

STABILITY OF BEAMS SUBJECT TO TORSION AND FRACTIONAL FACTORS

BY

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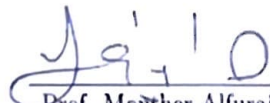
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Dedication
To My Mother & Father

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LIST OF SYMBOLS

Ω	A bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, where n is a positive integer.
$L^p(\Omega)$	The space of measurable functions on Ω such that $\ u\ _p < \infty$ with $1 \leq p < \infty$ and $\ u\ _p = \begin{cases} \left(\int_{\Omega} u(x) ^p dx \right)^{1/p} & \text{if } p < \infty, \\ \sup_{\Omega} u & \text{if } p = \infty. \end{cases}$
$L^1_{loc}(\Omega)$	The set of measurable functions u such that $u \in L^1(\Omega)$ for any compact subset K of Ω .
$\mathcal{D}(\Omega)$	The space of functions that have continuous derivatives of all orders with a compact support in Ω .
$C^1([0, \infty))$	All differentiable functions whose derivative is continuous on $[0, \infty)$.
$\ u\ $	The L^2 -norm of u .
$W^{m,p}(\Omega)$	$= \{u \in L^p(\Omega) \mid D^{\alpha}u \in L^p(\Omega), \text{ with } 1 \leq p \leq \infty, \alpha \in N^n, \alpha \leq m\}$.
$H^1(\Omega)$	The Sobolev space $W^{1,2}(\Omega)$.
$H^2(\Omega)$	The Sobolev space $W^{2,2}(\Omega)$.
u_x	The partial derivative of u with respect to x , $u_x = \frac{\partial u}{\partial x}$.

$$\nabla \quad \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \text{ for any } u \in W^{1,p}(\Omega).$$

$$\Delta \quad = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \text{ for any } u \in W^{2,p}(\Omega).$$

$$\Gamma(\alpha) \quad \text{The gamma function is defined as}$$

$$= \begin{cases} (\alpha - 1)!, & \text{for } \alpha \in \{1, 2, 3 \dots\}, \\ \int_0^\infty x^{\alpha-1} e^{-x} dx, & \text{for } \alpha > 0. \end{cases}$$

$$\Re(\alpha) \quad \text{The real part of } \alpha \text{ is given by } \Re(\alpha) = \Re(x + iy) = x.$$

DISSERTATION ABSTRACT

NAME: Banan Suliman Al-Homidan

TITLE OF STUDY: Stability of Beams Subject to Torsion and Fractional Factors

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This dissertation concentrates on three main issues: the dynamic of a Timoshenko beam under axial tension and torsional effect, a nonlinear Telegraph problem of fractional order and a Timoshenko beam also of non-integer order. The questions addressed here are the stability of the structure and the existence and uniqueness of solutions. In contrast to the integer order problems, the Galerkin method and the semi-group method seem not suitable to demonstrate the existence and uniqueness of a solution for fractional problems. We have used rather the notion of resolvents. Regarding the stability of the Timoshenko system and for the sake of economy and simplicity of implementation, we have reduced the number of frictional dampings in the original problem (one in each component) to only two. We kept the one in the equation corresponding to the torsion and adopted only one in either one of

the two equations of the Timoshenko system. The cases of effective axial loads or non effective axial loads have been studied separately as we have found that they present different characteristics. A new parameter has been obtained ensuring exponential stability when it is zero and a polynomial stability when it is different from zero. This covers the usual equal speeds of propagation and non-equal speeds of propagation as a special case. For both fractional order problems, we have established Mittag-Leffler and uniform stability under a lower order fractional term and under a fractional and/or viscoelastic damping. The main tool for the first problem is the multiplier technique. This energy method has been generalized to the fractional frameworks to cope with our problems.

ملخص الرسالة

الاسم: بنان سليمان الحميدان

عنوان الدراسة: استقرار عوارض خاضعة للإلتواء ولعوامل كسرية

التخصص: قسم الرياضيات

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تركز هذه الأطروحة على ثلاث قضايا رئيسية: ديناميكية عارضه من نوع تيموشينكو خاضعة لتوتر محوري وتأثير إلتوائي، ومشكلة التلغراف غير الخطية ذات الترتيب الكسري وعارضة تيموشينكو أيضاً ذات الترتيب الغير صحيح. الأسئلة المعالجة هنا هي استقرار الحلول ووجود الحلول ووحدانيتها. على نقيض المسائل ذات المشتقات الصحيحة، يبدو أن طريقة جاليركين وطريقة شبه المجموعة غير مناسبة لإثبات وجود ووحدانية حل المسائل الكسرية. بدلا من هاذة الطرق لقد استخدمنا فكرة العائلات المفككة. فيما يتعلق باستقرار نظام تيموشينكو ومن أجل الاقتصاد وبساطة التنفيذ، قمنا بتقليل عدد مخدمات الإحتكاك في المشكلة الأصلية (واحد في كل مكون) إلى اثنين فقط. لقد احتفظنا بمخدم الإحتكاك في المعادلة الخاصة بالإلتواء واعتمدنا واحدة فقط في إحدى معادلتين نظام تيموشينكو. تمت دراسة حالتي التأثير بفاعلية الحمولة المحورية من عدمه بشكل منفصل حيث أنه أتضح لنا أن لديهما خصائص مختلفة. كما أنه تم الحصول على معاميل جديد يضمن الاستقرار الأسّي عندما يكون صفراً واستقرار متعدد الحدود عندما يكون مختلفاً عن الصفر. بهذا تصبح حالة تساوي أو عدم تساوي سرعة الإنتشار المعروفة في نظام تيموتشنكو حالة خاصة من ماسبق. بالنسبة لمشكلتي الترتيب، قمنا بإثبات إستقرار النظام على شكل ميتاج-ليفلر تحت تأثير حد كسري ذو رتبة أقل من الرتبة الأساسية وتحت تخميد كسري و/أو لزج مرن. الأداة الرئيسية للتعامل مع المشكلة الأولى هي تقنية الطاقة. وقد تم تعميم هاذة التقنية لتتماشي مع الأطر الكسرية لمواجهة مشاكلنا.

CHAPTER 1

INTRODUCTION

1.1 The Problem

1.1.1 Bending Torsion Problem

In the last five decades, there has been an increasing focus on the control and stability of different kind of beams. This is primarily driven by their significance in various fields, including industry and medical research, such as applications in robot arms, suspended bridges, spacecraft, and blood vessels.

The Timoshenko theory was introduced in 1921. It was about a model which accurately captures the dynamic behavior of beams when both the rotational inertia and transverse shear are not negligible. This gives a better description than the well-known and commonly used Euler-Bernoulli model.

The transverse vibrations of a beam depends, in general, on its geometrical properties, such as the, length, size and shape, cross section, moment of inertia and shear coefficient. It also depends on its mechanical properties such as density,

Young's modulus and modulus of rigidity. The following model was derived in [1]

$$\begin{cases} \mu v_{tt}(x, t) - kAG (v_{xx}(x, t) - \phi_x(x, t)) = 0, & 0 < x < L, \ t > 0 \\ \rho I \phi_{tt}(x, t) - EI \phi_{xx}(x, t) - kAG (v_x(x, t) - \phi(x, t)) = 0, & 0 < x < L, \ t > 0 \\ w(0, t) = 0, \ \varphi(0, t) = 0, & t > 0 \end{cases}$$

where

$v(x, t)$ = transverse displacement,

$\phi(x, t)$ = rotational angle of the beam,

μ = mass density,

ρI = moment of mass inertia,

EI = rigidity coefficient,

kAG = shear modulus of elasticity,

L = length of the beam.

The problem is complemented by different kinds of boundary conditions. In addition, several internal and boundary controllers or damping devices have been incorporated and added to the structure to reduce the vibrations. Namely, structural damping, weak (frictional) damping, and viscoelastic damping have been utilized. Moreover, dissipation through coupling with a heat equation has been considered in the literature. It has been established that one of these damping may force the whole solution to decay exponentially to the rest state in the energy norm.

Indeed, in a bid to stabilize such structures, scientists and researchers came up

with several means and ways. Various kinds of controls, for instance in the form of dampers (described by frictional, structural, viscoelastic or thermal coupling damping) have been implemented either internally or at the boundary. In so doing, several arguments and methods have been deployed to establish the stability of the structures.

In this thesis, we want to shed some light on the vibration and buckling of beams of Timoshenko type subject to bending and torsion. Vibrations refer to the natural oscillations or wiggling of beams when disturbed, whereas buckling refers to when they collapse or deform under pressure. When discussing beams, such as those employed in constructions, bridges, or even common structures, the focus is often on their ability to bend or flex when subjected to loads. However, it is important to note that beams can also experience twisting or rotation in response to forces. Understanding how beams behave when they are bent and twisted is crucial for designing safe and efficient structures. For example, in tall buildings or long bridges, these bending and torsional movements can affect stability and performance. By studying these characteristics, engineers may have a clear idea on how to act so that the structures can withstand the forces they will encounter in the real world. To this end, an essential step to rely on the adopted model, is the analytic treatment of the stability. Determining sufficient conditions under which the system is stable is of paramount importance. Prior to this, it is natural to ascertain the existence and uniqueness of a solution in an appropriate underlying space.

Our main reference is the following problem:

$$\begin{cases} \rho I \phi_{tt} + c_3 \phi_t - EI \phi_{xx} - kAG(v_x - \phi) - K \psi_{xx} = 0, & x \in (0, L), t \in (0, \infty), \\ \mu v_{tt} + c_1 v_t - kAG(v_{xx} - \phi_x) + P v_{xx} = 0, & x \in (0, L), t \in (0, \infty), \\ -GJ \psi_{xx} - K \phi_{xx} + P(I_s/\mu) \psi_{xx} + I_s \psi_{tt} + c_2 \psi_t = 0, & x \in (0, L), t \in (0, \infty) \end{cases} \quad (1.1)$$

This system has been examined in the field of mechanical engineering. In contrast to Timoshenko beams, for which mathematicians have thoroughly investigated their dynamical behavior, providing analytical and numerical treatments, bending-torsion problems have not been well studied by mathematicians. The coupling of bending and torsion effects in (1.1), has remained largely unexplored in mathematical investigations.

Taking into account the bending and torsional deformations is more realistic since many structures, such as aircraft wings, turbine blades, and tall buildings, experience both of these deformations simultaneously. Neglecting the coupling between these two types of deformation can lead to inaccuracies in predicting the structural response. By considering beams that experience both bending and torsion, it becomes possible to make more precise estimates of the deflections, stresses, and natural frequencies in structural components. This is especially crucial in critical applications where accuracy is important, such as in aerospace engineering, electrical or electronic engineering and civil engineering (the design of buildings resistant to seismic activity).

The system (1.1) proficiently describes the vibrations of a bar (beam) tak-

ing into account both bending and rotational deformations. Unlike the Euler-Bernoulli model, which portrays only the bending effect, the Timoshenko model is suitable for thick structures.

1.1.2 Fractional Problems (Telegraph and Timoshenko)

In the last four decades, fractional differential equations and systems and their applications have received significant attention by scientists and engineers. This has resulted in a rapid progress of the theory of fractional calculus to accommodate their use in mathematics and other fields. A fractional differential equation involves derivatives of fractional order, unlike traditional differential equations that rely only on integer-order derivatives. A derivative of fractional order expands the concept of an integer-order derivative, connecting integer-order calculus with complex phenomena in various fields. Such equations have attracted considerable interest due to their capacity to represent a wide range of physical and biological phenomena with greater accuracy compared to traditional integer-order differential equations. They are commonly employed in a wide range of fields, including physics, engineering, biology, and economics and signal processing.

In the second chapter of this thesis, we examine two fractional problems. The first problem is nonlinear as a nonlinear term of power type is acting directly in the system. The model can be looked at as a generalization of the well-known Telegraph problem. Notably the order of the (frictional) damping in the integer order case, described by the velocity of the state, is half the order of the leading term (which is of the second order). This characteristic is conserved here when we

pass to the fractional case. In addition, to this lower order fractional term whose order is half of the one of the leading term, we consider a damping of viscoelastic type. For economy and also simplicity and easiness of implementation, only one at a time is effective. Our main objective is to study the impact of these terms on the dynamic of the structure despite the nonlinear factor. More details are given a little below. The aforementioned statement applies also to the second problem. It concerns a fractional version of the Timoshenko beam.

As explained above, fractional calculus provides a suitable platform for investigating numerous phenomena. Typically, particles spread randomly, a phenomenon observed in various fields such as engineering, biology, ecology, and medicine. They have been the subject of investigation by numerous researchers. Various formulations of diffusion are available in the market, along with a variety of treatment options.

For anomalous processes and disordered systems, where the Mean Squared Displacement MSD (measure of the deflection of the actual position of a particle as regards to a given reference position) exhibits nonlinear growth, the consideration of fractional derivatives becomes highly advocated. Complex diffusion phenomena are more accurately described in the fractional context. Indeed, it has been witnessed by a large number of researchers that fractional derivatives are more appropriate for disordered phenomena.

In the case of normal diffusion, the mean square displacement (MSD) exhibits a linear growth ($\langle v(t) \rangle \sim t$). This indicates that particles are moving in normal mode. For subdiffusion, where ($\langle v(t) \rangle \sim t^\alpha$, $0 < \alpha < 1$), the MSD grows slower

than in normal diffusion, indicating slower particle movement. On the other hand, when $\alpha = 2$, the MSD corresponds to the ballistic diffusion. It is important to note that the standard heat equation or the wave equation cannot sufficiently describe cases other than the normal diffusion and the ballistic diffusion.

The First Fractional Problem

Our first concern is a nonlinear fractional initial boundary value problem with an order ranging between 1 and 2 described in [2]

$$\left\{ \begin{array}{l} {}^C D^{2\alpha} v(x, t) + a {}^C D^\alpha v(x, t) + |v|^\gamma v = \Delta v(x, t) - b \int_0^t k(t-s) \Delta v(x, s) ds, \\ t > 0, x \in \Omega, 1/2 < \alpha < 1, \\ v(x, t) = 0, t > 0, x \in \partial\Omega, \\ v(x, 0) = v_0(x), v'(x, 0) = v_1(x), x \in \Omega. \end{array} \right. \quad (1.2)$$

The type of fractional derivative we consider in this model is the Caputo fractional derivative. In the equation of (1.2), k is a nonnegative function commonly known as the relaxation function whilst $v_0(x)$ and $v_1(x)$ represent the initial data of the state. The values of a and b are either 1 or 0, depending on their effectiveness in the equation. The value of γ is positive and will be determined later.

The correlation between the heat flux vector and the temperature gradient is well-known under the form

$$\mathbf{q} = -\kappa \nabla T$$

where κ represents the thermal conductivity. Having in mind the conservation of energy

$$\gamma \frac{\partial T}{\partial t} = -div \mathbf{q}$$

where γ represents the heat capacity, the fundamental equation for heat can be expressed as

$$\frac{\partial T}{\partial t} = \frac{k}{\gamma} \Delta T.$$

It is commonly acknowledged that the historical relationship between the gradient and the flux is given by

$$\mathbf{q} = -\frac{k}{\Gamma(\mu)} \frac{\partial}{\partial t} \int_0^t (t-s)^{\mu-1} \nabla T(s) ds = -k {}^{RL}D^\mu \nabla T, \quad 0 < \mu < 1.$$

Therewithal, we are driven to the fractional heat conduction equation

$${}^C D^\alpha T = \delta \Delta T$$

for $0 < \alpha < 1$ and $\delta > 0$.

On the other hand, for a material with memory, the relationship between the heat flux vector and the temperature gradient will incorporate a term of the form

$$\int_0^t g(t-s) \nabla T(s) ds.$$

The fractional equation

$${}^C D^{2\alpha} v(x, t) = \Delta v(x, t), \quad \frac{1}{2} < \alpha < 1 \quad (1.3)$$

is a fractional version that combines elements of both the heat equation and the wave equation. This case corresponds to the "super-diffusion", a concept that has been briefly discussed above. It is motivated by the observation that the mean square displacement increases non-linearly over time ($\langle v(t) \rangle \sim t^{2\alpha}$, $1 < 2\alpha < 2$) (see [3],[4]).

To make things simpler and more convenient, we will focus on the problem

$$\left\{ \begin{array}{l} {}^C D^\alpha ({}^C D^\alpha v(x, t)) + a {}^C D^\alpha v(x, t) + |v|^\gamma v = \Delta v(x, t) - b \int_0^t k(t-s) \Delta v(x, s) ds, \\ t > 0, \quad x \in \Omega, \quad 1/2 < \alpha < 1, \\ v(x, t) = 0, \quad t > 0, \quad x \in \partial\Omega, \\ v(x, 0) = v_0(x), \quad x \in \Omega. \end{array} \right. \quad (1.4)$$

Because the fractional composition ${}^C D^{\alpha+\beta} v(x, t) = {}^C D^\alpha ({}^C D^\beta v(x, t))$ is not correct in general, the two problems are not exactly the same. The expression in the right hand side is often referred to as a "sequential" fractional derivative.

The Second Fractional Problem

For beams with a non-negligible thickness and a significant deflection resulting from the shear, it is highly advisable to appeal to the Timoshenko model. This particular modeling approach, with its integer-order formulation involving second-order time derivatives, has garnered significant attention from a consider-

able number of researchers. As a result, numerous findings related to this model have been documented in the literature.

Due to some anomalous, chaotic, fractal and disordered considerations, the incorporation of the balance law alongside the constitutive relationship linking stress and strain introduces a nonlocal time-dependent term. The nonlocal term takes into account the entire history of the state. This characteristic hints at the suitability of fractional calculus for investigating such phenomena. There are different kinds of fractional derivatives in the market. According to several published investigations, a genuine fractional derivative must involve a singular kernel in its formulation.

In normal situation (the MSD is linear), damping devices are either incorporated in the material or added artificially inside the structure or at the boundary. These are expected to slow down vibrations or absorb shocks that could harm the structure. Wherefore, these dampers, often called "rotary" dampers in the market are crucial to, help protecting the structure. Placing ourselves in the anomalous context, we will specifically tackle the fractional problem [5]

$$\begin{cases} \rho_1 {}^C D^\alpha \varphi - k(\varphi_x + \psi)_x = 0, & 0 < x < 1, t > 0, 1 < \alpha < 2, \\ \rho_2 {}^C D^\alpha \psi - b\psi_{xx} + \int_0^t \omega(t-s)\psi_{xx}(s)ds + k(\varphi_x + \psi) = 0, & 0 < x < 1, t > 0, \end{cases} \quad (1.5)$$

with the given Dirichlet boundary conditions and initial values:

$$\begin{cases} \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, & t \geq 0, \\ \varphi(x, 0) = \varphi_0, \varphi_t(x, 0) = \varphi_1, \psi(x, 0) = \psi_0, \psi_t(x, 0) = \psi_1. \end{cases} \quad (1.6)$$

The symbol ${}^C D^\alpha$ represents the Caputo fractional derivative operator (defined below).

In fractional calculus, it is established that ${}^C D^\alpha ({}^C D^\alpha \varphi(x, t)) \neq {}^C D^{2\alpha} \varphi(x, t)$.

As a consequence, when we rewrite our system using the sequential forms ${}^C D^\alpha ({}^C D^\alpha \varphi)$ and ${}^C D^\alpha ({}^C D^\alpha \psi)$, additional terms would arise. The relationship between both forms is actually expressed by

$${}^C D^\alpha ({}^C D^\alpha \varphi(x, t)) = {}^C D^{2\alpha} \varphi(x, t) + \frac{\varphi'(x, 0)t^{1-2\alpha}}{\Gamma(2-2\alpha)}$$

where $\alpha > 1/2$ (which is our case). The appearing term will not effect the stability of the system.

We will focus on the simplified system provided by

$$\begin{cases} \rho_1 {}^C D^\alpha ({}^C D^\alpha \varphi) - k(\varphi_x + \psi)_x = 0, & 1/2 < \alpha < 1, \\ \rho_2 {}^C D^\alpha ({}^C D^\alpha \psi) - b\psi_{xx} + \int_0^t \omega(t-s)\psi_{xx}(s)ds + k(\varphi_x + \psi) = 0. \end{cases} \quad (1.7)$$

For $0 < x < 1$ and $t > 0$ under the given initial and boundary conditions in (1.6), it is evident that the two fractional derivatives are equal when $\varphi'(x, 0) = 0$. Otherwise, in terms of stability as was already mentioned above, it will remain un-

affected as long as $1 - 2\alpha < 0$. Several alternative fractional models for describing Timoshenko beams may be formulated under various conditions and in different situations. The equations at hand include fractional derivatives of Caputo type. We made this choice because of their practical suitability and convenience. These kinds of problems have primarily been addressed from an engineering viewpoint, focusing on aspects such as response determination, resonance, free vibrations, impact on mechanical responses, parametric analyses, steady-state response, transient response and more. Several analytical and semi-analytical methods have been utilized in the study (see [6], [7], [8]). These include Galerkin schemes, central difference schemes, direct numerical integration schemes, averaging techniques and different approximations of fractional derivatives. The methods used were contrasted with conventional approaches, adapted to the fractional scenario, like the Laplace transform and Fourier integral transform, which have played a crucial role. Error estimates and computational trials have been carried out to support the utility, efficiency, and precision of fractional models in portraying the behavior of beams.

We recall that, Timoshenko models with integer order (featuring second-order time derivatives of the states as primary terms) have been explored in various research works, incorporating fractional damping (see [9], [10]). The fractional damping can manifest as either frictional weak (characterized by a fractional derivative of the state) damping or strong (involving a fractional derivative of the Laplacian of the state) structural damping. In the latter scenario, it can be interpreted as a viscoelastic damping with a singular kernel. Specifically, it is

represented as the convolution of a singular kernel (of power type) with the time derivative of the Laplacian of the state. The former scenario corresponds to a distributed delay of the time derivative of the state (in the case of the Caputo derivative).

1.2 Literature Review

In this part of the document we are expected to report on the existing works related to our subject. Avoiding bias seems impossible without devoting several dozens of pages in this thesis to survey the huge number of papers on the Timoshenko system.

1.2.1 Bending Torsion Problem

The Timoshenko model was first introduced by Timoshenko in the year 1921 in the article [1], in the following form

$$\begin{cases} EI\phi_{xx} + \lambda C\Omega(\psi_x - \phi) - \frac{\rho I}{g}\phi_{tt} = 0, \\ \frac{\rho\Omega}{g}\psi_{tt} - \lambda C\Omega(\psi_{xx} - \phi_x) = 0 \end{cases} \quad (1.8)$$

where ϕ is the angular rotation, ψ is the displacement, EI is the flexural rigidity of the bar, Ω is the area of cross-section, $\frac{\rho}{g}$ is the density of the material, C is the modulus of rigidity to the material of the bar and λ is the constant characterizing the shape of the cross-section.

Amidst all the used techniques to investigate the stability question, perhaps

the most common ones are the multiplier technique (also known as the energy method) and the spectral theory. In this thesis, we shall consider the latter one.

The system (1.8) has been considered by a huge number of authors see, for instance [11] and [12]. The Timoshenko problem with two frictional dampings

$$\begin{cases} \rho_1 u_{tt} - k(u_x - \psi)_x + u_t = 0 \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(u_x - \psi) + \psi_t = 0 \end{cases}$$

has been considered in [13]. The author obtained exponential decay of the solution.

Here, and below by exponential (polynomial) decay or stability we mean that solutions are bounded by an exponentially (polynomially) decay function in an appropriate energy norm. In addition, the involved parameters are positive constants in general and are pertinent to the nature of the problem in the question.

The authors in [14], considered

$$\begin{cases} \rho w_{tt} = (K(w_x - \phi))_x, \\ I_\rho \phi_{tt} = (EI\phi_x)_x + K(w_x - \phi) - b(x)\phi_t = 0, \end{cases} \quad (1.9)$$

and proved the exponential stability of the system if and if we have equal speeds of propagation. In [15], the authors examined the same system (1.9). A uniform stability result is proven in the case of equal speeds of propagation. In the case of nonequal speeds of propagation they showed a lack of uniform stability.

In [16], the authors considered the Timoshenko problem

$$\begin{cases} \rho w_{tt} - K(w_x - \phi)_x = 0, \\ I_\rho \phi_{tt} - EI\phi_{xx} - K(w_x - \phi) = 0 \end{cases}$$

with the boundary conditions

$$\begin{cases} w(0, t) = 0, \quad \phi(0, t) = 0, \\ K\phi(L, t) - Kw_x(L, t) = \alpha w_t(L, t), \\ EI\phi_x(L, t) = -\beta \phi_t(L, t). \end{cases}$$

They proved a uniform stability result.

In [17] a Timoshenko beam with a locally distributed feedback has been treated. The author proved an exponential stability result without the use of the equal speeds of propagation assumption.

We can also mention the work in [15], which concerns the frictionally damped system

$$\begin{cases} a\phi_{tt} - k(\phi_x + \psi)_x = 0, \\ b\psi_{tt} - d\psi_{xx} + k(\phi_x + \psi) + h(x)\psi_t = 0, \end{cases}$$

where h is a continuous function and the two boundary cases

$$\phi(0, t) = \phi(1, t) = 0, \quad \psi(0, t) = \psi(1, t) = 0, \tag{1.10}$$

$$\phi_x(1, t) = 0, \quad \phi(0, t) = \psi(0, t) = \psi(1, t) = 0. \quad (1.11)$$

The authors also considered the case of boundary feedback

$$\begin{cases} \phi(0, t) = \phi(1, t) = 0, \\ \psi_x(0, t) = c\psi_t(0, t), \\ \phi_x(1, t) = -d\phi_t(1, t). \end{cases} \quad (1.12)$$

For the boundary conditions (1.10) and (1.11) and assuming that $h \geq h_0 > 0$ the problem is uniformly stable provided that the speeds of propagation are equal, that is $\frac{a}{k} = \frac{b}{d}$. The problem is not uniformly stable in the case $\frac{a}{k} \neq \frac{b}{d}$. For the boundary (1.12), they proved that under the condition $\frac{a}{k} = \frac{b}{d}$ the problem is uniformly stable otherwise it is not uniformly stable.

The authors in [18], looked at the system

$$\begin{cases} \rho\phi_{tt} - k(\phi_x + \psi)_x = 0, \\ \mu\psi_{tt} - d\psi_{xx} + k(\phi_x + \psi) + \int_0^t g(t - \tau)(a(x)(\psi_x(\tau))_x d\tau + b(x)h(\psi_t) = 0, \end{cases}$$

with the kernel g satisfying

$$g'(s) \leq -\xi(s)g(s)$$

and under other conditions on the functions a, b, h . They proved that the energy of the system is bounded above by a quantity, depending on g and h , which tends

to zero as time goes to infinity.

The Timoshenko system with an infinite memory of the following form:

$$\begin{cases} \rho_1 \phi_{tt} - k_1(\phi_x + \psi)_x = 0, & x \in (0, L), t \in (0, \infty) \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\phi_x + \psi) + \int_0^{+\infty} g(t-s) \psi_{xx}(s) ds = 0, & x \in (0, L), t \in (0, \infty) \end{cases} \quad (1.13)$$

has been investigated by many authors, as in [19] where the authors considered the system (1.13) when g decays the conditions

$$g(t) > 0, \quad \exists c_1, c_2, c_3 > 0, \quad -c_1 g(t) \leq g'(t) \leq c_2 g(t), \quad |g''(t)| \leq k_2 g(t), \quad \forall t \geq 0,$$

$$\bar{k}_2 := k_2 - \int_0^\infty g(s) ds > 0.$$

They studied both cases of equal and non-equal speeds of propagation. Exponential stability and polynomial stability were demonstrated, respectively.

The same problem has been addressed in [19], [20] with a kernel g satisfying

- the function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of class $C^1(\mathbb{R}^+)$ nonincreasing such that

$$g_0 := \int_0^\infty g(s) ds \in (0, k_2).$$

- There is a differentiable function $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that is nonincreasing, such that

$$g'(s) \leq -h(s)g(s), \quad \forall s \in \mathbb{R}^+.$$

As a matter of fact, many authors have considered Timoshenko beams with a viscoelastic damping of finite memory, that is

$$\begin{cases} \rho_1 \phi_{tt} - k_1(\phi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\phi_x + \psi) + \int_0^t g(t-s) \psi_{xx}(s) ds = 0, \end{cases} \quad (1.14)$$

with different assumptions on the kernel on g . Here is a brief overview of what has been accomplished. The first to consider the Timoshenko system with memory are the authors of [21], where the kernel is of polynomial type i.e.

$$\begin{cases} 0 < g(t) \leq b_0(1+t^{-p}), \\ -b_1 g(t)^{\frac{p+1}{p}} \leq g'(t) \leq -b_2 g(t)^{\frac{p+1}{p}}, \\ -b_3 |g'(t)|^{\frac{p+1}{p+1}} \leq g''(t) \leq -b_4 |g'(t)|^{\frac{p+2}{p+1}}. \end{cases}$$

Actually, exponential stability can be attained for exponentially decaying relaxation functions. Likewise, one can prove polynomial (power-type) stability when the relaxation function itself is polynomially decaying. The condition $\frac{k}{\rho_1} = \frac{b}{\rho_2}$, divides these two kinds of rates.

In [22], the authors assumed the following

- The function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is differentiable and

$$g(0) > 0, \quad 1 - \int_0^\infty g(s) ds = l > 0. \quad (1.15)$$

- There is a constant $\xi > 0$ and $1 \leq p \leq 3/2$ such that

$$g'(s) \leq -\xi g^p(s), \quad \forall s \in \mathbb{R}^+.$$

They obtained exponential decay in the case that $p = 1$ and polynomial decay in case $p \neq 1$.

In [23], the authors made the assumption

$$g'(s) \leq -h(t)g(s), \quad \forall s \in \mathbb{R}^+$$

where h is non increasing differentiable function in addition to (1.15). They showed exponential stability under the condition that $\rho_1 = \rho_2 = k_1 = k_2 = 1$.

In [24], the authors considered the condition $g'(t) \leq -H(g(t))$ where H is a function that meets certain regularity and convexity requirements. A general stability result has been established.

In [25], the authors treated the thermoelastic system

$$\rho_1 \phi_{tt} - k(\phi_x + \psi)_x = 0,$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) + \gamma\theta_x = 0,$$

$$\rho_3 \theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} = 0.$$

It consists of a coupling of the Timoshenko model with a heat equation. Exponential stability was demonstrated, given the condition of equal speeds of propagation.

Also, they proved that in case of non-equal speeds of propagation the system is not exponentially stable. Fourier's law makes an assumption that results in an unrealistic characteristic, namely that any sudden disturbance occurring at a specific point will be instantaneously perceived throughout the entire material. To remedy to this issue other alternatives have been proposed such as the system below termed "second sound" in [26]

$$\begin{cases} \rho_1 \phi_{tt} - \sigma(\phi_x, \psi)_x + \mu \phi_t = 0, \\ \rho_2 \psi_{tt} - b \psi_{xx} + k(\phi_x + \psi) + \beta \theta_x = 0, \\ \rho_3 \theta_t + \gamma q_x + \delta \psi_{tx} = 0, \\ \tau_0 q_t + q + \kappa \theta_x = 0. \end{cases}$$

This system has also been studied by several other researchers such as in [27].

Exponential stability holds provided that

$$X_0 = \left(\tau_0 - \frac{\rho_1}{\rho_3 \kappa} \right) \left(\rho_2 - \frac{b \rho_1}{\kappa} \right) - \frac{\tau_0 \rho_1 \delta^2}{\rho_3 \kappa} = 0.$$

For the case $X_0 \neq 0$, there is a lack of exponential stability.

For more details on Timoshenko systems, we refer the reader to the references [28], [29], [30], [31],[32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [43], [44], [45], [46], [47], [48].

The Bending–torsional dynamic response of a composite Timoshenko thin-walled beam with closed cross-section subjected to axial loading has been inves-

tigated in [49].

After figuring out the equations of the system, the dynamic of the structure has been studied in case of free vibrations. In particular, a special attention was given to the beam's response under different types of loads. Specific example were provided illustrating the effect of bending and twisting on the dynamic of the beam.

Employing some known principles, the authors in [50] also derived a similar system to (1.1). The beam in question, made of different materials, is prompted to bending and twisting due to the architecture of its layers. The axial forces were also taken into consideration in the model. The obtained results on the impact of the factors have been validated by employing some numerical software. Several other studies in the field of mechanical engineering have also addressed the system (1.1), such as [51].

A closely related problem to (1.1) had been discussed in [52]. Namely, it was about the system

$$\begin{cases} EI\theta_{xx} + kAG(w_x - \theta) + K\psi_{xx} - \rho I\theta_{tt} = 0, \\ kAG(w_{xx} - \theta_x) - \rho Aw_{tt} - Pw_{xx} = 0, \\ GJ\psi_{xx} + K\theta_{xx} - I_\alpha\psi_{tt} = 0. \end{cases} \quad (1.16)$$

Through numerical treatments, the authors investigated the impact of the aforementioned factors on the vibration and buckling of the structure. The current system differs slightly from the one we are considering here (1.1) observing that

the third equation does not include the term $P(I_s/\mu)\psi_{xx}$, as is the case in (1.1).

In addition, notice that the system is undamped.

1.2.2 Fractional Problems

In contrast to the previous subject (on Timoshenko systems of integer order), we could not find much to read about the fractional case. The difference in resources is huge and especially in mathematical treatments.

In [53], the authors derived the fractional heat conduction equation

$${}^C D^\alpha T = \delta \Delta T$$

where $0 < \alpha < 1$ and $\delta > 0$. A subdiffusive problem has been formulated in [54] based on the Rayleigh-Stokes problem

$$\frac{\partial w}{\partial t} = (1 + \gamma {}^{RL} D^\beta) \Delta w.$$

The authors indicated that the solutions of Stoke's first problem can be seen as a specific case of this one. Fractional calculus provides a suitable framework for investigating numerous phenomena in engineering and science such as in [55],[56], [57], [58], [53], [59], [60]. When dealing with a material exhibiting memory effects, the connection between the heat flux vector and the temperature gradient will

include a term of the following form:

$$\int_0^t g(t-s) \nabla T(s) ds.$$

This, together with the balance law, leads to a viscoelastic term. On the other hand the fractional equation

$${}^C D^{2\alpha} v(x, t) = \Delta v(x, t) \tag{1.17}$$

embraces all the orders ranging from 1 to 2 and hence interpolates the heat equation and the wave equation. It describes the super-diffusion case (see [61], [62]) when the Mean Square Displacement (MSD) grows nonlinearly in time. The importance of super-diffusion is highlighted in several studies, as seen in references such as [57], [59], [60], [63].

As mentioned earlier in the introduction, a linear growth of the MSD corresponds to the ordinary case. Subdiffusion occurs when particles move slower than in the ordinary case, while $\alpha = 2$ corresponds to the ballistic diffusion. Apart from the normal and ballistic diffusion, the other cases cannot be adequately described by the standard heat equation or the wave equation. In particular, the equation

$${}^C D^{2\alpha} v(x, t) = \Delta v(x, t) - b \int_0^t k(t-s) \Delta v(x, s) ds, \quad 1/2 < \alpha < 1, \tag{1.18}$$

interpolates the equation encountered in heat conduction [64]

$$u'(t) = \Delta u(t) - \int_0^t k(t-s)\Delta u(s)ds \quad (1.19)$$

and the familiar viscoelastic equation [65]

$$w''(t) = \Delta w(t) - \int_0^t k(t-s)\Delta w(s)ds. \quad (1.20)$$

Many researchers have explored the stability of these problems, treating them as parabolic problems with memory [66], [67], and as hyperbolic problems [68], [69], [70], [71], [72], [73].

In the viscoelastic case, researchers have found that the memory term introduces a weak damping effect, which is able by itself to drive the system to rest. The kernel plays a crucial role in determining the stability rate. Generally, the stability will be of the same type of kernel and cannot be better. Different stability rates, such as exponential, polynomial and arbitrary, can be found in the literature [68], [65], [66], [64], [67], [69], [70], [74], [71], [72], [73].

In [75], the following problem was investigated:

$$\partial_t^\alpha x(z, t) = \Delta x(z, t) + f(z, t), \quad z \in \Omega, \quad t > 0$$

where ∂_t^α is the Caputo fractional derivative of order $\alpha \in (1, 2)$. The authors in [76] explored mild solutions to the previous problem of fractional evolution systems in Banach spaces, along with addressing the controllability problem. They

derived a new representation of solution operators and introduced a new concept of mild solutions for the target equations, employing Laplace transform and Mainardi's Wright-type function. Subsequently, the authors established a new compactness result for the solution operators under the condition that the sine family is compact. Furthermore, controllability results for mild solutions were derived.

The authors in [77] investigated a fractional semilinear integro-differential equation of order $1 < \alpha < 2$ given by

$$\begin{cases} D_t^\alpha(t) = Au(t) + f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, & t \in [0, t] \\ u(0) + m(u) = u_0 \in X, & u'(0) + n(u) = u_0 \in X \end{cases}$$

where X is a Banach space and $A : D(A) \subset X \rightarrow X$ is an operator. By employing Krasnoselskii's fixed-point theorem and the contraction mapping principle, they constructed several families of operators to establish a proper definition of a mild solution and determine conditions for the existence of such solutions.

Similar types of problems have been addressed in [78] and [79]. In [78], the authors investigated the existence of positive mild solutions using Schauder's fixed-point theorem and Krasnoselskii's fixed-point theorem. Meanwhile, in [79], the authors studied the existence of local and global mild solutions employing the solution operator, measure of non-compactness and various fixed-point theorems associated with compact and non-compact semigroups.

Models of the form (1.4) are fascinating not only mathematically but also in

terms of their applications. They describe the vibrations of viscoelastic structures in anomalous media. Based on telegraph equation [55], [56], [74]

$$\frac{\partial^2 v}{\partial t^2} + c \frac{\partial v}{\partial t} = \Delta v, \quad (1.21)$$

the authors in [58] demonstrated that the law of iterated Brownian motion and the telegraph process with Brownian time can be described accurately using the fractional telegraph equation

$${}^C D^{2\alpha} v(x, t) + a {}^C D^\alpha v(x, t) = \Delta v(x, t), \quad 0 < \alpha < 1. \quad (1.22)$$

Hence, our current model (1.5) is a kind of nonlinear fractional telegraph equation with memory.

The Telegraph equation (1.21) is obviously dissipative. This is because of the inclusion of the first-order derivative, often referred to as frictional damping. In actual fact, the system energy decreases exponentially towards zero. This observation motivated us to explore whether the lower-order fractional derivative in (1.22) serves as a dissipative component in the fractional problem (1.5) or not. A positive answer to this question will prompt another question, namely, what would be the rate of convergence?

One can find some research on the well-posedness for basic fractional problems within the range of orders between 0 and 1 and also between 1 and 2. Unfortunately, there are only a few papers that address the precise form of our problem

(1.5) or even close to it. Nevertheless, three papers were identified: the paper [80], the paper [81] (pertaining to an infinite history) and the paper [82], which deals with abstract operators.

For problem (1.4), the case $\alpha = 1/2$ corresponds to the parabolic case [64], [67]. In case $\alpha = 1$, we recover the viscoelastic problem studied extensively by many authors [66], [60], [70], [74], [71], [72], [73].

1.3 Methodology

To prove the stability, that is the asymptotic convergence of the solutions to the equilibrium state in the energy norm, several ways have been developed. Some of the techniques employed are based on differential inequalities and integral inequalities.

The one adopted here is the so-called energy method or also known as the multiplier technique. It is based on exhibiting a differential inequality whose solutions decay to zero asymptotically (that is as time goes to infinity) in a certain norm. More precisely, its strategy consists first in determining the classical energy $E(t)$ associated to the system. If a certain component of the system is somehow damped then the corresponding term in the kinetic energy will appear in the derivative of the classical energy with negative sign (this is the case, for instance, for a frictional or a structural damping in which cases terms like $-||\phi_t||^2$, $-||\psi_t||^2$, $-||v_t||^2$, $-||\nabla\phi_t||^2$, $-||\nabla\psi_t||^2$, $-||\nabla v_t||^2$ arise). The appearance of such terms helps actually to decide on the dissipativeness of the system (in addition to the

well-posedness of the problem).

Then, realizing that there is a need to have other terms with negative sign (like the potential energy $-||\nabla\phi||^2, -||\nabla v||^2, -||\nabla\psi||^2$), we start putting forward extra functionals say $V_i(t), i = 1, 2, \dots, m$ whose derivatives, along solutions of the problem, will give rise to such terms. As we go along with this process, unfortunately some undesirable new terms appear in the differentiation process. Often, they are problematic and present an important challenge. This is due to difficulties estimating and evaluating these terms. It is very common that new terms of higher order (than the existing one in the energy) pop up in the calculation. They need a clear way to get rid of them for instance by cancellation against another suitably chosen functional. These situations have been encountered in our present work. In some cases, one is forced to pass to higher order energy functional.

As we move forward one must keep an eye on the equivalence between the newly constructed functional

$$L(t) = E(t) + \sum_{i=1}^m \lambda_i V_i(t), \quad \lambda_i > 0, \quad t \geq 0.$$

and the classical energy. The equivalence $\mathcal{L}(t) \sim E(t)$, means that there exist two positive constants N_1 and N_2 , verifying

$$N_1 E(t) \leq \mathcal{L}(t) \leq N_2 E(t), \quad \forall t \geq 0.$$

This is because, after all, it is the classical energy which needs to approach zero as

time evaluates. In doing so, this puts several restrictions on the type of multipliers to be utilized hence narrowing drastically the marge of manoeuvre.

The most commonly employed methods to prove existence and uniqueness of a solution for similar problems to ours, in an appropriate underlying space, are the Galerkin method and the semi-group method. Both are fine for the integer order case (in addition to some fixed point theorems). However, for non-integer problems, these methods are no longer applicable. We shall use rather the notion of resolvent families which is the counter part of the semi-group notation.

1.4 Preliminaries

To simplify the forthcoming exposition and repetition, we gathered some identities, inequalities, facts, and definitions in this subsection.

1. **Young's Inequality.** Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $\epsilon > 0$, we have

$$ab \leq \epsilon a^p + C_\epsilon b^q, \quad \forall a, b > 0,$$

where $C_\epsilon = \frac{1}{q(\epsilon p)^{p/q}}$. For $p = q = 2$, we have

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2.$$

2. **Hölder Inequality.** Assume $1 \leq p, q \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} = 1$. If

$f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\int_{\Omega} |fg| \leq \|f\|_p \|g\|_q.$$

Cauchy-Schwarz Inequality. For $p = q = 2$, and according to the Hölder inequality we recover the well-known inequality

$$\|fg\|_{L^1} \leq \|f\|_2 \|g\|_2.$$

3. **Poincaré Inequality.** Assume that $1 \leq p < \infty$ and Ω is a bounded smooth domain in $\mathbb{R}^n$. Then, there exists a positive constant C depending on Ω and p such that

$$\|f\|_{L^p(\Omega)} \leq C \|\nabla f\|_{L^p(\Omega)}, \quad \forall f \in W_0^{1,p}(\Omega) \cup W_*^{1,p}(\Omega)$$

where the zero here in $W_0^{1,p}(\Omega)$ represents the zero at the boundary and

$$W_*^{1,p}(\Omega) := \left\{ f \in W^{1,p}(\Omega) : \int_{\Omega} f = 0 \right\}.$$

4. **Green's Inequality.** Suppose that Ω is a smooth bounded domain in $\mathbb{R}^n$ with a smooth boundary. Then the following identity holds

$$\int_{\Omega} u \Delta v dx = - \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u \nabla v \cdot \nu ds, \quad \forall u \in H^1(\Omega) \text{ and } v \in H^2(\Omega)$$

where ν is the outer unit normal to $\partial\Omega$. For $u \in H_0^1(\Omega)$, the Green's formula reduces to

$$\int_{\Omega} u \Delta v dx = - \int_{\Omega} \nabla u \cdot \nabla v dx.$$

In the definition of the Sobolev space $W^{m,p}(\Omega)$, we use the "weak" partial derivatives of the function u

Definition 1.4.1 *Let $u \in L_{loc}^1(\Omega)$. We say that the function u is weakly partially differentiable with respect to x_k if there exists a function $v \in L_{loc}^1(\Omega)$ such that*

$$\int_{\Omega} u(x) \partial_k \phi(x) dx = - \int_{\Omega} v(x) \phi(x) dx, \quad \forall \phi \in \mathcal{D}(\Omega).$$

In the remaining part of this subsection, we present several formulas and inequalities concerning fractional derivatives, as referenced in [83] and [84]. Additionally, we introduce the fractional product rule.

Definition 1.4.2 *The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by*

$$I^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds, \quad \alpha > 0$$

for any measurable function φ provided that the right hand side exists. Here, $\Gamma(\alpha)$ is the usual Gamma function.

Definition 1.4.3 ([84]) *The fractional derivative of order α in the sense of Ca-*

puto is defined by

$${}^C D^\alpha \varphi(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \varphi'(s) ds, \quad 0 < \alpha < 1,$$

$${}^C D^\alpha \varphi(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \varphi''(s) ds, \quad 1 < \alpha < 2,$$

whereas the fractional derivative of order α in the sense of Riemann-Liouville is defined by

$${}^{RL} D^\alpha \varphi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \varphi(s) ds, \quad 0 < \alpha < 1$$

provided that the right hand sides exist. Here the primes denote time differentiation.

The relationship between both derivatives is

$${}^{RL} D^\alpha \varphi(t) = \frac{\varphi(0)t^{-\alpha}}{\Gamma(1-\alpha)} + {}^C D^\alpha \varphi(t), \quad 0 < \alpha < 1, \quad t > 0. \quad (1.23)$$

Next, we recall the one-parametric and two-parametric Mittag-Leffler functions [83]

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \Re(\alpha) > 0,$$

and

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0,$$

resp. It is to be remarked that $E_{\alpha,1}(z) \equiv E_\alpha(z)$.

Proposition 1.4.4 ([85]) *If $\varphi(t)$ is a differentiable function satisfying*

$${}^C D^\alpha \varphi(t) \leq -\gamma \varphi(t), \quad 0 < \alpha < 1, \quad t > 0$$

for some $\gamma > 0$, then $\varphi(t) \leq \varphi(0)E_\alpha(-\gamma t^\alpha)$, $t \geq 0$ whilst in the Riemann-Liouville scenario, the decay is of the form $t^{\alpha-1}E_{\alpha,\alpha}(-\gamma t^\alpha)$.

Proposition 1.4.5 ([83]) *For $\alpha, \beta > 0$, the two-parametric Mittag-Leffler functions enjoy the property*

$$\lambda t^\alpha E_{\alpha,\alpha+\beta}(\lambda t^\alpha) = E_{\alpha,\beta}(\lambda t^\alpha) - \frac{1}{\Gamma(\beta)}, \quad t \geq 0.$$

Proposition 1.4.6 ([83], p. 61) *For $\mu, \alpha, \beta > 0$, it holds that*

$$\frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} E_{\alpha,\beta}(\lambda s^\alpha) s^{\beta-1} ds = t^{\mu+\beta-1} E_{\alpha,\mu+\beta}(\lambda t^\alpha), \quad t > 0.$$

The next proposition deals with the permutation of the Riemann-Liouville fractional derivative with the integral.

Proposition 1.4.7 ([84] p. 99) *If $I^{1-\alpha}\varphi(t) \in C^1([0, \infty))$, $0 < \alpha < 1$ and $\psi(t)$ is a continuous function, then*

$${}^{RL}D^\alpha \int_0^t \varphi(t-s)\psi(s)ds = \int_0^t \psi(s){}^{RL}D^\alpha \varphi(t-s)ds + \psi(t) \lim_{t \rightarrow 0^+} I^{1-\alpha}\varphi(t), \quad t > 0.$$

A fractional version of the well-known product rule in the integer-order case is given next.

