

**ANALYSIS OF WAVE PROPAGATION IN POROUS
FUNCTIONALLY GRADED CARBON NANOTUBE-
REINFORCED BEAMS**

BY

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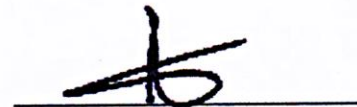
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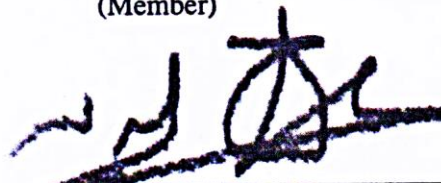
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*Dedicated to My Beloved
Family*

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LIST OF ABBREVIATIONS

FG	:	Functionally Graded
FGM	:	Functionally Graded Material
CNT	:	Carbon Nanotube
SWCNT	:	Single-Walled Carbon Nanotube
MWCNT	:	Multi-Walled Carbon Nanotube
CNTRC	:	Carbon Nanotube-Reinforced Composite
FG-CNTRC	:	Functionally Graded Carbon Nanotube-Reinforced Composite
FSDT	:	First-Order Shear Deformation Theory
HSDT	:	Higher-Order Shear Deformation Theory
GDE	:	Governing Differential Equation

ABSTRACT

Full Name : SAEED MOHAMMED SAEED AL-HOURI

Thesis Title : ANALYSIS OF WAVE PROPAGATION IN POROUS
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Functionally graded materials (FGMs) are a novel class of composite materials that offer exceptional performance owing to their gradual alteration in material composition throughout their structure. Scientists and engineers have been improving the microstructure of materials in pursuit of advanced materials. This evolution has led to the emergence of various nanomaterials in the research field. Carbon nanotubes (CNTs) are one of the nanofillers used in composite structures due to their excellent mechanical, thermal, and electrical attributes. The application of the concept of FGMs in the distribution of CNTs within polymer matrices bears the birth of functionally graded (FG) CNTs-reinforced composite (CNTRC) structures. This thesis aims to investigate the wave propagation behavior in porous infinite functionally graded carbon nanotube-reinforced composite (FG-CNTRC) beams. The beams comprise four patterns of single-walled carbon nanotubes (SWCNTs) distributed in the polymer matrix with the mixture rule used to estimate the material properties of FG-CNTR beams. Innovative to this study are three porosity models describing the porosity distributions within the matrix, nonlinear distributions of FG-CNTs based on exponential power-law models to achieve an optimal distribution of CNTs, and the Kerr elastic foundation to depict the effect of the surroundings on the beams. Integral first- and higher-order shear deformation theories (FSDT and HSDT) are considered to model the FG-CNTRC beams for analytically formulating the

dispersion of the waves with novel correction and shape functions to describe the distribution of shear stress and strain for each theory, respectively. The equations of motion are derived based on Hamilton's principle. The validity of the mathematical models is confirmed through numerical comparisons with previously published works. The thesis parametric investigation includes the influence of the dispersion patterns of CNTs, porosity models, and volume fraction of CNTs and porosity on wave propagation responses. In addition, the study examines the effects of the Kerr foundation and the nonlinear model on wave dispersion behavior.

Comparisons were conducted, and an excellent agreement was reached between the newly proposed models and earlier published research works. Analytical findings suggest that the arrangements of CNTs manipulate the rigidities of the beams, affecting the dispersion relations. Also, increasing the volume fractions of CNTs improves the stiffness of the beams, corresponding to faster wave velocities. Additionally, augmenting the volume fractions of the porosity and implementing a uniform porosity model leads to improved propagating velocities. Further, the impact of CNTs volume fractions, porosity models, and porosity volume fractions becomes more pronounced as the wavenumber increases. Furthermore, the nonlinear distribution of FG-CNTs greatly influences the wave propagation, with the impact varying depending on the wave type and distribution pattern of CNTs. Also, the foundation presence boosts wave velocities and influences only the bending waves, with interactions among the components of the Kerr foundation producing diverse effects on dispersion relations. Moreover, the two shear deformation theories predict identical wave propagation behavior of all beams at lower wavenumbers. However, they differ significantly in capturing the dispersion of propagating waves with higher wavenumbers.

ملخص الرسالة

الاسم الكامل : سعيد محمد سعيد علي محمد الحوري
عنوان الرسالة : تحليل انتشار الموجات في عوارض مسامية مركبة، ومسلحة بأنابيب الكربون النانوية تسليحاً متدرجاً وظيفياً
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المواد ذات الخواص المتدرجة وظيفياً (FGMs) هي نوع جديد من المواد المركبة التي تقدم أداءً استثنائياً، بسبب الاختلاف التدريجي في تكوين بنيتها. لقد عمل العلماء والمهندسون على تحسين البنية المجهرية للمواد سعياً وراء المواد الحديثة المتقدمة. ومنذ ذلك الحين، ظهرت أنواع مختلفة من المواد النانوية في مجال البحث. تعد الأنابيب النانوية الكربونية (CNTs) إحدى مواد الحشو المستخدمة في الهياكل المركبة نظراً لخصائصها الميكانيكية والحرارية والكهربائية الممتازة. إن تطبيق مفهوم المواد متدرجة الخواص في توزيع الأنابيب النانوية الكربونية في البوليمر ينتج منه ولادة مواد مركبة ذات تسليح متدرج بأنابيب الكربون النانوية (FG-CNTRC). تهدف هذه الأطروحة إلى دراسة انتشار الموجات في عوارض مركبة ومسلحة بأنابيب الكربون النانوية تسليحاً متدرجاً، وتحتوي على مسام في تركيبها وذات طول لا محدود. تتكون العوارض من أربعة أنماط تدرج من أنابيب الكربون النانوية أحادية الجدار (SWCNTs) موزعة في مصفوفة البوليمر مع استخدام قاعدة الخليط لتقدير خصائصها الميكانيكية. الجديد في هذه الدراسة، استخدام ثلاثة نماذج مسامية لوصف توزيعات المسام داخل مصفوفة البوليمر، والتوزيعات المتدرجة غير الخطية لأنابيب الكربون بناءً على نماذج أسية لتحقيق توزيع مثالي لأنابيب الكربون النانوية، وأساس كير المرن لتصوير تأثير البيئة المحيطة على العوارض. الدراسة تستخدم نظريتي الرتبة الأولى والرتبة العليا من تشوه القص المصنوعة بمعادلات تكاملية لتمثيل تشتت الموجات تمثيلاً تحليلياً مع معادلات جديدة لوصف توزيع إجهاد القص والانفعال لكلا النظريتين. معادلات الحركة الحاكمة تُشتق باستخدام مبدأ هاملتون. أيضاً، الرسالة تحقق من مصداقية وقابلية تطبيق النظريات المقترحة من خلال المقارنة مع نتائج الأعمال المنشورة سابقاً في هذا المجال. تدرس الأطروحة تأثير كمية وأنماط توزيع الأنابيب النانوية الكربونية في البوليمر، مع الأخذ في الاعتبار تأثير التوزيع الأسّي اللأخطي بالأنابيب وتواجد المسام في البوليمر وأخيراً أساس كير المرن.

أُجريت المقارنات بين النماذج المقترحة حديثاً والأعمال البحثية المنشورة سابقاً، وأظهرت توافقاً كبيراً بينهم. تشير النتائج التحليلية إلى أن ترتيبات الأنابيب النانوية الكربونية تتلاعب بصلابة العوارض، مما يؤثر على علاقات التشتت. كما أن زيادة نسبة وجود الأنابيب النانوية الكربونية تعمل على تحسين صلابة العوارض، مما

يتوافق مع موجات ذات سرعات أعلى. بالإضافة إلى ذلك، فإن زيادة النسب الحجمية للمسام مع تنفيذ توزيع متجانس يؤدي إلى تحسين سرعات الانتشار. فضلاً عن ذلك، يصبح تأثير زيادة نسبة وجود المسام والأنابيب النانوية الكربونية في البوليمر وتطبيق النماذج المتنوعة للمسام أكثر وضوحاً مع زيادة العدد الموجي. أيضاً، يؤثر التوزيع المتدرج غير الخطي للأنابيب النانوية الكربونية إلى حد بعيد على انتشار الموجة، مع اختلاف التأثير اعتماداً على نوع الموجة ونمط توزيع أنابيب الكربون. ثم إنَّ وجود الأساس يعزز سرعات انتشار الموجات ويؤثر فقط على موجات الانحناء، حيث تنتج التفاعلات بين مكونات أساس كر تأثيرات متنوعة على علاقات التشتت. علاوة على ذلك، تظهر نظريتنا تشوه القص المقترحتين سلوكاً متماثلاً لانتشار الموجات في جميع أنواع العوارض عندما تكون الأعداد الموجية منخفضة. ومع ذلك، فهي تختلف إلى حد كبير في التقاط تشتت الموجات عندما تكون الأعداد الموجية بقيم أعلى.

CHAPTER 1

INTRODUCTION

1.1 General

Materials of different varieties have been used in various engineering fields, such as construction, automobiles, aviation, and industrial products. Under practical circumstances, these materials are naturally subjected to different loadings and environmental conditions in their service life. Therefore, it is essential to develop their mechanical properties to prevent the harm brought on by these circumstances. Many new advanced materials have been developed recently, especially with the advancement of technology for manufacturing them. Among the material types, composite materials began replacing homogeneous ones in recent years. Composite materials combine two or more different types of materials while retaining the beneficial properties of each component. However, the sharp transition of properties at the interface in the laminate layers causes delamination failure or crack phenomena when the composite structures are subjected to harsh working conditions. A gradual variation in the material composition in one or more directions was the key to overcoming this disadvantage. This led to the development of a new advanced material called functionally graded material (FGM).

FGMs are the new family of composite materials that offer exceptional levels of performance and reliability. Their gradual changes in the material compositions are based on a specific mathematical function to provide adequate performance depending on the loads and environmental conditions. Metal and ceramic are often utilized materials in FGM. Another material type used as fiber support in polymer matrix composites is carbon nanotubes (CNTs). It is a hot topic in nanotechnology because of its remarkable qualities, which include high modulus of elasticity, tensile strength, and low-density characteristics. FG-CNTs-reinforced composite (CNTRC) structures were developed by implementing CNTs as functionally graded in the polymer matrix to improve the macro-mechanical properties of CNTRC structures with low content. As a result, the investigation of the FG-CNTRC structures is crucial and has piqued researchers' interest.

1.2 Motivations and Need for Research

FG-CNTRC has demonstrated a substantial promise for developing advanced, high-performing lightweight structures, a crucial part of modern life. Numerous research publications focused on analytical and numerical analyses to examine the bending, buckling, vibration, dynamic, and thermal responses of FG-CNTRC structures. The wave propagation analysis of FG-CNTRC beams is generally lacking despite the vast literature on these beams. Different volume fractions and nonlinear distributions of CNTs and any potential interactions between a structural member and surroundings should all be considered as part of a thorough wave propagation analysis.

1.3 Scope and Objectives of the Thesis

This thesis aims to derive the governing differential equations (GDE) of porous FG-CNTRC beam, solve the GDE analytically, and provide numerical examples for the solved problems. The specific objectives of the thesis include the following:

- Proposing simple and efficient shear deformation beam theories.
- Studying the wave propagation analysis of the FG-CNTRC beams.
- Investigating the effects of the distribution patterns of CNTs in the polymer matrix, CNTs volume fraction, nonlinearity of CNTs, foundations, and porosity.
- Discussing the frequencies and dispersion relations of the porous FG-CNTRC beam.

1.4 Thesis Layout

The body of the thesis contains six chapters to address the abovementioned objectives and research gaps, as follows:

- **Chapter 2** gives a background frame of the material and a review of the fundamental principles of beam theories, elastic foundations, wave analysis, calculus of variation, and analysis methodology.
- **Chapter 3** provides a literature review of functionally graded nanocomposites and examines the relevant publications on the mechanical, dynamic, and wave propagation

analyses of functionally graded carbon nanotube-reinforced composite structures. Also, it covers the influences of material properties, foundations, and porosity.

- **Chapter 4** is devoted to analyzing the porosity-dependent wave propagation in FG-CNTRC Beams utilizing an integral higher-order shear deformation theory. The study of this chapter is based on a published journal paper in the International Journal of Structural Stability and Dynamics.
- **Chapter 5** presents wave propagation analysis of composite beams reinforced with nonlinear FG-CNT distributions supported on Kerr elastic foundation utilizing an improved integral first-order shear deformation theory. The study of this chapter is based on an accepted ISI journal paper.
- **Chapter 6** summarizes the research's contributions and significant results and provides recommendations for future research in the thesis area.

CHAPTER 2

BACKGROUND

This chapter will discuss the main concepts of advanced materials, beam theories, elastic foundations, wave analysis, mathematical background, and analysis methodology.

2.1 Advanced Materials

There are intensifying demands for high-performance-weight ratio materials in various modern industries and branches of engineering and technology, such as aerospace structures, aeronautics, high-temperature thermal barrier coatings, and nanostructures. These demands need the structures to be designed according to their specific application, eventually leading to the development of composite materials to overcome the difficulties related to the weight of structures.

2.1.1 Composite Materials

Composites are heterogeneous materials that exhibit anisotropic characteristics, which show coupled deformations in all directions. Composite materials combine the beneficial properties of two or more different materials, which makes them preferable in many situations with almost unlimited applications. Those composite materials can produce lightweight materials with desirable characteristics such as strength, toughness, stiffness, magnetic transparency, heat resistance, excellent corrosion resistance, acoustical insulation, and high fatigue strength. Fibrous, particulate, and laminated composites are the three main classes of composites. Fibrous composites are composed of dispersed fibers in a matrix where the fibers resist all the loads, and the matrix assists in stress transfer, protects the fibers from detrimental environments, and binds them together. In particulate composites, the matrix materials contain some suspended particles, such as concrete, where cement binds sand and aggregates. The laminated composites consist of many stacked laminae or plies to form the structure, where each lamina contains fibers bonded by a matrix material, as in a carbon fiber [1, 2].

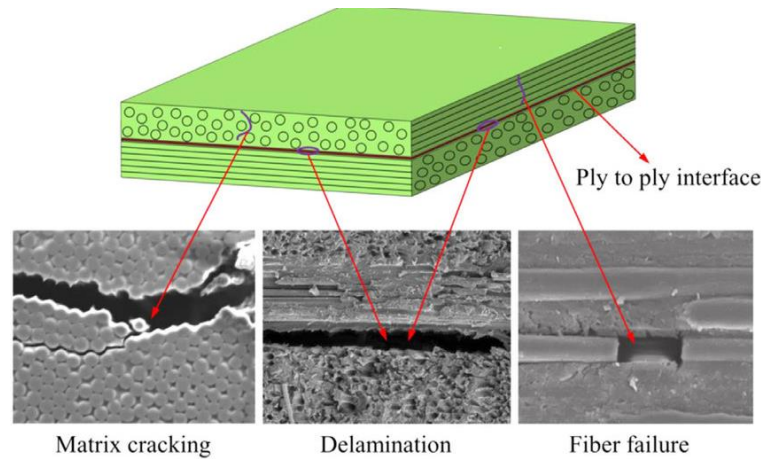


Figure 2.1: Failure mechanisms of carbon fiber-reinforced composite laminates [3].

In laminated composites, there are locations of discontinuity of the material properties between each lamina, which lead to weak interfaces between these layers in which the stress concentration eventually causes failure and delamination, as shown in **Figure 2.1**. Developing functionally graded materials (FGMs) solves the aforementioned problem [3].

2.1.2 Concept of FGMs

FGMs replace the abrupt change between the layers of the laminates with a smooth and continuous change in a material's properties. Depending on the loads and environmental circumstances, the material compositions of FGMs are gradually changed based on a mathematical function to give satisfactory performance. This solution eradicates interface problems like poor adhesion and stress concentrations.

The history of these materials dates back to Japan in 1984 when it was initially a result of a space plane project. Although the concept of FGMs appears to be a modern material invention, it is not. Nature inspired the development of these materials. For example, bones and skin have functional grading as a function of depth and location on the body to provide specific functionality, as shown in **Figure 2.2** [4].

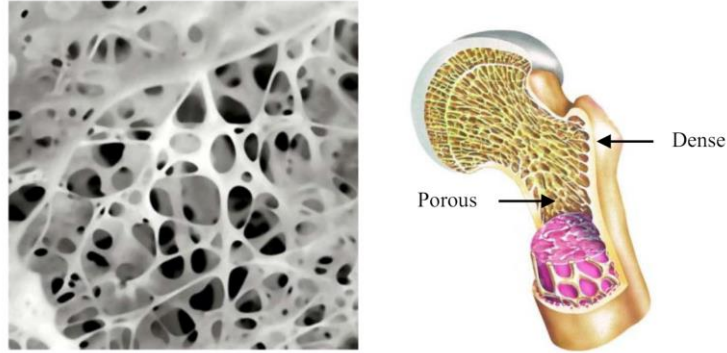


Figure 2.2: Functionally graded density in bone structure [4].

These materials have a variety of benefits that make them useful in potential applications. Some of these advantages are [1, 5]:

- Allow the elimination of stress concentrations.
- Provide the option to design for many complicated environments and to limit deformation, dynamic response, and other factors.
- Reduce the transverse stresses both in-plane and through-the-thickness.
- Improve the toughness and thermal properties.

Figure 2.3 compares FGM and laminated materials' thermal properties and material structures [6].

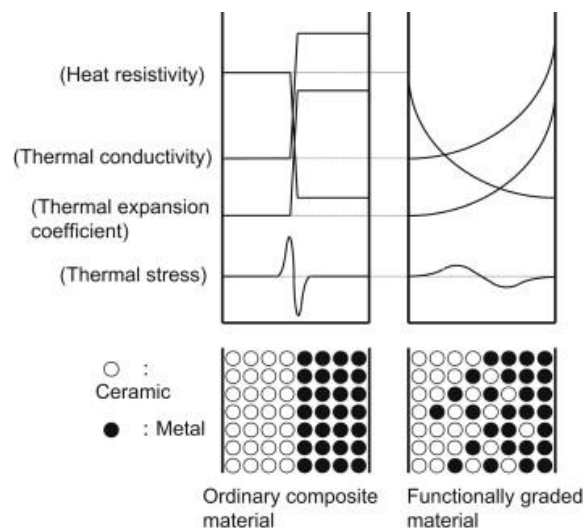


Figure 2.3: Comparison of the thermal properties and material structures of FGM and laminated materials [6].

2.1.3 Concept of FG-CNTRC Structures

Scientists have been altering the microstructures of materials for decades to pursue advancement in materials science. This pursuit bears progress in nanomaterials, allowing the development of nanofillers in composites. Nanofillers are becoming a crucial component of many composite applications—particularly when it comes to enhancing thermal, mechanical, and vibrational properties. Previous research on the qualities of nanofillers revealed that nanofillers should have the following properties:

- Outstanding mechanical properties.
- Large surface area and aspect ratio to enhance interfacial contact with the matrix.
- Spread evenly throughout the matrix, avoiding clustering [7].

Carbon nanotubes (CNTs) and graphene are among the nanofillers that have captivated researchers due to their unique structures, short diameter, and high aspect ratio. CNTs have exceptional mechanical and thermal characteristics with a modulus of elasticity of 1 TPa, tensile strength of 200 GPa, and thermal conductivity up to $600 \text{ Wm}^{-1}\text{K}^{-1}$ [8]. A nanotube is a nanometer structure with a tube shape having organic or inorganic nanotube types. CNT is a long, cylindrical, or tube-like structure made by rolling a graphene or graphite sheet (**Figure 2.4** [9]).

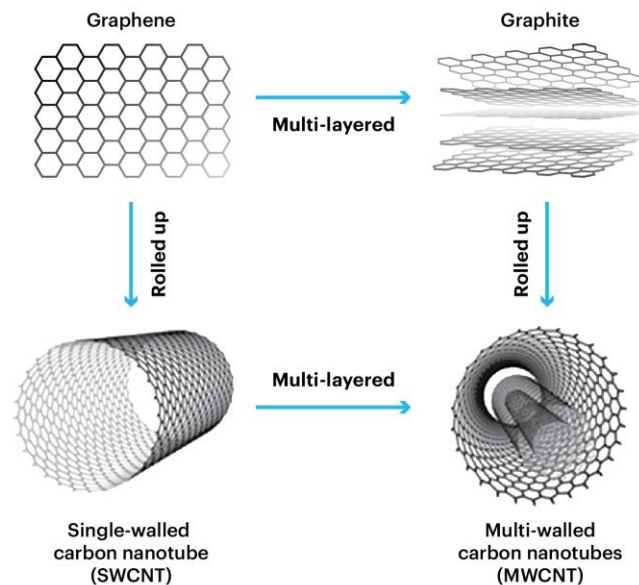


Figure 2.4: Schematic structure of graphene, graphite, SWCNT and MWCNT [9].

There are two types of CNTs: single-walled carbon nanotubes (SWCNTs) and multi-walled carbon nanotubes (MWCNTs), depicted in **Figure 2.4** [9]. SWCNT consists of a single rolled graphene sheet, while MWCNT is composed of many concentric rolled sheets of graphene or a rolled graphite sheet. The application field determines whether to utilize MWCNT or SWCNT on the structure. SWCNTs have three configurations based on chirality (Shown in **Figure 2.5**): the armchair, the zigzag, and the chiral configurations [1, 8].

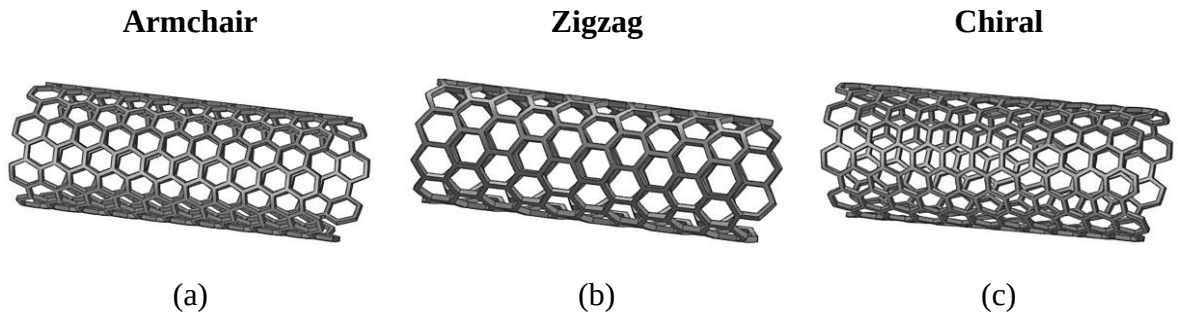


Figure 2.5: Schematic representation of different types of SWCNTs [11].

