

Innovative Methods and Structured Resolving Set Shapes in Determining Metric Dimensions of Strongly Regular Graphs

BY

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**INNOVATIVE METHODS AND
STRUCTURED RESOLVING SET
SHAPES IN DETERMINING
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STRONGLY REGULAR GRAPHS**

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MATHEMATICS

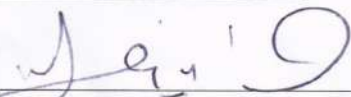
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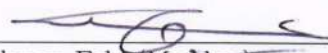
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This dissertation, written by **AMAL ABDULRAHMAN ALHUJAYLAN** under the direction of his dissertation adviser and approved by his dissertation committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS**.

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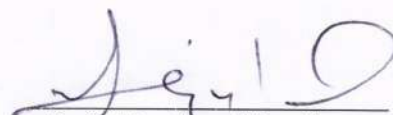

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Dedication

Norah and Mlikah

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LIST OF SYMBOLS

$V(G)$	Vertex set of graph G .
$E(G)$	Edge set of graph G .
n	Order of the graph (number of vertices).
$N_G(v)$	Neighborhood of vertex v in graph G .
d	Diameter of graph G .
$G[S]$	Induced subgraph of G with vertex set S .
$G_1 + G_2$	Join of two graphs.
$G_1 \square G_2$	Cartesian product of two graphs.
A_G	Adjacency matrix of graph G .
D_G	Distance matrix of graph G .
R	Resolving set of a graph.
$r(v R)$	Metric representation of vertex $v \in V(G)$.
$\dim(G)$	Metric dimension of a graph G .
$\mathcal{R}$	Shape of the resolving set.
G^m	Set of metric representations in which the sum of the 1-components equals m .

v_B^1	A metric representation where the value 1 appears in the positions specified by the set $B \subseteq \{1, \dots, \alpha\}$, and 2 elsewhere.
$G'_{\mathcal{R}}$	Helper graph constructed w.r.t. $\mathcal{R}$.
$k_{w_i}^*$	Degree of regularity of a resolving vertex w_i in the C.N.T.
m^*	Number of edges in the resolving set $\mathcal{R}$.
x_m	Number of vertices from G^m used in the C.N.T.
$y_m^{w_i}$	Number of vertices from G^m used in the C.N.T. related to w_i .
$z_m^{w_i}$	Number of vertices from G^m used in the C.N.T. for one μ column of w_i .
$\bar{z}_m^{w_i}$	Number of vertices from G^m used in the C.N.T. for one λ column of w_i .
γ	Total number of cells in the C.N.T.
γ_{w_i}	Total number of cells in columns associated with w_i .

DISSERTATION ABSTRACT

NAME: Amal Abdulrahman Alhujaylan

TITLE OF STUDY: Innovative Methods and Structured Resolving Set Shapes
in Determining Metric Dimensions of Strongly Regular
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This dissertation explores the metric dimension of strongly regular graphs (SRGs) by employing novel computational techniques. The metric dimension is a key parameter in graph theory, representing the minimum number of vertices required to uniquely determine all other vertices in a graph based on distances. Due to its complexity, determining the metric dimension of arbitrary graphs is a challenging NP-hard problem. This research focuses on developing a new method using the Common Neighborhood Table (C.N.T.) technique to simplify the process of identifying minimum resolving sets in SRGs.

The work begins with an exhaustive brute-force algorithm to compute metric dimensions, followed by the introduction of the C.N.T. method, which helps construct helper

graphs to verify minimum resolving sets. This method is applied to various SRGs with parameters (n, k, λ, μ) , yielding new insights into their resolving sets and metric dimensions. Several SRGs of varying sizes, up to 200 vertices, are examined, and results are compared with known values from the literature.

The findings reveal cases where previously reported upper bounds on the metric dimension were tightened, demonstrating the efficacy of the C.N.T. technique in improving the accuracy of existing results. These results have significant implications for the theoretical study of graph structures and practical applications such as network topology, cryptography, and combinatorial designs.

ملخص الرسالة

الاسم:

أمل عبدالرحمن الحجيلان

عنوان الدراسة:

طرق مبتكرة وأشكال محددة لمجموعات الحل في تحديد أبعاد القياس للرسم البيانية المنتظمة بشدة

التخصص:

الرياضيات

تاريخ الدرجة العلمية: 1 جمادى الأولى 1446 هـ

تستكشف هذه الأطروحة البعد المترى للرسم البيانية المنتظمة بقوة (SRGs) من خلال استخدام تقنيات حسابية جديدة. يُعد البعد المترى أحد المعايير الأساسية في نظرية الرسم البيانية، حيث يمثل الحد الأدنى لعدد الرؤوس اللازمة لتحديد جميع الرؤوس الأخرى في الرسم البياني بشكل فريد بناءً على المسافات. نظرًا لتعقيده، فإن تحديد البعد المترى للرسم البيانية العامة يمثل مشكلة صعبة من نوع NP-hard. يركز هذا البحث على تطوير طريقة جديدة باستخدام تقنية جدول الجوار المشترك (C.N.T.) لتبسيط عملية تحديد مجموعات الحلول الدنيا في الرسم البيانية المنتظمة بقوة. تبدأ الدراسة باستخدام خوارزمية البحث الشامل لحساب الأبعاد المترية، تليها إدخال طريقة C.N.T. التي تساعد في بناء رسوم بيانية مساعدة للتحقق من مجموعات الحلول الدنيا. يتم تطبيق هذه الطريقة على مجموعة متنوعة من الرسوم البيانية المنتظمة بقوة ذات المعلمات (n, k, λ, μ) ، مما يوفر رؤى جديدة حول مجموعات الحلول الخاصة بها وأبعادها المترية. تم فحص عدة رسوم بيانية منتظمة بقوة ذات أحجام مختلفة تصل إلى 200 رأس، وتمت مقارنة النتائج بالقيم المعروفة في الأدبيات.

كشفت النتائج عن حالات تم فيها تشديد الحدود العليا السابقة على البعد المترى، مما يظهر فعالية تقنية C.N.T. في تحسين دقة النتائج الحالية. لهذه النتائج تأثيرات كبيرة على الدراسات النظرية لهياكل الرسوم البيانية وكذلك التطبيقات العملية مثل تصميم الشبكات، التشفير، والتصاميم التوافقية.

CHAPTER 1

INTRODUCTION

Strongly regular graphs are of considerable interest across multiple disciplines due to their complex structures and broad application spectrum. These graphs are crucial in fields as varied as statistics, where they are used for experimental design, and quantum computation, where they aid in developing mutually unbiased bases (MUBs) and symmetric, informationally complete, positive operator-valued measures (SIC-POVMs)[2]. They also have significant roles in Euclidean geometry, coding theory, group theory, the theory of special functions, finite geometry, extremal combinatorics, and cryptography.

The metric dimension of a graph, a critical parameter in graph theory, is defined by the minimum number of vertices required to uniquely identify all other vertices based on distances. This concept finds applications in identifying network spread sources, graph labeling, and low-dimensional embedding of symbolic data. The associated decision problem, which determines whether the metric dimension of a graph does not exceed a given integer, is NP-complete, highlighting its computational complexity.

1.1 Preliminaries

This section lays the foundational groundwork by introducing essential terminology and concepts from **graph theory** that are pivotal for navigating the deeper discussions in this dissertation. The material covered here, encompassing definitions, lemmas, and theorems, is universally recognized and forms a core component of graph theory as illustrated in seminal texts such as those by Godsil and Royle [3], and Chartrand et al. [4]. These elements are not only foundational but integral to the comprehension and exploration of the complex topics that will be addressed in subsequent chapters of this dissertation.

A **graph** $G = (V, E)$ consists of a set $V(G)$ of vertices and a set $E(G)$ of edges, which are pairs of vertices. The **order** of the graph, denoted as n , represents the number of vertices, $|V(G)|$, and the **size** of the graph represents the number of edges, $|E(G)|$. In a **simple graph**, which has no loops (edges connected at both ends to the same vertex) or multiple edges between the same pair of vertices, the following definitions and properties apply:

- **Adjacency:** Two vertices u and v are **adjacent**, denoted $u \sim v$, if there is an edge (u, v) connecting them.
- **Neighborhood:** The **neighborhood** of a vertex v , denoted $N_G(v)$, includes all vertices adjacent to v . The **degree** of v , denoted $\deg(v)$, is the number of vertices in $N_G(v)$.
- **Path:** A **path** in graph G is a sequence of vertices $v_1, v_2, \dots, v_k$ such that each

consecutive pair v_i, v_{i+1} is connected by an edge.

- **Cycle:** A cycle C_α in a graph is a path $v_1, v_2, \dots, v_\alpha$ where v_α is also connected back to v_1 , forming a closed loop. This implies $v_\alpha \sim v_1$ and $\alpha \geq 3$.
- **Connectivity:** A graph is **connected** if there is a path between any pair of vertices.
- **Independent Edges:** Pairwise non-adjacent edges are called **independent edges**.

Definition 1.1 The **distance** between two vertices u and v in a connected graph G , denoted $d_G(u, v)$, is defined as the length of the shortest path connecting them. The distance function satisfies the following properties:

1. $d_G(u, v) \geq 0$ for all vertices u, v , and $d_G(u, u) = 0$.
2. The distance is symmetric: $d_G(u, v) = d_G(v, u)$.
3. The triangle inequality holds: for any vertex w , $d_G(u, w) \leq d_G(u, v) + d_G(v, w)$.

Additionally, the **diameter** of G , denoted $d(G)$, is the maximum distance between any pair of vertices in the graph. If the graph under discussion is clear, the notation $d(u, v)$ is used for the distance between two vertices, and d is used for the diameter:

$$d = \max_{u, v \in V(G)} d(u, v).$$

Definition 1.2 A **regular graph** is a graph where each vertex has the same number of neighbors i.e., every vertex has the same degree.

Definition 1.3 A **subgraph** H of a graph G consists of a subset $V(H) \subseteq V(G)$ and a subset $E(H) \subseteq E(G)$. An **induced subgraph** H is defined by $H = G[S]$ for $S \subseteq V(G)$, containing all edges in G that have both endpoints in S . A **spanning subgraph** H of G includes all vertices of G (i.e., $V(H) = V(G)$).

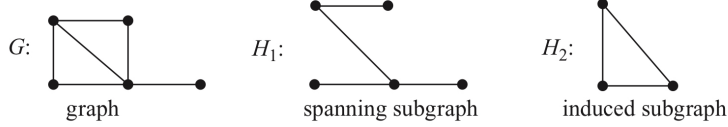


Figure 1.1: Types of Subgraphs.

Definition 1.4 Graph Isomorphism: Two graphs G and H are **isomorphic**, denoted $G \cong H$, if there is a bijection $f : V(G) \rightarrow V(H)$ that preserves adjacency, meaning $u \sim v$ in G if and only if $f(u) \sim f(v)$ in H .

Corollary 1.1 The isomorphism $f : V(G) \rightarrow V(H)$ preserves the distance between any two vertices. Specifically, for any vertices u and v in G , the distance between $f(u)$ and $f(v)$ in H is equal to the distance between u and v in G , i.e., $d_H(f(u), f(v)) = d_G(u, v)$.

Definition 1.5 Graph Invariant: A property or characteristic that does not change under graph isomorphism is called a **graph invariant**.

Definition 1.6 Vertex Labeling: A function $\ell : V \rightarrow \mathcal{L}$ that assigns labels from a label set $\mathcal{L}$ to the vertices of G is known as **vertex labeling**.

Definition 1.7 A **strongly regular graph** is defined by parameters (n, k, λ, μ) and is a regular graph where each pair of adjacent vertices has λ common neighbors and each pair of non-adjacent vertices has μ common neighbors.

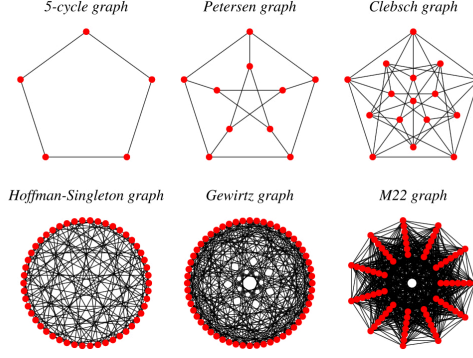


Figure 1.2: Example of a Strongly Regular Graph.

Definition 1.8 [5] A **resolving set** in a graph G with vertex set V is a subset $R \subseteq V$ such that for every pair of distinct vertices $u, v \in V$, there exists at least one vertex $w \in R$ where the distances $d(w, u)$ and $d(w, v)$ are unequal. This ensures that w can distinguish or "resolve" the vertices u and v based solely on their distances to w .

Formally, $R = \{w_1, \dots, w_\alpha\}$ is said to be resolving if the mapping:

$$r : V \longrightarrow \mathbb{R}^\alpha$$

$$u \mapsto (d(w_1, u), \dots, d(w_\alpha, u))$$

is injective. This means that the vector of distances from the vertices in R to any vertex u uniquely identifies u within G , where $d_G(w_i, u)$ represents the distance from w_i to u and $r(v|R)$ called the metric representation of v .

Definition 1.9 The **metric dimension** of a graph G , denoted $\dim(G)$, is defined as the minimum cardinality of a resolving set R within G . This parameter is crucial in various applications where unique spatial identification is required, such as network topology analysis, robotic navigation, and sensor placement.

A vertex v is a boundary vertex of a graph G if there exists a vertex u such that no neighbor of v is further away from u than v itself. The set of all boundary vertices of G is denoted by $\partial(G)$ and is called its boundary. This concept plays a crucial role in defining certain metric dimensions where boundary vertices significantly influence the resolving sets and their properties.

The *join* $G_1 + G_2$ of two graphs is defined as the graph whose vertex set is $V_1 \cup V_2$ and edge set $E(G_1 + G_2) = E_1 \cup E_2 \cup \{v_1v_2 : v_1 \in V_1, v_2 \in V_2\}$. This operation results in a graph that combines the individual edges of G_1 and G_2 along with additional edges connecting every vertex of G_1 to every vertex of G_2 , often increasing the metric dimension by integrating the complexities of both graphs.

The *Cartesian product* $G_1 \square G_2$ is another fundamental graph operation where the vertex set of the resulting graph is the Cartesian product $V_1 \times V_2$; two vertices (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \square G_2$ if and only if either $x_1 = y_1$ and $x_2y_2 \in E_2$, or $x_2 = y_2$ and $x_1y_1 \in E_1$. This definition is crucial for the study of metric dimension within product graphs because it affects how distances are computed and consequently how resolving sets are constructed.

These concepts highlight the importance of spatial relationships within graph structures and their practical relevance in theoretical and applied contexts.

Matrix representations in graph theory include:

Definition 1.10 The *adjacency matrix* A_G of a graph G is an $n \times n$ matrix defined

as:

$$A_G = [a_{ij}] \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.11 The **distance matrix** D_G of a graph G is an $n \times n$ matrix given by:

$$D_G = [d_{ij}] \quad \text{where} \quad d_{ij} \text{ is the shortest path distance between } v_i \text{ and } v_j.$$

Definition 1.12 A **permutation matrix** P is an $n \times n$ matrix used to rearrange the rows or columns of another matrix according to a specific permutation of its indices.

It is defined by:

$$P = [p_{ij}] \quad \text{where} \quad p_{ij} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{otherwise,} \end{cases}$$

with σ being a permutation of the set $\{1, 2, \dots, n\}$.

Lemma 1.1 G and H are isomorphic graphs if and only if a permutation matrix P exists such that $A_H = PA_G P^T$ where G and H are simple graphs.

We present several key results from **combinatorics** that are utilized throughout this dissertation.

Principle of Inclusion and Exclusion: Let V be our universe and let $A_1, A_2, \dots, A_\alpha$ be subsets of V (it is not forbidden that some set occurs more than once in the sequence). If I is a subset of the index set $\{1, \dots, \alpha\}$, we set $A_I = \cap_{i \in I} A_i$ with the convention that $A_\emptyset = V$ (intersecting more sets gives a smaller result; so

intersecting no sets at all should give the largest possible set).

Theorem 1.1 *Let $(A_1, \dots, A_n)$ be a family of subsets of V . Then the number of elements of V that are in none of the subsets A_i is $\sum_{I \subseteq \{1, \dots, n\}} (-1)^{|I|} |A_I|$.*

Lemma 1.2 *The number of n -tuples of non-negative integers $x_1, \dots, x_n$ with $x_1 + \dots + x_n = k$ is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.*

Theorem 1.2 *Some properties of combinatorics:*

$$1. \binom{n}{k} = \binom{n}{n-k}.$$

$$2. k \binom{n}{k} = n \binom{n-1}{k-1}.$$

$$3. \sum_{k=0}^n \binom{n}{k} = 2^n.$$

Theorem 1.3 *The number of selections of k objects from a set of n objects is given by the following:*

Repetitions	Order	No-Order
Allowed	n^k	$\binom{n+k-1}{k}$
Not allowed	$n(n-1) \dots (n-k+1)$	$\binom{n}{k}$

1.2 Literature Review

The concept of metric dimension was introduced independently by Slater in 1975 [6] and by Harary and Melter in 1976 [5]. They provided an equivalent formula for the

metric dimension of trees based on the number of leaves. In 1996, the authors of [7] proved that a graph G with $\dim(G) = 2$ cannot contain $K_{3,3}$ or K_5 as a subgraph.

Although determining the metric dimension of arbitrary graphs is computationally complex, exact formulae, upper bounds, and polynomial-time algorithms are known for specific types of graphs.

In 2000, the authors of [8] determined classes of graphs with metric dimensions equal to 1, $n - 2$, and $n - 1$. Moreover, they established the following inequality:

$$n - d^{\dim(G)} \leq \dim(G) \leq n - d, \quad (1.1)$$

where d denotes the diameter of G .

In the same year, the authors of [9] evaluated the metric dimension of the cycle C_n . In 2002, the authors of [10] found the metric dimension of a wheel W_n . In 2005, the authors of [11] provided the following useful bounds for the metric dimension:

- (1) $\dim(G) \leq |\partial(G)|$.
- (2) $\max\{\dim(G), \dim(H)\} \leq \dim(G \square H) \leq \min\{\dim(G) + |H|, \dim(H) + |G|\} - 1$.
- (3) $\dim(G) + \dim(H) \leq \dim(G + H)$.
- (4) $\dim(G) \leq \dim(G \square P_n) \leq \dim(G) + 1$.
- (5) $\dim(G) \leq \dim(G \square K_2) \leq \dim(G) + 1$.
- (6) $\dim(G \square K_n) \leq \dim(G) + n - 2$, for $n \geq 3$.

$$(7) \dim(G \square C_n) \leq \begin{cases} \dim(G) + 1 & \text{if } n \text{ is odd;} \\ \dim(G) + 2 & \text{if } n \text{ is even.} \end{cases}$$

In 2007, Hernando and colleagues investigated $\mathcal{G}_{\dim(G),d}$, the class of graphs defined by a fixed metric dimension $\dim(G)$ and diameter d . It is established that the minimum order of any graph within $\mathcal{G}_{\dim(G),d}$ equals $\dim(G) + d$. The first major contribution of their work was to provide a comprehensive characterization of graphs in $\mathcal{G}_{\dim(G),d}$ having the order $\dim(G) + d$ across all values of $\dim(G)$ and d . This characterization extended previous knowledge, which was limited to cases where $d \leq 2$ or $\dim(G) \leq 1$. Their second significant contribution was the determination of the maximum possible order for graphs within $\mathcal{G}_{\dim(G),d}$ for any given d and $\dim(G)$, where previously only a weak upper bound was available [12].

In 2008, the authors of [13] showed that the metric dimension of honeycomb networks is exactly three. The honeycomb networks $HG(n)$ correspond to a partial hexagonal tiling of $\mathbb{R}^2$, where $HG(1)$ is a single regular hexagon and $HG(n)$ consists of $n - 1$ layers of hexagons around a central one. Several other graph classes with metric dimensions of two or three were examined in [14]. The authors in [15], [16] computed the metric dimensions of complete regular bipartite graphs $K_{n,n}$ and $(n - 1)$ -regular bipartite graphs $G(n, n)$ with $n \geq 3$, respectively. In 2009, the authors of [17] introduced a characterization of graphs with metric dimension two, based on the distance partition of the vertex set of a graph G . A family of trees with metric dimension two was introduced in 2017 by the authors of [18]. A characterization of graphs G with $\dim(G) = n - 3$ was provided by the authors of [19].

The following Table 1.1 summarizes some well-known results on metric dimensions, along with relevant references.

Table 1.1: Metric Dimensions of Some Graph Families

Graph	G	$\dim(G)$	Note	Reference
Path	P_n	1		[8]
Cycle	C_n	2	$n > 2$	[9]
Odd Prism	D_n	2	n odd	[14]
Even Prism	D_n	3	n even	[14]
Wheel	W_n	3	$n = 3, 6$	[11]
Wheel	W_n	$\lfloor \frac{2n+2}{5} \rfloor$	$n \notin \{3, 6\}$	[20]
Fan	F_n	$\lfloor \frac{2n+2}{5} \rfloor$	$n \notin \{1, 2, 3, 6\}$	[11]
Antiprism	A_n	3		[14]
Petersen	$P_{n,2}$	3	$n \geq 5$	[14]
Hexagonal	$HX(n)$	3		[13]
Honeycomb	$HC(n)$	3		[13]
Tree	T	$l(T) - ex(T)$		[8]
Jahangir	J_{2n}	$\lfloor \frac{2n}{3} \rfloor$	$n \geq 4$	[14]
Grids in d dimension	$G_{n_1, \dots, n_d}$	d		[7],[21]
Class of $\dim(G) = n - 2$	$K_{s,t}$	$n - 2$	$s, t \geq 1$	[8]
	$K_s + \overline{K}_t$		$s \geq 1, t \geq 2$	
	$K_s + (K_1 \cup K_t)$		$s, t \geq 1$	
Complete	K_n	$n - 1$	$n \geq 2$	[8]

In 2012, Bailey and Meagher focused on the Grassmann graph $G_q(n, k)$, where the vertices represent k -subspaces of F_q^n , and adjacency is defined by the intersection of subspaces in a $(k - 1)$ -dimensional subspace for $k \geq 2$. They established an upper

bound on the metric dimension of this graph, equivalent to the count of 1-dimensional subspaces of F_q^n . Moreover, they formulated resolving sets of this dimension, particularly when $k + 1$ divides n , and provided alternative constructions for other scenarios [22].

Simultaneously, in 2012, researchers in [23] investigated the metric dimension of an extensive class of circulant graphs $C_n(1, 2, 3)$. Their findings indicated that these circulant graphs possess a metric dimension of 4 when $n \equiv 2, 3, 4, 5 \pmod{6}$. When $n \equiv 0 \pmod{6}$, merely 5 strategically chosen vertices are necessary to resolve the entire graph, leading to the conclusion that $\dim(C_n(1, 2, 3)) \leq 5$, except in cases where $n \equiv 1 \pmod{6}$, at which point $\dim(C_n(1, 2, 3)) \leq 6$.

In 2014, the authors of [24] delved into the maximum difference between the metric dimension $\dim(G)$ and the determining number $D(G)$ of graphs, as a function of the graph's order n . They utilized functions related to locating-dominating sets to establish bounds and performed exact calculations for specific families of graphs such as paths, cycles, complete graphs, and 2-dimensional grids. They defined the function $(\dim - D)(n)$ as the maximum value of $\dim(G) - D(G)$ for all graphs of order n , primarily focusing on two functions: λ , the minimum cardinality of a locating-dominating set, and $\lambda_{\mathcal{C}^*}$, for the class of twin-free graphs $\mathcal{C}^*$. Their key findings include:

1. $\lfloor \frac{2}{5}n \rfloor - 2 \leq (\dim - Det)(n) \leq n - 2$, for $n \geq 8$.
2. $(\dim - Det)(n) \leq (\lambda - Det)(n) \leq \lambda(n) = n - 1$.
3. $Det(G) \geq n - r$, where r is the order of G^* .
4. $(\dim - Det)(n) \leq (\lambda - Det)(n) \leq \lambda_{\mathcal{C}^*}(n) = n - \left\lceil \frac{\log_2 n}{2} \right\rceil + 1$, for $n \geq 4$.

In 2018, research by Bailey and others on Johnson graphs $J(n, k)$ and Kneser graphs $K(n, k)$ revealed that combinatorial objects such as projective planes and symmetric designs can form effective resolving sets for these graphs [25]. Moreover, they demonstrated that the metric dimension of a graph G can be bounded below by considering its automorphism group $Aut(G)$, where the least number of points forming a base for $Aut(G)$ is termed the determining number, $Det(G)$. This leads to the relationship:

$$Det(G) \leq \dim(G) \quad (1.2)$$

They also showed that any resolving set for the Kneser graph $K(n, k)$ serves as a resolving set for the Johnson graph $J(n, k)$, particularly when $n > 2k$, thus:

$$\dim(J(n, k)) \leq \dim(K(n, k)) \quad (1.3)$$

The metric dimensions for the Johnson graph $J(n, k)$ and Kneser graph $K(n, k)$ were also bounded as follows:

$$\dim(J(n, k)) \leq \left\lfloor \frac{k}{k+1}(n+1) \right\rfloor \quad (1.4)$$

$$\dim(K(n, k)) \leq \left\lceil \frac{n}{2k-1}(n+1) \right\rceil \left(\binom{2k-1}{k} - 1 \right) \quad (1.5)$$

Furthermore, they provided an algebraic approach to defining resolving sets, exemplified through an incidence matrix A of a set system, which leads to an effective method for identifying the blocks in symmetric designs as resolving sets for Johnson

graphs.