Proposition 1.4.8 ([86]) *Let $\varphi(t)$ and $\psi(t)$ be absolutely continuous functions on $[0, T]$. Then, for $t \in [0, T]$ and $0 < \alpha < 1$, we have*

$$\begin{aligned} & \varphi(t) {}^C D^\alpha \psi(t) + \psi(t) {}^C D^\alpha \varphi(t) = {}^C D^\alpha (\varphi \psi)(t) \\ & + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{\varphi'(\eta) d\eta}{(t-\eta)^\alpha} \int_0^\xi \frac{\psi'(s) ds}{(t-s)^\alpha}. \end{aligned}$$

Setting $f(t)$ equal to $g(t)$ will lead to the familiar inequality (instead of the standard chain rule)

$${}^C D^\alpha f^2(t) \leq 2f(t) {}^C D^\alpha f(t), \quad t \geq 0.$$

The vector version also holds.

A more general fraction chain rule is provided by the proposition below.

Proposition 1.4.9 *Let $v(t)$ be a real-valued differentiable function, then for $0 < \alpha < 1$*

$${}^C D^\alpha \left(\frac{|v|^{\rho+2}}{\rho+2} \right) \leq |v|^\rho v {}^C D^\alpha v.$$

Proof. The inequality results immediately by combining a result in [87]

$${}^C D^\alpha \left(\frac{|v|^{\rho+2}}{\rho+2} \right) \leq |v|^{\rho+1} {}^C D^\alpha |v|$$

and another one in [88]

$${}^C D^\alpha |v| \leq \operatorname{sgn}(v) {}^C D^\alpha v.$$

■

From now on, we shall drop the letter "C" referring to "Caputo" in ${}^C D^\alpha$, for notation convenience. We shall use the notation

$$(k \square v)(t) := \int_{\Omega} \int_0^t k(t-s) |v(t) - v(s)|^2 ds dx, \quad t \geq 0$$

and

$$\mathcal{T}_\chi(t) := \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\chi'(\eta) d\eta}{(t-\eta)^\alpha} \right)^2 dx, \quad t \geq 0.$$

It is well-admitted and documented that there are no semi-groups for fractional problem. The counterpart of semigroup is the concept of resolvents as defined below (also refer to [89], [90]).

The following abstract problem has been examined in [91]

$$\begin{cases} {}^C D^\gamma U(x, t) = \mathcal{P}U(x, t) + \int_0^t \mathcal{S}(t-s)U(x, s)ds + f(t, U(t)), & 0 < \gamma < 1, \\ U(x, 0) = U_0(x) \in X. \end{cases} \quad (1.24)$$

Definition 1.4.10 *The family of bounded linear operators $(R_\gamma(t))_{t \geq 0}$ determines a γ -resolvent for (1.24) if*

(a) *The mapping $R_\gamma(t) : [0, \infty) \rightarrow \mathcal{L}(X) := \mathcal{L}(X; X)$ is strongly continuous and $R_\gamma(0) = I$.*

(b) *$\forall z \in D(\mathcal{P}), R_\gamma(\cdot)z \in C([0, \infty); D(\mathcal{P})) \cap C^\gamma((0, \infty); X)$ designates the*

space of continuous functions z such that ${}^C D^\gamma z$ exists and is continuous and

$$\begin{aligned} {}^C D^\gamma R_\gamma(t)z &= \mathcal{P}R_\gamma(t)z + \int_0^t \mathcal{Q}(t-s)R_\gamma(s)zds \\ &= R_\gamma(t)\mathcal{P}u + \int_0^t R_\gamma(t-s)\mathcal{Q}(s)zds, \quad t \geq 0. \end{aligned}$$

The following family

$$R_\gamma(t) := \frac{1}{2\pi i} \int_\Gamma \mu^{\gamma-1} e^{\mu t} (\mu^\gamma I - \mathcal{P} - \hat{\mathcal{Q}}(\mu))^{-1} d\mu, \quad t \geq 0$$

is shown to be a γ -resolvent for a suitable path $\Gamma := \{te^{i\theta} : t \geq r\} \cup \{re^{i\zeta} : -\theta \leq \zeta \leq \theta\} \cup \{te^{-i\theta} : t \geq r\}$ oriented counterclockwise where $\pi/2 < \theta < \phi$ and r is determined in the proof. An associated family to $(R_\gamma(t))_{t \geq 0}$ is defined by

$$S_\gamma(t) := \frac{t^{1-\gamma}}{2\pi i} \int_\Gamma e^{\mu t} (\mu^\gamma I - \mathcal{P} - \hat{\mathcal{Q}}(\mu))^{-1} d\mu, \quad t \geq 0.$$

Before moving forward in this direction, it is worth recalling the following result in [82] about a similar abstract problem but of order between 1 and 2 (without transforming it into a system of two equations of half the order). Let $(X, \|\cdot\|)$ be a Banach space and $M, (N(t))_{t \geq 0}$ be closed linear operators defined on the domains $D(M)$ and $D(N(t)) \supseteq D(M)$ dense in X , respectively. We denote by $\|\cdot\|_1$ the graph norm in $D(M)$, $R(\nu, M) := (\nu I - M)^{-1}$ and $\rho(M)$ the resolvent of M . The

problem

$${}^CD^\beta v(x, t) = Mv(x, t) - \int_0^t N(t-s)v(x, s)ds, \quad 1 < \beta < 2$$

with initial data $(v(0) = v_0, v'(0) = 0)$ in $D(M)$, admits a classical solution $v \in C((0, \infty); D(M)) \cap C^1((0, \infty); X)$ such that $\frac{t^{1-\beta}}{\Gamma(2-\beta)} * (v - v_0) \in C^2((0, \infty); X)$ under the following assumptions

(S1) For some $0 < \theta_0 \leq \pi/2$ and every $\theta < \theta_0$, there is a constant $C > 0$ such that

$$\Sigma_{0, \beta\omega} := \{\nu \in \mathbf{C} : \nu \neq 0, |\arg(\nu)| < \beta\omega\} \subset \rho(M)$$

and $\|R(\nu, M)\| < C/|\nu|$, $\nu \in \Sigma_{0, \beta\omega}$ with $\omega = \theta + \pi/2$.

(S2) For each $v \in D(M)$, $N(\cdot)v$ is strongly measurable on $(0, \infty)$. There exists a locally integrable function $f(t)$ with Laplace transform $\hat{f}(\nu)$, $\Re(\nu) > 0$ and $\|N(t)v\| \leq f(t) \|v\|_1$, $t > 0$, $v \in D(M)$. In addition, $\hat{N} : \Sigma_{0, \pi/2} \rightarrow \mathcal{L}(D(M); X)$ (the space of bounded linear operators from $D(M)$ into X) has an analytic extension $\tilde{N}$ to $\Sigma_{0, \omega}$ verifying $\|\tilde{N}(\nu)v\| \leq \|\tilde{N}(\nu)\| \|v\|_1$, $v \in D(M)$ and $\|\tilde{N}(\nu)\| = O\left(\frac{1}{|\nu|}\right)$ as $|\nu| \rightarrow \infty$.

(S3) There exists a subspace S dense in $(D(M), \|\cdot\|_1)$ and $\tilde{C} > 0$ such that $M(S) \subseteq D(M)$, $\hat{N}(\nu)(S) \subseteq D(M)$, $\|M\hat{N}(\nu)v\| \leq \tilde{C} \|v\|$, $v \in S$ and $\nu \in \Sigma_{0, \omega}$.

It is proved in [91] that (1.24) possesses a unique solution under the above assumptions **(S2)**, **(S3)** and

(S1)' For some $\pi/2 < \phi < \pi$, there exists a constant $C = C(\phi) > 0$ such that

$$\Sigma_{0,\phi} := \{\nu \in \mathbf{C} : |\arg(\nu)| < \phi\} \subset \rho(\mathcal{P})$$

and $\|R(\nu, \mathcal{P})\| < C/|\nu|$, $\nu \in \Sigma_{0,\phi}$.

The function $f : (0, \infty) \times X \rightarrow X$ was assumed to be continuous and locally Lipschitz in the second variable uniformly, with respect to the first variable.

Theorem 1.4.11 ([91]) *For $U_0 \in \mathcal{D}(\mathcal{P})$, and under the condition **(S1)'**, **(S2)** and **(S3)**, the function*

$$U(t) := R_\gamma(t)U_0 \in C([0, \infty); \mathcal{D}(\mathcal{P})) \cap C^\gamma((0, \infty); X)$$

is a solution of (1.24) (with $f \equiv 0$).

In the case $f \equiv 0$, we have the following Theorem

Theorem 1.4.12 ([91]) *For $U_0 \in \mathcal{D}(\mathcal{P})$, and under the condition **(S1)'**, **(S2)** and **(S3)**, the function*

$$U(t) := R_\gamma(t)U_0 + \int_0^t (t-s)^{\gamma-1} S_\gamma(t-s) f(s, U(s)) ds$$

belongs to $C([0, \infty); \mathcal{D}(\mathcal{P})) \cap C^\gamma((0, \infty); X)$ and is a solution of (1.24) for $T > 0$.

1.5 Results Description

1.5.1 Bending Torsion Problem

One of our main focus in this dissertation is system (1.1) describing the motion of a beam of Timoshenko type. The main novelty in this model is the insertion of the torsion component and coupling it with the Timoshenko system combining in this way, bending, rotation and torsion. Although, this issue has been discussed from the engineering point of view, we are unable to find any mathematical study (well-posedness, stability, ...).

In addition to the aforementioned features, the presence of axial forces in the structure brings considerable complications in the mathematical analysis. This factor, too, is rarely studied by mathematicians. The usual methods and techniques do not apply in a systematic manner even with some modifications. It needs some drastic changes. The existence and uniqueness of a solution in an adequate space has been shown using the semi-group theory. Regarding the stability of the system, in view of the difficulties mentioned above, we have treated the case $P = 0$ and $P \neq 0$ separately.

We have examined the stability of the system with two dampings and determined the conditions under which exponential stability and polynomial stability occur. Furthermore, the system well-posedness is determined by using the semi-group theory.

In Chapter 2, we treat the case when the damping is absent in the first equation i.e. $c_3 = 0$. In the first section of this chapter, we consider the formulation of

the system (2.1) and also find the conditions that preserves the system's hyperbolicity. We study the well-posedness of the system using the semigroup theory. We also establish the exponential stability of the system. We consider both cases of effective and non-effective axial load P . In the first case where $P \neq 0$ we obtained exponential stability under the condition that EI is large or small (negative) P . This result is obtained without using the equal speeds of propagation. Moreover, in the case when $P = 0$, we obtain exponential stability without any condition on the system's parameters. We establish this result by benefiting from the second diffusion in the third equation. More precisely, we acquire a new Lyapunov functional that helps us to obtain this result.

In Chapter 3, we consider the absence of the damping in the second equation, i.e. $c_1 = 0$. In the first section, we discuss the well-posedness of the system. In the second section, we consider the case $P \neq 0$ and we prove exponential stability under the condition that $\eta_2 = 0$ (where η_2 is a relation involving the parameter of the system) and large EI or small (negative) P . Notably, the condition $\eta_2 = 0$ reduces to the equal speeds of propagation property when $K = P = 0$. In the case $\eta_2 \neq 0$, we get polynomial decay. In the final section of this chapter, we focus on the case $P = 0$. We obtain exponential decay in case $\eta_1 = 0$. In the case $\eta_1 \neq 0$, we find polynomial decay.

The second aim of this thesis is to consider two fractional problems discussed in the next part.

1.5.2 Non-Linear Fractional Telegraph Problem

In Chapter 4, we aim to investigate the effect of a nonlinear term, the impact of a fractional lower-order term and the role of a memory term on the stability of differential equations. We will analyze the impact of a fractional lower-order term and the memory term separately. In the integer case, it is well-known that having both of them simultaneously will increase the dissipation. Hence, the difficult issue is to focus on only one of them at a time. The lower-order term has been expressed as a fractional derivative of half-order of the leading fractional derivative mentioned in equation (1.22). It would be intriguing, particularly from a mathematical perspective, to consider different other orders. Below, we will demonstrate that this term induces a dissipative behavior. In the proof, we will demonstrate that the lower-order term alone, without considering the memory term, is enough to ensure the Mittag-Leffler stability of the system. This means that the solutions of the system will decay as the Mittag-Leffler function (in the energy norm).

Nevertheless, in the absence of the lower-order fractional term, the Mittag-Leffler stability can only be established for "small" kernels under a viscoelastic damping. This might not be due to a deficiency in the argument used but to the fact that the memory term alone does not have the ability to generate a significant amount of damping. The application of the product rule (as stated in Proposition 1.4.8 above) produces a complicated term which does not appear in the integer case. It is characterized by three singular integrals that involve functions with in

determined signs.

Furthermore, the stability we achieved is only of local character due to the difficulty to control the nonlinear terms. In reality, this particular characteristic is not exclusive to fractional problems, it is also observed in problems of integer orders. The purpose of this chapter is to propose a solution to address these challenges. Unfortunately, we were unable to compare our findings with previous ones since, to the best of our knowledge, this is the first work that treats this issue.

1.5.3 Fractional Timoshenko Beam

In Chapter 5, for problem (1.6)-(1.7), we shall first finalize the discussion on the existence and uniqueness of a solution by using the notion of resolvents (instead of semi-groups). Then, we will highlight the difficulty of treating the stability for this problem. The utilization of a fractional chain rule version gives rise to a problematic term in the fractional derivative of the “energy” functional. Even though we found a way to control it, a smallness condition on the involved kernel is still imposed. We obtain Mittag-Leffler- type stability provided that the kernel itself decays somehow in a similar manner.

Additionally, we demonstrate that the memory term alone can stabilize the system in a Mittag-Leffler fashion under the condition of equal speeds of propagation, that is $\frac{\rho_1}{k} = \frac{\rho_2}{b}$ in equation (1.7). The system exhibits Lyapunov stability or uniform stability when considering different speeds of propagation. This is established by passing to the second order energy.

CHAPTER 2

BENDING TORSION

WITHOUT DAMPING IN THE

ROTATION COMPONENT

In this chapter, we consider

$$\left\{ \begin{array}{l} \rho I \phi_{tt} + c_3 \phi_t - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} = 0, \quad x \in (0, L), \quad t \in (0, \infty), \\ \mu v_{tt} + c_1 v_t - kAG(v_{xx} - \phi_x) + P v_{xx} = 0, \quad x \in (0, L), \quad t \in (0, \infty), \\ -GJ\psi_{xx} - K\phi_{xx} + P(I_s/\mu)\psi_{xx} + I_s\psi_{tt} + c_2\psi_t = 0, \quad x \in (0, L), \quad t \in (0, \infty) \end{array} \right. \quad (2.1)$$

where x is the spatial variable (the distance from the origin), t is the time variable and L is the length of the beam. The translation of the bending in the z direction is indicated by $v(x, t)$. The function $\psi(x, t)$ is the rotation of the torsion around the x - axis. The representation of the rotation of the cross-section caused by bending alone (or the bending-induced slope) is indicated as $\phi(x, t)$. We assume that there

is a constant axial load P acting along the cross-section aligned with the x -axis. The load P can have either a positive or negative value, in order to incorporate tensions. The symbol ρ represents the density of the material. The second moment of area of the beam cross-section with respect to the x -axis is I . The symbol μ refers to the quantity of mass contained within a given length. The polar mass moment of inertia per unit length is defined with respect to the x -axis by I_s . The (positive) constants C_i , $i = 1, 2, 3$ stand for the coefficients of the (frictional) dampings of the bending, torsion and rotation components, respectively. The coefficients EI is the bending rigidity (note that E denotes Young's modulus) and K is the bending torsion coupling rigidity. The symbol kGA represents the shear rigidity of the composite beam. Moreover, A denotes the cross-sectional area, G represents the shear modulus and the shear correction is k . The torsional rigidity is denoted by GJ (where J stands for the polar moment of the area). We examine system (2.1) when c_3 is set to zero. This chapter is divided into four sections. In the first section, we give the derivation of the problem and make sure to conserve the hyperbolicity of the system. The second section addresses the well-posedness of the system in an adequate underlying space. The third section is dedicated to discussing the exponential stability of the system. The last section deals with an interesting special case when no axial loading is active on the structure. Notably, we obtain an unexpected stability result without imposing a standard condition. All our relations below hold for $t \geq 0$ and $x \in [0, L)$.

2.1 Formulation of the Model

The total energy corresponding to the system is given by

$$E_1(t) = \frac{1}{2} \int_0^L \left\{ EI(\phi_x)^2 - P \left[(v_x)^2 + (I_s/\mu)(\psi_x)^2 \right] + 2K\phi_x\psi_x + kAG(v_x - \phi)^2 \right. \\ \left. + GJ(\psi_x)^2 + \mu(v_t)^2 + I_s(\psi_t)^2 + \rho I(\phi_t)^2 \right\} dx.$$

For the hyperbolicity of the system we assume

$$P < kAG \quad \text{and} \quad P < \frac{GJ}{I_s} \mu. \quad (2.2)$$

If we differentiate the above functional with respect to time t

$$\frac{dE_1(t)}{dt} = \frac{1}{2} \int_0^L \left[EI \frac{d}{dt}(\phi_x)^2 - P \left(\frac{d}{dt}(v_x)^2 + \frac{d}{dt}[(I_s/\mu)(\psi_x)^2] \right) + 2K \frac{d}{dt}(\phi_x\psi_x) \right. \\ \left. + kAG \frac{d}{dt} \left[(v_x)^2 + (\phi)^2 - 2v_x\phi \right] + GJ \frac{d}{dt}(\psi_x)^2 + \mu \frac{d}{dt}(v_t)^2 + I_s \frac{d}{dt}(\psi_t)^2 + \rho I \frac{d}{dt}(\phi_t)^2 \right] dx$$

we find

$$\frac{dE_1(t)}{dt} = \int_0^L \left\{ EI\phi_x\phi_{xt} - P \left[v_x v_{xt} + (I_s/\mu)\psi_x\psi_{xt} \right] + K(\phi_x\psi_{tx} + \phi_{xt}\psi_x) \right. \\ \left. + kAG(v_x v_{xt} + \phi\phi_t - v_{xt}\phi - v_x\phi_t) + GJ\psi_x\psi_{xt} + \mu v_t v_{tt} + I_s\psi_t\psi_{tt} + \rho I\phi_t\phi_{tt} \right\} dx. \quad (2.3)$$

The identities below are direct applications of integration by parts

$$EI \int_0^L \phi_x\phi_{xt} \, dx = EI\phi_t\phi_x \Big|_0^L - EI \int_0^L \phi_t\phi_{xx} \, dx, \quad (2.4)$$

$$-P \int_0^L v_x v_{xt} dx = -P v_t v_x \Big|_0^L + P \int_0^L v_t v_{xx} dx, \quad (2.5)$$

$$-P(I_s/\mu) \int_0^L \psi_x \psi_{xt} dx = -P(I_s/\mu) \psi_t \psi_x \Big|_0^L + P(I_s/\mu) \int_0^L \psi_t \psi_{xx} dx, \quad (2.6)$$

$$K \int_0^L \phi_x \psi_{xt} dx = K \psi_t \phi_x \Big|_0^L - K \int_0^L \psi_t \phi_{xx} dx, \quad (2.7)$$

$$K \int_0^L \phi_{tx} \psi_x dx = K \phi_t \psi_x \Big|_0^L - K \int_0^L \phi_t \psi_{xx} dx, \quad (2.8)$$

$$kAG \int_0^L v_x v_{xt} dx = kAG v_t v_x \Big|_0^L - kAG \int_0^L v_t v_{xx} dx, \quad (2.9)$$

$$-kAG \int_0^L v_{xt} \phi dx = -kAG v_t \phi \Big|_0^L + kAG \int_0^L v_t \phi_x dx, \quad (2.10)$$

$$GJ \int_0^L \psi_x \psi_{xt} dx = GJ \psi_t \psi_x \Big|_0^L - GJ \int_0^L \psi_t \psi_{xx} dx. \quad (2.11)$$

Having in mind the aforesaid identities (2.4), (2.5), (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11) in (2.3), we infer

$$\begin{aligned} \frac{dE_1(t)}{dt} = & \int_0^L \left[-EI \phi_t \phi_{xx} + P \left(v_t v_{xx} + (I_s/\mu) \psi_t \psi_{xx} \right) - K \left(\psi_t \phi_{xx} + \phi_t \psi_{xx} \right) \right. \\ & + kAG \left(-v_t v_{xx} + \phi \phi_t + v_t \phi_x - v_x \phi_t \right) - GJ \psi_t \psi_{xx} + \mu v_t v_{tt} + I_s \psi_t \psi_{tt} + \rho I \phi_t \phi_{tt} \left. \right] dx \\ & + \underbrace{\left[EI \phi_t \phi_x \Big|_0^L - P v_t v_x \Big|_0^L - P(I_s/\mu) \psi_t \psi_x \Big|_0^L + K \psi_t \phi_x \Big|_0^L + K \phi_t \psi_x \Big|_0^L + kAG v_t v_x \Big|_0^L \right.} \\ & \left. \underbrace{-kAG v_t \phi \Big|_0^L + GJ \psi_t \psi_x \Big|_0^L}_{=I} \right]. \end{aligned} \quad (2.12)$$

A reordering in (2.12) leads to

$$\begin{aligned}
\frac{dE_1(t)}{dt} = \int_0^L & \left[\left(-EI\phi_{xx} - K\psi_{xx} + kAG(\phi - v_x) + \rho I\phi_{tt} \right) \phi_t \right. \\
& + \left(Pv_{xx} + kAG(\phi_x - v_{xx}) + \mu v_{tt} \right) v_t \\
& \left. + \left(P(I_s/\mu)\psi_{xx} - K\phi_{xx} - GJ\psi_{xx} + I_s\psi_{tt} \right) \psi_t \right] dx + I.
\end{aligned} \tag{2.13}$$

On the account of our boundary conditions, that is

$$\phi(0, t) = \phi(L, t) = v_x(0, t) = v_x(L, t) = \psi(0, t) = \psi(L, t) = 0$$

it is evident that $I = 0$. We are left with the relation

$$\frac{dE_1(t)}{dt} = -c_3\|\phi_t\|^2 - c_1\|v_t\|^2 - c_2\|\psi_t\|^2, \quad t \geq 0.$$

To have an idea of the second order energy $E_2(t)$, we differentiate system (2.1)

with respect to time

$$\begin{cases} \rho I\phi_{ttt} + c_3\phi_{tt} - EI\phi_{xxt} - kAG(v_x - \phi)_t - K\psi_{xxt} = 0, \\ \mu v_{ttt} + c_1v_{tt} - kAG(v_{xx} - \phi_x)_t + Pv_{xxt} = 0, \\ -GJ\psi_{xxt} - K\phi_{xxt} + P(I_s/\mu)\psi_{xxt} + I_s\psi_{ttt} + c_2\psi_{tt} = 0. \end{cases} \tag{2.14}$$

Multiplying the first equation of (2.14) by ϕ_{tt} , the second equation by v_{tt} and the last equation by ψ_{tt} , we find

$$\begin{cases} \rho I \phi_{tt} \phi_{ttt} + c_3 \phi_{tt}^2 - EI \phi_{tt} \phi_{xxt} - kAG(v_x - \phi)_t \phi_{tt} - K \phi_{tt} \psi_{xxt} = 0, \\ \mu v_{tt} v_{ttt} + c_1 v_{tt}^2 - kAG(v_{xx} - \phi_x)_t v_{tt} + P v_{xxt} v_{tt} = 0, \\ -GJ \psi_{xxt} \psi_{tt} - K \phi_{xxt} \psi_{tt} + P(I_s/\mu) \psi_{xxt} \psi_{tt} + I_s \psi_{ttt} \psi_{tt} + c_2 \psi_{tt}^2 = 0. \end{cases}$$

After, integrating the above equations over $(0, L)$ and summing them up, we obtain

$$\begin{aligned} & - \int_0^L \left(c_1 v_{tt}^2 + c_2 \psi_{tt}^2 + c_3 \phi_{tt}^2 \right) dx = \int_0^L \left[-EI \phi_{tt} \phi_{xxt} + P \left(v_{tt} v_{xxt} + (I_s/\mu) \psi_{tt} \psi_{xxt} \right) \right. \\ & - K \left(\psi_{tt} \phi_{xxt} + \phi_{tt} \psi_{xxt} \right) + kAG \left(-v_{xxt} v_{tt} - \phi_{xt} v_{tt} + v_{xt} \phi_{tt} + \phi_t \phi_{tt} \right) - GJ \psi_{xxt} \psi_{tt} \\ & \left. + \mu v_{ttt} v_{tt} + I_s \psi_{ttt} \psi_{tt} + \rho I \phi_{ttt} \phi_{tt} \right] dx. \end{aligned}$$

With the help of integration by parts, we can see that

$$\begin{aligned} & - \int_0^L \left(c_1 v_{tt}^2 + c_2 \psi_{tt}^2 + c_3 \phi_{tt}^2 \right) dx = \int_0^L \left[\frac{1}{2} \frac{d}{dt} EI \phi_{xt}^2 - P \frac{1}{2} \frac{d}{dt} (v_{xt}^2 + (I_s/\mu) \psi_{xt}^2) \right. \\ & \left. - K \frac{d}{dt} (\phi_{xt} \psi_{xt}) + \frac{1}{2} \frac{d}{dt} kAG(v_{xt} - \phi_t)^2 + \frac{1}{2} \frac{d}{dt} \left(GJ \psi_{xt}^2 + \mu v_{tt}^2 + I_s \psi_{tt}^2 + \rho I \phi_{tt}^2 \right) \right] dx. \end{aligned}$$

Prompted by this identity, we may define the second order energy by

$$\begin{aligned} E_2(t) = & \frac{1}{2} \int_0^L \left[EI \phi_{xt}^2 - P[v_{xt}^2 + (I_s/\mu) \psi_{xt}^2] - 2K(\phi_{xt} \psi_{xt}) + kAG(v_{xt} - \phi_t)^2 + GJ \psi_{xt}^2 \right. \\ & \left. + \mu v_{tt}^2 + I_s \psi_{tt}^2 + \rho I \phi_{tt}^2 \right] dx, \quad \forall t \geq 0 \end{aligned}$$

and its derivative is clearly equal to

$$\frac{dE_2}{dt} = -c_1||v_{tt}||^2 - c_2||\psi_{tt}||^2 - c_3||\phi_{tt}||^2, \quad t \geq 0.$$

Alternatively, we may compute the derivation of the energy by direct multiplication of the first equation of the system by ϕ_t , the second by v_t and the third by ψ_t . In the expression

$$\begin{aligned} E(t) = \frac{1}{2} \left[EI||\phi_x||_2^2 - P||v_x||_2^2 + \frac{I_s P}{\mu}||\psi_x||_2^2 + kAG||v_x - \phi||_2^2 + GJ||\psi_x||_2^2 \right. \\ \left. + \mu||v_t||_2^2 + I_s||\psi_t||_2^2 + \rho I||\phi_t||_2^2 + 2K \int_0^L \phi_x \psi_x dx \right], \quad t \geq 0 \end{aligned}$$

the part

$$T(t) = \frac{1}{2} \int_0^L \left[\mu(v_t)^2 + I_s(\psi_t)^2 + \rho I(\phi_t)^2 \right] dx, \quad t \geq 0$$

is called the kinetic energy whilst the potential energy for $t \geq 0$ is a equal to (the reaming part)

$$\begin{aligned} U(t) = \frac{1}{2} \int_0^L \left\{ EI(\phi_x)^2 + 2K\phi_x\psi_x + [GJ - P(I_s/\mu)](\psi_x)^2 - P(v_x)^2 \right. \\ \left. + kAG(v_x - \phi)^2 \right\} dx. \end{aligned}$$

We claim that $U(t) \geq 0$, under the condition

$$K < GJ - P(I_s/\mu), \quad K < EI. \quad (2.15)$$

To prove this claim, let us perform the decomposition

$$EI\phi_x^2 = EI\delta\phi_x^2 + EI(1-\delta)\phi_x^2, \quad 0 < \delta < 1$$

and rewrite $U(t)$ in the form

$$\begin{aligned} U(t) = \frac{1}{2} \int_0^L & \left[\underbrace{EI\delta\phi_x^2 + 2K\phi_x\psi_x + [GJ - P(I_s/\mu)]\psi_x^2}_{:=A_1} \right. \\ & \left. + \underbrace{EI(1-\delta)\phi_x^2 - P(v_x)^2 + kAG(v_x - \phi)^2}_{:=A_2} \right] dx. \end{aligned}$$

The quadratic expression A_1 is nonnegative provided that

$$\Delta_{A_1} = 4K^2 - 4\delta EI[GJ - P(I_s/\mu)] < 0,$$

which is obviously the case if $\delta = K/EI$ (the choice of δ is possible) since (2.15)

holds.

For A_2 , we first rewrite it in a tricky way, then apply Poincaré inequality to reach

$$\begin{aligned} A_2 &= \int_0^L \left[EI \left(1 - \frac{K}{EI} \right) \phi_x^2 + (kAG - P)(v_x - \phi)^2 - 2P(v_x - \phi)\phi - P\phi^2 \right] dx \\ &= \int_0^L \left[(EI - K)\phi_x^2 + (kAG - P)(v_x - \phi)^2 - 2P(v_x - \phi)\phi - P\phi^2 + M\phi^2 - M\phi^2 \right] dx \\ &\geq \int_0^L \left[\underbrace{[(EI - K) - (M + P)C_p]}_{\geq 0} \phi_x^2 + \underbrace{(kAG - P)(v_x - \phi)^2 - 2P(v_x - \phi)\phi + M\phi^2}_{:=A_3} \right] dx \end{aligned}$$

for some $M > 0$ to be determined in an adequate fashion. Actually, we want

to manage so as to make the first term and A_3 positive. To this end, we start imposing

$$(EI - K) - (M + P)C_p > 0 \iff EI - K > MC_p + PC_p$$

that is

$$M < \frac{EI - K - PC_p}{C_p}. \quad (2.16)$$

On the other hand,

$$\triangle_{A_3} = 4P^2 - 4(kAG - P)M < 0$$

necessitates

$$M > \frac{P^2}{kAG - P}. \quad (2.17)$$

Combining the conditions (2.16) and (2.17)

$$\frac{EI - K - PC_p}{C_p} > M > \frac{P^2}{kAG - P}$$

we observe that a choice of M is possible in case

$$P^2 C_p < (EI - K - PC_p)(kAG - P).$$

Solving for P we reach the condition

$$P < \frac{(EI - K)kAG}{kAGC_p + (EI - K)} =: P_*. \quad (2.18)$$

Hence, to preserve the positivity of the energy we must assume the condition in (2.18). Notice that the condition $P < \frac{EI-K}{C_p}$ is therefore automatically satisfied.

Having in mind (2.18), it is clear that

$$P < P_* < kAG. \quad (2.19)$$

This implies that the conditions for the system's hyperbolicity, as defined in (2.2) and the condition in (2.19), can be expressed as

$$P < \min \left\{ P_*, \frac{GJ}{I_s} \mu \right\}. \quad (2.20)$$

2.2 Well-Posedness of the Problem

In this section, we use the semigroup theory to study the existence and uniqueness of solutions for the bending torsion system (2.1) with $c_3 = 0$, that is,

$$\left\{ \begin{array}{l} \rho I \phi_{tt} - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} = 0, \\ \mu v_{tt} + c_1 v_t - [kAG - P](v_{xx} - \phi_x) + P\phi_x = 0, \\ -[GJ - P(I_s/\mu)]\psi_{xx} - K\phi_{xx} + I_s\psi_{tt} + c_2\psi_t = 0, \\ \phi(0, t) = \phi(L, t) = v_x(0, t) = v_x(L, t) = \psi(0, t) = \psi(L, t) = 0, \\ \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x). \end{array} \right. \quad (2.21)$$

Given the Neumann boundary conditions for v , the standard practice is to make the change of function

$$\bar{v}(x, t) := v(x, t) - \frac{1}{L} \int_0^L v(x, t) dx, \quad x \in [0, L], \quad t \geq 0 \quad (2.22)$$

so that

$$\int_0^L \bar{v}(x, t) dx = 0, \quad t \geq 0$$

and therefore, we can apply Poincaré inequality. Obviously

$$\bar{v}_t(x, t) = v_t(x, t) - \frac{1}{L} \int_0^L v_t(x, t) dx \quad \text{and} \quad \bar{v}_{tt}(x, t) = v_{tt}(x, t) - \frac{1}{L} \int_0^L v_{tt}(x, t) dx. \quad (2.23)$$

From the second equation in (2.21), we entail

$$\begin{aligned}\mu \int_0^L v_{tt} dx &= -c_1 \int_0^L v_t dx + [kAG - P] \int_0^L (v_{xx} - \phi_x) dx - P \int_0^L \phi_x dx \\ &= -c_1 \int_0^L v_t dx + [kAG - P] (v_x - \phi) \Big|_0^L - P \phi \Big|_0^L = -c_1 \int_0^L v_t dx.\end{aligned}\quad (2.24)$$

If we substitute (2.23) in (2.24), it appears that

$$\int_0^L v_{tt} dx = -\frac{c_1}{\mu} \int_0^L v_t dx = \frac{c_1 L}{\mu} (\bar{v}_t - v_t).$$

Hence

$$\bar{v}_{tt} = v_{tt} - \frac{c_1 L}{L\mu} (\bar{v}_t - v_t) \text{ or } v_{tt} = \bar{v}_{tt} + \frac{c_1}{\mu} (\bar{v}_t - v_t).$$

Consequently, the second equation of (2.21) becomes

$$\mu \left[\bar{v}_{tt} + \frac{c_1}{\mu} (\bar{v}_t - v_t) \right] + c_1 v_t - [kAG - P] (\bar{v}_x - \phi)_x + P \phi_x = 0$$

or simply

$$\mu \bar{v}_{tt} + c_1 \bar{v}_t - [kAG - P] (\bar{v}_x - \phi)_x + P \phi_x = 0.$$

For the first equation, it suffices to notice that $\bar{v}_x = v_x$ and use (2.22). This means that the above problem (2.21) may be recast in a very similar one by merely replacing v by $\bar{v}$. To avoid a complicated presentation with the bar notation, we shall drop the bar. We end up with the same problem with the advantage of being able to apply Poincaré inequality to the component v .

For the sake of preparing the ground for our well-posedness, let us consider

the Hilbert space:

$$\mathcal{H} := H_0^1(0, L) \times L^2(0, L) \times H_*^1(0, L) \times L_*^2(0, L) \times H_0^1(0, L) \times L^2(0, L)$$

where

$$L_*^2(0, L) := \left\{ \vartheta \in L^2(0, L) : \int_0^L \vartheta(x) dx = 0 \right\}, \quad H_*^1(0, L) = H^1(0, L) \cap L_*^2(0, L) \quad (2.25)$$

and

$$H_*^2(0, L) := H^2(0, L) \cap H_*^1(0, L). \quad (2.26)$$

The space $\mathcal{H}$ is endowed with the inner product

$$\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^L \left[\rho I u \tilde{u} + \mu w \tilde{w} + I_s z \tilde{z} + EI \phi_x \tilde{\phi}_x + [GJ - P(I_s/\mu)] \psi_x \tilde{\psi}_x \right. \quad (2.27)$$

$$\left. + K [\psi_x \tilde{\phi}_x + \phi_x \tilde{\psi}_x] + (kAG - P) [(v_x - \phi)(\tilde{v}_x - \tilde{\phi})] + P\phi\tilde{\phi} - P(v_x\tilde{\phi} + \phi\tilde{v}_x) \right] dx$$

and the norm

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 &= \rho I \|u\|^2 + \mu \|w\|^2 + I_s \|z\|^2 + EI \|\phi_x\|^2 + [GJ - P(I_s/\mu)] \|\psi_x\|^2 \\ &+ 2K \int_0^L \psi_x \phi_x dx + (kAG - P) \|v_x - \phi\|^2 + P \|\phi\|^2 - 2P \int_0^L v_x \phi dx \end{aligned} \quad (2.28)$$

for the vectors $U = (\phi, u, v, w, \psi, z)^T$ and $\tilde{U} = (\tilde{\phi}, \tilde{u}, \tilde{v}, \tilde{w}, \tilde{\psi}, \tilde{z})^T \in \mathcal{H}$. We can easily see that $\mathcal{H}$ equipped with the inner product (2.27) and the norm (2.28),

forms a Hilbert space under the condition (2.18).

Considering U with $u = \phi_t, w = v_t, z = \psi_t$, we can rewrite the system in (2.21) in the shorter and simpler form

$$\begin{cases} \frac{d}{dt}U(t) = \mathcal{A}U(t), & \forall t \geq 0, \\ U(0) = U_0 = (\phi_0, \phi_1, v_0, v_1, \psi_0, \psi_1)^T \end{cases} \quad (2.29)$$

where the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{H}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \frac{EI}{\rho I}D^2 - \frac{kAG}{\rho I} & 0 & \frac{kAG}{\rho I}D & 0 & \frac{K}{\rho I}D^2 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{kAG}{\mu}D & 0 & \frac{kAG-P}{\mu}D^2 & -\frac{c_1}{\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \frac{K}{I_s}D^2 & 0 & 0 & 0 & \frac{GJ-P(I_s/\mu)}{I_s}D^2 & -\frac{c_2}{I_s} \end{pmatrix} \begin{pmatrix} \phi \\ u \\ v \\ w \\ \psi \\ z \end{pmatrix} \quad (2.30)$$

where D denotes the partial derivative with respect to x . That is

$$\mathcal{A}U = \begin{pmatrix} u \\ \frac{EI}{\rho I}\phi_{xx} + \frac{kAG}{\rho I}(v_x - \phi) + \frac{K}{\rho I}\psi_{xx} \\ w \\ -\frac{c_1}{\mu}w + \frac{kAG}{\mu}(v_x - \phi)_x - \frac{P}{\mu}v_{xx} \\ z \\ \frac{K}{I_s}\phi_{xx} + \frac{GJ-P(I_s/\mu)}{I_s}\psi_{xx} - \frac{c_2}{I_s}z \end{pmatrix} \quad (2.31)$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{ U \in \mathcal{H} \mid \phi, \psi \in H^2(0, L) \cap H_0^1(0, L), \ v \in H_*^2(0, L), \right. \\ \left. u, z \in H_0^1(0, L), \ w \in H_*^1(0, L) \right\}.$$

In our next move, we shall state and prove an existence and uniqueness theorem for our problem. We recall here that problem (2.21) has been transformed into a similar one with v replaced by $\bar{v}$ and the bar is dropped. With that in mind, we shall continue to refer to (2.21). Recall the conditions specified in (2.15) and (2.20), that is

$$P < \min \left\{ P_*, \frac{GJ}{I_s} \mu \right\}, \quad K < GJ - P(I_s/\mu), \quad K < EI. \quad (2.32)$$

Theorem 2.2.1 *Let $U_0 \in \mathcal{H}$ and the condition (2.32) holds. Then, there exists a unique solution $U \in C(\mathbb{R}^+, \mathcal{H})$ to the Problem (2.21). Furthermore, if $U_0 \in \mathcal{D}(\mathcal{A})$, then we have a unique solution $U \in C(\mathbb{R}^+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. Our argument relies on the use of the Lumer-Phillips Theorem [92].

To begin with, we claim that the operator $\mathcal{A}$ is dissipative. Indeed, for every

$U \in \mathcal{D}(\mathcal{A})$, it is evident that

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \rho I \int_0^L u \left[\frac{EI}{\rho I} \phi_{xx} + \frac{kAG}{\rho I} (v_x - \phi) + \frac{K}{\rho I} \psi_{xx} \right] dx \\
&+ \mu \int_0^L w \left[-\frac{c_1}{\mu} w + \frac{kAG}{\mu} (v_x - \phi)_x - \frac{P}{\mu} v_{xx} \right] dx \\
&+ I_s \int_0^L z \left[\frac{K}{I_s} \phi_{xx} + \frac{GJ - P(I_s/\mu)}{I_s} \psi_{xx} - \frac{c_2}{I_s} z \right] dx + EI \int_0^L \phi_x u_x dx \\
&+ [GJ - P(I_s/\mu)] \int_0^L \psi_x z_x dx + K \int_0^L [\phi_x z_x + \psi_x u_x] dx \\
&+ (kAG - P) \int_0^L (w_x - u)(v_x - \phi) dx + P \int_0^L u \phi dx - P \int_0^L [w_x \phi + v_x u] dx.
\end{aligned}$$

After several integration by parts, our dissipativeness claim follow from

$$\begin{aligned}
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= -EI \int_0^L u_x \phi_x dx + kAG \int_0^L uv_x dx - kAG \int_0^L u \phi dx - K \int_0^L u_x \psi_x dx \\
&- c_1 \|w\|^2 - kAG \int_0^L v_x w_x dx + kAG \int_0^L \phi w_x dx + P \int_0^L v_x w_x dx - K \int_0^L z_x \phi_x dx \\
&- [GJ - P(I_s/\mu)] \int_0^L z_x \psi_x dx - c_2 \|z\|^2 + EI \int_0^L \phi_x u_x dx + [GJ - P(I_s/\mu)] \\
&\times \int_0^L \psi_x z_x dx + K \int_0^L \phi_x z_x dx + K \int_0^L \psi_x u_x dx + kAG \int_0^L w_x v_x dx \\
&- kAG \int_0^L w_x \phi dx - kAG \int_0^L uv_x dx + kAG \int_0^L u \phi dx - P \int_0^L w_x v_x dx + P \int_0^L w_x \phi dx \\
&+ P \int_0^L uv_x dx - P \int_0^L u \phi dx + P \int_0^L u \phi dx - P \int_0^L w_x \phi dx - P \int_0^L v_x u dx \\
&= -c_1 \|w\|^2 - c_2 \|z\|^2 \leq 0.
\end{aligned}$$

Second, we show that the operator $I - \mathcal{A}$ is surjective, that is for all $\mathcal{G} =$

$(g_1, g_2, g_3, g_4, g_5, g_6)^T \in \mathcal{H}$ there exists a unique $U = (\phi, u, v, w, \psi, z)^T \in \mathcal{D}(\mathcal{A})$,

such that

$$(I - \mathcal{A})U = \mathcal{G}.$$

Equivalently, this may be rewritten as

$$\left\{ \begin{array}{l} \phi - u = g_1, \\ u - \frac{EI}{\rho I} \phi_{xx} - \frac{kAG}{\rho I} (v_x - \phi) - \frac{K}{\rho I} \psi_{xx} = g_2, \\ v - w = g_3, \\ w + \frac{c_1}{\mu} w - \frac{kAG}{\mu} (v_x - \phi)_x + \frac{P}{\mu} v_{xx} = g_4, \\ \psi - z = g_5, \\ z - \frac{K}{I_s} \phi_{xx} - \frac{GJ-P(I_s/\mu)}{I_s} \psi_{xx} + \frac{c_2}{I_s} z = g_6. \end{array} \right. \quad (2.33)$$

For regular ϕ, v, ψ , the following holds true:

$$\left\{ \begin{array}{l} u = \phi - g_1 \in H_0^1(0, L), \\ w = v - g_3 \in H_*^1(0, L), \\ z = \psi - g_5 \in H_0^1(0, L). \end{array} \right. \quad (2.34)$$

Hence it suffices to solve for ϕ, v, ψ . Taking into account (2.34) in (2.33), leads to

$$\left\{ \begin{array}{l} \phi - g_1 - \frac{EI}{\rho I} \phi_{xx} - \frac{kAG}{\rho I} (v_x - \phi) - \frac{K}{\rho I} \psi_{xx} = g_2, \\ v - g_3 + \frac{c_1}{\mu} (v - g_3) - \frac{kAG}{\mu} (v_x - \phi)_x + \frac{P}{\mu} v_{xx} = g_4, \\ \psi - g_5 - \frac{K}{I_s} \phi_{xx} - \frac{GJ-P(I_s/\mu)}{I_s} \psi_{xx} + \frac{c_2}{I_s} (\psi - g_5) = g_6. \end{array} \right.$$

The forthcoming weak formulation is a direct consequence of Green's formula

$$\left\{ \begin{array}{l} \rho I \int_0^L \phi \tilde{\phi} dx + EI \int_0^L \phi_x \tilde{\phi}_x dx - kAG \int_0^L (v_x - \phi) \tilde{\phi} dx + K \int_0^L \psi_x \tilde{\phi}_x dx \\ = \rho I \int_0^L g_1 \tilde{\phi} dx + \rho I \int_0^L g_2 \tilde{\phi} dx, \quad \forall \tilde{\phi} \in H_0^1(0, L), \\ \mu \int_0^L v \tilde{v} dx + c_1 \int_0^L v \tilde{v} dx + kAG \int_0^L (v_x - \phi) \tilde{v}_x dx - P \int_0^L v_x \tilde{v}_x dx \\ = (\mu + c_1) \int_0^L g_3 \tilde{v} dx + \mu \int_0^L g_4 \tilde{v} dx, \quad \forall \tilde{v} \in H_*^1(0, L), \\ I_s \int_0^L \psi \tilde{\psi} dx + K \int_0^L \phi_x \tilde{\psi}_x dx + [GJ - P(I_s/\mu)] \int_0^L \psi_x \tilde{\psi}_x dx + c_2 \int_0^L \psi \tilde{\psi} dx \\ = (I_s + c_2) \int_0^L g_5 \tilde{\psi} dx + I_s \int_0^L g_6 \tilde{\psi} dx, \quad \forall \tilde{\psi} \in H_0^1(0, L). \end{array} \right.$$

The problem consists in finding $(\phi, v, \psi) \in H_0^1(0, L) \times H_*^1(0, L) \times H_0^1(0, L)$ such that for all $(\tilde{\phi}, \tilde{v}, \tilde{\psi}) \in H_0^1(0, L) \times H_*^1(0, L) \times H_0^1(0, L)$ the following holds

$$B((\phi, v, \psi), (\tilde{\phi}, \tilde{v}, \tilde{\psi})) = F(\tilde{\phi}, \tilde{v}, \tilde{\psi}) \quad (2.35)$$

where $B : [H_0^1(0, L) \times H_*^1(0, L) \times H_0^1(0, L)]^2 \longrightarrow \mathbb{R}$ is the bilinear functional defined by

$$\begin{aligned} B((\phi, v, \psi), (\tilde{\phi}, \tilde{v}, \tilde{\psi})) &:= \rho I \int_0^L \phi \tilde{\phi} dx + EI \int_0^L \phi_x \tilde{\phi}_x dx + K \int_0^L \psi_x \tilde{\phi}_x dx \\ &+ kAG \int_0^L (v_x - \phi)(\tilde{v}_x - \tilde{\phi}) dx + K \int_0^L \phi_x \tilde{\psi}_x dx + I_s \int_0^L \psi \tilde{\psi} dx + [GJ - P(I_s/\mu)] \\ &\times \int_0^L \psi_x \tilde{\psi}_x dx + c_2 \int_0^L \psi \tilde{\psi} dx + \mu \int_0^L v \tilde{v} dx + c_1 \int_0^L v \tilde{v} dx - P \int_0^L v_x \tilde{v}_x dx. \end{aligned}$$

In particular

$$\begin{aligned} B((\phi, v, \psi), (\phi, v, \psi)) &= \rho I \|\phi\|^2 + EI \|\phi_x\|^2 + kAG \|v_x - \phi\|^2 + 2K \int_0^L \phi_x \psi_x dx \\ &+ I_s \|\psi\|^2 + [GJ - P(I_s/\mu)] \|\psi_x\|^2 + c_2 \|\psi\|^2 + \mu \|v\|^2 + c_1 \|v\|^2 - P \|v_x\|^2. \end{aligned}$$