The nanofillers are dispersed uniformly or randomly throughout the matrix in the conventional nanocomposites, which do not exhibit spatially varying physical properties at the macroscopic level. Moreover, the content and proportion of nanofillers that can be incorporated into the polymer matrix ought to be limited since incorporating additional nanofillers will lead to inadequate filler dispersion, ultimately deteriorating the mechanical characteristics of nanocomposites. These issues have made it more difficult to further employ CNTs as reinforcing elements in nanocomposites. Enhancing the micromechanical properties of CNTs-reinforced composite structures with low CNTs content leads to the introduction of the concept of FGMs into conventional nanocomposite structures. The Chinese researcher Shen first proposed the idea of composite structures reinforced with FG-CNTs, where the volume fractions of CNTs are gradient arranged in the matrix to improve the interfacial bonding strength. FG-CNTs improve mechanical, electrical, and thermal properties and help reduce inter-laminar stresses caused by an uneven elastic modulus in conventional laminated composites [10, 11].

2.2 Effective Properties Models of FG-CNTRC

The FG-CNTRC is produced by functionally varying CNTs along the thickness of the structure in the polymer matrix. There are five types of CNTs distribution forms, shown in **Figure 2.6**, because of the manufacturing limitations, which are as follows: UD, FG-O, FG-X, FG-V, and FG- Λ [11].

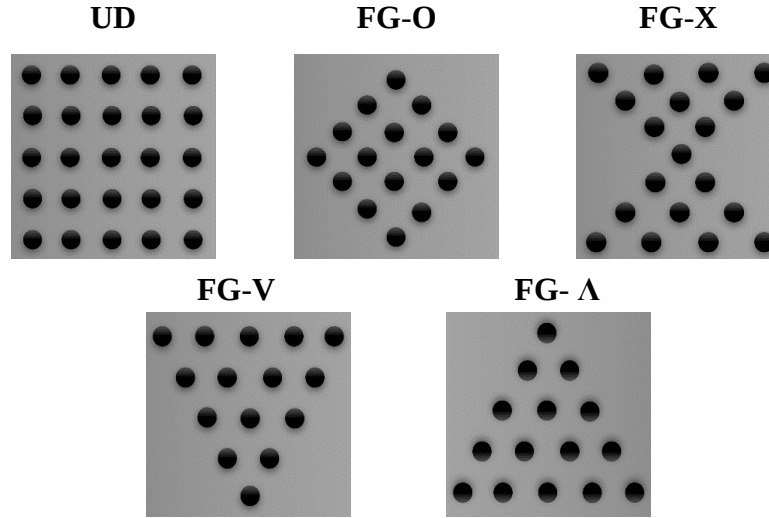


Figure 2.6: Cross sections of different distribution forms of CNTs.

Type UD represents the uniform distribution of the CNTs in the z -direction. FG-V and FG- Λ , the CNTs increase or decrease respectively through the thickness, forming a triangle shape. In type FG-X, the CNTs decrease gradually from the top until the center of the beam and then increase symmetrically to the bottom. Type FG-O has an opposite CNTs distribution compared to type FG-X. The volume fractions are the tool to describe the gradation of CNTs as mathematical functions. The different patterns of CNTs distributed along the beam cross-section are expressed in **Table 2.1** below as follows [11]:

Table 2.1: Expressions of CNTs' patterns distributed along the beam cross-section [11].

Distribution of CNTs	V_{cnt}
UD	$V_{cnt} = V_{cnt}^*$ (2-1a)
FGO	$V_{cnt} = 2 \left(1 - 2 \frac{ z }{h} \right) V_{cnt}^*$ (2-1b)
FGX	$V_{cnt} = 4 \frac{ z }{h} V_{cnt}^*$ (2-1c)
FGV	$V_{cnt} = \left(1 + \frac{2z}{h} \right) V_{cnt}^*$ (2-1d)
FGA	$V_{cnt} = \left(1 - \frac{2z}{h} \right) V_{cnt}^*$ (2-1e)

where:

- h : The total thickness of the member:
- V_{cnt}^* : The given volume fraction of CNTs, which can be calculated as:

$$V_{cnt}^* = \frac{W_{cnt}}{W_{cnt} + \left(\frac{\rho^{cnt}}{\rho^p} \right) (1 - W_{cnt})} \quad (2-2)$$

where:

- W_{cnt} : The mass fraction of CNTs.
- ρ^{cnt}, ρ^p : The density of CNTs and the polymer matrix, respectively.

The total volume fractions of FG-CNTRC structures are equal to the volume percentages of the matrix polymer and CNTs. They depend only on z because the gradation is applied only in the z -direction. At any location of the beam, the following holds:

$$V_{cnt} + V_p = 1 \quad (2-3)$$

where:

- V_p : The volume fraction of the polymer matrix.

Predicting the effective material properties of composites strengthened with CNTs is crucial. So, there are numerous micromechanical materials models proposed; some of them are present below [7]:

- Modified Halpin-Tsai model.
- The extended rule of mixture (EROM).
- Eshelby–Mori–Tanaka (EMT) approach.

The extended rule of the mixture has received considerable academic support [11]. So, it is the utilized model in this study.

2.3 Elastic Beams Theories

Beams are structural members whose cross-sectional dimensions are small compared to their lengths with a ratio of length to height of 10 or more (If the beam dimensions are length a , height h , and width b , then: $a > 10 * \text{the larger of } b \text{ \& } h$). They support both transverse and in-plane forces, which cause bending about the perpendicular axis of their lengths. Because of the structural configuration of beams, structural theories known as beam theories can predict the deformation and stress fields. These theories are derived from the 3D elasticity theory by simplifying certain stress and deformation state assumptions. The beam theories could be classified into the following two main categories [12, 13]:

- The classical beam theory, in which the effects of the transverse shear deformation are ignored.
- The shear deformation beam theories, where the transverse shear deformation effects are being considered.

The following subsections will discuss the theories describing the behavior of the beams.

2.3.1 Classical Beam Theory (CBT)

The classical beam theory (CBT), also referred to as the Euler-Bernoulli beam theory, is the most established and well-known beam theory. Its displacement field is formulated according to the assumptions stated in the following:

- The mid-surface deformation is small and can be neglected, making its strain even smaller and negligible.
- The mid-surface level is assumed to still be the mid-surface after deformation.
- Planes perpendicular to the mid-surface are assumed to remain perpendicular after deformation. This is equivalent to saying that the transverse shear strain is negligible, leading to neglecting the transverse normal strain.
- The beam is under transverse loading only, which hardly causes any normal stresses in the transverse direction. Hence, σ_z is neglected.

Omitting the negligible dimensions, CBT results in a displacement field that has only two variables expressed as follows [12 - 14]:

$$u(x, z, t) = u_0(x, t) - z \frac{\partial w_0(x, t)}{\partial x} \quad (2-4)$$

$$w(x, z, t) = w_0(x, t)$$

where:

- z : The distance from the neutral axis to any point in the beam.
- $u_0(x, t)$: The deformation at the mid-surface along x .
- $w_0(x, t)$: The transversal deformation and does not depend on the z coordinate.

Although CBT is adequate for slender beams, it does not represent the behavior of short and deep beams well.

2.3.2 Timoshenko Beam Theory

Another popular theory is the Timoshenko beam theory, alternatively called the first-order shear deformation beam theory (FSDT). The assumptions of Timoshenko's theory are identical to those of (CBT), except that plane sections no longer remain normal to the mid-plane after bending. Thus, transverse strain and shear deformations are taken into account. Timoshenko's beam theory states that the beam will experience a uniform shear distribution through thickness. A shear correction factor is needed to account for a constant shear

distribution. The shear strain angles will lead to additional terms in u . The displacement field becomes of three variables as follows [12, 13]:

$$\begin{aligned} u(x, z, t) &= u_0(x, t) + z\phi_x(x, t) \\ w(x, z, t) &= w_0(x, t) \end{aligned} \quad (2-5)$$

where:

- $\phi_x(x, t)$: The angles of deflection in xz -plane.

Because of the assumed uniform shear distribution across the beam's depth, Timoshenko beam theory is not preferred for the requirement of a shear correction factor.

2.3.3 Higher-Order Shear Deformation Theories (HSDT)

Higher-order shear deformation theories (HSDT) take into account cross-section warping and satisfy the upper and lower faces' condition for zero transverse shear stress without using a shear correction factor. The displacement field shown below is assumed to be of third degree in terms of z [12 - 14].

$$\begin{aligned} u(x, z, t) &= u_0(x, t) + z\phi_x(x, t) - cz^3 \left(\phi_x + \frac{\partial w}{\partial x} \right) \\ w(x, z, t) &= w_0(x, t) \end{aligned} \quad (2-6)$$

HSDBT provides more reliable solutions when compared with simpler theories. **Figure 2.7** [14] summarizes and illustrates the differences between CPT, FSDT, and TSDT for the plate or beam theories.

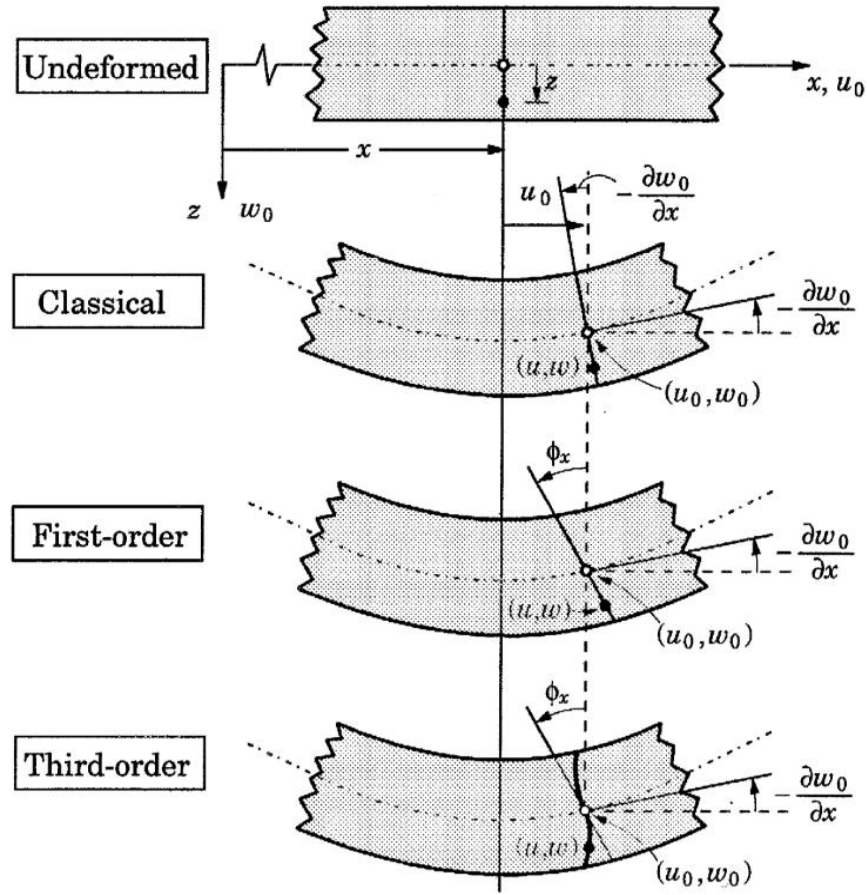


Figure 2.7: Geometries of a beam or plate based on (a) Undeformed state, (b) Classical (CPT), (c) FSDT, and (d) TSDT [14].

2.4 Elastic Foundations

Elastic foundations can be represented using various models, including the Winkler, Pasternak, and Kerr models. The Winkler model depicts the foundation as a tightly spaced, linearly independent spring [15, 16]. **Figure 2.8** shows a representation of the Winkler Foundation. As a result, the relationship between the applied pressure and the foundation's deflection curve is:

$$R_f = k_L w \quad (2-7)$$

where:

- R_f : The reaction force of the foundation.
- k_L : Winkler spring coefficient.

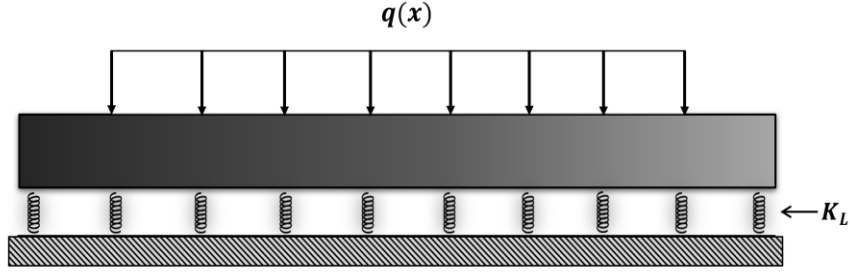


Figure 2.8: Winkler Foundation.

The Winkler-Pasternak foundation model assumes the existence of a shear layer between the spring and the structure to count for the shear effect between foundation particles. After accounting for the shear effect, the Pasternak equation is as follows [16]:

$$R_f = k_L w - k_s \nabla^2 w \quad (2-8)$$

where:

- k_s : The shear layer coefficient.
- ∇^2 : The Laplace operator.

The Winkler-Pasternak foundation is depicted in **Figure 2.9**.

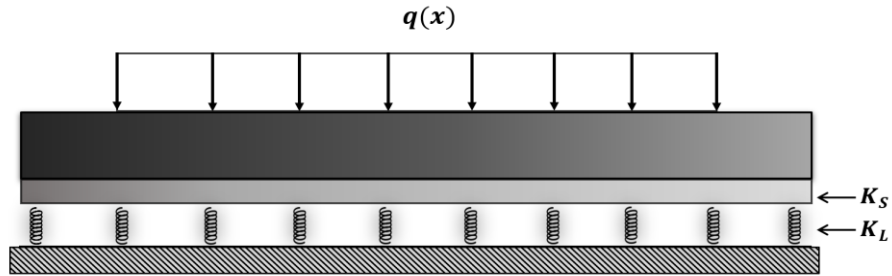


Figure 2.9: Winkler-Pasternak foundation.

The Kerr foundation model implements three different actions: the upper and lower springs and the shear layer actions, as shown in **Figure 2.10**. Thus, the foundation effect can be expressed as [16]:

$$R_f = \frac{k_L k_U}{k_L + k_U} w - \frac{k_s k_U}{k_L + k_U} \nabla^2 w \quad (2-9)$$

where:

- k_U : The upper spring coefficient.

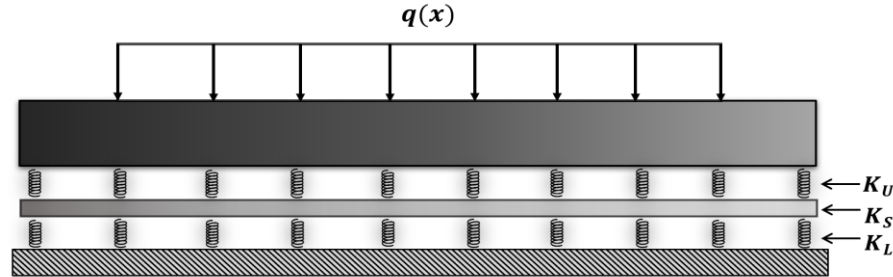


Figure 2.10: Kerr foundation.

The Winkler-Pasternak foundation emerges when the upper layer becomes infinitely stiff, which is equivalent to its removal [17, 18].

2.5 Wave Propagation Analysis

Wave propagation analysis is about performing spectral analysis to determine the behavior nature of waves. Spectral analysis produces two unique results: a spectrum relation (a relationship between wavenumber and frequency) and dispersion relations (relationships between phase and group velocities with frequency).

Table 2.2: Comparisons between structural dynamics and wave propagation.

Comparison Type	Structural Dynamics	Wave Propagation
Vibration Frequencies	Few hundreds Hz	Of the order of kHz or higher
Design Interest	Long term (Steady state) effect	Short term (Transient response) effect
Modes	Only first mode is considered	Higher order modes participate in amplifying the response
Phase Information	Not critical	Important due to the multi-modal phenomenon in wave propagation
Applications	Most structures	Impact and blast

Structural engineering has two types of dynamic analysis: structural dynamics and wave propagation. The distinction between structural dynamics and wave propagation may not always be readily apparent to scientists and engineers. Hence, **Table 2.2** above compares the two analyses to tell them apart [1].

2.5.1 Need for Wave Propagation Analysis

Wave propagation analysis plays a significant role in a variety of practical applications. For example [1, 19]:

- In impact loadings, where the elastic theories cannot effectively model the system's response.
- In the investigation of aerospace or military materials subjected to extreme loadings that cause permanent damage.
- In studying the characteristics of the waves dispersion under the ground to predict the behaviors of earthquakes.
- In detecting defects or damages and their location in the structures.
- In determining the elastic or viscoelastic properties of the materials.

2.5.2 Essential Components of a Wave

A mechanical wave is a disturbance of a medium around an equilibrium position that may be propagated. Mechanical waves need a medium to propagate, contrary to electromagnetic waves that can propagate in air and a vacuum. The following are some types of mechanical waves:

- Longitudinal waves are where the displacement of the medium exists in the same direction as the moving wave.
- Transverse waves are where the displacement of the medium exists in the perpendicular direction of the moving wave.

The one-dimensional mechanical wave that travels in the positive x-direction is expressed as [1]:

$$u_t(x, t) = A f(\kappa x - \omega t) \quad (2-10)$$

where:

- $u_t(x, t)$: The deformation of the traveling wave.
- A : The amplitude of the traveling wave.
- κ : The wavenumber.
- ω : The circular frequency.
- t : The time.

When this traveling wave reaches the end of a 1D medium, it will reflect and travel back. The expression of the reflected wave is described with g to represent the negative x-axis [1]:

$$u_r(x, t) = B g(\kappa x + \omega t) \quad (2-11)$$

where:

- $u_r(x, t)$: The deformation of the reflected wave.
- B : The amplitude of the reflected wave.

The superposition principle is applicable in linear behavior, so two or more waves could be added together. D'Alembert Solution emerges when the incident wave in Eq. (2-10) and reflected wave in Eq. (2-11) are superimposed as the following [1]:

$$u_t(x, t) + u_r(x, t) = A f(\kappa x - \omega t) + B g(\kappa x + \omega t) \quad (2-12)$$

If the two waves oscillate in time-harmonic, then they could be represented by a sine and cosine function as below:

$$u(x, y) = A \cos(\kappa x - \omega t) + B \cos(\kappa x + \omega t) \quad (2-13)$$

The components of a harmonic wave can be visualized by a periodic transverse wave in **Figure 2.11**, where:

- λ : The wavelength that describes the distance between two consecutive wave crests or troughs.

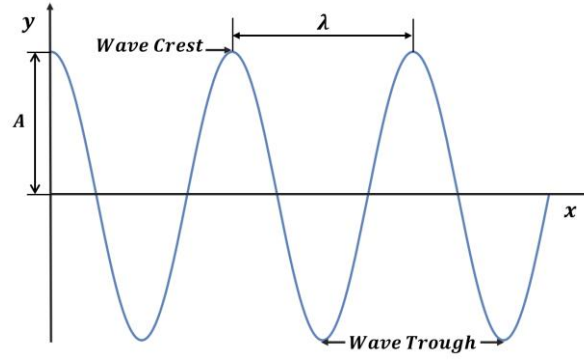


Figure 2.11: Periodic transverse wave.

The period of the wave denoted as T (sec) is reversely related to frequency f (Hz), expressed as:

$$T = \frac{1}{f} \quad (2-14)$$

2.5.3 Components of Wave Propagation

A propagating wave in a 1D medium can be also expressed as (Other than the form in (2-12)) [1]:

$$u_w(x, t) = A e^{i(\kappa x - \omega t)} \quad (2-15)$$

where:

- $u_w(x, t)$: The deformation of the wave.
- κ : The wavenumber.
- $(\kappa x - \omega t)$: The phase of the wave.
- ω : The circular frequency which related to frequency by the following:

$$\omega = 2 \pi f \quad (2-16)$$

The phase velocity (c_p) is the velocity of a wave with a constant phase, which can be derived by taking the derivative of the wave phase and equated to zero, as follows:

$$\frac{d(\kappa x - \omega t)}{dt} = \kappa \frac{dx}{dt} - \omega = 0 \quad (2-17)$$

$$\frac{dx}{dt} = \frac{\omega}{\kappa} = c_p$$

In wave propagation analysis, more than one harmonic wave is present, and these waves are superimposed to create what is known as a wave packet. Hence, the phase velocity is not enough to describe the dispersion relations, but an additional velocity is needed. **Figure 2.12** shows a wave packet with representations of both velocities. The superposition of two harmonic waves with very close frequencies and wavenumbers having the same amplitude can be expressed [1]:

$$u_w(x, t) = A e^{i(\kappa_1 x - \omega_1 t)} + A e^{i(\kappa_2 x - \omega_2 t)} \quad (2-18)$$

where:

$$\begin{aligned} \kappa_1 &= \kappa - d\kappa, & \kappa_2 &= \kappa + d\kappa \\ \omega_1 &= \omega - d\omega, & \omega_2 &= \omega + d\omega \end{aligned} \quad (2-19)$$

Substituting the wavenumbers and frequencies from Eq. (2-19) gives:

$$u(x, t) = A e^{i((\kappa - d\kappa)x - (\omega - d\omega)t)} + A e^{i((\kappa + d\kappa)x - (\omega + d\omega)t)} \quad (2-20)$$

Expanding and simplifying

$$u(x, t) = A e^{i((\kappa x - d\kappa x) - (\omega t - d\omega t))} + A e^{i((\kappa x - \omega t) + (d\kappa x - d\omega t))} \quad (2-21)$$

Taking the common factor in Eq. (2-21)

$$u(x, t) = A e^{i(\kappa x - \omega t)} (e^{-i(d\kappa x - d\omega t)} + e^{i(d\kappa x - d\omega t)}) \quad (2-22)$$

The second term in Eq. (2-22) (Inside the brackets) can be simplified by a cosine term.

$$u(x, t) = 2 A e^{i(\kappa x - \omega t)} \cos(d\kappa x - d\omega t) \quad (2-23)$$

Eq. (2-23) represents the response of two waves [1]:

- A carrier traveling wave ($e^{i(\kappa x - \omega t)}$) with a phase velocity same as in Eq. (2-17).
- A modulation or envelope traveling wave ($\cos(d\kappa x - d\omega t)$) with a velocity expressed in Eq. (2-24):

$$c_g = \frac{d\omega}{d\kappa} \quad (2-24)$$

where:

- c_g : The group velocity of the modulation.

Figure 2.12 shows the phase and group velocities of a 1D propagating wave packet. The wave packet travels with the group velocity, whereas the wave itself travels with the phase velocity.

