A FRAMEWORK FOR SOLVING LARGE-SCALE NONLINEAR EQUATIONS WITH CONVEX CONSTRAINTS

BY

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Dedication

I dedicate this work to my late grandmothers Mariam and Noor.

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First and foremost I thank the Almighty Allah who has given me countless gifts throughout my life and for enabling me to reach this milestone.

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List of Acronyms

CGPM Conjugate Gradient Projection Method

CGM Conjugate Gradient Method

MOPCGM Modified Optimal Perry Conjugate Gradient Method

SCGM Specrtal Conjugate Gradient Method

STTDFPM Specrtal Three Term Derivative-Free Projection Method

NSM Non-monotone Spectral Methods

THESIS ABSTRACT

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The objective of this thesis was to address the problem of large-scale nonlinear equations with convex constraints using a unified framework that includes various conjugate direction algorithms such as Conjugate Gradient Projection Method (CGPM), Modified Optimal Perry Conjugate Gradient Method (MOPCGM), and Specrtal Three Term Derivative-Free Projection Method (STTDFPM).

These algorithms produce sequences defined by specific formulations. The proposed unified framework introduced a safeguard condition on the line search direction, along with carefully chosen line search procedures, and computed the projection point accordingly. In this work, we introduced two other new algorithms based on the generalizations of CGPM and MOPCGM. All the new algorithms satisfied the sufficient descent condition. The framework was based on the generalizations and STTDFPM method due to their great performance in solving the problem.

We proved the algorithms' convergence under reliable restrictions, specifically when

the function is Lipschitz continuous for both the framework and the Generalized CGPM(GCGPM).

The operation of the proposed algorithms was verified through numerical experiments on functions defined over convex constraints. Additionally, the Generalized MOPCGM(GMOPCGM) exhibits a trust region property provided the nonlinear function is monotone. From the numerical experiments, the Generalized CGPM outperformed its counterparts in all aspects. The algorithms were also applied to compressed sensing to reconstruct the signals.

ملخص الرسالة

الاسم: اسم الطالب

عنوان الدراسة: عنوان الرسالة

التخصص: قسم الطلاب

تاريخ الدرجة العلمية: أدخل الشهر والسنة

خوارزميات يتضمن موحد إطار باستخدام المحدبة القيود ذات النطاق واسعة الخطية غير الأنظمة مشكلة معالجة هو الأطروحة هذه هدف كان الموحد الإطار قدم محددة. بصيغ محددة تسلسلات الخوارزميات هذه تنتج .STTDFPMو MOPCGMو CGPM مثل مختلفة مترافق اتجاه لذلك. وفقًا الإسقاط نقطة وحسب بعناية، مختارة خطية بحث إجراءات مع جنب إلى جنبًا الخط، عن البحث اتجاه في وقائيًا شرطًا المقترح شرط الجديدة الخوارزميات جميع استوفت .MOPCGMو CGPM تعميمات على تعتمدان أخريين جديدتين خوارزميتين قدمنا العمل، هذا في قيود ظل في الخوارزميات تقارب أثبتنا المشكلة. حل في الرائع لأدائها نظرًا STTDFPM وطريقة التعميمات إلى الإطار استند الكافي. الانحدار الخوارزميات تشغيل من التحقق تم .(GCGPM) المعمم CGPMو الإطار من لكل Lipschitz مستمرة الدالة تكون عندما وتحديدًا موثوقة، (GMOPCGM)MOPCGM يُظهر ذلك، إلى بالإضافة المحدبة. القيود عبر المحددة الوظائف على العددية التجارب خلال من المقترحة جميع في نظيراتها على المعممة CGPM تفوقت العددية، التجارب من رتيبة. الخطية غير الوظيفة تكون أن بشرط الثقة منطقة خاصية المعمم الإشارات. بناء لإعادة المضغوط الاستشعار على الخوارزميات تطبيق تم كما الجوانب.

CHAPTER 1

INTRODUCTION

Nonlinear equations represent a significant class of problems, closely related to optimization challenges, frequently emerging in various fields of science, technology, and industry. Over the past two decades, many problems have been extensively studied, highlighting their importance across disciplines [6]. These problems are often complex, requiring sophisticated methods to solve efficiently.

In this work, we address the problem of finding $x \in \Gamma \subset \mathbb{R}^n$ such that:

$$G(x) = 0, \tag{1.1}$$

where Γ is assumed to be a nonempty, closed, and convex subset of $\mathbb{R}^n$, and $G : \Gamma \to \mathbb{R}^n$ is a continuous and monotone function. Problems of this nature can be either smooth or non-smooth and appear in a variety of applications, such as Bregman distances [53], Monotone variational inequality problems [48, 70, 108], Non-negative matrix factorization [16], Phase retrieval [18], Constrained neural networks [25], Chemical equilibrium systems in thermodynamics [71, 104], Financial modeling and forecasting [36], Signal processing and signal restoration, especially in wavelet de-convolution and compressed sensing [4, 42], solid state physics, plasma, and fluid mechanics [22, 45].

Given the wide range of applications, developing efficient algorithms to solve (1.1) is crucial. Various methods have been proposed to solve these problems, with considerable research focusing on improving their accuracy and computational efficiency. Among such methods include methods involving determining the derivatives of $G(x)$ in (1.1) such as Newton and Newton-like methods in [53, 73]. However, they require $G(x)$ to be not only continuous but also differentiable(smooth) which may not be the case in some instances. Further, even if G is differentiable, it may be difficult to obtain derivatives of $G(x)$ in (1.1) and this requires large memory for computation and storing. This has steered mathematicians to devise methods that do not require $G(x)$ to be differentiable. Such methods are called derivative-free methods.

Many derivative-free methods have been proposed stemming from Conjugate gradient methods(CGM) and Spectral conjugate gradient methods.

In this work, we comprehensively studied three derivative-free methods and proposed a Framework that unifies them. They include the CGPM [109], the MOPCGM [83], and the STTDFPM [50]. Generalizations of CGPM and MOPCGM were proposed. Then we proposed a Framework that combines the Generalized CGPM, Generalized MOPCGM, and STTDFPM. This was a result of the common properties that were identified. One of the features is the adaptive scaling such that there is a balance between the robustness, aggressiveness, and accuracy of the algorithms. In relation to that, we verified that the proposed algorithms satisfy the sufficient descent condition and converge globally provided G is Lipschitz continuous and monotone. The numerical results were also recorded and showed that the Generalized CGPM outperformed its counterparts in every aspect, that is in terms of function evaluations, CPU time, and the number of iterations.

The thesis is arranged as follows: Chapter 2 reviews the literature of Derivative-Free methods, Chapter 3 details the generalizations of the MOPCGM and CGPM and also identifies the features of the two generalizations and STTDFPM, Chapter 4 proposes the Framework of the three derivative-free methods, Chapter 5 details the numerical experiments and Chapter 6 concludes with areas of further research.

CHAPTER 2

PRELIMINARIES AND LITERATURE REVIEW

We begin by looking at some of the useful terminologies and lemmas for this research.

2.1 Terminologies and preliminary information

Definition 2.1 *If* $\Gamma \subset \mathbb{R}^n$ *is closed and convex and* $G : \Gamma \to \mathbb{R}^n$ *a function. Then* G *is* monotone *on* Γ *if*

$$\langle G(x) - G(y),\ x - y\rangle \geq 0 \quad \textit{for every } x, y \in \Gamma. \tag{2.1}$$

Definition 2.2 *A mapping* $\Pi_\Gamma : \mathbb{R}^n \to \Gamma$ *is an orthogonal projection if*

$$\Pi_\Gamma(x) = \operatorname{argmin}\{||y - x||:\ y \in\ \Gamma\} \quad \textit{for each } x \in \mathbb{R}^n.$$

The following properties of projection operators are important in our research.

Lemma 2.1 *[50, 83, 95]If a nonempty set* $\Gamma \subset \mathbb{R}^n$ *is convex and closed, then we have.*

1. $(x - \Pi_\Gamma(x))^T(\Pi_\Gamma(x) - y) \geq 0$ *for each* $x \in \mathbb{R}^n, \quad y \in \Gamma$.

2. $||\Pi_\Gamma(x) - \Pi_\Gamma(y)|| \leq ||x - y||$ *for each* $x, y \in \mathbb{R}^n$.

2.2 Derivative free methods

Most of the iterative approaches rely on the merit function defined as follows;

$$\min f(x). \tag{2.2}$$

Where $f(x) = \frac{1}{2}||F(x)||^2$.

The steps to resolve (2.2) involve generating iterates x_k defined by

$$x_{k+1} = x_k + \alpha_k p_k,$$

where α_k is the step length determined by various appropriate line search methods and p_k is the search direction that must satisfy

$$F_k^T p_k \leq -\tau ||F_k||^2, \qquad \tau \geq 0. \tag{2.3}$$

(2.3) is called sufficient descent condition and as in [6]. Even though the algorithms relying on (2.2) are valid for resolving (1.1), they are also accompanied by some trade-offs on the other hand that include ensuring that all minimizers to be produced require the function $F(x)$ to be regular, they also need the level sets of the merit function to be bounded and thus the solution set of the system must be bounded and so without the regularity condition, it becomes difficult to prove the convergence of the points produced even if the sequence of merit function values accumulates to zero. Therefore, it is worthwhile to mention that some approaches can easily solve the problem (1.1) without defining any merit function similar to one in (2.2) see [86]. To overcome the above limitations and with motivation of the projection proximal point algorithm in [87] to obtain the solution of the set-valued monotone operators in a Hilbert space, the first work on projection-based algorithms was done by Solodov and Svaiter [86] in 1999 and it has played a key role in applying first-order optimization methods to solve (1.1) [50, 62].

Derivative-free methods have been widely suggested to solve (1.1) due to their easy implementation, small storage capacities, and good convergence properties [30, 72, 86]. The most suggested derivative-free approaches for (1.1) involve:

1. determining a descent direction for subsequent iterate to be close to the solution along the direction with some step size obtained using suitable methods.
2. constructing a hyperplane for the current iterate to be separated from the solution set due to convex separation.
3. projecting x_k onto the hyperplane for the algorithm's convergence.

If G is monotone and $z_k = x_k + \alpha_k p_k$, the hyperplane

$$H_k = \{x \in R^n : \langle G(z_k), x - z_k \rangle = 0\} \tag{2.4}$$

separates strictly x_k from Γ of (1.1).

The subsequent subsections review such methods that apply similar steps to solve (1.1).

2.2.1 Spectral Three-Term Derivative-Free Method

We initially review the concept of the spectral gradient method and its history and subsequent developments. Consider an unconstrained optimization problem.

$$\min F(x) \tag{2.5}$$

$F : \mathbf{R}^n \to \mathbf{R}$ is differentiable. The spectral gradient technique is expressed as

$$x_{k+1} = x_k + \tau^k p_k$$

where $\tau_1^k = \frac{s_{k-1}^T s_{k-1}}{y_{k-1}^T s_{k-1}}$ or $\tau_2^k = \frac{s_{k-1}^T y_{k-1}}{y_{k-1}^T y_{k-1}}$ are the spectral parameters,

$$s_{k-1} = x_k - x_{k-1}, \tag{2.6}$$

and

$$y_{k-1} = g_k - g_{k-1}, \tag{2.7}$$

$g_k = g(x_k) = \nabla F(x_k)$ and p_k is the descent search direction and $\tau_1^k \geq \tau_2^k$. It's worth noting that the longest spectral parameter τ_1^k is superior to the shortest one in most cases. see [28] and references there in.

The spectral gradient method was suggested in 1988 in [15] by Barzilai and Borwein to solve (2.5).

Nine years later, Raydan proposed a non-monotone line search approach that ensures global convergence when coupled with the method of Barzilai and Borwein [80, 17]. Birgin and Martinez [17] later in 2001 also proposed another Specrtal Conjugate Gradient Method (SCGM) and used it to solve (2.5) and it gave promising results. In 2003, Cruz and Raydan [27] modified the spectral gradient method to solve (1.1) systematically by choosing $\pm$ gradient as the search direction. In the same year again, William and Marcos suggested Non-monotone Spectral Methods (NSM) and used it to solve (1.1) [27].

In 2005, spectral gradient projection approach was suggested by Li and Zhou [106]. This method combines modified spectral gradient and projection techniques in [15] and [86] respectively. The search direction is given by

$$p_k = \begin{cases} -G(x_k) & k = 0 \\ -\tau_k G(x_k) & k \geq 1 \end{cases} \tag{2.8}$$

such that $\tau_k = \frac{||s_k||^2}{s_k^T w_k}$, and $w_k = y_k + rs_k$. Where s_k and y_k are as defined in (2.6) and (2.7) respectively, $r > 0$. The iterate x_{k+1} is given by

$$x_{k+1} = x_k - \frac{\langle G(z_k), x_k - z_k \rangle G(z_k)}{||G(z_k)||^2} \tag{2.9}$$

by projecting x_k onto the hyperplane H_k in (2.9) such that $z_k = x_k + \alpha_k p_k$ and $\alpha_k = \gamma^{m_k}$ such that m_k is the smallest positive integer such that

$$-G(z_k)^T p_k \geq \rho \gamma^m ||p_k||^2 \tag{2.10}$$

where $\rho,\ \gamma \in (0, 1)$

In 2006, William, Raydan, and Mario [55] used Barzilai and Borwein's method to solve the nonlinear systems without using any information concerning gradients of the functions involved and their results were really promising.

Later in 2008, Yu et al [103] suggested a Spectral gradient projection method

(SGPM) to solve (1.1) which does not require solving any sub-problems. This method also combines the modified spectral gradient method and projection method (MSGM) in [15] and [86] respectively. The direction in solving (1.1) is

$$p_k = \begin{cases} -G(x_k) & k = 0 \\ -\tau_k G(x_k) & k \geq 1 \end{cases} \tag{2.11}$$

such that $\tau_k = \frac{||s_k||^2}{s_k^T w_k}$ and $w_k = y_k + r s_k$ where s_k and y_k are as defined in (2.6) and (2.7) respectively, $r > 0$ and the initial point must belong to Γ with carefully chosen parameters.

$x_{k+1} = \Pi_\Gamma(x_k - \eta_k G(z_k))$, where $\eta_k = \frac{\alpha_k G(z_k)^T p_k}{||G(x_k)||^2}$, $z_k = x_k + \alpha_k p_k$ and $\alpha_k = \gamma^{m_k}$ such that m_k is the smallest positive integer m for which

$$-G(z_k)^T p_k \geq \rho \gamma^m ||p_k||^2 \tag{2.12}$$

where $\rho,\ \gamma \in (0, 1)$.

These algorithms in [106] and [103] are almost the same but the difference arises in the computation of x_{k+1}. In the latter, x_{k+1} is chosen to be the best approximation close to the solution by taking the orthogonal projection of (2.9).

In 2009, Yu et al [103], suggested a spectral gradient projection technique to solve (1.1). This method couples both the modified spectral gradient method and projection. p_k is as in (2.11) such that $\tau_k = \frac{||s_k||^2}{s_k^T w_k}$, and $w_k = y_k + \sigma s_k$, where s_k and y_k are defined as before in (2.6) and (2.7) respectively, $\sigma > 0$. This method is globally convergent G in (1.1) is Lipschitz continuous.

Xiao, Wang, and Hu proposed a spectral gradient method and applied it to solve (1.1). At every iteration, a spectral gradient approach is employed on the resulting problem without any information on derivatives. The convergence of their method is not different from the existing results and p_k is given by

$$p_k = -\tau_k G_k \quad k = 0, 1, \cdots$$

Where τ_k is the projection of the usual well-known spectral parameter on an interval $0 < \lambda_1 \leq \tau_k \leq \lambda_2$. see [99]

In 2019, Dai, Huang, and Liu [28] suggested a class of spectral gradient methods whose step length is the convex combination of τ_1^k and τ_2^k. The step length is given by

$$\eta_k = \eta_k \tau_1^k + (1 - \eta_k)\tau_2^k$$

such that $\eta_k \in [0, 1]$.

In 2019, Li et al [59] gave a three-term method stemming from the Quasi-Newton approach to solve the (2.5). However, the convergence of their method needs the function to be uniformly convex. Subsequently, Amini and Faramarzi proposed an SCGM and solved (2.5). They modified the above method in [59] to address the requirement of uniform convexity for the convergence of the method. see details in [7]

Again Liu, Feng, and Zou [63] proposed another SCGM to solve (2.5). This method combines both the spectral parameter and the conjugate parameter, leading to its name SCGM. The method was proved to be globally convergent provided the conjugate parameter bounds the conjugate parameter due to Fletcher and Reeves and

$$p_k = \begin{cases} -g_k & k = 0 \\ \\ -\beta_k g_k + \theta_k p_{k-1} & k \geq 1 \end{cases}$$

such that $\beta_k = -\frac{g_{k-1}^T p_{k-1}}{||g_{k-1}||^2} + \theta_k \frac{g_k^T p_{k-1}}{||g_k||^2}$ is the spectral parameter and θ_k is any conjugate parameter. When $\theta_k = \theta_k^{FR} = \frac{||g_k||^2}{||g_{k-1}||^2}$, we obtain spectral FR conjugate gradient.

Later in 2020, another Spectral Projection Method (SPM) employed to recover signals in (1.1) was suggested by Abubakar et al [5]. Their approach is also based on the Barzilai-Borwein gradient method [15] using the hyperplane projection approach of Solodov and Svaiter [86] which is an extension of the modified method by Liu and Duan [64]. The algorithm proved to be efficient in compressive sensing [5] there in.

The search direction is given by

$$p_k = \begin{cases} -G_k & k = 0 \\ \\ -\beta_k G_k & k \geq 1 \end{cases}$$

such that $\beta_k = \min\{\max\{\tau_{\min}, \tau_k\}, \min\{\tau_{\max}, \tau_k\}\}$, τ_k is the well known spectral parameter and $0 < \tau_{\min} < \tau_{\max}$.

In fact this method and one in [99] are closely related.

In 2023, Wang [94] proposed three-term Spectral CG methods to solve a (2.5) and inverse problems. However, the convergence of his method required function in the unconstrained problem to be strongly convex and I believe this is also another area that requires focus on how to relax such a requirement. This approach used the Hestenes-Stiefel conjugate parameter and p_k is given by

$$p_k = \begin{cases} -g_k & k = 0 \\ -\beta_k g_k + \theta_k p_{k-1} + \delta_k y_{k-1} & k \geq 1 \end{cases}$$

such that $\beta_k = \max\{1, 1 + (r - \frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}}) \frac{g_k^T s_{k-1}}{g_k^T y_{k-1}}\}$.

In 2023, Ibrahim, Alshahrani, and Al-Homidan [50] also proposed another spectral derivative free scheme that solves even when (1.1) is pseudomonotone. They were motivated by the fact that most of the methods that have been proposed to solve (1.1) are suitable when it is monotone [50]. They combined Solodov's and Svaiter's projection method along with the structure of Amini and Faramarzi's method [7]. Ibrahim Alshahrani and Al-Homidan [50] proved that their methods are also sufficient descent (2.3), and global convergence was independent of Lipschitz continuity provided the mild assumptions hold as stated in their work. The numerical results also showed that their methods outperformed the then-existing methods. They adopted the adaptive line search to determine the step length and the search direction is given by

$$p_k = \begin{cases} -G_k & k = 0 \\ -\beta_k G_k + \theta_k p_{k-1} - \eta_k y_{k-1} & k \geq 1 \end{cases}$$

where

$$\beta_k = \Pi_{[a,b]}(\frac{s_{k-1}^T y_{k-1}}{||y_{k-1}||^2}),$$

$$\theta_k = \frac{y_{k-1}^T G_k}{w_k},$$

$$\eta_k = \frac{p_{k-1}^T G_k}{w_k},$$

$$s_{k-1} = x_k - x_{k-1} + \lambda y_{k-1},$$

$$y_{k-1} = G_k - G_{k-1}$$

$$\Pi_{[a,b]}(t) = \max\{\max\{a, t\}, \min\{b, t\}\}$$

$$w_k = \max\{\phi||y_{k-1}||||p_{k-1}||, ||G_{k-1}||^2\},$$

$\phi > 0$ and $\lambda > 0$.

Therefore,

$$p_k = -\Pi_{[a,b]}(\frac{s_{k-1}^T y_{k-1}}{||y_{k-1}||^2})G_k + \frac{y_{k-1}^T G_k}{w_k} p_{k-1} - \frac{p_{k-1}^T G_k}{w_k} y_{k-1}.$$

when $||G_{k-1}||^2 \geq \phi||y_{k-1}||||p_{k-1}||$ for all k, then

$$p_k = -\Pi_{[a,b]}(\frac{s_{k-1}^T y_{k-1}}{||y_{k-1}||^2})G_k + \frac{y_{k-1}^T G_k}{||G_{k-1}||^2} p_{k-1} - \frac{p_{k-1}^T G_k}{||G_{k-1}||^2} y_{k-1}.$$

It was proved that the trust region condition holds and converges globally under mild assumptions, for details see [50]. The method was employed to compressive sensing.

2.2.2 Conjugate gradient projection methods

In this section, the main consideration is the CGPM by Zhang et al [109]

Conjugate gradient methods to (1.1) are defined by

$$x_{k+1} = x_k + \alpha_k p_k,$$

where α_k is determined via different line search methods and

$$p_k = \begin{cases} -G_k & k = 0 \\ \\ -G_k + \theta_k p_{k-1} & k \geq 1 \end{cases} \tag{2.13}$$

θ_k is a parameter that determines the nature of the conjugate gradient method. $\theta_k^{HS} = \frac{G(x_k)^T y_{k-1}}{p_{k-1}^T y_{k-1}}$, $\theta_k^{FR} = \frac{||G(x_k)||^2}{||G(x_{k-1})||^2}$, $\theta_k^{PRP} = \frac{G(x_k)^T y_{k-1}}{||G(x_{k-1})||^2}$, $\theta_k^{DY} = \frac{||G(x_k)||^2}{p_{k-1} y_{k-1}}$ by Hestenes–Stiefel (HS) [49], Fletcher and Reeves (FR) [43], Polak-Ribiére-Polyak (PRP) [79], Dai-Yuan (DY) [30] etc respectively where

$$y_{k-1} = G_{k+1} - G_k \tag{2.14}$$

. For any θ_k, a search direction has to satisfy a descent property by $G(x_k)^T p_k \leq 0$, for all k. The strength of a CGM entirely relies on θ_k.

Xiao and Zhu [100] improved the CGM to (1.1) applied to compressive sensing problem. To enhance the performance, Liu and Li [64] went ahead and studied the algorithm by improving the CGM approach.

As a result, extending Perry [77] and DL(Dai and Lio) [31] conjugate gradient parameters,their updates

$$\theta_k^p = \frac{(y_{k-1} - s_{k-1})^T G_k}{G_{k-1}^T y_{k-1}} \tag{2.15}$$

and

$$\theta_k^{DL} = \frac{G_k^T (y_{k-1} - r s_{k-1})}{G_{k-1}^T y_{k-1}} \tag{2.16}$$

respectively, where $s_{k-1} = x_k - x_{x-1}$ and $r > 0$. These modifications of FR conjugate gradient coefficients resulted in more stable computational performance than exhibited in FRCG [31, 77].

In 2008, Wanyou Cheng [23] proposed an algorithm to solve (1.1). The method combines PRP [79] approach and projection x_k onto H_k (Hyper plane) in (2.4). Because of monotonicity of G in (1.1), if $\bar{x}$ is a solution and $z_k = x_k + \alpha_k p_k$, by monotonicity of (1.1), we have

$$\langle G(z_k), \bar{x} - z_k \rangle \leq 0$$

Assume $x_k \notin \Gamma$, then

$$\langle G(x_k) - G(z_k), x_k - z_k \rangle > 0$$

Then (2.4) strictly separates x_k from Γ and x_{k+1} is obtained from

$$x_{k+1} = x_k - \frac{G(z_k)^T (x_k - z_k) G(z_k)}{||G(z_k)||^2}, \tag{2.17}$$

and

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k p_{k-1} & k \geq 1 \end{cases} \tag{2.18}$$

where $\theta_k = \theta_k^{PRP}$.

In 2009, Li and Li [57] suggested a family of modified PRP (MPRP) derivative-free techniques to solve (1.1) without considering the merit function, and their method is globally convergent. p_k is obtained by

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k^{PRP} p_{k-1} - \beta_k y_{k-1} & k \geq 1 \end{cases}$$

such that $\beta_k = \frac{G_k^T p_{k-1}}{||G_{k-1}||^2}$. α_k is found using an inexact line search and x_{k+1} is obtained using (2.9). This method is a result of the MPRP by Zhang et al [107]. Where p_k is

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k^{PRP}(I - \frac{G_k G_k^T}{||G_k||^1}) p_{k-1} & k \geq 1 \end{cases}$$

This is a two-term PRP technique that was suggested by Cheng [24]

Two decades back, three-term conjugate gradient approaches were invented on a large scope for solving (1.1) [57, 101].

In 2011, Andrei [8] also suggested an MPRP method to solve (2.5). This method was proven to satisfy both the conjugacy and sufficient descent condition (2.3). Numerically it gives promising results when compared to the then existing PRP methods.

In 2013, Ahookhosh et al [6] constructed two new PRP-based conjugate gradient procedures. The search direction p_k is defined by

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \alpha_{k-1}\theta_k p_{k-1} - \beta_k y_{k-1} & k \geq 1 \end{cases}$$

such that $\theta_k = \theta_k^{PRP}$, $\beta_k 1 = \frac{\alpha_{k-1}^2 G_k^T y_{k-1} ||p_{k-1}||^2}{||G_{k-1}||^4}$, $\beta_k 2 = \frac{\alpha_{k-1} G_k^T p_{k-1}}{||G_{k-1}||^2} + \frac{G_k^T y_{k-1} ||y_{k-1}||^2}{||G_{k-1}||^4}$, y_{k-1} defined as in (2.14) and x_{k+1} is computed using (2.17).

In the same year, Al-Baali, Narushima, and Yabe suggested [12] CGM that was employed to solve (2.5), and their methods gave promising outcomes.

In 2014, Zoltan and Sanja [76] proposed an FR-type direction that uses a combination of CG technique and projection approach. The direction is given by

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \alpha_{k-1}\theta_k p_{k-1} - \beta_k y_{k-1} & k \geq 1 \end{cases}$$

such that $\theta_k = \theta_k^{FR}$, $\beta_k 1 = \frac{\alpha_{k-1}^2 G_k^T y_{k-1}||p_{k-1}||^2}{||G_{k-1}||^2}$, $\beta_k 2 = \frac{\alpha_{k-1} G_k^T p_{k-1}}{||G_{k-1}||^2} + \frac{||G_k^{|}|^2}{||G_{k-1}||^4}$ and $y_{k-1} = G_k - G_{k-1}$. x_{k+1} is computed using (2.17). This method is also globally convergent.

In 2015, Dai, Chen, and Wen [32] recommended a derivative-free approach that associates Livieris and Pintela's modified Perry's CGM [69] and the Solodev's hyperplane projection method [86] and they used it to solve (1.1) and p_k is given by

$$p_k = \begin{cases} -G_k & k = 0 \\ -(I + \theta_k^{MP} \frac{G_k^T p_{k-1}}{||G_k||^2})G_k + \theta_k^{MP} p_{k-1} & k \geq 1 \end{cases}$$

where $\theta_k^{MP} = \frac{G_k^T(y_{k-1} - \alpha_{k-1}p_{k-1})}{p_k^T w_{k-1}}$ and $w_{k-1} = y_{k-1} - \gamma\alpha_{k-1}p_{k-1}$ such that $\gamma > 0$.

In 2015, Zhou and Wang [111] proposed a PRP-based residual method to solve (1.1). The approach was determined by swapping the gradients. of the PRP-Conjugate Gradient Method (CGM) with the residuals alongside the projection technique and global convergence was also achieved.

Again in the same year, Sun and Liu [91] proposed modified PRP methods. The three methods emanated from the term in PRP method that was suggested by

Cheng [23]. They are sufficiently descent without any line search and when the exact line search is satisfied, they become the original PRP approach. Furthermore, their convergence is global considering the wolf conditions to hold.

Again in 2018 Liu and Feng [62] gave a derivative-free algorithm and it was proved to be efficient in solving (1.1). The convergence was linear as in [62] and p_k is

$$p_k = \begin{cases} -G_k & k = 0 \\ -\beta_k G_k + \theta_k p_{k-1} & k \geq 1 \end{cases}$$

where $\theta_k = \theta_k^{PDY} = \frac{||G_k||^2}{p_{k-1}^T u_{k-1}}$, $\beta_k = a - \frac{G_k^T p_{k-1}}{p_{k-1}^T u_{k-1}}$ and $u_{k-1} = y_{k-1} + t_{k-1} p_{k-1}$ and $t_{k-1} = \max\{1, 1 - \frac{p_{k-1}^T y_{k-1}}{p_{k-1}^T u_{k-1}}\}$, $a > 0$. x_{k+1} is obtained by the orthogonal projection of $x_k - \frac{G(z_k)^T (x_k - z_k) G(z_k)}{||G(z_k)||^2}$ on to the zeros of (1.1).

In 2020, Dai and Zhu [35] modified the modified Hestenes-Stiefel method and used it to solve (1.1). It combines the famous hyperplane projection in [86] and the modified Hestenes-Stiefel in [34]. The search direction is obtained from

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k^{NHZ} p_{k-1} & k \geq 1 \end{cases}$$

where $\theta_k^{NHZ} = \frac{G_{k-1}^T y_{k-1}}{p_{k-1}^T w_{k-1}} - \eta \frac{||y_{k-1}||^2}{(p_{k-1}^T w_{k-1})^2} G_{k-1}^T p_{k-1}$, $w_{k-1} = y_{k-1} + \gamma \alpha_{k-1} p_{k-1}$ and $\gamma > 0$ and $\eta > 1/4$. Here we observe that there is a relationship between the search directions in [32] and [35].

In the same year, Zheng et al [109] also proposed another technique which was proved globally convergent provided the appropriate line search and projection step are chosen.

pseudo monotonicity and

$$p_k = \begin{cases} -G_k & k = 0 \\ \\ -\gamma G_k + \theta_k p_{k-1} + \tau a_k w_{k-1} & k \geq 1 \end{cases}$$

satisfies (2.3) such that $\gamma, \tau > 0$

$a_k = \frac{G_k^T p_{k-1}}{w_{k-1}^T p_{k-1}}$, $\theta_k = \frac{G_k^T p_{k-1} - 2a_k ||w_{k-1}||^2}{p_{k-1}^T w_{k-1}}$, $w_{k-1} = y_{k-1} + t_{k-1} p_{k-1}$ and $t_{k-1} = \max\{1, 1 - \frac{p_{k-1}^T y_{k-1}}{||p_{k-1}||^2}\}$

Their method was an extension of Al Baali's work together with the projection method to solve (1.1). The global convergence relies on the pseudo monotonicity and Lipschitz continuity of (1.1).

In 2021, Halilu et al [47] invented a two-term CGM coupled with the projection scheme of Solodov and Svaiter [86] to solve (1.1). In 2022, Liu et al [65] came up with a three-term CGPM to solve (1.1) and the direction is sufficiently descent and satisfies the trust region properties. The convergence of the scheme is also independent of the Lipschitz continuity of the function in (1.1). They used this method to restore the sparse signals. Because of the requirement for the function to be Lipschitz continuous for global convergence by some method, it motivated Liu et al to propose another method that is globally convergent and independent of the Lipschitz continuity.

In 2023, Ibrahim, Kimiaei, and Kumam [52] suggested a CGM. It is a result of Dai–Liao method. The global convergence was confirmed under a mild assumption of the function to be monotone.

Recently in 2024, Chankong et al [19] developed a class of CGPM that extends Wang's approach to solving (1.1) without using derivatives or its regularity. However, the convergence requires the function to be Lipschitz continuous and this also could be another area that may need focus.

2.2.3 Perry Conjugate Method

In 1977, Perry [77] a modified FR conjugate gradient method. The modification proved to be stable in computational performance and this was attributed to the replacement of the Hessian by the Newton-like matrix update of rank one and

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k p_{k-1} & k \geq 1 \end{cases} \tag{2.19}$$

where $\theta_k = \frac{(G_k - G_{k-1} - \alpha_{k-1} p_{k-1})^T G_k}{(G_k - G_{k-1})^T p_{k-1}}$

In 2013, Liu and Xu developed various Symmetric Perry CGM to resolve unconstrained problems.

In 2015, Dai, Chen, and Wen [32] suggested a derivative-free method for (1.1) resulting from modified Perry's CGM and projection approach. Their approach converges globally as long as the G in (1.1) is monotone and Lipschitz continuous. Where

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k^{MP} \frac{G_k^T p_{k-1}}{||G_K||^2} G_k + \theta_k^{MP} p_{k-1} & k \geq 1 \end{cases}$$

and $\theta_k^{MP} = \frac{G_k^T (y_{k-1} - s_{k-1}}{p_{k-1}^T v_{k-1}}$, $v_{k-1} = y_{k-1} + r s_{k-1}$, $s_{k-1} = z_{k-1} - x_{k-1}$, $z_{k-1} = x_{k-1} + \tau_{k-1} p_{k-1}$, $y_{k-1} = G(z_{k-1}) - G_{k-1}$.

In 2016, Livieris and Pintelas developed another Perry CGM and proved its convergence. This method corrects any loss of orthogonality that may occur due to an ill-conditioned problem, which can affect the method's convergence [68].

In 2017, Andrei [9] suggested an adaptive Perry CGM and applied the ideas of the Newton-like methods. The scaled Perry conjugate gradient is given by

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k s_{k-1} + \beta_k y_{k-1} & k \geq 1 \end{cases}$$

where $\theta_k = \frac{y_{k-1}^T G_k - \tau_k s_{k-1}^T G_k}{y_{k-1}^T s_{k-1}}$, $\beta_k = \frac{s_{k-1}^T G_k}{y_{k-1}^T s_{k-1}}$, $\tau_k = 1 + t_k(\frac{||y_{k-1}||^2}{y_{k-1}^T s_{k-1}} - \frac{y_{k-1}^T s_{k-1}}{||s_{k-1}||^2}) + \frac{y_{k-1}^T s_{k-1}}{||s_{k-1}||^2}$ and t_k is the usual spectral parameter. The convergence analysis of this approach was later corrected by Ou [74].

Later in 2018, Abubakar and Kumam [2] recommended an algorithm that was an extension of the Dai-Liao extension by Babaie and Reza [13]. This method is globally convergent and

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k p_{k-1} & k \geq 1 \end{cases},$$

where

$$\theta_k = \frac{G_k^T(y_{k-1} - \zeta_k s_{k-1})}{p_{k-1}^T y_{k-1}}$$

$$\zeta_k = o\frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}} - r\frac{s_{k-1}^T y_{k-1}}{||s_{k-1}||^2}$$

such that $o \geq 1/4,\ r \leq 0,\ s_{k-1}$ and y_{k-1} as in (2.6) and (2.14) respectively.