The linear form $F : H_0^1(0, L) \times H_*^1(0, L) \times H_0^1(0, L) \longrightarrow \mathbb{R}$ represents the expression

$$\begin{aligned} F(\tilde{\phi}, \tilde{v}, \tilde{\psi}) &:= \rho I \int_0^L g_1 \tilde{\phi} dx + \rho I \int_0^L g_2 \tilde{\phi} dx + (\mu + c_1) \int_0^L g_3 \tilde{v} dx + \mu \int_0^L g_4 \tilde{v} dx \\ &+ (I_s + c_2) \int_0^L g_5 \tilde{\psi} dx + I_s \int_0^L g_6 \tilde{\psi} dx. \end{aligned}$$

Set

$$\begin{aligned} \|(\phi, v, \psi)\|_{H_0^1 \times H_*^1 \times H_0^1}^2 &= \|\phi\|^2 + \|\phi_x\|^2 + \|\psi\|^2 + \|\psi_x\|^2 + \|v\|^2 + \|v_x\|^2 + \|v_x - \phi\|^2 \\ &= \|\phi\|_{H_0^1(0, L)}^2 + \|v\|_{H_*^1(0, L)}^2 + \|\psi\|_{H_0^1(0, L)}^2 + \|v_x - \phi\|^2. \end{aligned}$$

One of the requirements of the method is to prove that the operator B is continuous

$$\begin{aligned} |B((\phi, v, \psi), (\tilde{\phi}, \tilde{v}, \tilde{\psi}))| &\leq \rho I \int_0^L |\phi| |\tilde{\phi}| dx + EI \int_0^L |\phi_x| |\tilde{\phi}_x| dx + kAG \\ &\times \int_0^L |(v_x - \phi)| |(\tilde{v}_x - \tilde{\phi})| dx + K \int_0^L |\psi_x \tilde{\phi}_x + \phi_x \tilde{\psi}_x| dx + I_s \int_0^L |\psi| |\tilde{\psi}| dx \\ &+ [GJ - P(I_s/\mu)] \int_0^L |\psi_x| |\tilde{\psi}_x| dx + c_2 \int_0^L |\psi| |\tilde{\psi}| dx + \mu \int_0^L |v| |\tilde{v}| dx \\ &+ c_1 \int_0^L |v| |\tilde{v}| dx - P \int_0^L v_x \tilde{v}_x dx. \end{aligned}$$

The upcoming estimation results form direct utilization of Cauchy-Schwarz inequality

$$\begin{aligned}
|B((\phi, v, \psi), (\tilde{\phi}, \tilde{v}, \tilde{\psi}))| &\leq c\|\phi\|\|\tilde{\phi}\| + c\|v_x - \phi\|\|\tilde{v}_x - \tilde{\phi}\| \\
&+ c(\|\psi_x\| + \|\phi_x\|)(\|\tilde{\phi}_x\| + \|\tilde{\psi}_x\|) + c\|v\|^2\|\tilde{v}\|^2 + c\|v_x\|\|v_x\| \\
&\leq c(\|\phi\|_{H_0^1} + \|\psi\|_{H_0^1} + \|v\|_{H_*^1} + \|v_x - \phi\|) + (\|\tilde{\phi}\|_{H_0^1} + \|\tilde{\psi}\|_{H_0^1} + \|\tilde{v}\|_{H_*^1} + \|\tilde{v}_x - \tilde{\phi}\|) \\
&\leq c\|(\phi, v, \psi)\|_{H_0^1 \times H_*^1 \times H_0^1} \|(\tilde{\phi}, \tilde{v}, \tilde{\psi})\|_{H_0^1 \times H_*^1 \times H_0^1}
\end{aligned}$$

where $c > 0$ is a generic constant which may take different values at different occurrences and $(\phi, v, \psi), (\tilde{\phi}, \tilde{v}, \tilde{\psi}) \in [H_0^1(0, L) \times H_*^1(0, L) \times H_0^1(0, L)]^2$. Next, we show the continuity of F again by employing Cauchy-Schwarz inequality

$$\begin{aligned}
|F(\tilde{\phi}, \tilde{v}, \tilde{\psi})| &\leq \rho I \int_0^L |g_1| |\tilde{\phi}| dx + \rho I \int_0^L |g_2| |\tilde{\phi}| dx + (\mu + c_1) \int_0^L |g_3| |\tilde{v}| dx \\
&+ \mu \int_0^L |g_4| |\tilde{v}| dx + (I_s + c_2) \int_0^L |g_5| |\tilde{\psi}| dx + I_s \int_0^L |g_6| |\tilde{\psi}| dx \\
&\leq \rho I \|g_1\| \|\tilde{\phi}\| + \rho I \|g_2\| \|\tilde{\phi}\| + (\mu + c_1) \|g_3\| \|\tilde{v}\| + \mu \|g_4\| \|\tilde{v}\| \\
&+ (I_s + c_2) \|g_5\| \|\tilde{\psi}\|^2 + I_s \|g_6\| \|\tilde{\psi}\| \\
&\leq c \left(\|g_1\|_{H_0^1} + \|g_2\| + \|g_3\|_{H_*^1} + \|g_4\| + \|g_5\|_{H_0^1} + \|g_6\| \right) (\|\tilde{\phi}\|_{H_0^1} + \|\tilde{v}\|_{H_*^1} + \|\tilde{\psi}\|_{H_0^1}) \\
&\leq c \|(\tilde{\phi}, \tilde{v}, \tilde{\psi})\|_{H_0^1 \times H_*^1 \times H_0^1}.
\end{aligned}$$

Now that we are done with the continuity of B and F , we turn to the coercivity of B . For this purpose, we notice that from the definition of B , it is easy to see

that

$$\begin{aligned}
B((\phi, v, \psi), (\phi, v, \psi)) &= \rho I \|\phi\|^2 + EI \|\phi_x\|^2 + kAG \|v_x - \phi\|^2 + 2K \int_0^L \phi_x \psi_x dx \\
&+ I_s \|\psi\|^2 + [GJ - P(I_s/\mu)] \|\psi_x\|^2 + c_2 \|\psi\|^2 + \mu \|v\|^2 + c_1 \|v\|^2 - P \|v_x\|^2 \\
&\geq a_1 \|(\phi, v, \psi)\|_{H_0^1 \times H_*^1 \times H_0^1}^2
\end{aligned}$$

where $a_1 \geq 0$, $(\phi, v, \psi) \in H_0^1(0, L) \times H_*^1(0, L) \times H_0^1(0, L)$ and under the condition (2.18) which we recall here

$$P < \frac{(EI - K)kAG}{kAGC_p + (EI - K)}.$$

Our objective is achieved. By virtue of Lax-Milgram Lemma (see [93]), there exists

$$(\phi, v, \psi) \in H_0^1(0, L) \times H_*^1(0, L) \times H_0^1(0, L)$$

and in view of (2.34), it is clear that

$$(\phi, u, v, w, \psi, z) \in (H_0^1)^2 \times (H_*^1)^2 \times (H_0^1)^2.$$

Furthermore, if $(\tilde{\phi}, \tilde{v}, \tilde{\psi}) \equiv (0, \tilde{v}, 0) \in H_0^1 \times H_*^1 \times H_0^1$, then (2.35), reduces to

$$\begin{aligned}
&kAG \int_0^L (v_x - \phi) \tilde{v}_x dx + \mu \int_0^L v \tilde{v} dx + c_1 \int_0^L v \tilde{v} dx - P \int_0^L v_x \tilde{v}_x dx \\
&= (\mu + c_1) \int_0^L g_3 \tilde{v} dx + \mu \int_0^L g_4 \tilde{v} dx
\end{aligned}$$

and therefore

$$\begin{aligned}
& -kAG \int_0^L v_{xx} \tilde{v} dx + kAG \int_0^L \phi_x \tilde{v} dx + (\mu + c_1) \int_0^L v \tilde{v} dx + P \int_0^L v_{xx} \tilde{v} dx \\
& = (\mu + c_1) \int_0^L g_3 \tilde{v} dx + \mu \int_0^L g_4 \tilde{v} dx.
\end{aligned}$$

With the help of the second equation of (2.34), it appears that

$$\begin{aligned}
[kAG - P] \int_0^L v_{xx} \tilde{v} dx &= \int_0^L \left[kAG\phi_x + (\mu + c_1)(v - g_3) - \mu g_4 \right] \tilde{v} dx \\
&= \int_0^L \left[kAG\phi_x + (\mu + c_1)w - \mu g_4 \right] \tilde{v} dx, \quad \forall \tilde{v} \in H_*^1.
\end{aligned} \tag{2.36}$$

Moreover, for any $f_1 \in C_*^1(0, L) \subset H_*^1(0, L)$, it holds that

$$[kAG - P] \int_0^L v_{xx} f_1 dx = \int_0^L \left[kAG\phi_x + (\mu + c_1)w - \mu g_4 \right] f_1 dx, \quad \forall f_1 \in H_*^1.$$

This implies

$$[kAG - P]v_{xx} = kAG\phi_x + (\mu + c_1)w - \mu g_4 \in L^2(0, L)$$

which solves the fourth equation of (2.33). Hence, by the elliptic regularity theory,

we are lead to

$$v \in H_*^2(0, L).$$

Similarly, if $(\tilde{\phi}, \tilde{v}, \tilde{\psi}) \equiv (\tilde{\phi}, 0, 0) \in H_0^1 \times H_*^1 \times H_0^1$, then (2.35), becomes

$$\begin{aligned} & \rho I \int_0^L \phi \tilde{\phi} dx + EI \int_0^L \phi_x \tilde{\phi}_x dx - kAG \int_0^L (v_x - \phi) \tilde{\phi} dx + K \int_0^L \psi_x \tilde{\phi}_x dx \\ &= \rho I \int_0^L g_1 \tilde{\phi} dx + \rho I \int_0^L g_2 \tilde{\phi} dx \end{aligned}$$

or

$$\begin{aligned} & \rho I \int_0^L (\phi - g_1 - g_2) \tilde{\phi} dx - EI \int_0^L \phi_{xx} \tilde{\phi} dx - kAG \int_0^L v_x \tilde{\phi} dx + kAG \int_0^L \phi \tilde{\phi} dx \\ & - K \int_0^L \psi_{xx} \tilde{\phi} dx = 0. \end{aligned}$$

Appealing to the first equation of (2.34), we attain

$$\begin{aligned} EI \int_0^L \phi_{xx} \tilde{\phi} dx &= \int_0^L \left[\rho I (\phi - g_1 - g_2) - kAG(v_x - \phi) - K\psi_{xx} \right] \tilde{\phi} dx \\ &= \int_0^L \left[\rho I (u - g_2) - kAG(v_x - \phi) - K\psi_{xx} \right] \tilde{\phi} dx, \quad \forall \tilde{\phi} \in H_0^1. \end{aligned} \quad (2.37)$$

Hence (passing to C_0^1 functions)

$$EI\phi_{xx} = \rho Iu - \rho Ig_2 - kAG(v_x - \phi) - K\psi_{xx} \quad (2.38)$$

solves the second equation of (2.33). Likewise, if $(\tilde{\phi}, \tilde{v}, \tilde{\psi}) \equiv (0, 0, \tilde{\psi}) \in H_0^1 \times H_*^1 \times$

H_0^1 , then (2.35), gives

$$\begin{aligned} & K \int_0^L \phi_x \tilde{\psi} dx + I_s \int_0^L \psi \tilde{\psi} dx + [GJ - P(I_s/\mu)] \int_0^L \psi_x \tilde{\psi} dx + c_2 \int_0^L \psi \tilde{\psi} dx \\ &= (I_s + c_2) \int_0^L g_5 \tilde{\psi} dx + I_s \int_0^L g_6 \tilde{\psi} dx. \end{aligned}$$

Wherefore

$$\begin{aligned} & -K \int_0^L \phi_{xx} \tilde{\psi} dx - [GJ - P(I_s/\mu)] \int_0^L \psi_{xx} \tilde{\psi} dx + (I_s + c_2) \int_0^L (\psi - g_5) \tilde{\psi} dx \\ &= I_s \int_0^L g_6 \tilde{\psi} dx \end{aligned}$$

and according to the third equation of (2.34), we end up with

$$\begin{aligned} & [GJ - P(I_s/\mu)] \int_0^L \psi_{xx} \tilde{\psi} dx = \int_0^L [(I_s + c_2)(\psi - g_5) - K\phi_{xx} - I_s g_6] \tilde{\psi} dx \\ &= \int_0^L [(I_s + c_2)z - K\phi_{xx} - I_s g_6] \tilde{\psi} dx, \quad \forall \tilde{\psi} \in H_0^1. \end{aligned} \tag{2.39}$$

Thereupon, as above

$$[GJ - P(I_s/\mu)]\psi_{xx} = (I_s + c_2)z - K\phi_{xx} - I_s g_6 \tag{2.40}$$

which solves the sixth equation of (2.33). Combining (2.38) and (2.40), gives rise

to

$$\underbrace{\left[GJ - P(I_s/\mu) - \frac{K^2}{EI} \right]}_{\neq 0} \psi_{xx} = (I_s + c_2)z - I_s g_6 - \frac{\rho IK}{EI} u + \frac{\rho IK}{EI} g_2 + \frac{KkAG}{EI} (v_x - \phi) \in L^2(0, L).$$

Employing the elliptic regularity theory, we reach

$$\psi \in H^2(0, L) \cap H_0^1(0, L).$$

Hence, we can derive from (2.38) the following

$$\phi \in H^2(0, L) \cap H_0^1(0, L)$$

under the condition that $EIGJ - EIP(I_s/\mu) > K^2$, it is worth noting that this condition is already fulfilled by (2.15).

We conclude that there exists a unique $U = (\phi, u, v, w, \psi, z)^T \in \mathcal{D}(\mathcal{A})$, and $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contractions on $\mathcal{H}$. With this, the theorem's proof is complete. ■

2.3 Exponential Decay Result

In this section, we study the asymptotic behavior for the Timoshenko system (2.21) subject to torsional and axial effects. It should be recalled here that adding (unnecessary) damping means putting ourselves in a more comfortable situation from a mathematics point of view. From the physical and engineering point of view, this entails more complication and the procedure will be more costly. Therefore, the more interesting and challenging situation is when we seek the least possible damping and therefore the more economic one. This is our main objective in the subsequent results. We removed one (frictional) damping and

kept only two at time.

We shall associate to the system (2.21) the energy $E(t)$

$$E(t) = \frac{1}{2} \int_0^L \left\{ EI(\phi_x)^2 + 2K\phi_x\psi_x + [kAG - P](v_x - \phi)^2 - 2Pv_x\phi + P\phi^2 \right. \\ \left. + [GJ - P(I_s/\mu)](\psi_x)^2 + \mu(v_t)^2 + I_s(\psi_t)^2 + \rho I(\phi_t)^2 \right\} dx.$$

Our strategy consists in constructing a number of functionals to be added to $E(t)$ with certain weights. They are chosen in such a way that some targeted terms appear and at the same time some undesired terms that arise in the computation can be controlled by their derivatives. Therewithal, they will assist us in drawing a conclusion by attaining an appropriate differential inequality.

In this section and the rest of this dissertation, C_p will designated the Poincaré constant. Also, all our estimations are valid for all $t \geq 0$.

To maintain the hyperbolic character of the system, we assume that

$$kAG > P \text{ and } GJ > P(I_s/\mu) \text{ and } EI > K \text{ and } GJ - P(I_s/\mu) > K. \quad (2.41)$$

The derivative of the energy, along solutions of our system, is equal to

$$\frac{dE(t)}{dt} = -c_1||v_t||^2 - c_2||\psi_t||^2. \quad (2.42)$$

Let us start now the process of modifying further the classical energy. In due course of time, cleverly chosen functionals are put forward. We begin with the

rather standard functional

$$V_1(t) := -\rho I \int_0^L \phi \phi_t dx.$$

Lemma 2.3.1 *The functional $V_1(t)$ satisfies*

$$\frac{dV_1(t)}{dt} \leq -\rho I \|\phi_t\|^2 + \left(EI + \frac{kAGC_p}{2} + \frac{K}{2} \right) \|\phi_x\|^2 + \frac{K}{2} \|\psi_x\|^2 + \frac{kAG}{2} \|v_x - \phi\|^2$$

for all $t \geq 0$.

Proof. By utilizing the first equation (2.21) and employing integration by parts, we can derive

$$\begin{aligned} \frac{dV_1(t)}{dt} &= -\rho I \int_0^L \phi_t^2 dx - \int_0^L \phi \left[EI \phi_{xx} + kAG(v_x - \phi) + K \psi_{xx} \right] dx \\ &= -\rho I \|\phi_t\|^2 + EI \|\phi_x\|^2 - kAG \int_0^L \phi(v_x - \phi) dx + K \int_0^L \phi_x \psi_x dx. \end{aligned}$$

Combining Poincaré inequality and Young's inequality results in

$$\begin{aligned} \frac{dV_1(t)}{dt} &\leq -\rho I \|\phi_t\|^2 + EI \|\phi_x\|^2 + \frac{kAGC_p}{2} \|\phi_x\|^2 + \frac{kAG}{2} \|v_x - \phi\|^2 + \frac{K}{2} \|\phi_x\|^2 \\ &\quad + \frac{K}{2} \|\psi_x\|^2, \quad t \geq 0. \end{aligned} \tag{2.43}$$

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Lemma 2.3.2 *The functional*

$$V_2(t) := -\mu \int_0^L v_t \int_0^x \phi(t, y) dy dx + \mu \int_0^L v v_t dx + \frac{c_1}{2} \int_0^L v^2 dx$$

fulfills

$$\begin{aligned} \frac{dV_2(t)}{dt} &\leq \left[\frac{(\mu^2 + c_1^2)}{4\epsilon} + \mu \right] \|v_t\|^2 + \frac{L^2}{2} \epsilon \|\phi_t\|^2 + C_p \left[\epsilon + \frac{P^2}{2[kAG - P]} \right] \|\phi_x\|^2 \\ &\quad - \left[\frac{kAG - P}{2} \right] \|v_x - \phi\|^2, \epsilon > 0, \quad t \geq 0. \end{aligned} \quad (2.44)$$

Proof. In view of the second equation of (3.35), we infer

$$\begin{aligned} \frac{dV_2(t)}{dt} &= -\mu \int_0^L v_t \int_0^x \phi_t(t, y) dy dx - \int_0^L \left[-c_1 v_t + (kAG - P)(v_{xx} - \phi_x) - P\phi_x \right] \\ &\quad \times \int_0^x \phi(t, y) dy dx + \mu \|v_t\|^2 + \int_0^L v \left[-c_1 v_t + (kAG - P)(v_{xx} - \phi_x) - P\phi_x \right] dx \\ &\quad + c_1 \int_0^L v v_t dx \\ &= -\mu \int_0^L v_t \int_0^x \phi_t(t, y) dy dx + c_1 \int_0^L v_t \int_0^x \phi dy dx + (kAG - P) \int_0^L \phi(v_x - \phi) dx \\ &\quad - P \|\phi\|^2 + \mu \|v_t\|^2 - (kAG - P) \int_0^L v_x(v_x - \phi) dx + P \int_0^L \phi(v_x - \phi) dx + P \|\phi\|^2. \end{aligned}$$

By virtue of Young's inequality, Poincaré inequality and the fact that

$$\begin{aligned} \int_0^L \left(\int_0^x \phi_t dy \right)^2 dx &\leq \int_0^L \left(\int_0^x 1 dy \right) \left(\int_0^x \phi_t^2 dy \right) dx \\ &\leq \left(\int_0^L \phi_t^2 dy \right) \int_0^L x dx = \frac{L^2}{2} \int_0^L \phi_t^2 dy dx \end{aligned} \quad (2.45)$$

we can see that

$$\begin{aligned} \frac{dV_2(t)}{dt} &\leq \frac{\mu^2}{4\epsilon} \|v_t\|^2 + \frac{L^2}{2} \epsilon \|\phi_t\|^2 + \frac{c_1^2}{4\epsilon} \|v_t\|^2 + \epsilon C_p \|\phi_x\|^2 - (kAG - P) \|v_x - \phi\|^2 \\ &\quad + \mu \|v_t\|^2 + \epsilon_2 \|v_x - \phi\|^2 + \frac{P^2}{4\epsilon_2} \|\phi\|^2, \quad \epsilon, \epsilon_2 > 0. \end{aligned}$$

Choosing $\epsilon_2 := \frac{kAG - P}{2}$ completes the proof. ■

Lemma 2.3.3 *The functional*

$$V_3(t) := I_s \int_0^L \psi \psi_t dx + \frac{c_2}{2} \int_0^L \psi^2 dx - \rho I \int_0^L \psi \phi_t dx + \frac{I_s(EI - K)}{GJ - P(I_s/\mu)} \int_0^L \psi_t \phi dx$$

verifies the estimate

$$\begin{aligned} \frac{dV_3(t)}{dt} &\leq \left\{ I_s + \frac{(\rho I)^2}{4\epsilon} + \frac{[(EI - K)]^2}{4\epsilon [GJ - P(I_s/\mu)]^2} (I_s^2 + c_2^2) \right\} \|\psi_t\|^2 \\ &\quad - \left[\frac{GJ - P(I_s/\mu) - K}{2} \right] \|\psi_x\|^2 + 2\epsilon \|\phi_t\|^2 + \frac{(kAG)^2 C_p}{2[GJ - P(I_s/\mu) - K]} \|v_x - \phi\|^2 \\ &\quad - \left\{ \frac{K(EI - K)}{GJ - P(I_s/\mu)} - \epsilon C_p \right\} \|\phi_x\|^2, \quad \epsilon > 0, \quad t \geq 0. \end{aligned} \tag{2.46}$$

Proof. Direct computation, with the help of the first and the third equation

(2.21), leads to

$$\begin{aligned} \frac{dV_3(t)}{dt} &= I_s \|\psi_t\|^2 + \int_0^L \psi \left\{ [GJ - P(I_s/\mu)] \psi_{xx} + K \phi_{xx} - c_2 \psi_t \right\} dx + c_2 \int_0^L \psi \psi_t dx \\ &\quad - \rho I \int_0^L \psi_t \phi_t dx + \int_0^L \psi \left[-EI \phi_{xx} - kAG(v_x - \phi) - K \psi_{xx} \right] dx + \frac{I_s(EI - K)}{GJ - P(I_s/\mu)} \\ &\quad \times \int_0^L \psi_t \phi_t dx + \frac{EI - K}{GJ - P(I_s/\mu)} \int_0^L \phi \left\{ [GJ - P(I_s/\mu)] \psi_{xx} + K \phi_{xx} - c_2 \psi_t \right\} dx. \end{aligned}$$

The identity to come is derived from integrating by parts

$$\begin{aligned}\frac{dV_3(t)}{dt} &= I_s \|\psi_t\|^2 - \left[GJ - P(I_s/\mu) \right] \|\psi_x\|^2 - K \int_0^L \psi_x \phi_x dx - \rho I \int_0^L \psi_t \phi_t dx \\ &+ EI \int_0^L \psi_x \phi_x dx - kAG \int_0^L \psi(v_x - \phi) dx + K \|\psi_x\|^2 + \frac{I_s(EI - K)}{GJ - P(I_s/\mu)} \int_0^L \psi_t \phi_t dx \\ &- (EI - K) \int_0^L \phi_x \psi_x dx - \frac{K(EI - K)}{GJ - P(I_s/\mu)} \|\phi_x\|^2 - \frac{c_2(EI - K)}{GJ - P(I_s/\mu)} \int_0^L \phi \psi_t dx\end{aligned}$$

and the next one from an application of Poincaré and Young's inequality

$$\begin{aligned}\frac{dV_3(t)}{dt} &\leq I_s \|\psi_t\|^2 - \left[GJ - P(I_s/\mu) \right] \|\psi_x\|^2 + \frac{(\rho I)^2}{4\epsilon} \|\psi_t\|^2 + \epsilon \|\phi_t\|^2 \\ &+ \frac{(kAG)^2}{4\epsilon_3} \|v_x - \phi\|^2 + \epsilon_3 C_p \|\psi_x\|^2 + K \|\psi_x\|^2 + \frac{[I_s(EI - K)]^2}{4\epsilon [GJ - P(I_s/\mu)]^2} \|\psi_t\|^2 + \epsilon \|\phi_t\|^2 \\ &- \frac{K(EI - K)}{GJ - P(I_s/\mu)} \|\phi_x\|^2 + \frac{[c_2(EI - K)]^2}{4[GJ - P(I_s/\mu)]^2 \epsilon} \|\psi_t\|^2 + \epsilon C_p \|\phi_x\|^2, \quad \epsilon, \epsilon_3 > 0, \quad t \geq 0.\end{aligned}$$

Choosing $\epsilon_3 := \frac{GJ - P(I_s/\mu) - K}{2C_p}$ leads to the desired result. ■

Define

$$M_1 := \frac{GJ - P(I_s/\mu) - K}{GJ - P(I_s/\mu)}, \quad M_2 := \frac{kAG - P}{kAG} \quad (2.47)$$

and assume

$$C_p^2 P^2 < 2M_1 M_2^2 K(EI - K). \quad (2.48)$$

Before presenting the main results in this section, we recall the conditions specified in (2.15) and (2.20), that is

$$P < \min \left\{ P_*, \frac{GJ}{I_s} \mu \right\}, \quad K < GJ - P(I_s/\mu), \quad K < EI. \quad (2.49)$$

Theorem 2.3.4 *Let (ϕ, v, ψ) be the solution of system (2.21) subject to the conditions (2.49) and (2.48). Then, the functional $E(t)$ (and thereafter the classical energy) decays exponentially as time goes to infinity, that is there are two positive constants C and ξ independent of the initial data and t such that*

$$E(t) \leq CE(0)e^{-\xi t}, \quad \forall t \geq 0.$$

Remark 2.3.5 *It should be noticed that the system is exponentially stable regardless of the sign of P .*

Proof. To begin with, it is practical to construct the functional

$$\mathcal{L}(t) := NE(t) + \sum_{i=1}^3 \lambda_i V_i(t)$$

where N and λ_i , $i = 1, 2, 3$ will be determined in the course of the demonstration.

Having in mind the previous Lemmas 2.3.1, 2.3.2, and 2.3.3, it appears that

$$\begin{aligned} \frac{d\mathcal{L}(t)}{dt} &\leq A_1 \|v_t\|^2 + A_2 \|\psi_t\|^2 + A_3 \|\phi_t\|^2 + A_4 \|\phi_x\|^2 + A_5 \|\psi_x\|^2 + A_6 \|v_x - \phi\|^2 \\ &\quad - \epsilon_1 \|v_x\|^2 + \epsilon_1 \|v_x - \phi + \phi\|^2 - \epsilon_2 \|\phi\|^2 + \epsilon_2 \|\phi\|^2 \\ &\leq A_1 \|v_t\|^2 + A_2 \|\psi_t\|^2 + A_3 \|\phi_t\|^2 + A_4 \|\phi_x\|^2 + A_5 \|\psi_x\|^2 + (A_6 + 2\epsilon_1 c) \|v_x - \phi\|^2 \\ &\quad - (\epsilon_2 - 2\epsilon_1 c) \|\phi\|^2 - \epsilon_1 \|v_x\|^2 + \epsilon_2 \|\phi\|^2 \end{aligned}$$

where

$$\begin{aligned}
A_1 &:= -c_1 N + \left[\frac{(\mu^2 + c_1^2)}{4\epsilon} + \mu \right] \lambda_2, \\
A_2 &:= -c_2 N + \left\{ I_s + \frac{(\rho I)^2}{4\epsilon} + \frac{[(EI - K)]^2}{4\epsilon [GJ - P(I_s/\mu)]^2} (I_s^2 + c_2^2) \right\} \lambda_3, \\
A_3 &:= -\rho I \lambda_1 + \frac{L^2}{2} \epsilon \lambda_2 + 2\epsilon \lambda_3, \\
A_4 &:= \left[EI + \frac{kAGC_p}{2} + \frac{K}{2} \right] \lambda_1 + C_p \left[\epsilon + \frac{P^2}{2(kAG - P)} \right] \lambda_2 \\
&\quad - \left\{ \frac{K(EI - K)}{GJ - P(I_s/\mu)} - \epsilon C_p \right\} \lambda_3, \\
A_5 &:= \frac{K}{2} \lambda_1 - \left[\frac{GJ - P(I_s/\mu) - K}{2} \right] \lambda_3, \\
A_6 &:= \frac{kAG}{2} \lambda_1 - \left[\frac{kAG - P}{2} \right] \lambda_2 + \frac{(kAG)^2 C_p}{2[GJ - P(I_s/\mu) - K]} \lambda_3.
\end{aligned}$$

Taking N large enough in A_1, A_2 allows ϵ to be chosen as small as we wish.

Consequently, the coefficient A_3 is negative. For the other coefficients (to be negative), we need to select the different involved parameters in such a manner

that

$$\begin{cases}
\left[EI + \frac{kAGC_p}{2} + \frac{K}{2} \right] \lambda_1 + C_p \frac{P^2}{2[kAG - P]} \lambda_2 < \frac{K(EI - K)}{GJ - P(I_s/\mu)} \lambda_3, \\
K \lambda_1 < [GJ - P(I_s/\mu) - K] \lambda_3, \\
kAG \lambda_1 + \frac{(kAG)^2 C_p}{[GJ - P(I_s/\mu) - K]} \lambda_3 < (kAG - P) \lambda_2.
\end{cases}$$

Clearly, λ_1 may be picked small enough and hence we need to focus only with

$$\begin{cases} \frac{C_p P^2}{2[kAG-P]} \lambda_2 < \frac{K(EI-K)}{GJ-P(I_s/\mu)} \lambda_3, \\ \frac{(kAG)^2 C_p}{GJ-P(I_s/\mu)-K} \lambda_3 < (kAG-P) \lambda_2. \end{cases}$$

The above system of inequalities necessitates

$$\left(\frac{C_p P^2}{2[kAG-P]} \right) \left(\frac{(kAG)^2 C_p}{[GJ-P(I_s/\mu)-K]} \right) < (kAG-P) \left(\frac{K(EI-K)}{GJ-P(I_s/\mu)} \right)$$

which may be rewritten in the form

$$C_p^2 P^2 < 2 \left(\frac{GJ-P(I_s/\mu)-K}{GJ-P(I_s/\mu)} \right) \left(\frac{kAG-P}{kAG} \right)^2 K(EI-K). \quad (2.50)$$

Recalling (2.47), that is

$$M_1 := \frac{GJ-P(I_s/\mu)-K}{GJ-P(I_s/\mu)} \quad \text{and} \quad M_2 := \frac{kAG-P}{kAG}$$

the condition in (2.50), becomes

$$C_p^2 P^2 < 2M_1 M_2^2 K(EI-K).$$

Finally, it is notable that for large EI or small P , we get the needed result. ■

Remark 2.3.6 *In the case $P < 0$, we note that $M_1 M_2^2$ is bigger than in the case $P > 0$. Hence the condition takes place (range is bigger).*

2.4 Special Case: Absence of Axial Load

This section deals with the absence of axial forces and any kind of damping in the first component. Namely, of interest is the problem

$$\left\{ \begin{array}{l} \rho I \phi_{tt} - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} = 0, \\ \mu v_{tt} + c_1 v_t - kAG(v_{xx} - \phi_x) = 0, \\ -GJ\psi_{xx} - K\phi_{xx} + I_s \psi_{tt} + c_2 \psi_t = 0, \\ \phi(0, t) = \phi(L, t) = v_x(0, t) = v_x(L, t) = \psi(0, t) = \psi(L, t) = 0, \\ \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x). \end{array} \right. \quad (2.51)$$

The energy functional we shall adopt (classical energy) is

$$\begin{aligned} E(t) = & \frac{1}{2} \int_0^L \left[EI(\phi_x)^2 + 2K\phi_x\psi_x + kAG(v_x - \phi)^2 + GJ(\psi_x)^2 + \mu(v_t)^2 + I_s(\psi_t)^2 \right. \\ & \left. + \rho I(\phi_t)^2 \right] dx. \end{aligned} \quad (2.52)$$

In order to conserve the positivity of the energy, we make the following assumptions:

$$K < EI \quad \text{and} \quad K < GJ. \quad (2.53)$$

Recall the result already found in (2.13)

$$\frac{dE(t)}{dt} = -c_1 \|v_t\|^2 - c_2 \|\psi_t\|^2. \quad (2.54)$$

We introduce the new functional (notice the difference with the previous one)

$$V_1(t) := -\rho I \int_0^L \phi \phi_t dx + \mu \int_0^L \phi \left(\int_0^x v_t(t, y) dy \right) dx, \quad \forall t \geq 0. \quad (2.55)$$

Lemma 2.4.1 *The functional $V_1(t)$ satisfies, along solutions of our problem, the relation*

$$\frac{dV_1(t)}{dt} \leq -(\rho I - \epsilon) \|\phi_t\|^2 + \left(EI + \epsilon C_p + \frac{K}{4\epsilon_1} \right) \|\phi_x\|^2 + K\epsilon_1 \|\psi_x\|^2 + \frac{L^2}{4\epsilon} (\mu^2 + c_1^2) \|v_t\|^2$$

where $\epsilon, \epsilon_1 > 0$ and $t \geq 0$.

Proof. The rate of change of $V_1(t)$ may be expressed by

$$\begin{aligned} \frac{dV_1(t)}{dt} &= -\rho I \int_0^L \phi_t^2 dx - \int_0^L \phi \left(EI \phi_{xx} + kAG(v_x - \phi) + K\psi_{xx} \right) dx + \mu \int_0^L \phi_t \\ &\quad \times \left(\int_0^x v_t(t, y) dy \right) dx + \int_0^L \phi \left\{ \int_0^x \left[-c_1 v_t(t, y) + kAG(v_{xx} - \phi_x)(t, y) \right] dy \right\} dx \\ &= -\rho I \|\phi_t\|^2 + EI \|\phi_x\|^2 - kAG \int_0^L \phi(v_x - \phi) dx + K \int_0^L \phi_x \psi_x dx + \mu \int_0^L \phi_t \\ &\quad \times \left(\int_0^x v_t(t, y) dy \right) dx - c_1 \int_0^L \phi \int_0^x v_t(t, y) dy dx + kAG \int_0^L \phi(v_x - \phi) dx. \end{aligned}$$

The next relation is a simple consequence of Young's inequality and Poincaré inequality

$$\begin{aligned} \frac{dV_1(t)}{dt} &\leq -\rho I \|\phi_t\|^2 + EI \|\phi_x\|^2 + \frac{K}{4\epsilon_1} \|\phi_x\|^2 + K\epsilon_1 \|\psi_x\|^2 + \left(\frac{\mu^2}{4\epsilon} + \frac{c_1^2}{4\epsilon} \right) \\ &\quad \int_0^L \left(\int_0^x v_t(y) dy \right)^2 dx + \epsilon \|\phi_t\|^2 + \epsilon C_p \|\phi_x\|^2 \end{aligned} \quad (2.56)$$

where $\epsilon > 0$ and with the help of Cauchy-Schwarz inequality, it is obvious that

$$\begin{aligned} \int_0^L \left(\int_0^x v_t(y) dy \right)^2 dx &\leq \int_0^L \left[\left(\int_0^x 1 dy \right) \left(\int_0^x v_t^2 dy \right) \right] dx \\ &\leq \int_0^L x dx \int_0^x v_t^2 dy dx \leq L \int_0^L v_t^2 dx \int_0^L 1 dx \leq L^2 \|v_t\|^2. \end{aligned} \quad (2.57)$$

Merging (2.57) and (2.56) finishes the demonstration. ■

Lemma 2.4.2 *The functional*

$$V_2(t) := I_s \int_0^L \psi \psi_t dx + \frac{c_2}{2} \int_0^L \psi^2 dx - \rho I \int_0^L \psi \phi_t dx + \frac{I_s(EI - K)}{GJ} \int_0^L \psi_t \phi dx, \quad \forall t \geq 0$$

verifies the estimate

$$\begin{aligned} \frac{dV_2(t)}{dt} &\leq \left\{ I_s + \frac{1}{4\epsilon} \left[(\rho I)^2 + \frac{(EI - K)^2}{(GJ)^2} (I_s^2 + c_2) \right] \right\} \|\psi_t\|^2 - \left(GJ - K - C_p \epsilon_2 \right) \\ &\quad \times \|\psi_x\|^2 + 2\epsilon \|\phi_t\|^2 + \frac{(kAG)^2}{4\epsilon_2} \|v_x - \phi\|^2 - \left[\frac{K(EI - K)}{GJ} - C_p \epsilon \right] \|\phi_x\|^2 \end{aligned} \quad (2.58)$$

where $\epsilon, \epsilon_2 > 0$ and $t \geq 0$.

Proof. Imitating exactly the same steps as in the proof of Lemma 2.4.1, it is

evident that

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= I_s ||\psi_t||^2 + \int_0^L \psi \left(GJ\psi_{xx} + K\phi_{xx} - c_2\psi_t \right) dx + c_2 \int_0^L \psi\psi_t dx \\
&\quad - \rho I \int_0^L \psi_t\phi_t dx + \int_0^L \psi \left(-EI\phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} \right) dx \\
&\quad + \frac{I_s(EI - K)}{GJ} \int_0^L \psi_t\phi_t dx + \frac{EI - K}{GJ} \int_0^L \phi \left(GJ\psi_{xx} + K\phi_{xx} - c_2\psi_t \right) dx \\
&= I_s ||\psi_t||^2 - GJ ||\psi_x||^2 - K \int_0^L \psi_x\phi_x dx - \rho I \int_0^L \psi_t\phi_t dx \\
&\quad + EI \int_0^L \psi_x\phi_x dx - kAG \int_0^L \psi(v_x - \phi) dx + K ||\psi_x||^2 + \frac{I_s(EI - K)}{GJ} \int_0^L \psi_t\phi_t dx \\
&\quad - (EI - K) \int_0^L \phi_x\psi_x dx - \frac{K(EI - K)}{GJ} ||\phi_x||^2 - \frac{c_2(EI - K)}{GJ} \int_0^L \phi\psi_t dx
\end{aligned}$$

and

$$\begin{aligned}
\frac{dV_2(t)}{dt} &\leq I_s ||\psi_t||^2 - GJ ||\psi_x||^2 + \frac{(\rho I)^2}{4\epsilon} ||\psi_t||^2 + \epsilon ||\phi_t||^2 + \frac{(kAG)^2}{4\epsilon_2} ||v_x - \phi||^2 \\
&\quad + \epsilon_2 C_p ||\psi_x||^2 + K ||\psi_x||^2 + \frac{[I_s(EI - K)]^2}{4\epsilon(GJ)^2} ||\psi_t||^2 + \epsilon ||\phi_t||^2 - \frac{K(EI - K)}{GJ} ||\phi_x||^2 \\
&\quad + \frac{[c_2(EI - K)]^2}{4(GJ)^2\epsilon} ||\psi_t||^2 + \epsilon C_p ||\phi_x||^2
\end{aligned}$$

where $\epsilon > 0$ and $t \geq 0$. I

Remark 2.4.3 A close look at the proof shows that the exponential stability depends on the correct use of the bending torsion coupling rigidity K , which dispenses us from the equal speeds of propagation assumption. If $K = 0$, we cannot get $||\phi_x||^2$ with negative sign from $V_2(t)$, and we will be forced to assume equal speeds of propagation to get it. This is not surprising because, without coupling, we are left with the usual Timoshenko system which, as is well known, requires equal speeds of

propagation for the exponential stability.

In preparation of our next functional, we invoke the problem

$$\begin{cases} \omega(x, t) = - \int_0^x \phi(y, t) dy, \\ \omega(0, t) = 0, \end{cases} \quad x \in (0, L). \quad (2.59)$$

Lemma 2.4.4 *The functional*

$$V_3(t) := \mu \int_0^L v_t \omega dx + \mu \int_0^L v v_t dx + \frac{c_1}{2} \int_0^L v^2 dx, \quad \forall t \geq 0$$

complies with the relation

$$\frac{dV_3(t)}{dt} \leq \left[\frac{1}{4\epsilon} (\mu^2 + c_1^2) + \mu \right] \|v_t\|^2 + L^2 \epsilon \|\phi_t\|^2 + \epsilon C_p \|\phi_x\|^2 - kAG \|v_x - \phi\|^2 \quad (2.60)$$

where $\epsilon > 0$ and $t \geq 0$.

Proof. On account of the second equation of (2.51), a mere differentiation of

$V_3(t)$ gives

$$\begin{aligned} \frac{dV_3(t)}{dt} &= \mu \int_0^L v_t \omega_t dx + \int_0^L \omega \left[-c_1 v_t + kAG(v_{xx} - \phi_x) \right] dx \\ &\quad + \mu \int_0^L v_t^2 dx + \int_0^L v \left[-c_1 v_t + kAG(v_{xx} - \phi_x) \right] dx + c_1 \int_0^L v v_t dx \\ &= \mu \int_0^L v_t \omega_t dx - c_1 \int_0^L \omega v_t dx + kAG \int_0^L \phi(v_x - \phi) dx + \mu \|v_t\|^2 \\ &\quad - kAG \int_0^L v_x(v_x - \phi) dx. \end{aligned}$$

Poincaré inequality and Young's inequality yield

$$\frac{dV_3(t)}{dt} \leq \frac{\mu^2}{4\epsilon} \|v_t\|^2 + L^2\epsilon \|\phi_t\|^2 + \frac{c_1^2}{4\epsilon} \|v_t\|^2 + \epsilon C_p \|\phi_x\|^2 - kAG \|v_x - \phi\|^2 + \mu \|v_t\|^2$$

where $\epsilon > 0$ and $t \geq 0$. In the last step we profited from Cauchy-Schwarz inequality to derive the estimation

$$\begin{aligned} \int_0^L \omega_t^2 dx &= \int_0^L \left(- \int_0^x \phi_t(y, t) dy \right)^2 dx \leq \int_0^L \left[\left(\int_0^x 1 dy \right) \left(\int_0^x \phi_t^2 dy \right) \right] dx \\ &\leq \int_0^L x dx \int_0^x \phi_t^2 dy dx \leq L \int_0^L \phi_t^2 dx \int_0^L 1 dx \leq L^2 \|\phi_t\|^2. \end{aligned} \quad (2.61)$$

The proof is now complete. ■

Theorem 2.4.5 *Under the condition (2.53), the functional $E(t)$ (and subsequently the classical energy) of system (2.51) decays exponentially as time goes to infinity, that is there are two positive constants C and ξ independent of the initial data and t such that*

$$E(t) \leq CE(0)e^{-\xi t}, \quad \forall t \geq 0.$$

Proof. The functional we will rely on is

$$\mathcal{L}(t) := NE(t) + \sum_{i=1}^3 \lambda_i V_i(t), \quad t \geq 0$$

where N and λ_i , $i = 1, 2, 3$ will be determined in the course of the proof. Com-

binning the previous Lemmas 2.4.1, 2.4.2 and 2.4.4, we effortlessly find

$$\frac{d\mathcal{L}(t)}{dt} \leq A_1 \|v_t\|^2 + A_2 \|\psi_t\|^2 + A_3 \|\phi_t\|^2 + A_4 \|\phi_x\|^2 + A_5 \|\psi_x\|^2 + A_6 \|v_x - \phi\|^2$$

with

$$\begin{aligned} A_1 &:= -c_1 N + \frac{L^2}{4\epsilon} (\mu^2 + c_1^2) \lambda_1 + \left(\frac{\mu^2 + c_1^2}{4\epsilon} + \mu \right) \lambda_3, \\ A_2 &:= -c_2 N + \left\{ I_s + \frac{1}{4\epsilon} \left[(\rho I)^2 + \frac{(EI - K)^2}{(GJ)^2} (I_s^2 + c_2) \right] \right\} \lambda_2, \\ A_3 &:= -(\rho I - \epsilon) \lambda_1 + 2\epsilon \lambda_2 + \epsilon L^2 \lambda_3, \\ A_4 &:= \left(EI + \epsilon C_p + \frac{K}{4\epsilon_1} \right) \lambda_1 - \left[\frac{K(EI - K)}{GJ} - \epsilon C_p \right] \lambda_2 + \epsilon C_p \lambda_3, \\ A_5 &:= K \epsilon_1 \lambda_1 - \left(GJ - \epsilon_2 C_p - K \right) \lambda_2, \\ A_6 &:= \frac{(kAG)^2}{4\epsilon_2} \lambda_2 - kAG \lambda_3. \end{aligned}$$

Observe that a large value of N in A_1, A_2 allows us to ignore ϵ at this stage. On the same occasion, we postpone dealing with A_3 (as we can manage to have it negative). Regarding the remaining conditions

$$\begin{cases} \left(EI + \frac{K}{4\epsilon_1} \right) \lambda_1 < \frac{K(EI - K)}{GJ} \lambda_2, \\ K \epsilon_1 \lambda_1 < (GJ - \epsilon_2 C_p - K) \lambda_2, \\ \frac{(kAG)^2}{4\epsilon_2} \lambda_2 < kAG \lambda_3, \end{cases}$$

it is clear that we can pick λ_1 small and then λ_3 large. The latter observation

opens the door for ϵ_2 to be small. The fact that $K < EI$ and $K < GJ$ permits us to conclude. ■

CHAPTER 3

BENDING TORSION

PROBLEM WITHOUT

DAMPING IN THE BENDING

COMPONENT

In this chapter, we shall delve into a new situation where there is no damping in the second component. This means we omit the frictional damping which was active in the bending component in the originally derived model. After formulating the problem in an abstract setting and invoking an existence and uniqueness result, we center our goal towards the stability of the system.

