

**FINITE ELEMENT ANALYSIS OF ROTATING LINEARLY
AND EXPONENTIALLY TAPERED TIMOSHENKO BEAM
USING VIOLIN STRING SHAPE FUNCTION**

BY

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To my parents, siblings, my wife, and my sons Rakan and Ammar

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NOMENCLATURE

$[I]$	identity matrix
$[\mathcal{A}]$	rotational transformation matrix
$[\mathcal{A}_\varphi]$	partial derivative of $[\mathcal{A}]$ with respect to φ
$[\mathcal{A}_\psi]$	constant setting angle transformation
$[\tilde{\mathcal{A}}_I]$	skew symmetric matrix
L	total length of the beam (root to tip)
n	total number of elements
l^i	element length = L/n
i	refers to the i^{th} element
ξ^i	= x^i/l^i , non-dimensional elemental coordinates in the x -direction
A_0	cross sectional area of the beams' root
A^i	beam element cross sectional area
b^i	beam cross-section width at an arbitrary location x^i
h^i	beam cross-section depth at an arbitrary location x^i
b_0	cross-section width at an arbitrary at the root of the beam
h_0	cross-section depth at an arbitrary at the root of the beam
	i^{th} beam element
I_{yy}^i	cross-section area moment of inertia about the y -axis
I_{zz}^i	cross-section area moment of inertia about the z -axis
I_{yz}^i	product cross-section area moment of inertia about the yz -axes
I_0	cross-section area moment of inertia at the root of the beam
ν_y	taper parameter in the y -direction
ν_z	taper parameter in the z -direction
$\mathcal{R}_0$	hub radius
R	= $\mathcal{R}_0/L$, non-dimensional hub radius
$[B_j^i]$	elemental strain displacement matrix
$\{d^i\}$	elastic deformation vector
$\{e^i\}, \{\bar{e}^i\}$	generalized nodal displacement vector

F_P^i	centrifugal force
F_x^i, F_y^i, F_z^i	x , y , and z -components of the centrifugal force
E	modulus of elasticity
G	modulus of rigidity
ρ	mass density
ν	Poisson's ratio
κ	shear correction factor
$[k_{ee}^i]$	composite elemental elastic stiffness matrix
$[k_b^i]$	bending elemental stiffness matrix in the (xz) -plane
$[k_s^i]$	shear elemental stiffness matrix in the (xz) -plane
$[k_{c,w}^i]$	centrifugal elemental stiffness matrix in the (xz) -plane
$[m_{ee}^i]$	composite element elastic mass matrix
$[m_t^i]$	translational elemental mass matrix in the (xz) -plane
$[m_r^i]$	rotary inertia elemental mass matrix in the (xz) -plane
$[m_{rr}^i]$	element mass matrix associated with reference motion
$[m_{re}^i]$	element mass matrix representing the coupling between elastic deformations and reference motion
$[M^i]$	composite elemental mass matrix
$[M]$	global mass matrix of the beam
$\mathcal{L}^i$	$= \mathcal{T}^i - \mathcal{U}^i - \mathcal{V}^i$, Lagrangian
$\mathcal{V}^i$	potential energy of the i^{th} beam element
$\mathcal{T}^i$	kinetic energy of the i^{th} beam element
Π^i	strain energy of the i^{th} beam element
M_y^i, M_z^i	bending moments
$[\mathcal{N}^i]$	shape functions of the i^{th} beam element
p^i	point in the deformed state of the i^{th} beam element
P_0^i, P_1^i	points in the undeformed state of the i^{th} beam element before and after setting angles are made
Q_y^i, Q_z^i	shear forces
$\mathcal{R}_P^i$	global position vector of point P^i
r_g	$= \sqrt{I_{0yy}/A_0}$, radius of gyration of the cross-section
$r_{P_0}^i, r_{P_1}^i$	position vectors of point P_0^i and P_1^i , respectively
V^i	element volume
u^i	axial nodal displacement
w_b^i	bending nodal displacement in the z -direction
w_s^i	shear nodal displacement in the z -direction
x^i	elemental co-ordinate in the x -direction

y^i	elemental co-ordinate in the y -direction
z^i	elemental co-ordinate in the z -direction
(xyz)	local co-ordinate system
$(x'y'z')$	co-ordinate system obtained by rotating the (xyz) co-ordinate system about the x -axis by a setting angle ψ
$(x^iy^iz^i)$	elemental co-ordinate system
$(\bar{x}\bar{y}\bar{z})$	elemental co-ordinate system initial parallel to $(x^iy^iz^i)$ co-ordinate system
(XYZ)	global co-ordinate system
ϵ^i	element strain
γ^i	angle of distortion due to shear deformation
θ_y^i	element bending rotation about the z -axis
Φ	shear deformation parameter
τ^i	element shear stress
σ^i	element normal stress
φ	$= \Omega t$
ψ	setting angle
f	frequency, Hz
Ω	rate of spin of the hub
$\bar{\Omega}$	$= \Omega L^2 / \sqrt{EI_{0yy}/\rho A_0}$, spin parameter $\sqrt{2}$
$[]$	transpose of $[]$
$[]_E, []_T$	Euler-Bernoulli or Timoshenko quantities

THESIS ABSTRACT

NAME: AYMAN ZAYED MAALUWI
TITLE OF STUDY: FINITE ELEMENT ANALYSIS OF ROTATING LINEARLY AND EXPONENTIALLY TAPERED TIMOSHENKO BEAM USING VIOLIN STRING SHAPE FUNCTION
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A finite element analysis is developed on a rotating Timoshenko beam to get the transverse free vibration. The beam displacement is described using two degrees of freedom (DOFs) on each node: one transverse and one rotation. The beam is assumed to be linearly or exponentially tapered in two planes. The governing differential equation of motion for free vibrations of a rotating beam is obtained using Lagrangian's equations. Both the cubic Hermitian polynomial and violin string shape functions are used to solve the finite element model. A computer code is developed to drive the mass and stiffness matrices using a consistent mass formulation. The model can handle different classical boundary conditions, including clamped-free and hinged-free. After defining the eigenvalue problem, numerical solutions are derived and compared whenever possible with exact and numerical approximations published in the literature. The convergence of the violin string shape function and the effect of exponential taper are investigated. It is observed that the violin string shape functions has an advantage of providing better accuracy at higher spin parameter and gyration ratios. Also, investigating the exponentially tapered rotating beam reveal no critical taper ratio detected unlike the linearly tapered beam

ملخص الرسالة

الاسم: أيمن زايد معلوي
عنوان الدراسة: تحليل العناصر المحدودة لعتبة دوارة حسب نظرية تيموشينكو ذو مقطع عرضي يتغير خطيًا أو أسياً باستخدام دالة وتر الكمان
التخصص: الهندسة الميكانيكية
تاريخ الدرجة العلمية: ديسمبر 2023 م

يتم إجراء تحليل العناصر المحدودة على عتبة تيموشينكو الدوارة للحصول على الاهتزاز الحر المستعرض. توصف حركة العتبة باستخدام درجتين من الحرية (DOFs) في كل عقدة: حركة عرضية وحركة دوارة. افترض أن تكون العتبة مدببة خطيًا أو بشكل أسياً في سطحين مستويين. استخدمت معادلات لاغرانجيان للحصول على المعادلة التفاضلية الحاكمة لحركة للاهتزازات الحرة للعتبة الدوارة. أستخدم كل من دالة متعددة الحدود الهرميتية المكعبة (cubic Hermitian polynomial) ودالة وتر الكمان (violin string shape functions) لحل نموذج العناصر المحدودة. طوّر كود حاسوبي لإيجاد مصفوفات الكتلة والصلابة باستخدام صيغة كتلة متسقة. يمكن للنموذج التعامل مع ظروف الحدود الكلاسيكية المختلفة. بعد تعريف المشكل المميز، يتم اشتقاق الحلول العددية ومقارنتها كلما أمكن ذلك مع الحلول الصحيحة والتقديرات التقريبية المنشورة في أبحاث سابقة. بعد إيجاد الحلول للعتبة الدوارة، يدرس تقارب دالة وتر الكمان وتأثير تناقص العتبة الأسّي. يلاحظ أن دالة وتر الكمان توفر دقة أفضل عند سرعات الدوران الأعلى. أيضًا، تكشف نتائج دراسة العتبة الدوارة المدببة بشكل أسّي عن عدم وجود نسبة تفتق حرجة على عكس العتبة الدوارة المدببة بشكل خطي.

CHAPTER 1

INTRODUCTION

1.1 General

Numerous studies have been conducted in the engineering literature on the dynamic problems of rotating beams. This tremendous research attention can be attributed to the vast practical applications of rotating beams in many engineering structures. Throughout the years, the understanding of rotating beams has been used in the development of windmill turbine blades, robot manipulators, helicopter rotor wings, spinning space constructions, and propeller blades. However, determining the mode shapes and natural frequencies of rotating beams is the only way to design many engineering structures in the best possible way today because these rotating beam characteristics have a strong association with many engineering failure problems. As well as this, it is important to precisely find out the dynamic modal properties of rotating beams early on in the design process to make sure that the engineering structures being considered are less likely to break down and are safe for people to use. Therefore, a thorough investigation is needed to develop models to address the

engineering problems associated with rotating beams. Sohre [98] gives a guideline on how to identify the cause of a blade failure based on the failure mechanisms. It shows how to link the failure mechanism with possible causes. Sohre [98], there is always a high correlation between design and excessive stress and resonance. This summary of practical experience shows the importance of understanding the behavior of the rotating beams.

Many researchers have explored modeling issues regarding the behavior of rotating beams. The issues include stiffening, fatigue, vibration, thermal warping, and Coriolis effects, all of which impact the frequencies and mode shapes of the beam while rotating at certain angular speeds. To deal with these aerodynamic problems, rotatory beam models have a number of important parameters built in. These include shear deformation, root offset, rotary inertia, pre-twist, pre-lag, pre-cone, and cross-sections that are not symmetrical. While such parameters have helped formulate the equations of motion, various solutions have been suggested. These solutions include the Southwell method, the transfer matrix approach [37, 39], the method of integral equations [13], the Myklestad method [5, 42], the Rayleigh-Ritz method [17, 38], the Galerkin method [117, 23], the power series solution method [16, 20], the finite difference method [46], the finite element technique [21, 19, 2, 23, 24, 40], and the dynamic stiffness matrix method [28].

Even though these methods are different from one another, they are characterized by small deformations in rotating beams. This characterization distorts the numerical accuracy of the solutions and remains a significant challenge for modeling. Furthermore, the task of estimating the free vibration and stability characteristics of rotating

beams, particularly when accounting for chordwise and flapwise bending and torsion, is very challenging and approaches the limit of impossibility. Hence, most studies have focused on obtaining approximate solutions to simple rotating-beam problems.

In the last decade, the development of sophisticated computer hardware and software has enhanced the ability of researchers to solve free-vibration problems. There is a prevailing belief that contemporary software has the capability to precisely calculate the inherent frequencies and modal shapes of spinning beams. One reason for providing an accurate estimation is that polynomial and interpolating calculations are performed automatically. On top of that, there is a meager chance of human error when performing such calculations. Therefore, the present study employs a contemporary programming language to do finite element analysis in order to determine the natural frequency and mode shapes of rotating beams.

1.2 Literature Review

Beam models have been employed in most research publications on rotating structural components that focus on numerical and analytical models. However, this review will only focus on rotating beams and their associated vibration problems. Over the years, numerous scholarly publications have been dedicated to the study of rotating beams. Review papers on the vibration characteristics of rotating blades are Leissa [30] in 1981, Ramamurti and Balasubramanian [121] and Leissa et al. [123] in 1984.

Papers covers detailed surveys on rotating-beam dynamics were published by Kunz [32] in 1994, Bazoune [92] in 2005, Rafiee et al. [120] in 2017 and the most recent

review by Hoskoti [118] in 2023.

1.2.1 Beam Vibration History

The eighteenth century saw the exploration of beams' bending vibration theory. The theory pioneers were Daniel Bernoulli and Leonhard Euler [27]. The two Swiss mathematicians discovered the differential equation for calculating a slender beam's lateral vibration around 1750. Since its inception, engineers have frequently used the Euler-Bernoulli beam theory (EBT) to calculate the deflection and load-carrying properties of beams. The Bernoulli beam theory (EBT) may be seen as a simplified representation of the linear theory of elasticity. The assumption is made that the plane cross-sections maintain their planarity and that the slope of the beam angles is very small. Additionally, it is assumed that the plane cross-sections of the beam are oriented at a right angle to the neutral axis. Furthermore, it is assumed that the impacts of shear deflection and spinning inertia are neglected. A fourth-order differential equation with constant coefficients can represent the mathematical formulation that describes the behavior of the beam.

$$\frac{\partial^2}{\partial x^2} \left(EI^i \frac{\partial^2 w}{\partial x^2} \right) + \rho A^i \left(\frac{\partial w}{\partial x} \right) = 0 \quad (1.1)$$

While the theory estimates the number of cycles per second of slender beams' flexural vibrations in lower modes with adequate accuracy [92], it provides less accurate results for slender beams in higher modes. A century later, Rayleigh updated the EBT to include the rotary inertia effect [3]. Nonetheless, the enhanced version of the EBT failed to explain thick beams in higher modes. In 1921, Timoshenko provided a theory

by including the effects of transverse shear deformation and rotational bending [1]. The use of the Timoshenko beam theory (TBT) is appropriate in situations where higher modes of vibration are of interest or when the dimensions of the beam's cross-section are significant relative to its length. The impact of rotational inertia and shear deformation in such situations is dependent on the wavelength of vibration. The increase in mode order results in a more pronounced effect. When compared to EBT, TBT differs by having a higher level of flexibility. TBT assumes that the plane cross-sections of the beam, which are initially perpendicular to the neutral axis, maintain their planarity even after deformation, although they are no longer perpendicular to the neutral axis [1]. In the lower modes, the effect of the shear is not significant, but in the higher modes, the effect of the shear increases and decreases the frequency value [92]. Two partial differential equations are obtained as a result of the additional transverse-shear correction. They are expressed in terms of the rotation angle and the flexural displacement.

$$\frac{\partial}{\partial x} \left(\kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) \right) - \rho A^i \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x} \left(F_x \frac{\partial w}{\partial x} \right) = 0 \quad (1.2)$$

$$\frac{\partial}{\partial x} \left(E I^i \frac{\partial \theta}{\partial x} \right) + \kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) - \rho I^i \left(\frac{\partial^2 \theta}{\partial t^2} \right) = 0 \quad (1.3)$$

In the early nineties, Timoshenko [1] found that, compared to the correction due to rotary inertia, the correction due to shear is significantly larger. Similarly, Weaver et al. [112] reported that the shear correction was 3.2 times higher than the correction associated with rotary inertia. Many researchers in the literature have suggested various values of the shear correction factor. However, the values suggested by Cowper [81] and Przemieniecki [105] seem to be broadly acknowledged and often used in

academic literature.

1.2.2 Rotation Stiffening Effect

The effect of stiffening is more pronounced when a blade is rotating at speeds well above the fundamental critical speed. Lamb and Southwell [4] were the first to report rotation's stiffening effect. In an attempt to determine the crucial frequency for spinning components, the researchers incorporated the rotational stiffening effect using below mathematical expression:

$$\lambda_{Ri}^2 = \lambda_{Ni}^2 + \bar{\Omega}^2 \quad (1.4)$$

In (1.4), the symbol $\bar{\Omega}$ denotes the parameter representing the rotational speed, while λ_{Ni} and λ_{Ri} represent the disc non-rotating and rotating, respectively, non-dimensional natural frequencies of the i^{th} mode. Southwell and Gough [6] expanded upon prior research by incorporating a variety of blade configurations, such as the knife-edged wedge and the uniform cantilever. The utilization of Rayleigh's quotient was employed in order to make predictions regarding the fundamental natural frequency. This led to the derivation of the subsequent equation, which establishes an upper bound relation.

$$\bar{\omega}_{Ri}^2 = \frac{V_{\max}}{T_{\max}}$$

The variables $T_{\max}$ and $V_{\max}$ denote the maximum kinetic and potential energies, respectively. When analyzing rotating beams, the determination of potential energy requires the inclusion of both strain energy and the influence exerted by the centrifugal force field. The objective of this study is to determine the inherent natural frequency

and build an upper limit relationship, which is expressed mathematically.

$$\lambda_{Ri}^2 = \lambda_{Ni}^2 + S_i \bar{\Omega}^2 \quad (1.5)$$

where S_i is the Southwell's coefficient (SC) or a constant that corresponds to the i^{th} mode. Thus, the λ_{Ri} value can be obtained within lower and upper error limits. Southwell and Gough [6] found the S_1 value to be $S_1 = 1.193$ after solving the problem associated with rotating uniform cantilevered blades. However, for the knife-edged problem, the researchers found the values of $S_1 = 1.524$ and $S_1 = 1.455$ using a tabular and a graphical method, respectively. The practicality of Southwell and Gough [6] approach lies in its ability to derive the frequency prediction of a rotating beam by utilizing the natural frequency of a non-spinning beam and the associated spin parameter. The efficiency of the process lies in the rapid acquisition, interpretation, and visualization of data pertaining to the rotation of beams.

1.2.3 Rotating Beams

Euler-Bernoulli Beam

In rotating beam-vibration problems, Euler-Bernoulli beam is utilized due to its simplicity and accuracy. Euler-Bernoulli rotating beams has been investigated considering different effects using different solution approaches [118].