In 2019, researchers in [26] investigated the metric dimension of the non-component graph $\Gamma(\mathbb{V})$, associated with a finite vector space $\mathbb{V}$. They explored the relationship between a metric basis for $\Gamma(\mathbb{V})$ and a traditional vector space basis for $\mathbb{V}$. They demonstrated that a $\Gamma(\mathbb{V})$ -basis exhibits the exchange property similar to that of a $\mathbb{V}$ -basis, except when the underlying field contains exactly 2 elements and the dimension of $\mathbb{V}$ is three or greater. Furthermore, they established that a $\Gamma(\mathbb{V})$ -basis includes a $\mathbb{V}$ -basis when the field has three or more elements.

In 2021, the focus of [27] was on the metric dimension and metric independence number of the incidence graphs of symmetric designs, as well as their fractional counterparts. Their findings suggest a lower bound for the metric dimension of a symmetric design $\mathcal{P} = (X, B)$ to be at least $\lceil \frac{n}{k+1-\lambda} \rceil$.

In 2022, the authors of [28] developed a hybrid algorithm designed to solve the metric dimension problem (MDP) by integrating graph representation learning with a greedy repair policy. Extensive testing indicated that this hybrid algorithm offers superior computational accuracy and efficiency on random graphs. Their study also suggests that the attributes of complex networks and the design of graph neural networks significantly influence the learning process of the MDP.

Further in 2022, researchers in [29] introduced a novel family of infinite graphs, denoted as D_α , for $\alpha = 1, 2, \dots$, designed to embed all graphs with a metric dimension of α . This family was utilized to derive several extremal results concerning metric dimensions, such as establishing that the maximum degree of any graph with a metric dimension at most α is $3^\alpha - 1$. They also found that the maximum number of edges

in a graph with n vertices and metric dimension α is $\frac{n(3^\alpha-1)(1-o(1))}{2}$, and the largest n for which there exists a graph of metric dimension at most α containing $K_{n,n}$ as a subgraph is $2^\alpha - 1$. Additionally, the largest wheel subgraph in a graph of metric dimension at most α has 3^α vertices. They further demonstrated that the maximum possible chromatic number and degeneracy of any graph with a metric dimension at most α are 2^α and $\frac{3^\alpha-1}{2}$, respectively.

In 2023, the authors of [30] proposed a necessary and sufficient condition for deriving a graph G from its twin graph G^T and characterized all graphs with $\dim(G) = n-4$ and $n(G^T) = 4$ using this condition. For these graphs, they identified that:

1. $4 \leq n(G^T) \leq 7$ if $d = 3$, with the bounds being sharp;
2. $4 \leq n(G^T) \leq 9$ if $d = 2$.

In the same year, researchers introduced the notion of Levenshtein graphs, which are akin to Hamming graphs but utilize the edit distance instead of the Hamming distance. This study explored various properties of these graphs, including necessary and sufficient conditions for their shortest path distance to equate to the edit distance, and provided a characterization of their automorphism group and determining number. They also established bounds on the metric dimension of Levenshtein graphs.

These insights lead into the broader discussion on methods used to estimate the metric dimension of graphs, a problem which is known to be NP-hard. This issue is particularly pertinent in the study of strongly regular graphs. The development and optimization of algorithms for this purpose, such as brute force approaches, greedy algorithms, and integer linear programming (ILP), remain central themes in this area

of research. For instance, the greedy algorithm iteratively adds vertices that most reduce the number of unresolved vertices until all vertices in a graph are uniquely identified by their distances to the selected set.

Moreover, graph decomposition techniques and group theory have proven essential for understanding the automorphism groups of graphs, which in turn helps to approximate their metric dimensions. Specifically, these methods have shed light on the metric dimension of strongly regular graphs, providing both theoretical and practical insights into an otherwise challenging NP-hard problem.

Recent works have focused on deriving upper bounds for the metric dimension of strongly regular graphs using various computational methods:

Theorem 1.4 [31] *For a strongly regular graph with n vertices, the metric dimension is bound by:*

$$\dim(G) \leq 2\sqrt{n} \ln(n) \quad (1.6)$$

Theorem 1.5 [32] *If a strongly regular graph has n vertices and valency $k < n/2$, then:*

$$\dim(G) \leq \frac{4n \ln(n)}{k} \quad (1.7)$$

These studies align with the dissertation's goal of developing a novel method or opening a pathway to determine the metric dimension of strongly regular graphs, which is an important step in addressing this complex NP-hard problem. By integrating combinatorial approaches and optimization techniques such as integer programming, there is potential to make significant advances in this field.

The following table provides a synthesized overview of key studies that have con-

tributed to our understanding of the metric dimensions across a variety of graph families, with a particular focus on strongly regular graphs. These entries represent significant milestones in graph theory and metric dimension research, documenting both general properties and specific characteristics of strongly regular graphs. This compilation not only highlights the breadth of research conducted in this area but also pinpoints the complexity and challenges faced when dealing with strongly regular graphs. The table serves as a resourceful summary for comparing different approaches and outcomes in the metric dimension studies of these graph types.

Table 1.2: Metric Dimensions of Some Strongly Regular Graphs

(n, k, λ, μ)	No.	$\dim(G)$	Note	Ref.
$(15, 6, 1, 3)$	1	4	$J(6, 2)$	[33]
$(35, 16, 6, 8)$	3854	6	Includes $AJ(8, 4)$ Quotient Johnson graph	[33]
$(36, 14, 7, 4)$	1	6		[33]
$(36, 14, 4, 6)$	180	6		[33]
$(49, 24, 11, 12)$	2	7	Self-complementary from $7^2 : (3 \times D_{16})$	[34]
$(50, 7, 0, 1)$	1	11	Hoffman-Singleton graph	[34]
$(55, 18, 9, 4)$	1	8	$J(11, 2)$	[34]
$(56, 10, 0, 2)$	1	9	Gewirtz graph	[34]
$(64, 27, 10, 12)$	1	7	Affine polar graph $VO^-(6, 2)$	[34]
$(64, 18, 2, 6)$	1	10	From $2^6 : (3.S_6)$	[34]
$(66, 20, 10, 4)$	1	8	$J(12, 2)$	[34]
$(77, 16, 0, 4)$	1	11	M_{22} graph	[34]
$(81, 32, 13, 12)$	1	8	From $3^4 : GL(2, 3) : S_4$	[34]
$(81, 16, 7, 2)$	1	11	$H(2, 9)$	[34]
$(81, 40, 19, 20)$	1	7	Paley graph P_{81}	[34]
$(81, 40, 19, 20)$	1	8	Self-complementary from $3^5 : (4.S_5)$	[34]
$(100, 36, 14, 12)$	1	9	Hall-Janko graph	[34]
$(100, 22, 0, 6)$	1	≤ 14	Higman-Sims graph	[34]

1.3 Outline

This section provides a concise overview of the dissertation, outlining the contents and structure of each chapter. This introduction aims to familiarize the reader with the scope of research undertaken and the methodological approaches employed throughout

this study.

- **Chapter 1:** This introductory chapter establishes the groundwork by presenting fundamental concepts in graph theory and combinatorics. It includes a comprehensive review of existing literature on the metric dimension of graphs, discussing various approaches used in calculating this parameter.
- **Chapter 2:** This chapter utilizes a brute-force algorithm to determine the metric dimension or an upper bound for specific strongly regular graphs (SRGs). The algorithm analyzes adjacency matrices to identify distinct rows in the corresponding distance matrices. The results are validated against known metrics for graphs with up to 100 vertices, with newly discovered findings considered as upper bounds for larger graphs.
- **Chapter 3:** A novel method is introduced to identify a resolving set for a given SRG. The process involves constructing a new graph H , which contains the helper graph and its complement within the SRG. Establishing a one-to-one correspondence between this new graph H and the SRG confirms the assumed structure of the resolving set for the SRG.
- **Chapter 4:** This chapter addresses the computational challenges of determining the metric dimension, given the NP-hard complexity of the problem. It explores several combinatorial approaches and introduces an integer programming formulation to aid in predicting the selection of metric representations. These metric representations are necessary for constructing the Common Neighborhood Table (C.N.T.), a newly introduced tool in analyzing strongly regular graphs.

- **Chapter 5:** This chapter focuses on applying integer programming techniques to specific graph configurations, such as paths and cycles combined with isolated vertices. The objective is to optimize the process of constructing the Common Neighborhood Table (C.N.T.) and refine the assumptions used in calculating the metric dimension for known strongly regular graphs (SRGs).
- **Chapter 6:** This chapter relaxes some of the stringent constraints imposed in earlier models, introducing linear programming to aid in constructing the CNT more flexibly. This approach allows for a more nuanced exploration of the metric dimensions across various SRGs without the previously assumed parameter restrictions.
- **Chapter 7:** The concluding chapter summarizes the key findings of the dissertation and discusses potential avenues for future research. It reflects on the implications of the study's results for the field of graph theory and outlines prospective challenges and innovations that could advance understanding of metric dimensions in strongly regular graphs.

The structured approach in this dissertation not only advances theoretical knowledge in graph theory but also enhances practical methodologies for tackling complex problems associated with strongly regular graphs. The combination of rigorous analysis and innovative modeling provides a significant contribution to the field and sets the stage for future explorations.

CHAPTER 2

BRUTE-FORCE

DETERMINATION OF METRIC

DIMENSIONS IN STRONGLY

REGULAR GRAPHS

Brute-force search, or exhaustive search, also known as generate and test, is a very general problem-solving technique and algorithmic paradigm that consists of systematically checking all possible candidates for whether or not each candidate satisfies the problem statement. We used a brute-force algorithm to find an upper bound for $\dim(G)$, where G is a known strongly regular graph. A brute-force solution determining the exact metric dimension of a general graph, on the other hand, requires an exhaustive search over a very large solution space. For a fixed set of size $|R|$, there are $\binom{n}{|R|}$ subsets of vertices that must be considered. Since we are interested in the

smallest $|R|$ for which a subset of vertices of this size resolves the graph G , increasing values of $|R|$ starting at 1 must be tested until a solution is found.

Consider the adjacency $n \times n$ matrix A_G and the distance $n \times n$ matrix D_G of a graph $G = (V, E)$ on n vertices, and let $|R|$ be a positive integer. The metric representations of vertices of a graph $G = (V, E)$ is a one to one vector function from vertex set to $\mathbb{R}^{|R|}$. This means the rows of sub-matrix generated from all n rows and $|R|$ columns (of resolving vertices) of distance matrix D_G are mutually distinct. Hence, for a graph G , if there is a $n \times |R|$ sub-matrix, where $|R|$ satisfies $\min_{|R|} \{n \leq 2^{|R|} + |R|\}$, with mutually distinct rows then $\dim(G) = |R|$. We apply the algorithm on adjacency matrices of distance regular graphs with diameter 2 from [35], using MATLAB.

Algorithm 1 Evaluating Distance Matrix for Strongly Regular Graph

Input: $n \times n$ adjacency matrix of the strongly regular graph A_G

Output: $n \times n$ distance matrix of the strongly regular graph D_G

$$D_G = A_G$$

for $i = 1$ **to** n **do**

for $j = 1$ **to** n **do**

if $i \neq j$ **then**

if $A_G(i, j) = 0$ **then**

$D_G(i, j) = 2;$

end

end

end

end

D_G

Algorithm 2 Evaluating Distance Sub-Matrix for Strongly Regular Graph

Input: $n \times n$ distance matrix of the strongly regular graph D_G

Output: $n \times |R|$ distance sub-matrix of the strongly regular graph, Res

$B\{\} = [\quad]$;

$Res = [\quad]$;

for $\alpha = \min_{\alpha}\{n \leq \alpha + 2^{\alpha}\} : \alpha + 10$ **do**

for $|R| = \alpha$ **do**

for $k = 1:10000$ **do**

$B\{k\} = D_G(1 : n, [1 : |R|])$

if *number of unique rows of* $B\{k\} = n$ **then**

$Res := B\{k\}$

end

end

end

end

Res

$\dim(G) = |R|$

The brute-force algorithms implemented in this study have not only confirmed known results but have also identified several cases where the computed upper bounds are lower than previously reported values. These cases are highlighted in red in the tables below, demonstrating the potential of our computational approach to refine existing theoretical bounds.

Theorem 2.1 [36] *Let p be a prime satisfying $p \equiv 1 \pmod{4}$. Then the metric dimension of the Paley graph P_p satisfies:*

$$\lfloor \log_2(p) \rfloor \leq \dim(P_p) \leq 2\lfloor \log_2(p) \rfloor \quad (2.1)$$

Table 2.1: Metric Dimensions of Paley Graphs with Theoretical Bounds

Graph	(n, k, λ, μ)	$\dim(G)$	$2\sqrt{n} \log(n)$	$\frac{4n \log(n)}{k}$	$\lfloor \log_2(p) \rfloor$	$2\lfloor \log_2(p) \rfloor$
P_5	(5, 2, 0, 1)	2	16.1	7.18	2	4
P_9	(9, 4, 1, 2)	3	13.2	19.78	3	6
P_{13}	(13, 6, 2, 3)	4	18.5	22.23	3	7
P_{17}	(17, 8, 3, 4)	4	23.36	24.08	4	8
P_{25}	(25, 12, 5, 6)	5	32.19	26.82	4	9
P_{29}	(29, 14, 6, 7)	6	36.27	27.90	4	9
P_{37}	(37, 18, 8, 9)	5	43.93	29.69	5	10
P_{41}	(41, 20, 9, 10)	7	47.55	30.45	5	10
P_{49}	(49, 24, 11, 12)	7	54.49	31.7	5	11
P_{53}	(53, 26, 12, 13)	7	57.8	32.37	5	11
P_{61}	(61, 30, 14, 15)	7	64.22	27.86	5	11
P_{73}	(73, 36, 17, 18)	7	73.32	34.8	6	12
P_{81}	(81, 40, 19, 20)	7	79.1	35.60	6	12
P_{89}	(89, 44, 21, 22)	8	84.7	36.32	6	12
P_{97}	(97, 48, 23, 24)	8	90.11	36.98	6	13
P_{101}	(101, 50, 24, 45)	$\leq$ 9	92.76	38.84	6	13

Theorem 2.2 [37] *For $n \geq 6$, the class dimension of the Johnson scheme $J(n, 2)$ (and thus the metric dimension of the Johnson graph $J(n, 2)$ and Kneser graph $K(n, 2)$) is*

given by:

$$\dim(J(n, 2)) = \dim(K(n, 2)) = \frac{2}{3}(n - i) + i, \quad (2.2)$$

where

$$n \equiv i \pmod{3}, \quad i \in \{0, 1, 2\} \quad (2.3)$$

Table 2.2: Selected Metric Dimensions of $K(n, 2)$ and $J(n, 2)$ Graphs

G	(n, k, λ, μ)	$\dim(G)$	$\frac{2}{3}(n - i) + i$	$\bar{G}$	$(n, \bar{k}, \bar{\lambda}, \bar{\mu})$
$J(5, 2)$	$(10, 6, 3, 4)$	4	—	—	—
$J(6, 2)$	$(15, 8, 4, 4)$	4	4	$K(6, 2)$	$(15, 6, 1, 3)$
$J(7, 2)$	$(21, 10, 5, 4)$	5	5	$K(7, 2)$	$(21, 10, 3, 6)$
$J(8, 2)$	$(28, 12, 6, 4)$	6	6	$K(8, 2)$	$(28, 15, 6, 10)$
$J(9, 2)$	$(36, 14, 7, 4)$	6	6	$K(9, 2)$	$(36, 21, 10, 15)$
$J(10, 2)$	$(45, 16, 8, 4)$	7	7	$K(10, 2)$	$(45, 28, 15, 21)$

Theorem 2.3 [38] For all $m \geq 1$, the metric dimension of the square lattice graph

$H(2, m)$ is given by:

$$\dim(H(2, m)) = \left\lfloor \frac{2}{3}(2m - 1) \right\rfloor \quad (2.4)$$

Table 2.3: Selected $H(n, 2)$ Graph Dimensions

G	(n, k, λ, μ)	$\dim(G)$	$\lfloor \frac{2}{3}(2m - 1) \rfloor$
$H(2, 4)$	$(16, 6, 2, 2)$	4	4
$H(2, 5)$	$(25, 8, 3, 2)$	6	6
$H(2, 6)$	$(36, 10, 4, 2)$	7	7
$H(2, 7)$	$(49, 12, 5, 2)$	8	8
$H(2, 8)$	$(64, 14, 6, 2)$	10	10
$H(2, 9)$	$(81, 16, 7, 2)$	11	11

Theorem 2.4 [22] *Let $G_q(n, k)$ be a Grassmann graph, where $2 \leq k \leq n/2$. The metric dimension of $G_q(n, k)$ is bounded above by the Gaussian binomial coefficient:*

$$\dim(G_q(n, k)) \leq \begin{bmatrix} n \\ 1 \end{bmatrix}_q \quad (2.5)$$

where $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ represents the Gaussian binomial coefficient.

In Table 2.4, the value of $\dim(G_2(4, 2))$ aligns with the prior result reported in [33], while the remaining values have not yet been precisely determined.

Table 2.4: Metric Dimensions of Select Grassmann Graphs

G	(n, k, λ, μ)	$\dim(G)$	$2\sqrt{n} \log(n)$	$\frac{4n \log(n)}{k}$	$\begin{bmatrix} n \\ 1 \end{bmatrix}_q$
$G_2(4, 2)$	$(35, 18, 9, 9)$	6	42.07	27.65	15
$G_3(4, 2)$	$(130, 48, 20, 16)$	$\leq$ 10	110.99	52.73	40
$G_2(5, 2)$	$(155, 42, 17, 9)$	$\leq$ 13	125.5	74.45	31
$G_2(6, 2)$	$(651, 90, 33, 9)$	$\leq$ 20	330.6	187.44	63

Table 2.5 provides an upper bound for graphs that do not belong to a specific

family and have an order of 100 or more. The metric dimensions of these graphs have not been previously evaluated or studied, except for the Hall-Janko and Higman-Sims graphs. Our result for the Hall-Janko graph aligns with the findings reported in [34]. However, our calculations for the metric dimension of the Higman-Sims graph yield a value 13, which is lower than the upper bound of 14 reported in [34], as we have identified 100 distinct metric representations for this dimension.

Table 2.5: Metric Dimension of Some SRGs

G	(n, k, λ, μ)	$\dim(G)$
Hall-Janko graph	$(100, 36, 14, 12)$	9
Higman-Sims*	$(100, 22, 0, 6)$	$\leq$ 13
1 st subconstituent of McLaughlin graph	$(112, 30, 2, 10)$	$\leq$ 13
Zara graph on 126 vertices	$(126, 45, 12, 18)$	$\leq$ 10
2 nd subconstituent of McLaughlin graph	$(162, 105, 72, 60)$	$\leq$ 13
	$(176, 70, 18, 34)$	$\leq$ 13
Berlekamp-van Lint-Seidel graph	$(243, 22, 1, 2)$	$\leq$ 37
Delsarte graph	$(243, 110, 37, 60)$	$\leq$ 13
McLaughlin graph	$(275, 112, 30, 56)$	$\leq$ 15
$O_7(3)$ graph	$(378, 117, 36, 36)$	$\leq$ 15
$G_2(4)$ graph on 416 vertices	$(416, 100, 36, 20)$	$\leq$ 20
$U_6(2)$ graph	$(672, 176, 40, 48)$	$\leq$ 20

Note: Bailey, in [34], estimates the upper bound of the metric dimension of the Higman-Sims graph* to be less than or equal to 14.

2.1 Conclusion

In this chapter, we explored the brute-force approach for determining the metric dimension of strongly regular graphs (SRGs). By applying exhaustive search techniques on adjacency and distance matrices, we were able to compute upper bounds for $\dim(G)$, demonstrating the computational feasibility of such an approach despite its inherent complexity. The use of MATLAB and Julia for verification provided confidence in the accuracy of our results.

Through comparison with theoretical bounds from the literature, our calculations produced tighter upper bounds in several cases, highlighted in red in the tables. These findings contribute to the ongoing efforts in refining the metric dimension for various graph families, particularly strongly regular graphs such as Paley, Kneser, Johnson, Grassmann graphs, and others with higher orders.

Future work could focus on improving the efficiency of these algorithms, potentially by incorporating heuristic methods or advanced optimization techniques to reduce computational time. Additionally, extending this method to explore alternative graph families and real-world graph applications would further enhance its relevance and applicability in graph theory.

CHAPTER 3

NOVEL METHOD: CONSTRUCTING A HELPER GRAPH USING THE C.N.T. TECHNIQUE TO DETERMINE METRIC DIMENSION OF STRONGLY REGULAR GRAPHS

In this chapter, we present a novel method for determining the metric dimension $\dim(G)$ of strongly regular graphs (SRGs). This method is based on constructing a helper graph $G'_{\mathcal{R}}$, which is created after constructing a newly introduced structure, called the Common Neighborhood Table (C.N.T.).

The C.N.T. technique organizes all possible metric representations $r(u|R)$, denoted by v_B^1 , into disjoint sets G^m . These sets and their combinatorial properties are crucial for generating a new graph H , defined with a vertex set as G that contains the helper graph $G'_{\mathcal{R}}$ and its complement within G . Under specific conditions, the new graph H is isomorphic to the original SRG G , with parameters $SRG(n, k, \lambda, \mu)$, and leads to the identification of a resolving set for the graph.

This approach simplifies the process of determining the metric dimension and understanding the underlying structure of SRGs. The chapter begins by introducing the sets of metric representations and defining the C.N.T. technique, followed by a detailed exploration of its combinatorial foundations. The existence of the helper graph $G'_{\mathcal{R}}$ depends on satisfying a key condition (Condition 3.1), which ensures the applicability of the method to strongly regular graphs.

3.1 C.N.T. Technique

In this section, we introduce the Common Neighborhood Table (C.N.T.) technique, which plays a critical role in constructing a helper graph and determining the metric dimension of strongly regular graphs (SRGs).

Lemma 3.1 *For any strongly regular graph G , there are at most $\dim(G) + 2^{\dim(G)}$ distinct metric representations.*

Proof. Let $R = \{w_1, \dots, w_\alpha\}$ be a minimum resolving set for a strongly regular graph G , with $\dim(G) = \alpha$. Consider a vertex $v \in V \setminus R$. The metric representation

of v with respect to R is defined as:

$$r(v|R) := (d(v, w_1), \dots, d(v, w_\alpha))$$

Since G is strongly regular with diameter 2, we have:

$$0 < d_G(v, w_i) \leq 2, \quad \text{for } i = 1, \dots, \alpha$$

Thus, there are 2^α possible metric representations for vertices in $V \setminus R$. For each vertex $v = w_i \in R$, the metric representation has a 0 in the i^{th} position, and since the relations between resolving vertices are known, their metric representations are determined. Therefore, there are exactly α distinct metric representations for resolving vertices. Summing both, the total number of metric representations for all vertices in G is:

$$\dim(G) + 2^{\dim(G)}.$$

I

Note: For simplicity, we denote $|R| = \alpha$ throughout.

Definition 3.1 For any vertex $v \in V \setminus R$, let $v_B^1 := (a_1, a_2, \dots, a_\alpha)$ denote a metric representation where the value 1 appears in the positions specified by the set $B \subseteq \{1, \dots, \alpha\}$, and 2 elsewhere. In addition, $v^0 := (2, 2, \dots, 2)$ represents a special case where all components are equal to 2, indicating that the vertex is at distance 2 from all resolving elements in R . This vertex is known as the **farthest vertex**.

Remark 3.1.1 For a resolving set $R = \{w_1, \dots, w_\alpha\}$, the **relation** between the resolv-

ing vertices refers to their adjacency or non-adjacency. The total number of possible relations between the resolving vertices is $\binom{\alpha}{2}$.

Before constructing the helper graph, we first introduce the *Common Neighborhood Table (C.N.T.)*, which plays a critical role in finding the metric representations. The C.N.T. will serve as the foundation for generating the helper graph by capturing the relations with resolving vertices.

Definition 3.2 For a strongly regular graph $SRG(n, k, \lambda, \mu)$, the **Common Neighborhood Table (C.N.T.)** is a table consisting of $\binom{\alpha}{2}$ columns, each representing a possible relation (adjacent or non-adjacent) between two resolving vertices w_i and w_j ($i \neq j$). Every relation has either λ or μ common metric representations based on whether the vertices w_i and w_j are adjacent or non-adjacent.

Definition 3.3 The set $G^m := \{v_B^1\}$, where $2 \leq m \leq \alpha$, consists of all metric representations in which the sum of the 1-components is equal to m . Each metric representation in G^m appears in exactly $\binom{m}{2}$ distinct common neighborhoods. In other words, if a metric representation is used in one relation (selected in one column of the C.N.T.), it will also appear in $\binom{m}{2} - 1$ additional distinct relations.