3.1 Well-Posedness of the Problem

Here, in this section we investigate the existence and uniqueness of solutions to a Timoshenko beam subject to torsional as well as axial effects. The original model (1.1) is released from the damping in the bending component, namely

$$\left\{ \begin{array}{l} \rho I \phi_{tt} + c_3 \phi_t - EI \phi_{xx} - kAG(v_x - \phi) - K \psi_{xx} = 0, \\ \mu v_{tt} - [kAG - P](v_{xx} - \phi_x) + P \phi_x = 0, \\ -[GJ - P(I_s/\mu)] \psi_{xx} - K \phi_{xx} + I_s \psi_{tt} + c_2 \psi_t = 0, \\ \phi(0, t) = \phi(L, t) = v_x(0, t) = v_x(L, t) = \psi(0, t) = \psi(L, t) = 0, \\ \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x). \end{array} \right. \quad (3.1)$$

For this purpose, the instrumental tool is the semi-group theory. Beforehand (as in the previous problem), we make the change of function

$$\bar{v}(x, t) := v(x, t) - \frac{1}{L} \int_0^L v(x, t) dx, \quad x \in [0, L], t \geq 0. \quad (3.2)$$

The advantage of this new function is that its mean is equal to zero, i.e

$$\int_0^L \bar{v}(x, t) dx = 0, \quad t \geq 0.$$

Effortlessly, one can check that this new function satisfies the same equation and the whole system (3.1) remains unchanged. Proceeding as previously, we recall

$$\mathcal{H} := H_0^1(0, L) \times L^2(0, L) \times H_*^1(0, L) \times L_*^2(0, L) \times H_0^1(0, L) \times L^2(0, L)$$

as in (2.25), (2.26) which we equip with the inner product (2.27) and the norm (2.28). For the vector $U = (\phi, u, v, w, \psi, z)^T$, with $u = \phi_t, w = v_t, z = \psi_t$, we can remodel problem (3.1) as

$$\begin{cases} \frac{d}{dt}U(t) = \mathcal{A}U(t), & \forall t \geq 0, \\ U(0) = U_0 = (\phi_0, \phi_1, v_0, v_1, \psi_0, \psi_1)^T. \end{cases} \quad (3.3)$$

The operator $\mathcal{A}$ takes the form

$$\mathcal{A}U = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \frac{EI}{\rho I} D^2 - \frac{kAG}{\rho I} & -\frac{c_3}{\rho I} & \frac{kAG}{\rho I} D & 0 & \frac{K}{\rho I} D^2 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ -\frac{kAG}{\mu} D & 0 & \frac{kAG-P}{\mu} D^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \frac{K}{I_s} D^2 & 0 & 0 & 0 & \frac{GJ-P(I_s/\mu)}{I_s} D^2 & -\frac{c_2}{I_s} \end{pmatrix} \begin{pmatrix} \phi \\ u \\ v \\ w \\ \psi \\ z \end{pmatrix} \quad (3.4)$$

that is

$$\mathcal{A}U = \begin{pmatrix} u \\ -\frac{c_3}{\rho I}\phi_t + \frac{EI}{\rho I}\phi_{xx} + \frac{kAG}{\rho I}(v_x - \phi) + \frac{K}{\rho I}\psi_{xx} \\ w \\ \frac{kAG}{\mu}(v_x - \phi)_x - \frac{P}{\mu}v_{xx} \\ z \\ \frac{K}{I_s}\phi_{xx} + \frac{GJ-P(I_s/\mu)}{I_s}\psi_{xx} - \frac{c_2}{I_s}\psi_t \end{pmatrix} \quad (3.5)$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left\{ U \in \mathcal{H} \mid \phi, \psi \in H^2(0, L) \cap H_0^1(0, L), \ v \in H_*^2(0, L), \right. \\ \left. u, z \in H_0^1(0, L), \ w \in H_*^1(0, L) \right\}.$$

Recall, the condition (2.49), that is

$$P < \min \left\{ P_*, \frac{GJ}{I_s}\mu \right\}, \quad K < GJ - P(I_s/\mu), \quad K < EI. \quad (3.6)$$

Our existence and uniqueness result reads as follows

Theorem 3.1.1 *Let $U_0 \in \mathcal{H}$ and (3.6) holds, then there exists a unique solution $U \in C(\mathbb{R}^+, \mathcal{H})$ to the problem (3.1). Furthermore, if $U_0 \in \mathcal{D}(\mathcal{A})$, then we have a unique solution $U \in C(\mathbb{R}^+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

The proof is similar to that of Theorem 2.2.1.

3.2 Stability of the Structure

In this section, we focus our attention on the stability of system (3.1), and the decay rate of the energy associated with the system. Our main tool is the multiplier method. We achieve stability enjoying an exponential speeds in the case $\eta_2 = 0$ where

$$\eta_2 := \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{I_s} - \frac{kAG - P}{\mu} \right) - \frac{K^2}{\rho I I_s}.$$

This condition reminds us of a similar condition found in the literature [27]. In case $K = 0$ (no coupling of the system with the torsional equation), we recover the conventional equal speeds of propagation assumption.

3.2.1 Exponential Stability

As usual, selecting an appropriate Lyapunov functional relies on a careful choice of the multipliers. Actually, this is the essence of the energy method. In this subsection, we shall build such a Lyapunov functional with the aim to reach an exponential decay of the energy functional (modified classical energy). We begin with the functional

$$\begin{aligned} E_1(t) = & \frac{1}{2} \int_0^L \left\{ EI(\phi_x)^2 + 2K\phi_x\psi_x + (kAG - P)(v_x - \phi)^2 - 2Pv_x\phi + P\phi^2 \right. \\ & \left. + \left[GJ - P(I_s/\mu) \right] (\psi_x)^2 + \mu(v_t)^2 + I_s(\psi_t)^2 + \rho I(\phi_t)^2 \right\} dx. \end{aligned} \quad (3.7)$$

Owing to (2.13), we see that

$$\frac{dE_1(t)}{dt} = -c_3 \|\phi_t\|^2 - c_2 \|\psi_t\|^2, \quad t \geq 0. \quad (3.8)$$

Our first target is coming up with the term $-\|\phi_x\|^2$. For that we introduce our first functional

$$V_1(t) := \rho I \int_0^L \phi \phi_t dx + \frac{c_3}{2} \int_0^L \phi^2 dx - \frac{\mu k A G}{(k A G - P)} \int_0^L \phi \left(\int_0^x v_t(t, y) dy \right) dx, \quad \forall t \geq 0.$$

Lemma 3.2.1 *The functional $V_1(t)$ satisfies, along solutions of our problem, the relation*

$$\begin{aligned} \frac{dV_1(t)}{dt} \leq & \left[\rho I + \frac{\mu^2 (k A G)^2}{4\epsilon (k A G - P)^2} \right] \|\phi_t\|^2 - \left(\frac{EI}{2} - \frac{P k A G C_p}{k A G - P} \right) \|\phi_x\|^2 + \frac{K^2}{2EI} \|\psi_x\|^2 \\ & + \frac{\epsilon L^2}{2} \|v_t\|^2 \end{aligned} \quad (3.9)$$

where $\epsilon > 0$ and $t \geq 0$.

Proof. It is straightforward to come up with the identity

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \rho I \|\phi_t\|^2 + \int_0^L \phi \left[-c_3 \phi_t + EI \phi_{xx} + kAG(v_x - \phi) + K \psi_{xx} \right] dx \\
&+ c_3 \int_0^L \phi \phi_t dx - \frac{\mu kAG}{kAG - P} \int_0^L \phi_t \left(\int_0^x v_t(t, y) dy \right) dx \\
&- \frac{kAG}{kAG - P} \int_0^L \phi \int_0^x \left[(kAG - P)(v_{xx} - \phi_x)(t, y) dy - P \phi_x(t, y) dy \right] dx \\
&= \rho I \|\phi_t\|^2 - EI \|\phi_x\|^2 + kAG \int_0^L \phi(v_x - \phi) dx - K \int_0^L \phi_x \psi_x dx \\
&- \frac{\mu kAG}{kAG - P} \int_0^L \phi_t \left(\int_0^x v_t(t, y) dy \right) dx - kAG \int_0^L \phi(v_x - \phi) dx + \frac{PkAG}{kAG - P} \|\phi\|^2.
\end{aligned}$$

The assistance of Young's inequality and Poincaré inequality, ensures that

$$\begin{aligned}
\frac{dV_1(t)}{dt} &\leq \rho I \|\phi_t\|^2 - EI \|\phi_x\|^2 + \frac{EI}{2} \|\phi_x\|^2 + \frac{K^2}{2EI} \|\psi_x\|^2 \\
&+ \frac{\mu^2(kAG)^2}{4\epsilon(kAG - P)^2} \|\phi_t\|^2 + \epsilon \int_0^L \left(\int_0^x v_t(y) dy \right)^2 dx + \frac{PkAGC_p}{kAG - P} \|\phi_x\|^2, \quad \epsilon > 0.
\end{aligned} \tag{3.10}$$

This relation (3.10) together with the fact below

$$\begin{aligned}
\int_0^L \left(\int_0^x v_t(y) dy \right)^2 dx &\leq \int_0^L \left[\left(\int_0^x 1 dy \right) \left(\int_0^x v_t^2 dy \right) \right] dx \\
&\leq \int_0^L x \int_0^x v_t^2 dy dx \leq \int_0^L v_t^2 dx \int_0^L x dx \leq \frac{L^2}{2} \|v_t\|^2
\end{aligned} \tag{3.11}$$

guarantees the sought estimation (3.9) in the statement of the lemma. I

Lemma 3.2.2 *The functional*

$$V_2(t) := -\mu \int_0^L v_t v dx, \quad \forall t \geq 0$$

enjoys the property

$$\frac{dV_2(t)}{dt} \leq -\mu \|v_t\|^2 + \frac{3kAG - 2P}{2} \|v_x - \phi\|^2 + \frac{kAG}{2} \|\phi_x\|^2 - P \|\phi\|^2, \quad t \geq 0. \quad (3.12)$$

Proof. Multiple use of integration by parts leads to

$$\begin{aligned} \frac{dV_2(t)}{dt} &= -\mu \|v_t\|^2 - \mu \int_0^L v_{tt} v dx \\ &= -\mu \|v_t\|^2 - \int_0^L v \left[(kAG - P)(v_x - \phi)_x - P\phi_x \right] dx \\ &= -\mu \|v_t\|^2 + (kAG - P) \int_0^L v_x (v_x - \phi) dx - P \int_0^L v_x \phi dx \\ &= -\mu \|v_t\|^2 + (kAG - P) \int_0^L (v_x - \phi + \phi)(v_x - \phi) dx - P \int_0^L (v_x - \phi + \phi) \phi dx \\ &= -\mu \|v_t\|^2 + (kAG - P) \|v_x - \phi\|^2 + (kAG - P) \int_0^L \phi (v_x - \phi) dx \\ &\quad - P \int_0^L (v_x - \phi) \phi dx - P \|\phi\|^2 \end{aligned}$$

followed by Young's inequality and Poincaré inequality, we may arrive at

$$\begin{aligned} \frac{dV_2(t)}{dt} &\leq -\mu \|v_t\|^2 + (kAG - P) \|v_x - \phi\|^2 + \frac{(kAG - P)C_p}{2} \|\phi_x\|^2 \\ &\quad + \frac{(kAG - P)}{2} \|v_x - \phi\|^2 + \frac{P}{2} \|v_x - \phi\|^2 + \frac{PC_p}{2} \|\phi_x\|^2 - P \|\phi\|^2. \end{aligned}$$

The coming lemma is valid under the assumption

$$\eta_2 := \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{I_s} - \frac{kAG - P}{\mu} \right) - \frac{K^2}{\rho I I_s} = 0. \quad (3.13)$$

To avoid a cumbersome presentation, we incorporate the notation

$$M_3 := \frac{\mu}{kAG - P} \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right]$$

and

$$M_4 := \frac{I_s}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right).$$

Lemma 3.2.3 *Assuming that (3.13) is met, the functional*

$$V_3(t) := - \int_0^L (v_x - \phi) \phi_t dx - \int_0^L \phi_x v_t dx + M_3 \int_0^L \psi_x v_t dx + M_4 \int_0^L (v_x - \phi) \psi_t dx$$

possesses the property

$$\begin{aligned} \frac{dV_3(t)}{dt} &\leq \left[\frac{c_3^2}{4\epsilon(\rho I)^2} + 1 + \epsilon \right] \|\phi_t\|^2 - \left(\frac{kAG}{\rho I} - 2\epsilon \right) \|v_x - \phi\|^2 + \frac{P}{\mu} \left(1 + \frac{|M_3|}{2} \right) \|\phi_x\|^2 \\ &+ \frac{P|M_3|}{2\mu} \|\psi_x\|^2 + \frac{1}{4\epsilon I_s^2} M_4^2 (I_s^2 + c_2^2) \|\psi_t\|^2 \end{aligned} \quad (3.14)$$

where $\epsilon > 0$, $t \geq 0$.

Some immediate remarks are in order.

Remark 3.2.4 1. From the condition (3.13), we must have that

$$\frac{kAG - P}{\mu} > \frac{GJ - P(I_s/\mu)}{I_s} \text{ and } \frac{kAG - P}{\mu} > \frac{EI}{\rho I}$$

as supported by the numerical example in the paper [49] or

$$\frac{kAG - P}{\mu} < \frac{GJ - P(I_s/\mu)}{I_s} \text{ and } \frac{kAG - P}{\mu} < \frac{EI}{\rho I}.$$

2. If $K = 0$, then we infer from the condition $\eta_2 = 0$ in (3.13), that either

$$\frac{EI}{\rho I} = \frac{kAG - P}{\mu}$$

(we recover therefore the equal speeds of propagation for the classical Timoshenko system used in the already published papers to show exponential stability) and/or

$$\frac{kAG - P}{\mu} = \frac{GJ - P(I_s/\mu)}{I_s}$$

meaning that the second and third equations in (3.1) have equal speeds of propagation. Remarkably, it is not necessary that the speeds of propagation of the first and the third equation be the same unless both terms in (3.13) vanish in which case all speeds of propagation are identical.

Proof. [Proof of Lemma 3.2.3] On account of system (3.1) we can easily see

that

$$\begin{aligned}
\frac{dV_3(t)}{dt} = & - \int_0^L (v_x - \phi)_t \phi_t dx - \frac{1}{\rho I} \int_0^L (v_x - \phi) \left[-c_3 \phi_t + EI \phi_{xx} + kAG(v_x - \phi) \right. \\
& + K \psi_{xx} \left. \right] dx - \int_0^L \phi_{xt} v_t dx - \frac{1}{\mu} \int_0^L \phi_x \left[(kAG - P)(v_x - \phi)_x - P \phi_x \right] dx \\
& + \frac{\mu}{kAG - P} \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right] \int_0^L \psi_{xt} v_t dx \\
& + \frac{1}{kAG - P} \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right] \int_0^L \psi_x \\
& \times \left[(kAG - P)(v_x - \phi)_x - P \phi_x \right] dx + \frac{I_s}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L (v_x - \phi)_t \psi_t dx \\
& + \frac{1}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L (v_x - \phi) \left[(GJ - P(I_s/\mu)) \psi_{xx} + K \phi_{xx} - c_2 \psi_t \right] dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{dV_3(t)}{dt} = & - \int_0^L v_{xt} \phi_t dx + ||\phi_t||^2 - \frac{c_3}{\rho I} \int_0^L (v_x - \phi) \phi_t dx - \frac{EI}{\rho I} \int_0^L \phi_{xx} (v_x - \phi) dx \\
& - \frac{kAG}{\rho I} ||v_x - \phi||^2 - \frac{K}{\rho I} \int_0^L (v_x - \phi) \psi_{xx} dx + \int_0^L \phi_t v_{xt} dx + \frac{kAG - P}{\mu} \int_0^L \phi_{xx} \\
& \times (v_x - \phi) dx + \frac{P}{\mu} ||\phi_x||^2 - \frac{\mu}{kAG - P} \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right] \\
& \times \int_0^L \psi_t v_{xt} dx - \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right] \int_0^L \psi_{xx} (v_x - \phi) dx \\
& - \frac{P}{kAG - P} \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right] \int_0^L \psi_x \phi_x dx \\
& + \frac{I_s}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L \psi_t v_{xt} dx - \frac{I_s}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L \phi_t \psi_t dx \\
& + \frac{GJ - P(I_s/\mu)}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L \psi_{xx} (v_x - \phi) dx + \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \\
& \times \int_0^L \phi_{xx} (v_x - \phi) dx - \frac{c_2}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L (v_x - \phi) \psi_t dx
\end{aligned}$$

or, after arranging things

$$\begin{aligned}
\frac{dV_3(t)}{dt} &= \|\phi_t\|^2 - \frac{c_3}{\rho I} \int_0^L (v_x - \phi) \phi_t dx - \frac{kAG}{\rho I} \|v_x - \phi\|^2 + \frac{P}{\mu} \|\phi_x\|^2 \\
&- \left\{ \frac{\mu}{kAG - P} \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right] \right. \\
&- \frac{I_s}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left. \right\} \int_0^L \psi_t v_{xt} dx \\
&- \frac{P}{kAG - P} \left[\left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left(\frac{GJ - P(I_s/\mu)}{K} \right) - \frac{K}{\rho I} \right] \int_0^L \psi_x \phi_x dx \\
&- \frac{I_s}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L \phi_t \psi_t dx - \frac{c_2}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \int_0^L (v_x - \phi) \psi_t dx.
\end{aligned}$$

It is readily seen that the evaluation below is true

$$\begin{aligned}
\frac{dV_3(t)}{dt} &\leq \|\phi_t\|^2 + \frac{c_3^2}{4\epsilon(\rho I)^2} \|\phi_t\|^2 + \epsilon \|v_x - \phi\|^2 - \frac{kAG}{\rho I} \|v_x - \phi\|^2 + \frac{P}{\mu} \|\phi_x\|^2 \\
&+ \frac{P|M_3|}{2\mu} \|\phi_x\|^2 + \frac{P|M_3|}{2\mu} \|\psi_x\|^2 + \frac{1}{4\epsilon} \left[\frac{I_s}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \right]^2 \|\psi_t\|^2 + \epsilon \|\phi_t\|^2 \\
&+ \frac{1}{4\epsilon} \left[\frac{c_2}{K} \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \right]^2 \|\psi_t\|^2 + \epsilon \|v_x - \phi\|^2
\end{aligned}$$

where $\epsilon > 0$ for $t > 0$ and profiting from $\eta_2 = 0$. Some simple elementary operations show that the aforementioned expression is actually our η_2 defined before the Lemma 3.2.3, in (3.13) i.e.

$$\eta_2 = \left(\frac{EI}{\rho I} - \frac{kAG - P}{\mu} \right) \left[\frac{GJ - P(I_s/\mu)}{I_s} - \frac{kAG - P}{\mu} \right] - \frac{K^2}{\rho I I_s}.$$

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Our next functional will supply $-\|\psi_x\|^2$.

Lemma 3.2.5 *The functional*

$$V_4(t) := I_s \int_0^L \psi \psi_t dx + \frac{c_2}{2} \int_0^L \psi^2 dx, \quad \forall t \geq 0$$

verifies the estimate

$$\frac{dV_4(t)}{dt} \leq I_s \|\psi_t\|^2 - \left[\frac{GJ - P(I_s/\mu)}{2} \right] \|\psi_x\|^2 + \frac{K^2}{2[GJ - P(I_s/\mu)]} \|\phi_x\|^2, \quad t \geq 0. \quad (3.15)$$

Proof. A simple application of integration by parts and Young's inequality yields for $t \geq 0$

$$\begin{aligned} \frac{dV_4(t)}{dt} &= I_s \|\psi_t\|^2 + \int_0^L \psi \left[(GJ - P(I_s/\mu)) \psi_{xx} + K \phi_{xx} - c_2 \psi_t \right] dx + c_2 \int_0^L \psi \psi_t dx \\ &= I_s \|\psi_t\|^2 - [GJ - P(I_s/\mu)] \|\psi_x\|^2 - K \int_0^L \psi_x \phi_x dx \\ &\leq I_s \|\psi_t\|^2 - [GJ - P(I_s/\mu)] \|\psi_x\|^2 + \left[\frac{GJ - P(I_s/\mu)}{2} \right] \|\psi_x\|^2 \\ &\quad + \frac{K^2}{2[GJ - P(I_s/\mu)]} \|\phi_x\|^2. \end{aligned}$$

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Theorem 3.2.6 *Assume that $\eta_2 = 0$ and (3.6) holds. Then we have exponential decay of the energy provided that $2K \geq EI$. When $2K < EI$, we assume further that*

$$P < \frac{kAGEI}{2kAGC_p + EI}.$$

Proof. Assembling the previously introduced four functionals together with $E_1(t)$ in one functional as

$$\mathcal{L}_1(t) := NE_1(t) + \sum_{i=1}^4 \lambda_i V_i(t) \quad (3.16)$$

where N and λ_i , $i = 1, 2, 3, 4$ will be determined as we go along with the proof and utilizing the previous relations (3.9), (3.12), (3.14) and (3.15), we obtain

$$\frac{d\mathcal{L}_1(t)}{dt} \leq A_1 \|\phi_t\|^2 + A_2 \|\psi_t\|^2 + A_3 \|v_t\|^2 + A_4 \|\phi_x\|^2 + A_5 \|\psi_x\|^2 + A_6 \|v_x - \phi\|^2 - P\lambda_2 \|\phi\|^2$$

where

$$\begin{aligned} A_1 &:= -c_3 N + \left[\rho I + \frac{\mu^2 (kAG)^2}{4\epsilon (kAG - P)^2} \right] \lambda_1 + \left[\frac{c_3^2}{4\epsilon (\rho I)^2} + 1 + \epsilon \right] \lambda_3, \\ A_2 &:= -c_2 N + \frac{1}{4\epsilon I_s^2} M_4^2 (I_s^2 + c_2^2) \lambda_3 + I_s \lambda_4, \\ A_3 &:= \frac{\epsilon L^2}{2} \lambda_1 - \mu \lambda_2, \\ A_4 &:= - \left(\frac{EI}{2} - \frac{PkAGC_p}{kAG - P} \right) \lambda_1 + \frac{kAG}{2} \lambda_2 + \frac{P}{\mu} \left(1 + \frac{M_3}{2} \right) \lambda_3 + \frac{K^2}{2[GJ - P(I_s/\mu)]} \lambda_4, \\ A_5 &:= \frac{K^2}{2EI} \lambda_1 + \frac{PM_3}{2\mu} \lambda_3 - \left[\frac{GJ - P(I_s/\mu)}{2} \right] \lambda_4, \\ A_6 &:= \frac{(3kAG - 2P)}{2} \lambda_2 - \left(\frac{kAG}{\rho I} - 2\epsilon \right) \lambda_3. \end{aligned}$$

We realize that for a considerably large value of N , permitting ϵ to be small, the coefficients A_1 and A_2 may be made negative. Consequently, the coefficient A_3 may be considered negative as well. For the other coefficients (to be negative),

we need to select the parameters there in a way that

$$\begin{cases} \frac{kAG}{2}\lambda_2 + \frac{P}{\mu}\left(1 + \frac{M_3}{2}\right)\lambda_3 + \frac{K^2}{2[GJ-P(I_s/\mu)]}\lambda_4 < \left(\frac{EI}{2} - \frac{P kAGC_p}{kAG-P}\right)\lambda_1, \\ \frac{K^2}{2EI}\lambda_1 + \frac{PM_3}{2\mu}\lambda_3 < \left[\frac{GJ-P(I_s/\mu)}{2}\right]\lambda_4, \\ \frac{(3kAG-2P)}{2}\lambda_2 < \frac{kAG}{\rho I}\lambda_3. \end{cases}$$

A quick glimpse at these inequalities reveals that taking λ_2 small enough will simplify them to

$$\begin{cases} \frac{P}{\mu}\left(1 + \frac{M_3}{2}\right)\lambda_3 + \frac{K^2}{2[GJ-P(I_s/\mu)]}\lambda_4 < \left(\frac{EI}{2} - \frac{P kAGC_p}{kAG-P}\right)\lambda_1, \\ \frac{K^2}{EI}\lambda_1 + \frac{PM_3}{\mu}\lambda_3 < [GJ - P(I_s/\mu)]\lambda_4. \end{cases}$$

Another brief glance show that for λ_3 small enough the system is simplified to

$$\begin{cases} \frac{K^2}{2[GJ-P(I_s/\mu)]}\lambda_4 < \left(\frac{EI}{2} - \frac{P kAGC_p}{kAG-P}\right)\lambda_1, \\ \frac{K^2}{EI}\lambda_1 < [GJ - P(I_s/\mu)]\lambda_4, \end{cases}$$

which are possible if

$$\frac{K^4}{2EI[GJ - P(I_s/\mu)]} < [GJ - P(I_s/\mu)]\left[\frac{EI}{2} - \frac{P kAGC_p}{kAG - P}\right]$$

or

$$K^4(kAG - P) < EI[GJ - P(I_s/\mu)]^2[EI(kAG - P) - 2P kAGC_p].$$

Bearing in mind the fact that $K^2 < [GJ - P(I_s/\mu)]^2$, it suffices that the inequality

$$K^2(kAG - P) < EI[EI(kAG - P) - 2PkAGC_p] \quad (3.17)$$

holds. Clearly, the condition (3.17) is fulfilled for negative P . Now for positive P , we need

$$EI(kAG - P) - 2PkAGC_p > 0$$

that is

$$P < \frac{kAGEI}{2kAGC_p + EI}. \quad (3.18)$$

This condition is to be compared with one we assumed previously to keep the energy positive in (2.18), namely

$$P < \frac{(EI - K)kAG}{kAGC_p + (EI - K)}.$$

- If $2K < EI$, then

$$EI(kAG)^2C_p < 2(EI - K)C_p(kAG)^2$$

or after adding $EI(EI - K)kAG$ to both sides

$$kAGEI[kAGC_p + (EI - K)] < (EI - K)kAG[2kAGC_p + EI]$$

leading to

$$\frac{kAGEI}{2kAGC_p + EI} < \frac{(EI - K)kAG}{kAGC_p + (EI - K)}.$$

Hence to obtain exponential decay in this case it is enough to assume

$$P < \frac{kAGEI}{2kAGC_p + EI}.$$

- If $2K \geq EI$, then

$$P < \frac{(EI - K)kAG}{kAGC_p + (EI - K)} \leq \frac{kAGEI}{2kAGC_p + EI}.$$

which means that the condition in (3.18), is already guaranteed by the one in (2.18) (the condition on P that makes the energy nonnegative).

Under condition (3.18) and for large EI or small (or negative) P , all the coefficients A_i , $i = 1, 2, \dots, 6$ are negative suggesting that

$$\frac{d\mathcal{L}_1(t)}{dt} \leq -CE_1(t), \quad \forall t \geq 0 \quad (3.19)$$

for some positive constant C . In order to pass to $\mathcal{L}_1(t)$ and subsequently to $E_1(t)$ to conclude the exponential decay for $E_1(t)$, we need to show the equivalence $E_1(t) \sim \mathcal{L}_1(t)$. From the definition of $V_1(t)$, it is not difficult to demonstrate that

$$|V_1(t)| \leq \rho I \delta_1 \|\phi\|^2 + \frac{\rho I}{4\delta_1} \|\phi_t\|^2 + \frac{c_2}{2} \|\phi\|^2 + \frac{\mu k AG \delta_2}{k AG - P} \|\phi\|^2 + \frac{\mu k AGL^2}{8\delta_2(k AG - P)} \|v_t\|^2 \quad (3.20)$$

where $\delta_1, \delta_2 > 0$ are the constants which stem from Young's inequality. Likewise, for $V_2(t), V_3(t)$ and $V_4(t)$, we find

$$|V_2(t)| \leq \frac{\mu}{4\delta_3} \|v_t\|^2 + \mu\delta_3 \|v\|^2 \leq \frac{\mu}{4\delta_3} \|v_t\|^2 + 2\mu\delta_3 C_p \|v_x - \phi\|^2 + 2\mu\delta_3 C_p \|\phi\|^2 \quad (3.21)$$

$$\begin{aligned} |V_3(t)| &\leq \delta_4 \|v_x - \phi\|^2 + \frac{1}{4\delta_4} \|\phi_t\|^2 + \delta_5 \|\phi_x\|^2 + \frac{1}{4\delta_5} \|v_t\|^2 + |M_3| \delta_6 \|\psi_x\|^2 + \frac{|M_3|}{4\delta_6} \|v_t\|^2 \\ &+ \frac{|M_4|}{4\delta_7} \|v_x - \phi\|^2 + |M_4| \delta_7 \|\psi_t\|^2 \end{aligned} \quad (3.22)$$

and

$$|V_4(t)| \leq \frac{I_s C_p}{4\delta_8} \|\psi_x\|^2 + I_s \delta_8 \|\psi_t\|^2 + \frac{C_p c_2}{2} \|\psi_x\|^2 \quad (3.23)$$

for any $\delta_i > 0$, for $i = 3, 4, 5, 6, 7, 8$. Assembling (3.7), (3.20), (3.21), (3.22) and (3.23) in

$$\mathcal{L}_1(t) = NE(t) + \sum_{i=1}^4 \lambda_i V_i, \quad \forall t \geq 0 \quad (3.24)$$

steers us directly to

$$\begin{aligned}
\mathcal{L}_1(t) \leq & \left[\frac{N}{2} (EI + K) + \delta_5 \lambda_3 \right] \|\phi_x\|^2 + \left[\frac{N}{2} (GJ - P(I_s/\mu) + K) + |M_3| \delta_6 \lambda_3 \right. \\
& + \left. \left(\frac{I_s C_p}{4\delta_8} + \frac{C_p c_2}{2} \right) \lambda_4 \right] \|\psi_x\|^2 + \left[\frac{N}{2} (kAG - P) + 2\mu\delta_3 C_p \lambda_2 + \left(\delta_4 + \frac{|M_4|}{4\delta_7} \right) \lambda_3 \right] \\
& \times \|v_x - \phi\|^2 + NP \|v_x\|^2 + \left[NP + 2\mu\delta_3 C_p \lambda_2 + \left(\rho I \delta_1 + \frac{c_2}{2} + \frac{\mu kAG \delta_2}{kAG - P} \right) \lambda_1 \right] \|\phi\|^2 \\
& + \left[\frac{N\mu}{2} + \frac{\mu kAGL^2}{8\delta_2 (kAG - P)} \lambda_1 + \frac{\mu}{4\delta_3} \lambda_2 + \left(\frac{|M_3|}{4\delta_6} + \frac{1}{4\delta_5} \right) \lambda_3 \right] \|v_t\|^2 \\
& + \left[\frac{NI_s}{2} + |M_4| \delta_7 \lambda_3 + I_s \delta_8 \lambda_4 \right] \|\psi_t\|^2 + \left[\frac{N\rho I}{2} + \frac{\rho I}{4\delta_1} \lambda_1 + \frac{1}{4\delta_4} \lambda_3 \right] \|\phi_t\|^2.
\end{aligned}$$

This already ascertain the following part of the equivalence

$$\mathcal{L}_1(t) \leq N_1 E_1(t), \quad \forall t \geq 0 \quad (3.25)$$

where N_1 is a positive constant. On the flip side, the relations

$$\begin{aligned}
V_1(t) \geq & -\rho I \delta_1 \|\phi\|^2 - \frac{\rho I}{4\delta_1} \|\phi_t\|^2 + \frac{c_2}{2} \|\phi\|^2 - \frac{\mu kAG \delta_2}{kAG - P} \|\phi\|^2 - \frac{\mu kAGL^2}{8\delta_2 (kAG - P)} \|v_t\|^2 \\
& (3.26)
\end{aligned}$$

where $\delta_1, \delta_2 > 0$.

$$V_2(t) \geq -\frac{\mu}{4\delta_3} \|v_t\|^2 - 2\mu\delta_3 C_p \|v_x - \phi\|^2 - 2\mu\delta_3 C_p \|\phi\|^2, \quad \delta_3 > 0 \quad (3.27)$$

$$\begin{aligned}
V_3(t) \geq & -\delta_4 \|v_x - \phi\|^2 - \frac{1}{4\delta_4} \|\phi_t\|^2 - \delta_5 \|\phi_x\|^2 - \frac{1}{4\delta_5} \|v_t\|^2 - |M_3| \delta_6 \|\psi_x\|^2 - \frac{|M_3|}{4\delta_6} \|v_t\|^2 \\
& - \frac{|M_4|}{4\delta_7} \|v_x - \phi\|^2 - |M_4| \delta_7 \|\psi_t\|^2, \quad \delta_i > 0, \quad i = 4, \dots, 7
\end{aligned} \tag{3.28}$$

and

$$V_4(t) \geq -\frac{I_s C_p}{4\delta_8} \|\psi_x\|^2 - I_s \delta_8 \|\psi_t\|^2 - \frac{C_p c_2}{2} \|\psi_x\|^2, \quad \delta_8 > 0 \tag{3.29}$$

are able to demonstrate the other relation between $\mathcal{L}_1(t)$ and $E_1(t)$. Indeed, recalling that

$$\mathcal{L}_1(t) = N E_1(t) + \sum_{i=1}^4 \lambda_i V_i, \quad \forall t \geq 0 \tag{3.30}$$

we obviously have

$$\begin{aligned}
\mathcal{L}_1(t) \geq & \left[\frac{N}{2} (EI - K) - \delta_5 \lambda_3 \right] \|\phi_x\|^2 + \left[\frac{N}{2} (GJ - P(I_s/\mu) - K) - M_3 \delta_6 \lambda_3 \right. \\
& - \left(\frac{I_s C_p}{4\delta_8} + \frac{C_p c_2}{2} \right) \lambda_4 \left. \right] \|\psi_x\|^2 + \left[\frac{N}{2} (kAG - P) - \frac{2\mu \delta_3 C_p}{\lambda} - \left(\delta_4 + \frac{|M_4|}{4\delta_7} \right) \lambda_3 \right] \|v_x - \phi\|^2 \\
& - \frac{NP}{2\delta_9} \|v_x\|^2 + \left[\frac{N}{2} (P - P\delta_9) - \left(\rho I \delta_1 - \frac{c_2}{2} + \frac{\mu kAG \delta_2}{kAG - P} \right) \lambda_1 \right] \|\phi\|^2 \\
& + \left[\frac{N\mu}{2} - \frac{\mu kAGL^2}{8\delta_2(kAG - P)} \lambda_1 - \frac{\mu}{4\delta_3} \lambda_2 - \left(\frac{|M_3|}{4\delta_6} + \frac{1}{4\delta_5} \right) \lambda_3 \right] \|v_t\|^2 \\
& + \left[\frac{NI_s}{2} - |M_4| \delta_7 \lambda_3 - I_s \delta_8 \lambda_4 \right] \|\psi_t\|^2 + \left[\frac{N\rho I}{2} - \frac{\rho I}{4\delta_1} \lambda_1 - \frac{1}{4\delta_4} \lambda_3 \right] \|\phi_t\|^2.
\end{aligned}$$

For large N (or small lambdas λ_i , $i = 1, \dots, 4$) and any values of λ_i , $i = 1, \dots, 4$

we can see that

$$\mathcal{L}_1(t) \geq N_2 E_1(t), \quad \forall t \geq 0 \quad (3.31)$$

for some N_2 positive. Combining both (3.25) and (3.31), we achieve the equivalence

$$N_2 E_1(t) \leq \mathcal{L}_1(t) \leq N_1 E_1(t), \quad \forall t \geq 0.$$

Finally, recalling (3.19), the right side of the equivalence yields the exponential decay of $\mathcal{L}_1(t)$. This property is inherited by $E_1(t)$ through the left hand side of the equivalence. ■

3.2.2 Polynomial Stability

In this subsection, we discuss with the case $\eta_2 \neq 0$. Passing to the second order energy we could prove a polynomial stability

Theorem 3.2.7 *Suppose $\eta_2 \neq 0$ and (3.6) holds. Then, the energy $E_1(t)$ decays polynomially, more precisely there is positive constant C independent of the initial data such that*

$$E_1(t) \leq C \left(\frac{E_1(0) + E_2(0)}{t} \right), \quad \forall t > 0$$

for some $E_2(0) > 0$ determined in the proof.

We start by preparing the ground for the proof. Differentiating the equations in (3.1) with respect to time, it is evident that

$$\begin{cases} \rho I \phi_{ttt} + c_3 \phi_{tt} - EI \phi_{xxt} - kAG(v_{xt} - \phi_t) - K \psi_{xxt} = 0, \\ \mu v_{ttt} - [kAG - P](v_{xx} - \phi_x)_t + P \phi_{xt} = 0, \\ -[GJ - P(I_s/\mu)] \psi_{xxt} - K \phi_{xxt} + I_s \psi_{ttt} + c_2 \psi_{tt} = 0, \end{cases} \quad (3.32)$$

where $(x, t) \in (0, L) \times (0, +\infty)$. To discuss the case $\eta_2 \neq 0$, we define the second order modified energy by

$$\begin{aligned} E_2(t) := E_1(t, \phi_t, v_t, \psi_t) &= \frac{1}{2} \int_0^L \left\{ EI(\phi_{xt})^2 + 2K \phi_{xt} \psi_{xt} + (kAG - P)(v_x - \phi)_t^2 \right. \\ &\quad \left. - 2P v_{xt} \phi_t + P \phi_t^2 + [GJ - P(I_s/\mu)](\psi_{xt})^2 + \mu(v_{tt})^2 + I_s(\psi_{tt})^2 + \rho I(\phi_{tt})^2 \right\} dx. \end{aligned}$$

It is not difficult to achieve

$$\frac{dE_2(t)}{dt} = -c_3 \|\phi_{tt}\|^2 - c_2 \|\psi_{tt}\|^2, \quad t \geq 0. \quad (3.33)$$

Proof. The right functional to count on is

$$\mathcal{L}_2(t) := N(2E_1(t) + E_2(t)) + \sum_{i=1}^4 \lambda_i V_i(t).$$

Profiting from what we achieved earlier, we may build on to notice that there

exists a positive constant C such that

$$\begin{aligned} \frac{d\mathcal{L}_2(t)}{dt} &\leq -CE_1(t) - c_3N||\phi_t||^2 - c_2N||\psi_t||^2 - c_3N||\phi_{tt}||^2 - c_2N||\psi_{tt}||^2 \\ &\quad - \eta_2\lambda_3 \int_0^L \psi_t v_{xt} dx. \end{aligned} \quad (3.34)$$

The last term on the right-hand side of (3.34) may be treated as follows:

$$\begin{aligned} -\eta_2 \int_0^L \psi_t v_{xt} dx &= -\eta_2 \int_0^L \psi_t (v_x - \phi)_t dx - \eta_2 \int_0^L \psi_t \phi_t dx \\ &= -\eta_2 \frac{d}{dt} \int_0^L \psi_t (v_x - \phi) dx + \eta_2 \int_0^t \psi_{tt} (v_x - \phi) dx - \eta_2 \int_0^L \psi_t \phi_t dx. \end{aligned}$$

Next, soliciting Young's inequality, we reach the inequality

$$-\eta_2 \int_0^L \psi_t v_{xt} dx \leq -\eta_2 \frac{d}{dt} \int_0^L \psi_t (v_x - \phi) dx + \frac{\eta_2^2}{4\epsilon} \left(||\psi_{tt}||^2 + ||\psi_t||^2 \right) + \epsilon ||v_x - \phi||^2 + \epsilon ||\phi_t||^2$$

where $\epsilon > 0$. The above prompt us to address the new functional

$$\mathcal{L}_3(t) := \mathcal{L}_2(t) + \eta_2\lambda_3 \int_0^L \psi_t (v_x - \phi) dx, \quad t \geq 0.$$

From the relation (3.34), it is clear that

$$\begin{aligned} \frac{d\mathcal{L}_3(t)}{dt} &\leq -C_2E_1(t) - c_3N||\phi_t||^2 - c_2N||\psi_t||^2 - c_3N||\phi_{tt}||^2 - c_2N||\psi_{tt}||^2 \\ &\quad + \frac{\eta_2^2}{4\epsilon} \left(||\psi_{tt}||^2 + ||\psi_t||^2 \right) + \epsilon ||v_x - \phi||^2 + \epsilon ||\phi_t||^2, \quad t \geq 0. \end{aligned}$$

For small enough ϵ and large N , we see that

$$\frac{d\mathcal{L}_3(t)}{dt} \leq -C_3 E_1(t), \quad C_3 > 0.$$

It suffices now to integrate the above relation over $(0, t)$, keeping in mind that $E_1(t)$ is not increasing, to remark that

$$tE_1(t) \leq \int_0^t E_1(s)ds \leq C_3 \left(\mathcal{L}_3(0) - \mathcal{L}_3(t) \right) \leq C_3 \mathcal{L}_3(0).$$

Finally, we conclude that

$$E_1(t) \leq C \left(\frac{E_1(0) + E_2(0)}{t} \right), \quad \forall t > 0, \quad C > 0.$$

I

3.3 Special Case: Absence of Axial Load

In this subsection, we shall also place ourselves in a more economic and less complicated situation by disposing of the frictional damping in the bending component. In addition, we examine the case where there is no axial loading. The torsion equation will consequently be coupled only with the rotation caused by the bending. We study the Timoshenko system (2.1), with $P = 0$ and $c_1 = 0$ that

is

$$\left\{ \begin{array}{l} \rho I \phi_{tt} + c_3 \phi_t - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} = 0, \\ \mu v_{tt} - kAG(v_{xx} - \phi_x) = 0, \\ -GJ\psi_{xx} - K\phi_{xx} + I_s \psi_{tt} + c_2 \psi_t = 0, \\ \phi(0, t) = \phi(L, t) = v_x(0, t) = v_x(L, t) = \psi(0, t) = \psi(L, t) = 0, \\ \phi(x, 0) = \phi_0(x), \quad \phi_t(x, 0) = \phi_1(x), \quad v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \\ \psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x). \end{array} \right. \quad (3.35)$$

By virtue of the above assumption, the load P is equal to zero and thereupon the energy functional $E_1(t)$ in (3.7) reduces to

$$\begin{aligned} E_1(t) = \frac{1}{2} \int_0^L \left[EI(\phi_x)^2 + 2K\phi_x\psi_x + kAG(v_x - \phi)^2 + GJ(\psi_x)^2 + \mu(v_t)^2 \right. \\ \left. + I_s(\psi_t)^2 + \rho I(\phi_t)^2 \right] dx, \quad t \geq 0. \end{aligned} \quad (3.36)$$

Remember, the condition (2.53) that keeps the positivity of the energy, that is

$$K < EI \quad \text{and} \quad K < GJ. \quad (3.37)$$

Subsequently from (2.13), it appears that

$$\frac{dE_1(t)}{dt} = -c_3 \|\phi_t\|^2 - c_2 \|\psi_t\|^2, \quad t \geq 0. \quad (3.38)$$

Prior to stating and proving the result of this section, we shall prepare a number of lemmas where we introduce suitable functionals and estimate their derivatives.

Lemma 3.3.1 *For the functional*

$$V_1(t) := I_s \int_0^L \psi \psi_t dx + \frac{c_2}{2} \int_0^L \psi^2 dx - \rho I \int_0^L \psi \phi_t dx + \frac{I_s(EI - K)}{GJ} \int_0^L \psi_t \phi dx, \quad t \geq 0$$

the estimate below holds

$$\begin{aligned} \frac{dV_1(t)}{dt} &\leq \left\{ I_s + \frac{1}{4\epsilon} \left[(\rho I)^2 + \frac{(EI - K)^2}{(GJ)^2} (I_s^2 + c_2^2) \right] \right\} \|\psi_t\|^2 \\ &\quad - \left(GJ - K - \epsilon C_p - kAGC_p \epsilon_2 \right) \|\psi_x\|^2 + \left(2\epsilon + \frac{c_3^2}{4\epsilon} \right) \|\phi_t\|^2 + \frac{kAG}{4\epsilon_2} \|v_x - \phi\|^2 \\ &\quad - \left[\frac{K(EI - K)}{GJ} - C_p \epsilon \right] \|\phi_x\|^2, \quad \epsilon, \epsilon_2 > 0, \quad t \geq 0. \end{aligned} \quad (3.39)$$

Proof. With the first equation and the third equations of problem (3.35) in mind, it is readily seen that

$$\begin{aligned} \frac{dV_1(t)}{dt} &= I_s \|\psi_t\|^2 + \int_0^L \psi \left(GJ \psi_{xx} + K \phi_{xx} - c_2 \psi_t \right) dx + c_2 \int_0^L \psi \psi_t dx - \rho I \int_0^L \psi_t \phi_t dx \\ &\quad + \int_0^L \psi \left(c_3 \phi_t - EI \phi_{xx} - kAG(v_x - \phi) - K \psi_{xx} \right) dx + \frac{I_s(EI - K)}{GJ} \int_0^L \psi_t \phi_t dx \\ &\quad + \frac{(EI - K)}{GJ} \int_0^L \phi \left(GJ \psi_{xx} + K \phi_{xx} - c_2 \psi_t \right) dx \\ &= I_s \|\psi_t\|^2 - GJ \|\psi_x\|^2 - K \int_0^L \psi_x \phi_x dx - \rho I \int_0^L \psi_t \phi_t dx + c_3 \int_0^L \psi \phi_t dx \\ &\quad + EI \int_0^L \psi_x \phi_x dx - kAG \int_0^L \psi (v_x - \phi) dx + K \|\psi_x\|^2 + \frac{I_s(EI - K)}{GJ} \int_0^L \psi_t \phi_t dx \\ &\quad - (EI - K) \int_0^L \phi_x \psi_x dx - \frac{K(EI - K)}{GJ} \|\phi_x\|^2 - \frac{c_2(EI - K)}{GJ} \int_0^L \phi \psi_t dx. \end{aligned}$$

Thanks to Young's inequality and Poincaré inequality, we immediately secure the estimation

$$\begin{aligned} \frac{dV_1(t)}{dt} &\leq I_s \|\psi_t\|^2 - GJ \|\psi_x\|^2 + \frac{(\rho I)^2}{4\epsilon} \|\psi_t\|^2 + \epsilon \|\phi_t\|^2 + \frac{c_3^2}{4\epsilon} \|\phi_t\|^2 + \epsilon C_p \|\psi_x\|^2 \\ &+ \frac{kAG}{4\epsilon_2} \|v_x - \phi\|^2 + kAG\epsilon_2 C_p \|\psi_x\|^2 + K \|\psi_x\|^2 + \frac{[I_s(EI - K)]^2}{4\epsilon(GJ)^2} \|\psi_t\|^2 + \epsilon \|\phi_t\|^2 \\ &- \frac{K(EI - K)}{GJ} \|\phi_x\|^2 + \frac{[c_2(EI - K)]^2}{4(GJ)^2\epsilon} \|\psi_t\|^2 + \epsilon C_p \|\phi_x\|^2, \quad \epsilon > 0, \quad t \geq 0. \end{aligned}$$

To end the proof, it suffices to arrange the terms. ■

The second functional is crucial as it compensates the lack of damping we intentionally disposed of without affecting negatively the overall computation.

Lemma 3.3.2 *The functional*

$$V_2(t) := -\mu \int_0^L v_t v dx, \quad \forall t \geq 0$$

possesses the property below

$$\frac{dV_2(t)}{dt} \leq -\mu \|v_t\|^2 + \frac{3kAG}{2} \|v_x - \phi\|^2 + \frac{kAGC_p}{2} \|\phi_x\|^2, \quad \epsilon > 0, \quad t \geq 0. \quad (3.40)$$

Proof. For solutions of our system and in particular in view of the second equation of (3.35), using Young's inequality and Poincaré inequality, it is visible

that

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= -\mu||v_t||^2 - \int_0^L v \left[kAG(v_x - \phi)_x \right] dx \\
&= -\mu||v_t||^2 + kAG \int_0^L v_x(v_x - \phi) dx \\
&= -\mu||v_t||^2 + kAG||v_x - \phi||^2 + kAG \int_0^L \phi(v_x - \phi) dx \\
&\leq -\mu||v_t||^2 + \frac{3kAG}{2}||v_x - \phi||^2 + \frac{kAGC_p}{2}||\phi_x||^2.
\end{aligned}$$

■

The third functional is of great help as it supplies us with $-||v_x - \phi||^2$. In point of fact, such a term can be obtained by a multiplier acting on the first equation rather than the second equation as is our present strategy. Nevertheless, several attempts have proved fruitless by force of newly appearing terms difficult to control and sometimes higher order terms. The reader will certainly remark the wise choice of the different weights carefully chosen in order to dispose with a number of undesirable terms.

Lemma 3.3.3 *For solutions of our problem, the rate of change of the functional*

$$\begin{aligned}
V_3(t) &:= - \int_0^L \phi_t(v_x - \phi) dx - \int_0^L v_t \phi_x dx + \left[\frac{\mu}{\rho I k A G} \left(\frac{G J E I}{K} - K \right) - \frac{G J}{K} \right] \\
&\quad \times \int_0^L v_t \psi_x dx + \left(\frac{k A G I_s}{K \mu} \right) \left(\frac{\mu E I}{\rho I k A G} - 1 \right) \int_0^L \psi_t(v_x - \phi) dx, \quad \forall t \geq 0
\end{aligned}$$

and under the condition

$$\eta_1 := \left(\frac{GJ}{I_s} - \frac{kAG}{\mu} \right) \left(\frac{EI}{\rho I} - \frac{kAG}{\mu} \right) - \frac{K^2}{I_s \rho I} = 0 \quad (3.41)$$

obeys the relation

$$\begin{aligned} \frac{dV_3(t)}{dt} &\leq \left[\frac{c_3^2}{4(\rho I)^2 \epsilon} + \epsilon + 1 \right] \|\phi_t\|^2 - \left(\frac{kAG}{\rho I} - 2\epsilon \right) \|v_x - \phi\|^2 \\ &+ \frac{1}{4\epsilon} \left[\frac{c_2^2}{K^2} \left(\frac{EI}{\rho I} - \frac{kAG}{\mu} \right)^2 + \left(\frac{I_s EI}{K \rho I} - \frac{kAG I_s}{K \mu} \right)^2 \right] \|\psi_t\|^2 \end{aligned} \quad (3.42)$$

where $\epsilon > 0$ and $t \geq 0$.

