045,569 deaths. Its extraordinarily rapid spread during the peak infection time put a strain on the health care system [39]. Due to the harmful effects of the COVID-19 pandemic in many areas, many researchers have been exploring the pandemic as a classical compartmental model.

We first investigate the simplest classical *SIR* model,

$$\begin{aligned}\frac{dS(t)}{dt} &= \frac{-\beta S(t)I(t)}{N} \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - \gamma I(t) \\ \frac{dR(t)}{dt} &= \gamma I(t)\end{aligned}\tag{2.1}$$

In this model, $N = S(t) + I(t) + R(t)$ is the fixed total population size with neither immigration nor any demographic change. In the above *SIR* model, β refers to the rate of connections between the susceptible and infectious compartments, while γ represents individuals' transformations from the infectious class to the recovered class.

Since many infectious diseases have an incubation period, which is the time when the pathogens are present without showing symptoms, the extended *SEIR* model is discovered. Sometimes, individuals at this stage (in the latent period) can transmit the disease to others through either direct or indirect interactions. Exploring such epidemics shows that adding this exposed class to the *SIR* model will give more realistic results [7], [69].

Thus, by adding the exposed compartment to the *SIR* model, the following system represents the well-known *SEIR* model.

$$\begin{aligned}
\frac{dS(t)}{dt} &= \frac{-\beta S(t)I(t)}{N} \\
\frac{dE(t)}{dt} &= \frac{\beta S(t)I(t)}{N} - \sigma E(t) \\
\frac{dI(t)}{dt} &= \sigma E(t) - \gamma I(t) \\
\frac{dR(t)}{dt} &= \gamma I(t)
\end{aligned} \tag{2.2}$$

Where σ denotes the removal of individuals from an exposed class to be infected.

In an effort to provide a more reliable and realistic approach, numerous studies have further enhanced this model by adding new categories that reflect the behavior of the disease. However, in this chapter, we are only going to discuss the simplest ODE's *SEIR* model using the fourth-order Rung-Kutta (RK4) method.

2.2 Stability Analysis of Classical Epidemic Models

In this section, we explore the stability of the *SIR* and *SEIR* epidemic models based on their reproduction ratios and their equilibrium points.

2.2.1 Equilibrium Analysis

Equilibrium points give crucial insights into epidemic behavior, leading public health policies and manners that successfully minimize spread of the disease. Considering these principles is critical for creating strong epidemic models and planning for

prospective outbreaks. The epidemic model's equilibrium points provide information on the various outcomes of an epidemic, such as whether the disease will die out or persist in the population.

Definition 2.2.1 *A constant u^* is said to be the equilibrium point of the following system of ODEs:*

$$\frac{du(t)}{dt} = g(t, u), \quad u(t_0) \geq 0$$

if and only if u^ satisfies $g(t, u^*) = 0$.*

Definition 2.2.2 *u_0^* is said to be a disease-free equilibrium (DFE) when no infected individuals exist in the population.*

Definition 2.2.3 *u^* is said to be an endemic equilibrium (EE) when a disease remains in the population at a steady average rate, neither increasing nor decreasing over time.*

By using the above definitions, the disease-free equilibrium for the *SIR* model (2.1) occurs when $I(t) = 0$. Hence, $u_0^* = (N, 0, 0)$. Whereas, the endemic equilibrium occurs when $I(t) > 0$. Which implies that $S^* = \frac{\gamma N}{\beta}$. In similar way, the disease-free equilibrium for *SEIR* model (2.2) is $u_0^* = (N, 0, 0, 0)$ while the endemic equilibrium occurs at $S^* = \frac{\sigma E^* N}{\beta I^*}$ and $E^* = \frac{\gamma I^*}{\sigma}$. Based on these equilibrium, the analyzing stability of the epidemic model can be obtained using the eigenvalues of its Jacobian matrix at the equilibrium points.

Definition 2.2.4 *The Jacobian matrix is a square matrix that represents the partial derivatives of every equation with respect to other variables. $J = \frac{\partial g}{\partial u_i}$.*

Theorem 2.2.5 *Let J be the $(k \times k)$ Jacobian matrix for non-linear system of first ODEs, then the stability of the equilibrium has five different considerations based on the value of the eigenvalues of J [70],[80].*

- *The equilibrium will be stable when the real part of all eigenvalues is negative.*
- *The equilibrium will be unstable when the real part of all eigenvalues is positive.*
- *If at least one of the eigenvalues has a negative real part and at least one has a positive real part, the equilibrium will represent a saddle point.*
- *The equilibrium is called a node point if all eigenvalues are real.*
- *The equilibrium is called a spiral point if we have at least a complex conjugate pair of eigenvalues.*
- *The equilibrium will be a linear center when having a zero real part on a pair of complex eigenvalues.*

The Jacobian Matrix for SIR and $SEIR$ Model

For SIR model, we let

$$\begin{aligned}
 g1 &= \frac{-\beta S(t)I(t)}{N} \\
 g2 &= \frac{\beta S(t)I(t)}{N} - \gamma I(t) \\
 g3 &= \gamma I(t)
 \end{aligned} \tag{2.3}$$

The Jacobian matrix for the *SIR* model is evaluated as follows:

$$J = \begin{bmatrix} \frac{\partial g_1}{\partial S} & \frac{\partial g_1}{\partial I} & \frac{\partial g_1}{\partial R} \\ \frac{\partial g_2}{\partial S} & \frac{\partial g_2}{\partial I} & \frac{\partial g_2}{\partial R} \\ \frac{\partial g_3}{\partial S} & \frac{\partial g_3}{\partial I} & \frac{\partial g_3}{\partial R} \end{bmatrix}$$

Thus,

$$J_{u^*} = \begin{bmatrix} -\beta \frac{I^*}{N} & -\beta \frac{S^*}{N} & 0 \\ \beta \frac{I^*}{N} & \beta \frac{S^*}{N} - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix}$$

Now, we first analyze the Jacobian matrix at disease-free equilibrium $u_0^* = (N, 0, 0)$.

The Jacobian matrix at this point is represented as:

$$J_{u_0^*} = \begin{bmatrix} 0 & -\beta & 0 \\ 0 & \beta - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix}$$

To analyze the stability of the (DFE), we find the eigenvalues of the above matrix by

first evaluating the determinant as follows:

$$\det(J_{u_0^*} - \lambda) = \det \begin{bmatrix} -\lambda & -\beta & 0 \\ 0 & \beta - \gamma - \lambda & 0 \\ 0 & \gamma & -\lambda \end{bmatrix} = \lambda^2(\beta - \gamma - \lambda)$$

Hence, the corresponding eigenvalues at (DFE) are $\lambda_1 = 0$ (with multiplicity 2) and $\lambda_2 = \beta - \gamma$. Therefore, due to the stability theorem (2.2.5), the (DFE) would be stable if $\beta - \gamma < 0$ and unstable if $\beta - \gamma > 0$.

2.2.2 The Basic Reproduction Ratio

The fundamental reproduction number (R_0) is a crucial factor in determining whether infectious diseases spread or disappear in the population. This ratio refers to the average number of secondary cases caused by an infected individual in a susceptible class [31]. We rewrite the basic epidemic model as:

$$\frac{du}{dt} = \mathcal{F}(u) - \mathcal{V}(u), \quad (2.4)$$

where $\mathcal{F}$ represents the rate at which infections emerge. We let $\mathcal{V} = \mathcal{V}^- - \mathcal{V}^+$. Where $\mathcal{V}^-$ refers to the rate of transfer of individuals out of the compartment and $\mathcal{V}^+$ refers to the rate of transfer of individuals into the compartment. In the epidemic model, we let i be the number of compartments and j refer to the infected one. Then, we evaluate the following Jacobian matrices at (FDE) as follows:

$$F = \left[\frac{\partial \mathcal{F}(u_0^*)}{\partial u_i} \right],$$

and

$$V = \left[\frac{\partial \mathcal{V}(u_0^*)}{\partial u_j} \right].$$

The matrix FV^{-1} represents the next generation matrix.

Definition 2.2.6 *Based on the next generation method [83], the reproduction ratio R_0 is the largest eigenvalue of the next generation matrix FV^{-1} .*

Let $u = (S, I, R)^T$, we rewrite the basic SIR model as:

$$\frac{du}{dt} = \begin{bmatrix} 0 \\ \frac{\beta SI}{N} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{\beta SI}{N} \\ \gamma I \\ -\gamma I \end{bmatrix}.$$

Now, since the infected compartment is only I, the Jacobian matrices F and V are as follows:

$$F = \begin{bmatrix} \frac{\beta S}{N} \end{bmatrix},$$

and

$$V = \begin{bmatrix} \gamma \end{bmatrix}.$$

For the SIR model (2.1), the next generation matrix at (DFE) is represented as:

$$FV^{-1} = \begin{bmatrix} \frac{\beta}{\gamma} \end{bmatrix},$$

Hence, we conclude that $R_0 = \frac{\beta}{\gamma}$, and the infection will die out if $R_0 < 1$ or can spread through the population if $R_0 > 1$.

For the basic *SEIR* model (2.2), we found that

$$\mathcal{F} = \begin{bmatrix} \frac{\beta SI}{N} \\ 0 \end{bmatrix},$$

and

$$\mathcal{V} = \begin{bmatrix} \sigma E \\ -\sigma E + \gamma I \end{bmatrix}.$$

In similar manner to the *SIR* model, we found that the next generation matrix at (DFE) for the basic *SEIR* model, which is represented as:

$$FV^{-1} = \begin{bmatrix} 0 & \frac{\beta S^0}{N} \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \sigma & 0 \\ -\sigma & \gamma \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-\beta}{\gamma} & \frac{\beta}{\sigma} \\ 0 & 0 \end{bmatrix}.$$

Thus, $R_0 = \frac{\beta}{\gamma}$.

2.3 Numerical Method

Numerous methods have been used to solve non-linear ODEs. One of these methods is the Runge-Kutta method, which includes number of advanced techniques with different levels of accuracy. Researchers have developed the first, second, third, and fourth order Runge-Kutta method to solve linear and non-linear ODEs. The fourth-order Runge-Kutta (RK4) method has been widely used to numerically solve non-linear first-order initial value problems. Moreover, It is preferred to use RK4 due to its con-

vergent properties [9].

Consider the initial value problem,

$$\frac{du}{dt} = g(t, u), \quad u(t_0) = u_0. \quad (2.5)$$

The Runge-Kutta method considers different points in the domain and focuses on calculating the weighted average of the slopes at these points. [9]. As in [46], the general scheme of the Runge-Kutta method is presented as:

$$u^{n+1} = u^n + \sum_{i=1}^s \zeta_i k_i \quad (2.6)$$

$$k_1 = hg(t_0, u_0),$$

$$k_2 = hg(t_0 + hc_2, u_0 + ha_{21}k_1),$$

$$k_3 = hg(t_0 + hc_3, u_0 + h(a_{31}k_1 + a_{32}k_2)),$$

...

$$k_s = hg(t_0 + hc_s, u_0 + h(a_{s1}k_1 + a_{s2}k_2 + \dots + a_{s,s-1}k_{s-1})),$$

where s is the number of slopes, k_i are the coefficients that must be evaluated, and ζ_i are the weighted coefficients. Moreover, c_j usually satisfy the following condition:

$$c_j = \sum_{i=1}^{j-1} a_{ji}. \quad (2.7)$$

The following is the 4th order Runge-Kutta method used for the numerical approximation solution of non-linear ODEs.

$$u^{n+1} = u^n + h \left(\frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \right), \quad (2.8)$$

where,

$$\begin{aligned} k_1 &= g(t_0, u_0), \\ k_2 &= g\left(t_0 + \frac{h}{2}, u_0 + \frac{k_1 h}{2}\right), \\ k_3 &= g\left(t_0 + \frac{h}{2}, u_0 + \frac{k_2 h}{2}\right), \\ k_4 &= g(t_0 + h, u_0 + k_3 h). \end{aligned}$$

2.4 Experimental Result

In this section, we examine the fourth-order convergence of the Runge-Kutta method using the following ODE.

Example 1 (A test problem) *Consider the following first-order differential equation,*

$$\frac{du(t)}{dt} = -u(t) + \frac{t^3}{\Gamma(4)}, \quad (2.9)$$

$$u(0) = 0, \quad t > 0.$$

The exact solution of the above initial value problem is given as [22],

$$u(t) = t^4 E_{1,5}(-t),$$

where $E_{\alpha,\beta}$ represents the Mittag-Leffler function. Table 2.1 shows the errors and the order of convergence of the 4th order Rung-Kutta scheme (2.8) to the above ODE.

Step size	Error	Order of Conv.
0.1	2.51e-07	0
0.05	1.5691e-08	3.9997
0.025	9.8034e-10	4.0005
0.0125	6.1253e-11	4.0004
0.00625	3.8278e-12	4.0002

Table 2.1: The absolute error, convergence order of example 1 using numerical scheme (2.8)

The following figure shows both the exact and the numerical solution using the RK4 method.

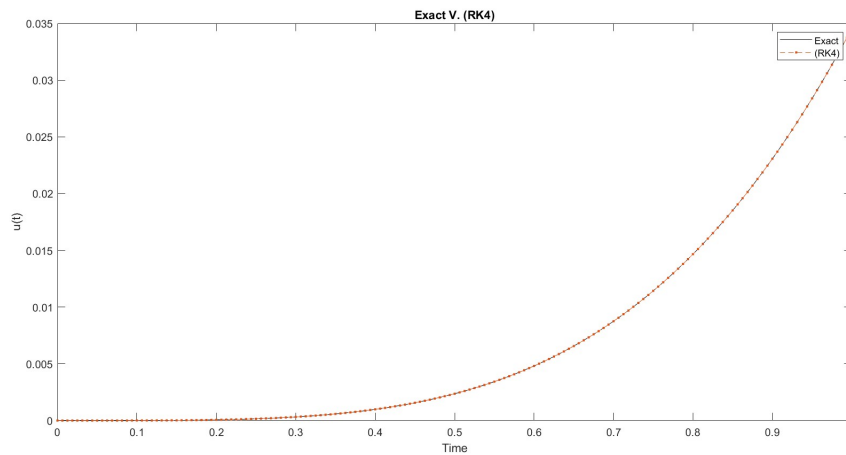


Figure 2.1: The exact solution and numerical (RK4) solution for example. 1

Example 2 (The *SEIR* Model) [94] Solve the basic *SEIR* model (2.2) with the following initial condition,

$$(S(0), E(0), I(0), R(0)) = (65251000, 500, 0, 0),$$

using the parameters, $\beta = 0.5$, $\sigma = 0.02$, $\gamma = 0.01$.

Figure 2.2 displays the simulation of the basic *SEIR* model for the Rung-Kutta method for 500 days.

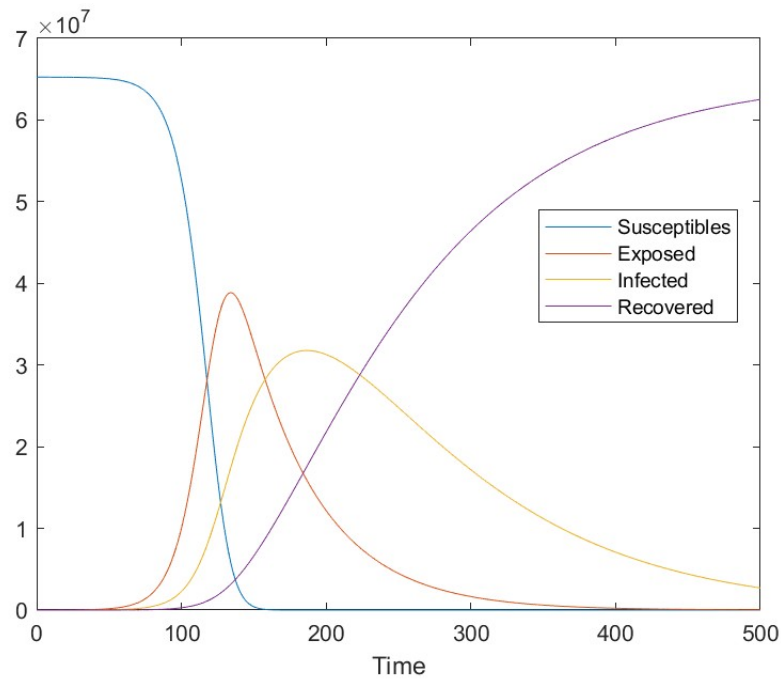


Figure 2.2: The simulation of the *SEIR* model (2.2) using Rung-Kutta scheme (2.8)

CHAPTER 3

FRACTIONAL EPIDEMIC MODELS

The integer order derivatives in traditional epidemic models, such as the classical epidemic model, are sufficient to describe the system dynamic. But in some cases they may not accurately capture the complex behavior of epidemics. Traditional epidemic models rely on integer-order derivatives, which may not adequately capture the inherent memory effects. Therefore, a class of models was introduced by incorporating fractional derivatives instead of classical derivatives that leading to more accurate and realistic modeling of epidemic phenomena. Fractional epidemic models have emerged as a more valuable tool in the field of epidemiology because they provide more nuanced and comprehensive understanding of disease dynamics. These models can capture complex, non-linear, and heterogeneous aspects of epidemic, which makes them a promising approach for improved disease modeling, forecasting, and decision-making in public health.

We start this chapter by describing some preliminaries of fractional calculus followed by describing the fractional epidemic model. We will propose numerical schemes to solve these models. The dynamics of these models shall be demonstrated through numerical experiments using proposed numerical schemes.

3.1 Preliminaries

In this section, we will provide some definitions of fractional calculus and their properties.

Definition 3.1.1 *The Gamma function [75] of a real number α is defined as:*

$$\Gamma(\alpha) = \int_0^{\infty} s^{\alpha-1} e^{-s} ds.$$

Remark 3.1.2 *The gamma function is the extension of the factorial of any real $\alpha > 0$.*

Definition 3.1.3 *The left and right-sided Riemann-Liouville integrals of order*

$\alpha \in (0, 1)$ satisfy the following equations, respectively,

$$I_{a,x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1} dt \quad (3.1)$$

$$I_{x,b}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1} dt \quad (3.2)$$

where, $f : [a, b] \rightarrow R$.

Definition 3.1.4 *The left-sided and right-sided Caputo derivative of order α is defined*

by,

$${}_CD_{a,x}^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x f^{(m)}(t)(x-t)^{m-\alpha-1} dt, \quad (3.3)$$

$${}_CD_{x,b}^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b f^{(m)}(t)(t-x)^{m-\alpha-1} dt, \quad (3.4)$$

where, $m-1 < \alpha \leq m$, m is an integer.

Remarks 3.1.5 If $\alpha = m$, then,

$${}_CD_{a,x}^\alpha f(x) = f^{(m)}(x).$$

Lemma 3.1.6 [61] Let $g : [t_0, \infty) \times \Omega \rightarrow R^n$, where $\Omega \in R^n$ and $u(t_0) = 0$ is the initial condition for the following Caputo fractional-order dynamical system,

$${}_CD_{t_0,t}^\alpha u(t) = g(t, u), \quad t_0 > 0,$$

where $0 < \alpha \leq 1$. Then, the examined system has a unique solution if $g(t, u)$ satisfies the Lipschitz condition with respect to u .

Definition 3.1.7 A constant u^* is said to be the equilibrium point of the following Caputo fractional system:

$${}_CD_{t_0,t}^\alpha u(t) = g(t, u), \quad u(t_0) \geq 0,$$

if and only if u^* satisfies $g(t, u^*) = 0$.

Lemma 3.1.8 [74] Let $0 < \alpha \leq 1$ and $u \in R$ for the following Caputo fractional

order system,

$${}_CD_{t_0,t}^\alpha u(t) = g(u), \quad u(t_0) = u_0.$$

The equilibrium points are said to be locally asymptotically stable if and only if all eigenvalues $\tilde{\lambda}$ of the Jacobian matrix $J = \frac{\partial g}{\partial u}$ satisfy $|\arg(\tilde{\lambda})| > \frac{\alpha\pi}{2}$.

Definition 3.1.9 [75] For $\alpha > 0$, $\beta > 0$, and $x \in C$, the one parameter Mittag Leffler function is represented as,

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)},$$

while the two parameters Mittag Leffler function is defined as,

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}$$

Remark 3.1.10 When $\beta = 1$, $E_{\alpha,1}(x) = E_\alpha(x)$.

Remark 3.1.11 For $\alpha = \beta = 1$, the Mittag Leffler function $E_{1,1}(x) = e^x$.

Definition 3.1.12 [10] Let $u \in H^1(a, b)$, $b > a$, and $\alpha \in [0, 1]$. The formula for fractional α -order Atangana-Baleanu-Caputo (ABC) derivatives of u is defined by

$$({}_{ABC}D_{a,t}^\alpha u)(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t u'(s) E_\alpha \left[\frac{-\alpha}{1-\alpha} (t-s)^\alpha \right] dt, \quad (3.5)$$

where, $M(\alpha)$ is a normalization function that satisfies the condition $M(0) = M(1) = 1$.

Definition 3.1.13 [82] *The associated fractional (ABC) integral is defined as,*

$${}_{ABC}I_{0,t}^{\alpha}u(t) = \frac{1-\alpha}{M(\alpha)}u(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds. \quad (3.6)$$

3.2 Some Fractional Epidemic Models

Recently, fractional calculus has been widely used to simulate biological mathematical models for many infectious diseases. Certainly, numerous studies claim that employing fractional calculus to demonstrate epidemic models is more effective than using classical epidemic models ([43] , [96], [90], [37] , [18]). Since the classical or traditional epidemic model described by a set of ODE's assumes that the rates of infection and recovery are instantaneous and memory-less, the fractional calculus is introduced to incorporate memory effects and long-range interactions. The idea of using fractional calculus basically represents the epidemic model by replacing the integer-order derivatives in the classical model by a non-integer-order derivatives in the differential equations.

3.2.1 Fractional *SEIR* Epidemic Model

José et al. [19] presented simulation of a deterministic *SEIR* model (1.3) by considering the birth rate Λ , the natural mortality rate μ and the average virus-related fatality rate ϵ . We are considering the fractional extension of this model by utilizing

fractional time derivative. The fractional version of the model (1.3) is as follows:

$$\begin{aligned}
D_{0,t}^\alpha S(t) &= \Lambda - \left(\mu^\alpha + \frac{\beta I}{N} \right) S, \\
D_{0,t}^\alpha E(t) &= \frac{\beta SI}{N} - (\sigma^\alpha + \mu^\alpha) E, \\
D_{0,t}^\alpha I(t) &= \sigma^\alpha E - (\gamma^\alpha + \mu^\alpha + \epsilon^\alpha) I, \\
D_{0,t}^\alpha R(t) &= \gamma^\alpha I - \mu^\alpha R,
\end{aligned} \tag{3.7}$$

with the following initial conditions,

$$S(0) = S_0 \geq 0, \quad E(0) = E_0 \geq 0, \quad I(0) = I_0 \geq 0.$$

In this fractional *SEIR* epidemic model, the birth rate Λ , the natural mortality rate μ and the related virus fatality rate ϵ are considered. Moreover, the model is based on the following assumptions:

- The initial population size is N_0 , while $N(t)$ is the total population that includes the classes of susceptible $S(t)$, exposed $E(t)$, infected $I(t)$, and recovered $R(t)$.

Hence,

$$N(t) = S(t) + E(t) + I(t) + R(t) \leq N_0$$

- β is the rate of transmission between exposed and susceptible individuals. People in the susceptible class therefore join the exposed class at a rate of $\frac{\beta I}{N}$.
- The natural death rate is considered in this model and is denoted by μ .
- σ represents the rate of change after moving from being exposed to the infectious.

- γ is the recovery rate of infectious individuals.

Stability Analysis of The Fractional *SEIR* Model

We evaluate the basic reduction ratio R_0 for the fractional *SEIR* model in a similar way to evaluating the R_0 for the basic *SEIR* model using the next generation method.

Thus, the following represents the next generation matrix at (DFE),

$$FV^{-1} = \begin{bmatrix} 0 & \frac{\beta\lambda}{N\mu^\alpha} \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} (\sigma^\alpha + \mu^\alpha) & 0 \\ -\sigma^\alpha & (\gamma^\alpha + \mu^\alpha) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\beta\Lambda\sigma^\alpha}{N\mu^\alpha(\sigma^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)} & \frac{\beta\Lambda}{N\mu^\alpha(\gamma^\alpha + \mu^\alpha)} \\ 0 & 0 \end{bmatrix}.$$

It is obvious that,

$$R_0 = \frac{\beta\Lambda\sigma^\alpha}{N\mu^\alpha(\sigma^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)}.$$

As a result, the infection will perish if $\beta\Lambda\sigma^\alpha < N\mu^\alpha(\sigma^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)$, or can spread through the population if $\beta\Lambda\sigma^\alpha > N\mu^\alpha(\sigma^\alpha + \mu^\alpha)(\gamma^\alpha + \mu^\alpha)$.