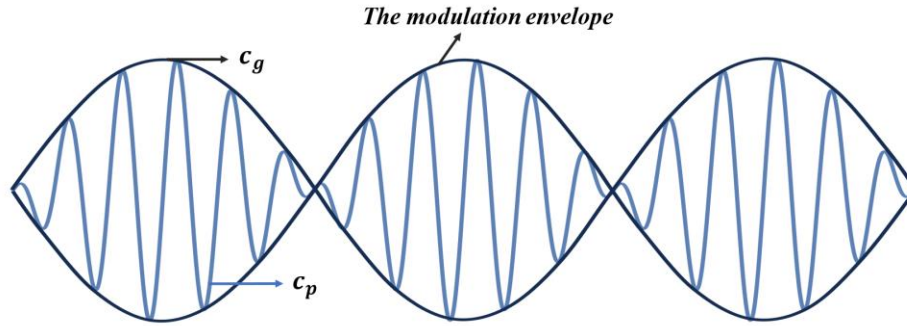


Figure 2.12: Phase and group velocities of 1D wave propagating.

The group and phase velocity can be related to each other as a function of the wavenumber:

$$c_g = \frac{d\omega}{d\kappa} = \frac{d(c_p \kappa)}{d\kappa} = c_p + \kappa \frac{dc_p}{d\kappa} \quad (2-25)$$

Or they are related as a function of the circular frequency:

$$c_g = \frac{d\omega}{d\kappa} = \frac{d\omega}{d\left(\frac{\omega}{c_p}\right)} = \frac{1}{\frac{d}{d\omega}\left(\frac{\omega}{c_p}\right)} = \frac{1}{\frac{1}{c_p} - \frac{\omega}{c_p^2} \frac{dc_p}{d\omega}} = \frac{1}{\frac{1}{c_p} \left(1 - \frac{\omega}{c_p} \frac{dc_p}{d\omega}\right)} \quad (2-26)$$

From Eq. (2-25) and Eq. (2-26), the group velocity is equal to phase velocity if the phase velocity is not a function of the frequency or wavenumber. In this case, the waves are called non-dispersive waves, where they maintain their form while propagating.

Alternatively, if the phase velocity is a function of the frequency or wavenumber, the group velocity will not be the same as the phase velocity. This case, called dispersive waves, leads to the waves' shape constantly changing as they propagate. **Figure 2.13** shows the behavior of non-dispersive and dispersive waves [1].

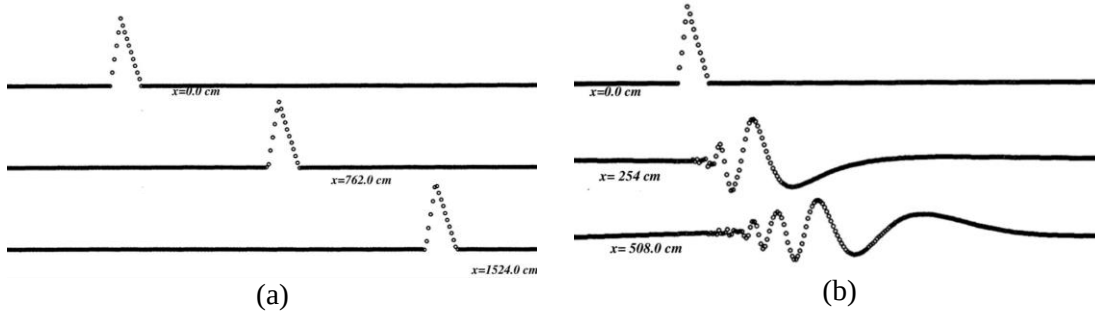


Figure 2.13: Representation of (a) a non-dispersive wave and (b) a dispersive wave [1].

2.6 Concepts in Calculus of Variation

In this section, some concepts of variational calculus that will be used in the formulations are explained briefly.

2.6.1 First Variation

It is a differential operator that acts on functionals. Functional is a function that has a domain of admissible functions and a range of real numbers. By equating the derivatives to zero, one may determine the extreme values of a function. However, the extreme values of functional are when its first variation is zero.

2.6.2 Fundamental Lemma of Calculus of Variations

In a simple form, let $f(x)$ be continuous in $[a, b]$ and let be $\eta(x)$ an arbitrary function on $[a, b]$, such that η, η', η'' are continuous and $\eta(a) = \eta(b) = 0$, then if

$$\int_a^b f(x) \cdot \eta(x) dx = 0 \quad (2-27)$$

For all $\eta(x)$, then $f(x) = 0$ on $[a, b]$. The fundamental lemma of calculus of variations will be used to obtain the governing motion equations from the integral form of the Hamilton principle.

2.7 Hamilton's Principle

The fundamental concept in classical mechanics, Hamilton's Principle, also called the Principle of Least Action, states that the path that minimizes the difference between the kinetic energy, external work, and potential energy is the actual path taken by a physical system as it transitions between two states:

$$\delta \int_{t_1}^{t_2} L dt = \delta \int_{t_1}^{t_2} (U + V - K) dt = 0 \quad (2-28)$$

where:

- δ : The differential operator (The first variation).
- L : The Lagrangian function for the system, which is equal to $(U + V - K)$.
- $U, V, \text{ and } K$: The strain energy, external work, and kinetic energy, respectively.

The following expression gives the kinetic energy of a continuous body:

$$K = \frac{1}{2} \int \rho \left(\frac{\partial u_i}{\partial t} \right)^2 d\Omega, \quad i = 1, 2, 3 \quad (2-29)$$

$$\{u_1, u_2, u_3\} = \{u, v, w\}$$

where:

- ρ : The density of the material.
- u, v, w : The displacement field in x, y, z axes.
- Ω : The volume of the material.

The strain energy can be written as:

$$U = \frac{1}{2} \int \sigma_{ij} \varepsilon_{ij} d\Omega \quad (2-30)$$

$$\{\varepsilon\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}\} = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}\}$$

$$\{\sigma\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\} = \{\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}\}$$

where:

- $\{\sigma\}$ and $\{\varepsilon\}$: The stress and strain vectors, respectively.

The external work has the following expression:

$$V = \int q_0 w dS \quad (2-31)$$

where:

- q_0 : The distributed transverse loads.
- S : The surface on which the distributed loads act.

2.8 Analysis Methodology

The formulation and governing equations derivation work will go through the stages discussed in the following subsections.

2.8.1 Kinematics

Kinematics describes the displacement field of the points in the beam. Kinematics is formulated based on the chosen beam theory to describe the behavior of the beam through which the waves propagate.

2.8.2 Strain-Displacement Relations

The components of the strains are derived in terms of displacement's unknowns using linear strain-displacement relationships. 3D infinitesimal strain for small deformation is expressed in an indicial notation as [12]:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3. \quad (2-32)$$

$$\{x_1, x_2, x_3\} = \{x, y, z\}$$

2.8.3 Stress-Strain Relations

Constitutive equations describe the materials and how they respond to external stimuli as applied forces. One of the fundamental constitutive equations of linear elasticity is Generalized Hook's Law. The stress-strain relations of linear elastic anisotropic solids can be written as [1]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} & Q_{56} \\ Q_{61} & Q_{62} & Q_{63} & Q_{64} & Q_{65} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} \quad (2-33)$$

$$\{\sigma\} = [Q]\{\varepsilon\}$$

where:

- $[Q]$: The symmetrical stiffness properties matrix, where $Q_{ij} = Q_{ji}$.

The stiffness matrix in Eq. (2-33) is a general matrix for anisotropic material with 21 independent elastic constants. For the beams under study, the stiffness matrix reduces to:

$$\begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} \quad (2-34)$$

2.8.4 Governing Equations

The motion equations of the FG-CNTRC beams are derived based on Hamilton's principle, expressed in Eq. (2-28), after substituting the energy virtual variations (δU , δW , and δK) as functions of the middle displacements. Expressing all energy variables in terms of the beam theory unknowns is essential for the solution process. This process involves utilizing the calculus of variations, integration by parts, and equating the coefficients of middle displacements to zero.

2.8.5 Dispersion Relations

The analytical solutions for wave propagation analysis in an infinite solid are given as follows [1]:

$$u_{0j}(x, y, z, t) = U_j e^{i(\kappa_1 x + \kappa_2 y + \kappa_3 z - \omega t)} \quad (2.35)$$

where:

- u_{0j} : The solutions of the middle displacements.
- U_j : The amplitudes of the propagating waves.
- j : The number of unknown middle displacements.
- $\kappa_1, \kappa_2, \kappa_3$: The wavenumbers in x, y, and z axes.

For wave propagation analysis in an infinite beam element, the equation reduced to [1, 19]:

$$u_{0j}(x, t) = U_j e^{i(\kappa_1 x - \omega t)}, \quad j = (1, 2, 3) \quad (2.36)$$

The middle displacement solutions are then substituted in the derived governing equations to construct the characteristic equation that can be written as a matrix form:

$$([K] - \omega^2 [M]) \{\Lambda\} = 0 \quad (2.37)$$

where:

- K : The stiffness matrix contains the terms related to strain energy and external work.
- M : The mass matrix contains the terms related to kinetic energy.
- Λ : The vector that consists of the amplitudes of the waves.

The system of equations in (2-37) is an eigenvalue problem, which has two solutions:

- The vector of the waves' amplitudes $\{\Lambda\}$ is zero, which gives the trivial solution. This means the beam will not vibrate, which is incorrect.
- The determinant of the matrix in Eq. (2-37) is zero, expressed as:

$$|[K] - \omega^2 [M]| = 0 \quad (2.38)$$

The solutions of the equations in (2-38) are called the eigenvalues, and they are equal to the square of the circular frequencies. Taking the square root of the eigenvalues gives these frequencies as functions of the wavenumbers. For beams in this study, three frequencies are found corresponding to the theory's unknowns, as shown below:

$$\omega_j = \omega_j(\kappa_1), \quad (j = 1, 2, 3) \quad (2-39)$$

where ω_1 , ω_2 , and ω_3 represent the frequencies of bending (flexural), longitudinal, and shear modes, respectively. The wave frequencies f_j are also functions of wavenumbers, defined as:

$$f_j = \frac{\omega_j(\kappa_1)}{2\pi} \quad (2-40)$$

Finally, the propagating phase C_{pj} and group velocity C_{gj} velocities can be determined as functions of the wavenumber from their relations with circular frequencies as follows:

$$C_{pj} = \frac{\omega_j(\kappa_1)}{\kappa_1}, \quad C_{gj} = \frac{d\omega_j(\kappa_1)}{d\kappa_1} \quad (2-41)$$

2.8.6 Results Validation

The analytical results are verified by comparing with earlier publications of similar works. Most researchers in this field follow this procedure, and it is used to ensure:

- The accuracy of the calculations.
- The reliability of the findings.
- The recognition and honor of the outstanding work of early researchers.

CHAPTER 3

LITERATURE REVIEW

The literature review of the prior works done in the field is conducted to identify the gaps that need to be filled. The literature covers the mechanical and dynamic analysis of FG-CNTRC structures in general and wave propagation in specific. In addition, it covers various effects of material gradation and associated properties, foundations, and porosity.

3.1 Introduction

Since Dr. Ijima [20, 21], a Japanese scientist, synthesized multi-walled Carbon nanotubes (CNTs) in 1991, the nanocomposites field has gained more research investigations. CNTs have extraordinary mechanical properties [22], lightweight, high thermal characteristics, and good electrical conductivity [23], so they are regarded as one of the most exceptional options for reinforcing polymer matrix nanocomposites to replace conventional fiber reinforcement [24, 25] in various structures (beams, plates, and shells). However, subjecting the nanocomposite structures to severe working conditions results in abrupt changes in the composite's characteristics at the interface, which may cause delamination failure [26]. Inspired by the idea of functionally graded materials (FGMs), Shen first suggested changing the distributions of CNTs functionally in the polymer due to the importance of the amount and configuration of CNTs and interfacial bonding strength [27]. As a result, a new advanced material called functionally graded carbon nanotube-reinforced composites (FG-CNTRC) was developed [7, 8, 10, 11], which highly reduces the interface issue and stress concentration [28]. Aerospace structural parts, pressure vessel valves, acoustics, structural components of automotive, submarines, Micro/Nano-Electro-Mechanical Systems (MEMS/NEMS), medical equipment, and rockets are among the engineering fields where FG-CNTRC structures find application. Therefore, many researchers investigated the mechanical responses of these structures.

3.2 Mechanical Analysis of FG-CNTRC Structures

In recent years, the bending behaviors of the FG-CNTRC beams, plates, arches, and shells have been extensively researched and investigated. Khadir et al. [29] investigated the influence of thickness stretching on the mechanical responses, including bending behavior, of the FG-CNTRC laminated plates when placed on an elastic substrate by employing a newly proposed quasi-3-D HSDT. There was a reduction in the displacement field's number of unknowns to four, though it was greater than five in other models. Regarding nonlinear bending responses, Zhang et al. [30] analyzed FG-CNTRC shallow arches resting on elastic substrates under uniform radial pressure. The arches had either pinned or clamped ends. The Reddy shear deformation theory (RSDT) and von Karman formulation formed the basis of the analysis. Another study [31] investigated the nonlinear bending response of the FG-CNTRC plate in a thermal environment supported by elastic foundations considering a negative Poisson's ratio. The temperature-dependent properties of the material were estimated by employing Voigt models. The negative Poisson ratio considerably influenced the bending behavior of the FG-CNTRC plates. By applying a numerical technique, the large deflection solution, as well as the dimensionless load-deflection behaviors of the FG-CNTRC plate, were introduced by Cho [32]. The problem formulation was derived from the nonlinear theory of von Karman and a hierarchical model. The introduction of CNTs led to a considerable reduction in central deflection, which became more pronounced with higher volume fractions of CNTs. In another research, Kumar and Sarangi analyzed bending in the smart FG-CNTRC beam coupled with piezoelectric materials [33]. Throughout the thickness of the beam, there was a variation in the material characteristics, which were determined using the mixture rule. The smart FG-CNTRC beam was analyzed by developing a finite element model. The results revealed that combined piezoelectric materials directly impacted the bending responses. For axially FG-CNTRC nanobeams, the bending analysis was performed [34] by modifying an analytical model, and a new HSDT was employed to satisfy shear throughout the beam's thickness. The mechanical properties of these nanobeams were estimated using both the mixture rule and molecular dynamic simulation. The arrangement of CNTs was in the axial direction along the beam's length, which is based on novel exponential graded functions. Further, some other studies investigate nonlinear bending [35 – 39],

buckling [40 – 43], and thermal responses [44] for additional investigation of the FG-CNTRC structures.

3.3 Vibration Analysis of FG-CNTRC Structures

Researchers have also put a great effort into implementing new theories to predict the behavior of FG-CNTRC under vibration. Mirzaei and Kiani [45] studied the natural vibration characteristics of a rectangular CNTRC plate featuring a central rectangular cutout using the FSDT and Ritz methods. It was found that utilizing FG-CNTs across the thickness enhanced the fundamental frequency. Furthermore, the frequencies are proportional to the volume fractions of CNTs and might be improved with additional reinforcement of CNTs. Duc et al. [46] analyzed the static behavior and natural vibration characteristics of a rectangular FG-CNTRC plate, which was supported by a Winkler-Pasternak elastic substrate. The Hamilton principle and FSDT served as a foundation for developing the mathematical model, and the Navier solution was used to find the center deflection and the natural frequency parameters. Yas and Heshmati [47] explored FG-CNTRC beams' vibrational properties with a random orientation of CNTs subjected to a moving load. The findings demonstrated that the dynamic behavior of the beams was influenced by some parameters such as CNTs arrangement, moving load's velocity, and boundary conditions. Yas and Heshmati further worked with Pourasghar and Kamarian [48] to investigate cylindrical panels' natural vibration and dynamic response based on the theory of 3D elasticity.

Singh and Sahoo [49] collaborated to investigate the static and free vibration responses of the FG-CNTRC sandwich plate using the trigonometric shear deformation theory (TrSDT) and Navier method. The parametric study detailed the deflection, stress, frequency, volume fraction, and CNT dispersion of the FG-CNTRC sandwich plate. Utilizing the first-order discrete layer formulation and the Navier approach, the dynamic properties of the FG-CNTRC sandwich plate containing a viscoelastic core were analyzed by Zhai et al. [50], considering a total of five different stacking sequences. The kinematic relations were derived using the FSDT, and then the governing equations were obtained using the Hamilton principle. Furthermore, the vibration frequencies resulting from various stacking sequences were investigated to obtain the best dynamic properties. In another article by Chen et al. [51],

the dynamic instability analysis of the FG-CNTRC hybrid plate subjected to periodic loads was presented in detail. The Galerkin approach with a reduced eigenfunction transform derived the governing equations (Mathieu-form). The outcomes revealed the significant impact of dynamic loading, the volume fractions of CNTs, and bending stresses on the dynamic instability. By employing the TrSDT, research was carried out by [52] to analyze the dynamic behaviors of the FG-CNTRC beam subjected to various dynamic loadings. The obtained governing equations were solved using the Gram-Schmidt-Ritz technique. Moreover, the Newmark integration was utilized to investigate the beam's dynamic responses when subjected to dynamic loadings. In another study, Houalef et al. [53] examined the free vibration behavior of the FG-CNTRC beam according to a refined third-order shear deformation Finite element (FE) beam theory. Lagrange's approach obtained the governing differential equations and then solved them numerically using the differential quadrature finite element method (QFEM). A recent article by Chalak et al. [54] performed a study on the free vibration behaviors of the FG-CNTRC beam exposed to a hygro-thermal environment. The analysis followed the FE high-order zigzag theory and employed temperature-dependent material properties. A total of five CNTs distribution patterns were considered. The CNT's gradation rule significantly affected the stress dispersion throughout the beam's thickness.

3.4 Wave Propagation Analysis of FG-CNTRC Structures

Lately, some researchers have focused on wave propagation analysis of various structures as it plays a significant role in a variety of practical applications. For instance, wave dispersion is crucial in predicting the system responses to distinct loading scenarios [55]. Moreover, the aerospace and military commonly apply the wave propagation principle to analyze the long-term damage to CNTs composites from extreme loadings. Additionally, it is among the most effective techniques for examining the fracture's dispersion. Furthermore, the vibration responses result basically from the wave propagation, which is highly significant for the mechanical behavior analysis of the structures. Also, the wave propagation characteristics of FG-CNTRC structures give a more precise perception of the dynamic behavior at high frequencies.

Ebrahimi and Rostami [56] investigated the wave dispersion of FG-CNTRC beams on a Winkler foundation and compared various shear deformation theories with previous investigations. The governing equations were derived from the Hamilton principle and then solved analytically. The study considers four different reinforcement patterns. For the first time, Ebrahimi and Seyfi [57] explored the effect of agglomeration factor and nano-filler waviness on wave dispersion of beams enhanced by multi-walled CNTs (MWCNTs) in nanocomposite materials. The beams were built using an improved HSDT and supported by an elastic Winkler-Pasternak foundation. The micromechanical model was generated by the law of mixing and a novel version of the Halpin-Tsai models. Another study by Ebrahimi et al. [58] studied the thermomechanical wave propagation in hybrid composite beams using an improved trigonometric beam theory with CNTs and carbon fibers (CF) as reinforcements in the matrix. Accounting for fluid and temperature factors, Chen et al. [59] examined the wave propagation of a fluid-conveying pipe strengthened by FG-CNTRC. Hamilton's principle and HSDT were used to derive the equations of motion. The influences of several parameters, such as the CNTs' volume fractions, temperature, flow velocity, and the CNTs' patterns, were addressed. The results indicated increased wave velocity as the volume fraction of CNTs increased. In another research, the temperature-dependent wave propagation in FG-CNTRC plates based on HSDT was researched by Ding et al. [60]. The governing equations were generated by Reddy's plate model and Lagrange's principle. The analysis demonstrated a relationship between the wave propagation properties of the plates and CNTs volume fraction and distribution pattern in the matrix. Utilizing the Bernoulli beam theory, Seyf et al. [61] observed the wave propagation in FG-CNTRC beams resting on an elastic medium. Across the beam length, CNTs were graded using a power-law function. Modeling the FG-CNTRC beams was based on the Euler-Bernoulli beam theory, while the governing equations were derived using the Hamilton principle. The solution reviewed the mechanical response's sensitivity to various parameters, including the power index, wave number, Winkler coefficient, and Pasternak coefficient. As a first attempt, Janghorban and Nami [62] examined the wave propagation behavior in FG-CNTs-reinforced nanocomposite plates using the second-order shear deformation theory (SSDT). A total of four CNT arrangements were accounted for in the study. An analytical approach was used to obtain the frequency and phase velocity.

A recent article by Hao and Li [63] presented the wave dispersion analysis of infinite piezoelectric FG-CNTRC sandwich plates. The kinematic governing equations were derived by applying the TrSDT and Hamilton principles. In [64], the wave propagation behaviors of FG-CNTRC beams were investigated, resting on elastic foundations in thermal environments by applying various shear deformation theories. In recent research [65], by applying HSDT, the wave propagation behaviors of the FG-CNTRC beams on elastic foundations were investigated. The Euler-Lagrange approach was employed to obtain the governing equations, which were then solved using the eigenvalue technique.

3.5 Material Properties Impact on FG-CNTRC Structures

Porosities have a considerable impact on the composites because they alter the effective properties of the composed materials. The presence of the porosities in materials could be caused as a byproduct or imperfect during the manufacturing processes [66]. The current fabrication methods are cumbersome, time-consuming, and inadequate for mass production of FG-CNTRC [7], which raises problems related to improper dispersion of CNTs and imperfect CNTRC beams involving porosities due to the variations in solidification temperatures of constituent materials during the fabrication. Also, some intentionally engineer porosity in materials for specific purposes such as reducing weight, conducting electricity, providing thermal resistance, and absorbing energy impacts [67]. In recent research, Ong et al. [68] examined the vibration responses of the porous viscoelastic FG-CNTRC beam. Furthermore, simply supported beams with four CNTs' distribution patterns and three porosity models were considered in the study. The derivation of the governing equations was done through an energy method that considered the viscoelastic beams and layers. Then those equations were solved using a proposed series expansion approach. Also, Alsubaie et al. [69] investigated the vibration response for porous FG-CNTRC beams on a viscoelastic Winkler-Pasternak foundation using HSDT. This study examines four reinforcement patterns of CNTs with three different forms of porosity distributions. According to the results, introducing a damping coefficient can enhance vibration performance with high spring constant factors. Furthermore, the beam's fundamental frequency increases with the rising porosity volume fractions, suggesting that porosity influences the vibrational properties of FG-CNTR beams. Some studies attempted

to investigate the response of FG-CNTRC-containing porosity [70, 71], but none in wave propagation analysis.

