

**ANALYTICITY THEOREMS FOR
FUNCTIONS THAT ARE HOLOMORPHIC
ALONG VECTOR FIELDS**

BY

ALHUSSAIN ALKHUNAIZI

A Thesis Presented to the
COLLEGE OF GRADUATE AND INTERDISCIPLINARY STUDIES
KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICS

KING FAHD UNIVERSITY
OF PETROLEUM & MINERALS

2024/05

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

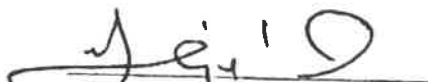
This thesis, written by **ALHUSSAIN ALKHUNAIZI** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

Thesis Committee


Dr. Jae-Cheon Joo (Adviser)


Dr. Adel Khalfallah (Member)


Dr. Muhammad Imran Qureshi (Member)


Prof. Montier Alfuraidan
Department Chairman


Prof. Suliman Al-Homidan
Dean of Graduate Studies

Date



©Alhussain Alkhunaizi
2024

*To my esteemed advisor, Dr. Jae-Cheon Joo, for your unwavering
guidance and support throughout this journey*

ACKNOWLEDGMENTS

I express my acknowledgment and obligation to Dr. Jae-Cheon Joo for his great effort, assistance, guidance, and patience. Without him, I would not have reached this point. His deep knowledge inspired me and motivated me to study harder. I am obligated to my thesis committee members, Dr. Adel Khalfallah and Dr. Muhammad Imran Qureshi for devoting their valuable time to reviewing my work. A special gratitude to Professor Monther Alfuraidan, the chairman of the Mathematics Department. His Relentless efforts in advancing the department are greatly appreciated.

I am grateful for my mother, she has been encouraging and supporting me to study and succeed in my academic journey. There are no words that can express how important is her role. I am thankful for my brother who always supported me. Finally, I extend my appreciation to all who have had an impact or played a role in my thesis preparation and writing.

TABLE OF CONTENTS

ACKNOWLEDGMENTS	vi
ABSTRACT (ENGLISH)	ix
ABSTRACT (ARABIC)	x
CHAPTER 1 INTRODUCTION	1
1.1 Motivation	1
1.2 Organization	6
CHAPTER 2 PRELIMINARIES	7
2.1 Holomorphic Functions	7
2.1.1 Properties of Functions of Several Complex Variables	7
2.1.2 Unique Continuation Property	8
2.2 Vector Fields	10
2.2.1 Contracting Vector Fields	10
CHAPTER 3 MAIN RESULTS	16
3.1 Diagonal Vector Fields	18

3.2 Contracting Vector Fields	22
CHAPTER 4 AN EXAMPLE OF NON-CONTRACTING VECTOR	
FIELD CASE	30
4.1 Non-Contracting Case	30
4.2 Special Case Proof	31
4.3 Future Work	36
REFERENCES	37
VITAE	39

THESIS ABSTRACT

NAME: Alhussain Alkhunaizi

TITLE OF STUDY: Analyticity Theorems for Functions That Are Holomorphic Along Vector fields

MAJOR FIELD: Mathematics

DATE OF DEGREE: 2024/05

This thesis investigates some analyticity theorems for functions of several complex variables. We introduce Forelli's theorem and demonstrate its generalizations. Mainly, in the view of vector fields. The case of contracting vector fields is already resolved and the results will be explained. An investigation of a non-contracting case will be discussed.

ملخص الرسالة

الاسم: الحسين الخنيزي

عنوان الدراسة: نظريات في تحليل الدوال المجسمة على طول حقول المتجهات

التخصص: الرياضيات

تاريخ الدرجة العلمية: 2024/05

تبحث هذه الأطروحة في بعض النظريات التحليلية لدوال متعددة معقدة المتغيرات. نقدم نظرية فورييلي ونوضح تعميماتها. خاصة، في ضوء حقول المتجهات. لقد تم بالفعل حل حالة تعاقد الحقول المتجهة وسيتم شرح النتائج. سيتم التحقيق في قضية عدم التعاقد مناقشة.

CHAPTER 1

INTRODUCTION

1.1 Motivation

The theory of functions of several complex variables is an extension of complex functions of one complex variable to multi-dimensions. Some theorems are extendable to higher dimensions, such as Cauchy's integral formula and the maximum modulus principle. On the other hand, some results fail on the higher dimensions. For example, a famous result in the one variable complex analysis due to Riemann is that any nonempty open simply connected proper subset of the complex plane is biholomorphic to the unit disk. It turns out that the statement fails in higher dimensions. The first counter-example was introduced by H. Poincaré. He proved that the two sets

$$\Delta^2 = \{(\zeta, \eta) \in \mathbb{C}^2 : |\zeta| < 1, |\eta| < 1\},$$

and

$$B^2 = \{(\zeta, \eta) \in \mathbb{C}^2 : |\zeta|^2 + |\eta|^2 < 1\}$$

are not biholomorphic, although both domains are not only simply connected, but both are convex.

Now let us recall the definition of a holomorphic function.

Definition: Let $X \subseteq \mathbb{C}^n$ ($n \geq 1$) be open. A function $\phi = u + iv : X \rightarrow \mathbb{C}$ is *holomorphic* if it is C^1 smooth and for $m = 1, 2, \dots, n$, we have

$$\frac{\partial \phi}{\partial \bar{z}_m} = 0.$$

Remark: $\frac{\partial \phi}{\partial \bar{z}_m} = 0$ is equivalent to the Cauchy-Riemann equations:

$$u_{x_m} = v_{y_m}, u_{y_m} = -v_{x_m}.$$

Definition: Let $w = (w_1, \dots, w_n) \in \mathbb{C}^n$. Let $D^n(w, A) := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i - w_i| < A, i = 1, 2, 3, \dots, n\}$ and let $X \subseteq \mathbb{C}^n$ ($n \geq 1$) be an open set. A function $\phi : X \rightarrow \mathbb{C}$ is *analytic* if for every $w \in X$, there is $A > 0$ so that $\bar{D}^n(w, A) \subseteq X$ and

$$\phi(z) = \sum_{j_1, \dots, j_n \geq 0} a_{j_1, \dots, j_n} (z_1 - w_1)^{j_1} \cdots (z_n - w_n)^{j_n}$$

for every $z \in D^n(w, A)$.

Remark: A function is analytic if and only if it is holomorphic.

One of the main results in this field is Hartogs' theorem on separate analyticity [1],

which was introduced by Hartogs in 1906. The theorem states that $\phi \in C^1$ condition can be dropped because the holomorphicity of ϕ in each variable separately ensures that $\phi \in C^1$. In the real case, the existence of partial derivatives does not imply continuity of the function. However, by Hartogs' analyticity theorem, continuity is not necessarily assumed in the definition of analyticity. This result of Hartogs is one of the main results in the concern of complex analyticity of a function of several complex variables.

Another important result in this concern is Forelli's theorem.

Theorem 1.1 ([2],[3]) *Let $\phi : \mathbb{B}^n \rightarrow \mathbb{C}$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$. If we have:*

- $\phi \in C^\infty(0)$.
- $\forall v \in \mathbb{B}^n$, we have $\phi(\mu v)$ is holomorphic in $\mu \in \mathbb{C}$.

Then ϕ is holomorphic.

Remark: $\phi \in C^\infty(0)$ means for every integer r , there is an open set U_r containing the origin satisfying $\phi \in C^r(U_r)$.

Forelli proved his result in 1977 [2]. Later, independently, Stoll proved the same result in 1980 in his paper on parabolic manifolds and applied it [3]. The main idea of the proof is based on proving that the formal power series of ϕ is convergent on a neighborhood of the origin and does not contain anti-holomorphic terms. Forelli established a closed form for the terms of the formal power series of ϕ and worked out to prove that this formal power series converges to the function ϕ around the origin.