It has been established how rotating uniform and tapered Euler-Bernoulli beams can help simplify a problem that was very complicated in its early stages [7, 9, 8, 10, 12, 11, 13, 21, 14–19].

the effect of a setting angle ψ was first introduced by Lo and Renbarger [7]. The parameter denoted as the setting angle ψ pertains to the angular displacement at which the beam is affixed to the outer edge of the hub at its base, relative to the plane of rotation. Typically, it is counterclockwise and positive. In their study, Lo and Renbarger [7] introduced a partial differential equation that describes the dynamics of a rotating, uniform Euler-Bernoulli beam that is rigidly attached to a hub. The selection of the mid-plane of the beam as the reference angle for the rotational plane was random. The aircraft lies within the rotational plane while ψ is equal to 0° and is positioned outside the rotational plane when ψ is equal to 90° . The Coriolis effect vanishes for zero setting angles and becomes small for low rotational speeds. Hence, the effect can be neglected for low rotational speed [7]. According to the equation below, Lo and Renbarger [7] showed that the frequency parameters exhibit a relationship with the frequency at $\psi = 0^\circ$ for every given setting angle:

$$\lambda_{Ri}^2 = \lambda_{Ri0}^2 - \bar{\Omega}^2 \sin^2 \psi \quad (1.6)$$

When Lo and Renbarger's [7] proposed expression is used in the Southwell equation, in any mode under consideration, the difference between in-plane and out-of-plane Southwell coefficients equals one. Naguleswaran [86] demonstrated that this characteristic holds true for slender beams with constant inertia and stiffness properties. Bazoune [27] later confirmed that this formula is also valid for rotating tapered Timoshenko beams. Lo [9] simplified the previous problem by assuming that every blade part, except for the extreme connected to the rotating shaft, was rigid. Lo [9] discovered that $S_i = \cos^2 \psi$, resulting in an expression similar to the above formula. This

formula implies that all modes have the same SC values, contradicting the Southwell expression, in which the value of S_i depends on the mode. Besides, the literature demonstrates that variables such as the hub radius parameter R ¹ has a significant influence on SC. Lo [9] disregarded the Coriolis acceleration when addressing the issue of the rigid blade and proposed a solution in the phase plane. The observed disparity in the frequency estimation was determined to be around 0.7% after accounting for the inclusion of the Coriolis acceleration. The result reached by Lo [9] is that the percentage error in frequency may be represented as a mathematical function involving three variables: the setting angle denoted as ψ , the frequency ratio expressed as $\lambda_{Ni}/\bar{\Omega}$, and the ratio between the vibration amplitude and the beam length represented by the variable L . Carnegie [8] derived an equation to quantify the increase in potential energy resulting from the centrifugal forces exerted on a horizontal, uniform blade attached to a rigid rotating hub. The beam has a setting angle $-\psi$. In the investigation, the researcher assumed that the effects of rotary inertia and shear distortion were absent, the deformation was small, and the Coriolis effect was negligible. Then, using Rayleigh's method to calculate the frequency problem, Carnegie [8] estimated the upper bound value for the symmetrical cross-section blade's non-rotating fundamental frequency parameter. Carnegie [8] estimated the parameter to be 3.53 for the non-rotating beam, which is 0.4% more than the exact value of 22 of 3.51602 [8]. Also, Carnegie [8] discovered that for a 9-inch steel blade, the lead-lag and fundamental flapping frequencies were 13.1 and 11.3 times more than the non-rotating frequency when spinning at 1000 rad/s. Given that the setting angle will increase

¹A non-dimensional quantity $R = \mathcal{R}_0/L$.

from 0 to 90° , the rise in frequency is 16%. The estimations presented by Carnegie [8] and the expression proposed by Lo and Renbarger [7] are in excellent agreement with each other. With the formula proposed by Lo and Renbarger [7], the lead-lag frequencies were 12.97 times higher than the non-rotating frequency.

Furthermore, many investigators have researched the impact of a tip mass on various forms of rotating beams. For example, Austin and Pan [113] focused on the influence of a tip mass on an Euler-Bernoulli beam's frequencies. The beam was assumed to rotate at constant angular speed, and the tip mass was assumed to possess the effect of rotary inertia. They found that the higher the angular speed, the higher the frequencies, and vice versa. Jones [13] aimed to address the effect of a tip mass determination on the frequencies of a rotating Euler-Bernoulli beam. By employing the integral equations approach, the investigator derived lower-bound approximations for the eigenvalue problem. The study showed that the square of the tip-to-beam mass ratio increased the expected lower limits for the first two lowest modes. Using a finite element analysis, Hao [21] developed a model for rotating prismatic beams. The finite element model takes into account the influence of hub radius, tip mass, and setting angle. In the analysis, a third-order polynomial was used to approximate the displacement field. The formulation of the displacement field used in this study was derived from the Euler-Bernoulli beam theory, using components of identical length. In the present study, the equation for the stiffness matrices and elemental mass was produced by the researcher. Additionally, the influence of the tip mass was included in the expression that encompasses the mass and centrifugal stiffness matrices. The model of Hao [21] showed evidence of a stiffening effect produced by a large hub radius

and/or rotational speed. While he found that the setting angle significantly impacted the fundamental mode, he noticed that the impact was insignificant on higher modes. The researcher observations can be attributed to the constancy of $\Omega^2 \sin^2 \psi$, irrespective of the mode under consideration. This factor had a more pronounced influence on modes characterized by small frequency parameters. The results of Hao [21] also indicated that the fundamental frequency was inversely proportional to the tip mass, provided that the rotational speed was fixed. For high tip mass values, a linear correlation was seen between the rotational speed parameter and the fundamental frequency. However, no linear relationship could be found for a low tip mass value. The relationship was found to have a concave parabolic shape. Therefore, an upper bound estimate could be used to predict the fundamental frequency for a zero-tip mass. A reduction in the second and third frequency parameters is observed when the rotational rates are low and the tip mass increases. Conversely, at high rotational speeds, an increase in tip mass would lead to an elevation of these frequency parameters.

The reported outcomes may be explained by considering the influence of the tip mass on the mass matrix, as shown by the investigation conducted by Hao [21]. Investigation has shown that this particular factor has a more pronounced impact on reducing frequencies while operating at lower rotational speeds. However, as the rotational speed increases, the stiffness and frequency are observed to increase. The other results showed the fluctuation of the fundamental frequency parameters for different setting angles. When the motion is not in-plane or out-of-plane, a three-dimensional vibration is presumed since it contains flapwise and edgewise components. Thus, the component must be identified. Hence, the component under consideration needs to

be specified. Hao [21] neglected the Coriolis effect and did not specify or consider the flapwise component.

Hoa et al. [14] obtained high eigenvalues because of the sign error in the component of the centrifugal stiffness matrix. The study included obtaining and comparing further data with the specific scenario of a rotating beam, which had a different radius and no tip mass, in order to assess any differences. The comparison results showed that the exact values of the predicted values were almost the same. Bhat [17] contributed to the rotating beam literature by extending the problem addressed by Hoa et al. [14]. The researcher used the Rayleigh-Ritz technique to evaluate the beams' natural frequencies. Bhat [17] looked into the problem by finding eigenvalues and eigenvectors and adding tip mass to mass and stiffness matrices. Their results were the same as those of Hoa et al. [14].

Hodges and Rutkowski [15] utilized Legendre polynomial form functions and implemented a finite element technique to examine spinning beam vibration frequencies. This method required the expression of the displacement function in terms of a full set of Legendre polynomials. Precision depends on the number of series terms n and elements i utilized. Hodges and Rutkowski [15] indicated that the standard element shape functions would be obtained when $n=3$. They reported three frequency parameters for tapered cantilevers and rotating uniform beams at different values of n and i and found the optimal solution at $n = 15$ and $i = 1$. This method provided accurate results for rotating beams with simple geometries, and it could be enhanced to provide accurate results for rotating beams with geometric non-linearity. Overall, the limitation of the method of Hodges and Rutkowski [15] is that it can only provide

precise outcomes for the flapping vibration of slender beams.

Furthermore, a group of researchers used Hamilton's technique to formulate a solution to the problem of rotating beams with a prescribed variation of mass distribution and flexural rigidity. In their study, Wright et al. [16] utilized the approach of Frobenius, employing a power series solution, to estimate the mode shapes and frequencies of a beam. A characteristic expression was obtained, and the square root of the eigenvalues represented the beam's natural frequencies. They predicted numerical values for different boundary conditions, such as hinged-free, to allow for a flapping mode whose rotation speed matches the fundamental frequency [16]. The approach presented yielded precise outcomes in comparison to the exact solution; nevertheless, its applicability is limited to simpler structures. Besides, if there are many terms in the series solution, a sophisticated computer would be needed to find the root of the equation.

Many researchers were interested in rotating tapered beams [21, 17, 14]. Hashemi et al. [114] employed a dynamic stiffness matrix (DSM) technique to perform the vibration analysis of uniform and linearly tapered beams. In contrast, Liu and Yeh [22] used the method proposed by Garlekin [66] to estimate the non-uniform rotating beams' bending frequencies. The investigator also used this same method to examine the impact of several variables on non-uniform beams' pure bending vibration, such as the rotating speed, setting angle, tip mass, elastic root restraints, and taper ratio. Storti and Aboelnaga [115] examined the deflection of a straight, uniform beam fastened to a rotating hub. Hodges and Rutkowski's work was expanded to include rotating tapered beams by Wang and Werely [117]. They used a spectral finite element

method (SFEM) and a wave propagation technique to find the answer to the differential equation. They finally obtained a linear ordinary differential equation (ODE), which was solved using the Frobenius technique. The results of Wang and Werely [117] were compared with those of Hodges and Rutkowski [42], and they were found to be in good agreement.

Khulief [19] used the finite analysis approach to expand upon the research conducted by Bhat [17] and Hoa [21] in the field of vibration analysis. Khulief derived symbolic expressions for the stiffness and mass matrices of linearly tapered beams. In contrast to Bhat [17] and Hoa [21], Khulief [19] assumed that the impact on rotary inertia by the tip mass and hub radius is negligible. They discovered that the first and second bending frequencies had different stiffening effects for fixed and hinged-free beams with different taper ratios. To determine the Euler-Bernoulli beam's free bending frequencies, Banerjee [28] employed a dynamic stiffness technique using the Frobenius method of solution. The results obtained were in agreement with those obtained with other numerical techniques.

In the analysis conducted by Chung and Yoo [99], the finite element method was utilized to study a rotating cantilever beam. The decision was made to use stretch deformation instead of axial deformation, resulting in the creation of three linear partial differential equations, two of which were coupled through stretch and chordwise deformations. The deformation was assumed to be cubic polynomials. The use of 100 finite element beam elements, which was higher than other studies, yielded good results compared to the exact solution in their convergence analysis for bending modes. However, more elements were required to attain accurate results for the stretching

mode. In addition, the natural frequencies were analyzed, and it was discovered that stretch deformation led to differences between chordwise and flapwise motions, specifically the veering phenomena and the divergence instability.

Sahu [119] modified the differential equation for rotating Euler-Bernoulli beam bending vibrations to incorporate an exponentially tapered beam. Sahu [119] provided a solution considering fixed-free boundary conditions. The author provided the equation but didn't provide numerical solution. The governing equation of tapered cantilever rotating beam in [119], the answer was assumed as a linear combination of non-rotating uniform cantilever beam's shape functions

Nourifar et al.[97] expanded the work of Sahu [119] where the Euler-Bernoulli with a circular, exponentially tapered rotating beam. The equations of motion were solved using the differential transform method (DTM). Based on the author knowledge, this is the only paper published for rotating exponentially tapered beams. Dealing with a circular beam only made the geometrical properties easier since the taper is possible in a one direction and there is no difference between breadth and width. Such an exponential beam model will better understand more complex beams.

Kurt and Kaya [95] studied transverse (out-of-plane) vibrations of a rotating Euler-Bernoulli tapered nanobeam. The governing partial differential equation was solved using the differential transform method (DTM). Kurt and Kaya [95] considered rotational speed, hub radius, small-scale parameters, and taper ratios. The work of Kurt and Kaya [95] was built on Eringen [94], where Eringen developed the nonlocal theory, which includes small-scale effects. According to the findings of Kurt and Kaya [95] in their study on flapwise vibrations, a rise in the small-scale parameter leads to an

increase in the first dimensionless natural frequency, whereas the second dimensionless natural frequency drops.

Timoshenko Beam

Many researchers have investigated the influence of rotary inertia and shear deformation on the analysis of rotating beams. This consideration is because these two parameters help improve the accuracy of the solutions for rotating beams' vibration [20]. For example, Ansari [51] used Galerkin's and Perturbation methods to investigate the rotary inertia effect on rotating slender beams' eigenvalues and extensional tensile force. The results showed that the rotary inertia in analyzing rotating slender beams increased the extensional tensile force by 10%. Similarly, Wang and Guan [117] examined the impact of rotary inertia, shear deformation, and loading on rotating beams' flexural frequencies and found that the impact of rotary inertia could be considered negligible for rotating beams with low or high width-to-height ratios w/h . Additionally, it was shown that the effect of shear deformation was more significant compared to the influence of rotating inertia. Regarding low-order vibration modes, Ansari [51] reported that the influence of shear deformation could be considered insignificant for low w/h . However, for high-order vibration modes, the investigator noted that the influence of shear deformation was significant and should be considered in the analysis.

Du et al. [20] used a Newtonian method to derive the equations of motion for the vibration of a rotating beam. In their study, they expanded upon the Frobenius solution [16] by including additional factors such as the tip mass, rotational inertia, and shear deformation. In the case of a cantilever Euler-Bernoulli beam, it was observed

that the fundamental frequency exhibited an inverse relationship with the tip mass. This effect was in agreement with the findings of Bhat [17] and Hoa [21]. The frequency and tip mass were directly proportional for higher modes and speed parameters greater than 1. However, a contrasting pattern in the frequencies was seen for rotating speeds of 0 and 1. Naguleswaran [87] pointed out some inconsistencies in typographical errors in the work of Du et al. [20] and suggested that an accurate result can be obtained if the frequency equation is correctly computed. In contrast to Du et al.'s use of the bisection method, Naguleswaran [87] came to the conclusion that the iteration method was more effective and accurate. Wang et al. [117] used Galerkin's technique and shifted Legendre polynomials to predict the frequencies of a rotating uniform Timoshenko beam attached to a rigid hub. The investigator undertook to include the setting angle ψ in the Timoshenko beam equations. The simulation findings provided empirical support for the presence of stiffening resulting from rotational forces. It was discovered that at higher modes, there is a corresponding increase in both rotary inertia and shear deformation effects on the natural frequencies. According to the investigation by Wang et al. [117], a rise in the slenderness ratio caused the frequency ratio between Timoshenko and Euler-Bernoulli to converge towards unity.

The results of Du et al. [20] were consistent with the outcome indicating the merger between TBT and EBT. Wang et al. [117] also found that shear deformation has a bigger effect on a stationary beam than rotary inertia. However, for rotating beams characterized by higher mode frequencies, the relative effects may be reversed as a result of centrifugal stiffening effects. In their study, Wang et al. [117] demonstrated that the frequencies of lower modes in Timoshenko beams are mainly influenced by

the setting angle, in contrast to higher modes.

Furthermore, Yokoyama [2] employed the Lagrangian function and the finite element technique to obtain the vibration characteristics for the rotating beam discussed in Wang et al. [117]. The investigator presented the mass matrices and elemental stiffness expressions, including rotary inertia effects and shear deformation.

The work of Schilhansl [10] and Wright et al. [16] was extended by Lee and Kuo [22] to account for the rotary inertia and shear deformation impacts. They found that the estimate of the upper bound value contained corrections depending on the inverse of the slenderness ratio (r_{gy}/L) and the shear deformation parameter Φ . However, when Lee and Kuo [22] ignored these effects, they noticed that a cantilever beam's SC agreed with that provided by Schilhansl [10]. The elastic end restraint was found to have a significant impact on the fundamental bending mode. Lee and Kuo [22] concluded that at the root, the translational flexibility effect was more significant than the rotational flexibility effect.

Nagraj [88] extended the work of Peters [116] to Timoshenko beams. The investigator also obtained an expression that indicated the relationship between the maximum static deflection and the fundamental natural frequency for rotating Timoshenko beams. The results were in agreement with the outcome presented in the literature. During the late nineties, researchers attempted to include the taper effect. Bazoune and Khulief [40] pioneered the vibrational analysis of a rotating doubly tapered Timoshenko beam using a finite element analysis. They accounted for the centrifugal stiffening effects and utilized the consistent mass approach to arrive at the expressions for the stiffness matrices and the finite element mass [40]. Bazoune and

Khulief [40] also presented the simulation results generated for different rotational speeds, taper ratios, and vibration modes for both hinged-free and fixed-free boundary conditions. The investigators found that the natural frequencies had a linear relationship with the taper ratio. The critical taper ratio was reported after the trend reversal of a rotating tapered beam's frequencies. Bazoune and Khulief [40] concluded that the softening effect due to the decline in the cross-sectional area was dominated by the stiffening effect caused by rotation. Khulief and Bazoune [26] later considered the effect of setting angle and hub radius and accounted for different combinations of free-end, hinged, and fixed conditions, as an extension of their previous work. Their simulation results indicated in-plane and out-of-plane vibrations. Their prediction outcomes also showed that the in-plane frequencies were low [26]. Years later, the method of Yokoyama [2] was extended by Mulmule et al. [24] to include depth taper. The results of Mulmule et al. [24] for uniform Timoshenko beams agreed with those of Yokoyama [2] because the two groups of researchers used similar methods. Given that Mulmule et al. [24] used $\kappa = 0.83$ as the shear correction factor in their analysis, their results would be different from those presented by Yokoyama [2], who used $\kappa = 0.85$ [24]. Nonetheless, prediction results indicated evidence of a stiffening effect when the speed of rotation and/or the hub radius increased. The findings related to the variation of the flapping frequency and the two lead-lag parameters as a function of the spin, hub radius, and taper ratio parameters have been described using various graphical predictions. Mulmule et al. [24] concluded that the taper ratio was indirectly proportional to the second frequency parameter. However, this conclusion was inconsistent with the results presented, [92]. The predictions indicated that the

frequency parameters of the second mode for uniform beams were greater than those of tapered beams, provided that the hub radius was neglected [24]. Vindo Kumar and Ganguli [93] introduced violin string shape functions. Traditional finite element analysis uses polynomial form functions from the governing equation of a non-rotating Timoshenko beam.

$$\frac{\partial}{\partial x} \left(\kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) \right) = 0 \quad (1.7a)$$

$$\frac{\partial}{\partial x} \left(E I^i \frac{\partial \theta}{\partial x} \right) + \kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) = 0 \quad (1.7b)$$

Equation (1.7) is a special case of equations (1.2) and (1.3) where centrifugal and inertial terms are neglected. After decoupling equations (1.7) and solving for $w(x)$ and $\theta(x)$, this results in the typical shape functions given by

$$w(x) = a_3 + (a_0 - \frac{2EI}{GA\kappa}a_2)x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 \quad (1.8)$$

$$\theta(x) = a_0 + a_1x + a_2x^2 \quad (1.9)$$

Equations (1.8) and (1.9) are the typically used equations when solving rotating Timoshenko beams. Vinod Kumar and Ganguli [93] proposed to use equations (1.2) and (1.3) by eliminating the inertial term only, which yields

$$\frac{\partial}{\partial x} \left(\kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) \right) + \frac{\partial}{\partial x} \left(F_x \frac{\partial w}{\partial x} \right) = 0 \quad (1.10)$$

$$\frac{\partial}{\partial x} \left(E I^i \frac{\partial \theta}{\partial x} \right) + \kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) = 0 \quad (1.11)$$

Vinod Kumar and Ganguli [93] used equations (1.10) and (1.11) to create shape functions for Timoshenko beams. These functions take into account the centrifugal force F_x . Getting an analytical solution for F_x is difficult. A naive assumption of $F_x = \text{constant}$ is inappropriate throughout the whole beam. The authors suggested that F_x is constant in each element. Consider the spinning beam as a succession of violin strings for determining shape functions. Outboard elements have less stress than inboard elements. Vinod Kumar and Ganguli [93] used shape functions for uniformly rotating Timoshenko beams. They mention that the tapered beams were investigated, but there was no result shown for tapered beams. The fundamental mode converged faster with the presented novel shape functions than with conventional polynomial shape functions at low and high rotating speeds. The violin string shape functions at high speeds converges faster than the cubic shape functions for uniform beams with different slenderness ratios. The study of Vinod Kumar and Ganguli [93] did not address hub radius, setting angle, or the Coriolis effect. Vinod Kumar and Ganguli [93] did not compare model findings to other literature.