Definition 3.4 The set G_{ind} contains the metric representations of vertices that do not appear in any common neighborhood of pairs of resolving vertices $w_i w_j$, for $i, j \in \{1, \dots, \alpha\}$. Formally, $G_{ind} = G^1 \cup G^0$, where G^1 represents metric representations with exactly one 1-component, and G^0 contains only $v^0 = (2, 2, \dots, 2)$.

Lemma 3.2 *Let G be a strongly regular graph $SRG(n, k, \lambda, \mu)$. If $\lambda < \alpha - 2$, where $\alpha = |R|$ is the size of a resolving set R , then there must exist non-adjacent vertices in R .*

Proof. Assume, for the sake of contradiction, that all vertices in the resolving set $R = \{w_1, w_2, \dots, w_\alpha\}$ are adjacent. By the property of a strongly regular graph (SRG), any two adjacent vertices share exactly λ common neighbors. However, since the cardinality of R is α , the number of common neighbors between any two adjacent vertices would need to be $\alpha - 2$, as there are $\alpha - 2$ other vertices in the resolving set that could be common neighbors. This creates a contradiction when $\lambda < \alpha - 2$, as the number of common neighbors required ($\alpha - 2$) exceeds the available number of λ common neighbors. Therefore, it is impossible for all vertices in R to be adjacent, and there must exist non-adjacent vertices in R . ■

Corollary 3.1 *The set G^2 captures all possible relations between the vertices of the resolving set $R = \{w_1, w_2, \dots, w_\alpha\}$, as each metric representation occurs exactly once in the C.N.T. Therefore, the size of G^2 is given by $|G^2| = \binom{\alpha}{2}$, corresponding to the total number of possible relations between the vertices in R .*

Proof. Consider any metric representation $v_B^1 \in G^2$, where $B = \{i, j\}$ and $i, j \in \{1, \dots, \alpha\}$. This represents all possible pairs of distinct elements from the set $\{1, \dots, \alpha\}$. Since each pair of resolving vertices corresponds to one unique relation in the C.N.T., the total number of such pairs is given by $\binom{\alpha}{2}$. Hence, the number of metric representations in G^2 is exactly $\binom{\alpha}{2}$. ■

Definition 3.5 *The set $G_{\lambda,\mu}$ consists of metric representations chosen from the sets G^m , where $2 \leq m \leq \alpha$, that appear in the Common Neighborhood Table (C.N.T.). These representations are determined by the adjacency and non-adjacency relations between resolving vertices in a strongly regular graph. The common neighborhoods corresponding to these relations are defined by the parameters λ and μ of the SRG.*

Definition 3.6 *The set G_{comp} consists of metric representations selected from G_{ind} to complete the Common Neighborhood Table (C.N.T.) while ensuring that each resolving vertex has the correct degree k and that the total number of vertices in the graph matches the graph order n . These elements are used to fill any remaining positions in the C.N.T. after selecting suitable elements from G^m ($2 \leq m \leq \alpha$).*

Corollary 3.2 *The following inequality holds:*

$$|G_{\lambda,\mu}| + |G_{comp}| \leq 2^{\dim(G)}.$$

Proof. From Lemma 3.1, we know that the total number of distinct metric representations for vertices not in the resolving set R is bounded by $2^{\dim(G)}$. The set $G_{\lambda,\mu}$ contains the metric representations selected for filling the Common Neighborhood Table (C.N.T.) based on adjacency and non-adjacency conditions, while the set G_{comp} contains the remaining metric representations required to meet the degree k and order n of the graph. Since the α elements of the resolving set R are not counted in this bound, the combined size of the sets $G_{\lambda,\mu}$ and G_{comp} must satisfy the inequality:

$$|G_{\lambda,\mu}| + |G_{comp}| \leq 2^{\dim(G)}.$$

■

Corollary 3.3 *For any strongly regular graph G , the following inequality holds:*

$$n \leq \dim(G) + 2^{\dim(G)}.$$

Proof. For any connected graph G , the number of vertices n and the metric dimension $\dim(G)$ satisfy the inequality:

$$n - d^{\dim(G)} \leq \dim(G) \leq n - d,$$

where d is the diameter of the graph. In the case of a strongly regular graph (SRG), the diameter $d = 2$. By considering the lower bound of the metric dimension, we obtain the inequality:

$$n \leq \dim(G) + 2^{\dim(G)}.$$

Thus, the total number of vertices n is bounded by the metric dimension $\dim(G)$ and the number of distinct metric representations $2^{\dim(G)}$. ■

3.1.1 Properties of Partitioned Sets in G^m

In this subsection, we will explore the properties of the partitioned sets G^m , which consist of metric representations characterized by the number of 1-components. We will examine the relationships between these sets, their symmetries, and how they distribute over the Common Neighborhood Table (C.N.T.). By studying these properties, we gain a deeper understanding of the structure of the C.N.T. and its role in

resolving sets and constructing strongly regular graphs.

Lemma 3.3 *For any integer $m = 0, 1, \dots, \lfloor \frac{\alpha}{2} \rfloor$, the sizes of the sets $G^{\alpha-m}$ and G^m are equal, i.e., $|G^{\alpha-m}| = |G^m|$.*

Proof. This result follows directly from the symmetry of binomial coefficients, specifically:

$$\binom{\alpha}{\alpha-m} = \binom{\alpha}{m}.$$

■

Lemma 3.4 *Let $w_i \in R$, where $i \in \{1, \dots, \alpha\}$. The number of elements $v_B^1 \in G^m$, where the index $i \in B$ and $|B| = m$, is given by $\binom{\alpha-1}{m-1}$. Moreover, the sets G^m and $G^{\alpha-m+1}$ contain the same number of such elements.*

Proof. For any element in G^m , the metric representation contains m positions, one of which is occupied by the index i . Removing the i^{th} position leaves $\alpha - 1$ choices for the remaining $m - 1$ positions. Hence, the number of such elements is given by:

$$\binom{\alpha-1}{m-1}.$$

Now, using the symmetry of binomial coefficients:

$$\binom{\alpha-1}{m-1} = \binom{\alpha-1}{\alpha-m},$$

which shows that G^m and $G^{\alpha-m+1}$ contain the same number of elements v_B^1 , where the index $i \in B$.

■

Lemma 3.5 *The number of elements $v_B^1 \in G^m$, where the indices $i, j \in B$ and $|B| = m$, is given by $\binom{\alpha-2}{m-2}$. Moreover, the sets G^m and $G^{\alpha+2-m}$ contain the same number of elements with indices i and j included in B .*

Proof. For any element $v_B^1 \in G^m$, the metric representation contains m positions, two of which are occupied by the indices i and j . Removing the positions corresponding to i and j leaves $\alpha - 2$ choices for the remaining $m - 2$ positions. Hence, the number of such elements is given by:

$$\binom{\alpha - 2}{m - 2}.$$

Using the symmetry of binomial coefficients:

$$\binom{\alpha - 2}{m - 2} = \binom{\alpha - 2}{\alpha - m},$$

which shows that G^m and $G^{\alpha-m+2}$ contain the same number of elements where both indices i and j are present in the set B . ■

Lemma 3.6 *The maximum number of vertices with metric representations connected to a single resolving vertex $w_i \in R$, for $i = 1, \dots, \alpha$, that can be used in the Common Neighborhood Table (C.N.T.) is $2^{\alpha-1}$. Moreover, the maximum number of vertices with metric representations connected simultaneously to both w_i and w_j , where $i \neq j$, is $2^{\alpha-2}$.*

Proof. We use the binomial sum identity:

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

To compute the maximum number of vertices connected to a single vertex w_i , we sum over all possible subsets of size $m - 1$ from the $\alpha - 1$ other vertices in the resolving set:

$$k_{max} = \sum_{m=1}^{\alpha} \binom{\alpha - 1}{m - 1}.$$

Shifting the index, this becomes:

$$k_{max} = \sum_{m=0}^{\alpha-1} \binom{\alpha - 1}{m} = 2^{\alpha-1}.$$

This gives the maximum number of vertices connected to w_i .

Similarly, to compute the maximum number of vertices connected to both w_i and w_j , we sum over all possible subsets of size $m - 2$ from the $\alpha - 2$ other vertices:

$$\eta_{max} = \sum_{m=2}^{\alpha} \binom{\alpha - 2}{m - 2}.$$

Again, shifting the index, we get:

$$\eta_{max} = \sum_{m=0}^{\alpha-2} \binom{\alpha - 2}{m} = 2^{\alpha-2}.$$

Thus, the maximum number of vertices connected to both w_i and w_j is $2^{\alpha-2}$. ■

Lemma 3.7 *The metric representation $v_{k_1 \dots i \dots k_m}^1$ appears in $m - 1$ distinct common neighborhoods (relations) involving the resolving vertex w_i , where $i \in \{1, \dots, \alpha\}$. In other words, if a metric representation $v_{k_1 \dots k_m}^1$ appears in one relation (one column of the C.N.T.) related to w_i , the number of additional appearances across all*

pairs $w_i w_j$, where $j \in \{1, \dots, \alpha\}$ and $j \neq i$, is $m - 1$. This is equivalent to $(\alpha - 1) - (\text{number of components with two in } v_{k_1 \dots i \dots k_m}^1)$, which counts how many distinct common neighborhoods involve w_i .

Proof. Without loss of generality (WLOG), assume $i = k_i$, meaning the index i corresponds to one of the positions in the metric representation $v_{k_1 \dots k_m}^1$. This representation has a 1 in m positions, one of which corresponds to w_i . The remaining $m - 1$ positions correspond to other vertices $w_{k_1}, w_{k_2}, \dots, w_{k_{i-1}}, w_{k_{i+1}}, \dots, w_{k_m}$.

In the Common Neighborhood Table (C.N.T.), each column represents a relation (adjacency or non-adjacency) between resolving vertices w_i and w_j . For each pair $w_i w_j$, if j is among the indices $k_1, k_2, \dots, k_{i-1}, k_{i+1}, \dots, k_m$, the metric representation $v_{k_1 \dots k_m}^1$ will appear in the corresponding relation.

Since there are m total positions in the metric representation and one of these is occupied by w_i , the remaining $m - 1$ positions must correspond to distinct relations involving w_i and other vertices w_j . Therefore, the metric representation appears in $m - 1$ distinct common neighborhoods (relations) related to w_i .

If the representation $v_{k_1 \dots k_m}^1$ appears in one relation between w_i and w_j , the number of remaining appearances across other relations $w_i w_j$ (for $j \neq i$) is:

$$(\alpha - 1) - (\text{number of components with two in } v_{k_1 \dots k_m}^1),$$

which simplifies to $m - 1$, as the number of components with value 2 corresponds to the number of positions where the representation does not involve w_i . ■

Lemma 3.8 *For any two distinct metric representations v_B^1 and v_C^1 , where $|B| = |C| = m$ and $|B \cap C| = k$, they share exactly $\binom{k}{2}$ common columns (relations) in the Common Neighborhood Table (C.N.T.).*

Proof. Let v_B^1 and v_C^1 be two distinct metric representations with $|B| = |C| = m$ and $|B \cap C| = k$, meaning they share k common positions. Each of these common positions corresponds to a shared neighborhood relation in the C.N.T.

Since the C.N.T. is built from the pairwise relations between resolving vertices w_i and w_j , and the positions in $B \cap C$ correspond to pairs of resolving vertices $w_i w_j$, the metric representations v_B^1 and v_C^1 will appear in exactly $\binom{k}{2}$ common columns (relations) of the C.N.T.

Thus, the number of common columns where both v_B^1 and v_C^1 appear is determined by the number of distinct pairs that can be formed from the k shared indices, which is $\binom{k}{2}$. ■

Lemma 3.9 *In any C.N.T., consider a set of metric representations $v_{B_1}^1, v_{B_2}^1, \dots, v_{B_{\binom{\alpha-2}{m-2}}}^1$ such that for any $i \neq j$, $|B_i| = |B_j| = m$ and $|B_i \cap B_j| = k$. The total number of distinct metric representations with these intersections in the C.N.T. is given by $\binom{\alpha-k}{m-k}$.*

Proof. Given that $|B_i| = m$ and $|B_i \cap B_j| = k$, the number of remaining distinct positions (those not in the intersection) is $m - k$. These remaining positions must be filled from the remaining $\alpha - k$ vertices, since the other k vertices are already fixed in the intersection.

Thus, the total number of ways to assign the remaining $m - k$ positions from $\alpha - k$ available vertices is given by the binomial coefficient:

$$\binom{\alpha - k}{m - k}.$$

This represents the total number of metric representations in the C.N.T. that share k common components with each other.

Note: This calculation assumes no vertices are removed or excluded from the set, and the distribution of G^m across the C.N.T. remains complete. ■

3.1.2 Existence of the Helper Graph $G'_{\mathcal{R}}$

In this subsection, we formalize the construction of the helper graph $G'_{\mathcal{R}}$, which plays a crucial role in the determination of the metric dimension of strongly regular graphs. This graph is built from the sets $G_{\lambda,\mu}$, G_{comp} , and the resolving set R , ensuring that all vertices of the strongly regular graph are uniquely labeled by their metric representations. We define the vertex set and edge set of $G'_{\mathcal{R}}$, and show under what conditions this helper graph exists by verifying the relationship between the graph order n , the resolving set R , and the disjoint sets $G_{\lambda,\mu}$ and G_{comp} .

Definition 3.7 *The helper graph $G'_{\mathcal{R}} = (V, E')$, where:*

$$V(G'_{\mathcal{R}}) := G_{\lambda,\mu} \cup R \cup G_{comp} \subseteq \left(\bigcup_{m=0}^{|R|} G^m \right) \cup R,$$

$$E'(G'_{\mathcal{R}}) := \{(v_B^1, w_i) : i \in B\} \cup \{w_i \sim w_j \mid \text{if the C.N.T column of } w_i w_j \text{ has } \lambda \text{ vertices}\}.$$

Proposition 3.1 *If there exists a set $G_{\lambda,\mu}$ that satisfies the following condition:*

$$n = |R| + |G_{\lambda,\mu}| + |G_{comp}|, \quad (3.1)$$

then, $G'_{\mathcal{R}}$ exists.

Proof. Consider the sets $G_{\lambda,\mu}$ and G_{comp} with respect to the resolving set $R = \{w_1, \dots, w_\alpha\}$. Since they satisfy condition (3.1), it follows that they label exactly all vertices of the strongly regular graph (because n is the total number of vertices). Additionally, we know that $G_{\lambda,\mu}$, G_{comp} , and R are disjoint sets. This implies that the labeling is one-to-one, meaning that each label is unique. In our case, the label is the metric representation:

$$r(v|R) := (d_G(v, w_1), \dots, d_G(v, w_\alpha)),$$

which is unique for each vertex. Thus, we generate the labeled graph $G'_{\mathcal{R}} = (V, E')$, where $E' \subseteq E$, since the edges between vertices in sets $G_{\lambda,\mu}$ and G_{comp} , as well as between those vertices and R , are defined based on the C.N.T. ■

Remark 3.1.2 *The helper graph $G'_{\mathcal{R}}$ is a subgraph of the strongly regular graph $SRG(n, k, \lambda, \mu)$ and satisfies the parameters (k, λ, μ) only for the resolving vertices $w_i \in R$, where $i \in \{1, \dots, \alpha\}$. The structure of $G'_{\mathcal{R}}$ is critical in determining the resolving set's effect on the overall properties of the strongly regular graph.*

3.2 Existence of an Isomorphic Strongly Regular

Graph $SRG(n, k, \lambda, \mu)$

In this section, we explore the conditions under which an isomorphic strongly regular graph $SRG(n, k, \lambda, \mu)$ can be constructed. The construction is based on the existence of a helper graph $G'_{\mathcal{R}}$ and a bijection map that preserves adjacency relations between the vertices. We define the structure of the induced resolving subgraph and prove that, under certain conditions, an isomorphic strongly regular graph labeled with respect to the resolving set can be obtained.

Definition 3.8 *The **induced resolving subgraph**, denoted by $\mathcal{R} = (R, E'')$, is a subgraph of a strongly regular graph $SRG(n, k, \lambda, \mu)$ induced by the resolving set R . The vector representations of all vertices in $SRG(n, k, \lambda, \mu)$ are derived with respect to this resolving set R .*

Lemma 3.10 *If there exists a bijective map $f : V(G'_{\mathcal{R}}) \longrightarrow V(SRG(n, k, \lambda, \mu))$ that preserves adjacency relations, then there exists a strongly regular graph $SRG(n, k, \lambda, \mu)$ whose vertices are labeled with respect to the resolving set R . Moreover, the adjacency matrix of $SRG(n, k, \lambda, \mu)$ can be written as the sum of the adjacency matrix of the helper graph $G'_{\mathcal{R}}$ and the adjacency matrix of its complement $\overline{G'_{\mathcal{R}}}$.*

Proof. Let $f : V(G'_{\mathcal{R}}) \longrightarrow V(SRG(n, k, \lambda, \mu))$ be a bijective map that preserves adjacency relations. This means that for any pair of vertices $u, v \in V(G'_{\mathcal{R}})$,

$$(u \sim v \text{ in } G'_{\mathcal{R}}) \iff (f(u) \sim f(v) \text{ in } SRG(n, k, \lambda, \mu)).$$

Since f is bijective and adjacency-preserving, $G'_{\mathcal{R}}$ and $SRG(n, k, \lambda, \mu)$ have identical adjacency structures under the labeling provided by f . Thus, the graph $G'_{\mathcal{R}}$ can be viewed as a labeled subgraph of $SRG(n, k, \lambda, \mu)$, with its vertices corresponding to those of $SRG(n, k, \lambda, \mu)$ under the mapping f .

Moreover, the adjacency matrix $A(SRG(n, k, \lambda, \mu))$ of the strongly regular graph can be expressed as the sum of the adjacency matrix of the helper graph $G'_{\mathcal{R}}$, denoted by $A(G'_{\mathcal{R}})$, and the adjacency matrix of its complement $\overline{G'_{\mathcal{R}}}$, denoted by $A(\overline{G'_{\mathcal{R}}})$:

$$A(SRG(n, k, \lambda, \mu)) = A(G'_{\mathcal{R}}) + A(\overline{G'_{\mathcal{R}}}).$$

This decomposition follows directly from the definition of the complement graph $\overline{G'_{\mathcal{R}}}$, as every edge in $\overline{G'_{\mathcal{R}}}$ corresponds to a non-edge in $G'_{\mathcal{R}}$, and vice versa. Thus, the adjacency relations of $SRG(n, k, \lambda, \mu)$ are fully captured by the adjacency relations of $G'_{\mathcal{R}}$ and its complement.

Finally, because f preserves adjacency relations, the labeling of the vertices in $SRG(n, k, \lambda, \mu)$ reflects the resolving set R , ensuring that the structure of $G'_{\mathcal{R}}$ aligns with the resolving set properties. |

3.3 Resolving Set of a Strongly Regular Graph and Its Subgraphs

In this section, we investigate the relationship between the resolving set of a strongly regular graph (SRG) G and its subgraph H . Specifically, we establish sufficient

conditions under which a subgraph H of G shares the same resolving set as G . These results enhance our understanding of the stability of the metric dimension when a graph is decomposed into subgraphs or extended with additional vertices and edges.

Given a strongly regular graph $G = (V, E) = \text{SRG}(n, k, \lambda, \mu)$, we construct a spanning bipartite subgraph $H_{R, \bar{R}}$ using a vertex subset $R \subseteq V$. The following theorem provides a sufficient condition under which R , a resolving set for a specific spanning subgraph $H = G[R] \cup H_{R, \bar{R}}$, also serves as a resolving set for the entire graph G . The decomposition of G into induced subgraphs and their respective properties is key to this result.

Theorem 3.1 *Let $G = (V, E) = \text{SRG}(n, k, \lambda, \mu)$ be a strongly regular graph, and let $H_{R, \bar{R}} = (V, E')$ be a spanning bipartite subgraph of G , where $R \subseteq V$, and E' consists of all edges between R and $V \setminus R$. Suppose G can be decomposed as*

$$G = G[R] \cup H_{R, \bar{R}} \cup H_{\bar{R}},$$

where:

- $G[R]$ is the subgraph induced by R ,
- $H_{R, \bar{R}}$ contains all edges between R and $V \setminus R$,
- $H_{\bar{R}}$ contains the remaining edges of G between vertices in $V \setminus R$.

If R is a resolving set for the spanning subgraph

$$H = G[R] \cup H_{R,\overline{R}},$$

then R is also a resolving set for G .

Proof. To prove that R is a resolving set for G , we need to show that the metric representation $r(v \mid R)_G$ is unique for all $v \in V(G)$. Since R is a resolving set for H , we know:

$$r(v \mid R)_H \text{ is unique for all } v \in V(H).$$

Now, we will demonstrate that for all $v \in V(G)$:

$$r(v \mid R)_G = r(v \mid R)_H \iff d_G(v, w_i) = d_H(v, w_i), \quad \forall i \in \{1, \dots, |R|\}.$$

The inclusion of edges $E(G) \setminus (E(H_{R,\overline{R}}) \cup E(G[R]))$ only affects distances $d_H(v, w_i)$ for $v \in V \setminus R$. We consider the following cases:

i. For $w_j, w_i \in R$, the distances remain unchanged:

$$d_H(w_j, w_i) = d_G(w_j, w_i), \quad \forall i, j \in \{1, \dots, |R|\},$$

because $G[R]$ is an induced subgraph of G .

ii. For $v \in V \setminus R$ and $w_i \in R$, the distance is determined by adjacency in H :

$$d_G(v, w_i) = \begin{cases} 1, & \text{if } w_i \sim v \text{ in } H, \\ 2, & \text{if } w_i \not\sim v \text{ in } H. \end{cases}$$

The edges $E(G) \setminus E(H)$ are restricted to $V \setminus R$ and do not affect the shortest paths between v and w_i .

Hence, for all $v \in V(G)$:

$$r(v \mid R)_G = r(v \mid R)_H.$$

Since $r(v \mid R)_G$ is unique for all $v \in V(G)$, it follows that R is a resolving set for G . ■

3.4 Metric Dimension of $K(6, 2)$ and $J(6, 2)$ Using the Common Neighborhood Table (C.N.T.) Method

In this section, we introduce a novel approach for determining the metric dimension of strongly regular graphs (SRGs) using the Common Neighborhood Table (C.N.T) and the helper graph. We demonstrate the effectiveness of this method by applying it to two well-known graphs: $K(6, 2)$ and its complement $J(6, 2)$. Our findings reveal that the resolving sets of $K(6, 2)$ and $J(6, 2)$ may have different structures, which implies

that their minimum resolving sets are not necessarily of the same form.

The C.N.T. method allows us to explore various induced subgraphs to identify potential resolving sets and to construct the corresponding helper graph for these subgraphs. The process includes calculating the minimum possible size of the resolving set, examining all non-isomorphic graphs of that size, and determining whether the C.N.T. can be formed. The method not only provides a systematic approach to solving this NP-hard problem for specific cases but also opens the door to future generalizations.

The steps are as follows:

1. We calculate the minimum α that satisfies the condition for a resolving set as described in inequality (1.1).
2. We list all non-isomorphic graphs with order α .
3. For each specific shape of a resolving set R , we attempt to construct the C.N.T. and analyze the helper graph.
4. If the helper graph exists, we check for an isomorphism with the SRG.
5. If no isomorphism is found, we discard that particular structure of R and proceed with another.
6. If an isomorphism exists, we conclude that the metric dimension of the SRG is α , the size of the resolving set.

We will now proceed step by step, beginning with the analysis of $K(6, 2)$ and its resolving-induced subgraphs.

3.4.1 $K(6, 2)$

We begin by analyzing the strongly regular graph $K(6, 2)$, which has parameters $(15, 6, 1, 3)$. According to equation (1.1), we first calculate the minimum value of α , which represents the size of the smallest resolving set for this graph. Based on our method, we determine that the minimum α is 4.

The next step is to list all non-isomorphic graphs on 4 vertices, which represent potential resolving-induced subgraphs. There are 11 such graphs, which are illustrated in Figure (3.1).

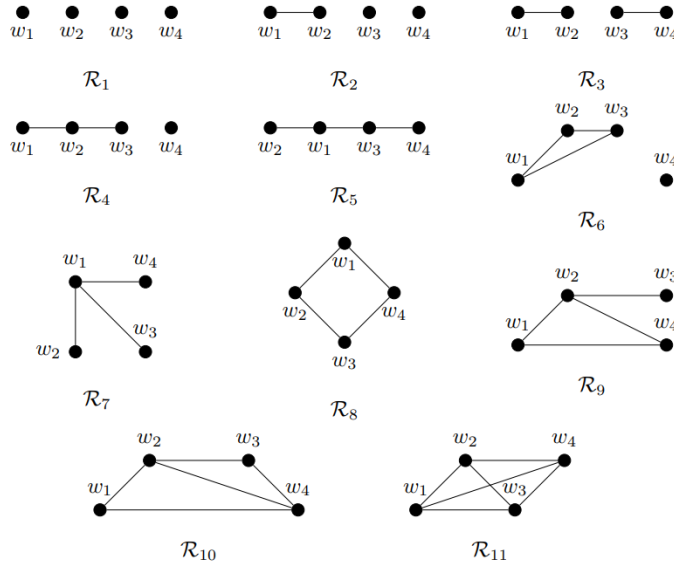


Figure 3.1: Non-isomorphic Graphs on 4 Vertices

For each graph, we will examine whether it can be used as a resolving set R for $K(6, 2)$ by constructing the common neighborhood table (C.N.T.). If the C.N.T. can be constructed and the corresponding helper graph is valid, we will determine the structure of the resolving-induced subgraph and check for an isomorphism with

$K(6, 2)$. If an isomorphism exists, we conclude that the given graph represents a valid resolving set for $K(6, 2)$.