Now, we explore the stability of the fractional *SEIR* model represented in (3.7) by investigating its equilibrium. To find the equilibrium points for the above system, we solve the following system,

$$D_{0,t}^\alpha S^*(t) = 0,$$

$$D_{0,t}^\alpha E^*(t) = 0,$$

$$D_{0,t}^\alpha I^*(t) = 0,$$

$$D_{0,t}^\alpha R^*(t) = 0.$$

The disease-free equilibrium (DFE) is represented as:

$$u_0^* = \left(\frac{\Lambda}{\mu}, 0, 0, 0 \right).$$

Whereas by using (2.2.3), the endemic equilibrium (EE) is represented as:

$$u^* = \left(\frac{N(\sigma^\alpha \gamma^\alpha + \sigma^\alpha \mu^\alpha + \mu^\alpha \gamma^\alpha + \mu^{2\alpha})}{\beta \sigma}, \frac{\Lambda \beta \sigma^\alpha - N \mu^\alpha (\sigma^\alpha \gamma^\alpha + \mu^\alpha \sigma^\alpha + \mu^\alpha \gamma^\alpha + \mu^{2\alpha})}{\sigma^\alpha (\sigma^\alpha + \mu^\alpha) \beta}, \right. \\ \left. \frac{\Lambda \beta \sigma^\alpha - N \mu^\alpha (\sigma^\alpha \gamma^\alpha + \mu^\alpha \sigma^\alpha + \mu^\alpha \gamma^\alpha + \mu^{2\alpha})}{\beta (\sigma^\alpha \gamma^\alpha + \sigma^\alpha \mu^\alpha + \mu^\alpha \gamma^\alpha + \mu^{2\alpha})}, \right. \\ \left. \frac{\gamma^\alpha (\Lambda \beta \sigma^\alpha - \mu^\alpha N \sigma^\alpha \gamma^\alpha - \mu^{2\alpha} N \sigma^\alpha - \mu^{2\alpha} N \gamma^\alpha - \mu^{3\alpha} N)}{\beta \mu^\alpha (\sigma^\alpha \gamma^\alpha + \mu^\alpha \sigma^\alpha + \mu^\alpha \gamma^\alpha + \mu^{2\alpha})} \right).$$

The Jacobian matrix for the above fractional *SEIR* model is evaluated using the MAPLE program as follows:

$$J_u = \begin{bmatrix} -\mu^\alpha - \frac{\beta I}{N} & 0 & -\frac{\beta S}{N} & 0 \\ \frac{\beta I}{N} & -\sigma^\alpha - \mu^\alpha & \frac{\beta S}{N} & 0 \\ 0 & \sigma^\alpha & -\gamma^\alpha - \mu^\alpha & 0 \\ 0 & 0 & \gamma^\alpha & -\mu^\alpha \end{bmatrix}.$$

Thus, the Jacobian matrix at (DFE) is shown as:

$$J_{u_0^*} = \begin{bmatrix} -\mu^\alpha & 0 & -\frac{\beta \Lambda}{N \mu^\alpha} & 0 \\ 0 & -\sigma^\alpha - \mu^\alpha & \frac{\beta \Lambda}{N \mu^\alpha} & 0 \\ 0 & \sigma^\alpha & -\gamma^\alpha - \sigma^\alpha & 0 \\ 0 & 0 & \gamma^\alpha & -\mu^\alpha \end{bmatrix}.$$

Hence, the corresponding eigenvalues for the Jacobian matrix at (FDE) are:

$$\lambda_{1,2} = -\mu^\alpha,$$

$$\lambda_{3,4} = \frac{-N\mu^\alpha(\sigma^\alpha + \gamma^\alpha + 2\mu^\alpha) \pm \sqrt{(N\sigma^\alpha\mu^\alpha - N\mu^\alpha\gamma^\alpha)^2 + 4N\Lambda\beta\sigma^\alpha\mu^\alpha}}{2N\mu^\alpha}.$$

Referring to theorem (2.2.5), the trajectory behavior of the (FDE) will be stable when,

$$\sqrt{(N\sigma^\alpha\mu^\alpha - N\mu^\alpha\gamma^\alpha)^2 + 4N\Lambda\beta\sigma^\alpha\mu^\alpha} < N\mu^\alpha(\sigma^\alpha + \gamma^\alpha + 2\mu^\alpha).$$

3.2.2 Fractional *SEIRNP* Epidemic Model

The fractional *SEIRNP* epidemic model is a mathematical model which is used to simulate the spread of an infectious disease within a certain population [41]. The fractional *SEIRNP* model is described by the following set of coupled ordinary differential equations,

$$\begin{aligned} D_{0,t}^\alpha S(t) &= \mu^\alpha(1 - S) - \frac{\beta SI}{N} \\ D_{0,t}^\alpha E(t) &= \frac{\beta SI}{N} - (\sigma^\alpha + \mu^\alpha)E, \\ D_{0,t}^\alpha I(t) &= \sigma^\alpha E - (\gamma^\alpha + \mu^\alpha)I, \\ D_{0,t}^\alpha R(t) &= \gamma^\alpha I - \mu^\alpha R, \\ D_{0,t}^\alpha N(t) &= \mu^\alpha(1 - N), \\ D_{0,t}^\alpha P(t) &= \phi\gamma^\alpha I - \omega^\alpha P, \end{aligned} \tag{3.8}$$

with the following initial conditions,

$$S(0) = S_0 \geq 0, \quad E(0) = E_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad N(0) = N_0 \geq 0, \quad P(0) = P_0 \geq 0.$$

In this model, both government intervention $N(t)$ and public perception $P(t)$ are considered, and they are added as a compartment to $SEIR$ model. The above model is based on the following assumptions:

- The total population N in the above system is classified into $S(t), E(t), I(t), R(t)$ which indicate the classes of susceptible individuals, exposed individuals, infected individuals, and recovered individuals, respectively. Thus,

$$N(t) = S(t) + E(t) + I(t) + R(t).$$

- β is the transmission rate between susceptible and exposed individuals. Thus, individuals in the susceptible class enter the exposed class by the rate $\frac{\beta I}{N}$.
- Exposed individuals become infected at the rate σ , while the recovery rate for infectious individuals is referred to by γ .
- μ represents the natural death. Moreover, it is assumed that the birth rate is equal to the death rate.
- The growth of public perception increases at rate $\phi\gamma$ and decreases at rate ω . In this growth, ϕ is the proportion of severe individuals, while ω^{-1} is the mean time of public reaction.

- The mean of the latent period is σ^{-1} , while the mean of infectious duration is the value γ^{-1} .

Remarks 3.2.1 *The parameter is raised to the fractional power α in the model to confirm the consistency in the units for the left and right hand sides [26], [86], [8].*

3.2.3 Fractional SEI_AI_SHRD Epidemic Model

The fractional SEI_AI_SHRD epidemic model proposed by Biala and Khaliq in [13] is an extension of the classical epidemic model with more compartments to capture the complexity of COVID-19 transmission and the various stages of infection. The compartments included in this model are susceptible, exposed-asymptomatic, infected-symptomatic, infected-hospitalized, and recovered-dead. The SEI_AI_SHRD model is as follows,

$$\begin{aligned}
D_{0,t}^\alpha S(t) &= \mu^\alpha - \frac{S(t)}{N} (\beta_A I_A(t) + \beta_S I_S(t)) - \mu^\alpha S(t), \\
D_{0,t}^\alpha E(t) &= \frac{S(t)}{N} (\beta_A I_A(t) + \beta_S I_S(t)) - (\sigma^\alpha + \mu_1^\alpha) E(t), \\
D_{0,t}^\alpha I_A(t) &= \eta \sigma^\alpha E(t) - (\gamma_1^\alpha + \mu_2^\alpha) I_A(t), \\
D_{0,t}^\alpha I_S(t) &= (1 - \eta) \sigma^\alpha E(t) - (\gamma_2^\alpha + \mu_3^\alpha) I_S(t), \\
D_{0,t}^\alpha H(t) &= \gamma_1^\alpha I_A(t) + \gamma_2^\alpha I_S(t) - (\rho^\alpha + \mu_4^\alpha) H(t), \\
D_{0,t}^\alpha R(t) &= \rho^\alpha H(t) - \mu^\alpha R(t), \\
D_{0,t}^\alpha D(t) &= \mu_1^\alpha E(t) + \mu_2^\alpha I_A(t) + \mu_3^\alpha I_S(t) + \mu_4^\alpha H(t).
\end{aligned} \tag{3.9}$$

The non-negative initial conditions associated with the system are,

$$\begin{aligned} S(0) &= S_0, \quad E(0) = E_0, \quad I_A(0) = I_{A0}, \quad I_S(0) = I_{S0}, \\ H(0) &= H_0, \quad R(0) = R_0, \quad D(0) = D_0. \end{aligned}$$

Let N represents the population size and,

$$N = S(t) + E(t) + I_A(t) + I_S(t) + H(t) + R(t) + D(t).$$

This model is described under the following assumptions:

- The transmission rate between the susceptible population and infected individuals with asymptomatic disease is denoted by β_A . The rate of interaction between susceptible and infected symptomatic is represented by β_S . The asymptomatic individual has hosted the pathogens and the symptomatic individual has infectious symptoms.
- σ^{-1} denotes the latent period for the exposed population, while η is the fraction of the exposed class that moves to the asymptomatic class after the latent period, and $(1 - \eta)$ is the remaining symptomatic population.
- The population moves from the asymptomatic infectious and symptomatic infectious class to exist in the hospitalized compartment at the rate γ_1 and γ_2 , respectively.
- The recovery rate of the hospitalized individuals is ρ .

- μ represents the natural birth rate, where μ_1, μ_2, μ_3 , and μ_4 indicate the mortality rate for the exposed class, asymptomatic infected class, symptomatic infected class, hospitalized class, respectively.

3.2.4 Stability Analysis for The Fractional $SEI_A I_S HRD$ Model

By considering the system of fractional epidemic model that is displayed in (3.9), it is obvious that the free-disease equilibrium $u_0^* = (N, 0, 0, 0, 0, 0, 0)$. To evaluate the Basic Reproduction Ratio R_0 , we also used the next generation method. We consider the following,

$$D_{0,t}^\alpha E(t) = \frac{S(t)}{N} (\beta_A I_A(t) + \beta_S I_S(t)) - (\sigma^\alpha + \mu_1^\alpha) E(t),$$

$$D_{0,t}^\alpha I_A(t) = \eta \sigma^\alpha E(t) - (\gamma_1^\alpha + \mu_2^\alpha) I_A(t),$$

$$D_{0,t}^\alpha I_S(t) = (1 - \eta) \sigma^\alpha E(t) - (\gamma_2^\alpha + \mu_3^\alpha) I_S(t),$$

$$D_{0,t}^\alpha H(t) = \gamma_1^\alpha I_A(t) + \gamma_2^\alpha I_S(t) - (\rho^\alpha + \mu_4^\alpha) H(t).$$

We let,

$$\mathcal{F}_1 = \frac{S}{N} (\beta_A I_A + \beta_S I_S),$$

$$\mathcal{F}_2 = 0,$$

$$\mathcal{F}_3 = 0,$$

$$\mathcal{F}_4 = 0,$$

and,

$$\mathcal{V}_1 = (\sigma^\alpha + \mu_1^\alpha) E,$$

$$\mathcal{V}_2 = (\gamma_1^\alpha + \mu_2^\alpha) I_A - \eta \sigma^\alpha E,$$

$$\mathcal{V}_3 = (\gamma_2^\alpha + \mu_3^\alpha) I_S - (1 - \eta) \sigma^\alpha E,$$

$$\mathcal{V}_4 = (\rho^\alpha + \mu_4^\alpha) H - \gamma_1^\alpha I_A - \gamma_2^\alpha I_S.$$

To evaluate the next generation matrix FV^{-1} , we found the following Jacobian matrices:

$$F = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial E} & \frac{\partial \mathcal{F}_1}{\partial I_A} & \frac{\partial \mathcal{F}_1}{\partial I_S} & \frac{\partial \mathcal{F}_1}{\partial H} \\ \frac{\partial \mathcal{F}_2}{\partial E} & \frac{\partial \mathcal{F}_2}{\partial I_A} & \frac{\partial \mathcal{F}_2}{\partial I_S} & \frac{\partial \mathcal{F}_2}{\partial H} \\ \frac{\partial \mathcal{F}_3}{\partial E} & \frac{\partial \mathcal{F}_3}{\partial I_A} & \frac{\partial \mathcal{F}_3}{\partial I_S} & \frac{\partial \mathcal{F}_3}{\partial H} \\ \frac{\partial \mathcal{F}_4}{\partial E} & \frac{\partial \mathcal{F}_4}{\partial I_A} & \frac{\partial \mathcal{F}_4}{\partial I_S} & \frac{\partial \mathcal{F}_4}{\partial H} \end{bmatrix},$$

and,

$$V = \begin{bmatrix} \frac{\partial \mathcal{V}_1}{\partial E} & \frac{\partial \mathcal{V}_1}{\partial I_A} & \frac{\partial \mathcal{V}_1}{\partial I_S} & \frac{\partial \mathcal{V}_1}{\partial H} \\ \frac{\partial \mathcal{V}_2}{\partial E} & \frac{\partial \mathcal{V}_2}{\partial I_A} & \frac{\partial \mathcal{V}_2}{\partial I_S} & \frac{\partial \mathcal{V}_2}{\partial H} \\ \frac{\partial \mathcal{V}_3}{\partial E} & \frac{\partial \mathcal{V}_3}{\partial I_A} & \frac{\partial \mathcal{V}_3}{\partial I_S} & \frac{\partial \mathcal{V}_3}{\partial H} \\ \frac{\partial \mathcal{V}_4}{\partial E} & \frac{\partial \mathcal{V}_4}{\partial I_A} & \frac{\partial \mathcal{V}_4}{\partial I_S} & \frac{\partial \mathcal{V}_4}{\partial H} \end{bmatrix}.$$

Evaluating these Jacobian matrices at $u_0^* = (N, 0, 0, 0, 0, 0, 0)$ results in the following,

$$F_{u_0^*} = \begin{bmatrix} 0 & \beta_A & \beta_S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and,

$$V_{u_0^*} = \begin{bmatrix} \sigma^\alpha + \mu_1^\alpha & 0 & 0 & 0 \\ -\eta\sigma^\alpha & \gamma_1^\alpha + \mu_2^\alpha & 0 & 0 \\ -(1-\eta)\sigma^\alpha & 0 & \gamma_2^\alpha + \mu_3^\alpha & 0 \\ 0 & -\gamma_1^\alpha & -\gamma_2^\alpha & \rho^\alpha + \mu_4^\alpha \end{bmatrix}.$$

The next generation matrix is evaluated as,

$$FV^{-1} = \begin{bmatrix} \frac{\beta_A \eta \sigma^\alpha}{(\sigma^\alpha + \mu_1^\alpha)(\gamma_1^\alpha + \mu_2^\alpha)} - \frac{\beta_S(\eta-1)\sigma^\alpha}{(\sigma^\alpha + \mu_1^\alpha)(\gamma_2^\alpha + \mu_3^\alpha)} & \frac{\beta_A}{\gamma_1^\alpha + \mu_2^\alpha} & \frac{\beta_S}{\gamma_2^\alpha + \mu_3^\alpha} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the basic reproduction ratio based on the next generation method is the largest eigenvalue of the above matrix. Hence,

$$R_0 = \frac{\beta_A \eta \sigma^\alpha (\gamma_2^\alpha + \mu_3^\alpha) - \beta_S(\eta-1)\sigma^\alpha (\gamma_1^\alpha + \mu_2^\alpha)}{(\sigma^\alpha + \mu_1^\alpha)(\gamma_1^\alpha + \mu_2^\alpha)(\gamma_2^\alpha + \mu_3^\alpha)}.$$

We may also analyze the stability of the fractional $SEI_A I_S HRD$ by evaluating the Jacobian matrix at (FDE) as follows,

$$J_{u^*} = \begin{bmatrix} -\mu^\alpha & 0 & -\frac{S\beta_A}{N} & -\frac{S\beta_S}{N} & 0 & 0 & 0 \\ 0 & -\sigma^\alpha - \mu_1^\alpha & \frac{S\beta_A}{N} & \frac{S\beta_S}{N} & 0 & 0 & 0 \\ 0 & \eta\sigma^\alpha & -\gamma_1^\alpha - \mu_2^\alpha & 0 & 0 & 0 & 0 \\ 0 & (1-\eta)\sigma^\alpha & 0 & -\gamma_2^\alpha - \mu_3^\alpha & 0 & 0 & 0 \\ 0 & 0 & \gamma_1^\alpha & \gamma_2^\alpha & -\rho^\alpha + \mu_4^\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^\alpha & -\mu^\alpha & 0 \\ 0 & \mu_1^\alpha & \mu_2^\alpha & \mu_3^\alpha & \mu_4^\alpha & 0 & 0 \end{bmatrix}$$

Then, we determine the eigenvalues of the above Jacobian matrix to assess stability using the Theorem.(2.2.5).

3.3 Numerical Methods for Fractional Caputo IVPs

In this section we shall develop a class of numerical schemes to solve the fractional epidemic models (3.8) and (3.9). These schemes are based on the Lagrange interpolation of the non-linear function.

Definition 3.3.1 *Let consider grid points $\{t_j, j \in S\}$, where $S = t_0, t_1, t_2, \dots, t_n$.*

Then, the general formula of i^{th} Lagrange polynomial of order n is defined as

$$g_i(t) = \prod_{j=0, j \neq i}^n \frac{t - t_j}{t_i - t_j} \quad (3.10)$$

and the following operator defined the projection of Lagrange interpolation

$$Lg = \sum_{i=0}^m g(t_i)g_i \quad (3.11)$$

We consider the fractional initial value problem as,

$$D_{0,t}^\alpha u(t) = g(t, u(t)), \quad 0 < t < b < \infty, \quad 0 < \alpha \leq 1, \quad (3.12)$$

with the initial condition $u(0) = u_0$.

The non-linear function g satisfies the Lipschitz condition,

$$\|g(t, u_1(t)) - g(t, u_2(t))\| \leq L \|u_1(t) - u_2(t)\|, \quad L > 0$$

3.3.1 Approximation of Fractional RL-Integrals to Solve Fractional IVPs

Let $g : [0, b] \rightarrow R$ and $\alpha \in (0, 1]$. We consider the following fractional initial value problem:

$$\begin{cases} ({}_C D_{0,t}^\alpha u)(t) = g(t, u(t)), & 0 < t < b < \infty \\ u(0) = u_0 \end{cases}$$

After applying the fractional Riemann-Louisville integral to the above Caputo fractional derivative, we obtain the following formula:

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t g(s, u(s))(t-s)^{\alpha-1} ds. \quad (3.13)$$

3.3.2 Fractional Forward Euler Method

Fractional forward Euler method to approximate the fractional integral

$${}_0^{RL} I_t^{(\alpha)} g(t) = {}_{RL} D_{0,t}^{(-\alpha)}(g(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds$$

is developed. Let $g : [0, b] \rightarrow \mathbb{R}$, and consider the $n+1$ grid points $t_0, t_1, t_2, \dots, t_n$, where $t_j = jh, h = \frac{b-0}{n}$. Approximate $g(t) = g(t_i) = g_i$ on the interval $[t_j, t_{j+1}]$, $0 \leq j \leq n$ and evaluate the integral at $t = t_n$ as,

$${}_0^{RL}I_t^{(\alpha)} g(t_n) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_n - s)^{\alpha-1} g(s) ds$$

$$\begin{aligned} {}_0^{RL}I_t^{(\alpha)} g(t_n) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_n - s)^{\alpha-1} g_0 ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_n - s)^{\alpha-1} g_1 ds \\ &+ \dots + \frac{1}{\Gamma(\alpha)} \int_{t_{n-1}}^{t_n} (t_n - s)^{\alpha-1} g_{n-1} ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_n - s)^{\alpha-1} g_j ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n g_j \left[\frac{(t_n - s)^\alpha}{-\alpha} \right]_{t_j}^{t_{j+1}} \\ &= \frac{-1}{\alpha \Gamma(\alpha)} \sum_{j=0}^n g_j \left[(t_n - t_{j+1})^\alpha - (t_n - t_j)^\alpha \right] \\ &= \frac{h^\alpha}{\alpha \Gamma(\alpha)} \sum_{j=0}^n g_j \left[(n - j)^\alpha - (n - (j + 1))^\alpha \right] \end{aligned}$$

The solution $u(t)$ in (3.13) approximated at $t = t_{n+1}$ is ,

$$u_{n+1} = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} g(t_j, u_j), \quad (3.14)$$

where,

$$b_{j,n+1} = \frac{h^\alpha}{\alpha} \left[(n - j + 1)^\alpha - (n - j)^\alpha \right]. \quad (3.15)$$

3.3.3 Fractional Trapezoidal Rule

Here we develop fractional Trapezoidal rule to approximate the integral,

$${}^{RL}_0 I_t^{(\alpha)} g(t) = {}^{RL} D_{0,t}^{(-\alpha)}(g(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds$$

Let $g : [0, b] \rightarrow R$, and consider the $n + 1$ grid points $t_0, t_1, t_2, \dots, t_n$, where $t_k = kh$, $h = \frac{b-0}{n}$. On each sub-interval, $[t_k, t_{k+1}]$, $k = 0, 1, 2, \dots, n - 1$ approximate $g(t)$ by a linear Lagrange polynomial $P^{k+1}(t)$ is defined as,

$$\begin{aligned} P^1(t) &= \frac{t-t_1}{t_0-t_1}g(t_0) + \frac{t-t_0}{t_1-t_0}g(t_1) = \frac{1}{h}[(t_1-t)g(t_0) + (t-t_0)g(t_1)], \\ P^2(t) &= \frac{t-t_2}{t_1-t_2}g(t_1) + \frac{t-t_1}{t_2-t_1}g(t_2) = \frac{1}{h}[(t_2-t)g(t_1) + (t-t_1)g(t_2)], \\ P^3(t) &= \frac{t-t_3}{t_2-t_3}g(t_2) + \frac{t-t_2}{t_3-t_2}g(t_3) = \frac{1}{h}[(t_3-t)g(t_2) + (t-t_2)g(t_3)], \\ &\vdots \quad \vdots \quad \vdots \\ P^n(t) &= \frac{t-t_n}{t_{n-1}-t_n}g(t_{n-1}) + \frac{t-t_{n-1}}{t_n-t_{n-1}}g(t_n) = \frac{1}{h}[(t_n-t)g(t_{n-1}) + (t-t_{n-1})g(t_n)]. \end{aligned}$$

Then,

$$\begin{aligned} P(t) &= \frac{1}{h}[(t_1-t)g(t_0) + ((t-t_0) + (t_2-t))g(t_1) + ((t-t_1) + (t_3-t))g(t_2) \\ &+ \dots + ((t-t_{n-2}) + (t_n-t))g(t_{n-1}) + (t-t_{n-1})g(t_n)] \\ &= \frac{1}{h}(t_1-t)g(t_0) + \frac{1}{h} \sum_{j=0}^{n-1} [(t-t_{j-1}) + (t_{j+1}-t)]g(t_j) + \frac{1}{h}(t-t_{n-1})g(t_n). \end{aligned}$$

Using this interpolating polynomial, we evaluate the integral,

$${}_0^{RL}I_t^{(\alpha)}g(t_n) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_n - s)^{\alpha-1} g(s) ds$$

as,

$$\begin{aligned} {}_0^{RL}I_t^{(\alpha)}g(t_n) &= \frac{1}{h\Gamma(\alpha)} \int_0^{t_1} (t_n - t)^{\alpha-1} (t_1 - t) g(t_0) dt \\ &+ \frac{1}{h\Gamma(\alpha)} \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (t_n - t)^{\alpha-1} (t - t_{j-1}) g(t_j) dt \\ &+ \frac{1}{h\Gamma(\alpha)} \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} (t_{j+1} - t) g(t_j) dx \\ &+ \frac{1}{h\Gamma(\alpha)} \int_{t_{n-1}}^{t_n} (t_n - t)^{\alpha-1} (t - t_{n-1}) g(t_n) dt \\ &= \frac{1}{h\Gamma(\alpha)} \sum_{k=0}^n a_{k,n} g(t_k), \end{aligned}$$

where, $a_{0,n} = \int_0^{t_1} (t_n - t)^{\alpha-1} (t_1 - t) dt$, $a_{n,n} = \int_{t_{n-1}}^{t_n} (t_n - t)^{\alpha-1} (t - t_{n-1}) dt$,

and $j = 1, 2, \dots, n-1$, $a_{j,n}$ are,

$$a_{j,n} = \int_{t_{j-1}}^{t_j} (t_n - t)^{\alpha-1} (t_{j-1} - t) dt + \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} (t - t_{j+1}) dt.$$

These integrals are solve to obtain the following formula,

$$a_{j,n+1} = \frac{h^{1+\alpha}}{\alpha(\alpha+1)} \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j=0 \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1} & 1 \leq j \leq n, \\ 1, & j=n+1 \end{cases} \quad (3.16)$$

Using piecewise linear interpolation of $g(t, u)$, we obtain the solution of (3.13) at $t = t_{n+1}$ as,

$$u_{n+1} = u_0 + \frac{1}{h\Gamma(\alpha)} \sum_{j=0}^{n+1} a_{j,n+1} g(t_j, u_j), \quad (3.17)$$

where $a_{j,n+1}$ are given by (3.16). This is an implicit method where we need to compute u_{n+1} in order to compute $g(t_{n+1}, u_{n+1})$ at $t = t_{n+1}$. We use the Euler method to predict u_{n+1}^P ,

$$u_{n+1}^P = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} g(t_j, u_j), \quad (3.18)$$

with,

$$b_{j,n+1} = \frac{h^\alpha}{\alpha} [(n-j+1)^\alpha - (n-j)^\alpha],$$

and get the following predictor-corrector method,

$$u_{n+1} = u_0 + \frac{1}{h\Gamma(\alpha)} \sum_{j=0}^n a_{j,n+1} g(t_j, u_j) + a_{n+1,n+1} g(t_{n+1}, u_{n+1}^P). \quad (3.19)$$

3.3.4 Fractional Simpson's Rule

The same idea is extended to develop the predictor-corrector Simpson's rule. Consider,

$${}^{RL}_0 I_t^{(\alpha)} g(t) = {}^{RL}_0 D_{0,t}^{(-\alpha)}(g(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds,$$

where, $g : [0, b] \rightarrow R$. We consider the $n+1$ grid points $t_0, t_1, t_2, \dots, t_n$, where $t_j = jh$,

$$h = \frac{b-0}{n}.$$

On each subinterval $[t_j, t_{j+1}]$, $j = 0, 1, 2, \dots, n-1$, approximate $g(t)$ by a quadratic

Lagrange polynomial $P^j(x)$ defined in $(t_j, g_j), (t_{j+\frac{1}{2}}, g_{j+\frac{1}{2}}), (t_{j+1}, g_{j+1})$, where $t_j = jh$.