The requirement that $\phi \in C^\infty(0)$ in Forelli's theorem turned out to be necessary

and it is not possible to relax it. A counter-example is the following:

$$\phi(z_1, z_2) = \begin{cases} \frac{\bar{z}_2}{\bar{z}_1} z_1^{k+2}, & \text{if } z_1 \neq 0 \\ 0, & \text{if } z_1 = 0 \end{cases} \quad (1.1)$$

Then ϕ is C^k smooth, satisfying the second condition but is not holomorphic.

This suggests that the generalization of the theorem occurs in the second condition. The statement that $\phi(\mu v)$ is holomorphic in the variable μ can be viewed as covering the unit ball by disjoint lines that intersect at a single point (the origin).

In 2006, E. M. Chirka proved a general case that is valid only in $\mathbb{C}^2$, by considering foliation [7]. In 2013, Joo, Kim, and Schmalz proved that it is still valid in all dimensions with the foliation by holomorphic discs (Forelli's case here considered as foliation by lines intersecting at the origin) [5].

Definition: Let L be a smooth manifold, then a *foliation* of L is a partition of L (an equivalence relation) into a collection of sub-manifolds that are connected. Each sub-manifold of a foliation is called a leaf.

Theorem 1.2 ([5],[7]) *Let $X \subseteq \mathbb{C}^n$ be a domain that has a C^1 radial foliation by holomorphic discs at $x \in X$. If $\phi : X \rightarrow \mathbb{C}$ satisfies the following:*

- $\phi \in C^\infty(x)$.
- ϕ is holomorphic along the leaves.

Then ϕ is holomorphic on X .

On the other hand, the condition that $\phi(\mu v)$ is holomorphic in μ can be viewed in terms of vector fields as: $\bar{S}\phi = 0$, where

$$S = \sum_{j=1}^k z_j \frac{\partial}{\partial z_j}.$$

Remark: ϕ is holomorphic along the integral curves of $S \iff \bar{S}\phi = 0$.

At this stage, it is natural to ask whether or not S can be replaced by other vector fields.

In 2009, K.-T. Kim, E. A. Poletsky, and G. Schmalz proved a generalization for diagonal vector fields when the ratios of eigenvalues are positive (Forelli's case is when all eigenvalues are 1) [6]. The idea of the proof is similar in the sense that it depends on the convergence of the formal power series.

In 2016, J.-C. Joo, K.-T. Kim, and G. Schmalz succeeded in proving that the conclusion holds for all contracting vector fields under the same condition that the ratios of eigenvalues are positive [4]. The idea of the proof is different from the previous results, it is difficult to prove the convergence as the previous results. Instead, the new approach is to use the unique continuation property to show the holomorphicity of the function. The main goal of this thesis is to review [4] extensively and to apply the idea of the proof to an example of a non-contracting vector field.

1.2 Organization

Chapter 2 is devoted to introducing some basics that will be used throughout the next chapters.

Chapter 3 will demonstrate the main results, with a detailed proof of Theorem 3.3 and a simplified proof based on its idea to fit Theorem 3.2.

Chapter 4 is intended to provide a proof of a non-contracting case where we develop the scheme in Theorem 3.3 to fit the case.

CHAPTER 2

PRELIMINARIES

2.1 Holomorphic Functions

2.1.1 Properties of Functions of Several Complex Variables

In this section, we expand the discussion of the properties of functions of several complex variables.

Theorem 2.1 (Cauchy's integral formula)

Let $w = (w_1, \dots, w_n) \in \mathbb{C}^n$. Let $D^n(w, A) := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i - w_i| < A, i = 1, 2, 3, \dots, n\}$ and let $X \subseteq \mathbb{C}^n$ be an open set such that $\bar{D}^n(w, A) \subseteq X$. If a function $\phi : X \rightarrow \mathbb{C}$ is holomorphic, then

$$\phi(z) = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1 - w_1| = A} \cdots \oint_{|\zeta_n - w_n| = A} \frac{\phi(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_n \cdots d\zeta_1$$

for every $z \in D^n(w, A)$.

This is an example of a result that is extended from the one variable case. On the other hand, we will see some results that fail in higher dimensions.

In Chapter 1, we mentioned that the Riemann mapping theorem fails in higher dimensions since Δ^2 and B^2 are not biholomorphic. This counter-example can be extended in general on $\mathbb{C}^n$ ($n \geq 2$).

Theorem 2.2 *The sets Δ^n and B^n are not biholomorphic for $n \geq 2$.*

Moreover, the set of biholomorphic equivalence classes of simply connected domains in $\mathbb{C}^n$ is uncountable.

According to Picard's theorem in one complex variable, an entire function is either constant or assumes all points on the complex plane except at most one point. As a consequence, if a proper subset $U \subset \mathbb{C}$ such that $\mathbb{C} - U$ contains a nonempty open subset. Then U is not biholomorphic to $\mathbb{C}$. The statement is not true in higher dimensions.

Theorem 2.3 *There is a proper subset $U \subset \mathbb{C}^2$ such that $\mathbb{C}^2 - U$ contains a nonempty open subset with U is biholomorphic to $\mathbb{C}^2$*

2.1.2 Unique Continuation Property

The unique continuation property (usually called analytic continuation), is a method to extend the definition of a holomorphic function to a larger domain.

Theorem 2.4 *For an open set $S \subseteq \mathbb{C}$, and $\phi : S \rightarrow \mathbb{C}$ being analytic. The set $A := \{z \in S : \phi(z) = 0\}$ is totally disconnected or ϕ is constant.*

Proof: If A is empty, then H is vacuously totally disconnected. Assume that H is nonempty, and let $z_0 \in A$. Then $\phi(z_0) = 0$. As ϕ is analytic, there is an $R > 0$ such that $\phi = \sum_{n=1}^{\infty} a_n(z - z_0)^n$ on $D(z_0, R)$. Since $\phi(z_0) = 0$, it follows that $a_0 = 0$. Let $\phi(z) = (z - z_0)^m \psi(z)$, such that $\psi(z_0) \neq 0$. So $\psi(z) = \sum_{n=1}^{\infty} a_{n+m}(z - z_0)^n$. Since $\psi(z)$ is continuous, we can find $\delta > 0$ such that $|\psi(z) - a_m| < \epsilon$ whenever $|z - z_0| < \delta$. Taking $\epsilon = \frac{|a_m|}{2}$ implies that $|\psi(z) - a_m| < \frac{|a_m|}{2}$ and hence $\psi(z) \neq 0$ on $D(z_0, \delta)$. As a consequence, $\phi \neq 0$ on $D(z_0, \delta)$. QED

Theorem 2.5 *Let ϕ be an analytic function on an open nonempty set S in $\mathbb{C}$ and let ψ_1 and ψ_2 be two analytic functions on a connected open set S' containing S such that $\phi = \psi_1 = \psi_2$ on S . Then $\psi_1 = \psi_2$ on S' .*

Proof: Let $h(z) = \psi_2(z) - \psi_1(z)$. Then $h(z) \equiv 0$ on S . By Theorem 2.4 and the fact that S cannot be totally disconnected, being a part of a connected set S' . We conclude that $h(z) \equiv 0$ on S' as it is constant and vanishes somewhere. QED

That is, an analytic function has a unique extension. In particular, if an analytic function vanishes on a nonempty open connected set S , then it must be identically zero on any open set containing S .

The result can be extended further, if two analytic functions agree on a ray L , then they are equal on any open connected set containing L .

2.2 Vector Fields

For a general manifold M , a vector field can be described as an assignment to a vector for every point in M . In our context, we are interested in the case where $M = \mathbb{C}^n$.