I. Resolving Subgraph for $K(6, 2)$ Based on $\mathcal{R}_1$ (Four Isolated Vertices)

Let us first consider the trivial graph $\mathcal{R}_1$ with 4 isolated vertices as the potential resolving set. The C.N.T. and degree table for $K(6, 2)$ with respect to $\mathcal{R}_1$ are shown in Table 3.1.

Table 3.1: Common Neighborhoods & Degree Tables of $SRG(15, 6, 1, 3)$ w.r.t. $\mathcal{R}_1$.

Common Neighborhoods Table of $SRG(15, 6, 1, 3)$						Degree Table of R			
$\mathbf{w_1 w_2}$	$\mathbf{w_1 w_3}$	$\mathbf{w_1 w_4}$	$\mathbf{w_2 w_3}$	$\mathbf{w_2 w_4}$	$\mathbf{w_3 w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{12}^1	v_{13}^1	v_{14}^1	v_{123}^1	v_{124}^1	v_{34}^1	v_{12}^1	v_{12}^1	v_{13}^1	v_{14}^1
v_{123}^1	v_{123}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{134}^1	v_{13}^1	v_{123}^1	v_{123}^1	v_{134}^1
v_{124}^1	v_{134}^1	v_{124}^1	v_{23}^1	v_{24}^1	v_{234}^1	v_{14}^1	v_{124}^1	v_{134}^1	v_{124}^1
						v_{123}^1	v_{234}^1	v_{234}^1	v_{234}^1
						v_{124}^1	v_{23}^1	v_{23}^1	v_{24}^1
						v_{134}^1	v_{24}^1	v_{34}^1	v_{34}^1

From the C.N.T., we observe that $G_{1,3}$ with respect to the trivial graph $\mathcal{R}_1$ can be constructed as $G^2 \cup G^3$, and the complementary graph is v^0 . Therefore, the total number of vertices in the graph is given by:

$$\underbrace{4}_{|R|} + \underbrace{10}_{|G_{\lambda, \mu}|} + \underbrace{1}_{|G_{comp}|} = 15 = n$$

Thus, the vertex set of the helper graph is:

$$V(G'_{\mathcal{R}_1}) = G^3 \cup G^2 \cup R \cup G^0.$$

The corresponding vertex mapping function f from $G'_{\mathcal{R}_1}$ to $K(6, 2)$ is shown in Table 3.2.

$V(G'_{\mathcal{R}_1})$	$f(V(G'_{\mathcal{R}_1}))$	$V(G'_{\mathcal{R}_1})$	$f(V(G'_{\mathcal{R}_1}))$
w_1	$\{0, 1\}$	v_{12}^1	$\{3, 4\}$
w_2	$\{0, 2\}$	v_{13}^1	$\{2, 4\}$
w_3	$\{0, 3\}$	v_{14}^1	$\{2, 3\}$
w_4	$\{0, 4\}$	v_{23}^1	$\{1, 4\}$
v_{123}^1	$\{4, 5\}$	v_{24}^1	$\{1, 3\}$
v_{124}^1	$\{3, 5\}$	v_{34}^1	$\{1, 2\}$
v_{134}^1	$\{2, 5\}$	v^0	$\{0, 5\}$
v_{234}^1	$\{1, 5\}$		

Table 3.2: Mapping from $V(G'_{\mathcal{R}_1})$ to $V(K(6, 2))$

Therefore, we conclude that the trivial graph $\mathcal{R}_1$ is a valid resolving-induced subgraph of $K(6, 2)$, and the newly defined graph H , which contains both the helper graph and its complement, is isomorphic to $K(6, 2)$. Consequently, the metric dimension of $K(6, 2)$ is 4.

II. Resolving Subgraph for $K(6, 2)$ Based on $\mathcal{R}_2$ (One Edge and Two Isolated Vertices)

Next, we consider the graph $\mathcal{R}_2$, which consists of 2 isolated vertices and one edge. We attempt to form the C.N.T. for $K(6, 2)$ with respect to this configuration. The C.N.T. and degree table for $\mathcal{R}_2$ are shown in Table 3.3.

Table 3.3: Common Neighborhoods & Degree Tables of $SRG(15, 6, 1, 3)$ w.r.t. $\mathcal{R}_2$.

Common Neighborhoods Table of $SRG(15, 6, 1, 3)$						Degree Table of R			
$\mathbf{w_1 w_2}$	$\mathbf{w_1 w_3}$	$\mathbf{w_1 w_4}$	$\mathbf{w_2 w_3}$	$\mathbf{w_2 w_4}$	$\mathbf{w_3 w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1
	v_{134}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{134}^1	v_{134}^1	v_{234}^1	v_{143}^1	v_{134}^1
	v_{13}^1	v_{14}^1	v_{23}^1	v_{24}^1	v_{234}^1	v_{13}^1	v_{23}^1	v_{13}^1	v_{234}^1
						v_{14}^1	v_{24}^1	v_{13}^1	v_{14}^1
						w_2	w_1	v_{23}^1	v_{24}^1
						v_1^1	v_2^1	v_3^1	v_4^1

From the C.N.T., we construct the graph $G_{1,3}$ with respect to $\mathcal{R}_2$. The resulting set is $\{v_{1234}^1, v_{134}^1, v_{234}^1, v_{13}^1, v_{14}^1, v_{23}^1, v_{24}^1\}$, and the complementary graph is G^1 . Therefore, the total number of vertices in the graph is given by:

$$\underbrace{4}_{|R|} + \underbrace{7}_{|G_{\lambda, \mu}|} + \underbrace{4}_{|G_{comp}|} = 15 = n$$

Thus, the vertex set of the helper graph is:

$$V(G'_{\mathcal{R}_2}) = G^4 \cup \{v_{134}^1, v_{234}^1\} \cup (G^2 \setminus \{v_{12}^1, v_{34}^1\}) \cup G^1 \cup R.$$

Despite constructing the Common Neighborhood Table (C.N.T.) for $\mathcal{R}_2$, a bijective map cannot be established between the helper graph $G'_{\mathcal{R}_2}$ and $K(6, 2)$. Specifically, certain vertices, such as v_{134}^1 , which represent the common neighborhood of w_1 , w_3 , and w_4 , cannot exist within the adjacency structure of $K(6, 2)$. This leads to the conclusion that $\mathcal{R}_2$ is not a valid resolving-induced subgraph of $K(6, 2)$.

Corollary 3.4 *$\mathcal{R}_2$ is not a resolving subgraph of $K(6, 2)$.*

Proof. Let $O = \{0, 1, 2, 3, 4, 5\}$, which is the set of elements for $K(6, 2)$, and let the vertex set of $K(6, 2)$ be represented by pairs $\{i, j\}$ where $i, j \in O$. The adjacency in $K(6, 2)$ is defined by disjointness, meaning two vertices (pairs of sets) are adjacent if and only if they are disjoint.

Now, consider the vertices $w_1 = \{a, b\}$, $w_2 = \{c, d\}$, $w_3 = \{a, c\}$, $w_4 = \{a, d\}$, which form the resolving set $\mathcal{R}_2$. For completeness, the set $O \setminus (w_1 \cup w_2 \cup w_3 \cup w_4)$ contains two remaining elements, denoted $\{e, f\}$.

Next, we attempt to construct the common neighborhood vertices such as v_{134}^1 and v_{234}^1 . For v_{134}^1 , it must satisfy the following two conditions:

- It must be disjoint from w_1 , w_3 , and w_4 ,
- It must intersect with w_2 .

However, upon examination, $w_2 \subseteq \cup_{i=1,3,4} w_i$, which implies that no set can simultaneously be disjoint from w_1 , w_3 , and w_4 , while intersecting w_2 . Therefore, v_{134}^1 cannot exist. Similarly, v_{234}^1 also cannot be constructed for the same reasons.

Since constructing these common neighborhood vertices is impossible, the Common Neighborhood Table (C.N.T.) for $\mathcal{R}_2$ cannot be fully completed. As a result, the helper graph $G'_{\mathcal{R}_2}$ cannot be constructed, and consequently, $\mathcal{R}_2$ is not a valid resolving-induced subgraph of $K(6, 2)$.

I

III. Resolving Subgraph for $K(6, 2)$ Based on $\mathcal{R}_4$ (Path of Length 2 and One Isolated Vertex)

In this case, we analyze the subgraph $\mathcal{R}_4$, which consists of a path of length 2 where w_1 is connected to w_2 , w_2 is connected to w_3 , and w_4 is an isolated vertex. We construct the Common Neighborhood Table (C.N.T.) and degree tables to verify whether a bijection can be established between the helper graph $G'_{\mathcal{R}_4}$ and $K(6, 2)$.

We begin by constructing the C.N.T. for $\mathcal{R}_4$. The relationships between the vertices in terms of their common neighborhoods are shown in the table below.

Table 3.4: Common Neighborhoods & Degree Tables of $(SRG(15, 6, 1, 3))$ w.r.t. $\mathcal{R}_4$.

Common Neighborhoods Table of $SRG(15, 6, 1, 3)$						Degree Table of R			
$\mathbf{w_1 w_2}$	$\mathbf{w_1 w_3}$	$\mathbf{w_1 w_4}$	$\mathbf{w_2 w_3}$	$\mathbf{w_2 w_4}$	$\mathbf{w_3 w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{124}^1	v_{134}^1	v_{124}^1	v_{234}^1	v_{124}^1	v_{134}^1	v_{124}^1	v_{124}^1	v_{134}^1	v_{124}^1
	v_{13}^1	v_{134}^1		v_{234}^1	v_{234}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{134}^1
	w_2	v_{14}^1		v_{24}^1	v_{34}^1	v_{13}^1	v_{24}^1	v_{13}^1	v_{234}^1
						v_{14}^1	w_1	v_{34}^1	v_{14}^1
						w_2	w_3	w_2	v_{24}^1
						v_1^1	v_2^1	v_3^1	v_{34}^1

Based on the table, the helper graph $G'_{\mathcal{R}_4}$ is constructed as follows:

$$G_{1,3} = (G^3 \setminus \{v_{123}^1\}) \cup (G^2 \setminus \{v_{12}^1, v_{23}^1\})$$

and

$$G_{comp} = (G^1 \setminus \{v_4^1\}) \cup G^0.$$

The total number of vertices in the graph is given by:

$$\underbrace{4}_{|R|} + \underbrace{7}_{|G_{\lambda,\mu}|} + \underbrace{4}_{|G_{comp}|} = 15 = n.$$

Thus, the vertex set for $G'_{\mathcal{R}_4}$ is:

$$V(G'_{\mathcal{R}_4}) = (G^3 \setminus \{v_{123}^1\}) \cup (G^2 \setminus \{v_{12}^1, v_{23}^1\}) \cup (G^1 \setminus \{v_4^1\}) \cup G^0 \cup R.$$

Next, we define the mapping f from $G'_{\mathcal{R}_4}$ to $K(6, 2)$, as shown in Table 3.5. This mapping establishes a bijection between the vertices of the helper graph and $K(6, 2)$, ensuring that the structure of the helper graph corresponds with the original graph.

$V(G'_{\mathcal{R}_4})$	$f(V(G'_{\mathcal{R}_4}))$	$V(G'_{\mathcal{R}_4})$	$f(V(G'_{\mathcal{R}_4}))$
w_1	$\{a, b\}$	v_{14}^1	$\{d, e\}$
w_2	$\{c, d\}$	v_{24}^1	$\{b, e\}$
w_3	$\{a, e\}$	v_{34}^1	$\{b, d\}$
w_4	$\{a, c\}$	v_1^1	$\{c, e\}$
v_{124}^1	$\{e, f\}$	v_2^1	$\{a, f\}$
v_{134}^1	$\{d, f\}$	v_3^1	$\{b, c\}$
v_{234}^1	$\{b, f\}$	v^0	$\{a, d\}$
v_{13}^1	$\{c, f\}$		

Table 3.5: Mapping from $V(G'_{\mathcal{R}_4})$ to $V(K(6, 2))$

Thus, the bijection defined in the above table guarantees that $\mathcal{R}_4$ is a valid resolving subgraph of $K(6, 2)$. Additionally, a new graph H , containing both the helper graph and its complement in $K(6, 2)$, can be constructed to fully establish the isomorphism. Therefore, the metric dimension of $K(6, 2)$ is 4, which corresponds to the minimum size of the resolving set.

IV. Resolving Subgraph for $K(6, 2)$ Based on $\mathcal{R}_5$ (Path of Length 3)

In this case, we analyze the subgraph $\mathcal{R}_5$, which consists of a path of length 3 where w_2 is connected to w_1 , w_1 is connected to w_3 , and w_3 is connected to w_4 . We construct

the Common Neighborhood Table (C.N.T.) and degree tables to verify whether the helper graph $G'_{\mathcal{R}_5}$ can be mapped bijectively to $K(6, 2)$, ensuring that their structures align.

The relationships between the vertices in terms of their common neighborhoods are shown in the table below.

Table 3.6: Common Neighborhoods & Degree Tables of $SRG(15, 6, 1, 3)$ w.r.t. $\mathcal{R}_5$.

Common Neighborhoods Table of $SRG(15, 6, 1, 3)$						Degree Table of R			
$\mathbf{w_1w_2}$	$\mathbf{w_1w_3}$	$\mathbf{w_1w_4}$	$\mathbf{w_2w_3}$	$\mathbf{w_2w_4}$	$\mathbf{w_3w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{124}^1	v_{13}^1	v_{124}^1	v_{234}^1	v_{124}^1	v_{234}^1	v_{124}^1	v_{124}^1	v_{234}^1	v_{124}^1
		v_{14}^1	v_{23}^1	v_{234}^1		v_{13}^1	v_{234}^1	v_{13}^1	v_{234}^1
		w_4	w_1	v_{24}^1		v_{14}^1	v_{23}^1	v_{23}^1	v_{14}^1
						w_2	v_{24}^1	w_2	v_{24}^1
						w_3	w_1	w_4	w_3
						v_1^1	v_2^1	v_3^1	v_4^1

From the table, the helper graph $G_{1,3}$ corresponding to $\mathcal{R}_5$ is constructed as:

$$G_{1,3} = \{v_{124}^1, v_{234}^1, v_{13}^1, v_{14}^1, v_{23}^1, v_{24}^1\},$$

and

$$G_{comp} = G^1 \cup G^0.$$

The total number of vertices in the graph is given by:

$$\underbrace{4}_{|R|} + \underbrace{6}_{|G_{\lambda,\mu}|} + \underbrace{5}_{|G_{comp}|} = 15 = n.$$

Thus, the vertex set of the helper graph is:

$$V(G'_{\mathcal{R}_5}) = \{v_{124}^1, v_{234}^1\} \cup (G^2 \setminus \{v_{12}^1, v_{34}^1\}) \cup R \cup G^1 \cup G^0.$$

Despite constructing the Common Neighborhood Table (C.N.T.) for $\mathcal{R}_5$, it turns out that certain vertices, such as v_{234}^1 , cannot exist within the structure of $K(6, 2)$. Specifically, the adjacency conditions of $K(6, 2)$, where vertices represent disjoint pairs, are not satisfied for some vertices in $G'_{\mathcal{R}_5}$. As a result, $\mathcal{R}_5$ is not a valid resolving-induced subgraph of $K(6, 2)$.

Corollary 3.5 $\mathcal{R}_5$ is not a resolving subgraph of $K(6, 2)$.

Proof. The reasoning is identical to that of Corollaries 3.4 and 3.5. In the case of $\mathcal{R}_5$, the adjacency structure of the helper graph $G'_{\mathcal{R}_5}$ fails to meet the necessary conditions for $K(6, 2)$. Specifically, certain vertices, such as v_{234}^1 , cannot be constructed due to the disjointness conditions, as previously discussed. Therefore, $\mathcal{R}_5$ does not serve as a valid resolving subgraph of $K(6, 2)$. ■

V. Resolving Subgraph for $K(6, 2)$ Based on $\mathcal{R}_7$ (Star of 4 Vertices)

In this case, we analyze the subgraph $\mathcal{R}_7$, which consists of a star on 4 vertices, where w_1 is connected to w_2 , w_3 , and w_4 . We attempt to form the Common Neighborhood

Table (C.N.T.) for $K(6, 2)$ with respect to this star configuration, $\mathcal{R}_7$. The table of common neighborhoods and degree table is given below.

Table 3.7: Common Neighborhoods & Degree Tables of $SRG(15, 6, 1, 3)$ w.r.t. $\mathcal{R}_7$.

Common Neighborhoods Table of $SRG(15, 6, 1, 3)$						Degree Table of R			
$\mathbf{w_1w_2}$	$\mathbf{w_1w_3}$	$\mathbf{w_1w_4}$	$\mathbf{w_2w_3}$	$\mathbf{w_2w_4}$	$\mathbf{w_3w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{12}^1	v_{13}^1	v_{14}^1	v_{23}^1	v_{24}^1	v_{34}^1	v_{12}^1	v_{12}^1	v_{13}^1	v_{14}^1
			v_{234}^1	v_{234}^1	v_{234}^1	v_{13}^1	v_{23}^1	v_{23}^1	v_{24}^1
			w_1	w_1	w_1	v_{14}^1	v_{24}^1	v_{34}^1	v_{34}^1
						w_2	v_{234}^1	v_{234}^1	v_{234}^1
						w_3	w_1	w_1	w_1
						w_4	v_2^1	v_3^1	v_4^1

Consequently, $G_{1,3}$ with respect to star S_4 is obtained as $G_{1,3} = G^2 \cup \{v_{234}^1\}$ and $G_{comp} = G^1 \setminus \{v_1^1\} \cup G^0$. Therefore, the total number of vertices in the graph is given by:

$$\underbrace{4}_{|R|} + \underbrace{7}_{|G_{\lambda,\mu}|} + \underbrace{4}_{|G_{comp}|} = 15 = n$$

$$V(G'_{\mathcal{R}_7}) = \{v_{234}^1\} \cup G^2 \cup R \cup (G^1 \setminus \{v_1^1\}) \cup G^0$$

Corollary 3.6 $\mathcal{R}_7$ is not a resolving subgraph of $K(6, 2)$.

Proof. The same reasoning used in Corollary 3.4 applies here. Briefly, the structure of $\mathcal{R}_7$ (a star graph) and the disjointness condition of $K(6, 2)$ imply that no valid

common neighborhood, such as v_{234}^1 , can exist that satisfies the adjacency rules of $K(6, 2)$. Therefore, $\mathcal{R}_7$ cannot serve as a resolving subgraph of $K(6, 2)$. ■

VI. Resolving Subgraph for $K(6, 2)$ Based on $\mathcal{R}_8$ (Cycle C_4)

In this case, we consider $\mathcal{R}_8$, the cycle graph C_4 , where the vertices w_1, w_2, w_3, w_4 form a cycle, i.e., w_1 is connected to w_2 , w_2 to w_3 , w_3 to w_4 , and w_4 back to w_1 .

The Common Neighborhood Table (C.N.T.) and Degree Table for the strongly regular graph $SRG(15, 6, 1, 3)$ with respect to the cycle graph $\mathcal{R}_8$ are given in Table 3.8.

Table 3.8: Common Neighborhoods & Degree Tables of $SRG(15, 6, 1, 3)$ w.r.t. $\mathcal{R}_8$.

Common Neighborhoods Table of $SRG(15, 6, 1, 3)$						Degree Table of R			
$\mathbf{w_1w_2}$	$\mathbf{w_1w_3}$	$\mathbf{w_1w_4}$	$\mathbf{w_2w_3}$	$\mathbf{w_2w_4}$	$\mathbf{w_3w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{12}^1	v_{13}^1	v_{14}^1	v_{23}^1	v_{24}^1	v_{34}^1	v_{12}^1	v_{12}^1	v_{13}^1	v_{14}^1
	w_2			w_1		v_{13}^1	v_{23}^1	v_{23}^1	v_{24}^1
	w_4			w_3		v_{14}^1	v_{24}^1	v_{34}^1	v_{34}^1
						w_2	w_1	w_2	w_1
						w_4	w_3	w_4	w_3
						v_1^1	v_2^1	v_3^1	v_4^1

Consequently, the helper graph $G'_{\mathcal{R}_8}$ with respect to the cycle graph C_4 is obtained as $G_{1,3} = G^2$, and the complementary graph is $G_{comp} = G^1 \cup G^0$. Therefore, the total number of vertices in the graph is given by:

$$\underbrace{4}_{|R|} + \underbrace{6}_{|G_{\lambda,\mu}|} + \underbrace{5}_{|G_{comp}|} = 15 = n$$

Thus, the vertex set of the helper graph is:

$$V(G'_{\mathcal{R}_8}) = G^2 \cup R \cup G^1 \cup G^0.$$

Next, we define the mapping f from $G'_{\mathcal{R}_8}$ to $K(6, 2)$, as shown in Table 3.9. This mapping defines a bijection between the two graphs.

$V(G'_{\mathcal{R}_8})$	$f(V(G'_{\mathcal{R}_8}))$	$V(G'_{\mathcal{R}_8})$	$f(V(G'_{\mathcal{R}_8}))$
w_1	$\{0, 1\}$	v_{24}^1	$\{1, 4\}$
w_2	$\{2, 3\}$	v_{34}^1	$\{1, 3\}$
w_3	$\{0, 4\}$	v_1^1	$\{2, 4\}$
w_4	$\{2, 5\}$	v_2^1	$\{0, 5\}$
v_{12}^1	$\{4, 5\}$	v_3^1	$\{1, 2\}$
v_{13}^1	$\{3, 5\}$	v_4^1	$\{0, 3\}$
v_{14}^1	$\{3, 4\}$	v^0	$\{0, 2\}$
v_{23}^1	$\{1, 5\}$		

Table 3.9: Mapping from $V(G'_{\mathcal{R}_8})$ to $V(K(6, 2))$

Thus, the bijection defined in the above table guarantees that $\mathcal{R}_8$, which is a cycle C_4 , is a valid resolving subgraph of $K(6, 2)$. A new graph H , containing the helper graph and its complement, can be created and made isomorphic to $K(6, 2)$. Therefore, the metric dimension of $K(6, 2)$ is 4, which corresponds to the minimum size of the

resolving set.

VII. Non-Resolving Subgraphs for $K(6, 2)$ Based on $\mathcal{R}_3, \mathcal{R}_6, \mathcal{R}_9, \mathcal{R}_{10}, \mathcal{R}_{11}$

In this subsection, we demonstrate that the graphs $\mathcal{R}_3, \mathcal{R}_6, \mathcal{R}_9, \mathcal{R}_{10}, \mathcal{R}_{11}$ cannot serve as the shape of a resolving set for any strongly regular graph with parameters $(15, 6, 1, 3)$. Specifically, it is impossible to construct the Common Neighborhood Table (C.N.T.) for these cases.

We define $A_{w_i w_j}^m$ as the set of vertices from G^m that can be filled based on the relationship between the vertices w_i and w_j , and therefore, $|A_{w_i w_j}^m| = \binom{\alpha-2}{m-2}$. Furthermore, we calculate the number of vertices that remain independent from these sets, given by the following formula:

$$\sum_{m=0}^{\alpha} |G^m \setminus \left(\cup_{i,j} A_{w_i w_j}^m \right)|. \quad (3.2)$$

Below, we explore the impossibility of constructing the C.N.T. for each case:

$\mathcal{R}_3$ (Two Disjoint Edges): In this case, two columns have $\lambda = 1$, i.e., $|w_1 w_2| = |w_3 w_4| = 1$. The sets $A_{w_1 w_2}^m$ and $A_{w_3 w_4}^m$ are disjoint for $m < 4$. We compute how many vertices are independent of these sets in G^m by calculating:

$$|G^4 \setminus (A_{w_1w_2}^4 \cup A_{w_3w_4}^4)| = 1 - (1 + 1 - 1) = 0,$$

$$|G^3 \setminus (A_{w_1w_2}^3 \cup A_{w_3w_4}^3)| = 4 - (2 + 2 - 0) = 0,$$

$$|G^2 \setminus (A_{w_1w_2}^2 \cup A_{w_3w_4}^2)| = 6 - (1 + 1 - 0) = 4.$$

Although there are 4 independent vertices in G^2 , they are insufficient to fill the remaining columns. Thus, no valid C.N.T. can be constructed for $\mathcal{R}_3$.