For $j = 0$, we take $t_0, t_{\frac{1}{2}}, t_1$ points in the interval $[t_0, t_1]$ and obtain,

$$\begin{aligned} P^1(t) &= \frac{(t - t_{\frac{1}{2}})(t - t_1)}{(t_0 - t_{\frac{1}{2}})(t_0 - t_1)} g(t_0) + \frac{(t - t_0)(t - t_1)}{(t_{\frac{1}{2}} - t_0)(t_{\frac{1}{2}} - t_1)} g(t_{\frac{1}{2}}) + \frac{(t - t_0)(t - t_{\frac{1}{2}})}{(t_1 - t_0)(t_1 - t_{\frac{1}{2}})} g(t_1) \\ &= \frac{2}{h^2} (t - t_{\frac{1}{2}})(t - t_1) g(t_0) - \frac{4}{h^2} (t - t_0)(t - t_1) g(t_{\frac{1}{2}}) + \frac{2}{h^2} (t - t_0)(t - t_{\frac{1}{2}}) g(t_1). \end{aligned}$$

For $j = 1$, we take $t_1, t_{\frac{3}{2}}, t_2$ points in the interval $[t_1, t_2]$ and obtain,

$$\begin{aligned} P^2(t) &= \frac{(t - t_{\frac{3}{2}})(t - t_2)}{(t_1 - t_{\frac{3}{2}})(t_1 - t_2)} g(t_1) + \frac{(t - t_1)(t - t_2)}{(t_{\frac{3}{2}} - t_1)(t_{\frac{3}{2}} - t_2)} g(t_{\frac{3}{2}}) + \frac{(t - t_1)(t - t_{\frac{3}{2}})}{(t_2 - t_1)(t_2 - t_{\frac{3}{2}})} g(t_2) \\ &= \frac{2}{h^2} (t - t_{\frac{3}{2}})(t - t_2) g(t_1) - \frac{4}{h^2} (t - t_1)(t - t_2) g(t_{\frac{3}{2}}) + \frac{2}{h^2} (t - t_1)(t - t_{\frac{3}{2}}) g(t_2). \end{aligned}$$

Similarly for any $j = 2, 3, \dots, n-2$, we take $t_j, t_{j+\frac{1}{2}}, t_{j+1}$ points in the interval $[t_j, t_{j+1}]$ and obtain,

$$\begin{aligned}
P^{j+1}(t) &= \frac{(t - t_{j+\frac{1}{2}})(t - t_{j+1})}{(t_j - t_{j+\frac{1}{2}})(t_j - t_{j+1})}g(t_j) + \frac{(t - t_j)(t - t_{j+1})}{(t_{j+\frac{1}{2}} - t_j)(t_{j+\frac{1}{2}} - t_{j+1})}g(t_{j+\frac{1}{2}}) \\
&\quad + \frac{(t - t_j)(t - t_{j+\frac{1}{2}})}{(t_{j+1} - t_j)(t_{j+1} - t_{j+\frac{1}{2}})}g(t_{j+1}) \\
&= \frac{2}{h^2}(t - t_{j+\frac{1}{2}})(t - t_{j+1})g(t_j) - \frac{4}{h^2}(t - t_j)(t - t_{j+1})g(t_{j+\frac{1}{2}}) \\
&\quad + \frac{2}{h^2}(t - t_j)(t - t_{j+\frac{1}{2}})g(t_{j+1}).
\end{aligned}$$

For $j = n-1$, we take $t_{n-1}, t_{n-\frac{1}{2}}, t_n$ points in the interval $[t_{n-1}, t_n]$ and obtain,

$$\begin{aligned}
P^n(t) &= \frac{(t - t_{n-\frac{1}{2}})(t - t_n)}{(t_{n-1} - t_{n-\frac{1}{2}})(t_{n-1} - t_n)}g(t_{n-1}) + \frac{(t - t_{n-1})(t - t_n)}{(t_{n-\frac{1}{2}} - t_{n-1})(t_{n-\frac{1}{2}} - t_n)}g(t_{n-\frac{1}{2}}) \\
&\quad + \frac{(t - t_{n-1})(t - t_{n-\frac{1}{2}})}{(t_n - t_{n-1})(t_n - t_{n-\frac{1}{2}})}g(t_n) \\
&= \frac{2}{h^2}(t - t_{n-\frac{1}{2}})(t - t_n)g(t_{n-1}) - \frac{4}{h^2}(t - t_{n-1})(t - t_n)g(t_{n-\frac{1}{2}}) \\
&\quad + \frac{2}{h^2}(t - t_{n-1})(t - t_{n-\frac{1}{2}})g(t_n).
\end{aligned}$$

Combining all these polynomials, we get,

$$\begin{aligned}
P(t) &= \frac{2}{h^2} \left((t - t_{\frac{1}{2}})(t - t_1)g(t_0) + [(t - t_0)(t - t_{\frac{1}{2}}) + (t - t_{\frac{3}{2}})(t - t_2)]g(t_1) + [(t - t_1)(t - t_{\frac{3}{2}}) \right. \\
&\quad \left. + (t - t_{\frac{5}{2}})(t - t_3)]g(t_2) + \dots + [(t - t_{n-2})(t - t_{n-\frac{3}{2}}) + (t - t_{n-\frac{1}{2}})(t - t_n)]g(t_{n-1}) \right. \\
&\quad \left. + (t - t_{n-1})(t - t_{n-\frac{1}{2}})g(t_n) \right) - \frac{4}{h^2} \left((t - t_0)(t - t_1)g(t_{\frac{1}{2}}) \right. \\
&\quad \left. + (t - t_1)(t - t_2)g(t_{\frac{3}{2}}) + \dots + (t - t_{n-2})(t - t_{n-1})g(t_{n-\frac{3}{2}}) + (t - t_{n-1})(t - t_n)g(t_{n-\frac{1}{2}}) \right),
\end{aligned}$$

which further simplified as,

$$P(t) = \frac{2}{h^2} \left((t - t_{\frac{1}{2}})(t - t_1)g(t_0) + \frac{2}{h^2} \sum_{j=1}^{n-1} [(t - t_{j-1})(t - t_{j-\frac{1}{2}}) + (t - t_{j+\frac{1}{2}})(t - t_{j+1})]g(t_j) \right. \\ \left. + (t - t_{n-1})(t - t_{n-\frac{1}{2}})g(t_n) \right) - \frac{4}{h^2} \sum_{j=0}^{n-1} (t - t_j)(t - t_{j+1})g(t_{j+\frac{1}{2}}).$$

Multiplying the factors, we get,

$$P(t) = \frac{2}{h^2} \left((t^2 - \frac{3}{2}ht + \frac{1}{2}h^2)g(t_0) + \sum_{j=1}^{n-1} [t^2 + (\frac{3}{2} - 2j)ht + (j^2 - \frac{3}{2} + \frac{1}{2})h^2]g(t_j) \right. \\ \left. + \sum_{j=1}^{n-1} [t^2 - (\frac{3}{2} + 2j)ht + (j^2 + \frac{3}{2}j + \frac{1}{2})h^2]g(t_j) + (t^2 + (\frac{3}{2}h - 2nh))t \right. \\ \left. + (n^2 - \frac{3}{4}n + \frac{1}{2})h^2)g(t_n) \right) - \frac{4}{h^2} \sum_{j=0}^{n-1} (t^2 - (2j+1)ht + (j^2 + j)h^2)g(t_{j+\frac{1}{2}}).$$

Next, we replace $g(t)$ by $P(t)$ in the integral,

$${}_0^{RL}I_t^{(\alpha)}g(t_n) = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - t)^{\alpha-1}g(t)dx,$$

and compute the integral,

$$\begin{aligned}
{}_0^{RL}I_t^{(\alpha)} g(t_n) &= \frac{2}{h^2 \Gamma(\alpha)} \left(\int_{t_0}^{t_1} (t_n - t)^{\alpha-1} \left(t^2 - \frac{3}{2}ht + \frac{1}{2}h^2 \right) g(t_0) dt \right. \\
&\quad + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (t_n - t)^{\alpha-1} \left[t^2 + \left(\frac{3}{2} - 2j \right) ht + \left(j^2 - \frac{3}{2}j + \frac{1}{2} \right) h^2 \right] g(t_j) dx \\
&\quad + \sum_{j=1}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} \left[t^2 - \left(\frac{3}{2} + 2j \right) ht + \left(j^2 + \frac{3}{2}j + \frac{1}{2} \right) h^2 \right] g(t_j) dt \\
&\quad + \int_{t_{n-1}}^{t_n} (t_n - t)^{\alpha-1} \left[t^2 + \left(\frac{3}{2}h - 2nh \right) t + \left(n^2 - \frac{3}{2}n + \frac{1}{2} \right) h^2 \right] g(t_n) dt \Big) \\
&\quad - \frac{4}{h^2 \Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} \left(t^2 - (2j+1)ht + (j^2 + j)h^2 \right) g(t_{j+\frac{1}{2}}) dt.
\end{aligned}$$

Integrate over each subinterval and simplify to get,

$${}_0^{RL}I_t^{(\alpha)} g(t_n) = \sum_{j=0}^n a_{j,n} g(t_j) + \sum_{j=0}^{n-1} \tilde{a}_{j,n} g(t_{j+\frac{1}{2}}),$$

where [22],

$$a_{n,j} = \frac{h^\alpha}{\Gamma(\alpha+3)} \begin{cases} 4[(n+1)^{2+\alpha} - n^{2+\alpha}] - (\alpha+2)[3(n+1)^{\alpha+1} + n^{1+\alpha}] \\ \quad + (\alpha+1)(\alpha+2)n^\alpha, & j=0 \\ -(\alpha+2)[(n+1-j)^{1+\alpha} + 6(n-j)^{1+\alpha} + (n-1-j)^{1+\alpha}] \\ \quad + 4[(n+1-j)^{2+\alpha} - (n-1-j)^{2+\alpha}], & 1 \leq j \leq n-1 \\ 2-\alpha, & j=n \end{cases}$$

and,

$$\tilde{a}_{j,n} = \frac{4h^\alpha}{\Gamma(\alpha+3)} \left(2[(n-j)^{2+\alpha} - (n-j-1)^{2+\alpha}] + (\alpha+2)[(n-j+1)^{1+\alpha} + (n-j)^{1+\alpha}] \right), \quad 0 \leq j \leq n-1. \quad (3.20)$$

Fractional Simpson's Predictor Corrector Method for FIVPs:

The solution $u(t)$ in (3.13) approximated at $t = t_{n+1}$ is,

$$u_{n+1} = u_0 + \frac{1}{h^2\Gamma(\alpha)} \sum_{j=0}^{n+1} a_{j,n+1} g(t_j, u_j) + \frac{1}{h^2\Gamma(\alpha)} \sum_{j=0}^n \tilde{a}_{j,n+1} g(t_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}),$$

where,

$$a_{n,j} = \frac{h^{\alpha+2}}{2 \prod_{i=0}^2 (\alpha+i)} \begin{cases} 4[(n+2)^{2+\alpha} + (n+1)^{2+\alpha}] - (\alpha+2)[3(n+2)^{1+\alpha} + (n+1)^{1+\alpha}] \\ + (\alpha+1)(\alpha+2)(n+1)^\alpha, & j=0 \\ -(\alpha+2)[(n+2-j)^{1+\alpha} + 6(n+1-j)^{1+\alpha} + (n-1)^{1+\alpha}] \\ + 4[(n+2-j)^{2+\alpha} - (n-j)^{2+\alpha}], & 1 \leq j \leq n \\ 2-\alpha, & n+1 \end{cases}$$

$$\begin{aligned} \tilde{a}_{j,n} = \frac{4h^{\alpha+2}}{2 \prod_{i=0}^2 (\alpha+i)} & \left(2[(n-j)^{2+\alpha} - (n-j-1)^{2+\alpha}] \right. \\ & \left. + (\alpha+2)[(n-j+1)^{1+\alpha} + (n-j)^{1+\alpha}] \right), \quad 0 \leq j \leq n \end{aligned}$$

This is an implicit method where u_{n+1} is needed to compute $g(t_{n+1}, u_{n+1})$ at $t = t_{n+1}$.

We use the fractional Euler method to predict u_{n+1}^P as,

$$u_{n+1}^P = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} g(t_j, u_j),$$

where, $b_{j,n+1} = \frac{h^\alpha}{\alpha} [(n-j+1)^\alpha - (n-j)^\alpha]$.

We also need to compute the unknowns $u_{n+\frac{1}{2}}$, we again use the fractional Euler method,

$$u_{n+\frac{1}{2}} = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \tilde{b}_{j,n+1} g(t_j, u_j), \quad (3.21)$$

where,

$$\tilde{b}_{j,n+1} = \frac{h^\alpha}{\alpha} [(n-j+\frac{1}{2})^\alpha - (n-j-\frac{1}{2})^\alpha].$$

Then,

$$u_{n+1} = u_0 + \frac{1}{h^2 \Gamma(\alpha)} \sum_{j=0}^n a_{j,n+1} g(t_j, u_j) + a_{n+1,n+1} g(t_{n+1}, u_{n+1}^P) + \frac{1}{h^2 \Gamma(\alpha)} \sum_{j=0}^n \tilde{a}_{j,n+1} g(t_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}). \quad (3.22)$$

3.3.5 Fractional Fourth Order Method

On each sub-interval, $[t_j, t_{j+1}]$, $j = 0, 1, 2, \dots, n-1$, we approximate the non-linear function $g(t)$ using the cubic Lagrange polynomial $P^j(t)$ defined in $(t_j, g_j), (t_{j+\frac{1}{3}}, g_{j+\frac{1}{3}}), (t_{j+\frac{2}{3}}, g_{j+\frac{2}{3}}), (t_{j+1}, g_{j+1})$, where, $t_k = kh$.

For $j = 0$, we take the points $t_0, t_{\frac{1}{3}}, t_{\frac{2}{3}}, t_1$ in the interval $[t_0, t_1]$ and obtain,

$$\begin{aligned}
P^1(t) &= \frac{(t - t_{\frac{1}{3}})(t - t_{\frac{2}{3}})(t - t_1)}{(t_0 - t_{\frac{1}{3}})(t_0 - t_{\frac{2}{3}})(t_0 - t_1)}g(t_0) + \frac{(t - t_0)(t - t_{\frac{2}{3}})(t - t_1)}{(t_{\frac{1}{3}} - t_0)(t_{\frac{1}{3}} - t_{\frac{2}{3}})(t_{\frac{1}{3}} - t_1)}g(t_{\frac{1}{3}}) \\
&+ \frac{(t - t_0)(t - t_{\frac{1}{3}})(t - t_1)}{(t_{\frac{2}{3}} - t_0)(t_{\frac{2}{3}} - t_{\frac{1}{3}})(t_{\frac{2}{3}} - t_1)}g(t_{\frac{2}{3}}) + \frac{(t - t_0)(t - t_{\frac{1}{3}})(t - t_{\frac{2}{3}})}{(t_1 - t_0)(t_1 - t_{\frac{1}{3}})(t_1 - t_{\frac{2}{3}})}g(t_1) \\
&= \frac{-9}{2h^3}(t - t_{\frac{1}{3}})(t - t_{\frac{2}{3}})(t - t_1)g(t_0) + \frac{27}{2h^3}(t - t_0)(t - t_{\frac{2}{3}})(t - t_1)g(t_{\frac{1}{3}}) \\
&- \frac{27}{2h^3}(t - t_0)(t - t_{\frac{1}{3}})(t - t_1)g(t_{\frac{2}{3}}) + \frac{9}{2h^3}(t - t_0)(t - t_{\frac{1}{3}})(t - t_{\frac{2}{3}})g(t_1)
\end{aligned}$$

For $j = 1$, we take the points $t_1, t_{\frac{4}{3}}, t_{\frac{5}{3}}, t_2$ in the interval $[t_1, t_2]$ and obtain,

$$\begin{aligned}
P^2(t) &= \frac{(t - t_{\frac{4}{3}})(t - t_{\frac{5}{3}})(t - t_2)}{(t_1 - t_{\frac{4}{3}})(t_1 - t_{\frac{5}{3}})(t_1 - t_2)}g(t_1) + \frac{(t - t_1)(t - t_{\frac{5}{3}})(t - t_2)}{(t_{\frac{4}{3}} - t_1)(t_{\frac{4}{3}} - t_{\frac{5}{3}})(t_{\frac{4}{3}} - t_2)}g(t_{\frac{4}{3}}) \\
&+ \frac{(t - t_1)(t - t_{\frac{4}{3}})(t - t_2)}{(t_{\frac{5}{3}} - t_1)(t_{\frac{5}{3}} - t_{\frac{4}{3}})(t_{\frac{5}{3}} - t_2)}g(t_{\frac{5}{3}}) + \frac{(t - t_1)(t - t_{\frac{4}{3}})(t - t_{\frac{5}{3}})}{(t_2 - t_1)(t_2 - t_{\frac{4}{3}})(t_2 - t_{\frac{5}{3}})}g(t_2) \\
&= \frac{-9}{2h^3}(t - t_{\frac{4}{3}})(t - t_{\frac{5}{3}})(t - t_2)g(t_1) + \frac{27}{2h^3}(t - t_1)(t - t_{\frac{5}{3}})(t - t_2)g(t_{\frac{4}{3}}) \\
&- \frac{27}{2h^3}(t - t_1)(t - t_{\frac{4}{3}})(t - t_2)g(t_{\frac{5}{3}}) + \frac{9}{2h^3}(t - t_1)(t - t_{\frac{4}{3}})(t - t_{\frac{5}{3}})g(t_2).
\end{aligned}$$

Similarly $j = 2, 3, \dots, n-2$, we take the points $t_j, t_{j+\frac{1}{3}}, t_{j+\frac{2}{3}}, t_{j+1}$ in the interval $[t_j, t_{j+1}]$ and obtain,

$$\begin{aligned}
P^{j+1}(t) &= \frac{(t - t_{j+\frac{1}{3}})(t - t_{j+\frac{2}{3}})(t - t_{j+1})}{(t_j - t_{j+\frac{1}{3}})(t_j - t_{j+\frac{2}{3}})(t_j - t_{j+1})} g(t_j) \\
&\quad + \frac{(t - t_j)(t - t_{j+\frac{2}{3}})(t - t_{j+1})}{(t_{j+\frac{1}{3}} - t_j)(t_{j+\frac{1}{3}} - t_{j+\frac{2}{3}})(t_{j+\frac{1}{3}} - t_{j+1})} g(t_{j+\frac{1}{3}}) \\
&\quad + \frac{(t - t_j)(t - t_{j+\frac{1}{3}})(t - t_{j+1})}{(t_{j+\frac{2}{3}} - t_j)(t_{j+\frac{2}{3}} - t_{j+\frac{1}{3}})(t_{j+\frac{2}{3}} - t_{j+1})} g(t_{j+\frac{2}{3}}) \\
&\quad + \frac{(t - t_j)(t - t_{j+\frac{1}{3}})(t - t_{j+\frac{2}{3}})}{(t_{j+1} - t_j)(t_{j+1} - t_{j+\frac{1}{3}})(t_{j+1} - t_{j+\frac{2}{3}})} g(t_{j+1}) \\
&= \frac{-9}{2h^3} (t - t_{j+\frac{1}{3}})(t - t_{j+\frac{2}{3}})(t - t_{j+1}) g(t_j) + \frac{27}{2h^3} (t - t_j)(t - t_{j+\frac{2}{3}})(t - t_{j+1}) g(t_{j+\frac{1}{3}}) \\
&\quad - \frac{27}{2h^3} (t - t_j)(t - t_{j+\frac{1}{3}})(t - t_{j+1}) g(t_{j+\frac{2}{3}}) + \frac{9}{2h^3} (t - t_j)(t - t_{j+\frac{1}{3}})(t - t_{j+\frac{2}{3}}) g(t_{j+1}).
\end{aligned}$$

For $j = n-1$, we take the points $t_{n-1}, t_{n-\frac{2}{3}}, t_{n-\frac{1}{3}}, t_n$ in the interval $[t_{n-1}, t_n]$ and obtain,

$$\begin{aligned}
P^n(t) &= \frac{(t - t_{n-\frac{2}{3}})(t - t_{n-\frac{1}{3}})(t - t_n)}{(t_{n-1} - t_{n-\frac{2}{3}})(t_{n-1} - t_{n-\frac{1}{3}})(t_{n-1} - t_n)} g(t_{n-1}) \\
&\quad + \frac{(t - t_{n-1})(t - t_{n-\frac{1}{3}})(t - t_n)}{(t_{n-\frac{2}{3}} - t_{n-1})(t_{n-\frac{2}{3}} - t_{n-\frac{1}{3}})(t_{n-\frac{2}{3}} - t_n)} g(t_{n-\frac{2}{3}}) \\
&\quad + \frac{(t - t_{n-1})(t - t_{n-\frac{2}{3}})(t - t_n)}{(t_{n-\frac{1}{3}} - t_{n-1})(t_{n-\frac{1}{3}} - t_{n-\frac{2}{3}})(t_{n-\frac{1}{3}} - t_n)} g(t_{n-\frac{1}{3}}) \\
&\quad + \frac{(t - t_{n-1})(t - t_{n-\frac{2}{3}})(t - t_{n-\frac{1}{3}})}{(t_n - t_{n-1})(t_n - t_{n-\frac{2}{3}})(t_n - t_{n-\frac{1}{3}})} g(t_n).
\end{aligned}$$

Using $t_{n-1} = (n-1)h$, $t_{n-\frac{2}{3}} = (n-\frac{2}{3})h$, $t_{n-\frac{1}{3}} = (n-\frac{1}{3})h$, $t_n = nh$.

$$\begin{aligned}
P^n(t) &= \frac{-9}{2h^3}(t-t_{n-\frac{2}{3}})(t-t_{n-\frac{1}{3}})(t-t_n)g(t_{n-1}) + \frac{27}{2h^3}(t-t_{n-1})(t-t_{n-\frac{1}{3}})(t-t_n)g(t_{n-\frac{2}{3}}) \\
&\quad - \frac{27}{2h^3}(t-t_{n-1})(t-t_{n-\frac{2}{3}})(t-t_n)g(t_{n-\frac{1}{3}}) + \frac{9}{2h^3}(t-t_{n-1})(t-t_{n-\frac{2}{3}})(t-t_{n-\frac{1}{3}})g(t_n).
\end{aligned} \tag{3.23}$$

Combining these polynomials, we get,

$$\begin{aligned}
P(t) &= \frac{9}{2h^3} \left(- (t-t_{\frac{1}{3}})(t-t_{\frac{2}{3}})(t-t_1)g(t_0) \right. \\
&\quad + [(t-t_0)(t-t_{\frac{1}{3}})](t-t_{\frac{2}{3}}) - (t-t_{\frac{4}{3}})(t-t_{\frac{5}{3}})(t-t_2)]f(t_1) \\
&\quad + [(t-t_1)(t-t_{\frac{4}{3}})(t-t_{\frac{5}{3}}) - (t-t_{\frac{7}{3}})(t-t_{\frac{8}{3}})(t-t_3)]g(t_2) + \dots \\
&\quad + [(t-t_{n-2})(t-t_{n-\frac{5}{3}})(t-t_{n-\frac{4}{3}}) - (t-t_{n-\frac{2}{3}})(t-t_{n-\frac{1}{3}})](t-t_n)g(t_{n-1}) \\
&\quad + (t-t_{n-1})(t-t_{n-\frac{2}{3}})(t-t_{n-\frac{1}{3}})g(t_n) \Big) \\
&\quad + \frac{27}{2h^3} \left((t-t_0)(t-t_{\frac{2}{3}})(t-t_1)g(t_{\frac{1}{3}}) - (t-t_0)(t-t_{\frac{1}{3}})(t-t_1)g(t_{\frac{2}{3}}) + (t-t_{\frac{5}{3}})(t-t_2)g(t_{\frac{4}{3}}) \right. \\
&\quad - (t-t_1)(t-t_{\frac{4}{3}})(t-t_2)g(t_{\frac{5}{3}}) + \dots + (t-t_{n-1})(t-t_{n-\frac{1}{3}})(t-t_n)g(t_{n-\frac{2}{3}}) \\
&\quad \left. - (t-t_{n-1})(t-t_{n-\frac{2}{3}})(t-t_n)g(t_{n-\frac{1}{3}}) \right).
\end{aligned}$$

By writing in summation form, we get,

$$\begin{aligned}
P(t) = & \frac{9}{2h^3} \left(-(t - t_{\frac{1}{3}})(t - t_{\frac{2}{3}})(t - t_1)g(t_0) + \right. \\
& \sum_{j=1}^{n-1} [(t - t_{j-1})(t - t_{j-\frac{2}{3}})(t - t_{j-\frac{1}{3}}) - (t - t_{j+\frac{1}{3}})(t - t_{j+\frac{2}{3}})(t - t_{j+1})]g(t_j) \\
& + (t - t_{n-1})(t - t_{n-\frac{2}{3}})(t - t_{n-\frac{1}{3}})g(t_n) \Big) \\
& + \frac{27}{2h^3} \sum_{j=0}^{n-1} [(t - t_j)(t - t_{j+\frac{2}{3}})(t - t_{j+1})g(t_{j+\frac{1}{3}}) - (t - t_j)(t - t_{j+\frac{1}{3}})(t - t_{j+1})g(t_{j+\frac{2}{3}})],
\end{aligned}$$

which further simplifies to,

$$\begin{aligned}
P(t) = & \frac{9}{2h^3} \left((-t^3 + 2ht^2 - \frac{11}{9}h^2t + \frac{2}{9}h^3)g(t_0) \right. \\
& + \sum_{j=1}^{n-1} [t^3 - (3j-2)ht^2 + (3j^2 - 4j + \frac{11}{9})h^2t - (j^3 - 2j^2 + \frac{11}{9})h^2t \\
& - (j^3 - 2j^2 + \frac{11}{9}j - \frac{2}{9})h^3]g(t_j) \\
& - [t^3 - (3j+2)ht^2 + (3j^2 + 4j + \frac{11}{9})h^2t - (j^3 + 2j^2 + \frac{11}{9}j + \frac{2}{9})h^3]g(t_j) \\
& + [t^3 - (3n-2)ht^2 + (3n^2 - 4n + \frac{11}{9})h^2t - (n^3 - 2n^2 + \frac{11}{9}n - \frac{2}{9})h^3]g(t_n) \Big) \\
& + \frac{27}{2h^3} \sum_{j=0}^{n-1} [t^3 - (3j + \frac{5}{3})ht^2 + (3j^2 + \frac{10}{3}j + \frac{2}{3})h^2t - (j^3 + \frac{5}{3}j^2 + \frac{2}{3}j)h^3]g(t_{j+\frac{1}{3}}) \\
& - \frac{27}{2h^3} \sum_{j=0}^{n-1} [t^3 - (3j + \frac{4}{3})ht^2 + (3j^2 + \frac{8}{3}j + \frac{1}{3})h^2t - (j^3 + \frac{4}{3}j^2 + \frac{1}{3}j)h^3]g(t_{j+\frac{2}{3}}).
\end{aligned} \tag{3.24}$$

Now we replace $g(t)$ by $P(t)$ in the integral,

$${}_0^{RL}I_t^{(\alpha)}g(t_n) = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - t)^{\alpha-1}g(t)dt \tag{3.25}$$

and compute the integral,

$$\begin{aligned}
{}_0^{RL}I_t^{(\alpha)}g(t_n) &= \frac{9}{2h^3\Gamma(\alpha)} \left(\int_{t_0}^{t_1} (t_n - t)^{\alpha-1} \left(-t^3 + 2ht^2 - \frac{11}{9}h^2t + \frac{2}{9}h^3 \right) g(t_0) dt \right. \\
&+ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} (t_n - t)^{\alpha-1} \left[t^3 - (3j-2)ht^2 + (3j^2 - 4j + \frac{11}{9})h^2t \right. \\
&- \left. (j^3 - 2j^2 + \frac{11}{9}j - \frac{2}{9})h^3 \right] g(t_j) dt \\
&- \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} \left[t^3 - (3j+2)ht^2 + (3j^2 + 4j + \frac{11}{9})h^2t \right. \\
&- \left. (j^3 + 2j^2 + \frac{11}{9}j + \frac{2}{9})h^3 \right] g(t_j) dt \\
&+ \int_{t_{n-1}}^{t_n} (t_n - t)^{\alpha-1} \left[t^3 - (3n-2)ht^2 + (3n^2 - 4n + \frac{11}{9})h^2t \right. \\
&- \left. (n^3 - 2n^2 + \frac{11}{9}n - \frac{2}{9})h^3 \right] g(t_n) dt \Big) \\
&+ \frac{27}{2h^3\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} \left[t^3 - (3j + \frac{5}{3})ht^2 + (3j^2 + \frac{10}{3}j + \frac{2}{3})h^2t \right. \\
&- \left. (j^3 + \frac{5}{3}j^2 + \frac{2}{3}j)h^3 \right] g(t_{j+\frac{1}{3}}) dt \\
&- \frac{27}{2h^3\Gamma(\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - t)^{\alpha-1} \left[t^3 - (3j + \frac{4}{3})ht^2 + (3j^2 + \frac{8}{3}j + \frac{1}{3})h^2t \right. \\
&- \left. (j^3 + \frac{4}{3}j^2 + \frac{1}{3}j)h^3 \right] g(t_{j+\frac{2}{3}}) dt.
\end{aligned}$$