Definition: Let $G \subseteq \mathbb{C}^n$ be a domain. A *vector field* is a vector-valued function $f : G \rightarrow \mathbb{C}^n$.

A standard way of representing a general vector field S is

$$S = \sum_{j=1}^n f_j(z) \frac{\partial}{\partial z_j}.$$

If each function f_j is holomorphic, we say that S is a holomorphic vector field.

Definition: Let $S = \sum_{j=1}^n f_j(z) \frac{\partial}{\partial z_j}$ be a vector field in $\mathbb{C}^n$. A *complex integral curve* of S is a solution to the system

$$z'_j(\mu) = f_j(z(\mu)), \mu \in \mathbb{C}$$

for $j = 1, 2, 3, \dots, n$.

2.2.1 Contracting Vector Fields

In this section, we introduce the normal form of contracting vector fields proved by Poincaré and Dulac. For more details, the reader is invited to read [10].

Definition: Let $S = \sum_{j=1}^n h_j(z) \frac{\partial}{\partial z_j}$ be a holomorphic vector field and let $h(z) = (h_1(z), \dots, h_n(z))$. If

- $h_j(0) = 0$ for $j = 1, 2, \dots, n$.
- All eigenvalues of $Jh(0)$ (the Jacobian matrix of h) have positive real parts.

Then we say that S is *contracting at the origin*.

Based on the theory of vector fields, all contracting vector fields can be represented by the Poincaré-Dulac normal form. We give a brief introduction to the Poincaré-Dulac theorem.

Definition: Let $\kappa_1, \dots, \kappa_n$ be the eigenvalues of a matrix T . The eigenvalues are said to be *resonant* if there is some $j \in \{1, 2, 3, \dots, n\}$ satisfying $\kappa_j = \sum_{t=1}^n m_t \kappa_t$ for some integers $m_1, \dots, m_n \geq 0$ with $m_1 + \dots + m_n \geq 2$. Such a relation is called a resonance.

Lemma 2.1 *Let ρ be a non-trivial vector valued polynomial such that $\rho(0) = \rho'(0) = 0$. Then the equation $z' = Tz$ can be transformed into $\zeta' = T\zeta + v(\zeta) + R$ by the change of variables $\zeta = z + \rho(z)$, where $v(\zeta) = \frac{\partial \rho}{\partial \zeta} T\zeta - T\rho(\zeta)$ and R denotes the terms of higher orders.*

Definition: Let $L_T \rho := \frac{\partial \rho}{\partial z} Tz - T\rho(z)$. The homological equation associated with the linear operator T is given by the equation

$$L_T \rho = v,$$

where ρ is the unknown vector field and v is the known vector field.

Lemma 2.2 *If the eigenvalues of T are non-resonant, then $L_T \rho = v$ is solvable in the class of formal power series for every vector field v with no constant term or a linear part at the origin. Moreover, if there is no resonance of order k , then $L_T \rho = v$ is solvable for any homogeneous polynomial vector field of degree k in the class of homogeneous polynomial vector fields of degree k .*

Theorem 2.6 *Let $\zeta' = T\zeta + v_r(\zeta) + R(\zeta)$, where $R(\zeta)$ represents the terms of higher order. If the eigenvalues of T are non-resonant, then the equation can be reduced to the form $y' = Ty$.*

Proof: By Lemma 2.2, the equation $L_T \rho_r = v_r$ is solvable for each $r \geq 2$. By Lemma 2.1 and the substitution $y = \zeta - \rho_r(\zeta)$, we get $\zeta' = y' + \frac{\partial \rho_r}{\partial \zeta} \zeta'$, which implies that

$$\begin{aligned} y' &= -\frac{\partial \rho_r}{\partial \zeta} \zeta' + \zeta' \\ &= -\frac{\partial \rho_r}{\partial \zeta} \zeta' + T\zeta + v_r(\zeta) + R(\zeta) \\ &= -\frac{\partial \rho_r}{\partial \zeta} (T\zeta + v_r(\zeta) + R(\zeta)) + T\zeta + v_r(\zeta) + R(\zeta) \\ &= -\frac{\partial \rho_r}{\partial \zeta} T\zeta + T(y + \rho_r(\zeta)) + v_r(\zeta) + R_1(\zeta), \end{aligned}$$

where $R_1(\zeta) = R(\zeta) - \frac{\partial \rho_r}{\partial \zeta} v_r(\zeta) - \frac{\partial \rho_r}{\partial \zeta} R(\zeta)$, which are all of order more than r . Finally, we get

$$y' = Ty - L_T \rho_r(\zeta) + v_r(\zeta) + R_1(\zeta)$$

$$= Ty + R_1(\zeta)$$

$$= Ty + R_2(y).$$

This annihilates terms of degree r as $R_2(y)$ contains only terms of higher orders. Successive substitutions in the same way annihilate the terms of orders $2, 3, \dots$, so the remaining part is the linear part. QED

This is the case when eigenvalues are non-resonant. Next, we describe the general case when we possibly have resonant relations.

Let $\kappa_1, \dots, \kappa_n$ be resonant. Let e_f be an element of the eigenbasis corresponding to the eigenvalues $\kappa_1, \dots, \kappa_n$. In the coordinates x_i with respect to the basis e_i , we define $x^m = x_1^{m_1} \cdots x_n^{m_n}$ to be a monomial in terms of the coordinates x_i .

Definition: The monomial $x^m e_f$ is said to be *resonant* if $\kappa_f = \sum_{t=1}^n m_t \kappa_t$ for some integers $m_1, \dots, m_n \geq 0$ with $m_1 + \cdots + m_n \geq 2$.

Definition: Let $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{C}^n$. The *convex hull* of κ is the intersection of all convex sets containing the points $\kappa_1, \dots, \kappa_n$ in the complex plane.

Definition: $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{C}^n$ belongs to the *Poincaré domain* if the convex hull of $\kappa_1, \dots, \kappa_n$ in the complex plane does not contain the origin.

Now we are ready to introduce the Poincaré-Dulac theorem.

Theorem 2.7 *If the eigenvalues of a holomorphic vector field belong to the Poincaré domain at a singular point. Then on a neighborhood of that point, the field is equivalent to a polynomial vector field by a biholomorphic change of variables. Moreover, The monomials of degree greater than one are resonant.*

It can be shown that when the eigenvalues are in the Poincaré domain, the number of resonant terms is finite. The non-resonant terms can be annihilated as in Theorem 2.6. The main idea relies on these two facts, the rest is finding the suitable change of variables.

Since the eigenvalues of $Jh(0)$ have positive real parts for a contracting vector field, it follows that the convex hull of the eigenvalues cannot contain the origin and hence it satisfies the hypothesis of Poincaré-Dulac theorem. Therefore, every contracting vector field can take the following form on a neighborhood of the origin:

Definition: Let S be a vector field that is contracting at the origin, then S can be transformed by a biholomorphic change of coordinates near zero into the following form, called the *Poincaré-Dulac normal form* [10]:

$$S = \sum_{j=1}^n (\kappa_j z_j + g_j(z)) \frac{\partial}{\partial z_j},$$

such that each $g_j(z)$ is a polynomial satisfying $g_j(0) = 0$ depending only on $z_1, \dots, z_{j-1}$.

A special class of contracting vector fields that will be considered in our discussion is the aligned type. Therefore, it would be appropriate to give a definition.

Definition: Let S be a contracting vector field at the origin. Then we call S an *aligned vector field* if the ratios of eigenvalues are positive.