$\mathcal{R}_6$ (K_3 with One Isolated Vertex): Here, three columns correspond to the complete graph K_3 , meaning $|w_1w_2| = |w_1w_3| = |w_2w_3| = 1$. We need to fill the columns involving the isolated vertex w_4 (i.e., w_1w_4, w_2w_4, w_3w_4) with vertices independent of $A_{w_1w_2}^m, A_{w_1w_3}^m, A_{w_2w_3}^m$. We compute:

$$|G^4 \setminus (A_{w_1w_2}^4 \cup A_{w_1w_3}^4 \cup A_{w_2w_3}^4)| = 1 - (1 + 1 + 1 - 1 - 1 - 1 + 1) = 0,$$

$$|G^3 \setminus (A_{w_1w_2}^3 \cup A_{w_1w_3}^3 \cup A_{w_2w_3}^3)| = 4 - (2 + 2 + 2 - 1 - 1 - 1 + 1) = 0,$$

$$|G^2 \setminus (A_{w_1w_2}^2 \cup A_{w_1w_3}^2 \cup A_{w_2w_3}^2)| = 6 - (1 + 1 + 1 - 0 - 0 - 0 + 0) = 3.$$

While G^2 provides some vertices, it is insufficient to complete the remaining columns for w_1w_4, w_2w_4, w_3w_4 . Thus, no valid C.N.T. can be constructed for $\mathcal{R}_6$.

$\mathcal{R}_9$ (K_3 with One Edge): In this case, four columns correspond to $\lambda = 1$, which are $|w_1w_2|, |w_1w_4|, |w_2w_3|, |w_2w_4| = 1$. These columns are filled by the resolving

vertices. We now need to fill the columns involving the edge w_3 , which include w_1w_3, w_2w_3, w_3w_4 .

We compute the number of remaining independent vertices in the following sets:

$$|G^4 \setminus (A_{w_1w_2}^4 \cup A_{w_1w_4}^4 \cup A_{w_2w_4}^4)| = 1 - (1 + 1 + 1 - 1 - 1 - 1 + 1) = 0,$$

$$|G^3 \setminus (A_{w_1w_2}^3 \cup A_{w_1w_4}^3 \cup A_{w_2w_4}^3)| = 4 - (2 + 2 + 2 - 1 - 1 - 0) = 0,$$

$$|G^2 \setminus (A_{w_1w_2}^2 \cup A_{w_1w_4}^2 \cup A_{w_2w_4}^2)| = 6 - (1 + 1 + 1 - 0 - 0 - 0) = 3.$$

Even though G^2 provides some independent vertices, it is insufficient to fill all the remaining columns. Therefore, the Common Neighborhood Table (C.N.T.) for $\mathcal{R}_9$ cannot be completed, making it an invalid resolving subgraph for $K(6, 2)$.

$\mathcal{R}_{10}$ (K_4 Minus One Edge): In this case, $\mathcal{R}_{10}$ represents the complete graph K_4 with one missing edge between w_1 and w_3 . The degree table shows that the degrees $k_{w_i} \leq 4$, which means the Common Neighborhood Table (C.N.T.) is incomplete. While some columns can be filled, the overall table does not provide a sufficient structure for constructing a full resolving set. The table for $\mathcal{R}_{10}$ is shown below:

Table 3.10: Common Neighborhoods & Degree Tables of $SRG(15, 6, 1, 3)$ w.r.t. $\mathcal{R}_{10}$.

Common Neighborhoods Table of $SRG(15, 6, 1, 3)$						Degree Table of R			
$\mathbf{w_1w_2}$	$\mathbf{w_1w_3}$	$\mathbf{w_1w_4}$	$\mathbf{w_2w_3}$	$\mathbf{w_2w_4}$	$\mathbf{w_3w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
w_3	w_2	w_3	w_1	w_1	w_1	w_2	w_1	w_1	w_1
				w_4		w_3	w_3	w_2	w_3
				v_{24}^1		w_4	v_{24}^1	w_4	v_{24}^1
						v_1^1	v_2^1	v_3^1	v_4^1

Since the missing edge w_1w_3 prevents the completion of the C.N.T, $\mathcal{R}_{10}$ cannot be a valid resolving-induced subgraph of $SRG(15, 6, 1, 3)$.

$\mathcal{R}_{11}$ (Complete Graph K_4): Since $1 = \lambda < |R| - 2 = 2$, we cannot build a C.N.T. based on K_4 . As per Lemma 3.2, for a strongly regular graph $SRG(n, k, \lambda, \mu)$, if $\lambda < \alpha - 2$, where $\alpha = |R|$ is the size of a resolving set, then there must exist non-adjacent vertices in R . However, the complete graph K_4 has no non-adjacent vertices by definition. Therefore, $\mathcal{R}_{11}$, being a complete graph K_4 , cannot be a valid resolving-induced subgraph of $SRG(15, 6, 1, 3)$.

3.4.2 $J(6, 2)$

In this section, we shift our focus to the complement of $K(6, 2)$, denoted $J(6, 2)$, which is a strongly regular graph with parameters $(15, 8, 4, 4)$. Using the same Common Neighborhood Table (C.N.T.) and helper graph method employed for $K(6, 2)$, we aim to determine the metric dimension of $J(6, 2)$.

As with $K(6, 2)$, the resolving set R for $J(6, 2)$ is defined as the smallest set of vertices such that all other vertices are uniquely identified by their distances to the vertices in R . The process for determining this set follows a similar series of steps as before:

1. Calculate the minimum size α of a resolving set for $J(6, 2)$.
2. Identify non-isomorphic graphs of order α that could serve as resolving-induced subgraphs.
3. Construct the C.N.T. for each candidate subgraph and analyze whether it constitutes a valid resolving set based on its adjacency structure.
4. Check if a bijection can be established between the helper graph for the candidate subgraph and $J(6, 2)$. If so, the subgraph is a valid resolving set.

Through this method, we establish that the minimum resolving set for $J(6, 2)$ contains 4 vertices. We now proceed by listing all non-isomorphic graphs on 4 vertices (see Figure (3.1)) and examining their potential as resolving-induced subgraphs. Following the same approach as for $K(6, 2)$, we will construct the C.N.T., verify the helper graph, and determine if a bijection to $J(6, 2)$ exists.

I. Resolving Subgraph for $J(6, 2)$ Based on $\mathcal{R}_3$ (Two Independent Edges)

In this case, we analyze the resolving subgraph $\mathcal{R}_3$, which consists of two independent edges. The Common Neighborhood Table (C.N.T.) and degree tables for $SRG(15, 8, 4, 4)$ with respect to $\mathcal{R}_3$ are provided in Table (3.11).

Table 3.11: Common Neighborhoods & Degree Tables of $(SRG(15, 8, 4, 4))$ w.r.t. $\mathcal{R}_3$.

Common Neighborhoods Table of $SRG(15, 8, 4, 4)$						Degree Table of R			
$\mathbf{w_1 w_2}$	$\mathbf{w_1 w_3}$	$\mathbf{w_1 w_4}$	$\mathbf{w_2 w_3}$	$\mathbf{w_2 w_4}$	$\mathbf{w_3 w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1
v_{123}^1	v_{123}^1	v_{124}^1	v_{123}^1	v_{124}^1	v_{134}^1	v_{123}^1	v_{123}^1	v_{123}^1	v_{124}^1
v_{124}^1	v_{134}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{234}^1	v_{124}^1	v_{124}^1	v_{134}^1	v_{134}^1
v_{12}^1	v_{13}^1	v_{14}^1	v_{23}^1	v_{24}^1	v_{34}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{234}^1
						v_{12}^1	v_{12}^1	v_{13}^1	v_{14}^1
						v_{13}^1	v_{23}^1	v_{23}^1	v_{24}^1
						v_{14}^1	v_{24}^1	v_{34}^1	v_{34}^1
						w_2	w_1	w_4	w_3

Consequently, the graph $G_{4,4}$ with respect to $\mathcal{R}_3$ is obtained as $G_{4,4} = G^4 \cup G^3 \cup G^2$, and the complementary graph G_{comp} is empty. The total number of vertices in the graph is given by:

$$\underbrace{4}_{|R|} + \underbrace{11}_{|G_{\lambda,\mu}|} + \underbrace{0}_{|G_{comp}|} = 15 = n.$$

Thus, the vertex set of the helper graph $G'_{\mathcal{R}_3}$ is:

$$V(G'_{\mathcal{R}_3}) = G^4 \cup G^3 \cup G^2 \cup R.$$

The mapping f from $G'_{\mathcal{R}_3}$ to $J(6, 2)$, as detailed in Table (3.12), shows a clear one-to-one correspondence between the vertex sets of both graphs.

$V(G'_{\mathcal{R}_3})$	$f(V(G'_{\mathcal{R}_3}))$	$V(G'_{\mathcal{R}_3})$	$f(V(G'_{\mathcal{R}_3}))$
w_1	$\{0, 1\}$	v_{234}^1	$\{2, 3\}$
w_2	$\{0, 2\}$	v_{12}^1	$\{1, 2\}$
w_3	$\{3, 4\}$	v_{13}^1	$\{1, 4\}$
w_4	$\{3, 5\}$	v_{14}^1	$\{1, 5\}$
v_{1234}^1	$\{0, 3\}$	v_{23}^1	$\{2, 4\}$
v_{123}^1	$\{0, 4\}$	v_{24}^1	$\{2, 5\}$
v_{124}^1	$\{0, 5\}$	v_{34}^1	$\{4, 5\}$
v_{134}^1	$\{1, 3\}$		

Table 3.12: Mapping from $V(G'_{\mathcal{R}_3})$ to $V(J(6, 2))$

This bijection confirms that $\mathcal{R}_3$, formed by two independent edges, is a valid resolving-induced subgraph of $J(6, 2)$. Hence, the metric dimension of $J(6, 2)$ is 4, corresponding to the smallest possible resolving set size $\alpha = 4$, as established by equation (1.1).

II. Resolving Subgraph for $J(6, 2)$ Based on $\mathcal{R}_5$ (Path of Length 3)

The Common Neighborhood Table (C.N.T.) and degree table for the strongly regular graph $SRG(15, 8, 4, 4)$ with respect to the path graph $\mathcal{R}_5$ are shown in Table (3.13).

Table 3.13: Common Neighborhoods & Degree Tables of $(SRG(15, 8, 4, 4))$ w.r.t. $\mathcal{R}_5$.

Common Neighborhoods Table of $SRG(15, 8, 4, 4)$						Degree Table of R			
$\mathbf{w_1 w_2}$	$\mathbf{w_1 w_3}$	$\mathbf{w_1 w_4}$	$\mathbf{w_2 w_3}$	$\mathbf{w_2 w_4}$	$\mathbf{w_3 w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1	v_{1234}^1
v_{123}^1	v_{123}^1	v_{124}^1	v_{123}^1	v_{124}^1	v_{134}^1	v_{123}^1	v_{123}^1	v_{123}^1	v_{124}^1
v_{124}^1	v_{134}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{234}^1	v_{124}^1	v_{124}^1	v_{134}^1	v_{134}^1
v_{12}^1	v_{13}^1	w_3	w_1	w_3	v_{34}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{234}^1
						v_{12}^1	v_{12}^1	v_{13}^1	v_{24}^1
						v_{13}^1	v_{24}^1	v_{34}^1	v_{34}^1
						w_2	v_2^1	w_1	v_4^1
						w_3	w_2	w_4	w_3

Consequently, the vertex set of graph $G'_{\mathcal{R}_5}$ is constructed as $G_{4,4} = G^4 \cup G^3 \cup G^2 \setminus \{v_{14}^1, v_{23}^1\}$ and $G_{comp} = \{v_2^1, v_4^1\}$. The total number of vertices in the graph is:

$$\underbrace{4}_{|R|} + \underbrace{9}_{|G_{\lambda,\mu}|} + \underbrace{2}_{|G_{comp}|} = 15 = n.$$

Thus, the vertex set of the helper graph is:

$$V(G'_{\mathcal{R}_5}) = G^4 \cup G^3 \cup (G^2 \setminus \{v_{14}^1, v_{23}^1\}) \cup R.$$

The mapping f from $G'_{\mathcal{R}_5}$ to $J(6, 2)$ is shown in Table (3.14), which provides a one-to-one correspondence between the vertex sets of the two graphs.

$V(G')$	$f(V(G'))$	$V(G')$	$f(V(G'))$
w_1	$\{0, 1\}$	v_{234}^1	$\{2, 3\}$
w_2	$\{0, 2\}$	v_{12}^1	$\{0, 5\}$
w_3	$\{1, 3\}$	v_{13}^1	$\{1, 5\}$
w_4	$\{3, 4\}$	v_{24}^1	$\{2, 4\}$
v_{1234}^1	$\{0, 3\}$	v_{34}^1	$\{3, 5\}$
v_{123}^1	$\{1, 2\}$	v_2^1	$\{2, 5\}$
v_{124}^1	$\{0, 4\}$	v_4^1	$\{4, 5\}$
v_{134}^1	$\{1, 4\}$		

Table 3.14: Mapping from $V(G'_{\mathcal{R}_5})$ to $V(J(6, 2))$

This mapping demonstrates a bijective correspondence between $G'_{\mathcal{R}_5}$ and $J(6, 2)$. Thus, $\mathcal{R}_5$, corresponding to the path graph P_4 , forms a valid resolving-induced subgraph of $J(6, 2)$. Therefore, the metric dimension of $J(6, 2)$ is confirmed to be 4, matching the minimum resolving set size indicated by equation (1.1).

III. Resolving Subgraph for $J(6, 2)$ Based on $\mathcal{R}_{10}$ (K_4 with One Missing Edge)

The Common Neighborhood Table (C.N.T.) and degree table for the strongly regular graph $SRG(15, 8, 4, 4)$ with respect to the induced subgraph $\mathcal{R}_{10}$, which represents the complete graph K_4 with one missing edge (w_1w_3) , are shown in Table (3.15).

Table 3.15: Common Neighborhoods & Degree Tables of $(SRG(15, 8, 4, 4))$ w.r.t. $\mathcal{R}_{10}$.

Common Neighborhoods Table of $SRG(15, 8, 4, 4)$						Degree Table of R			
$\mathbf{w_1 w_2}$	$\mathbf{w_1 w_3}$	$\mathbf{w_1 w_4}$	$\mathbf{w_2 w_3}$	$\mathbf{w_2 w_4}$	$\mathbf{w_3 w_4}$	$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$
v_{123}^1	v_{123}^1	v_{124}^1	v_{123}^1	v_{124}^1	v_{134}^1	v_{123}^1	v_{123}^1	v_{123}^1	v_{124}^1
v_{124}^1	v_{134}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{234}^1	v_{124}^1	v_{124}^1	v_{134}^1	v_{134}^1
v_{12}^1	w_2	v_{14}^1	v_{23}^1	w_1	v_{34}^1	v_{134}^1	v_{234}^1	v_{234}^1	v_{234}^1
w_3	w_4	w_3	w_1	w_3	w_1	v_{12}^1	v_{12}^1	v_{23}^1	v_{14}^1
						v_{14}^1	v_{23}^1	v_{34}^1	v_{34}^1
						w_2	v_2^1	w_1	v_4^1
						w_3	w_1	w_2	w_1
						w_4	w_3	w_4	w_3

Consequently, the graph $G_{4,4}$ with respect to $\mathcal{R}_{10}$ is obtained as $G_{4,4} = G^3 \cup (G^2 \setminus \{v_{13}^1, v_{24}^1\})$ and $G_{comp} = \{v_2^1, v_4^1\} \cup G^0$. The total number of vertices in the graph is:

$$\underbrace{4}_{|R|} + \underbrace{8}_{|G_{\lambda, \mu}|} + \underbrace{3}_{|G_{comp}|} = 15 = n.$$

Thus, the vertex set of the helper graph is:

$$V(G'_{\mathcal{R}_{10}}) = G^3 \cup (G^2 \setminus \{v_{13}^1, v_{24}^1\}) \cup \{v_2^1, v_4^1, v^0\} \cup R.$$

Corollary 3.7 $\mathcal{R}_{10}$ is not a resolving induced subgraph of $J(6, 2)$.

Proof. The graph $G'_{\mathcal{R}_{10}}$ cannot form a subgraph that has a bijection to $J(6, 2)$. Recall that in $J(6, 2)$, two sets $\{a, b\}$ and $\{c, d\}$ are adjacent if and only if $|\{a, b\} \cap \{c, d\}| = 1$. Now, consider the vertices w_1, w_2, w_3, w_4 in $\mathcal{R}_{10}$, which form the graph K_4 with one missing edge (namely, $w_1 w_3$). The possible configurations for these vertices could be $\{a, c\}, \{a, b\}, \{b, d\}, \{a, d\}$, among others.

To maintain the resolving property, a vertex $v_{lts}^1 = \{x_1, x_2\}$ should exist such that $|w_l \cup w_s \cup w_t| = 3$, implying that there is an intersection of exactly one element in the common neighborhoods of these vertices. Applying the Principle of Inclusion-Exclusion (PIE) to this scenario, we compute:

$$|w_l \cap w_t \cap w_s| = |w_l \cup w_t \cup w_s| - \sum_{i=l,t,s} |w_i| + \sum_{i \neq j=l,t,s} |w_i \cap w_j| = 3 - 6 + 3 = 0.$$

This result shows that no common element exists across the chosen vertices. Therefore, there is no vertex that satisfies the adjacency conditions in $J(6, 2)$.

Additionally, vertices such as v_{124}^1 and v_{234}^1 cannot exist because their corresponding sets w_2 and w_4 are disjoint, violating the adjacency condition. Given these contradictions, there is no bijective correspondence between $J(6, 2)$ and the vertex set $V(G'_{\mathcal{R}_{10}})$. As a result, $\mathcal{R}_{10}$ cannot serve as a resolving-induced subgraph for $J(6, 2)$. ■

IV. Shapes That Cannot Form Resolving Sets

Corollary 3.8 *The graphs $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_4, \mathcal{R}_6, \mathcal{R}_7, \mathcal{R}_9$, and $\mathcal{R}_{11}$ are not resolving-induced subgraphs of $SRG(15, 8, 4, 4)$.*

Proof. Each of the graphs $\mathcal{R}_i$, where $i \in \{1, 2, 4, 6, 7, 9, 11\}$, cannot serve as the

shape of a resolving set for a strongly regular graph with parameters $(15, 8, 4, 4)$ because the Common Neighborhood Table (C.N.T.) cannot be properly constructed for these cases. The specific issues for each graph are as follows:

- $\mathcal{R}_1$: Although the table was constructed, the C.N.T. does not satisfy the necessary conditions. The total number of vertices needed to fill the table's columns exceeds 15. From Lemma 3.6, the maximum number of vertices connected to a single resolving vertex is $2^{\alpha-1} = 8$, and the maximum number of vertices connected simultaneously to two resolving vertices is $2^{\alpha-2} = 4$. Hence, the inclusion of all G^m for $m \in \{1, \dots, 4\}$ would require a minimum of 19 vertices in $V(G'_{\mathcal{R}_1})$, which is more than the 15 vertices available.
- $\mathcal{R}_2$: Similar to $\mathcal{R}_1$, while the table was constructed, the C.N.T. does not meet the conditions. The number of vertices required to fill the columns exceeds 15. Given $\eta_{\max} = 4$ and $k_{\max} = 8$, constructing G^4, G^3, G^2 , and excluding specific vertices such as v_1^1 and v_2^1 , results in $|V(G'_{\mathcal{R}_2})| = 17$, again exceeding the total available vertices.
- $\mathcal{R}_4$: The table was built, but the degree condition fails. Specifically, $\deg(w_2) = 9$, which exceeds the parameter $k = 8$, making this subgraph invalid for resolving purposes.
- $\mathcal{R}_6$: Although the table was constructed, the C.N.T. does not satisfy the required conditions (3.1), rendering this graph invalid.
- $\mathcal{R}_7$: The degrees of the vertices in the table do not align with the expected

parameters. In particular, $\deg(w_1) = 9$ exceeds $k = 8$, while the other vertices w_2, w_3, w_4 have degrees of 7, which are less than $k = 8$.

- $\mathcal{R}_8$: Since $\lambda = 4 = \eta_{\max}$, filling the C.N.T. requires using G^4, G^3 , and G^2 . However, we encounter an issue where:

$$|G^3 \setminus (A_{w_1 w_3}^3 \cup A_{w_2 w_4}^3)| = 4 - (2 + 2 - 0) = 0.$$

This means there are no available vertices in G^3 to complete the necessary columns, leaving the table incomplete.

- $\mathcal{R}_9$: The table was built, but the degrees of certain vertices fail to meet the required conditions. Specifically, $\deg(w_1) = \deg(w_4) = 7$ and $\deg(w_3) = 6$, all of which are less than $k = 8$.
- $\mathcal{R}_{11}$: The table was built, but the degree of each vertex w_i is 7, which is less than $k = 8$, thereby violating the degree conditions.

In all these cases, the C.N.T. cannot be constructed in a way that satisfies the necessary conditions for a resolving set in $SRG(15, 8, 4, 4)$. ■

3.5 Conclusion

Based on the extensive analysis of the Common Neighborhood Table (C.N.T.) method applied to $K(6, 2)$ and $J(6, 2)$, we can summarize the key results as follows:

3.5.1 For $K(6, 2)$

Theorem 3.2 [1] *The shape of a minimum resolving set for $K(6, 2)$, where $\dim(K(6, 2)) = 4$, must be one of the following subgraphs:*

- *A trivial graph consisting of four isolated vertices.*
- *A path on three vertices with one isolated vertex.*
- *A cycle on four vertices (i.e., C_4).*

3.5.2 For $J(6, 2)$

Theorem 3.3 [1] *The shape of a minimum resolving set for $J(6, 2)$, where $\dim(J(6, 2)) = 4$, must either be:*

- *Two independent edges.*
- *A path on four vertices.*

CHAPTER 4

INDEPENDENT EDGES AND ISOLATED VERTICES: A STRUCTURED APPROACH TO METRIC DIMENSION IN STRONGLY REGULAR GRAPHS USING C.N.T.

In this chapter, we focus on the application of a combinatorial approach to identify specific subgraphs of strongly regular graphs (SRGs) with parameters (n, k, λ, μ) . Building on the method introduced in the previous chapter, we restrict the shape of the resolving set R to independent edges and isolated vertices. This assumption

simplifies the process of determining the metric dimension of SRGs by employing the Common Neighborhood Table (C.N.T.) to identify a corresponding helper graph, $G'_{\mathcal{R}}$, under specific structural conditions.

We first enumerate the possible subgraphs $G'_{\mathcal{R}}$ within SRGs, recognizing that the number of non-isomorphic graphs grows rapidly with n . Given the computational complexity, our focus shifts toward identifying the simplest resolving sets based on specific structures like independent edges or isolated vertices. To further streamline the construction, we introduce a combinatorial system using integer programming to predict the number of metric representations derived from the parameters of $G_{\lambda,\mu}$, aiming to minimize the metric dimension of G' .

This chapter will explore the enumeration of all candidates for $G'_{\mathcal{R}}$ as subgraphs of SRGs with different vertex counts, followed by the presentation of the linear system necessary for solving the existence of these graphs. We conclude by discussing how specific shapes of resolving sets can be used to determine the minimum resolving set of SRGs.

4.1 Enumeration of Subgraphs Generated by the Common Neighborhood Table

The method outlined in the previous chapter begins with an assumption regarding the shape of R , referred to as $\mathcal{R}$. This method is structured into two main components. For clarity, let's recall our notation: G represents a strongly regular graph with parameters (n, k, λ, μ) , and $G'_{\mathcal{R}}$ is the graph generated by the Common Neighborhood

Table (C.N.T.) that fulfills a specific condition. The first component involves determining $G'_{\mathcal{R}}$, which is based solely on the parameters of the strongly regular graph. The second component explores the graph's structure by establishing a one-to-one correspondence between the vertex set $V(G'_{\mathcal{R}})$ and the vertex set $V(G)$ of the strongly regular graph G .

In this section, we will focus on the first component by enumerating all possible candidates for $G'_{\mathcal{R}}$ as subgraphs of $SRG(n, k, \lambda, \mu)$. However, due to the rapid increase in the number of non-isomorphic graphs on n vertices, this enumeration becomes computationally expensive. The number of non-isomorphic graphs on small vertex sets grows exponentially, as shown below:

- $n = 1$: 1 graph (a single vertex with no edges).
- $n = 2$: 2 graphs (one with no edges, one with a single edge).
- $n = 3$: 4 graphs.
- $n = 4$: 11 graphs.
- $n = 5$: 34 graphs.
- $n = 6$: 156 graphs.
- $n = 7$: 1,044 graphs.
- $n = 8$: 12,346 graphs.

The sequence continues and is registered in the OEIS (Online Encyclopedia of Integer Sequences) as A000088 [39]. For larger values of n , the numbers quickly become

large, making it computationally challenging to check all possible shapes. Given that constructing the C.N.T. is already a complex task, we reduce the complexity by focusing on specific shapes of $G'_{\mathcal{R}}$ where $\mathcal{R}$ is composed of independent edges and isolated vertices. Our primary motivation in selecting these shapes is to ensure that resolving vertices appear within the C.N.T., and no resolving vertices are missed.