Evaluating these integrals results in,

$$a_{j,n} = \frac{h^\alpha}{2\Gamma(\alpha+4)} \left\{ \begin{array}{ll} 2(n-1)^{\alpha+1}(27n^2 + 9n\alpha - 27n - 4\alpha + \alpha^2 + 6) \\ + n^\alpha(2\alpha^3 - 11n\alpha^2 + 12\alpha^2 + 36n^2\alpha \\ - 55n\alpha + 22\alpha - 66n - 54n^3 + 12 + 108n^2), & j = 0 \\ 54(n+1-j)^{\alpha+3} - 18(\alpha+3)(n+1-j)^{\alpha+2} \\ + 2(\alpha+2)(\alpha+3)(n+1-j)^{\alpha+1} \\ - 54(n-1-j)^{\alpha+3} - 18(\alpha+3)(n-1-j)^{\alpha+2} \\ - 2(\alpha+2)(\alpha+3)(n-1-j)^{\alpha+1} \\ - 4(\alpha+3)(n-j)^\alpha(\alpha^2 + 3\alpha + 18j^2 - 36jn + 2 + 18n^2), & 1 \leq j \leq n-1 \\ 2(\alpha^2 - 4\alpha + 6), & j = n \end{array} \right. \quad (3.26)$$

$$\tilde{a}_{j,n} = \frac{h^\alpha}{\Gamma(\alpha+4)} \left\{ \begin{array}{ll} -12(18(n-j-1)^{\alpha+3} + 8(\alpha+3)(n-j-1)^{\alpha+2} \\ + (\alpha+2)(\alpha+3)(n-j-1)^{\alpha+1}) \\ + 24(9(n-j)^{\alpha+3} - 5(\alpha+3)(n-j)^{\alpha+2} \\ + (\alpha+2)(\alpha+3)(n-j)^{\alpha+1}), & 1 \leq j \leq n-1 \end{array} \right. \quad (3.27)$$

$$\bar{a}_{j,n} = \frac{h^\alpha}{\Gamma(\alpha+4)} \begin{cases} 24[9(n-j-1)^{\alpha+3} + 5(\alpha+3)(n-j-1)^{\alpha+2} + (\alpha+2)(\alpha+3)(n-j-1)^{\alpha+1}] \\ -12[18(n-j)^{\alpha+3} - 5(\alpha+3)(n-j)^{\alpha+2} + (\alpha+2)(\alpha+3)(n-j)^{\alpha+1}], \\ 1 \leq j \leq n-1. \end{cases} \quad (3.28)$$

Fractional Fourth Order Predictor Corrector Method for FIVPs:

The approximated solution $u(t)$ at $t = t_{n+1}$ is represented as,

$$\begin{aligned} u_{n+1} = u_0 &+ \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^{n+1} a_{j,n+1} g(t_j, u_j) + \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \tilde{a}_{j,n+1} g(t_{j+\frac{1}{3}}, u_{j+\frac{1}{3}}) \\ &+ \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \bar{a}_{j,n+1} g(t_{j+\frac{2}{3}}, u_{j+\frac{2}{3}}). \end{aligned} \quad (3.29)$$

$$a_{j,n+1} = \frac{h^{\alpha+3}}{\prod_{i=0}^3 (\alpha + i)} \left\{ \begin{array}{ll} \begin{array}{l} 2(n-1)^{\alpha+1}(27n^2 + 9n\alpha - 27n - 4\alpha + \alpha^2 + 6) \\ + n^{\alpha}(2\alpha^3 - 11n\alpha^2 + 12\alpha^2 + 36n^2\alpha - 55n\alpha + 22\alpha \\ - 66n - 54n^3 + 12 + 108n^2), \end{array} & j = 0 \\ \begin{array}{l} 54(n+1-j)^{\alpha+3} - 18(\alpha+3)(n+1-j)^{\alpha+2} \\ + 2(\alpha+2)(\alpha+3)(n+1-j)^{\alpha+1} \\ - 54(n-1-j)^{\alpha+3} - 18(\alpha+3)(n-1-j)^{\alpha+2} \\ - 2(\alpha+2)(\alpha+3)(n-1-j)^{\alpha+1} \\ - 4(\alpha+3)(n-j)^{\alpha}(\alpha^2 + 3\alpha + 18j^2 - 36jn + 2 + 18n^2), \end{array} & 1 \leq j \leq n \\ 2(\alpha^2 - 4\alpha + 6), & j = n+1 \end{array} \right. \quad (3.30)$$

$$\tilde{a}_{j,n} = \frac{h^{\alpha+3}}{\prod_{i=0}^3 (\alpha + i)} \left\{ \begin{array}{l} -12[18(n-j-1)^{\alpha+3} + 8(\alpha+3)(n-j-1)^{\alpha+2}] \\ -12[(\alpha+2)(\alpha+3)(n-j-1)^{\alpha+1}] \\ +24[9(n-j)^{\alpha+3} - 5(\alpha+3)(n-j)^{\alpha+2}] \\ +24[(\alpha+2)(\alpha+3)(n-j)^{\alpha+1}], \end{array} \right. \quad 1 \leq j \leq n \quad (3.31)$$

$$\bar{a}_{j,n} = \frac{h^{\alpha+3}}{\prod_{i=0}^3 (\alpha + i)} \begin{cases} 24(9(n-j-1)^{\alpha+3} + 5(\alpha+3)(n-j-1)^{\alpha+2} \\ +(\alpha+2)(\alpha+3)(n-j-1)^{\alpha+1}) \\ -12(18(n-j)^{\alpha+3} - 8(\alpha+3)(n-j)^{\alpha+2} \\ +(\alpha+2)(\alpha+3)(n-j)^{\alpha+1}). \end{cases} \quad 1 \leq j \leq n \quad (3.32)$$

This is an implicit method. We need u_{n+1} to compute $g(t_{n+1}, u_{n+1})$ at $t = t_{n+1}$. We use the fractional Euler method to compute u_{n+1}^P . so,

$$u_{n+1}^P = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} g(t_j, u_j) \quad (3.33)$$

where,

$$b_{j,n+1} = \frac{h^\alpha}{\alpha} \left[(n-j+1)^\alpha - (n-j)^\alpha \right] \quad (3.34)$$

We also need to calculate the unknowns $u_{n+\frac{1}{3}}$ and $u_{n+\frac{2}{3}}$. We again use the fractional Euler method,

$$u_{n+\frac{1}{3}} = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \tilde{b}_{j,n+1} g(t_j, u_j), \quad (3.35)$$

$$u_{n+\frac{2}{3}} = u_0 + \frac{1}{\Gamma(\alpha)} \bar{b}_{j,n+1} g(t_j, u_j), \quad (3.36)$$

where,

$$\tilde{b}_{j,n+1} = \frac{h^\alpha}{\alpha} \left[(n-j+\frac{2}{3})^\alpha - (n-j-\frac{1}{3})^\alpha \right], \quad (3.37)$$

$$\bar{b}_{j,n+1} = \frac{h^\alpha}{\alpha} \left[(n-j+\frac{1}{3})^\alpha - (n-j-\frac{2}{3})^\alpha \right], \quad (3.38)$$

and then,

$$\begin{aligned} u_{n+1} = & u_0 + \frac{1}{2h^3\Gamma(\alpha)} \left[\sum_{j=0}^n a_{j,n+1}g(t_j, u_j) + a_{n+1,n+1}g(t_{n+1}, u_{n+1}^P) \right] \\ & + \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \tilde{a}_{j,n+1}g(t_{j+\frac{1}{3}}, u_{j+\frac{1}{3}}) + \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \bar{a}_{j,n+1}g(t_{j+\frac{2}{3}}, u_{j+\frac{2}{3}}) \end{aligned} \quad (3.39)$$

3.4 Numerical Methods for Fractional Atangana-Baleanu-Caputo (ABC) IVPs

In this section, we shall discover some numerical schemes to solve FIVPs for Atangana-Baleanu-Caputo fractional derivative. We rewrite the fractional initial value problem for (ABC) of order α , where $0 < \alpha \leq 1$,

$$\begin{cases} ({}_{ABC}D_{0,t}^\alpha u)(t) = g(t, u(t)), & 0 < t < b < \infty \\ u(t_0) = u_0, \end{cases} \quad (3.40)$$

where, $u : [0, b] \rightarrow R$.

The $({}_{ABC}D_{0,t}^\alpha u)(t)$ represents the Atangana-Baleanu-Caputo fractional (ABC) as in Eq.(3.5). After applying the (ABC) integral as in Eq.(3.6) to the above fractional

initial value problem, we obtain,

$$u(t) = u_0 + \frac{1-\alpha}{M(\alpha)}g(t, u(t)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}g(s, u(s))ds. \quad (3.41)$$

3.4.1 Left Rectangle Rule for Atangana-Baleanu-Caputo (ABC) IVPs

Considering the above initial value problem (3.40) and its solution in Eq.(3.41), it is found that the solution has the term ${}_0^{RL}I_t^{(\alpha)}g(t)$. Thus, a similar approach to what we have done for fractional Caputo IVPs. Hence, the solution $u(t)$ at $t = t_{n+1}$ will be presented as

$$u_{n+1} = u_0 + \frac{1-\alpha}{M(\alpha)}g(t_n, u(t_n)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1}g(t_j, u_j) \quad (3.42)$$

Where, b as in Eq.(3.15)

3.4.2 Fractional Trapezoidal Rule for Atangana-Baleanu-Caputo (ABC) IVPs

We again use the same method for ${}_0^{RL}I_t^{(\alpha)}g(t_n)$ that was evaluated before in the previous section based on the fractional trapezoidal rule to obtain the following scheme:

$$u_{n+1} = u_0 + \frac{1-\alpha}{M(\alpha)}g(t_n, u(t_n)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \sum_{j=0}^n a_{j,n+1}g(t_j, u_j) + a_{n+1,n+1}g(t_{n+1}, u_{n+1}^P) \quad (3.43)$$

Where $a_{j,n+1}$ as in (3.16) and the predictor corrector u_{n+1}^P refer to Eq.(3.18).

3.4.3 Fractional Simpson's Rule for Atangana-Baleanu-Caputo IVPs

After applying fractional Simpson's predictor corrector method to Eq.(3.41) and in similar terms to what we have obtained in the Caputo section, the following scheme attains the solution $u(t)$ at $t = t_{n+1}$

$$\begin{aligned} u_{n+1} = u_0 + \frac{1-\alpha}{M(\alpha)}g(t_n, u(t_n)) + \frac{\alpha}{h^\alpha M(\alpha)\Gamma(\alpha)} \sum_{j=0}^n a_{j,n+1}g(t_j, u_j) \\ + a_{n+1,n+1}g(t_{n+1}, u_{n+1}^P) + \frac{1}{h^2\Gamma(\alpha)} \sum_{j=0}^n \tilde{a}_{j,n+1}g(t_{j+\frac{1}{2}}, u_{j+\frac{1}{2}}) \end{aligned} \quad (3.44)$$

Again, u_{n+1}^P and $u_{j+\frac{1}{2}}$ are the predictor correctors as in Eqs.(3.18) and (3.21).

3.4.4 Fractional Fourth Order Method for Atangana-Baleanu-Caputo (ABC) IVPs

We use a similar technique for solving fractional Caputo derivatives, and after discretizing, we obtain

$$u_{n+1} = u_0 + \frac{1-\alpha}{M(\alpha)}g(t_n, u(t_n)) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_0^{t_n} g(t)(t_n - t)^{\alpha-1}dt \quad (3.45)$$

Since we have the same integral term ${}_0^{RL}I_t^{(\alpha)}g(t)$ as what we evaluated before, we use the same evaluation. Thus, we replace $g(t)$ by $P(x)$ as in Eq.(3.24). Hence, the

following numerical scheme represents the solution $u(t)$ for the fractional (ABC) initial value problem (3.40) approximated at $t = t_{n+1}$.

$$u_{n+1} = u_0 + \left(\frac{1-\alpha}{M(\alpha)} g(t_n, u(t_n)) \right) + \frac{\alpha}{M(\alpha)} \left[\frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^{n+1} a_{j,n+1} g(t_j, u_j) \right. \\ \left. + \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \tilde{a}_{j,n+1} g(t_{j+\frac{1}{3}}, u_{j+\frac{1}{3}}) + \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \bar{a}_{j,n+1} g(t_{j+\frac{2}{3}}, u_{j+\frac{2}{3}}) \right] \quad (3.46)$$

where $a_{j,n+1}$, $\tilde{a}_{j,n+1}$ and $\bar{a}_{j,n+1}$ are displayed in Eqs.(3.30), (3.31), and (3.26) respectively. Once again, using the corrector method, the fractional fourth-order (ABC) scheme that provides the solution $u(t)$ in equation (3.40) at $t = t_{n+1}$ is represented as

$$u_{n+1} = u_0 + \left(\frac{1-\alpha}{M(\alpha)} g(t_n, u(t_n)) \right) + \frac{\alpha}{M(\alpha)} \left[\frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n a_{j,n+1} g(t_j, u_j) + a_{n+1,n+1} g(t_{n+1}, u_{n+1}^P) \right. \\ \left. + \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \tilde{a}_{j,n+1} g(t_{j+\frac{1}{3}}, u_{j+\frac{1}{3}}) + \frac{1}{2h^3\Gamma(\alpha)} \sum_{j=0}^n \bar{a}_{j,n+1} g(t_{j+\frac{2}{3}}, u_{j+\frac{2}{3}}) \right] \quad (3.47)$$

where u_{n+1}^P , $u_{n+\frac{1}{3}}$ and $u_{n+\frac{2}{3}}$ as in Eqs.(3.33), (3.35) and (3.36), respectively.

3.4.5 Error Analysis

Lemma 3.4.1 *Let $u(t)$ be the solution in (3.13). By assuming $g(t, u)$ is bounded and τ is sufficiently small, the following inequality holds:*

$$|u(t + \tau) - u(t)| \leq M_1 \tau^{\gamma(\alpha)}, \quad t \in [0, T - \tau]$$

where,

$$\gamma(\alpha) = \begin{cases} \alpha, & 0 < \alpha < 1 \\ 1, & \alpha \geq 1 \end{cases}$$

and M_1 is a constant that is independent of τ .

Proof. As in [22], by the assumption that g is bounded, there exists a positive constant N such that $|g| \leq N$. By (3.13), we obtain

$$u(t + \tau) - u(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t+\tau} g(s, u(s)) (t + \tau - s)^{\alpha-1} ds - \frac{1}{\Gamma(\alpha)} \int_0^t g(s, u(s)) (t - s)^{\alpha-1} ds$$

Thus

$$\begin{aligned} |u(t + \tau) - u(t)| &\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^t ((t + \tau - s)^{\alpha-1} - (t - s)^{\alpha-1}) g(s, u(s)) ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left| \int_t^{t+\tau} (t + \tau - s)^{\alpha-1} g(s, u(s)) ds \right| \\ &\leq \frac{N}{\Gamma(\alpha)} \int_0^t |((t + \tau - s)^{\alpha-1} - (t - s)^{\alpha-1})| ds + \frac{N}{\Gamma(\alpha)} \int_t^{t+\tau} |(t + \tau - s)^{\alpha-1}| ds \\ &= \frac{N}{\Gamma(\alpha + 1)} |\tau^\alpha - ((t + \tau)^\alpha - t^\alpha)| + \frac{N\tau^\alpha}{\Gamma(\alpha + 1)} \\ &\leq \frac{2N\tau^\alpha}{\Gamma(\alpha + 1)} + \frac{N}{\Gamma(\alpha + 1)} |(t + \tau)^\alpha - t^\alpha| \end{aligned}$$

Now we discuss the following cases in order to prove the following:

$$|(t + \tau)^\alpha - t^\alpha| \leq \begin{cases} \tau^\alpha, & 0 < \alpha < 1 \\ \alpha T^{\alpha-1} \tau, & \alpha \geq 1 \end{cases} \quad (3.48)$$

Case(1): if $\alpha \geq 1$, then by using the mean value theorem, we obtain,

$$\begin{aligned} |(t + \tau)^\alpha - t^\alpha| &\leq \alpha\tau(t + \theta)^{\alpha-1} \\ &\leq \alpha T^{\alpha-1}\tau \end{aligned}$$

Case(2): If $0 < \alpha < 1$, we prove the following inequality:

$$y^\alpha - x^\alpha \leq (y - x)^\alpha, \quad 0 \leq x \leq y$$

which is equivalent to

$$1 - x^\alpha \leq (1 - x)^\alpha, \quad 0 \leq x \leq 1 \quad (3.49)$$

Let $f(x) = 1 - x^\alpha - (1 - x)^\alpha$, then $f'(x) = \alpha((1 - x)^{\alpha-1} - x^{\alpha-1})$.

Thus, we can achieve the possible maximum value of $f(x)$ at $x = 0, 1$, or $\frac{1}{2}$. It is obvious that $f(0) = f(1) = 0$ and $f(\frac{1}{2}) = 1 - 2^{1-\alpha}$, so that $f(x) \leq 0$, which implies the inequality (3.49).

Therefore, $(t + \tau)^\alpha - t^\alpha \leq \tau^\alpha$ holds. Thus, the inequality (3.48) holds.

As a result, we obtain the following:

$$|u(t + \tau) - u(t)| \leq \begin{cases} \frac{3N\tau^\alpha}{\Gamma(\alpha+1)}, & 0 < \alpha < 1 \\ \frac{2N\tau^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha NT^{\alpha-1}\tau}{\Gamma(\alpha+1)}, & \alpha \geq 1 \end{cases}$$

Using the assumption that τ is sufficiently small, we obtain

$$|u(t + \tau) - u(t)| \leq M_1 \tau^{\gamma(\alpha)}$$

Where, $M_1 = \max \left\{ \frac{3N}{\Gamma(\alpha+1)}, \frac{2N+\alpha NT^{\alpha-1}}{\Gamma(\alpha+1)} \right\}$, which completed the proof. ■

Lemma 3.4.2 [22] *Let $u(t)$ be the solution in (3.13), and assume that $g(t, u(t))$ is a continuous function and satisfies the Lipschitz condition with respect to t and the second variable u with a Lipschitz constant K . If τ is sufficiently small, then the following inequality holds:*

$$\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} g(t, u(t)) dt - \frac{\tau^\alpha}{\alpha} \sum_{j=0}^n b_{j,n+1} g(t_j, u(t_j)) \right| \leq M_2 \tau^{\gamma(\alpha)}$$

Proof. To prove this lemma, we use the product rectangle formula and Lemma (3.4.2). Also, the Lipschitz condition is used with some simplification.

$$\begin{aligned} & \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} g(t, u(t)) dt - \frac{\tau^\alpha}{\alpha} \sum_{j=0}^n b_{j,n+1} g(t_j, u(t_j)) \right| \\ &= \left| \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (g(t, u(t)) - g(t_j, u(t_j))) dt \right| \\ &\leq \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} (|g(t, u(t)) - g(t, u(t_j))| + |g(t, u(t_j)) - g(t_j, u(t_j))|) dt \\ &\leq K \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - t)^{\alpha-1} ((t - t_j) + M_1(t - t_j)^{\gamma(\alpha)}) dt \\ &\leq K(1 + M_1) \tau^{\gamma(\alpha)} \frac{t_{n+1}^\alpha}{\alpha} \end{aligned}$$

The proof is completed by letting $M_2 = K(1 + M_1)^{\frac{t_{n+1}^\alpha}{\alpha}}$. |

Theorem 3.4.3 *If $g \in C^4([0, T])$, then for each subinterval $[t_k, t_{k+1}]$ the error of the correspond numerical solution of $\tilde{g}$ is of order $O(\Delta t^4)$.*

Proof. Let $\tilde{g}$ be the Lagrange interpolation that is displayed in Eq.(3.10), and $\tilde{u}$ be the corresponding numerical solution. Then,

$$\begin{aligned}
& |[D_{0,t}^\alpha u(t)]_{t=t_n} - [D_{0,t}^\alpha \tilde{u}(t)]_{t=t_n}| \\
& \leq \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} |g(s) - \tilde{g}(s)| ds \\
& \leq \max_{t \in [0, t_n]} |g^4(t)| \frac{1}{24\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} \prod_{j=0}^3 |s - s_j^{(k)}| ds \\
& \leq \max_{t \in [0, t_n]} |g^4(t)| \frac{(t_{k+1} - t_k)^4}{24\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} ds \\
& = \max_{t \in [0, t_n]} |g^4(t)| \frac{t_n^\alpha}{24\Gamma(\alpha + 1)} \Delta t^4.
\end{aligned}$$
|

3.4.6 Numerical Experiment

We discover the following examples to test our numerical methods for solving FDEs.

Example 3 *Consider the following fractional differential equation*

$${}_C D_{0,t}^\alpha u(t) = t^\mu \tag{3.50}$$

with its corresponding exact solution as

$$u(t) = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \alpha)} t^{\mu - \alpha}$$

Let $\mu = 4$, then by applying the left rectangle rule presented in (3.17), the Table (3.1)

displays the errors and convergence order for different values of α .

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$		$\alpha = 0.9$	
h	Error	Order	Error	Order	Error	Order	Error	Order
0.1000	1.27e-02	0.0000	7.19e-02	0.0000	2.62e-01	0.0000	3.86e-01	0.0000
0.0500	4.13e-03	1.6201	2.73e-02	1.3952	1.19e-01	1.1357	1.87e-01	1.0443
0.0250	1.30e-03	1.6643	1.01e-02	1.4336	5.31e-02	1.1655	8.91e-02	1.0714
0.0125	4.03e-04	1.6937	3.69e-03	1.4566	2.34e-02	1.1813	4.20e-02	1.0852
0.0063	1.23e-04	1.7145	1.33e-03	1.4710	1.03e-02	1.1897	1.97e-02	1.0923

Table 3.1: The errors and the orders of convergence for example 3 using the left rectangle scheme that is represented in (3.17)

Example 4 Consider the following fractional differential equation

$${}_CD_{0,t}^\alpha u(t) = \sin(t) \quad (3.51)$$

Using scheme (3.19), the following table shows the error and the order of convergence of using the fractional trapezoidal rule to solve the above FDEs.

	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$		$\alpha = 0.9$	
h	Error	Order	Error	Order	Error	Order	Error	Order
0.1000	3.88e-04	0.0000	5.0601e-04	0.0000	4.46e-04	0.0000	4.15e-04	0.0000
0.0500	1.06e-04	1.8751	1.30e-04	1.9562	1.12e-04	1.9912	1.04e-04	1.9967
0.0250	2.83e-05	1.9008	3.33e-05	1.9704	2.81e-05	1.9951	2.60e-05	1.9982
0.0125	7.49e-06	1.9193	8.44e-06	1.9797	7.04e-06	1.9973	6.51e-06	1.9991
0.0063	1.96e-06	1.9335	2.13e-06	1.9859	1.76e-06	1.9984	1.3e-06	1.9995

Table 3.2: The errors and the orders of convergence for example 4 using the Trapezoidal scheme that is represented in (3.19)

3.4.7 Numerical Results

Left Rectangle Rule

In this numerical experiment, scheme (3.17) is used to discover the *SEIR* model that is displayed in (3.7) with involving the natural birth and death parameter μ . We use the same parameters and initial values in [94] to obtain the following results in Fig.3.1.

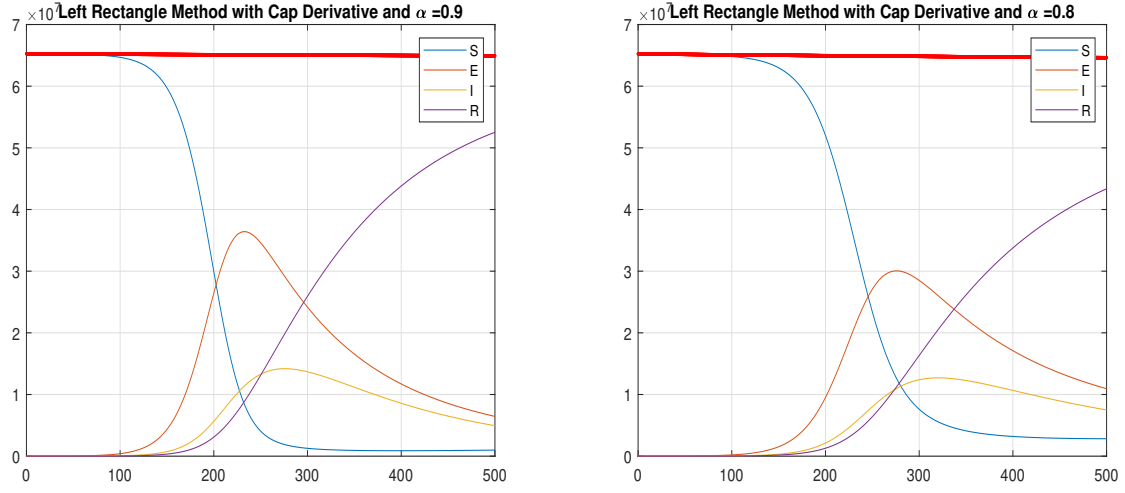


Figure 3.1: Fractional $SEIR$ model with birth and death using the Left Rectangle Rule

Fractional Trapezoidal Rule

Now we consider the basic $SEIR$ model displayed in (3.7) as a fractional model. We numerically solve the model using the fractional trapezoidal scheme (3.19) for two different values of α with the following chosen initial values and parameters,

$$S(0) = 1000, \quad E(0) = 20, \quad I(0) = 40, \quad R(0) = 10$$

$$\frac{\beta}{N} = 0.01, \quad \sigma = 0.02, \quad \gamma = 0.1$$

The simulation of the $SEIR$ model presented in Fig.3.2

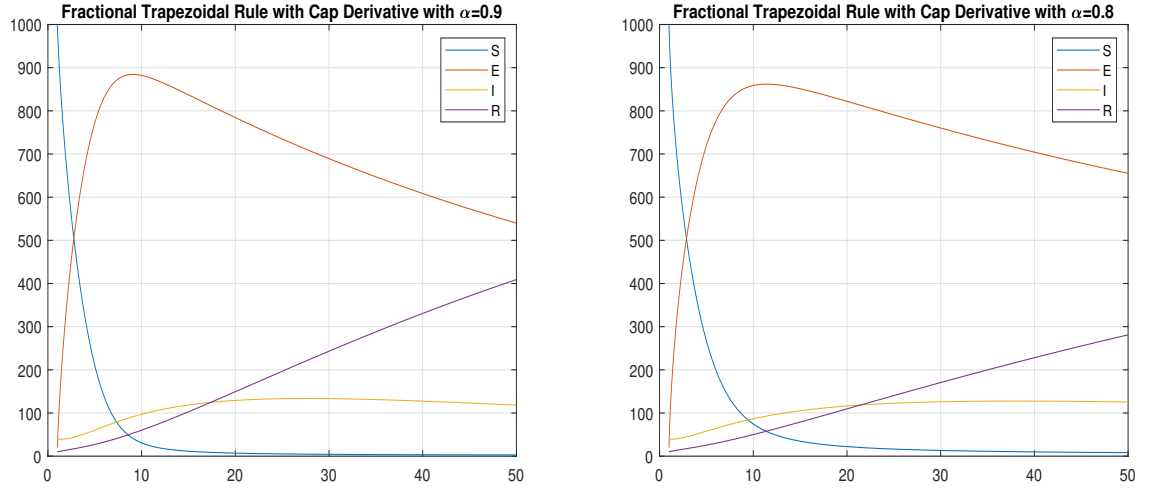


Figure 3.2: Fractional $SEIR$ model without considering natural birth or death using Trapezoidal Rule

Fractional Simpson's Rule

Now we apply the scheme (3.22) in order to simulate the fractional $SEIRNP$ model, which is represented in (3.8). We use the same parameters and initial values that Furati et al. used in [41]. The numerical results for two different values of α for this fractional Simpson's method are displayed in Fig.3.3

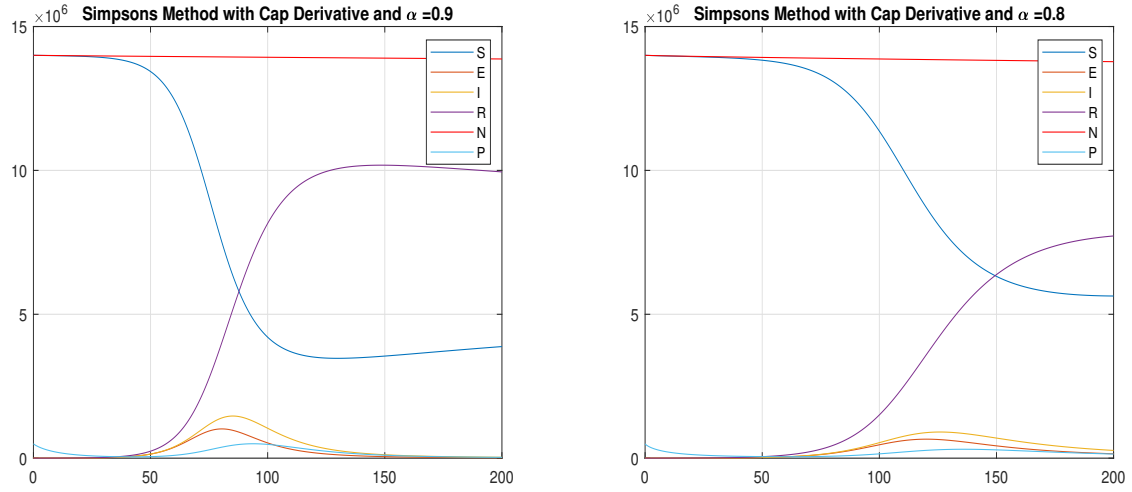


Figure 3.3: Fractional $SEIRNP$ model using Simpson's Rule

Fractional Fourth Order Method

In the Fig.3.4, the fractional $SEI_A I_S HRD$ model is used as in (3.9) with the same parameters and initial values as in [13]. These graphs illustrate the numerical analysis of the extended $SEIR$ model using the fourth-order method represented in scheme (3.39).