For various values of o and r, the search direction changes to different search directions. see [2]. The step length is obtained using the basic line search also known as the backtracking method. Because the basic line search may not guarantee to obtain x_k such that $\|G(x_k)\| - \|G(x_{k-1})\| \leq 0$, the convergence may be very slow hence making the algorithm very inefficient. To solve this problem, Ibrahim et al [52] devised a new derivative-free projection approach by introducing another search direction defined by

$$p_k = \begin{cases} -G_k & k = 0 \\ -G_k + \theta_k p_{k-1} & k \geq 1 \end{cases}$$

such that

$$\theta_k = \frac{G_k^T(y_{k-1} - \zeta_k s_{k-1})}{p_{k-1}^T y_{k-1}^*}$$

$$\zeta_k = o\frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}} - r\frac{s_{k-1}^T y_{k-1}^*}{||s_{k-1}||^2}$$

$$j_{k-1} = \max\{1, 1 - \frac{y_{k-1}^T p_{k-1}}{||p_{k-1}||^2}\}$$

$$y_{k-1}^* = y_{k-1} + j_{k-1} p_{k-1}$$

such that $o > 0$, $r \leq 0$, $s_{k-1} = x_k - x_{k-1}$ and $y_{k-1} = G_k - G_{k-1}$

Yao, He, and Shi upgraded the Perry CGM using adaptive parameter choice. They proposed an advanced Perry update matrix. The goal was to suppress the non-symmetric Perry matrix and

$$p_k = \begin{cases} -G_k & k = 0 \\ -\theta_k G_k & k \geq 1 \end{cases}$$

where $\theta_k = I - \frac{s_{k-1} y_{k-1}^T}{y_{k-1}^T s_{k-1}} - \frac{y_{k-1} s_{k-1}^T}{y_{k-1}^T s_{k-1}} - t - \frac{s_{k-1} y_{k-1}^T}{s_{k-1}^T s_{k-1}}$ and $t = 2\frac{y_{k-1}^T y_{k-1}}{y_{k-1}^T s_{k-1}}$

Perry CGM has undergone various modifications and improvements to solve both regular and non-differentiable optimization problems [60, 61]

In 2019, Waziri et al [97] also proposed an adaptive class of Perry CGM to solve (1.1). They are a combination of the modified versions of Perry conjugate methods and projection technique and the convergent is global provided some conditions are satisfied. The parameter θ_k^{MP} that is adaptive is given by

$$\theta_k^{MP} = \frac{G_k^T(z_{k-1} - \zeta_k s_{k-1})}{p_{k-1}^T z_{k-1}} \tag{2.20}$$

$$\zeta_k = o\frac{||z_{k-1}||^2}{s_{k-1}^T z_{k-1}} - r\frac{s_{k-1}^T z_{k-1}}{||s_{k-1}||^2} \tag{2.21}$$

such that $o > 1/4$, $r < 1/4$, $s_{k-1} = x_k - x_{k-1}$ and $z_{k-1} = y_{k-1} + \rho\frac{\max\{0, \tau_{k-1}\}}{s_{k-1}^T v_{k-1}} v_{k-1}$, and $\rho \in (0,3)$, v_{k-1} is any vector and $\tau_{k-1} = 2s_{k-1}^T(G_k + G_{k-1}) + 2(g_{k-1} - g_k)$, g is the merit function as defined in (2.2)

In 2021, Awwal et al [11] recommended a Perry-type derivative method and used it to solve (1.1). The scheme depends on the BFGS method with an upgraded scheme of Perry's parameter. For this,

$$p_k = \begin{cases} -G_k & k = 0 \\ -\gamma_k G_k + \theta_k s_{k-1} + \beta_k \rho_{k-1} & k \geq 1 \end{cases}$$

where $\gamma_k = a + 2\frac{G_k^T \rho_{k-1} G_k^T s_{k-1}}{||G_k||^t \tau_{k-1}^T s_{k-1}}$, $\rho_{k-1} = G_k - G_{k-1} + r s_{k-1}$, $r > 0$, $\tau_{k-1} = \rho_{k-1} + (\max\{1, 1 - \frac{\rho_{k-1}^T s_{k-1}}{||s_{k-1}||^2}\}) s_{k-1}$, $\theta_k = \frac{G_k^T(\rho_{k-1} - s_{k-1})}{\tau_{k-1}^T s_{k-1}}$, $\beta_k = \frac{s_{k-1}^T G_k}{\tau_{k-1}^T s_{k-1}}$ and $a > 0$.

The major drawback of Wazir's method was the reliance on o and r in ζ_k of (2.21). This motivated Sabi'u et al [83] to propose another two-term Perry CGM as a result of obtaining the optimal choice of ζ_k. This was achieved by minimizing the difference between the large eigenvalue and the smallest eigenvalue of the update matrix associated with the search direction and this got rid of dependence on the parameters o and r. Therefore,

$$p_k = \begin{cases} -G_k & k = 0 \\ \\ -M_k G_k + \theta_k^{MOP} p_{k-1} & k \geq 1 \end{cases}$$

such that $M_k = I - \theta_k^{MOP} \frac{G_k^T p_{k-1}}{||G_k||^2}$ and $\theta_k^{MOP} = \frac{(y_{k-1} - \zeta_k^* s_{k-1})^T G_k}{p_{k-1}^T y_{k-1}}$ and $\zeta_k^* = \frac{s_{k-1}^T y_{k-1}}{||s_{k-1}||^2}$. The method is globally convergent provided the problem in (1.1) is monotone and Lipschitz continuous. The method gave promising results in compressive sensing. This method relied on Kafaki and Ghanbari's work which is an inclusion of Hager and Zhang's formula of 2015 [46] and Dai and Kou's formula [29] as special cases. Using Dai and Liao's (DL) method of (2.5) in [1] whose conjugate gradient parameter is given by

$$\beta_k^{DL} = \frac{g_k^T y_{k-1}}{p_{k-1}^T y_{k-1}} - \zeta \frac{g_k^T s_{k-1}}{p_{k-1}^T y_{k-1}} \tag{2.22}$$

where g_k is the gradient of the unconstrained optimization problem, s_{k-1} and y_{k-1} in (2.6) and (2.7) respectively and for sure when $\zeta = 0$, we arrive at $\beta_k^{HS} = \frac{g_k^T y_{k-1}}{p_{k-1}^T y_{k-1}}$ due to Hestenes and Stiefel [49]. Also in comparison with Hager and Zhang's (HZ) conjugate parameter

$$\beta_k^{HZ} = \frac{g_k^T y_{k-1}}{p_{k-1}^T y_{k-1}} - 2 \frac{||y_{k-1}||^2}{p_{k-1}^T y_{k-1}} \frac{g_k^T p_{k-1}}{p_{k-1}^T y_{k-1}} \tag{2.23}$$

such that $\zeta = 2 \frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}}$. DL method is regarded to be highly numerically efficient, however, sometimes it does not achieve a descent direction and it also depends on the parameter ζ which has no optimal value. This was the driving force for Kafaki and Ghanbari to suggest their methods in [13] to determine the appropriate values for ζ. This method was later adopted by Waziri et al and extended to solve (1.1) with the conjugate parameter as defined in (2.20).

Therefore, Sabi'u et al used this approach and extended it to determine the best value $t_k = \zeta$ and solved (1.1). From (2.19) and its parameter, we obtain

$$p_k = -Q_k G_k \tag{2.24}$$

such that $Q_k = I - \frac{y_{k-1}s_{k-1}^T}{y_{k-1}^T s_{k-1}} + t_k \frac{s_{k-1}s_{k-1}^T}{y_{k-1}^T s_{k-1}}$ and $y_{k-1} = w_{k-1} + \gamma s_{k-1}$, $\gamma \in (0,1)$ and $w_{k-1} = G_k - G_{k-1}$, s_{k-1} defined as in (2.6). To ensure the Q_k is symmetric, we introduce $\tilde{Q}_k = (Q_k^T + Q_k)/2$ such that (2.24) becomes

$$p_k = -\tilde{Q}_k G_k \tag{2.25}$$

such that $\tilde{Q}_k = I - \frac{y_{k-1}s_{k-1}^T}{2y_{k-1}^T s_{k-1}} - \frac{s_{k-1}y_{k-1}^T}{2y_{k-1}^T s_{k-1}} + t_k \frac{s_{k-1}s_{k-1}^T}{y_{k-1}^T s_{k-1}}$. As (1.1) is monotone, then $y_{k-1}^T s_{k-1} > 0$, thus y_{k-1} and s_{k-1} are not zero vectors. So, there is some set of mutually orthogonal vectors $\{u_k^i\}_{i=1}^{n-2}$ for which $s_k^T u_k^i = y_k^T u_k^i = 0$ for avery $i \in \{1, \cdots n-2\}$ and $||u_k^i||= 1$. Consequently

$$\tilde{Q}_k u_k^i = u_k^i, \qquad \forall i \in \{1, \cdots, n-2\}$$

Implying that $\{u_k^i\}$ are eigenvectors of $\tilde{Q}$ with the corresponding characteristic root 1 for each u_k^i. So, we can determine the remaining two eigenvectors $\tilde{Q}_k$ say η_k^+ and η_k^-. We know that the $tr(\tilde{Q})$ of a square square matrix is the sum of its eigenvalues. Therefore

$$tr(\tilde{Q}_k) = n - 1 + t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}}$$

hence

$$n - 2 + \eta_k^- + \eta_k^+ = n - 1 + t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}}$$

therefore

$$\eta_k^- + \eta_k^+ = 1 + t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}} \tag{2.26}$$

Using Frobenius norm on $\tilde{Q}^T \tilde{Q}$, we have

$$tr(\tilde{Q}_k^T \tilde{Q}_k) = n - 3/2 + \frac{||s_{k-1}||^2 ||y_{k-1}||2}{2(y_{k-1}^T s_{k-1})^2)} + t_k^2 \frac{||s_{k-1}||^4}{(y_{k-1}^T s_{k-1})^2}$$

and so

$$n - 2 + {\eta_k^-}^2 + {\eta_k^+}^2 = n - 3/2 + \frac{||s_{k-1}||^2 ||y_{k-1}||2}{2(y_{k-1}^T s_{k-1})^2)} + t_k^2 \frac{||s_{k-1}||^4}{(y_{k-1}^T s_{k-1})^2}$$

then

$${\eta_k^-}^2 + {\eta_k^+}^2 = 1/2 + \frac{||s_{k-1}||^2 ||y_{k-1}||2}{2(y_{k-1}^T s_{k-1})^2)} + t_k^2 \frac{||s_{k-1}||^4}{(y_{k-1}^T s_{k-1})^2} \tag{2.27}$$

Now from (2.26) and (2.27), we obtain

$$\eta_k^- \eta_k^+ = \frac{1}{4} + t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}} - \frac{1}{4} \frac{||s_{k-1}||^2 ||y_{k-1}||^2}{(y_{k-1}^T s_{k-1})^2} \tag{2.28}$$

Using (2.26) and (2.27) we get

$$\eta_k^{\pm} = \frac{(1 + t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}}) \pm \sqrt{(t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}} - 1)^2 + \frac{||s_{k-1}||^2 ||y_{k-1}||^2}{(y_{k-1}^T s_{k-1})^2} - 1}}{2} \tag{2.29}$$

we find that $\eta_k^+ > 0$ and whenever $t_k > \frac{1}{4}(\frac{||y_{k-1}||^2}{y_{k-1}^T s_{k-1}} - \frac{y_{k-1}^T s_{k-1}}{||s_{k-1}||^2})$, then $\eta^- > 0$. Then $\tilde{Q}_k$ is positive definite which ensures the sufficiency descent condition is satisfied. Sabi'u et al minimized the square of the difference difference $(\eta^+ - \eta^-)^2$ to get t_k. But

$$(\eta^+ - \eta^-)^2 = (t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}} - 1)^2 + \frac{||s_{k-1}||^2 ||y_{k-1}||^2}{(y_{k-1}^T s_{k-1})^2} - 1$$

thus $t_k^* = \frac{s_{k-1}^T y_{k-1}}{||s_{k-1}||^2}$ is the optimal value of the parameter t_k and thus obtaining θ_k^{MOP}.

2.3 Inertial derivative-free methods

The initial method was suggested in 1964 by Polyak [78] to solve a continuously differentiable convex optimization problem. The inertial step is added into an iterative scheme purposely to increase the speed of convergence. Various studies have proved that iterative schemes that incorporated the inertial step to solve nonlinear problems exhibited better numerical performance basically CPU time and number of iterations than those without that step. This has moved mathematicians to develop various iterative methods with the inertial step. (see, for instance [84, 81, 93, 40, 10, 50] and the references there in). However, there is no sufficient literature regarding the study of the impact of the inertial step on conjugate gradient algorithms to solve nonlinear systems of (1.1). This approach has recently gained interest as evidenced by the literature.

In 2021, by the motivation of the inertial technique due to its role in the increase of the convergence rate, Abubakar et al [3] proposed a derivative-free approach combined an inertial step to solving (1.1) and it gave promising results. Inertial extrapolation is an approach in which an extra step called the inertial step is included in the existing step(s) of an iterative method [3]. If we have two initial points x_o and x_1, then the inertial term can be defined by

$$y_k = x_k + \phi_k(x_k - x_{k-1})$$

such that the sequence $\{\phi_k\}_{k=1}^{\infty}$ satisfies some condition. Many methods have been proposed combining the Inertial step with the DFPM. see [3, 54]. The crosscutting feature among these techniques is that they are stable. Nonetheless, their global convergence was determined after assuming that G is monotone and Lipschitz continuous [50]. To suppress these assumptions, Yin et al. [102] developed a family derivative-free inertial methods to solve (1.1) and proved the global convergence by neglecting the Lipschitz continuity of the underlying mapping.

Their work, Abubakar et al [3] used the projection Solodov and Svaiter's approach in [86] coupled with Brazilian and Bowein-like spectral parameters suggested by Abubakar et al in [5]. Abubakar and others applied the framework of Amini and Faramarzi's CGM in [7].

In one of the schemes they proposed, Ibrahim et al [50] Incorporated an inertial technique which is regarded to be a realistic technique that expedites an optimization method, and indeed its application roots back to the heavy ball method [78]. The inertial approach has captured the interest of researchers [50] and has been widely used in convex optimization problems, especially when the objective function is smooth [92, 56]. Concerning that, the two methods spectral conjugate gradient and Inertial technique have been extended to solve nonconvex but smooth unconstrained problems [98].

CHAPTER 3

GENERALIZED DERIVATIVE-FREE METHODS

In this Chapter, we give a generalized result for the Perry conjugate gradient and Conjugate gradient projection techniques.

We also detail the convergence analysis of algorithms provided the following assumptions hold:

(A1) Γ is nonempty

(A2) G is Lipschitz continuous. Then there exists some nonzero constant L such that for any $x, y \in \mathbf{R}^n$ then $||G(y) - G(x)|| \leq L||y - x||$

3.1 Generization of MOPCGM

From $s_{k-1} = x_k - x_{k-1}$, $g_k = \nabla G(x_k)$ and $y_{k-1} = g_k - g_{k-1}$. Also assume that B_{k-1} is positive-definite, then

$$B_k = B_{k-1} - \frac{B_{k-1}s_{k-1}(B_{k-1}s_{k-1})^T}{s_{k-1}^T B_{k-1}s_{k-1}} + \frac{y_{k-1}y_{k-1}}{s_{k-1}^T y_{k-1}}$$

It is a rank 2 matrix update called DFP.

Now, using the Quasi-Newton approach as used in [77, 83] we propose

$$p_k = -\tilde{Q}_k G_k \tag{3.1}$$

such that

$$\tilde{Q}_k = \lambda I - \lambda \frac{y_{k-1} s_{k-1}^T}{2 y_{k-1}^T s_{k-1}} - \lambda \frac{s_{k-1} y_{k-1}^T}{2 y_{k-1}^T s_{k-1}} + t_k \frac{s_{k-1} s_{k-1}^T}{y_{k-1}^T s_{k-1}} \tag{3.2}$$

$\lambda > 0$ and $t_k > 0$. Because G is monotone, then $s_{k-1}^T y_{k-1} > 0$ for all $x \neq x^*$. This implies that both s_{k-1} and y_{k-1} are nonzero vectors. Let D be spanned by $\{s_{k-1}, y_{k-1}\}$ and a be any vector in $\mathbb{R}^n$ such that $a^T D \neq 0$, then

$$a^T \tilde{Q}_k a = t_k \frac{(a^T s_{k-1})^2}{y_{k-1}^T s_{k-1}} > 0$$

and this shows that $\tilde{Q}_k$ is positive definite.

But $\tilde{Q}_k$ is rank 2 matrix update, then eigenvalue λ is of multiplicity $n-2$. Now we need to determine the other two eigenvalues η_k^- and η_k^+ since $\tilde{Q}_k$ is full rank based on the symmetric property of $\tilde{Q}_k$. Therefore we can find a set of vectors that are mutually orthogonal $\{u_k^i\}_{i=1}^{n-2}$ such that

$$\tilde{Q}_k u_k^i = \lambda u_k^i, \quad i = 1, \cdots, n-2$$

and satisfy ${u_k^i}^T D = 0$ for $i = 1, \cdots, n-2$.

Therefore $\{u_k^i\}_{i=1}^{n-2}$ are eigenvectors of $\tilde{Q}_k$ with the corresponding eigenvalue λ for every u_k^i. We can now determine the remaining 2 eigenvalues of $\tilde{Q}_k$ that is η_k^+ and η_k^-. The following lemmas are crucial.

Lemma 3.1 *Let $\tilde{Q}_k$ be defined as in (3.2), then*

$$\boldsymbol{tr}(\tilde{Q}_k) = \lambda(n-1) + t_k \frac{||s_{k-1}||^2}{s_{k-1}^T y_{k-1}}$$

Proof. Using the linearity property of the trace of a matrix, it is the sum of all eigenvalues of a matrix. So we have

$$\mathbf{tr}(\tilde{Q}_k) = \mathbf{tr}(\lambda I) - \mathbf{tr}(\lambda \frac{y_{k-1}s_{k-1}^T}{2y_{k-1}^T s_{k-1}}) - \mathbf{tr}(\lambda \frac{s_{k-1}y_{k-1}^T}{2y_{k-1}^T s_{k-1}}) + \mathbf{tr}(t_k \frac{s_{k-1}s_{k-1}^T}{y_{k-1}^T s_{k-1}})$$

but $\mathbf{tr}(\lambda I) = n\lambda$. Let $A = y_{k-1}s_{k-1}^T$, $A^T = s_{k-1}y_{k-1}^T$ and $B = s_{k-1}s_{k-1}^T$ then $\mathbf{tr}(A) = \mathbf{tr}(A^T) = y_{k-1}^T s_{k-1}$ and $\mathbf{tr}(B) = ||s_{k-1}||^2$. Then

$$\mathbf{tr}(\tilde{Q}_k) = \lambda(n-1) + t_k \frac{||s_{k-1}||^2}{y_{k-1}^T s_{k-1}}$$

∎

Lemma 3.2 *Let $\tilde{Q}_k$ be defined as in (3.2), then*

$$\boldsymbol{tr}(\tilde{Q}_k^T \tilde{Q}_k) = \lambda^2(n - \frac{3}{2}) + \frac{\lambda^2}{2} \frac{||s_{k-1}||^2 ||y_{k-1}||^2}{(s_{k-1}^T y_{k-1})^2} + t_k^2 \frac{||s_{k-1}||^4}{(s_{k-1}^T y_{k-1})^2}$$

Proof. From

$$\tilde{Q}_k^T \tilde{Q}_k = \left(\lambda I - \lambda \frac{y_{k-1}s_{k-1}^T}{2y_{k-1}^T s_{k-1}} - \lambda \frac{s_{k-1}y_{k-1}^T}{2y_{k-1}^T s_{k-1}} + t_k \frac{s_{k-1}s_{k-1}^T}{y_{k-1}^T s_{k-1}}\right)^T \left(\lambda I - \lambda \frac{y_{k-1}s_{k-1}^T}{2y_{k-1}^T s_{k-1}} - \lambda \frac{s_{k-1}y_{k-1}^T}{2y_{k-1}^T s_{k-1}} + t_k \frac{s_{k-1}s_{k-1}^T}{y_{k-1}^T s_{k-1}}\right)$$

$$\tilde{Q}_k^T\tilde{Q}_k = \left(\lambda I - \lambda\frac{s_{k-1}y_{k-1}^T}{2y_{k-1}^Ts_{k-1}} - \lambda\frac{y_{k-1}s_{k-1}^T}{2y_{k-1}^Ts_{k-1}} + t_k\frac{s_{k-1}s_{k-1}^T}{y_{k-1}^Ts_{k-1}}\right)$$
$$\left(\lambda I - \lambda\frac{y_{k-1}s_{k-1}^T}{2y_{k-1}^Ts_{k-1}} - \lambda\frac{s_{k-1}y_{k-1}^T}{2y_{k-1}^Ts_{k-1}} + t_k\frac{s_{k-1}s_{k-1}^T}{y_{k-1}^Ts_{k-1}}\right)$$

$$\tilde{Q}_k^T\tilde{Q}_k = \lambda^2 I - \lambda^2\frac{3s_{k-1}y_{k-1}^T}{4y_{k-1}^Ts_{k-1}} - \lambda^2\frac{3y_{k-1}s_{k-1}^T}{4y_{k-1}^Ts_{k-1}} + t_k\lambda\frac{s_{k-1}s_{k-1}^T}{y_{k-1}^Ts_{k-1}} + \lambda^2\frac{||s_{k-1}||^2y_{k-1}y_{k-1}^T}{4(y_{k-1}^Ts_{k-1})^2}$$
$$-t_k\lambda\frac{||s_{k-1}||^2y_{k-1}s_{k-1}^T}{2(y_{k-1}^Ts_{k-1})^2}+\lambda^2\frac{||y_{k-1}||^2s_{k-1}s_{k-1}^T}{4(y_{k-1}^Ts_{k-1})^2}-\lambda t_k\frac{||s_{k-1}||^2s_{k-1}y_{k-1}^T}{2(y_{k-1}^Ts_{k-1})^2}+t_k^2\frac{||s_{k-1}||^2s_{k-1}s_{k-1}^T}{(y_{k-1}^Ts_{k-1})^2}$$

we have seen that $\mathbf{tr}(y_{k-1}s_{k-1}^T) = y_{k-1}^Ts_{k-1}$ and $\mathbf{tr}(s_{k-1}s_{k-1}^T) = ||s_{k-1}||^2$. Therefore

properties of trace of matrices

$$\mathbf{tr}(\tilde{Q}_k^T\tilde{Q}_k) = \mathbf{tr}(\lambda^2 I)-\mathbf{tr}(\lambda^2\frac{3s_{k-1}y_{k-1}^T}{4y_{k-1}^Ts_{k-1}})-\mathbf{tr}(\lambda^2\frac{3y_{k-1}s_{k-1}^T}{4y_{k-1}^Ts_{k-1}})+\mathbf{tr}(t_k\lambda\frac{s_{k-1}s_{k-1}^T}{y_{k-1}^Ts_{k-1}})+\mathbf{tr}(\lambda^2\frac{||s_{k-1}||^2y_{k-1}y_{k-1}^T}{4(y_{k-1}^Ts_{k-1})^2})$$
$$-\mathbf{tr}(t_k\lambda\frac{||s_{k-1}||^2y_{k-1}s_{k-1}^T}{2(y_{k-1}^Ts_{k-1})^2})+\mathbf{tr}(\lambda^2\frac{||y_{k-1}||^2s_{k-1}s_{k-1}^T}{4(y_{k-1}^Ts_{k-1})^2})-\mathbf{tr}(\lambda t_k\frac{||s_{k-1}||^2s_{k-1}y_{k-1}^T}{2(y_{k-1}^Ts_{k-1})^2})+\mathbf{tr}(t_k^2\frac{||s_{k-1}||^2s_{k-1}s_{k-1}^T}{(y_{k-1}^Ts_{k-1})^2})$$

Therefore

$$\mathbf{tr}(\tilde{Q}_k^T\tilde{Q}_k) = n\lambda^2 - \lambda^2\frac{3}{4} - \lambda^2\frac{3}{4} + t_k\lambda\frac{||s_{k-1}||^2}{y_{k-1}^Ts_{k-1}} + \lambda^2\frac{||s_{k-1}||^2||y_{k-1}||^2}{4(y_{k-1}^Ts_{k-1})^2}$$
$$- t_k\lambda\frac{||s_{k-1}||^2}{2(y_{k-1}^Ts_{k-1})} + \lambda^2\frac{||y_{k-1}||^2||s_{k-1}||^2}{4(y_{k-1}^Ts_{k-1})^2} - \lambda t_k\frac{||s_{k-1}||^2}{2(y_{k-1}^Ts_{k-1})} + t_k^2\frac{||s_{k-1}||^4}{(y_{k-1}^Ts_{k-1})^2}$$

$$\mathbf{tr}(\tilde{Q}_k^T\tilde{Q}_k) = \lambda^2(n - \frac{3}{2}) + \lambda^2\frac{||s_{k-1}||^2||y_{k-1}||^2}{2(y_{k-1}^Ts_{k-1})^2} + t_k^2\frac{||s_{k-1}||^4}{(y_{k-1}^Ts_{k-1})^2}$$

for all λ. ∎

Lemma 3.3 *The product of eigenvalues η^+ and η^- is given by* $\eta^+\eta^- = \frac{\lambda^2}{4} - \frac{\lambda^2}{4}\left(\frac{||s_{k-1}||||y_{k-1}||}{s_{k-1}^T y_{k-1}}\right)^2 + \lambda t_k \frac{||s_{k-1}||^2}{s_{k-1}^T y_{k-1}}$.

Proof. Using Lemma 3.1 and the usual sum of all eigenvalues as the trace of a matrix, we get

$$\lambda(n-2) + \eta_k^- + \eta_k^+ = \lambda(n-1) + t_k \frac{||s_{k-1}||^2}{s_{k-1}^T y_{k-1}}.$$

Also using Lemma 3.2, we obtain

$$\lambda^2(n-2) + {\eta_k^-}^2 + {\eta_k^+}^2 = \lambda^2(n - \frac{3}{2}) + \frac{\lambda^2}{2}\frac{||s_{k-1}||^2||y_{k-1}||^2}{(s_{k-1}^T y_{k-1})^2} + t_k^2 \frac{||s_{k-1}||^4}{(s_{k-1}^T y_{k-1})^2}.$$

Implying that

$$ {\eta_k^-}^2 + {\eta_k^+}^2 = \frac{\lambda^2}{2} + \frac{\lambda^2}{2}\frac{||s_{k-1}||^2||y_{k-1}||^2}{(s_{k-1}^T y_{k-1})^2} + t_k^2 \frac{||s_{k-1}||^4}{(s_{k-1}^T y_{k-1})^2}. \tag{3.3}$$

Let $a = \frac{||s_{k-1}||^2}{s_{k-1}^T y_{k-1}}$ and $b = \frac{||s_{k-1}||||y_{k-1}||}{s_{k-1}^T y_{k-1}}$, then we obtain

$$\eta_k^- + \eta_k^+ = \lambda + a t_k \tag{3.4}$$

and

$${\eta_k^-}^2 + {\eta_k^+}^2 = \frac{\lambda^2}{2} + \frac{\lambda^2}{2} b^2 + a^2 t_k^2 \tag{3.5}$$

respectively.

Now from (3.4) and (3.5), we obtain

$$\eta_k^- \eta_k^+ = \frac{\lambda^2}{4} - \frac{\lambda^2}{4} b^2 + \lambda a t_k. \tag{3.6}$$

Remark 3.1 *when $\lambda = 1$, then we obtain*

$$\eta_k^- \eta_k^+ = \frac{1}{4} - \frac{1}{4} b^2 + a t_k$$

which is the case in MOPCGM [83].

∎

Lemma 3.4 *Let $\tilde{Q}_k$ be defined as in (3.2), then the optimal value of t is $t_k^* = \lambda \frac{s_{k-1}^T y_{k-1}}{||s_{k-1}||^2}$.*

From (3.4) and (3.6), we obtain

$$\eta_k^2 - (\lambda + at)\eta_k + \left[\frac{\lambda^2}{4} - \frac{\lambda^2}{4} b^2 + a t_k \lambda\right] = 0. \tag{3.7}$$

This means

$$\eta_k^{\pm} = \frac{(\lambda + a t_k) \pm \sqrt{(\lambda + at)^2 - 4(\frac{\lambda^2}{4} - \frac{\lambda^2}{4} b^2 + a t_k \lambda)}}{2}. \tag{3.8}$$

For positive definiteness of the matrix $\tilde{Q}_k$, by applying some algebra, $t > \frac{\lambda}{4a}(b^2 - 1)$. Consequently, we obtain

$$t > \frac{\lambda}{4}\left(\frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}} - \frac{s_{k-1}^T y_{k-1}}{||s_{k-1}||^2}\right).$$

Now to obtain the optimal value of t_k, we minimize the square of the difference between η_k^+ and η_k^- so that the condition number is small, then

$$(\eta_k^+ - \eta_k^-)^2 = (\lambda + at)^2 - 4(\frac{\lambda^2}{4} - \frac{\lambda^2}{4}b^2 + at_k\lambda). \tag{3.9}$$

Therefore, the optimal value of t is $\frac{\lambda}{a}$.

That is

$$t_k^* = \lambda \frac{s_{k-1}^T y_{k-1}}{||s_{k-1}||^2}.$$

Remark 3.2 *For $\lambda = 1$, we obtain the Optimal Perry parameter $t_k^* = \frac{s_{k-1}^T y_{k-1}}{||s_{k-1}||^2}$ obtained by Subu'i et al [83].*

We propose a new search direction is given by

$$p_k = \begin{cases} -G_k & k = 0 \\ -M_k G_k + \theta_k^{GMOP} p_{k-1} & k \geq 1 \end{cases} \tag{3.10}$$

such that $M_k = \lambda + \theta_k^{GMOP} \frac{G_k^T p_{k-1}}{||G_k||^2}$ and $\theta_k^{GMOP} = \frac{(v_{k-1} - t_k^* s_{k-1})^T G_k}{p_{k-1}^T v_{k-1}}$ and $t_k^* = \lambda \frac{s_{k-1}^T v_{k-1}}{||s_{k-1}||^2}$, $\lambda > 0$, $v_k = y_{k-1} + \tau s_{k-1}$ and $\tau > 0$. (2.3).

Remark 3.3 *To adaptively choose λ depending on the problem and the current iterate, we define λ_k as the projection of two quantities onto the interval $[\alpha_{\min}, \alpha_{\max}]$, where $\alpha_{\min} > 0$. This ensures that λ_k remains bounded between $\alpha_{\min}$ and $\alpha_{\max}$, adjusting itself dynamically based on the progress of the optimization.*

We define:

$$\lambda_k = \Pi_{[\alpha_{\min}, \alpha_{\max}]} \left(\frac{\|v_{k-1}\|^2}{s_{k-1}^\top v_{k-1}}, \frac{s_{k-1}^\top v_{k-1}}{\|s_{k-1}\|^2} \right),$$

The projection operator $\Pi_{[\alpha_{\min}, \alpha_{\max}]}(\cdot)$ projects the values onto the interval $[\alpha_{\min}, \alpha_{\max}]$.

$\frac{\|v_{k-1}\|^2}{s_{k-1}^\top v_{k-1}}$ *measures how much the* $G(x)$ *changes relative to the step size. A large value indicates a significant change in gradients, suggesting that a larger* λ_k *may be beneficial.*

$\frac{s_{k-1}^\top v_{k-1}}{\|s_{k-1}\|^2}$ *measures the alignment between the step and the change in* $G(x)$*, relative to the step length. A smaller value suggests a smaller* λ_k*.*

$$\lambda_k = \Pi_{[\alpha_{\min},\alpha_{\max}]}\left(\max\left(\frac{\|v_{k-1}\|^2}{s_{k-1}^\top v_{k-1}}, \frac{s_{k-1}^\top v_{k-1}}{\|s_{k-1}\|^2}\right)\right) \tag{3.11}$$

This ensures that λ_k *is bounded within* $[\alpha_{\min}, \alpha_{\max}]$*, with* $\alpha_{\min} > 0$ *to prevent the step size from becoming too small.*

- $\alpha_{\min}$ *ensures that* λ_k *does not become too small, which could cause slow convergence.* - $\alpha_{\max}$ *limits* λ_k *from growing too large, preventing instability.*

This adaptive scheme allows λ_k *to tune itself according to the characteristics of the problem, without requiring manual adjustments for each new optimization problem. The dynamic nature of* λ_k *ensures a balance between stability and fast progress.*

Using (3.11), (3.10) becomes

$$p_k = \begin{cases} -G_k & k = 0 \\ -M_k G_k + \theta_k^{GMOP} p_{k-1} & k \geq 1 \end{cases} \tag{3.12}$$

such that $M_k = \lambda_k + \theta_k^{GMOP} \frac{G_k^T p_{k-1}}{||G_k||^2}$ and

$$\theta_k^{GMOP} = \frac{(v_{k-1} - t_k^* s_{k-1})^T G_k}{p_{k-1}^T v_{k-1}}, \tag{3.13}$$

$$t_k^* = \lambda_k \frac{s_{k-1}^T v_{k-1}}{||s_{k-1}||^2}, \tag{3.14}$$

and $\lambda_k > 0$ defined in (3.3), $v_k = y_{k-1} + \tau s_{k-1}$ and $\tau > 0$. In lemma 3.5, we verified that the p_k of (3.10) satisfies (2.3).

Lemma 3.5 *Let $\{p_k\}$ and G_k be produced by algorithm 1. Let $\alpha_{min} > 0$, then $G_k^T p_k \leq -\alpha_{min}||G_k||^2$.*

Proof. To verify this, we multiply G_k^T in (3.10) and we obtain

$$G_k^T p_k = -\lambda_k ||G_k||^2 - \theta_k^{GMOP} \frac{G_k^T p_{k-1}}{||G_k||^2} ||G_k||^2 + \theta_k^{GMOP} G_k^T p_{k-1},$$

$$G_k^T p_k = -\lambda_k ||G_k||^2,$$

but from (3.11) $\lambda_k \geq \alpha_{min}$. It implies that

$$G_k^T p_k \leq -\alpha_{min} ||G_k||^2, \tag{3.15}$$

∎

From the Lemma 3.5 above, the descent condition generally does not depend on t_k.

3.1.1 Generalized MOPCGM Algorithm

Algorithm 1 Generalized MOPCGM

Input: Function G, initial guess $x_0 \in \mathbb{R}^n$, $\lambda_o > 0$, tolerance $\epsilon = \text{Tol}$, $\rho \in (0,1)$, parameters τ, $\beta > 0$, $\eta > 0$, $\zeta > 0$, $\alpha_{min} > 0$, $\alpha_{max} > \alpha_{min}$, $\gamma, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in (0,2)$, $0 < \zeta_1 \leq \zeta_2$, projection function Π_Γ on convex set Γ
Output: Solution x^*
Initialization: Set $k \leftarrow 0$, $\lambda \leftarrow \lambda_o$, $z_k \leftarrow x_0$, initialize α_k
while $\|G_k\| > \epsilon$ **do**
    if $\|G(x_{k+1})\| < \|G(x_k)\|$ **then**
        $\lambda_{k+1} \leftarrow \lambda_k$
        Else $\lambda_{k+1} \leftarrow \Pi_{[\alpha_{min}, \alpha_{max}]}(\max(\frac{\|v_{k-1}\|^2}{s_{k-1}^T v_{k-1}}, \frac{s_{k-1}^T v_{k-1}}{\|s_{k-1}\|^2}))$
    end if
    Determine p_k as in (3.10)
    Adjust $\alpha_k = \max\{\rho^i \beta\}$ such that

$$G(x_k + \alpha_k p_k)^T p_k \leq -\zeta \alpha_k \|p_k\|^2 \Pi_{[\zeta_1, \zeta_2]}(\|G(x_k + \alpha_k p_k)\|) \tag{3.16}$$

    if $z_k = x_k + \alpha_k p_k \in \Gamma$ and $\|G(z_k)\| < \epsilon$ **then**
        $x^* \leftarrow z_k$
        break
    end if
    Compute $\mu_k \leftarrow \frac{G(z_k)^T (x_k - z_k)}{\|G(z_k)\|^2}$
    Update $x_{k+1} \leftarrow \Pi_\Gamma(x_k - \gamma \mu_k G(z_k))$
    Compute $s_{k-1} = z_{k-1} - x_{k-1}$ and $v_{k-1} = G(x_k) - G(x_{k-1}) + \tau s_{k-1}$
    Compute θ_k^{MOP} and t^* using (3.14) and (3.13) respectively.
    if $\|f_{k+1}\| < \|f_k\|$ **then**
        $\gamma = \min(\gamma \cdot \gamma_1, \gamma_2)$
    else
        $\gamma = \max(\gamma \cdot \gamma_3, \gamma_4)$
        break
    end if
    if $\|p_k\| \approx 0$ **then**
        $x^* \leftarrow x_k$
        break
    end if
    Set $x_k \leftarrow x_{k+1}$
end while
Return x^*

3.1.2 Convergence Analysis of Generalized MOPCGM

The next Lemma examines that the algorithm is well defined.

Lemma 3.6 *Let G be Lipschitz continuous, then*

$$\alpha_k \geq \min\{\eta, \frac{\alpha_{min}\rho}{L+\zeta\zeta_2}\frac{||G_k||^2}{||p_k||^2}\}.$$

Proof. With the line search in Algorithm 1, Let α_k be the optimal step length that satisfies (3.16), then $\tilde{\alpha}_k = \alpha_k/\rho$ violets (3.16). So,

$$G(x_k + \tilde{\alpha}_k p_k)^T p_k > -\zeta\tilde{\alpha}_k||p_k||^2\Pi_{[\zeta_1,\zeta_2]}(\|G(x_k + \tilde{\alpha}_k p_k)\|)$$

and $G_k^T p_k \leq -\alpha_{min}||G_k||^2$.

$$||G_k||^2 \leq \frac{1}{\alpha_{min}}\left[(G(x_k + \tilde{\alpha}_k p_k) - G_k)^T p_k - G(x_k + \tilde{\alpha}_k p_k)^T p_k\right]$$

$$||G_k||^2 \leq \frac{1}{\alpha_{min}}[(G(x_k + \tilde{\alpha}_k p_k) - G_k)^T p_k + \zeta\tilde{\alpha}_k||p_k||^2\zeta_2]$$

$$||G_k||^2 \leq \frac{\alpha_k(L+\zeta_2\zeta)}{\rho\alpha_{min}}||p_k||^2$$

This completes the proof. ∎

Lemma 3.7 *The direction p_k produced by Algorithm 1 meets the trust region property*

$$\alpha_{min}\|G(x_k)\| \leq \|p_k\| \leq \kappa\|G(x_k)\|.$$

Proof.