Remark: Since $\frac{\kappa_i}{\kappa_j} > 0$, we can write $\kappa_j = r_j e^{i\theta}$ for some $\theta \in \mathbb{R}$ and $r_j > 0$, for $j = 1, \dots, n$. Therefore, taking $e^{i\theta}$ as a common factor, we may assume that $\kappa_k > 0$ for

every $k = 1, \dots, n$.

CHAPTER 3

MAIN RESULTS

We begin with a discussion of a formal power series A that satisfies $\bar{S}A = 0$, where S is an aligned vector field. We use standard multi-index symbols, i.e, for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n} \text{ and } |\alpha| := \alpha_1 + \cdots + \alpha_n.$$

In [4], we have the following result that will be used in the proofs of Theorem 3.2 and Theorem 3.3:

Proposition 3.1 ([4]) *Let $S = \sum_{j=1}^n (\kappa_j z_j + g_j(z)) \frac{\partial}{\partial z_j}$ be a contracting vector field. If $A = \sum A_{\alpha,\beta} z^\alpha \bar{z}^\beta$ is a formal power series such that $\bar{S}A = 0$, then $A_{\alpha,\beta} = 0$ whenever $|\beta| > 0$.*

Proof: For $\beta_1, \dots, \beta_{n-1} = 0$ and $\beta_n \neq 0$, consider the term of degree $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$ in $\bar{S}A$. Let $W_t = \kappa_t z_t + g_t(z)$, then the components can be expressed as $\bar{W}_t \frac{\partial}{\partial \bar{z}_t}$. For $t < n$, the terms must contain $\bar{z}_t$ and the term $\bar{g}_n$ also contains $\bar{z}_t$ terms. Hence the coefficient of the desired term is $\kappa_n \beta_n A_{\alpha,\beta}$, which is zero if and only $A_{\alpha,\beta} = 0$ as S is contracting and $\beta_n \neq 0$.

We complete the proof by induction with respect to the lexicographical ordering defined by

$$\beta < \beta' \iff \exists t : \beta_k = \beta'_k \forall k < t \text{ and } \beta_t < \beta'_t.$$

The base step $(0, \dots, \beta_n)$ is done. Assume that $A_{\alpha, \beta} = 0$ for all $\beta < \beta'$, we show that $A_{\alpha, \beta'} = 0$.

Consider the term of degree $(\alpha_1, \dots, \alpha_n, \beta'_1, \dots, \beta'_n)$. Assume that $\beta_t \neq 0$ and $\beta_k = 0$ for $k < t$. The g_k term in $\bar{W}_k \frac{\partial}{\partial \bar{z}_k}$ with $k \geq t$ has terms that contain $\bar{z}_k$ with $k < t$ or increase the order in $\bar{z}_t, \dots, \bar{z}_n$. In both cases, they cannot produce terms with degree $(\alpha_1, \dots, \alpha_n, \beta'_1, \dots, \beta'_n)$. Therefore, the required coefficient is given by

$$(\kappa_t \beta'_t + \dots + \kappa_n \beta'_n) A_{\alpha, \beta'}.$$

Again, this coefficient vanishes if and only if $A_{\alpha, \beta'} = 0$ as S is contracting. QED

In general, this result is not sufficient to conclude about the holomorphicity of a smooth function ϕ from the fact that its formal power series does contain anti-holomorphic terms. However, if we assume that ϕ is real-analytic, then it has a power series representation and we can conclude about the function itself.

Theorem 3.1 ([4]) *If $\phi : \mathbb{B}^n \rightarrow \mathbb{C}$ is real-analytic and $\bar{S}\phi = 0$ at every point for a contracting vector field S , then ϕ is holomorphic.*

3.1 Diagonal Vector Fields

Definition: Let S be a vector field such that any function $\phi : \mathbb{B}^n \rightarrow \mathbb{C}$ that is of $C^\infty(0)$ and satisfying $\bar{S}\phi = 0$ is holomorphic around the origin. Then we will say that S is a *Forelli vector field*.

Theorem 3.2 ([6])

Let $S = \sum_{j=1}^n \kappa_j z_j \frac{\partial}{\partial z_j}$, where $\kappa_1, \dots, \kappa_n$ are complex numbers satisfying $\frac{\kappa_i}{\kappa_j} > 0$.

Then S is a Forelli vector field.

The original approach proves the convergence of the power series. However, we will present here a simplified proof of Theorem 3.3 to fit our case to compare the complexity of domains in both cases.

Let $\mathbb{H} := \{ \mu \in \mathbb{C} : \operatorname{Re}(\mu) > 0 \}$. We will call a sequence $(\mu_k) \subseteq \mathbb{H}$ an *admissible sequence* if

$$\lim_{k \rightarrow \infty} e^{-\kappa_j \operatorname{Re}(\mu_k)} = 0$$

for every $j = 1, 2, 3, \dots, n$. It is clear that a sequence is admissible if and only if $\operatorname{Re}(\mu_k) \rightarrow \infty$.

Proposition 3.2 Let ϕ be bounded and holomorphic on $\mathbb{H}$ and let $\kappa_1, \dots, \kappa_n$ be positive real numbers. If

$$\lim_{k \rightarrow \infty} |\phi(\mu_k)| e^{\kappa_j \operatorname{Re}(\mu_k)} = 0$$

for $j = 1, \dots, n$, an integer $l \geq 0$, and an admissible sequence (μ_k) . Then $\phi \equiv 0$ on $\mathbb{H}$.

Proof: If the statement is proved for any fixed κ_j , it is easy to see that the argument is valid for others, so let $\psi_l(\mu) := \phi(\mu)e^{\kappa_1 l(\mu)}$.

Claim 1: ψ_l is bounded on $\mathbb{H}$ for every l .

Assume to the contrary that $|\psi_l(\mu_k)| \rightarrow \infty$ for some integer $l \geq 0$ and a sequence $(\mu_k) \subseteq \mathbb{H}$. Since ϕ is bounded, it follows that $e^{\kappa_1 l \operatorname{Re}(\mu_k)} \rightarrow \infty$, so $e^{-\kappa_1 l \operatorname{Re}(\mu_k)} \rightarrow 0$ and hence (μ_k) is admissible. A contradiction as $|\psi_l(\mu_k)| \rightarrow 0$ for any admissible sequence.

Claim 2: There is a constant $\delta_0 > 0$ such that $e^{-\kappa_1 l \operatorname{Re}(\mu_k)} > \delta_0$ on $\partial\mathbb{H}$.

This is obvious as $\partial\mathbb{H} = \{\mu \in \mathbb{C} : \operatorname{Re}(\mu) = 0\}$.

Let

$$M := \sup_{\mu \in \mathbb{H}} |\phi(\mu)|,$$

and

$$C := \frac{1}{\delta_0}.$$

Then

$$|\psi_l(\mu)| \leq MC^l$$

for $\mu \in \partial\mathbb{H}$. As each ψ_l is bounded and holomorphic on $\mathbb{H}$, by the maximum modulus principle for unbounded domains (known as Phragmén–Lindelöf principle), we have

$$|\psi_l(\mu)| \leq MC^l$$

on $\mathbb{H}$. Let Γ be a connected open subset of $\mathbb{H}$ such that $\Gamma \subseteq \{ \mu \in \mathbb{H} : e^{-\kappa_1 \operatorname{Re}(\mu_k)} < \frac{1}{2C} \}$.

Hence we have

$$|\phi(\mu)| \leq \frac{M}{2^l} \rightarrow 0$$

on Γ . Finally, we have $\phi(\mu) = 0$ on Γ and the unique continuation property implies that $\phi \equiv 0$ on $\mathbb{H}$. QED

Proof of Theorem 3.2: Assume that all eigenvalues are negative and take the form $-\kappa_j$, where $\kappa_j > 0$. Let V be an open neighborhood around the origin that is sufficiently small such that $\phi \in C^2$. We consider the complex flow curve that passes through $\beta = (\beta_1, \dots, \beta_n)$.