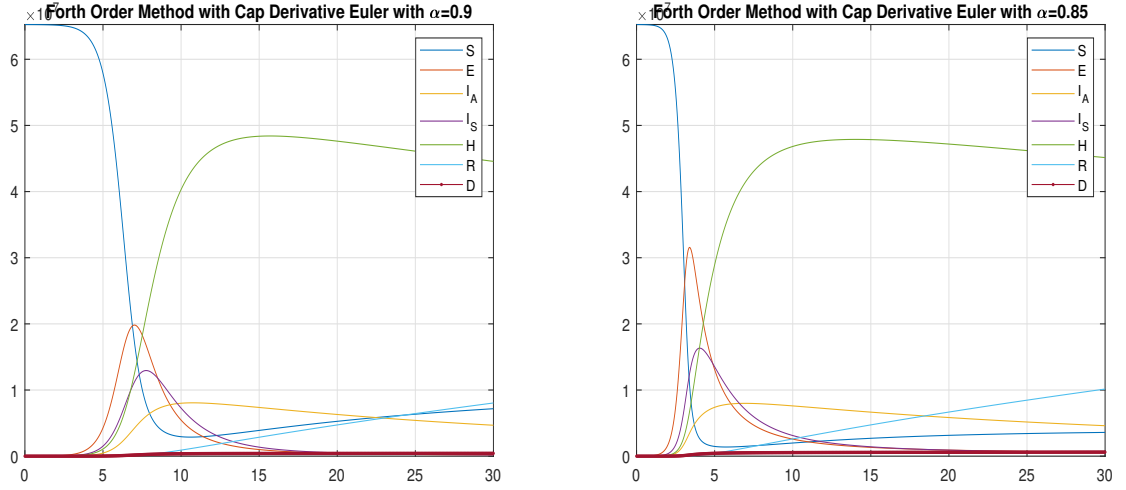


Figure 3.4: Fractional $SEI_A I_S HRD$ model using fourth order method

The following graphs display the simulation of fractional epidemic models for Atangana-Baleanu-Caputo (ABC) with the same models and parameters used in the previous figures. These figures display the numerical results using the fractional left rectangle rule, trapezoidal rule, Simpson's rule, and fourth-order method for (ABC) epidemic models using schemes (3.42), (3.43), (3.45) and (3.47), respectively.

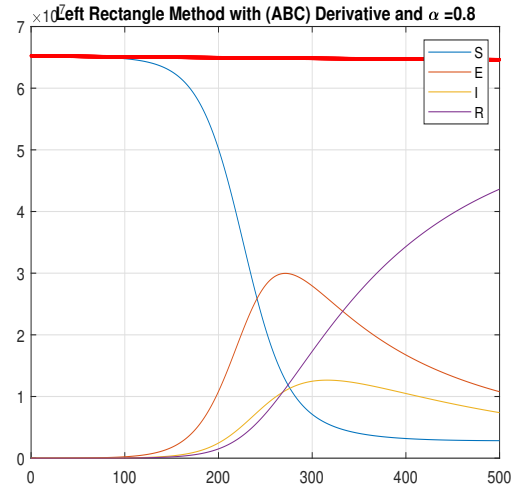
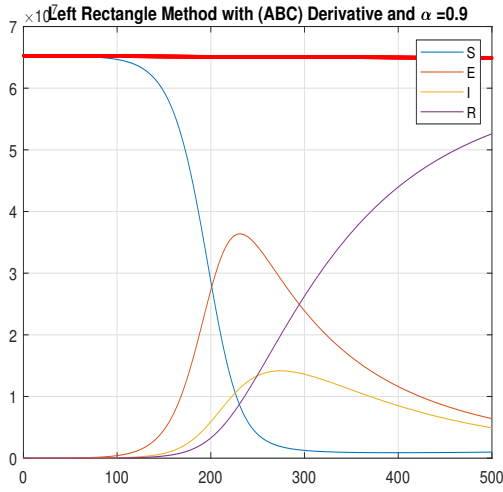


Figure 3.5: Fractional (ABC) $SEIR$ model with consideration of birth and death

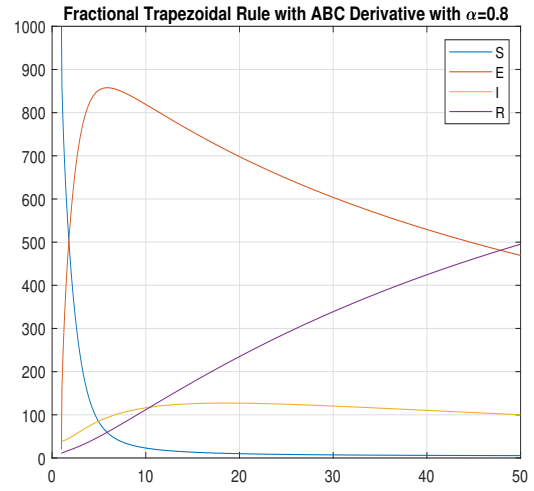
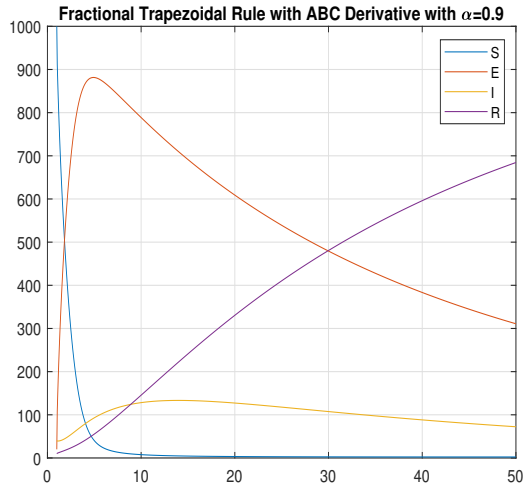


Figure 3.6: Fractional (ABC) $SEIR$ model without considering birth and death

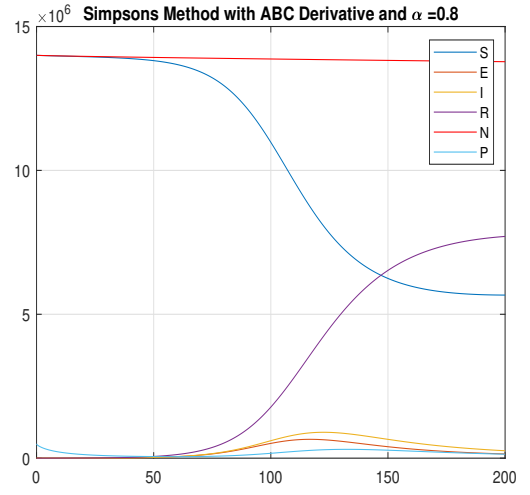
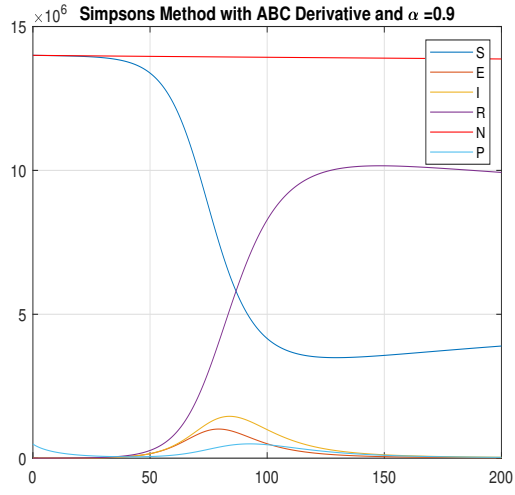


Figure 3.7: Fractional (ABC) $SEIRNP$ model

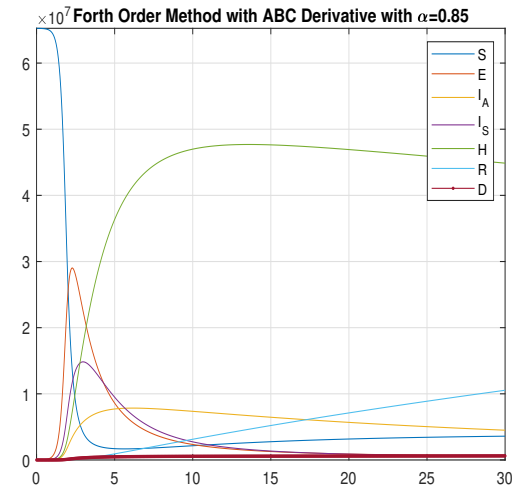
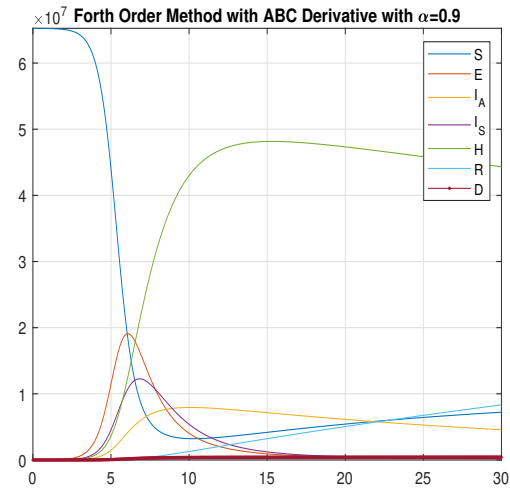


Figure 3.8: Fractional (ABC) $SEI_A I_S HRD$ model

CHAPTER 4

DIFFUSIVE EPIDEMIC MODEL

Several important qualitative features are better modeled by non-local fractional operators. A number of studies have demonstrated the usefulness of the fractional partial differential equations because of their ability to capture the aspects of non-locality and spatial heterogeneity in contrast to the corresponding classical partial differential equations. Space fractional diffusive models can capture long-range interactions, while the classical diffusion equation assumes local interactions that may not capture the full dynamics of diseases with long-range transmission patterns [87].

During the study of fractional differential equations, the most crucial task of researchers is to find the exact solution of the dynamical systems. The exact solution to most of the problems is not known, and therefore one must search for a numerical solution. Because of the complexity of fractional operators, it is always a challenging task for researchers to find numerical solutions to the non-linear systems.

Diffusive epidemic models are mathematical frameworks that provide insight into the influence of spatial influence on the spread of infectious diseases. The diffusion term is incorporated into epidemic models to account for the movement of individuals

in space, which can significantly affect the dynamics of disease transmission. Researchers have developed various modeling techniques to understand and predict the spread of infectious diseases such as COVID-19.

Most of the epidemic models consist of a system of non-linear ordinary differential equations. These epidemic models lack a realistic approach, which is due to the large mobility of people within a country or even worldwide. The mixing of populations takes place without differentiating individuals in different places. Since the spread of the disease may be faster in one part of the population and slower in the other part, therefore, the spatial effect cannot be ignored while studying the spread of the disease. Al-Showaikh and Twizell [5] extend the *SEIR* model by adding second-order space derivative (diffusion) terms in the model. Since then, several researchers have considered epidemic models that include diffusion terms.

In this chapter, we shall develop an L -stable numerical scheme to investigate the disease dynamics by solving epidemic models with classical along with fractional diffusion.

The most popular approach used to develop time-stepping schemes for reaction-diffusion systems is exponential time differencing (ETD). This approach, originally developed for the classical reaction-diffusion problems, has now been extended to the fractional reaction-diffusion problems. The exact solution of the spatially discretized problem obtained by using Duhamel's principle is a Volterra integral equation involving matrix exponential functions and a non-linear function $F(u, t)$. Several numerical schemes are developed by approximating the non-linear function by interpolating polynomials and matrix exponential functions by rational functions, such as Padé

approximations. We shall develop an L -stable time-stepping scheme based on the positivity-preserving-rational approximation of the matrix-exponential function.

The space fractional Laplacian operator is defined in several ways depending on the context in which it is applied. The spectral fractional Laplacian considered in this work is discretized using the Fourier spectral approach presented by Bueno-Orovio et al. [17]. On a bounded domain with homogeneous Dirichlet or homogeneous Neumann boundary conditions, the fractional Laplacian is defined spectrally using eigenvalues and corresponding eigenfunctions of the classical Laplacian operator on the domain. The advantage of this approach is that it results in a diagonal representation of the fractional Laplacian operator.

4.1 The Description of Diffusive Epidemic Models

We shall consider two epidemic models with diffusion, SIR and $SEIR$ models.

The following SIR model with modified saturated incidence rate and without delay factor is considered [93],

$$\begin{aligned}\frac{dS}{dt} &= \eta - \nu S(x, t) - \frac{\beta S(x, t)I(x, t)}{1 + \sigma_1 S(x, t) + \sigma_2 I(x, t)}, \\ \frac{dI}{dt} &= \frac{\beta S(x, t)I(x, t)}{1 + \sigma_1 S(x, t) + \sigma_2 I(x, t)} - (\nu + \delta + r)I(x, t), \\ \frac{dR}{dt} &= rI(x, t) - \nu R(x, t),\end{aligned}$$

where $S(x, t)$, $I(x, t)$, and $R(x, t)$ are the number of susceptible, infected, and recovered individuals, respectively at location x and time t . The parameter η is the recruitment

rate, ν the rate of natural death, δ is the rate of death due to disease, r is the rate of recovery, and β is the infection coefficient. The modified incidence rate is,

$$\frac{\beta S(x, t)I(x, t)}{1 + \sigma_1 S(x, t) + \sigma_2 I(x, t)},$$

with inhibitory effects measures $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$.

Nauman et al. [3] extended this model by including diffusion terms in this model,

$$\begin{aligned}\frac{\partial S}{\partial t} &= D_S \Delta S(x, t) + \eta - \nu S(x, t) - \frac{\beta S(x, t)I(x, t)}{1 + \sigma_1 S(x, t) + \sigma_2 I(x, t)}, \\ \frac{\partial I}{\partial t} &= D_I \Delta I(x, t) + \frac{\beta S(x, t)I(x, t)}{1 + \sigma_1 S(x, t) + \sigma_2 I(x, t)} - (\nu + \delta + r)I(x, t), \\ \frac{\partial R}{\partial t} &= D_R \Delta R(x, t) + rI(x, t) - \nu R(x, t),\end{aligned}$$

where D_S , D_I , D_R are the diffusion coefficients. In this work, we shall consider an extension of this model by replacing the classical Laplacian with the fractional Laplacian,

$$\begin{aligned}\frac{\partial S}{\partial t} &= -D_S(-\Delta)^{\frac{\alpha}{2}} S(x, t) + \eta - \nu S(x, t) - \frac{\beta S(x, t)I(x, t)}{1 + \sigma_1 S(x, t) + \sigma_2 I(x, t)}, \\ \frac{\partial I}{\partial t} &= -D_I(-\Delta)^{\frac{\alpha}{2}} I(x, t) + \frac{\beta S(x, t)I(x, t)}{1 + \sigma_1 S(x, t) + \sigma_2 I(x, t)} - (\nu + \delta + r)I(x, t), \\ \frac{\partial R}{\partial t} &= -D_R(-\Delta)^{\frac{\alpha}{2}} R(x, t) + rI(x, t) - \nu R(x, t),\end{aligned}\tag{4.1}$$

where $1 < \alpha \leq 2$.

A diffusive *SEIR* model for the transmission dynamics of measles considered by Al-Showaikh and Twizel [5] determines the geographic spread of the disease in a pop-

ulation that has not been vaccinated against it. We have also considered an extension of this model by combining fractional Laplacian instead of the classical Laplacian.

$$\begin{aligned}
\frac{\partial S}{\partial t} &= -D_S(-\Delta)^{\frac{\alpha}{2}} S(x, t) + \mu N - (\mu + \beta I)S \\
\frac{\partial E}{\partial t} &= -D_E(-\Delta)^{\frac{\alpha}{2}} E(x, t) + \beta IS - (\mu + \sigma)E \\
\frac{\partial I}{\partial t} &= -D_I(-\Delta)^{\frac{\alpha}{2}} I(x, t) + \sigma E - (\mu + \gamma)I \\
\frac{\partial R}{\partial t} &= -D_R(-\Delta)^{\frac{\alpha}{2}} R(x, t) + \gamma I - \mu R
\end{aligned} \tag{4.2}$$

Since R is not present in the first three equations, we shall solve the system only for S , E , and I and compute $R = N - S - E - I$, where N is the total population under consideration.

The exact solution to most of the problems is unknown and therefore we must search for a numerical solution. Because of the complexity of fractional operators, it is always a challenging task for researchers to find numerical solutions of non-linear systems. The numerical schemes we use must approximate the solution that preserves all the traits that hold for the solution of the epidemic model, such as positivity, boundness, stability properties of steady states, etc. The positivity of the numerical solution is an important property of the solution of epidemic models. Many modern and classical methods fail to preserve the positivity property.

4.2 Reaction Diffusion Equation and Spatial Discretization

We consider the following initial boundary value problem,

$$\frac{\partial u}{\partial t} = -\mathcal{K}(-\Delta)^{\frac{\alpha}{2}} u + F(u, t), \quad (x, t) \in \Omega \times (0, T], \quad (4.3)$$

with initial condition,

$$u(x, 0) = u_0(x), \quad x \in \Omega \subset \mathbb{R},$$

and having either Dirichlet or Neumann boundary conditions. The non-linear term $F(u, t)$ satisfies the Lipschitz condition, that is,

$$\|F(u, t) - F(\tilde{u}, t)\| \leq L\|u - \tilde{u}\|,$$

for some $L > 0$. The initial condition function u_0 can be non-smooth or we can have mismatched initial-boundary conditions.

The spectral definition of the fractional Laplacian is most useful for developing numerical methods. It requires computing the eigenvalues and eigenfunctions of the classical Laplacian. Straightforward summation results in fractional Laplacian discretization in multidimensional cases. Explicit analytical formulas are available to compute the eigenvalues and corresponding eigenfunctions of the Laplacian.

Definition 4.1 Assume that the Laplacian operator $(-\Delta)$ has a complete set of orthonormal eigenfunctions ϕ_n with corresponding eigenvalues λ_n on a bounded domain $\Omega \subset \mathbb{R}$ with the homogeneous boundary conditions, then,

$$(-\Delta)\phi_n = \lambda_n\phi_n.$$

Let

$$u = \sum_{n=0}^{\infty} \hat{u}_n \phi_n, \quad \hat{u}_n = \langle u, \phi_n \rangle, \quad \sum_{n=0}^{\infty} |\hat{u}_n|^2 |\lambda_n|^\alpha < \infty.$$

Then

$$(-\Delta)^{\frac{\alpha}{2}} u = \sum_{n=0}^{\infty} \hat{u}_n \lambda_n^{\frac{\alpha}{2}} \phi_n.$$

The eigenvalues and eigenfunctions of the classical Laplace operator depend on the choice of the boundary conditions. For $x \in \Omega = (a, b)$, the eigenvalues and eigenvectors of the Laplacian with homogeneous Dirichlet boundary conditions are given by,

$$\begin{aligned} \lambda_n &= \left(\frac{(n+1)\pi}{L} \right)^2, \\ \phi_n &= \sqrt{\frac{2}{L}} \sin \left(\frac{(n+1)\pi(x_n - a)}{L} \right), \end{aligned}$$

respectively, where $L = b - a$ is the length of the interval. Similarly, the eigenvalues and eigenvectors corresponding to the homogeneous Neumann boundary conditions

are,

$$\begin{aligned}\lambda_n &= \left(\frac{n\pi}{L}\right)^2, \\ \phi_n &= \sqrt{\frac{2}{L}} \cos\left(\frac{n\pi(x_n - a)}{L}\right),\end{aligned}$$

respectively. We approximate the fractional Laplacian operator by the truncated series using finite number of the orthonormal trigonometric functions ϕ_n ,

$$(-\Delta)^{\frac{\alpha}{2}} u \approx \sum_{n=0}^{N_x-1} \hat{u}_n \lambda_n^{\frac{\alpha}{2}} \phi_n. \quad (4.4)$$

The Fourier coefficients $\hat{u}_n$ in (4.4) are computed using highly efficient algorithms known as "Discrete Fourier Transforms." The computational efficiency of these algorithms is due to the use of the Fast Fourier Transform (FFT) in computing matrix vector products.

For periodic boundary conditions, please see the reference [91] pages 23-46. For the one dimensional problems, we represent equation (5.1) in Fourier space by applying the Fourier transform [17], [95],

$$\frac{d}{dt} \mathcal{F}\{u(x, t)\} + A \mathcal{F}\{u(x, t)\} = \mathcal{F}\{F(u, t)\}, \quad (4.5)$$

where the discretization matrix is $A = \mathcal{K} \text{diag}[(\lambda_n)^\alpha]$. Denoting $\mathcal{F}\{u(x, t)\} = \hat{u}(t)$, $\mathcal{F}\{F(u, t)\} = \hat{F}(u, t)$, we can write (4.5) as,

$$\frac{d}{dt} \hat{u}(t) + A \hat{u}(t) = \hat{F}(u, t). \quad (4.6)$$

We compute the coefficients $\hat{u}$ using Fourier transform and the inverse Fourier transforms to reconstruct the solution $u(x, t)$ in the physical space. Based on the specified homogeneous boundary conditions, discrete sine or cosine transforms and their inverses are applied [16].

4.3 Time-Stepping Schemes

We develop a third order L -stable numerical method by considering the following abstract initial value problem

$$\begin{aligned}\hat{u}_t + A\hat{u} &= \hat{F}(u, t), \quad t \in (0, T], \\ u(0) &= u_0.\end{aligned}\tag{4.7}$$

Using the integrating factor, we write the exact solution of (4.7) as:

$$\hat{u}(t) = e^{-tA}\hat{u}_0 + \int_0^t e^{-(t-s)A}\hat{F}(u(s), s)ds.\tag{4.8}$$

For some fixed k_o , let $0 < k \leq k_o$ be some fixed time step with $t_n = nk$. Evaluating (4.8) at $t = t_n$, we get,

$$\hat{u}(t_n) = e^{-t_n A}\hat{u}_0 + \int_0^{t_n} e^{-(t_n-s)A}\hat{F}(u(s), s)ds,\tag{4.9}$$

Similarly evaluating (4.8) at $t = t_{n+1} = t_n + k$, we get,

$$\hat{u}(t_n + k) = e^{-(t_n+k)A}\hat{u}_0 + \int_0^{t_n+k} e^{-(t_n+k-s)A}\hat{F}(u(s), s)ds,$$

which is simplified as,

$$\begin{aligned}\hat{u}(t_n + k) &= e^{-(t_n+k)A}\hat{u}_0 + \int_0^{t_n} e^{-(t_n+k-s)A}\hat{F}(u(s), s)ds + \int_{t_n}^{t_n+k} e^{-(t_n+k-s)A}\hat{F}(u(s), s)ds \\ &= e^{-kA}\hat{u}(t_n) + \int_{t_n}^{t_n+k} e^{-(t_n+k-s)A}\hat{F}(u(s), s)ds.\end{aligned}$$

Now we apply the transformation $s - t_n = k\tau$ or $s = t_n + k\tau$ and $ds = k d\tau$ and get the recurrence formula,

$$\hat{u}_{n+1} = e^{-kA}\hat{u}_n + k \int_0^1 e^{-k(1-\tau)A}\hat{F}(u(t_n + k\tau), t_n + k\tau)d\tau, \quad (4.10)$$

where $\hat{u}_n = \hat{u}(t_n)$. The computational challenges associated with approximating the recurrence formula (4.10) include an accurate and efficient approximation of matrix exponential functions and integral terms. Approximating the integral is quite challenging because it involves a non-linear function of the unknown function $\hat{u}(x, t)$. The computational accuracy of the solution depends on how well we approximate these matrix exponential functions and the integral terms in (4.10). We employ the exponential time-differencing (ETD) approach to approximate the integral.

4.3.1 A Third Order Method

Third order scheme is developed using the following steps.

Step:1 Approximate $\hat{F}(u(t), t)$ by a constant $\hat{F}_n = \hat{F}(u(t_n), t_n)$ for all $t \in [t_n, t_{n+1}]$ and compute the integral in the recurrence formula (4.10) and label it as $\hat{a}_n$,

$$\hat{a}_n \approx \Phi_0\left(\frac{kA}{2}\right)\hat{u}_n + \frac{k}{2}\Phi_1\left(\frac{kA}{2}\right)\hat{F}_n, \quad (4.11)$$

where $\Phi_0(kA) = e^{-kA}$, $\Phi_1(kA) = (e^{-kA} - I)(-kA)^{-1}$.