Sachin and Reddy [33] studied out-of-plane vibration (flap-wise motion) and in-plane vibration (chord-wise motion) of a rotating beam with different hub radius, slenderness ratio, and speed. The impact of the slenderness ratio is clear, as previous studies have shown. Natural frequencies rise with slenderness. The authors shed the

spotlight on a veering phenomenon that occurs between natural frequencies. Also, it was shown that the veering region shifted with changes in the slenderness ratio. Yardimoglua and Inman [35] simulated a fir-tree root on a finite element model with a rectangular beam with a pre-twist, including centrifugal force field and root flexibility effects. Yardimoglua and Inmanb [35] considered 7 degrees of freedom for each node and coupled bending-bending-torsion vibration. A minimal effect was observed between rigid and flexible roots. Yardimoglua and Inmanb [35] conducted a study on a beam with a rectangular cross-section. The beam also has a pre-twisted configuration and includes the effects of centrifugal force fields and root flexibility. This configuration was meant to simulate the features of a fir tree root. The researchers used a finite element model for their analysis. Yardimoglua and Inmanb [35] investigated a system with seven degrees of freedom per node, specifically focusing on the coupled bending-bending-torsion vibration. There was a little impact detected when comparing stiff and flexible roots. Later, Yardimoglu [101] used a semi-symbolic computer program to evaluate a finite element beam model. The result was compared with the commercial Finite Element Analysis program "ABAQUS". An interesting concept was introduced by Yardimoglu [101], where the author was trying to have a beam profile that has equal stress across the whole beam. The author used the axial load equation given by Bickford [110] as

$$\frac{d}{dz}[A(z)\sigma(z)] + q(z) = 0 \quad (1.12)$$

where the axial force due to centrifugal force due to constant angular velocity

acting on the beam element is $q(z) = \rho A(z)\Omega^2 z$. Solving equation (1.12) with respect to $A(z)$ at a constant stress along the beam $\sigma(z) = \sigma_0$ yields:

$$A(z) = A_0 e^{\left(-\frac{\rho\Omega^2}{2\sigma_0} z^2\right)} \quad (1.13)$$

From equation (1.13), the taper ratio equals $0.5\rho\Omega^2/\sigma_0$. To get the depth and breadth taper ratio, they are related by

$$\frac{\rho\Omega^2}{2\sigma_0} = \nu_z + \nu_y \quad (1.14)$$

Li and Zhang [34] studied a rotating axially FG tapered beam using the B-spline method. They neglected the damping on the beam. The study focused on the rotating beam's coupled chord-wise bending and stretching (B-S) vibration. It was shown that when the B-S coupling effect is included, there are significant differences when the angular velocity is increased, especially for the fundamental frequencies. When coupling is neglected at the 60 spin parameter, the fundamental frequency is 12.087, while when coupling is included, it drops to 8.1723. This shows a drop of 32.38% when the B-S coupling is included. Also, as Sachin and Reddy observed [33], there are veering regions when veering the spin parameter. Almuslmani and Ganesan [100] examined a tapered composite driveshaft using hierarchical finite element analysis. The model took into account the axis of rotation at the center of the beam, as it was a driveshaft. This resulted in the analysis being more focused on rotordynamic problems than structural vibrations.

The vast majority of published models on rotating beams deal with uniform or

linearly tapered formulations. To date, there has been a lack of substantial focus on exponentially tapered models applied to rotating beams in comparison to the attention given to rotating uniform and linearly tapered beams.

1.3 Motivation and Objectives

The review of the existing research indicates a lack of focus on two primary topics. The first topic under consideration is a rotating beam with an exponentially tapered profile. The majority of the existing literature primarily concentrates on two types of beams: uniform beams and linearly tapered beams. The second topic is the utilization of other shape functions, such as violin string, as an alternative to the often employed cubic (polynomial) shape functions. The cubic shape function is extensively employed in various applications, encompassing both rotating and non-rotating scenarios. The objective of this study is to conduct a modal analysis of a rotating Timoshenko beam with linear and exponential tapering using a discrete mathematical model and the Finite Element Method (FEM). This investigation is intended to include other real-life rotating blade features, such as angular settings and hub radius.

This thesis achieves both the formulation and the generalized eigenvalue problem solution. Throughout the process of analysis, the following are taken into consideration:

- Develop a finite beam element model capable of simulating double tapering, either in a linear or exponential manner.
- Both rotating inertia and shear deformation will be taken into consideration in

the beam element.

- The influence of centrifugal force on stiffening behavior will be taken into consideration.
- The beam setting angle and hub radius are also considered in the finite element formulation.
- Study the convergence of the violin string shape functions under different conditions.

1.4 Methodology

The following subjects will be addressed during the stages of the analysis:

Chapter 2 covers the crucial assumptions that serve as the foundation for the equations of motion derivations. The orientation of the beam is illustrated by the introduction of a collection of transformation matrices. A mathematical framework consisting of two sets of objects can be employed for modeling a multibody system. One set of objects consists of the rigid hub, whereas the other set comprises the flexible beam. The equations of motion are derived through the application of a Lagrangian technique involving the kinetic and potential energy of the system.

Chapter 3 presents a comprehensive exploration of the fundamental principles underlying the linear or exponential tapered beam element. Initially, the geometric properties of the beam are established. Afterwards, the beam element model undergoes a process of discretization, followed by the subsequent construction of shape

functions. Then, the elemental stiffness, inertia, and centrifugal inertia matrices are obtained and assessed symbolically, incorporating elements such as spin rate, shear deformation parameters, setting angle, and taper ratios. Ultimately, the equation of motion for the whole beam is derived utilizing the well-established standard approach of finite element assembly.

The simulation results are presented and analyzed in Chapter 4. The exposition commences with a comprehensive introduction to Mathematica code, encompassing many aspects such as system matrices, input data, numerical integration, and eigenvalue solutions. The current model's correctness and fidelity are demonstrated and validated through the reproduction of numerous degenerate examples and compared with those of earlier researchers. The subsequent section of the study involves the computation of the modal properties of the rotating beam and their subsequent comparison with existing findings in the academic literature. Tabular and graphical representations are utilized to display natural frequencies, encompassing both two-dimensional and three-dimensional plots. The charts shown in this analysis emphasize and examine the impacts of taper ratios, setting angle, and other important variables.

Chapter 5 of the document combines the final remarks and suggestions for potential avenues of research in the future.

CHAPTER 2

THEORY

During an analysis of dynamic systems, it is of utmost importance to formulate equations of motion. The process at hand involves the creation of a dynamic model, which is achieved in the present chapter through the utilization of the energy method and a Lagrangian function. These tools are employed to deduce the equations of motion for a spinning, tapered Timoshenko beam. The relationship between the motion of a reference point and the deformations of an elastic material is characterized by an assembly of elemental inertia matrices that remain constant across time. The matrices presented above are formulated using a collection of time-invariant variables, including scalars and matrices. These variables are influenced by several parameters, such as rotary inertia, the displacement field, tapering in the (xy) - and (xz) -planes, shear deformation, setting angle, and hub radius. This approach facilitates the straightforward incorporation of many effects, such as angle adjustment, tapering in the (xy) - and (xz) -planes, and centrifugal forces.

2.1 Model Description

Figure 2.1 depicts the rotating beam model studied in this thesis. The beam is attached to the outside edge of a rotating rigid hub with a radius of $\mathcal{R}_0$. The origin is on the hub's rotation axis in the global Cartesian coordinate system (XYZ) . In this system, the Z -axis aligns with the hub's rotating axis, which rotates with an angular velocity of $\Omega_Z = \dot{\phi}$. The locally specified (xyz) Cartesian coordinate system. The beam assembly is fixed to the root and aligned with the global (XYZ) coordinate system. From the global (XYZ) coordinate system, the origin is moved by $\mathcal{R}_0$ in the radial X -direction.

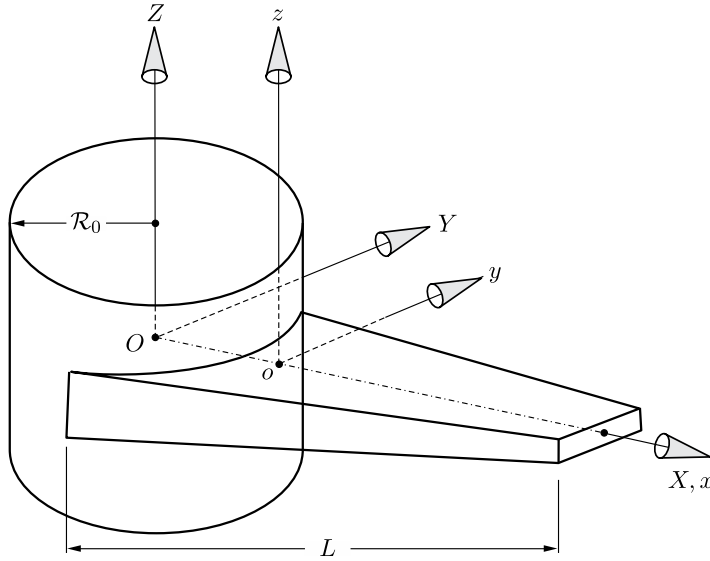


Figure 2.1: Tapered beam in undeformed state before the setting angle ψ is applied

The beam centerline initially aligns with the x -axis and X -axis. Obtaining the Cartesian coordinate system $(x'y'z')$ involves rotating the (xyz) coordinate system around the x -axis by an angle ψ , known as the setting angle. Figure 2.2 illustrates this rotation. It aligns the y' and z' axes with the cross-section's principal axes.

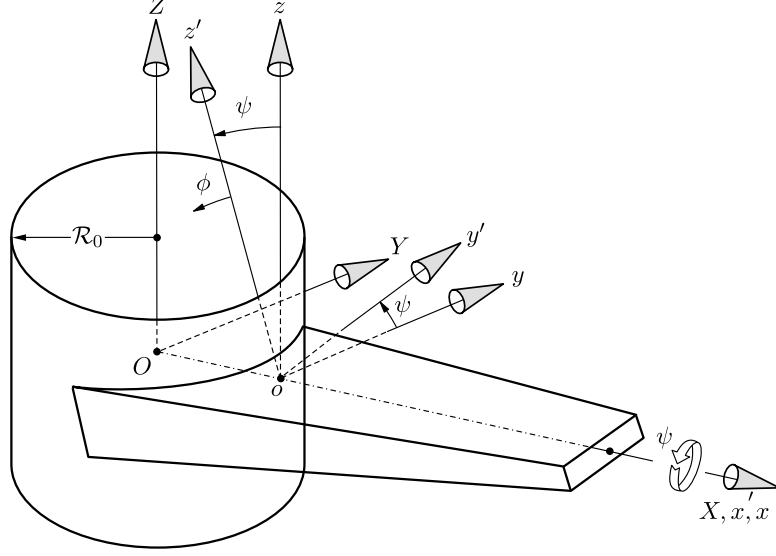


Figure 2.2: Tapered beam in undeformed state after the setting angle ψ is applied.

2.2 Assumptions

The present formulation is based on several assumptions, which include that

1. The deformations or displacements are small
2. The beam material is linearly elastic, isotropic, and homogeneous.
3. The neutral axis of the beam is initially perpendicular to the plane cross-sections, but following deformation, it is no longer perpendicular.
4. The displacement of points along its centerline is causing the deflection of the beam.
5. Damping, Coriolis, aerodynamic forces, warping of the beam's cross-section, and thermal stresses are neglected
6. It is assumed that the hub is rigid
7. The beam edges tangent to the hub are neglected.

2.3 Transformation Matrices

2.3.1 Transformation Matrix due to Setting Angle

A transformation allows angular modifications before elastic deformation to orient Beam's cross-section. To generate the $(x'y'z')$ coordinate system, rotate the (xyz) coordinate system by ψ around the x -axis, as shown in Figure 2.2. It includes a constant single-axis rotation around the x -axis and may be represented as

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [\mathcal{A}_\psi] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad (2.1)$$

The rotational transformation that rotates an object by an angle ψ around the x -axis is represented by the symbol $[\mathcal{A}_\psi]$. $[\mathcal{A}_\psi]$ is given by

$$[\mathcal{A}_\psi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \psi & \sin \psi \\ 0 & -\sin \psi & \cos \psi \end{bmatrix} \quad (2.2)$$

2.3.2 Transformation Matrix due to Rotation

By evaluating its relative position to the (xyz) coordinate system and the body-fixed coordinate system. The global position of an infinitesimal volume at P^i on the i^{th} beam element may be calculated. The transnational and rotational orientation of the body's coordinate system (xyz) relative to the global coordinate system (XYZ) are represented by $\mathcal{R}_0$ and φ , respectively. The rotational transformation matrix, $[\mathcal{A}]$, relates the body coordinate system (xyz) to the global coordinate system (XYZ) , as

indicated in the equation.

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \{\mathcal{R}_0\} + [\mathcal{A}] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \quad (2.3)$$

The $\{\mathcal{R}_0\} = [X_0 \ 0 \ 0]^T$ represents the position of the origin of the body coordinate system with respect to the global coordinate system. It should be noted that the quantity $[\mathcal{A}]$ exhibits implicit time dependence, with $\dot{\varphi} = \Omega_Z$ denoting the hub angular velocity. The following is a mathematical representation of the expression $[\mathcal{A}]$:

$$[\mathcal{A}] = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.4)$$

2.4 Generalized Co-ordinates

For identifying the location of a specific point within the model, it is crucial to build a collection of generalized coordinates capable of determining the overall position of any infinitesimally small volume on the object.

Two sets of generalized coordinates are needed to determine the infinitesimal volume position on the i^{th} element of the system. They are the rigid and elastic degrees of freedom described by the rotational motion of the reference frame and the elastic deformation of the beam, respectively. At nodal positions on the element, infinitesi-

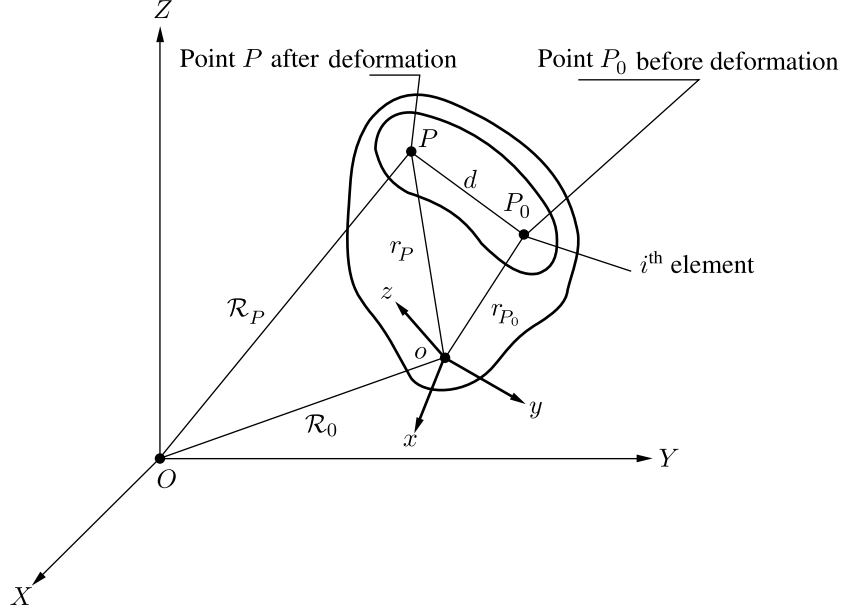


Figure 2.3: An arbitrary location of the i^{th} beam element prior to and following deformation.

mal volume rotations and translations represent elastic degrees of freedom. The Finite Element Method (FEM) estimates the location and orientation of a small volume on the i^{th} beam element.

2.5 The Shape Functions and displacement Field

In figure 2.3, the position of a point (denoted as P^i) on the i^{th} beam element can be determined globally using

$$\{\mathcal{R}_P^i\} = \{\mathcal{R}_0\} + [\mathcal{A}]\{r_P^i\} \quad (2.5)$$

The term $\{\mathcal{R}_0\}$ refers to the body coordinate system origin of in relation to the global coordinate system, specifically as previously defined by the vector $[X_0 \ 0 \ 0]^T$.