Researchers also tried to investigate other than linear distribution patterns of CNTs by implementing exponential order in the equations of the volume fractions. For instance, Zerrouki et al. [72] concentrated on the bending responses of reinforced composite beams reinforced with novel exponential distributions of FG-CNTs using HSDT. The paper discussed the impacts of the CNTs' volume fraction, the exponent order, and the aspect ratio on bending behavior. The analytical results showed that the nonlinear distribution of CNTs has a notable influence on mechanical properties. Furthermore, raising the degree exponent of the CNTs stiffens the X-beam while weakening the O-beam in terms of bending resistance. Also, Zerrouki et al. [73] extended the study of the nonlinear distribution of FG-CNTs to the vibration behavior of FG-CNTRC. The analytical results concluded that nonlinearity significantly affects the FG-CNTRC beam's strength. As the exponent order increases, the fundamental frequency rises in the X-beam but falls in the O- and V-beams. The fundamental frequency decreases with increasing aspect ratio and increases with increasing volume fractions of CNTs.

3.6 Summary of The Literature Review

Based on the literature review presented in the previous sections, there are some notable gaps in the wave propagation analysis of FG-CNTRC beams regarding some scenarios to be considered. The following highlights the missing analysis opportunities:

- The investigation of wave propagation is lacking compared to the bending, buckling, and vibration of FG-CNTRC structures.
- There is no wave propagation analysis considering the porosity presence in FG-CNTRC beams.
- The nonlinear distributions of CNTs have not been studied thoroughly in mechanical analysis in FG-CNTRC structure, let alone in wave propagation.
- The effect of the Kerr foundation in wave propagation analysis of FG-CNTRC beams was not considered in the research.

CHAPTER 4

FIRST STUDY

This chapter presents the first published paper: Analyzing the Porosity-Dependent Wave Propagation in FG-CNTRC Beams Utilizing an Integral Higher-Order Shear Deformation Theory published by the International Journal of Structural Stability and Dynamics.

4.1 Introduction

The literature review in Chapter 3 identifies some gaps in wave propagation analysis of FG-CNTRC beams, notably the absence of analysis considering the porosity effect, and the neglect of longitudinal and shear waves despite their structural relevance, with no previous application of the three-unknown integral HSDT. Hence, this study investigates the wave propagation of porous FG-CNTRC beams, considering different types of waves employing an HSDT with an integral form. The study considers the bending, longitudinal, and shear waves of FG-CNTRC beams using an HSDT with an integral form and three unknowns. The beams are composed of four pattern distributions of SWCNTs with the rule of mixture utilized to estimate their material properties. The governing equations are derived based on Hamilton's principle with the stiffness and mass matrices formulated by a generalized solution of harmonic wave propagation. Three different porosity models are assumed to describe the porosity distributions within the matrix. The circular frequency-wave number relationship is obtained by solving the eigenvalue problem. The influences of multiple parameters on wave propagation, such as the configurations and volume fractions of CNTs, porosity models, and porosity volume fraction, are discussed.

4.2 Mathematical Model of FG-CNTRC Beams

4.2.1 Configuration and Properties of CNTRC Beams

The FG-CNTRC beam is produced using a combination of the SWCNTs and polymer matrix. **Figure 4.1** shows a typical beam with length (L) and thickness (h) to represent the beams in the study.

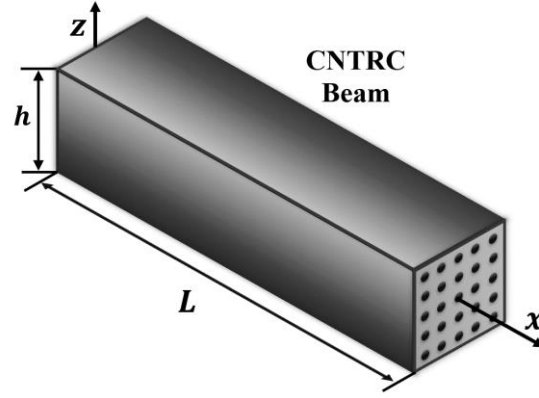


Figure 4.1: Geometry of a UD-beam.

This study assumes only four different CNTs' distribution forms out of the five described in Section 2.2 through the beam' cross-section, as shown in **Figure 4.2**.

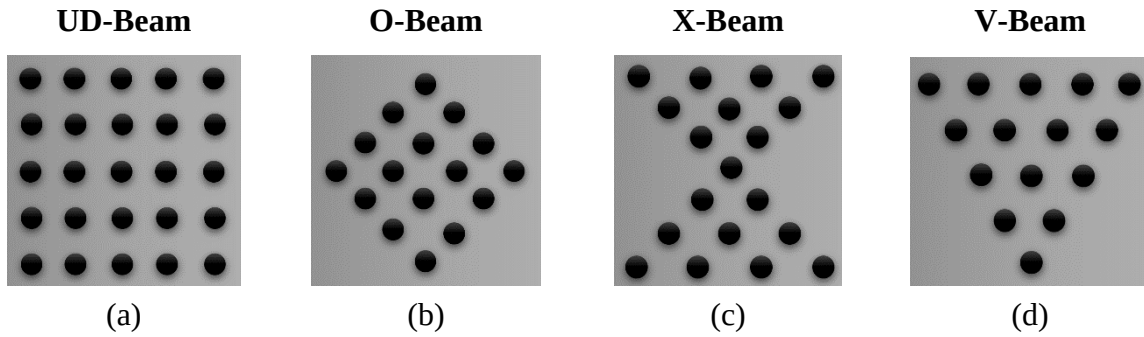


Figure 4.2: Cross-sections of four distribution forms of SWCNTs.

The mathematical functions used for describing the patterns of CNTs are expressed in terms of the CNTs volume fractions, as shown in **Table 4.1** [74].

Table 4.1: Expressions of SWCNT's patterns distributed along the beam cross-section.

Distribution of CNTs	V_{cnt}	
UD	$V_{cnt} = V_{cnt}^*$	(4-1a)
FGO	$V_{cnt} = 2 \left(1 - 2 \frac{ z }{h} \right) V_{cnt}^*$	(4-1b)
FGX	$V_{cnt} = 4 \frac{ z }{h} V_{cnt}^*$	(4-1c)
FGV	$V_{cnt} = \left(1 + \frac{2z}{h} \right) V_{cnt}^*$	(4-1d)

where:

- V_{cnt}^* : The given volume fraction of CNTs.

At any location of the beam, the equation of the volume fractions of CNTs (V_{cnt}) and polymer (V_p) holds:

$$V_{cnt} + V_p = 1 \quad (4-2)$$

The rule of mixture is the micromechanical model utilized to evaluate the material properties of the FG-CNTRC beams, which are expressed as follows:

$$E_{11} = \eta_1 V_{cnt} E_{11}^{cnt} + V_p E^p \quad (4-3)$$

$$\frac{\eta_3}{G_{12}} = \frac{V_{cnt}}{G_{12}^{cnt}} + \frac{V_p}{G^p} \quad (4-4)$$

$$v = V_{cnt} v^{cnt} + V_p v^p \quad (4-5)$$

$$\rho = V_{cnt} \rho^{cnt} + V_p \rho^p \quad (4-6)$$

where:

- E_{11} , G_{12} , v , and ρ : The modulus of elasticity, the shear modulus, Poisson's ratios, and the density of materials, respectively (cnt stands for CNTs and p for polymer matrix).
- η_1, η_2 , and η_3 : The size-dependent efficiency parameters of SWCNTs.

4.2.2 Porosity Models of The Polymer Matrix

Although the manufacturing process produces imperfect CNTs-Polymer matrix beams that have inevitable porosities spreading through the thickness, some design these porosities intentionally to reduce the weight. In either case, the existence of the porosities significantly affects the response of the composites by altering their effective properties. This change is calculated based on the assumed porosity volume fraction, denoted by a . Three types of porosity distribution are considered in this study, as shown in **Figure 4.3**.

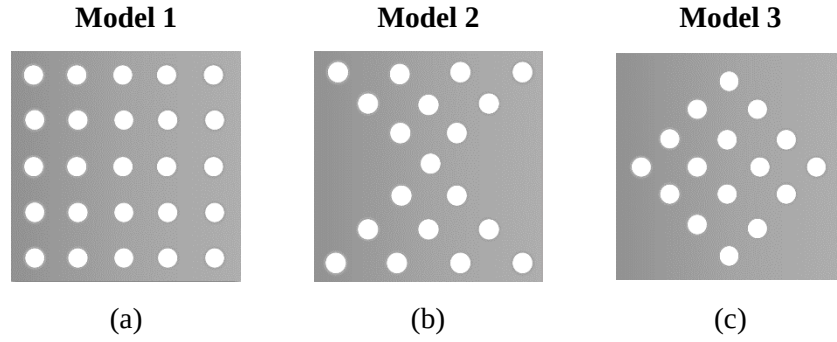


Figure 4.3: Porosity distributions for the three models.

The material's effective properties get altered based on the volume fractions and distributions of the porosity through the thickness. The porosities are solely on the polymer matrix, so its material properties are the only ones directly affected. The porosities' models are expressed in **Table 4.2** [69].

Table 4.2: Expressions of the material property based on the three models of the porosity distributions.

The Porosity Models	Material Property (P)	
Model 1	$P \times (1 - a)$	(4-7a)
Model 2	$P \times \left(1 - a \frac{2 z }{h}\right)$	(4-7b)
Model 3	$P \times \left(1 - \frac{a}{2} \left(1 - \frac{2 z }{h}\right)\right)$	(4-7c)

where:

- P : The material properties of the polymer (E^p, ν^p, ρ^p).

The porosity volume fraction (a) of zero represents a beam without a porosity presence.

4.3 Formulations and Governing Equations Derivation

4.3.1 Kinematics

The theory used to describe the beam behavior is an HSDT with an integral form that involves u_0 , w_0 and θ as the three unknowns. The integral form of HSDT is advantageous because it employs a single undetermined integral term in the displacement field that simplifies the overall complexity to capture accurate shear rotations. The displacement field is expressed as follows:

$$\begin{aligned} u &= u_0(x, t) - z \frac{\partial w}{\partial x} + k_1 f(z) \int \theta(x, t) dx \\ w &= w_0(x, t) \end{aligned} \quad (4-8)$$

where:

- u_0 , w_0 : The beam's displacements in the x and z directions, respectively, at the middle beam thickness ($z = 0$).
- $\theta(x, t)$: The rotation of the beam.

The shape function $f(z)$, provided as a combination of functions, describes the distributions of transverse shear stresses and strains through the thickness of the beam.

$$f(z) = z - \left(\left(\frac{1}{n} \right) z^n \left(\frac{2}{h} \right)^{(n-1)} \right) + \frac{h}{10 \pi^n} \sin \left(\frac{\pi}{h} z \right) \quad (4-9)$$

where:

- n : An odd integer.

4.3.2 Strain-Displacement Relations

The components of the strains are derived in terms of the unknown middle displacements using linear strain-displacement relationships. 3D infinitesimal strain for small deformation is discussed in Section 2.8.2. The normal ε_x and shear γ_{xz} strains are the only present based on the assumptions.

$$\varepsilon_x = \frac{\partial u}{\partial x} = \varepsilon_x^0 - z \varepsilon_x^b + f(z) \varepsilon_x^s \quad (4-10)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = g(z) \gamma_{xz}^s \quad (4-11)$$

where:

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_x^b \\ \varepsilon_x^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 w_0}{\partial x^2} \\ k_1 \theta \\ k_1 \int \theta dx \end{Bmatrix} \quad (4-12)$$

and

$$g(z) = \frac{df(z)}{dz} = 1 - \left(\frac{2}{h}\right)^{n-1} z^{n-1} + \frac{1}{10} \pi^{1-n} \cos\left(\frac{\pi}{h} z\right) \quad (4-13)$$

By applying a wave propagation solution, the undetermined integrals could be simplified to the following relation:

$$\int \theta(x, t) dx = A' \frac{\partial \theta(x, t)}{\partial x} \quad (4-14)$$

where, A' and k_1 can be obtained as functions of the wavenumber κ_1 as follows:

$$A' = -\frac{1}{\kappa_1^2} = -\frac{1}{k_1} \quad (4-15)$$

This simplification is a result of the property of the exponential functions in the derivative and integral that were employed in the general solution of wave propagation.

4.3.3 Stress-Strain Relations

The stresses are found by using the stress-strain constitutive relations discussed in Section 2.8.3. Based on the assumptions and displacement field, only σ_x and τ_{xz} are present, expressed as:

$$\sigma_x = Q_{11}(z) \varepsilon_x, \quad \tau_{xz} = Q_{55}(z) \gamma_{xz} \quad (4-16)$$

where:

- Q_{11}, Q_{55} : The components of the stiffness properties matrix are given as:

$$Q_{11}(z) = \frac{E_{11}}{1 - \nu^2}, \quad Q_{55}(z) = G_{12} \quad (4-17)$$

4.3.4 Governing Equations

The motion equations of the FG-CNTRC beams are determined based on the Hamilton's principle, stated as:

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K) dt = 0 \quad (4-18)$$

where:

- δ : The differential operator (The first variation).
- $U, V, \text{ and } K$: The total strain energy, work done by external forces, and the kinetic energy, respectively. $\delta U, \delta V, \delta K$ are expressed in terms of the three middle displacements.

In this study, only the strain and kinetic energies are present. The strain energy variation is expressed as follows:

$$\begin{aligned} \delta U &= \int_0^L \int_{-h/2}^{h/2} (\sigma_x \delta \varepsilon_x + \tau_{xz} \delta \gamma_{xz}) dz dx \\ \delta U &= \int_0^L \left(N \frac{\partial \delta u_0}{\partial x} - M_b \frac{\partial^2 \delta w_0}{\partial x^2} + M_s k_1 \delta \theta + Q A' k_1 \frac{\partial \delta \theta}{\partial x} \right) dx \end{aligned} \quad (4-19)$$

where:

- $N, M_b, M_s, \text{ and } Q$: The stress resultants, expressed as:

$$\begin{Bmatrix} N \\ M_b \\ M_s \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_x \begin{Bmatrix} 1 \\ z \\ f \end{Bmatrix} dz, \quad Q = \int_{-h/2}^{h/2} g \tau_{xz} dz \quad (4-20)$$

The kinetic energy variation has the following form:

$$\begin{aligned}\delta K &= \int_0^L \int_{-h/2}^{h/2} \rho(z) \left(\frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} + \frac{\partial w_0}{\partial t} \frac{\partial \delta w_0}{\partial t} \right) dz dx \\ &= - \int_0^L \int_{-h/2}^{h/2} \rho(z) \left(\frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 w_0}{\partial t^2} \delta w_0 \right) dz dx\end{aligned}\quad (4-21)$$

Or by expanding and simplifying the expression in (4-21)

$$\begin{aligned}\delta K &= - \int_0^L \left(-I_1 \left(\frac{\partial^2 u_0}{\partial t^2} \delta u_0 - \frac{\partial^2 w_0}{\partial t^2} \delta w_0 \right) + I_2 \left(\frac{\partial^2 u_0}{\partial t^2} \frac{\partial \delta w_0}{\partial x} + \frac{\partial^3 w_0}{\partial t^2 \partial x} \delta u_0 \right) \right. \\ &\quad - I_3 \left(A' k_1 \frac{\partial^2 u_0}{\partial t^2} \frac{\partial \delta \theta}{\partial x} + A' k_1 \frac{\partial^3 \theta}{\partial t^2 \partial x} \delta u_0 \right) - I_4 \frac{\partial^3 w_0}{\partial t^2 \partial x} \frac{\partial \delta w_0}{\partial x} \\ &\quad + I_5 \left(A' k_1 \frac{\partial^3 w_0}{\partial t^2 \partial x} \frac{\partial \delta \theta}{\partial x} + A' k_1 \frac{\partial^3 \theta}{\partial t^2 \partial x} \frac{\partial \delta w_0}{\partial x} \right) \\ &\quad \left. - I_6 (A' k_1)^2 \frac{\partial^3 \theta}{\partial t^2 \partial x} \frac{\partial \delta \theta}{\partial x} \right) dx\end{aligned}\quad (4-22)$$

where:

- I_i ($i = 1, 2, 3, 4, 5, 6$): The mass moments of inertia are defined as:

$$\begin{Bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{Bmatrix} = \int_{-h/2}^{h/2} \rho(z) \begin{Bmatrix} 1 \\ z \\ f(z) \\ z^2 \\ z f(z) \\ f(z)^2 \end{Bmatrix} dz\quad (4-23)$$

The above energy virtual variations (δU and δK) are substituted into Hamilton's principle as functions of the three middle displacements, and by utilizing the calculus of variations, the motion equations are obtained. This process involves integration by parts and equating the coefficients of the δu_0 , $\delta \theta$, and δw_0 to zero, which leads to the following equations:

$$\begin{aligned}\delta u_0: \quad & \frac{\partial N}{\partial x} - I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^3 w_0}{\partial t^2 \partial x} - I_3 A' k_1 \frac{\partial^3 \theta}{\partial t^2 \partial x} = 0 \\ \delta w_0: \quad & \frac{\partial^2 M_b}{\partial x^2} - I_1 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^3 u_0}{\partial t^2 \partial x} + I_4 \frac{\partial^4 w_0}{\partial t^2 \partial x^2} - I_5 A' k_1 \frac{\partial^4 \theta}{\partial t^2 \partial x^2} = 0\end{aligned}\quad (4-24)$$

$$\delta\theta: M_s k_1 - \frac{\partial Q}{\partial x} A' k_1 - I_3 A' k_1 \frac{\partial^3 u_0}{\partial t^2 \partial x} + I_5 A' k_1 \frac{\partial^4 w_0}{\partial t^2 \partial x^2} - I_6 A'^2 k_1^2 \frac{\partial^4 \theta}{\partial t^2 \partial x^2} = 0$$

The stress resultants in the motion equations can be expressed in the form of middle displacements and components of material stiffnesses as in Eq. (4-25):

$$\begin{Bmatrix} N \\ M_b \\ M_s \end{Bmatrix} = \begin{bmatrix} A_{11} & B_{11} & D_{11} \\ B_{11} & E_{11} & F_{11} \\ D_{11} & F_{11} & G_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 w_0}{\partial x^2} \\ k_1 \theta \end{Bmatrix} \quad (4-25)$$

$$Q = A_{55}^s k_1 A' \frac{\partial \theta}{\partial x}$$

where:

- $A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, G_{11}, A_{55}^s$: The beam stiffnesses are defined as:

$$\begin{Bmatrix} A_{11} \\ B_{11} \\ D_{11} \\ E_{11} \\ F_{11} \\ G_{11} \end{Bmatrix} = \int_{-h/2}^{h/2} Q_{11} \begin{Bmatrix} 1 \\ z \\ f(z) \\ z^2 \\ z f(z) \\ f(z)^2 \end{Bmatrix} dz \quad (4-26)$$

$$A_{55}^s = \int_{-h/2}^{h/2} Q_{55} g(z)^2 dz$$

After substituting the resultants-displacements relations, the governing equations as functions of unknown displacements are expressed in (4-27):

$$\begin{aligned} A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11} \frac{\partial^3 w_0}{\partial x^3} + D_{11} k_1 \frac{\partial \theta}{\partial x} - I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^3 w_0}{\partial t^2 \partial x} + I_3 \frac{\partial^3 \theta}{\partial t^2 \partial x} &= 0 \\ B_{11} \frac{\partial^3 u_0}{\partial x^3} - E_{11} \frac{\partial^4 w_0}{\partial x^4} + F_{11} k_1 \frac{\partial^2 \theta}{\partial x^2} - I_1 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^3 u_0}{\partial t^2 \partial x} + I_4 \frac{\partial^4 w_0}{\partial t^2 \partial x^2} \\ &+ I_5 \frac{\partial^4 \theta}{\partial t^2 \partial x^2} = 0 \end{aligned} \quad (4-27)$$

$$\begin{aligned} D_{11} k_1 \frac{\partial u_0}{\partial x} - F_{11} k_1 \frac{\partial^2 w_0}{\partial x^2} + G_{11} k_1^2 \theta - A_{55}^s \frac{\partial^2 \theta}{\partial x^2} + I_3 \frac{\partial^3 u_0}{\partial t^2 \partial x} - I_5 \frac{\partial^4 w_0}{\partial t^2 \partial x^2} \\ - I_6 \frac{\partial^4 \theta}{\partial t^2 \partial x^2} = 0 \end{aligned}$$

4.3.5 Dispersion Relations

The analytical solutions for the wave propagation problem in an infinite beam are:

$$\begin{pmatrix} u_0(x, t) \\ w_0(x, t) \\ \theta(x, t) \end{pmatrix} = \begin{pmatrix} U_m e^{i(\kappa_1 x - \omega t)} \\ W_m e^{i(\kappa_1 x - \omega t)} \\ \theta_m e^{i(\kappa_1 x - \omega t)} \end{pmatrix} \quad (4-28)$$

where:

- U_m , W_m , and θ_m : The waves' amplitudes.
- ω : The circular frequency.
- κ_1 : The wavenumber in the directions of the x-axis.
- i : The imaginary number ($\sqrt{-1}$).

If the displacement solutions are plugged into the governing equations, the following eigenvalue problem can be derived and rewritten into a matrix form as follows:

$$([K] - \omega^2 [M]) \{\Lambda\} = 0 \quad (4-29)$$

where:

- $[K]$, $[M]$: The stiffness and mass matrices (3×3), respectively.
- Λ : The vector of the waves' amplitudes (3×1).

where the elements of the symmetric matrices are:

$$\begin{pmatrix} K_{11} \\ K_{12} \\ K_{13} \\ K_{22} \\ K_{23} \\ K_{33} \end{pmatrix} = \begin{pmatrix} A_{11} \kappa_1^2 \\ -i B_{11} \kappa_1^3 \\ -i D_{11} k_1 \kappa_1 \\ -E_{11} \kappa_1^4 \\ -F_{11} k_1 \kappa_1^2 \\ -G_{11} k_1^2 - A_{55}^s \kappa_1^2 \end{pmatrix} \quad (4-30)$$

and

$$\begin{pmatrix} M_{11} \\ M_{12} \\ M_{13} \\ M_{22} \\ M_{23} \\ M_{33} \end{pmatrix} = \begin{pmatrix} I_1 \\ -i I_2 \kappa_1 \\ -i I_3 \kappa_1 \\ -I_1 - I_4 \kappa_1^2 \\ -I_5 \kappa_1^2 \\ -I_6 \kappa_1^2 \end{pmatrix} \quad (4-31)$$

This eigenvalue problem has a nontrivial solution only if the determinant of $[[K] - \omega^2[M]]$ is zero (As discussed in Section 2.8.5). The circular frequencies are solved numerically as functions of the wavenumbers by finding the eigenvalues that equal the square of the frequencies. Three circular frequencies are found corresponding to the three-unknown theory, as shown below:

$$\omega_j = \omega_j(\kappa_1), \quad f_j(\kappa_1) = \frac{\omega_j(\kappa_1)}{2\pi} \quad (j = 1, 2, 3) \quad (4-32)$$

where ω_1 , ω_2 , and ω_3 are the frequencies of bending, longitudinal, and shear modes, respectively. The phase velocity C_{pj} and the group velocity C_{gj} can be determined from the circular wave frequencies as follows:

$$C_{pj} = \frac{\omega_j(\kappa_1)}{\kappa_1}, \quad C_{gj} = \frac{d\omega_j(\kappa_1)}{d\kappa_1} \quad (4-33)$$

The dimensionless circular frequencies, used for validation purposes, are expressed:

$$\bar{\omega} = \omega L \sqrt{\frac{I_{00}}{A_{110}}} \quad (4-34)$$

$$I_{00} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^0 dz, \quad A_{110} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E_p}{1 - \nu_p^2} dz \quad (4-35)$$

where:

- A_{110} and I_{00} : The stiffness and mass moments of inertia of a beam made of only a pure polymer matrix (Without porosity presence), respectively.