Step:2 We approximate $\hat{F}(u(t), t) \approx 2\hat{F}_n^a - \hat{F}_n$, for all $t \in [t_n, t_{n+1}]$, evaluate the recurrence formula (4.10) and denote the answer by $\hat{b}_n$,

$$\hat{b}_n \approx \Phi_0(kA)\hat{u}_n + k\Phi_1(kA)(2\hat{F}_n^a - \hat{F}_n), \quad (4.12)$$

where $\hat{F}_n^a = \hat{F}(a_n, t_n + k/2)$, and $a_n = \mathcal{F}^{-1}\{\hat{a}_n\}$.

Step:3 We approximate using the interpolation,

$$\hat{F}(u(t), t) \approx \hat{F}_n + (t - t_n)(-3\hat{F}_n + 4\hat{F}_n^a - \hat{F}_n^b) + \frac{(t - t_n)^2}{2}(\hat{F}_n - 2\hat{F}_n^a + \hat{F}_n^b),$$

for $t \in [t_n, t_{n+1}]$ and evaluate the integral. After re-arranging the terms, we obtain the following formula to compute $\hat{u}_{n+1}$,

$$\begin{aligned} \hat{u}_{n+1} &= \Phi_0(kA)\hat{u}_n + k\Phi_1(kA)\hat{F}_n + k\Phi_2(kA)(-3\hat{F}_n + 4\hat{F}_n^a - \hat{F}_n^b) \\ &+ 4k\Phi_3(kA)(\hat{F}_n - 2\hat{F}_n^a + \hat{F}_n^b), \end{aligned} \quad (4.13)$$

where,

$$\begin{aligned}\hat{F}_n^b &= \hat{F}(b_n, t_n + k), \text{ and } b_n = \mathcal{F}^{-1}\{\hat{b}_n\}, \\ \Phi_2(kA) &= (e^{-kA} - I + kA)(-kA)^{-2}, \\ \Phi_3(kA) &= \left(e^{-kA} - I + kA - \frac{k^2 A^2}{2}\right)(-kA)^{-3},\end{aligned}$$

and compute $u_{n+1} = \mathcal{F}^{-1}\{\hat{u}_{n+1}\}$.

The formulas (4.13) is rewritten by sorting with respect to $\hat{F}_n$, $\hat{F}_n^a$ and $\hat{F}_n^b$ as,

$$\hat{u}_{n+1} = \Psi_0(kA)\hat{u}_n + \Psi_1(kA)\hat{F}_n + \Psi_2(kA)\hat{F}_n^a + \Psi_3(kA)\hat{F}_n^b, \quad (4.14)$$

where,

$$\begin{aligned}\Psi_0(kA) &= \Phi_0(kA), \\ \Psi_1(kA) &= k(4I - kA - e^{-kA}(4I + 3kA + k^2 A^2))(kA)^{-3}, \\ \Psi_2(kA) &= 4k(-2I + kA + e^{-kA}(2I + kA))(kA)^{-3}, \\ \Psi_3(kA) &= k(4I - 3kA + k^2 A^2 - e^{-kA}(4I + kA))(kA)^{-3},\end{aligned}$$

with,

$$\hat{a}_n \approx \Phi_0\left(\frac{kA}{2}\right)\hat{u}_n + \frac{k}{2}\Phi_1\left(\frac{kA}{2}\right)\hat{F}_n, \quad (4.15)$$

and,

$$\hat{b}_n \approx \Phi_0(kA)\hat{u}_n + k\Phi_1(kA)(2\hat{F}_n^a - \hat{F}_n). \quad (4.16)$$

This Runge-Kutta type time-stepping scheme can easily be implemented to solve problems when A is a scalar or a nonsingular diagonal matrix. However, for non-diagonal matrix A , computing $(-A)^{-1}$ and $(-A)^{-3}$ can lead to computational difficulties when the eigenvalues of the matrix A are close to zero. Efficient computation of matrix exponential functions e^{-kA} and $e^{-kA/2}$ is also a challenge for non-diagonal matrices.

We approximate the matrix exponential functions using third-order single-pole rational approximations [71], [72], which also results in the cancellation of the matrix inverses $(-A)^{-1}$ and $(-A)^{-3}$.

4.3.2 A Third Order L -Stable Method

First we derive a third order L -acceptable restricted Padé approximation of e^{-z} by considering,

$$e^{-z} \approx \frac{a_0 + a_1x + a_2x^2}{(1 + bx)^3} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6}.$$

Then,

$$\begin{aligned}
a_0 + a_1x + a_2x^2 &= (1 + bx)^3 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) \\
&= 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + 3bx - 3bx^2 + \frac{3}{2}bx^3 - \frac{1}{2}bx^4 + 3b^2x^2 \\
&\quad - 3b^2x^3 + \frac{3}{2}b^2x^4 - \frac{1}{2}b^2x^5 + b^3x^3 - b^3x^4 + \frac{1}{2}b^3x^5 - \frac{1}{6}b^3x^6 \\
&= -\frac{1}{6}b^3x^6 + \left(-\frac{1}{2}b^2 + \frac{1}{2}b^3\right)x^5 + \left(-\frac{1}{2}b + \frac{3}{2}b^2 - b^3\right)x^4 \\
&\quad + \left(-\frac{1}{6} + \frac{3}{2}b - 3b^2 + b^3\right)x^3 + \left(\frac{1}{2} - 3b + 3b^2\right)x^2 + (-1 + 3b)x + 1.
\end{aligned}$$

By comparing the coefficients of x , we get,

$$b^3 - 3b + \frac{3}{2}b - \frac{1}{6} = 0 \rightarrow \boxed{b_1 = 2.405149578502864}$$

$$a_1 = 3b - 1 \rightarrow \boxed{a_1 = 6.21544873550858}$$

$$a_2 = 3b^2 - 3b + \frac{1}{2} \rightarrow \boxed{a_2 = 10.6387847494089}$$

A third order L -acceptable rational approximation is,

$$R_{3L}(z) = \frac{(1 + 6.2154487355z + 10.6387847494z^2)}{(1 + 2.4051495785z)^3}. \quad (4.17)$$

Third order accuracy of the approximation (4.17) is shown by using Taylor series expansions,

$$e^{-z} = 1 - z + 0.5z^2 - 0.1666666667z^3 + 0.04166666666z^4 + O(z^4),$$

$$R_{3L}(z) = 1 - z + 0.5z^2 - 0.1666666667z^3 + 6.43863383063z^4 + O(z^4)$$

Thus,

$$\|e^{-z} - R_{3L}(z)\| = O(z^4).$$

The rational approximation (4.17) satisfies $|R_{3L}(z)| < 1$ for all $z > 0$ and $R_{3L}(z) \rightarrow 0$ as $z \rightarrow \infty$. Using the third order restricted Padé approximation (4.17), the method (4.14) is modified as:

$$\hat{u}_{n+1} = R_{3L}(kA)\hat{u}_n + \Psi_1(kA)\hat{F}_n + \Psi_2(kA)\hat{F}_n^a + \Psi_3(kA)\hat{F}_n^b, \quad (4.18)$$

where Ψ_1 , Ψ_2 and Ψ_3 become,

$$\begin{aligned} \Psi_1(kA) &= k\left(\frac{1}{6}I - 24.5519605332kA\right)(I + 2.4051495785kA)^{-3}, \\ \Psi_2(kA) &= 4k\left(\frac{1}{6}I + 13.9131757838kA\right)(I + 2.405149578kA)^{-3}, \\ \Psi_3(kA) &= k\left(\frac{1}{6}I - 24.3852938666kA + 13.91317578383k^2A^2\right)(I + 2.4051495785kA)^{-3}, \end{aligned}$$

and,

$$\hat{a}_n = R_{3L}\left(\frac{kA}{2}\right)\hat{u}_n + \Phi_1\left(\frac{kA}{2}\right)\hat{F}_n, \quad (4.19)$$

$$\hat{b}_n = R_{3L}(kA)\hat{u}_n + \Phi_1(kA)\left(2\hat{F}_n^a - \hat{F}_n\right), \quad (4.20)$$

with,

$$\begin{aligned}
R_{3L} \left(\frac{kA}{2} \right) &= (I + 3.107724367754kA + 2.659696187352k^2A^2)(I + 1.202574789251kA)^{-3}, \\
\Phi_1 \left(\frac{kA}{2} \right) &= k(0.5I + 1.678862183877kA + 1.739146972978k^2A^2)(1 + 1.202574789251kA)^{-3}, \\
\Phi_1(kA) &= k(1.0I + 6.7154487355kA + 13.91317578k^2A^2)(I + 2.4051495785kA)^{-3}.
\end{aligned}$$

Algorithm:2 For L -stable method with Dirichlet boundary conditions.

1. Input data: $L = b - a$, α , N , $\mathcal{K}$, T , and u (initial condition)
2. Compute eigenvalues $\lambda = \left[\frac{(1 : N)\pi}{L} \right]^\alpha$
3. Define Spacial mesh $x = a + (1 : N)h$, $h = \frac{b-a}{N}$
4. Define $Z = k\mathcal{K}\lambda$, and

$$R_{t1} = 1 + 3.107724367754Z + 2.659696187352Z.^2$$

$$P_{t1} = k(0.5 + 1.678862183877Z + 1.739146972979Z^2)$$

$$P_{t2} = k(1 + 6.715448735509Z + 13.91317578383Z^2)$$

$$d_{t1} = (1 + 1.202574789251Z)^3$$

$$d_{t2} = (1 + 2.405149578503Z)^3$$

$$R_1 = 1 + 6.215448735509Z + 10.638784749409Z.^2$$

$$P_1 = k\left(\frac{1}{6} - 24.5519605332389Z\right)$$

$$P_2 = 4k\left(\frac{1}{6} + 13.91317578383Z\right)$$

$$P_3 = k\left(\frac{1}{6} - 24.385293866572Z + 13.91317578383Z^2\right)$$

5. For $n = 1 : \frac{T}{k}$ do the following

$$u_{dst} = dst(u)$$

$$f_u = f(u, t_n)$$

$$a_n = idst((R_{t1} \cdot u_{dst} + P_{t1} \cdot dst(f_u))/d_{t1})$$

$$f_n^a = f(a_n, t_{n+1})$$

$$b_n = idst((R_1 \cdot u_{dst} + P_{t2} \cdot dst(2f_n^a - f_u))/d_{t2})$$

$$f_b^n = f(b_n, t_{n+1})$$

$$u = idst((R_1 \cdot u_{dst} + P_1 \cdot dst(f_u) + P_2 \cdot dst(f_n^a) + P_3 \cdot dst(f_b^n))/d_{t2})$$

Remark: For problems with homogeneous Neumann boundary conditions, the Fourier transforms dst and $idst$ are replaced by dct and $idct$ respectively. The eigenvalues are computed as,

$$\lambda = \left[\frac{(0 : N-1)\pi}{L} \right]^\alpha,$$

and $x = a + (0 : N-1)h + \frac{h}{2}$ with $h = \frac{b-a}{N+1}$. The amplification symbols of the two rational approximations used are shown in Figure 4.1. The approximation $R_{3A}(z)$ is A -acceptable and behaves like the A -stable Crank-Nicolson approximation. The approximation $R_{3L}(z)$ approaches zero asymptotically without crossing the x -axis.

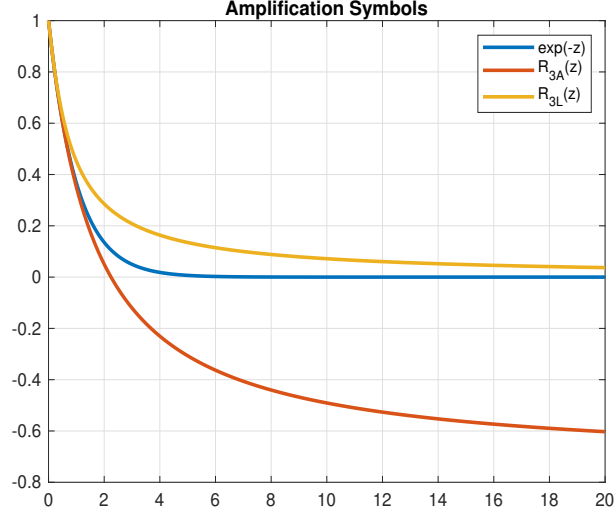


Figure 4.1: Amplification symbols of the rational approximations $R_{3A}(z)$ and $R_{3L}(z)$.

4.4 Stability Regions and Linear Error Analysis

The stability region of a Runge-Kutta type numerical method is the region in the complex plane where the solution of the numerical method remains stable and bounded.

The stability regions typically depend on the coefficients of the method. To find the stability regions of the newly developed schemes, we consider the following non-linear autonomous ODE [25],

$$u_t = cu + F(u), \quad (4.21)$$

with a non-linear function F of the complex valued function u . Stability regions are investigated by considering linearized autonomous non-linear ODE. For that we assume that $u_0 = u(t_0)$ be a fixed point of (4.21) such that $cu_0 + F(u_0) = 0$. Let u be a perturbation of the fixed point u_0 and let $\Lambda = F'(u_0)$, then linearization about this

fixed point results in the following equation,

$$u_t = cu + \Lambda u. \quad (4.22)$$

The fixed point u_0 is said to be stable if $\text{Re}(c + \Lambda) < 0$ [25]. It is naturally important for a numerical method to satisfy this property, to cover as much dynamics as possible. The regions of stability are determined by applying numerical methods to equation (4.22) and taking into account the recurrence formula u_{n+1}/u_n . Assume that $x = \Lambda k$ and $y = ck$ using the time step size k . Applying the new L -stable method to solve (4.22), we get the ratio,

$$\mathcal{R}_L(x, y) = \frac{u_{n+1}}{u_n} = \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{Den},$$

with,

$$Den = (0.390625 + 0.9395115541y)^6(0.2 + 0.4810299157y)^6(0.4 + 0.4810299157y)^3,$$

$$\begin{aligned} c_0 = & 0.0001220703125(5.94042098433y^{14} + 40.5186836105y^{13} + 126.947818128y^{12} \\ & + 242.293945793y^{11} + 314.922297003y^{10} + 295.059776063y^9 + 205.638996451y^8 \\ & + 108.369291460y^7 + 43.4210993737y^6 + 13.1717289787y^5 + 2.97960261381y^4 \\ & + 0.487693762910y^3 + 0.0546321364210y^2 + 0.00375145903942y \\ & + 0.000119209289551), \end{aligned}$$

$$\begin{aligned} c_1 = & 0.0001220703125(-7.76875586177y^{13} - 15.1850530377y^{12} + 17.3923134489y^{11} \\ & + 93.3261693951y^{10} + 149.397872433y^9 + 139.653577309y^8 + 87.5725098821y^7 \\ & + 38.9096817917y^6 + 12.5229961039y^5 + 2.92316349703y^4 + 0.485406029174y^3 \\ & + 0.0546321364208y^2 + 0.00375145903942y + 0.000119209289551), \end{aligned}$$

$$\begin{aligned} c_2 = & 0.0001220703125(-7.76875586194y^{13} + 25.9424775008y^{12} + 174.055414151y^{11} \\ & + 359.318636444y^{10} + 412.778481916y^9 + 306.220019756y^8 + 1562.60897188y^7 \\ & + 56.4879602887y^6 + 14.6274556865y^5 + 2.70878605183y^4 + .354215169650y^3 \\ & + 0.0318915356757y^2 + 0.00187572951971y + 0.0000596046447754), \end{aligned}$$

$$\begin{aligned} c_3 = & 0.0001220703125(15.5375117240y^{12} + 34.0267758314y^{11} + 2.05691028727y^{10} \\ & - 74.8275223989y^9 - 120.876307668y^8 - 101.620075906y^7 - 54.1585747363y^6 \\ & - 19.3131677392y^5 - 4.62211048982y^4 - .714532529874y^3 - 0.0641679470800y^2 \\ & - 0.00242010141661y + 0.0000198682149251). \end{aligned}$$

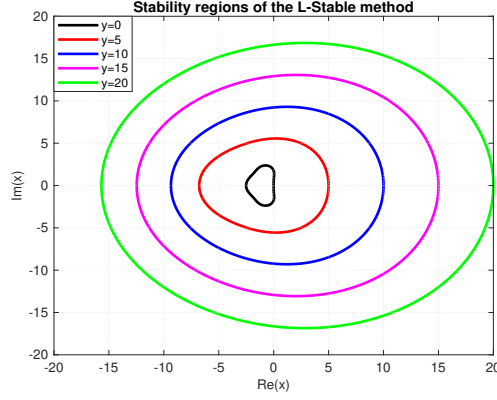


Figure 4.2: Stability regions of the third order L-stable method.

The stability regions of the two new schemes are plotted for $y = 0, 5, 10, 20$ and are shown in figure 4.2.

4.4.1 Linear Error Analysis

Linear error analysis is a technique performed to analyze and quantify errors introduced by numerical methods when approximating solutions of mathematical equations. It provides insights into the accuracy and stability of the numerical method.

The third-order convergence of the methods is proved by considering the linearization of $\mathbf{F}(\mathbf{u})$ by $\Lambda\mathbf{u}$. We perform the linear error analysis of the numerical scheme (4.18) using the exact solution,

$$\mathbf{u}(t_{n+1}) = e^{k(\Lambda-A)}\mathbf{u}(t_n), \quad (4.23)$$

of the linearized semi-discretized system,

$$\frac{d\mathbf{u}}{dt} = -A\mathbf{u} + \Lambda\mathbf{u}. \quad (4.24)$$

4.4.2 Error Analysis: Third Order L -stable Method

By applying (4.18) to (4.24) we obtain,

$$\begin{aligned}
\mathbf{u}_{n+1} = & ((I + 2.4051495785kA))^{-3} [(I + 6.2154487355kA + 10.6387847494k^2A^2)] u_n \\
& + ((I + 2.4051495785kA))^{-3} \left[k\Lambda \left(\frac{1}{6}I - 24.5519605332kA \right) \right] u_n \\
& + ((I + 2.4051495785kA))^{-3} \left[4k\Lambda \left(\frac{1}{6}I + 13.9131757838kA \right) \right] a_n \\
& + ((I + 2.4051495785kA))^{-3} \left[k\Lambda \left(\frac{1}{6}I - 24.3852938666kA + 13.9131757838k^2A^2 \right) \right] b_n,
\end{aligned} \tag{4.25}$$

where $\mathbf{a}_n$ and $\mathbf{b}_n$ are,

$$\begin{aligned}
\mathbf{a}_n = & (I + 1.20257479kA)^{-3} \times \\
& \left[I + 3.10772437kA + 2.65969619k^2A^2 + 0.5k\Lambda + 1.67886218k^2\Lambda A + 1.73914697k^3\Lambda A^2 \right] u_n, \\
\mathbf{b}_n = & (I + 2.4051495785kA)^{-3} [I + 6.215448736kA + 10.63878475k^2A^2] u_n + \\
& (I + 2.4051495785kA)^{-3} [k\Lambda + 6.715448736k^2\Lambda A + 13.91317578k^3\Lambda A^2] (2a_n - u_n),
\end{aligned}$$

Using the following Taylor series expansions,

$$\begin{aligned}
\frac{1}{(1 + 1.2025747893z)^3} &= 1 - 3.607724367z + 8.677116738z^2 - 17.39146972z^3 \\
&\quad + 31.37181455z^4 + O(z^5), \\
\frac{1}{(1 + 2.4051495785z)^3} &= 1 - 7.215448734z + 34.70846696z^2 - 139.1317578z^3 \\
&\quad + 501.9490329z^4 + O(z^5),
\end{aligned}$$

equation (4.25) is simplified as,

$$\begin{aligned}\mathbf{u}_{n+1} &= \left[I + k(\Lambda - A) + k^2(0.5\Lambda^2 - \Lambda A + 0.5A^2) + 0.1666666667k^3(\Lambda^3 - 3\Lambda^2A + 3\Lambda A^2 - A^3) \right. \\ &\quad \left. + k^4(38.63180297A^2\Lambda^2 - 25.71286866A\Lambda^3 + \dots) \right] \mathbf{u}_n.\end{aligned}\tag{4.26}$$

We rewrite the exact solution (4.23) using the Taylor series expansion of $e^{k(\Lambda-A)}$

$$\begin{aligned}\mathbf{u}_{n+1} &= \left[I + k(\Lambda - A) + k^2(0.5\Lambda^2 - \Lambda A + 0.5A^2) + 0.1666666667k^3(\Lambda^3 - 3\Lambda^2A + 3\Lambda A^2 - A^3) \right. \\ &\quad \left. + k^4(0.04166666666\Lambda^4 - 0.1666666667\Lambda^3A + 0.25\Lambda^2A^2 - \dots) + \mathcal{O}(k^5) \right] \mathbf{u}_n\end{aligned}\tag{4.27}$$

It follows from (4.26) and (4.27) that the local truncation error is

$$\mathcal{E}_{n+1} = \left\| \mathbf{u}_{n+1} - e^{-k(A-\Lambda)} \mathbf{u}_n \right\| = \mathcal{O}(k^4).\tag{4.28}$$

4.5 Numerical Experiments

Numerical experiments conducted in this section, chapter consist of three problems: a test problem with an exact solution available. This problem is considered to compute the order of convergence of the scheme. The second and third problems are space-fractional diffusive epidemic models, *SIR* and *SEIR*. The effects of space-fractional diffusion of the solution profile are shown in figures 4.6, and 4.7-4.10.

4.5.1 A Test Problem

We consider the following test problem with a known exact solution to compute the convergence results of our scheme (see, Zhai et al. [104]).

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\mathcal{K}_\alpha(\Delta)^{\frac{\alpha}{2}} + f(u, x, t), & 0 < x < 1, \quad t > 0, \quad 1 < \alpha \leq 2, \\
u(x, 0) &= 1, & 0 < x < 1, \\
u(0, t) &= 0, \quad u(1, t) = 0, & t > 0, \\
f(u, x, t) &= \frac{\mathcal{K}_\alpha}{4} t^\alpha (3[1 + (2\pi)^\alpha] \sin(2\pi x) - [1 + (6\pi)^\alpha] \sin(6\pi x)) + \alpha t^{\alpha-1} \sin^3(3\pi x) - \mathcal{K}_\alpha u
\end{aligned} \tag{4.29}$$

The exact solution of the problem (4.5.1) is

$$u(x, t) = t^\alpha \sin^3(2\pi x). \tag{4.30}$$

Problem (4.5.1) is solved for different values of α . Graphs of the exact and the numerical solution are shown in figure 4.3. The exact solution (4.30) of (4.5.1) involves t^α and therefore remains the same at $T = 1$ for all values of α . It will be different for different values of α when $T \neq 1$. To show the accuracy of our numerical scheme, we have computed the numerical solutions for different values of α at $T = 0.5$. The corresponding graphs of the exact and the numerical solutions are given in figure 4.4. Reliability of the numerical solution is shown in figure 4.5 by plotting the time evolution graphs for $0 \leq t \leq 1$. Third-order convergence of the fully discrete scheme is shown in table 4.1. Central processing unit time (CPU) recorded for each iteration to show the computational efficiency of the methods.

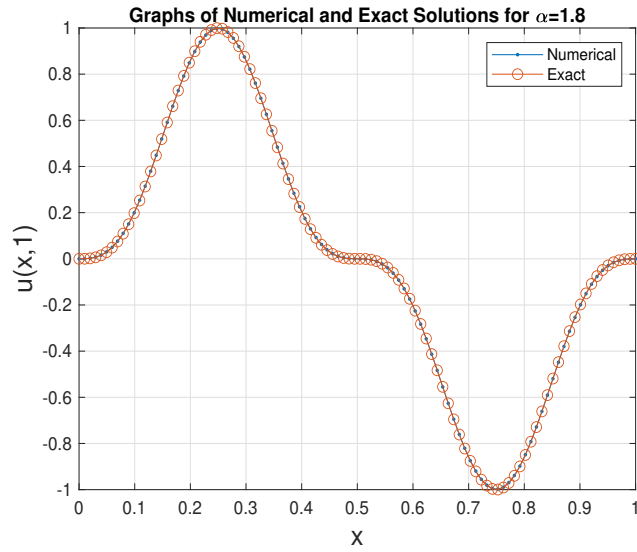


Figure 4.3: Problem 4.5.1: Graph of exact and numerical solution at $T = 1$.

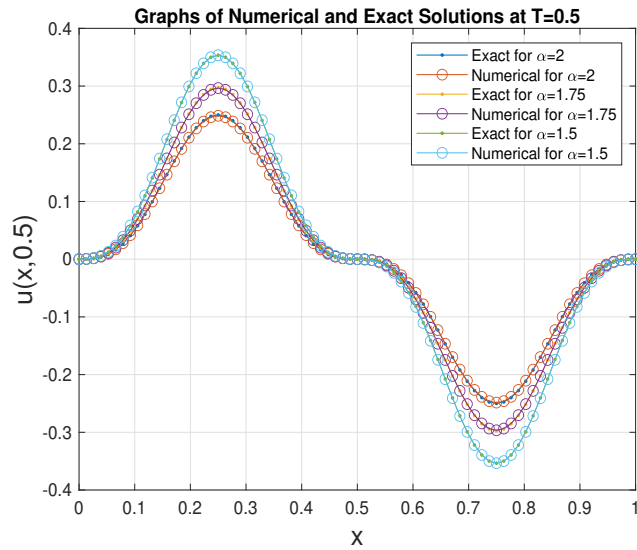


Figure 4.4: Problem 4.5.1: Graphs of the exact and numerical solutions at $T = 0.5$ for different values of α .

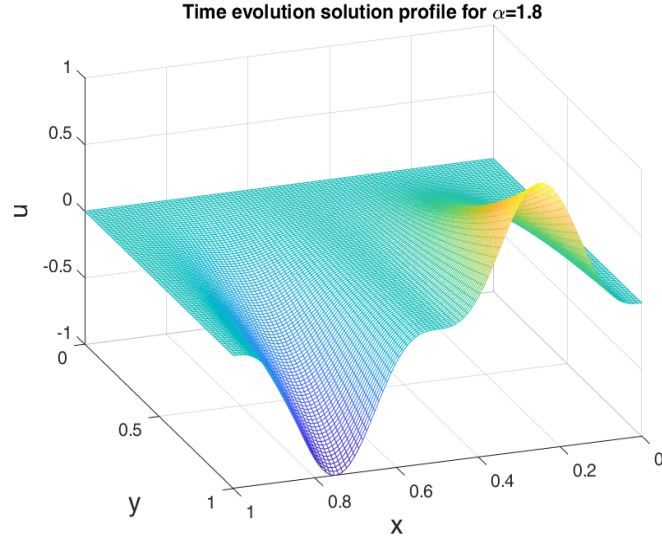


Figure 4.5: Problem 4.5.1: Time evolution graph of the numerical solution.