From

$$s_{k-1} = z_{k-1} - x_{k-1} = \alpha_{k-1} p_{k-1}$$

Also $|v_{k-1}^T s_{k-1}| \leq (L\gamma + \tau)||s_{k-1}||^2$, then $|t_k^{GMOP}| = \lambda_k \frac{|v_{k-1}^T s_{k-1}|}{||s_{k-1}||^2} \leq \alpha_{max}(L\gamma + \tau)$.

Therefore

$$|\theta_k^{GMOP}| = |\frac{v_{k-1}^T G_k}{p_{k-1}^T v_{k-1}} - \lambda_k \frac{s_{k-1}^T v_{k-1}}{||s_{k-1}||^2} \frac{s_{k-1}^T G_k}{p_{k-1}^T v_{k-1}}|,$$

$$|\theta_k^{GMOP}| \leq \frac{|v_{k-1}^T G_k|}{\tau \alpha_{k-1} ||p_{k-1}||^2} + \frac{\alpha_{max}(L\gamma + \tau)||s_{k-1}||^2}{||s_{k-1}||^2} \frac{|s_{k-1}^T G_k|}{\tau \alpha_{k-1} ||p_{k-1}||^2},$$

$$|\theta_k^{GMOP}| \leq \frac{||v_{k-1}|| ||G_k||}{\tau \alpha_{k-1} ||p_{k-1}||^2} + \frac{\alpha_{max}(L\gamma + \tau)||s_{k-1}||^2}{||s_{k-1}||^2} \frac{||s_{k-1}|| ||G_k||}{\tau \alpha_{k-1} ||p_{k-1}||^2},$$

$$|\theta_k^{GMOP}| \leq \frac{(L\gamma + \tau)\alpha_{k-1} ||p_{k-1}|| ||G_k||}{\tau \alpha_{k-1} ||p_{k-1}||^2} + \alpha_{max} \frac{(L\gamma + \tau)||s_{k-1}||^2}{||s_{k-1}||^2} \frac{\alpha_{k-1} ||p_{k-1}|| ||G_k||}{\tau \alpha_{k-1} ||p_{k-1}||^2},$$

$$|\theta_k^{GMOP}| \leq \frac{(L\gamma + \tau)||G_k||}{\tau ||p_{k-1}||} + \alpha_{max}(L\gamma + \tau) \frac{||G_k||}{\tau ||p_{k-1}||},$$

$$|\theta_k^{GMOP}| \leq (1 + \alpha_{max}) \frac{(L\gamma + \tau)||G_k||}{\tau ||p_{k-1}||}, \tag{3.17}$$

Now

$$||p_k|| \leq |\lambda_k - \theta_k^{GMOP} \frac{G_k^T p_{k-1}}{||G_k||^2}| ||G_k|| + |\theta_k^{GMOP}| ||p_{k-1}||,$$

$$||p_k|| \leq |\lambda - \theta_k^{GMOP}| ||G_k|| + |\theta_k^{GMOP} \frac{G_k^T p_{k-1}}{||G_k||^2}| ||p_{k-1}||.$$

But also

$$|(\lambda_k - \theta_k^{GMOP} \frac{G_k^T p_{k-1}}{||G_k||^2}) G_k + \theta_k^{GMOP} p_{k-1}| \leq \alpha_{max} ||G_k|| + |\theta_k^{GMOP}| \frac{G_k^T p_{k-1}}{||G_k||^2} | ||G_k|| + |\theta_k^{GMOP}| ||p_{k-1}||.$$

$$|(\lambda_k - \theta_k^{GMOP} \frac{G_k^T p_{k-1}}{||G_k||^2}) G_k + \theta_k^{GMOP} p_{k-1}| \leq \alpha_{max} ||G_k|| + |\theta_k^{GMOP}| \frac{\|G_k\| \|p_{k-1}\|}{||G_k||^2} | ||G_k|| + |\theta_k^{GMOP}| ||p_{k-1}||.$$

It implies that

$$\|p_k\| \leq \alpha_{max}||G_k|| + |\theta_k^{GMOP}| \|p_{k-1}\| + |\theta_k^{GMOP}| ||p_{k-1}||,$$

Therefore,

$$\|p_k\| \leq \alpha_{max}||G_k|| + 2|\theta_k^{GMOP}| \|p_{k-1}\|.$$

Using (3.17)

$$\|p_k\| \leq \alpha_{max}||G_k|| + 2(1+\alpha_{max})\frac{(L\gamma+\tau)||G_k||}{\tau||p_{k-1}||}\|p_{k-1}\|.$$

$$\|p_k\| \leq \alpha_{max}||G_k|| + 2(1+\alpha_{max})\frac{(L\gamma+\tau)||G_k||}{\tau}.$$

We finally get

$$\|p_k\| \leq \left(\alpha_{max} + 2(1+\alpha_{max})\frac{(L\gamma+\tau)}{\tau}\right)\|G_k\|. \tag{3.18}$$

From (3.18), we conclude that

$$||p_k|| \leq \kappa||G_k||, \tag{3.19}$$

where

$$\kappa = [\alpha_{max} + 2(1+\alpha_{max})\frac{(L\gamma+\tau)}{\tau}] \tag{3.20}$$

From lemma 3.5

$$\alpha_{min}||G_k||^2 \leq -G_k^T p_k \leq ||G_k||||p_k||. \tag{3.21}$$

Therefore, we complete the proof by combining (3.19) and (3.21).

∎

Lemma 3.8 *Suppose all the assumptions A1 and A2 hold, then*

$$\lim_{k\to\infty} \alpha_k ||p_k||= 0$$

Proof. Beginning from the line search (3.16) $G(z_k)^T p_k \leq -\zeta\alpha_k ||p_k||^2 \|G(z_k)\|$

Therefore

$$G(z_k)^T (x_k - z_k) = -\alpha_k G(x_k + \alpha_k p_k)^T p_k$$

$$G(z_k)^T (x_k - z_k) \geq \zeta\alpha_k^2 ||p_k||^2 \|G(z_k)\|$$

$$G(z_k)^T (x_k - z_k) \geq \zeta ||x_k - z_k||^2 \|G(z_k)\| \tag{3.22}$$

Now, we apply the monotonicity of G and A1. Therefore there is $x^* \in \Gamma$ such that $G(x^*) = 0$

$$G(z_k)^T (x_k - x^*) = G(z_k)^T (x_k - z_k + z_k - x^*)$$

$$G(z_k)^T (x_k - x^*) = G(z_k)^T (x_k - z_k) + G(z_k)^T (z_k - x^*)$$

$$G(z_k)^T (x_k - x^*) \geq G(z_k)^T (x_k - z_k) + G(x^*)^T (z_k - x^*)$$

$$G(z_k)^T (x_k - x^*) \geq G(z_k)^T (x_k - z_k) \tag{3.23}$$

This implies that

$$G(z_k)^T (x_k - x^*) \geq \zeta ||x_k - z_k||^2 \|G(z_k)\| \tag{3.24}$$

Applying the non-expansive property of the projection operator, we get

$$||x_{k+1} - x^*||^2 = ||\Pi_\Gamma(x_k - \gamma\mu_k G(z_k)) - x^*||^2,$$

$$||x_{k+1} - x^*||^2 \leq ||(x_k - \gamma\mu_k G(z_K)) - x^*||^2,$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - 2\gamma\mu_k G(z_k)^T(x_k - x^*) + \gamma^2\mu_k^2||G(z_k)||^2.$$

using (3.23), we obtain

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - 2\gamma\mu_k G(z_k)^T(x_k - z_k) + \gamma^2\mu_k^2||G(z_k)||^2.$$

But $\mu_k^2 = \frac{(G(z_k)^T(x_k - z_k))^2}{||G(z_k)||^4}$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - \gamma(2-\gamma)\frac{(G(z_k)^T(x_k - z_k))^2}{||G(z_k)||^2}.$$

Using (3.22), we obtain

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - \gamma(2-\gamma)\zeta^2||x_k - z_k||^4. \tag{3.25}$$

Implying that

$$0 \leq ||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 \tag{3.26}$$

So, the sequence $\{\|x_k - x^*\|\}$ is bounded below and non-increasing, hence $\{x_k\}$ is convergent.

From (3.25) we have

$$||x_k - x^*||^2 \leq ||x_0 - x^*||^2 - \gamma(2-\gamma)\zeta^2 \sum_{j=0}^{k} ||x_j - z_j||^4.$$

This means that

$$\gamma(2-\gamma)\zeta^2 \sum_{j=0}^{k} ||x_j - z_j||^4 \leq ||x_0 - x^*||^2 - ||x_k - x^*||^2 \leq ||x_0 - x^*||^2,$$

hence

$$\gamma(2-\gamma)\zeta^2 \sum_{j=0}^{\infty} ||x_j - z_j||^4 \leq ||x_0 - x^*||^2 < \infty \tag{3.27}$$

This completes the proof. ∎

Theorem 3.4 *Suppose x_k is generated by Algorithm 1, then*

$$\lim_{k\to\infty} \inf ||G_k|| = 0 \tag{3.28}$$

Proof. Suppose that there is some $\epsilon > 0$ such that $||G_k|| > \epsilon$ for all k. Using (3.21) together with this, we obtain

$$\alpha_{min}\epsilon \leq \|p_k\| \qquad \forall k. \tag{3.29}$$

From (3.29), we have $\alpha_k \to 0$ as $k \to \infty$.

Applying the line search in (3.16) there is $\bar{\alpha}_k$ (that is if α_k is the optimal step-size that satisfies the line search, then $\bar{\alpha}_k = \frac{\alpha_k}{\rho}$ violets the line search) such that

$$-G(x_k + \bar{\alpha}_k \zeta p_k)^T p_k < \bar{\alpha}_k \zeta ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(\|G(x_k + \bar{\alpha}_k p_k)\|) \leq \bar{\alpha}_k \zeta ||p_k||^2 \zeta_2 \tag{3.30}$$

Since x_k and p_k are bounded, we can select sub-sequences that converge to their accumulation points. Let $\bar{x}$ and $\bar{p}$ be the accumulation points of x_k and p_k respectively, then as k approaches infinity and by continuity of G in (3.30), we have

$$-G(\bar{x})^T\bar{p} \leq 0$$

Also using (3.15) as k approaches infinity, we have

$$-G(\bar{x})^T\bar{p} \geq 0$$

This is for sure a contradiction.

Therefore

$$\lim_{k\to\infty} \inf||G(x_k)||= 0.$$

Hence global convergence. This completes the proof.

∎

We can provide an alternative proof that depends on the Lipschitz continuity of G. Assume G is Lipschitz continuous.

Proof. We have two cases to consider.

1. Case 1

   Suppose $\lim_{k\to\infty} \inf\|p_k\|= 0$. Now when we apply (3.15), it means there is $\Omega \in \mathbb{R}_+$ such that

   $$\|G_k\|\leq \Omega\|p_k\|, \ \forall \ k$$

   and taking the limits concludes (3.28).

2. Case II

Let $\lim_{k\to\infty} \inf\|p_k\| \neq 0$. Applying the line search in (3.16) there is $\tilde{\alpha}_k$ (that is if α_k is the optimal step-size that satisfies the line search, then $\tilde{\alpha}_k = \frac{\alpha_k}{\rho}$ violets the line search) such that

$$-G(x_k + \tilde{\alpha}_k p_k)^T p_k < \zeta\tilde{\alpha}_k ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(\|G(x_k + \tilde{\alpha}_k p_k)\|) \leq \tilde{\alpha}_k \zeta ||p_k||^2 \zeta_2 \quad (3.31)$$

But using sufficient descent condition (3.28), Triangle inequality, Cauchy-Schwartz inequality and Lipschitz continuity of G, we have

$$\alpha_{min}||G_k||^2 \leq -G_k^T p_k = (G(x_k + \tilde{\alpha}_k p_k) - G_k)^T p_k - G(x_k + \tilde{\alpha}_k p_k)^T p_k,$$

$$\alpha_{min}||G_k||^2 \leq L\tilde{\alpha}_k ||p_k||^2 + \zeta\tilde{\alpha}_k ||p_k||^2 \zeta_2,$$

$$\alpha_{min}||G_k||^2 \leq \tilde{\alpha}_k ||p_k||^2 (L + \zeta\zeta_2)\,.$$

We arrive at

$$\frac{\rho\alpha_{min}}{||p_k||(L + \zeta\zeta_2)}\|G_k\| \leq \alpha_k ||p_k||.$$

Consequently, we have

$$||G_k||^2 \leq \alpha_k ||p_k|| \frac{\alpha_{min}||p_k||(L + \zeta\zeta_2)}{\rho\alpha_{min}}.$$

Taking the limits in k implies that

$$\lim_{k\to\infty} \inf||G_k|| = 0.$$

Hence global convergence. This completes the proof.

3.2 Generalization of CGPM

The CG parameter given by Hager and Zhang (HZ) [46] is a particular case of Dai and Liao (DL) parameter [33]. It happens when $t = 2\frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}}$ in (2.23) which may be considered to be an adaptive type of the D-L scheme in (2.22).

We now propose a generalized parameter related to H-Z parameter t_k to be

$$t_k = \lambda \frac{||y_{k-1}||^2}{s_{k-1}^T y_{k-1}}, \tag{3.32}$$

such that $\lambda > 0$. The proposed search direction is

$$p_k = \begin{cases} -G_k & k = 0 \\ -\lambda_k G_k + \theta_k^{GCGM} p_{k-1} + \tau a_k w_{k-1} & k \geq 1 \end{cases}, \tag{3.33}$$

such that $\lambda_k > 0$,

$$\theta_k^{GCGM} = \frac{G_k^T w_{k-1}}{p_{k-1}^T w_{k-1}} - \lambda_k \frac{||w_{k-1}||^2}{p_{k-1}^T w_{k-1}} \frac{G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}}, \tag{3.34}$$

where $\lambda_k = \Pi_{[\alpha_{min}, \alpha_{max}]}(\max(\frac{\|w_{k-1}\|^2}{s_{k-1}^T w_{k-1}}, \frac{s_{k-1}^T w_{k-1}}{\|s_{k-1}\|^2}))$ as used in [82] and

$$\Pi_{[\alpha_{min}, \alpha_{max}]}(x) = \max(\alpha_{min}, \min(x, \alpha_{max})),$$

$a_k = \frac{G_k^T p_{k-1}}{w_{k-1}^T p_{k-1}}$, $\tau > 0$, $w_{k-1} = y_{k-1} + r_k p_{k-1}$, $y_{k-1} = G(x_k) - G(x_{k-1})$, $s_{k-1} = x_k - x_{k-1}$ and $r_k = 1 + \max\{0, -\frac{G_k^T p_{k-1}}{w_{k-1}^T p_{k-1}}\}$

The next lemma verifies that (3.33) satisfies the sufficient descent condition.

Lemma 3.9 *Let* $\{p_k\}$ *and* $G(x_k)$ *be produced by the algorithm 2. Let* $0 \leq \tau \leq 1$*, and* $\alpha_{min} \geq \frac{1+\tau}{2}$ *, then (3.33) meets the sufficient descent condition* $G_k^T p_k \leq -\xi \|G_k\|^2$*, where* $\xi \geq 0$*.*

The condition holds for $k = 0$.

$$G_k^T p_k = -\lambda_k ||G_k||^2 + \theta_k^{GCGM} G_k^T p_{k-1} + \tau a_k G_k^T w_{k-1}.$$

$$G_k^T p_k = -\lambda_k ||G_k||^2 + \frac{G_k^T w_{k-1}}{p_{k-1}^T w_{k-1}} G_k^T p_{k-1} - \lambda_k \frac{||w_{k-1}||^2}{p_{k-1}^T w_{k-1}} \frac{G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}} G_k^T p_{k-1} + \tau \frac{G_k^T p_{k-1}}{w_{k-1}^T p_{k-1}} G_k^T w_{k-1}.$$

$$G_k^T p_k = -\lambda_k ||G_k||^2 + (1+\tau) \frac{G_k^T w_{k-1}}{p_{k-1}^T w_{k-1}} G_k^T p_{k-1} - \lambda_k \frac{||w_{k-1}||^2}{p_{k-1}^T w_{k-1}} \frac{G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}} G_k^T p_{k-1}.$$

$$G_k^T p_k = -\lambda_k ||G_k||^2 + \frac{2\sqrt{\lambda_k}(1+\tau)}{2\sqrt{\lambda_k}} \frac{G_k^T w_{k-1}}{p_{k-1}^T w_{k-1}} G_k^T p_{k-1} - \lambda_k \frac{||w_{k-1}||^2}{p_{k-1}^T w_{k-1}} \frac{G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}} G_k^T p_{k-1}.$$

$$G_k^T p_k = -\lambda_k ||G_k||^2 + \frac{2\sqrt{\lambda_k}(1+\tau)}{2\sqrt{\lambda_k}} \frac{G_k^T w_{k-1} G_k^T p_{k-1} p_{k-1}^T w_{k-1}}{(p_{k-1}^T w_{k-1})^2} - \lambda_k \frac{||w_{k-1}||^2 (G_k^T p_{k-1})^2}{(p_{k-1}^T w_{k-1})^2}.$$

$$G_k^T p_k = -\lambda_k ||G_k||^2 + \frac{(1+\tau)}{2\sqrt{\lambda_k}} \frac{G_k^T (p_{k-1}^T w_{k-1}) 2\sqrt{\lambda_k} w_{k-1} (G_k^T p_{k-1})}{(p_{k-1}^T w_{k-1})^2} - \lambda_k \frac{||w_{k-1}||^2 (G_k^T p_{k-1})^2}{(p_{k-1}^T w_{k-1})^2}.$$

$$G_k^T p_k \leq -\lambda_k ||G_k||^2 + \frac{(1+\tau)^2}{4\lambda_k} \frac{||G_k||^2 (p_{k-1}^T w_{k-1})^2 + 4\lambda_k ||w_{k-1}||^2 (G_k^T p_{k-1})^2}{2(p_{k-1}^T w_{k-1})^2} - \lambda_k \frac{||w_{k-1}||^2 (G_k^T p_{k-1})^2}{(p_{k-1}^T w_{k-1})^2}.$$

$$G_k^T p_k \leq -\lambda_k ||G_k||^2 + \frac{(1+\tau)^2}{4\lambda_k} ||G_k||^2.$$

But $0 < \alpha_{min} \leq \lambda_k \leq \alpha_{max}$, then

$$G_k^T p_k \leq -\alpha_{min}\left(1 - \frac{(1+\tau)^2}{4\alpha_{min}^2}\right) ||G_k||^2. \tag{3.35}$$

This completes the proof.

3.2.1 Generalized CGPM Algorithm

Algorithm 2 Generalized CGPM

Input: Function G, initial guess $x_0 \in \mathbb{R}^n$, $\lambda_o > 0$, tolerance $\epsilon = \text{Tol}$, $\rho \in (0, 1)$, parameters $\tau > 0$, $\eta > 0$, $\zeta > 0$, $\gamma, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in (0, 2)$, $0 < \alpha_{min} \leq \alpha_{max}$, $0 < \zeta_1 \leq \zeta_2$, projection function Π_Γ on convex set Γ
Output: Solution x^*
Initialization: Set $k \leftarrow 0$, $z_k \leftarrow x_0$, $\lambda \leftarrow \lambda_o$, initialize α_k
while $\|G_k\| > \epsilon$ **do**
  if $\|G(x_{k+1})\| < \|G(x_k)\|$ **then**
    $\lambda \leftarrow \lambda$
    Else $\lambda \leftarrow \Pi_{[\alpha_{min}, \alpha_{max}]}(\max(\frac{\|w_{k-1}\|^2}{s_{k-1}^T w_{k-1}}, \frac{s_{k-1}^T w_{k-1}}{\|s_{k-1}\|^2}))$
  end if
  Compute θ_k^{GCGPM} using (3.34).
  Determine p_k as in (3.33)
  Adjust $\alpha_k = \max\{\rho^i \eta \mid i = 0, 1, 2, \dots\}$ such that

$$G(x_k + \alpha_k p_k)^T p_k \leq -\zeta \alpha_k \|p_k\|^2 \Pi_{[\zeta_1, \zeta_2]}(\|G(x_k + \alpha_k p_k)\|) \tag{3.36}$$

  if $z_k = x_k + \alpha_k p_k \in \Gamma$ and $\|G(z_k)\| < \epsilon$ **then**
    $x^* \leftarrow z_k$
    break
  end if
  Compute $\mu_k \leftarrow \frac{G(z_k)^T (x_k - z_k)}{\|G(z_k)\|^2}$
  Update $x_{k+1} \leftarrow \Pi_\Gamma(x_k - \gamma \mu_k G(z_k))$
  Compute $v_{k-1} = G(x_k) - G(x_{k-1}) + r_k p_{k-1}$
  if $\|f_{k+1}\| < \|f_k\|$ **then**
    $\gamma = \min(\gamma \cdot \gamma_1, \gamma_2)$
  else
    $\gamma = \max(\gamma \cdot \gamma_3, \gamma_4)$
    break
  end if
  if $\|p_k\| \approx 0$ **then**
    $x^* \leftarrow x_k$
    break
  end if
  Set $x_k \leftarrow x_{k+1}$
end while
Return x^*

3.2.2 Convergence Analysis for the generalized CGPM

Assuming all the conditions A1 and A2 are met, then we can analyze the convergence of algorithm 1.

Lemma 3.10 *Let G be Lipschitz continuous. Let $0 \leq \tau \leq 1$, and $\alpha_{min} \geq \frac{1+\tau}{2}$, Then*

$$\alpha_k \geq \min\{\eta, \frac{\rho[4\alpha_{min}^2 - (1+\tau)^2]}{4\alpha_{min}[L+\zeta\zeta_2]} \frac{||G_k||^2}{||p_k||^2}\}$$

Proof. Let α_k be the optimal step length that satisfies (3.36), then, $\tilde{\alpha}_k = \alpha_k/\rho$ violets (3.36). Therefore,

$$G(x_k + \tilde{\alpha}_k p_k)^T p_k > -\zeta\tilde{\alpha}_k ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(||G(x_k + \tilde{\alpha}_k p_k)||)$$

But $G_k^T p_k \leq -\alpha_{min}[1 - \frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2$. So

$$||G_k||^2 \leq -\frac{4\alpha_{min}}{[4\alpha_{min}^2 - (1+\tau)^2]} G_k^T p_k$$

$$||G_k||^2 \leq \frac{4\alpha_{min}}{[4\alpha_{min}^2 - (1+\tau)^2]}[(G(x_k + \tilde{\alpha}_k p_k) - G_k)^T p_k - G(x_k + \tilde{\alpha}_k p_k)^T p_k]$$

$$||G_k||^2 \leq \frac{4\alpha_{min}}{[4\alpha_{min}^2 - (1+\tau)^2]}[(G(x_k + \tilde{\alpha}_k p_k) - G_k)^T p_k + \zeta\tilde{\alpha}_k ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(\|G(x_k + \tilde{\alpha}_k p_k)\|)],$$

$$||G_k||^2 \leq \frac{4\alpha_k\alpha_{min}}{\rho[4\alpha_{min}^2 - (1+\tau)^2]}[L+\zeta\zeta_2]||p_k||^2.$$

∎

Lemma 3.11 *Suppose all the assumptions A1 and A2 hold, then*

$$\lim_{k\to\infty} \alpha_k ||p_k|| = 0$$

Proof. Beginning from the line search (3.36) $G(z_k)^T p_k \leq -\zeta\alpha_k ||p_k||^2 ||G(z_k)||$

Therefore

$$G(z_k)^T (x_k - z_k) = -\alpha_k G(x_k + \alpha_k p_k)^T p_k$$

$$G(z_k)^T (x_k - z_k) \geq \zeta\alpha_k^2 ||p_k||^2 ||G(z_k)||$$

$$G(z_k)^T(x_k - z_k) \geq \zeta ||x_k - z_k||^2 ||G(z_k)|| \tag{3.37}$$

Now, we apply the monotonicity of G and A1. Therefore there is $x^* \in \Gamma$ such that $G(x^*) = 0$

$$G(z_k)^T(x_k - x^*) = G(z_k)^T(x_k - z_k + z_k - x^*)$$

$$G(z_k)^T(x_k - x^*) = G(z_k)^T(x_k - z_k) + G(z_k)^T(z_k - x^*)$$

$$G(z_k)^T(x_k - x^*) \geq G(z_k)^T(x_k - z_k) + G(x^*)^T(z_k - x^*)$$

$$G(z_k)^T(x_k - x^*) \geq G(z_k)^T(x_k - z_k) \tag{3.38}$$

This implies that

$$G(z_k)^T(x_k - x^*) \geq \zeta ||x_k - z_k||^2 ||G(z_k)|| \tag{3.39}$$

Applying the non-expansive property of the projection operator, we get

$$||x_{k+1} - x^*||^2 = ||\Pi_\Gamma(x_k - \gamma\mu_k G(z_k)) - x^*||^2$$

$$||x_{k+1} - x^*||^2 \leq ||(x_k - \gamma\mu_k G(z_K)) - x^*||^2$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - 2\gamma\mu_k G(z_k)^T(x_k - x^*) + \gamma^2\mu_k^2 ||G(z_k)||^2$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - 2\gamma\mu_k G(z_k)^T(x_k - z_k) + \gamma^2\mu_k^2 ||G(z_k)||^2 \tag{3.40}$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - 2\gamma \frac{(G(z_k)^T(x_k - z_k))^2}{\|G(z_k)\|^2} + \gamma^2 \frac{(G(z_k)^T(x_k - z_k))^2}{\|G(z_k)\|^2} \tag{3.41}$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - \gamma(2-\gamma) \frac{(G(z_k)^T(x_k - z_k))^2}{||G(z_k)||^2}$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - \gamma(2-\gamma)(\zeta ||x_k - z_k||^4) \tag{3.42}$$

Inequalities (3.40) and (3.42) follow from (3.37) and (3.38) respectively.

From (3.42), we obtain

$$0 \leq ||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2$$

This implies that $\{||x_0 - x^*||\}$ is a decreasing sequence and bounded below. This means that $\{x_k\}$ is convergent. From (3.42) we settle that

$$||x_k - x^*||^2 \leq ||x_0 - x^*||^2 - \gamma(2-\gamma)\zeta^2 \sum_{j=0}^{k} (||x_j - z_j||^4. \tag{3.43}$$

$$\sum_{k=0}^{\infty} ||x_k - z_k||^4 \leq ||x_0 - x^*||^2 < \infty.$$

This completes the proof. ∎

Theorem 3.5 *Suppose x_k is produced by Algorithm 2 and $\alpha_{min} > \frac{(1+\tau)}{2}$ as defined in algorithm 2, then*

$$\lim_{k\to\infty} \inf ||G_k|| = 0. \tag{3.44}$$

Proof. We have two cases to consider.

1. Case 1

   Suppose $\lim_{k\to\infty} \inf\|p_k\| = 0$. Now when we apply (3.35), it means there is $\Omega \in \mathbb{R}_+$ such that

   $$\|G_k\| \leq \Omega \|p_k\|, \ \forall\ k$$

   and taking the limits concludes (3.44).

2. Case II

   Let $\lim_{k\to\infty} \inf\|p_k\| \neq 0$. Applying the line search in (3.36) there is $\tilde{\alpha}_k$ (that is if α_k is the optimal step-size that satisfies the line search, then $\tilde{\alpha}_k = \frac{\alpha_k}{\rho}$ violets the line search) such that

   $$-G(x_k + \tilde{\alpha}_k p_k)^T p_k < \zeta \tilde{\alpha}_k ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(\|G(x_k + \tilde{\alpha}_k p_k)\|) \leq \tilde{\alpha}_k \zeta ||p_k||^2 \zeta_2 \tag{3.45}$$

But using sufficient descent condition (3.44), Triangle inequality, Cauchy-Schwartz inequality and Lipschitz continuity of G, we have

$$\alpha_{min}[1-\frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2 \leq -G_k^T p_k = (G(x_k+\tilde{\alpha}_k p_k)-G_k)^T p_k - G(x_k+\tilde{\alpha}_k p_k)^T p_k,$$

$$\alpha_{min}[1-\frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2 \leq L\tilde{\alpha}_k||p_k||^2+\zeta\tilde{\alpha}_k||p_k||^2\zeta_2,$$

$$\alpha_{min}[1-\frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2 \leq \tilde{\alpha}_k||p_k||^2(L+\zeta\zeta_2)\,.$$

We arrive at

$$\frac{\rho[4\alpha_{min}^2-(1+\tau)^2]}{4||p_k||\alpha_{min}(L+\zeta\zeta_2)}\|G_k\| \leq \alpha_k||p_k||.$$

Consequently, we have

$$||G_k||^2 \leq \alpha_k||p_k||\frac{4\alpha_{min}||p_k||(L+\zeta\zeta_2)}{\rho[4\alpha_{min}^2-(1+\tau)^2]}.$$

Taking the limits in k implies that

$$\lim_{k\to\infty}\inf||G_k||=0.$$

Hence global convergence. This completes the proof.

∎

3.2.3 Features of GCGPM, GMOPCGM, and STTDFPM

In this section we describe the features associated with GCGPM, GMOPCGM, and STTDFPM.

The most common features associated with the methods mentioned above include

Where $\Phi_k = \Pi_{[\eta_1,\eta_2]}(G(z_k))$, $\lambda_k = \Pi_{[\alpha_{min},\alpha_{max}]}(\max(\frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}, \frac{s_{k-1}^T y_{k-1}}{\|y_{k-1}\|^2}))$

From table A.1, we can deduce the similarities among the three methods. In [82], the spectral parameter considered is $\beta_k = \Pi_{[\eta_1,\eta_2]}(\frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}})$. But $\frac{\|s_{k-1}\|^2}{s_{k-1}^T y_{k-1}}$ is

Properties	GCGPM	GMOPCGM	STTDFPM
line search method	$G(x_k+\alpha_k p_k)^T p_k \leq -\zeta\alpha_k\|p_k\|^2\Phi_k$	$G(x_k+\alpha_k p_k)^T p_k \leq -\zeta\alpha_k\|p_k\|^2\Phi_k$	$G(x_k+\alpha_k p_k)^T p_k \leq -\zeta\alpha_k\|p_k\|^2\Phi_k$
Search direction	$-\lambda_k G_k + \theta_k^{GCGPM} p_{k-1} + \gamma\frac{G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}} w_{k-1}$	$-\lambda_k G_k + \theta_k^{GMOP}\frac{G_k^T p_{k-1}}{\|\|G_K\|\|^2} G_k + \theta_k^{GMOP} p_{k-1}$	$-\beta_k G_k + \theta_k^{STTDFPM} p_{k-1} - \eta_k y_{k-1}$
Conjugate Parameter	$\theta_k^{GCGPM} = \frac{(w_{k-1}-t_k^{GCGPM} p_{k-1})^T G_k}{p_{k-1}^T w_{k-1}}$	$\theta_k^{GMOP} = \frac{(v_{k-1}-t_k^{GMOP} s_{k-1})^T G_k}{p_{k-1}^T v_{k-1}}$	$\theta_k^{STTDFPM} = \frac{y_{k-1}^T G_k}{u_k}$
Parameter t	$t_k^{GCGPM} = \lambda_k\frac{\|\|w_{k-1}\|\|^2}{p_{k-1}^T w_{k-1}}$	$t_k^{GMOP} = \lambda_k\frac{s_{k-1}^T v_{k-1}}{\|\|s_{k-1}\|\|^2}$	$t=0$
Sufficient descent	Yes	Yes	Yes
DL Class	Yes	Yes	Yes
Hestenes–Stiefel (HS) Class	No	No	Yes
Number of Terms	Three Term	Two Term	Three Term
Spectral Conjugate gradient	Yes	Yes	Yes
Assumptions (A1) and (A2)	Yes	Yes	Yes
Monotonicity of G	Yes	Yes	Yes
Projection of x_k to Γ	Yes	Yes	Yes
Globally convergent	Yes	Yes	Yes

Table 3.1: Comparison of Properties of Different Optimization Methods

always considered to be more superior compared to $\frac{s_{k-1}^T y_{k-1}}{\|y_{k-1}\|^2}$. So replacing β_k with λ_k will not greatly affect the performance of STTDFPM.

The DL class of methods exploits the Dai-Liao (DL) conjugate parameter, which is defined as:

$$\theta_k^{DL} = \theta_k^{HS} - t\frac{G(x_k)^\top s_{k-1}}{p_{k-1}^\top y_{k-1}}, \tag{3.46}$$

where θ_k^{HS} is the Hestenes-Stiefel (HS) conjugate parameter [49], and $t \geq 0$ is a controllable parameter that influences the method's behavior. The parameter t has attracted significant interest in the literature because different values of t lead to distinct methods with varying performance characteristics. For example, when $t = 0$, the DL parameter simplifies to the HS parameter, placing methods like the STTDFPM into this category.

Specific choices for the parameter t lead to well-known methods. For instance:

- When

  $$t = 2\frac{\|y_{k-1}\|^2}{s_{k-1}^\top y_{k-1}}, \tag{3.47}$$

  we obtain CGPM.

- When

  $$t = \frac{s_{k-1}^\top y_{k-1}}{\|s_{k-1}\|^2}, \tag{3.48}$$

we derive the Modified Optimal Perry Conjugate Gradient Method (MOPCGM).

The table A.1 above compares the properties of three algorithms, that is GCGPM, GMOPCGM and STTDFPM.

- **Line search criteria**: Each method uses a different line search strategy to determine step size. However, these line search strategies are closely related.
  - **CGPM:** Incorporates a line search criterion based on the condition $G(x_k + \alpha_k p_k)^\top p_k \leq -\zeta \alpha_k \|p_k\|^2 \|G(x_k + \alpha_k p_k)\|$, ensuring that the step taken minimizes along the gradient.
  - **MOPCGM:** Uses a similar line search condition but excludes the norm of $G(x_k + \alpha_k p_k)$, simplifying the condition slightly.
  - **STTDFPM:** Includes an additional factor β_k in the condition, which could adaptively modify the step size.
- **Search direction**: There are variations in the way $G(x_k)$ and previous search direction p_{k-1} are combined.
- **Conjugate parameters**: Each method's parameterization affects the overall update rules.
  - **CGPM:** The conjugate parameter θ_k^{CGPM} involves both $w_{k-1} = y_{k-1} + \gamma s_{k-1}$, $\gamma > 0$ and t_k^{CGPM}, as well as their relation to the $G(x_k)$ as defined in (3.47). w_{k-1} replaces y_{k-1}
  - **MOPCGM:** The conjugate parameter θ_k^{MOP} follows a similar formula but replaces y_{k-1} with w_{k-1} and is defined in (3.47).

– **STTDFPM:** The conjugate parameter is as in (3.46) when $t = 0$.

- **Global convergence**: All methods are guaranteed to converge under certain assumptions, ensuring robustness across different problem types.

We can see from that table that all the algorithms use a similar line search to ensure that the descent direction is adquate.

All the algorithms meet the sufficient descent condition and are DL class.

GCGPM and STTDFPM are three term spectural conjugate method while GMOPCGM is a two term SCGM.

Each algorithm projects the point x_k to the constraint set Γ.

All algorithms are globally convergent and this ensures the stability and robustness of the algorithms.

CHAPTER 4

FRAMEWORK

4.1 Framework

In the Framework, we mean a unification of all the three algorithms that we have discussed in the table A.1 in chapter 3. Table A.1 brings out a general picture of the common features regarding the three algorithms, that is, the generalized Conjugate gradient method(GCGM), Generalised Modified Optimal Perry conjugate gradient method(GMOPCGM), and STTDFPM all fall in the DL class and are Spectral Gradient methods. In this chapter, we proposed a framework that encompasses all three methods GCGM, GMOPCGM, and STTDFPM.

Our main goal here is to write all three algorithms to be in one algorithm. This is achieved by writing the search direction of the framework as the convex combination of the individual search directions and line searches respectively. In addition to that, we also proposed a new search direction and line search for the framework. We proved that the framework is sufficiently descent and then proved the global convergence under some assumptions. Here is the general algorithm of the framework.

4.1.1 General Algorithm of the Framework

Algorithm 3 General Framework

1: **Input:** Function G, initial guess $x_0 \in \mathbb{R}^n$, tolerance $\epsilon = \text{Tol}$, $\rho \in (0,1)$, tuning parameters, projection function Π_Γ on convex set Γ
2: **Output:** Solution x^*
3: **Search direction:** Find the search direction p_k
4: **Line Search:** Find the line search α_k using the appropriate method.
5: **Update:** Update x_k
6: **Convergence:** Check convergence. If it converges, stop. Otherwise, go to step 3.
7: **Return** x^*

In the following section, we proposed the search direction for the framework by expressing it as a convex combination of the directions of the three methods discussed before.