4.4 Numerical Results and Discussion

In this section, two main parts are considered: Results validation to verify the accuracy of the model and Parametric studies to examine the influences of various factors. The wave propagation behavior of porous FG-CNTR beams is presented based on the listed mechanical properties in **Table 4.3**, retrieved from [74]. The armchair (10, 10) SWCNTs and Polymethyl methacrylate (PMMA) are used as the reinforcement and the polymer matrix, respectively. The efficiency parameters of CNTs (η_i) and their associated given volume fractions (V_{cnt}^*) are shown in **Table 4.4** below [74].

Table 4.3: Mechanical properties of CNTs and polymer matrix.

Materials	$E_{11}(GPa)$	$E_{22}(GPa)$	$G_{12}(GPa)$	$\rho(kg/m^3)$	ν
Reinforcement	600	10	17.2	1400	0.19
Polymer Matrix	2.5	2.5	$\frac{E_p}{2(1 + \nu_p)}$	1190	0.3

Table 4.4: Volume fractions of CNTs and their corresponding efficiency parameters.

Case	V_{cnt}^*	η_1	η_2 and η_3
1	0.12	1.2833	1.0566
2	0.17	1.3414	1.7101
3	0.28	1.3238	1.7380

4.4.1 Results Validation

This paper's mathematical models are numerically validated by comparing with two investigation methods: free vibration and wave propagation results of earlier publications of similar works.

Table 4.5 shows the dimensionless fundamental frequency comparisons between the present work (Having three different values of n substituted in the shape function) with Wattanasakulpong and Ungbhakorn [74], Alsubaie et al. [69], S.H. Tagrara et al. [75], and Gawah et al. [76]. The comparisons in the first section of **Table 4.5** ignore the foundation and

porosity effects on the references and consider only FSDT and TSDT [74]. Also, the second section of **Table 4.5** demonstrates fundamental frequency results, considering only the porosity effect based on the porosity model 1. According to **Table 4.5**, the current theory results with $n = 3$ are nearly close to the HSDT published theories. Furthermore, as the value of n increases, the fundamental frequency approaches the results of FSDT. Also, the results in [74] were obtained from a general form solution of different shear deformation theories, which did not account for shear correction factors for the FSDT case.

Table 4.5: Comparison of the dimensionless fundamental frequencies ($L/h = 15$, $Vcnt = 0.12$, $\kappa_1 = \frac{\pi}{L}$).

Source	$a = 0$			
	UD	O	X	V
FSDT^(a)	0.9976	0.7628	1.1485	0.8592
TSDT^(a)	0.9745	0.7453	1.1152	0.8441
Ref^(b)	0.9750	0.7446	1.1164	0.8444
Ref^(c)	0.9745	0.7554	1.1033	0.8442
Ref^(d)	0.9749	0.7446	1.1163	0.8442
Present ($n = 3$)	0.9746	0.7454	1.1154	0.8442
Present ($n = 5$)	0.9770	0.7499	1.1172	0.8458
Present ($n = 7$)	0.9803	0.7529	1.1214	0.8480

Source	$a = 0.2$			
	UD	O	X	V
Ref^(b)	1.0321	0.7917	1.1767	0.8968
Present ($n = 3$)	1.0317	0.7925	1.1754	0.8966
Present ($n = 5$)	1.0344	0.7979	1.1773	0.8985
Present ($n = 7$)	1.0382	0.8016	1.1821	0.9010

FSDT^(a) and TSDT^(a) Taken from Wattanasakulpong and Ungbhakorn [74].

Ref^(b) Taken from Alsubaie et al [69].

Ref^(c) Taken from Gawah et al [76].

Ref^(d) Taken from S.H. Tagrara et al [75].

Figure 4.4 shows the phase velocity-wave number of the bending wave in the UD beam, ignoring the porosity and foundation effects. The phase velocity curves of Euler, Reddy,

trigonometric, and exponential theories are retrieved from Zhang et al. [65]. The present curves were plotted with different values of n in the shape function to examine the change in the phase velocity behavior and how to approach the other theories. Based on **Figure 4.4**, the present study theory with n equal to 3 has the same behavior as Reddy theory from Zhang et al. [65]. Also, as the value of n increases, the change rate of the phase velocity decreases, which generates slower phase velocities compared to the other theories at the same value of the wavenumber.

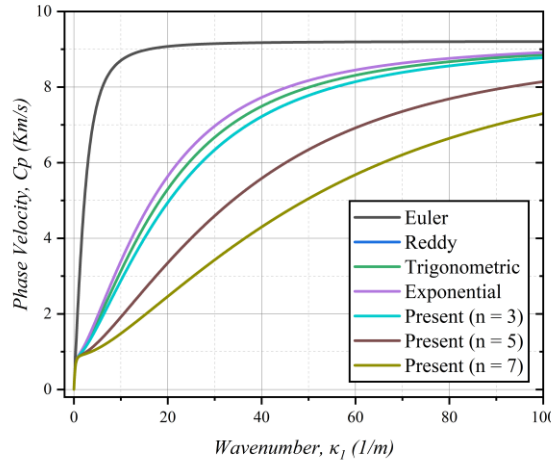


Figure 4.4: Phase velocity vs wavenumber of UD-Beam bending waves for comparative analysis ($a = 0$, $V_{cnt} = 0.12$).

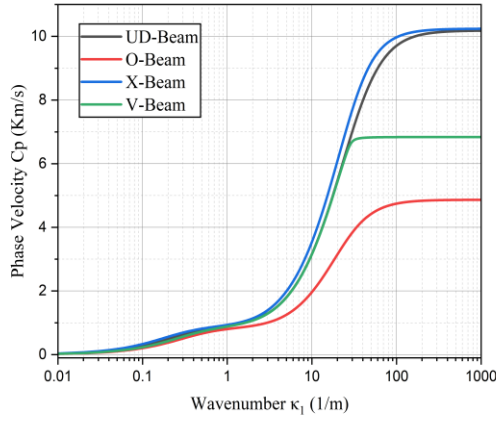
4.4.2 Parametric Studies

An impact study of several parameters in the wave propagation of porous FG-CNTRC beams is presented. The parameters include the arrangements and volume fractions of CNTs, porosity volume fractions, and porosity models. For the parametric study, all analyses are conducted with n equal to 3 in the shape function.

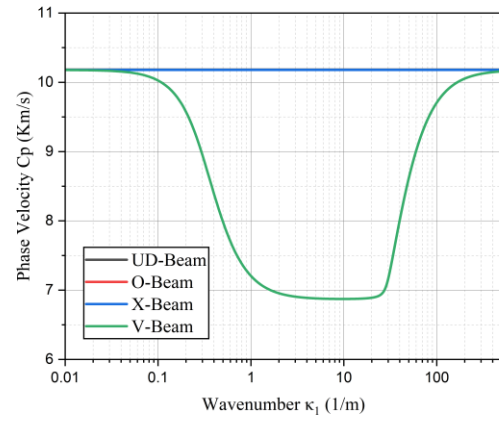
4.4.2.1 Effect of CNTs Distributions

Figure 4.5 shows the variations in the phase velocities with the wavenumbers of porous FG-CNTRC beams for different distributions of CNTs in the polymer matrix. **Figure 4.5 (a)** demonstrates that the phase velocities of the bending waves for all distributions rise as the wavenumber increases and then tend to level off. The change in the phase velocities is higher in the lower values of wavenumbers. In **Figure 4.5 (b)**, the longitudinal waves are

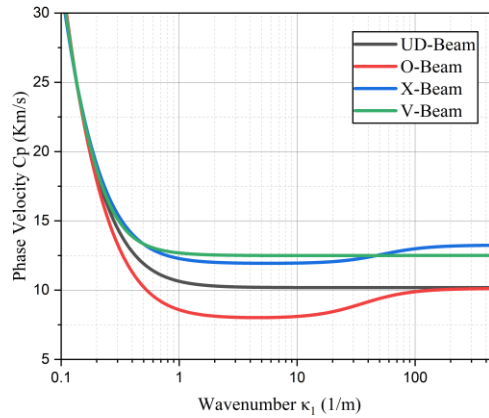
constant for all beam types except for the V-beam because of its unsymmetric distribution of CNTs, where its phase velocity in lower values of wavenumber decreases initially before growing to approach the values of other distributions as wavenumber increases. In contrast to bending waves, the phase velocities of shear waves decrease as the wavenumber increases, and the phase velocities change sharply when the wavenumber is less than one, as depicted in **Figure 4.5 (c)**. Also, the phase velocities of both O and X-beams slightly increase afterward before approaching constant values. Generally, the X-beam generally has the highest phase velocity, whereas the O-beam has the lowest for bending and shear waves.



(a) Bending waves.



(b) Longitudinal waves.

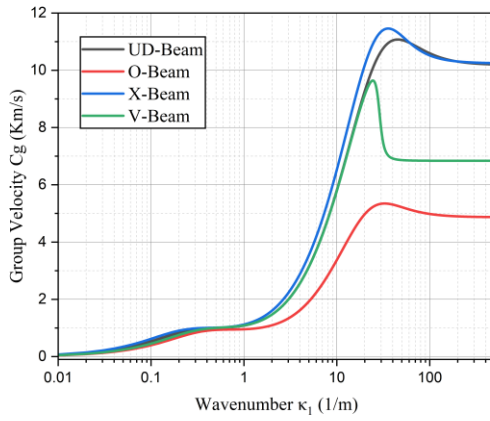


(c) Shear waves.

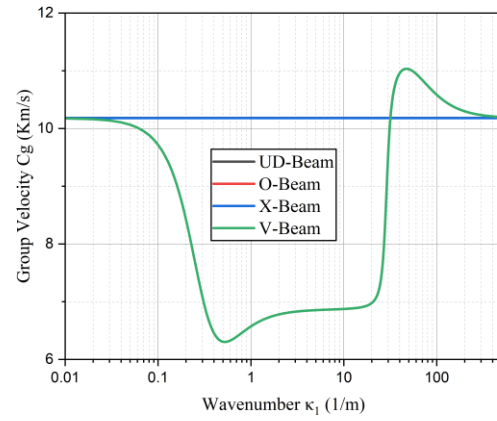
Figure 4.5: Phase velocities vs wavenumbers of different beam types ($V_{cnt} = 0.12$, $a = 0.25$, Porosity model 1).

In **Figure 4.6**, the relations between the group velocities and the wavenumbers of the porous beams with different distributions of CNTs are plotted. **Figure 4.6 (a)** illustrates

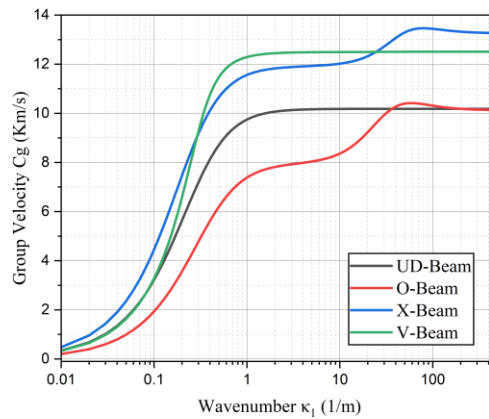
that the group velocities of the bending waves ascend as the wavenumber increases until reaching peak values and then balance off. At lower wavenumbers, the group velocities vary the most. For longitudinal waves, the group velocity of the V-beam is the only one to have nonconstant behavior, where at lower wavenumbers, its group velocity initially reduces, increases to approach a peak, and then reaches the values of other distributions, as shown in **Figure 4.6 (b)**. In contrast to the phase velocities of the shear wave, the group velocities increase as the wavenumber increases, and they change sharply when the wavenumber is less than 1, as pictured in **Figure 4.6 (c)**. Likewise, the group velocities of both O and X-beams slightly increase later and then stabilize. Again, the X-beam has the highest group velocity, whereas the O-beam has the lowest for bending and shear waves.



(a) Bending waves.



(b) Longitudinal waves.



(c) Shear waves.

Figure 4.6: Group velocities vs wavenumbers of different beam types ($V_{cnt} = 0.12$, $a = 0.25$, Porosity model 1).

4.4.2.2 Effect of CNTs Volume Fractions

Figure 4.7 depicts how the volume fractions of CNTs affect the dispersion relations with porosity volume fractions. **Figure 4.7** indicates that when the volume fractions of CNTs and porosity increase, so do both velocities.

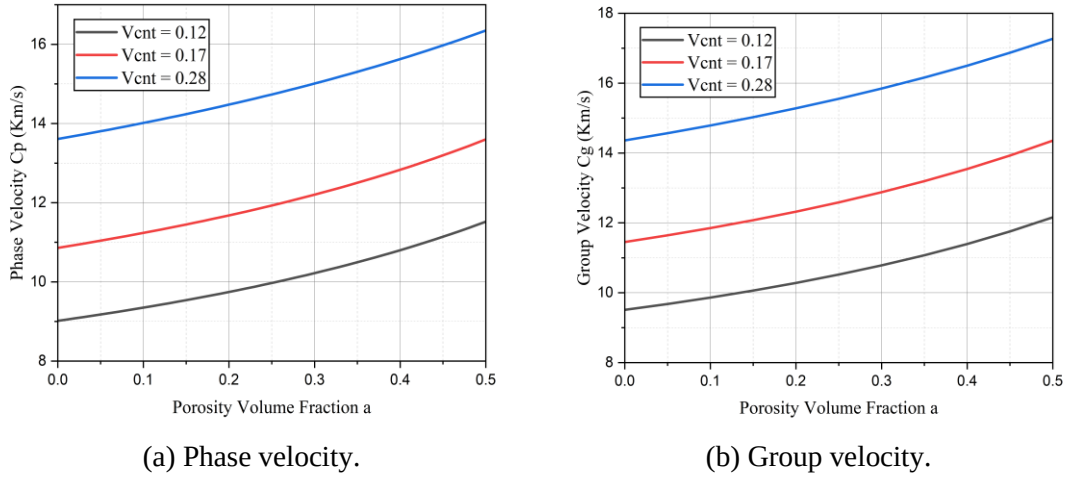


Figure 4.7: Impact of different volume fractions of CNTs and the porosity ($\kappa_1 = 100$ 1/m, X-Beam, Porosity model 1, Bending wave).

The impact of the volume fractions of CNTs on the dispersion relations of the X-beam is illustrated in **Figure 4.8**. It shows that the effect related to the volume fractions of CNTs increases as the wavenumber increases while maintaining the same behavior.

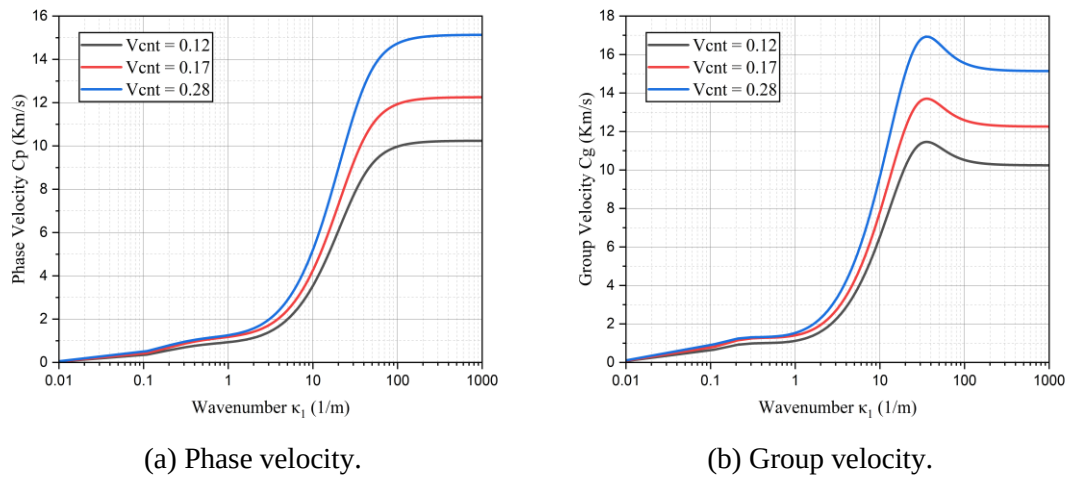


Figure 4.8: Dispersion relations of different volume fractions of CNTs ($a = 0.25$, $V_{cnt} = 0.12$, X-Beam, Porosity model 1, Bending wave).

4.4.2.3 Effect of Porosity Models and Volume Fractions

Figure 4.9 shows the effect of the porosity model on the relationship between phase and group velocities with porosity volume fractions. The porosity model 1 has a considerable impact that increases the velocities of the propagating waves compared to the other models.

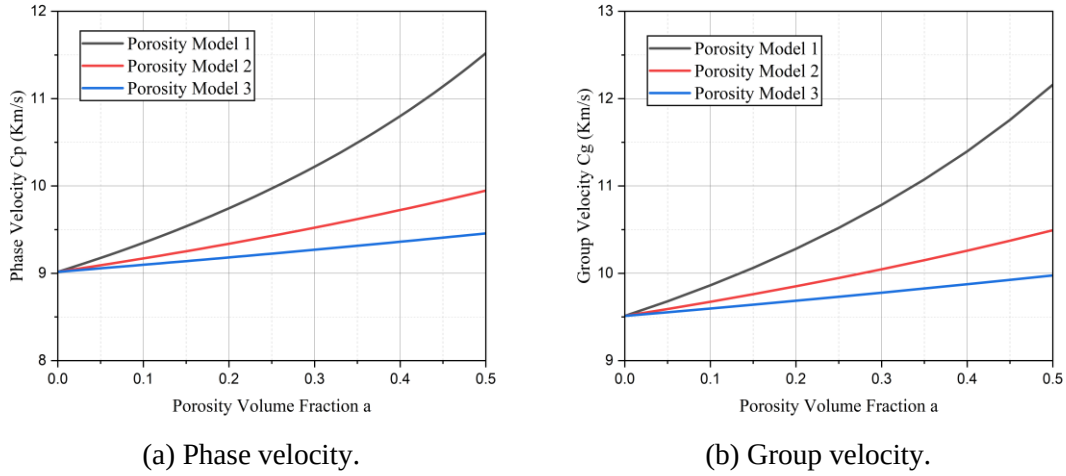


Figure 4.9: Impact of the porosity volume fractions and models ($\kappa_1 = 100$ 1/m, $V_{cnt} = 0.12$, X-Beam, Bending wave).

In **Figure 4.10** and **Figure 4.11**, the influence of the porosity models and volume fractions in the wave propagation of the X-beam are illustrated as a typical behavior. Also, the effect of porosity models and volume fractions increases as the wavenumber increases while retaining the same behavior.

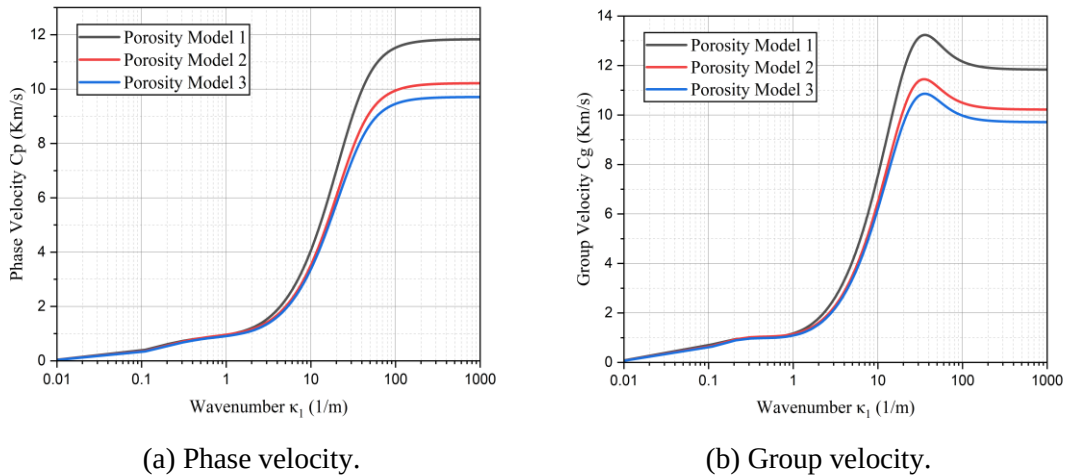
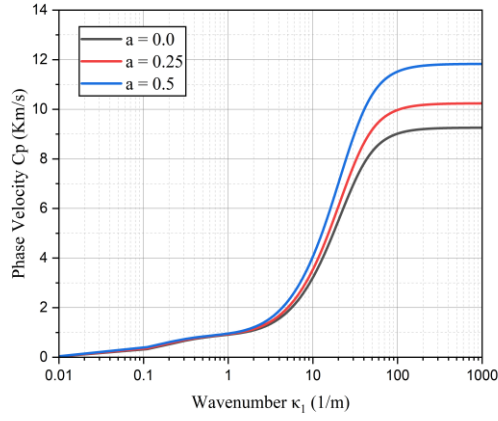
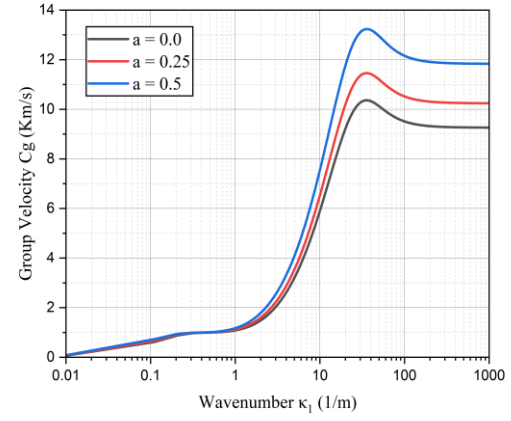


Figure 4.10: Dispersion relations of different porosity models ($a = 0.5$, $V_{cnt} = 0.12$, X-Beam, Bending wave).