Table 4.1: Problem 4.5.1: Convergence results of the L -stable scheme

α	h	k	Error	Order	CPU-time
1.50	0.01000	0.01000	7.8012e-05	0.0000	0.0465
	0.00500	0.00500	1.3601e-05	2.5200	0.0507
	0.00250	0.00250	2.0758e-06	2.7119	0.1339
	0.00125	0.00125	2.9012e-07	2.8389	0.3872
	0.00063	0.00063	3.8487e-08	2.9142	1.1520
	0.00031	0.00031	4.9610e-09	2.9556	3.2679
1.75	0.01000	0.01000	1.2144e-04	0.0000	0.0713
	0.00500	0.00500	2.5012e-05	2.2795	0.0573
	0.00250	0.00250	4.3520e-06	2.5229	0.1320
	0.00125	0.00125	6.6477e-07	2.7107	0.4229
	0.00063	0.00063	9.3023e-08	2.8372	1.0596
	0.00031	0.00031	1.2351e-08	2.9130	3.3617
1.95	0.01000	0.01000	1.5054e-04	0.0000	0.0468
	0.00500	0.00500	3.5519e-05	2.0835	0.0499
	0.00250	0.00250	7.0037e-06	2.3424	0.1224
	0.00125	0.00125	1.1813e-06	2.5677	0.4034
	0.00063	0.00063	1.7681e-07	2.7402	1.0796
	0.00031	0.00031	2.4441e-08	2.8548	3.2478

4.5.2 Fractional Diffusive *SIR* Epidemic Model

Here we consider the fractional diffusive *SIR* system (4.1).

$$\begin{aligned}
\frac{\partial S}{\partial t} &= -D_S(-\Delta)^{\frac{\alpha}{2}}S(x,t) + \eta - \nu S(x,t) - \frac{\beta S(x,t)I(x,t)}{1 + \sigma_1 S(x,t) + \sigma_2 I(x,t)}, \\
\frac{\partial I}{\partial t} &= -D_I(-\Delta)^{\frac{\alpha}{2}}I(x,t) + \frac{\beta S(x,t)I(x,t)}{1 + \sigma_1 S(x,t) + \sigma_2 I(x,t)} - (\nu + \delta + r)I(x,t), \\
\frac{\partial R}{\partial t} &= -D_R(-\Delta)^{\frac{\alpha}{2}}R(x,t) + rI(x,t) - \nu R(x,t),
\end{aligned} \tag{4.31}$$

where $1 < \alpha \leq 2$.

The initial conditions are,

$$\begin{cases} S(x,0) = 1.5x, & I(x,0) = 0.5x, & R(x,0) = 0.5x, & 0 \leq x \leq 0.5 \\ S(x,0) = 1.5(1-x), & I(x,0) = 0.5(1-x), & R(x,0) = 0.5(x-1), & 0.5 \leq x \leq 1 \end{cases}$$

The parameter values taken for this experiment are:

$$D_S = D_I = D_R = 0.1, \quad \eta = 0.25, \quad \nu = 0.04, \quad \beta = 10, \quad \sigma_1 = \sigma_2 = 0.5, \quad \delta = 0.01, \quad r = 24.$$

The domain we consider is: $0 \leq x \leq 1$, $0 \leq t \leq 100$. The numerical experiments are performed using $\Delta x = 0.01$ and $\Delta t = 0.01$.

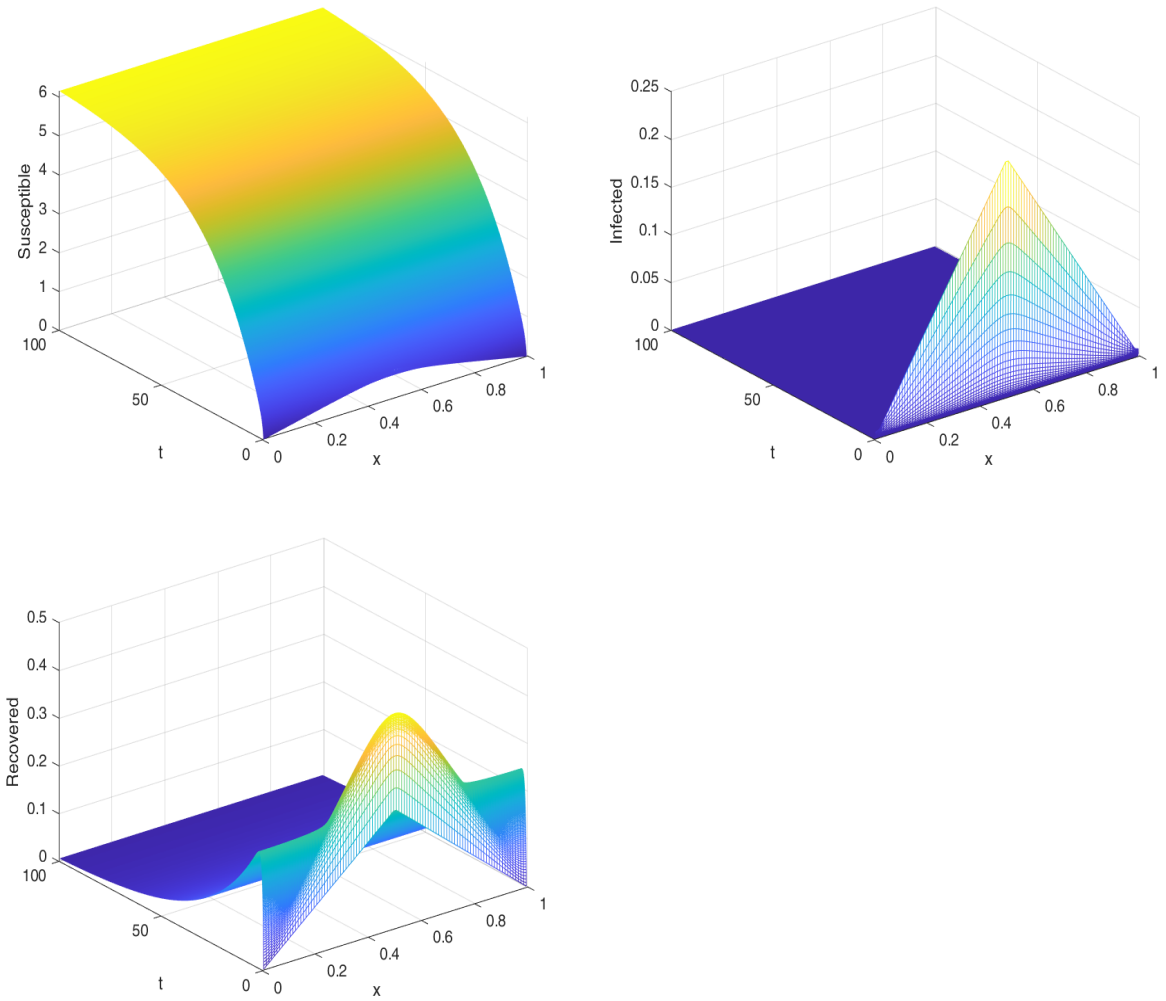


Figure 4.6: SIR Model: Susceptible, Infected and Recovered with $\alpha = 1.5$

4.5.3 Fractional Diffusive *SEIR* Epidemic Model

Here we shall numerically solve the fractional diffusive *SEIR* model (4.2). This model determines the geographic spread of the disease in a population which has not been

vaccinated against it.

$$\begin{aligned}
\frac{\partial S}{\partial t} &= -D_S(-\Delta)^{\frac{\alpha}{2}}S(x,t) + \mu N - (\mu + \beta I)S, \\
\frac{\partial E}{\partial t} &= -D_E(-\Delta)^{\frac{\alpha}{2}}E(x,t) + \beta IS - (\mu + \sigma)E, \\
\frac{\partial I}{\partial t} &= -D_I(-\Delta)^{\frac{\alpha}{2}}I(x,t) + \sigma E - (\mu + \gamma)I, \\
\frac{\partial R}{\partial t} &= -D_R(-\Delta)^{\frac{\alpha}{2}}R(x,t) + \gamma I - \mu R.
\end{aligned} \tag{4.32}$$

Since R is not present in first three equations, we shall solve the system only for S , E , and I and compute $R = \mathcal{N} - S - E - I$, where $\mathcal{N}$ is the total population under consideration. The total population taken for the numerical experiments is $\mathcal{N} = 5 \times 10^7$. The initial conditions are,

$$\begin{cases} S(x_i, 0) = 31250(i - 1), \ E(x_i, 0) = 125(i - 1), \ R(x_i, 0) = 75(i - 1), & 0 \leq i \leq \frac{N+1}{2} \\ S(x_i, 0) = 31250(N - i), \ E(x_i, 0) = 125(N - i), \ R(x_i, 0) = 75(N - i), & 0 \leq i \leq \frac{N+1}{2} \end{cases}$$

The parameter values taken for this experiment are:

$D_S = D_E = D_I = D_R = 0.01$, $\mu = 0.022$, $\beta = 5 \times 10^{-4}$, $\sigma = 45.6$, $\gamma = 73$. The domain we consider is: $0 \leq x \leq 1$, $0 \leq t \leq 0.1$. The numerical experiments are performed using $\Delta x = 0.025$ and $\Delta t = 0.001$.

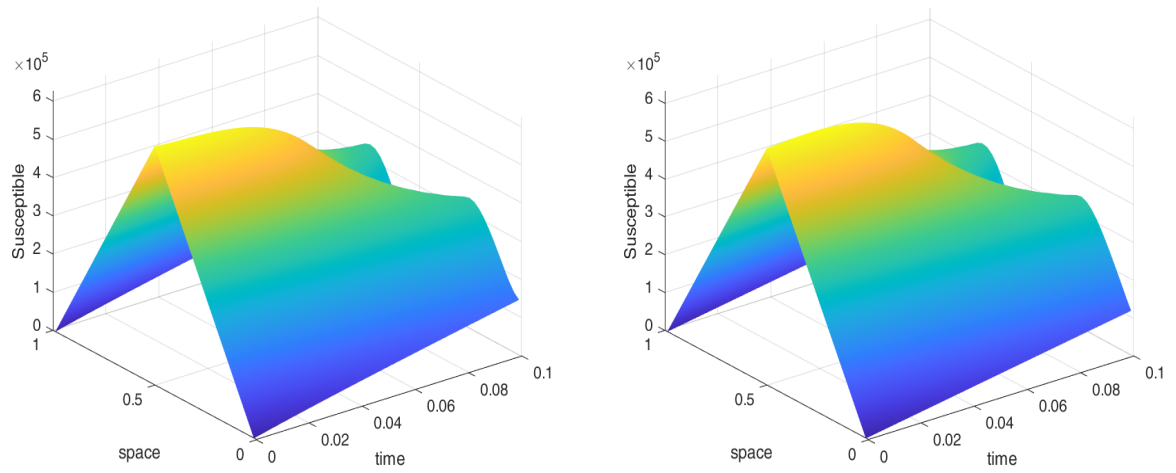


Figure 4.7: *SEIR* Model: Susceptible for $\alpha = 2$ (Left) and $\alpha = 1.2$ (Right)

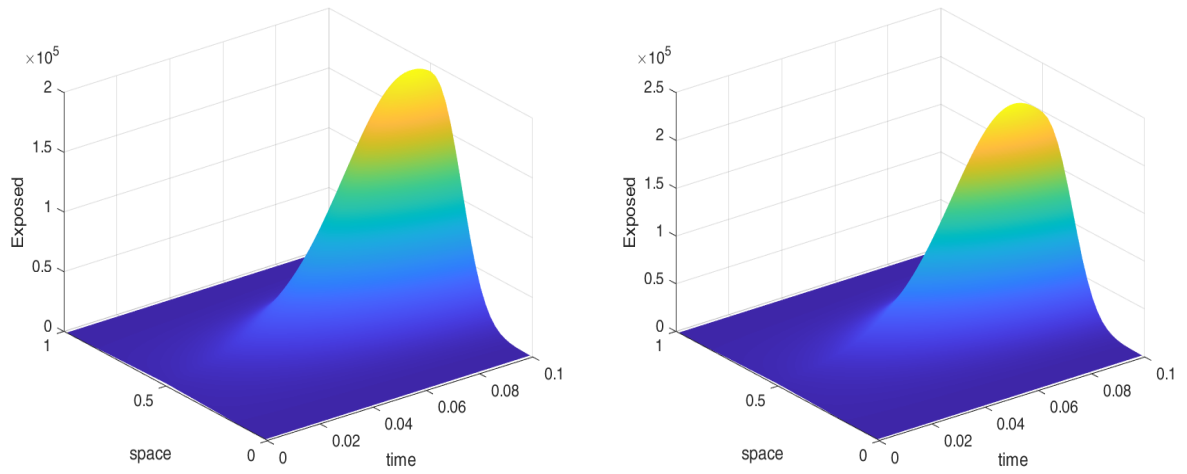


Figure 4.8: *SEIR* Model: Exposed for $\alpha = 2$ (Left) and $\alpha = 1.2$ (Right)

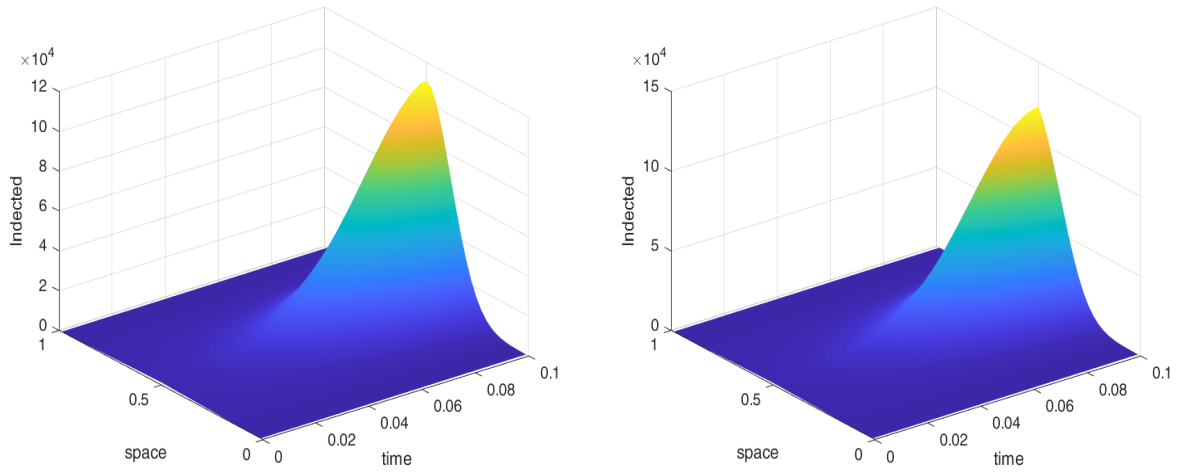


Figure 4.9: *SEIR* Model: Infected for $\alpha = 2$ (Left) and $\alpha = 1.2$ (Right)

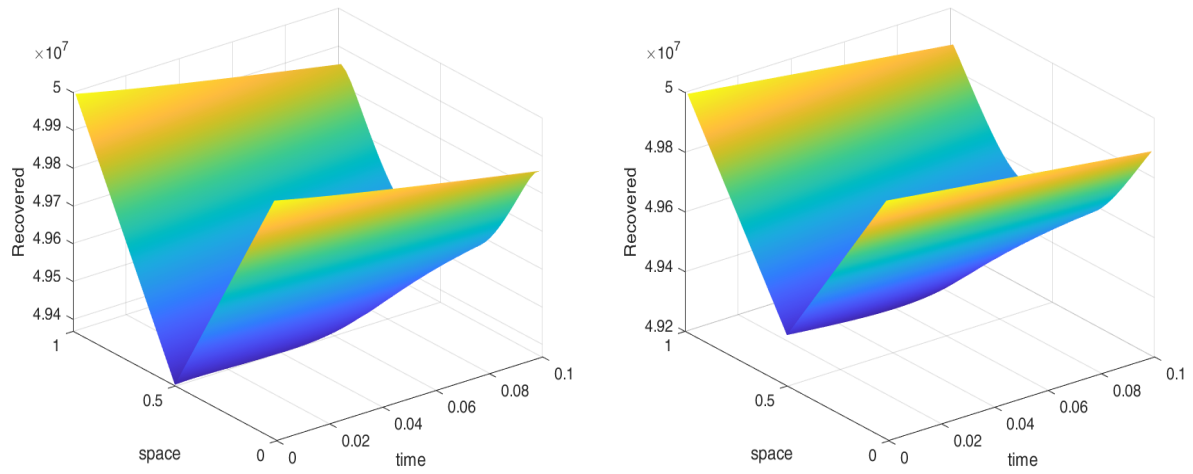


Figure 4.10: *SEIR* Model: Recovered for $\alpha = 2$ (Left) and $\alpha = 1.2$ (Right)

CHAPTER 5

TIME-SPACE-FRACTIONAL DIFFUSIVE EPIDEMIC MODELS

The time-space fractional equations have the ability to capture the memory-dependent, non-Fickian, and non-local behavior in several complex systems. These equations help us to better understand the underlying transport mechanism and reaction processes. The study of these equations brings interesting mathematical and computational challenges in this active research field.

Fractional partial differential equations have gained popularity due to their use in many application areas. Numerous studies have shown that the fractional partial differential equations are more useful in understanding aspects of non-locality and spatial heterogeneity compared to the corresponding integer-order ones. Most fractional models cannot be solved analytically; therefore, efficient numerical methods are crucial to solving such problems. Recently, there has been a growing interest in devising effective and reliable numerical approaches to solve time-space fractional

reaction-diffusion systems.

Modeling of anomalous diffusion, non-local interaction, and memory effects using the fractional differential operators results in more complex models than their classical integer-order counterparts. The complex interplay between the time and space fractional derivatives makes the task of developing stable and convergent numerical schemes quite challenging. The increased computational cost is another challenge, especially when solving multi-dimensional systems. Efficient memory and storage management techniques also play a crucial role in developing such numerical schemes.

The matrix transfer technique (MTT) is a powerful mathematical method developed to discretize the fractional Laplacian. Ilic et al. [51] presented the Matrix Transfer Technique (MTT) to discretize the fractional spectral Laplacian operator with homogeneous boundary conditions. This technique was extended to solve problems with non-homogeneous boundary conditions in [52]. Rahman et al. in [78] were able to find improved numerical solutions for the fractional advection-dispersion equation and fractional diffusion equation using the MTT. Khaliq et al. [35] have used MTT for spatial discretization and second-order implicit predictor-corrector schemes for time integration. Fourth-order time-stepping schemes with the MTT approach for space-fractional reaction-diffusion equations were derived by Alzahrani et al. [4]. Zheng et al. [105] used MTT to solve inhomogeneous linear space fractional problems. Zhai [104] developed a high-accuracy spectral method for space-fractional diffusion equations. Based on the Chelyshkov operation matrix, Ding and Zhang [29] developed a numerical method for solving a type of time-space fractional reaction diffusion equation. The fast and high-order IMEX method is presented by Kamran in [55] to solve

a non-linear time-space fractional reaction-diffusion equation.

In this chapter, we have utilized the matrix transfer technique to efficiently discretize the space fractional Laplacian operator. By utilizing the discrete Fourier sine and Fourier cosine and their inverse transforms, we can not only avoid computing the matrix of eigenvectors and its inverse but also avoid computing matrix vector products. The spacial discretization results in a system of time Caputo fractional differential equations. A predictor-corrector scheme is developed using fractional Euler and Adams methods to solve the semi-discretized system of time fractional differential equations in time.

5.1 Problem Statement and Spatial Discretization

We consider the fractional reaction-diffusion equation with Caputo time fractional derivative,

$${}_CD_{0,t}^\beta u(x,t) = -\mathcal{K}(-\Delta)^{\frac{\alpha}{2}} u(x,t) + F(x,t,u(x,t)), \quad x \in \Omega, \quad t \in (0, T], \quad (5.1)$$

with initial condition,

$$u(x, 0) = u_0(x), \quad x \in \Omega \subset \mathbb{R},$$

and homogeneous Neumann or homogeneous Dirichlet boundary conditions. The diffusion constant is denoted by $\mathcal{K}$ and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator of order α , $1 < \alpha \leq 2$ defined in the bounded domain Ω in $\mathbb{R}$. The Caputo derivative

${}_CD_{0,t}^\beta u(x,t)$ of order β , $0 < \beta < 1$, is defined as,

$${}_CD_{0,t}^\beta u(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial u(x,s)}{\partial s} ds.$$

5.1.1 Matrix Transfer Technique

The matrix transfer technique (MTT) provides a systematic way to approximate the fractional Laplacian using matrix operations. This technique has gained popularity due to its effectiveness in solving fractional differential equations. We use this technique to approximate the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$. The classical Laplacian $(-\Delta)$, discretized using a compact fourth-order finite difference method, is expressed as the product of diagonally dominant tridiagonal matrices. These matrices are then factorized using their eigenvalues and eigenvectors.

First, we consider equation (5.1) with the classical Laplacian,

$${}_CD_{0,t}^\beta u(x,t) = -\mathcal{K}(-\Delta)u(x,t) + F(x,t,u(x,t)), \quad (x,t) \in \Omega \times (0,T], \quad (5.2)$$

$$u(x,0) = u_0(x), \quad x \in \Omega = [0,L] \subset \mathbb{R}.$$

We describe the matrix transfer technique for the equation (5.2). Let $h = \frac{L}{N}$ be the spatial step with $x_i = ih$, $i = 0, 1, 2, \dots, N$ and $u_i(t) = u(x_i, t)$ for $i = 1, 2, \dots, N-1$.

We approximate the classical Laplacian using the following fourth-order approximation [30],

$$\Delta = \frac{\partial^2}{\partial x^2} = -\frac{1}{h^2} \frac{\delta_x^2}{1 + \frac{1}{12}\delta_x^2} \quad (5.3)$$

where δ_x^2 represents the second-order central difference operator,

$$\delta_x^2 u_i = u_{i-1} - 2u_i(t) + u_{i+1}.$$

Applying the approximation (5.3), we obtain the system of differential equations,

$${}_C D_{0,t}^\beta u_i(t) = -\frac{\mathcal{K}}{h^2} \left[\frac{-u_{i-1}(t) + 2u_i(t) - u_{i+1}}{\frac{1}{12}u_{i-1}(t) + \frac{5}{6}u_i(t) + \frac{1}{12}u_{i+1}(t)} \right] + F(x_i, t, u_i), \quad i = 1, 2, \dots, N-1,$$

which can be written in matrix form,

$$\begin{aligned} {}_C D_{0,t}^\beta \mathbf{u}(t) &= -\frac{\mathcal{K}}{h^2} (B^{-1}C) \mathbf{u}(t) + \mathbf{F}(t, \mathbf{u}(t)), \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \tag{5.4}$$

where $\mathbf{u}(t) = [u_1(t), u_2(t), \dots, u_{N-1}(t)]^T$, $\mathbf{u}_0 = [u_1(0), u_2(0), \dots, u_{N-1}(0)]^T$,

$$\mathbf{F}(t, \mathbf{u}) = [F(x_1, t, u_1(t)), F(x_2, t, u_2(t)), F(x_3, t, u_3(t)), \dots, F(x_{N-1}, t, u_{N-1}(t))]^T,$$

$$B = \text{tirdiagonal} \begin{bmatrix} 1 & 5 & 1 \\ 12 & 6 & 12 \end{bmatrix} \quad \text{and} \quad C = \text{tridiagonal} \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}.$$

The diagonalizable tridiagonal matrices can be written as $B = P\Lambda_B P^{-1}$ and $C = P\Lambda_C P^{-1}$, where Λ_B and Λ_C are the diagonal matrices of the eigenvalues of B and C , respectively. Since both B and C have the same upper and lower diagonals, they have the same eigenvectors (see [84]). The orthogonal matrix P consists of the corresponding eigenvectors of B or C . Now we can approximate the classical Laplacian

as

$$-\Delta \approx \frac{B^{-1}C}{h^2} = \frac{1}{h^2} P \Lambda P^{-1},$$

where $\Lambda = \Lambda_B^{-1} \Lambda_C$. The fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is approximated as [109],

$$(-\Delta)^{\frac{\alpha}{2}} \approx \frac{1}{h^\alpha} P \Lambda^{\frac{\alpha}{2}} P^{-1}.$$

The equation (5.1) is written as,

$${}_C D_{0,t}^\beta \mathbf{u} = -A^{\frac{\alpha}{2}} \mathbf{u}(t) + F(t, \mathbf{u}(t)), \quad (5.5)$$

where $A^{\frac{\alpha}{2}} = \frac{\mathcal{K}}{h^\alpha} P \Lambda^{\frac{\alpha}{2}} P^{-1}$.

5.1.2 Eigenvalues and Eigenvectors

Eigenvalues and corresponding eigenvectors depend on the boundary conditions. For the homogeneous Dirichlet boundary conditions, the matrix of eigenvalues $\Lambda^{\frac{\alpha}{2}}$ and the matrix P of corresponding eigenvectors are given as [51], [52],

$$\Lambda^{\frac{\alpha}{2}} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N-1}), \quad (5.6)$$

$$P = [\xi_1, \xi_2, \dots, \xi_{N-1}], \quad (5.7)$$

where,

$$\begin{aligned}\lambda_i &= 2^\alpha \left[\frac{\sin^2 \left(\frac{i\pi}{2N} \right)}{1 - \frac{1}{3} \sin^2 \left(\frac{i\pi}{2N} \right)} \right]^{\frac{\alpha}{2}}, \\ \xi_i &= \left[\sin \left(\frac{i\pi}{N} \right), \sin \left(\frac{2i\pi}{N} \right), \dots, \sin \left(\frac{(N-1)i\pi}{N} \right) \right]^T, \quad i = 1, 2, \dots, N-1.\end{aligned}$$

For the homogeneous Neumann boundary conditions, the matrices $\Lambda^{\frac{\alpha}{2}}$ and P are,

$$\Lambda^{\frac{\alpha}{2}} = \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_N), \quad P = (\xi_0, \xi_1, \dots, \xi_N),$$

where,

$$\begin{aligned}\lambda_i &= 2^\alpha \left[\frac{\cos^2 \left(\frac{i\pi}{2N} \right)}{1 - \frac{1}{3} \cos^2 \left(\frac{i\pi}{2N} \right)} \right]^{\frac{\alpha}{2}}, \\ \xi_i &= \left[1, \cos \left(\frac{i\pi}{N} \right), \dots, \cos \left(\frac{(N-1)i\pi}{N} \right) \right]^T, \quad i = 0, 1, 2, \dots, N-1.\end{aligned}$$

5.2 Time-Stepping Schemes

In this section, we present the $1 + \beta$ order time-stepping scheme to solve the semi-discretized system of non-linear fractional differential equations (5.5). Applying the fractional integral,

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds$$

on both sides of (5.5), we obtain the following Volterra integral equation as an exact solution [27], [28],

$$\begin{aligned}\mathbf{u}(t) &= \mathbf{u}_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} [-A^{\frac{\alpha}{2}} \mathbf{u}(s) + \mathbf{F}(s, \mathbf{u}(s))] ds, \\ &= \mathbf{u}_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathbf{H}(s, \mathbf{u}(s)) ds,\end{aligned}\tag{5.8}$$

where for simplicity of notation, we use $\mathbf{H}(s, \mathbf{u}(s)) = -A^{\frac{\alpha}{2}} \mathbf{u}(s) + \mathbf{F}(s, \mathbf{u}(s))$. For a given positive integer M , we consider the partition of the interval $[0, T]$ with the temporal nodes $t_m = m\Delta t$, $m = 0, 1, 2, \dots, M$, where $\Delta t = \frac{T}{M}$ is the temporal step size. Using these temporal nodes, we can write (5.8) as,

$$\mathbf{u}(t_{m+1}) = \mathbf{u}_0 + \frac{1}{\Gamma(\beta)} \sum_{j=0}^m \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^{\beta-1} \mathbf{H}(s, \mathbf{u}(s)) ds,\tag{5.9}$$

The following fractional forward Euler scheme is used as predictor scheme and fractional Adams scheme is used as corrector scheme in developing the fractional predictor-corrector scheme.