In the i^{th} beam element undeformed state, any point can be denoted as P_0^i and

is located at position vector $\{r_{P_0}^i\} = [x_0 \ 0 \ 0]^T$ in the body coordinate system (xyz) . After angular adjustments are made, this point becomes P_1^i and its position vector $\{r_{P_1}^i\}$ in the local coordinate system (xyz) is determined by

$$\{r_{P_1}^i\} = [\mathcal{A}_\psi] \{r_{P_0}^i\} \quad (2.6)$$

In equation 2.6, the symbol i as a superscript represents the i -th beam element, whereas $[\mathcal{A}_\psi]$ refers to a constant transformation matrix, as defined in equation 2.2. Let P^i denote the position of point P_1^i after the deformation. The mathematical formulation for the spatial coordinates of point P^i within the i^{th} beam element, denoted as $\{r_P^i\}$, is as follows.

$$\{r_P^i\} = \{r_{P_1}^i\} + \{d^i\} \quad (2.7)$$

In the deformed and undeformed states, vectors $\{r_P^i\}$ and $\{r_{P_1}^i\}$ represent the origin of the body coordinate system used to point P^i and P_1^i displacement. The finite element approach discretizes nodal points on the continuum to represent the deformation vector $\{d^i\}$ in finite-dimensional space. Elastic deformations at any point P^i can be expressed using nodal coordinates $\{e^i\}$ and shape functions $[\mathcal{N}^i]$, where $\{e^i\}$ represents the element's elastic coordinates in the (x, y, z) coordinate system [80].

$$\{d^i\} = [\mathcal{N}^i] \{e^i\} \quad (2.8)$$

where $[\mathcal{N}^i]$ is the shape function matrix. The nodal displacement vector $\{e^i\}$, has a

dimension of $(n_{dof} \times 1)$ where n_{dof} is elastic degrees of freedom total number of of the i^{th} element.

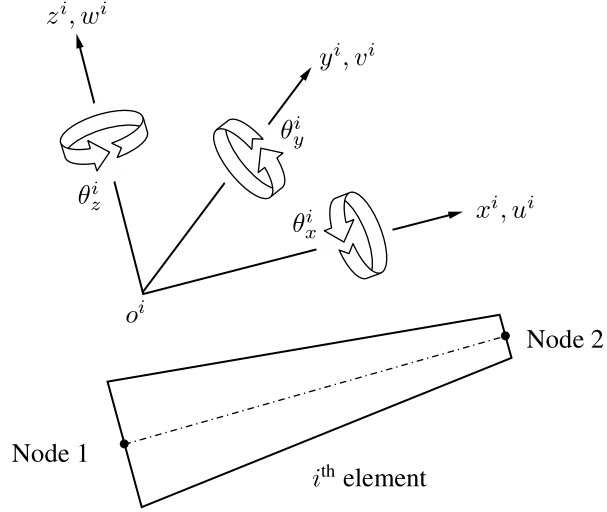


Figure 2.4: The generalized coordinates of the i^{th} beam element

The diagram presented in Figure 2.4 illustrates a standard two-node, two-dimensional beam element with a length denoted as l^i . In this configuration, each node is equipped with four degrees of freedom. The vector $\{e^i\}$ is given by.

$$\{e^i\}_{4 \times 1} = \begin{bmatrix} w_1^i & \theta_{y1}^i & w_2^i & \theta_{y2}^i \end{bmatrix}^T \quad (2.9)$$

where the nodal transnational displacements in the z -directions are (w_1^i, w_2^i) ; $(\theta_{y1}^i, \theta_{y2}^i)$ are the nodal rotational displacements in the (xz) .

The substitution of equations (2.7) and (2.8) into equation (2.5) results in

$$\{\mathcal{R}_P^i\} = \{\mathcal{R}_0\} + [\mathcal{A}] \left(\{r_{P_1}^i\} + [\mathcal{N}^i] \{e^i\} \right) \quad (2.10)$$

2.6 Strain Energy

Linear structural analysis uses a similar formula for rotating tapered Timoshenko beam strain energy. The local specification of elastic coordinates prevents stiffness coupling between elastic deformations and reference motion. The cross-section orientation, denoted by a setting angle ψ , creates a stiffness coupling in elastic coordinates.

The point P^i , positioned x^i from the leftmost end of the i^{th} deformed element, will be investigated. The displacements along the x , y , and z -axes are represented by U^i , V^i , and W^i . When the i^{th} element deforms, the cross-sectional area is translated in the x , y , and z directions, indicated as u^i , v^i , and w^i . There are also tiny rotations around the x , z , and y -axes, represented as θ_x^i , θ^i , and θ_z^i .

The strain energy of the i^{th} element can be expressed as follows:

$$\mathcal{U}^i = \frac{1}{2} \int_{V^i} \left(\sigma_{xx}^i \varepsilon_{xx}^i + \sigma_{yy}^i \varepsilon_{yy}^i + \sigma_{zz}^i \varepsilon_{zz}^i + \tau_{xy}^i \gamma_{xy}^i + \tau_{yz}^i \gamma_{yz}^i + \tau_{xz}^i \gamma_{xz}^i \right) dV^i \quad (2.11)$$

The related stress components and strain at point P^i of the i^{th} element of the infinitesimal volume V^i are denoted by ε^i , γ^i , σ^i , and τ^i . The generalized Hooke's law,

which applies to a homogeneous and isotropic material, is

$$\begin{aligned}
\sigma_{xx}^i &= \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)\varepsilon_{xx}^i + \nu(\varepsilon_{yy}^i + \varepsilon_{zz}^i) \right] \\
\sigma_{yy}^i &= \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)\varepsilon_{yy}^i + \nu(\varepsilon_{zz}^i + \varepsilon_{xx}^i) \right] \\
\sigma_{zz}^i &= \frac{E}{(1+\nu)(1-2\nu)} \left[(1-\nu)\varepsilon_{zz}^i + \nu(\varepsilon_{xx}^i + \varepsilon_{yy}^i) \right] \\
\tau_{xy}^i &= G\gamma_{xy}^i \\
\tau_{yz}^i &= G\gamma_{yz}^i \\
\tau_{xz}^i &= G\gamma_{xz}^i
\end{aligned} \tag{2.12}$$

Shear modulus of rigidity, Young's modulus of elasticity, and Poisson's ratio of the material are represented by variables G , E , and ν . In an isotropic material, these quantities relate.

$$E/G = 2(1 + \nu) \tag{2.13}$$

The substitution of equations (2.12) and (2.13) into equation (2.11) results in

$$\begin{aligned}
\mathcal{U}^i &= \frac{1}{2} \int_{V^i} \left(\frac{1}{E} \left[\sigma_{xx}^{i2} + \sigma_{yy}^{i2} + \sigma_{zz}^{i2} - 2\nu(\sigma_{xx}^i \sigma_{yy}^i + \sigma_{xx}^i \sigma_{zz}^i + \sigma_{yy}^i \sigma_{zz}^i) \right] \right. \\
&\quad \left. + \frac{1}{G} (\tau_{xy}^{i2} + \tau_{yz}^{i2} + \tau_{xz}^{i2}) \right) dV^i
\end{aligned} \tag{2.14}$$

Equation (2.14) can be written as

$$\mathcal{U}^i = \frac{1}{2} \{e^i\}^T [k_{ee}^i] \{e^i\} \tag{2.15}$$

The $[k_{ee}^i]$ represents the elastic stiffness matrix for each element, defined with respect to the body coordinate system.

The general form of Equation (2.15) is

$$u^i = \frac{1}{2} \begin{Bmatrix} \varphi \\ e^i \end{Bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & [k_{ee}^i] \end{bmatrix} \begin{Bmatrix} \varphi \\ e^i \end{Bmatrix} \quad (2.16)$$

The bending planes, or (xz) planes, are assumed to be aligned with the principal axes of the beam's cross-section. Furthermore, it is assumed that the x -axis and the beam's centroidal axis coincide.

2.6.1 Bending Strain Energy

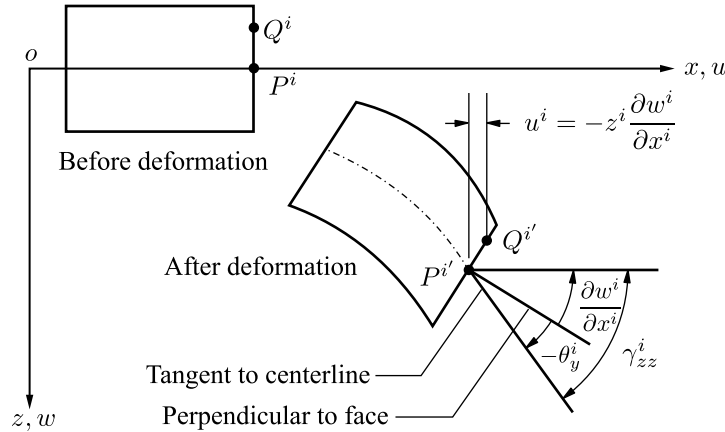


Figure 2.5: Planar deformation in the (xz) -plane

In cases when the dimensions of the beam's cross-section are not significantly smaller than its length, or when there is a need to analyze higher bending modes, it becomes imperative to account for the influences of rotating inertia and shear deformation. This may be achieved by using the Timoshenko Beam Theory (TBT). The figure shown in Figure 2.5 illustrates the bending deformation of the i^{th} element inside the (xz) -plane. The kinematic relations pertaining to the i^{th} element are stated in a manner consistent with the given context.

$$\begin{aligned}
U^i &= -z^i \left(\frac{\partial w^i}{\partial x^i} \right) \\
V^i &= 0
\end{aligned}
\tag{2.17}$$

$$W^i = w^i$$

The translations w^i may be decomposed into two components: w_b^i , which arises from bending, and w_s^i , which arises from shear deformation.

$$w^i = w_b^i + w_s^i \tag{2.18}$$

Differentiating of the above relation with respect to the element co-ordinate x^i result in

$$\frac{\partial w^i}{\partial x^i} = \frac{\partial w_b^i}{\partial x^i} + \frac{\partial w_s^i}{\partial x^i} = -\theta^i + \gamma_{xz}^i \tag{2.19}$$

where

$$\theta^i = -\frac{\partial w_b^i}{\partial x^i} \tag{2.20}$$

Additionally, the convention of utilizing the right-hand corkscrew rule is implemented.

The equation representing the strain component associated with bending deformation, excluding zero values, may be expressed as follows.

$$\varepsilon_{xx}^i = \frac{\partial U^i}{\partial x^i} = -z^i \frac{\partial^2 w_b^i}{\partial x^{i^2}} \tag{2.21}$$

The bending deformation strain energy is given by

$$\mathcal{U}_b^i = \frac{1}{2} \int_{V^i} \{\varepsilon_{xx}^i\}^T \{\sigma_{xx}^i\} dV^i \quad (2.22)$$

The previous equation can be rewritten using equation (2.12) as

$$\mathcal{U}_b^i = \frac{E}{2} \int_{V^i} \{\varepsilon_{xx}^i\}^T \{\varepsilon_{xx}^i\} dV^i \quad (2.23)$$

When equation (2.21) is substituted into (2.23),

$$\mathcal{U}_b^i = \frac{E}{2} \int_0^{l^i} \int_{A^i} \left(z^{i^2} \left(\frac{\partial^2 w_b^i}{\partial x^{i^2}} \right)^2 \right) dA^i dx^i \quad (2.24)$$

or

$$\mathcal{U}_b^i = \frac{E}{2} \int_0^{l^i} \int_{A^i} \left(z^{i^2} \left(\frac{\partial \theta^i}{\partial x^i} \right)^2 \right) dA^i dx^i \quad (2.25)$$

Noting

$$I_{yy}^i = \int z^{i^2} dA^i \equiv \text{Cross-section area moment of inertia about the } y\text{-axis} \quad (2.26)$$

the bending deformation strain energy expressed as

$$\mathcal{U}_b^i = \frac{E}{2} \int_0^{l^i} \left(I_{yy}^i \left(\frac{\partial \theta^i}{\partial x^i} \right)^2 \right) dx^i \quad (2.27)$$

where the generalized co-ordinates $\{e^i\}$ may be used to define $\frac{\partial \theta^i}{\partial x^i}$

$$\frac{\partial \theta^i}{\partial x^i} = \frac{\partial}{\partial x^i} ([\mathcal{N}_\theta^i] \{e^i\}) = [B_\theta^i] \{e^i\} \quad (2.28)$$

where the differential operator $[B_\theta^i]$, related to bending deformations, is provided by

$$[B_\theta^i] = \frac{\partial}{\partial x^i} ([\mathcal{N}_\theta^i]) \quad (2.29)$$

and $[\mathcal{N}_\theta^i]$ is the elemental shape function due to bending deformations in the (xz) -plane.

Equation (2.27) can be written as

$$\mathcal{U}_b^i = \frac{1}{2} \{e^i\}^T \left(\int_0^{l^i} \begin{pmatrix} [B_\theta^i]^T EI_{yy} [B_\theta^i] \end{pmatrix} dx^i \right) \{e^i\} \quad (2.30)$$

in compact form,

$$\mathcal{U}_b^i = \frac{1}{2} \{e^i\}^T ([k_b^i]) \{e^i\} \quad (2.31)$$

where in which $[k_b^i]$ represents the bending stiffness matrix of the element that is resulted from bending deformations in the (xz) -plane. The expression of $[k_b^i]$ is given by

$$[k_b^i] = \int_0^{l^i} [B_\theta^i]^T EI_{yy} [B_\theta^i] dx^i \quad (2.32)$$

Equation (2.31) in compact form

$$\mathcal{U}_b^i = \frac{1}{2} \{e^i\}^T [k_b^i] \{e^i\} \quad (2.33)$$

2.6.2 Strain Energy due to Shear Deformation

The (xz) -plane shear strain is given by

$$\gamma_{xz}^i = \frac{\partial w^i}{\partial x^i} - \theta^i \quad (2.34)$$

and the strain energy

$$\mathcal{U}_s^i = \frac{1}{2} \int_{V^i} (\tau_{xz}^i \gamma_{xz}^i) dV^i \quad (2.35)$$

The shear stress τ_{pq}^i is defined as the ratio of the shear force applied to a particular cross-sectional area. The dimensionless shear correction factors κ_z account for the nonuniform distribution of shear stress throughout the cross-section.

$$\tau_{xz}^i = \kappa_z G \gamma_{xz}^i \quad (2.36)$$

The shear correction factor κ_z is dependant upon the geometric characteristics of the cross-sectional shape. The literature has derivations of multiple values for the shear correction factor κ according to different cross-sectional forms. In their publication, Cowper [81] introduced a formula:

$$\kappa = \frac{6(1+\nu)}{7+6\nu}, \quad \text{for solid circular cross-section} \quad (2.37)$$

$$\kappa = \frac{10(1+\nu)}{(12+11\nu)}, \quad \text{for solid rectangular cross-section} \quad (2.38)$$

Stephen and Levinson [82] formulae

$$\kappa = \frac{6(1+\nu)^2}{(7+12\nu+4\nu^2)}, \quad \text{for solid circular cross-section} \quad (2.39)$$

$$\kappa = \frac{5(1+\nu)}{(6+5\nu)}, \quad \text{for solid rectangular cross-section} \quad (2.40)$$

The strain energy $\mathcal{U}_s^i$ can be rewritten in the form

$$\mathcal{U}_s^i = \frac{1}{2} \int G(\kappa_z \gamma_{xz}^i)^2 dV^i \quad (2.41)$$

which can be written as

$$\mathcal{U}_s^i = \frac{1}{2} \int_0^{l^i} \left(\left(\frac{\partial w^i}{\partial x^i} - \theta^i \right)^T \kappa_z G A^i \left(\frac{\partial w^i}{\partial x^i} - \theta^i \right) \right) dx^i \quad (2.42)$$

Let

$$\frac{\partial w^i}{\partial x^i} - \theta^i = \left(\frac{\partial}{\partial x^i} [\mathcal{N}_w^i] - [\mathcal{N}_\theta^i] \right) \{e^i\} \quad (2.43)$$

where $[B_w^i]$ is a differential operator related to the z -direction translational deformations provided by

$$[B_w^i] = \frac{\partial}{\partial x^i} [\mathcal{N}_w^i] \quad (2.44)$$

and $[\mathcal{N}_w^i]$ is the shape function due to translational deformation in the z -direction.