(a) Phase velocity.



(b) Group velocity.

Figure 4.11: Dispersion relations of different porosity volume fractions ($V_{cnt} = 0.12$, X-Beam, Porosity model 1, Bending wave).

CHAPTER 5

SECOND STUDY

This chapter presents the second paper: Wave Propagation Analysis of Composite Beams Reinforced with Nonlinear FG-CNT Distributions Supported on Kerr Elastic Foundation Utilizing an Improved Integral First-Order Shear Deformation Theory.

5.1 Introduction

Based on the literature review in Chapter 3, there are some notable gaps in the wave propagation analysis of FG-CNTRC beams, including the limited research work on other waves, an absence of different foundation models besides Winkler and Winkler-Pasternak foundations, reliance on linear distribution models of CNTs, with no prior use of an improved FSDT with integral form. Therefore, for the first time, this study analyzes the wave propagation behavior of the FG-CNTRC beams supported on the Kerr substrate based on a nonlinear distribution of CNTs and an integral FSDT. A novel correction function is proposed for the shear strain to enhance the accuracy of the results. The proposed model is validated by comparing its outcomes with the results found in the literature. The mechanical properties of constituents are determined using the mixture rule, while the Hamilton principle derives the governing equations of motion. The parametric investigation includes the influence of the dispersion patterns and volume fractions of CNTs on wave propagation responses. In addition, the effect of the Kerr foundation and the degree of nonlinearity of CNTs on wave dispersion behavior are examined.

5.2 Mathematical Model of an FG-CNTRC Beam

5.2.1 Configuration and Properties of Nonlinear FG-CNTs

Figure 5.1 illustrates a beam with a uniform (UD) distribution of CNTs, supported by a Kerr foundation, with a length (L) and a thickness (h). The foundation parameters are k_U , k_s and k_L which represent the upper spring, shear layer, and lower spring coefficients, respectively.

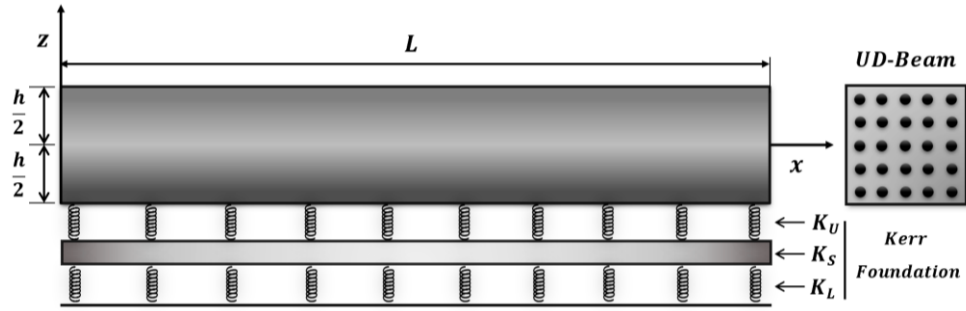


Figure 5.1: Geometry of a UD beam in Kerr foundation.

The FG-CNTRC beam is created by using a mixture of SWCNTs and a polymer. This study assumes different nonlinear distribution forms of CNTs across the beams' cross-sections: FG-O, FG-X, and FG-V, as demonstrated in **Table 5.1**.

Table 5.1: Cross-sections of different nonlinear SWCNTs' distribution forms.

Distribution of CNTs	$0 < n < 1$	$n = 1$	$1 < n$
FG-O			
FG-X			
FG-V			

For the case of ($n = 0$), the distribution of CNTs becomes uniform in the polymer matrix, resulting in the UD Beam, as shown in **Figure 5.1**. The mathematical functions that describe the distribution patterns of CNTs are defined in terms of exponential power-law volume fractions of CNTs, as illustrated in **Table 5.2** [72, 73].

Table 5.2: Expressions of nonlinear patterns of CNTs distributed in the thickness.

CNT's Patterns	V_{cnt}	
UD	$V_{cnt} = V_{cnt}^*$	(5-1a)
FGO	$V_{cnt} = (n + 1) \left(1 - 2 \frac{ z }{h}\right)^n V_{cnt}^*$	(5-1b)
FGX	$V_{cnt} = (n + 1) \left(2 \frac{ z }{h}\right)^n V_{cnt}^*$	(5-1c)
FGV	$V_{cnt} = (n + 1) \left(\frac{1}{2} + \frac{z}{h}\right)^n V_{cnt}^*$	(5-1d)

where:

- n : The exponent degree of the power-law distributions of CNTs.
- V_{cnt}^* : The given volume fraction of CNTs.

The relationship between the volume fractions of CNTs (V_{cnt}) and the polymer matrix (V_p) in Eq. (5-2) holds for each position of the beam:

$$V_{cnt} + V_p = 1 \quad (5-2)$$

The law of mixture is employed to assess the effective material properties of FG-CNTRC beams. The following is an expression for the material properties:

$$E_{11} = \eta_1 V_{cnt} E_{11}^{cnt} + V_p E^p \quad (5-3)$$

$$\frac{\eta_3}{G_{12}} = \frac{V_{cnt}}{G_{12}^{cnt}} + \frac{V_p}{G^p} \quad (5-4)$$

$$v = V_{cnt} v^{cnt} + V_p v^p \quad (5-5)$$

$$\rho = V_{cnt}\rho^{cnt} + V_p\rho^p \quad (5-6)$$

where:

- E_{11} , G_{12} , ν , and ρ : The modulus of elasticity, the shear modulus, Poisson's ratios, and the density of materials, respectively (cnt stands for CNTs and p for polymer matrix).
- η_1 , η_2 , and η_3 : The size-dependent efficiency of SWCNTs.

5.2.2 The Kerr Foundation

This study aims to analyze the response of the FG-CNTRC beams on a Kerr elastic foundation that implements three different actions: the actions of the upper and lower springs and the shear interaction, as shown in **Figure 5.1**. Thus, the foundation effect can be implemented based on the transverse middle displacement w_0 as:

$$R_f = \left(\frac{k_L k_U}{k_L + k_U} w_0(x, t) - \frac{k_s k_U}{k_L + k_U} w_{0,xx}(x, t) \right) \quad (5-7)$$

where:

- R_f : The reaction force of the foundation.
- k_L , k_U , and k_s : The parameters of the lower spring, upper spring, and shear layer, respectively.
- $f_{\overline{xxx} \overline{yyy}}^{i \ j}$: It indicates a derivate of function f with respect to some variable $\left(\frac{\partial^{(i+j)} f}{\partial x^{(i)} \partial y^{(j)}} \right)$.

The repeated symbols represent higher-order derivatives for each variable.

The Winkler-Pasternak foundation emerges when the upper layer becomes infinitely stiff, which is equivalent to its removal; thus, the reaction force becomes as follows:

$$R_f = (k_L w_0(x, t) - k_s w_{0,xx}(x, t)) \quad (5-8)$$

To facilitate the discussion and comparisons, the foundation parameters are defined as:

$$k_U = \beta_U \left(\frac{A_{110}}{L^2} \right), \quad k_L = \beta_L \left(\frac{A_{110}}{L^2} \right) \quad (5-9)$$

$$k_s = \beta_s A_{110} \quad (5-10)$$

where:

- β_L , β_U , and β_s : The constant factors of the lower spring, upper spring, and shear layer, respectively.

5.3 Formulations and Governing Equations Derivation

5.3.1 Kinematics

Based on Timoshenko's theory, an improved FSDT with an undetermined integral form is used to describe the beam behavior. The displacement field involves u_0 , w_0 , and θ as the unknowns, expressed as follows:

$$\begin{aligned} u &= u_0(x, t) - z k_1 \int \theta(x, t) dx \\ w &= w_0(x, t) \end{aligned} \quad (5-11)$$

where:

- $u_0(x, t)$, $w_0(x, t)$: The middle-plane displacements ($z = 0$) in the x and z directions, respectively.
- $\theta(x, t)$: The beam rotation.

5.3.2 Strain-Displacement Relations

The linear strain-displacement relationships are used to derive the components of the strains in terms of displacement unknowns. The 3D infinitesimal strain for small deformation was presented in Section 2.8.2. The strain contains only ε_x and γ_{xz} based on the assumptions of the displacement field shown below.

$$\varepsilon_x = u_{,x} = u_{0,x} - z k_1 \theta \quad (5-12)$$

$$\gamma_{xz} = C_f(u_{,z} + w_{0,x}) = C_f(w_{0,x} - k_1 \int \theta dx) \quad (5-13)$$

where:

- C_f : The shear correction function. Novel to this study, the correction function in Eq. (5-14) is employed and defined as:

$$C_f = \frac{\pi}{r} \cos\left(\frac{\pi z}{h}\right) \quad (5-14)$$

$$r = 2.425$$

Applying the analytical solution of wave propagation simplifies the undetermined integrals to the relation illustrated in Eq. (5-15):

$$\int \theta(x, t) dx = A' \frac{\partial \theta(x, t)}{\partial x} \quad (5-15)$$

where A' and k_1 are related as functions of the wavenumber κ_1 as follows:

$$A' = -\frac{1}{\kappa_1^2}, \quad k_1 = \kappa_1^2 \quad (5-16)$$

The characteristics of the exponential functions employed in the general wave propagation solution led to this simplification.

5.3.3 Stress-Strain Relations

The 3D constitutive relations are used to determine the stresses and were defined previously in Section 2.8.3. Based on the assumptions, the only relevant stresses for this study are σ_x and τ_{xz} , which correspond to non-zero components of the strains, defined as the following:

$$\sigma_x = Q_{11}(z) \varepsilon_x, \quad \tau_{xz} = Q_{55}(z) \gamma_{xz} \quad (5-17)$$

$$Q_{11}(z) = \frac{E_{11}(z)}{1 - \nu(z)^2}, \quad Q_{55}(z) = G_{12}(z) \quad (5-18)$$

5.3.4 Governing Equations

The motion equations are derived based on the Hamilton's principle, stated as:

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K) dt = 0 \quad (5-19)$$

where:

- δ : The first variation operator.

- $U, V, \text{ and } K$: The strain energy, work of the external forces, and the kinetic energy, respectively.

The three middle-plane displacements are used to express the energy variations. The strain energy variation is found as follows:

$$\begin{aligned}\delta U &= \int_0^L \int_{-h/2}^{h/2} (\sigma_x \delta \varepsilon_x + \tau_{xz} \delta \gamma_{xz}) dz dx \\ &= \int_0^L \left(N \delta u_{0,x} - k_1 M_b \delta \theta + Q (\delta w_{0,x} - A' k_1 \delta \theta_{,x}) \right) dx\end{aligned}\quad (5-20)$$

where:

- $N, M_b, \text{ and } Q$: The stress resultants, expressed as:

$$\begin{Bmatrix} N \\ M_b \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_x \begin{Bmatrix} 1 \\ z \end{Bmatrix} dz, \quad Q = \int_{-h/2}^{h/2} \tau_{xz} dz \quad (5-21)$$

The virtual external work done by the Kerr foundation can be written as:

$$\delta V = \int_0^L \left(\frac{k_L k_U}{k_L + k_U} w_0 - \frac{k_s k_U}{k_L + k_U} w_{0,xx} \right) \delta w_0 dx \quad (5-22)$$

The variation of kinetic energy takes the following form in Eq. (5-23):

$$\begin{aligned}\delta K &= \int_0^L \int_{-h/2}^{h/2} \rho(z) (\dot{u} \delta \dot{u} + \dot{w}_0 \delta \dot{w}_0) dz dx \\ &= - \int_0^L \int_{-h/2}^{h/2} \rho(z) (\ddot{u} \delta u + \ddot{w}_0 \delta w_0) dz dx\end{aligned}\quad (5-23)$$

By expanding and simplifying, it could be written as represented in Eq. (5-24)

$$\begin{aligned}\delta K &= - \int_0^L \left(I_0 (\ddot{u}_0 \delta u_0 + \ddot{w}_0 \delta w_0) - I_1 (A' k_1 \ddot{w}_0 \delta \theta_{,x} + A' k_1 \delta \ddot{\theta}_{,x} \delta u_0) \right. \\ &\quad \left. + I_2 (A' k_1)^2 \delta \ddot{\theta}_{,x} \delta \theta_{,x} \right) dx\end{aligned}\quad (5-24)$$

where:

- $\overset{i}{\ddot{f}}$: It indicates a derivate of function f with respect to the time $\left(\frac{\partial^{(i)}f}{\partial t^{(i)}}\right)$. The repeated dots indicate higher-order derivatives.
- I_i ($i = 0, 1, 2$): The mass inertia terms, expressed as:

$$\begin{Bmatrix} I_0 \\ I_1 \\ I_2 \end{Bmatrix} = \int_{-h/2}^{h/2} \rho(z) \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz \quad (5-25)$$

The motion equations are then determined with the use of the calculus of variations when the energy virtual variations (δU , δV , and δK) in Eqs. (5-20), (5-22), and (5-24) are substituted into Hamilton's principle as functions of the three middle displacements. After integrating by parts and setting the coefficients of δu_0 , δw_0 , and $\delta \theta$ to zero, the following three equations are finally obtained:

$$\begin{aligned} \delta u_0: \quad N_{,x} &= I_0 \ddot{u}_0 - I_1 A' k_1 \ddot{\theta}_{,x} \\ \delta w_0: \quad Q_{,x} - \frac{k_L k_U}{k_L + k_U} w_0 + \frac{k_s k_U}{k_L + k_U} w_{0,xx} &= I_0 \ddot{w}_0 \\ \delta \theta: \quad M_b k_1 - Q_{,x} A' k_1 &= I_1 A' k_1 \ddot{u}_{0,x} - I_2 (A' k_1)^2 \ddot{\theta}_{,xx} \end{aligned} \quad (5-26)$$

The expression of the stress resultants in the motion equations as a combination of displacements and material stiffness components is shown in Eq. (5-27):

$$\begin{aligned} \begin{Bmatrix} N \\ M_b \end{Bmatrix} &= \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{Bmatrix} u_{0,x} \\ -k_1 \theta \end{Bmatrix} \\ Q &= A_{55} (w_{0,x} - A' k_1 \theta_{,x}) \end{aligned} \quad (5-27)$$

where:

- A_{11} , B_{11} , D_{11} , and A_{55} : The stiffnesses of the beam, defined as:

$$\begin{Bmatrix} A_{11} \\ B_{11} \\ D_{11} \end{Bmatrix} = \int_{-h/2}^{h/2} Q_{11} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz, \quad A_{55} = \int_{-h/2}^{h/2} C_f^2 Q_{55} dz \quad (5-28)$$

After substituting the resultants-displacements relations, the governing equations are obtained in Eq. (5-29) as functions of middle displacements only:

$$\begin{aligned}
A_{11} u_{0,xx} - B_{11} k_1 \theta_{,x} &= I_0 \ddot{u}_0 + I_1 \ddot{\theta}_{,x} \\
A_{55} (w_{0,xx} + \theta_{,xx}) - \frac{k_L k_U}{k_L + k_U} w_0 + \frac{k_s k_U}{k_L + k_U} w_{0,xx} &= I_0 \ddot{w}_0 \\
B_{11} k_1 u_{0,x} - D_{11} k_1^2 \theta + A_{55} (w_{0,xx} + \theta_{,xx}) &= -I_1 \ddot{u}_{0,x} + I_2 \ddot{\theta}_{,xx}
\end{aligned} \tag{5-29}$$

5.3.5 Dispersion Relations

The analytical solutions of middle-plane displacements for wave propagation in an infinite beam are as follows:

$$\begin{Bmatrix} u_0(x, t) \\ w_0(x, t) \\ \theta(x, t) \end{Bmatrix} = \begin{Bmatrix} U_m e^{i(\kappa_1 x - \omega t)} \\ W_m e^{i(\kappa_1 x - \omega t)} \\ \theta_m e^{i(\kappa_1 x - \omega t)} \end{Bmatrix} \tag{5-30}$$

where:

- U_m , W_m , and θ_m : The amplitudes of the waves.
- i : The imaginary number ($\sqrt{-1}$).
- κ_1 : The wavenumber in the x-axis.
- ω : The circular frequency.

If the displacement solutions in Eq. (5-30) are inserted into Eq. (5-29), the eigenvalue problem is produced and rewritten as a matrix form:

$$([K] - \omega^2 [M]) \{\Lambda\} = 0 \tag{5-31}$$

where:

- $[K]$, $[M]$: The stiffness and mass matrices (3×3), respectively.
- $\{\Lambda\}$: The vector of the amplitudes of the waves (3×1).

The detailed representations of the above matrices' equations in Eq. (5-31) are expressed as:

$$\left(\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{12} & M_{22} & M_{23} \\ M_{13} & M_{23} & M_{33} \end{bmatrix} \right) \begin{Bmatrix} U_m \\ W_m \\ \theta_m \end{Bmatrix} = 0 \tag{5-32}$$

where the following describes the components of the symmetric stiffness matrix:

$$\begin{Bmatrix} K_{11} \\ K_{12} \\ K_{13} \\ K_{22} \\ K_{23} \\ K_{33} \end{Bmatrix} = \begin{Bmatrix} A_{11}\kappa_1^2 \\ 0 \\ i B_{11}\kappa_1\kappa_1 \\ -\frac{k_L k_U}{k_L + k_U} - \frac{k_s k_U}{k_L + k_U}\kappa_1^2 - A_{55}\kappa_1^2 \\ -A_{55}\kappa_1^2 \\ -D_{11}\kappa_1^2 - A_{55}\kappa_1^2 \end{Bmatrix} \quad (5-33)$$

and for the elements of the symmetric mass matrix

$$\begin{Bmatrix} M_{11} \\ M_{12} \\ M_{13} \\ M_{22} \\ M_{23} \\ M_{33} \end{Bmatrix} = \begin{Bmatrix} I_0 \\ 0 \\ i I_1 \kappa_1 \\ -I_0 \\ 0 \\ -I_2 \kappa_1^2 \end{Bmatrix} \quad (5-34)$$

This eigenvalue problem has a non-trivial solution only if the determinant of $[[K] - \omega^2[M]]$ is equal to zero (As discussed in Section 2.8.5). The circular frequencies are numerically solved in terms of wavenumbers by taking the square root of the eigenvalues. There are three circular and wave frequencies, which correlate to the three unknown theory displacements, as indicated below:

$$\omega_j = \omega_j(\kappa_1), \quad (j = 1, 2, 3) \quad (5-35)$$

$$f_j(\kappa_1) = \frac{\omega_j(\kappa_1)}{2\pi} \quad (5-36)$$

where ω_1 , ω_2 , and ω_3 represent the frequencies of bending, longitudinal, and shear wave modes, respectively. The phase velocity C_{pj} and the group velocity C_{gj} are found from the circular frequencies relations:

$$C_{pj} = \frac{\omega_j(\kappa_1)}{\kappa_1}, \quad C_{gj} = \frac{d\omega_j(\kappa_1)}{d\kappa_1} \quad (5-37)$$

For validation purposes, the dimensionless circular frequencies are utilized, stated as follows:

$$\bar{\omega} = \omega L \sqrt{\frac{I_{00}}{A_{110}}} \quad (5-38)$$

$$I_{00} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho^p dz, \quad A_{110} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E_p}{1 - \nu_p^2} dz \quad (5-39)$$

where:

- I_{00} and A_{110} : The inertia and stiffness of a beam made completely of pure polymer matrix, respectively.

5.4 Numerical Results and Discussion

This section contains two main parts: results validation, which verifies the model's accuracy, and parametric studies, which investigate the effects of various parameters. The results of this study are numerically obtained based on the armchair (10, 10) SWCNTs and polymethyl methacrylate (PMMA), which serve as the reinforcement and polymer matrix, respectively. The mechanical properties of the CNTs and matrix are stated in **Table 5.3** [74].

Table 5.3: Mechanical properties of CNTs and polymer matrix.

Materials	$E_{11}(GPa)$	$E_{22}(GPa)$	$G_{12}(GPa)$	$\rho(kg/m^3)$	ν
Reinforcement	600	10	17.2	1400	0.19
Polymer Matrix	2.5	2.5	$\frac{E_p}{2(1 + \nu_p)}$	1190	0.3

The CNTs-given volume fractions (V_{cnt}^*) and the corresponding size-dependent efficiency parameters (η_i) are given in **Table 5.4** below [74].

Table 5.4: Volume fractions of CNTs and their corresponding efficiency parameters.

Case	V_{cnt}^*	η_1	η_2 and η_3
1	0.12	1.2833	1.0566
2	0.17	1.3414	1.7101
3	0.28	1.3238	1.7380

5.4.1 Results Validation

The mathematical models are numerically validated by comparing their outcomes with the free vibration results in earlier publications. **Table 5.5** compares the dimensionless fundamental frequency of the present work with that of previous studies conducted by Wattanasakulpong and Ungbhakorn [74], Tagrara et al. [75], Alsubaie et al. [69], and Gawah et al. [76]. The comparisons in **Table 5.5** disregard the nonlinearity of CNTs volume fraction in the current work because the references only consider the linear case with $n = 1$. The left part in **Table 5.5** presents the fundamental frequency results for beams without foundation, while the right part displays the results for beams supported by the Winkler-Pasternak foundation. **Table 5.5** indicates that while the present theory comprises an FSDT, it closely matches the more advanced published HSDT theories, showing only minor discrepancies. This accuracy is achieved by incorporating the novel shear correction function, leading to improved results.

Table 5.5: Comparison of dimensionless fundamental frequencies ($L/h = 15$, $V_{cnt} = 0.12$, $\lambda = \frac{\pi}{L}$).

Theory	$\beta_L = \beta_s = 0.0$				$\beta_U = \infty, \beta_L = 0.1, \beta_s = 0.02$			
	UD	O	X	V	UD	O	X	V
FSDT^(a)	0.9976	0.7628	1.1485	0.8592	1.1339	0.9339	1.2688	1.0142
TSDT^(a)	0.9745	0.7453	1.1152	0.8441	1.1137	0.9198	1.2387	1.0014
Ref^(b)	0.9749	0.7446	1.1163	0.8442	1.1140	0.9192	1.2397	1.0015
Ref^(c)	0.9750	0.7446	1.1164	0.8444	1.1141	0.9192	1.2399	1.0017
Ref^(d)	0.9745	0.7554	1.1033	0.8442	1.1137	0.9279	1.2280	1.1015
Present	0.9750	0.7557	1.1035	0.8444	1.1141	0.9282	1.2282	1.0017

FSDT^(a) and TSDT^(a) Taken from Wattanasakulpong and Ungbhakorn [74].

Ref^(b) Taken from Tagrara et al. [75].

Ref^(c) Taken from Alsubaie et al. [69].