4.2 Metric Dimension and Construction of Subgraphs

The construction of $G'_{\mathcal{R}}$ begins by generating the Common Neighborhood Table (C.N.T.). To assist in this process, we employ a combinatorial system based on integer programming, which helps predict the number of metric representations from each set.

We present a linear system that must be solved to establish the existence of $G_{\lambda,\mu}$, leading to a potential spanning subgraph G' corresponding to $\mathcal{R}$, the resolving induced subgraph of $SRG(n, k, \lambda, \mu)$. The size of R , denoted α , represents the minimum number for which the system has a solution, and must satisfy $n \leq 2^\alpha + \alpha$. Under these conditions, the resulting G' has $\dim(G') \leq \alpha$. We assume that the resolving set R consists solely of independent edges, isolated vertices, or both.

Let G be the strongly regular graph $SRG(n, k, \lambda, \mu)$. Assume that we can construct the common neighborhood table (C.N.T.) under the assumption that R is a resolving set of G with the specific shape of independent edges and isolated vertices. The following identities and formulas apply:

Fact 1 *Given the parameters (n, k, λ, μ) and the context of the C.N.T., let:*

- γ represent the total number of cells,
- m^* represent the number of independent edges in $\mathcal{R}$,
- $K = (k_{w_1}^*, \dots, k_{w_\alpha}^*)$ represent the degrees of resolving vertices within the C.N.T.,
and
- γ_{w_i} represent the total number of cells in columns associated with w_i .

Under these assumptions, the following identities hold:

1. *The total number of cells in the C.N.T. is:*

$$\gamma = m^* \lambda + \left(\binom{\alpha}{2} - m^* \right) \mu \quad (4.1)$$

2. *If w_i is an end vertex of one of the independent edges of $\mathcal{R}$, then:*

$$k_{w_i}^* = \begin{cases} k - 2 & \text{if } v_i^1 \in G_{comp}, \\ k - 1 & \text{if } v_i^1 \notin G_{comp}. \end{cases}$$

3. *If w_i is an isolated vertex in $\mathcal{R}$, then:*

$$k_{w_i}^* = \begin{cases} k - 1 & \text{if } v_i^1 \in G_{comp}, \\ k & \text{if } v_i^1 \notin G_{comp}. \end{cases}$$

4. The number of cells γ_{w_i} in the C.N.T. associated with w_i is:

$$\gamma_{w_i} = \begin{cases} (\alpha - 1)\mu & \text{if } w_i \text{ is an isolated vertex,} \\ \lambda - (\alpha - 2)\mu & \text{if } w_i \text{ is an end vertex of an independent edge.} \end{cases}$$

Corollary 4.1 *Let R be a resolving set with $|R| = \alpha$, where each vertex $w_i \in R$ is adjacent to at most one vertex $w_j \in R$. Then, the total number of cells γ in the C.N.T. satisfies the following conditions based on the relationship between λ and μ :*

i- If $\mu > \lambda$:

$$\left\lfloor \frac{\alpha}{2} \right\rfloor \lambda + \left(\binom{\alpha}{2} - \left\lfloor \frac{\alpha}{2} \right\rfloor \right) \mu \leq \gamma \leq \binom{\alpha}{2} \mu$$

ii- If $\mu < \lambda$:

$$\left\lfloor \frac{\alpha}{2} \right\rfloor \lambda + \left(\binom{\alpha}{2} - \left\lfloor \frac{\alpha}{2} \right\rfloor \right) \mu \geq \gamma \geq \binom{\alpha}{2} \mu$$

iii- If $\mu = \lambda$:

$$\left\lfloor \frac{\alpha}{2} \right\rfloor \lambda + \left(\binom{\alpha}{2} - \left\lfloor \frac{\alpha}{2} \right\rfloor \right) \mu = \gamma = \binom{\alpha}{2} \mu$$

Proof. The proof follows directly from the expression for γ derived in Fact 4.1.

Recall that:

$$\gamma = m^* \lambda + \left(\binom{\alpha}{2} - m^* \right) \mu,$$

where m^* represents the number of independent edges in $\mathcal{R}$.

- In case (i), when $\mu > \lambda$, the largest contribution to γ comes from the pairs of vertices where μ is applied, resulting in the upper bound $\binom{\alpha}{2} \mu$. The lower bound

comes from applying λ to the maximum number of independent edges $\lfloor \frac{\alpha}{2} \rfloor$, followed by applying μ to the remaining pairs.

- In case (ii), when $\mu < \lambda$, the largest contribution to γ comes from the pairs where λ is applied, resulting in the upper bound $\lfloor \frac{\alpha}{2} \rfloor \lambda + ((\binom{\alpha}{2} - \lfloor \frac{\alpha}{2} \rfloor)) \mu$. The lower bound comes from applying μ to all pairs of vertices, yielding $\binom{\alpha}{2} \mu$.

- In case (iii), when $\mu = \lambda$, the equality holds throughout, and γ is exactly $\binom{\alpha}{2} \mu$, as both λ and μ contribute equally.

Thus, the bounds for γ are established as stated in the corollary. ■

4.3 Integer Programming Approach to Subgraph Construction

To formalize the construction of $G'_{\mathcal{R}}$, we employ integer programming. This approach helps predict the number of metric representations selected from each set based on the Common Neighborhood Table (C.N.T.).

The integer system must be solved to determine the existence of $G_{\lambda, \mu}$ and, consequently, a possible spanning subgraph G' , with respect to $\mathcal{R}$. The size of the resolving set R , denoted α , must satisfy the constraint $n \leq 2^\alpha + \alpha$, and the resulting graph G' will have $\dim(G') \leq \alpha$.

To construct the system, we define the following parameters:

- Let x_m represent the number of elements chosen from the set G^m to fill the C.N.T.

- Let $y_m^{w_i}$ represent the number of elements chosen from G^m to fill $(\alpha - 1)$ columns related to w_i .
- Let $z_m^{w_i}$ represent the number of elements selected from G^m to fill columns associated with non-adjacent vertices of w_i .
- Let $\bar{z}_m^{w_i}$ represent the number of elements chosen from G^m to fill columns associated with adjacent vertices of w_i .

The following relationships hold:

$$\mu = \sum_{m=2}^{\alpha} z_m^{w_i} \quad \forall i \in \{1, \dots, \alpha\}, \quad \lambda = \sum_{m=2}^{\alpha} \bar{z}_m^{w_i} \quad \forall i \in \{1, \dots, \alpha\}$$

Proposition 4.1 [1] *The integer system determining $|G_{\lambda, \mu}|$ is as follows:*

$$\begin{aligned} |G_{\lambda, \mu}| &= \sum_{m=2}^{\alpha} x_m \\ \gamma &= \sum_{m=2}^{\alpha} \binom{m}{2} x_m \\ \gamma_{w_i} &= \sum_{m=2}^{\alpha} (m-1) y_m^{w_i} \quad \forall i \in \{1, \dots, \alpha\} \\ x_m &= \frac{1}{m} \sum_{i=1}^{\alpha} y_m^{w_i} \quad \forall m \in \{2, \dots, \alpha\} \end{aligned}$$

Moreover, $y_m^{w_i}$ is defined as:

$$y_m^{w_i} = \begin{cases} \frac{\alpha-1}{m-1} z_m^{w_i} & \text{if } w_i \text{ is an isolated vertex in } R, \\ \frac{1}{m-1} (\bar{z}_m^{w_i} + (\alpha-2) z_m^{w_i}) & \text{otherwise.} \end{cases}$$

Finally, the degree of each vertex $k_{w_i}^*$ in the C.N.T. is:

$$k_{w_i}^* = \sum_{m=2}^{\alpha} y_m^{w_i} \quad \forall i \in \{1, \dots, \alpha\}.$$

The system satisfies the following conditions:

1. $0 \leq x_m \leq \binom{\alpha}{m}, \forall m \in \{2, \dots, \alpha\}$
2. $0 \leq y_m^{w_i} \leq \binom{\alpha-1}{m-1}, \forall m, i \in \{2, \dots, \alpha\}$
3. $0 \leq z_m^{w_i}, \bar{z}_m^{w_i} \leq \binom{\alpha-2}{m-2}, \forall m, i \in \{2, \dots, \alpha\}$
4. $z_m^{w_i} \leq y_m^{w_i} \leq x_m$
5. $k_{w_i}^* \in \{k, k-1, k-2\}$

Proof. The value $|G_{\lambda, \mu}|$ is determined by summing the values of x_m over all m , specifically:

$$|G_{\lambda, \mu}| = \sum_{m=2}^{\alpha} x_m.$$

The total number of cells in the C.N.T., γ , is derived by summing the contributions of all pairs of vertices:

$$\gamma = \sum_{m=2}^{\alpha} \binom{m}{2} x_m.$$

Next, γ_{w_i} , representing the number of cells in the C.N.T. associated with vertex w_i , is:

$$\gamma_{w_i} = \sum_{m=2}^{\alpha} (m-1) y_m^{w_i}.$$

The relationship between x_m and $y_m^{w_i}$ follows from the fact that each term $y_m^{w_i}$ is repeated m times across the vertices, resulting in the equation:

$$x_m = \frac{1}{m} \sum_{i=1}^{\alpha} y_m^{w_i}.$$

Finally, we distinguish between cases where w_i is an isolated vertex in R or part of an independent edge. In the former case, $y_m^{w_i}$ depends on $z_m^{w_i}$, while in the latter case, it depends on both $\bar{z}_m^{w_i}$ and $z_m^{w_i}$.

It is straightforward to see that the remaining inequalities follow naturally. ■

Assumption: In this framework, we assume that if a set G^m is used in the C.N.T., it is used completely, meaning that all elements of the set are employed. This assumption is introduced to simplify the extensive calculations required.

Remark: This assumption provides a foundation for the method while reducing computational complexity. However, it also opens the possibility for future improvement of the technique by relaxing this condition. Allowing partial use of the set G^m in constructing the C.N.T. may lead to more efficient and refined results.

Theorem 4.1 *For a strongly regular graph $SRG(n, k, \lambda, \mu)$, the existence of a solution to the linear system described in Proposition 4.1 implies the existence of the set $G_{\lambda, \mu}$ and the graph $G'_{\mathcal{R}} = (V, E')$ with $\dim(G') \leq |R|$, where equality holds when R is the minimum resolving set of G' .*

4.4 Algorithm

In this section, we present a novel method for calculating the size of the minimum resolving set, which consists of independent edges, isolated vertices, or both, for a strongly regular graph $SRG(n, k, \lambda, \mu)$. The algorithm follows these steps:

1. **Initialize the resolving set size:** Begin with the minimum value of α such that $n \leq 2^\alpha + \alpha$. This ensures that the set R is large enough to potentially resolve the graph.
2. **Define the shape of the resolving set:** Assume the shape of the resolving set $R = \{w_1, \dots, w_\alpha\}$, which can take one of the following $(\lfloor \frac{\alpha}{2} \rfloor + 1)$ forms:
 - (a) Isolated vertices.
 - (b) One edge with isolated vertices.
 - (c) $\vdots$ (other combinations of independent edges and isolated vertices).
 - (d) $\lfloor \frac{\alpha}{2} \rfloor$ independent edges if α is even, or $\lfloor \frac{\alpha-1}{2} \rfloor$ independent edges and one isolated vertex if α is odd.
3. **Compute key parameters:** Calculate the parameters γ , γ_{w_i} , and $k_{w_i}^*$ for each $i \in \{1, \dots, \alpha\}$, which will assist in constructing the Common Neighborhood Table (C.N.T.) and verifying the resolving set properties.
4. **Solve the system of equations:** Using the linear system described in Proposition 4.1, solve for each possible case corresponding to the different forms of the resolving set R . This is accomplished using Algorithm 3.

5. **Increase resolving set size if no solution:** If there is no solution for the current value of α , increase the size of α by 1 and repeat the process. This ensures that we find the minimum resolving set.
6. **Construct C.N.T. when a solution is found:** For the first $\alpha = \alpha_0$ where a solution is found, construct the Common Neighborhood Table (C.N.T.), which helps visualize and validate the resolving set.
7. **Find the complement graph G_{comp} :** After constructing the C.N.T., identify the complement graph G_{comp} , which is formed by vertices not in the resolving set and is essential for finalizing the construction of G' .
8. **Write the vertex set of the induced subgraph:** Finally, express the vertex set of the spanning subgraph G' as:

$$V(G') = G_{\lambda,\mu} \cup G_{comp} \cup R,$$

where $G_{\lambda,\mu}$ represents the subgraph corresponding to the system solution.

Algorithm 3 Finding the size of selections from each G^m

Input: Graph parameters $SRG(n, k, \lambda, \mu)$

Output:

- (i) Size of the resolving set $|R|$ and number of independent edges m^* ;
- (ii) Existence of the farthest vertex v^0 ;
- (iii) Size of selections from G^m , x_m ;
- (iv) Size of selections from G^m associated with w_i , $y_m^{w_i}$;
- (v) Size of selections from G^m associated with w_i that fill μ common neighborhoods,
 $z_m^{w_i}$;
- (vi) Size of selections from G^m associated with w_i that fill λ common neighborhoods,
 $\bar{z}_m^{w_i}$.

1 begin

2 $\alpha \leftarrow$ minimum value such that $n \leq 2^\alpha + \alpha$ $R \leftarrow \{w_1, w_2, \dots, w_\alpha\}$ $m^* \leftarrow$
 Number of independent edges

3 end

Algorithm 4 Finding the size of selections from each G^m (Part 2)

Input: Graph parameters $SRG(n, k, \lambda, \mu)$ **Output:** m^* , x_m , $y_m^{w_i}$, $z_m^{w_i}$ and $\bar{z}_m^{w_i}$

```
4 if  $\alpha \bmod 2 = 0$  then
5   | Maximum Independent Edges :=  $\lfloor \alpha/2 \rfloor$ 
6 else
7   | Maximum Independent Edges :=  $\lfloor (\alpha - 1)/2 \rfloor$ 
8 for  $m^* = 0$  to Maximum Independent Edges do
9   | compute  $\gamma$  and for  $i = 1$  to  $\alpha$  do
10  |   | compute  $\gamma_{w_i}$ , and  $k_{w_i}^*$  for each  $w_i$ 
11  |   | while true do
12  |   |   | Try solving the linear system for each shape of  $R$  if solution found then
13  |   |   |   | break
14  |   |   | else
15  |   |   |   |  $\alpha := \alpha + 1$  update  $R$  to include new element Recompute Possible Shapes
```

4.5 Computations: minimum Resolving Sets and Metric Dimensions of Strongly Regular Graphs

In this section, we compute $G'_{\mathcal{R}}$ for parameters (n, k, λ, μ) and compare the evaluated metric dimension $\dim(G)$ of some strongly regular graphs (SRGs) discussed in Chapter 2 or found in the literature, with the size of the minimum resolving set R , under the following assumptions:

1. The resolving set consists of independent edges and isolated vertices.
 2. The condition $z_m^{w_i} = 0$ or $z_m^{w_i} = \binom{\alpha - 2}{m - 2}$ for $m \geq 3$.
- (4.2)

The results are compiled in tables, which include the graph's minimum resolving set that adheres to these assumptions, along with the original reference for its metric dimension. Our experiments were performed on SRGs with up to 200 vertices. Any SRG family with more than 64 vertices did not yield a $G'_{\mathcal{R}}$ graph under these assumptions (4.2). The mathematical programming formulations were automatically generated and solved using the HIGHS solver. The computations were carried out on systems with Intel processors (3.47 GHz and 3.46 GHz) and 96 GB of RAM. The results can be classified into four categories:

1. If the linear system has no solution, then $G'_{\mathcal{R}}$, respecting the assumptions in (4.2), does not exist. This indicates that no strongly regular graph with the same parameters can have the assumed $\mathcal{R}$ under the constraints, providing a proof of non-existence of such a resolving induced subgraph of G .

2. If the system has a solution and the size of R equals the metric dimension of a graph with the same parameters, it suggests that the minimum resolving set following the assumed shape may be the actual minimum resolving set. Thus, the structure of the strongly regular graph G can be used to map $G'_{\mathcal{R}}$, and its complement $\bar{G}'_{\mathcal{R}}$, to the SRG (n, k, λ, μ) via an isomorphism f .
3. If the size of R is greater than the metric dimension $\dim(G)$, as in cases like $(36, 21, 10, 15)$ and $(64, 36, 20, 20)$, it implies that the minimum resolving set cannot match the assumed shape. To maintain such a shape, the resolving set must satisfy $\dim(G) < |R|$.
4. If the size of R is less than the metric dimension, there is no one-to-one correspondence between $V(G'_{\mathcal{R}})$ and $V(G)$. Additionally, we explore cases where $|R|$ equals $\dim(G)$, which aligns with the previous three observations.

The following tables summarize the results of our computations, detailing the minimum resolving set sizes and shapes for various strongly regular graphs (SRGs) under the given assumptions. The graphs are categorized by their metric dimension and resolving set characteristics, with comparisons to known results from the literature.

Table 4.1: Results of Fully Classified Graphs

# Graphs	$\dim(G)$	$\text{SRG}(n, k, \lambda, \mu)$	$ R $	Shape of R
1	2	$\text{SRG}(5, 2, 0, 1)$	2	Isolated vertices
1	3	$\text{SRG}(9, 4, 1, 2)$	3	One edge
1	3	$\text{SRG}(10, 3, 0, 1)$	3	Isolated vertices
1	4	$\text{SRG}(13, 6, 2, 3)$	4	One or two edges
1	4	$\text{SRG}(15, 6, 1, 3)$	4	Isolated vertices
2	4	$\text{SRG}(16, 6, 2, 2)$	4	Two edges
1	4	$\text{SRG}(16, 5, 0, 2)$	4	Isolated vertices
1	4	$\text{SRG}(17, 8, 3, 4)$	4	Two edges
1	5	$\text{SRG}(21, 10, 3, 6)$	5	Isolated vertices
10	5	$\text{SRG}(26, 10, 3, 4)$	5	Isolated vertices
1	5	$\text{SRG}(27, 10, 1, 5)$	5	Isolated vertices
40	5	$\text{SRG}(29, 14, 6, 7)$	5	One or two edges
1	6	$\text{SRG}(36, 21, 10, 15)$	7	Isolated vertices

Table 4.2: Results of Specific Graphs with More Than 45 Vertices or not Completely Classified

Graph	$\dim(G)$	$\text{SRG}(n, k, \lambda, \mu)$	$ R $	Shape of R
P_{41}	7	$\text{SRG}(41, 20, 9, 10)$	7	Isolated vertices
Gewirtz graph	9	$\text{SRG}(56, 10, 0, 2)$	9	Isolated vertices
$PSP(6, 2)$	6	$\text{SRG}(63, 30, 13, 15)$	6	Isolated vertices
$2^6 : (S_3 \times GL(3, 2))$	9	$\text{SRG}(64, 21, 8, 6)$	7	Isolated vertices
Affine polar graph $VO^-(6, 2)$	7	$\text{SRG}(64, 27, 10, 12)$	6	Isolated vertices
Complement of $VO^-(6, 2)$	7	$\text{SRG}(64, 36, 20, 20)$	8	Four edges

Some of the graphs referenced in Table (4.3) are described as follows:

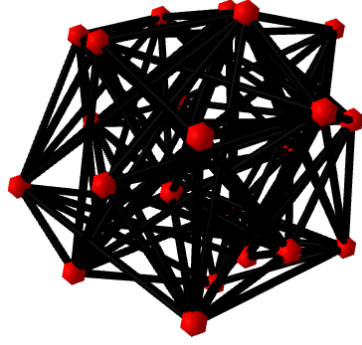
- (1) $\text{SRG}(49, 24, 11, 12)$: A self-complementary graph from the group $7^2 : (3 \times D_{16})$, which has a unique structure defined by its symmetry properties.
- (2) $\text{SRG}(50, 7, 0, 1)$: The Hoffman-Singleton graph, a well-known strongly regular graph with parameters $(50, 7, 0, 1)$, which plays a key role in graph theory due to its distinct properties.
- (3) $\text{SRG}(50, 42, 35, 36)$: The complement of the Hoffman-Singleton graph, which complements the original graph by interchanging edges and non-edges.

- (4) $SRG(56, 45, 36, 36)$: Intersection-2 graph of a quasisymmetric 2-(21,6,4) design with intersection numbers 0,2.
- (5) $SRG(64, 18, 2, 6)$: A graph from the group $2^6 : (3.S_6)$, associated with a specific automorphism group.
- (6) $SRG(64, 42, 26, 30)$: The complement of a graph derived from $2^6 : (S_3 \times GL(3, 2))$, another notable structure in the realm of group theory and graph automorphisms.

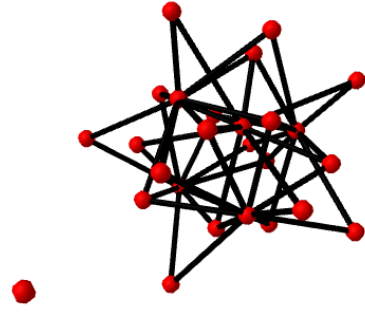
Table 4.3: Graphs with No Resolving Set R Under Assumptions (No Solution to Linear System)

#	$G=SRG(n, k, \lambda, \mu)$	$dim(G)$	(#)	$G=SRG(n, k, \lambda, \mu)$	$dim(G)$
1	$SRG(25, 8, 3, 2)$	6	21	$SRG(45, 12, 3, 3)$	8
15	$SRG(25, 12, 5, 6)$	5	57	$SRG(45, 12, 3, 3)$	7
4	$SRG(28, 12, 6, 4)$	6	1	$SRG(45, 16, 8, 4)$	7
1	$SRG(29, 14, 6, 7)$	6	1	$H(2,7):= SRG(49, 12, 5, 2)$	8
227	$SRG(35, 18, 9, 9)$	6	1	$(1):= SRG(49, 24, 11, 12)$	7
3854	$SRG(35, 16, 6, 8)$	6	1	$(2):=SRG(50, 7, 0, 1)$	11
1	$SRG(36, 10, 4, 2)$	7	1	$(3):= SRG(50, 42, 35, 36)$	11
32548	$SRG(36, 15, 6, 6)$	6	1	$J(11, 2):=SRG(55, 18, 9, 4)$	8
1	$SRG(36, 14, 7, 4)$	6	1	$(4):=SRG(56, 45, 36, 36)$	Not yet evaluated
180	$SRG(36, 14, 4, 6)$	5	1	$P_{61}:SRG(61, 30, 14, 15)$	7
6760	$SRG(37, 18, 8, 9)$	5	1	$(5):=SRG(64, 18, 2, 6)$	10
27	$SRG(40, 12, 2, 4)$	7	1	$(6):= SRG(64, 42, 26, 30)$	9
1	$SRG(40, 12, 2, 4)$	8			

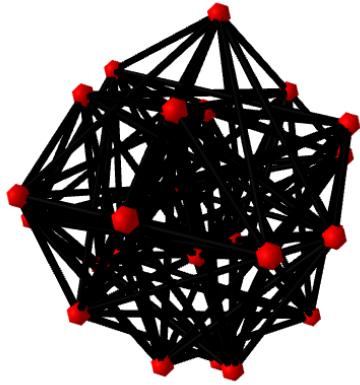
The following figures visually illustrate the G'_R subgraphs for various strongly regular graphs (SRGs) based on their parameters. These representations help provide insight into the structures and computations described earlier in this section:



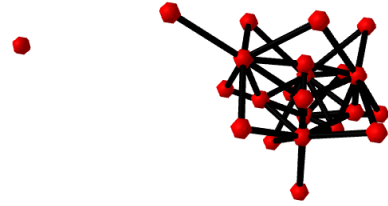
An example of a graph with parameters $SRG(26, 10, 3, 4)$.



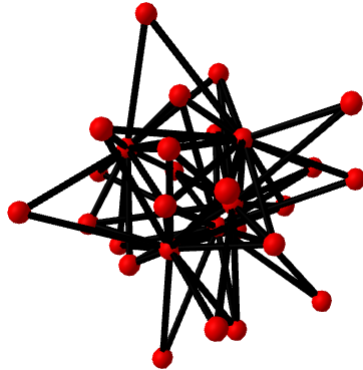
G'_R of $SRG(26, 10, 3, 4)$.



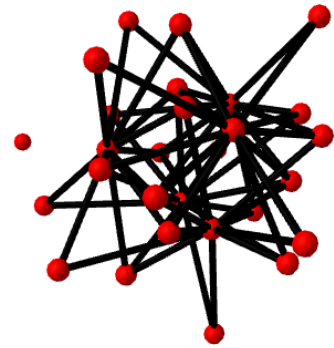
An example of a graph with parameters $SRG(27, 10, 1, 5)$.



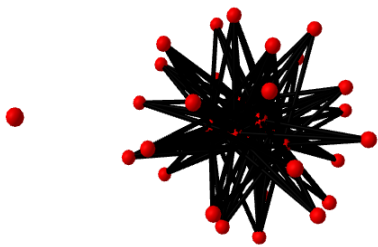
G'_R of $SRG(27, 10, 1, 5)$.