The flow curve is given by

$$\mu \rightarrow z(\beta, \mu),$$

such that

$$z_j(\beta, \mu) = \beta_j e^{-\kappa_j \mu}.$$

Since S is contracting, we can find an open neighborhood of the origin $V' \subseteq V$ such that for any $\beta \in V'$ we have the image of $z(\beta, t)$, $t \geq 0$ is contained in V .

Let $\psi(\beta, \mu) := \phi(z(\beta, \mu))$. By hypothesis, ψ is holomorphic in μ in the region $\mathbb{H}$. Let $\beta_0 = (\beta_1, \dots, \beta_n) \in V' - \{0\}$ be fixed. Since $\phi \in C^2(V)$, we get

$$\frac{\partial}{\partial \bar{\mu}} \left(\frac{\partial \psi}{\partial \bar{\beta}_j} \right) = \frac{\partial}{\partial \bar{\beta}_j} \left(\frac{\partial \psi}{\partial \bar{\mu}} \right) = 0$$

for each $j = 1, \dots, n$. Therefore, $\frac{\partial \psi}{\partial \bar{\beta}_j}$ is a bounded holomorphic function in the variable

μ on $\mathbb{H}$.

By the chain rule, we have

$$\frac{\partial \psi}{\partial \beta_n} = e^{-\kappa_n \bar{\mu}} \frac{\partial \phi}{\partial \bar{z}_n}.$$

The formal power series of ϕ contains no $\bar{z}$ terms by Proposition 3.1. Therefore, the function $\frac{\partial \phi}{\partial \bar{z}_n}$ is $C^\infty(0)$ with a trivial formal power series. Thus,

$$\left| \frac{\partial \phi}{\partial \bar{z}_n}(z) \right| |z_j^{-l}| \rightarrow 0$$

as $z \rightarrow 0$ for every $l \geq 0$ and $j = 1, \dots, n$. Since $e^{-\kappa_n \operatorname{Re}(\mu)}$ is bounded on $\mathbb{H}$ by some $C > 0$. We have

$$\left| \frac{\partial \psi}{\partial \beta_n}(\mu_k) \right| |z_j(\beta_0, \mu_k)^{-l}| \leq C \left| \frac{\partial \phi}{\partial \bar{z}_n}(z(\beta_0, \mu_k)) \right| |z_j(\beta_0, \mu_k)^{-l}| \rightarrow 0$$

for any admissible sequence (μ_k) in $\mathbb{H}$. Hence the function $\frac{\partial \psi}{\partial \beta_n}$ satisfies the hypothesis of Proposition 3.2 and so $0 = \frac{\partial \psi}{\partial \beta_n} = e^{-\kappa_n \bar{\mu}} \frac{\partial \phi}{\partial \bar{z}_n}$, and so $\frac{\partial \phi}{\partial \bar{z}_n} = 0$ over the flow $\{z(\beta_0, \mu) : \mu \in \mathbb{H}\}$. Repeating the process for $z_{n-1}, \dots, z_1$ yields that $\bar{\partial} \phi = 0$. Since β_0 is arbitrary on V' , it follows that ϕ is holomorphic on V' . QED

Conversely, if $\frac{\kappa_i}{\kappa_j}$ is not a positive real number, the unique continuation property fails and a counter-example can always be constructed.

Example: Let $\kappa_1 = 1 + i$ and $\kappa_2 = 1 - i$. Let $S = \kappa_1 z_1 \frac{\partial}{\partial z_1} + \kappa_2 z_2 \frac{\partial}{\partial z_2}$. S is contracting as $\operatorname{Re}(\kappa_1) = \operatorname{Re}(\kappa_2) = 1 > 0$, but not aligned. We will construct a

counter-example for which the unique continuation property fails.

The flow is given by $(C_1 e^{\kappa_1 \mu}, C_2 e^{\kappa_2 \mu})$.

Let $D(\kappa_1, \kappa_2) = \{\mu \in \mathbb{C} : |e^{-\kappa_1 \mu}| < 1, |e^{-\kappa_2 \mu}| < 1\}$. For $\mu = x + iy$, we have

$\operatorname{Re}(\kappa_1) = x - y$ and $\operatorname{Re}(\kappa_2) = x + y$.

So we have $D(\kappa_1, \kappa_2) = \{\mu \in \mathbb{C} : e^{-(x-y)} < 1, e^{-(x+y)} < 1\} = \{\mu \in \mathbb{C} : x - y >$

$0, x + y > 0\}$. A sequence (μ_j) is admissible $\iff x_j - y_j \rightarrow \infty, x_j + y_j \rightarrow \infty$.

Consider the function $\phi(\mu) = e^{-\mu^2} = e^{-(x^2 - y^2) - 2ixy}$. So for an admissible sequence (μ_j) we have

$$e^{l(x_j - y_j)} |\phi(\mu_j)| = e^{-(x_j^2 - y_j^2) + l(x_j - y_j)} = e^{(x_j - y_j)(l - (x_j + y_j))} \rightarrow 0,$$

and similarly

$$e^{l(x_j + y_j)} |\phi(\mu_j)| = e^{-(x_j^2 - y_j^2) + l(x_j + y_j)} = e^{(x_j + y_j)(l - (x_j - y_j))} \rightarrow 0.$$

However $\phi(\mu) \not\equiv 0$ and hence the unique continuation fails.

3.2 Contracting Vector Fields

Theorem 3.3 ([4])

Let S be an aligned contracting vector field. Then S is a Forelli vector field.

While the region that is considered in the previous theorem is just a half plane, the case here is more complicated as we have polynomials with possible singularities when

defining the region without the appropriate restrictions on the domain. We will first set up the domains carefully depending on the behavior of the polynomials inside the region. Let $P_1, \dots, P_n$ be nontrivial polynomials in the single variable μ . Let $\kappa_1, \dots, \kappa_n$ be positive real numbers.

Consider the region

$$D(P, \kappa) := \{ \mu \in \mathbb{C} : |P_j(\mu)|e^{-\kappa_j \operatorname{Re}(\mu)} < 1, j = 1, 2, 3, \dots, n \} .$$

A sequence (μ_k) will be called *admissible* if

$$\lim_{k \rightarrow \infty} |P_j(\mu_k)e^{-\kappa_j \mu_k}| = 0$$

for $j = 1, \dots, n$.

Let $E_r := \{ \mu \in \mathbb{R} : \mu > r \}$. For a given $D(P, \kappa)$, we can find a sufficiently large r such that $E_r \subseteq D(P, \kappa)$. Let $D^*(P, \kappa)$ be the connected component that contains E_r .

Proposition 3.3 ([4])

If ϕ is a bounded holomorphic function on $D(P, \kappa)$ and

$$\lim_{k \rightarrow \infty} \left| \frac{\phi(\mu_k)e^{\kappa_j l \mu_k}}{P_j^l(\mu_k)} \right| = 0$$

for every nonnegative integer l and for any admissible sequence whose members are different from zeros of $P_1, \dots, P_n$, then $\phi \equiv 0$ on $D^(P, \kappa)$.*

Proof: We may assume that ϕ is continuous on the boundary of $D(P, \kappa)$ by multiplying each polynomial P_j with $C > 1$ to extend the region.

There are $\epsilon > 0$ and $A > 0$ such that $|P_j| > \epsilon$ for all $\mu \in \mathbb{C}$ with $\text{Re}(\mu) > A$.