Remark 3.3.4 1. Observe that by virtue of the condition $\eta_1 = 0$, we must

have

$$\frac{kAG}{\mu} \geq \frac{GJ}{I_s} \text{ and } \frac{kAG}{\mu} \geq \frac{EI}{\rho I}$$

which case is supported by the numerical example in the paper [49] or

$$\frac{kAG}{\mu} \leq \frac{GJ}{I_s} \text{ and } \frac{kAG}{\mu} \leq \frac{EI}{\rho I}$$

2. If $K = 0$, then the condition η_1 in (3.41) reminds us of the equal speeds of propagation property. The equal speeds of propagation property is between the first and the second equations. Here it does not necessarily involves the first and the second equation in the system. A close look shows that it can

be between the second and the third equation delineating the torsion i.e.

$$\frac{\mu}{kAG} = \frac{I_s}{GJ}.$$

Actually, when $K = 0$, the third equation becomes completely independent of the first two equations.

3. If $K = 0$, the choice of $V_3(t)$ would not be possible because of the division by K in the calculation.

Proof. [Proof of Lemma 3.3.3] On the account of the first and the second equation in (3.35), we may write

$$\begin{aligned} \frac{dV_3(t)}{dt} = & -\frac{1}{\rho I} \int_0^L \left[-c_3 \phi_t + EI \phi_{xx} + kAG(v_x - \phi) + K \psi_{xx} \right] (v_x - \phi) dx \\ & - \int_0^L \phi_t (v_x - \phi)_t dx - \frac{kAG}{\mu} \int_0^L [(v_x - \phi)_x] \phi_x dx - \int_0^L v_t \phi_{tx} dx \\ & + \frac{kAG}{\mu} \left[\frac{\mu}{\rho I kAG} \left(\frac{GJ EI}{K} - K \right) - \frac{GJ}{K} \right] \int_0^L [(v_x - \phi)_x] \psi_x dx \\ & + \left[\frac{\mu}{\rho I kAG} \left(\frac{GJ EI}{K} - K \right) - \frac{GJ}{K} \right] \int_0^L v_t \psi_{tx} dx \\ & + \frac{1}{I_s} \left(\frac{kAG I_s}{K \mu} \right) \left(\frac{\mu EI}{\rho I kAG} - 1 \right) \int_0^L \left[GJ \psi_{xx} + K \phi_{xx} - c_2 \psi_t \right] (v_x - \phi) dx \\ & + \left(\frac{kAG I_s}{K \mu} \right) \left(\frac{\mu EI}{\rho I kAG} - 1 \right) \int_0^L \psi_t (v_x - \phi)_t dx. \end{aligned}$$

Then, making use of integration by parts, reveals that

$$\begin{aligned}
\frac{dV_3(t)}{dt} = & \frac{c_3}{\rho I} \int_0^L \phi_t(v_x - \phi)dx - \frac{EI}{\rho I} \int_0^L \phi_{xx}(v_x - \phi)dx - \frac{kAG}{\rho I} \|v_x - \phi\|^2 \\
& - \frac{K}{\rho I} \int_0^L \psi_{xx}(v_x - \phi)dx - \int_0^L \phi_t v_{xt} dx + \|\phi_t\|^2 + \frac{kAG}{\mu} \int_0^L (v_x - \phi)\phi_{xx} dx + \int_0^L v_{xt}\phi_t dx \\
& - \left[\left(\frac{GJEI}{K\rho I} - \frac{K}{\rho I} \right) - \frac{kAGGJ}{\mu K} \right] \int_0^L (v_x - \phi)\psi_{xx} dx - \left[\frac{\mu}{\rho IkAG} \left(\frac{GJEI}{K} - K \right) - \frac{GJ}{K} \right] \\
& \times \int_0^L v_{xt}\psi_t dx + \left(\frac{kAGGJ}{K\mu} \right) \left(\frac{\mu EI}{\rho IkAG} - 1 \right) \int_0^L \psi_{xx}(v_x - \phi)dx + \left(\frac{kAG}{\mu} \right) \\
& \times \left(\frac{\mu EI}{\rho IkAG} - 1 \right) \int_0^L \phi_{xx}(v_x - \phi)dx - \left(\frac{c_2 kAG}{K\mu} \right) \left(\frac{\mu EI}{\rho IkAG} - 1 \right) \int_0^L \psi_t(v_x - \phi)dx \\
& + \left(\frac{kAGI_s}{K\mu} \right) \left(\frac{\mu EI}{\rho IkAG} - 1 \right) \int_0^L \psi_t v_{xt} dx - \left(\frac{kAGI_s}{K\mu} \right) \left(\frac{\mu EI}{\rho IkAG} - 1 \right) \int_0^L \psi_t \phi_t dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{dV_3(t)}{dt} = & \frac{c_3}{\rho I} \int_0^L \phi_t(v_x - \phi)dx + \left[-\frac{EI}{\rho I} + \frac{kAG}{\mu} + \frac{EI}{\rho I} - \frac{kAG}{\mu} \right] \int_0^L \phi_{xx}(v_x - \phi)dx \\
& - \frac{kAG}{\rho I} \|v_x - \phi\|^2 + \left[-\frac{K}{\rho I} - \frac{GJEI}{K\rho I} + \frac{K}{\rho I} + \frac{kAGGJ}{\mu K} + \frac{GJEI}{\rho IK} - \frac{kAGGJ}{K\mu} \right] \\
& \times \int_0^L \psi_{xx}(v_x - \phi)dx + \|\phi_t\|^2 - \left(\frac{c_2 EI}{\rho IK} - \frac{c_2 kAG}{K\mu} \right) \int_0^L \psi_t(v_x - \phi)dx \\
& + \underbrace{\left[-\frac{\mu JEI}{\rho IkAK} + \frac{K\mu}{\rho IkAG} + \frac{GJ}{K} + \frac{I_s EI}{\rho IK} - \frac{kAGI_s}{K\mu} \right]}_{\eta_1=0} \int_0^L \psi_t v_{xt} dx \\
& - \left(\frac{I_s EI}{\rho IkAG} - \frac{I_s kAG}{K\mu} \right) \int_0^L \psi_t \phi_t dx
\end{aligned} \tag{3.43}$$

and thus

$$\begin{aligned} \frac{dV_3(t)}{dt} &\leq \frac{c_3^2}{4(\rho I)^2\epsilon} \|\phi_t\|^2 + \epsilon \|v_x - \phi\|^2 - \frac{kAG}{\rho I} \|v_x - \phi\|^2 + \|\phi_t\|^2 + \frac{1}{4\epsilon} \left(\frac{c_2 EI}{\rho IK} - \frac{c_2 kAG}{K\mu} \right)^2 \\ &\times \|\psi_t\|^2 + \epsilon \|v_x - \phi\|^2 + \frac{1}{4\epsilon} \left(\frac{I_s EI}{K\rho I} - \frac{kAG I_s}{K\mu} \right)^2 \|\psi_t\|^2 + \epsilon \|\phi_t\|^2, \quad \epsilon > 0, \quad t \geq 0. \end{aligned}$$

In (3.43), we assumed that

$$-\frac{\mu GJEI}{\rho I kAGK} + \frac{K\mu}{\rho I kAG} + \frac{GJ}{K} + \frac{I_s EI}{\rho IK} - \frac{kAG I_s}{K\mu} = 0$$

which is equivalent to

$$\eta_1 = \frac{GJ}{I_s} \left(\frac{EI}{\rho I} - \frac{kAG}{\mu} \right) - \frac{kAG}{\mu} \left(\frac{EI}{\rho I} - \frac{kAG}{\mu} \right) - \frac{K^2}{\rho I I_s} = 0.$$

I

Now, we are ready to state and prove the result of this subsection.

Theorem 3.3.5 *Let $\eta_1 = 0$ and (3.37) holds. The functional $E_1(t)$ (and hence the classical energy) decays exponentially as time goes to infinity, that is, there are two positive constants C and ξ independent of the initial data and t such that*

$$E_1(t) \leq CE_1(0)e^{-\xi t}, \quad \forall t \geq 0.$$

Proof. First we present our reference functional

$$\mathcal{L}_1(t) := NE_1(t) + \sum_{i=1}^3 \lambda_i V_i(t)$$

where N and λ_i , $i = 1, 2, 3$ will be determined as we go along the proof. Invoking the previous relations (3.39), (3.40) and (3.42) we end up with the shorter form

$$\frac{d\mathcal{L}_1(t)}{dt} \leq A_1 \|\phi_t\|^2 + A_2 \|\psi_t\|^2 + A_3 \|v_t\|^2 + A_4 \|\phi_x\|^2 + A_5 \|\psi_x\|^2 + A_6 \|v_x - \phi\|^2$$

where

$$\begin{aligned} A_1 &:= -c_3 N + \left(2\epsilon + \frac{c_3^2}{4\epsilon}\right) \lambda_1 + \left[\frac{c_3^2}{4(\rho I)^2 \epsilon} + \epsilon + 1\right] \lambda_3, \\ A_2 &:= -c_2 N + \left\{ I_s + \frac{1}{4\epsilon} \left[(\rho I)^2 + \frac{(EI - K)^2}{(GJ)^2} (I_s^2 + c_2^2) \right] \right\} \lambda_1 + \frac{1}{4\epsilon} \left[\frac{c_2^2}{K^2} \left(\frac{EI}{\rho I} - \frac{kAG}{\mu} \right)^2 \right. \\ &\quad \left. + \left(\frac{I_s EI}{K \rho I} - \frac{kAG I_s}{K \mu} \right)^2 \right] \lambda_3, \\ A_3 &:= -\mu \lambda_2, \\ A_4 &:= -\left[\frac{K(EI - K)}{GJ} - C_p \epsilon \right] \lambda_1 + \frac{kAG C_p}{2} \lambda_2, \\ A_5 &:= -\left(GJ - K - \epsilon C_p - kAG C_p \epsilon_2 \right) \lambda_1, \\ A_6 &:= \frac{kAG}{4\epsilon_2} \lambda_1 + \frac{3kAG}{2} \lambda_2 - \left(\frac{kAG}{\rho I} - 2\epsilon \right) \lambda_3. \end{aligned}$$

Proceeding with the selection of the involved parameters, it can be noticed that considering N large enough in A_1, A_2 releases the limitation on $\epsilon > 0$. Regarding

the rest of the coefficients we need to ascertain that

$$\begin{cases} \frac{kAGC_p}{2}\lambda_2 < \frac{K(EI-K)}{GJ}\lambda_1 \\ K + kAGC_p\epsilon_2 < GJ, \\ \frac{1}{4\epsilon_2}\lambda_1 + \frac{3}{2}\lambda_2 < \frac{1}{\rho I}\lambda_3. \end{cases}$$

Clearly, λ_3 may be picked large enough and in the same opportunity ϵ_2 small enough (recall that $K < GJ$). Moreover, λ_2 may be selected small. The rest of the proof is similar to the one of Theorem 3.2.6. ■

Absence of Axial Loading: Case of Non-Equal Speeds of Propagation

As is customary for Timoshenko beams, we shall imitate the same procedure and upgrade the energy to the second order. This can be performed by differentiating the equations (with respect to time) of the original system and employing suitable multipliers. Our reference system becomes

$$\begin{cases} \rho I \phi_{ttt} + c_3 \phi_{tt} - EI \phi_{xxt} - kAG(v_{xt} - \phi_t) - K \psi_{xxt} = 0, \\ \mu v_{ttt} - kAG(v_{xxt} - \phi_{xt}) = 0, \\ -GJ \psi_{xxt} - K \phi_{xxt} + I_s \psi_{ttt} + c_2 \psi_{tt} = 0. \end{cases} \quad (3.44)$$

To discuss the case $\eta_1 \neq 0$, we define the second order energy

$$\begin{aligned} E_2(t) := E_1(t, \phi_t, v_t, \psi_t) &= \frac{1}{2} \int_0^L \left[EI(\phi_{xt})^2 + 2K\phi_{xt}\psi_{xt} + kAG(v_{xt} - \phi_t)^2 \right. \\ &\quad \left. + GJ(\psi_{xt})^2 + \mu(v_{tt})^2 + I_s(\psi_{tt})^2 + \rho I(\phi_{tt})^2 \right] dx, \quad t \geq 0. \end{aligned}$$

Then, it is not difficult to see that

$$\frac{dE_2(t)}{dt} = -c_3||\phi_{tt}||^2 - c_2||\psi_{tt}||^2, \quad t \geq 0. \quad (3.45)$$

Theorem 3.3.6 *In case $\eta_1 \neq 0$ and under the condition (3.37), the energy $E_1(t)$ decays polynomially, namely there is constant C independent of the initial data such that*

$$E_1(t) \leq C \left(\frac{E_1(0) + E_2(0)}{t} \right), \quad \forall t > 0.$$

Proof. The rate of change we shall examine is the one of the functional

$$\mathcal{L}_2(t) := N(2E_1(t) + E_2(t)) + \sum_{i=1}^3 \lambda_i V_i(t), \quad t \geq 0.$$

As usual, here N and $\lambda_i > 0$ will be fixed later. From the above argument $\eta_1 = 0$ we know that there exists a positive constant C such that

$$\begin{aligned} \frac{d\mathcal{L}_2(t)}{dt} &\leq -CE_1(t) - c_3N||\phi_t||^2 - c_2N||\psi_t||^2 - c_3N||\phi_{tt}||^2 - c_2N||\psi_{tt}||^2 \\ &+ \eta_1\lambda_3 \int_0^L \psi_t v_{xt} dx. \end{aligned} \quad (3.46)$$

Using the second equation of (3.44), we now treat the last term on the right-hand side of (3.46) as follows:

$$\begin{aligned} \eta_1 \int_0^L \psi_t v_{xt} dx &= \eta_1 \int_0^L \psi_t (v_x - \phi)_t dx + \eta_1 \int_0^L \psi_t \phi_t dx \\ &= \eta_1 \frac{d}{dt} \int_0^L \psi_t (v_x - \phi) dx - \eta_1 \int_0^t \psi_{tt} (v_x - \phi) dx + \eta_1 \int_0^L \psi_t \phi_t dx. \end{aligned}$$

With the assistance of Young's inequality and imitating the steps resulting in use of the Cauchy-Schwarz inequality and

$$\begin{aligned} \int_0^L \left(\int_0^x \phi_t(y, t) dy \right)^2 dx &\leq \int_0^L \left[\left(\int_0^x 1 dy \right) \left(\int_0^x \phi_t^2 dy \right) \right] dx \\ &\leq \int_0^L x \int_0^x \phi_t^2 dy dx \leq \int_0^L \phi_t^2 dx \int_0^L x dx \leq \frac{L^2}{2} \|\phi_t\|^2. \end{aligned} \quad (3.47)$$

It holds that

$$\eta_1 \int_0^L \psi_t v_{xt} dx \leq \eta_1 \frac{d}{dt} \int_0^L \psi_t (v_x - \phi) dx + \frac{\eta_1^2}{4\epsilon} \left(\|\psi_{tt}\|^2 + \|\psi_t\|^2 \right) + \epsilon \|v_x - \phi\| + \epsilon \|\phi_t\|^2$$

where $\epsilon > 0$. A new functional is in order

$$\mathcal{L}_3(t) := \mathcal{L}_2(t) - \eta_1 \lambda_3 \int_0^L \psi_t (v_x - \phi) dx, \quad t \geq 0.$$

Effortlessly, we find

$$\begin{aligned} \frac{d\mathcal{L}_3(t)}{dt} &\leq -C_2 E_1(t) - c_3 N \|\phi_t\|^2 - c_2 N \|\psi_t\|^2 - c_3 N \|\phi_{tt}\|^2 - c_2 N \|\psi_{tt}\|^2 \\ &\quad + \frac{\eta_1^2}{4\epsilon} \left(\|\psi_{tt}\|^2 + \|\psi_t\|^2 \right) + \epsilon \|v_x - \phi\| + \epsilon \|\phi_t\|^2, \quad t \geq 0. \end{aligned}$$

Choosing N large enough and ϵ small, the estimation reduces to

$$\frac{d\mathcal{L}_3(t)}{dt} \leq -C_2 E_1(t), \quad t \geq 0.$$

It suffices now to integrate the previous relation over $(0, t)$ and utilize the fact

that $E_1(t)$ is nonincreasing to infer that

$$tE_1(t) \leq \int_0^t E_1(s)ds \leq \frac{1}{C} \left(\mathcal{L}_3(0) - \mathcal{L}_3(t) \right) \leq \frac{\mathcal{L}_3(0)}{C}$$

which means

$$E_1(t) \leq \frac{1}{C} \left(\frac{E_1(0) + E_2(0)}{t} \right), \quad \forall t > 0.$$

I

CHAPTER 4

STABILITY OF A NONLINEAR FRACTIONAL TELEGRAPH PROBLEM

In this chapter, we examine a fractional equation with an order ranging from 1 to 2. This equation can be seen as an interpolation of the heat equation and the wave equation. The problem is characterized by a nonlinearity since it involves a power type term. Our objective is to shed some light on the impact of the lower order derivative and the viscoelastic term on the stability of the system. More precisely, we explore the potential of stabilizing the system using either a lower-order fractional term or a memory term that involves the Laplacian. We demonstrate a global Mittag-Leffler stability result in case a fractional frictional damping is active. We also prove a local Mittag-Leffler stability result when the material is viscoelastic in case of "small" relaxation functions. In contrast to the

problems involving integer-order, remarkably the present case presents additional significant challenges. As a matter of fact, they are expected as they are intrinsic to the fractional derivative nature. Notably, one is faced with many difficulties caused by their non-local character, in addition to the invalidity of some crucial properties known in the integer case, such as the product rule. We use several properties in fractional calculus. Furthermore, we introduce new Lyapunov-type functionals in this chapter in the multiplier technique framework.

We consider the problem (see [2])

$$\left\{ \begin{array}{l} {}^C D^\alpha ({}^C D^\alpha v(x, t)) + a {}^C D^\alpha v(x, t) + |v|^\gamma v = \Delta v(x, t) - b \int_0^t k(t-s) \Delta v(x, s) ds, \\ t > 0, \ x \in \Omega, \ 1/2 < \alpha < 1, \\ v(x, t) = 0, \ t > 0, \ x \in \partial\Omega, \\ v(x, 0) = v_0(x), \ x \in \Omega. \end{array} \right. \quad (4.1)$$

Our assumptions on the kernel k are:

(**K₁**) $k \in L^1_{loc}(R^+)$ is a nonnegative continuous function satisfying $\hat{k}(\lambda) = O(|\lambda|^{-1})$ and $I^{1-\alpha}k(t) \in C^1([0, \infty))$,

$${}^{RL}D^\alpha k(t) \leq -C_k k(t), \ t > 0$$

and

$$\bar{k} := \int_0^\infty k(s) ds < 1$$

for some positive constant C_k .

To deal with the nonlinearity we need to assume that

(**K₂**) $0 < \gamma < \frac{n}{n-2} - 1 = \frac{2}{n-2}$ if $n \geq 3$ and $\gamma > 0$ if $n = 1, 2$.

This assumption will allow us to use the embedding $H_0^1(\Omega) \subset L^q(\Omega)$ when $2 \leq q \leq 2n/(n-2)$ if $n \geq 3$ and $q \geq 2$ if $n = 1, 2$. For the kernel function, it is easy to see from Proposition 1.4.4 that

$$k(t) \leq Kt^{\alpha-1}E_{\alpha,\alpha}(-C_k t^\alpha), \quad t \geq 0 \quad (4.2)$$

for some $K > 0$. In addition, Proposition 1.4.5 and Proposition 1.4.6 imply that such kernels are summable and

$$\int_0^t k(s)ds \leq K \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-C_k s^\alpha)ds = Kt^\alpha E_{\alpha,\alpha+1}(-C_k t^\alpha) \leq K/C_k, \quad t > 0.$$

In fact, this family of functions $t^{\alpha-1}E_{\alpha,\alpha}(-C_k t^\alpha)$ satisfies (K_1). It is also worth noting that the arguments below are valid for initial data in $H_0^1(\Omega)$.

4.1 Well-Posedness of the Problem

It is evident that equation (4.1) can be reformulated as a system of two equations of order α

$$\begin{cases} {}^C D^\alpha v(x, t) = w(x, t), \\ {}^C D^\alpha w(x, t) = \Delta v(x, t) - aw(x, t) - b \int_0^t k(t-s) \Delta v(x, s) ds - |v|^\gamma v. \end{cases}$$

This is an advantage of using rather the sequential form of the fractional derivative.

Alternatively, in matrix form we obtain

$$\begin{cases} {}^C D^\alpha U = \mathcal{A}_1 U - \int_0^t \mathcal{A}_2(t-s)U(s)ds + f(t, U(t)), & \forall t \geq 0. \\ U(t) = (v, w)(t) \end{cases} \quad (4.3)$$

where the operator $\mathcal{A}_1$, $\mathcal{A}_2$ and $f(t, U(t))$ is defined by

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I \\ \Delta & -aI \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & 0 \\ bk(t-s)\Delta & 0 \end{pmatrix} \quad \text{and} \quad f(t, U(t)) = \begin{pmatrix} 0 \\ -|v|^\gamma(x, t)v(x, t) \end{pmatrix}.$$

Note that, if $v'(t)$ is continuous nearby zero then $w(0) = {}^C D^\alpha v(0) = 0$.

From Theorem (1.4.12), there exists a unique local solution, to our problem, in the space

$$\mathcal{H} := \left\{ w \in C([0, T], \mathcal{D}(\mathcal{A}_1)) \cap C^1([0, T], H_0^1(\Omega)) : t^{-\alpha} * w_t \in C^1([0, T], L^2(\Omega)) \right\},$$

$$\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}_2) := H^2(\Omega) \cap H_0^1(\Omega).$$

In the present situation our operator is the well-known Laplacian which fortunately satisfies the above conditions in an appropriate space. Therefore, we can profit from this result to ensure the well-posedness of our problem (4.1).

4.2 Stability Under an Active Lower Order Term

In this section, we present and prove our first stability result. It concerns the case $a = 1$ and $b = 0$. That is we shall examine the case when the only actor is the lower order term. Trying in this way to have an idea on its effect on the dynamic of the structure. Let

$$E_0(t) := \frac{1}{2} (\|D^\alpha v\|^2 + \|\nabla v\|^2) + \frac{1}{\gamma + 2} \|v(t)\|_{\gamma+2}^{\gamma+2}, \quad t \geq 0$$

by comparison with the conventional classical energy in the integer order case.

With the help of (4.1) and integration by parts, we obtain

$$\begin{aligned} D^\alpha E_0(t) &= \int_{\Omega} D^\alpha v D^\alpha (D^\alpha v) dx - \frac{\alpha}{2\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\ &+ \int_{\Omega} \nabla v \cdot D^\alpha \nabla v dx - \frac{\alpha}{2\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 dx \\ &+ \frac{1}{\gamma+2} D^\alpha \|v(t)\|_{\gamma+2}^{\gamma+2} \\ &= -\|D^\alpha v\|^2 + \frac{1}{\gamma+2} D^\alpha \|v(t)\|_{\gamma+2}^{\gamma+2} - \int_{\Omega} |v|^\gamma v D^\alpha v dx \\ &- \frac{\alpha}{2\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 + \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 \right] dx. \end{aligned}$$

From Proposition 1.4.8, it appears that

$$\begin{aligned} D^\alpha E_0(t) &\leq -\|D^\alpha v\|^2 - \frac{\alpha}{2\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\times \left[\left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 + \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 \right] dx \end{aligned} \quad (4.4)$$

for $t \geq 0$. We will make the assumption that our initial data (v_0, v_1) belongs to the space $(H_0^1(\Omega))^2$ and begin the process of searching for an appropriate functional to utilize in order to achieve our stability result. In the following lemma, we introduce a functional whose fractional derivative is capable of supplying two missing terms to come up with the full expression of $E_0(t)$ in the right hand side of (4.4).

Lemma 4.2.1 *For the functional*

$$V_0(t) := \int_{\Omega} v D^{\alpha} v dx, \quad t \geq 0$$

it holds that

$$\begin{aligned} D^{\alpha} V_0(t) &\leq \left(1 + \frac{C_p}{2}\right) \|D^{\alpha} v\|^2 - \frac{1}{2} \|\nabla v\|^2 - \|v(t)\|_{\gamma+2}^{\gamma+2} \\ &+ \frac{\alpha}{2\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[C_p \left| \int_0^{\xi} \frac{[\nabla v(s)]' ds}{(t-s)^{\alpha}} \right|^2 + \left(\int_0^{\xi} \frac{[D^{\alpha} v(s)]' ds}{(t-s)^{\alpha}} \right)^2 \right] dx \end{aligned}$$

where C_p is the Poincaré constant and $t \geq 0$.

Proof. As $D^{\alpha} v$ is absolutely continuous, we may apply Proposition 1.4.8 to derive, for $t \geq 0$

$$\begin{aligned} D^{\alpha} V_0(t) &= \|D^{\alpha} v\|^2 + \int_{\Omega} v D^{\alpha} (D^{\alpha} v) dx - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^{\xi} \frac{v'(\eta) d\eta}{(t-\eta)^{\alpha}} \right) \\ &\times \left(\int_0^{\xi} \frac{[D^{\alpha} v(s)]' ds}{(t-s)^{\alpha}} \right) dx \\ &= \|D^{\alpha} v\|^2 - \|\nabla v\|^2 - \int_{\Omega} v D^{\alpha} v dx - \|v(t)\|_{\gamma+2}^{\gamma+2} \\ &- \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^{\xi} \frac{v'(\eta) d\eta}{(t-\eta)^{\alpha}} \right) \left(\int_0^{\xi} \frac{[D^{\alpha} v(s)]' ds}{(t-s)^{\alpha}} \right) dx. \end{aligned}$$

This leads to

$$\begin{aligned} D^\alpha V_0(t) &\leq \left(1 + \frac{C_p}{2}\right) \|D^\alpha v\|^2 - \frac{1}{2} \|\nabla v\|^2 - \|v(t)\|_{\gamma+2}^{\gamma+2} \\ &+ \frac{\alpha}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[C_p \left| \int_0^\xi \frac{[\nabla v(s)]' ds}{(t-s)^\alpha} \right|^2 + \left(\int_0^\xi \frac{[D^\alpha v(s)]' ds}{(t-s)^\alpha} \right)^2 \right] dx \end{aligned}$$

for $t \geq 0$, which completes the demonstration. ■

Theorem 4.2.2 *The problem (4.1) with $a = 1$ and $b = 0$ is Mittag-Leffler stable.*

This means that there are positive constants B_0 (which depends on $E_0(0)$) and ν_0 such that

$$E_0(t) \leq B_0 E_\alpha(-\nu_0 t^\alpha), \quad t \geq 0.$$

Proof. Let

$$\mathcal{L}_0(t) := N E_0(t) + \lambda V_0(t), \quad t \geq 0$$

for certain positive constants λ and N to be determined. Therefore, in view of Proposition 1.4.9, we can write

$$\begin{aligned} D^\alpha \mathcal{L}_0(t) &\leq -N \|D^\alpha v\|^2 + \lambda \left(1 + \frac{C_p}{2}\right) \|D^\alpha v\|^2 - \frac{\lambda}{2} \|\nabla v\|^2 - \lambda \|v(t)\|_{\gamma+2}^{\gamma+2} \\ &- \frac{\alpha N}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 + \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 \right] dx \\ &+ \frac{\alpha \lambda}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[C_p \left| \int_0^\xi \frac{[\nabla v(s)]' ds}{(t-s)^\alpha} \right|^2 + \left(\int_0^\xi \frac{[D^\alpha v(s)]' ds}{(t-s)^\alpha} \right)^2 \right] dx. \end{aligned}$$

Hence

$$\begin{aligned}
D^\alpha \mathcal{L}_0(t) &\leq \left[\lambda \left(1 + \frac{C_p}{2} \right) - N \right] \|D^\alpha v\|^2 - \frac{\lambda}{2} \|\nabla v\|^2 - \lambda \|v(t)\|_{\gamma+2}^{\gamma+2} \\
&+ \frac{\alpha}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[(C_p \lambda - N) \left| \int_0^\xi \frac{[\nabla v(s)]' ds}{(t-s)^\alpha} \right|^2 \right. \\
&\left. + (\lambda - N) \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 \right] dx
\end{aligned} \tag{4.5}$$

for $t \geq 0$. To obtain $-C_1 E_0(t)$ with some positive constant C_1 in the right hand side of (4.5), we need

$$\begin{cases} \lambda \left(1 + \frac{C_p}{2} \right) < N, \\ C_p \lambda \leq N, \\ \lambda \leq N. \end{cases}$$

Clearly, all three conditions are fulfilled for

$$N > \max \left\{ C_p, 1 + \frac{C_p}{2} \right\} \lambda.$$

Consequently

$$D^\alpha \mathcal{L}_0(t) \leq -C_1 E_0(t), \quad t \geq 0$$

for some positive constant C_1 and as $\mathcal{L}_0(t)$ and $E_0(t)$ are equivalent (for some values of λ), we obtain

$$D^\alpha \mathcal{L}_0(t) \leq -C_2 \mathcal{L}_0(t), \quad t \geq 0$$

for some positive constant C_2 . The conclusion can be reached from Proposition

1.4.4 and the equivalence of $\mathcal{L}_0(t)$ and $E_0(t)$. I

4.3 Stability Under an Active Viscoelastic Damping

In addition, to the questions raised above on whether the lower order term plays the role of an efficient damper or not, one might ask another question. Namely, we might ask the same questions for a viscoelastic term. A priori this seems to be not straightforward to answer.

4.3.1 A Fractional Energy-Like Functional

Before beginning the discussion of this matter, the same material should be prepared.

Proposition 4.3.1 *If $I^{1-\alpha}k(t) \in C^1([0, \infty))$, $0 < \alpha < 1$, then the identity stated below holds for all regular functions v (differentiable H^1 functions)*

$$\begin{aligned} \int_{\Omega} D^{\alpha} \nabla v \cdot \int_0^t k(t-s) \nabla v(s) ds dx &= \frac{1}{2} \left\{ ({}^{RL}D^{\alpha} k \square \nabla v) + \left(\int_0^t k(s) ds \right) D^{\alpha} \|\nabla v\|^2 \right\} \\ &- \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^{\xi} \frac{k(\eta) d\eta}{(t-\eta)^{\alpha}} \int_0^{\xi} \frac{[\|\nabla v(s)\|^2]'}{(t-s)^{\alpha}} ds - \frac{1}{2} D^{\alpha} (k \square \nabla v)(t) \\ &+ \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^{\xi} \frac{[(\nabla v)(\eta)]' d\eta}{(t-\eta)^{\alpha}} \cdot \left[\int_0^{\xi} (t-s)^{-\alpha} \right. \\ &\times \left. \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right] dx, \quad t \geq 0. \end{aligned}$$

Proof. By virtue of Proposition 1.4.8 and the identity

$$\begin{aligned} (k \square \nabla v)(t) &= \int_{\Omega} \int_0^t k(t-s) |\nabla v(t) - \nabla v(s)|^2 ds dx = \|\nabla v\|^2 \int_0^t k(t-s) ds \\ &+ \int_0^t k(t-s) \|\nabla v(s)\|^2 ds - 2 \int_{\Omega} \nabla v \cdot \int_0^t k(t-s) \nabla v(s) ds dx \end{aligned}$$

it is evident that

$$\begin{aligned} D^{\alpha}(k \square \nabla v)(t) &= \left(D^{\alpha} \int_0^t k(t-s) ds \right) \|\nabla v\|^2 + \left(\int_0^t k(t-s) ds \right) D^{\alpha} \|\nabla v\|^2 \\ &- \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^{\xi} \frac{k(\eta) d\eta}{(t-\eta)^{\alpha}} \int_0^{\xi} \frac{[\|\nabla v(s)\|^2]' ds}{(t-s)^{\alpha}} \\ &+ D^{\alpha} \int_0^t k(t-s) \|\nabla v(s)\|^2 ds - 2 \int_{\Omega} D^{\alpha} \nabla v \cdot \int_0^t k(t-s) \nabla v(s) ds dx \\ &- 2 \int_{\Omega} \nabla v \cdot D^{\alpha} \int_0^t k(t-s) \nabla v(s) ds dx + \frac{2\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\times \int_0^{\xi} \frac{(\nabla v)'(\eta) d\eta}{(t-\eta)^{\alpha}} \cdot \left(\int_0^{\xi} (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right) dx. \end{aligned} \quad (4.6)$$

In addition, notice that

$$\begin{aligned} ({}^{RL}D^{\alpha} k \square \nabla v)(t) &= \|\nabla v\|^2 \int_0^t {}^{RL}D^{\alpha} k(t-s) ds + \int_0^t {}^{RL}D^{\alpha} k(t-s) \|\nabla v(s)\|^2 ds \\ &- 2 \int_{\Omega} \nabla v \cdot \int_0^t {}^{RL}D^{\alpha} k(t-s) \nabla v(s) ds dx. \end{aligned} \quad (4.7)$$

Also, by Proposition 1.4.7, we see that

$$D^{\alpha} \int_0^t k(t-s) ds = \int_0^t {}^{RL}D^{\alpha} k(t-s) ds + \lim_{t \rightarrow 0^+} I^{1-\alpha} k(t). \quad (4.8)$$

Again, using Proposition 1.4.7 and the summability of k , we infer that

$$\begin{aligned}
& D^\alpha \int_0^t k(t-s) \|\nabla v(s)\|^2 ds \\
&= {}^{RL}D^\alpha \int_0^t k(t-s) \|\nabla v(s)\|^2 ds - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(\int_0^t k(t-s) \|\nabla v(s)\|^2 ds \right) \Big|_{t=0} \\
&= \int_0^t {}^{RL}D^\alpha k(t-s) \|\nabla v(s)\|^2 ds + \|\nabla v(t)\|^2 \lim_{t \rightarrow 0^+} I^{1-\alpha} k(t) \tag{4.9}
\end{aligned}$$

and

$$\begin{aligned}
& \int_\Omega \nabla v \cdot D^\alpha \int_0^t k(t-s) \nabla v(s) ds dx \\
&= \int_\Omega \nabla v \cdot \int_0^t {}^{RL}D^\alpha k(t-s) \nabla v(s) ds dx + \|\nabla v(t)\|^2 \lim_{t \rightarrow 0^+} I^{1-\alpha} k(t). \tag{4.10}
\end{aligned}$$

Combining the previous relationships (4.8)-(4.10) with (4.6), we obtain

$$\begin{aligned}
D^\alpha (k \square \nabla v)(t) &= \|\nabla v\|^2 \int_0^t {}^{RL}D^\alpha k(t-s) ds + \left(\int_0^t k(t-s) ds \right) D^\alpha \|\nabla v\|^2 \\
&+ \int_0^t {}^{RL}D^\alpha k(t-s) \|\nabla v(s)\|^2 ds - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \\
&\times \int_0^\xi \frac{[\|\nabla v(s)\|^2]' ds}{(t-s)^\alpha} - 2 \int_\Omega D^\alpha \nabla v \cdot \int_0^t k(t-s) \nabla v(s) ds dx \\
&- 2 \int_\Omega \nabla v \cdot \int_0^t {}^{RL}D^\alpha k(t-s) \nabla v(s) ds dx + \frac{2\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \int_0^\xi \frac{[(\nabla v)(\eta)]' d\eta}{(t-\eta)^\alpha} \cdot \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right) dx. \tag{4.11}
\end{aligned}$$

Moreover, considering the relationships (4.7) and (4.11), we can derive the follow-

ing identity for $t \geq 0$

$$\begin{aligned}
D^\alpha(k \square \nabla v)(t) &= ({}^{RL}D^\alpha k \square \nabla v) + \left(\int_0^t k(s) ds \right) D^\alpha \|\nabla v\|^2 \\
&- \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \int_0^\xi \frac{[\|\nabla v(s)\|^2]' ds}{(t-s)^\alpha} \\
&- 2 \int_\Omega D^\alpha \nabla v \cdot \int_0^t k(t-s) \nabla v(s) ds dx + \frac{2\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \int_0^\xi \frac{[(\nabla v)(\eta)]' d\eta}{(t-\eta)^\alpha} \cdot \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right) dx. \quad (4.12)
\end{aligned}$$

■

The multiplications of the equation in (4.1) by $D^\alpha v$ (when $a = 0$ and $b = 1$) yields

$$\begin{aligned}
&\int_\Omega D^\alpha v D^\alpha (D^\alpha v) dx + \int_\Omega |v|^\gamma v D^\alpha v dx \\
&= - \int_\Omega D^\alpha \nabla v \cdot \nabla v dx + \int_\Omega D^\alpha \nabla v \cdot \int_0^t k(t-s) \nabla v(s, x) ds dx, \quad t \geq 0 \quad (4.13)
\end{aligned}$$

and by the above Proposition 1.4.8 , we see that

$$\begin{aligned}
&\int_\Omega D^\alpha v D^\alpha (D^\alpha v) dx + \int_\Omega |v|^\gamma v D^\alpha v dx + \int_\Omega D^\alpha \nabla v \cdot \nabla v dx - \frac{1}{2} D^\alpha \|\nabla v\|^2 \int_0^t k(s) ds \\
&+ \frac{1}{2} D^\alpha(k \square \nabla v)(t) = \frac{1}{2} ({}^{RL}D^\alpha k \square \nabla v) - \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \\
&\times \int_0^\xi \frac{[\|\nabla v(s)\|^2]' ds}{(t-s)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{[(\nabla v)(\eta)]' d\eta}{(t-\eta)^\alpha} \cdot \left[\int_0^\xi (t-s)^{-\alpha} \right. \\
&\times \left. \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right] dx, \quad t \geq 0. \quad (4.14)
\end{aligned}$$

This identity motivated and justified our choice of energy-like functional

$$E_1(t) := \frac{1}{2} \left[\|D^\alpha v\|^2 + \left(1 - \int_0^t k(s) ds\right) \|\nabla v\|^2 + (k \square \nabla v)(t) \right] + \frac{1}{\gamma + 2} \|v(t)\|_{\gamma+2}^{\gamma+2}, \quad t \geq 0.$$

4.3.2 Stability Driven by the Memory Term

Here, in the absence of the fractional damping, only the viscoelastic term is active on the system. Without loss of generality, we pick $b = 1$.

Lemma 4.3.2 *The functional $E_1(t)$, satisfies along the solutions of (4.1), for $t \geq 0$, the following estimation*

$$\begin{aligned} D^\alpha E_1(t) &\leq \frac{1}{2} ({}^{RL}D^\alpha k \square \nabla v) - \frac{1}{2} I^{1-\alpha} k(t) \|\nabla v\|^2 - \frac{\alpha}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\times \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx - \frac{\alpha}{4\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 dx \\ &+ \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right|^2 dx. \end{aligned}$$

Proof. It is evident that appealing to Proposition 1.4.8, we find

$$\begin{aligned}
D^\alpha E_1(t) &= \frac{1}{2} D^\alpha \left[\|D^\alpha v\|^2 + \left(1 - \int_0^t k(s) ds\right) \|\nabla v\|^2 + (k \square \nabla v)(t) \right] \\
&+ \frac{1}{\gamma + 2} D^\alpha \|v(t)\|_{\gamma+2}^{\gamma+2} \\
&= \int_\Omega D^\alpha v D^\alpha (D^\alpha v) dx - \frac{\alpha}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\
&+ \left(1 - \int_0^t k(s) ds\right) \left[\int_\Omega \nabla v \cdot D^\alpha (\nabla v) dx - \frac{\alpha}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \right. \\
&\times \left. \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 dx \right] - \frac{1}{2} \left(D^\alpha \int_0^t k(s) ds \right) \|\nabla v\|^2 + \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi \frac{[\|\nabla v(s)\|^2]' ds}{(t-s)^\alpha} \right) \int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} + \frac{1}{2} D^\alpha (k \square \nabla v)(t) + \frac{1}{\gamma + 2} D^\alpha \|v(t)\|_{\gamma+2}^{\gamma+2}.
\end{aligned}$$

Also, owing to

$$\left(D^\alpha \int_0^t k(s) ds \right) = \left({}^{RL}D^\alpha \int_0^t k(s) ds \right) = I^{1-\alpha} k(t) \quad (4.15)$$

and equation (4.14), we can write

$$\begin{aligned}
D^\alpha E_1(t) &= - \int_\Omega |v|^\gamma v D^\alpha v dx - \int_\Omega D^\alpha \nabla v \cdot \nabla v dx + \int_\Omega D^\alpha \nabla v \cdot \nabla v dx \\
&\times \int_0^t k(t-s) \nabla v(s, x) ds dx - \frac{\alpha}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\
&+ \left(1 - \int_0^t k(s) ds\right) \left[\int_\Omega \nabla v \cdot D^\alpha (\nabla v) dx - \frac{\alpha}{2\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \right. \\
&\times \left. \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 dx \right] - \frac{1}{2} I^{1-\alpha} k(t) \|\nabla v\|^2 + \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi \frac{[\|\nabla v(s)\|^2]' ds}{(t-s)^\alpha} \right) \int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} + \frac{1}{2} D^\alpha (k \square \nabla v)(t) + \frac{1}{\gamma + 2} D^\alpha \|v(t)\|_{\gamma+2}^{\gamma+2}.
\end{aligned}$$

Employing the Proposition 4.3.1, again results in

$$\begin{aligned}
D^\alpha E_1(t) &= \frac{1}{2}({}^{RL}D^\alpha k \square \nabla v) - \frac{1}{2}I^{1-\alpha}k(t) \|\nabla v\|^2 + \frac{1}{\gamma+2}D^\alpha \|v(t)\|_{\gamma+2}^{\gamma+2} \\
&\quad - \int_{\Omega} |v|^\gamma v D^\alpha v dx - \frac{\alpha}{2\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\
&\quad - \frac{\alpha}{2\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 dx + \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\quad \times \int_0^\xi \frac{[(\nabla v)(\eta)]' d\eta}{(t-\eta)^\alpha} \cdot \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right] dx, \quad t \geq 0.
\end{aligned}$$

We conclude the desired result using Proposition 1.4.9 and the evaluation

$$\begin{aligned}
&\int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \cdot \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right) dx \\
&\leq \frac{1}{4} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 dx + \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\quad \times \left| \int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) \nabla v(\tau) d\tau \right)' ds \right|^2 dx.
\end{aligned}$$

I

The last term in the estimation of $D^\alpha E_1(t)$ needs to be controlled somehow. This term does not appear in the integer order case (second order) and, whatever evaluation we come up with, its weight is going to be considerable with compared with the functional with which we shall perturb $E_1(t)$. To remedy to this issue, we suggest a new functional in the next lemma.

Lemma 4.3.3 *For the functional*

$$V_1(t) := \int_{\Omega} \left| \int_0^t k(t-s) \nabla v(s) ds \right|^2 dx, \quad t \geq 0$$

it hold that

$$\begin{aligned}
D^\alpha V_1(t) &\leq 2\epsilon_1 \bar{k} \int_{\Omega} \left(\int_0^t k(t-s) |\nabla v(s)|^2 ds \right) dx + \frac{K}{\epsilon_1} (|^{RL}D^\alpha k| \square \nabla v)(t) \\
&+ \frac{[KE_\alpha(-C_k t^\alpha)]^2}{\epsilon_1} \|\nabla v\|^2 - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left| \int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta k(\eta-\tau) \nabla v(\tau) d\tau \right)' d\eta \right|^2 dx
\end{aligned}$$

for $\epsilon_1 > 0$, $t \geq 0$.

Proof. Relying on Proposition 1.4.8, we can write

$$\begin{aligned}
D^\alpha V_1(t) &= 2 \int_{\Omega} \left(\int_0^t k(t-s) \nabla v(s) ds \right) \cdot D^\alpha \left(\int_0^t k(t-s) \nabla v(s) ds \right) dx \\
&- \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta k(\eta-\tau) \nabla v(\tau) d\tau \right)' d\eta \right|^2 dx
\end{aligned}$$

and with the help of (1.23), we can derive the following

$$\begin{aligned}
D^\alpha V_1(t) &= 2 \int_{\Omega} \left(\int_0^t k(t-s) \nabla v(s) ds \right) \\
&\times \left[^{RL}D^\alpha \left(\int_0^t k(t-s) \nabla v(s) ds \right) dx - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(\int_0^t k(t-s) \nabla v(s) ds \right) \Big|_{t=0} \right] dx \\
&- \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta k(\eta-\tau) \nabla v(\tau) d\tau \right)' d\eta \right|^2 dx.
\end{aligned}$$

Furthermore, the integrability of k and Proposition 1.4.7 lead to

$$\begin{aligned}
D^\alpha V_1(t) &= 2 \int_{\Omega} \left(\int_0^t k(t-s) \nabla v(s) ds \right) \cdot \left[\int_0^t {}^{RL}D^\alpha k(t-s) [\nabla v(s) - \nabla v(t)] ds \right. \\
&\quad \left. + \nabla v(t) \int_0^t {}^{RL}D^\alpha k(s) ds + \nabla v(t) \lim_{t \rightarrow 0^+} I^{1-\alpha} k(t) \right] dx \\
&\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta k(\eta-\tau) \nabla v(\tau) d\tau \right)' d\eta \right|^2 dx \\
&= 2 \int_{\Omega} \left(\int_0^t k(t-s) \nabla v(s) ds \right) \cdot \int_0^t {}^{RL}D^\alpha k(t-s) [\nabla v(s) - \nabla v(t)] ds dx \\
&\quad + 2 I^{1-\alpha} k(t) \int_{\Omega} \left(\int_0^t k(t-s) \nabla v(s) ds \right) \cdot \nabla v(t) dx \\
&\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta k(\eta-\tau) \nabla v(\tau) d\tau \right)' d\eta \right|^2 dx.
\end{aligned}$$

For $\epsilon_1 > 0$, utilizing Young's inequality, we see that

$$\begin{aligned}
D^\alpha V_1(t) &\leq 2\epsilon_1 \bar{k} \int_{\Omega} \left(\int_0^t k(t-s) |\nabla v(s)|^2 ds \right) dx + \frac{1}{\epsilon_1} \left(\int_0^t |{}^{RL}D^\alpha k|(s) ds \right) \\
&\quad (|{}^{RL}D^\alpha k| \square \nabla v)(t) + \frac{[I^{1-\alpha} k(t)]^2}{\epsilon_1} \|\nabla v\|^2 - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\quad \times \left| \int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta k(\eta-\tau) \nabla v(\tau) d\tau \right)' d\eta \right|^2 dx. \tag{4.16}
\end{aligned}$$

The relation below is a direct applications of Proposition 1.4.6 and (4.2)

$$\begin{aligned}
I^{1-\alpha} k(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} k(s) ds \\
&\leq \frac{K}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-C_k(t-s)^\alpha) s^{-\alpha} ds = K E_\alpha(-C_k t^\alpha). \tag{4.17}
\end{aligned}$$

Hence

$$\begin{aligned} \int_0^t |{}^{RL}D^\alpha k(s)| ds &= - \int_0^t DI^{1-\alpha} k(s) ds = - [I^{1-\alpha} k(s)]_0^t \\ &= I^{1-\alpha} k(0) - I^{1-\alpha} k(t) \leq K. \end{aligned} \quad (4.18)$$

To derive the next evaluation, it suffices to employ (4.17) and (4.18) in (4.16)

$$\begin{aligned} D^\alpha V_1(t) &\leq 2\epsilon_1 \bar{k} \int_\Omega \left(\int_0^t k(t-s) |\nabla v(s)|^2 ds \right) dx + \frac{K}{\epsilon_1} (|{}^{RL}D^\alpha k| \square \nabla v)(t) \\ &+ \frac{[KE_\alpha(-C_k t^\alpha)]^2}{\epsilon_1} \|\nabla v\|^2 - \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\times \left| \int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta k(\eta-\tau) \nabla v(\tau) d\tau \right)' d\eta \right|^2 dx \end{aligned}$$

where $\epsilon_1 > 0$ and $t \geq 0$. I

The next lemma introduces the second functional.