Ref^(d) Taken from Gawah et al. [76].

Another comparison of the dimensionless fundamental frequency is made with Zerrouki et al. [73], considering the nonlinearity of FG-CNTs, as shown in **Table 5.6**. According to **Table 5.6**, the present study results are relatively close to the Zerrouki et al. [73]. Furthermore, as

the value of the exponent increases, the fundamental frequencies of the X-beam rise while the fundamental frequencies of the O-beam and V-beam decline.

Table 5.6: Comparison of the dimensionless frequencies with nonlinearity effect ($L/h = 15$, $V_{cnt} = 0.12$, $\lambda = \frac{\pi}{L}$).

$\beta_L = \beta_s = 0.0$						
N	O		X		V	
	<i>Present</i>	<i>Ref^(a)</i>	<i>Present</i>	<i>Ref^(a)</i>	<i>Present</i>	<i>Ref^(a)</i>
0.5	0.8529	0.8440	1.0556	1.0609	0.9111	0.9087
1.0	0.7557	0.7453	1.1035	1.1152	0.8444	0.8447
1.5	0.6780	0.6684	1.1352	1.1530	0.7840	0.7875
2.0	0.6153	0.6069	1.1575	1.1812	0.7313	0.7375
2.5	0.5639	0.5569	1.1741	1.2031	0.6861	0.6941

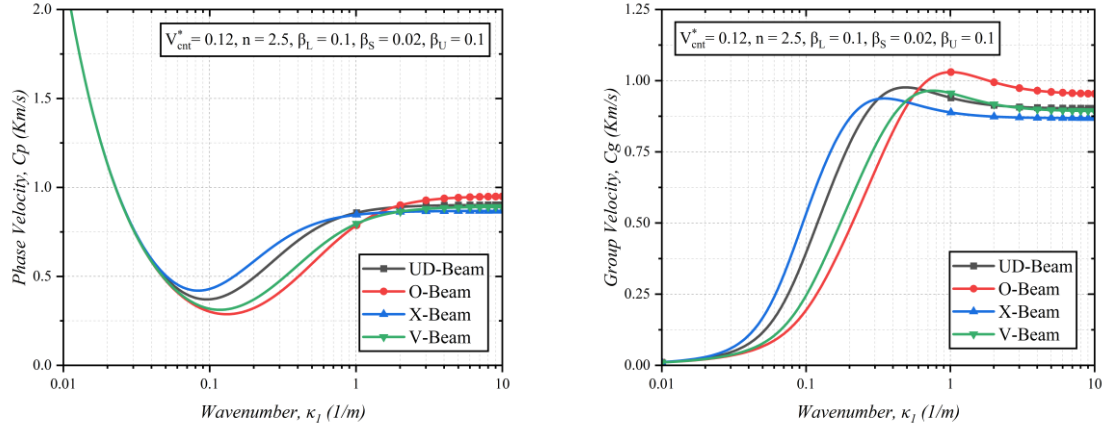
Ref^(a) Taken from Zerrouki et al. [73].

5.4.2 Parametric Analysis

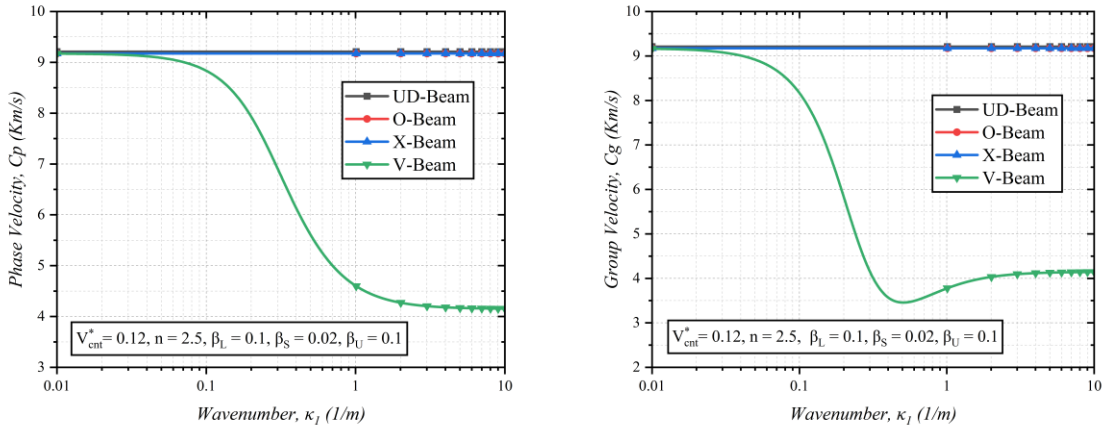
The parametric wave propagation analysis of nonlinear FG-CNTs-reinforced composite beams is presented. Several examples are used to study the impacts of CNT's nonlinear arrangement and volume fractions and the Kerr foundation on various dispersion relations.

5.4.2.1 Effect of The CNTs Patterns and Volume Fractions

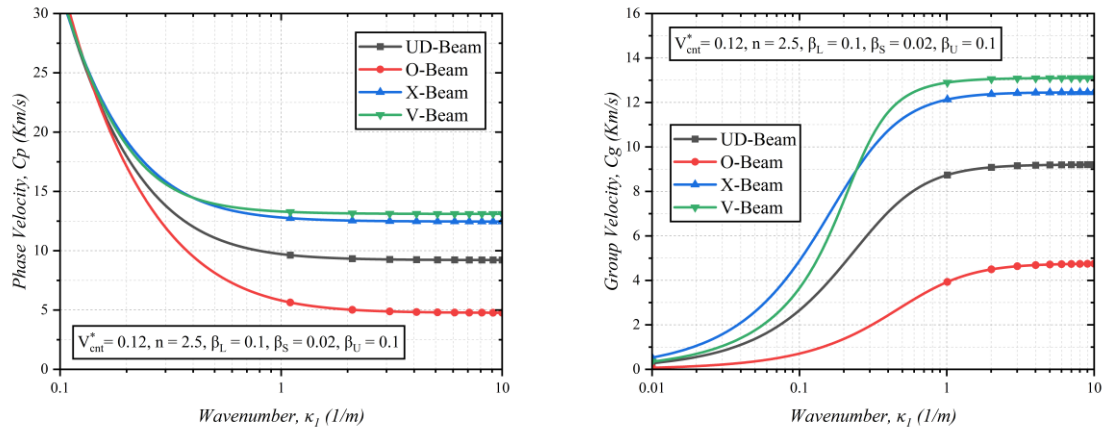
The wave velocities propagate differently based on the distribution patterns of CNTs. **Figure 5.2** presents comparison graphs of the nonlinear configurations of CNTs in the FG-CNTRC beams to evaluate the behavior of phase and group velocities. **Figure 5.2 (a)** indicates that the phase velocities of the bending waves descend sharply initially as the wavenumber increases before rising slightly and stabilizing, whereas the group velocities ascend as the wavenumber grows until attaining peak values and then balance off. Also, both velocities exhibit a reversal behavior where the beams of higher velocities initially switch to having slightly slower overall velocities. Additionally, at lower wavenumbers, the X-beam has the highest phase and group velocities of bending waves, whereas as the wavenumber grows, the O-beam becomes slightly higher.



(a) Bending waves.



(b) Longitudinal waves.



(c) Shear waves.

Figure 5.2: Phase (Left) and group (Right) velocities versus wavenumbers of different CNTs patterns.

For longitudinal waves, both velocities of the V-beam are the only ones to have a declining behavior due to unsymmetric arrangement of CNTs in the matrix, as shown in **Figure 5.2 (b)**. In parallel to the behavior of bending waves' velocities, the phase velocities of the shear waves descend sharply as the wavenumber increases before reaching stable values, while the group velocities increase as the wavenumber grows until converging to constant values, as pictured in **Figure 5.2 (c)**. Besides, the X-beam shows the highest shear wave velocities at lower wavenumbers, but the V-beam rapidly surpasses it at higher wavenumbers. It can be concluded generally from **Figure 5.2** that the X-beam exhibits high bending rigidity, whereas the V-beam possesses the highest shear rigidity. Additionally, the symmetric distribution of CNTs does not affect the beam's axial rigidity, while the asymmetric arrangement of the V-beam reduces it.

The volume fractions of CNTs affect the dispersion relations of FG-CNTRC beams. All beams exhibit nearly identical behavior in terms of volume fractions of CNTs. Hence, only the X-beam is used to represent the other beams. **Figure 5.3** illustrates the dispersion relations at different volume fractions of CNTs for the X-beam. Generally, the volume fractions of CNTs do not affect both velocities at low wavenumbers, while its impact becomes more pronounced as the wavenumber increases. As anticipated, it is evident that increasing the volume fractions of CNTs leads to higher wave velocities, thanks to enhanced stiffness and the increase of the material rigidity.

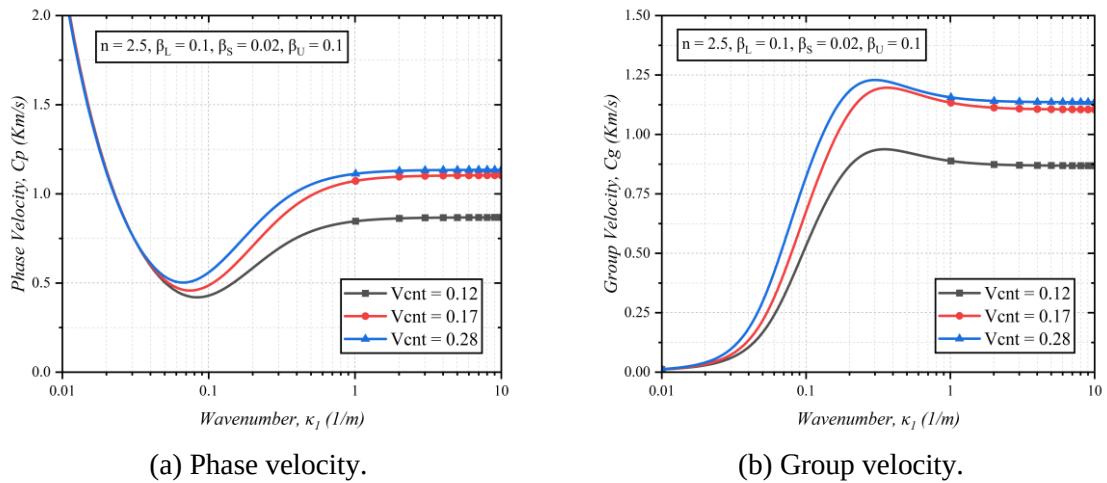
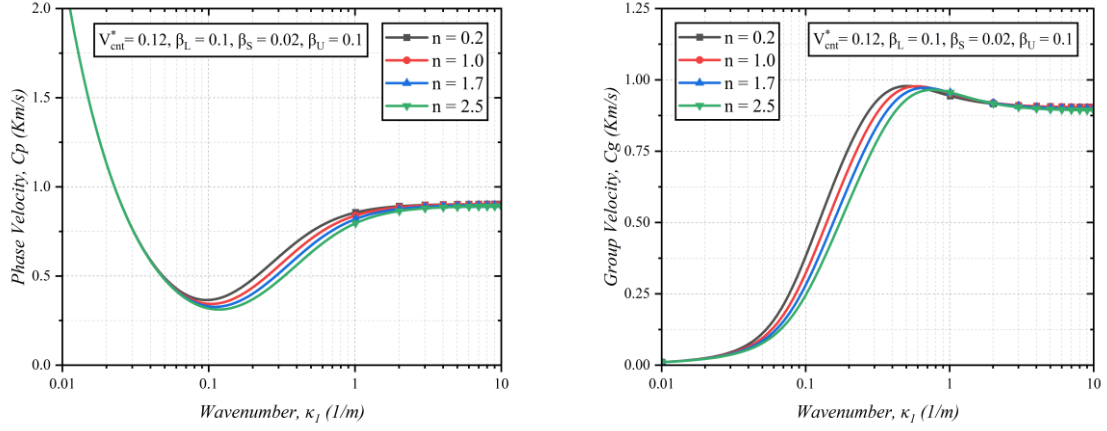


Figure 5.3: Dispersion relations of different CNTs volume fractions (X-Beam, Bending waves).

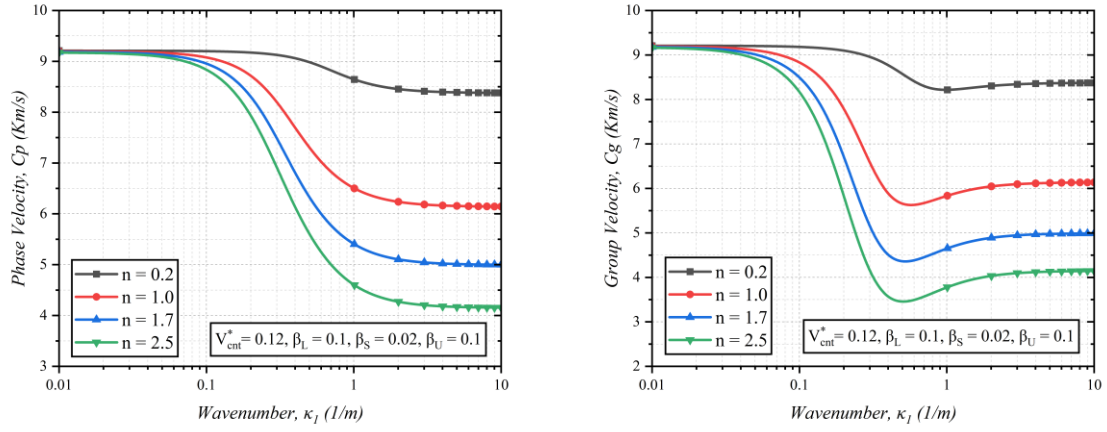
5.4.2.2 Effect of The Nonlinear CNTs Distributions

The introduction of the nonlinearity in the arrangement of CNTs affects the wave propagation of beams with FG-CNTs distributions (O-beam, X-beam, V-beam). Only the V-beam is considered in detail because it is being influenced the most by the exponent degree. **Figure 5.4** illustrates the dispersion relations of V-beam with different nonlinearity degrees. **Figure 5.4 (a)** shows that the phase and group velocities of the bending waves are affected the most by the exponent degree when the wavenumber is in the range of 0.05 and 1 $1/m$. Also, as the exponent degree increases, both velocities tend to decline. The longitudinal waves shown in **Figure 5.4 (b)** indicate that the nonlinearity of CNTs has an incredible effect on the wave velocities as the wavenumber increases due to an unsymmetric nature of the V-beam distribution. Furthermore, both velocities grow and linearize as the values of the exponent degree decrease, approaching the behavior of symmetric beams. In contrast to bending and longitudinal waves, the shear waves propagate faster as the degree of nonlinearity increases. Also, as the wavenumber increases, both velocities increase, as depicted in **Figure 5.4 (c)**.

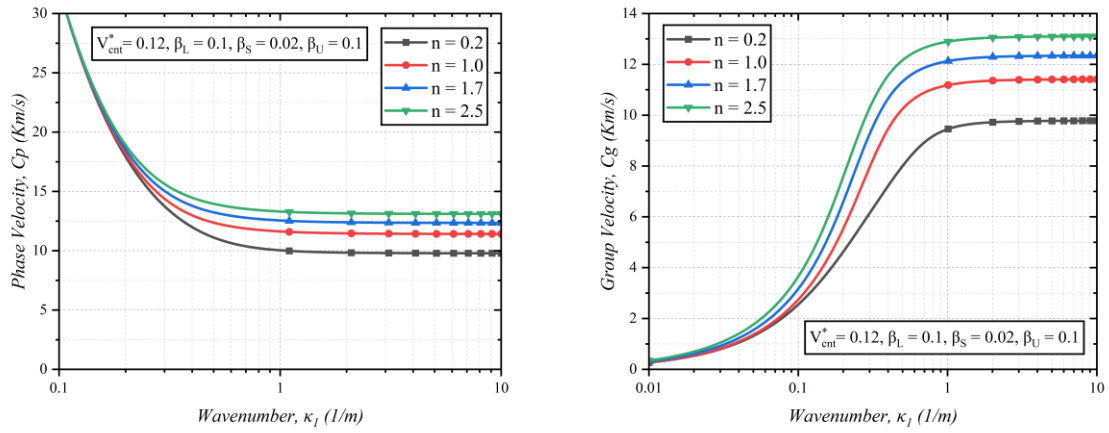
The FG-CNTRC beams have different behaviors regarding the effect of the nonlinear distributions of CNTs. In **Figure 5.5**, the relations of the phase and group velocities versus the exponent degree of CNTs are plotted to compare the influence of nonlinearity with each beam type and to capture their behavior. Also, all curves are based on a wavenumber of 0.3 $1/m$, with the UD beam being the baseline for comparison of other beams' behaviors. **Figure 5.5 (a)** indicates that both bending wave velocities of the X-beam increase slightly with increasing the degree of nonlinearity, whereas the O-beam and V-beam velocities decrease. All pattern forms of CNTs in **Figure 5.5 (b)** exhibit constant longitudinal waves' velocities with negligible change regarding the nonlinearity effect except for the V-beam, whose unsymmetric distributions of CNTs causes its velocities to decline as values of exponent degree increase. For shear waves, as the exponent degree increases, the phase and group velocities of X- and V- beams grow while the O-beam velocities decline, as depicted in **Figure 5.5 (c)**. All velocities of the beams approach the behavior of the UD-beam as the exponent degree decreases.



(a) Bending waves.

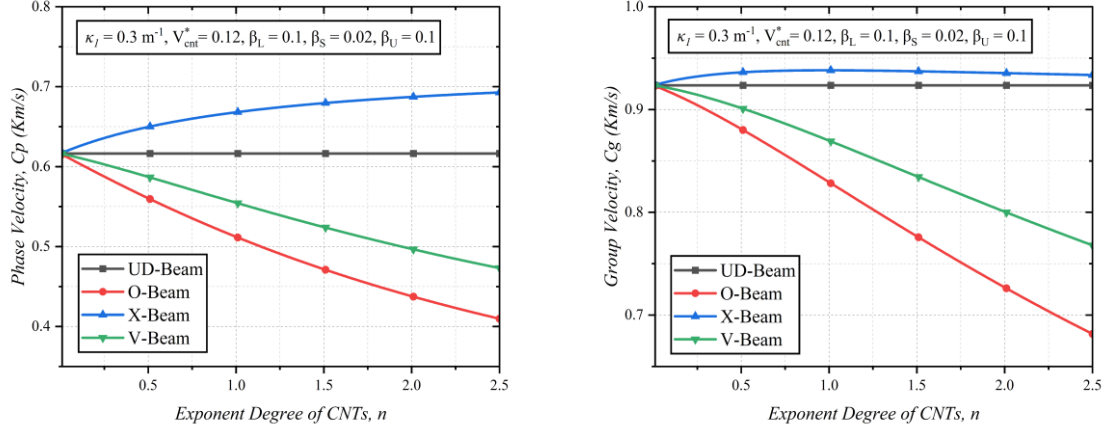


(b) Longitudinal waves.

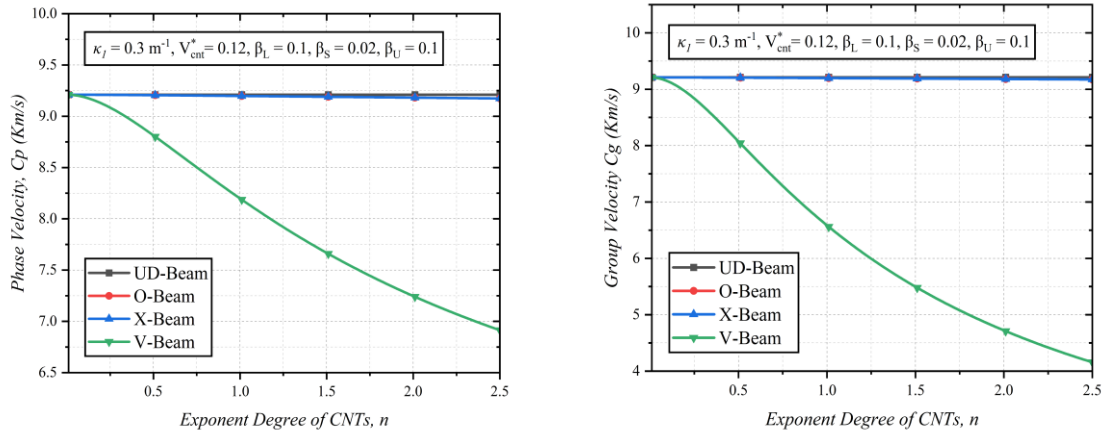


(c) Shear waves.

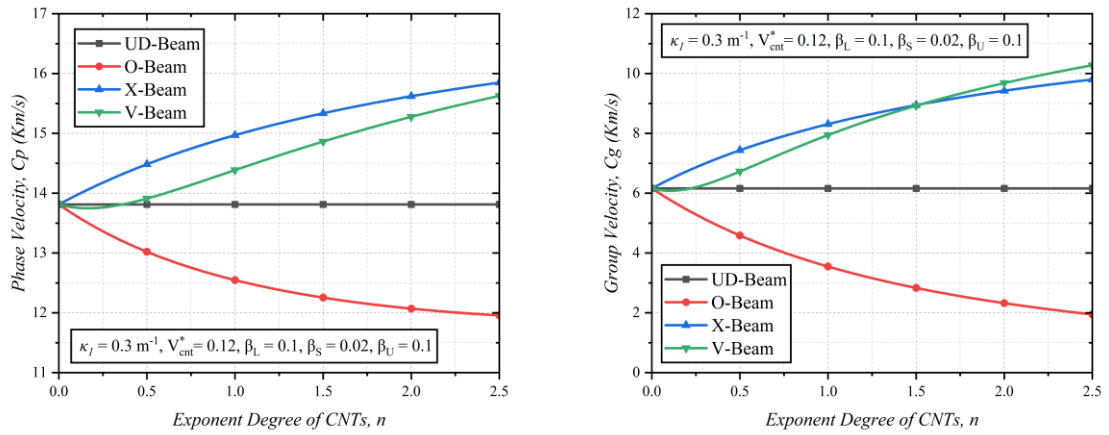
Figure 5.4: Phase (Left) and group (Right) velocities versus wavenumbers of different exponent degrees (V-Beam).



(a) Bending waves.



(b) Longitudinal waves.



(c) Shear waves.

Figure 5.5: Phase (Left) and group (Right) velocities versus exponent degrees of different CNTs patterns.