4.1.2 The line search for the Framework

The proposed search direction of the framework given by

$$p_k = \begin{cases} -G_k & k = 0 \\ -(\lambda_k + \Lambda_k)G_k + \Theta_k p_{k-1} + \upsilon \Phi_k^{STTDFPM} y_{k-1} + \nu\tau\Phi_k^{GCGPM} w_{k-1} & k \geq 1 \end{cases} \tag{4.1}$$

such that $\Theta_k = (\delta\theta_k^{GMOP} + \upsilon\theta_k^{STTDFPM} + \nu\theta_k^{GCGPM})$, $\Lambda_k = \delta\theta_k^{GMOP}\frac{G_k^T p_{k-1}}{||G_k||^2}$, $\Phi_k^{GCGPM} = \frac{G_k^T s_{k-1}}{s_{k-1}^T w_{k-1}}$, $\Phi_k^{STTDFPM} = -\frac{p_{k-1}^T G_k}{u_k}$, $u_k = \max\{\phi||y_{k-1}||||p_{k-1}||, ||G_{k-1}||^2\}$, $\delta, \upsilon, \nu \in [0,1]$ and $\delta + \upsilon + \nu = 1$.

Lemma 4.1 *Let $\{p_k\}$ and $G(x_k)$ be produced by the algorithm 4. Let $0 \leq \tau \leq 1$, $\nu \in [0,1]$ and $\alpha_{min} \geq \sqrt{\nu}\frac{1+\tau}{2}$, then (4.1) is meets the sufficiently descent condition.*

Proof. It is clear if $k=0$. On the other hand,

$$G_k^T p_{k-1} = -\lambda_k ||G_k||^2 - \Lambda_k \|G_k\|^2 + \Theta_k G_k^T p_{k-1} + \upsilon \Phi_k^{STTDFPM} G_k^T y_{k-1} + \nu \Phi_k^{GCGPM} G_k^T w_{k-1}$$

But

$$-\Lambda_k ||G_k||^2 = -\delta \theta_k^{GMOP} G_k^T p_{k-1}. \tag{4.2}$$

Also

$$\Theta_k G_k^T p_{k-1} = (\delta \theta_k^{GMOP} + \upsilon \theta_k^{STT} + \nu \theta_k^{CDPM}) G_k^T p_{k-1},$$

$$\Theta_k G_k^T p_{k-1} = \delta \theta_k^{GMOP} G_k^T p_{k-1} + \upsilon \theta_k^{STT} G_k^T p_{k-1} + \nu \theta_k^{CDPM} G_k^T p_{k-1}. \tag{4.3}$$

Summing (4.2) and (4.3), we obtain

$$-\Lambda_k ||G_k||^2 + \Theta_k G_k^T p_{k-1} = \upsilon \theta_k^{STTDFPM} G_k^T p_{k-1} + \nu \theta_k^{GCGPM} G_k^T p_{k-1}. \tag{4.4}$$

But

$$\theta_k^{GCGPM} = \frac{(w_{k-1} - t_k^{GCGPM} p_{k-1})^T G_k}{p_{k-1}^T w_{k-1}},$$

$$\theta_k^{GMOP} = \frac{(v_{k-1} - t_k^{GMOP} s_{k-1})^T G_k}{p_{k-1}^T v_{k-1}},$$

$$\theta_k^{STTDFPM} = \frac{y_{k-1}^T G_k}{u_k}.$$

Substituting the parameters in to (4.9)

$$-\Lambda_k ||G_k||^2 + \Theta_k G_k^T p_{k-1} = \upsilon \frac{y_{k-1}^T G_k}{u_k} G_k^T p_{k-1} + \nu \frac{(w_{k-1} - t_k^{GCGPM} p_{k-1})^T G_k}{p_{k-1}^T w_{k-1}} G_k^T p_{k-1}.$$

Also $t_k^{CGPM} = \lambda_k \frac{||w_{k-1}||^2}{p_{k-1}^T w_{k-1}}$, implying that

$$-\Lambda_k||G_k||^2+\Theta_k G_k^T p_{k-1} = \upsilon\frac{y_{k-1}^T G_k}{u_k}G_k^T p_{k-1}+\nu\frac{w_{k-1}^T G_k G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}}-\nu\lambda_k\frac{||w_{k-1}||^2}{p_{k-1}^T w_{k-1}}\frac{p_{k-1}^T G_k}{p_{k-1}^T w_{k-1}}G_k^T p_{k-1}.$$

$$-\Lambda_k||G_k||^2+\Theta_k G_k^T p_{k-1} = \upsilon\frac{y_{k-1}^T G_k}{u_k}G_k^T p_{k-1}+\nu\frac{w_{k-1}^T G_k G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}}-\nu\lambda_k\frac{\|w_{k-1}\|^2 p_{k-1}^T G_k}{(p_{k-1}^T w_{k-1})^2}G_k^T p_{k-1}. \tag{4.5}$$

On the other hand,

$$r\Phi_k^{STTDFPM}G_k^T y_{k-1}+\nu\Phi_k^{GCGPM}G_k^T w_{k-1} = -\upsilon\frac{p_{k-1}^T G_k}{u_k}G_k^T y_{k-1}+\nu\tau\frac{G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}}G_k^T w_{k-1}. \tag{4.6}$$

Adding (4.5) to (4.6), we get

$$\begin{aligned}-\Lambda_k||G_k||^2&+\Theta_k G_k^T p_{k-1}+\upsilon\Phi_k^{STTDFPM}G_k^T y_{k-1}+\nu\Phi_k^{GCGPM}G_k^T w_{k-1} = \\ &\nu(1+\tau)\frac{G_k^T p_{k-1}}{p_{k-1}^T w_{k-1}}G_k^T w_{k-1}-\nu\lambda_k\frac{\|w_{k-1}\|^2(p_{k-1}^T G_k)^2}{(p_{k-1}^T w_{k-1})^2}.\end{aligned}$$

We multiply $p_{k-1}^T w_{k-1}$ up and down in the first term on the right-hand side.

$$\begin{aligned}-\Lambda_k||G_k||^2&+\Theta_k G_k^T p_{k-1}+\upsilon\Phi_k^{STTDFPM}G_k^T y_{k-1}+\nu\Phi_k^{GCGPM}G_k^T w_{k-1} = \\ &\nu(1+\tau)\frac{G_k^T p_{k-1}p_{k-1}^T w_{k-1}G_k^T w_{k-1}}{(p_{k-1}^T w_{k-1})^2}-\nu\lambda_k\frac{\|w_{k-1}\|^2(p_{k-1}^T G_k)^2}{(p_{k-1}^T w_{k-1})^2}\end{aligned}$$

$$\begin{aligned}-\Lambda_k||G_k||^2&+\Theta_k G_k^T p_{k-1}+\upsilon\Phi_k^{STTDFPM}G_k^T y_{k-1}+\nu\Phi_k^{GCGPM}G_k^T w_{k-1} = \\ &\nu(1+\tau)\frac{G_k^T p_{k-1}p_{k-1}^T w_{k-1}G_k^T w_{k-1}}{(p_{k-1}^T w_{k-1})^2}-\nu\lambda_k\frac{\|w_{k-1}\|^2(p_{k-1}^T G_k)^2}{(p_{k-1}^T w_{k-1})^2}\end{aligned}$$

$$- \Lambda_k||G_k||^2 + \Theta_k G_k^T p_{k-1} + \upsilon \Phi_k^{STTDFPM} G_k^T y_{k-1} + \nu \Phi_k^{GCGPM} G_k^T w_{k-1} = \nu \frac{(1+\tau)}{\sqrt{2\lambda_k}} \frac{G_k^T p_{k-1} p_{k-1}^T \sqrt{2\lambda_k} w_{k-1} G_k^T w_{k-1}}{(p_{k-1}^T w_{k-1})^2} - \nu \lambda_k \frac{\|w_{k-1}\|^2 (p_{k-1}^T G_k)^2}{(p_{k-1}^T w_{k-1})^2}.$$

Using Young's inequality,

$$- \Lambda_k||G_k||^2 + \Theta_k G_k^T p_{k-1} + \upsilon \Phi_k^{STTDFPM} G_k^T y_{k-1} + \nu \Phi_k^{GCGPM} G_k^T w_{k-1} \leq \nu \left(\frac{(1+\tau)^2}{4\lambda_k} \frac{(\|G_k\|^2 (w_{k-1}^T p_{k-1})^2}{(p_{k-1}^T w_{k-1})^2} + \frac{2\lambda_k \|w_{k-1}\|^2 (G_k^T p_{k-1})^2}{2(p_{k-1}^T w_{k-1})^2} \right) - \nu \lambda_k \frac{\|w_{k-1}\|^2 (p_{k-1}^T G_k)^2}{(p_{k-1}^T w_{k-1})^2},$$

$$-\Lambda_k||G_k||^2 + \Theta_k G_k^T p_{k-1} + \upsilon \Phi_k^{STTDFPM} G_k^T y_{k-1} + \nu \Phi_k^{GCGPM} G_k^T w_{k-1} \leq \nu \frac{(1+\tau)^2}{4\lambda_k} \|G_k\|^2 \tag{4.7}$$

Adding $-\lambda_k \|G_k\|^2$ to (4.7), it follows that

$$G_k^T p_k \leq -\lambda_k (1 - \nu \frac{(1+\tau)^2}{4\lambda_k^2}) \|G_k\|^2. \tag{4.8}$$

But $\alpha_{min} \leq \lambda_k$, we obtain

$$G_k^T p_k \leq -\alpha_{min} (1 - \nu \frac{(1+\tau)^2}{4\alpha_{min}^2}) \|G_k\|^2. \tag{4.9}$$

This completes the proof. ∎

4.1.3 Algorithm for the Framework

Algorithm 4 Framework

Input: Function G, initial guess $x_0 \in \mathbb{R}^n$, $\lambda_o > 0$, tolerance $\epsilon = \text{Tol}$, $\rho \in (0,1)$, parameters σ, σ_1, σ_2, $\phi > 0$ $\beta > 0$, $\eta > 0$, $\zeta > 0$, $\gamma, \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in (0,2)$, $0 < \alpha_{min} \leq \alpha_{max}$, $0 < \zeta_1 \leq \zeta_2$, τ, $\delta, \upsilon, \nu \in [0,1]$, projection function Π_Γ on convex set Γ
Output: Solution x^*
Initialization: Set $k \leftarrow 0$, $z_k \leftarrow x_0$, $\lambda \leftarrow \lambda_o$, initialize α_k
while $\|G_k\| > \epsilon$ **do**
  if $\|G(x_{k+1})\| < \|G(x_k)\|$ **then**
    $\lambda \leftarrow \lambda$
    Else $\lambda \leftarrow \Pi_{[\alpha_{min},\alpha_{max}]}(\max(\frac{\|w_{k-1}\|^2}{s_{k-1}^T w_{k-1}}, \frac{s_{k-1}^T w_{k-1}}{\|s_{k-1}\|^2}))$
  end if
  Determine p_k as in (4.1)
  Adjust $\alpha_k = \max\{\rho^i \eta \mid i = 0, 1, 2, \dots\}$ such that

$$G(x_k + \alpha_k p_k)^T p_k \leq -\zeta \alpha_k \|p_k\|^2 \Pi_{[\zeta_1,\zeta_2]}(\|G(x_k + \alpha_k p_k)\|) \tag{4.10}$$

  if $z_k = x_k + \alpha_k p_k \in \Gamma$ and $\|G(z_k)\| < \epsilon$ **then**
    $x^* \leftarrow z_k$
    break
  end if
  Compute $\mu_k \leftarrow \frac{G(z_k)^T (x_k - z_k)}{\|G(z_k)\|^2}$
  Compute $s_{k-1} = z_{k-1} - x_{k-1}$ and $\upsilon_{k-1} = G(x_k) - G(x_{k-1}) + \sigma_1 s_{k-1}$
  Update $x_{k+1} \leftarrow \Pi_\Gamma(x_k - \gamma \mu_k G(z_k))$
  if $\|f_{k+1}\| < \|f_k\|$ **then**
    $\gamma = \min(\gamma \cdot \gamma_1, \gamma_2)$
  else
    $\gamma = \max(\gamma \cdot \gamma_3, \gamma_4)$
    break
  end if
  if $\|p_k\| \approx 0$ **then**
    $x^* \leftarrow x_k$
    break
  end if
  Set $x_k \leftarrow x_{k+1}$
end while
Return x^*

We have seen that the Framework is sufficient descent, now we discuss its convergence.

4.1.4 Convergence Analysis for the Framework

Assuming all the conditions A1 and A2 in chapter 3 are met, then we can analyze the convergence of algorithm 4.

Lemma 4.2 *Let G be Lipschitz continuous. Let $0 \leq \tau \leq 1$, and $\alpha_{min} \geq \frac{1+\tau}{2}$, Then*

$$\alpha_k \geq \min\{\eta, \frac{\rho[4\alpha_{min}^2 - (1+\tau)^2]}{4\alpha_{min}[L+\zeta\zeta_2]}\frac{||G_k||^2}{||p_k||^2}\}$$

Proof. Let α_k be the optimal step length that satisfies (4.10), then, $\tilde{\alpha}_k = \alpha_k/\rho$ violets (4.10). Therefore,

$$G(x_k + \tilde{\alpha}_k p_k)^T p_k > -\zeta\tilde{\alpha}_k ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(||G(x_k+\tilde{\alpha}_k p_k)||)$$

But $G_k^T p_k \leq -\alpha_{min}[1 - \nu\frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2$. So

$$||G_k||^2 \leq -\frac{4\alpha_{min}}{[4\alpha_{min}^2 - \nu(1+\tau)^2]}G_k^T p_k$$

$$||G_k||^2 \leq \frac{4\alpha_{min}}{[4\alpha_{min}^2 - \nu(1+\tau)^2]}[(G(x_k+\tilde{\alpha}_k p_k) - G_k)^T p_k - G(x_k+\tilde{\alpha}_k p_k)^T p_k],$$

$$||G_k||^2 \leq \frac{4\alpha_{min}}{[4\alpha_{min}^2 - \nu(1+\tau)^2]}[(G(x_k+\tilde{\alpha}_k p_k) - G_k)^T p_k + \zeta\tilde{\alpha}_k ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(\|G(x_k+\tilde{\alpha}_k p_k)\|)],$$

$$||G_k||^2 \leq \frac{4\alpha_k\alpha_{min}}{\rho[4\alpha_{min}^2 - \nu(1+\tau)^2]}[L+\zeta\zeta_2]||p_k||^2.$$

∎

Lemma 4.3 *Suppose all the assumptions A1 and A2 hold, then*

$$\lim_{k\to\infty} \alpha_k ||p_k|| = 0$$

Proof. Beginning from the line search (4.10) $G(z_k)^T p_k \leq -\zeta\alpha_k ||p_k||^2 ||G(z_k)||$

Therefore

$$G(z_k)^T(x_k - z_k) = -\alpha_k G(x_k + \alpha_k p_k)^T p_k$$

$$G(z_k)^T(x_k - z_k) \geq \zeta \alpha_k^2 ||p_k||^2 ||G(z_k)||$$

$$G(z_k)^T(x_k - z_k) \geq \zeta ||x_k - z_k||^2 ||G(z_k)|| \tag{4.11}$$

Now, we apply the monotonicity of G and A1. Therefore we can find $x^* \in \Gamma$ with $G(x^*) = 0$

$$G(z_k)^T(x_k - x^*) = G(z_k)^T(x_k - z_k + z_k - x^*)$$

$$G(z_k)^T(x_k - x^*) = G(z_k)^T(x_k - z_k) + G(z_k)^T(z_k - x^*)$$

$$G(z_k)^T(x_k - x^*) \geq G(z_k)^T(x_k - z_k) + G(x^*)^T(z_k - x^*)$$

$$G(z_k)^T(x_k - x^*) \geq G(z_k)^T(x_k - z_k) \tag{4.12}$$

This implies that

$$G(z_k)^T(x_k - x^*) \geq \zeta ||x_k - z_k||^2 ||G(z_k)|| \tag{4.13}$$

Applying the non-expansive property of the projection operator, we get

$$||x_{k+1} - x^*||^2 = ||\Pi_\Gamma(x_k - \gamma\mu_k G(z_k)) - x^*||^2$$

$$||x_{k+1} - x^*||^2 \leq ||(x_k - \gamma\mu_k G(z_K)) - x^*||^2$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - 2\gamma\mu_k G(z_k)^T(x_k - x^*) + \gamma^2\mu_k^2 ||G(z_k)||^2$$

$$||x_{k+1} - x^*||^2 \leq ||x_k - x^*||^2 - 2\gamma\mu_k G(z_k)^T(x_k - z_k) + \gamma^2\mu_k^2 ||G(z_k)||^2 \tag{4.14}$$

$$||x_{k+1}-x^*||^2 \leq ||x_k-x^*||^2 - 2\gamma\frac{(G(z_k)^T(x_k-z_k))^2}{\|G(z_k)\|^2} + \gamma^2\frac{(G(z_k)^T(x_k-z_k))^2}{\|G(z_k)\|^2} \quad (4.15)$$

$$||x_{k+1}-x^*||^2 \leq ||x_k-x^*||^2 - \gamma(2-\gamma)\frac{(G(z_k)^T(x_k-z_k))^2}{||G(z_k)||^2}$$

$$||x_{k+1}-x^*||^2 \leq ||x_k-x^*||^2 - \gamma(2-\gamma)(\zeta||x_k-z_k||^4) \quad (4.16)$$

Inequalities (4.14) and (4.16) follow from (4.11) and (4.12) respectively. From (4.16), we obtain

$$0 \leq ||x_{k+1}-x^*||^2 \leq ||x_k-x^*||^2$$

This implies that $\{||x_0-x^*||\}$ is non-increasing and bounded below. This implies that $\{x_k\}$ is convergent. From (4.16) it settles that

$$||x_k-x^*||^2 \leq ||x_0-x^*||^2 - \gamma(2-\gamma)\zeta^2\sum_{j=0}^{k}(||x_j-z_j||^4. \quad (4.17)$$

$$\sum_{k=0}^{\infty}||x_k-z_k||^4 \leq ||x_0-x^*||^2 < \infty.$$

This completes the proof. ∎

Theorem 4.1 *Suppose x_k is produced by Algorithm 4, then*

$$\lim_{k\to\infty}\inf||G_k|| = 0. \quad (4.18)$$

Proof. We have two cases to consider.

1. Case 1

   Suppose $\lim_{k\to\infty}\inf\|p_k\| = 0$. Now when we apply (4.9), it means there is

$\Omega \in \mathbb{R}_+$ such that

$$\|G_k\| \leq \Omega \|p_k\|, \ \forall \ k$$

and taking the limits concludes (4.18).

2. Case II

Let $\lim_{k\to\infty} \inf \|p_k\| \neq 0$. Applying the line search in (4.10) there is $\tilde{\alpha}_k$ (that is if α_k is the optimal step-size that satisfies the (4.10), then $\tilde{\alpha}_k = \frac{\alpha_k}{\rho}$ violets the line search) such that

$$-G(x_k+\tilde{\alpha}_k p_k)^T p_k < \zeta \tilde{\alpha}_k ||p_k||^2 \Pi_{[\zeta_1,\zeta_2]}(\|G(x_k+\tilde{\alpha}_k p_k)\|) \leq \tilde{\alpha}_k \zeta ||p_k||^2 \zeta_2 \quad (4.19)$$

But using sufficient descent condition (4.9), Triangle inequality, Cauchy-Schwartz inequality and Lipschitz continuity of G,

$$\alpha_{min}[1-\nu\frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2 \leq -G_k^T p_k = (G(x_k+\tilde{\alpha}_k p_k)-G_k)^T p_k - G(x_k+\tilde{\alpha}_k p_k)^T p_k,$$

$$\alpha_{min}[1-\nu\frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2 \leq L\tilde{\alpha}_k ||p_k||^2 + \zeta \tilde{\alpha}_k ||p_k||^2 \zeta_2,$$

$$\alpha_{min}[1-\nu\frac{(1+\tau)^2}{4\alpha_{min}^2}]||G_k||^2 \leq \tilde{\alpha}_k ||p_k||^2 (L+\zeta\zeta_2)\,.$$

We arrive at

$$\frac{\rho[4\alpha_{min}^2 - \nu(1+\tau)^2]}{4||p_k||\alpha_{min}(L+\zeta\zeta_2)}\|G_k\| \leq \alpha_k ||p_k||.$$

Consequently, for $\alpha_{min} \neq \pm\frac{\sqrt{\nu}(1+\tau)}{2}$, we have

$$||G_k||^2 \leq \alpha_k ||p_k|| \frac{4\alpha_{min}||p_k||(L+\zeta\zeta_2)}{\rho[4\alpha_{min}^2 - \nu(1+\tau)^2]}.$$

Taking the limits in k implies that,

$$\lim_{k \to \infty} \inf ||G_k|| = 0.$$

Hence global convergence and this completes the proof.

$\blacksquare$

CHAPTER 5

NUMERICAL EXPERIMENTS

5.1 Algorithms Testing

In this chapter, we studied and compared the performance of the Framework, GCGPM, and GMOPCGM with the other three methods. That is STTDFPM [[50], Algorithm 1], MOPCGM [[83], Algorithm 2.1], and CGPM [[109], Algorithm 2.1] without changing their line searches or parameters.

The comparison was conducted by considering the number of iterations, CPU time, and function evaluation number.

The Algorithms were applied to Julia and tested with the same initial values.

The algorithms were set in such a way that iteration stopped when $Tol \leq 10^{-11}$ or the iterations exceeded 2000. The number of iterations, number of function evaluations, and CPU time were represented by IT, FE, and CPU, respectively, and the norm of G was also recorded to notice the accuracy of the algorithms. 19 test problems from different sources were used. All the experiments were conducted with three distinct dimensions, that is 10^3, 10^4, and 5×10^4. 14 different initial points were used for each dimension in the experiment.

We carefully selected the best parameters that seemed to suit each proposed algorithm for better performance.

For GCGPM, we selected the following parameters $\tau = 0.001$, $\eta = 0.6$, $\lambda_o = 1.0$, $\rho = 0.5$, $\zeta = 0.1$, $\zeta_1 = 1.0$, $\zeta_2 = 1.0$, $\alpha_{min} = 0.55$, $\alpha_{max} = 1.9$, $\gamma = 1.8$, $\gamma_1 = 1.1$, $\gamma_2 = 1.7$, $\gamma_3 = 1.05$, $\gamma_4 = 1.05$

For GMOPCGM, we selected the following parameters

$\tau = 1.0$, $\rho = 0.8$, $\beta = 0.5$, $\zeta = 0.0001$, $\alpha_{min} = 0.1$, $\alpha_{max} = 2.0$, $\lambda_o = 1.0$, $\gamma = 1.1$, $\gamma_2 = 1.8$, $\gamma_3 = 1.0$, $\gamma_4 = 1.0$, $\zeta_1 = 1.0$, $\zeta_2 = 1.0$

For the Framework, we selected the following parameters

$\sigma_1 = 2.50$, $\rho = 0.8$, $\eta = 0.5$, $\zeta = 10^{-4}$, $\alpha_{min} = 0.55$, $\alpha_{max} = 1.9$, $\lambda_o = 1.0$, $\gamma = 1.8$, $\gamma_1 = 1.1$, $\gamma_2 = 1.8$, $\gamma_3 = 0.85$, $\gamma_4 = 1.0$, $\zeta_1 = 1.0$, $\zeta_2 = 1.0$, $\phi = 50.0$, $\delta = 1/3$, $\upsilon = 1/3$, $\nu = 1/3$, $\tau = 0.0001 = \sigma_2$, $\sigma = 0.6$.

The following are the initial points used for the experiments;

$0 = (0, \cdots, 0)^T$, $0.2 = (8.2, \cdots, 8.2)^T$, $0.4 = (\frac{2}{5}, \cdots, \frac{2}{5})^T$, $0.5 = (\frac{1}{2}, \cdots, \frac{1}{2})^T$, $0.6 = (\frac{3}{5}, \cdots, \frac{3}{5})^T$, $0.8 = (\frac{4}{5}, \cdots, \frac{4}{5})^T$, $1.0 = (1.0, \cdots, 1.0)^T$, $1.1 = (\frac{11}{10}, \cdots, \frac{11}{10})^T$, $1 - 1/m = (1 - \frac{1}{m}, \cdots, 1 - \frac{1}{m})^T$, $1/m = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{m})^T$, $(k-1)/m = (0, \frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m})^T$, $1/m = (\frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m})^T$, $1/3^k = (\frac{1}{3}, \frac{1}{3^2}, \cdots, \frac{1}{3^m})^T$, $k/m = (\frac{1}{m}, \frac{2}{m}, \cdots, 1)^T$.

All algorithms were terminated if either of the following was met;

1. $\|G(x_k)\| < Tol$

2. $p_k < 0.1Tol$

3. $k > 2000$

$\|G(x_k)\|$ represents the usual norm of G at x_k and $Tol = 10^{-11}$.

For a better assessment and comparison of the performance of the various schemes, we employed the Moré and Dolan performance profile in [39]. The performance profile of the three new algorithms and their counterparts are represented in Figures

5.1-5.3. As in the figures, the vertical axes depict the chances that a certain solver outperforms the rest of the competing algorithms.

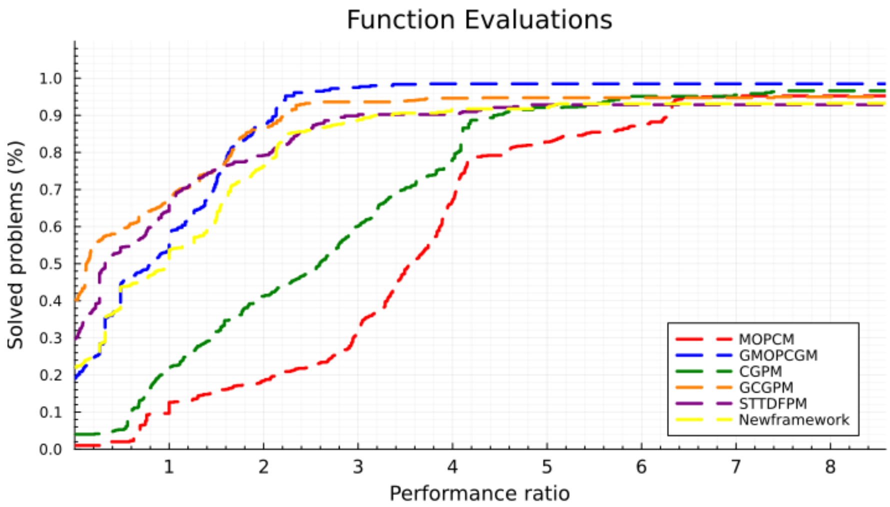

Figure 5.1: Profile of function evaluations

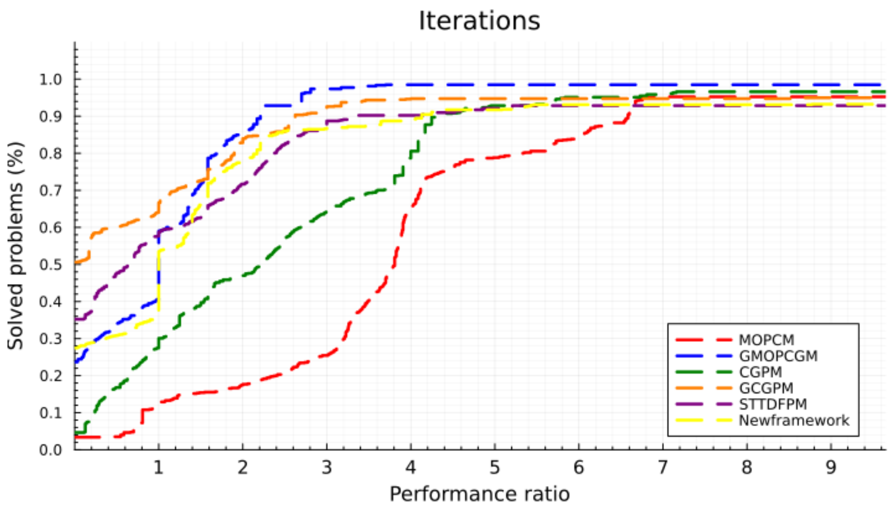

Figure 5.2: Profile of iterations

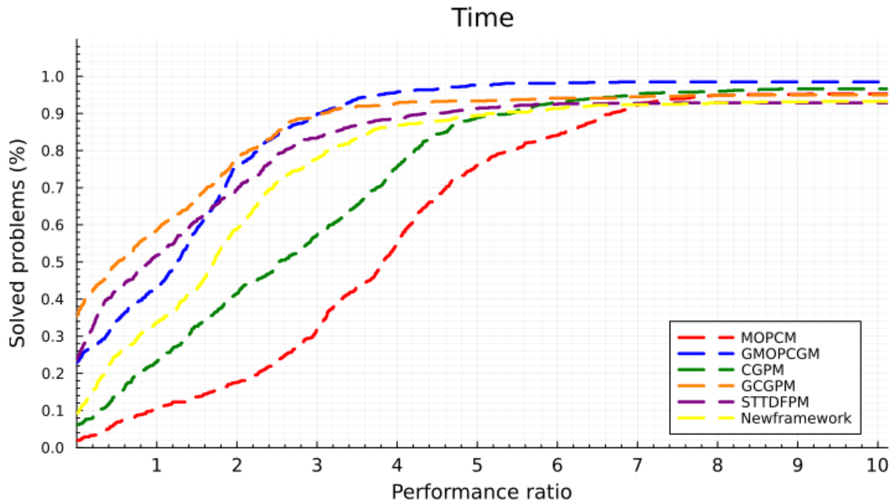

Figure 5.3: Profile of time

From Figures 5.1-5.3, it was observed that GCGPM outperformed its counterparts where it won by 40% 50%, and 38% as depicted in Figures 5.1, 5.2 and 5.3 respectively. This is because the four methods namely GCGPM, STTDFPM, GMOPCGM, and the Framework share the same properties and so there should be a small deviation in their performance. This can be witnessed from the figures. However, as a surprise, the GMOPCGM and the Framework were outperformed by STTDFPM though they both won MOPCGM and CGPM.

For further details concerning the individual outputs can be found in tables in appendix A

The following are the problems used in the experiment.

Problem 5.1(Problem 4.1 in [82]) $G(x)$ is

$G(x_i) = 2x_i - \sin x_i, \quad i = 1, 2, \cdots, n$ and $\Gamma = [-2, \infty]$.

Problem 5.2(Problem 10 in [55])

$G(x_i) = \log(|x_i|+1) - \frac{x_i}{n}, \quad , i = 1, 2, \cdots, n.$ and $\Gamma = \mathbb{R}$.

Problem 5.3(Problem 4.1 in[109])

$G(x_i) = \exp(x_i) - 1, \quad , i = 1, 2, \cdots, n.$ and $\Gamma = \mathbb{R}$.

Problem 5.4(Problem 4.5 [82])The general interpretation $G(x)$ defined as

$G(x_i) = 4x_i + (x_{i+1} - 2x_i) - \frac{x_{i-1}^2}{3}, \quad , i = 1, 2, \cdots, n-1.$

$G(x_n) = 4x_n + (x_{n-1} - 2x_n) - \frac{x_{n-1}^2}{3}$

and $\Gamma = \mathbb{R}$.

Problem 4.5(Problem 4.4 in [109]) Exponential problem $G(x)$ defined as

$G(x_1) = x_1 - \exp\cos(\frac{x_1+x_2}{n+1}),$

$G(x_i) = x_i - \exp\cos(\frac{x_{i-1}+x_i+x_{i+1}}{n+1}), \quad i = 2, \cdots, n-1.$

$G(x_n) = x_n - \exp\cos(\frac{x_{n-1}+x_n}{n+1})$

and $\Gamma = \mathbb{R}$

Problem 5.6(Problem 4.4 [109]) Exponential problem $G(x)$ defined as

$G(x_1) = x_1 + \sin(x_1) - 1,$

$G(x_i) = -x_{i-1} + 2x_i + \sin(x_i) - 1, \quad i = 2, \cdots, n-1.$

$G(x_n) = x_n + \sin(x_n) - 1$

Problem 5.7(Problem 19 in [55])Zero Jacobian function $G(x)$ defined as

$G(x_1) = \sum_{j=1}^{n} x_j^2$

$G(x_i) = -2x_1x_i, \quad for \quad i = 2, \cdots, n$

and $\Gamma = \mathbb{R}$

Problem 5.8(Problem 14 [88]) The general interpretation of $G(x)$ defined as

$G(x_1) = x_1(x_1^2 + x_2^2) - 1$

$G(x_i) = x_i(x_{i-1}^2 + 2x_i^2 + x_{i+1}^2) - 1, \quad for \quad i = 2, \cdots, n-1$

$G(x_n) = x_n(x_{n-1}^2 + x_n^2)$

and $\Gamma = \mathbb{R}$

Problem 5.9(Problem 12 [55])Trigexp function $G(x)$ defined as

$G(x_1) = 3x_1^3 + 2x_2 - 5 + \sin(|x_1 - x_2|)\sin(|x_1 + x_2|)$

$G(x_i) = -x_{i-1}\exp(x_{i-1} - x_i) + x_1(4 + 3x_i^3) + 2x_{i+1} - 5 + \sin(|x_i - x_{i+1}|)\sin(|x_i + x_{i+1}|), \quad for \;\; i = 2, \cdots, n-1$

$G(x_n) = -x_{n-1}\exp(x_{n-1} - x_n) + 4x_n - 3$

and $\Gamma = \mathbb{R}$

Problem 5.10(Problem 2 [88]) Complementary problem $G(x)$ defined as

$G(x_i) = (x_i - 1)^2 - 1.01, \quad for \; i = 1, \cdots, n$

and $\Gamma = \mathbb{R}$

Problem 5.11(Problem 4 [88]) Complementary problem and $G(x)$ defined as

$G(x_i) = \frac{i}{n}\exp x_i - 1, \quad for \; i = 1, \cdots, n$

and $\Gamma = \mathbb{R}$

Problem 5.12(Problem 11 [51])

$G(x_i) = x_i - \sin(|x_i - 1|), \quad for \; i = 1, \cdots, n$

and $\Gamma = \mathbb{R}$

Problem 5.13(Problem 4.5 in [96])

$G(x_i) = 2x_i - \sin(|x_i - 1|), \quad for \; i = 1, \cdots, n$

and $\Gamma = \mathbb{R}$

Problem 5.14(Problem 6 [88])

$G(x_i) = x_i - 2\sin(|x_i - 1|), \quad for \; i = 1, \cdots, n$

and $\Gamma = \mathbb{R}$

Problem 5.15(Problem 11 [88])

$G(x_i) = (\exp x_i)^2 + 3\sin x_i \cos x_i - 1, \quad for \; i = 1, \cdots, n$

and $\Gamma = \mathbb{R}$

Problem 5.16(Problem 5 [97]) The singular function $G(x)$ defined as

$G(x_1) = 2.5x_1 + x_2 - 1$

$G(x_i) = x_{i-1} + 2.5x_i + x_{i+1} - 1, \quad for\ i = 2, \cdots, n-1$

$G(x_n) = x_{n-1} + 2.5x_n - 1$

Problem 5.17(Problem 1 [110]) $G(x)$ defined as

$G(x_i) = 2x_i - \sin(|x_i|), \quad for\ i = 1, \cdots, n$

Problem 5.18(Problem 32 [55]) Minimal function $G(x)$ is defined as

$G(x_i) = 0.5\{\log x_i + \exp x_i - \sqrt{(\log x_i - \exp x_i)^2 - 10^{-10}}\}, \quad for\ i = 1, \cdots, n$

Problem 5.19(Problem 4.11 [58]) $G(x)$ defined by

$G(x_i) = 2(10^{-5})(x_i - 1) + 4x_i \sum_{j=1}^{n} x_j^2 - x_i, \quad for\ i = 1, \cdots, n$

Problem 5.20(Problem 4.6 [83]) $G(x)$ defined by

$G(x_i) = x_i(\cos(x_i - 1/n)(\sin x_i - 1 - (1 - x_i)^2 - 1/n \sum_{j=1}^{n} x_j), \quad for\ i = 1, \cdots, n$

5.2 Signal Restoration

Signal restoration refers to recuperating/recovering an original signal from degraded observed signals[21]. Signal restoration is a real-world problem that includes but is not limited to dequantization [66, 67], denoising [38, 75, 105], deblurring [14, 20].

Signal restoration includes large-scale inverse problems in which a multidimensional signal x is to be obtained from the observation of data y consisting of signals related to it either mathematically or physically. Both the original signal x and the observed y are taken to lie in some real Hilbert spaces which may be independent [26, 90].