Lemma 4.3.4 *The fractional rate of change of the functional*

$$V_2(t) := \int_\Omega v D^\alpha v dx, \quad t \geq 0$$

is in accordance with

$$\begin{aligned} D^\alpha V_2(t) &\leq \|D^\alpha v\|^2 - (1 - \epsilon_2) \|\nabla v\|^2 - \|v(t)\|_{\gamma+2}^{\gamma+2} \\ &+ \frac{\bar{k}}{4\epsilon_2} \int_\Omega \left(\int_0^t k(t-s) |\nabla v(s)|^2 ds \right) dx + \frac{\alpha \epsilon_3 C_p}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi \frac{\nabla v'(s) ds}{(t-s)^\alpha} \right|^2 dx \\ &+ \frac{\alpha}{4\epsilon_3 \Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(s)]' ds}{(t-s)^\alpha} \right)^2 dx \end{aligned}$$

for $\epsilon_2, \epsilon_3 > 0$ and $t \geq 0$, where C_p is the Poincaré constant.

Proof. Proposition 1.4.8 and integration by parts, imply that

$$\begin{aligned}
D^\alpha V_2(t) &= \|D^\alpha v\|^2 + \int_{\Omega} v D^\alpha (D^\alpha v) dx \\
&\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha v(s)]' ds}{(t-s)^\alpha} \right) dx \\
&= \|D^\alpha v\|^2 + \int_{\Omega} v \left[\Delta v - \int_0^t k(t-s) \Delta v(s) ds - |v|^\gamma v \right] dx \\
&\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha v(s)]' ds}{(t-s)^\alpha} \right) dx \\
&= \|D^\alpha v\|^2 - \|\nabla v\|^2 + \int_{\Omega} \nabla v \cdot \int_0^t k(t-s) \nabla v(s) ds dx - \|v(t)\|_{\gamma+2}^{\gamma+2} \\
&\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha v(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

Moreover, a direct use of Young's inequality and Poincaré inequality, leads to

$$\begin{aligned}
D^\alpha V_2(t) &\leq \|D^\alpha v\|^2 - \|\nabla v\|^2 + \epsilon_2 \|\nabla v\|^2 + \frac{\bar{k}}{4\epsilon_2} \int_{\Omega} \left(\int_0^t k(t-s) |\nabla v(s)|^2 ds \right) dx \\
&\quad - \|v(t)\|_{\gamma+2}^{\gamma+2} + \frac{\alpha \epsilon_3 C_p}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi \frac{\nabla v'(s) ds}{(t-s)^\alpha} \right|^2 dx \\
&\quad + \frac{\alpha}{4\epsilon_3 \Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(s)]' ds}{(t-s)^\alpha} \right)^2 dx
\end{aligned}$$

where $\epsilon_2, \epsilon_3 > 0$ and $t \geq 0$. I

Lemma 4.3.5 *For the functional*

$$V_3(t) := \int_0^t k(t-s) \|\nabla v(s)\|^2 ds, \quad t \geq 0$$

the estimation to come is true

$$D^\alpha V_3(t) \leq -C_k V_3(t) + \left(\lim_{t \rightarrow 0^+} I^{1-\alpha} k(t) \right) \|\nabla v\|^2, \quad t \geq 0.$$

Proof. By Proposition 1.4.5 and (K_1) , it is straightforward to entail that

$$\begin{aligned} D^\alpha V_3(t) &:= D^\alpha \left(\int_0^t k(t-s) \|\nabla v(s)\|^2 ds \right) \\ &= \int_0^t {}^{RL}D^\alpha k(t-s) \|\nabla v(s)\|^2 ds + \left(\lim_{t \rightarrow 0^+} I^{1-\alpha} k(t) \right) \|\nabla v\|^2 \\ &\leq -C_k \int_0^t k(t-s) \|\nabla v(s)\|^2 ds + \left(\lim_{t \rightarrow 0^+} I^{1-\alpha} k(t) \right) \|\nabla v\|^2. \end{aligned}$$

■

As we have removed the lower-order fractional term (considered in the previous case), which we found acts as genuine damper and was responsible for the appearance of the nice term $-\|D^\alpha v\|^2$, we need to find a way to fill this gap. The next suggested functional will take care of this matter.

Lemma 4.3.6 *If the exponent γ obeys condition $(\mathbf{K}_2)$, then for the given functional*

$$V_4(t) := - \int_{\Omega} D^\alpha v \int_0^t k(t-s) [v(t) - v(s)] ds dx, \quad t \geq 0,$$

we have, along solutions of problem (4.1)

$$\begin{aligned}
D^\alpha V_4(t) &\leq \epsilon_5 \|\nabla v\|^2 + \bar{k} \left(\frac{1}{4\epsilon_5} + \frac{C_p}{4\epsilon_6} + 1 \right) (k \square \nabla v)(t) + \left(\epsilon_4 - \frac{k_0}{2} \right) \|D^\alpha v\|^2 \\
&+ \epsilon_6 C_1 E_1^{\gamma+1}(t) - \frac{K C_p}{2k_0} ({}^{RL}D^\alpha k \square \nabla v)(t) + \frac{C_4 E_\alpha(-C_k t^\alpha)}{\epsilon_4} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right)^2 dx + \frac{\alpha \epsilon_7}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\
&+ \frac{\alpha}{4\epsilon_7 \Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right)^2 dx,
\end{aligned}$$

where $\epsilon_i > 0$, $i = 4, 5, 6, 7$, $t \geq t_0 > 0$ and $k_0 = \int_0^{t_0} k(s) ds$ for some $t_0 > 0$.

Proof. Differentiating $V_4(t)$ with the help of Proposition 1.4.8, it holds that

$$\begin{aligned}
D^\alpha V_4(t) &= - \int_\Omega D^\alpha (D^\alpha v) \int_0^t k(t-s) [v(t) - v(s)] ds dx \\
&- \int_\Omega D^\alpha v D^\alpha \int_0^t k(t-s) [v(t) - v(s)] ds dx + \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right) dx.
\end{aligned}$$

Along solutions of (4.1), we find

$$\begin{aligned}
D^\alpha V_4(t) &= - \int_\Omega \left[\Delta v - \int_0^t k(t-s) \Delta v(s) ds - |v|^\gamma v \right] \int_0^t k(t-s) [v(t) \\
&- v(s)] ds dx - \int_\Omega D^\alpha v \left[D^\alpha \left(v(t) \int_0^t k(t-s) ds \right) - D^\alpha \int_0^t k(t-s) v(s) ds \right] dx \\
&+ \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \\
&\times \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right) dx.
\end{aligned}$$

Next, we apply Proposition 1.4.7

$$\begin{aligned}
D^\alpha V_4(t) &= \int_{\Omega} \left[\left(1 - \int_0^t k(s) ds \right) \nabla v + \int_0^t k(t-s) [\nabla v(t) - \nabla v(s)] ds \right] \\
&\times \int_0^t k(t-s) [\nabla v(t) - \nabla v(s)] ds dx \\
&- \int_{\Omega} D^\alpha v \left[D^\alpha \left(v(t) \int_0^t k(t-s) ds \right) - \int_0^t {}^{RL}D^\alpha k(t-s) v(s) ds - v(t) I^{1-\alpha} k(0) \right] dx \\
&+ \int_{\Omega} |v|^\gamma v \int_0^t k(t-s) [v(t) - v(s)] ds dx \\
&+ \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \\
&\times \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right) dx.
\end{aligned}$$

Hence

$$\begin{aligned}
D^\alpha V_4(t) &= \left(1 - \int_0^t k(s) ds \right) \int_{\Omega} \nabla v \cdot \int_0^t k(t-s) [\nabla v(t) - \nabla v(s)] ds dx \\
&+ \int_{\Omega} \left| \int_0^t k(t-s) [\nabla v(t) - \nabla v(s)] ds \right|^2 dx + \int_{\Omega} |v|^\gamma v \int_0^t k(t-s) [v(t) - v(s)] ds dx \\
&- \int_{\Omega} D^\alpha v \left[D^\alpha \left(v(t) \int_0^t k(t-s) ds \right) - \int_0^t {}^{RL}D^\alpha k(t-s) v(s) ds \right] dx \\
&+ I^{1-\alpha} k(0) \int_{\Omega} v D^\alpha v dx + \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \\
&\times \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right] dx. \tag{4.19}
\end{aligned}$$

Working out each term separately, an application of Proposition 1.4.8 guarantees

the identity

$$D^\alpha \left(v(t) \int_0^t k(t-s)ds \right) = \left(\int_0^t k(s)ds \right) D^\alpha v(t) + v(t) \left[\int_0^t {}^{RL}D^\alpha k(t-s)ds \right. \\ \left. + I^{1-\alpha}k(0) \right] - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right).$$

Therefore, utilizing integration by parts, we obtain

$$\begin{aligned} & - \int_\Omega D^\alpha v D^\alpha \left(v(t) \int_0^t k(s)ds \right) dx = - \int_\Omega v D^\alpha v \left[\int_0^t {}^{RL}D^\alpha k(t-s)ds \right. \\ & \left. + I^{1-\alpha}k(0) \right] dx - \left(\int_0^t k(s)ds \right) \|D^\alpha v\|^2 + \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega D^\alpha v \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ & \left(\int_0^\xi \frac{k(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right) dx \\ & \leq - \int_\Omega v D^\alpha v \int_0^t {}^{RL}D^\alpha k(t-s)ds dx - I^{1-\alpha}k(0) \int_\Omega v D^\alpha v dx \\ & - \left(\int_0^t k(s)ds \right) \|D^\alpha v\|^2 + \epsilon_4 \|D^\alpha v\|^2 + \frac{\alpha^2}{4\epsilon_4 \Gamma^2(1-\alpha)} \int_\Omega \left[\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \right. \\ & \left. \times \left(\int_0^\xi \frac{k(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right) \right]^2 dx \end{aligned} \quad (4.20)$$

where $\epsilon_4 > 0$, $t \geq 0$. The Cauchy-Schwarz inequality can be utilized on the last term in equation (4.20) as follows

$$\begin{aligned} & \int_\Omega \left[\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right) \right]^2 dx \\ & \leq \int_\Omega \left[\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta)d\eta}{(t-\eta)^\alpha} \right)^2 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right)^2 \right] dx \\ & \leq \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta)d\eta}{(t-\eta)^\alpha} \right)^2 \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right)^2 dx. \end{aligned}$$

In view of assumption $(\mathbf{K}_1)$, (4.17), Proposition 1.4.6 and the fact that

$E_\alpha(-C_k t^\alpha) \leq C_2 t^{-\alpha}$ (for some positive constant C_2) away from zero

$$\begin{aligned} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta)d\eta}{(t-\eta)^\alpha} \right)^2 &\leq [K\Gamma(1-\alpha)E_\alpha(-C_k t^\alpha)]^2 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\leq K^2 \Gamma^2(1-\alpha) E_\alpha^2(-C_k t^\alpha) \frac{t^\alpha}{\alpha} \leq C_3 E_\alpha(-C_k t^\alpha), \end{aligned}$$

for $t \geq t_0 > 0$ where $C_3 := K^2 \Gamma^2(1-\alpha) C_2 / \alpha$. Then, we end up with the following

$$\begin{aligned} & - \int_\Omega D^\alpha v D^\alpha \left(v(t) \int_0^t k(s) ds \right) dx \leq - \left(\int_0^t k(s) ds \right) \|D^\alpha v\|^2 \\ & - \int_\Omega v D^\alpha v \int_0^t {}^{RL} D^\alpha k(t-s) ds dx + \epsilon_4 \|D^\alpha v\|^2 - I^{1-\alpha} k(0) \int_\Omega v D^\alpha v dx \\ & + \frac{C_4 E_\alpha(-C_k t^\alpha)}{\epsilon_4} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right)^2 dx, \end{aligned} \quad (4.21)$$

for $\epsilon_4 > 0$, $t \geq t_0 > 0$ where $C_4 = \alpha C_3 / 4 \Gamma^2(1-\alpha)$. Furthermore, we can easily see that

$$\int_\Omega \nabla v \cdot \int_0^t k(t-s) [\nabla v(t) - \nabla v(s)] ds dx \leq \epsilon_5 \|\nabla v\|^2 + \frac{\bar{k}}{4\epsilon_5} (k \square \nabla v)(t) \quad (4.22)$$

where $\epsilon_5 > 0$ and the embedding mentioned above ($H_0^1(\Omega) \subset L^q(\Omega)$ when $2 \leq q \leq 2n/(n-2)$ if $n \geq 3$ and $q \geq 2$ if $n = 1, 2$) insure that

$$\begin{aligned} & \int_\Omega |v|^\gamma v \int_0^t k(t-s) [v(t) - v(s)] ds dx \leq \epsilon_6 \int_\Omega |v|^{2(\gamma+1)} dx + \frac{\bar{k} C_p}{4\epsilon_6} (k \square \nabla v)(t) \\ & \leq \epsilon_6 C_1 \left(\int_\Omega |\nabla v|^2 dx \right)^{\gamma+1} + \frac{\bar{k} C_p}{4\epsilon_6} (k \square \nabla v)(t) \leq \epsilon_6 C_1 E_1^{\gamma+1}(t) + \frac{\bar{k} C_p}{4\epsilon_6} (k \square \nabla v)(t) \end{aligned} \quad (4.23)$$

where $\epsilon_6 > 0$ and C_1 is the embedding constant. Notice that employing (4.18), it appears that

$$\begin{aligned}
& \int_{\Omega} D^{\alpha} v \int_0^t {}^{RL}D^{\alpha} k(t-s) v(s) ds dx - \int_{\Omega} v D^{\alpha} v \int_0^t {}^{RL}D^{\alpha} k(t-s) ds dx \\
&= - \int_{\Omega} D^{\alpha} v \int_0^t {}^{RL}D^{\alpha} k(t-s) (v(t) - v(s)) ds dx \\
&\leq \frac{k_0}{2} \|D^{\alpha} v\|^2 + \frac{C_p}{2k_0} \left(\int_0^t |{}^{RL}D^{\alpha} k|(s) ds \right) (|{}^{RL}D^{\alpha} k| \square \nabla v)(t) \\
&\leq \frac{k_0}{2} \|D^{\alpha} v\|^2 - \frac{C_p}{2k_0} I^{1-\alpha} k(0) ({}^{RL}D^{\alpha} k \square \nabla v)(t) \\
&\leq \frac{k_0}{2} \|D^{\alpha} v\|^2 - \frac{KC_p}{2k_0} ({}^{RL}D^{\alpha} k \square \nabla v)(t). \tag{4.24}
\end{aligned}$$

Finally

$$\begin{aligned}
& \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^{\xi} \frac{[D^{\alpha} v(\eta)]' d\eta}{(t-\eta)^{\alpha}} \right) \left[\int_0^{\xi} (t-s)^{-\alpha} \right. \\
& \times \left. \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right] dx \\
&\leq \epsilon_7 \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^{\xi} \frac{[D^{\alpha} v(\eta)]' d\eta}{(t-\eta)^{\alpha}} \right)^2 dx \tag{4.25} \\
&+ \frac{1}{4\epsilon_7} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^{\xi} (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right]^2 dx
\end{aligned}$$

where $\epsilon_7 > 0$. Gathering all the previous relations (4.21)-(4.25) in (4.19), it is

clear that

$$\begin{aligned}
D^\alpha V_4(t) &\leq \epsilon_5 \|\nabla v\|^2 + \bar{k} \left(\frac{1}{4\epsilon_5} + \frac{C_p}{4\epsilon_6} + 1 \right) (k \square \nabla v)(t) + \left(\epsilon_4 - \frac{k_0}{2} \right) \|D^\alpha v\|^2 \\
&- \frac{KC_p}{2k_0} ({}^{RL}D^\alpha k \square \nabla v)(t) + \epsilon_6 C_1 E_1^{\gamma+1}(t) + \frac{C_4 E_\alpha(-C_k t^\alpha)}{\epsilon_4} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right)^2 dx + \frac{\alpha \epsilon_7}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx \\
&+ \frac{\alpha}{4\epsilon_7 \Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s)-v(\tau)] d\tau \right)' ds \right]^2 dx
\end{aligned}$$

where $t \geq t_0 > 0$ I

Another term has emerged that also requires control. To address this new issue,

we introduce the functional

$$V_5(t) := \int_\Omega \left(\int_0^t k(t-s) [v(t)-v(s)] ds \right)^2 dx, \quad t \geq 0.$$

Lemma 4.3.7 *The functional $V_5(t)$ mentioned above satisfies*

$$\begin{aligned}
D^\alpha V_5(t) &\leq \epsilon_8 \|D^\alpha v\|^2 + C_p \left(\epsilon_9 \bar{k} + \frac{\bar{k}^3}{\epsilon_8} + \frac{\epsilon_{10} \alpha \bar{k}}{\Gamma(1-\alpha)} \right) (k \square \nabla v)(t) \\
&+ \frac{C_p K}{\epsilon_9} (|{}^{RL}D^\alpha k| \square \nabla v)(t) + \frac{C_5 E_\alpha(-C_k t^\alpha)}{\epsilon_{10}} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right)^2 dx \\
&- \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s)-v(\tau)] d\tau \right)' ds \right]^2 dx
\end{aligned}$$

where $\epsilon_8, \epsilon_9, \epsilon_{10} > 0$ and $t \geq 0$.

Proof. First, we α -differentiate $V_5(t)$

$$\begin{aligned} D^\alpha V_5(t) &= 2 \int_{\Omega} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) D^\alpha \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) dx \\ &\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right)^2 dx. \end{aligned}$$

Then, in view of Proposition 1.4.5 and Proposition 1.4.6, it is evident that

$$\begin{aligned} &\int_{\Omega} D^\alpha \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) dx \\ &= \int_{\Omega} D^\alpha \left(v(t) \int_0^t k(t-s) ds \right) dx - \int_{\Omega} D^\alpha \left(\int_0^t k(t-s) v(s) ds \right) dx \\ &= \left(\int_0^t k(s) ds \right) D^\alpha v(t) + v(t) \left[\int_0^t {}^{RL}D^\alpha k(t-s) ds + I^{1-\alpha} k(0) \right] \\ &\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right) \\ &\quad - \int_{\Omega} D^\alpha \left(\int_0^t k(t-s) v(s) ds \right) dx. \end{aligned}$$

Moreover

$$\begin{aligned} D^\alpha V_5(t) &= 2 \left(\int_0^t k(s) ds \right) \int_{\Omega} D^\alpha v(t) \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) dx \\ &\quad + 2 \int_{\Omega} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) v(t) \left[\int_0^t {}^{RL}D^\alpha k(t-s) ds + I^{1-\alpha} k(0) \right] dx \\ &\quad - \frac{2\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \right) \\ &\quad \times \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right) dx - 2 \int_{\Omega} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \\ &\quad D^\alpha \left(\int_0^t k(t-s) v(s) ds \right) dx - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^\xi (t-s)^{-\alpha} \right. \\ &\quad \times \left. \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right]^2 dx. \end{aligned}$$

From (4.9), it is clear that

$$\begin{aligned}
D^\alpha V_5(t) &= 2 \left(\int_0^t k(s) ds \right) \int_\Omega D^\alpha v(t) \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) dx \\
&+ 2 \int_\Omega \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) v(t) \left[\int_0^t {}^{RL}D^\alpha k(t-s) ds + I^{1-\alpha} k(0) \right] dx \\
&- \frac{2\alpha}{\Gamma(1-\alpha)} \int_\Omega \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \right) \\
&\times \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right) dx - 2 \int_\Omega \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \\
&\times \left({}^{RL}D^\alpha \int_0^t k(t-s) v(s) ds + v(t) I^{1-\alpha} k(0) \right) dx - \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right)' ds \right]^2 dx
\end{aligned}$$

Furthermore, utilizing Proposition 1.4.5, Young's inequality and Poincaré inequality, results in

$$\begin{aligned}
D^\alpha V_5(t) &\leq \epsilon_8 \|D^\alpha v\|^2 + \frac{\bar{k}^3 C_p}{\epsilon_8} (k \square \nabla v)(t) \\
&+ 2 \int_\Omega \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \int_0^t {}^{RL}D^\alpha k(t-s) [v(t) - v(s)] ds dx \\
&- \frac{2\alpha}{\Gamma(1-\alpha)} \int_\Omega \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \right) \\
&\times \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right) dx - \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^\xi (t-s)^{-\alpha} \right. \\
&\times \left. \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right]^2 dx
\end{aligned} \tag{4.26}$$

where $\epsilon_8 > 0$. In addition, we can evaluate

$$\begin{aligned}
& 2 \int_{\Omega} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \int_0^t {}^{RL}D^\alpha k(t-s) [v(t) - v(s)] ds dx \\
& \leq \epsilon_9 \bar{k} \int_{\Omega} \int_0^t k(t-s) [v(t) - v(s)]^2 ds dx \\
& + \frac{1}{\epsilon_9} \left(\int_0^t |{}^{RL}D^\alpha k(s)| ds \right) \int_{\Omega} \int_0^t |{}^{RL}D^\alpha k(t-s)| [v(t) - v(s)]^2 ds dx \\
& \leq \epsilon_9 \bar{k} C_p (k \square \nabla v)(t) + \frac{C_p}{\epsilon_9} \left(\int_0^t |{}^{RL}D^\alpha k(s)| ds \right) (|{}^{RL}D^\alpha k| \square \nabla v)(t).
\end{aligned}$$

The inequality to come is a straightforward application of (4.18)

$$\begin{aligned}
& 2 \int_{\Omega} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \int_0^t {}^{RL}D^\alpha k(t-s) [v(t) - v(s)] ds dx \\
& \leq \epsilon_9 \bar{k} C_p (k \square \nabla v)(t) + \frac{C_p K}{\epsilon_9} (|{}^{RL}D^\alpha k| \square \nabla v)(t).
\end{aligned} \tag{4.27}$$

At this stage the relations (4.27) in (4.26), imply that

$$\begin{aligned}
D^\alpha V_5(t) & \leq \epsilon_8 \|D^\alpha v\|^2 + \frac{\bar{k}^3 C_p}{\epsilon_8} (k \square \nabla v)(t) + \epsilon_9 \bar{k} C_p (k \square \nabla v)(t) \\
& + \frac{C_p K}{\epsilon_9} (|{}^{RL}D^\alpha k| \square \nabla v)(t) - \frac{2\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \left(\int_0^t k(t-s) [v(t) - v(s)] ds \right) \\
& \times \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{k(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right) dx - \frac{\alpha}{\Gamma(1-\alpha)} \int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
& \times \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right]^2 dx
\end{aligned}$$

where $\epsilon_8, \epsilon_9 > 0$, $t \geq 0$. Finally, similar argument to the one in (4.21), brings a

$C_5 = \frac{\alpha C_3}{\Gamma(1-\alpha)}$ such that

$$\begin{aligned} D^\alpha V_5(t) &\leq \epsilon_8 \|D^\alpha v\|^2 + C_p \left(\epsilon_9 \bar{k} + \frac{\bar{k}^3}{\epsilon_8} + \frac{\epsilon_{10} \alpha \bar{k}}{\Gamma(1-\alpha)} \right) (k \square \nabla v)(t) \\ &+ \frac{C_p K}{\epsilon_9} (|{}^{RL}D^\alpha k| \square \nabla v)(t) + \frac{C_5 E_\alpha(-C_k t^\alpha)}{\epsilon_{10}} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{v'(s) ds}{(t-s)^\alpha} \right)^2 dx \\ &- \frac{\alpha}{\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right)^2 dx \end{aligned}$$

for $t \geq 0$. I

The following is the main result of this part of the thesis showing the local Mittag-Leffler stability of the system.

Theorem 4.3.8 *Assume that k satisfies $(\mathbf{K}_1)$ and γ satisfies $(\mathbf{K}_2)$. Then, the solution to problem (4.1) (with $a = 0$) is locally Mittag-Leffler stable, that is (for certain values of the initial data) there exist positive constants C and ξ such that*

$$E_1(t) \leq C E_\alpha(-\xi t^\alpha), \quad t \geq 0$$

for small K (or alternatively large C_k).

Proof. We construct the functional

$$\mathcal{L}_1(t) := N E_1(t) + \sum_{i=1}^5 \lambda_i V_i(t), \quad t > 0$$

for some positive constants $N, \lambda_i, i = 1, \dots, 5$ to be determined later. The evaluations obtained in Lemma 4.3.3 to Lemma 4.3.7 with $N_1 = N$, guarantee the

relation

$$\begin{aligned}
D^\alpha \mathcal{L}_1(t) &\leq A_1 \|\nabla v\|^2 + A_2 \|D^\alpha v\|^2 + A_3 V_3(t) - \lambda_2 \|v(t)\|_{\gamma+2}^{\gamma+2} + \epsilon_6 \lambda_4 C_1 E_1^{\gamma+1}(t) \\
&+ A_4 ({}^{RL}D^\alpha k \square \nabla v)(t) + A_5 (k \square \nabla v)(t) + A_6 \frac{\alpha}{4\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi \frac{[D^\alpha v(\eta)]' d\eta}{(t-\eta)^\alpha} \right)^2 dx + A_7 \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left| \int_0^\xi \frac{[\nabla v(\eta)]' d\eta}{(t-\eta)^\alpha} \right|^2 dx \quad (4.28) \\
&+ A_8 \frac{\alpha}{4\Gamma(1-\alpha)} \int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s k(s-\tau) [v(s) - v(\tau)] d\tau \right)' ds \right]^2 dx
\end{aligned}$$

where

$$\begin{aligned}
A_1 &:= \frac{[KE_\alpha(-C_k t^\alpha)]^2}{\epsilon_1} N - (1 - \epsilon_2) \lambda_2 + K \lambda_3 + \epsilon_5 \lambda_4, \\
A_2 &:= \lambda_2 + \left(\epsilon_4 - \frac{k_0}{2} \right) \lambda_4 + \epsilon_8 \lambda_5, \\
A_3 &:= 2\epsilon_1 \bar{k} N + \frac{\bar{k}}{4\epsilon_2} \lambda_2 - C_k \lambda_3, \\
A_4 &:= \frac{N}{2} - \frac{KN}{\epsilon_1} - \frac{KC_p}{2k_0} \lambda_4 - \frac{C_p K}{\epsilon_9} \lambda_5, \\
A_5 &:= \bar{k} \left(\frac{1}{4\epsilon_5} + \frac{C_p}{4\epsilon_6} + 1 \right) \lambda_4 + \bar{k} C_p \left(\epsilon_9 + \frac{\bar{k}^2}{\epsilon_8} + \frac{\epsilon_{10} \alpha}{\Gamma(1-\alpha)} \right) \lambda_5, \\
A_6 &:= -2N + \frac{\lambda_2}{\epsilon_3} + 4\epsilon_7 \lambda_4, \\
A_7 &:= \frac{\alpha (4\epsilon_3 C_p \lambda_2 - N)}{4\Gamma(1-\alpha)} + C_p E_\alpha(-C_k t^\alpha) \left(\frac{C_4 \lambda_4}{\epsilon_4} + \frac{C_5 \lambda_5}{\epsilon_{10}} \right), \\
A_8 &:= \frac{\lambda_4}{\epsilon_7} - 4\lambda_5.
\end{aligned}$$

The approach involves selecting the parameters in such a way that it results in the appearance of the term $-CE_1(t)$, $C > 0$, in addition to $\epsilon_6 \lambda_4 C_1 E_1^{\gamma+1}(t)$, on the right-hand side of (4.28). Notice that as $E_\alpha(-C_k t^\alpha)$ approaches zero, we can

disregard this term by selecting a sufficiently large t_0 . Additionally, we can also disregard ϵ_4 and ϵ_{10} . Let $\epsilon_1 = 1$, $\epsilon_2 = 1/2$ and recall that $\bar{k} \leq K/C_k$. Therefore, for small values of K (or large values of C_k) and consequently small $\epsilon_5, \epsilon_6, \epsilon_8, \epsilon_9$, it is enough to request that

$$\left\{ \begin{array}{l} \frac{\lambda_2}{\epsilon_3} + 4\epsilon_7\lambda_4 \leq 2N, \\ 4\epsilon_3C_p\lambda_2 \leq N, \\ \lambda_2 < \lambda_4 \frac{k_0}{2}, \\ \frac{\lambda_4}{\epsilon_7} \leq 4\lambda_5. \end{array} \right.$$

Choose $\epsilon_3 = \epsilon_7 = 1/4$, $\lambda_5 = \lambda_4$, $\lambda_2 = \frac{k_0\lambda_4}{4}$ and $N \geq \lambda_4 \max \left\{ C_p \frac{k_0}{4}, k_0 + 1 \right\}$. In the end, we reverse our approach and choose the remaining (skipped and ignored) parameters in order to guarantee the fractional inequality

$$D^\alpha \mathcal{L}_1(t) \leq -C_6 E_1(t) + M_4 C_1 E_1^{\gamma+1}(t), \quad t > t_0$$

for some $C_6 > 0$. Owing to the equivalence of $\mathcal{L}_1(t)$ and $E_1(t)$

$$D^\alpha \mathcal{L}_1(t) \leq -C_7 \mathcal{L}_1(t) + C_8 \mathcal{L}_1^{\gamma+1}(t), \quad C_7, C_8, \quad t > t_0.$$

Let $\bar{t}, \omega > 0$ be real numbers (we can assume that $t_0 < \bar{t}$) such that $\mathcal{L}_1(t_0) < \omega$ and $C_8 \mathcal{L}_1^\gamma(t) \leq \omega < C_7$, $0 \leq t \leq \bar{t}$ so that

$$D^\alpha \mathcal{L}_1(t) \leq -(C_7 - \omega) \mathcal{L}_1(t), \quad t_0 \leq t \leq \bar{t}.$$

Clearly

$$\mathcal{L}_1(t) \leq \mathcal{L}_1(t_0)E_\alpha(-(C_7 - \omega)(t - t_0)^\alpha), \quad t_0 \leq t \leq \bar{t}.$$

Assume that there exists $t_0 \leq t_1 \leq \bar{t}$ such that $\mathcal{L}_1(t_1) = \omega$, then as

$$E_\alpha(-(C_7 - \omega)(t - t_0)^\alpha) \leq 1,$$

we must have

$$\omega = \mathcal{L}_1(t_1) \leq \mathcal{L}_1(t_0)E_\alpha(-(\chi - \omega)(t_1 - t_0)^\alpha) < \omega.$$

This is a contradiction. Starting from $\bar{t}$, we can continue the process as long as

we desire. The proof is complete. █

CHAPTER 5

FRACTIONALLY AND VISCOELASTICALLY DAMPED FRACTIONAL TIMOSHENKO BEAM

This chapter is concerned with a fractional Timoshenko system of order between one and two. We address the question of well-posedness in an appropriate space when the rotational component is viscoelastic or subject to a viscoelastic controller. To this end we use the notion of alpha-resolvent. Moreover, we prove that the memory term alone may stabilize the system in a Mittag-Leffler fashion. The system is Lyapunov stable or uniformly stable in the case of different speeds of propagation.

We will consider the system (see[5])

$$\left\{ \begin{array}{l} \rho_1 {}^C D^\alpha ({}^C D^\alpha \varphi) - k(\varphi_x + \psi)_x = 0, \quad 1/2 < \alpha < 1, \\ \rho_2 {}^C D^\alpha ({}^C D^\alpha \psi) - b\psi_{xx} + \int_0^t \omega(t-s)\psi_{xx}(s)ds + k(\varphi_x + \psi) = 0, \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = 0, \quad t \geq 0, \\ \varphi(x, 0) = \varphi_0, \quad \varphi_t(x, 0) = \varphi_1, \quad \psi(x, 0) = \psi_0, \quad \psi_t(x, 0) = \psi_1. \end{array} \right. \quad (5.1)$$

5.1 Well-Posedness of the Problem

In this section, we complete the analysis of existence and uniqueness issues. We shall assume that our initial data satisfy $U_0 \in \mathcal{D}(\mathcal{A}_1)$ (this is defined below and involves the knowledge of fractional derivatives of the states at zero). Our problem (5.1) may be rewritten as a system of fractional order α , $1/2 < \alpha < 1$. As a matter of fact, letting $U = (\varphi, \tilde{\varphi}, \psi, \tilde{\psi})^T$, $\tilde{\varphi} = {}^C D^\alpha \varphi$, $\tilde{\psi} = {}^C D^\alpha \psi$ it is evident that

$$\left\{ \begin{array}{l} {}^C D^\alpha U = \mathcal{A}_1 U - \int_0^t \mathcal{A}_2(t-s)U(s)ds \\ U(0) = U_0 = (\varphi_0, \tilde{\varphi}_0, \psi_0, \tilde{\psi}_0)^T \end{array} \right. \quad (5.2)$$

where the operator $\mathcal{A}_1$ and $\mathcal{A}_2$ defined as

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I_d & 0 & 0 \\ k\partial_x^2/\rho_1 & 0 & k\partial_x/\rho_1 & 0 \\ 0 & 0 & 0 & I_d \\ -k\partial_x/\rho_2 & 0 & b\partial_x^2/\rho_2 - kI_d/\rho_2 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_2(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \omega(t)\partial_x^2/\rho_2 & 0 \end{pmatrix}.$$

We set

$$\mathcal{H} = H_0^1(0, 1) \times L^2(0, 1) \times H_0^1(0, 1) \times L^2(0, 1)$$

and

$$\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(\mathcal{A}_2) = \left\{ U = \left(\varphi, \tilde{\varphi}, \psi, \tilde{\psi} \right)^T \in \mathcal{H} : \varphi, \psi \in H^2(0, 1) \cap H_0^1(0, 1), \tilde{\varphi}, \tilde{\psi} \in H_0^1(0, 1) \right\}.$$

From Theorem 1.4.11, we can see that if we assume that $U_0 \in \mathcal{D}(\mathcal{A}_1)$, it follows that we have a classical solution

$$U \in C(\mathbb{R}^+, \mathcal{D}(\mathcal{A}_1)) \cap C^1(\mathbb{R}^+, \mathcal{H})$$

satisfying $t^{-\alpha} * \varphi_t \in C^1((0, \infty), L^2(0, 1))$ and $t^{-\alpha} * \psi_t \in C^1((0, \infty), L^2(0, 1))$.

Bearing in mind that, in our case, the Laplacian is the infinitesimal generator of an analytic semigroup, the conditions **(S1)'**, **(S2)** and **(S3)** are fulfilled.

5.2 Mittag-Leffler and Uniform Stability

The kernel ω is assumed to satisfy the following condition:

(W) The kernel $\omega : (0, \infty) \rightarrow (0, \infty)$ is a nonnegative continuous function satisfying $I^{1-\alpha}\omega(t) \in C^1([0, \infty))$,

$${}^{RL}D^\alpha \omega(t) \leq -C_\omega \omega(t), \quad t > 0$$

and

$$\bar{\omega} := \int_0^\infty \omega(s)ds < b$$

for some positive constant C_ω .

Proposition 5.2.1 *Assuming (W), we have*

$$\omega(t) \leq \tilde{\omega} t^{\alpha-1} E_{\alpha,\alpha}(-C_\omega t^\alpha), t \geq 0$$

and

$$\int_0^t \omega(s)ds \leq \tilde{\omega} \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-C_\omega s^\alpha)ds = \tilde{\omega} t^\alpha E_{\alpha,\alpha+1}(-C_\omega t^\alpha) \leq \tilde{\omega}/C_\omega, \quad t > 0$$

for some $\tilde{\omega} > 0$.

Proof. The proofs follow by a direct application of Propositions 1.4.4-1.4.6. ■

Before addressing the stability issue, we need to prove two identities.

5.2.1 A Useful Functional

We recall the proposition 4.3.1

Proposition 5.2.2 *If $I^{1-\alpha}w(t) \in C^1([0, \infty))$, $0 < \alpha < 1$ and $v \in H_0^1(0, 1)$ is a*

differentiable function, then the identity stated below holds

$$\begin{aligned}
& \int_0^1 D^\alpha v_x \int_0^t w(t-s)v_x(s)dsdx = \frac{1}{2} \left\{ ({}^{RL}D^\alpha w \square v_x) + \left(\int_0^t w(s)ds \right) D^\alpha \|v_x\|^2 \right\} \\
& - \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{w(\eta)d\eta}{(t-\eta)^\alpha} \int_0^\xi \frac{[\|v_x(s)\|^2]' ds}{(t-s)^\alpha} - \frac{1}{2} D^\alpha (w \square v_x)(t) \\
& + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{[(v_x)(\eta)]' d\eta}{(t-\eta)^\alpha} \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s w(s-\tau)v_x(\tau)d\tau \right)' ds \right] dx, \quad t \geq 0.
\end{aligned} \tag{5.3}$$

Proposition 5.2.3 For functions v and ω that are absolutely continuous, with $I^{1-\alpha}\omega$, where $0 < \alpha < 1$ is also absolutely continuous, the following holds for $t \geq 0$

$$\begin{aligned}
D^\alpha \int_0^t \omega(t-s)[v(t)-v(s)]ds &= \int_0^t {}^{RL}D^\alpha \omega(t-s)(v(t)-v(s))ds \\
&+ \left(\int_0^t \omega(s)ds \right) D^\alpha v(t) - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right).
\end{aligned}$$

Proof. It is evident, based on Propositions 1.4.7 and 1.4.8, that we can have

$$\begin{aligned}
D^\alpha \int_0^t \omega(t-s)[v(t)-v(s)]ds &= D^\alpha \left(v(t) \int_0^t \omega(t-s)ds \right) - D^\alpha \int_0^t \omega(t-s)v(s)ds \\
&= \left(\int_0^t \omega(s)ds \right) D^\alpha v(t) + v(t) \left[\int_0^t {}^{RL}D^\alpha \omega(t-s)ds + I^{1-\alpha}\omega(0) \right] \\
&- \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{v'(s)ds}{(t-s)^\alpha} \right) \\
&- \int_0^t {}^{RL}D^\alpha \omega(t-s)v(s)ds - v(t)I^{1-\alpha}\omega(0), \quad t \geq 0.
\end{aligned}$$

I

In the next proposition, we define the energy and we find its derivative.

Proposition 5.2.4 *Differentiate the energy functional*

$$E_1(t) = \frac{1}{2} \left[\rho_1 \|D^\alpha \varphi\|^2 + \rho_2 \|D^\alpha \psi\|^2 + \left(b - \int_0^t \omega(s) ds \right) \|\psi\|^2 + k \|\varphi_x + \psi\|^2 \right] \\ + \frac{1}{2} (\omega \square \psi_x)(t), \quad t \geq 0$$

we attain

$$D^\alpha E_1(t) \leq -\frac{I^{1-\alpha} \omega(t)}{2} \|\psi_x\|^2 - \frac{C_\omega}{2} (\omega \square \psi_x)(t) + \frac{\alpha}{b\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ \times \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' ds \right)^2 dx \\ - \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) - \rho_2 \mathcal{T}_{D^\alpha \psi}(t) - k \mathcal{T}_{\varphi_x + \psi}(t) - \frac{b}{2} \mathcal{T}_{\psi_x}(t), \quad t \geq 0. \quad (5.4)$$

Proof. Having in mind (5.3), we have along the solutions of (5.1)

$$D^\alpha E_1(t) = \rho_1 \int_0^1 D^\alpha \varphi D^\alpha (D^\alpha \varphi) dx - \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \int_0^1 D^\alpha \psi D^\alpha (D^\alpha \psi) dx - \rho_2 \mathcal{T}_{D^\alpha \psi}(t) \\ - \frac{1}{2} \left(D^\alpha \int_0^t \omega(s) ds \right) \|\psi_x\|^2 + \frac{1}{2} \left(b - \int_0^t \omega(s) ds \right) D^\alpha \|\psi_x\|^2 \\ + \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{\omega(\eta) d\eta}{(t-\eta)^\alpha} \int_0^\xi \frac{[(\psi_x)^2(s)]' ds}{(t-s)^\alpha} dx \\ + k \int_0^1 (\varphi_x + \psi) D^\alpha (\varphi_x + \psi) dx - k \mathcal{T}_{\varphi_x + \psi}(t) + \frac{1}{2} ({}^{RL} D^\alpha \omega \square \psi_x)(t) \\ + \frac{1}{2} \left(\int_0^t \omega(t-s) ds \right) D^\alpha \|\psi_x\|^2 - \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{\omega(\eta) d\eta}{(t-\eta)^\alpha} \\ \int_0^\xi \frac{[(\psi_x)^2(s)]' ds}{(t-s)^\alpha} dx - \int_0^1 D^\alpha \psi_x \int_0^t \omega(t-s) \psi_x(s) ds dx + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ \times \left(\int_0^\xi \frac{[\psi_x(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' ds \right) dx.$$

It is easy to see that from (5.1), we can derive

$$\begin{aligned} & \rho_1 \int_0^1 D^\alpha \varphi D^\alpha (D^\alpha \varphi) dx + \rho_2 \int_0^1 D^\alpha \psi D^\alpha (D^\alpha \psi) dx = -k \int_0^1 (\varphi_x + \psi) D^\alpha (\varphi_x + \psi) dx \\ & - b \int_0^1 \psi_x D^\alpha \psi_x dx + \int_0^1 D^\alpha \psi_x \int_0^t \omega(t-s) \psi_x(s) ds dx. \end{aligned}$$

Moreover,

$$\begin{aligned} D^\alpha E_1(t) = & -\frac{1}{2} \left(D^\alpha \int_0^t \omega(s) ds \right) \|\psi_x\|^2 + \frac{1}{2} ({}^{RL}D^\alpha \omega \square \psi_x)(t) - \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) - \rho_2 \mathcal{T}_{D^\alpha \psi}(t) \\ & - b \mathcal{T}_{\psi_x}(t) - k \mathcal{T}_{\varphi_x + \psi}(t) + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\psi_x(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left[\int_0^\xi (t-s)^{-\alpha} \right. \\ & \times \left. \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' ds \right] dx, \quad t > 0. \end{aligned} \quad (5.5)$$

Since the final expression in equation (5.5) can be estimated by

$$\begin{aligned} & \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\psi_x(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' ds dx \\ & \leq \frac{b\Gamma(1-\alpha)}{2\alpha} \mathcal{T}_{\psi_x}(t) \\ & + \frac{1}{b} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' ds \right]^2 dx \end{aligned}$$

it is evident that, for $t > 0$,

$$\begin{aligned} D^\alpha E_1(t) \leq & -\frac{I^{1-\alpha} \omega(t)}{2} \|\psi_x\|^2 - \frac{C_\omega}{2} (\omega \square \psi_x)(t) - \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) - \rho_2 \mathcal{T}_{D^\alpha \psi}(t) - k \mathcal{T}_{\varphi_x + \psi}(t) \\ & - \frac{b}{2} \mathcal{T}_{\psi_x}(t) + \frac{\alpha}{b\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ & \times \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' ds \right]^2 dx \end{aligned} \quad (5.6)$$

where the assumption **(W)** has been employed. We do not know the sign of the right-hand side of (5.6), therefore this is a sensitive situation. ■

5.2.2 Equal Speeds of Propagation

This section will introduce several functionals and their fractional derivatives, with which we will modify our energy functional.

We introduce the first functional

$$V_1(t) := \left\| \int_0^t \omega(t-s) \psi_x(s) ds \right\|^2 + \int_0^t \omega(t-s) \|\psi_x(s)\|^2 ds, \quad t \geq 0.$$

Lemma 5.2.5 *The functional $V_1(t)$ satisfies, along solutions of our problem, the relation*

$$\begin{aligned} D^\alpha V_1(t) &\leq -\frac{C_\omega}{2} \int_0^t \omega(t-s) \|\psi_x(s)\|^2 ds + \frac{4\tilde{\omega}^2}{C_\omega} (|^{RL}D^\alpha \omega| \square \psi_x)(t) \\ &+ \left(\frac{4\tilde{\omega} [\tilde{\omega} E_\alpha(-C_\omega t^\alpha)]^2}{C_\omega} + \tilde{\omega} \right) \|\psi_x\|^2 - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\times \left[\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-\tau) \psi_x(\tau) d\tau \right)' d\eta \right]^2 dx, \quad t \geq 0. \end{aligned}$$

Proof. By the Proposition 1.4.8, we derive

$$\begin{aligned} D^\alpha V_1(t) &= 2 \int_0^1 \left(\int_0^t \omega(t-s) \psi_x(s) ds \right) D^\alpha \left[\int_0^t \omega(t-s) \psi_x(s) ds \right] dx \\ &- \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-\tau) \psi_x(\tau) d\tau \right)' d\eta \right)^2 dx \\ &+ D^\alpha \int_0^t \omega(t-s) \|\psi_x(s)\|^2 ds, \quad t \geq 0. \end{aligned}$$

The relation between the Riemann-Liouville fractional derivative and the Caputo fractional derivative yields

$$D^\alpha \left(\int_0^t \omega(t-s) \psi_x(s) ds \right) = {}^{RL} D^\alpha \left(\int_0^t \omega(t-s) \psi_x(s) ds \right) dx - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \left(\int_0^t \omega(t-s) \psi_x(s) ds \right) \Big|_{t=0}.$$

Furthermore, the summability of ω and Proposition 1.4.7 result in

$$\begin{aligned} D^\alpha V_1(t) &= 2 \int_0^1 \left(\int_0^t \omega(t-s) \psi_x(s) ds \right) \left[\int_0^t {}^{RL} D^\alpha \omega(t-s) \psi_x(s) ds + \psi_x(t) \lim_{t \rightarrow 0^+} I^{1-\alpha} \omega(t) \right] dx \\ &\quad - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' d\eta \right)^2 dx \\ &\quad + \int_0^t {}^{RL} D^\alpha \omega(t-s) \|\psi_x(s)\|^2 ds + \|\psi_x(t)\|^2 \lim_{t \rightarrow 0} I^{1-\alpha} \omega(t). \end{aligned}$$

According to the definition of the Riemann-Liouville fractional derivative and Young's inequality, it can be deduced that

$$\begin{aligned} D^\alpha V_1(t) &\leq \frac{2\bar{\omega}}{C_\omega} \int_0^1 \left[\int_0^t {}^{RL} D^\alpha \omega(t-s) [\psi_x(s) - \psi_x(t)] ds + \psi_x(t) I^{1-\alpha} \omega(t) \right]^2 dx \\ &\quad + \frac{C_\omega}{2} \int_0^t \omega(t-s) \|\psi_x(s)\|^2 ds - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ &\quad \times \left[\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-\tau) \psi_x(\tau) d\tau \right)' d\eta \right]^2 dx - C_\omega \int_0^t \omega(t-s) \|\psi_x(s)\|^2 ds \\ &\quad + \|\psi_x(t)\|^2 \lim_{t \rightarrow 0} I^{1-\alpha} \omega(t), \quad t \geq 0. \end{aligned} \tag{5.7}$$

Benefiting from Proposition 1.4.6, we get

$$I^{1-\alpha}\omega(t) \leq \frac{\tilde{\omega}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-C_\omega(t-s)^\alpha) s^{-\alpha} ds = \tilde{\omega} E_\alpha(-C_\omega t^\alpha) \leq \tilde{\omega}, \quad (5.8)$$

and using

$$\psi_x^2(s) \leq 2[\psi_x(s) - \psi_x(t)]^2 + 2\psi_x^2(t),$$

the equation (5.2.14) results in

$$\begin{aligned} D^\alpha V_1(t) &\leq \frac{-C_\omega}{2} \int_0^t \omega(t-s) \|\psi_x(s)\|^2 ds + \left(\frac{2\bar{\omega} [\tilde{\omega} E_\alpha(-C_\omega t^\alpha)]^2}{C_\omega} + \tilde{\omega} \right) \|\psi_x\|^2 \\ &+ \frac{4\bar{\omega}}{C_\omega} \left(\int_0^t |{}^{RL}D^\alpha \omega(t-s)| ds \right) \int_0^1 \int_0^t |{}^{RL}D^\alpha \omega(t-s)| [\psi_x(s) - \psi_x(t)]^2 ds dx \\ &- \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-\tau) \psi_x(\tau) d\tau \right)' d\eta \right)^2 dx \end{aligned}$$

where $t \geq 0$. The stated estimate is derived by observing that

$$0 \leq \int_0^t |{}^{RL}D^\alpha \omega(s)| ds = I^{1-\alpha}\omega(0) - I^{1-\alpha}\omega(t) \leq \tilde{\omega}. \quad (5.9)$$

Our objective is achieved. I

The following lemma presents our second functional.