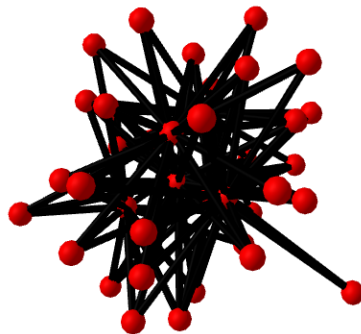
G'_R of $SRG(29, 14, 6, 7)$.



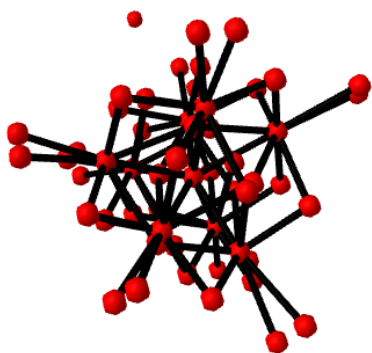
G'_R of $SRG(29, 14, 6, 7)$.



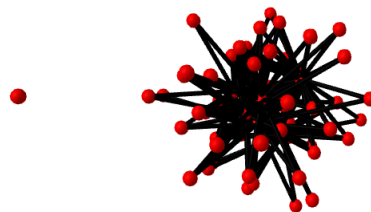
$G'_\mathcal{R}$ of $SRG(36, 21, 10, 15)$.



$G'_\mathcal{R}$ of $SRG(41, 20, 9, 10)$.



$G'_\mathcal{R}$ of $SRG(56, 10, 0, 2)$.



$G'_\mathcal{R}$ of $SRG(63, 30, 13, 15)$.

CHAPTER 5

MODELING C.N.T. FOR PATHS AND CYCLES

This chapter explores the application of the Common Neighborhood Table (C.N.T.) technique to specific configurations within strongly regular graphs, particularly focusing on paths and cycles. We aim to model the C.N.T. in cases where resolving sets are structured as either paths or cycles, with or without isolated vertices. By using integer programming, we optimize the process of constructing the C.N.T., which in turn assists in refining assumptions and enhancing accuracy in metric dimension calculations. The chapter systematically addresses various configurations, offering a robust framework for understanding the role of paths and cycles in determining the metric dimension of strongly regular graphs.

5.1 Path-Based Configurations of Resolving Sets in Strongly Regular Graphs

In this section, we explore configurations where the resolving set $\mathcal{R}$ is structured as a singular path or as a union of paths, possibly combined with isolated vertices. This approach simplifies the complexity involved in resolving sets while retaining the intricate structures that are pivotal in understanding their properties within strongly regular graphs.

Recall of Path Definition: A path, denoted by P_α , is a sequence of vertices $v_1, v_2, \dots, v_\alpha$ where each pair of consecutive vertices v_i and v_{i+1} is connected by an edge. This was briefly introduced in Chapter 1. A distinctive property of paths is that all vertices have a degree of two, except for the end vertices which have a degree of one.

For any two vertices u and v in $V(P_\alpha)$, the number of common neighbors in the path P_α naturally arises from the distance between u and v , and is given by:

$$N_{P_\alpha}(u, v) = \begin{cases} 0 & \text{if } v \sim u, \\ 1 & \text{if } v \not\sim u \text{ and } d_{P_\alpha}(u, v) = 2, \\ 0 & \text{if } v \not\sim u \text{ and } d_{P_\alpha}(u, v) > 2. \end{cases} \quad (5.1)$$

From this understanding, it becomes evident that the selection of vertices in the resolving set $\mathcal{R}$ influences the values of λ and μ in the Common Neighborhood Table (C.N.T.), specifically impacting μ . In the structure of P_α , we consider vertex pairs

(u, v) where $d_{P_\alpha}(u, v) = 2$. This distance corresponds to the count of P_3 subpaths within P_α , which are simply sets of three consecutive vertices.

Calculating the number of such subpaths, P_3 , involves determining the possible consecutive triples in P_α . This count is $\alpha - 2$, derived from $\alpha - 3 + 1$, which represents the transition from one vertex triplet to the next along the path. Consequently, the C.N.T. features $\alpha - 2$ columns, each populated by $\mu - 1$ vertices, corresponding to pairs of vertices that are two edges apart. Meanwhile, the number of λ columns, which reflect direct adjacency in the graph, totals $\alpha - 1$, accounting for each edge in the path.

The degree $k_{w_i}^*$ of a vertex w_i within the Common Neighborhood Table depends on its position within the resolving set $\mathcal{R}$ —whether it is isolated, an end vertex, or an interior vertex. Each vertex w_i corresponds to $\alpha - 1$ columns in the C.N.T., reflecting the relationships dictated by the path structure.

The value of γ_{w_i} , which represents the total count of cells in the columns associated with vertex w_i , varies based on the vertex's position:

- If w_i is an end vertex of P_α , it is associated with one column containing λ cells (representing an adjacent vertex), one column containing $\mu - 1$ cells (representing a vertex exactly two steps away), and the remaining columns filled with μ cells (representing vertices further away).
- If w_i is an interior vertex:
 - If the distance from one end is less than two, w_i is linked to two columns each containing λ cells (one for each adjacent vertex), one column with $\mu - 1$ cells, and the other columns filled with μ cells.

- If the distance from both ends is two or more, w_i is associated with two columns each containing λ cells, two columns with $\mu - 1$ cells, and the remaining columns filled with μ cells.
- For isolated vertices, there are no columns containing λ cells, as there are no directly adjacent vertices.

For the path configurations of $\mathcal{R}$ considered, we will compute the parameters γ , γ_{w_i} , and $k_{w_i}^*$. Subsequently, we will introduce minor modifications to the system outlined in Proposition 4.1.

5.1.1 Path P_α

In this subsection, we develop a linear system to establish the existence of $G_{\lambda,\mu}$ and, consequently, a potential helper graph $G'_{\mathcal{R}}$. It is assumed that the resolving graph $\mathcal{R}$ under consideration is solely composed of a single path.

Corollary 5.1 *Assume $SRG(n, k, \lambda, \mu)$ is a strongly regular graph. Within the context of the C.N.T, let γ represent the total number of cells, $K = (k_{w_1}^*, \dots, k_{w_\alpha}^*)$ denote a vector indicating the degrees of resolving vertices within the C.N.T, and γ_{w_i} signify the total number of cells in columns associated with w_i in the C.N.T. The following statements hold:*

1. *The total number of cells is given by:*

$$\gamma = (\alpha - 1)\lambda + (\alpha - 2)(\mu - 1) + \left(\binom{\alpha}{2} - (2\alpha - 3) \right) \mu. \quad (5.2)$$

2. For an end vertex w_i of the path, the degree $k_{w_i}^*$ is determined by:

$$k_{w_i}^* = \begin{cases} k - 2, & \text{if } v_i^1 \in G_{comp}, \\ k - 1, & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.3)$$

3. For an interior vertex w_i :

$$k_{w_i}^* = \begin{cases} k - 3, & \text{if } v_i^1 \in G_{comp}, \\ k - 2, & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.4)$$

4. The total number of cells associated with w_i is:

$$\gamma_{w_i} = \begin{cases} \lambda + (\mu - 1) + (\alpha - 3)\mu, & \text{if } w_i \text{ is an end vertex,} \\ 2\lambda + (\mu - 1) + (\alpha - 4)\mu, & \text{if } w_i \text{ is an interior vertex} \\ & \text{and } d(w_i, w_1) < 2 \text{ or } d(w_i, w_\alpha) < 2, \\ 2\lambda + 2(\mu - 1) + (\alpha - 5)\mu, & \text{if } w_i \text{ is an interior vertex} \\ & \text{and } d(w_i, w_1) \geq 2 \text{ and } d(w_i, w_\alpha) \geq 2. \end{cases} \quad (5.5)$$

5.1.2 Independent Equal Paths

In this subsection, we consider resolving sets $\mathcal{R}$ composed solely of independent paths of equal order β or a combination of independent paths of equal order β and isolated vertices.

Lemma 5.1 *The number of all possible configurations for paths of a specific order, where $3 \leq \beta < \alpha$, is given by:*

$$\sum_{\beta=3}^{\alpha-1} \left\lfloor \frac{\alpha}{\beta} \right\rfloor. \quad (5.6)$$

Proof. Let β represent the fixed order of each path considered within the resolving set $\mathcal{R}$. We assume $\mathcal{R}$ is composed of independent paths of order β and potentially isolated vertices:

$$\mathcal{R} = \bigcup_{i=1}^m P_{\beta} \cup \{\text{isolated vertices}\}.$$

Define $\mathcal{F}_{\beta}$ as the set of all possible configurations of $\mathcal{R}$ for a given β :

$$\mathcal{F}_{\beta} := \{\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_l\}.$$

Here, l denotes the number of distinct configurations possible for this specific order β . Each configuration starts with the maximum number of non-overlapping paths:

- $\mathcal{R}_1$ includes $\left\lfloor \frac{\alpha}{\beta} \right\rfloor$ paths and c isolated vertices, where $c \leq \beta - 1$.
- $\mathcal{R}_2$ to $\mathcal{R}_{\left\lfloor \frac{\alpha}{\beta} \right\rfloor}$ decrease in the number of paths by one in each subsequent configuration, increasing the number of isolated vertices by β .

The last configuration, $\mathcal{R}_l$, contains one path of order β and $(\alpha - \beta)$ isolated vertices, if β divides α exactly. Otherwise, the isolated vertices adjust to $\alpha - m\beta$, where m is the number of paths in the previous configuration.

Thus, the total number of all possible configurations for all allowable β is the sum of the maximum configurations for each β , $|\mathcal{F}_{\beta}| = \left\lfloor \frac{\alpha}{\beta} \right\rfloor$. ■

To calculate the parameter γ , the C.N.T. is decomposed into three parts: columns within P_β , columns between vertices from different P_β paths or between path vertices and isolated vertices, and columns within isolated vertices. Denote the number of columns in each part by x_1 , x_2 , and x_3 , respectively:

$$\begin{aligned} x_1 &= m^* \gamma_{P_\beta}, \\ x_2 &= \frac{\beta m^*}{2} (2\alpha - \beta(m^* + 1)), \\ x_3 &= \binom{\alpha - m^* \beta}{2}. \end{aligned}$$

The vector $K = (k_{w_1}^*, \dots, k_{w_\alpha}^*)$, representing the degrees of resolving vertices within the C.N.T., and the total number of cells in columns associated with each w_i , denoted by γ_{w_i} , remain unchanged, as detailed in equations (5.3)–(5.5).

Corollary 5.2 *Let $SRG(n, k, \lambda, \mu)$ be a strongly regular graph. In the context of the C.N.T., let γ represent the total number of cells, m^* denote the count of independent paths of the same size β in $\mathcal{R}$, $K = (k_{w_1}^*, \dots, k_{w_\alpha}^*)$ indicate the degrees of resolving vertices within the C.N.T., and γ_{w_i} signify the total number of cells in columns associated with w_i . The following statements hold:*

1.

$$\begin{aligned}
\gamma &= m^* \left((\beta - 1)\lambda + (\beta - 2)(\mu - 1) + \left(\binom{\beta}{2} - (2\beta - 3) \right) \mu \right) \\
&\quad + \binom{\alpha - m^*\beta}{2} \mu + \frac{\beta m^*}{2} (2\alpha - \beta(m^* + 1)) \mu \\
&= \lambda m^*(\beta - 1) + (\mu - 1)m^*(\beta - 2) \\
&\quad + \mu \left(\binom{\beta}{2} - (2\beta - 3) + \binom{\alpha - m^*\beta}{2} + \frac{\beta m^*}{2} (2\alpha - \beta(m^* + 1)) \right).
\end{aligned}$$

2. If w_i is an end vertex of the path P_β ,

$$k_{w_i}^* = \begin{cases} k - 2, & \text{if } v_i^1 \in G_{comp}, \\ k - 1, & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.7)$$

3. If w_i is an interior vertex of the path P_β ,

$$k_{w_i}^* = \begin{cases} k - 3, & \text{if } v_i^1 \in G_{comp}, \\ k - 2, & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.8)$$

4. If w_i is an isolated vertex in $\mathcal{R}$,

$$k_{w_i}^* = \begin{cases} k - 1, & \text{if } v_i^1 \in G_{comp}, \\ k, & \text{if } v_i^1 \notin G_{comp}. \end{cases}$$

5.

$$\gamma_{w_i} = \begin{cases} \lambda + (\mu - 1) + (\alpha - 3)\mu, & \text{if } w_i \text{ is an end vertex of } P_\beta, \\ 2\lambda + (\mu - 1) + (\alpha - 4)\mu, & \text{if } d(w_i, w_j) < 2, \text{ for } j = 1 \text{ or } j = \beta, \\ 2\lambda + 2(\mu - 1) + (\alpha - 5)\mu, & \text{if } d(w_i, w_j) \geq 2, \text{ for } j = 1, \beta, \\ (\alpha - 1)\mu, & \text{if } w_i \text{ is isolated.} \end{cases} \quad (5.9)$$

5.1.3 Independent Paths Configuration

In this subsection, we investigate configurations where the resolving set $\mathcal{R}$ consists of independent paths of various or, potentially combined with isolated vertices.

In order to derive the parameter γ , we decompose the Common Neighborhood Table (C.N.T.) into three parts:

1. Columns within each P_{β_i} are calculated using Equation (5.2).
2. Columns that join vertices from different paths or paths and isolated vertices are accounted for as μ columns. The total number of such columns is calculated as $\sum_{i=1}^{m^*} \beta_i (\alpha - \sum_{j=1}^i \beta_j)$.
3. The number of columns involving only isolated vertices is given by $\binom{\alpha - \sum_{i=1}^{m^*} \beta_i}{2}$.

Corollary 5.3 *Let $SRG(n, k, \lambda, \mu)$ be a strongly regular graph. In the context of the Common Neighborhood Table (C.N.T.), let γ represent the total number of cells, m^**

denote the count of independent cycles in $\mathcal{R}$ with sizes:

$$2 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{m^*} \leq (\alpha - 1),$$

satisfying

$$\sum_{i=1}^{m^*} \beta_i \leq \alpha.$$

Let $K = (k_{w_1}^*, \dots, k_{w_\alpha}^*)$ be a vector indicating the degrees of resolving vertices within the C.N.T., and let γ_{w_i} signify the total number of cells in columns associated with w_i .

The following statements hold:

1.

$$\gamma = \sum_{i=1}^{m^*} \gamma_{P_{\beta_i}} + \left(\alpha - \sum_{i=1}^{m^*} \beta_i \right) \mu + \mu \sum_{i=1}^{m^*} \beta_i \left(\alpha - \sum_{j=1}^i \beta_j \right), \quad (5.10)$$

where

$$\gamma_{P_{\beta_i}} = (\beta_i - 1)\lambda + (\beta_i - 2)(\mu - 1) + \left(\binom{\beta_i}{2} - (2\beta_i - 3) \right) \mu. \quad (5.11)$$

2. If w_i is an end vertex of the path P_β ,

$$k_{w_i}^* = \begin{cases} k - 2, & \text{if } v_i^1 \in G_{comp}, \\ k - 1, & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.12)$$

3. If w_i is an interior vertex of the path P_β ,

$$k_{w_i}^* = \begin{cases} k - 3, & \text{if } v_i^1 \in G_{comp}, \\ k - 2, & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.13)$$

4. If w_i is an isolated vertex in $\mathcal{R}$,

$$k_{w_i}^* = \begin{cases} k - 1, & \text{if } v_i^1 \in G_{comp}, \\ k, & \text{if } v_i^1 \notin G_{comp}. \end{cases}$$

5.

$$\gamma_{w_i} = \begin{cases} \lambda + (\mu - 1) + (\alpha - 3)\mu, & \text{if } w_i \text{ is an end vertex of } P_{\beta_i}, \\ 2\lambda + (\mu - 1) + (\alpha - 4)\mu, & \text{if } d(w_i, w_j) < 2, \text{ for } j = 1 \text{ or } j = \beta_i, \\ 2\lambda + 2(\mu - 1) + (\alpha - 5)\mu, & \text{if } d(w_i, w_j) \geq 2, \text{ for } j = 1, \beta_i, \\ (\alpha - 1)\mu, & \text{if } w_i \text{ is isolated.} \end{cases} \quad (5.14)$$

5.2 Cycle-Based Configurations of Resolving Sets in Strongly Regular Graphs

Recall from Chapter 1 that a cycle, denoted as C_α , is a closed trail in a graph where the only repeated vertices are the initial and terminal vertices, forming a loop of length

α . For our discussions, we focus on cycles where $\alpha > 5$ to ensure that each vertex has a well-defined neighborhood distinct from its adjacent vertices.

For any two vertices u and v in $V(C_\alpha)$, their common neighborhood is determined as follows:

$$N_{C_\alpha}(u, v) = \begin{cases} 0 & \text{if } v \sim u, \\ 1 & \text{if } v \not\sim u \text{ and } d_{C_\alpha}(u, v) = 2, \\ 0 & \text{if } v \not\sim u \text{ and } d_{C_\alpha}(u, v) > 2. \end{cases} \quad (5.15)$$

This structure suggests that the metric parameters λ and μ in the Common Neighborhood Table (C.N.T.) are influenced predominantly by μ . Specifically, vertices that are two edges apart contribute to the count of $\mu - 1$ columns in the C.N.T. Given the cyclic arrangement, each vertex w_i in C_α is uniquely associated with two other vertices $w_{(i-2) \bmod \alpha}$ and $w_{(i+2) \bmod \alpha}$, forming pairs that are counted twice as we complete one cycle. Therefore, there are α unique pairs contributing to α columns with $\mu - 1$ vertices.

Furthermore, the degree $k_{w_i}^*$ for each vertex w_i depends on its position in C_α :

- If w_i is an isolated vertex, there are no columns with λ vertices.
- If w_i is part of the cycle, each w_i is associated with two columns that contain λ vertices, reflecting the two adjacent vertices in the cycle. Additionally, each w_i contributes to two columns with $\mu - 1$ vertices due to the vertices exactly two steps away.

This configuration influences the calculations of γ , γ_{w_i} , and $k_{w_i}^*$ within the C.N.T.

Subsequent adjustments to the system as defined in Proposition 4.1 are tailored to accommodate the specific cyclic structure of $\mathcal{R}$.

5.2.1 Cycle-Based Configurations: Single Cycle C_α

In this subsection, we focus on resolving induced subgraphs $\mathcal{R}$ that are solely comprised of a single cycle. This consideration simplifies the analysis while preserving the generality for cycles of length α .

Corollary 5.4 *Let $SRG(n, k, \lambda, \mu)$ be a strongly regular graph. Within the framework of the Common Neighborhood Table (C.N.T), let γ denote the total number of cells, $K = (k_{w_1}^* \dots k_{w_\alpha}^*)$ represent the degrees of resolving vertices within the C.N.T, and γ_{w_i} indicate the total number of cells in columns associated with w_i in the C.N.T. The following conditions hold:*

1. For γ :

$$\gamma = \begin{cases} 3\lambda, & \text{if } \alpha = 3, \\ 4\lambda + 2(\mu - 2), & \text{if } \alpha = 4, \\ \alpha\lambda + \alpha(\mu - 1) + \left(\binom{\alpha}{2} - 2\alpha\right)\mu, & \text{if } \alpha \geq 5. \end{cases} \quad (5.16)$$

2. For $k_{w_i}^*$:

$$k_{w_i}^* = \begin{cases} k - 3 & \text{if } v_i^1 \in G_{comp}, \\ k - 2 & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.17)$$

3. For γ_{w_i} :

$$\gamma_{w_i} = \begin{cases} 2\lambda, & \text{if } \alpha = 3, \\ 2\lambda + (\mu - 2), & \text{if } \alpha = 4, \\ 2\lambda + 2(\mu - 1) + (\alpha - 5)\mu, & \text{if } \alpha \geq 5. \end{cases} \quad (5.18)$$

5.2.2 Independent Uniform Cycles

We consider configurations where the resolving induced subgraph $\mathcal{R}$ consists solely of independent cycles of the same size or a combination of independent cycles of the same size and isolated vertices.

Lemma 5.2 *The number of all possible configurations for cycles with lengths $3 \leq \beta < \alpha$ is given by:*

$$\sum_{\beta=3}^{\alpha-1} \left\lfloor \frac{\alpha}{\beta} \right\rfloor \quad (5.19)$$

Proof. Similarly, see Lemma (5.1). ■

Corollary 5.5 *Let $SRG(n, k, \lambda, \mu)$ denote a strongly regular graph. Within the Common Neighborhood Table (C.N.T.), let γ represent the total number of cells, m^* denote the number of independent cycles of equal size β in $\mathcal{R}$, $K = (k_{w_1}^* \dots k_{w_\alpha}^*)$ represent the vector of degrees of resolving vertices within the C.N.T., and γ_{w_i} denote the total number of cells in columns associated with each w_i . The following assertions hold:*

1. For the total number of cells γ :

$$\gamma = m^* \gamma_{C_\beta} + \binom{\alpha - m^* \beta}{2} \mu + \frac{m^* \beta}{2} (2\alpha - \beta(m^* + 1)) \mu, \quad (5.20)$$

where

$$\gamma_{C_\beta} = \begin{cases} 3\lambda, & \text{if } \beta = 3, \\ 4\lambda + 2(\mu - 2), & \text{if } \beta = 4, \\ \beta\lambda + \beta(\mu - 1) + \left(\binom{\beta}{2} - 2\beta\right)\mu, & \text{if } \beta \geq 5. \end{cases} \quad (5.21)$$

2. For vertices w_i in C_β :

$$k_{w_i}^* = \begin{cases} k - 3 & \text{if } v_i^1 \in G_{comp}, \\ k - 2 & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.22)$$

3. For isolated vertices w_i in $\mathcal{R}$:

$$k_{w_i}^* = \begin{cases} k - 1 & \text{if } v_i^1 \in G_{comp}, \\ k & \text{if } v_i^1 \notin G_{comp}. \end{cases}$$

4. For γ_{w_i} :

$$\gamma_{w_i} = \begin{cases} 2\lambda + (\alpha - 3)\mu, & \text{if } w_i \in C_3, \\ 2\lambda + (\mu - 2) + (\alpha - 4)\mu, & \text{if } w_i \in C_4, \\ 2\lambda + 2(\mu - 1) + (\alpha - 5)\mu, & \text{if } w_i \in C_\beta, \beta \geq 5, \\ (\alpha - 1)\mu, & \text{if } w_i \notin C_\beta. \end{cases} \quad (5.23)$$

5.2.3 Independent Cycles

This subsection explores configurations where the resolving induced subgraph $\mathcal{R}$ consists exclusively of independent cycles of varying sizes, potentially combined with isolated vertices. We analyze how these configurations influence the structure of the Common Neighborhood Table (C.N.T)

Corollary 5.6 *Let $SRG(n, k, \lambda, \mu)$ denote a strongly regular graph. Within the C.N.T. context:*

- γ represents the total number of cells.
- m^* denotes the number of independent cycles within $\mathcal{R}$, with sizes arranged as $3 \leq \beta_1 \leq \beta_2 \leq \dots \leq \beta_{m^*} \leq (\alpha - 1)$ and $\sum_{i=1}^{m^*} \beta_i \leq \alpha$.
- $K = (k_{w_1}^* \dots k_{w_\alpha}^*)$ indicates the degrees of resolving vertices within the C.N.T.
- γ_{w_i} denotes the number of cells associated with each w_i .

The following conditions are quantified:

1. Total γ is computed as:

$$\gamma = \sum_{i=1}^{m^*} \gamma_{C_{\beta_i}} + \left(\alpha - \sum_{i=1}^{m^*} \beta_i \right) \mu + \mu \sum_{i=1}^{m^*} \beta_i \left(\alpha - \sum_{j=1}^i \beta_j \right), \quad (5.24)$$

where

$$\gamma_{C_{\beta_i}} = \begin{cases} 3\lambda & \text{if } \beta_i = 3, \\ 4\lambda + 2(\mu - 2) & \text{if } \beta_i = 4, \\ \beta_i\lambda + \beta_i(\mu - 1) + \left(\binom{\beta_i}{2} - 2\beta_i\right)\mu & \text{if } \beta_i \geq 5. \end{cases} \quad (5.25)$$

2. Degree of resolving vertices w_i in cycle C_{β_i} :

$$k_{w_i}^* = \begin{cases} k - 3 & \text{if } v_i^1 \in G_{comp}, \\ k - 2 & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (5.26)$$

3. For isolated vertices within $\mathcal{R}$:

$$k_{w_i}^* = \begin{cases} k - 1 & \text{if } v_i^1 \in G_{comp}, \\ k & \text{if } v_i^1 \notin G_{comp}. \end{cases}$$

4. Values of γ_{w_i} for different vertices are:

$$\gamma_{w_i} = \begin{cases} 2\lambda + (\alpha - 3)\mu, & w_i \in C_3, \\ 2\lambda + (\mu - 2) + (\alpha - 4)\mu, & w_i \in C_4, \\ 2\lambda + 2(\mu - 1) + (\alpha - 5)\mu, & w_i \in C_{\beta_i}, \beta_i \geq 5, \\ (\alpha - 1)\mu, & w_i \notin C_{\beta_i}, i \in \{1, \dots, m^*\}. \end{cases} \quad (5.27)$$

5.2.4 Path/Cycle-Based Configurations and Their Impact on the C.N.T.

Building upon the configurations of $\mathcal{R}$ as composed of independent cycles and isolated vertices, we introduce a detailed model. This model extends the mathematical formulations introduced in the previous chapters and is delineated in Proposition 5.1 for cycles of length $\alpha \geq 5$.