The observed signal is given by

$$y = Hx + k \tag{5.1}$$

such that x is the signal that we want to recover from y, k is called the additive noise, H is a given operator representing the observation process like blurring or degradation, [85, 89]. For instance, for a blurred version of x then H is a convolution matrix. If $x = [x_1, x_2, x_3, x_4, x_5]^T$ is a signal of length 5 and $d = [1, 0, -1]$ is an edge detection filter(Kernel), then the corresponding Convolution matrix $H = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$. This is an example in the 1 dimension. The observation from x is $y = Hx = [x_1 - x_3, x_2 - x_4, x_3 - x_5]^T$. In general for 1 dimension $x = [x_1, \cdots, x_n]$ and the filter(Kernel) $h = [h_1, \cdots, h_k]$ such that $k < n$, then

$$y_i = x_i h_1 + \cdots + x_{i+k} h_k$$

The resulting vector

$$y = \begin{bmatrix} h_1 & h_2 & \cdots & h_k & 0 & \cdots & 0 \\ 0 & h1 & h_2 & \cdots & h_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & h_1 & h_2 & \cdots & h_k \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

is a convolution result, as if the filter h slid over x one position at a time.

In 2D, it utilizes the sliding window operation where the filter H moves(slides) across the input signal x. For every placement of the filter H on the input signal x, the inner product is computed of the filter and the overlapping signal components eg pixels for the input image.

$$y = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} h_1x_1 + h_2x_2 + h_4x_3 + h_5x_4 & h_2x_1 + h_3x_2 + h_5x_3 + h_6x_4 \\ h_4x_1 + h_5x_2 + h_7x_3 + h_8x_4 & h_5x_1 + h_6x_2 + h_8x_3 + h_9x_4 \end{bmatrix}$$

The problem of restoring the original signal x from the observed signal y is an inverse problem [41, 85].

Equation (5.1) can be restated into an objective function

$$T(x) = R(y, Hx) + \eta L(x), \tag{5.2}$$

$R(y, Hx)$ caters for x and y disagreement, regularization term $L(x)$ and η is the regularization parameter.

Different techniques have been developed to resolve (5.1). For example, Multilevel approach [41], majorization-minimization[85], Accelerated Projected Gradient Method APGM[37] and many more.

In [85], defined $R(y, Hx) = ||y - Hx||_2^2$ to be the mean square error and the regularization term $L(x)$ has to be of l_1 norm due to sparsity of x. Then $||x||_1 = \sum_{i=1}^{n} |x_i|$ and $||u||_2^2 = \sum_{i=1}^{n} u_i^2$. Therefore (5.2) becomes

$$T(x) = ||y - Hx||_2^2 + \eta||x||_1 \tag{5.3}$$

so to obtain x, its through minimizing/possibly solving (5.3) called l_1 regularized linear inverse problem or the penalized least-squares problem.

The presence of the l_1 term may lead to small components of x to become exactly zero, thus leading to sparse solutions [42].

Yin et al in [102], evaluated the performance of their technique by obtaining the sparse solution x for

$$\min_{x \in \mathbb{R}^m} \frac{1}{2}||y - Hx||^2 + \eta||x||_1 \tag{5.4}$$

and (5.4) is assumed to be under-determined linear systems of equations such that $H \in \mathbb{R}^{p \times m}$ and $y \in \mathbb{R}^n$ and $p << m$. In [42, 99, 102], they let $x = s - r$ where s and r are non-negative vectors such that $s_i = (x_i)_+$ and $r_i = (-x_i)_+$ for every $i = 1, \cdots, m$ such that $(\cdot)_+$ denotes the positive part operator defined as $(u)_+ = \max\{0, u\}$. Thus we have $||x||_1 = \langle E_n, s + r \rangle$ where $E_n = (1, \cdots, 1)$.

Therefore (5.4) can be written as

$$\min_{x \in \mathbb{R}^m} \frac{1}{2}||y - H(s - r)||^2 + \eta\langle E_m, s + r \rangle, r \geq 0, s \geq 0 \tag{5.5}$$

Let g be a shift vector such that $s \longleftarrow s + g$ and $r \longleftarrow r + g$. we see that this shift vector increases and term $2\eta E_n^T g \geq 0$. For sure it means that at the solution of (5.5), $s_i = 0$ or $r_i = 0$ for all i following the definition of the positive operator. Now from (5.4), $\eta\langle E_m, s + r\rangle = \eta\langle E_{2m}, z\rangle$ such that $z = \begin{pmatrix} s \\ r \end{pmatrix} \in \mathbb{R}^{2m}$.

Also from $||y - H(s-r)||^2 = \langle y - H(s-r), y - H(s-r)\rangle$. Therefore

$$||y - H(s-r)||^2 = ||y||^2 - y^T H(s-r) - (s-r)^T H^T y + ||H(s-r)||^2$$

let $a = H^T y$, then

$$||y - H(s-r)||^2 = ||y||^2 - a^T(s-r) - (s-r)^T a + ||H(s-r)||^2$$

Also

$$||H(s-r)||^2 = \langle H(s-r), H(s-r)\rangle = \langle Hs, H(s-r)\rangle - \langle Hr, H(s-r)\rangle$$

$$||H(s-r)||^2 = s^T H^T H s - s^T H^T H r - (r^T H^T H s - r^T H^T H r)$$

$$||H(s-r)||^2 = s^T H^T H s - s^T H^T H r - r^T H^T H s + r^T H^T H r$$

$$||H(s-r)||^2 = \begin{pmatrix} s^T & r^T \end{pmatrix} \begin{pmatrix} H^T H & -H^T H \\ -H^T H & H^T H \end{pmatrix} \begin{pmatrix} s \\ r \end{pmatrix}$$

$$||H(s-r)||^2 = \begin{pmatrix} s \\ r \end{pmatrix}^T \begin{pmatrix} H^T H & -H^T H \\ -H^T H & H^T H \end{pmatrix} \begin{pmatrix} s \\ r \end{pmatrix}$$

$$||H(s-r)||^2 = z^T Q z$$

Again

$$-a^T(s-r) - (s-r)^T a = 2\begin{pmatrix} -a \\ a \end{pmatrix}^T z$$

and

$$\eta\langle E_{2m}, z\rangle = \eta E_{2m}^T z$$

Therefore

$$\frac{1}{2}||y-Hx||^2+\eta||x||_1=\frac{1}{2}(||y||^2+2\begin{pmatrix}-a\\ a\end{pmatrix}^T z+z^TQz)+\eta E_{2m}^T z$$

$$\frac{1}{2}||y-Hx||^2+\eta||x||_1=\frac{1}{2}||y||^2+\frac{1}{2}z^TQz+(\eta E_{2m}^T+\begin{pmatrix}-a\\ a\end{pmatrix}^T)z$$

$$\frac{1}{2}||y-Hx||^2+\eta||x||_1=\frac{1}{2}||y||^2+\frac{1}{2}z^TQz+\begin{pmatrix}\eta E_m-a\\ \eta E_m+a\end{pmatrix}^T z$$

Let $d=\begin{pmatrix}\eta E_n-H^Ty\\ \eta E_n+H^Ty\end{pmatrix}$

Because $||y||^2\geq 0$ is the observation, then

$$\frac{1}{2}||y-Hx||^2+\eta||x||_1\geq\frac{1}{2}z^TQz+d^Tz$$

Problem (5.4) can be redefined by

$$\min\frac{1}{2}z^TQz+d^Tz,\quad z\geq 0 \tag{5.6}$$

such that $z=\begin{pmatrix}s\\ r\end{pmatrix}$, $d=\begin{pmatrix}\eta E_n-H^Ty\\ \eta E_n+H^Ty\end{pmatrix}$, $Q=\begin{pmatrix}H^TH & -H^TH\\ -H^TH & H^TH\end{pmatrix}$.

From [44, 99] the function G is monotone and continuous.

It was also noted in [82] that z satisfies (5.6) if and only if it satisfies (5.7)

$$G(z)=\min\{z, Qz+d\}=0 \tag{5.7}$$

The 'min' is interpreted to be a point-wise minimum and G is monotone and continuous as proved in [53].

Therefore problem (5.7) can be interpreted to be in the form of (1.1). In the experiment, we considered compressive sensing where we aimed at reconstructing a sparse signal from the observed signal. Firstly, we defined a sparse signal x_{true} of length $n = 2^{12}$ and sparsity $k = 2^9$, that is k are non-zero elements in the signal that are randomly selected.

A Sensing Matrix H of size $m \times n$ was obtained randomly. Where $m < n$ and in this case we considered $m = 2^{11}$. The Gaussian noise was also determined whose components were produced normal distributions $N(\mu = 0, \sigma = 0.01)$. The observation y was considered to be the sum of the noise and the product of the sensing matrix and the original signal.

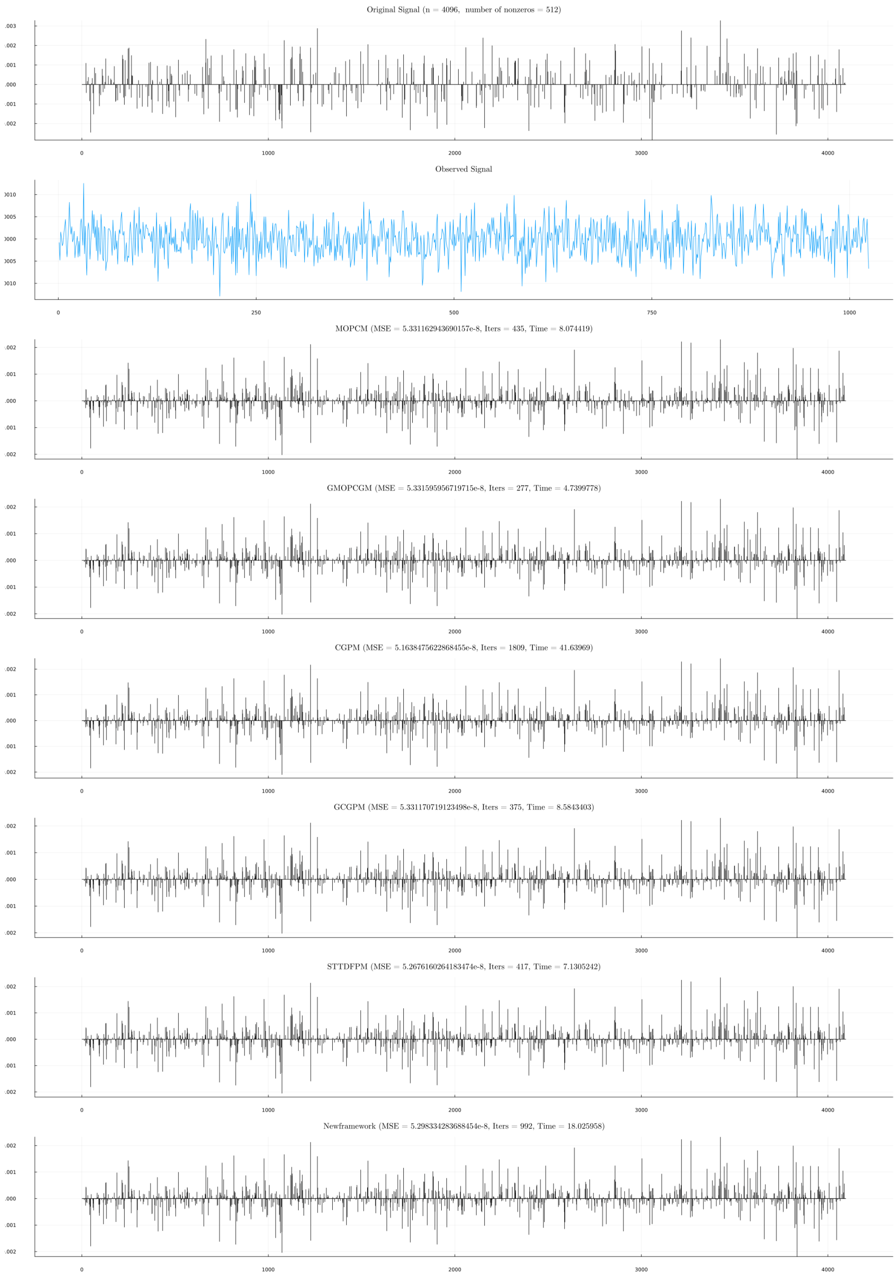

Figure 5.4: Reconstructed signals

In the application of the GCGPM, GMOPCGM, and the Framework in compressed sensing, the iteration process began with the initial point $x_o = H^T y$ of length n, and the experiment was terminated when $\|G(x_k)\| < 10^{-5}$.

The parameters for the Framework and GCGPM remained the same.

For GMOPCGM, we selected the following parameters.

$\tau = 1.05$, $\rho = 0.8$, $\beta = 0.5$, $\zeta = 0.0001$, $\alpha_{min} = 0.1$, $\alpha_{max} = 2.0$, $\lambda_o = 1.0$, $\gamma = 1.8$, $\gamma_1 = 1.1$, $\gamma_2 = 1.8$, $\gamma_3 = 0.85$, $\gamma_4 = 1.0$, $\zeta_1 = 1.0$, $\zeta_2 = 1.0$.

To consolidate the applicability of our schemes in recovering sparse signals, we compared them with STTDFPM in [50], CGPM in [109] and MOPCGM in [83]. The comparison helped us to evaluate how best our methods perform relative to their counterparts. A good performance implies a small mean square error given by,

$$MSE = \frac{\|x_{true} - x_{recovered}\|}{n}.$$

So, for fairness to all solvers, we conducted 10 experiments and determined the means for the Iterations, means for the function, and mean times besides the mean square errors. The results were recorded in table 5.1 below.

Algorithm	**Iterations**	**Function Evals**	**Time**	**MSE**
MOPCM	421.7	1267.1	7.5477	5.23×10^{-8}
GMOPCGM	**269.0**	**808.8**	**4.5182**	5.23×10^{-8}
CGPM	1708.4	5468.7	39.6073	5.06×10^{-8}
GCGPM	363.2	1089.6	8.0243	5.23×10^{-8}
STTDFPM	398.2	831.7	6.8776	5.15×10^{-8}
Newframework	955.2	3106.4	17.3467	5.20×10^{-8}

Table 5.1: Comparison of algorithm performance based on iterations, function evaluations, time, and MSE.

As we can see from the table, it is clear that CGPM gives a small mean square error. However, GMOPCGM took less time, had few iterations, and had few function evaluations. Figure 5.4 shows the quality of the reconstructed signals by the solvers.

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Conclusion

In this work, we proposed two algorithms, including GCGPM and GMOPCGM, that resulted from generalizing CGPM and MOPCGM, respectively. The framework was, in turn, the result of the convex combination of GCGPM, GMOPCGM, and STTDFPM. The methods were proved to be globally convergent. GCGPM exhibited better performance in solving the high-dimensional systems of nonlinear equations, where it won in every aspect. GMOPCGM was the best in signal reconstruction with the shortest time, few iterations, and few function evaluations. Therefore, GCGPM generally stands out among the six schemes we worked with.

6.2 Areas of further Research

we have formulated a generalization of the Modified Optimal Perry conjugate gradient and Conjugate Gradient projection methods. Then we gave a framework that combined the two methods with spectral three-term derivative-free methods. We studied some of the properties of the Generalized Methods and the Framework.

However, further, we point out some feasible areas that may require attention for further research.

Trust region property of the Generalized Conjugate gradient projection method

Due to time constraints, we did not determine whether the search direction satisfied the property of the trust region. If not, it would be natural to ask ”How can we make it satisfy such a wonderful property that limits the step size from being more aggressive or being very short which may severely affect its convergence and robustness in general?” One way to do that is to replace the previous search direction in the correction term $r_k p_{k-1}$ of the change in values of G at points x_k and x_{k-1} with the change in values x or by defining s_{k-1} in a way like in GMOPCGM.

Adaptive scaling

In these methods, we suggested spectral conjugate ways of scaling the search direction update matrix that is adaptive to balance the rate of convergence, accuracy, and robustness of the algorithm such that it can handle various problems without adjusting the parameters because not all problems can be handled with same values of the parameters. A question that can arise is ”is there any other way(s) we can adaptively scale the search direction”

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APPENDIX A

TABLES OF RESULTS

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F1	0.0	1000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		3.48E-12	26	95	0	1	5	7.79E-12	17	70	0	0	4	0	0	4	0	1	5
	0.4		9.62396E-12	11	35	0	1	4	5.48346E-12	16	64	0	0	3	3.41151E-12	7	22	0	1	4
	0.5		9.94084E-12	11	35	0	1	4	6.62617E-12	16	64	0	0	3	4.82237E-12	7	22	0	1	4
	0.6		8.90002E-12	11	35	0	1	4	7.6356E-12	16	64	0	0	3	5.62004E-12	7	22	0	1	4
	0.8		1.77727E-12	11	35	0	1	4	9.2168E-12	16	64	0	0	3	1.68589E-12	7	22	0	1	4
	1.0		1.5146E-12	13	43	0	1	4	1.63156E-12	17	68	0	0	3	0	0	4	0	1	4
	1.1		2.25061E-12	14	47	0	1	4	1.67438E-12	17	68	0	0	3	0	0	4	0	1	4
	1 - 1/m		1.55691E-12	13	43	0	1	4	1.63102E-12	17	68	0	0	3	0	0	4	0	1	4
	1/k		2.7179E-12	17	57	0	8	25	7.84911E-12	50	200	0	0	3	8.97779E-12	28	85	6.79134E-12	1	4
	(k-1)/m		8.69654E-12	48	170	0	6	19	5.6226E-12	52	208	0	0	3	6.45777E-12	32	99	8.14695E-12	40	129
	1/m		3.16227E-12	9	29	0	1	4	3.56041E-12	13	52	0	0	3	3.16233E-13	5	19	0	39	125
	$1/3^k$		3.77727E-12	10	32	6.72878E-17	0	4	7.26003E-12	44	176	1.46196E-17	0	3	5.68303E-12	9	28	6.72878E-17	1	4
	k/m		2.48338E-12	49	174	0	6	19	5.89289E-12	52	208	0	0	3	6.42807E-12	32	99	9.39511E-12	0	4
F2	0.0	1000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		2.52527E-12	18	56	0	5	16	6.21497E-12	19	72	0	6	21	-	-	-	0	2	7
	0.4		5.1698E-12	12	38	0	1	4	3.03032E-12	16	63	0	2	9	0	0	3	0	1	4
	0.5		8.20012E-12	12	38	0	1	4	4.8892E-12	16	63	0	2	9	0	0	3	0	1	4
	0.6		1.24159E-12	13	41	0	1	4	7.34433E-12	16	63	0	2	9	0	0	3	0	1	4
	0.8		2.51124E-12	13	41	0	1	4	2.33588E-12	17	67	0	2	9	0	0	3	0	1	4
	1.0		4.60161E-12	13	41	0	2	7	4.1176E-12	17	67	0	2	9	0	0	3	0	2	7
	1.1		6.06767E-12	13	41	0	2	7	5.32412E-12	17	67	0	2	9	0	0	3	0	2	7
	1 - 1/m		4.58758E-12	13	41	0	2	7	4.11058E-12	17	67	0	2	9	0	0	3	0	2	7
	1/k		5.44465E-12	13	41	0	5	16	6.78402E-12	47	187	7.12238E-14	5	18	8.66031E-12	45	135	9.63089E-12	47	152
	(k-1)/m		1.73016E-12	13	41	0	8	25	8.88651E-12	73	291	0	2	9	6.87084E-12	52	158	8.59636E-12	44	142
	1/m		3.47225E-12	9	29	0	1	4	3.40911E-12	10	39	0	0	3	0	0	3	0	1	4
	$1/3^k$		7.73083E-12	17	53	3.13807E-16	8	28	4.86342E-12	47	187	3.84298E-16	1	6	6.72286E-12	45	137	8.90298E-12	17	54
	k/m		1.73568E-12	13	41	0	8	25	8.29486E-12	72	287	0	2	9	6.88853E-12	52	158	8.65065E-12	44	142

Table A.1: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F3	0.0	1000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		1.13751E-12	14	50	0	1	13	1.98011E-12	16	75	0	1	15	0	0	12	0	1	13
	0.4		7.0006E-12	15	53	0	1	4	2.63313E-12	16	64	0	0	3	0	0	4	0	1	4
	0.5		1.93798E-12	18	64	0	2	8	2.71036E-12	16	64	0	0	3	0	0	4	0	2	8
	0.6		5.56116E-12	20	72	0	2	8	2.66121E-12	16	64	0	0	3	0	0	4	0	2	8
	0.8		5.5401E-12	22	80	0	1	5	1.89585E-12	16	64	0	1	7	0	0	4	0	1	5
	1.0		3.51786E-12	23	84	0	1	5	3.0474E-12	17	70	0	1	7	0	0	4	0	1	5
	1.1		3.72851E-12	21	78	0	1	5	3.0474E-12	17	70	0	1	7	0	0	4	0	1	5
	1 - 1/m		3.55999E-12	23	84	0	1	5	3.0474E-12	17	70	0	1	7	0	0	4	0	1	5
	1/k		5.13373E-12	33	115	0	6	20	8.18987E-12	49	198	0	1	7	7.9172E-12	50	152	7.93677E-12	39	127
	(k-1)/m		5.23874E-12	61	220	0	8	26	5.75671E-12	62	250	9.62986E-12	15	47	7.65506E-12	55	167	5.88419E-12	45	146
	1/m		3.14571E-12	9	29	0	1	4	3.55296E-12	13	52	0	0	3	3.30018E-13	5	19	0	1	4
	$1/3^k$		4.6145E-12	21	70	3.14018E-16	0	4	5.66004E-12	48	192	2.22045E-16	0	3	2.22045E-16	0	4	3.14018E-16	0	4
	k/m		5.31508E-12	61	220	0	8	26	9.0133E-12	61	247	7.67475E-12	16	50	7.77063E-12	55	167	7.16232E-12	45	146
F4	0.0	1000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		9.00003E-12	97	389	0	3	11	9.04458E-12	15	85	0	1	6	0	2	11	0	3	11
	0.4		7.85633E-12	92	371	0	1	6	7.00582E-12	13	78	0	0	4	0	0	5	0	1	6
	0.5		9.97857E-12	92	371	0	1	6	9.32786E-12	13	78	0	0	4	0	0	5	0	1	6
	0.6		8.8836E-12	93	375	0	1	6	1.10657E-12	14	84	0	0	4	0	0	5	0	1	6
	0.8		8.93562E-12	94	379	0	2	11	5.41978E-12	12	71	0	0	4	0	0	5	0	2	11
	1.0		8.43271E-12	95	383	0	2	11	1.6227E-12	13	77	0	0	4	0	0	5	0	2	11
	1.1		9.43617E-12	95	383	0	2	11	2.36699E-12	13	77	0	0	4	0	0	5	0	2	11
	1 - 1/m		8.42284E-12	95	383	0	2	11	1.61592E-12	13	77	0	0	4	0	0	5	0	2	11
	1/k		9.47025E-12	116	465	9.98874E-12	12	57	8.16996E-12	65	389	0	10	45	6.2004E-12	35	109	7.35066E-12	11	55
	(k-1)/m		8.37718E-12	113	453	4.64888E-12	17	86	9.15529E-12	88	531	8.47213E-12	28	135	7.87682E-12	7	23	5.68934E-12	12	61
	1/m		9.99439E-12	72	291	0	1	6	1.58276E-12	11	66	0	0	4	0	0	5	0	1	6
	$1/3^k$		9.45902E-12	107	429	1.8968E-14	0	5	8.12121E-12	24	143	1.73123E-15	0	4	6.09615E-16	0	5	1.8968E-14	0	5
	k/m		8.3807E-12	113	453	4.64366E-12	17	86	8.33945E-12	96	576	8.59572E-12	28	135	7.85897E-12	7	23	5.60808E-12	12	61

Table A.2: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F5	0.0	1000	8.56643E-12	12	38	5.72968E-12	35	107	6.33354E-12	17	68	6.17907E-12	8	24	8.58048E-12	7	22	5.74372E-12	35	107
	0.2		1.72733E-12	13	41	5.08369E-12	36	110	2.03628E-12	18	72	2.38737E-13	8	27	7.80809E-12	7	24	5.09773E-12	36	110
	0.4		7.30253E-12	12	38	4.88708E-12	35	107	5.39264E-12	17	68	5.26625E-12	8	24	7.31658E-12	7	22	4.90112E-12	35	107
	0.5		6.99358E-12	12	38	4.67643E-12	35	107	5.16795E-12	17	68	5.04156E-12	8	24	6.99358E-12	7	22	4.69047E-12	35	107
	0.6		6.68463E-12	12	38	4.46578E-12	35	107	4.92921E-12	17	68	4.81686E-12	8	24	6.67058E-12	7	22	4.47982E-12	35	107
	0.8		6.05268E-12	12	38	9.1703E-12	34	104	4.46578E-12	17	68	4.36748E-12	8	24	6.05268E-12	7	22	9.21243E-12	34	104
	1.0		5.42073E-12	12	38	8.21535E-12	34	104	4.00235E-12	17	68	3.90405E-12	8	24	5.43477E-12	7	22	8.24344E-12	34	104
	1.1		5.09773E-12	12	38	7.73788E-12	34	104	3.76361E-12	17	68	3.67935E-12	8	24	5.11177E-12	7	22	7.76596E-12	34	104
	1 - 1/m		5.42073E-12	12	38	8.22939E-12	34	104	4.00235E-12	17	68	3.91809E-12	8	24	5.42073E-12	7	22	8.25748E-12	34	104
	1/k		8.54815E-12	12	38	5.70804E-12	35	107	9.1568E-12	38	152	6.15964E-12	8	24	8.55692E-12	7	22	5.72815E-12	35	107
	(k-1)/m		7.05355E-12	12	38	4.71068E-12	35	107	9.8646E-12	39	156	5.08599E-12	8	24	7.06173E-12	7	22	4.72809E-12	35	107
	1/m		8.56643E-12	12	38	5.71564E-12	35	107	6.33354E-12	17	68	6.17907E-12	8	24	8.56643E-12	7	22	5.74372E-12	35	107
	$1/3^k$		8.56499E-12	12	38	5.72847E-12	35	107	6.50405E-12	40	160	6.17795E-12	8	24	8.56482E-12	7	22	5.74284E-12	35	107
	k/m		7.05062E-12	12	38	4.7087E-12	35	107	9.87241E-12	39	156	5.08365E-12	8	24	7.05872E-12	7	22	4.72601E-12	35	107
F6	0.0	1000	8.93156E-12	155	623	9.71799E-12	26	107	2.00118E-12	11	55	1.75542E-12	10	40	4.69047E-12	11	34	9.7531E-12	26	107
	0.2		8.67878E-12	155	622	9.71799E-12	27	110	7.80809E-12	13	63	1.77648E-12	10	39	2.6893E-12	13	40	9.7531E-12	27	110
	0.4		8.90347E-12	147	591	5.93331E-12	25	103	6.21418E-13	11	55	3.32827E-12	9	36	9.47223E-12	11	34	5.95437E-12	25	103
	0.5		9.84438E-12	134	539	4.6975E-12	23	95	1.20773E-12	10	50	3.58105E-12	8	32	5.97895E-12	10	33	4.71154E-12	23	95
	0.6		8.86134E-12	146	587	4.42014E-12	25	103	6.17907E-13	11	55	3.13166E-12	8	32	8.49622E-12	11	34	4.43418E-12	25	103
	0.8		9.85842E-12	152	611	4.42365E-12	26	107	2.48567E-12	11	55	5.54712E-13	9	40	1.93798E-12	12	37	4.44472E-12	26	107
	1.0		9.99183E-12	155	623	6.25631E-12	26	107	5.37158E-12	11	55	2.12757E-12	10	40	3.24401E-12	12	37	6.27737E-12	26	107
	1.1		8.51728E-12	157	631	6.68463E-12	26	107	7.36573E-12	11	55	3.47573E-12	10	40	3.89E-12	12	37	6.69867E-12	26	107
	1 - 1/m		9.97077E-12	155	623	6.24928E-12	26	107	5.35051E-12	11	55	2.11352E-12	10	40	3.25103E-12	12	37	6.26333E-12	26	107
	1/k		8.80998E-12	155	623	7.42622E-12	38	190	7.94789E-12	36	244	3.67904E-12	31	149	9.93882E-12	44	137	9.20385E-12	49	247
	(k-1)/m		9.39991E-12	152	611	9.0243E-12	35	174	4.64088E-12	45	293	4.05327E-12	26	122	7.65408E-12	64	198	5.08176E-12	39	195
	1/m		8.91752E-12	155	623	9.69692E-12	26	107	2.00118E-12	11	55	1.75542E-12	10	40	4.85197E-12	11	34	9.73203E-12	26	107
	$1/3^k$		8.85698E-12	155	623	2.78311E-12	42	203	3.85011E-12	38	257	4.85072E-12	24	112	7.92819E-12	61	189	3.59985E-12	40	202
	k/m		9.40107E-12	152	611	9.89172E-12	34	165	4.39082E-12	45	293	2.2557E-12	20	94	5.74855E-12	65	203	2.96859E-12	39	195

Table A.3: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F8	0.0	1000	9.0449E-12	155	623	7.58013E-12	34	173	7.03731E-12	49	342	4.13856E-13	20	109	7.54797E-12	64	210	6.11255E-12	38	195
	0.2		8.87306E-12	268	1304	1.2047E-12	38	198	9.4422E-12	61	443	9.08169E-12	85	424	7.35183E-12	72	240	5.577E-12	35	186
	0.4		9.42147E-12	593	2321	5.14797E-12	94	467	8.16241E-12	59	412	8.86548E-12	92	446	9.42922E-12	208	632	9.78545E-12	120	677
	0.5		9.79924E-12	540	2097	7.97373E-12	47	245	1.60118E-12	56	391	6.14559E-12	132	629	9.60529E-12	109	347	6.03E-13	38	204
	0.6		9.26853E-12	691	2835	1.92393E-12	41	211	9.70343E-12	520	3275	7.30627E-12	131	626	9.94783E-12	227	680	8.84227E-12	28	151
	0.8		9.67312E-12	642	2538	5.93579E-12	40	210	3.59598E-12	51	357	8.88755E-12	72	345	9.59838E-12	205	620	9.9041E-12	71	362
	1.0		9.58799E-12	609	2446	7.22003E-12	39	201	7.45506E-12	55	387	7.40082E-12	132	631	9.81016E-12	224	680	7.16231E-12	36	188
	1.1		9.54796E-12	555	2205	9.96003E-12	44	231	2.59891E-12	51	359	7.54634E-12	130	621	9.5103E-12	216	654	5.13794E-12	31	164
	1 - 1/m		9.95369E-12	590	2340	2.7036E-12	41	214	7.95124E-12	51	359	8.00975E-12	132	632	9.54206E-12	223	675	9.57426E-12	84	427
	1/k		9.35022E-12	400	1519	5.93258E-12	39	208	7.05602E-12	54	373	5.99479E-12	31	156	9.75001E-12	115	358	9.85137E-12	29	163
	(k-1)/m		9.56506E-12	594	2341	3.61775E-12	67	341	6.36133E-12	56	395	9.49224E-12	133	637	9.6292E-12	218	662	8.34353E-12	99	565
	1/m		9.89158E-12	410	1544	2.12635E-12	42	214	7.14829E-12	57	398	6.28879E-12	28	149	9.623E-12	128	396	8.88285E-12	29	151
	$1/3^k$		8.76605E-12	149	599	5.44365E-12	39	199	2.32289E-12	52	362	4.40923E-12	36	206	9.82399E-12	71	227	9.21914E-12	28	147
	k/m		9.58763E-12	646	2555	9.17303E-12	113	566	6.81665E-12	57	402	9.58465E-12	133	638	9.94468E-12	215	653	8.96545E-12	107	608
F9	0.0	1000	9.52562E-12	646	2939	NaN		22	251 6.48202E-12	182	1682	4.77383E-12	32	225	6.7523E-12	88	359	NaN	2000	39849
	0.2		NaN	2000	15996	NaN	11	204	NaN	2001	52003	NaN	2001	38003	NaN	2001	38003	NaN	2000	39984
	0.4		9.58225E-12	624	2827	NaN	11	190	5.92469E-12	234	2149	NaN	2001	37946	8.5287E-12	65	264	NaN	2000	39970
	0.5		9.5936E-12	667	3016	5.51836E-12	67	529	8.135E-12	96	900	5.15038E-12	56	347	9.00279E-12	89	361	7.05035E-12	36	258
	0.6		9.7629E-12	614	2794	8.50988E-12	65	513	8.12924E-12	200	1845	NaN	2001	37974	9.37E-12	74	301	6.1703E-12	39	279
	0.8		9.48122E-12	604	2744	5.18804E-12	60	450	5.85153E-12	185	1710	6.32393E-12	34	230	NaN	2001	37929	9.59244E-12	35	244
	1.0		0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	1.1		9.40871E-12	604	2743	9.62676E-12	61	452	5.75289E-12	192	1773	8.29995E-12	34	229	5.6803E-12	59	243	3.87798E-12	40	275
	1 - 1/m		9.70043E-12	463	2112	9.08445E-12	48	357	9.66048E-12	107	995	2.41919E-12	24	167	8.59023E-12	40	167	7.87838E-12	30	210
	1/k		9.43397E-12	664	3012	285.0120454	2000	6055	5.92465E-12	180	1664	3.77346E-12	41	270	9.7306E-12	83	334	265.7400073	2000	6008
	(k-1)/m		9.83249E-12	403	1883	5.45065E-12	71	516	4.00808E-12	36	355	NaN	2001	37934	8.59558E-12	60	247	6.6601E-12	49	340
	1/m		9.81199E-12	631	2860	NaN	19	270	6.47581E-12	183	1691	8.17705E-12	31	217	6.25826E-12	88	359	NaN	2000	39866
	$1/3^k$		9.37892E-12	630	2859	NaN	11	178	6.3616E-12	180	1664	7.98256E-12	29	201	9.8306E-12	92	374	NaN	2000	39898
	k/m		8.57987E-12	370	1741	7.59032E-12	78	611	4.0671E-12	36	355	NaN	2001	37934	9.50689E-12	54	223	1.80695E-12	43	304

Table A.4: Comparison of Optimization Methods

f	x_0	**m**	**MOPCM**			**GMOPCGM**			**CGPM**			**GCGPM**			**STTDFPM**			**Framework**		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F10	0.0	1000	9.38797E-12	150	597	12.76295552	11	34	7.33062E-12	33	182	1.69924E-12	14	51	1.444402965	2001	4003	4.58515E-12	35	138
	0.2		9.38797E-12	151	600	12.76295552	12	38	5.77883E-12	29	156	7.72383E-13	15	59	0.612986817	2001	4003	4.58515E-12	36	141
	0.4		8.32068E-12	139	557	9.12817E-12	33	135	7.80809E-12	29	168	3.02634E-12	10	38	3.34934E-12	14	41	8.71389E-12	34	137
	0.5		9.90055E-12	150	602	5.82096E-12	31	126	8.93858E-12	26	150	8.0047E-13	10	44	2.53482E-12	13	41	9.80225E-12	34	139
	0.6		9.67586E-12	148	594	4.95028E-12	34	138	6.20013E-12	30	175	2.77356E-12	9	35	5.55414E-12	12	39	4.95028E-12	34	138
	0.8		8.71389E-12	144	578	7.80809E-12	34	138	6.73378E-12	30	176	2.15565E-12	9	35	9.98481E-12	13	39	7.83618E-12	34	138
	1.0		8.6858E-12	137	550	8.74198E-12	31	125	8.14513E-12	28	164	1.50264E-12	9	35	3.76361E-12	12	38	5.80692E-12	32	131
	1.1		8.51728E-12	122	490	7.35871E-12	33	133	4.1147E-12	27	158	3.42657E-12	9	35	4.32535E-12	11	35	6.59335E-12	29	119
	1 - 1/m		8.77006E-12	137	550	8.4892E-12	31	125	8.2645E-12	28	164	1.50264E-12	9	35	3.98831E-12	12	38	5.86309E-12	32	131
	1/k		8.25313E-12	151	602	5.6826E-12	52	203	6.86107E-12	42	231	1.49527E-12	15	60	4.964659457	2001	4008	6.81706E-12	35	137
	(k-1)/m		9.53464E-12	160	642	9.22846E-12	35	142	4.24421E-12	37	214	4.41766E-12	12	47	9.63278E-12	68	206	7.722E-12	35	142
	1/m		9.05093E-12	147	585	14.72214594	11	34	8.99476E-12	30	164	2.46461E-12	13	47	1.729315567	2001	4003	8.14513E-12	34	134
	$1/3^k$		8.99685E-12	145	577	6.25302E-12	50	204	8.59002E-12	37	206	6.07259E-12	14	54	2.270046787	2001	4007	7.1771E-12	37	147
	k/m		8.68082E-12	160	642	6.47366E-12	34	137	4.4855E-12	38	218	4.63198E-12	12	47	8.87972E-12	68	206	6.73737E-12	38	154
F11	0.0	1000	2.06748E-12	26	80	7.49445E-12	33	102	9.22669E-12	52	204	1.39029E-12	11	33	8.66712E-12	57	175	4.88325E-12	32	99
	0.2		6.17369E-12	672	5091	8.7039E-12	59	378	5.23657E-12	88	818	4.23038E-12	51	466	7.33844E-12	58	217	7.06042E-12	128	898
	0.4		2.73856E-12	68	237	7.54286E-12	38	115	8.62481E-12	55	219	7.22076E-12	36	109	5.74137E-12	57	174	8.96194E-12	35	108
	0.5		1.3821E-12	81	280	6.21166E-12	36	110	8.14816E-12	59	238	5.57188E-12	34	103	7.58473E-12	59	178	8.9483E-12	58	191
	0.6		9.72129E-12	92	316	5.49179E-12	38	116	7.80793E-12	60	242	5.08979E-12	37	112	8.00974E-12	58	178	8.56145E-12	37	119
	0.8		3.87594E-12	101	347	9.6065E-12	36	111	7.47859E-12	66	267	8.43808E-13	10	34	7.26465E-12	59	175	8.90581E-12	34	107
	1.0		2.59416E-12	106	363	5.5423E-12	42	129	7.94426E-12	61	247	3.70922E-13	10	34	7.19335E-12	56	170	6.81124E-12	63	208
	1.1		2.61597E-12	96	333	9.03344E-12	39	120	5.93064E-12	59	240	8.66424E-12	28	85	8.28006E-12	57	173	4.76282E-12	59	195
	1 - 1/m		1.75194E-12	102	351	7.99305E-12	40	123	4.74518E-12	61	247	3.71064E-13	10	34	6.94076E-12	56	170	6.49832E-12	41	129
	1/k		1.12856E-12	25	77	4.88196E-12	36	109	6.12801E-12	59	232	9.88854E-12	10	30	9.58405E-12	58	177	4.81636E-12	35	106
	(k-1)/m		1.25187E-12	103	356	7.53398E-12	36	113	6.75064E-12	63	257	9.90772E-12	26	79	8.13477E-12	58	174	6.40881E-12	36	116
	1/m		1.88928E-12	26	80	7.2286E-12	35	107	7.13339E-12	56	220	1.38522E-12	11	33	9.08415E-12	58	176	4.73044E-12	34	104
	$1/3^k$		3.09224E-12	34	104	7.08132E-12	32	99	8.23509E-12	56	220	3.36342E-13	10	33	8.34778E-12	57	176	8.88425E-12	34	106
	k/m		5.89911E-12	109	374	4.69437E-12	36	113	6.68538E-12	63	257	6.89699E-12	27	82	7.97405E-12	58	174	9.59323E-12	36	116