5.2.1 Fractional Forward Euler Scheme

Assuming $\mathbf{H}(s, \mathbf{u}(s)) = \mathbf{H}(t_j, \mathbf{u}(t_j)) = \mathbf{H}_j$ over each subinterval $[t_j, t_{j+1}]$ and integrating (5.9), we get [62], [64],

$$\mathbf{u}_{m+1} = \mathbf{u}_0 + \sum_{j=0}^m a_{j,m+1} \mathbf{H}_j,\tag{5.10}$$

where

$$a_{j,m+1} = \frac{\Delta t^\beta}{\Gamma(\beta+1)} [(m-j+1)^\beta - (m-j)^\beta], \quad 0 \leq j \leq m$$

5.2.2 Fractional Adams Scheme

Here we approximate $\mathbf{H}(s, \mathbf{u}(s))$ by piecewise linear Lagrange interpolating polynomial.

In each subinterval $[t_j, t_{j+1}]$, we approximate $\mathbf{H}(s, \mathbf{u}(s))$ by linear Lagrange polynomial

$P^j(s)$ as,

$$P^j(s) = \frac{1}{\Delta t} [(t_{j+1} - s)\mathbf{H}(t_j, \mathbf{u}(t_j)) + (s - t_j)\mathbf{H}(t_{j+1}, \mathbf{u}(t_{j+1}))]. \quad (5.11)$$

Substituting (5.11) into (5.9), integrating using integration by parts, we get,

$$\mathbf{u}_{m+1} = \mathbf{u}_0 + \sum_{j=0}^{m+1} b_{j,m+1} \mathbf{H}_j, \quad (5.12)$$

where

$$b_{j,m+1} = \frac{\Delta t^\beta}{\Gamma(\beta+2)} \begin{cases} m^{\beta+1} - (m-\beta)(m+1)^\beta, & m=0, \\ (m-j+2)^{\beta+2} - 2(m-j+1)^{\beta+1} + (m-j)^{\beta+1}, & 1 \leq j \leq m, \\ 1, & j = m+1 \end{cases}$$

and $\mathbf{H}_j = \mathbf{H}(t_j, \mathbf{u}(t_j))$.

5.2.3 Predictor-Corrector Scheme

We develop a predictor-corrector scheme by predicting $\mathbf{u}_{m+1}$ using the fractional Euler scheme (5.10) and then using it in the fractional Adams scheme (5.12) [27], [28], [64] as,

$$\begin{aligned}\mathbf{u}_{m+1}^p &= \mathbf{u}_0 + \sum_{j=0}^m a_{j,m+1} \left[-A^{\frac{\alpha}{2}} \mathbf{u}_j + \mathbf{F}(t_j, \mathbf{u}_j) \right], \\ \mathbf{u}_{m+1} &= \mathbf{u}_0 + \sum_{j=0}^m b_{j,m+1} \left[-A^{\frac{\alpha}{2}} \mathbf{u}_j + \mathbf{F}(t_j, \mathbf{u}_j) \right] + b_{m+1,m+1} \left[-A^{\frac{\alpha}{2}} \mathbf{u}_{m+1}^p + \mathbf{F}(t_{m+1}, \mathbf{u}_{m+1}^p) \right],\end{aligned}\tag{5.13}$$

5.2.4 Computationally Efficient Algorithm using Discrete Fourier Transforms

The discrete Fourier Sine Transform (DFST) and the Discrete Fourier Cosine Transform (DFCT) are very useful tools in solving partial differential equations (PDEs) when spectral methods are used for spatial discretization. The use of discrete Fourier transforms to compute matrix vector products requires less computational cost and also saves memory space [97].

Definition 5.1 *The discrete Fourier sine transform $\mathcal{F}_{ds}\{\mathbf{u}_j\}$ of a vector with real entries is defined as,*

$$\mathcal{F}_{ds}\{\mathbf{u}_j\} = \sum_{i=1}^{N-1} u_i \sin\left(\frac{ij\pi}{N}\right), \quad j = 1, 2, \dots, N-1.$$

A similar definition exists for the discrete Fourier cosine transform. The symmetric matrix P of eigenvectors obtained in the factorization of the discretization matrix

$A^{\frac{\alpha}{2}} = \frac{\mathcal{K}}{h^\alpha} P \Lambda^{\frac{\alpha}{2}} P^{-1}$ in (5.5) consists of either $\sin\left(\frac{ij\pi}{N}\right)$ functions or $\cos\left(\frac{ij\pi}{N}\right)$ functions, depending on the boundary conditions. The matrix-vector products for computing $A^{\frac{\alpha}{2}} \mathbf{u} = P \Lambda^{\frac{\alpha}{2}} P^{-1} \mathbf{u}$ can be efficiently computed using the discrete Fourier Sine or Cosine transforms and their inverses without explicitly computing the matrices P and P^{-1} as follows:

- Compute $P^{-1} \mathbf{u} = \mathcal{F}_{ds}^{-1}\{\mathbf{u}\}$
- Compute the product $\Lambda^{\frac{\alpha}{2}} P^{-1} \mathbf{u} = \Lambda^{\frac{\alpha}{2}} \mathcal{F}_{ds}^{-1}\{\mathbf{u}\}$ by component-wise multiplication
- Compute $A^{\frac{\alpha}{2}} \mathbf{u} = P \Lambda^{\frac{\alpha}{2}} P^{-1} \mathbf{u} = \mathcal{F}_{ds} \left(\Lambda^{\frac{\alpha}{2}} \mathcal{F}_{ds}^{-1}\{\mathbf{u}\} \right)$

The DFST and DFCT are efficiently computed using fast Fourier transform (FFT) algorithms [65]. The computational cost of calculating the matrix-vector product $A^{\frac{\alpha}{2}} \mathbf{u}$ is reduced from $\mathcal{O}(N^2)$ to only $\mathcal{O}(N \log N)$, where N is the spatial mesh size. The matrix-vector product is computed using DFST or DFCT, which does not introduce additional computational errors into the scheme [55].

The following algorithm is constructed for straightforward implementation of the scheme (5.13) using DFST to solve problems with homogeneous Dirichlet boundary conditions. For problems with homogeneous Neumann boundary conditions, we shall use DFCT.

Fractional Predictor-Corrector Method Algorithm:

- Compute matrix of eigenvalues $\Lambda^{\frac{\alpha}{2}}$

- Compute the integral coefficients $a_{j,m+1}$, $b_{j,m+1}$
- Compute $\tilde{\mathbf{u}}_0 = \mathcal{F}_{ds}^{-1}(\mathbf{u}_0)$, set $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}_0$
- For $m=1:M$ do the following:

Set $\tilde{\mathbf{u}}_p = \tilde{\mathbf{u}}$ and compute $\tilde{\mathbf{F}} = \mathcal{F}_{ds}^{-1}(\mathbf{F}(t, \mathbf{u}))$

Compute $\tilde{\mathbf{u}}_p = \tilde{\mathbf{u}}_0 + \sum_{j=0}^m a_{j,m+1} [-\Lambda^{\frac{\alpha}{2}} \tilde{\mathbf{u}}_p + \tilde{\mathbf{F}}]$

Compute $\tilde{\mathbf{v}} = \mathcal{F}_{ds}(\tilde{\mathbf{u}}_p)$

Compute $\tilde{\mathbf{F}}_p = \mathcal{F}_{ds}^{-1}(\mathbf{F}(t, \tilde{\mathbf{v}}))$

Compute $\tilde{\mathbf{u}} = \left[\tilde{\mathbf{u}}_0 + \sum_{j=0}^m b_{j,m+1} [-\Lambda^{\frac{\alpha}{2}} \tilde{\mathbf{u}} + \tilde{\mathbf{F}}] + b_{m+1,m+1} [-\Lambda^{\frac{\alpha}{2}} \tilde{\mathbf{v}} + \tilde{\mathbf{F}}_p] \right],$

Compute $\mathbf{u} = \mathcal{F}_{ds}(\tilde{\mathbf{u}})$

5.3 Stability and Convergence

The fractional Euler method and Adams method were originally developed for time fractional ordinary differential equations; see [64] and references therein. The applicability of these methods is extended to the time-space fractional non-linear reaction-diffusion equations. We assume that the function $\mathbf{F}(s, \mathbf{u}(s))$ satisfies the Lipschitz condition with respect to $\mathbf{u}(s)$, that is, there exists a constant L such that,

$$|\mathbf{F}(s, \mathbf{u}(s)) - \mathbf{F}(s, \mathbf{v}(s))| \leq L|\mathbf{u}(s) - \mathbf{v}(s)|.$$

The following theorem states the stability of the fractional forward Euler method (5.10).

Theorem 5.1 *Suppose that $\mathbf{u}_j, j = 1, 2, \dots, m+1$ are solutions of the fractional forward Euler method (5.10) and $\mathbf{F}(s, \mathbf{u}(s))$ satisfies the Lipschitz condition with respect to $\mathbf{u}$ with a Lipschitz constant L on the existing interval of its unique solution. Then the fractional forward Euler method (5.10) is unconditionally stable.*

Proof. Let $\mathbf{u}_m$ and $\mathbf{u}_m$ be the two solutions of the fractional forward Euler method (5.10) corresponding to the initial conditions $\mathbf{u}_0$ with Lipschitz constant L and $\mathbf{u}_0$ and $\mathcal{E}_m = \mathbf{u}_m - \mathbf{u}_m$ be the error for $m = 0, 1, \dots, M$, then,

$$\mathcal{E}_{m+1} = \mathcal{E}_0 - \sum_{j=0}^m a_{j,m+1} A^{\frac{\alpha}{2}} \mathcal{E}_j + \sum_{j=0}^m a_{j,m+1} [F(t_j, \mathbf{u}_j) - F(t_j, \mathbf{u}_j)]$$

Since the function $\mathbf{F}(s, \mathbf{u}(s))$ satisfies the Lipschitz condition with respect to $\mathbf{u}$, therefore we get,

$$\|\mathcal{E}_{m+1}\| \leq \|\mathcal{E}_0\| + K \sum_{j=0}^m a_{j,m+1} \|\mathcal{E}_j\| + L \sum_{j=0}^m a_{j,m+1} \|\mathcal{E}_j\|,$$

where $K = \|A^{\frac{\alpha}{2}}\|$. Then using Lemma 3.3.1 from [64], we get,

$$\|\mathcal{E}_{m+1}\| \leq C \|\mathcal{E}_0\|.$$

I

The stability of the fractional Adams method can be proved similarly. The convergence of the fractional forward Euler method is given in [64] by proving the following theorem.

Theorem 5.2 *Assume that $\mathbf{u}(t_{m+1})$ is the exact solution of (5.8) and $\mathbf{u}_{m+1}$ is the solution of the method (5.10). Let $\mathbf{F}(t, \mathbf{u}(t))$ satisfies the Lipschitz condition with respect to $\mathbf{u}$ with a Lipschitz constant L and $F, u \in C^1[0, T]$. Then we have*

$$|\mathbf{u}(t_{m+1}) - \mathbf{u}_{m+1}| \leq C\Delta t.$$

The error bounds for the fractional Adams method can easily be obtained using Lemma 3.3.1 [64]. The detailed error analysis of the fractional Adams method is given by Diethelm et al. [27], [28]. Error estimates are proven for small enough T using mathematical induction.

5.4 Numerical Experiments

In this section, we present numerical experiments of two types of examples. One is a non-epidemic example with exact solution available. The second example is the time-space fractional *SIR* model.

5.4.1 A Test Example

We consider the two-dimensional homogeneous time-space-fractional diffusion equation with homogeneous Dirichlet boundary conditions,

$${}_CD_{0,t}^\beta u(x, y, t) = -\mathcal{K}(-\Delta)^{\alpha/2} u, \quad (x, y, t) \in \Omega \times [0, T],$$

$$u(x, y, 0) = \sin(n\pi x) \sin(m\pi y)$$

where $\Omega = [0, 1] \times [0, 1]$, $1 < \alpha \leq 2$, and $0 \leq \beta \leq 1$. The exact solution of this problem is given as [102], [55],

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{E}_{\beta}(-\lambda_{n,m}^{\frac{\alpha}{2}} t^{\beta}) c_{n,m} \Phi_{n,m}(x, y)$$

where $\mathbf{E}_{\beta}$ is the Mittag-Leffler function and $\lambda_{n,m}^{\frac{\alpha}{2}}$ are the eigenvalues with corresponding eigenvectors $\Phi(x, y)$ of the two-dimensional fractional Laplacian with homogeneous Dirichlet boundary conditions. The coefficients $c_{n,m}$ are defined by,

$$c_{n,m} = \int_0^1 \int_0^1 u_0(x, y) \Phi_{n,m}(x, y) dx dy$$

Let $u_0(x, y) = \sin(4\pi x) \sin(4\pi y)$. The coefficient $c_{n,m}$ are,

$$c_{n,m} = \begin{cases} 1/2, & n = m = 4 \\ 0, & n, m \neq 4 \end{cases}$$

The exact solution of the problem takes the form,

$$u(x, y, t) = \mathbf{E}_{\beta}(-(32)^{\frac{\alpha}{2}} \pi^{\alpha} t^{\beta}) \sin(4\pi x) \sin(4\pi y)$$

5.4.2 Time-Space Fractional *SIR* Model

Mathematical epidemiology has become an important research topic these days. Several epidemic models, their stability, and numerical solutions are given in the literature

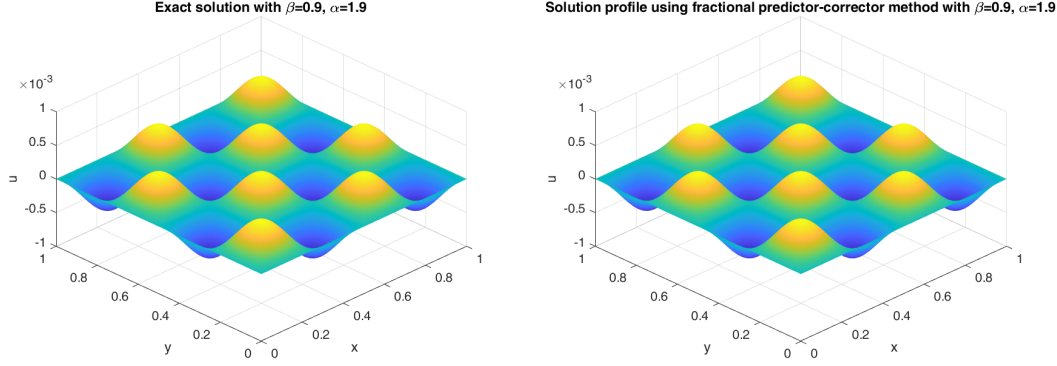


Figure 5.1: Problem 5.4.1: Solution profile comparison using Euler method and Adams method with $\beta = 0.9$ and $\alpha = 1.9$.

Table 5.1: Problem 5.4.1: Convergence results in time for $\alpha = 1.95$ and different values of β with $h = 0.025$.

β, α	k	Error	Order	Ext-Error	Ext-Order	CPU Time
$\beta = 0.65$ $\alpha = 1.95$	0.02500	3.575e-04	1.63	5.807e-06	0.00	0.0365
	0.01250	1.146e-04	1.64	9.659e-07	2.59	0.0699
	0.00625	3.662e-05	1.65	1.636e-07	2.56	0.1668
	0.00313	1.169e-05	1.65	2.798e-08	2.55	0.6351
$\beta = 0.75$ $\alpha = 1.95$	0.02500	2.013e-04	1.73	2.969e-06	0.00	0.0346
	0.01250	6.017e-05	1.74	4.646e-07	2.68	0.0645
	0.00625	1.794e-05	1.75	7.518e-08	2.63	0.1782
	0.00313	5.342e-06	1.75	1.236e-08	2.60	0.5968
$\beta = 0.85$ $\alpha = 1.95$	0.02500	9.158e-05	1.84	1.216e-06	0.00	0.0323
	0.01250	2.553e-05	1.84	1.767e-07	2.78	0.0633
	0.00625	7.102e-06	1.85	2.705e-08	2.71	0.1633
	0.00313	1.973e-06	1.85	4.190e-09	2.69	0.6539
$\beta = 0.95$ $\alpha = 1.95$	0.02500	2.230e-05	1.94	2.529e-07	0.00	0.0313
	0.01250	5.797e-06	1.94	3.340e-08	2.92	0.0645
	0.00625	1.504e-06	1.95	4.570e-09	2.87	0.1655
	0.00313	3.896e-07	1.95	5.884e-10	2.96	0.6430

[20], [32], [33], [34].

Most of the epidemic models are consist of a system of non-linear ordinary differential equations. These models do not provide a realistic approach due to the large mobility of people within a country or even worldwide, as they lack the spatial context of the environment. In other words, the population is mixing in such a way that there is no difference between individuals in one place and individuals in the other place. However, the actual scenario is that the spread of disease may be faster in one part of the population and slower in another. Therefore, when studying the spread of disease, the spatial effect cannot be ignored.

The following extended *SIR* model with fractional time-space derivatives is consider as a test problem [3]

$$\begin{aligned}
{}_C D_{0,t}^\beta S(x,t) &= -D_S(-\Delta)^{\frac{\alpha}{2}} S(x,t) + \eta^\beta - \nu^\beta S(x,t) - \frac{\mathcal{B}^\beta S(x,t)I(x,t)}{1 + \sigma_1^\beta S(x,t) + \sigma_2^\beta I(x,t)} \\
{}_C D_{0,t}^\beta I(x,t) &= -D_I(-\Delta)^{\frac{\alpha}{2}} I(x,t) - (\nu^\beta + \gamma^\beta + r^\beta)I(x,t) + \frac{\mathcal{B}^\beta S(x,t)I(x,t)}{1 + \sigma_1^\beta S(x,t) + \sigma_2^\beta I(x,t)}, \\
{}_C D_{0,t}^\beta R(x,t) &= -D_R(-\Delta)^{\frac{\alpha}{2}} R(x,t) + r^\beta I(x,t) - \nu^\beta R(x,t),
\end{aligned} \tag{5.14}$$

where D_S , D_I , D_R are the diffusion coefficients of susceptible $S(x,t)$, infected $I(x,t)$ and recovered $R(x,t)$ respectively. The other parameters are: recruitment rate η , natural death rate ν , death due to disease δ , recovery rate r and infection coefficient β . The quotient

$$\frac{\mathcal{B}^\beta S(x,t)I(x,t)}{1 + \sigma_1^\beta S(x,t) + \sigma_2^\beta I(x,t)}$$

is the modified incident with inhibitory effect $\sigma_1 \geq 0$, $\sigma_2 \geq 0$.

The time-space fractional schemes developed in this chapter are applied to solve the time-space fractional *SIR* system (5.14) using the following parameter values:

	value
η	0.25
ν	0.04
σ_1	0.5
σ_2	0.5
r	24
$\mathcal{B}$	10
δ	0.01

Table 5.2: Parameter values for problem 5.4.2

The system (5.14) is solved using the initial conditions,

$$S(x, 0) = \begin{cases} 1.5x, & 0 \leq x \leq 0.5 \\ 1.5(1 - x) & 0.5 \leq x \leq 1, \end{cases}$$

$$I(x, 0) = \begin{cases} 0.5x, & 0 \leq x \leq 0.5 \\ 0.5(1 - x) & 0.5 \leq x \leq 1, \end{cases}$$

$$R(x, 0) = \begin{cases} 0.5x, & 0 \leq x \leq 0.5 \\ 0.5(1 - x) & 0.5 \leq x \leq 1, \end{cases}$$

and the homogeneous Neumann boundary conditions. The system (5.14) is solved for two different values of the time derivative β , and the effect of the time fractional derivative is shown in the figures 5.2, 5.3, and 5.4.

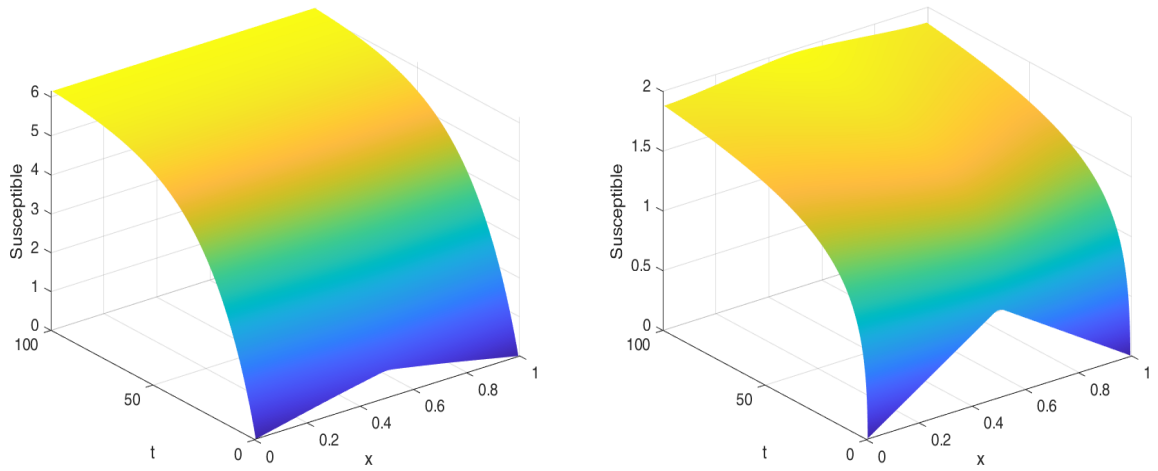


Figure 5.2: Susceptible for $\beta = 1$ (Left) and $\beta = 0.5$ (Right)

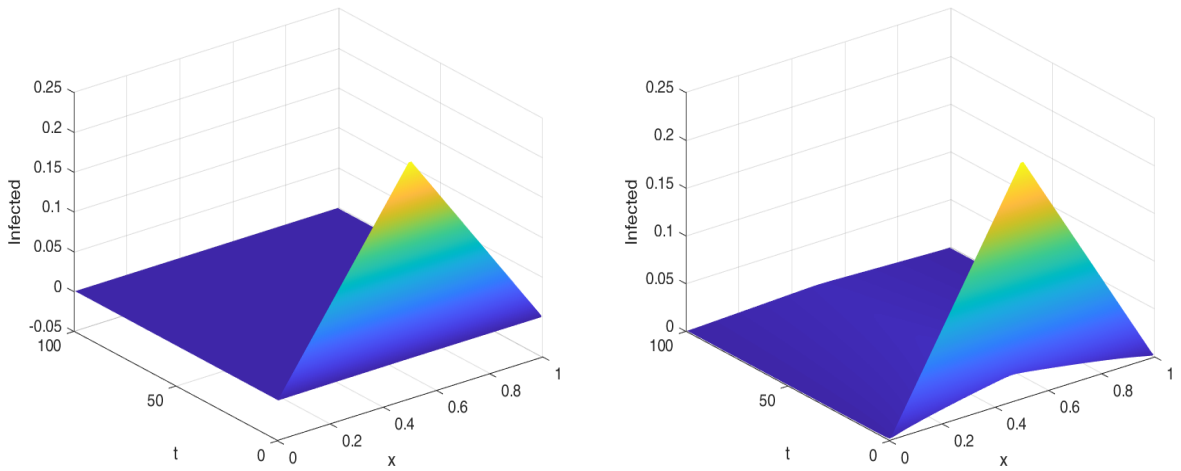


Figure 5.3: Infected for $\beta = 1$ (Left) and $\beta = 0.5$ (Right)

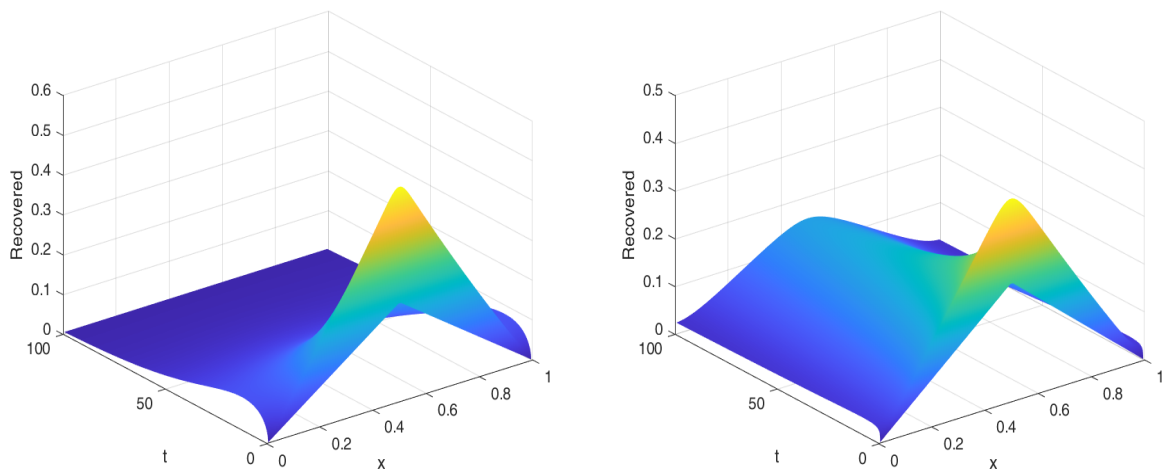


Figure 5.4: Recovered for $\beta = 1$ (Left) and $\beta = 0.5$ (Right)

CHAPTER 6

CONCLUSION AND FUTURE WORK

In this study we have:

- Conducted stability analysis of some epidemic models.
- Solved classical epidemic models using the well-known fourth-order Runge-Kutta method.
- Developed some high-order predictor-corrector schemes for fractional epidemic models.
- Developed a third-order highly efficient L -stable numerical scheme for space fractional diffusive epidemic models.
- Developed a computationally efficient predictor-corrector type scheme for time-space fractional diffusive epidemic models.

- Richardson extrapolation is utilized to further enhance the accuracy of our scheme.

Our future plan is to:

- Develop fourth-order A -stable and L -stable numerical schemes for space-fractional reaction-diffusion systems.
- Develop third and fourth order schemes for time-space fractional reaction-diffusion systems.
- Implement our schemes to investigate the dynamics of some other dynamical models in 2- and 3-dimension.
- Implement our time fractional numerical schemes to investigate the behavior of models such as the chaotic financial system and memcapacitor-based circuit chaotic oscillator.

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