Equation (2.43) may be substituted into (2.41) to get

$$\mathcal{U}_s^i = \frac{1}{2} \{e^i\}^T \left(\int_0^{l^i} \left(([B_w^i] - [\mathcal{N}_\theta^i])^T \kappa_z G A^i ([B_w^i] - [\mathcal{N}_\theta^i]) \right) dx^i \right) \{e^i\} \quad (2.45)$$

alternatively this may be expressed compact form as

$$\mathcal{U}_s^i = \frac{1}{2} \{e^i\}^T \left(k_s^i \right) \{e^i\} \quad (2.46)$$

where

$$[k_s] = \int_0^{l^i} ([B_w^i] - [\mathcal{N}_\theta^i])^T \kappa_z G A^i ([B_w^i] - [\mathcal{N}_\theta^i]) dx^i \quad (2.47)$$

Individual elastic stiffness matrices' contributions are combined to create the elemental elastic stiffness matrix, which is a composite matrix.

$$[k_{ee}^i] = [k_b^i] + [k_s^i] \quad (2.48)$$

where

$[k_b^i] \equiv$ elemental bending stiffness matrix in the (xz) -plane

$[k_s^i] \equiv$ elemental shear stiffness, matrix in the (xz) -plane

2.7 The Model velocity Expression

Finding the velocity vector of the infinitesimal volume at position P^i inside an element is required for determining the i^{th} element kinetic energy. The velocity vector of point P^i can be obtained by differentiating the position vector $\{\mathcal{R}_P^i\}$ with respect to time.

$$\{\dot{\mathcal{R}}_P^i\} = \{\dot{\mathcal{R}}_0\} + [\dot{\mathcal{A}}] \left(\{r_{P_1}^i\} + [\mathcal{N}^i] \{e^i\} \right) + [\mathcal{A}] \left(\{\dot{r}_{P_1}^i\} + [\mathcal{N}^i] \{\dot{e}^i\} \right) \quad (2.49)$$

simplifying this equation to get

$$\{\dot{\mathcal{R}}_P^i\} = \{\dot{\mathcal{R}}_0\} + [\dot{\mathcal{A}}] \{r_{P_1}^i\} + [\dot{\mathcal{A}}][\mathcal{N}^i] \{e^i\} + [\mathcal{A}][\mathcal{N}^i] \{\dot{e}^i\} \quad (2.50)$$

provided that the magnitude of the vector $\{r_{P_1}^i\}$ remains constant, then its time derivative can be assumed to be zero. The direction of the vector is handled by the transformation $[\mathcal{A}]$. To proceed, apply the differential operator

$$\frac{d(\)}{dt} = \frac{\partial(\)}{\partial t} + \{\Omega\} \times (\) \quad (2.51)$$

for the vector $\{\mathcal{R}_0\}$, one can write

$$\frac{d\{\mathcal{R}_0\}}{dt} = \frac{\partial\{\mathcal{R}_0\}}{\partial t} + \{\Omega\} \times \{\mathcal{R}_0\} = \{\Omega\} \times \{\mathcal{R}_0\} = [\tilde{\Omega}]\{\mathcal{R}_0\} \quad (2.52)$$

In general, the tilde symbol placed atop a matrix $[\tilde{\Omega}]$ signifies that the components of a vector $\{\Omega\}$ are utilized to construct a skew symmetric matrix $[\tilde{\Omega}]$, as stated in the work by Nikravesh [108].

$$[\tilde{\Omega}] = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}, \quad \{\Omega\} = \begin{Bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{Bmatrix} \quad (2.53)$$

For the present case, the skew symmetric matrix $[\tilde{\Omega}]$ and vector $\{\Omega\}$ are, respectively, given by

$$[\tilde{\Omega}] = \begin{bmatrix} 0 & -\Omega_Z & 0 \\ \Omega_Z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \{\Omega\} = \begin{Bmatrix} 0 \\ 0 \\ \Omega_Z \end{Bmatrix} \quad (2.54)$$

Where $[\tilde{\Omega}]$

$$[\tilde{\Omega}] = \Omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\varphi}[\widetilde{\mathcal{A}}_I] \quad (2.55)$$

For the ease of the writ up, subscript z is dropped from Ω_Z . $[\widetilde{\mathcal{A}}_I]$, the skew symmetric matrix, is expressed by

$$[\widetilde{\mathcal{A}}_I] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (2.56)$$

Taking the derivative of the matrix $[\mathcal{A}]$ with respect to time yields

$$[\mathcal{A}] = \dot{\varphi}[\mathcal{A}_\varphi] \quad (2.57)$$

where $[\mathcal{A}_\varphi]$ is expressed by

$$[\mathcal{A}_\varphi] = \begin{bmatrix} -\sin \varphi & -\cos \varphi \\ \cos \varphi & -\sin \varphi \end{bmatrix} \quad (2.58)$$

Equation (2.50) may be obtained by substituting equations (2.6), (2.52), (2.55), and (2.57).

$$\{\dot{\mathcal{R}}_P^i\} = \begin{bmatrix} [\widetilde{\mathcal{A}}_I]\{\mathcal{R}_0\} + [\mathcal{A}_\varphi][\mathcal{A}_0]\{r_{P_0}^i\} + [\mathcal{A}_\varphi][\mathcal{N}^i]\{e^i\} & [\mathcal{A}][\mathcal{N}^i] \end{bmatrix} \begin{Bmatrix} \dot{\varphi} \\ \dot{e}^i \end{Bmatrix} \quad (2.59)$$

The compact form is

$$\{\dot{\mathcal{R}}_{P^i}^i\} = \left[\begin{bmatrix} [\mathcal{D}^i] & [\mathcal{A}][\mathcal{N}^i] \end{bmatrix} \right] \begin{Bmatrix} \dot{\varphi} \\ \dot{e}^i \end{Bmatrix} \quad (2.60)$$

where

$$[\mathcal{D}^i] = [\widetilde{\mathcal{A}}_I]\{\mathcal{R}_0\} + [\mathcal{A}_\varphi][\mathcal{A}_0]\{r_{P_0}^i\} + [\mathcal{A}_\varphi][\mathcal{N}^i]\{e^i\} \quad (2.61)$$

2.8 Kinetic Energy

In order to get the i^{th} element kinetic energy, the kinetic energy of the infinitesimal volume located at point P^i is integrated across the whole volume of the beam, denoted as V^i .

$$\mathcal{T}^i = \frac{1}{2} \int_{V^i} \rho \{\dot{\mathcal{R}}_P^i\}^T \{\dot{\mathcal{R}}_P^i\} dV^i \quad (2.62)$$

the mass density of the i^{th} element is denoted by ρ . $\mathcal{T}^i$ may be represented in vector-matrix format by using the velocity equation (2.60)

$$\mathcal{T} = \frac{1}{2} \begin{Bmatrix} \dot{\varphi} \\ \dot{e}^i \end{Bmatrix}^T \left(\int_{V^i} \rho \begin{bmatrix} [\mathcal{D}^i]^T [\mathcal{D}^i] & [\mathcal{D}^i]^T [\mathcal{A}] [\mathcal{N}^i] \\ [\mathcal{N}^i]^T [\mathcal{A}]^T [\mathcal{D}^i] & [\mathcal{N}^i]^T [\mathcal{N}^i] \end{bmatrix} dV^i \right) \begin{Bmatrix} \dot{\varphi} \\ \dot{e}^i \end{Bmatrix} \quad (2.63)$$

The lower right sub-matrix of equation (2.63) is simplified by using the orthonormality property of the matrix $[\mathcal{A}]$, especially $[\mathcal{A}]^T [\mathcal{A}] = [\mathbf{I}]$. A more direct expression of the previously described equation would be as follows:

$$\mathcal{T}^i = \frac{1}{2} \{\dot{q}^i\}^T [M^i] \{\dot{q}^i\} \quad (2.64)$$

The vector of generalized coordinates, represented as $\{q^i\}$, is formally defined as $\{q^i\} = [\varphi^T \{e^i\}^T]^T$. In this case, the elastic nodal displacement vector of the i th element is shown by the set $\{e^i\}$, while the reference coordinates are indicated by the symbol φ . The elemental mass matrix is represented by the matrix $[M^i]$, which may be expressed symbolically as follows.

$$[M^i] = \begin{bmatrix} [m_{rr}^i] & [m_{re}^i] \\ [m_{er}^i] & [m_{ee}^i] \end{bmatrix} \quad (2.65)$$

The transpose of $[m_{er}^i]$ is equivalent to $[m_{re}^i]$. The *reference* and *elastic* coordinates are denoted by the subscripts r and e , respectively. When comparing equations (2.63) and (2.65) to get the elements of the mass matrix $[M^i]$

$$[m_{rr}^i] = \int_{V^i} \rho [\mathcal{D}^i]^T [\mathcal{D}^i] dV^i \quad (2.66)$$

$$[m_{re}^i] = [m_{er}^i]^T = \int_{V^i} \rho [\mathcal{D}^i]^T [\mathcal{A}] [\mathcal{N}^i] dV^i \quad (2.67)$$

$$[m_{ee}^i] = \int_{V^i} \rho [\mathcal{N}^i]^T [\mathcal{N}^i] dV^i \quad (2.68)$$

The sub-matrices $[m_{rr}^i]$ and $[m_{re}^i]$ are implicitly time-dependent, unlike $[m_{ee}^i]$. This is because they are determined by the generalized coordinates, which leads to a model with varying inertia. By using equation (2.61) to expand equations (2.66) and (2.67), the following is obtained:

$$[m_{rr}^i] = \int_{V^i} \rho \left(\begin{aligned} &\{\mathcal{R}_0\}^T \{\mathcal{R}_0\} + 2\{\mathcal{R}_0\}^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{A}_\varphi] [\mathcal{N}^i] \{e^i\} \\ &+ 2\{\mathcal{R}_0\}^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{A}_\varphi] [\mathcal{A}_0] \{r_{P_0}^i\} + \{e^i\}^T [\mathcal{N}^i]^T [\mathcal{I}_0] [\mathcal{N}^i] \{e^i\} \\ &+ 2\{e^i\}^T [\mathcal{N}^i]^T [\mathcal{I}_0] [\mathcal{A}_0] \{r_{P_0}^i\} + \{r_{P_0}^i\}^T [\mathcal{A}_0]^T [\mathcal{I}_0] [\mathcal{A}_0] \{r_{P_0}^i\} \end{aligned} \right) dV^i \quad (2.69)$$

$$[m_{re}^i] = \int_{V^i} \rho \left(\begin{aligned} &\{\mathcal{R}_0\}^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{A}] [\mathcal{N}^i] + \{e^i\}^T [\mathcal{N}^i]^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{N}^i] \\ &+ \{r_{P_0}^i\}^T [\mathcal{A}_0]^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{N}^i] \end{aligned} \right) dV^i \quad (2.70)$$

where

$$[\mathcal{I}_0] = [\mathcal{A}_\varphi]^T [\mathcal{A}_\varphi] = [\widetilde{\mathcal{A}}_I]^T [\widetilde{\mathcal{A}}_I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (2.71)$$

and

$$[\widetilde{\mathcal{A}}_I]^T = [\mathcal{A}_\varphi]^T [\mathcal{A}] \quad (2.72)$$

The equations (2.56) and (2.58) provide the expressions for $[\widetilde{\mathcal{A}}_I]$ and $[\mathcal{A}_\varphi]$, respectively. $[m_{rr}^i]$ and $[m_{re}^i]$ can be expressed as

$$\begin{aligned}
[m_{rr}^i] &= [\mathbf{I}_{\mathcal{R}_0 \mathcal{R}_0}^i] + 2[m_{\mathcal{R}_0 e}^i] \{e^i\} + 2[I_{\mathcal{R}_0 r_{P_0}}^i] \\
&+ \{e^i\}^T [m_{\mathcal{R}_0}^i] \{e^i\} + 2\{e^i\}^T [m_{r_{P_0} e}^i] + [\mathbf{I}_{r_{P_0} r_{P_0}}^i]
\end{aligned} \tag{2.73}$$

and

$$[m_{re}^i] = [m_{\mathcal{R}_0}^i] + \{e^i\}^T [\widetilde{C}^i] + [m_{r_{P_0}}^i] \tag{2.74}$$

where

$$[\mathbf{I}_{\mathcal{R}_0 \mathcal{R}_0}^i] = \int_{V^i} \rho \{\mathcal{R}_0\}^T \{\mathcal{R}_0\} dV^i \tag{2.75}$$

$$[m_{\mathcal{R}_0 e}^i] = \int_{V^i} \rho \{\mathcal{R}_0\}^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{A}_\varphi] [\mathcal{N}^i] dV^i \tag{2.76}$$

$$[\mathbf{I}_{\mathcal{R}_0 r_{P_0}}^i] = \int_{V^i} \rho \{\mathcal{R}_0\}^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{A}_\varphi] \{r_{P_0}^i\} dV^i \tag{2.77}$$

$$[m_t^i] = \int_{V^i} \rho [\mathcal{N}^i]^T [\mathcal{I}_0] [\mathcal{N}^i] dV^i \tag{2.78}$$

$$[m_{r_{P_0} e}^i] = \int_{V^i} \rho [\mathcal{N}^i]^T [\mathcal{I}_0] [\mathcal{A}_0] \{r_{P_0}^i\} dV^i \tag{2.79}$$

$$[\mathbf{I}_{r_{P_0} r_{P_0}}^i] = \int_{V^i} \rho \{r_{P_0}^i\}^T [\mathcal{A}_0]^T [\mathcal{I}_0] [\mathcal{A}_0] \{r_{P_0}^i\} dV^i \tag{2.80}$$

$$[m_{\mathcal{R}_0}^i] = \int_{V^i} \rho \{\mathcal{R}_0\}^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{A}] [\mathcal{N}^i] dV^i \tag{2.81}$$

$$[\widetilde{C}^i] = \int_{V^i} \rho [\mathcal{N}^i]^T [\widetilde{\mathcal{A}}_I] [\mathcal{N}^i] dV^i \tag{2.82}$$

$$[m_{r_{P_0}}^i] = \int_{V^i} \rho \{r_{P_0}^i\}^T [\mathcal{A}_0]^T [\widetilde{\mathcal{A}}_I]^T [\mathcal{N}^i] dV^i \tag{2.83}$$

$$[m_{ee}^i] = \int_{V^i} \rho [\mathcal{N}^i]^T [\mathcal{N}^i] dV^i \tag{2.84}$$

The composite mass matrix is represented by the sub-matrix $[m_{ee}^i]$, which may be expressed as:

$$[m_{ee}^i] = [m_t^i] + [m_r^i] \tag{2.85}$$

defined in relation to the axes that coordinate the element. Observe that the components of $[m_{ee}^i]$ are defined as follows and are time invariant.

$$[m_t^i] \equiv \text{elemental transnational mass matrix in the } (xz)\text{-plane}$$

$$[m_r^i] \equiv \text{elemental rotary inertia mass matrix in the } (xz)\text{-plane}$$

To calculate the kinetic energy of a beam, we can use the expression for $\mathcal{T}^i$ which can be written as:

$$\mathcal{T}^i = \frac{1}{2} \dot{\varphi}^T [m_{rr}^i] \dot{\varphi} + \dot{\varphi}^T [m_{re}^i] \{\dot{e}^i\} + \frac{1}{2} \{\dot{e}^i\}^T [m_{cc}^i] \{\dot{e}^i\} \quad (2.86)$$

The kinetic energy of each beam element must then be added together to get the total kinetic energy $\mathcal{T}$ of the beam. This can be done by using the following equation:

$$\mathcal{T} = \sum_{j=1}^n \mathcal{T}^i = \frac{1}{2} \{\dot{q}\}^T [M] \{\dot{q}\} \quad (2.87)$$

In such a setting, the vector $\{q\}$ represents the generalized co-ordinates of the beam. It consists of the elastic nodal displacements vector $\{e\}$, which includes every component of the beam, and the reference co-ordinate φ . The mass matrix of the beam, $[M]$, may be written as this.

$$[M] = \begin{bmatrix} [M_{rr}] & [M_{re}] \\ [M_{er}] & [M_{ee}] \end{bmatrix} \quad (2.88)$$

where the mass $[M]$ elements are provided by

$$[M_{rr}] = \sum_{i=1}^n \left(\int_{V_i} \rho [\mathcal{D}^i]^T [\mathcal{D}^i] dV^i \right) \quad (2.89)$$

$$[M_{re}] = [M_{er}]^T = \sum_{i=1}^n \left(\int_{V_i} \rho [\mathcal{D}^i]^T [\mathcal{A}] [\mathcal{N}^i] dV^i \right) \quad (2.90)$$

$$[M_{ee}] = \sum_{i=1}^n \left(\int_{V_i} \rho [\mathcal{N}^i]^T [\mathcal{N}^i] dV^i \right) \quad (2.91)$$

2.9 Potential Energy

The i^{th} element's transverse deformation's potential energy may be represented as follows: it is only reliant on the element's lateral deflection w^i .

$$\mathcal{V}^i = \frac{1}{2} \int_0^{k^2} F_z^i w^i dx^3 \quad (2.92)$$

where w^i is the z -direction nodal displacement. The z -component of the force F_P^i is represented by the force F_z^i . The global reference Z -axis must be taken into consideration when evaluating the formula F_z^i . Figure 2.6 shows the equation for the radial component of centrifugal force per unit volume with the beam at a distance x^i .

$$F_r^i = \rho A^i \Omega^2 (\mathcal{R}_0 + r_P^i) \quad (2.93)$$

$w^i \cos(\psi)$ along the global Z -axis and $-w^i \sin(\psi)$ along the global Y -axis are the two components of the projection of w^i onto the z -axis.

The force F_r^i may be decomposed into two constituent components.

$$F_X^i \approx F_r^i \approx \rho A^i \Omega^2 (\mathcal{R}_0 + r_P^i) \quad (2.94)$$

$$F_Y^i = F_X^i \left[-w^i \sin(\psi) / (\mathcal{R}_0 + r_P^i) \right] = -w^i \rho A^i \Omega^2 \sin(\psi) \quad (2.95)$$

Additionally, the force denoted as F_Y^i may be decomposed into two constituent components along the elemental y -axis and z -axis.

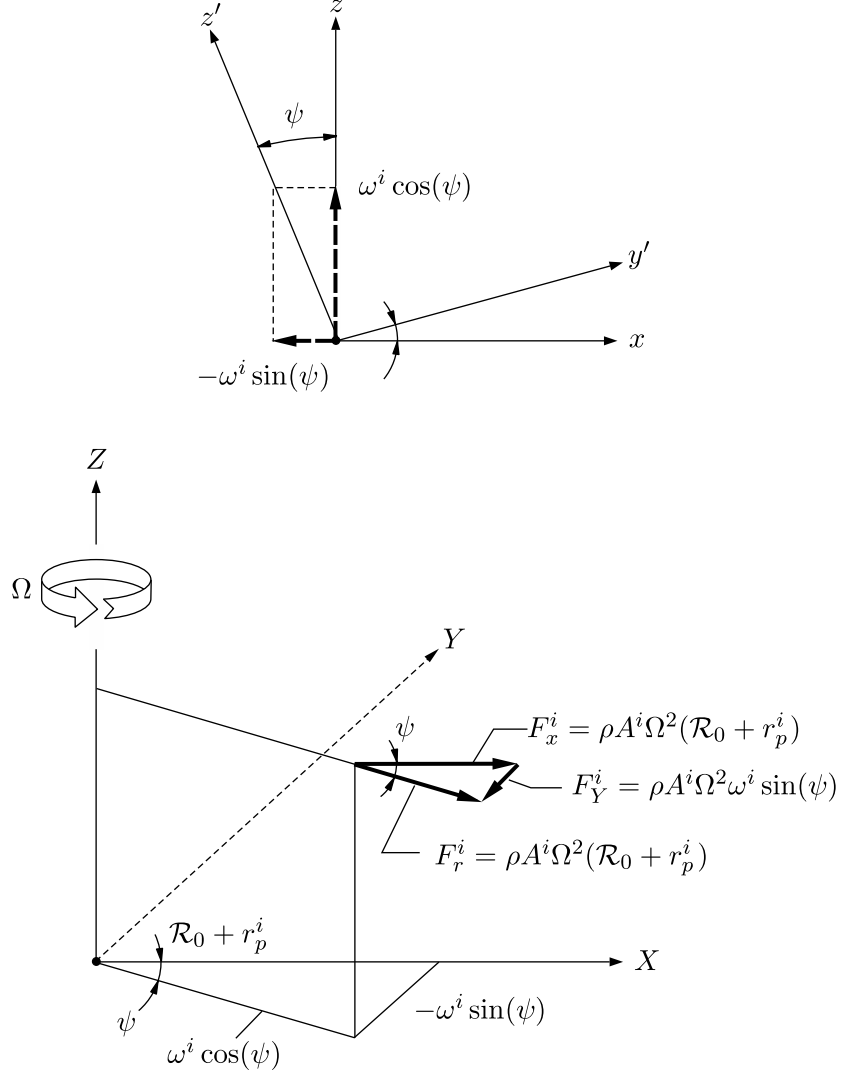


Figure 2.6: Force and displacements components.