Table A.5: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F12	0.0	1000	9.99885E-12	155	623	6.25279E-12	26	107	5.36631E-12	11	55	2.11879E-12	10	40	3.25454E-12	12	37	6.26859E-12	26	107
	0.2		9.8356E-12	149	597	6.25279E-12	27	110	2.63488E-12	11	53	1.5079E-12	11	43	3.55999E-12	13	39	6.26859E-12	27	110
	0.4		8.86837E-12	146	587	4.42189E-12	25	103	6.1264E-13	11	55	3.13693E-12	8	32	8.49797E-12	11	34	4.43594E-12	25	103
	0.5		9.83911E-12	134	539	4.69574E-12	23	95	1.20246E-12	10	50	3.58456E-12	8	32	5.97368E-12	10	33	4.70452E-12	23	95
	0.6		8.90523E-12	147	591	5.93155E-12	25	103	6.19662E-13	11	55	3.33529E-12	9	36	9.47223E-12	11	34	5.94911E-12	25	103
	0.8		9.66006E-12	152	611	5.9614E-12	26	107	1.44822E-12	11	55	7.96959E-13	9	40	7.55531E-12	11	36	5.97895E-12	26	107
	1.0		8.9298E-12	155	623	9.71974E-12	26	107	2.00118E-12	11	55	1.75542E-12	10	40	4.69574E-12	11	34	9.75134E-12	26	107
	1.1		9.13343E-12	156	627	5.36982E-12	26	107	3.61265E-12	10	49	3.19837E-12	10	40	6.89177E-12	12	37	5.38737E-12	26	107
	1 - 1/m		8.9105E-12	155	623	9.69868E-12	26	107	1.99942E-12	11	55	1.74488E-12	10	40	4.8625E-12	11	34	9.73203E-12	26	107
	1/k		9.86388E-12	155	623	6.21984E-12	26	107	5.43453E-12	60	322	2.05554E-12	10	40	7.22611E-12	63	191	6.23654E-12	26	107
	(k-1)/m		9.42076E-12	152	611	6.1213E-12	26	107	5.95597E-12	64	344	5.40154E-13	9	40	8.04981E-12	65	199	5.32338E-12	26	107
	1/m		9.97252E-12	155	623	6.24753E-12	26	107	5.34875E-12	11	55	2.11352E-12	10	40	3.24401E-12	12	37	6.26684E-12	26	107
	$1/3^k$		9.99102E-12	155	623	6.24996E-12	26	107	4.12743E-12	56	299	2.11792E-12	10	40	6.94929E-12	54	165	6.26756E-12	26	107
	k/m		9.41762E-12	152	611	6.13038E-12	26	107	7.88651E-12	59	318	5.41905E-13	9	40	8.0504E-12	65	199	5.33119E-12	26	107
F13	0.0	1000	9.68288E-12	99	399	2.75952E-12	5	26	8.94209E-12	15	89	6.87421E-12	33	132	6.86017E-12	15	49	2.46109E-12	5	26
	0.2		9.72501E-12	113	455	2.75952E-12	6	30	2.6261E-12	18	106	5.5752E-12	34	136	5.98246E-12	17	52	2.46109E-12	6	30
	0.4		7.61851E-12	95	383	6.3195E-14	4	26	3.54594E-12	16	96	4.94325E-12	32	128	5.21359E-12	15	47	5.61733E-14	4	26
	0.5		9.26158E-12	97	391	3.93213E-13	4	26	7.44999E-12	16	96	4.53249E-12	33	132	4.09012E-12	15	49	3.51083E-13	4	26
	0.6		7.94151E-12	99	399	1.24635E-12	5	26	1.77999E-12	17	102	7.02869E-12	33	132	6.22822E-12	15	49	1.1024E-12	5	26
	0.8		9.91459E-12	100	403	5.84905E-12	5	26	2.8192E-12	17	102	4.90463E-12	34	136	4.22002E-12	16	50	5.22763E-12	5	26
	1.0		7.73086E-12	102	411	7.14806E-12	5	29	3.74957E-12	17	102	5.5752E-12	34	136	5.6068E-12	16	50	6.41429E-12	5	29
	1.1		9.03689E-12	102	411	5.16795E-12	5	26	3.00176E-12	16	95	5.5752E-12	34	136	6.98305E-12	16	50	4.73963E-12	5	26
	1 - 1/m		7.72383E-12	102	411	7.11997E-12	5	29	3.74255E-12	17	102	5.5752E-12	34	136	5.60329E-12	16	50	6.38972E-12	5	29
	1/k		9.52002E-12	99	399	2.7082E-12	5	26	8.8069E-12	56	334	6.76695E-12	33	132	6.68273E-12	53	163	2.01786E-12	5	26
	(k-1)/m		9.40882E-12	99	399	9.26946E-12	9	49	4.81558E-12	53	318	6.72317E-12	33	132	9.99247E-12	56	172	5.45666E-14	5	31
	1/m		9.65128E-12	99	399	2.73143E-12	5	26	8.88241E-12	15	89	6.85315E-12	33	132	6.8391E-12	15	49	2.43301E-12	5	26
	$1/3^k$		9.67356E-12	99	399	2.75771E-12	5	26	9.7711E-12	46	275	6.86692E-12	33	132	6.55709E-12	46	142	2.45928E-12	5	26
	k/m		9.42175E-12	99	399	8.2213E-12	9	49	4.80897E-12	53	318	6.77112E-12	33	132	9.84601E-12	56	172	4.99361E-14	5	31

Table A.6: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F14	0.0	1000	8.08896E-12	98	395	9.72501E-13	9	51	1.53072E-12	15	89	8.64367E-12	39	156	5.7683E-12	16	52	9.72501E-13	9	51
	0.2		8.8157E-12	102	407	9.72501E-13	10	54	5.42073E-12	16	90	5.93682E-12	39	155	9.3985E-12	17	54	9.72501E-13	10	54
	0.4		9.68288E-12	94	379	1.6852E-12	8	41	2.9807E-12	15	90	9.74256E-12	38	152	8.00821E-12	16	50	1.69573E-12	8	41
	0.5		7.93799E-12	93	375	1.46753E-12	9	46	1.72382E-12	15	90	6.29492E-12	38	152	5.03102E-12	16	50	1.47806E-12	9	46
	0.6		9.99534E-12	89	359	1.04272E-12	9	46	4.92921E-12	14	84	5.29785E-12	37	148	4.31481E-12	15	49	1.05325E-12	9	46
	0.8		8.66474E-12	92	371	3.7917E-12	9	46	9.8198E-12	14	84	5.66297E-12	38	152	9.41957E-12	15	49	3.8233E-12	9	46
	1.0		8.43302E-12	95	383	4.64483E-12	9	49	2.82622E-12	15	90	6.42834E-12	39	156	3.60563E-12	16	52	4.68345E-12	9	49
	1.1		9.11061E-12	96	387	1.04272E-12	10	50	2.65419E-12	15	89	7.3903E-12	37	148	4.22002E-12	16	52	1.05325E-12	10	50
	1 - 1/m		8.41196E-12	95	383	4.6343E-12	9	49	2.82622E-12	15	90	6.4178E-12	39	156	3.58807E-12	16	52	4.6659E-12	9	49
	1/k		8.00713E-12	98	395	9.50224E-13	9	51	6.19149E-12	79	473	8.59386E-12	39	156	6.37322E-12	54	166	9.55043E-13	9	51
	(k-1)/m		9.41633E-12	95	383	9.78516E-12	9	46	7.79906E-12	81	485	5.87278E-12	38	155	6.10266E-12	57	175	9.91767E-12	9	46
	1/m		8.08194E-12	98	395	9.61968E-13	9	51	1.5237E-12	15	89	8.64367E-12	39	156	5.7683E-12	16	52	9.72501E-13	9	51
	$1/3^k$		8.08382E-12	98	395	9.61957E-13	9	51	6.05309E-12	78	467	8.63531E-12	39	156	7.31845E-12	51	157	9.72772E-13	9	51
	k/m		9.39906E-12	95	383	9.77105E-12	9	46	7.45066E-12	81	485	5.95639E-12	38	155	6.08788E-12	57	175	9.88244E-12	9	46
F15	0.0	1000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		0	1	9	0	1	21	0	1	26	0	0	19	0	0	20	0	1	21
	0.4		9.20541E-12	48	195	0	1	7	3.92511E-12	13	91	0	0	5	0	0	6	0	1	7
	0.5		0	1	4	0	1	7	0	1	3	0	0	5	0	0	3	0	1	7
	0.6		0	1	4	0	1	4	4.1147E-12	13	91	0	0	5	0	0	6	0	1	4
	0.8		7.54127E-12	49	199	0	1	7	0	1	4	0	0	3	0	0	6	0	1	7
	1.0		7.40786E-12	49	199	0	1	4	4.14278E-12	14	100	0	0	6	0	0	6	0	1	4
	1.1		6.71271E-12	49	199	0	1	5	0	1	5	0	0	6	0	2	13	0	1	5
	1 - 1/m		7.40084E-12	49	199	0	1	4	4.14278E-12	14	100	0	0	6	0	0	6	0	1	4
	1/k		7.90271E-12	44	179	3.51293E-12	12	73	1.99744E-12	51	367	0	8	48	0	20	72	2.58358E-12	12	73
	(k-1)/m		5.63061E-12	49	199	3.19281E-12	13	79	6.46267E-12	56	403	0	9	52	0	36	121	2.62616E-12	13	79
	1/m		6.50206E-12	39	159	0	5	34	2.63313E-12	11	77	0	0	5	0	0	6	0	10	64
	$1/3^k$		9.85472E-12	42	171	5.45675E-12	10	61	6.63018E-12	49	350	9.42055E-16	0	5	3.14018E-16	0	6	5.44009E-12	10	61
	k/m		5.63558E-12	49	199	3.21346E-12	13	79	9.29694E-12	58	417	0	8	46	0	28	96	2.63205E-12	13	79

Table A.7: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F16	0.0	1000	9.97541E-12	556	2252	2.71428E-12	71	334	9.18418E-12	461	2896	6.25761E-12	89	428	9.31188E-12	179	542	9.1995E-12	35	179
	0.2		9.22478E-12	551	2207	4.52521E-12	82	386	8.67971E-12	235	1496	8.5119E-12	90	429	9.58698E-12	170	516	8.62699E-12	41	209
	0.4		9.88033E-12	563	2305	9.44743E-12	80	381	9.14534E-12	169	1088	8.67259E-12	91	432	9.17772E-12	169	516	6.2982E-12	37	189
	0.5		9.7908E-12	514	2051	4.30752E-12	39	197	9.0793E-12	283	1793	7.34109E-12	84	400	9.8949E-12	171	518	4.63494E-12	38	195
	0.6		9.31576E-12	505	2024	1.8587E-12	42	209	8.69318E-12	328	2069	8.91305E-12	87	417	9.55484E-12	171	518	8.55925E-12	39	199
	0.8		9.35333E-12	471	1854	7.29284E-12	87	414	8.70299E-12	259	1643	7.52062E-12	87	417	9.11263E-12	168	509	6.83367E-12	43	219
	1.0		9.52549E-12	504	2020	6.09476E-12	73	355	9.36134E-12	195	1246	6.21361E-12	90	430	9.43351E-12	162	491	4.79415E-12	34	174
	1.1		9.97883E-12	524	2098	8.91866E-12	67	326	9.58916E-12	194	1239	8.4886E-12	94	442	9.89247E-12	160	485	7.83485E-12	37	190
	1 - 1/m		9.86994E-12	475	1853	5.44728E-12	65	320	9.92533E-12	195	1246	7.70817E-12	92	438	9.43712E-12	162	491	8.72027E-12	35	179
	1/k		9.54002E-12	566	2334	7.45925E-12	50	253	9.33405E-12	99	663	7.48819E-12	89	423	9.22771E-12	178	540	5.90888E-12	40	204
	(k-1)/m		9.45427E-12	601	2467	8.98838E-12	61	299	9.95601E-12	292	1853	5.502E-12	95	452	9.87802E-12	170	515	4.88094E-12	38	191
	1/m		9.51708E-12	577	2359	9.55508E-12	41	207	9.16973E-12	461	2896	6.3748E-12	89	428	9.35055E-12	178	541	9.64005E-12	40	204
	$1/3^k$		9.75127E-12	580	2388	4.94838E-12	38	194	9.73055E-12	475	2981	8.63193E-12	90	427	9.93039E-12	163	498	5.05156E-12	35	179
	k/m		9.65476E-12	507	1999	9.81407E-12	57	278	9.61818E-12	292	1853	9.60204E-12	92	438	9.27358E-12	174	526	6.70168E-12	40	204
F17	0.0	1000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		9.53428E-12	104	419	0	1	5	2.00656E-12	15	89	0	0	4	0	1	8	0	1	5
	0.4		7.43464E-12	92	371	0	1	6	5.5993E-12	13	78	0	0	4	0	0	5	0	1	6
	0.5		9.33003E-12	92	371	0	1	6	7.10737E-12	13	78	0	0	4	0	0	5	0	1	6
	0.6		8.21254E-12	93	375	0	1	6	8.69053E-12	13	78	0	0	4	0	0	5	0	1	6
	0.8		8.09174E-12	94	379	0	1	6	1.12806E-12	14	84	0	0	4	0	0	5	0	1	6
	1.0		7.49987E-12	95	383	0	1	6	1.49987E-12	14	84	0	0	4	0	0	5	0	1	6
	1.1		8.32493E-12	95	383	0	1	6	1.71042E-12	14	84	0	0	4	0	0	5	0	1	6
	1 - 1/m		7.49172E-12	95	383	0	1	6	1.49786E-12	14	84	0	0	4	0	0	5	0	1	6
	1/k		9.54817E-12	84	339	7.35308E-13	3	21	8.80443E-12	111	666	0	0	4	0	0	5	9.23617E-15	3	21
	(k-1)/m		7.97887E-12	93	375	0	1	6	9.02318E-12	132	792	0	0	4	0	0	5	0	1	6
	1/m		9.99287E-12	72	291	0	1	6	1.58177E-12	11	66	0	0	4	0	0	5	0	1	6
	$1/3^k$		9.04974E-12	80	323	0	1	6	9.89661E-12	105	630	9.20322E-16	0	4	1.14518E-16	0	5	0	1	6
	k/m		7.99126E-12	93	375	0	1	6	9.06577E-12	132	792	0	0	4	0	0	5	0	1	6

Table A.8: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F18	1000	0.2	7.07784E-12	17	53	4.72558E-12	35	107	3.62318E-12	20	77	9.62671E-12	11	33	NaN	NaN	NaN	NaN	2000	39967
	0.4		4.08661E-12	20	72	7.07784E-12	31	96	9.67586E-12	15	60	9.39499E-12	8	25	7.44297E-13	8	27	7.08486E-12	31	96
	0.5		5.48392E-12	19	68	7.21125E-12	26	81	2.27502E-12	16	64	2.59802E-13	8	27	2.9491E-13	8	27	7.2253E-12	26	81
	0.6		3.83383E-12	17	60	9.64777E-12	30	93	2.43652E-12	16	64	9.13519E-12	8	24	5.1539E-12	8	24	9.66884E-12	30	93
	0.8		4.40259E-12	11	36	5.82096E-12	32	98	1.92394E-12	16	64	4.3113E-12	7	21	2.05735E-12	7	21	5.83501E-12	32	98
	1.0		2.90697E-12	1	5	4.95028E-12	3	11	2.97719E-12	3	12	4.42365E-13	1	6	2.87888E-13	1	7	4.9573E-12	3	11
	1.1		4.83091E-12	11	35	4.94325E-12	31	95	6.78293E-12	14	55	4.67643E-12	7	21	3.15975E-13	7	24	4.94325E-12	31	95
	1 - 1/m		3.14571E-12	9	29	7.72383E-12	25	77	3.55296E-12	13	52	6.27035E-12	6	18	3.30018E-13	5	19	7.7449E-12	25	77
	1/k		1.99926E-12	67	241	7.61946E-12	35	111	9.80281E-12	57	234	NaN	2001	38005	7.73605E-12	58	178	4.73892E-12	33	105
	1/m		3.41253E-12	16	57	9.63373E-12	32	101	2.56993E-12	16	67	4.83091E-12	8	27	5.8982E-13	9	32	9.67586E-12	32	101
F19	0.0	1000	9.59873E-12	134	529	0.005923898	11	34	6.37172E-12	28	143	5.85168E-12	15	52	0.001770847	0	3	6.49943E-12	36	134
	0.2		9.59873E-12	135	537	0.005923898	12	54	8.44487E-12	30	175	6.27342E-13	17	80	0.000746299	1	22	6.48637E-12	37	154
	0.4		9.59873E-12	136	540	0.005923898	13	52	8.44487E-12	31	174	6.27342E-13	17	73	0.000748077	1	15	6.48637E-12	38	152
	0.5		8.96853E-12	120	487	0.005923898	12	47	8.44487E-12	31	177	6.27342E-13	17	74	0.000828328	2	19	6.48637E-12	37	147
	0.6		9.59873E-12	140	563	0.005923898	12	48	8.44487E-12	33	193	6.27342E-13	17	74	0.000747086	1	16	6.48637E-12	37	148
	0.8		9.59873E-12	137	549	0.005923898	13	57	8.44487E-12	31	182	5.85761E-13	17	75	0.00074674	1	17	6.48637E-12	38	157
	1.0		9.59873E-12	140	565	0.005923898	12	49	8.44487E-12	33	202	6.27232E-13	17	76	0.00074658	1	17	6.48637E-12	37	149
	1.1		9.59873E-12	137	549	0.005923898	12	49	8.01951E-12	27	186	6.27342E-13	17	76	0.00074653	1	18	6.48637E-12	37	149
	1 - 1/m		9.59873E-12	140	565	0.005923898	12	49	8.44487E-12	33	202	6.25257E-13	17	76	0.00074658	1	17	6.48637E-12	37	149
	1/m		8.83457E-12	127	507	6.16415E-12	37	140	4.54017E-12	26	145	2.16355E-13	10	40	0.087220877	2001	4003	8.57269E-12	25	100
F20	0.0	1000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		0	1	5	0	1	8	0	1	7	0	0	7	0	0	3	0	1	4
	0.4		9.03772E-12	92	369	4.28076E-12	13	61	5.07708E-12	15	85	8.51382E-12	42	167	0	3	14	4.72083E-12	14	66
	0.5		7.40317E-12	81	325	2.83481E-12	13	63	1.43077E-12	16	92	8.46036E-12	44	176	4.09216E-12	17	55	8.19949E-12	12	62
	0.6		9.83404E-12	97	390	4.41589E-12	13	64	2.03849E-12	15	86	8.2508E-12	41	163	7.18205E-12	14	44	5.04456E-12	11	53
	0.8		7.90611E-12	93	374	2.87672E-12	13	62	5.95722E-12	14	81	6.48111E-12	43	171	5.20281E-12	18	56	2.31091E-12	12	60
	1.0		9.85499E-12	80	322	4.49971E-12	14	67	1.30504E-12	15	88	5.81993E-12	43	173	8.66052E-12	15	49	3.86164E-12	13	65
	1.1		8.68147E-12	97	391	2.91863E-12	11	55	2.87672E-12	14	82	6.56494E-12	44	177	8.85852E-12	15	49	2.91863E-12	11	55
	1 - 1/m		8.09471E-12	81	326	4.49971E-12	14	67	1.32599E-12	15	88	7.84324E-12	41	164	9.85499E-12	15	49	3.84069E-12	13	65
	1/k		7.98669E-12	99	393	NaN	29	357	9.52698E-12	125	735	6.25005E-12	56	225	2.679809016	2001	4021	6.24378E-12	20	87
	(k-1)/m		7.86625E-12	99	397	9.12356E-12	29	142	7.6292E-12	115	685	5.97642E-12	43	173	7.70242E-12	61	183	2.18151E-12	49	197
	1/m		8.13662E-12	98	387	2.837198079	11	34	5.55906E-12	18	89	9.43588E-12	48	188	0.2893672	2001	4003	5.72671E-12	22	93
	$1/3^k$		1.12746E+77	2000	15832	0.643201406	11	34	3.00703E-12	56	237	5.41819E-12	32	107	0.621897133	2001	4020	2.96313E-12	81	279
	k/m		7.66416E-12	99	397	5.05355E-12	29	145	6.86756E-12	119	709	6.35077E-12	45	183	9.96526E-12	59	177	4.74034E-12	39	171

Table A.9: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F1	0.0	10000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		1.10096E-12	27	98	0	1	5	3.94075E-12	18	74	0	0	4	0	0	4	0	1	5
	0.4		3.04336E-12	12	38	0	1	4	2.77443E-12	17	68	0	0	3	4.85466E-12	7	24	0	1	4
	0.5		3.14357E-12	12	38	0	1	4	3.3526E-12	17	68	0	0	3	6.86235E-12	7	24	0	1	4
	0.6		2.81443E-12	12	38	0	1	4	3.86334E-12	17	68	0	0	3	7.99745E-12	7	24	0	1	4
	0.8		5.62021E-12	11	35	0	1	4	4.66337E-12	17	68	0	0	3	5.33124E-12	7	22	0	1	4
	1.0		4.78958E-12	13	43	0	1	4	5.15945E-12	17	68	0	0	3	0	0	4	0	1	4
	1.1		7.11705E-12	14	47	0	1	4	5.29487E-12	17	68	0	0	3	0	0	4	0	1	4
	1 - 1/m		4.80297E-12	13	43	0	1	4	5.15928E-12	17	68	0	0	3	0	0	4	0	1	4
	1/k		2.77267E-12	17	57	0	8	25	7.84832E-12	50	200	0	0	3	9.08244E-12	28	85	6.64795E-12	40	129
	(k-1)/m		3.0279E-12	65	231	0	6	19	6.80485E-12	54	216	0	0	3	7.93333E-12	34	105	6.88987E-12	41	132
	1/m		1E-11	8	26	0	1	4	7.03687E-12	12	48	0	0	3	4.5E-12	5	18	0	1	4
	$1/3^k$		3.77727E-12	10	32	3.22963E-16	0	4	7.26003E-12	44	176	7.01701E-17	0	3	5.68303E-12	9	28	3.22963E-16	0	4
	k/m		1.43432E-12	74	257	0	6	19	6.83687E-12	54	216	0	0	3	7.92967E-12	34	105	6.21157E-12	42	135
F2	0.0	10000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		6.99371E-12	18	56	0	5	16	2.73088E-12	20	76	0	6	21	NaN	NaN	NaN	0	2	7
	0.4		1.48755E-12	13	41	0	1	4	8.8809E-12	16	63	0	1	6	0	0	3	0	1	4
	0.5		2.35344E-12	13	41	0	1	4	2.30903E-12	17	67	0	2	9	0	0	3	0	1	4
	0.6		3.55236E-12	13	41	0		1	4 3.46355E-12	17	67	0	2	9	0	0	3	0	1	4
	0.8		7.14912E-12	13	41	0	1	4	6.81609E-12	17	67	0	2	9	0	0	3	0	1	4
	1.0		1.30993E-12	14	44	0	2	7	1.93159E-12	18	71	0	2	9	0	0	3	0	2	7
	1.1		1.73178E-12	14	44	0	2	7	2.48665E-12	18	71	0	2	9	0	0	3	0	2	7
	1 - 1/m		1.30993E-12	14	44	0	2	7	1.93159E-12	18	71	0	2	9	0	0	3	0	2	7
	1/k		5.11443E-12	13	41	0	5	16	8.74753E-12	50	199	5.8826E-13	5	18	7.82584E-12	45	135	7.94847E-12	47	151
	(k-1)/m		4.95025E-12	13	41	0	8	25	4.86658E-12	58	231	0	2	9	6.26084E-12	56	168	7.0366E-12	47	151
	1/m		1.0213E-12	9	29	0	1	4	4.04081E-12	8	31	0	0	3	0	0	3	0	1	4
	$1/3^k$		2.02978E-12	16	49	4.96462E-16	8	28	6.44516E-12	47	187	4.96467E-16	1	6	7.03018E-12	45	137	8.15884E-12	18	57
	k/m		4.95185E-12	13	41	0	8	25	4.71438E-12	58	231	0	2	9	6.26297E-12	56	168	7.08797E-12	47	151

Table A.10: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F3	0.0	10000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		3.59712E-12	14	50	0	1	13	6.26166E-12	16	75	0	1	15	0	0	12	0	1	13
	0.4		2.22045E-12	16	56	0	1	4	8.32667E-12	16	64	0	0	3	0	0	4	0	1	4
	0.5		6.12843E-12	18	64	0	2	8	8.57092E-12	16	64	0	0	3	0	0	4	0	2	8
	0.6		1.75415E-12	21	75	0	2	8	8.39329E-12	16	64	0	0	3	0	0	4	0	2	8
	0.8		1.75415E-12	23	83	0	1	5	5.9952E-12	16	64	0	1	7	0	0	4	0	1	5
	1.0		1.11022E-12	24	87	0	1	5	9.63674E-12	17	70	0	1	7	0	0	4	0	1	5
	1.1		1.17684E-12	22	81	0	1	5	9.63674E-12	17	70	0	1	7	0	0	4	0	1	5
	1 - 1/m		1.11022E-12	24	87	0	1	5	9.63674E-12	17	70	0	1	7	0	0	4	0	1	5
	1/k		5.09748E-12	33	115	0	6	20	5.19397E-12	46	186	0	1	7	7.65214E-12	50	152	9.34744E-12	39	126
	(k-1)/m		1.26213E-12	77	277	0	8	26	9.75686E-12	65	262	8.04764E-12	17	53	6.8671E-12	59	177	8.50377E-12	47	152
	1/m		9.99201E-12	8	26	0	1	4	7.03881E-12	12	48	0	0	3	4.52971E-12	5	18	0	1	4
	$1/3^k$		4.6145E-12	21	70	3.65532E-15	0	4	5.66004E-12	48	192	6.66134E-16	0	3	4.96507E-16	0	4	3.65532E-15	0	4
	k/m		1.04821E-12	77	277	0	8	26	9.9586E-12	58	235	8.55103E-12	17	53	6.87734E-12	59	177	8.6951E-12	47	152
F4	0.0	10000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		8.08231E-12	101	405	0	3	11	2.65422E-12	16	91	0	1	6	0	2	11	0	3	11
	0.4		9.66469E-12	95	383	0	1	6	2.05592E-12	14	84	0	0	4	0	0	5	0	1	6
	0.5		8.96106E-12	96	387	0	1	6	2.73735E-12	14	84	0	0	4	0	0	5	0	1	6
	0.6		7.97775E-12	97	391	0	1	6	3.49927E-12	14	84	0	0	4	0	0	5	0	1	6
	0.8		8.02447E-12	98	395	0	2	11	1.59048E-12	13	77	0	0	4	0	0	5	0	2	11
	1.0		7.57284E-12	99	399	0	2	11	5.13144E-12	13	77	0	0	4	0	0	5	0	2	11
	1.1		8.47397E-12	99	399	0	2	11	7.48508E-12	13	77	0	0	4	0	0	5	0	2	11
	1 - 1/m		7.57195E-12	99	399	0	2	11	5.12929E-12	13	77	0	0	4	0	0	5	0	2	11
	1/k		9.46903E-12	116	465	5.03424E-12	9	46	6.64537E-12	56	335	0	10	45	2.92884E-12	36	110	2.60906E-12	11	55
	(k-1)/m		8.22498E-12	107	429	6.61088E-12	18	91	7.01936E-12	91	549	3.78123E-12	31	149	2.27022E-12	2	11	4.84616E-12	10	51
	1/m		8.12322E-12	69	279	0	1	6	5.39042E-12	10	60	0	0	4	0	0	5	0	1	6
	$1/3^k$		9.45902E-12	107	429	2.13387E-14	0	5	8.12121E-12	24	143	1.97569E-15	0	4	6.87705E-16	0	5	2.13387E-14	0	5
	k/m		8.2253E-12	107	429	6.61361E-12	18	91	8.43669E-12	85	514	5.49545E-12	30	145	2.05809E-12	2	11	3.91288E-12	9	46

Table A.11: Comparison of Optimization Methods

f	x_0	**m**	**MOPCM**			**GMOPCGM**			**CGPM**			**GCGPM**			**STTDFPM**			**Framework**		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F5	0.0	10000	2.70894E-12	13	41	7.9492E-12	36	110	3.19744E-12	18	72	3.55271E-13	8	27	2.66454E-13	7	25	7.9492E-12	36	110
	0.2		5.4623E-12	13	41	7.06102E-12	37	113	6.4837E-12	18	72	7.54952E-13	8	27	5.77316E-13	7	25	7.06102E-12	37	113
	0.4		2.30926E-12	13	41	6.75016E-12	36	110	2.75335E-12	18	72	3.10862E-13	8	27	2.22045E-13	7	25	6.79456E-12	36	110
	0.5		2.22045E-12	13	41	6.4837E-12	36	110	2.62013E-12	18	72	3.10862E-13	8	27	9.99201E-12	7	24	6.4837E-12	36	110
	0.6		2.13163E-12	13	41	6.17284E-12	36	110	2.4869E-12	18	72	2.66454E-13	8	27	9.54792E-12	7	24	6.21725E-12	36	110
	0.8		1.90958E-12	13	41	5.59552E-12	36	110	2.26485E-12	18	72	2.66454E-13	8	27	8.61533E-12	7	24	5.63993E-12	36	110
	1.0		1.73195E-12	13	41	5.01821E-12	36	110	2.04281E-12	18	72	2.22045E-13	8	27	7.72715E-12	7	24	5.01821E-12	36	110
	1.1		1.59872E-12	13	41	4.70735E-12	36	110	1.90958E-12	18	72	2.22045E-13	8	27	7.28306E-12	7	24	4.75175E-12	36	110
	1 - 1/m		1.73195E-12	13	41	5.01821E-12	36	110	2.04281E-12	18	72	2.22045E-13	8	27	7.72715E-12	7	24	5.01821E-12	36	110
	1/k		2.71247E-12	13	41	7.93467E-12	36	110	9.91663E-12	38	152	3.78301E-13	8	27	2.69697E-13	7	25	7.96584E-12	36	110
	(k-1)/m		2.23646E-12	13	41	6.53159E-12	36	110	5.03914E-12	41	164	3.03344E-13	8	27	2.25677E-13	7	25	6.55656E-12	36	110
	1/m		2.70894E-12	13	41	7.9492E-12	36	110	3.19744E-12	18	72	3.55271E-13	8	27	2.66454E-13	7	25	7.9492E-12	36	110
	$1/3^k$		2.70891E-12	13	41	7.94912E-12	36	110	7.74431E-12	36	144	3.55309E-13	8	27	2.66574E-13	7	25	7.94902E-12	36	110
	k/m		2.23629E-12	13	41	6.53074E-12	36	110	5.03775E-12	41	164	3.03574E-13	8	27	2.25214E-13	7	25	6.55542E-12	36	110
F6	0.0	10000	9.34808E-12	161	647	3.70814E-12	28	115	6.32827E-12	11	55	5.55112E-12	10	40	6.01741E-12	11	36	3.73035E-12	28	115
	0.2		9.05942E-12	161	646	3.70814E-12	29	118	1.39888E-12	14	68	5.61773E-12	10	39	8.50431E-12	13	40	3.73035E-12	29	118
	0.4		9.30367E-12	153	615	6.51701E-12	26	107	1.96509E-12	11	55	4.88498E-13	9	40	2.02061E-12	12	37	6.53921E-12	26	107
	0.5		8.54872E-12	141	567	5.17364E-12	24	99	3.81917E-12	10	50	5.32907E-13	8	36	3.17524E-12	11	34	5.19584E-12	24	99
	0.6		9.25926E-12	152	611	4.88498E-12	26	107	1.95399E-12	11	55	9.90319E-12	8	32	1.84297E-12	12	37	4.88498E-12	26	107
	0.8		8.54872E-12	159	639	4.87388E-12	27	111	7.86038E-12	11	55	1.75415E-12	10	40	6.12843E-12	12	37	4.87388E-12	27	111
	1.0		8.68194E-12	162	651	6.87228E-12	27	111	9.76996E-13	12	60	6.72795E-12	10	40	4.17444E-12	12	39	6.90559E-12	27	111
	1.1		8.90399E-12	163	655	7.34968E-12	27	111	1.33227E-12	12	60	5.10703E-13	10	44	5.01821E-12	12	39	7.39409E-12	27	111
	1 - 1/m		8.68194E-12	162	651	6.87228E-12	27	111	9.76996E-13	12	60	6.72795E-12	10	40	4.17444E-12	12	39	6.90559E-12	27	111
	1/k		9.32026E-12	161	647	8.1963E-12	36	176	8.43857E-12	38	258	2.57018E-12	28	132	8.84882E-12	53	162	9.21889E-12	35	176
	(k-1)/m		9.84134E-12	158	635	8.87195E-12	35	174	7.08656E-12	40	258	3.82505E-12	21	98	9.77148E-12	66	201	7.93346E-12	40	199
	1/m		9.32587E-12	161	647	3.70814E-12	28	115	6.32827E-12	11	55	5.50671E-12	10	40	6.08402E-12	11	36	3.73035E-12	28	115
	$1/3^k$		9.31935E-12	161	647	4.75401E-12	37	185	3.9363E-12	38	258	2.71483E-12	27	128	6.73358E-12	52	162	5.46696E-12	39	196
	k/m		9.84157E-12	158	635	7.97332E-12	34	168	7.8573E-12	40	258	3.46021E-12	21	98	7.26582E-12	67	205	6.64934E-12	38	189