Let $D(P, \kappa, A) := D(P, \kappa) \cap \{ \mu \in \mathbb{C} : \text{Re}(\mu) > A \}$. By re-ordering, we may write

$$\frac{\deg(P_1)}{\kappa_1} \geq \frac{\deg(P_j)}{\kappa_j}$$

for all $j = 1, 2, 3, \dots, n$. For sufficiently large μ , there are constants C_j, C'_j such that for $d_j = \deg(P_j)$ we have

$$C_j |\mu|^{d_j} \leq |P_j(\mu)| \leq C'_j |\mu|^{d_j},$$

so we have

$$(|P_1| e^{-\kappa_1 \text{Re}(\mu)})^{\frac{\kappa_j}{\kappa_1}} \geq C_1^{\frac{\kappa_j}{\kappa_1}} |\mu|^{d_1 \frac{\kappa_j}{\kappa_1}} e^{-\kappa_j \text{Re}(\mu)} \geq C_1^{\frac{\kappa_j}{\kappa_1}} \frac{1}{C'_j} |P_j(\mu)| e^{-\kappa_j \text{Re}(\mu)}.$$

For an admissible sequence $(\mu_k) \subseteq D(P, \kappa, A)$, we have $\text{Re}(\mu_k) \rightarrow \infty$ since $|P_j| > \epsilon$.

This means that whenever

$$|P_1(\mu_k)| e^{-\kappa_1 \text{Re}(\mu_k)} \rightarrow 0, \text{ we have } |P_j(\mu_k)| e^{-\kappa_j \text{Re}(\mu_k)} \rightarrow 0 \text{ for } j = 1, \dots, n.$$

Let $\psi_l(\mu) := \phi(\mu) \frac{e^{\kappa_1 l \mu}}{\bar{P}_1(\mu)^l}$.

Then ψ_l is holomorphic on $D(P, \kappa, A)$.

Claim 1: ψ_l is bounded on $D(P, \kappa, A)$ for every integer $l \geq 0$.

Assume to the contrary that there is an admissible sequence (μ_k) in $D(P, \kappa, A)$ and

$l \geq 0$ such that $\psi_l(\mu_k) \rightarrow \infty$ as $k \rightarrow \infty$. Then as ϕ is bounded, we have

$$\lim_{k \rightarrow \infty} \frac{e^{\kappa_1 \operatorname{Re}(\mu_k)}}{|P_1(\mu_k)|} = \infty,$$

which means

$$\lim_{k \rightarrow \infty} |P_1(\mu_k)| e^{-\kappa_1 \operatorname{Re}(\mu_k)} = 0.$$

Therefore, (μ_k) is admissible and so by hypothesis we have $|\psi_l(\mu_k)| \rightarrow 0$. A contradiction.

Claim 2: There is $\delta_0 > 0$ such that $|P_1(\mu_k)| e^{-\kappa_1 \operatorname{Re}(\mu_k)} > \delta_0$ for any $\mu \in \partial D(P, \kappa) \cap \{ \mu \in \mathbb{C} : \operatorname{Re}(\mu) \geq A \}$.

If $\lim_{k \rightarrow \infty} |P_1(\mu_k)| e^{-\kappa_1 \operatorname{Re}(\mu_k)} = 0$ for some admissible sequence $(\mu_k) \subseteq \partial D(P, \kappa) \cap \{ \mu \in \mathbb{C} : \operatorname{Re}(\mu) \geq A \}$. Then as $P_j(\mu)$ has no zeros whenever $\operatorname{Re}(\mu) > A$, we conclude that $\operatorname{Re}(\mu_k) \rightarrow \infty$. So we have

$$\lim_{k \rightarrow \infty} |P_j(\mu_k)| e^{-\kappa_j \operatorname{Re}(\mu_k)} = 0$$

for $j = 1, \dots, n$. However, by definition of $D(P, \kappa)$, we have

$$|P_j(\mu_k)| e^{-\kappa_j \operatorname{Re}(\mu_k)} = 1$$

for some j since $\mu_k \in \partial D(P, \kappa)$. A contradiction.

Now we have

$$|P_1(\mu_k)| e^{-\kappa_1 \operatorname{Re}(\mu_k)} \geq \epsilon e^{-\kappa_1 A}$$

for each $\mu \in D(P, \kappa_1) \cap \{ \mu \in \mathbb{C} : \operatorname{Re}(\mu) = A \}$.

Let

$$M := \sup_{\mu \in D(P, \kappa)} |\phi(\mu)|,$$

and

$$C := \max\left\{\frac{1}{\delta_0}, \epsilon^{-1} e^{\kappa_1 A}\right\} .$$

Then

$$|\psi_l(\mu)| \leq MC^l$$

for $\mu \in \partial D(P, \kappa, A)$. As each ψ_l is bounded and holomorphic on $D(P, \kappa, A)$, by maximum modulus principle for unbounded domains, we have

$$|\psi_l(\mu)| \leq MC^l$$

on $D(P, \kappa, A)$.

Let Γ be a connected open set such that:

- $\Gamma \subseteq \{ \mu \in D(P, \kappa, A) : |P_1(\mu_k)| e^{-\kappa_1 \operatorname{Re}(\mu_k)} < \frac{1}{2C} \}$.
- $E_B := \{ \mu \in \mathbb{R} : \mu > B \} \subseteq \Gamma$ for some $B > A$.

Thus, for every $\mu \in \Gamma$, we have

$$|\phi(\mu)| \leq \frac{M}{2^l} \rightarrow 0.$$

Hence the unique continuation property implies that $\phi \equiv 0$ on $D^*(P, \kappa)$. QED

Proof of Theorem 3.3: Let V be an open neighborhood around the origin that is sufficiently small such that $\phi \in C^2$.

For a contracting aligned vector field, we have

$$S = \sum_{j=1}^n (\kappa_j z_j + g_j(z)) \frac{\partial}{\partial z_j},$$

we may assume without loss of generality that $\kappa_j > 0$ for every $j = 1, \dots, n$. With the following:

- $0 < \kappa_1 < \dots < \kappa_n$
- $g_j(e^{-\kappa_1 \mu}, \dots, e^{-\kappa_{j-1} \mu}) = e^{-\kappa_{j-1} \mu} g_j(z_1, \dots, z_{j-1})$

For every $j = 1, \dots, n$. The complex flow curve that passes through $\beta = (\beta_1, \dots, \beta_n)$ can be written as

$$\mu \rightarrow z(\beta, \mu) = (z_1(\beta, \mu), \dots, z_n(\beta, \mu))$$

such that

$$z_j(\beta, \mu) = e^{-\kappa_j \mu} (\beta_j + h_j(\beta, \mu))$$

where h_j is a holomorphic polynomial in the variables β, μ and for every $j = 1, \dots, n$

- $h_1 \equiv 0$.
- $h_j(\beta, 0) = 0$.
- $h_j(\cdot, \mu)$ depends only on $\beta_1, \dots, \beta_{j-1}$.

Since S is contracting, we can find an open neighborhood of the origin $V' \subseteq V$ such that for any $\beta \in V'$ we have the image of $z(\beta, t)$, $t \geq 0$ is contained in V .

Let $\psi(\beta, \mu) := \phi(z(\beta, \mu))$. By hypothesis, ψ is holomorphic in μ in the region $D_\beta := D(P_\beta, \kappa)$. Where

$$P_\beta(\cdot) = (\beta_1 + g_1(\beta, \cdot), \dots, \beta_n + g_n(\beta, \cdot)).$$

Let $\beta_0 = (\beta_1, \dots, \beta_n) \in V' - \{0\}$ be fixed. Since $\phi \in C^2(V)$, we get

$$\frac{\partial}{\partial \bar{\mu}} \left(\frac{\partial \psi}{\partial \beta_j} \right) = \frac{\partial}{\partial \beta_j} \left(\frac{\partial \psi}{\partial \bar{\mu}} \right) = 0$$

for each $j = 1, \dots, n$. Therefore, $\frac{\partial \psi}{\partial \beta_j}$ is a bounded holomorphic function in the variable μ on D_{β_0} .