Generally, the nonlinear distribution of CNTs significantly influences the wave propagation behavior of the beams, although the extent of this impact varies based on the

type of waves and the pattern forms of the CNTs. This effect arises from the fact that nonlinearity introduces a parabolic distribution of CNTs, leading to the relocation of CNTs by either concentrating or spreading them around the three linear arrangements. For the X-beam, a higher exponent degree leads to a greater concentration of CNTs at the top and bottom edges of the cross-section, which ultimately enhances the bending rigidity of the beam. In contrast, for the O-beam, a higher exponent degree causes CNTs to accumulate more in the center of the beam, thereby reducing its stiffness and rigidity. For the V-beam, the exponent degree further accentuates its asymmetric nature.

5.4.2.3 Effect of the Kerr Foundation

The foundation affects only the velocities of the bending waves. All beams have nearly the same behavior regarding the foundation effect, so only the X-beam is considered a representative typical example for other beams to examine the influence of the Kerr foundation on wave propagation. **Figure 5.6** illustrates the impact of the shear layer on the wave propagation of the X-beam, and it shows that as the shear layer coefficient increases, both velocities increase while retaining the same behavior.

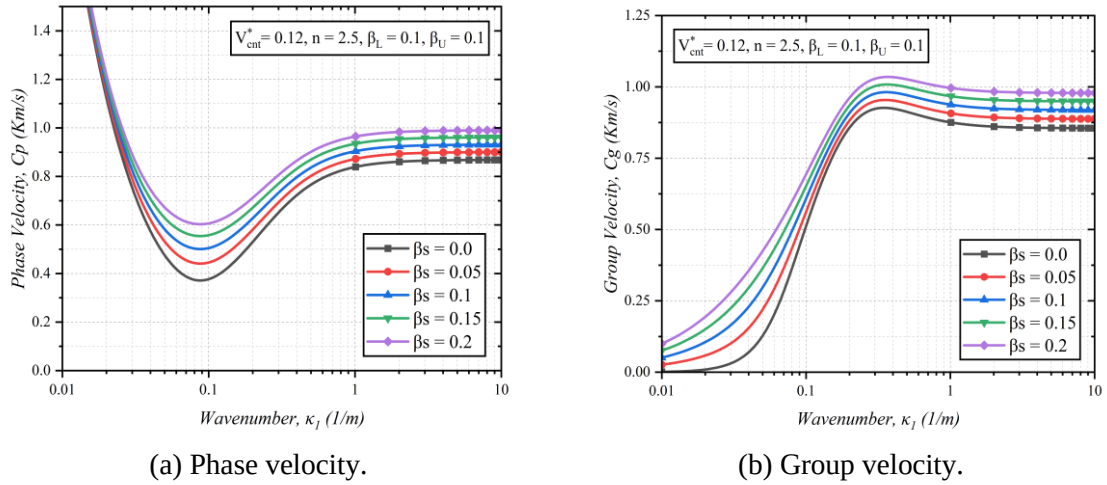


Figure 5.6: Dispersion relations of different shear layer coefficients (X-Beam, Bending waves).

For the lower spring case, **Figure 5.7** shows its effect on the dispersion relations of the X-beam. Both velocities are significantly impacted by the lower spring coefficient at lower wavenumbers, but this effect diminishes as the wavenumber rises. Additionally, as the coefficient of the lower spring increases, the values of phase velocities grow, contradicting

the group velocities' behavior. Also worth noting is that the phase velocities change their behavior from growing to declining as the coefficient of the lower spring equals more than zero.

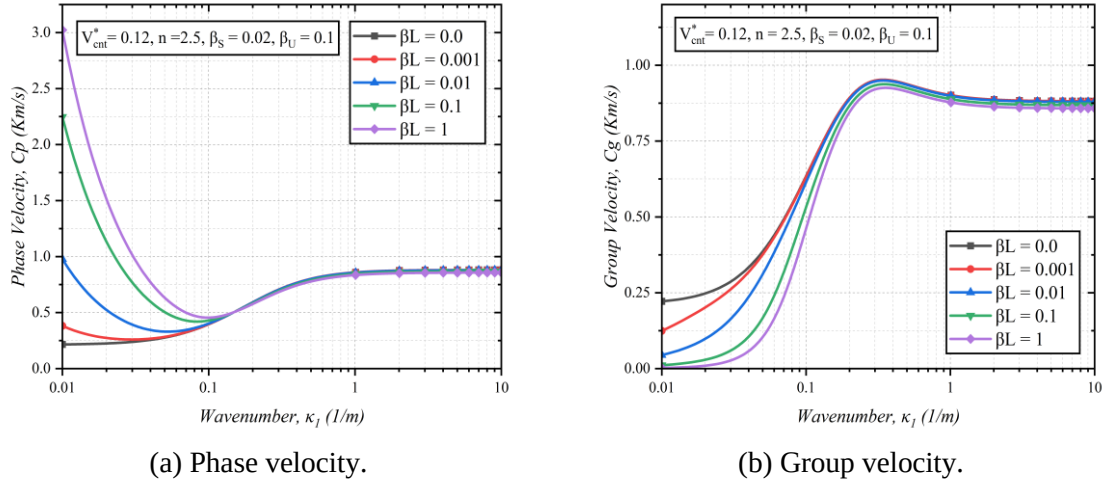


Figure 5.7: Dispersion relations of different lower spring coefficients (X-Beam, Bending waves).

The impact of the upper spring on the dispersion relations of the X-beam is illustrated in **Figure 5.8**. As in the lower spring case, at lower wavenumbers, the upper spring coefficient substantially affects both velocities; however, this effect becomes less pronounced as the wavenumber increases. In addition, as the coefficient of the upper spring grows, the values of phase velocities rise, which contradicts the behavior of group velocities.

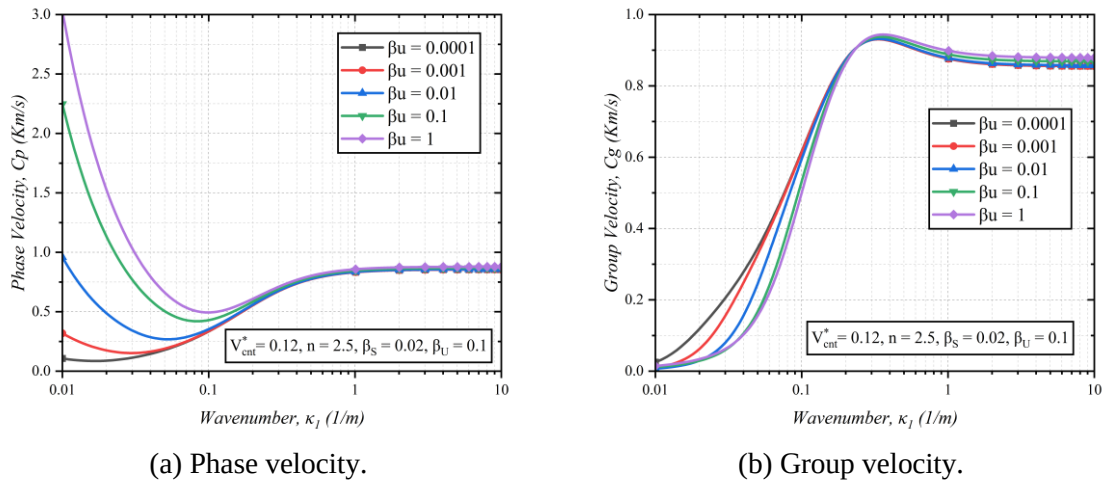


Figure 5.8: Dispersion relations of different upper spring coefficients (X-Beam, Bending waves).

In the previous cases, the observed increase in wave phase velocities due to the enhanced stiffness of the foundation can be attributed to the fact that as the stiffness of the medium grows, the structural stiffness of the beams also increases.

The Various foundation types have different effects on the behavior of the propagating waves. Although the Kerr substrate is the focus of this study, other foundation types are included to compare the behavior of the current foundation with other ones. The other foundation cases include no elastic foundation, Winkler, Pasternak, and Winkler-Pasternak substrates. **Figure 5.9** represents the influence of the foundation type on dispersion relations. As can be seen in **Figure 5.9 (a)**, the Kerr foundation has the third-highest phase velocities after the Winkler and Winkler-Pasternak foundations. Similarly, this confirms that the presence of a substrate or medium boosts the stiffness of the beams, thereby leading to higher phase velocities. On the other hand, the group velocities of the Kerr foundation are significantly low compared to those in the no-elastic foundation case and the Pasternak foundation, as shown in **Figure 5.9 (b)**.

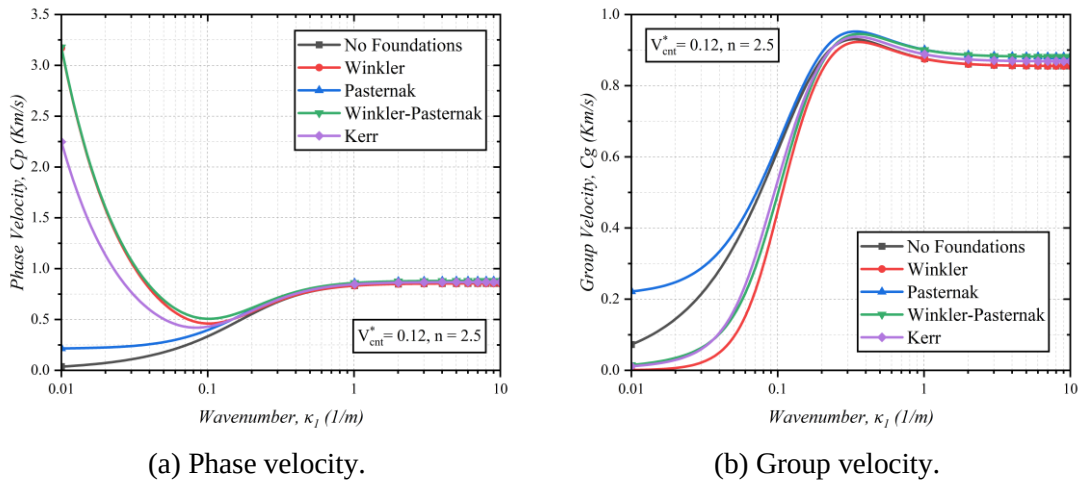


Figure 5.9: Dispersion relations of different types of foundations (X-Beam, Bending waves).

The interactions of the foundation springs and the shear layer with each other induce different effects on the dispersion relations. **Figure 5.10** depicts how the interactions of the upper and lower springs' coefficients affect the relationship of the wave velocities while the shear layer coefficient is constant. Both springs enhance their impact on the propagating waves as they become stiffer, increasing the phase velocities and decreasing the group velocities.

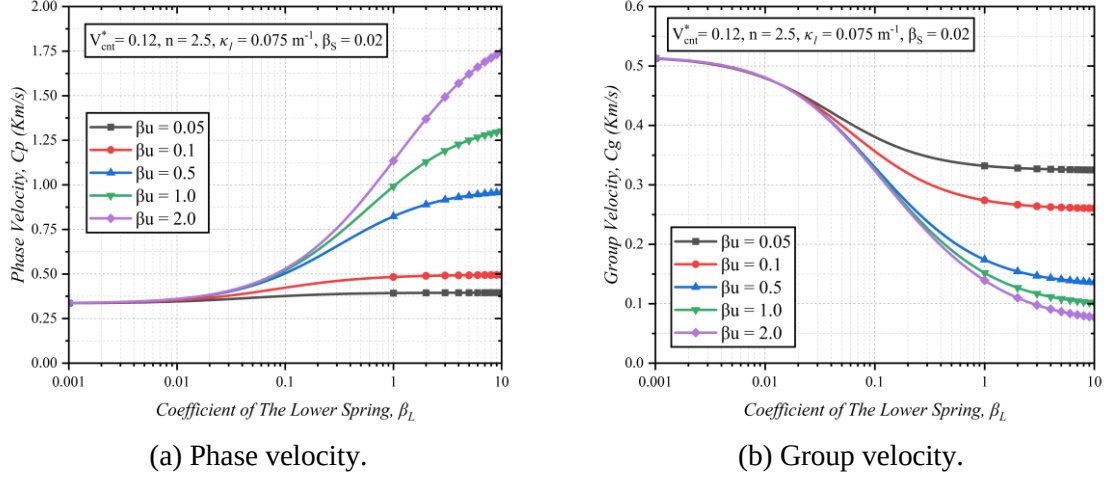


Figure 5.10: Interaction impact of the upper and lower springs (X-Beam, Bending waves).

In **Figure 5.11**, the interaction effects of the lower spring and shear layer on the wave velocities are illustrated. **Figure 5.11** indicates that the shear layer tends to improve the dispersions of the wave, yet as the lower spring stiffness increases, the shear layer gradually loses its impact before becoming ineffective.

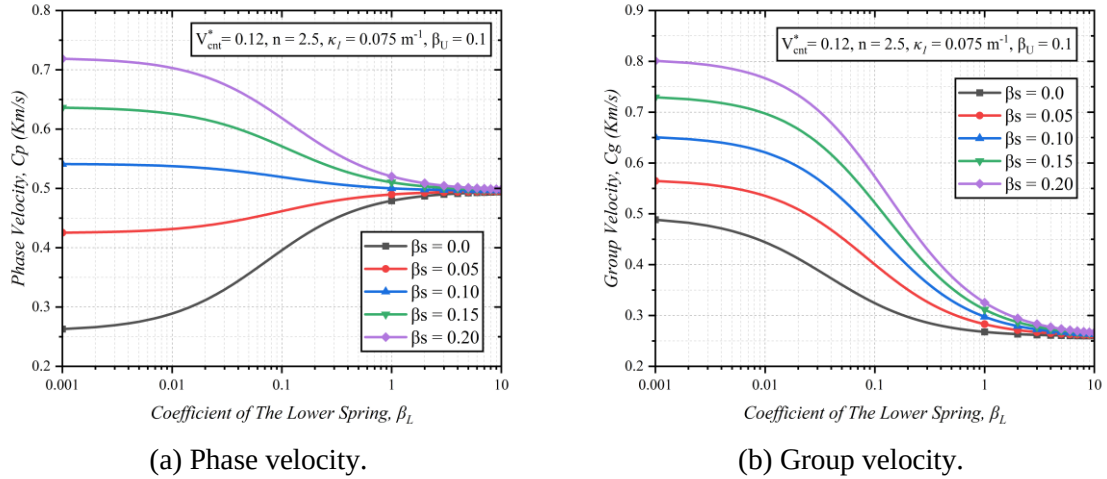
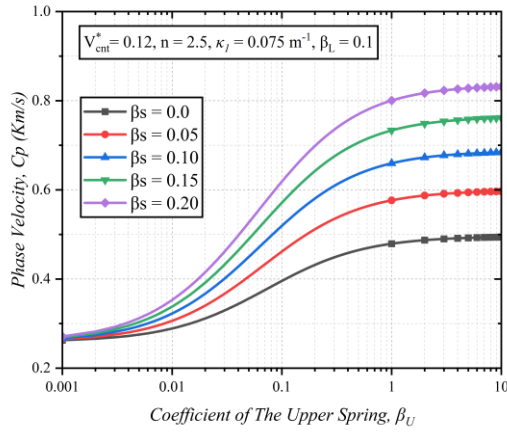


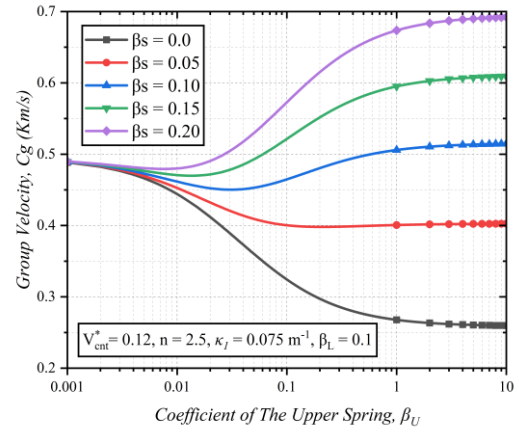
Figure 5.11: Interaction impact of the lower spring and the shear layer (X-Beam, Bending waves).

Figure 5.12 illustrates the interaction of the upper spring and shear layer on the propagating waves. The phase velocities are enhanced when the shear layer and upper spring become more rigid, as depicted in **Figure 5.12 (a)**. Also, the group velocities reduce as the stiffness of the upper spring grows at lower values of the shear layer coefficient. However, the behavior of the group velocities flips as the shear layer's stiffness

increases. As can be concluded, as the stiffness of the two springs and the shear layer of the Kerr foundation varies, its behavior changes and approaches that of other substrate types, leading to these intriguing results of the dispersion relations.



(a) Phase velocity.



(b) Group velocity.

Figure 5.12: Interaction impact of the upper spring and the shear layer (X-Beam, Bending waves).

CHAPTER 6

CONCLUDING REMARKS

This chapter highlights the novelty and contributions of the research, the significant results of the analyses, and recommendations for future research in this area.

6.1 Contributions of The Thesis

This thesis investigated the wave propagation of FG-CNTRC beams considering the following for the first time:

- Integral high-order and first-order shear theory (HSDT and FSDT) with three variables were introduced to describe the displacement fields.
- A new shape function was used to represent the transverse shear stress and strain distributions over the thickness of the beam in the case of HSDT.
- In FSDT, a novel correction function was utilized in the transverse shear stress and strain distributions to approach the fundamental frequency results of HSDT.
- The unprecedented inclusion of the porosity in wave propagation analysis was done with three models to describe the distributions of porosity within the matrix.
- The nonlinear distributions of FG-CNTs were incorporated in the wave propagation analysis using an exponent degree in the volume fraction expressions.
- The Kerr elastic foundation was introduced in the wave propagation analysis.

6.2 Conclusions of The Studies

This section presents the conclusions of the conducted studies and compares the results of the implemented theories in each study. The two studies are Analysis of Porosity-Dependent Wave Propagation in FG-CNTRC Beams Utilizing an Integral HSDT and Wave Propagation Analysis of Composite Beams Reinforced with Nonlinear FG-CNT Distributions Supported on Kerr Elastic Foundation Utilizing an Improved Integral FSDT.

6.2.1 First Study

This study has been conducted to investigate the wave propagation behavior of FG-CNTRC beams, considering the porosity effect and utilizing an integral three-variable HSDT. The analysis considered the bending, longitudinal, and shear waves of FG-CNTRC. The study used four different types of SWCNT arrangements and three porosity models to characterize the porosity patterns of the matrix. Hamilton's principle and the generalized wave propagation solution were employed to derive and solve the governing equations. The paper analyzed and discussed the influences of the patterns and volume fractions of CNTs, porosity models, and volume fractions. Through numerical investigations, the following remarks were found:

- The behavior of the present shape function with $n = 3$ is the same as that of HSDT for fundamental frequency and wave propagation analysis. Additionally, compared to the other theories, the wave propagation behavior changes significantly as the value of n grows while fundamental frequency results approach the FSDT ones.
- For all distributions of CNTs, the phase and group velocities of the bending waves rise with wavenumber, and both velocities show greater variability at lower wavenumber values.
- All pattern types of CNTs exhibit constant longitudinal wave phase and group velocities except for the V-beam pattern because of its unsymmetric distributions of CNTs.
- The phase velocities of shear waves decline as the wavenumber grows, whereas group velocities exhibit the reverse behavior.
- The X-beam has the highest phase and group velocities, while the O-beam has the lowest for bending and shear waves.
- The phase and group velocities become faster by raising the porosity and CNTs volume fractions.
- Compared to the other models, the porosity model 1 significantly affects phase and group velocities.
- The effects of volume fractions of CNTs, porosity models, and porosity volume fractions amplify as the wavenumber increases while maintaining the same behavior.

6.2.2 Second Study

This study investigated the wave propagation behavior of composite beams reinforced with nonlinear distribution patterns of FG-CNTs supported by Kerr substrate and utilizing an improved integral FSDT with three variables. The bending, longitudinal, and shear propagating waves of FG-CNTRC were discussed in the analysis. Four distinct SWCNT configurations were employed in the study. The governing equations were derived and solved using the wave propagation solution and Hamilton's principle. The parametric study examined how the volume fractions and dispersion patterns of the CNTs affected the wave propagation responses. Furthermore, the study examined the impact of the Kerr foundation and the nonlinearity degree of the model on the behavior of the waves. Numerical analysis revealed the following findings:

- The implementation of FSDT simplifies the derivation process without sacrificing accuracy. Also, incorporating an integral form allows for precise shear rotations while introducing a novel corrective function enables a better distribution of shear stresses and strains. Hence, the fundamental frequency results from this improved integral FSDT theory closely match those obtained from HSDT.
- The reinforcement arrangements of CNTs manipulate the axial, flexural, and shear rigidities of the FG-CNTRC beams ultimately affecting the dispersion relations.
- Reinforcing the beams by providing more volume fractions of CNTs improves the stiffness of the beams and results in higher propagating velocities.
- The nonlinear distributions of CNTs significantly impact the wave propagation of the FG-CNTRC beams, depending on the wave type and pattern forms of CNTs. This is due to the parabolic distribution, which relocates the CNTs by either concentrating or spreading them, thereby altering the stiffness of the beams.
- Supporting the FG-CNTRC beams with a foundation boosts their stiffness, leading to higher phase velocities. Hence, as the stiffness of the foundation grows, the phase velocities increase. Notably, only the bending waves are affected by the presence of the foundation.
- The interactions of the foundation two springs and the shear layer induce different effects on the dispersion relations. These results arise from the alterations in the stiffness of the

Kerr foundation's components, causing its behavior to resemble that of other substrate types.

6.2.3 Comparisons of The Two Studies

Based on the results of the two theories conducted in each study, the following notes could be noted:

- The two shear deformation theories predict identical wave propagation behavior at lower wavenumbers. However, they differ significantly in capturing the behavior of propagating waves with higher wavenumbers.
- HSDT gives higher values of the waves' velocities as the wavenumbers increase. On the other hand, FSDT has slower velocities, and all beams have nearly identical behavior.
- The V-beam's longitudinal waves of HSDT decrease initially before growing to approach the values of other distributions as the wavenumber increases. However, FSDT has longitudinal waves of V-beam with only declining behavior until reaching a stable value.
- There is a minimal disparity between the two theories in accurately depicting the propagation velocities of shear waves even at higher wavenumbers except for O-beam.

6.3 Suggestions for Further Work

Several recommendations are suggested based on the present analysis. Some remarks are presented for the different parametric studies, while others are for the potential improvement of the theory to describe wave propagation behavior better.

- The nonlinear distribution of FG-CNTs could be extended to the porosity models with functionally graded patterns.
- The viscoelastic behavior could be included in the Kerr Foundation.
- The nonlocal theories (Gradient elasticity theories) could be applied instead of local theory (Theory of elasticity) to embed microstructural properties of CNTs in nanobeams.
- The introduction of different mechanical materials models. For example, the Eshelby–Mori–Tanaka model accounts for the agglomeration effect of CNTs.
- The investigation of wave propagation in FG-CNTRC beams with MWCNTs type.

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