Lemma 5.2.6 *The functional*

$$V_2(t) := - \int_0^1 (\rho_1 \varphi D^\alpha \varphi + \rho_2 \psi D^\alpha \psi) dx, \quad t > 0,$$

fulfills

$$\begin{aligned} D^\alpha V_2(t) &\leq -\rho_1 \|D^\alpha \varphi\|^2 - \rho_2 \|D^\alpha \psi\|^2 + k \|\varphi_x + \psi\|^2 + \left(\epsilon_2 + b - \int_0^t \omega(s) ds \right) \|\psi_x\|^2 \\ &+ \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t) + 2\rho_1 \mathcal{T}_{\varphi_x + \psi}(t) + (2\rho_1 + \rho_2) \mathcal{T}_{\psi_x}(t) + \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t) \end{aligned}$$

where $\epsilon_2 > 0$, and $t \geq 0$.

Proof. Differentiating with respect to α the functional $V_2(t)$, leads to

$$\begin{aligned} D^\alpha V_2(t) &= -\rho_1 \|D^\alpha \varphi\|^2 + \frac{\alpha \rho_1}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\varphi'(\eta) d\eta}{(t-\eta)^\alpha} \right) \\ &\times \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx + k \|\varphi_x + \psi\|^2 - \rho_2 \|D^\alpha \psi\|^2 + \left(b - \int_0^t \omega(s) ds \right) \|\psi_x\|^2 \\ &+ \int_0^1 \psi_x \int_0^t \omega(t-s) [\psi_x(t) - \psi_x(s)] ds dx \\ &+ \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\psi'(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \psi(s)]' ds}{(t-s)^\alpha} \right) dx. \quad (5.10) \end{aligned}$$

After that we have the estimations

$$\int_0^1 \psi_x \int_0^t \omega(t-s) [\psi_x(t) - \psi_x(s)] ds dx \leq \epsilon_2 \|\psi_x\|^2 + \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t), \quad \epsilon_2 > 0,$$

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\varphi'(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx \leq \mathcal{T}_\varphi(t) + \mathcal{T}_{D^\alpha \varphi}(t)$$

and

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\psi'(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \psi(s)]' ds}{(t-s)^\alpha} \right) dx \leq \mathcal{T}_\psi(t) + \mathcal{T}_{D^\alpha \psi}(t).$$

Taking into account the previous estimations, we can see that (5.10) becomes

$$\begin{aligned} D^\alpha V_2(t) &\leq -\rho_1 \|D^\alpha \varphi\|^2 - \rho_2 \|D^\alpha \psi\|^2 + k \|\varphi_x + \psi\|^2 + \left(b - \int_0^t \omega(s) ds\right) \|\psi_x\|^2 \\ &+ \epsilon_2 \|\psi_x\|^2 + \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t) + \rho_1 [\mathcal{T}_\varphi(t) + \mathcal{T}_{D^\alpha \varphi}(t)] + \rho_2 [\mathcal{T}_\psi(t) + \mathcal{T}_{D^\alpha \psi}(t)], \quad t \geq 0. \end{aligned}$$

Finally, due to the relation

$$\mathcal{T}_\varphi(t) \leq \mathcal{T}_{\varphi_x}(t) \leq 2\mathcal{T}_{\varphi_x + \psi}(t) + 2\mathcal{T}_{\psi_x}(t) \quad (5.11)$$

it appears that

$$\begin{aligned} D^\alpha V_2(t) &\leq -\rho_1 \|D^\alpha \varphi\|^2 - \rho_2 \|D^\alpha \psi\|^2 + k \|\varphi_x + \psi\|^2 + \left(\epsilon_2 + b - \int_0^t \omega(s) ds\right) \|\psi_x\|^2 \\ &+ \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t) + 2\rho_1 \mathcal{T}_{\varphi_x + \psi}(t) + (2\rho_1 + \rho_2) \mathcal{T}_{\psi_x}(t) + \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t), \quad t \geq 0. \end{aligned}$$

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Our third functional is defined below.

Lemma 5.2.7 *If*

$$V_3(t) := -\rho_2 \int_0^1 D^\alpha \psi \int_0^t \omega(t-s) (\psi(t) - \psi(s)) ds dx, \quad t \geq 0,$$

then, for $t > t_0 > 0$,

$$\begin{aligned}
D^\alpha V_3(t) &\leq \rho_2 \left(\epsilon_2 + \frac{2\alpha\epsilon_1}{\Gamma(1-\alpha)} - \omega_0 \right) \|D^\alpha \psi\|^2 + b\epsilon_2 \|\psi_x\|^2 + k\epsilon_2 \|\varphi_x + \psi\|^2 \\
&+ \bar{\omega} \left(1 + \frac{k+b}{4\epsilon_2} \right) (\omega \square \psi_x)(t) - \frac{\tilde{\omega}\rho_2}{4\epsilon_2} ({}^{RL}D^\alpha \omega \square \psi_x)(t) + \frac{\rho_2 C_2 E_\alpha(-C_\omega t^\alpha)}{\epsilon_1} \mathcal{T}_\psi(t) \\
&+ \rho_2 \epsilon_3 \mathcal{T}_{D^\alpha \psi}(t) + \frac{\alpha\rho_2}{2\epsilon_3 \Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right)^2 dx,
\end{aligned}$$

where $\omega_0 = \int_0^{t_0} \omega(s) ds$ and $\epsilon_i > 0$, $i = 1, 2, 3$.

Proof. Applying the operator D^α to $V_3(t)$ and (5.1), we get

$$\begin{aligned}
D^\alpha V_3(t) &= -\rho_2 \int_0^1 D^\alpha \psi D^\alpha \int_0^t \omega(t-s) (\psi(t) - \psi(s)) ds dx \\
&- b \int_0^1 \psi_{xx} \int_0^t \omega(t-s) (\psi(t) - \psi(s)) ds dx + \int_0^1 \left(\int_0^t \omega(t-s) \psi_{xx}(s) ds \right) \\
&\times \int_0^t \omega(t-s) (\psi(t) - \psi(s)) ds dx + k \int_0^1 (\varphi_x + \psi) \int_0^t \omega(t-s) (\psi(t) - \psi(s)) ds dx \\
&+ \frac{\alpha\rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left[\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right] \\
&\times \left(\int_0^\xi \frac{[D^\alpha \psi(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

Multiple use of integration by parts and Proposition 5.2.3, result in

$$\begin{aligned}
D^\alpha V_3(t) = & -\rho_2 \int_0^1 D^\alpha \psi \int_0^t {}^{RL}D^\alpha \omega(t-s) (\psi(t) - \psi(s)) ds dx - \rho_2 \left(\int_0^t \omega(s) ds \right) \\
& \times \|D^\alpha \psi\|^2 + \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 D^\alpha \psi \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{\psi'(s) ds}{(t-s)^\alpha} \right) dx \\
& + \int_0^1 \left(\int_0^t \omega(t-s) (\psi_x(t) - \psi_x(s)) ds \right)^2 dx + \left(b - \int_0^t \omega(s) ds \right) \\
& \times \int_0^1 \psi_x \int_0^t \omega(t-s) (\psi_x(t) - \psi_x(s)) ds dx + k \int_0^1 (\varphi_x + \psi) \\
& \times \int_0^t \omega(t-s) (\psi(t) - \psi(s)) ds dx + \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
& \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right) \left(\int_0^\xi \frac{[D^\alpha \psi(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned} \tag{5.12}$$

Studying each term separately and using assumption **(W)**, the relation (5.9) and

Applying Young's inequality, we reach

$$\begin{aligned}
& \int_0^1 D^\alpha \psi \int_0^t {}^{RL}D^\alpha \omega(t-s) [\psi(t) - \psi(s)] ds dx \leq \epsilon_2 \|D^\alpha \psi\|^2 + \frac{\tilde{\omega}}{4\epsilon_2} (|{}^{RL}D^\alpha \omega| \square \psi_x)(t) \\
& \leq \epsilon_2 \|D^\alpha \psi\|^2 - \frac{\tilde{\omega}}{4\epsilon_2} ({}^{RL}D^\alpha \omega \square \psi_x)(t), \quad \epsilon_2 > 0,
\end{aligned} \tag{5.13}$$

$$\int_0^1 \psi_x \int_0^t \omega(t-s) (\psi_x(t) - \psi_x(s)) ds dx \leq \epsilon_2 \|\psi_x\|^2 + \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t), \quad \epsilon_2 > 0, \tag{5.14}$$

$$\int_0^1 \left[\int_0^t \omega(t-s) (\psi_x(t) - \psi_x(s)) ds \right]^2 dx \leq \bar{\omega} (\omega \square \psi_x)(t), \tag{5.15}$$

$$\begin{aligned}
& \int_0^1 (\varphi_x + \psi) \int_0^t \omega(t-s) (\psi(t) - \psi(s)) ds dx \\
& \leq \epsilon_2 \|\varphi_x + \psi\|^2 + \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t), \quad \epsilon_2 > 0,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
& \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 D^\alpha \psi \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{\psi'(s)ds}{(t-s)^\alpha} \right) dx \\
& \leq \frac{\alpha\epsilon_1}{\Gamma(1-\alpha)} \|D^\alpha \psi\|^2 \\
& + \frac{\alpha}{4\epsilon_1\Gamma(1-\alpha)} \int_0^1 \left[\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{\psi'(s)ds}{(t-s)^\alpha} \right) \right]^2 dx, \quad \epsilon_1 > 0
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
& \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \left[\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{\psi'(s)ds}{(t-s)^\alpha} \right) \right]^2 dx \\
& \leq \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 \left[\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right)^2 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\psi'(s)ds}{(t-s)^\alpha} \right)^2 \right] dx \\
& \leq \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right)^2 \mathcal{T}_\psi(t).
\end{aligned}$$

Subsequently, by noting that $E_\alpha(-C_\omega t^\alpha) \leq C_1 t^{-\alpha}$ (where C_1 is a positive constant) for values far from zero, we obtain

$$\begin{aligned}
\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right)^2 & \leq [\Gamma(1-\alpha)E_\alpha(-C_\omega t^\alpha)]^2 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
& \leq C_2 E_\alpha(-C_\omega t^\alpha), \quad t > t_0 > 0
\end{aligned}$$

where $C_2 := C_1 \Gamma^2(1-\alpha)/\alpha$. Moreover (5.17), becomes

$$\begin{aligned}
& \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^1 D^\alpha \psi \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta)d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{\psi'(s)ds}{(t-s)^\alpha} \right) dx \\
& \leq \frac{\alpha\epsilon_1}{\Gamma(1-\alpha)} \|D^\alpha \psi\|^2 + \frac{C_2 E_\alpha(-C_\omega t^\alpha)}{2\epsilon_1} \mathcal{T}_\psi(t), \quad t > t_0 > 0.
\end{aligned} \tag{5.18}$$

For the last term, we see that

$$\begin{aligned}
& \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right) \\
& \times \left(\int_0^\xi \frac{[D^\alpha \psi(s)]' ds}{(t-s)^\alpha} \right) dx \leq \epsilon_3 \mathcal{T}_{D^\alpha \psi}(t) + \frac{\alpha}{2\epsilon_3 \Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
& \times \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right)^2 dx, \quad \epsilon_3 > 0. \quad (5.19)
\end{aligned}$$

The equations (5.13)-(5.19), when substituted into (5.12), result in

$$\begin{aligned}
D^\alpha V_3(t) & \leq \rho_2 \left(\epsilon_2 + \frac{2\alpha\epsilon_1}{\Gamma(1-\alpha)} - \int_0^{t_0} \omega(s) ds \right) \|D^\alpha \psi\|^2 + b\epsilon_2 \|\psi_x\|^2 + k\epsilon_2 \|\varphi_x + \psi\|^2 \\
& + \bar{\omega} \left(1 + \frac{k+b}{4\epsilon_2} \right) (\omega \square \psi_x)(t) - \frac{\tilde{\omega}\rho_2}{4\epsilon_2} ({}^{RL}D^\alpha \omega \square \psi_x)(t) + \frac{\rho_2 C_2 E_\alpha(-C_\omega t^\alpha)}{\epsilon_1} \mathcal{T}_\psi(t) \\
& + \rho_2 \epsilon_3 \mathcal{T}_{D^\alpha \psi}(t) + \frac{\alpha\rho_2}{2\epsilon_3 \Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
& \times \left[\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right]^2 dx, \quad t > t_0 > 0.
\end{aligned}$$

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Our fourth function is responsible for controlling the last term in the computation of $D^\alpha V_3(t)$. Define

$$V_4(t) := \int_0^1 \left(\int_0^t \omega(t-s) [\psi(t) - \psi(s)] ds \right)^2 dx, \quad t \geq 0.$$

Lemma 5.2.8 *The fractional derivative of the functional $V_4(t)$ mentioned above*

satisfies

$$\begin{aligned}
D^\alpha V_4(t) &\leq \bar{\omega} \epsilon_2 \|D^\alpha \psi\|^2 + \bar{\omega} \left(1 + \frac{\bar{\omega}}{\epsilon_2} + \frac{2\alpha\epsilon_1}{\Gamma(1-\alpha)}\right) (\omega \square \psi_x)(t) + \tilde{\omega} (|{}^{RL}D^\alpha \omega| \square \psi_x)(t) \\
&+ \frac{C_2 E_\alpha(-C_\omega t^\alpha)}{\epsilon_1} \mathcal{T}_{\psi_x}(t) - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right)^2 dx, \quad \epsilon_1, \epsilon_2 > 0, \quad t > t_0 > 0.
\end{aligned}$$

Proof. A fractional differentiation of order α leads to

$$\begin{aligned}
D^\alpha V_4(t) &= 2 \int_0^1 \left(\int_0^t \omega(t-s) [\psi(t) - \psi(s)] ds \right) \\
&\times D^\alpha \left(\int_0^t \omega(t-s) [\psi(t) - \psi(s)] ds \right) dx - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right)^2 dx.
\end{aligned}$$

Considering Proposition 5.2.3, it is clear that

$$\begin{aligned}
D^\alpha V_4(t) &= 2 \int_0^1 \left(\int_0^t \omega(t-s) [\psi(t) - \psi(s)] ds \right) \int_0^t {}^{RL}D^\alpha \omega(t-s) \\
&\times (\psi(t) - \psi(s)) ds dx + 2 \int_0^1 \left(\int_0^t \omega(t-s) [\psi(t) - \psi(s)] ds \right) dx \left(\int_0^t \omega(s) ds \right) \\
&\times \int_0^1 D^\alpha \psi(t) dx - \frac{2\alpha}{\Gamma(1-\alpha)} \int_0^1 \left(\int_0^t \omega(t-s) [\psi(t) - \psi(s)] ds \right) \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi \frac{\omega(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{\psi'(s) ds}{(t-s)^\alpha} \right) dx - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right)^2 dx. \tag{5.20}
\end{aligned}$$

This identity (5.20) can be evaluated using Young's inequality and a similar ar-

gument as for equation (5.18) mentioned earlier, in the following manner

$$\begin{aligned}
D^\alpha V_4(t) &\leq \bar{\omega} \epsilon_2 \|D^\alpha \psi\|^2 + \bar{\omega} \left(1 + \frac{\bar{\omega}}{\epsilon_2} + \frac{2\alpha\epsilon_1}{\Gamma(1-\alpha)}\right) (\omega \square \psi_x)(t) + \tilde{\omega} (|^{RL} D^\alpha \omega| \square \psi_x)(t) \\
&+ \frac{C_2 E_\alpha(-C_\omega t^\alpha)}{\epsilon_1} \mathcal{T}_{\psi_x}(t) - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\times \left(\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right)^2 dx, \quad \epsilon_1, \epsilon_2 > 0, \quad t > t_0 > 0.
\end{aligned}$$

■

The fractional differentiation of our fifth functional results in the term $-\|\varphi_x + \psi\|^2$.

Lemma 5.2.9 *For the functional*

$$\begin{aligned}
V_5(t) &= \rho_2 \int_0^1 D^\alpha \psi (\varphi_x + \psi) dx + \frac{b\rho_1}{k} \int_0^1 \psi_x D^\alpha \varphi dx \\
&- \frac{\rho_1}{k} \int_0^1 D^\alpha \varphi \int_0^t \omega(t-s) \psi_x(s) ds dx, \quad t \geq 0
\end{aligned}$$

we have the following result, for $\epsilon_2, \epsilon_3 > 0$

$$\begin{aligned}
D^\alpha V_5(t) &\leq \frac{k}{12} [\varphi_x^2(1) + \varphi_x^2(0)] - k \|\varphi_x + \psi\|^2 + \rho_2 \|D^\alpha \psi\|^2 + \frac{\rho_1^2 \tilde{\omega}}{4k^2} E_\alpha(-C_\omega t^\alpha) \|\psi_x\|^2 \\
&+ [\epsilon_2 + \tilde{\omega} E_\alpha(-C_\omega t^\alpha)] \|D^\alpha \varphi\|^2 + \frac{\rho_1^2 \tilde{\omega}}{4\epsilon_2 k^2} (|^{RL} D^\alpha \omega| \square \psi_x)(t) + \left(\frac{b\rho_1}{k} - \rho_2 \right) \\
&\times \int_0^1 D^\alpha \psi_x D^\alpha \varphi dx + \frac{\rho_1}{k} \left(b + \frac{1}{\epsilon_3} \right) \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t) + \rho_2 \mathcal{T}_{\varphi_x + \psi}(t) + \frac{b\rho_1}{k} \mathcal{T}_{\psi_x}(t) \\
&+ \frac{3}{k} \left[\left(b\psi_x - \int_0^t \omega(t-s) \psi_x(s) ds \right)^2 (1) + \left(b\psi_x - \int_0^t \omega(t-s) \psi_x(s) ds \right)^2 (0) \right] \\
&+ \frac{\alpha\rho_1\epsilon_3}{2k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left[\int_0^\eta \omega(\eta-s) \psi_x(s) ds \right]' d\eta \right)^2 dx, \quad t \geq 0.
\end{aligned}$$

Proof. Evidently, as stated in Proposition 1.4.8 and the equations of the system,

we reach

$$\begin{aligned}
D^\alpha V_5(t) &= \int_0^1 (\varphi_x + \psi) \left[b\psi_{xx} - \int_0^t \omega(t-s)\psi_{xx}(s)ds - k(\varphi_x + \psi) \right] dx \\
&+ \rho_2 \int_0^1 D^\alpha \psi D^\alpha (\varphi_x + \psi) dx - \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha \psi(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \\
&\times \left(\int_0^\xi \frac{(\varphi_x + \psi)'(s) ds}{(t-s)^\alpha} \right) dx + \frac{b\rho_1}{k} \int_0^1 D^\alpha \psi_x D^\alpha \varphi dx + \frac{b}{k} \int_0^1 \psi_x k(\varphi_x + \psi)_x dx \\
&- \frac{\alpha b\rho_1}{k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\psi_x(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx \\
&- \frac{1}{k} \int_0^1 k(\varphi_x + \psi)_x \int_0^t \omega(t-s)\psi_x(s)ds dx - \frac{\rho_1}{k} \int_0^1 D^\alpha \varphi D^\alpha \int_0^t \omega(t-s)\psi_x(s)ds dx \\
&+ \frac{\alpha \rho_1}{k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left[\int_0^\eta \omega(\eta-s)\psi_x(s)ds \right]' d\eta \right) \\
&\times \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

It easy to see that by integration by parts, we obtain

$$\begin{aligned}
D^\alpha V_5(t) &= \left[\varphi_x \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right) \right]_0^1 - k \|\varphi_x + \psi\|^2 + \rho_2 \|D^\alpha \psi\|^2 \\
&+ \left(\frac{b\rho_1}{k} - \rho_2 \right) \int_0^1 D^\alpha \psi_x D^\alpha \varphi dx - \frac{\rho_1}{k} \int_0^1 D^\alpha \varphi D^\alpha \int_0^t \omega(t-s)\psi_x(s)ds dx \\
&- \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha \psi(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{(\varphi_x + \psi)'(s) ds}{(t-s)^\alpha} \right) dx \\
&- \frac{\alpha b\rho_1}{k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\psi_x(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx \\
&+ \frac{\alpha \rho_1}{k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left[\int_0^\eta \omega(\eta-s)\psi_x(s)ds \right]' d\eta \right) \\
&\times \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

Young's inequality, help in writing

$$\begin{aligned}
D^\alpha V_5(t) &\leq \frac{k}{12} [\varphi_x^2(1) + \varphi_x^2(0)] - k \|\varphi_x + \psi\|^2 + \rho_2 \|D^\alpha \psi\|^2 \\
&+ \left(\frac{b\rho_1}{k} - \rho_2 \right) \int_0^1 D^\alpha \psi_x D^\alpha \varphi dx + \rho_2 \mathcal{T}_{D^\alpha \psi}(t) + \rho_2 \mathcal{T}_{\varphi_x + \psi}(t) \\
&+ \frac{3}{k} \left[\left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (1) + \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (0) \right] \\
&+ \frac{b\rho_1}{k} \mathcal{T}_{\psi_x}(t) + \frac{b\rho_1}{k} \mathcal{T}_{D^\alpha \varphi}(t) - \frac{\rho_1}{k} \int_0^1 D^\alpha \varphi D^\alpha \int_0^t \omega(t-s)\psi_x(s)ds dx \\
&+ \frac{\alpha\rho_1}{k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left[\int_0^\eta \omega(\eta-s)\psi_x(s)ds \right]' d\eta \right) \\
&\times \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

Thereupon

$$\begin{aligned}
&\frac{\rho_1}{k} \int_0^1 D^\alpha \varphi D^\alpha \int_0^t \omega(t-s)\psi_x(s)ds dx \\
&= \frac{\rho_1}{k} \int_0^1 D^\alpha \varphi \left[\int_0^t {}^{RL}D^\alpha \omega(t-s)\psi_x(s)ds + \psi_x(t) \lim_{t \rightarrow 0^+} I^{1-\alpha} \omega(t) \right] dx \\
&\leq \frac{\rho_1}{k} \int_0^1 D^\alpha \varphi \left[\int_0^t {}^{RL}D^\alpha \omega(t-s) (\psi_x(s) - \psi_x(t)) ds + \psi_x(t) I^{1-\alpha} \omega(t) \right] dx \\
&\leq \epsilon_2 \|D^\alpha \varphi\|^2 + \frac{\rho_1^2 \tilde{\omega}}{4\epsilon_2 k^2} (|{}^{RL}D^\alpha \omega| \square \psi_x)(t) + \tilde{\omega} E_\alpha(-C_\omega t^\alpha) \|D^\alpha \varphi\|^2 \\
&+ \frac{\rho_1^2 \tilde{\omega}}{4k^2} E_\alpha(-C_\omega t^\alpha) \|\psi_x\|^2,
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\alpha\rho_1}{k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left[\int_0^\eta \omega(\eta-s)\psi_x(s)ds \right]' d\eta \right) \\
& \times \left(\int_0^\xi \frac{[D^\alpha\varphi(s)]' ds}{(t-s)^\alpha} \right) dx \leq \frac{\rho_1}{k\epsilon_3} \mathcal{T}_{D^\alpha\varphi}(t) \\
& + \frac{\alpha\rho_1\epsilon_3}{2k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left[\int_0^\eta \omega(\eta-s)\psi_x(s)ds \right]' d\eta \right)^2 dx.
\end{aligned}$$

Moreover

$$\begin{aligned}
D^\alpha V_5(t) & \leq \frac{k}{12} [\varphi_x^2(1) + \varphi_x^2(0)] - k \|\varphi_x + \psi\|^2 + \rho_2 \|D^\alpha\psi\|^2 + \frac{\rho_1^2\tilde{\omega}}{4k^2} E_\alpha(-C_\omega t^\alpha) \|\psi_x\|^2 \\
& + [\epsilon_2 + \tilde{\omega} E_\alpha(-C_\omega t^\alpha)] \|D^\alpha\varphi\|^2 + \frac{\rho_1^2\tilde{\omega}}{4\epsilon_2 k^2} (|^{RL}D^\alpha\omega| \square\psi_x)(t) + \left(\frac{b\rho_1}{k} - \rho_2 \right) \\
& \times \int_0^1 D^\alpha\psi_x D^\alpha\varphi dx + \frac{\rho_1}{k} \left(b + \frac{1}{\epsilon_3} \right) \mathcal{T}_{D^\alpha\varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha\psi}(t) + \rho_2 \mathcal{T}_{\varphi_x+\psi}(t) + \frac{b\rho_1}{k} \mathcal{T}_{\psi_x}(t) \\
& + \frac{3}{k} \left[\left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (1) + \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (0) \right] \\
& + \frac{\alpha\rho_1\epsilon_3}{2k\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \left[\int_0^\eta \omega(\eta-s)\psi_x(s)ds \right]' d\eta \right)^2 dx
\end{aligned} \tag{5.21}$$

for $\epsilon_2, \epsilon_3 > 0$. ■

In order to address the boundary terms in the relation (5.21), the following functional is required

$$V_6(t) = \rho_2 \int_0^1 m(x) D^\alpha\psi \left(b\psi_x(t) - \int_0^t \omega(t-s)\psi_x(s)ds \right) dx, \quad t \geq 0,$$

with $m(x) = 2 - 4x$. It is clear that $m(0) = -m(1) = 2$.

Lemma 5.2.10 *The functional $V_6(t)$ satisfies along solutions of equation (5.1)*

$$\begin{aligned}
D^\alpha V_6(t) \leq & - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (1) - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (0) \\
& + \left[4b^2 + \frac{kb^2}{2\epsilon_4} + \rho_2^2 \tilde{\omega} E_\alpha(-C_\omega t^\alpha) \right] \|\psi_x\|^2 + 2k(\epsilon_4 + \epsilon_2) \|\varphi_x + \psi\|^2 \\
& + [2b\rho_2 + \epsilon_2 + \tilde{\omega} E_\alpha(-C_\omega t^\alpha)] \|D^\alpha \psi\|^2 + \bar{\omega} \left(4 + \frac{k}{2\epsilon_2} \right) (\omega \square \psi_x)(t) \\
& + \frac{\tilde{\omega} \rho_2^2}{\epsilon_2} (|^{RL} D^\alpha \omega| \square \psi_x)(t) + \rho_2 b^2 \mathcal{T}_{\psi_x}(t) + 4\rho_2 \left(1 + \frac{1}{\epsilon_3} \right) \mathcal{T}_{D^\alpha \psi}(t) \\
& + \frac{\alpha \rho_2 \epsilon_3}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau)\psi_x(\tau)d\tau \right)' ds \right)^2 dx
\end{aligned}$$

for $t \geq 0$ and $\epsilon_i > 0$, $i = 2, 3, 4$.

Proof. A straight forwarded applications of applying Proposition 1.4.8 yields

$$\begin{aligned}
D^\alpha V_6(t) = & \rho_2 \int_0^1 m(x) D^\alpha (D^\alpha \psi) \left(b\psi_x(t) - \int_0^t \omega(t-s)\psi_x(s)ds \right) dx \\
& + \rho_2 \int_0^1 m(x) D^\alpha \psi \left(bD^\alpha \psi_x(t) - D^\alpha \int_0^t \omega(t-s)\psi_x(s)ds \right) dx \\
& - \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 m(x) \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[D^\alpha \psi(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \\
& \times \left(\int_0^\xi (t-s)^{-\alpha} \left(b\psi_x(s) - \int_0^s \omega(s-\sigma)\psi_x(\sigma)d\sigma \right)' ds \right) dx,
\end{aligned}$$

and utilizing the second equation in our system, we get

$$\begin{aligned}
D^\alpha V_6(t) &\leq \int_0^1 m(x) \left(b\psi_{xx} - \int_0^t \omega(t-s)\psi_{xx}(s)ds \right) \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right) dx \\
&- k \int_0^1 m(x)(\varphi_x + \psi) \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right) dx + b\rho_2 \int_0^1 m(x)D^\alpha\psi D^\alpha\psi_x(t)dx \\
&- \rho_2 \int_0^1 m(x)D^\alpha\psi D^\alpha \int_0^t \omega(t-s)\psi_x(s)ds dx + \rho_2 b^2 \mathcal{T}_{\psi_x}(t) + 4\rho_2 \left(1 + \frac{1}{\epsilon_3} \right) \mathcal{T}_{D^\alpha\psi}(t) \\
&+ \frac{\alpha\rho_2\epsilon_3}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau)\psi_x(\tau)d\tau \right)' ds \right)^2 dx
\end{aligned} \tag{5.22}$$

where $\epsilon_3 > 0$, $t \geq 0$. Next, employing (5.2.3) and (5.9), integrations by parts and

Young's inequality, we attain

$$\begin{aligned}
&\rho_2 \int_0^1 m(x)D^\alpha\psi D^\alpha \int_0^t \omega(t-s)\psi_x(s)ds dx \\
&= \rho_2 \int_0^1 m(x)D^\alpha\psi \left[\int_0^t {}^{RL}D^\alpha \omega(t-s)\psi_x(s)ds + \psi_x(t) \lim_{t \rightarrow 0^+} I^{1-\alpha}\omega(t) \right] dx \\
&= \rho_2 \int_0^1 m(x)D^\alpha\psi \left[\int_0^t {}^{RL}D^\alpha \omega(t-s) (\psi_x(s) - \psi_x(t)) ds + \psi_x(t)I^{1-\alpha}\omega(t) \right] dx \\
&\leq \epsilon_2 \|D^\alpha\psi\|^2 + \frac{\tilde{\omega}\rho_2^2}{\epsilon_2} (|{}^{RL}D^\alpha\omega| \square \psi_x)(t) + \tilde{\omega}E_\alpha(-C_\omega t^\alpha) (\|D^\alpha\psi\|^2 + \rho_2^2 \|\psi_x\|^2),
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
&\int_0^1 m(x)(\varphi_x + \psi) \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right) dx \\
&\leq 2(\epsilon_4 + \epsilon_2) \|\varphi_x + \psi\|^2 + \frac{b^2}{2\epsilon_4} \|\psi_x\|^2 + \frac{\bar{\omega}}{2\epsilon_2} (\omega \square \psi_x)(t),
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
& \int_0^1 m(x) \left(b\psi_{xx} - \int_0^t \omega(t-s)\psi_{xx}(s)ds \right) \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right) dx \\
&= - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (1) - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (0) \\
&\quad - \frac{1}{2} \int_0^1 m'(x) \left[\left(b - \int_0^t \omega(s)ds \right) \psi_x + \int_0^t \omega(t-s) (\psi_x(t) - \psi_x(s)) ds \right]^2 dx \\
&\leq - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (1) - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (0) \\
&\quad + 4b^2 \|\psi_x\|^2 + 4\bar{\omega}(\omega \square \psi_x)(t), \tag{5.25}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 m(x) D^\alpha \psi D^\alpha \psi_x(t) dx = \frac{1}{2} \int_0^1 m(x) \frac{d}{dx} (D^\alpha \psi)^2 dx \\
&= \frac{1}{2} [m(x) (D^\alpha \psi)^2]_0^1 - \frac{1}{2} \int_0^1 m'(x) (D^\alpha \psi)^2 dx \leq 2 \|D^\alpha \psi\|^2. \tag{5.26}
\end{aligned}$$

Using the earlier estimations (5.23)-(5.25) in (5.26) yields the subsequent result for $\epsilon_i > 0$, where i ranges from 2 to 4

$$\begin{aligned}
D^\alpha V_6(t) &\leq - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (1) - \left(b\psi_x - \int_0^t \omega(t-s)\psi_x(s)ds \right)^2 (0) \\
&\quad + \left[4b^2 + \frac{kb^2}{2\epsilon_4} + \rho_2^2 \tilde{\omega} E_\alpha(-C_\omega t^\alpha) \right] \|\psi_x\|^2 + 2k(\epsilon_4 + \epsilon_2) \|\varphi_x + \psi\|^2 + \bar{\omega} \left(4 + \frac{k}{2\epsilon_2} \right) \\
&\quad \times (\omega \square \psi_x)(t) + [2b\rho_2 + \epsilon_2 + \tilde{\omega} E_\alpha(-C_\omega t^\alpha)] \|D^\alpha \psi\|^2 + \frac{\tilde{\omega}\rho_2^2}{\epsilon_2} (|{}^{RL}D^\alpha \omega| \square \psi_x)(t) \\
&\quad + \rho_2 b^2 \mathcal{T}_{\psi_x}(t) + 4\rho_2 \left(1 + \frac{1}{\epsilon_3} \right) \mathcal{T}_{D^\alpha \psi}(t) + \frac{\alpha\rho_2\epsilon_3}{2\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\
&\quad \times \left[\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau)\psi_x(\tau)d\tau \right)' ds \right]^2 dx, \quad t \geq 0.
\end{aligned}$$

I

The derivative of the following functional gives us the boundary terms necessary to regulate the remaining boundary terms mentioned in Lemma 5.2.9. Let

$$V_7(t) := \rho_1 \int_0^1 m(x) \varphi_x D^\alpha \varphi dx, \quad t \geq 0.$$

Lemma 5.2.11 *The functional $V_7(t)$ satisfies*

$$\begin{aligned} D^\alpha V_7(t) &\leq 2\rho_1 \|D^\alpha \varphi\|^2 - k [\varphi_x^2(1) + \varphi_x^2(0)] + 6k \|\varphi_x + \psi\|^2 \\ &+ 7k \|\psi_x\|^2 + 8\rho_1 \mathcal{T}_{\varphi_x + \psi}(t) + 8\rho_1 \mathcal{T}_{\psi_x}(t) + 4\rho_1 \mathcal{T}_{D^\alpha \varphi}(t), \quad t \geq 0. \end{aligned}$$

Proof. Following a similar approach to the previous Lemma 5.2.9, it is clear that

$$\begin{aligned} D^\alpha V_7(t) &= \rho_1 \int_0^1 m(x) D^\alpha \varphi_x D^\alpha \varphi dx + \rho_1 \int_0^1 m(x) \varphi_x D^\alpha (D^\alpha \varphi) dx \\ &- \frac{\alpha \rho_1}{\Gamma(1-\alpha)} \int_0^1 m(x) \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\varphi_x(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx \\ &\leq -\frac{\rho_1}{2} \int_0^1 m'(x) (D^\alpha \varphi)^2 dx + k \int_0^1 m(x) \varphi_x (\varphi_x + \psi)_x dx + 4\rho_1 \mathcal{T}_{\varphi_x}(t) + 4\rho_1 \mathcal{T}_{D^\alpha \varphi}(t) \end{aligned}$$

or

$$\begin{aligned} D^\alpha V_7(t) &\leq 2\rho_1 \|D^\alpha \varphi\|^2 + \frac{k}{2} [m(x) \varphi_x^2]_0^1 - \frac{k}{2} \int_0^1 m'(x) \varphi_x^2 dx + k \|\varphi_x\|^2 \\ &+ k \|\psi_x\|^2 + 4\rho_1 \mathcal{T}_{\varphi_x}(t) + 4\rho_1 \mathcal{T}_{D^\alpha \varphi}(t), \quad t \geq 0. \end{aligned}$$

Having in mind the inequality $\|\varphi_x\|^2 \leq 2\|\varphi_x + \psi\|^2 + 2\|\psi_x\|^2$, we ential

$$\begin{aligned} D^\alpha V_7(t) &\leq 2\rho_1 \|D^\alpha \varphi\|^2 - k [\varphi_x^2(1) + \varphi_x^2(0)] + 6k \|\varphi_x + \psi\|^2 + 7k \|\psi_x\|^2 \\ &+ 8\rho_1 \mathcal{T}_{\varphi_x + \psi}(t) + 8\rho_1 \mathcal{T}_{\psi_x}(t) + 4\rho_1 \mathcal{T}_{D^\alpha \varphi}(t), \quad t \geq 0. \end{aligned}$$

I

For our coming functional, we shall need solutions of the problem

$$\begin{cases} -w_{xx} = \psi_x, & x \in (0, 1), \\ w(0) = w(1) = 0. \end{cases}$$

The last functional is

$$V_8(t) := \int_0^1 (\rho_1 w D^\alpha \varphi + \rho_2 \psi D^\alpha \psi) dx, \quad t \geq 0.$$

Lemma 5.2.12 *The functional $V_8(t)$ above fulfills the following for $\epsilon_2, \epsilon_5 > 0$ and $t \geq 0$*

$$\begin{aligned} D^\alpha V_8(t) &\leq (\rho_1 \epsilon_5 + \rho_2) \|D^\alpha \psi\|^2 + \frac{\rho_1}{4\epsilon_5} \|D^\alpha \varphi\|^2 + \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t) \\ &+ \left[\epsilon_2 - \left(b - \int_0^t \omega(s) ds \right) \right] \|\psi_x\|^2 + (\rho_1 + \rho_2) \mathcal{T}_{\psi_x}(t) + \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t). \end{aligned}$$

Proof. With the help of Proposition 1.4.8, we infer that

$$\begin{aligned}
D^\alpha V_8(t) &= \rho_1 \int_0^1 D^\alpha w D^\alpha \varphi dx + \rho_1 \int_0^1 w D^\alpha (D^\alpha \varphi) dx + \rho_2 \int_0^1 \psi D^\alpha (D^\alpha \psi) dx \\
&+ \rho_2 \|D^\alpha \psi\|^2 - \frac{\alpha \rho_1}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[w(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \varphi(s)]' ds}{(t-s)^\alpha} \right) dx \\
&- \frac{\alpha \rho_2}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{[\psi(\eta)]' d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[D^\alpha \psi(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

Given the solutions of equation (5.1), one can write

$$\begin{aligned}
D^\alpha V_8(t) &\leq \rho_1 \delta_5 \|D^\alpha w\|^2 + \frac{\rho_1}{4\delta_5} \|D^\alpha \varphi\|^2 + k \int_0^1 w(\varphi_x + \psi)_x dx \\
&+ \rho_2 \|D^\alpha \psi\|^2 + \int_0^1 \psi \left[b\psi_{xx} - \int_0^t \omega(t-s)\psi_{xx}(s)ds - k(\varphi_x + \psi) \right] dx \\
&+ \rho_1 \mathcal{T}_w(t) + \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_\psi(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t).
\end{aligned}$$

Since $\|D^\alpha w\|^2 \leq \|D^\alpha \psi\|^2$ (by performing the operation D^α on both sides, multiplying by $D^\alpha w$ and then integrating by parts), it can be concluded that

$$\begin{aligned}
D^\alpha V_8(t) &\leq \rho_1 \delta_5 \|D^\alpha \psi\|^2 + \frac{\rho_1}{4\delta_5} \|D^\alpha \varphi\|^2 - k \int_0^1 w_x(\varphi_x + \psi) dx + \rho_2 \|D^\alpha \psi\|^2 \\
&+ \int_0^1 \psi \left[\left(b - \int_0^t \omega(s) ds \right) \psi_{xx} + \int_0^t \omega(t-s) (\psi_{xx}(t) - \psi_{xx}(s)) ds - k(\varphi_x + \psi) \right] dx \\
&+ (\rho_1 + \rho_2) \mathcal{T}_{\psi_x}(t) + \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t).
\end{aligned}$$

Moreover,

$$\begin{aligned}
D^\alpha V_8(t) &\leq (\rho_1 \delta_5 + \rho_2) \|D^\alpha \psi\|^2 + \frac{\rho_1}{4\delta_5} \|D^\alpha \varphi\|^2 \\
&\quad - \int_0^1 \psi_x \left[\left(b - \int_0^t \omega(s) ds \right) \psi_x + \int_0^t \omega(t-s) (\psi_x(t) - \psi_x(s)) ds \right] dx \\
&\quad + (\rho_1 + \rho_2) \mathcal{T}_{\psi_x}(t) + \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t).
\end{aligned}$$

Hence

$$\begin{aligned}
D^\alpha V_8(t) &\leq (\rho_1 \epsilon_5 + \rho_2) \|D^\alpha \psi\|^2 + \frac{\rho_1}{4\epsilon_5} \|D^\alpha \varphi\|^2 + \frac{\bar{\omega}}{4\epsilon_2} (\omega \square \psi_x)(t) \\
&\quad + \left[\epsilon_2 - \left(b - \int_0^t \omega(s) ds \right) \right] \|\psi_x\|^2 + (\rho_1 + \rho_2) \mathcal{T}_{\psi_x}(t) + \rho_1 \mathcal{T}_{D^\alpha \varphi}(t) + \rho_2 \mathcal{T}_{D^\alpha \psi}(t)
\end{aligned}$$

where $\epsilon_2, \epsilon_5 > 0$. I

We are now ready to present and proof our first theorem.

Theorem 5.2.13 *Assume that the initial data satisfy $U_0 \in \mathcal{D}(\mathcal{A}_1)$ and ω satisfies assumption **(W)**. If $\frac{\rho_1}{k} = \frac{\rho_2}{b}$, then the solution of (5.1), goes to rest in a Mittag-leffler manner provided that $\tilde{\omega}$ is small enough, i.e., there exist two positive constants C and ξ (depending on $E_1(0)$) such that*

$$E_1(t) \leq C E_\alpha(-\xi t^\alpha), \quad t \geq 0.$$

Proof. First, we present our reference functional

$$\mathcal{L}_1(t) := N E_1(t) + \sum_{i=1}^8 \lambda_i V_i(t), \quad t \geq 0,$$

where N and λ_i are positive constants to be defined later. Having in mind all the previous lemmas above, we calculate $D^\alpha \mathcal{L}_1(t)$. The proof aims to derive a fractional differential inequality of the form

$$D^\alpha \mathcal{L}_1(t) \leq -C_3 \mathcal{L}_1(t), \quad t > t_0 > 0. \quad (5.27)$$

Proposition 1.4.4 enables one to derive the estimation in the theorem but only on (t_0, ∞) . By utilizing a continuity argument, we derive a comparable one on the interval $[0, t_0]$. First, note that by increasing the time t_0 , we may make the terms in $E_\alpha(-C_\omega t^\alpha)$ as small as we like. As a result, ϵ_1 may be ignored for the time being. For small $\tilde{\omega}$, hence small ϵ_2

$$\begin{aligned} \frac{d\mathcal{L}_1(t)}{dt} &\leq A_1 \|\psi_x\|^2 + A_2 \|D^\alpha \varphi\|^2 + A_3 \|D^\alpha \psi\|^2 + A_4 \|\phi_x + \psi\|^2 + A_5 \mathcal{T}_{D^\alpha \varphi}(t) \\ &+ A_6 \mathcal{T}_{D^\alpha \psi}(t) + A_7 \mathcal{T}_{\varphi_x + \psi}(t) + A_8 \mathcal{T}_{\psi_x}(t) \\ &+ A_9 \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) \psi_x(\tau) d\tau \right)' ds \right)^2 dx \\ &+ A_{10} \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-\eta)^{-\alpha} \right. \\ &\times \left. \left(\int_0^\eta \omega(\eta-s) (\psi(\eta) - \psi(s)) ds \right)' d\eta \right)^2 dx + A_{11} \left[\left(b\psi_x - \int_0^t \omega(t-s) \psi_x(s) ds \right)^2 (1) \right. \\ &\left. + \left(b\psi_x - \int_0^t \omega(t-s) \psi_x(s) ds \right)^2 (0) \right] + A_{12} [\varphi_x^2(1) + \varphi_x^2(0)] \end{aligned}$$

where

$$A_1 := (b - \omega_0)\lambda_2 + \left(4b^2 + \frac{kb^2}{2\epsilon_4}\right)\lambda_6 + 7k\lambda_7 - b\lambda_8,$$

$$A_2 := -\rho_1\lambda_2 + 2\rho_1\lambda_7 + \frac{\rho_1}{4\epsilon_5}\lambda_8,$$

$$A_3 := -\rho_2\lambda_2 - \omega_0\rho_2\lambda_3 + \rho_2\lambda_5 + 2b\rho_2\lambda_6 + (\rho_1\epsilon_5 + \rho_2)\lambda_8,$$

$$A_4 := k\lambda_2 - k\lambda_5 + 2k\epsilon_4\lambda_6 + 6k\lambda_7,$$

$$A_5 := -\rho_1N + \rho_1\lambda_2 + \frac{\rho_1}{k}\left(b + \frac{1}{\epsilon_3}\right)\lambda_5 + 4\rho_1\lambda_7 + \rho_1\lambda_8,$$

$$A_6 := -\rho_2N + \rho_2\lambda_2 + \rho_2\epsilon_3\lambda_3 + \rho_2\lambda_5 + 4\rho_2\left(1 + \frac{1}{\epsilon_3}\right)\lambda_6 + \rho_2\lambda_8,$$

$$A_7 := -kN + 2\rho_1\lambda_2 + \rho_2\lambda_5 + 8\rho_1\lambda_7,$$

$$A_8 := -\frac{b}{2}N + (2\rho_1 + \rho_2)\lambda_2 + \frac{b\rho_1}{k}\lambda_5 + \rho_2b^2\lambda_6 + 8\rho_1\lambda_7 + (\rho_1 + \rho_2)\lambda_8,$$

$$A_9 := \frac{N}{b} - \lambda_1 + \frac{\rho_1\epsilon_3}{2k}\lambda_5 + \frac{\rho_2\epsilon_3}{2}\lambda_6,$$

$$A_{10} := \frac{\rho_2}{2\epsilon_3}\lambda_3 - \lambda_4,$$

$$A_{11} := \frac{3}{k}\lambda_5 - \lambda_6,$$

$$A_{12} := \frac{k}{12}\lambda_5 - k\lambda_7.$$

We realize that for a considerable large value of N , the parameter ϵ_3 , may be large, the coefficients A_5 , A_6 , A_7 and A_8 may be rendered negative. Consequently, the coefficient A_{10} may be considered negative. Taking $\lambda_1 = \frac{2N}{b}$, will make A_9 negative. For the other coefficients to be negative (note that the coefficients A_{11} , A_{12} , can be zero not necessarily negative) we need to select the parameters

there in a way that

$$\left\{ \begin{array}{l} (b - \omega_0)\lambda_2 + \left(4b^2 + \frac{kb^2}{2\epsilon_4}\right)\lambda_6 + 7k\lambda_7 < b\lambda_8, \\ 2\rho_1\lambda_7 + \frac{\rho_1}{4\epsilon_5}\lambda_8 < \rho_1\lambda_2, \\ \rho_2\lambda_5 + 2b\rho_2\lambda_6 + (\rho_1\epsilon_5 + \rho_2)\lambda_8 < \rho_2\lambda_2 + \omega_0\rho_2\lambda_3, \\ k\lambda_2 + 2k\epsilon_4\lambda_6 + 6k\lambda_7 < k\lambda_5, \\ \frac{\rho_2}{2\epsilon_3}\lambda_3 < \lambda_4, \\ \frac{3}{k}\lambda_5 \leq \lambda_6, \\ \frac{k}{12}\lambda_5 \leq k\lambda_7. \end{array} \right. \quad (5.28)$$

Pick $\lambda_2 = \frac{\lambda_5}{4}$, $\lambda_6 = \frac{3\lambda_5}{k}$, $\epsilon_4 = \frac{k}{26}$ and $\lambda_7 = \frac{\lambda_5}{12}$. There remains

$$\left\{ \begin{array}{l} \left[\frac{b-\omega_0}{4} + \frac{7k}{12} + \frac{51b^2}{k} \right] \lambda_5 < b\lambda_8, \\ \frac{\lambda_8}{\epsilon_5} < \frac{\lambda_5}{3}, \\ 3\left(\frac{1}{4} + \frac{2b}{k}\right)\lambda_5 + \left(\frac{\rho_1}{\rho_2}\epsilon_5 + 1\right)\lambda_8 < \omega_0\lambda_3, \\ \frac{\rho_2\lambda_3}{2\epsilon_3} < \lambda_4. \end{array} \right.$$

Choosing a large value of ϵ_3 , we are left with

$$\left\{ \begin{array}{l} \left[\frac{b-\omega_0}{4} + \frac{7k}{12} + \frac{51b^2}{k} \right] \lambda_5 < b\lambda_8, \\ \frac{\lambda_8}{\epsilon_5} < \frac{\lambda_5}{3}, \\ 3\left(\frac{1}{4} + \frac{2b}{k}\right)\lambda_5 + \left(\frac{\rho_1}{\rho_2}\epsilon_5 + 1\right)\lambda_8 < \omega_0\lambda_3. \end{array} \right.$$

After that, for a large value of λ_3 such that the second relation is satisfied (with ϵ_5 being large), only the following remains:

$$\left[\frac{b - \omega_0}{4} + \frac{7k}{12} + \frac{51b^2}{k} \right] \lambda_5 < b\lambda_8.$$

For a large λ_8 or small enough λ_5 , we get the needed result. I

5.2.3 Non-Equal Speeds of Propagation

In real-world, the rates of propagation may not be identical. Therefore, it is not physically sound to assume equal speeds of propagation. We will need the following result for the basic fractional differential equation:

$$D^\alpha x(t) = f(t, x(t))$$

with 0 as an equilibrium. Let $\mathcal{K}$ be the class of strictly increasing continuous functions $h : [0, +\infty) \rightarrow [0, +\infty)$ satisfying $h(0) = 0$.