By the chain rule, we have

$$\frac{\partial \psi}{\partial \bar{\beta}_n} = e^{-\kappa_n \bar{\mu}} \frac{\partial \phi}{\partial \bar{z}_n}.$$

The formal power series of ϕ contains no $\bar{z}$ terms by Proposition 3.1. Therefore, the function $\frac{\partial \phi}{\partial \bar{z}_n}$ is $C^\infty(0)$ with a trivial formal power series. Thus,

$$\left| \frac{\partial \phi}{\partial \bar{z}_n}(z) \right| |z_j^{-l}| \rightarrow 0$$

as $z \rightarrow 0$ for every $l \geq 0$ and $j = 1, \dots, n$. Since $e^{-\kappa_n \text{Re}(\mu)}$ is bounded on D_{β_0} by some $C > 0$. We have

$$\left| \frac{\partial \psi}{\partial \bar{\beta}_n}(\mu_k) \right| |z_j(\beta_0, \mu_k)^{-l}| \leq C \left| \frac{\partial \phi}{\partial \bar{z}_n}(z(\beta_0, \mu_k)) \right| |z_j(\beta_0, \mu_k)^{-l}| \rightarrow 0$$

for any admissible sequence (μ_k) in D_{β_0} . Hence the function $\frac{\partial\psi}{\partial\beta_n}$ satisfies the hypothesis of Proposition 3.3 and so $0 = \frac{\partial\psi}{\partial\beta_n} = e^{-\kappa_n\bar{\mu}} \frac{\partial\phi}{\partial\bar{z}_n}$, and so $\frac{\partial\phi}{\partial\bar{z}_n} = 0$ over the flow $\{z(\beta_0, \mu) : \mu \in D_{\beta_0}^*\}$, where $D_{\beta_0}^*$ is the unbounded component of D_{β_0} that contains the positive real line. Repeating the process for $z_{n-1}, \dots, z_1$ yields that $\bar{\partial}\phi = 0$. Since β_0 is arbitrary on V' , it follows that F is holomorphic on V' . QED

The unique continuation property fails in the diagonal case when the alignedness condition is dropped and so it is meaningless to ask about the general contracting case.

To summarize, a contracting holomorphic vector field S is a Forelli vector field if and only if S is aligned. This gives a full classification of contracting vector fields.

CHAPTER 4

AN EXAMPLE OF NON-CONTRACTING VECTOR FIELD CASE

4.1 Non-Contracting Case

If the vector field is not contracting, we do not have a nice form such as the Poincaré-Dulac normal form and we have no results in this direction. Indeed, if S is non-contracting, it usually fails to be a Forelli vector field.

Example: Let $S = z \frac{\partial}{\partial z} - w \frac{\partial}{\partial w}$, then S is not contracting. The integral curves are $\sigma_1 = C_1 e^\mu$ and $\sigma_2 = C_2 e^{-\mu}$.

The function $\phi(z, w) = \bar{w}z$ is not holomorphic. However, $\bar{S}\phi = 0$ as $\bar{z}w = \bar{C}_1 \bar{C}_2 = \text{Constant}$.

The proof of Theorem 3.3 could be modified to fit some vector fields that are not

contracting. In the next section, we will try to simulate the main ideas to fit a very specific non-contracting vector field.

4.2 Special Case Proof

Let $S = z^2 \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}$. Then S is a non-contracting vector field.

In this section, we will show that S is indeed a Forelli vector field. First, we observe that the integral curves of S are:

$$\sigma_1 = \frac{1}{\mu + \kappa_1}$$

$$\sigma_2 = \kappa_2 e^{-\mu}.$$

We need to develop a formal power series analysis similar to Proposition 3.1.

Proposition 4.1 *Let $S = z^2 \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}$. If $A = \sum A_{\alpha, \beta, a, b} z^\alpha w^\beta \bar{z}^a \bar{w}^b$ is a formal power series such that $\bar{S}A = 0$, then $A_{\alpha, \beta, a, b} = 0$ whenever $a + b > 0$.*

Proof: The same argument in Proposition 3.1 is applicable here. If $a + b > 0$, consider the term of degree (α, β, a, b) in $\bar{S}A$. Its coefficient has to be $(a + b)A_{\alpha, \beta, a, b}$, which is zero if and only if $A_{\alpha, \beta, a, b} = 0$. QED

Theorem 4.1 *If $\phi : \mathbb{B}^2 \rightarrow \mathbb{C}$ is real analytic and $\bar{S}\phi = 0$ at every point for $S = z^2 \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}$, then ϕ is holomorphic.*

Next, we define the appropriate region that will be used throughout the proof:

$$\text{Let } D = \{\mu \in \mathbb{C} : \left| \frac{1}{\mu + \kappa_1} \right| < 1, |\kappa_2 e^{-\mu}| < 1\}.$$

Remark: $D = \{\mu \in \mathbb{C} : \left| \frac{1}{\mu + \kappa_1} \right| < 1\} \cap \{\mu \in \mathbb{C} : |\kappa_2 e^{-\mu}| < 1\} = (\mathbb{C} - \bar{D}(-\kappa_1, 1)) \cap \{\mu \in \mathbb{C} : \text{Re}(-\mu) > \ln(|\kappa_2|)\}$ if $\kappa_2 \neq 0$ and $D = \mathbb{C} - \bar{D}(-\kappa_1, 1)$ if $\kappa_2 = 0$. Geometrically, D is a half-plane minus a closed disc or the whole plane minus a closed disc. In both cases, D is connected.

Definition: Let (μ_k) be a sequence in D , we say that (μ_k) is an *admissible sequence* if $\lim_{k \rightarrow \infty} \left| \frac{1}{\mu_k + \kappa_1} \right| = \lim_{k \rightarrow \infty} |\kappa_2 e^{-\mu_k}| = 0$.

It is obvious that $\lim_{k \rightarrow \infty} \left| \frac{1}{\mu_k + \kappa_1} \right| = 0 \iff \mu_k \rightarrow \infty$ and $\lim_{k \rightarrow \infty} |\kappa_2 e^{-\mu_k}| = 0 \iff \text{Re}(\mu_k) \rightarrow \infty$. Therefore, a sequence is admissible if and only if $\text{Re}(\mu_k) \rightarrow \infty$.

In other words, $\kappa_2 e^{-\mu}$ determines the admissible sequences.

Proposition 4.2 *If ϕ is a bounded holomorphic function on D and*

$$\lim_{k \rightarrow \infty} \left| \frac{\phi(\mu_k)}{\sigma_1^l(\mu_k)} \right| = \lim_{k \rightarrow \infty} \left| \frac{\phi(\mu_k)}{\sigma_2^l(\mu_k)} \right| = 0$$

for any admissible sequence whose members are different from zeros of σ_1, σ_2 . Then $\phi \equiv 0$ on D .

Proof: The function $\psi_l = \kappa_2 \phi(\mu) e^{l\mu}$ is holomorphic on D .

Claim 1 : ψ_l is bounded for every integer l .

If ψ_l is unbounded, then there is an admissible sequence such that $\lim_{k \rightarrow \infty} |\kappa_2 e^{l\mu_k}| = \infty$, then $\lim_{k \rightarrow \infty} |e^{-l\mu_k}| = 0$ so (μ_k) is admissible and hence $\psi_l \rightarrow 0$. A contradiction.