Table A.12: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F8	0.0	10000	8.9536E-12	153	615	1.50201E-12	41	212	8.48138E-12	51	359	4.78203E-12	37	244	6.72604E-12	80	261	3.82173E-12	32	165
	0.2		9.38159E-12	152	618	9.79227E-12	45	234	6.75787E-12	55	401	8.55385E-12	93	486	9.6615E-12	77	259	2.97009E-12	28	155
	0.4		9.81387E-12	667	2715	3.75944E-12	42	222	7.90505E-12	49	342	7.06462E-12	40	225	9.92599E-12	194	608	4.33849E-12	38	195
	0.5		9.20428E-12	563	2191	8.25441E-12	33	169	8.11182E-12	48	335	8.08054E-12	129	615	7.60308E-12	63	209	9.19376E-12	24	132
	0.6		9.60487E-12	633	2568	9.08596E-12	43	224	9.29498E-12	53	370	9.14665E-12	123	587	9.54761E-12	227	683	1.81727E-12	42	222
	0.8		9.5967E-12	594	2332	9.78262E-12	40	215	2.22632E-12	54	378	9.23428E-12	67	322	9.44703E-12	195	591	9.98943E-12	65	359
	1.0		9.95095E-12	583	2227	8.84619E-12	43	223	4.42772E-12	55	387	9.74457E-12	126	602	9.17536E-12	187	571	9.4529E-12	67	342
	1.1		9.74568E-12	574	2248	7.19619E-12	48	247	9.256E-12	58	408	6.90943E-12	129	617	9.85254E-12	193	589	7.22175E-12	25	135
	1 - 1/m		9.25384E-12	579	2205	4.63099E-12	39	203	6.20801E-12	55	387	9.10206E-12	126	602	9.26608E-12	215	648	9.03363E-12	72	367
	1/k		9.74627E-12	345	1299	8.01703E-12	34	175	9.39584E-12	59	412	9.04382E-12	41	258	8.53277E-12	75	243	9.62924E-12	31	163
	(k-1)/m		9.76758E-12	724	2657	8.72019E-12	46	241	8.56175E-12	55	389	7.69128E-12	130	623	9.43388E-12	218	661	8.7106E-12	112	623
	1/m		9.6046E-12	346	1306	2.12675E-12	44	229	8.00135E-12	49	343	4.77706E-12	52	367	9.28967E-12	73	230	6.64848E-12	45	246
	$1/3^k$		8.67785E-12	147	591	9.16336E-12	40	207	3.42408E-12	54	375	4.06832E-12	36	269	7.89688E-12	68	217	7.49986E-12	72	365
	k/m		9.4459E-12	583	2280	6.40213E-12	44	223	6.99492E-12	57	403	7.44305E-12	130	623	9.53832E-12	213	647	5.41289E-12	46	242
F9	0.0	10000	9.80097E-12	652	2967	NaN	38	590	7.4657E-12	164	1519	3.49401E-12	28	196	9.68998E-12	65	271	NaN	2000	39878
	0.4		9.37404E-12	645	2924	NaN	11	190	8.50879E-12	235	2157	NaN	2001	37946	NaN	2001	37909	NaN	2000	39970
	0.5		2.04123E-11	2000	8354	3.66016E-12	57	423	7.93386E-12	77	728	NaN	2001	37962	9.64347E-12	70	283	4.35017E-12	38	268
	0.6		9.69179E-12	656	2990	6.50994E-12	55	415	7.1044E-12	180	1664	NaN	2001	37974	7.75173E-12	59	241	3.78714E-12	33	238
	0.8		9.98784E-12	640	2911	6.61058E-12	72	552	7.04814E-12	170	1574	9.74753E-12	30	203	8.56623E-12	63	258	2.20692E-12	33	237
	1.0		0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	1.1		2.21579E-11	2000	8326	NaN	34	363	6.42546E-12	176	1628	7.26352E-12	27	185	6.68418E-12	56	232	7.92713E-12	43	297
	1 - 1/m		9.98929E-12	442	2018	9.48014E-12	51	390	5.88917E-12	78	730	2.49433E-12	24	162	5.04836E-12	37	155	3.45032E-12	29	209
	1/k		9.69613E-12	653	2971	NaN	15	203	7.16613E-12	164	1519	6.2952E-12	30	210	9.14106E-12	71	288	NaN	2000	39942
	(k-1)/m		8.90881E-12	360	1702	4.18198E-12	80	638	4.87404E-12	37	365	NaN	2001	37934	8.72454E-12	59	243	2.79029E-12	44	302
	1/m		9.80034E-12	652	2967	NaN	74	1255	7.46638E-12	164	1519	8.46089E-12	29	201	7.64742E-12	70	292	NaN	2000	39862
	$1/3^k$		9.53734E-12	650	2961	6.94045E-12	65	478	7.02995E-12	165	1528	5.53898E-12	31	209	7.79689E-12	82	334	NaN	2000	39949
	k/m		9.41793E-12	351	1664	7.35348E-12	65	477	4.87204E-12	37	365	NaN	2001	37934	9.39723E-12	61	251	3.31533E-12	42	297

Table A.13: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F10	0.0	10000	9.08162E-12	156	621	40.3600091	11	34	9.08162E-12	34	188	5.37348E-12	14	51	4.567603229	2001	4003	6.4837E-12	36	142
	0.2		9.08162E-12	157	624	40.3600091	12	40	7.19425E-12	31	170	2.44249E-12	16	59	1.938434516	2001	4003	6.4837E-12	37	145
	0.4		9.70335E-12	144	577	5.75096E-12	35	143	9.70335E-12	30	174	9.57012E-12	10	38	4.26326E-12	14	43	5.50671E-12	36	145
	0.5		9.4369E-12	156	626	8.28226E-12	32	130	4.35207E-12	28	162	2.53131E-12	11	44	8.01581E-12	13	41	6.19504E-12	36	147
	0.6		9.17044E-12	154	618	6.92779E-12	35	142	7.63833E-12	31	181	8.77076E-12	9	35	3.59712E-12	13	40	7.01661E-12	35	142
	0.8		8.28226E-12	150	602	4.9738E-12	36	146	8.37108E-12	31	182	6.81677E-12	9	35	2.53131E-12	14	42	4.9738E-12	36	146
	1.0		8.28226E-12	143	574	5.50671E-12	33	133	3.9968E-12	30	176	4.75175E-12	9	35	4.75175E-12	12	40	8.14904E-12	33	135
	1.1		9.88098E-12	127	510	4.61853E-12	35	141	5.06262E-12	28	164	2.44249E-13	9	39	5.4623E-12	11	37	9.30367E-12	30	123
	1 - 1/m		8.28226E-12	143	574	5.50671E-12	33	133	3.9968E-12	30	176	4.75175E-12	9	35	4.84057E-12	12	40	8.23785E-12	33	135
	1/k		8.47801E-12	155	617	5.7672E-12	52	212	5.79455E-12	42	230	9.16043E-12	48	186	6.942239527	2001	4009	5.68959E-12	38	152
	(k-1)/m		8.72376E-12	166	666	5.18808E-12	37	150	5.08697E-12	38	220	3.50042E-13	12	51	7.7491E-12	71	215	4.49122E-12	37	150
	1/m		8.4599E-12	156	621	41.00431219	11	34	7.4607E-12	34	188	7.43849E-12	14	51	4.657922598	2001	4003	6.30607E-12	36	142
	$1/3^k$		8.83663E-12	155	617	7.58561E-12	42	173	9.0578E-12	54	282	5.77684E-12	13	48	4.99759511	2001	4009	4.76825E-12	36	142
	k/m		8.64226E-12	166	666	5.68449E-12	39	157	5.06871E-12	38	220	3.43316E-13	12	51	7.6871E-12	71	215	9.84606E-12	36	146
F11	0.0	10000	5.98159E-12	18	56	4.51697E-12	36	110	8.52982E-12	59	232	9.59657E-12	12	36	9.42557E-12	60	183	8.9094E-12	34	104
	0.2		5.31966E-12	81	348	NaN	20	351	8.59235E-12	91	884	5.54499E-12	86	710	7.58435E-12	63	232	6.36776E-12	120	924
	0.4		7.27645E-12	83	292	5.28598E-12	41	130	9.94895E-12	61	243	9.45192E-12	36	109	5.6862E-12	61	186	5.69831E-12	66	232
	0.5		5.00737E-12	92	323	7.28616E-12	42	131	5.83377E-12	63	253	5.59048E-12	33	100	6.55476E-12	61	186	8.74728E-12	57	190
	0.6		1.19914E-12	107	372	8.00095E-12	39	119	6.14962E-12	56	229	6.06978E-12	38	115	7.91468E-12	57	177	6.47486E-12	36	120
	0.8		5.23271E-12	112	390	5.0872E-12	37	113	5.8927E-12	66	267	1.21688E-12	13	40	7.26017E-12	62	184	6.34899E-12	34	108
	1.0		1.11145E-12	120	414	4.61298E-12	39	129	5.5465E-12	62	251	1.85392E-12	11	34	6.51649E-12	59	179	7.05671E-12	62	219
	1.1		1.57718E-12	107	375	4.6334E-12	44	148	5.49879E-12	59	246	7.98228E-12	14	43	8.59072E-12	61	183	6.976E-12	36	114
	1 - 1/m		1.41521E-12	113	393	4.77364E-12	39	129	8.66958E-12	61	247	1.85918E-12	11	34	6.49912E-12	59	179	8.86098E-12	41	140
	1/k		2.46538E-12	38	116	8.55318E-12	37	112	9.43177E-12	58	228	4.41481E-13	11	36	7.87955E-12	58	177	5.86459E-12	37	115
	(k-1)/m		6.05856E-12	123	426	8.12104E-12	39	121	5.69068E-12	68	277	3.48851E-13	13	43	7.46011E-12	61	183	9.20826E-12	64	215
	1/m		5.98164E-12	18	56	9.92645E-12	37	112	8.54206E-12	59	232	9.59583E-12	12	36	9.4155E-12	60	183	8.62643E-12	36	109
	$1/3^k$		6.99679E-12	18	56	9.61342E-12	33	101	8.75539E-12	60	236	4.72225E-12	12	36	9.13861E-12	60	183	8.93694E-12	34	104
	k/m		2.80838E-12	130	447	5.38662E-12	40	125	7.48989E-12	68	277	3.77786E-13	13	43	7.44671E-12	61	183	5.38376E-12	40	129

Table A.14: Comparison of Optimization Methods
%labeltab:comparison

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F12	0.0	10000	8.68194E-12	162	651	6.87783E-12	27	111	9.49241E-13	12	60	6.7002E-12	10	40	4.17999E-12	12	39	6.91114E-12	27	111
	0.2		8.54317E-12	156	625	6.87783E-12	28	114	8.33222E-12	11	53	4.76841E-12	11	43	4.57967E-12	13	41	6.91114E-12	28	114
	0.4		9.26481E-12	152	611	4.85723E-12	26	107	1.93734E-12	11	55	9.91984E-12	8	32	1.85407E-12	12	37	4.89053E-12	26	107
	0.5		8.55427E-12	141	567	5.16809E-12	24	99	3.80251E-12	10	50	5.27356E-13	8	36	3.19744E-12	11	34	5.19029E-12	24	99
	0.6		9.30367E-12	153	615	6.53366E-12	26	107	1.95954E-12	11	55	4.88498E-13	9	40	2.05391E-12	12	37	6.54476E-12	26	107
	0.8		8.38773E-12	159	639	6.56142E-12	27	111	4.57967E-12	11	55	2.52021E-12	10	40	4.02456E-12	12	37	6.58917E-12	27	111
	1.0		9.33142E-12	161	647	3.71925E-12	28	115	6.32827E-12	11	55	5.55112E-12	10	40	6.03961E-12	11	36	3.73035E-12	28	115
	1.1		9.55347E-12	162	651	5.91194E-12	27	111	6.38378E-13	11	54	4.71845E-13	10	44	8.85958E-12	12	39	5.93414E-12	27	111
	1 - 1/m		9.33142E-12	161	647	3.71925E-12	28	115	6.32827E-12	11	55	5.54001E-12	10	40	6.05627E-12	11	36	3.73035E-12	28	115
	1/k		8.66606E-12	162	651	6.87673E-12	27	111	6.34647E-12	64	342	6.68242E-12	10	40	6.58474E-12	57	174	6.89983E-12	27	111
	(k-1)/m		9.84431E-12	158	635	6.74148E-12	27	111	8.73792E-12	62	328	1.71069E-12	10	40	7.34231E-12	68	208	5.86384E-12	27	111
	1/m		8.68194E-12	162	651	6.87783E-12	27	111	9.49241E-13	12	60	6.7002E-12	10	40	4.17999E-12	12	39	6.91114E-12	27	111
	$1/3^k$		8.68127E-12	162	651	6.87722E-12	27	111	4.61995E-12	62	333	6.7166E-12	10	40	8.74076E-12	50	153	6.91047E-12	27	111
	k/m		9.84397E-12	158	635	6.74234E-12	27	111	7.86336E-12	68	368	1.71124E-12	10	40	7.34242E-12	68	208	5.86463E-12	27	111
F13	0.0	10000	9.70335E-12	103	415	8.72635E-12	5	26	4.55191E-12	16	95	8.99281E-12	34	136	8.99281E-12	16	50	7.78266E-12	5	26
	0.2		9.73666E-12	117	471	8.72635E-12	6	30	8.30447E-12	18	106	7.29417E-12	35	140	6.83897E-12	17	54	7.78266E-12	6	30
	0.4		7.62723E-12	99	399	1.9984E-13	4	26	1.80966E-12	17	102	6.4615E-12	33	132	5.9508E-12	15	49	1.77636E-13	4	26
	0.5		9.28146E-12	101	407	1.24345E-12	5	26	3.79696E-12	17	102	5.93969E-12	34	136	5.36238E-12	16	50	1.11022E-12	5	26
	0.6		7.9714E-12	103	415	3.94129E-12	5	26	5.62883E-12	17	102	9.20375E-12	34	136	8.17124E-12	16	50	3.4861E-12	5	26
	0.8		9.9476E-12	104	419	8.23785E-12	5	29	8.91509E-12	17	102	6.42819E-12	35	140	4.82947E-12	16	52	7.34968E-12	5	29
	1.0		7.74936E-12	106	427	5.55112E-14	5	31	1.89848E-12	18	108	7.29417E-12	35	140	6.41709E-12	16	52	3.33067E-14	5	31
	1.1		9.05942E-12	106	427	7.27196E-12	5	29	9.49241E-12	16	95	7.29417E-12	35	140	7.9714E-12	16	52	6.67244E-12	5	29
	1 - 1/m		7.74936E-12	106	427	3.33067E-14	5	31	1.89848E-12	18	108	7.29417E-12	35	140	6.41709E-12	16	52	3.33067E-14	5	31
	1/k		9.68539E-12	103	415	8.64024E-12	5	26	6.06141E-12	57	341	8.97472E-12	34	136	8.36575E-12	52	160	7.5768E-12	5	26
	(k-1)/m		9.43255E-12	103	415	8.26419E-12	10	54	5.65411E-12	55	330	8.82719E-12	34	136	7.21327E-12	59	181	1.65908E-13	5	31
	1/m		9.70335E-12	103	415	8.72635E-12	5	26	4.55191E-12	16	95	8.99281E-12	34	136	8.99281E-12	16	50	7.76046E-12	5	26
	$1/3^k$		9.70242E-12	103	415	8.72577E-12	5	26	5.24588E-12	54	323	8.99196E-12	34	136	7.49632E-12	45	139	7.78217E-12	5	26
	k/m		9.43387E-12	103	415	8.16654E-12	10	54	5.65263E-12	55	330	8.83363E-12	34	136	7.20253E-12	59	181	1.64619E-13	5	31

Table A.15: Comparison of Optimization Methods

f	x_0	**m**	**MOPCM**			**GMOPCGM**			**CGPM**			**GCGPM**			**STTDFPM**			**Framework**		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F14	0.0	10000	7.67164E-12	102	411	3.07532E-12	10	51	4.84057E-12	15	89	6.05072E-12	41	164	8.20455E-12	17	53	3.07532E-12	10	51
	0.2		8.35998E-12	106	423	3.07532E-12	11	54	2.17604E-12	17	96	8.84848E-12	40	159	4.74065E-12	18	57	3.07532E-12	11	54
	0.4		9.17044E-12	98	395	5.32907E-12	8	41	9.42579E-12	15	90	6.82787E-12	40	160	9.01501E-12	16	52	5.36238E-12	8	41
	0.5		7.53841E-12	97	391	4.64073E-12	9	46	5.4512E-12	15	90	9.38138E-12	39	156	5.65104E-12	16	52	4.67404E-12	9	46
	0.6		9.49241E-12	93	375	3.29736E-12	9	46	1.96509E-12	15	90	7.89369E-12	38	152	6.12843E-12	16	50	3.33067E-12	9	46
	0.8		8.22675E-12	96	387	5.07372E-12	9	49	3.94129E-12	15	90	8.43769E-12	39	156	4.77396E-12	16	52	5.10703E-12	9	49
	1.0		8.00471E-12	99	399	1.37668E-12	10	51	8.9373E-12	15	90	9.58122E-12	40	160	5.10703E-12	17	53	1.37668E-12	10	51
	1.1		8.61533E-12	100	403	3.29736E-12	10	50	8.39329E-12	15	89	5.16254E-12	39	156	5.9841E-12	17	53	3.33067E-12	10	50
	1 - 1/m		8.00471E-12	99	399	1.37668E-12	10	51	8.9373E-12	15	90	9.58122E-12	40	160	5.12923E-12	17	53	1.37668E-12	10	51
	1/k		7.6779E-12	102	411	3.03446E-12	10	51	6.87803E-12	80	479	6.05704E-12	41	164	6.68094E-12	54	166	3.07582E-12	10	51
	(k-1)/m		8.92924E-12	99	399	1.21735E-12	10	51	5.96751E-12	85	509	8.78652E-12	39	159	8.72023E-12	59	181	1.23409E-12	10	51
	1/m		7.67164E-12	102	411	3.04201E-12	10	51	4.84057E-12	15	89	6.05072E-12	41	164	8.20455E-12	17	53	3.07532E-12	10	51
	$1/3^k$		7.67115E-12	102	411	3.07516E-12	10	51	7.14596E-12	78	467	6.05015E-12	41	164	7.27492E-12	51	157	3.07531E-12	10	51
	k/m		8.92762E-12	99	399	1.21711E-12	10	51	5.94067E-12	85	509	8.79859E-12	39	159	8.71815E-12	59	181	1.23348E-12	10	51
F15	0.0	10000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		0	1	9	0	1	21	0	1	26	0	0	19	0	0	20	0	1	21
	0.4		8.85958E-12	50	203	0	1	7	1.13243E-12	14	98	0	0	5	0	0	6	0	1	7
	0.5		0	1	4	0	1	7	0	1	3	0	0	5	0	0	3	0	1	7
	0.6		0	1	4	0	1	4	1.22125E-12	14	98	0	0	5	0	0	6	0	1	4
	0.8		7.21645E-12	51	207	0	1	7	0	1	4	0	0	3	0	0	6	0	1	7
	1.0		7.10543E-12	51	207	0	1	4	1.22125E-12	15	107	0	0	6	0	0	6	0	1	4
	1.1		6.43929E-12	51	207	0	1	5	0	1	5	0	0	6	0	2	13	0	1	5
	1 - 1/m		7.08322E-12	51	207	0	1	4	1.22125E-12	15	107	0	0	6	0	0	6	0	1	4
	1/k		7.90696E-12	44	179	3.51282E-12	12	73	7.0298E-12	42	304	0	14	79	0	20	72	2.57616E-12	12	73
	(k-1)/m		9.79691E-12	50	203	1.01344E-12	14	85	3.99526E-12	55	396	0	8	46	0	16	56	8.31283E-12	13	79
	1/m		6.79456E-12	37	151	0	5	34	8.9706E-12	10	70	0	0	5	0	0	6	NaN	NaN	NaN
	$1/3^k$		9.85472E-12	42	171	5.45675E-12	10	61	6.63018E-12	49	350	3.54577E-15	0	5	2.51215E-15	0	6	5.44009E-12	10	61
	k/m		9.7982E-12	50	203	1.01384E-12	14	85	9.72729E-12	59	424	0	8	46	0	16	56	8.31451E-12	13	79

Table A.16: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F16	0.0	10000	9.56025E-12	577	2380	8.73125E-12	88	424	8.54464E-12	485	3043	5.04599E-12	91	436	8.98167E-12	167	508	6.85897E-12	39	197
	0.2		9.54962E-12	571	2296	9.39128E-12	102	477	8.94922E-12	230	1465	5.06966E-12	88	421	9.37966E-12	159	485	6.28562E-12	41	209
	0.4		9.6096E-12	558	2305	8.22105E-12	93	438	9.14239E-12	183	1173	7.05419E-12	89	425	9.93516E-12	168	508	3.69572E-12	46	235
	0.5		9.7421E-12	523	2139	8.00955E-12	59	294	8.76642E-12	159	1027	6.92458E-12	85	407	9.88343E-12	166	504	5.11361E-12	40	204
	0.6		9.66146E-12	599	2525	6.72809E-12	65	323	9.79151E-12	277	1754	7.91165E-12	86	412	9.8056E-12	164	498	7.91984E-12	41	206
	0.8		9.62611E-12	506	2017	6.36304E-12	68	334	8.85528E-12	241	1531	7.43568E-12	87	417	9.23121E-12	160	486	4.91874E-12	43	219
	1.0		9.73887E-12	508	1993	6.60693E-12	87	418	8.68309E-12	229	1457	6.94102E-12	92	435	9.64765E-12	155	472	2.47316E-12	44	224
	1.1		9.91643E-12	443	1707	7.29999E-12	92	442	8.18581E-12	54	377	7.70659E-12	88	422	9.98664E-12	157	478	2.98804E-12	43	219
	1 - 1/m		9.93719E-12	528	2134	5.06098E-12	63	306	8.77037E-12	229	1457	7.73652E-12	90	429	9.66421E-12	155	472	7.3728E-12	42	214
	1/k		9.78956E-12	541	2208	6.65139E-12	51	258	8.9169E-12	268	1706	8.88189E-12	88	419	9.77711E-12	173	522	3.02451E-12	43	221
	(k-1)/m		9.64959E-12	499	2000	5.09765E-12	51	259	9.00338E-12	302	1914	5.41734E-12	92	437	9.37301E-12	164	498	7.3919E-12	41	210
	1/m		9.48237E-12	534	2173	8.95169E-12	54	272	8.59497E-12	484	3037	5.1666E-12	89	427	9.58081E-12	167	507	2.5059E-12	45	229
	$1/3^k$		9.16658E-12	441	1679	4.84832E-12	47	235	9.68395E-12	477	2993	6.20537E-12	90	428	9.10158E-12	172	519	8.07483E-12	40	206
	k/m		9.78957E-12	477	1858	4.6102E-12	51	259	8.92613E-12	302	1914	8.18475E-12	87	417	9.36807E-12	164	498	6.35169E-12	48	244
F17	0.0	10000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		8.56208E-12	108	435	0	1	5	6.34529E-12	15	89	0	0	4	0	1	8	0	1	5
	0.4		9.14594E-12	95	383	0	1	6	1.64317E-12	14	84	0	0	4	0	0	5	0	1	6
	0.5		8.37866E-12	96	387	0	1	6	2.08573E-12	14	84	0	0	4	0	0	5	0	1	6
	0.6		7.37512E-12	97	391	0	1	6	2.55032E-12	14	84	0	0	4	0	0	5	0	1	6
	0.8		9.95429E-12	97	391	0	1	6	3.56725E-12	14	84	0	0	4	0	0	5	0	1	6
	1.0		9.22619E-12	98	395	0	1	6	4.74301E-12	14	84	0	0	4	0	0	5	0	1	6
	1.1		7.47605E-12	99	399	0	1	6	5.40883E-12	14	84	0	0	4	0	0	5	0	1	6
	1 - 1/m		9.22519E-12	98	395	0	1	6	4.74237E-12	14	84	0	0	4	0	0	5	0	1	6
	1/k		9.55063E-12	84	339	7.0734E-13	3	21	8.86308E-12	111	666	0	0	4	0	0	5	8.89229E-15	3	21
	(k-1)/m		9.8223E-12	96	387	0	1	6	8.72298E-12	138	828	0	0	4	0	0	5	0	1	6
	1/m		8.12309E-12	69	279	0	1	6	5.39009E-12	10	60	0	0	4	0	0	5	0	1	6
	$1/3^k$		9.04974E-12	80	323	0	1	6	9.89661E-12	105	630	4.4173E-15	0	4	5.49653E-16	0	5	0	1	6
	k/m		9.82383E-12	96	387	0	1	6	8.73955E-12	138	828	0	0	4	0	0	5	0	1	6

Table A.17: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F18	0.2	10000	2.26485E-12	18	56	6.57252E-12	36	110	1.79856E-12	21	81	5.32907E-13	11	36	NaN	NaN	NaN	NaN	2000	39967
	0.4		1.28786E-12	21	75	9.85878E-12	32	99	4.88498E-12	16	64	5.9952E-13	8	28	2.28706E-12	9	27	9.85878E-12	32	99
	0.5		1.73195E-12	20	71	4.37428E-12	28	87	7.19425E-12	16	64	8.21565E-13	8	27	8.88178E-13	8	27	4.44089E-12	28	87
	0.6		1.24345E-12	18	63	5.9508E-12	32	99	7.70495E-12	16	64	5.32907E-13	8	27	7.32747E-12	8	26	5.9508E-12	32	99
	0.8		1.37668E-12	12	39	8.08242E-12	33	101	6.08402E-12	16	64	2.44249E-13	7	24	6.50591E-12	7	21	8.12683E-12	33	101
	1.0		9.19265E-12	1	5	6.86118E-12	4	14	9.41469E-12	3	12	1.39888E-12	2	6	9.10383E-13	2	7	6.92779E-12	4	14
	1.1		1.5099E-12	12	38	6.88338E-12	32	98	3.41949E-12	15	59	2.44249E-13	7	24	1.02141E-12	8	24	6.86118E-12	32	98
	1 - 1/m		9.99201E-12	8	26	5.55112E-12	24	74	7.01661E-12	12	48	1.9984E-12	6	18	9.99201E-12	5	16	5.57332E-12	24	74
	1/k		2.34235E-12	82	294	4.64766E-12	35	112	5.37038E-12	59	242	1.13414E-12	12	39	6.35129E-12	60	184	NaN	2000	39954
	1/m		2.24265E-12	12	39	5.83977E-12	34	107	2.44249E-12	16	68	7.99361E-13	8	30	1.95399E-12	7	24	5.92859E-12	34	107
F19	0.0	10000	9.74299E-12	118	466	0.018711123	11	34	7.57649E-12	30	159	8.61377E-13	12	45	0.005599741	0	3	9.10704E-12	33	125
	0.2		9.74299E-12	119	474	0.018711123	12	54	9.22448E-12	28	167	3.1225E-13	15	72	0.002359994	1	22	9.10704E-12	34	145
	0.4		8.61759E-12	111	459	0.018711123	12	50	4.74698E-12	29	205	3.12337E-13	15	68	0.002360555	1	18	9.10704E-12	34	141
	0.5		8.18711E-12	115	467	0.018711123	12	50	9.22448E-12	29	181	3.1225E-13	15	69	0.002360353	1	19	9.10721E-12	34	141
	0.6		9.13609E-12	122	498	0.018711123	12	51	9.22448E-12	31	195	3.12337E-13	16	78	0.002360243	1	19	9.10721E-12	34	142
	0.8		9.74299E-12	125	514	0.018711123	12	52	5.79389E-12	27	186	3.12337E-13	15	70	0.002360133	1	20	9.10704E-12	34	143
	1.0		9.05838E-12	128	537	0.018711123	12	52	9.21832E-12	31	203	3.1225E-13	15	71	0.002360083	1	21	9.10704E-12	34	143
	1.1		9.74299E-12	125	514	0.018711123	12	53	9.22448E-12	31	203	3.1225E-13	15	71	0.002360067	1	21	9.10721E-12	34	144
	1 - 1/m		9.05803E-12	128	537	0.018711123	12	52	9.22448E-12	31	203	3.12337E-13	15	71	0.002360083	1	21	9.10721E-12	34	143
	1/m		8.21452E-12	122	485	0.105359107	11	34	9.67308E-12	27	147	3.33934E-13	10	39	0.033465693	2001	4003	9.25059E-12	28	110
F20	0.0	10000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		0	1	5	0	1	8	0	2	18	0	0	7	0	0	4	0	1	6
	0.4		8.6194E-12	96	385	2.89292E-12	12	58	1.8708E-12	16	91	6.10522E-12	44	175	0	4	15	4.7158E-12	14	69
	0.5		9.81033E-12	84	337	6.83301E-12	13	63	5.44359E-12	16	92	6.00257E-12	46	184	5.80408E-12	18	56	4.48082E-12	13	64
	0.6		9.34719E-12	101	406	2.00313E-12	14	66	7.69313E-12	15	86	5.47326E-12	43	171	9.3107E-12	14	46	4.8778E-12	11	56
	0.8		7.49464E-12	97	390	7.16382E-12	13	62	2.20162E-12	15	87	8.71524E-12	44	175	7.52431E-12	19	57	5.57592E-12	12	60
	1.0		9.21487E-12	84	338	2.06929E-12	15	69	4.91429E-12	15	88	7.82545E-12	44	177	4.3485E-12	16	52	9.54568E-12	13	65
	1.1		8.22243E-12	101	407	6.63452E-12	11	55	1.07685E-12	15	88	8.84756E-12	45	181	4.45115E-12	16	52	6.70068E-12	11	55
	1 - 1/m		9.28103E-12	84	338	2.06929E-12	15	69	4.91429E-12	15	88	7.82545E-12	44	177	4.41466E-12	16	52	9.54568E-12	13	65
	1/k		7.53293E-12	105	415	NaN	30	487	6.65169E-12	129	750	8.01366E-12	51	207	2.639572733	2001	4021	6.68853E-12	23	97
	(k-1)/m		7.83698E-12	104	417	5.09386E-12	26	130	7.06839E-12	68	406	6.88998E-12	42	179	8.31236E-12	69	206	NaN	NaN	NaN
	1/m		8.75173E-12	107	421	0.966857719	11	34	3.98801E-12	22	108	7.12733E-12	49	189	0.091950557	2001	4003	2.06929E-12	24	98
	$1/3^k$		9.61704E-12	139	520	0.626131515	11	34	8.23989E-12	108	492	NaN	2001	37142	0.532493538	2001	4004	NaN	NaN	NaN
	k/m		7.81369E-12	104	417	5.17925E-12	26	130	6.53059E-12	75	448	6.92557E-12	43	178	7.09793E-12	68	203	NaN	2000	39813

Table A.18: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F1	0.0	50000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		2.46182E-12	27	98	0	2	10	2.55461E-12	20	85	0	0	4	0	0	4	0	1	5
	0.4		6.80516E-12	12	38	0	1	4	6.20382E-12	17	68	0	0	3	2.4123E-13	7	25	0	1	4
	0.5		7.02924E-12	12	38	0	1	4	7.49665E-12	17	68	0	0	3	3.40993E-13	7	25	0	1	4
	0.6		6.29326E-12	12	38	0	1	4	8.63869E-12	17	68	0	0	3	3.97397E-13	7	25	0	1	4
	0.8		1.25672E-12	12	38	0	1	4	1.66842E-12	18	72	0	0	3	5.36446E-12	7	24	0	1	4
	1.0		1.07098E-12	14	46	0	1	4	1.8459E-12	18	72	0	0	3	0	0	4	0	1	4
	1.1		1.59142E-12	15	50	0	1	4	1.89435E-12	18	72	0	0	3	0	0	4	0	1	4
	1 - 1/m		1.07158E-12	14	46	0	1	4	1.84589E-12	18	72	0	0	3	0	0	4	0	1	4
	1/k		2.77773E-12	17	57	0	8	25	7.84824E-12	50	200	0	0	3	9.09179E-12	28	85	6.6227E-12	40	129
	(k-1)/m		2.12063E-12	77	273	0	6	19	5.71195E-12	56	224	0	0	3	6.90445E-12	36	111	8.51932E-12	43	138
	1/m		4.47214E-12	8	26	0	1	4	3.14699E-12	12	48	0	0	3	4.47214E-12	5	16	0	1	4
	$1/3^k$		3.77727E-12	10	32	1.28503E-15	0	4	7.26003E-12	44	176	2.79198E-16	0	3	5.68303E-12	9	28	1.28503E-15	0	4
	k/m		2.7164E-12	72	257	0	6	19	5.71731E-12	56	224	0	0	3	6.90381E-12	36	111	9.91697E-12	42	135
F2	0.0	50000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		1.53914E-12	19	59	0	5	16	6.05726E-12	20	76	0	6	21	NaN	NaN	NaN	0	2	7
	0.4		3.27688E-12	13	41	0	1	4	3.17758E-12	17	67	0	1	6	0	0	3	0	1	4
	0.5		5.21322E-12	13	41	0	1	4	5.11392E-12	17	67	0	2	9	0	0	3	0	1	4
	0.6		7.84465E-12	13	41	0	1	4	7.6957E-12	17	67	0	2	9	0	0	3	0	1	4
	0.8		1.58879E-12	14	44	0	1	4	2.43283E-12	18	71	0	2	9	0	0	3	0	1	4
	1.0		2.87968E-12	14	44	0	2	7	4.26987E-12	18	71	0	2	9	0	0	3	0	2	7
	1.1		3.82303E-12	14	44	0	2	7	5.56077E-12	18	71	0	2	9	0	0	3	0	2	7
	1 - 1/m		2.92933E-12	14	44	0	2	7	4.26987E-12	18	71	0	2	9	0	0	3	0	2	7
	1/k		5.08776E-12	13	41	0	5	16	8.19104E-12	50	199	6.74462E-13	5	18	7.75365E-12	45	135	8.92002E-12	46	148
	(k-1)/m		1.11483E-12	15	46	0	8	25	9.92816E-12	59	235	0	2	9	8.75325E-12	57	171	6.43822E-12	49	158
	1/m		4.46847E-12	8	26	0	1	4	2.28389E-12	7	27	0	0	3	0	0	3	0	1	4
	$1/3^k$		3.05521E-12	19	58	6.66122E-16	8	28	6.60737E-12	47	187	6.28028E-16	1	6	7.05814E-12	45	137	8.27028E-12	18	57
	k/m		1.11477E-12	15	46	0	8	25	9.8898E-12	59	235	0	2	9	8.75387E-12	57	171	6.46348E-12	49	158

Table A.19: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F3	0.0	50000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		8.09306E-12	14	50	0	2	20	3.5252E-12	20	105	0	1	15	0	0	12	0	1	13
	0.4		4.91542E-12	16	56	0	1	4	2.97904E-12	17	68	0	0	3	0	0	4	0	1	4
	0.5		1.34057E-12	19	67	0	2	8	3.07834E-12	17	68	0	0	3	0	0	4	0	2	8
	0.6		3.9224E-12	21	75	0	2	8	3.02869E-12	17	68	0	0	3	0	0	4	0	2	8
	0.8		3.9224E-12	23	83	0	1	5	2.18463E-12	17	68	0	1	7	0	0	4	0	1	5
	1.0		2.48253E-12	24	87	0	1	5	3.4259E-12	18	74	0	1	7	0	0	4	0	1	5
	1.1		2.63149E-12	22	81	0	1	5	3.4259E-12	18	74	0	1	7	0	0	4	0	1	5
	1 - 1/m		2.53218E-12	24	87	0	1	5	3.4259E-12	18	74	0	1	7	0	0	4	0	1	5
	1/k		5.06918E-12	33	115	0	6	20	7.59891E-12	44	178	0	1	7	7.62851E-12	50	152	9.36048E-12	39	126
	(k-1)/m		5.26137E-12	85	308	0	8	26	7.4126E-12	66	266	8.67517E-12	18	56	9.5862E-12	60	180	7.8506E-12	49	159
	1/m		4.46856E-12	8	26	0	1	4	3.12799E-12	12	48	0	0	3	4.46856E-12	5	16	0	1	4
	$1/3^k$		4.61451E-12	21	70	8.77568E-15	0	4	5.6596E-12	48	192	1.64673E-15	0	3	1.11022E-15	0	4	8.77568E-15	0	4
	k/m		5.26272E-12	85	308	0	8	26	9.49904E-12	61	246	8.78169E-12	18	56	9.58935E-12	60	180	7.88027E-12	49	159
F4	0.0	50000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		9.63088E-12	103	413	0	3	11	8.39321E-12	17	98	0	1	6	0	2	11	0	3	11
	0.4		8.40701E-12	98	395	0	1	6	4.59719E-12	14	84	0	0	4	0	0	5	0	1	6
	0.5		7.79495E-12	99	399	0	1	6	6.1209E-12	14	84	0	0	4	0	0	5	0	1	6
	0.6		9.50629E-12	99	399	0	1	6	7.82461E-12	14	84	0	0	4	0	0	5	0	1	6
	0.8		9.56196E-12	100	403	0	2	11	3.55643E-12	13	77	0	0	4	0	0	5	0	2	11
	1.0		9.0238E-12	101	407	0	2	11	1.06481E-12	14	83	0	0	4	0	0	5	0	2	11
	1.1		7.37124E-12	102	411	0	2	11	1.55321E-12	14	83	0	0	4	0	0	5	0	2	11
	1 - 1/m		9.02359E-12	101	407	0	2	11	1.06472E-12	14	83	0	0	4	0	0	5	0	2	11
	1/k		9.46892E-12	116	465	3.30883E-12	9	46	8.64752E-12	53	317	0	10	45	8.74381E-12	37	113	5.83834E-12	11	57
	(k-1)/m		8.63527E-12	103	413	4.23068E-12	19	96	9.85519E-12	90	543	5.67588E-12	32	156	3.89586E-15	2	11	4.48808E-13	6	36
	1/m		9.33833E-12	66	267	0	1	6	2.41055E-12	10	60	0	0	4	0	0	5	0	1	6
	$1/3^k$		9.45902E-12	107	429	1.72317E-13	0	5	8.12121E-12	24	143	1.57717E-14	0	4	5.54083E-15	0	5	1.72317E-13	0	5
	k/m		8.63539E-12	103	413	4.23116E-12	19	96	8.80029E-12	90	543	6.88044E-12	32	156	3.54162E-15	2	11	4.13467E-13	6	36