The components of F_Y^i along the various axes may be determined as follows.

$$F_y^i = F_Y^i \cos \psi = -w^i \rho A^i \Omega^2 \sin(\psi) \cos(\psi), \quad (2.96)$$

$$F_z^i = -F_Y^i \sin \psi = w^i \rho A^i \Omega^2 \sin(\psi)^2 \quad (2.97)$$

Equation (2.97) may be substituted into (2.92) to get

$$\mathcal{V}^i = -\frac{1}{2}\{e^i\}^T \left(\int_0^{l^i} [\mathcal{N}^i]^T \rho A^i \Omega^2 \sin(\psi)^2 [\mathcal{N}^i] dx^i \right) \{e^i\} \quad (2.98)$$

or

$$\mathcal{V}^i = -\frac{1}{2}\{e^i\}^T \left([m_t^i] \Omega^2 \sin(\psi)^2 \right) \{e^i\} \quad (2.99)$$

The translational elemental mass matrix in the z -direction is $[m_t^i]$. The equation for $[m_t^i]$ is

$$[m_t^i] = \int_0^{l^i} \rho A^i [\mathcal{N}_w^i]^T [\mathcal{N}_w^i] dx^i \quad (2.100)$$

The shape functions required to mathematically characterize the elastic deformations of the beam along the z -direction are represented by the matrix $[\mathcal{N}_w^i]$. The equation (2.99) can be expressed as

$$\mathcal{V}^i = \frac{1}{2}\{e^i\}^i \left([m_t^i]^2 \sin(\psi)^2 \right) \{e^i\} \quad (2.101)$$

2.10 Rotation Strain Energy

A centrifugal force generated by the hub's rotation generates an axial force F_P^i in the beam. As seen in Figure 2.7, the computation of the shift in the horizontal projection of the i^{th} element of length ds^i when the beam bends in the (xz) -plane may be done by

$$ds^i - dx^i = \sqrt{(dx^i)^2 + \left(\frac{\partial w^i}{\partial x^i} \right)^2} dx^i - dx^i \approx \frac{1}{2} \left(\frac{\partial w^i}{\partial x^i} \right)^2 dx^i \quad (2.102)$$

It is feasible to ignore the higher order components in the element's change in axial direction if infinitesimal displacements are assumed. The strain energy increment resulting from the centrifugal force F_P^i acting on a point P^i situated inside the infinitesimal volume V^i , positioned at a distance x^i from the left end of the i^{th} element, may be mathematically represented as follows:

$$U_{c,w}^i = \frac{1}{2} \int_0^{l^i} \left(\frac{\partial w^i}{\partial x^i} \right)^T F_P^i \left(\frac{\partial w^i}{\partial x^i} \right) dx^i \quad (2.103)$$

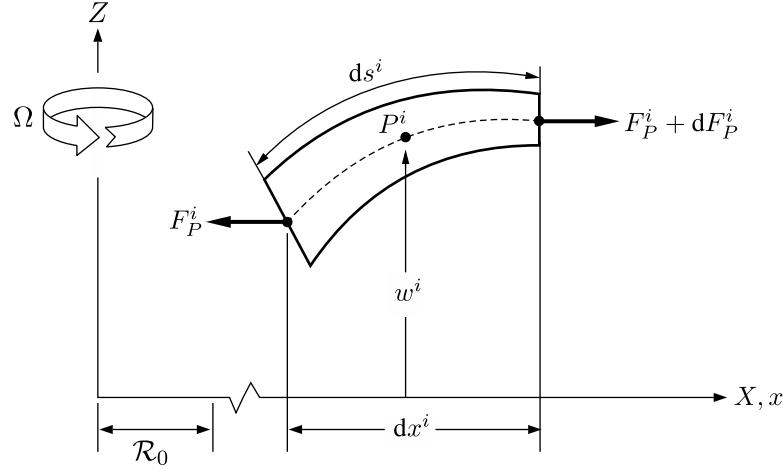


Figure 2.7: Deflection due to rotation in the (xz) -plane.

The following is the equation for the total strain energy of the rotating i^{th} element:

$$\mathcal{U}_c^i = \frac{1}{2} \int_0^{l^i} \left(\left(\frac{\partial w^i}{\partial x^i} \right)^T F_P^i \left(\frac{\partial w^i}{\partial x^i} \right) \right) dx^i \quad (2.104)$$

The expression for $\frac{\partial w^i}{\partial x^i}$ may be written in terms of, $\{e^i\}$, the elastic generalized coordinates, as

$$\frac{\partial w^i}{\partial x_j} = \frac{\partial}{\partial x^i} ([\mathcal{N}_w^i] \{e^i\}) = [B_w^i] \{e^i\} \quad (2.105)$$

Substitution of equation (2.105) into (2.104) gives

$$\mathcal{U}_c^i = \frac{1}{2} \{e^i\}^T \left(\int_0^{l^i} \left([B_v^i]^T F_P^i [B_v^i] + [B_w^i]^T F_P^i [B_w^i] \right) dx^i \right) \{e^i\} \quad (2.106)$$

in compact form

$$\mathcal{U}_c^i = \frac{1}{2} \{e^i\}^T \left([k_c^i] \right) \{e^i\} \quad (2.107)$$

in which

$[k_c^i] \equiv$ elemental centrifugal stiffness matrix in the xz -plane

The respective expression of $[k_c^i]$ are given by

$$[k_c^i] = \int_0^{l^i} [B_w^i]^T F_P^i [B_w^i] dx^i \quad (2.108)$$

The following chapter will provide the explicit formulation of F_P^i . Equation (2.107) can be written in more compact form as

$$\mathcal{U}_c^i = \frac{1}{2} \{e^i\}^T [k_c^i] \{e^i\} \quad (2.109)$$

The composite stiffness matrix $[k^i]$ is formed by combining the elemental stiffness matrix resulting from the centrifugal force field $[k_c^i]$ with the elemental elastic stiffness matrix $[k_{ee}^i]$.

$$[k^i] = [k_{ee}^i] + [k_c^i] \quad (2.110)$$

The elemental strain energy $\mathcal{U}^i$ is expressed as

$$\mathcal{U}^i = \mathcal{U}_e^i + \mathcal{U}_c^i \quad (2.111)$$

The variable $\mathcal{U}_e^i$ represents the elastic strain energy resulting from all elastic deformations, while $\mathcal{U}_c^i$ denotes the increase in strain energy caused by rotation. The expression for the strain energy $\mathcal{U}^i$ can be formulated as follows.

$$\mathcal{U}^i = \frac{1}{2} \begin{Bmatrix} \varphi \\ e^i \end{Bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & [k^i] \end{bmatrix} \begin{Bmatrix} \varphi \\ e^i \end{Bmatrix} \quad (2.112)$$

The total of the strain energy of the elements in the beam will provide the strain energy $\mathcal{U}$. The total strain energy may be expressed as a function of the number of elements, denoted by n .

$$\mathcal{U} = \sum_{j=1}^n \mathcal{U}^j = \frac{1}{2} \{q\}^T [K] \{q\} \quad (2.113)$$

where

$$[K] = [K_{ee}] + \Omega^2 [K_c] \quad (2.114)$$

The variables $[K_c]$ and $[K_{ee}]$ represent the constructed centrifugal stiffness matrix and the assembled elastic stiffness matrices, respectively.

2.11 Equations of Motion

The equations of motion at the elemental level are derived by the use of the Lagrangian method, which may be mathematically represented as follows.

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}^i}{\partial \{\dot{q}^i\}} \right) - \frac{\partial \mathcal{L}^i}{\partial \{q^i\}} = 0 \quad (2.115)$$

The Lagrangian function, denoted as $\mathcal{L}^i$, is defined in terms of the generalized coordinate vector $\{q^i\}$ and the corresponding vector of generalized forces $\{Q^i\}$, as introduced before. An expression for the elemental Lagrangian function is as follows.

$$\mathcal{L}^i = \mathcal{T}^i - \mathcal{U}^i - \mathcal{V} \quad (2.116)$$

The previously neglected elemental potential energy is represented by the variable $\mathcal{V}^i$. Substituting the formulas for kinetic (2.86) and strain (2.112) energies into the Lagrangian function, one can get.

$$\mathcal{L}^i = \frac{1}{2}\dot{\varphi}^T[m_{rr}^i]\dot{\varphi} + \dot{\varphi}^T[m_{re}^i]\{\dot{e}^i\} + \frac{1}{2}\{\dot{e}^i\}^T[m_{ee}^i]\{\dot{e}^i\} - \frac{1}{2}\{e^i\}^T[k^i]\{e^i\} \quad (2.117)$$

2.11.1 Equations of Motion Associated with Reference Motion

The equation of motion is given by applying the Lagrangian to the generalized coordinate φ (reference motion).

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}^i}{\partial \dot{\varphi}} \right) - \frac{\partial \mathcal{L}^i}{\partial \varphi} = 0 \quad (2.118)$$

By conducting the requisite algebraic procedures and differentiating with regard to time and generalized coordinates φ , one can derive the following expression.

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}^i}{\partial \dot{\varphi}} \right) &= [m_{rr}^i]\ddot{\varphi} + 2 \left([m_{\mathcal{R}_0e}^i]\{\dot{e}^i\} + \{\dot{e}^i\}^T[m_t^i]\{e^i\} + \{\dot{e}^i\}^T[m_{r_{P_0e}}^i] \right) \dot{\varphi} \\ &\quad + \left([m_{\mathcal{R}_0}^i] + \{e^i\}^T[\widetilde{C}_i^i] + [m_{r_{P_0}}] \right) \{\ddot{e}^i\} + \{\dot{e}^i\}^T[\widetilde{C}^i]\{\dot{e}^i\} \end{aligned} \quad (2.119)$$

and

$$\frac{\partial \mathcal{L}^i}{\partial \varphi} = \dot{\varphi}^T \left([\mathbf{I}_{\mathcal{R}_0}^{i*}] \{e^i\} + [\mathbf{I}_{\mathcal{R}_0 r_{P_0}}^{i*}] \right) \dot{\varphi} + [m_{\mathcal{R}_0 e}^i] \{\dot{e}^i\} \dot{\varphi} \quad (2.120)$$

where differentiation with respect to φ is indicated by $*$ inside a matrix. The expressions of $[m_{\mathcal{R}_0 e}^{i*}]$ and $[\mathbf{I}_{\mathcal{R}_0 r_{P_0}}^{i*}]$ are given by

$$[m_{\mathcal{R}_0 e}^{i*}] = \int_{V^i} \rho \{\mathcal{R}_0\}^T [\mathcal{A}_\varphi] [\mathcal{N}^i] dV^i \quad (2.121)$$

$$[\mathbf{I}_{\mathcal{R}_0 r_{P_0}}^{i*}] = \int_{V^i} \rho \{\mathcal{R}_0\}^T [\mathcal{A}_\varphi] [\mathcal{A}_0] \{r_{P_0}^i\} dV^i \quad (2.122)$$

Substituting equations (2.119) and (2.120) into (2.118) yields the hub-beam equation of motion for the generalized co-ordinate φ .

$$[m_{rr}^i] \ddot{\varphi} + [m_{re}^i] \{\ddot{e}^i\} = \{Q_{ext}^i\}_\varphi^2 + \{f_\varphi^i\} \quad (2.123)$$

Vectors $\{Q_{ext}^i\}$ and $\{f_\varphi^i\}$ represent external forces and quadratic velocity terms, including centripetal and Coriolis forces. This is done by differentiating the Lagrangian function with regard to time and the generalized coordinate φ . The source of vector $\{f_\varphi\}$ is

$$\begin{aligned} \{f_\varphi^i\} = & - \left([m_{\mathcal{R}_0 e}^i] \{\dot{e}^i\} + 2 \{\dot{e}^i\}^T [m_t^i] \{e^i\} + 2 \{\dot{e}^i\}^T [m_{r_{\mathcal{R}_0 e}}^i] \right) \Omega \\ & + \left([m_{\mathcal{R}_0 e}^{i*}] \{e^i\} + [\mathbf{I}_{\mathcal{R}_0 r_{P_0}}^{i*}] \right) \Omega^2 - \{e\}^T [\widetilde{C}^i] \{\dot{e}^i\} \end{aligned} \quad (2.124)$$

2.11.2 Equations of Motion Associated with Elastic Co-ordinates

The rotating Timoshenko beam equations of motion with generalized co-ordinates $\{e^i\}$ and rotation at i^{th} are as follows:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}^i}{\partial \{\dot{e}^i\}} \right) - \frac{\partial \mathcal{L}^i}{\partial \{e^i\}} = 0 \quad (2.125)$$

By conducting the requisite algebraic procedures and differentiating with regard to time and generalized coordinates $\{e^i\}$, one can get the following expression

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}^i}{\partial \{\dot{e}^i\}} \right) = [m_{re}^i]^T \ddot{\varphi} + [m_{ee}^i] \{\ddot{e}^i\} \quad (2.126)$$

$$\frac{\partial \mathcal{L}^i}{\partial \{e^i\}} = \left([m_{\mathcal{R}_0e}^i]^T + [m_t^i]^T \{e^i\} + [m_{r_{P_0}}^i] \right) \Omega^2 - [k^i] \{e^i\} \quad (2.127)$$

Equations (2.126) and (2.127) are substituted into (2.125) to get

$$[m_{re}^i]^T \ddot{\varphi} + [m_{ee}^i] \{\ddot{e}^i\} + [k^i] \{e^i\} = 0 \quad (2.128)$$

In vector matrix notation, equations (2.123) and (2.128) may be concatenated into one equation.

$$\begin{bmatrix} [m_{ee}^i] & [m_{re}^i] \\ [m_{re}^i] & [m_{ee}^i] \end{bmatrix} \begin{Bmatrix} \ddot{\varphi} \\ \{\ddot{e}^i\} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & [k^i] \end{bmatrix} \begin{Bmatrix} \varphi \\ \{e^i\} \end{Bmatrix} = \begin{Bmatrix} \{Q_{ext}^i\}_\varphi + f_\varphi^i \\ \{Q_{ext}^i\}_e + f_e^i \end{Bmatrix} \quad (2.129)$$

This model of multibody elastodynamics gives us equations (2.129) for the differential equations that govern a stiff hub-elastic beam element. Under consideration are the generalized coordinates φ and $\{e^i\}$. $\ddot{\varphi}$ and $\ddot{e}^i$, respectively, represent

the accelerations of the reference and elastic beam motions.

The sub-matrix $[m_{re}^i]$ shows how the reference motion and elastic deformations are coupled in terms of inertia. The sub-matrices $[m_{rr}^i]$ and $[m_{ee}^i]$ show the inertia of the reference motion and the elastic generalized co-ordinate mass matrix, respectively.

Using equation (2.110), we get the sub-matrix $[k^i]$, which shows how the beam element's stiffness has changed due to elastic deformation and taking rotational effects into account.

2.11.3 Equations of Motion for Constant Hub Speed

At zero angular acceleration, $\ddot{\varphi}$, the hub's angular velocity remains constant. The elastic coordinates $\{e^i\}$ may be used to express the equations of motion in this situation.

$$[m_{ee}^i]\{\ddot{e}^i\} + ([k^i] - \Omega^2[m_t^i])\{e^i\} = 0 \quad (2.130)$$

The centrifugal inertia force is obtained by differentiating the Lagrangian with regard to time and the generalized coordinate $\{e^i\}$.

The Lagrangian function for the beam may be established by utilizing equations (2.87) and (2.113) to sum the energies of the individual elements throughout the whole beam. The governing differential equation of the whole beam is

$$[M_{ee}]\{\ddot{e}\} + ([K_{ee}] + [K_c] - \Omega^2[M_t])\{e\} = 0 \quad (2.131)$$

The location of the beam is on a plane that makes an angle ψ with regard to the

cross-section's overall orientation. The equations of motion in this specific case need modifications to account for the setting angle's impact. In order to account for the rotating velocity and the setting angle, the beam's potential energy must be included.

2.12 Governing Equations of Motion

Using differentiation and algebraic manipulation, equation (2.116) can be used to write the Lagrangian function at the element level in a compact form. This is done by combining the kinetic energy, potential energy, and strain energy expressions from equations (2.87), (2.112), and (2.101) into Lagrange's equation. The standard finite element assembly procedure may derive the equation of motion for the whole elastic beam at constant hub speed based on the beam's node numbering.

$$[M_{ee}]\{\ddot{e}\} + ([K_{ee}] + [K_c] - \Omega^2[M_t] \sin(\psi)^2)\{e\} = 0 \quad (2.132)$$

Equation (2.132) is the matrix form of the differential equations governing the spinning tapered Timoshenko beam. An equation may be shortened as

$$[M_{ee}]\{\ddot{e}\} + [K^*]\{e\} = 0 \quad (2.133)$$

where

$$[K^*] = [K_{ee}] + [K_c] - \Omega^2[M_t] \sin(\psi)^2 \quad (2.134)$$

CHAPTER 3

FINITE ELEMENT

FORMULATION

In dynamic analysis, creating an accurate mathematical model is crucial for getting reliable simulation results. Because of this, large and complicated structures are often modeled using the Finite Element Method (FEM). This approach is well-known for its adaptability and ability to provide precise outcomes.