Claim 2 : There is a constant $\delta_0 > 0$ such that $|\kappa_2|e^{-\operatorname{Re}(\mu)} > \delta_0$ on ∂D .

If such δ_0 does not exist, then there is a sequence $(\mu_k) \subseteq \partial D$ such that $|\kappa_2|e^{-\operatorname{Re}(\mu_k)} \rightarrow 0$ and so $\lim_{k \rightarrow \infty} \left| \frac{1}{\mu + \kappa_1} \right| = 0$ also. The definition of the boundary implies that at least one of the two limits is 1. A contradiction.

Let $M = \sup |\phi(\mu)|$ on D and $C = \frac{1}{\delta_0}$. Then $|\psi_l(\mu)| \leq MC^l$ on ∂D . By the maximum modulus principle of unbounded domains, the inequality holds on D .

Let Γ be a connected open set such that

$$\Gamma \subseteq \left\{ \mu \in D : |\kappa_2 e^{-\mu}| < \frac{1}{2c} \right\}.$$

Therefore, for every $\mu \in \Gamma$ we have $|\phi(\mu)| \leq \frac{M}{2^l} \rightarrow 0$ as $l \rightarrow \infty$ and hence $\phi \equiv 0$ on Γ .

Finally, by the unique continuation property, we conclude that $\phi \equiv 0$ on D .

QED

Now we are ready to complete the proof.

Let $g(\mu, \kappa) := \phi(\kappa_1 e^{-\mu}, \frac{1}{\mu + \kappa_2}) = \phi(w, z)$, then the flow is given by $\beta = (\beta_1, \beta_2) = (\kappa_1, \frac{1}{\kappa_2})$. So $g(\mu, \beta) := \phi(\beta_1 e^{-\mu}, \frac{1}{\mu + \frac{1}{\beta_2}}) = \phi(w, z)$.

For an open neighborhood V around 0 such that $\phi \in C^2$ and a fixed nonzero $\beta = (\beta_1, \beta_2) \in V$, we have

$$\frac{\partial}{\partial \bar{\mu}} \left(\frac{\partial g}{\partial \bar{\beta}} \right) = \frac{\partial}{\partial \bar{\beta}} \left(\frac{\partial g}{\partial \bar{\mu}} \right) = 0$$

$\implies \frac{\partial g}{\partial \bar{\beta}}$ is holomorphic in μ .

$$\frac{\partial g}{\partial \bar{\beta}_2} = \frac{\partial \phi}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \bar{\beta}_2} + \frac{\partial \phi}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \bar{\beta}_2} + \frac{\partial \phi}{\partial w} \frac{\partial w}{\partial \bar{\beta}_2} + \frac{\partial \phi}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial \bar{\beta}_2} = \frac{-\beta_2^2}{(\beta_2 \mu + 1)^2} \frac{\partial \phi}{\partial \bar{z}}.$$

On the region D , we have $\left| \frac{1}{\mu + \frac{1}{\beta_2}} \right| \leq C$ for some $C > 0$ and so

$$\left| \frac{\beta_2^2}{(\beta_2 \mu + 1)^2} \right| \leq C^2.$$

The formal power series of ϕ contains no $\bar{\zeta}$ terms by Proposition 4.1. There-

fore, the function $\frac{\partial\phi}{\partial\bar{z}}$ is $C^\infty(0)$ with a trivial formal power series. Thus,

$$\left| \frac{\partial\phi}{\partial\bar{z}}(\zeta) \right| |z^{-l}| \rightarrow 0$$

and

$$\left| \frac{\partial\phi}{\partial\bar{z}}(\zeta) \right| |w^{-l}| \rightarrow 0$$

as $\zeta \rightarrow 0$. So we have

$$\left| \frac{\partial g}{\partial\beta_1}(\mu_k) \right| |z(\kappa, \mu_k)^{-l}| \leq C^2 \left| \frac{\partial\phi}{\partial\bar{z}}(\zeta(\beta_0, \mu_k)) \right| |z(\kappa, \mu_k)^{-l}| \rightarrow 0,$$

and

$$\left| \frac{\partial g}{\partial\bar{\kappa}_1}(\mu_k) \right| |w(\kappa, \mu_k)^{-l}| \leq C^2 \left| \frac{\partial\phi}{\partial\bar{z}}(\zeta(\beta_0, \mu_k)) \right| |w(\kappa, \mu_k)^{-l}| \rightarrow 0.$$

Note that we can find $t > 0$ such that $z(\beta, t)$ is contained in V , and so on D , we apply Proposition 4.2 to get

$$\frac{\partial g}{\partial\bar{\kappa}_1} = \frac{\partial\phi}{\partial\bar{z}} = 0.$$

Finally, we have

$$0 = \frac{\partial g}{\partial\bar{\mu}} = \frac{\partial\phi}{\partial z} \frac{\partial z}{\partial\bar{\mu}} + \frac{\partial\phi}{\partial\bar{z}} \frac{\partial\bar{z}}{\partial\bar{\mu}} + \frac{\partial\phi}{\partial w} \frac{\partial w}{\partial\bar{\mu}} + \frac{\partial\phi}{\partial\bar{w}} \frac{\partial\bar{w}}{\partial\bar{\mu}} = -\beta_1 e^{-\bar{\mu}} \frac{\partial\phi}{\partial\bar{w}}.$$

QED

4.3 Future Work

We have seen that $S = z^2 \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}$ is a Forelli vector field by modifying the proof of Theorem 3.3.

The main step is to establish the unique continuation property for the integral curves of S . The main feature that is common between S and contracting aligned vector fields is the existence of a representative over the integral curves that determine the admissible sequences.

It is worth mentioning that the alignedness condition in Theorem 3.3 is used to show the existence of a representative. This suggests that a main question that would open new doors toward a general result is to determine a condition that guarantees the existence of a representative, and investigate whether all vector fields that have a representative are necessarily Forelli vector fields?

There are normal forms that could apply to some classes of non-contracting vector fields. The suggested direction is to decide on the applicability of the scheme in Theorem 3.3.

REFERENCES

- [1] F. Hartogs. *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten*. Math. Ann. 62 (1906) 1–88.
- [2] F. Forelli. *Pluriharmonicity in terms of harmonic slices*. Math. Scand. 41 (1977), 358–364.
- [3] W. Stoll. *The characterization of strictly parabolic manifold*. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Serie 4. 7 (1980), 87–154.
- [4] J.-C. Joo, K.-T. Kim, and G. Schmalz. *On the generalization of Forelli’s theorem*. Math. Ann. 365 (2016), 1187–1200.
- [5] J.-C. Joo, K.-T. Kim, and G. Schmalz. *A generalization of Forelli’s theorem*. Math. Ann. 355 (2013), no. 3, 1171–1176.
- [6] K.-T. Kim, E. A. Poletsky, and G. Schmalz. *Functions holomorphic along holomorphic vector fields*. J. Geom. Anal. 19 (2009), no. 3, 655–666.
- [7] E. M. Chirka. *Variations of Hartogs’ theorem*. Proc. Steklov Inst. Math. 253 (2006), 212–224.

- [8] S. Krantz. *Function Theory of Several Complex Variables*. American Mathematical Soc. (2001).
- [9] H. Rossi and R. Gunning. *Analytic Functions of Several Complex Variables*. American Mathematical Soc. (2009).
- [10] V. I. Arnol'd. *Geometric methods in the theory of ordinary differential equations*. Springer-Verlag, Berlin (1983).

VITAE

- Name: Alhussain Alkhunaizi
- Nationality: Saudi
- Date of Birth: 1999/01/27
- Email: *alhussain221@gmail.com*
- Permenant Address: Eastern Province, Qatif, AL Washim st.
- Bachelor in Mathematics at KFUPM (2016-2022)