Table A.20: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F5	0.0	50000	6.05738E-12	13	41	7.84481E-12	37	113	7.1497E-12	18	72	7.94411E-13	8	27	4.96507E-13	7	25	7.84481E-12	37	113
	0.2		1.19162E-12	14	44	6.9511E-12	38	116	2.28393E-12	19	76	1.68812E-12	9	27	1.19162E-12	8	25	6.9511E-12	38	116
	0.4		5.16367E-12	13	41	6.65319E-12	37	113	6.15668E-12	18	72	6.9511E-13	8	27	4.96507E-13	7	25	6.65319E-12	37	113
	0.5		4.96507E-12	13	41	6.35529E-12	37	113	5.85878E-12	18	72	6.9511E-13	8	27	5.95808E-13	7	25	6.35529E-12	37	113
	0.6		4.76647E-12	13	41	6.05738E-12	37	113	5.56088E-12	18	72	5.95808E-13	8	27	4.96507E-13	7	25	6.15668E-12	37	113
	0.8		4.26996E-12	13	41	5.56088E-12	37	113	5.06437E-12	18	72	5.95808E-13	8	27	3.97205E-13	7	25	5.56088E-12	37	113
	1.0		3.87275E-12	13	41	4.96507E-12	37	113	4.56786E-12	18	72	4.96507E-13	8	27	3.97205E-13	7	25	4.96507E-12	37	113
	1.1		3.57485E-12	13	41	4.66716E-12	37	113	4.26996E-12	18	72	4.96507E-13	8	27	3.97205E-13	7	25	4.66716E-12	37	113
	1 - 1/m		3.87275E-12	13	41	4.96507E-12	37	113	4.56786E-12	18	72	4.96507E-13	8	27	3.97205E-13	7	25	4.96507E-12	37	113
	1/k		6.06759E-12	13	41	7.80066E-12	37	113	8.27001E-12	38	152	8.51835E-13	8	27	6.02809E-13	7	25	7.82786E-12	37	113
	(k-1)/m		4.99989E-12	13	41	6.42602E-12	37	113	8.63013E-12	38	152	6.78916E-13	8	27	5.05067E-13	7	25	6.44974E-12	37	113
	1/m		6.05738E-12	13	41	7.84481E-12	37	113	7.1497E-12	18	72	7.94411E-13	8	27	5.95808E-13	7	25	7.84481E-12	37	113
	$1/3^k$		6.05736E-12	13	41	7.84475E-12	37	113	8.49493E-12	34	136	7.94385E-13	8	27	4.96418E-13	7	25	7.84484E-12	37	113
	k/m		4.99907E-12	13	41	6.42595E-12	37	113	8.62686E-12	38	152	6.78675E-13	8	27	5.04936E-13	7	25	6.44921E-12	37	113
F6	0.0	50000	9.97979E-12	165	663	8.29166E-12	28	115	8.19236E-13	12	60	5.46158E-13	10	44	2.30876E-12	12	37	8.34131E-12	28	115
	0.2		9.73153E-12	165	662	8.29166E-12	30	122	1.48952E-12	14	68	5.95808E-13	10	43	7.74551E-12	13	42	8.34131E-12	29	118
	0.4		9.97979E-12	157	631	5.06437E-12	27	111	4.39409E-12	11	55	1.09232E-12	10	40	4.51821E-12	12	37	5.16367E-12	27	111
	0.5		9.13573E-12	145	583	4.02171E-12	25	103	8.53992E-12	10	50	1.19162E-12	9	36	7.1497E-12	11	34	4.07136E-12	25	103
	0.6		9.93014E-12	156	627	3.77345E-12	27	111	4.36926E-12	11	55	1.04266E-12	9	36	4.12101E-12	12	37	3.77345E-12	27	111
	0.8		9.13573E-12	163	655	3.77345E-12	28	115	9.93014E-13	12	60	3.9224E-12	10	40	5.61053E-12	12	39	3.77345E-12	28	115
	1.0		9.28468E-12	166	667	5.36227E-12	28	115	2.18463E-12	12	60	6.9511E-13	10	44	9.33433E-12	12	39	5.36227E-12	28	115
	1.1		9.53293E-12	167	671	5.75948E-12	28	115	2.97904E-12	12	60	1.14197E-12	11	44	1.83708E-12	13	40	5.75948E-12	28	115
	1 - 1/m		9.28468E-12	166	667	5.36227E-12	28	115	2.18463E-12	12	60	6.9511E-13	10	44	9.33433E-12	12	39	5.36227E-12	28	115
	1/k		9.97547E-12	165	663	6.66983E-12	37	178	6.53086E-12	38	258	1.61598E-12	33	156	5.25595E-12	55	168	8.50781E-12	43	214
	(k-1)/m		8.74693E-12	163	655	5.01396E-12	37	182	8.26034E-12	41	262	6.54998E-12	23	107	9.10388E-12	68	205	6.62971E-12	39	193
	1/m		9.97979E-12	165	663	8.29166E-12	28	115	8.19236E-13	12	60	5.46158E-13	10	44	2.25911E-12	12	37	8.34131E-12	28	115
	$1/3^k$		9.97837E-12	165	663	7.52798E-12	34	169	6.14635E-12	35	237	2.06944E-12	25	120	5.09359E-12	52	160	3.76949E-12	38	189
	k/m		8.74698E-12	163	655	6.24241E-12	40	193	8.19766E-12	41	262	6.51772E-12	23	107	4.27221E-12	65	199	6.67712E-12	39	193

Table A.21: Comparison of Optimization Methods

f	x_0	**m**	**MOPCM**			**GMOPCGM**			**CGPM**			**GCGPM**			**STTDFPM**			**Framework**		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F8	0.0	50000	9.76481E-12	151	607	5.82119E-12	35	177	7.58086E-12	56	391	4.3219E-12	40	279	9.76379E-12	71	232	6.1002E-12	57	289
	0.2		9.09253E-12	149	606	2.13299E-12	55	317	5.94028E-12	56	415	8.58919E-12	100	570	8.51582E-12	67	225	4.58911E-12	37	195
	0.4		9.97364E-12	544	2099	9.77277E-12	35	182	5.6997E-12	52	363	8.19542E-12	38	230	9.77794E-12	206	641	2.00851E-12	33	176
	0.5		9.87699E-12	582	2326	1.54539E-12	39	205	7.64551E-12	53	370	8.89919E-12	126	600	9.65767E-12	116	370	8.61711E-12	34	183
	0.6		9.69859E-12	589	2326	2.79319E-12	54	285	2.52991E-12	60	421	9.48789E-12	120	573	9.67184E-12	217	661	7.46798E-12	32	173
	0.8		9.45015E-12	570	2187	1.80008E-12	44	240	3.53983E-12	56	392	6.99371E-12	65	312	9.84555E-12	190	576	6.34036E-12	36	193
	1.0		9.57645E-12	551	2116	6.13998E-12	45	238	8.52424E-12	52	366	9.03939E-12	126	603	9.65277E-12	220	665	1.70929E-12	45	237
	1.1		9.71917E-12	521	2001	4.15074E-12	45	231	9.16625E-12	54	380	8.29208E-12	129	617	9.03838E-12	144	444	9.4943E-12	72	368
	1 - 1/m		9.82213E-12	547	2088	8.72484E-12	42	221	8.95124E-12	50	352	6.10471E-12	128	612	9.21907E-12	211	639	9.65153E-12	55	283
	1/k		9.91193E-12	301	1139	5.33768E-12	50	261	5.16176E-12	58	406	3.15308E-12	41	287	7.78342E-12	71	227	6.56794E-12	43	220
	(k-1)/m		9.55195E-12	578	2232	9.92038E-12	48	250	5.26669E-12	54	382	8.17371E-12	128	615	9.17908E-12	214	652	7.81264E-12	32	174
	1/m		9.7961E-12	299	1136	5.51445E-12	34	174	9.07973E-12	54	377	5.40005E-12	45	307	9.66767E-12	84	268	7.00522E-12	33	169
	$1/3^k$		9.46595E-12	145	583	9.91025E-12	44	224	6.33381E-12	51	354	5.37262E-12	61	483	6.48776E-12	78	255	4.81992E-12	39	204
	k/m		9.4435E-12	581	2274	4.99074E-12	47	248	9.80949E-12	60	424	8.46242E-12	128	615	9.11396E-12	212	645	8.29996E-12	34	182
F9	0.0	50000	9.64753E-12	661	3005	NaN	19	244	9.60972E-12	152	1410	7.1432E-12	32	214	8.18696E-12	83	360	NaN	2000	39926
	0.4		6.52501E-11	2000	8330	NaN	11	190	6.29358E-12	225	2068	NaN	2001	37946	8.32664E-12	58	235	NaN	2000	39970
	0.5		7.79034E-11	2000	8339	7.72555E-12	55	407	9.65004E-12	58	556	NaN	2001	37962	7.04114E-12	71	280	6.60894E-12	39	278
	0.6		3.18796E-11	2000	8362	4.49727E-12	62	461	8.45534E-12	165	1528	NaN	2001	37974	8.04094E-12	63	254	8.55644E-12	36	251
	0.8		3.18806E-11	2000	8344	NaN	16	222	8.48178E-12	159	1474	3.9276E-12	32	220	6.07071E-12	73	300	5.21623E-12	46	329
	1.0		0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	1.1		6.7791E-11	2000	8314	7.09273E-12	77	547	7.75173E-12	158	1465	2.29856E-12	26	179	8.1105E-12	53	219	2.17646E-12	54	366
	1 - 1/m		5.46568E-11	2000	8213	6.50484E-12	44	330	9.60179E-12	49	465	6.12156E-12	19	130	8.82758E-12	28	119	9.20792E-12	21	152
	1/k		4.43004E-11	2000	8347	NaN	23	273	9.5665E-12	151	1401	6.34873E-12	33	234	5.98073E-12	68	273	NaN	2000	39879
	(k-1)/m		4.36664E-11	2000	8251	4.94663E-12	74	568	4.20675E-12	38	375	3.15321E-12	44	311	4.13859E-12	69	283	2.55093E-12	64	465
	1/m		4.42962E-11	2000	8348	NaN	19	244	9.6094E-12	151	1401	3.57371E-12	38	254	9.63178E-12	87	376	NaN	2000	39926
	$1/3^k$		9.63578E-12	655	2981	NaN	32	548	9.60977E-12	152	1410	3.95608E-12	25	170	3.79898E-12	70	286	NaN	2000	39970
	k/m		8.55545E-12	345	1640	9.71823E-12	87	697	4.17421E-12	38	375	3.16485E-12	44	311	6.32668E-12	61	251	5.88245E-12	66	444

Table A.22: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F10	0.0	50000	9.33433E-12	160	637	90.24772393	11	34	7.94411E-12	35	194	3.47555E-13	14	55	10.21347132	2001	4003	6.45459E-12	37	146
	0.2		9.33433E-12	162	644	5.26297E-12	37	155	4.66716E-12	35	213	5.46158E-12	16	59	4.334471349	2001	4003	6.45459E-12	38	149
	0.4		9.73153E-12	148	593	5.70983E-12	36	147	8.53992E-12	31	180	5.46158E-13	10	42	9.53293E-12	14	43	5.46158E-12	37	149
	0.5		9.53293E-12	160	642	8.24201E-12	33	134	9.73153E-12	28	162	5.66018E-12	11	44	7.10005E-12	13	43	6.10703E-12	37	151
	0.6		9.33433E-12	158	634	7.0504E-12	36	146	6.70284E-12	32	187	5.46158E-13	9	39	8.04341E-12	13	40	7.0504E-12	36	146
	0.8		8.34131E-12	154	618	4.86577E-12	37	150	7.29865E-12	32	188	3.47555E-13	9	39	5.66018E-12	14	42	4.86577E-12	37	150
	1.0		8.34131E-12	147	590	5.46158E-12	34	137	8.93712E-12	30	176	3.47555E-13	9	39	2.13498E-12	13	41	8.14271E-12	34	139
	1.1		9.93014E-12	131	526	4.66716E-12	36	145	4.51821E-12	29	170	5.46158E-13	9	39	2.53218E-12	12	38	9.33433E-12	31	127
	1 - 1/m		8.34131E-12	147	590	5.46158E-12	34	137	8.93712E-12	30	176	3.47555E-13	9	39	2.13498E-12	13	41	8.14271E-12	34	139
	1/k		8.28981E-12	159	633	5.04366E-12	51	203	6.31633E-12	40	223	8.94818E-13	14	57	11.61149127	2001	4010	7.13759E-12	46	183
	(k-1)/m		8.74966E-12	170	682	4.52915E-12	39	157	4.41928E-12	39	226	7.80746E-13	12	51	7.28541E-12	73	221	9.96535E-12	37	150
	1/m		8.73852E-12	160	637	90.53687476	11	34	7.74551E-12	35	194	3.47555E-13	14	55	10.2538722	2001	4003	6.45459E-12	37	146
	$1/3^k$		8.54283E-12	160	637	5.38292E-12	60	246	4.24888E-12	43	237	5.26302E-12	14	53	10.42185058	2001	4009	5.85913E-12	37	146
	k/m		8.73324E-12	170	682	5.95507E-12	40	161	4.41468E-12	39	226	7.75101E-13	12	51	7.27589E-12	73	221	9.92738E-12	37	150
F11	0.0	50000	1.64666E-12	62	216	6.51655E-12	34	106	6.78585E-12	61	240	9.37848E-12	13	39	6.47908E-12	63	192	8.1856E-12	34	107
	0.2		5.54342E-12	108	401	5.07471E-12	58	360	6.92193E-12	122	1252	9.84659E-12	91	813	8.54173E-12	63	232	5.60042E-12	183	1436
	0.4		3.66755E-12	97	341	4.57346E-12	42	130	5.31541E-12	65	259	7.76875E-12	37	112	5.55855E-12	63	192	4.29593E-12	57	195
	0.5		7.28869E-12	114	397	7.71549E-12	38	119	4.87318E-12	68	273	6.24526E-12	34	103	6.47889E-12	63	192	6.45063E-12	38	119
	0.6		3.44577E-12	111	390	7.48861E-12	40	122	7.12646E-12	65	264	6.11166E-12	39	118	8.13903E-12	63	191	7.97228E-12	45	163
	0.8		4.71904E-12	117	411	8.51372E-12	36	110	8.34782E-12	68	276	2.40808E-12	13	40	8.20139E-12	62	186	6.66178E-12	54	183
	1.0		3.65742E-12	114	403	7.68589E-12	40	122	5.82966E-12	70	327	7.32678E-13	11	37	6.72519E-12	61	185	4.86414E-12	69	283
	1.1		1.94611E-12	122	427	5.77553E-12	48	204	5.18382E-12	60	257	9.03277E-13	11	37	8.16928E-12	63	189	9.44507E-12	62	207
	1 - 1/m		3.60464E-12	114	403	7.3549E-12	40	122	5.85589E-12	72	336	7.32461E-13	11	37	6.72285E-12	61	185	5.5616E-12	66	273
	1/k		4.55344E-12	56	195	9.64129E-12	35	108	6.6446E-12	61	240	4.43069E-13	12	39	9.44276E-12	62	189	9.34643E-12	63	205
	(k-1)/m		1.2173E-12	124	435	6.4902E-12	41	131	7.24595E-12	67	277	1.29746E-12	14	43	6.94668E-12	63	189	6.34309E-12	41	137
	1/m		1.53375E-12	70	239	6.3066E-12	36	111	6.78202E-12	61	240	9.37797E-12	13	39	6.48821E-12	63	192	7.92615E-12	36	112
	$1/3^k$		1.20153E-12	65	224	7.42462E-12	36	112	6.51114E-12	61	240	4.76459E-12	13	39	6.47593E-12	63	192	6.93097E-12	34	107
	k/m		1.20811E-12	124	435	8.97019E-12	42	134	5.41996E-12	65	269	1.31842E-12	14	43	6.94549E-12	63	189	6.32599E-12	41	137

Table A.23: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F12	0.0	50000	9.28468E-12	166	667	5.34986E-12	28	115	2.12257E-12	12	60	7.07522E-13	10	44	9.34674E-12	12	39	5.37469E-12	28	115
	0.2		9.12331E-12	160	641	4.58028E-12	30	122	1.00543E-12	13	63	4.84094E-13	11	47	1.70054E-12	14	42	5.36227E-12	29	118
	0.4		9.91772E-12	156	627	3.8231E-12	27	111	4.33202E-12	11	55	1.00543E-12	9	36	4.10859E-12	12	37	3.78586E-12	27	111
	0.5		9.13573E-12	145	583	4.02171E-12	25	103	8.50268E-12	10	50	1.1792E-12	9	36	7.13729E-12	11	34	4.04653E-12	25	103
	0.6		9.96737E-12	157	631	5.0892E-12	27	111	4.38167E-12	11	55	1.09232E-12	10	40	4.58028E-12	12	37	5.0892E-12	27	111
	0.8		8.96195E-12	163	655	5.12643E-12	28	115	5.70983E-13	12	60	5.63535E-12	10	40	8.99919E-12	12	37	5.13885E-12	28	115
	1.0		9.97979E-12	165	663	8.31649E-12	28	115	7.81998E-13	12	60	5.83396E-13	10	44	2.25911E-12	12	37	8.34131E-12	28	115
	1.1		8.49027E-12	167	671	4.58028E-12	28	115	1.42746E-12	11	54	1.05508E-12	11	44	3.3266E-12	13	40	4.62993E-12	28	115
	1 - 1/m		9.97979E-12	165	663	8.31649E-12	28	115	7.81998E-13	12	60	5.83396E-13	10	44	2.25911E-12	12	37	8.34131E-12	28	115
	1/k		9.27646E-12	166	667	5.354E-12	28	115	8.86384E-12	55	295	7.03547E-13	10	44	9.72195E-12	52	159	5.36646E-12	28	115
	(k-1)/m		8.74918E-12	163	655	5.24689E-12	28	115	8.42694E-12	62	330	3.82554E-12	10	40	7.06012E-12	70	214	4.564E-12	28	115
	1/m		9.28468E-12	166	667	5.36227E-12	28	115	2.12257E-12	12	60	7.07522E-13	10	44	9.34674E-12	12	39	5.37469E-12	28	115
	$1/3^k$		9.28453E-12	166	667	5.34983E-12	28	115	8.34211E-12	45	238	7.07505E-13	10	44	8.97213E-12	47	144	5.37457E-12	28	115
	k/m		8.74912E-12	163	655	5.24714E-12	28	115	9.39239E-12	63	336	3.8259E-12	10	40	7.06014E-12	70	214	4.56419E-12	28	115
F13	0.0	50000	9.16055E-12	106	427	8.71369E-12	5	29	1.63847E-12	17	101	8.31649E-12	35	140	7.249E-12	16	52	7.74551E-12	5	29
	0.2		9.18538E-12	120	483	8.71369E-12	7	38	6.35529E-12	17	101	6.72767E-12	36	144	6.30564E-12	18	55	7.74551E-12	6	33
	0.4		9.58258E-12	101	407	4.46856E-13	4	26	4.04653E-12	17	102	5.98291E-12	34	136	5.51123E-12	16	50	3.97205E-13	4	26
	0.5		8.76335E-12	104	419	2.78044E-12	5	26	8.49027E-12	17	102	5.51123E-12	35	140	4.34443E-12	16	52	2.48253E-12	5	26
	0.6		7.52208E-12	106	427	8.813E-12	5	26	2.01085E-12	18	108	8.53992E-12	35	140	6.57872E-12	16	52	7.79516E-12	5	26
	0.8		9.38398E-12	107	431	2.48253E-14	5	31	3.22729E-12	18	108	5.98291E-12	36	144	4.46856E-12	17	53	2.48253E-14	5	31
	1.0		9.75636E-12	108	435	1.24127E-13	5	31	4.24513E-12	18	108	6.72767E-12	36	144	5.90843E-12	17	53	7.4476E-14	5	31
	1.1		8.53992E-12	109	439	2.48253E-14	5	31	3.40107E-12	17	101	6.72767E-12	36	144	7.42278E-12	17	53	2.48253E-14	5	31
	1 - 1/m		9.75636E-12	108	435	7.4476E-14	5	31	4.24513E-12	18	108	6.72767E-12	36	144	5.95808E-12	17	53	7.4476E-14	5	31
	1/k		9.15755E-12	106	427	8.65559E-12	5	29	9.46119E-12	54	323	8.32933E-12	35	140	8.22142E-12	54	166	7.69954E-12	5	29
	(k-1)/m		8.90558E-12	106	427	5.38087E-12	11	59	4.70317E-12	57	342	8.17157E-12	35	140	6.76381E-12	61	187	3.69585E-13	5	31
	1/m		9.16055E-12	106	427	8.71369E-12	5	29	1.63847E-12	17	101	8.31649E-12	35	140	7.249E-12	16	52	7.74551E-12	5	29
	$1/3^k$		9.16037E-12	106	427	8.71332E-12	5	29	8.59115E-12	54	323	8.31646E-12	35	140	7.22438E-12	44	136	7.74547E-12	5	29
	k/m		8.90582E-12	106	427	5.36792E-12	11	59	4.70308E-12	57	342	8.1727E-12	35	140	6.76169E-12	61	187	3.68916E-13	5	31

Table A.24: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F14	0.0	50000	9.4088E-12	104	419	6.87662E-12	10	51	1.36539E-12	16	95	6.38011E-12	42	168	6.50424E-12	17	55	6.87662E-12	10	51
	0.2		7.59655E-12	109	435	4.39409E-12	11	56	4.14583E-12	19	105	9.3095E-12	41	163	4.74164E-12	19	58	6.87662E-12	11	54
	0.4		8.31649E-12	101	407	5.03954E-12	8	44	2.65631E-12	16	96	7.1497E-12	41	164	9.0116E-12	17	53	5.03954E-12	8	44
	0.5		9.25985E-12	99	399	4.36926E-12	9	49	1.58882E-12	16	96	9.90531E-12	40	160	5.70983E-12	17	53	4.41891E-12	9	49
	0.6		8.61439E-12	96	387	7.37313E-12	9	46	4.39409E-12	15	90	8.29166E-12	39	156	4.89059E-12	16	52	7.4476E-12	9	46
	0.8		7.4476E-12	99	399	1.06749E-12	10	51	8.813E-12	15	90	8.86265E-12	40	160	4.66716E-12	17	53	1.06749E-12	10	51
	1.0		9.80601E-12	101	407	3.07834E-12	10	51	2.55701E-12	16	96	4.79129E-12	42	168	4.02171E-12	17	55	3.07834E-12	10	51
	1.1		7.79516E-12	103	415	7.37313E-12	10	50	2.35841E-12	16	95	5.43675E-12	40	160	4.79129E-12	17	55	7.4476E-12	10	50
	1 - 1/m		9.80601E-12	101	407	3.07834E-12	10	51	2.55701E-12	16	96	4.79129E-12	42	168	4.09618E-12	17	55	3.07834E-12	10	51
	1/k		9.39531E-12	104	419	6.84175E-12	10	51	5.70197E-12	82	491	6.37222E-12	42	168	6.26179E-12	54	166	6.87783E-12	10	51
	(k-1)/m		8.09484E-12	102	411	2.72173E-12	10	51	6.61466E-12	87	521	9.25988E-12	40	163	6.54038E-12	61	187	2.75887E-12	10	51
	1/m		9.4088E-12	104	419	6.87662E-12	10	51	1.36539E-12	16	95	6.38011E-12	42	168	6.50424E-12	17	55	6.87662E-12	10	51
	$1/3^k$		9.40868E-12	104	419	6.87658E-12	10	51	5.88183E-12	80	479	6.37942E-12	42	168	6.81405E-12	51	157	6.87673E-12	10	51
	k/m		8.09458E-12	102	411	2.72157E-12	10	51	6.61105E-12	87	521	9.26259E-12	40	163	6.54029E-12	61	187	2.75874E-12	10	51
F15	0.0	50000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		0	1	9	0	1	21	0	1	26	0	0	19	0	0	20	0	1	21
	0.4		5.95808E-12	52	211	0	1	7	2.53218E-12	14	98	0	0	5	0	0	6	0	1	7
	0.5		0	1	4	0	1	7	0	1	3	0	0	5	0	0	3	0	1	7
	0.6		0	1	4	0	1	4	2.68114E-12	14	98	0	0	5	0	0	6	0	1	4
	0.8		8.88747E-12	52	211	0	1	7	0	1	4	0	0	3	0	0	6	0	1	7
	1.0		8.68887E-12	52	211	0	1	5	2.73079E-12	15	107	0	0	6	0	0	6	0	1	4
	1.1		7.94411E-12	52	211	0	1	5	0	1	5	0	0	6	0	2	13	0	1	5
	1 - 1/m		8.68887E-12	52	211	0	1	5	2.73079E-12	15	107	0	0	6	0	0	6	0	1	4
	1/k		7.90695E-12	44	179	1.0227E-12	15	89	9.29735E-12	52	373	0	15	85	0	20	72	2.57599E-12	12	73
	(k-1)/m		6.62734E-12	52	211	2.26695E-12	14	85	4.95471E-12	58	418	0	8	46	0	16	56	1.86059E-12	14	85
	1/m		9.97979E-12	35	143	NaN	NaN	NaN	4.02171E-12	10	70	0	0	5	0	0	6	NaN	NaN	NaN
	$1/3^k$		9.85466E-12	42	171	5.45675E-12	10	61	6.61371E-12	49	350	1.349E-14	0	5	6.28037E-15	0	6	5.44009E-12	10	61
	k/m		6.62786E-12	52	211	2.26668E-12	14	85	9.79034E-12	61	439	0	8	46	0	16	56	1.8606E-12	14	85

Table A.25: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE	Norm	IT	FE
F16	0.0	50000	9.88984E-12	520	2098	7.95693E-12	54	270	9.21928E-12	250	1595	5.44532E-12	85	403	9.99474E-12	167	506	2.40516E-12	38	195
	0.2		9.66009E-12	739	3174	2.69425E-12	57	286	8.95956E-12	183	1176	7.83889E-12	90	430	9.77792E-12	158	482	2.74178E-12	44	225
	0.4		9.37271E-12	506	2060	8.54135E-12	42	212	9.92637E-12	188	1202	6.344E-12	86	412	9.79274E-12	163	494	8.46584E-12	46	234
	0.5		9.73583E-12	460	1774	9.34644E-12	45	223	9.10641E-12	196	1254	8.98107E-12	90	428	9.82583E-12	158	479	8.12931E-12	41	209
	0.6		9.71686E-12	511	2034	9.35539E-12	86	422	9.30456E-12	222	1414	5.93877E-12	87	416	9.38542E-12	157	476	1.85446E-12	45	227
	0.8		9.29788E-12	462	1779	9.97724E-12	78	390	9.97444E-12	214	1364	7.14715E-12	86	410	9.30973E-12	155	471	5.08722E-12	44	224
	1.0		9.9966E-12	461	1814	9.14728E-12	88	423	8.97023E-12	179	1147	7.87888E-12	78	371	9.35516E-12	153	465	6.47289E-12	43	220
	1.1		9.61639E-12	464	1817	9.61594E-12	76	381	8.83879E-12	53	370	8.3206E-12	64	305	9.52199E-12	138	419	5.11912E-12	43	220
	1 - 1/m		9.95685E-12	430	1634	8.32673E-12	83	411	8.96815E-12	179	1147	4.85247E-12	79	376	9.40496E-12	153	465	6.46486E-12	43	220
	1/k		9.28015E-12	607	2522	7.48954E-12	45	226	8.66859E-12	304	1928	6.06261E-12	87	417	9.53139E-12	167	506	6.55226E-12	40	204
	(k-1)/m		9.63973E-12	562	2343	8.73938E-12	102	486	9.20407E-12	262	1663	8.36698E-12	87	418	9.09464E-12	159	482	6.09079E-12	42	214
	1/m		9.14547E-12	544	2203	7.06901E-12	61	306	9.22109E-12	250	1595	9.16177E-12	81	387	8.98892E-12	168	509	7.52429E-12	43	220
	$1/3^k$		9.81611E-12	504	1991	8.77188E-12	59	287	8.6934E-12	268	1706	6.62827E-12	88	421	9.7781E-12	163	494	4.6659E-12	44	224
	k/m		9.27264E-12	546	2255	9.58186E-12	94	450	9.27498E-12	261	1657	8.35631E-12	87	418	9.07942E-12	159	482	9.58815E-12	53	271
F17	0.0	50000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		7.44788E-12	111	447	0	2	10	1.18257E-12	16	95	0	0	4	0	1	8	0	1	5
	0.4		7.95576E-12	98	395	0	1	6	3.67423E-12	14	84	0	0	4	0	0	5	0	1	6
	0.5		9.98401E-12	98	395	0	1	6	4.66382E-12	14	84	0	0	4	0	0	5	0	1	6
	0.6		8.78819E-12	99	399	0	1	6	5.70268E-12	14	84	0	0	4	0	0	5	0	1	6
	0.8		8.65892E-12	100	403	0	1	6	7.97662E-12	14	84	0	0	4	0	0	5	0	1	6
	1.0		8.02557E-12	101	407	0	1	6	9.84208E-13	15	90	0	0	4	0	0	5	0	1	6
	1.1		8.90847E-12	101	407	0	1	6	1.12237E-12	15	90	0	0	4	0	0	5	0	1	6
	1 - 1/m		8.0254E-12	101	407	0	1	6	9.84181E-13	15	90	0	0	4	0	0	5	0	1	6
	1/k		9.55085E-12	84	339	7.0489E-13	3	21	8.86832E-12	111	666	0	0	4	0	0	5	8.86215E-15	3	21
	(k-1)/m		8.54464E-12	99	399	0	1	6	8.8072E-12	142	852	0	0	4	0	0	5	0	1	6
	1/m		9.3383E-12	66	267	0	1	6	2.41052E-12	10	60	0	0	4	0	0	5	0	1	6
	$1/3^k$		9.04974E-12	80	323	0	1	6	9.89661E-12	105	630	1.75759E-14	0	4	2.187E-15	0	5	0	1	6
	k/m		8.54491E-12	99	399	0	1	6	8.80672E-12	142	852	0	0	4	0	0	5	0	1	6

Table A.26: Comparison of Optimization Methods

f	x_0	m	MOPCM			GMOPCGM			CGPM			GCGPM			STTDFPM			Framework		
			Norm	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**	**Norm**	**IT**	**FE**
F18	0.2	50000	5.06437E-12	18	56	6.55389E-12	37	113	4.02171E-12	21	81	1.19162E-12	12	36	NaN	NaN	NaN	NaN	2000	39967
	0.4		2.87974E-12	21	75	9.68188E-12	33	102	1.68812E-12	17	68	1.34057E-12	9	28	5.26297E-12	9	27	9.73153E-12	33	102
	0.5		3.87275E-12	20	71	9.88049E-12	28	87	2.58184E-12	17	68	1.83708E-12	9	27	2.08533E-12	9	27	9.93014E-12	28	87
	0.6		2.68114E-12	18	63	5.75948E-12	33	102	2.78044E-12	17	68	1.19162E-12	9	27	2.97904E-13	8	27	5.75948E-12	33	102
	0.8		3.07834E-12	12	39	7.94411E-12	34	104	2.23428E-12	17	68	5.46158E-13	7	24	6.55389E-12	7	23	7.99376E-12	34	104
	1.0		2.03568E-12	2	8	6.75249E-12	5	17	3.37625E-12	4	16	3.12799E-12	2	6	2.03568E-12	2	7	6.75249E-12	5	17
	1.1		3.37625E-12	12	38	6.80214E-12	33	101	7.64621E-12	15	59	5.46158E-13	7	24	2.28393E-12	8	24	6.90144E-12	33	101
	1 - 1/m		4.41891E-12	8	26	5.66018E-12	23	71	3.07834E-12	12	48	9.43363E-13	5	18	4.51821E-12	5	16	5.61053E-12	23	71
	1/k		5.96971E-12	98	352	NaN	12	211	5.85996E-12	62	253	6.68623E-12	10	33	7.70371E-12	57	176	NaN	2000	39987
	1/m		5.11402E-12	12	39	4.66716E-12	33	105	3.37625E-12	17	72	1.98603E-13	9	33	8.53992E-12	9	32	4.66716E-12	33	105
F19	0.0	50000	7.51752E-12	130	516	0.041622745	11	34	9.55533E-12	24	125	4.79139E-12	13	46	0.012519738	2001	4003	3.94034E-12	24	91
	0.2		7.51752E-12	131	524	0.041622745	12	54	7.38438E-12	26	158	3.16184E-12	14	65	0.005277103	1	22	8.12904E-12	24	107
	0.4		7.51752E-12	140	577	0.041622745	12	52	7.38438E-12	29	194	3.1695E-12	14	64	0.005277354	1	20	8.1304E-12	24	105
	0.5		7.51752E-12	137	564	0.041622745	12	53	7.38447E-12	30	203	3.16106E-12	14	64	0.005277263	1	21	8.1304E-12	24	106
	0.6		7.51752E-12	135	548	0.041622745	12	53	7.38447E-12	29	187	3.16746E-12	14	65	7.48465E-12	13	55	8.1304E-12	24	106
	0.8		7.51752E-12	131	524	0.041622745	12	54	7.47893E-12	26	158	3.16164E-12	14	65	0.005277165	1	22	8.1304E-12	24	107
	1.0		7.51752E-12	131	524	0.041622745	12	54	7.47902E-12	26	158	3.16193E-12	14	65	0.005277142	1	22	8.1304E-12	24	107
	1.1		7.51752E-12	131	524	0.041622745	12	54	7.47893E-12	26	158	3.16126E-12	14	65	0.005277136	1	22	8.1304E-12	24	107
	1 - 1/m		7.51752E-12	131	524	0.041622745	12	54	7.47844E-12	26	158	3.16164E-12	14	65	0.005277142	1	22	8.12904E-12	24	107
F20	0.0	50000	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
	0.2		0	1	5	9.65582E-12	12	65	0	4	38	0	0	7	0	0	4	0	1	6
	0.4		7.58489E-12	99	397	7.94708E-12	12	58	4.33057E-12	16	91	6.91161E-12	45	179	0	2	8	1.88221E-12	15	71
	0.5		8.62036E-12	87	349	6.10566E-12	13	66	1.22418E-12	17	98	6.84527E-12	47	188	4.69276E-12	18	58	9.87008E-12	13	64
	0.6		8.32451E-12	104	418	4.33057E-12	14	66	1.66795E-12	16	92	6.25358E-12	44	175	3.59096E-12	15	49	1.9638E-12	12	58
	0.8		9.06413E-12	99	398	6.4015E-12	13	65	5.07019E-12	15	87	9.80375E-12	45	179	5.87615E-12	19	59	4.92227E-12	12	63
	1.0		8.17659E-12	87	350	4.4785E-12	15	69	1.07626E-12	16	94	8.83462E-12	45	181	9.72216E-12	16	52	8.47243E-12	13	68
	1.1		9.95167E-12	103	415	5.80981E-12	11	58	2.40757E-12	15	88	9.95167E-12	46	185	9.95167E-12	16	52	5.80981E-12	11	58
	1 - 1/m		8.17659E-12	87	350	4.4785E-12	15	69	1.07626E-12	16	94	8.83462E-12	45	181	9.72216E-12	16	52	8.47243E-12	13	68
	1/k		7.97513E-12	107	422	NaN	29	444	9.91214E-12	115	666	7.99852E-12	52	208	2.633535303	2001	4021	5.96028E-12	25	104
	(k-1)/m		9.33021E-12	107	429	5.50355E-12	17	84	7.67777E-12	128	763	6.22106E-12	45	183	6.22528E-12	70	209	NaN	2000	39836
	1/m		9.80375E-12	102	399	0.435314402	11	34	8.9162E-12	23	111	7.65123E-12	54	207	0.041139251	2001	4003	7.43697E-12	25	101
	k/m		9.32414E-12	107	429	5.45452E-12	17	84	7.84025E-12	128	763	6.08025E-12	45	183	9.96189E-12	69	206	NaN	NaN	NaN

Table A.27: Comparison of Optimization Methods

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- 2022, **Publication**, A Simple Algorithm for Prime Factorization and Primality Testing
  https://onlinelibrary.wiley.com/doi/10.1155/2022/7034529?msockid=3a27cac78d1566e71ebcdfc88c1c6728
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