The beam model is discretized into of finite elements, with the interconnected nodes' translations and rotations representing the model's degrees of freedom. The beam elements can be linearly or exponentially tapered in two planes. Each individual element possesses a distinct length, and the overall beam length is determined by summation of total number of these element lengths. The hub's outside perimeter is fixedly attached to the base of the beam. The model additionally takes into account the hub's spinning, shear deformation, hub radius, setting angle, tapering, and (xy) - and (xz) -plane spinning.

To determine the elemental mass and stiffness matrices of the rotating beam, it is essential to take into account the geometric characteristics of the model. The task at hand entails the formulation of precise mathematical representations for the changeable beam cross-sectional area. The formulas should be formulated with the i^{th} beam element arbitrary elemental coordinates x^i , y^i , and z^i , along with additional parameters that characterize the geometry of the beam.

3.1 Model Geometric Properties

One of the objectives of this thesis is to use both linear and exponential beam profiles. The linear tapered beam is illustrated on 3.1, and the exponential tapered beam is illustrated on 3.2. In order to achieve both cases, the following geometrical properties are presented for both profiles:

$$L = \sum_{i=1}^n l^i \quad (3.1)$$

$$\nu_y = 1 - b_L/b_0 \quad (3.2)$$

$$\nu_z = 1 - h_L/h_0 \quad (3.3)$$

In the equation, i represents the element, l^i its length, n the total number of elements, and L the overall beam length. In the (xy) and (xz) planes, the taper ratios for beam depth and breadth are given by ν_y and ν_z .

The geometrical relations can be established at any position $P^i(x^i, y^i, z^i)$ of the i^{th} beam element.

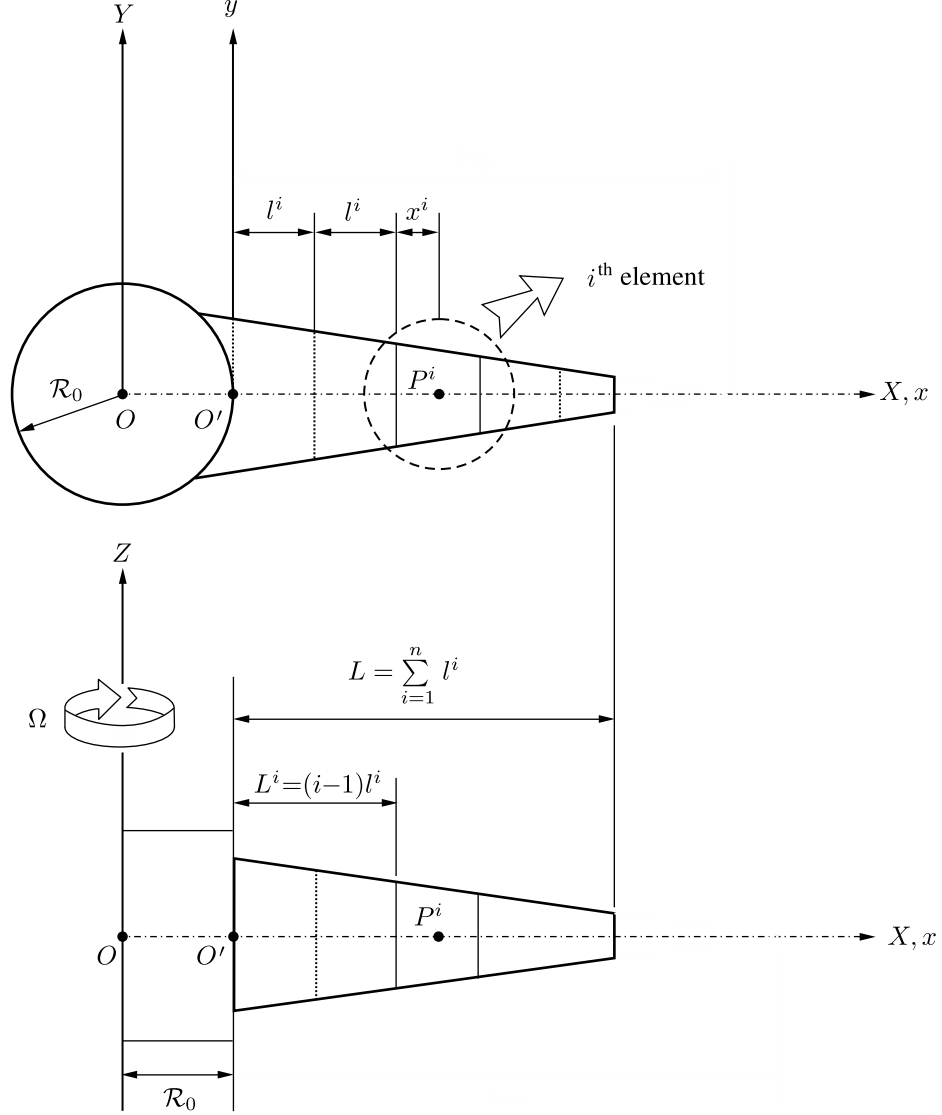


Figure 3.1: Linearly tapered rotating beam in two planes.

- For liner profile beam:

$$b^i = b_o \left(1 - \frac{\nu_y \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right) \quad (3.4)$$

$$h^i = h_o \left(1 - \frac{\nu_z \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right) \quad (3.5)$$

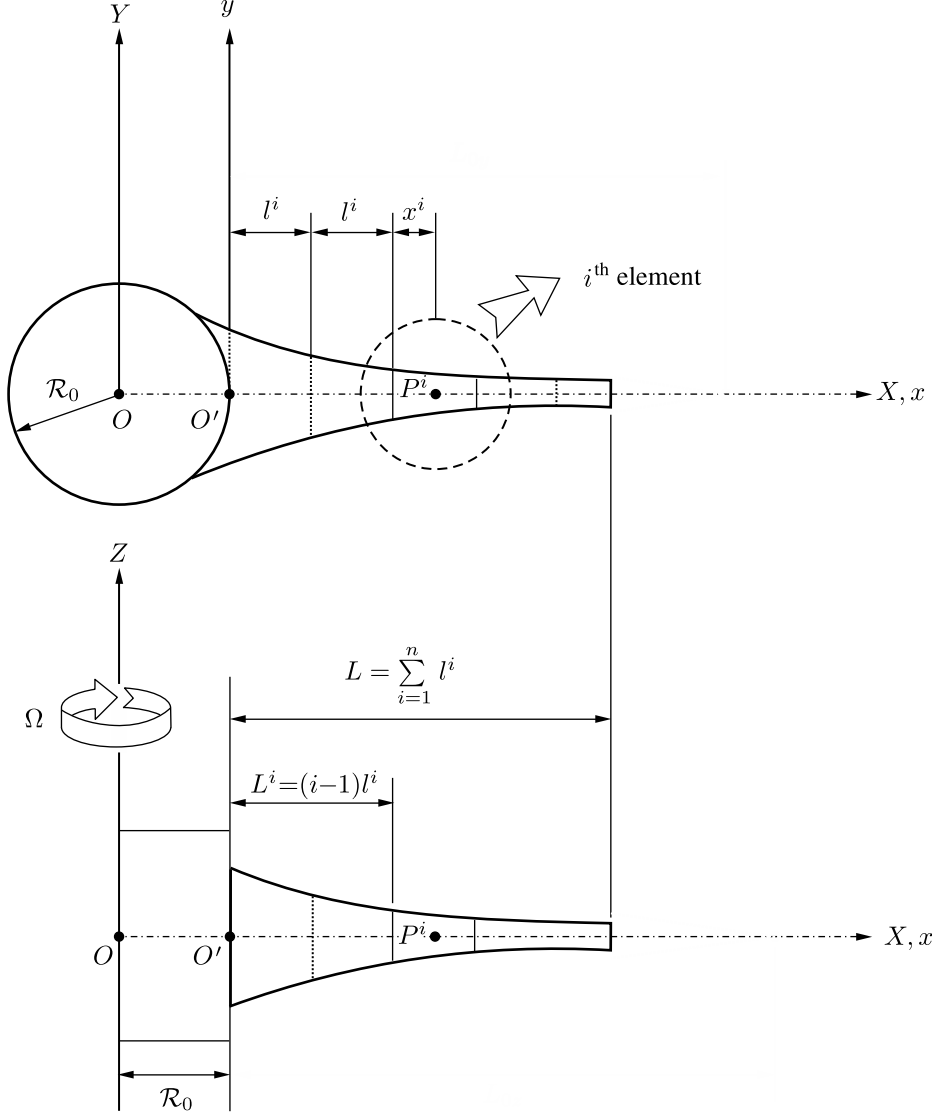


Figure 3.2: Exponentially tapered rotating beam in two planes.

- For exponential profile beam:

$$b^i = b_o e^{\frac{\nu_y((1-i)L - nx^i)}{Ln}} \quad (3.6)$$

$$h^i = h_o e^{\frac{\nu_z((1-i)L - nx^i)}{Ln}} \quad (3.7)$$

The variables b_o and h_o represent the beam's root cross-section width and depth, respectively. Similarly, at any given position x^i , the width and depth of the i^{th} beam element are denoted by b^i and h^i , respectively. For each point x^i , the cross-sectional

area of the i^{th} beam element is expressed as

$$A^i = b^i h^i \quad (3.8)$$

- For liner profile beam:

Equation (3.4) and (3.5) are substituted into (3.8) to obtain the variable cross-section area.

$$A^i = A_0 \left(1 - \frac{\nu_y \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right) \left(1 - \frac{\nu_z \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right) \quad (3.9)$$

- For exponential profile beam:

The variable cross-section area is obtained when equations (3.6) and (3.7) are substituted into (3.8).

$$A^i = A_0 e^{\left(\frac{\nu_y((1-i)L - nx^i)}{Ln} + \frac{\nu_z((1-i)L - nx^i)}{Ln} \right)} \quad (3.10)$$

where A_0 is the cross-sectional area of the beam's root. From equations (3.9) and (3.10) it clear that the area can be reduced either linearly or exponentially. Figures 3.3 can illustrate the shape of the beam if unity A_0 and L is assumed. Additionally, the results of various taper ratios are shown.

A beam element's cross-section area moment of inertia around its principal axis is expressed as

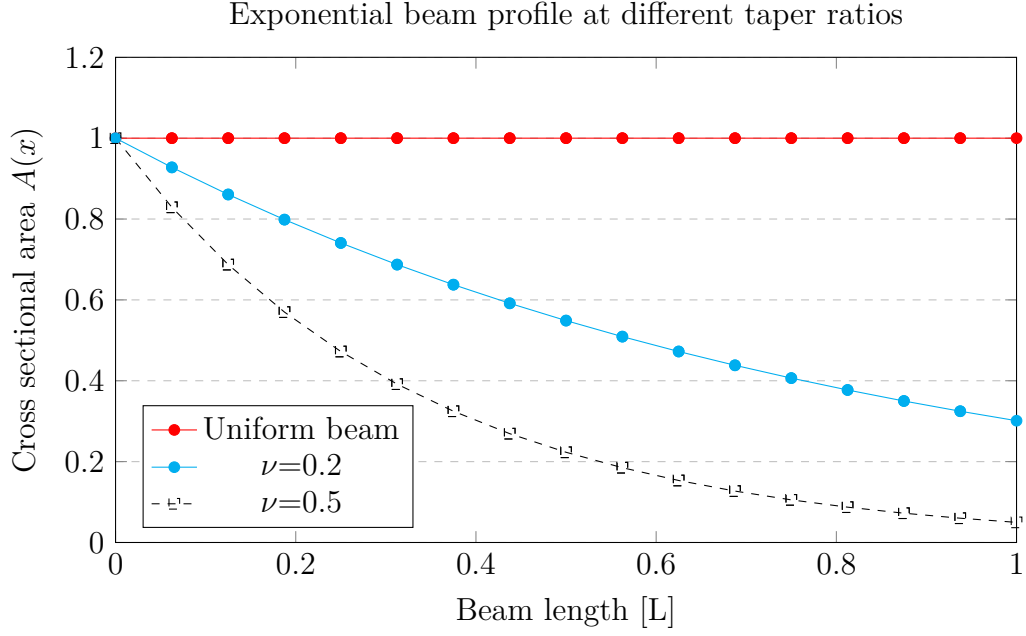


Figure 3.3: Cross-section area change in exponential beam profile

$$I_{yy}^i = \frac{1}{12} b^i h^{i3} \quad (3.11)$$

$$I_{zz}^i = \frac{1}{12} h^i b^{i3} \quad (3.12)$$

- For liner profile beam:

For the linearly tapered beam, equation (3.4) and (3.5) are substituted into (3.11) and (3.12) to obtain cross-section area moment of inertia.

$$I_{yy}^i = I_{oyy} \left(1 - \frac{\nu_y \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right) \left(1 - \frac{\nu_z \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right)^3 \quad (3.13)$$

$$I_{zz}^i = I_{ozz} \left(1 - \frac{\nu_y \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right)^3 \left(1 - \frac{\nu_z \left(\frac{(i-1)L}{n} + x^i \right)}{L} \right) \quad (3.14)$$

- For exponential profile beam:

For the exponentially tapered beam, equation (3.6) and (3.7) are substituted into (3.11) and (3.12) to obtain cross-section area moment of inertia.

$$I_{yy}^i = I_{o_{yy}} e^{\left(\frac{\nu_y((1-i)L-nx^i)}{Ln} + 3 \frac{\nu_z((1-i)L-nx^i)}{Ln} \right)} \quad (3.15)$$

$$I_{zz}^i = I_{o_{zz}} e^{\left(3 \frac{\nu_y((1-i)L-nx^i)}{Ln} + \frac{\nu_z((1-i)L-nx^i)}{Ln} \right)} \quad (3.16)$$

where the cross-section area moment of inertia at the beam's root around the principal axis is denoted by $I_{o_{yy}}$.

3.2 Transformation of Coordinate

Figure 3.4 illustrates how to rotate the beam's cross-sectional area around its centroidal axis in the (yz) -plane at a given angle ψ to generate a coordinate transformation that has the following relationship.

$$\begin{Bmatrix} y^i \\ z^i \end{Bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{bmatrix} \begin{Bmatrix} y' \\ z' \end{Bmatrix} \quad (3.17)$$

Consequently,

$$I_{yy}^i = I_{y'y'}^i \cos^2 \psi + I_{z'z'}^i \sin^2 \psi - I_{y'z'}^i \sin 2\psi \quad (3.18)$$

$$I_{zz}^i = I_{y'y'}^i \sin^2 \psi + I_{z'z'}^i \cos^2 \psi + I_{y'z'}^i \sin 2\psi \quad (3.19)$$

3.3 The Model of Beam Element

A beam element with two nodes and two degrees of freedom—nodal rotation θ^i and nodal translation w^i —is shown in Figure 3.4. The nodal displacement vector, denoted by $\{e^i\}$, is measured with respect to the element axes.

$$\{e^i\}_{4 \times 1} = \begin{bmatrix} w_1^i & \theta_1^i & w_2^i & \theta_2^i \end{bmatrix}^T \quad (3.20)$$

The subscripts 1 and 2 are used to indicate the node number. In the aforementioned equation, the variables (w_1^i, w_2^i) represent the translational displacements of the nodes in the vertical direction (z -direction). Similarly, the variables (θ_1^i, θ_2^i) represent the rotational displacements of the nodes in the plane formed by the x and z axes.

In accordance with the conventional finite element methodology, the elastic deformations of a specific point P^i belonging to the i^{th} beam element can be mathematically represented as follows:

$$\{d^i\}_{4 \times 1} = [\mathcal{N}^i]_{1 \times 4} \{e^i\}_{4 \times 1} \quad (3.21)$$

where $\{d^i\}$ stands for the i^{th} beam element elastic deformation vector, while $[\mathcal{N}^i]$ represents a shape functions matrix. Equation (3.21), applies to all shape functions $[\mathcal{N}^i]$ used for modeling structural elements. In this thesis, two sets of translation and rotational bending deformations shape functions are going to be used which are:

- Conventional cubic Hermitian polynomials. This is going to be referred as later as cubic shape function or polynomial.

- The violin string shape function or exponential shape function.

Both of these shape functions have a shear deformation component.

3.4 Relations of Kinematic

The following formulation may be used to get the kinematic equations governing the bending deformations of a Timoshenko beam element in the (xz) -plane:

$$\begin{aligned} U^i &= -z^i \left(\frac{\partial w^i}{\partial x^i} \right) \\ V^i &= 0 \\ W^i &= w^i \end{aligned} \tag{3.22}$$

3.5 Derivation of violin string shape functions

The cubic shape functions are already established in many literature [40, 2]. In this section, the focus will be given to drive the violin string shape functions. The following is the governing equation for the free vibrations of a rotating Timoshenko beam:

$$\frac{\partial}{\partial x} \left(\kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) \right) - \rho A^i \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x} \left(F_x \frac{\partial w}{\partial x} \right) = 0 \tag{3.23}$$

$$\frac{\partial}{\partial x} \left(E I^i \frac{\partial \theta}{\partial x} \right) + \kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) - \rho I^i \left(\frac{\partial^2 \theta}{\partial t^2} \right) = 0 \tag{3.24}$$

When determining the bending deformations shape functions in the (xz) -plane, consideration is given to the i^{th} finite beam element shown in Figure 3.4. Conventional cubic Hermitian polynomials form functions are used in the conventional approach to finite element analysis. These shape functions are produced using the governing equation of the non-rotating Timoshenko beam, which is provided by

$$\frac{\partial}{\partial x} \left(\kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) \right) = 0 \quad (3.25)$$

$$\frac{\partial}{\partial x} \left(E I^i \frac{\partial \theta}{\partial x} \right) + \kappa G A^i \left(\frac{\partial w}{\partial x} - \theta \right) = 0 \quad (3.26)$$

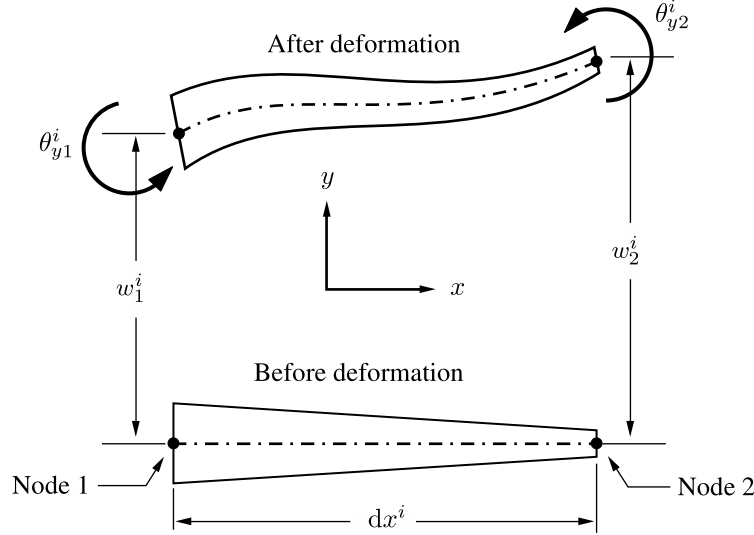


Figure 3.4: The i^{th} beam element bending deformation in the (xy) -plane.