Proposition 5.1 *The integer system to determine $|G_{\lambda,\mu}|$ is formulated as follows:*

$$|G_{\lambda,\mu}| = \sum_{m=2}^{\alpha} x_m, \quad (5.28)$$

$$\gamma = \sum_{m=2}^{\alpha} \binom{m}{2} x_m, \quad (5.29)$$

$$\gamma_{w_i} = \sum_{m=2}^{\alpha} (m-1) y_m^{w_i}, \quad \text{for all } i \in \{1, \dots, \alpha\}, \quad (5.30)$$

$$x_m = \frac{1}{m} \sum_{i=1}^{\alpha} y_m^{w_i}, \quad \text{for all } m \in \{2, \dots, \alpha\}, \quad (5.31)$$

$$y_m^{w_i} = \begin{cases} \frac{\alpha-1}{m-1} z_m^{w_i}, & \text{if } w_i \text{ is an isolated vertex,} \\ \frac{1}{m-1} \left(2\bar{z}_m^{w_i} + (\alpha-5) z_m^{w_i} \right. \\ \quad \left. + 2 \left(\mu - \sum_{l=2}^{m-1} z_l^{w_i} - \sum_{l=m+1}^{\alpha} z_l^{w_i} \right) \right), & \text{if } w_i \text{ is in a cycle.} \end{cases} \quad (5.32)$$

$$k_{w_i}^* = \sum_{m=2}^{\alpha} y_m^{w_i}, \quad \text{for all } i \in \{1, \dots, \alpha\}. \quad (5.33)$$

The following conditions are enforced:

1. $0 \leq x_m \leq \binom{\alpha}{m}$, for all $m \in \{2, \dots, \alpha\}$.
2. $0 \leq y_m^{w_i} \leq \binom{\alpha-1}{m-1}$, for all $m, i \in \{2, \dots, \alpha\}$.

3. $0 \leq z_m^{w_i}, \bar{z}_m^{w_i} \leq \binom{\alpha-2}{m-2}$, for all $m, i \in \{2, \dots, \alpha\}$.
4. $z_m^{w_i} \leq y_m^{w_i} \leq x_m$.
5. $k_{w_i}^*$ values must be one of the following: $\{k, k-1, k-2, k-3\}$.

5.2.5 Algorithm for Determining minimum Resolving Sets

This section outlines our novel method for calculating the minimum resolving set, consisting of independent cycles or paths, for a distance-transitive strongly regular graph, $SRG(n, k, \lambda, \mu)$. The algorithm proceeds as follows:

1. Initialize by setting the smallest α such that $n \leq 2^\alpha + \alpha$.
2. Hypothesize a resolving set $R = \{w_1, \dots, w_\alpha\}$.
3. Calculate the parameters γ , γ_{w_i} , and $k_{w_i}^*$ for each $i \in \{1, \dots, \alpha\}$.
4. Employ the derived system to find a solution for the current configuration of $\mathcal{R}$.
5. If a solution is unattainable, increment α by one and reevaluate.
6. Once a solution is confirmed for $\alpha = \alpha_0$, proceed to construct the Common Neighborhood Table (C.N.T.).
7. Identify and record G_{comp} , the complement graph within the resolving structure.
8. Define the vertex set of the helper graph $G'_{\mathcal{R}}$ as $V(G'_{\mathcal{R}}) = G_{\lambda, \mu} \cup R \cup G_{comp}$.

This algorithmic approach ensures systematic exploration and verification of potential resolving sets, adjusting for complexity and size constraints inherent to strongly regular graphs.

CHAPTER 6

LINEAR PROGRAMMING MODEL FOR C.N.T.

As discussed earlier, one of the main challenges in constructing the Common Neighborhood Table (C.N.T.) is handling the constraints on $z_m^{w_i}$, which we previously assumed to be either zero or $\binom{|R|-2}{m-2}$. These constraints significantly influence the metric dimension, as the minimum resolving set is directly impacted. In this chapter, we introduce a linear programming approach to facilitate the construction of the C.N.T. for a given graph $\mathcal{R}$.

6.1 Lattice Representation of C.N.T. Constraints

Definition 6.1 *Define the lattice $(\mathcal{S}, \vee, \wedge)$, where:*

$$\mathcal{S} = \left\{ \left(z_2^{w_i w_j}, z_3^{w_i w_j}, \dots, z_{|R|}^{w_i w_j} \right) \in \mathbb{Z}^{|R|-1} \mid 0 \leq z_m^{w_i w_j} \leq \binom{|R|-2}{m-2}, \sum_{m=2}^{|R|} z_m^{w_i w_j} \leq 2^{|R|-2} \right\}.$$

The join ($\vee$) and meet ($\wedge$) operations for any tuples $T^{(1)}, T^{(2)} \in \mathcal{S}$ are defined as:

$$T^{(1)} \vee T^{(2)} = \left(\max(z_2^{(1)}, z_2^{(2)}), \max(z_3^{(1)}, z_3^{(2)}), \dots, \max(z_{|R|}^{(1)}, z_{|R|}^{(2)}) \right),$$

$$T^{(1)} \wedge T^{(2)} = \left(\min(z_2^{(1)}, z_2^{(2)}), \min(z_3^{(1)}, z_3^{(2)}), \dots, \min(z_{|R|}^{(1)}, z_{|R|}^{(2)}) \right).$$

Moreover, $(\mathcal{S}, \vee, \wedge)$ is a bounded lattice with the least element $(0, \dots, 0)$ and the greatest element $\left(\binom{|R|-2}{0}, \binom{|R|-2}{1}, \dots, \binom{|R|-2}{|R|-2} \right)$.

The tuple $z^{w_i w_j}$ can be described as a $(|R| - 1)$ -dimensional vector, where each component $m \in \{2, \dots, |R|\}$ represents the number of elements in the corresponding G^{m+1} needed to fill the column $w_i w_j$ in the C.N.T. Each component satisfies:

$$0 \leq z_m^{w_i w_j} \leq \binom{|R| - 2}{m - 2}, \quad \text{for } m \in \{2, \dots, |R|\}.$$

The L_1 -norm of $z^{w_i w_j}$, denoted $|z^{w_i w_j}|_1$, represents either:

- μ : the number of resolving vertices in the column $w_i w_j$ of the C.N.T. when $w_i \not\sim w_j$, or
- λ : the number of resolving vertices when $w_i \sim w_j$.

6.2 Linear Optimization Programming Problem

Objective Function

Minimize: $Z = 0$.

Constraints

For $w_i, w_j \in R$, the constraints are as follows:

$$k_{w_i}^* = \sum_{m=2}^{|R|} \left(\sum_{j \neq i} \frac{1}{m-1} z_m^{w_i w_j} \right), \quad \forall i \in \{1, \dots, |R|\}, \quad (6.1)$$

$$\mu = \sum_{m=2}^{|R|} z_m^{w_i w_j}, \quad \text{if } w_i \not\sim w_j, \quad (6.2)$$

$$\lambda = \sum_{m=2}^{|R|} z_m^{w_i w_j}, \quad \text{if } w_i \sim w_j, \quad (6.3)$$

$$0 \leq z_m^{w_i w_j} \leq \binom{|R| - 2}{m - 2}, \quad \forall m \in \{2, \dots, |R|\}. \quad (6.4)$$

6.3 Column Representation in the C.N.T.

For a graph $\mathcal{R}$, we calculate the number of resolving vertices (w) in each column $w_i w_j$ of the C.N.T. The goal is to find the tuples $z^{w_i w_j}$, where $i, j \in \{1, \dots, |R|\}, i \neq j$, such

that:

$$|z^{w_i w_j}|_1 = \begin{cases} \lambda - \# \text{ of resolving vertices in column } w_i w_j, & \text{if } w_i \sim w_j, \\ \mu - \# \text{ of resolving vertices in column } w_i w_j, & \text{if } w_i \not\sim w_j. \end{cases} \quad (6.5)$$

6.4 Vector Representation for Conformance

To ensure the solution adheres to the structure of the C.N.T., we define the vector

VZ^{w_i} such that:

$$|VZ^{w_i}|_1 = \left| \left(\dots, \sum_{j=1}^{|R|-1} \frac{z_m^{w_i w_j}}{m-1}, \dots \right) \right|_1 \quad (6.6)$$

$$= k_{w_i}^*, \quad (6.7)$$

where:

$$k_{w_i}^* = \begin{cases} k - |N(w_i)| - 1, & \text{if } v_i^1 \in G_{comp}, \\ k - |N(w_i)|, & \text{if } v_i^1 \notin G_{comp}. \end{cases} \quad (6.8)$$

6.5 Graph Representation in the C.N.T.

We also define the vector VX to represent the overall structure of the graph $\mathcal{R}$:

$$|VX|_1 = \left| \left(\dots, \frac{1}{\binom{m}{2}} \sum_{i,j}^{\binom{|R|}{2}} z_m^{w_i w_j}, \dots \right) \right|_1 \quad (6.9)$$

$$= n - |R| - |G_{comp}|, \quad (6.10)$$

$$= |G_{\lambda, \mu}|. \quad (6.11)$$

6.6 Example: Application to Hamming Graph

$$H(2, 5)$$

The Hamming graph $H(2, 5)$ has parameters $(25, 8, 3, 2)$. Based on the calculations in Chapter 2, its metric dimension, $\dim(H(2, 5))$, is 6. However, as discussed in Chapter 4, the Common Neighborhood Table (C.N.T.) does not exist under the constraints that the minimum resolving set is either a trivial graph or composed of independent edges with $z_m^{w_i}$ either being zero or full. Using linear programming, we find the following solution for $\mathcal{R}$: three independent edges. The solution for $z^{w_i w_j}$ is:

$$\begin{aligned} z^{w_1 w_2} &= (0, 3, 0, 0, 0), & z^{w_1 w_3} &= (0, 2, 0, 0, 0), & z^{w_1 w_4} &= (1, 1, 0, 0, 0), & z^{w_1 w_5} &= (0, 2, 0, 0, 0), \\ z^{w_1 w_6} &= (0, 2, 0, 0, 0), & z^{w_2 w_3} &= (0, 2, 0, 0, 0), & z^{w_2 w_4} &= (0, 2, 0, 0, 0), & z^{w_2 w_5} &= (0, 2, 0, 0, 0), \\ z^{w_2 w_6} &= (1, 1, 0, 0, 0), & z^{w_3 w_4} &= (1, 2, 0, 0, 0), & z^{w_3 w_5} &= (1, 1, 0, 0, 0), & z^{w_3 w_6} &= (1, 1, 0, 0, 0), \\ z^{w_4 w_5} &= (0, 2, 0, 0, 0), & z^{w_4 w_6} &= (1, 1, 0, 0, 0), & z^{w_5 w_6} &= (0, 3, 0, 0, 0). \end{aligned}$$

This leads to the following configuration for the C.N.T.:

Table 6.1: Common Neighborhoods of $SRG(25, 8, 3, 2)$

$\mathbf{w_1 w_2}$	$\mathbf{w_1 w_3}$	$\mathbf{w_1 w_4}$	$\mathbf{w_1 w_5}$	$\mathbf{w_1 w_6}$	$\mathbf{w_2 w_3}$	$\mathbf{w_2 w_4}$	$\mathbf{w_2 w_5}$	$\mathbf{w_2 w_6}$
v_{123}^1	v_{123}^1	v_{124}^1	v_{156}^1	v_{156}^1	v_{123}^1	v_{124}^1	v_{125}^1	v_{256}^1
v_{124}^1	v_{136}^1	v_{14}^1	v_{125}^1	v_{136}^1	v_{234}^1	v_{234}^1	v_{256}^1	v_{26}^1
v_{125}^1								
$\mathbf{w_3 w_4}$	$\mathbf{w_3 w_5}$	$\mathbf{w_3 w_6}$	$\mathbf{w_4 w_5}$	$\mathbf{w_4 w_6}$	$\mathbf{w_5 w_6}$			
v_{234}^1	v_{345}^1	v_{136}^1	v_{345}^1	v_{456}^1	v_{456}^1			
v_{345}^1	v_{35}^1	v_{36}^1	v_{456}^1	v_{46}^1	v_{156}^1			
v_{34}^1					v_{256}^1			

Table 6.2: Degree Table of R

$\mathbf{w_1}$	$\mathbf{w_2}$	$\mathbf{w_3}$	$\mathbf{w_4}$	$\mathbf{w_5}$	$\mathbf{w_6}$
$v_{123}^1, v_{124}^1, v_{125}^1$	$v_{123}^1, v_{124}^1, v_{125}^1$	$v_{123}^1, v_{136}^1, v_{234}^1$	$v_{124}^1, v_{234}^1, v_{345}^1$	$v_{125}^1, v_{156}^1, v_{256}^1$	$v_{136}^1, v_{156}^1, v_{256}^1$
$v_{136}^1, v_{156}^1, v_{14}^1$	$v_{234}^1, v_{256}^1, v_{26}^1$	$v_{345}^1, v_{34}^1, v_{35}^1$	$v_{34}^1, v_{456}^1, v_{46}^1$	$v_{345}^1, v_{35}^1, v_{456}^1$	$v_{26}^1, v_{36}^1, v_{46}^1$
w_2, v_1^1	w_1, v_2^1	w_4, v_{36}^1	w_3, v_4^1	w_6, v_5^1	w_5, v_{456}^1

The resolved graph components are as follows:

$$G_{3,2} = \{v_{123}^1, v_{124}^1, v_{125}^1, v_{136}^1, v_{156}^1, v_{234}^1, v_{256}^1, v_{345}^1, v_{456}^1, v_{14}^1, v_{26}^1, v_{34}^1, v_{35}^1, v_{36}^1, v_{46}^1\}.$$

Thus, the complementary graph is given by:

$$G_{comp} = G^1 \setminus \{v_3^1, v_6^1\}.$$

Finally, we verify:

$$\underbrace{6}_R + \underbrace{15}_{|G_{\lambda,\mu}|} + \underbrace{4}_{|G_{comp}|} = 25 = n.$$

6.7 Conclusion

The existence of tuples $z^{w_i w_j}$ satisfying the above constraints implies that the helper graph $G'_{\mathcal{R}}$ exist for $SRG(n, k, \lambda, \mu)$.

CHAPTER 7

FUTURE WORK

This chapter outlines potential future directions based on the findings and methodologies presented in this thesis. The proposed research aims to extend the study of Common Neighborhood Tables (C.N.T.) and resolving sets in strongly regular graphs (SRG).

7.1 Listing Small SRGs

We aim to enumerate all $SRG(n, k, \lambda, \mu)$ for small values of n with resolving sets of size $|R| \leq 5$. This will involve systematically analyzing their structures and verifying the existence of minimum resolving sets.

7.2 Extending Methodologies

We intend to generalize the approach used for $K(6, 2)$ to other graphs. By analyzing their structures, we can determine possible resolving induced subgraphs with resolv-

ing sets of cardinality $|R| = \dim(G)$. This process will help identify patterns and invariants across different families of SRGs.

7.3 Preservation of Resolving Set Shapes

An important question is whether the shape of the resolving set is preserved across families of strongly regular graphs, such as P_n , for varying values of n . Investigating this could reveal deeper connections between graph structure and metric dimension.

7.4 Larger SRGs and Complex Configurations

For larger values of n , we aim to identify all $SRG(n, k, \lambda, \mu)$ corresponding to various configurations like stars, wheels, and fans. This analysis will leverage the integer programming techniques introduced in Chapter 3.

7.5 C.N.T. as a Combinatorial Design

The influence of removing a vertex from the full C.N.T. on other columns suggests that the C.N.T. might be viewed as a combinatorial design. We propose studying the indices of metric representations, such as v_B^i , in relation to:

- **Balanced Incomplete Block Design (BIBD):** Parameters (n, b, r, k, λ) , where:

- n : Number of elements,

- b : Number of blocks,
 - r : Number of blocks each element appears in,
 - k : Size of each block,
 - λ : Number of blocks in which each pair of elements appears.
- **Triple Balanced Block Design (TBBD):** Extending BIBD by considering triplets of elements with parameters $(n, b, r, k, \lambda_2, \lambda_3)$, where λ_3 represents the number of blocks containing each triplet of elements.
 - **Steiner Systems:** Exploring systems $S(t, k, n)$, which consist of n elements with blocks of size k , such that every t -subset of the n elements appears in exactly one block.

7.6 Impact of λ and μ on Metric Dimension

Finally, we will investigate how the difference between λ and μ influences the construction of the C.N.T. and whether this affects the metric dimension $|R|$. Specifically, we aim to analyze how close $\dim(G)$ is to $\min_{|R|} \{n \leq 2^{|R|+|R|}\}$ and how this relationship varies with graph parameters.

REFERENCES

- [1] M. Alfuraidan, A. Alhujaylan, and M. Alshahrani, “Metric dimension of strongly regular graphs: A combinatorial optimization constructive approach using resolving sets: Metric dimension of strongly regular graphs: A combinatorial optimization constructive approach using resolving sets,” submitted 2024.
- [2] A. E. Brouwer and H. Van Maldeghem, *Strongly Regular Graphs*, ser. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2022.
- [3] C. Godsil and G. F. Royle, *Algebraic graph theory*. Springer Science & Business Media, 2001, vol. 207.
- [4] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & Digraphs*. CRC Press, 2010, vol. 39.
- [5] F. Harary and R. A. Melter, “On the metric dimension of a graph,” *Ars Combinatoria*, vol. 2, pp. 191–195, 1976.
- [6] P. J. Slater, “Leaves of trees,” *Congr. Numer*, vol. 14, no. 549-559, p. 37, 1975.
- [7] S. Khuller, B. Raghavachari, and A. Rosenfeld, “Landmarks in graphs,” *Discrete applied mathematics*, vol. 70, no. 3, pp. 217–229, 1996.

- [8] G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann, “Resolvability in graphs and the metric dimension of a graph,” *Discrete Applied Mathematics*, vol. 105, no. 1-3, pp. 99–113, 2000.
- [9] G. Chartrand, C. Poisson, and P. Zhang, “Resolvability and the upper dimension of graphs,” *Computers & Mathematics with Applications*, vol. 39, no. 12, pp. 19–28, 2000.
- [10] B. Shanmukha, B. Sooryanarayana, and K. Harinath, “Metric dimension of wheels,” *Far East J. Appl. Math.*, vol. 8, no. 3, pp. 217–229, 2002.
- [11] C. Hernando, M. Mora, I. M. Pelayo, C. Seara, J. Cáceres, and M. L. Puertas, “On the metric dimension of some families of graphs,” *Electronic notes in discrete mathematics*, vol. 22, pp. 129–133, 2005.
- [12] C. Hernando, M. Mora, I. Pelayo, C. Seara, and D. Wood, “Extremal graph theory for metric dimension and diameter,” *Electronic Notes in Discrete Mathematics*, vol. 29, pp. 339–343, 2007, european Conference on Combinatorics, Graph Theory and Applications. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S1571065307001370>
- [13] P. Manuel, R. Bharati, I. Rajasingh *et al.*, “On minimum metric dimension of honeycomb networks,” *Journal of Discrete algorithms*, vol. 6, no. 1, pp. 20–27, 2008.
- [14] I. Javaid, M. T. Rahim, and K. Ali, “Families of regular graphs with constant metric dimension,” *Utilitas mathematica*, vol. 75, no. 1, pp. 21–33, 2008.

- [15] S. Saputro, E. T. Baskoro, A. Salman, and D. Suprijanto, “The metric dimensions of a complete n -partite graph and its cartesian product with a path,” *Journal of Combinatorial Mathematics and Combinatorial Computing*, vol. 71, p. 283, 2009.
- [16] M. Bača, E. T. Baskoro, A. Salman, S. W. Saputro, and D. Suprijanto, “The metric dimension of regular bipartite graphs,” *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, pp. 15–28, 2011.
- [17] G. Sudhakara and H. A. Kumar, “Graphs with metric dimension two-a characterization,” *Advances and Applications in Discrete Mathematics*, vol. 4, no. 2, pp. 169–186, 2009.
- [18] A. Behtoei, A. Davoodi, M. Jannesari, and B. Omoomi, “A characterization of some graphs with metric dimension two,” *Discrete Mathematics, Algorithms and Applications*, vol. 9, no. 02, 2017.
- [19] M. Jannesari and B. Omoomi, “Characterization of n -vertex graphs with metric dimension $n - 3$,” *Mathematica Bohemica*, vol. 139, no. 1, pp. 1–23, 2014.
- [20] P. Buczowski, G. Chartrand, C. Poisson, and P. Zhang, “On k -dimensional graphs and their bases,” *Periodica Mathematica Hungarica*, vol. 46, no. 1, pp. 9–15, 2003.
- [21] R. A. Melter and I. Tomescu, “Metric bases in digital geometry,” *Computer vision, graphics, and image Processing*, vol. 25, no. 1, pp. 113–121, 1984.
- [22] R. F. Bailey and K. Meagher, “On the metric dimension of Grassmann graphs,” *Discrete Mathematics and Theoretical Computer Science*, vol. Vol. 13

- no. 4, no. 4, pp. 97–104, Jan. 2012, special issue in honor of Laci Babai’s 60th birthday: Combinatorics, Groups, Algorithms, and Complexity. [Online]. Available: <https://inria.hal.science/hal-00990476>
- [23] M. Imran, A. Baig, S. A. U. H. Bokhary, and I. Javaid, “On the metric dimension of circulant graphs,” *Applied Mathematics Letters*, vol. 25, no. 3, pp. 320–325, 2012. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0893965911004083>
- [24] D. Garijo, A. González, and A. Márquez, “The difference between the metric dimension and the determining number of a graph,” *Applied Mathematics and Computation*, vol. 249, pp. 487–501, 2014. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0096300314014003>
- [25] “Resolving sets for johnson and kneser graphs,” *European Journal of Combinatorics*, vol. 34, no. 4, pp. 736–751, 2013.
- [26] U. Ali, S. A. Bokhary, K. Wahid, and G. Abbas, “On resolvability of a graph associated to a finite vector space,” *Journal of Algebra and Its Applications*, vol. 18, no. 02, p. 1950029, 2019.
- [27] L. Tang, S. Zhou, J. Chen, and Z. Zhang, “Metric dimension and metric independence number of incidence graphs of symmetric designs,” *Discrete Applied Mathematics*, vol. 291, pp. 43–50, 2021. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0166218X20305199>

- [28] J. Wu, L. Wang, and W. Yang, “Learning to compute the metric dimension of graphs,” *Applied Mathematics and Computation*, vol. 432, p. 127350, 2022. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0096300322004246>
- [29] J. Geneson, S. Kaustav, and A. Labelle, “Extremal results for graphs of bounded metric dimension,” *Discrete Applied Mathematics*, vol. 309, pp. 123–129, 2022. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0166218X21004674>
- [30] J. Wang, F. Tian, Y. Liu, J. Pang, and L. Miao, “On graphs of order n with metric dimension $n-4$,” *Graph. Comb.*, vol. 39, no. 2, mar 2023. [Online]. Available: <https://doi.org/10.1007/s00373-023-02627-x>
- [31] L. Babai, “On the complexity of canonical labeling of strongly regular graphs,” *SIAM Journal on Computing*, vol. 9, no. 1, pp. 212–216, 1980.
- [32] —, “On the order of uniprimitive permutation groups,” *Annals of Mathematics*, vol. 113, no. 3, pp. 553–568, 1981.
- [33] J. Kratica, D. Cvetkovic, M. Cangalovic, V. Kovacevic-Vujcic, and J. Kojic, “The metric dimension of strongly regular graphs,” *Proceedings of SYMOPIS*, pp. 341–344, 2008.
- [34] R. F. Bailey, “The metric dimension of small distance-regular and strongly regular graphs,” *AUSTRALASIAN JOURNAL OF COMBINATORICS*, vol. 62, no. 1, pp. 18–34, 2015.

- [35] “MS Windows NT online repository of distance-regular graphs,” <https://www.distanceregular.org/index.html>, accessed: 2023-2-18.
- [36] G. Fijavž and B. Mohar, “Rigidity and separation indices of paley graphs,” *Discrete Mathematics*, vol. 289, no. 1, pp. 157–161, 2004. [Online]. Available: <https://www.sciencedirect.com/science/article/pii/S0012365X04003747>
- [37] R. Bailey and P. Cameron, “Base size, metric dimension and other invariants of groups and graphs,” *Bull Lond Math Soc*, vol. 43, 03 2011.
- [38] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood, “On the metric dimension of cartesian products of graphs,” *SIAM journal on discrete mathematics*, vol. 21, no. 2, pp. 423–441, 2007.
- [39] N. J. A. Sloane, “The on-line encyclopedia of integer sequences,” 2024, number of graphs on n unlabeled nodes, sequence A000088. [Online]. Available: <https://oeis.org/A000088>

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