Proposition 5.2.14 [94] *Assume that f is a nonnegative bounded function such that $I^\alpha f$, $0 < \alpha < 1$ is also bounded. Then, $\liminf_{t \rightarrow \infty} f(t) = 0$.*

Proposition 5.2.15 [95] *If there exist a Lyapunov function $Z(x(t), t)$ and two functions $\vartheta_1(\cdot)$ and $\vartheta_2(\cdot)$ in $\mathcal{K}$ such that, for all $x \neq 0$, $\vartheta_1(\|x(t)\|) \leq Z(x(t), t) \leq \vartheta_2(\|x(t)\|)$ and $D^\alpha Z(x(t), t) \leq 0$, $0 < \alpha \leq 1$, then the equilibrium is Lyapunov uniformly stable.*

Theorem 5.2.16 *Assume that $U_0 \in \mathcal{D}(\mathcal{A}_1)$ and the speeds of propagation are not necessarily equal. Then, we have that $\liminf_{t \rightarrow \infty} E_1(t) = 0$. If $\psi_x(0) = 0$, then the system is Lyapunov uniformly stable for small values of $\tilde{\omega}$.*

We must find a Lyapunov function with a fractional derivative that is not positive. Revisiting the previous proof, we start from the stage where we neglected the term (by assuming equal speeds of propagation)

$$\left(\frac{b\rho_1}{k} - \rho_2 \right) \int_0^1 D^\alpha \psi_x D^\alpha \varphi dx$$

in $V_5(t)$, that is

$$D^\alpha \mathcal{L}_1(t) \leq -C_4 E_1(t) + \lambda_5 \left(\frac{b\rho_1}{k} - \rho_2 \right) \int_0^1 D^\alpha \psi_x D^\alpha \varphi dx, \quad t > t_0 > 0.$$

Due to this higher order term, we need to pass to energy of higher order. We express $E_1(t)$ as $E_1(t) = E_1(t, \varphi, \psi)$ to incorporate the influence of φ and ψ and we introduce the energy referred to as the '2n α -order' energy

$$E_2(t) := E_1(t, D^\alpha \varphi, D^\alpha \psi), \quad t \geq 0.$$

Thus

$$\begin{aligned} E_2(t) := & \frac{1}{2} \left[\rho_1 \|D^\alpha (D^\alpha \varphi)\|^2 + \rho_2 \|D^\alpha (D^\alpha \psi)\|^2 + \left(b - \int_0^t \omega(s) ds \right) \|D^\alpha \psi_x\|^2 \right. \\ & \left. + k \|D^\alpha (\varphi_x + \psi)\|^2 + (\omega \square D^\alpha \psi_x)(t) \right], \quad t \geq 0. \end{aligned}$$

The fractional derivative of $E_2(t)$ is given by

$$\begin{aligned}
D^\alpha E_2(t) &= \rho_1 \int_0^1 D^\alpha (D^\alpha \varphi) D^\alpha [D^\alpha (D^\alpha \varphi)] dx + \rho_2 \int_0^1 D^\alpha (D^\alpha \psi) D^\alpha [D^\alpha (D^\alpha \psi)] dx \\
&\quad - \frac{1}{2} I^{1-\alpha} \omega(t) \|D^\alpha \psi_x\|^2 + \left(b - \int_0^t \omega(s) ds \right) \int_0^1 D^\alpha \psi_x D^\alpha (D^\alpha \psi_x) dx \\
&\quad + k \int_0^1 D^\alpha (\varphi_x + \psi) D^\alpha [D^\alpha (\varphi_x + \psi)] dx + \frac{1}{2} D^\alpha (\omega \square D^\alpha \psi_x)(t) \\
&\quad - \left(b - \int_0^t \omega(s) ds \right) \mathcal{T}_{D^\alpha(\psi_x)} - \rho_1 \mathcal{T}_{D^\alpha(D^\alpha \varphi)} - \rho_2 \mathcal{T}_{D^\alpha(D^\alpha \psi)} - k \mathcal{T}_{D^\alpha(\varphi_x + \psi)} \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[(D^\alpha \psi_x)^2(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

Making use of (5.1), we see that

$$\begin{aligned}
D^\alpha E_2(t) &= -k \int_0^1 D^\alpha (D^\alpha \varphi_x) D^\alpha (\phi_x + \psi) dx - b \int_0^1 D^\alpha (D^\alpha \psi_x) D^\alpha (\psi_x) dx \\
&\quad + \int_0^1 D^\alpha (D^\alpha \psi_x) D^\alpha \left(\int_0^t w(t-s) \psi_x(s) dx \right) dx - k \int_0^1 D^\alpha (D^\alpha \psi) D^\alpha (\phi_x + \psi) dx \\
&\quad - \frac{1}{2} I^{1-\alpha} \omega(t) \|D^\alpha \psi_x\|^2 + \left(b - \int_0^t \omega(s) ds \right) \int_0^1 D^\alpha \psi_x D^\alpha (D^\alpha \psi_x) dx \\
&\quad + k \int_0^1 D^\alpha (\varphi_x + \psi) D^\alpha [D^\alpha (\varphi_x + \psi)] dx + \frac{1}{2} D^\alpha (\omega \square D^\alpha \psi_x)(t) \\
&\quad - \left(b - \int_0^t \omega(s) ds \right) \mathcal{T}_{D^\alpha(\psi_x)} - \rho_1 \mathcal{T}_{D^\alpha(D^\alpha \varphi)} - \rho_2 \mathcal{T}_{D^\alpha(D^\alpha \psi)} - k \mathcal{T}_{D^\alpha(\varphi_x + \psi)} \\
&\quad + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi \frac{\omega(\eta) d\eta}{(t-\eta)^\alpha} \right) \left(\int_0^\xi \frac{[(D^\alpha \psi_x)^2(s)]' ds}{(t-s)^\alpha} \right) dx.
\end{aligned}$$

Having in mind Proposition 5.2.2, we infer that

$$\begin{aligned}
& \frac{1}{2} D^\alpha (\omega \square D^\alpha \psi_x) = \frac{1}{2} ({}^{RL} D^\alpha \omega \square D^\alpha \psi_x)(t) \\
& - \int_0^1 D^\alpha (D^\alpha \psi_x) \int_0^t \omega(t-s) D^\alpha \psi_x(s) ds dx + \frac{1}{2} \left(\int_0^t \omega(t-s) ds \right) D^\alpha \|D^\alpha \psi_x\|^2 \\
& - \frac{\alpha}{2\Gamma(1-\alpha)} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{\omega(\eta) d\eta}{(t-\eta)^\alpha} \int_0^\xi \frac{[\|D^\alpha \psi_x(s)\|^2]' ds}{(t-s)^\alpha} \\
& + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \int_0^\xi \frac{[(D^\alpha \psi_x)(\eta)]' d\eta}{(t-\eta)^\alpha} \\
& \times \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) D^\alpha \psi_x(\tau) d\tau \right)' ds \right) dx
\end{aligned}$$

The coming relation is found by utilizing Young's inequality

$$\begin{aligned}
D^\alpha E_2(t) & \leq \frac{1}{2} ({}^{RL} D^\alpha \omega \square D^\alpha \psi_x)(t) - \frac{1}{2} I^{1-\alpha} \omega(t) \|D^\alpha \psi_x\|^2 + (\epsilon_6 - b) \mathcal{T}_{D^\alpha \psi_x} \\
& - \rho_1 \mathcal{T}_{D^\alpha(D^\alpha \varphi)} - \rho_2 \mathcal{T}_{D^\alpha(D^\alpha \psi)} - k \mathcal{T}_{D^\alpha(\varphi_x + \psi)} \\
& + \frac{\alpha}{2\epsilon_6 \Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^\xi (t-s)^{-\alpha} \left(\int_0^s \omega(s-\tau) D^\alpha \psi_x(\tau) d\tau \right)' ds \right)^2 dx
\end{aligned} \tag{5.29}$$

where $\epsilon_6 > 0$, $t \geq 0$.

Proposition 5.2.17 *Suppose that k and f are two continuous functions defined on the interval $(0, +\infty)$ such that*

$$I^{1-\alpha} k(t), I^{1-\alpha} f(t) \in C^1([0, \infty)).$$

Then,

$$\begin{aligned} \left(\int_0^t k(s) ds \right) D^\alpha f(t) &= \int_0^t k(t-s) [D^\alpha f(t) - D^\alpha f(s)] ds \\ &- \int_0^t {}^{RL}D^\alpha k(t-s) [f(t) - f(s)] ds + I^{1-\alpha} k(t) f(t) - f(0) I^{1-\alpha} k(t) - k(t) I^{1-\alpha} f(0) \end{aligned}$$

where $t \geq 0$.

Proof. We can easily see that

$$\begin{aligned} &\int_0^t k(t-s) [D^\alpha f(t) - D^\alpha f(s)] ds - \int_0^t {}^{RL}D^\alpha k(t-s) [f(t) - f(s)] ds \\ &= \left(\int_0^t k(s) ds \right) D^\alpha f(t) - \int_0^t k(t-s) D^\alpha f(s) ds - [I^{1-\alpha} k(t) - I^{1-\alpha} k(0)] f(t) \\ &+ \int_0^t {}^{RL}D^\alpha k(t-s) f(s) ds. \end{aligned}$$

By Proposition (1.4.7), it can be deduced that

$${}^{RL}D^\alpha \int_0^t k(t-s) f(s) ds = \int_0^t {}^{RL}D^\alpha k(t-s) f(s) ds + f(t) I^{1-\alpha} k(0),$$

and conversely, the connection between the two derivatives provides

$$\begin{aligned} {}^{RL}D^\alpha \int_0^t k(s) f(t-s) ds &= \int_0^t k(s) {}^{RL}D^\alpha f(t-s) ds + k(t) I^{1-\alpha} f(0) \\ &= \int_0^t k(t-s) \left[D^\alpha f(s) + \frac{s^{-\alpha} f(0)}{\Gamma(1-\alpha)} \right] ds + k(t) I^{1-\alpha} f(0) \\ &= \int_0^t k(t-s) D^\alpha f(s) ds + \frac{f(0)}{\Gamma(1-\alpha)} \int_0^t k(s) (t-s)^{-\alpha} ds + k(t) I^{1-\alpha} f(0) \\ &= \int_0^t k(t-s) D^\alpha f(s) ds + f(0) I^{1-\alpha} k(t) + k(t) I^{1-\alpha} f(0). \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^t {}^{RL}D^\alpha k(t-s)f(s)ds + f(t)I^{1-\alpha}k(0) &= \int_0^t k(t-s)D^\alpha f(s)ds + f(0)I^{1-\alpha}k(t) \\ &+ k(t)I^{1-\alpha}f(0), \end{aligned}$$

and

$$\begin{aligned} &\int_0^t k(t-s) [D^\alpha f(t) - D^\alpha f(s)] ds - \int_0^t {}^{RL}D^\alpha k(t-s) [f(t) - f(s)] ds \\ &= \left(\int_0^t k(s)ds \right) D^\alpha f(t) - I^{1-\alpha}k(t)f(t) + f(0)I^{1-\alpha}k(t) + k(t)I^{1-\alpha}f(0). \quad (5.30) \end{aligned}$$

■

By multiplying the equation (5.30) by $g(t)$, integrating over the interval $(0, 1)$, and utilizing Young's inequality, we derive the following result.

Corollary 5.2.18 *Let us suppose that k and f are both continuous functions on the interval $(0, +\infty)$ such that*

$$I^{1-\alpha}k(t), f(t) \in C^1(0, \infty)$$

and f , $D^\alpha f$ and g are all in $L^2(0, 1)$. Hence

$$\begin{aligned} &\left(\int_0^t k(s)ds \right) \int_0^1 g(t) (D^\alpha f) dx \leq 2\varepsilon \|g(t)\|^2 + \frac{1}{\varepsilon} \left(\int_0^t k(s)ds \right) (k \square D^\alpha f)(t) \\ &+ \frac{1}{\varepsilon} \left(\int_0^t |{}^{RL}D^\alpha k| ds \right) \times (|{}^{RL}D^\alpha k| \square f)(t) + \frac{1}{\varepsilon} [I^{1-\alpha}k(t)]^2 \|f\|^2 + \frac{1}{\varepsilon} [I^{1-\alpha}k(t)]^2 \|f(0)\|^2 \\ &+ \frac{1}{4\varepsilon} k^2(t) \|I^{1-\alpha}f(0)\|^2, \quad t \geq 0. \end{aligned}$$

In the case that $g = D^\alpha f$, this corollary gives the following:

Corollary 5.2.19 *Suppose that k and f are both continuous functions on $(0, +\infty)$*

with the property that $I^{1-\alpha}k(t)$ and $f(t)$ belong to $C^1(0, \infty)$. Then,

$$\begin{aligned} & \left(\int_0^t k(s) ds - 2\varepsilon \right) \|D^\alpha f\|^2 \leq \frac{1}{\varepsilon} \left(\int_0^t k(s) ds \right) (k \square D^\alpha f)(t) + \frac{1}{\varepsilon} \left(\int_0^t |{}^{RL}D^\alpha k| ds \right) \\ & \times (|{}^{RL}D^\alpha k| \square f)(t) + \frac{1}{\varepsilon} [I^{1-\alpha}k(t)]^2 \|f\|^2 + \frac{1}{\varepsilon} [I^{1-\alpha}k(t)]^2 \|f(0)\|^2 + \frac{1}{4\varepsilon} k^2(t) \|I^{1-\alpha}f(0)\|^2 \end{aligned}$$

where $t \geq 0$.

We recommend using the new functional, to control the last term in the energy

$E_2(t)$

$$V_9(t) := \left\| \int_0^t \omega(t-s) D^\alpha \psi_x(s) ds \right\|^2 + \int_0^t \omega(t-s) \|D^\alpha \psi_x(s)\|^2 ds, \quad t \geq 0.$$

Lemma 5.2.20 *The functional $V_9(t)$ verifies*

$$\begin{aligned} D^\alpha V_9(t) & \leq -\frac{C_\omega}{2} \int_0^t \omega(t-s) \|D^\alpha \psi_x(s)\|^2 ds + \frac{4\tilde{\omega}^2}{C_\omega} (|{}^{RL}D^\alpha \omega| \square D^\alpha \psi_x)(t) \\ & + \left(\frac{4\tilde{\omega} [\tilde{\omega} E_\alpha(-C_\omega t^\alpha)]^2}{C_\omega} + \tilde{\omega} \right) \|D^\alpha \psi_x\|^2 - \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \\ & \times \left[\int_0^\xi (t-\eta)^{-\alpha} \left(\int_0^\eta \omega(\eta-\tau) D^\alpha \psi_x(\tau) d\tau \right)' d\eta \right]^2 dx, \quad t \geq 0. \end{aligned}$$

The proof is similar to the one of Lemma 5.2.9.

Proof. [Proof of Theorem 5.2.16] Combining the estimations of (5.29) and $D^\alpha V$

with the above corollaries, for

$$\mathcal{L}_2(t) := N(E_1(t) + E_2(t)) + \sum_{i=1}^9 \lambda_i V_i(t),$$

assuming a large value of N , we get

$$D^\alpha \mathcal{L}_2(t) \leq -C_5 E_1(t) + C_6 [E_\alpha(-C_\omega t^\alpha)]^2 \|\psi_x(0)\|^2, \quad t > t_0 > 0$$

for small $\tilde{\omega}$ and large t_0 for some $C_5, C_6 > 0$. Applying the operator I^α to both sides results in

$$\begin{aligned} \mathcal{L}_2(t) - \mathcal{L}_2(0) &= \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t-s)^{\alpha-1} D^\alpha \mathcal{L}_2(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} D^\alpha \mathcal{L}_2(s) ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_0} (t-s)^{\alpha-1} D^\alpha \mathcal{L}_2(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} \{-C_5 E_1(t) + C_6 [E_\alpha(-C_\omega t^\alpha)]^2 \|\psi_x(0)\|^2\} ds. \end{aligned}$$

It should be noted that due to the continuity of $D^\alpha \mathcal{L}_2(s)$ (where all constants C_i , $i = 7, \dots, 12$ are positive),

$$\int_0^{t_0} (t-s)^{\alpha-1} D^\alpha \mathcal{L}_2(s) ds \leq \int_0^{t_0} (t_0-s)^{\alpha-1} |D^\alpha \mathcal{L}_2(s)| ds \leq C_7 t_0^\alpha.$$

Hence

$$C_5 I_{t_0}^\alpha E_1(t) \leq \mathcal{L}_2(0) + \frac{C_7 t_0^\alpha}{\Gamma(\alpha)} + C_6 I^\alpha [E_\alpha(-C_\omega t^\alpha)]^2 \|\psi_x(0)\|^2,$$

and according to Proposition 1.4.6, for $t \geq t_0$, we have

$$I^\alpha [E_\alpha(-C_\omega t^\alpha)]^2 \leq I^\alpha [C_8 t^{-\alpha} E_\alpha(-C_\omega t^\alpha)] \leq C_9 E_\alpha(-C_\omega t^\alpha) \leq C_{10}.$$

Thus,

$$C_{11} E_1(t) + I_{t_0}^\alpha E_1(t) \leq C_{12}.$$

The first conclusion follows from Proposition 5.2.14 and the second one from Proposition 5.2.3. ■

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Conclusion

In this dissertation, we considered a mechanical engineering problem that occurs often in applications. It consists of a beam with a significant rotation effect due to bending. It is adequately delineated by a Timoshenko type model. This structure is also subject to a considerable torsion. This latter factor is described by a coupling of the Timoshenko system with a new (third) equation. The derived model has been discussed by relatively few researchers treating rather the "engineering" side. However, we are unable to locate a mathematical analysis for such systems. In addition, the model takes into account the possibility of axial forces acting on the structure. This brings considerable mathematical difficulties and regrettably they are not frequently (or systematically) investigated in mathematics papers.

We first established an existence and uniqueness result in a suitable underlying

space with the helps of the semi-group theory. Then we turned to the stability issue focusing in particular on specifying the rate of decay. In this regards, we obtained sufficient conditions ensuring exponential and polynomial stability.

Notably, we came up with a condition resembling very much the equal speeds of propagation condition often encountered in the study of Timoshenko system. In a point of fact, this newly obtained condition comprises the equals speeds of propagation condition as a special case. Unfortunately, it was shown that this condition is not practically reasonable and therefore the one we obtained too. It was therefore pivotal to investigate the rather plausible situation in the absence of this condition. To this end, we have been inclined to pass to the second order energy to accommodate some higher order terms. The price to pay is a polynomial rate instead of an exponential one. In order to have a further insight on the system, we explored also the case of axial loading. The main methods used in all our proofs is the energy method. Due to the aforementioned features of the problem it was not a straightforward task to come up with appropriate functionals.

Remarkably, we found that (when ignoring P) the equal speeds of propagation condition does not concern necessarily the first two equations but may involve the second and third equation therefore allowing the speeds of propagation of the rotation component to be free and independent of the other two.

Non-Integer Fractional Damping and Viscoelastic Term

A derivative that is half the order of the leading term has been observed to have the ability to bring the system to rest. The rate achieved and anticipated follows the Mittag-Leffler pattern, which is a characteristic of fractional derivatives. The

stability is global in time. Moreover, this term functions as a damping term, similar to its integer-order counterpart. When dealing with a memory term in a viscoelastic material, a significant challenge arises in assessing the expression

$$\int_{\Omega} \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}} \left(\int_0^{\xi} (t-\eta)^{-\alpha} \left(\int_0^s k(s-\tau) |\nabla v(\tau)| d\tau \right)' d\eta \right)^2 dx$$

present in the derivative of the energy. In response to this situation, a new functional $V_1(t)$ has been introduced. In terms of nonlinearity, we have made additional assumptions on the exponent of the nonlinearity in order to take advantage of an embedding result. Regrettably, the fractional inequality derived is not the same as the one associated with the lower-order fractional term. To extract some information about the concerned functional we needed to examine small kernels and employ a strategy to address the resulting nonlinearity. The price to pay is a stability of local character only.

Fractional Timoshenko

In this thesis, we considered the dynamics of a Timoshenko beam operating in anomalous media. The model is obtained from the classical Timoshenko model by substituting the integer-order derivatives with fractional derivatives ranging from one to two. In fact, sequential fractional derivatives are more appropriate for the multiplier technique. It has been demonstrated that a viscoelastic damping mechanism affecting the rotational part can lead the structure to a state of equilibrium at a rate described by a Mittag-Leffler function when the propagation speeds are equal and the relaxation function meets a fractional inequality whose

solutions are limited by Mittag-Leffler functions. This extends a comparable instance of exponential stability observed in the classical (integer-order) case. The case when the speeds of propagation are not equal, the situation is much more complicated and differs from the classical situation. We have achieved uniform Lyapunov stability without a specific rate.

6.2 Future Work

Viscoelastic Effect

In this thesis, we have reduced the number of frictional dampings from three to only two. One in any of the two components of the Timoshenko beam and one in the torsional component. It is natural to ask whether it would be possible to weaken further the damping mechanisms by considering viscoelastic damping in one of the component, like

$$\left\{ \begin{array}{l} \rho I \phi_{tt} - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} = 0, \\ \mu v_{tt} + \int_0^t g(t - \tau) v_{xx}(\tau) d\tau - (kAG - P)(v_{xx} - \phi_x) + P\phi_x = 0, \\ -[GJ - P(I_s/\mu)]\psi_{xx} - K\phi_{xx} + I_s\psi_{tt} + c_2\psi_t = 0, \end{array} \right.$$

$$\left\{ \begin{array}{l} \rho I \phi_{tt} + \int_0^t g(t - \tau) \phi_{xx}(\tau) d\tau - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} = 0, \\ \mu v_{tt} - [kAG - P](v_{xx} - \phi_x) + P\phi_x = 0, \\ -[GJ - P(I_s/\mu)]\psi_{xx} - K\phi_{xx} + I_s\psi_{tt} + c_2\psi_t = 0 \end{array} \right.$$

and

$$\begin{cases} \rho I \phi_{tt} - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} = 0, \\ \mu v_{tt} - (kAG - P)(v_{xx} - \phi_x) = 0, \\ -[GJ - P(I_s/\mu)]\psi_{xx} - K\phi_{xx} + I_s\psi_{tt} - \int_0^t g(t - \tau)\psi_{xx}(\tau)d\tau = 0. \end{cases}$$

Thermal Coupling Effect

Another kind of weaker damping than the frictional one consists in considering a thermoelastic bending-torsion beam. This consists in coupling the Timoshenko beam with a heat equation, like

$$\begin{cases} \rho I \phi_{tt} - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} + \sigma\theta_x = 0, \\ \mu v_{tt} - (kAG - P)(v_{xx} - \phi_x) + P\phi_x = 0, \\ -[GJ - P(I_s/\mu)]\psi_{xx} - K\phi_{xx} + I_s\psi_{tt} + c_2\psi_t = 0, \\ \rho_1\theta_t - \rho_2\theta_{xx} + \sigma\phi_{xt} = 0. \end{cases}$$

The coupling here is with the first component. Theoretically, we might also examine the coupling with the second or even the third component. Another coupling

is the second sound type of the form

$$\left\{ \begin{array}{l} \rho I \phi_{tt} - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} + \sigma\theta_x = 0, \\ \mu v_{tt} - (kAG - P)(v_{xx} - \phi_x) + P\phi_x = 0, \\ -[GJ - P(I_s/\mu)]\psi_{xx} - K\phi_{xx} + I_s\psi_{tt} + c_2\psi_t = 0, \\ a\theta_t + q_t + \delta\phi_{xt} = 0, \\ bq_t + \alpha q + \theta_x = 0. \end{array} \right.$$

Also, to avoid the infinite speeds of propagation, one might consider the thermoleastic problem of type III

$$\left\{ \begin{array}{l} \rho I \phi_{tt} - EI \phi_{xx} - kAG(v_x - \phi) - K\psi_{xx} + \alpha\theta_x = 0, \\ \mu v_{tt} - (kAG - P)(v_{xx} - \phi_x) + P\phi_x = 0, \\ -[GJ - P(I_s/\mu)]\psi_{xx} - K\phi_{xx} + I_s\psi_{tt} + c_2\psi_t = 0, \\ a\theta_{tt} - b\theta_{xx} + \delta\phi_{ttx} + \gamma\theta_{txx} = 0. \end{array} \right.$$

In addition, for these problems and the ones studied in this thesis, it is interesting to investigate the corresponding nonlinear case by replacing the frictional damping ϕ_t , v_t and ψ_t by $h(\phi_t)$, $h(v_t)$ or $h(\psi_t)$. Of course, the time dependent context $f(t)h(\phi_t)$, $f(t)h(v_t)$ or $f(t)h(\psi_t)$, is also worthy of investigation as it appears often in applications. The competition between a nonlinear source in one of the components with the existing dampings offers a different venue of interesting investigations as well.

Fractional Problems

The investigation of the kind of fractional problems addressed here is in its infancy stage. It is quite challenging and therefore any contribution in this regards would certainly be welcome. It is in our intention to look at the following cases:

- Different relationships between the lower order and the leading order. That is, the case when the lower order is not exactly half of the leading order

$$D^\alpha(D^\alpha u(t, x)) + {}^C D^\beta u(t, x) + |u|^\gamma u = \Delta u(t, x) - b \int_0^t k(t-s) \Delta u(s, x) ds.$$

- The fractional thermoleastic models.
- The case of nonlinear fractional damping $f(t)h(D^\alpha u(t, x))$.
- The case of viscoelastic damping is still on open problem. It deserves to be revisited.
- The arguments developed in this work to prove the stability are versatile in the sense that they apply to many evolution fractional problems. However, the question of well-posedness needs to be fixed first. This seems to be also a challenging issue.
- The bending-torsion problem would be more interesting and more challenging in the fractional framework.

- All the aforementioned issues may be addressed for others kinds for fractional derivatives.

REFERENCES

- [1] S. P. Timoshenko, “On the correction for shear of the differential equation for transverse vibrations of prismatic bars.” *Phil Mag Ser*, vol. 41, pp. 744–764, 1921.
- [2] B. Al-Homidan and N.-e. Tatar, “Stabilization of a non-linear fractional problem by a non-integer frictional damping and a viscoelastic term,” *Fractal and Fractional*, vol. 7, no. 5, p. 367, 2023.
- [3] P. Barthelemy, J. Bertolotti, and D. S. Wiersma, “A lévy flight for light,” *Nature*, vol. 453, no. 7194, pp. 495–498, 2008.
- [4] I. Goychuk, V. O. Kharchenko, and R. Metzler, “Molecular motors pulling cargos in the viscoelastic cytosol: how power strokes beat subdiffusion,” *Physical Chemistry Chemical Physics*, vol. 16, no. 31, pp. 16 524–16 535, 2014.
- [5] B. Al-Homidan and N.-e. Tatar, “Fractional timoshenko beam with a viscoelastically damped rotational component,” *AIMS Mathematics*, vol. 8, no. 10, pp. 24 632–24 662, 2023.

- [6] R. Ansari, M. F. Oskouie, F. Sadeghi, and M. Bazdid-Vahdati, “Free vibration of fractional viscoelastic timoshenko nanobeams using the nonlocal elasticity theory,” *Physica E: Low-Dimensional Systems and Nanostructures*, vol. 74, pp. 318–327, 2015.
- [7] F. P. Pinnola, R. Barretta, F. Marotti de Sciarra, and A. Pirrotta, “Analytical solutions of viscoelastic nonlocal timoshenko beams,” *Mathematics*, vol. 10, no. 3, p. 477, 2022.
- [8] A. Pirrotta, S. Cutrona, S. D. Lorenzo, and A. D. Matteo, “Fractional viscoelastic timoshenko beam deflection via single equation,” *International Journal for Numerical Methods in Engineering*, vol. 104, no. 9, pp. 869–886, 2015.
- [9] A. K. Jha and S. S. Dasgupta, “Mathematical modeling of a fractionally damped nonlinear nanobeam via nonlocal continuum approach,” *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, vol. 233, no. 19-20, pp. 7101–7115, 2019.
- [10] M. Klanner, M. S. Prem, and K. Ellermann, “Steady-state harmonic vibrations of viscoelastic timoshenko beams with fractional derivative damping models,” *Applied Mechanics*, vol. 2, no. 4, pp. 797–819, 2021.
- [11] D.-X. Feng, D.-H. Shi, and W.-T. Zhang, “Feedback stabilization of timoshenko beam with boundary dissipation,” *IFAC Proceedings Volumes*, vol. 29, no. 1, pp. 2191–2196, 1996.

- [12] D. Feng, D. Shi, and W. Zhang, “Boundary feedback stabilization of timoshenko beam with boundary dissipation,” *Science in China Series A: Mathematics*, vol. 41, pp. 483–490, 1998.
- [13] C. A. Raposo, J. Ferreira, M. Santos, and N. Castro, “Exponential stability for the timoshenko system with two weak dampings,” *Applied Mathematics Letters*, vol. 18, no. 5, pp. 535–541, 2005.
- [14] A. Soufyane and A. Whebe, “Uniform stabilization for the timoshenko beam by a locally distributed damping,” *Electronic Journal of Differential Equations (EJDE)*[*electronic only*], vol. 2003, pp. Paper–No, 2003.
- [15] F. Ammar-Khodja, S. Kerbal, and A. Soufyane, “Stabilization of the nonuniform timoshenko beam,” *Journal of mathematical analysis and applications*, vol. 327, no. 1, pp. 525–538, 2007.
- [16] J. U. Kim and Y. Renardy, “Boundary control of the timoshenko beam,” *SIAM journal on control and optimization*, vol. 25, no. 6, pp. 1417–1429, 1987.
- [17] D.-H. Shi and D.-X. Feng, “Exponential decay of timoshenko beam with locally distributed feedback,” *IMA Journal of Mathematical Control and Information*, vol. 18, no. 3, pp. 395–403, 2001.
- [18] A. Guesmia and S. A. Messaoudi, “General energy decay estimates of timoshenko systems with frictional versus viscoelastic damping,” *Mathematical Methods in the Applied Sciences*, vol. 32, no. 16, pp. 2102–2122, 2009.

- [19] J. E. M. Rivera and H. D. F. Sare, “Stability of timoshenko systems with past history,” *Journal of Mathematical Analysis and Applications*, vol. 339, no. 1, pp. 482–502, 2008.
- [20] A. Guesmia and S. A. Messaoudi, “A general stability result in a timoshenko system with infinite memory: a new approach,” *Mathematical Methods in the Applied Sciences*, vol. 37, no. 3, pp. 384–392, 2014.
- [21] F. Ammar-Khodja, A. Benabdallah, J. M. Rivera, and R. Racke, “Energy decay for timoshenko systems of memory type,” *Journal of Differential Equations*, vol. 194, no. 1, pp. 82–115, 2003.
- [22] A. Guesmia and S. A. Messaoudi, “On the control of a viscoelastic damped timoshenko-type system,” *Applied mathematics and computation*, vol. 206, no. 2, pp. 589–597, 2008.
- [23] S. A. Messaoudi and M. I. Mustafa, “A stability result in a memory-type timoshenko system,” *Dynamic systems and applications*, vol. 18, no. 3, p. 457, 2009.
- [24] ———, “A general stability result in a memory-type timoshenko system.” *Communications on Pure & Applied Analysis*, vol. 12, no. 2, 2013.
- [25] J. E. M. Rivera and R. Racke, “Mildly dissipative nonlinear timoshenko systems—global existence and exponential stability,” *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 248–278, 2002.

- [26] S. A. Messaoudi, M. Pokojovy, and B. Said-Houari, “Nonlinear damped timoshenko systems with second sound—global existence and exponential stability,” *Mathematical Methods in the Applied Sciences*, vol. 32, no. 5, pp. 505–534, 2009.
- [27] M. Santos, D. A. Júnior, and J. M. Rivera, “The stability number of the timoshenko system with second sound,” *Journal of Differential Equations*, vol. 253, no. 9, pp. 2715–2733, 2012.
- [28] A. Djebabla and N.-E. Tatar, “Exponential stabilization of the timoshenko system by a thermo-viscoelastic damping,” *Journal of Dynamical and Control Systems*, vol. 16, pp. 189–210, 2010.
- [29] A. Djebabla and N.-e. Tatar, “Stabilization of the timoshenko beam by thermal effect,” *Mediterranean journal of mathematics*, vol. 7, pp. 373–385, 2010.
- [30] H. D. Fernández Sare and R. Racke, “On the stability of damped timoshenko systems: Cattaneo versus fourier law,” *Archive for Rational Mechanics and Analysis*, vol. 194, pp. 221–251, 2009.
- [31] C. Giorgi and F. Vegni, “Uniform energy estimates for a semilinear evolution equation of the mindlin-timoshenko beam with memory,” *Mathematical and computer modelling*, vol. 39, no. 9-10, pp. 1005–1021, 2004.
- [32] A. E. Green and P. Naghdi, “A re-examination of the basic postulates of thermomechanics,” *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, vol. 432, no. 1885, pp. 171–194, 1991.

- [33] ———, “A re-examination of the basic postulates of thermomechanics,” *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, vol. 432, no. 1885, pp. 171–194, 1991.
- [34] A. Green and P. Naghdi, “Thermoelasticity without energy dissipation,” *Journal of elasticity*, vol. 31, no. 3, pp. 189–208, 1993.
- [35] A. E. Green and P. Naghdi, “A unified procedure for construction of theories of deformable media. i. classical continuum physics,” *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, vol. 448, no. 1934, pp. 335–356, 1995.
- [36] A. Guesmia and S. A. Messaoudi, “General energy decay estimates of timoshenko systems with frictional versus viscoelastic damping,” *Mathematical Methods in the Applied Sciences*, vol. 32, no. 16, pp. 2102–2122, 2009.
- [37] Z. Liu and C. Peng, “Exponential stability of a viscoelastic timoshenko beam,” *Advances in Mathematical Sciences and Applications*, vol. 8, pp. 343–351, 1998.
- [38] S. A. Messaoudi and M. I. Mustafa, “A stability result in a memory-type timoshenko system,” *Dynamic systems and applications*, vol. 18, no. 3, p. 457, 2009.
- [39] S. A. Messaoudi and B. Said-Houari, “Energy decay in a timoshenko-type system with history in thermoelasticity of type iii,” 2009.

- [40] A. Soufyane, “Exponential stability of the linearized nonuniform timoshenko beam,” *Nonlinear Analysis: Real World Applications*, vol. 10, no. 2, pp. 1016–1020, 2009.
- [41] —, “Stabilisation de la poutre de timoshenko,” *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, vol. 328, no. 8, pp. 731–734, 1999.
- [42] S. Shoukui, “Boundary stabilization of nonuniform timoshenko beam,” *Applied Mathematics-A Journal of Chinese Universities*, vol. 14, pp. 467–474, 1999.
- [43] D.-H. Shi, S. Hou, and D.-X. Feng, “Feedback stabilization of a timoshenko beam with an end mass,” *International Journal of Control*, vol. 69, no. 2, pp. 285–300, 1998.
- [44] M. d. L. Santos *et al.*, “Decay rates for solutions of a timoshenko system with a memory condition at the boundary,” in *Abstract and Applied Analysis*, vol. 7. Hindawi, 2002, pp. 531–546.
- [45] C. A. Raposo, W. D. Bastos, and M. d. L. Santos, “A transmission problem for the timoshenko system,” *Computational & Applied Mathematics*, vol. 26, pp. 215–234, 2007.
- [46] J. E. M. Rivera and R. Racke, “Timoshenko systems with indefinite damping,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1068–1083, 2008.

- [47] J. E. Muñoz Rivera and R. Racke, “Global stability for damped timoshenko systems,” 2002.
- [48] O. Morgül, “Boundary control of a timoshenko beam attached to a rigid body: planar motion,” *International journal of control*, vol. 54, no. 4, pp. 763–791, 1991.
- [49] J. Li, R. Shen, H. Hua, and X. Jin, “Bending–torsional coupled dynamic response of axially loaded composite timoshenko thin-walled beam with closed cross-section,” *Composite Structures*, vol. 64, no. 1, pp. 23–35, 2004.
- [50] M. Kaya and O. O. Ozgumus, “Flexural–torsional-coupled vibration analysis of axially loaded closed-section composite timoshenko beam by using dtm,” *Journal of Sound and Vibration*, vol. 306, no. 3-5, pp. 495–506, 2007.
- [51] M. S. Sari, W. G. Al-Kouz, and R. Al-Waked, “Bending–torsional-coupled vibrations and buckling characteristics of single and double composite timoshenko beams,” *Advances in Mechanical Engineering*, vol. 11, no. 3, p. 1687814019834452, 2019.
- [52] C. Mei, “Free and forced wave vibration analysis of axially loaded materially coupled composite timoshenko beam structures,” 2005.
- [53] Y. Povstenko and Y. Povstenko, *Fractional heat conduction and related theories of thermoelasticity*. Springer, 2015.
- [54] F. Shen, W. Tan, Y. Zhao, and T. Masuoka, “The rayleigh–stokes problem for a heated generalized second grade fluid with fractional derivative model,”

- Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1072–1080, 2006.
- [55] T. Atanackovic, S. Pilipovic, and D. Zorica, “A diffusion wave equation with two fractional derivatives of different order,” *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 20, p. 5319, 2007.
- [56] J. Chen, F. Liu, and V. Anh, “Analytical solution for the time-fractional telegraph equation by the method of separating variables,” *Journal of Mathematical Analysis and Applications*, vol. 338, no. 2, pp. 1364–1377, 2008.
- [57] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. World Scientific, 2022.
- [58] E. Orsingher and L. Beghin, “Time-fractional telegraph equations and telegraph processes with brownian time,” *Probability Theory and Related Fields*, vol. 128, no. 1, pp. 141–160, 2004.
- [59] T. Sandev and Ž. Tomovski, “The general time fractional wave equation for a vibrating string,” *Journal of Physics A: Mathematical and Theoretical*, vol. 43, no. 5, p. 055204, 2010.
- [60] W. R. Schneider and W. Wyss, “Fractional diffusion and wave equations,” *Journal of Mathematical Physics*, vol. 30, no. 1, pp. 134–144, 1989.
- [61] P. Barthelemy, J. Bertolotti, and D. S. Wiersma, “A lévy flight for light,” *Nature*, vol. 453, no. 7194, pp. 495–498, 2008.

- [62] I. Goychuk, V. O. Kharchenko, and R. Metzler, “Molecular motors pulling cargos in the viscoelastic cytosol: how power strokes beat subdiffusion,” *Physical Chemistry Chemical Physics*, vol. 16, no. 31, pp. 16 524–16 535, 2014.
- [63] V. E. Tarasov, *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media*. Springer Science & Business Media, 2011.
- [64] A. Lunardi, “On the linear heat equation with fading memory,” *SIAM Journal on Mathematical Analysis*, vol. 21, no. 5, pp. 1213–1224, 1990.
- [65] M. Fabrizio and A. Morro, *Mathematical problems in linear viscoelasticity*. SIAM, 1992.
- [66] C. Giorgi and V. Pata, “Asymptotic behavior of a nonlinear hyperbolic heat equation with memory,” *Nonlinear Differential Equations and Applications NoDEA*, vol. 8, pp. 157–171, 2001.
- [67] I. Munteanu, “Stabilization of semilinear heat equations, with fading memory, by boundary feedbacks,” *Journal of Differential Equations*, vol. 259, no. 2, pp. 454–472, 2015.
- [68] M. Cavalcanti, V. Domingos Cavalcanti, J. Prates Filho, and J. Soriano, “Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping,” 2001.
- [69] V. Pata, “Exponential stability in linear viscoelasticity,” *Quarterly of applied mathematics*, vol. 64, no. 3, pp. 499–513, 2006.

- [70] N.-e. Tatar, “Polynomial stability without polynomial decay of the relaxation function,” *Mathematical methods in the applied sciences*, vol. 31, no. 15, pp. 1874–1886, 2008.
- [71] —, “On a large class of kernels yielding exponential stability in viscoelasticity,” *Applied mathematics and computation*, vol. 215, no. 6, pp. 2298–2306, 2009.
- [72] —, “Arbitrary decays in linear viscoelasticity,” *Journal of mathematical physics*, vol. 52, no. 1, 2011.
- [73] —, “A new class of kernels leading to an arbitrary decay in viscoelasticity,” *Mediterranean journal of mathematics*, vol. 10, pp. 213–226, 2013.
- [74] —, “Nonexistence results for a fractional problem arising in thermal diffusion in fractal media,” *Chaos, Solitons & Fractals*, vol. 36, no. 5, pp. 1205–1214, 2008.
- [75] Y. Zhou and J. W. He, “New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$,” *Evol. Equ. Control Theory*, vol. 10, no. 3, pp. 491–509, 2021.
- [76] I. Kim, K.-H. Kim, and S. Lim, “An l_q (l_p)-theory for the time fractional evolution equations with variable coefficients,” *Advances in Mathematics*, vol. 306, pp. 123–176, 2017.
- [77] X.-B. Shu and Q. Wang, “The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$,”

- Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 2100–2110, 2012.
- [78] B. Zhu, L. Liu, and Y. Wu, “Local and global existence of mild solutions for a class of semilinear fractional integro-differential equations,” *Fractional Calculus and Applied Analysis*, vol. 20, no. 6, pp. 1338–1355, 2017.
- [79] X. Wang and X. Shu, “The existence of positive mild solutions for fractional differential evolution equations with nonlocal conditions of order $1 < \alpha < 2$,” *Advances in Difference Equations*, vol. 2015, pp. 1–15, 2015.
- [80] A. M. El-Sayed and M. A. Herzallah, “Continuation and maximal regularity of an arbitrary (fractional) order evolutionary integro-differential equation,” *Applicable Analysis*, vol. 84, no. 11, pp. 1151–1164, 2005.
- [81] R. Ponce, “Bounded mild solutions to fractional integro-differential equations in banach spaces,” in *Semigroup Forum*, vol. 87, no. 2. Springer, 2013, pp. 377–392.
- [82] R. P. Agarwal, J. P. C. dos Santos, and C. Cuevas, “Analytic resolvent operator and existence results for fractional integro-differential equations,” *J. Abstr. Differ. Equ. Appl.*, vol. 2, no. 2, pp. 26–47, 2012.
- [83] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin *et al.*, *Mittag-Leffler functions, related topics and applications*. Springer, 2020.

- [84] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.
- [85] Y. Li, Y. Chen, and I. Podlubny, “Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized mittag-leffler stability,” *Computers & Mathematics with Applications*, vol. 59, no. 5, pp. 1810–1821, 2010.
- [86] A. A. Alikhanov, “Boundary value problems for the diffusion equation of the variable order in differential and difference settings,” *Applied Mathematics and Computation*, vol. 219, no. 8, pp. 3938–3946, 2012.
- [87] H. Dai and W. Chen, “New power law inequalities for fractional derivative and stability analysis of fractional order systems,” *Nonlinear Dynamics*, vol. 87, pp. 1531–1542, 2017.
- [88] S. Zhang, Y. Yu, and H. Wang, “Mittag-leffler stability of fractional-order hopfield neural networks,” *Nonlinear Analysis: Hybrid Systems*, vol. 16, pp. 104–121, 2015.
- [89] M. V. Kukushkin, “Abstract fractional calculus for m-accretive operators,” *arXiv preprint arXiv:1901.06118*, 2019.
- [90] A. A. Shkalikov, “Perturbations of self-adjoint and normal operators with discrete spectrum,” *Russian Mathematical Surveys*, vol. 71, no. 5, p. 907, 2016.

- [91] J. P. C. Dos Santos, “Fractional resolvent operator with α (0, 1) and applications,” *Frac. Diff. Calc.*, vol. 9, pp. 187–208, 2019.
- [92] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Springer Science & Business Media, 2012, vol. 44.
- [93] H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*. Springer, 2011, vol. 2, no. 3.
- [94] J. A. Gallegos, M. A. Duarte-Mermoud, N. Aguila-Camacho, and R. Castro-Linares, “On fractional extensions of barbalat lemma,” *Systems & Control Letters*, vol. 84, pp. 7–12, 2015.
- [95] M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos, and R. Castro-Linares, “Using general quadratic lyapunov functions to prove lyapunov uniform stability for fractional order systems,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 22, no. 1-3, pp. 650–659, 2015.

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- (2) Banan, Al-Homidan, and Nasser-eddine Tatar. *Stabilization of a non-linear fractional problem by a non-integer frictional damping and a viscoelastic term*. Fractal and Fractional, 7(5), (2023): 367.
- (3) Hassan, Jamilu Hashim, Nasser-eddine Tatar, and Banan Al-Homidan. *Mittag-Leffler stability and Lyapunov stability for a problem arising in porous media*. Fractional Calculus and Applied Analysis, (2024): 1-22.