The solution of Equation (3.25) and Equation (3.26) is a typical cubic shape functions. Where:

$$w^i(x^i) = a_3 + \left(a_0 - \frac{2EI}{GA^i \kappa} a_2 \right) x^i + \frac{a_1}{2} x^{i^2} + \frac{a_2}{3} x^{i^3} \quad (3.27)$$

$$\theta^i(x^i) = a_0 + a_1 x^i + a_2 x^{i^2} \quad (3.28)$$

where the four constants, $a_0 - a_3$, can be computed based on the boundary conditions.

Vinod Kumar and Ganguli [111] used below governing equations of motion

$$F_x(x) \frac{d^2 w}{dx^2} + \frac{d}{dx} \left(G A^i \kappa \left(\frac{dw}{dx} - \theta_y \right) \right) = 0 \quad (3.29)$$

$$G A^i \kappa \left(\frac{dw}{dx} - \theta \right) + \frac{d}{dx} \left(E I(x) \frac{d\theta}{dx} \right) = 0 \quad (3.30)$$

Getting the analytical solution to Equation (3.29) is made more difficult by the existence of $F_x(x)$. However, throughout the whole beam, the premise that $F_x(x) = F_x = \text{constant}$ is too simple. As a result, the beam is split up into several finite elements, with the assumption that F_x remains constant across each element.

The finite element method divides the beam into a set of N elements of uniform length. By assuming that the tension $F_x(x)$ is constant F_x^i within the i^{th} element, the equation for the rotating beam is reduced to that of the violin string. Global and local coordinates for an i^{th} element along the beam relate to each other as

$$X = R_0 + x^i + x'$$

where the distance from the element's left edge to its root is expressed as $x^i = (i-1)l^i$. As a consequence, there is increased stress in the inboard parts and decreased tension in the outboard elements.

$$F_x(x) = F_x^i = \int_{x^i}^L \rho A^i(x) \Omega^2 x dx = \sum_{j=1}^{j=N} \int m_j(x') \Omega^2 x' dx' \quad (3.31)$$

Assuming $E^i I(x) = EI^i$ and $\rho A^i(x) = \rho A^i$ within an element as constant, Equation (3.29) and Equation (3.30) give

$$F_x^i \frac{d^2 w}{dx'^2} + GA^i \kappa \left(\frac{d^2 w}{dx'^2} - \frac{d\theta}{dx'} \right) = 0 \quad (3.32)$$

$$GA^i \kappa \left(\frac{dw}{dx'} - \theta \right) + EI^i \frac{d^2 \theta}{dx'^2} = 0 \quad (3.33)$$

Equation (3.33) is decoupled from Equation (3.32) in order to eliminate the cou-

pling between w and θ , yielding

$$\frac{d^4 w}{dx^4} - \frac{F_x^i}{EI^i \left(1 + \frac{F_x^i}{\kappa GA^i}\right)} \frac{d^2 w}{dx^2} = 0 \quad (3.34)$$

Or it can be reduced to

$$\frac{d^4 w}{dx^4} - S^{i2} \frac{d^2 w}{dx^2} = 0 \quad (3.35)$$

Where

$$S^i = \sqrt{\frac{F_x^i}{EI^i \left(1 + \frac{F_x^i}{\kappa GA^i}\right)}} \quad (3.36)$$

The solution of which is

$$w^i(x^i) = \frac{e^{S^i x^i}}{S^{i2}} a_0 + \frac{e^{-S^i x^i}}{S^{i2}} a_1 + a_2 + a_3 x^i \quad (3.37)$$

or in matrix form

$$w^i(x^i) = [\chi^i] \{a_j\} \quad (3.38)$$

where

$$[\chi^i] = \begin{bmatrix} \frac{e^{S^i x^i}}{S^{i2}} & \frac{e^{-S^i x^i}}{S^{i2}} & 1 & x^i \end{bmatrix} \quad (3.39)$$

And

$$[a_j] = \begin{bmatrix} a_0 & a_1 & a_3 & a_4 \end{bmatrix} \quad (3.40)$$

It is assumed that the shear strain is constant along the finite element, irrespective of the axial coordinate x^i , as stated by Narayanaswami [77].

$$\gamma_{xz}^i = \text{const.} = \gamma_0 \quad (3.41)$$

The shearing force Q_i and bending moment M_y^i are related by

$$\frac{dM_y^i}{dx^i} - Q_z^i = 0 \quad (3.42)$$

where the relationship between bending moment and curvature is

$$M_y^i = -EI_{yy}^i \frac{\partial \theta_y^i}{\partial x^i} \quad (3.43)$$

Shear force and transverse shear strain are related by

$$Q_y^i = \kappa G A^i \gamma_{xz}^i \quad (3.44)$$

The shear correction factor, represented by the symbol κ , is used to account for the nonuniform dispersion of shear stress throughout the A^i cross-sectional area. This is the mathematical expression for the bending-induced slope.

$$\theta_y^i = \frac{\partial w}{\partial x} - \gamma_{xz}^i \quad (3.45)$$

Where

$$\gamma_{xz}^i = -\frac{EI^i}{\kappa G A^i} \frac{d^2 \theta}{dx^2} = -\frac{EI^i}{\kappa G A^i} \frac{d^3 w}{dx^3} \quad (3.46)$$

This result in

$$\theta_y^i = \frac{dw}{dx} + \frac{EI^i}{\kappa G A^i} \frac{d^3 w}{dx^3} \quad (3.47)$$

After rearranging and doing the derivatives

$$\theta_y^i = \left(\frac{e^{S^i x^i}}{S^{i2}} + \frac{1}{12} S^i e^{S^i x^i} l^{i2} \Phi \right) a_0 + \left(-\frac{e^{-S^i x^i}}{S^i} - \frac{1}{12} S^i e^{-S^i x^i} l^{i2} \Phi \right) a_1 + a_3 \quad (3.48)$$

or in matrix form

$$\theta^i(x^i) = [\Theta^i] \{a_j\} \quad (3.49)$$

where

$$[\Theta^i] = \begin{bmatrix} \left(\frac{e^{S^i x^i}}{S^i} + \frac{1}{12} S^i e^{S^i x^i} l^{i2} \Phi \right) & \left(-\frac{e^{-S^i x^i}}{S^i} - \frac{1}{12} S^i e^{-S^i x^i} l^{i2} \Phi \right) & 0 & 1 \end{bmatrix} \quad (3.50)$$

And

$$[a_j] = \begin{bmatrix} a_0 & a_1 & a_3 & a_4 \end{bmatrix} \quad (3.51)$$

Where

$$\Phi_y = \frac{12EI_{yy}^i}{\kappa_z GA^i l^{i2}} \quad (3.52)$$

The boundary conditions needed for the coefficients a_j to be defined in terms of bending deformations and slopes are:

$$\begin{aligned} w^i(0) &= w_1^i & \text{and} & & w^i(l^i) &= w_2^i \\ \theta^i(0) &= \theta_{y1}^i & \text{and} & & \theta^i(l^i) &= \theta_{y2}^i \end{aligned} \quad (3.53)$$

and using them with (3.37) and (3.48) equations yields

$$\begin{pmatrix} w_1^i \\ \theta_{y1}^i \\ w_2^i \\ \theta_{y2}^i \end{pmatrix} = \begin{bmatrix} \frac{1}{S^i} & \frac{1}{S^i} & 1 & 0 \\ \frac{1}{S^i} + \frac{1}{12}S^i l^i \Phi & -\frac{1}{S^i} - \frac{1}{12}S^i l^i \Phi & 0 & 1 \\ \frac{e^{S^i l^i}}{S^i} & \frac{e^{-S^i l^i}}{S^i} & 1 & l^i \\ \frac{e^{S^i l^i}}{S^i} + \frac{1}{12}S^i e^{S^i l^i} l^i \Phi & -\frac{e^{-S^i l^i}}{S^i} - \frac{1}{12}S^i e^{-S^i l^i} l^i \Phi & 0 & 1 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad (3.54)$$

or in more compact form

$$\{\delta^i\} = [\mathbf{A}]\{a_j\} \quad (3.55)$$

from which

$$\{a_j\} = \{\mathbf{A}\}^{-1}\{\delta^2\} \quad (3.56)$$

Determining the value of a_j , substituting it into the formula $w^i \left(\xi^i = \frac{x^i}{l^i} \right)$, and simplifying the result yields

$$w^i(\xi^i) = \mathcal{N}_{w1}^i w_1^i + \mathcal{N}_{w2}^i \theta_1^i + \mathcal{N}_{w3}^i w_2^i + \mathcal{N}_{w4}^i \theta_2^i \quad (3.57)$$

Where

$$\mathcal{N}_{w1}^i = \frac{e^{-S^i l^i \xi} \left(-12e^{S^i l^i} + 12e^{2S^i l^i \xi} + e^{S^i(l^i + l^i \xi)} \left(-12 + 12S^i(l^i - l^i \xi) + S^i l^i \right) \right) + e^{S^i l^i \xi} \left(12 + 12S^i(l^i - l^i \xi) + S^i l^i \right)}{24 + 12S^i l^i + S^i l^i \xi + e^{S^i l^i} \left(-24 + 12S^i l^i + S^i l^i \xi \right)} \quad (3.58a)$$

$$\begin{aligned}
& \left(12e^{-S^i l^i \xi} \left(-12e^{S^i l^i} + 12e^{S^i(l^i + 2l^i \xi)} + 2S^i e^{S^i(l^i + l^i \xi)} l^i \xi (12 + S^{i^2} l^{i^2} \Phi) \right. \right. \\
& \quad - e^{2S^i l^i} (-12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi) - e^{2S^i l^i \xi} (12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi) \\
& \quad + e^{S^i(2l^i + l^i \xi)} (-12 + 12S^i(l^i - l^i \xi) + S^{i^3} l^{i^2} (l^i - l^i \xi) \Phi) \\
& \quad \left. \left. + e^{S^i l^i \xi} (12 + 12S^i(l^i - l^i \xi) + S^{i^3} l^{i^2} (l^i - l^i \xi) \Phi) \right) \right) \\
\mathcal{N}_{w2} = & \frac{}{(S^i(-1 + e^{S^i l^i})(12 + S^{i^2} l^{i^2} \Phi) \\
& (24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi)))}
\end{aligned} \tag{3.58b}$$

$$\begin{aligned}
& e^{-S^i l^i \xi} \left(12e^{S^i l^i} - 12e^{2S^i l^i \xi} + e^{S^i(l^i + l^i \xi)} (-12 + 12S^i l^i \xi + S^{i^3} l^{i^3} \xi \Phi) \right. \\
& \quad \left. + e^{S^i l^i \xi} (12 + 12S^i l^i \xi + S^{i^3} l^{i^3} \xi \Phi) \right) \\
\mathcal{N}_{w3} = & \frac{}{24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi)}
\end{aligned} \tag{3.58c}$$

$$\begin{aligned}
& \left(12e^{-S^i l^i \xi} \left(-12e^{2S^i l^i} + 12e^{2S^i l^i \xi} - 2S^i e^{S^i(l^i + l^i \xi)} (l^i - l^i \xi) \right. \right. \\
& \quad (12 + S^{i^2} l^{i^2} \Phi) + e^{S^i(l^i + 2l^i \xi)} (-12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi) \\
& \quad + e^{S^i l^i} (12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi) - e^{S^i(2l^i + l^i \xi)} (-12 + 12S^i l^i \xi + S^{i^3} l^{i^3} \xi \Phi) \\
& \quad \left. \left. - e^{S^i l^i \xi} (12 + 12S^i l^i \xi + S^{i^3} l^{i^3} \xi \Phi) \right) \right) \\
\mathcal{N}_{w4} = & \frac{}{(S^i(-1 + e^{S^i l^i})(12 + S^{i^2} l^{i^2} \Phi) \\
& (24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi)))}
\end{aligned} \tag{3.58d}$$

Similarly, to get $\theta_y^i(\xi^i)$, insert a_j into the formula and simplify.

$$\theta_y^i(\xi^i) = \mathcal{N}_{\theta_1^i}^i w_1^i + \mathcal{N}_{\theta_2^i}^i \theta_1^i + \mathcal{N}_{\theta_3^i}^i w_2^i + \mathcal{N}_{\theta_4^i}^i \theta_2^i \tag{3.59}$$

Where

$$\mathcal{N}_{\theta 1} = \frac{S^i e^{-S^i l^i \xi^i} (-1 + e^{S^i l^i \xi^i}) (-e^{S^i l^i} + e^{S^i l^i \xi^i}) (12 + S^{i^2} l^{i^2} \Phi)}{24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi)} \quad (3.60a)$$

$$\begin{aligned} & e^{-S^i l^i \xi^i} (e^{S^i l^i} - e^{S^i l^i \xi^i}) (12 - 12e^{S^i (l^i + l^i \xi^i)} \\ & + e^{S^i l^i} (-12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi) + e^{S^i l^i \xi^i} (12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi)) \\ \mathcal{N}_{\theta 2} = & \frac{(-1 + e^{S^i l^i}) (24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi))}{(3.60b)} \end{aligned}$$

$$\mathcal{N}_{\theta 3} = \frac{S^i e^{-S^i l^i \xi^i} (-1 + e^{S^i l^i \xi^i}) (-e^{S^i l^i} + e^{S^i l^i \xi^i}) (12 + S^{i^2} l^{i^2} \Phi)}{24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi)} \quad (3.60c)$$

$$\begin{aligned} & e^{-S^i l^i \xi^i} (-1 + e^{S^i l^i \xi^i}) (-12e^{2S^i l^i} + 12e^{S^i l^i \xi^i} + e^{S^i (l^i + l^i \xi^i)} \\ & (-12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi) + e^{S^i l^i} (12 + 12S^i l^i + S^{i^3} l^{i^3} \Phi)) \\ \mathcal{N}_{\theta 4} = & \frac{(-1 + e^{S^i l^i}) (24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i^3} l^{i^3} \Phi))}{(3.60d)} \end{aligned}$$

Equations $\mathcal{N}_w$ and $\mathcal{N}_\theta$ allow us to represent the kinematic relations provided by equation (3.22) as

$$\begin{aligned}
& (e^{-S^i l^i \xi^i} l^i (144(-1 + e^{S^i l^i}) (e^{S^i l^i} (\theta_{y1}^i - \theta_{y2}^i) + e^{2S^i l^i \xi^i} (-\theta_{y1}^i + \theta_{y2}^i) - e^{S^i l^i \xi^i} (\theta_{y1}^i + \theta_{y2}^i) + e^{S^i l^i (1+\xi^i)} (\theta_{y1}^i + \theta_{y2}^i)) \\
& -144S^i (e^{S^i l^i \xi^i} (w_1^i - w_2^i) + e^{S^i l^i (2+\xi^i)} (-w_1^i + w_2^i) + e^{2S^i l^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) - e^{2S^i l^i \xi^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) \\
& -e^{S^i l^i} (w_1^i - w_2^i + l^i \theta_{y2}^i) + e^{S^i l^i (1+2\xi^i)} (w_1^i - w_2^i + l^i \theta_{y2}^i)) + 12S^{i2} e^{S^i l^i \xi^i} (-1 + e^{S^i l^i})^2 l^{i2} (\theta_{y1}^i + \theta_{y2}^i) \Phi \\
& -12S^{i3} l^{i2} (2e^{S^i l^i \xi^i} (w_1^i - w_2^i) - 2e^{S^i l^i (2+\xi^i)} (w_1^i - w_2^i) + e^{2S^i l^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) - e^{2S^i l^i \xi^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) \\
& -e^{S^i l^i} (w_1^i - w_2^i + l^i \theta_{y2}^i) + e^{S^i l^i (1+2\xi^i)} (w_1^i - w_2^i + l^i \theta_{y2}^i)) \Phi + S^{i5} e^{S^i l^i \xi^i} (-1 + e^{2S^i l^i}) l^{i4} (w_1^i - w_2^i) \Phi^2)) \\
U^i = \zeta & \left(\frac{((-1 + e^{S^i l^i}) (12 + S^{i2} l^{i2} \Phi) (24 + 12S^i l^i + S^{i3} l^{i3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i3} l^{i3} \Phi)))}{(-1 + e^{S^i l^i}) (12 + S^{i2} l^{i2} \Phi) (24 + 12S^i l^i + S^{i3} l^{i3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i3} l^{i3} \Phi)))} \right)
\end{aligned} \tag{3.61a}$$

$$V^i = 0 \tag{3.61b}$$

$$\begin{aligned}
& e^{-S^i l^i \xi^i} \left(-144(-1 + e^{S^i l^i})(e^{S^i l^i} - e^{S^i l^i \xi^i})(-1 + e^{S^i l^i \xi^i})(\theta_{y1}^i - \theta_{y2}^i) \right. \\
& -144S^{i2} e^{S^i l^i \xi^i} (-1 + e^{2S^i l^i}) l^i (w_1^i (-1 + \xi^i) - w_2^i \xi^i) - 144S^i (e^{2S^i l^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) + e^{2S^i l^i \xi^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) \\
& - e^{S^i l^i} (w_1^i - w_2^i + l^i \theta_{y2}^i) - e^{S^i l^i (1+2\xi^i)} (w_1^i - w_2^i + l^i \theta_{y2}^i) + e^{S^i l^i \xi^i} (w_1^i + w_2^i - l^i \theta_{y1}^i + l^i \theta_{y1}^i \xi^i + l^i \theta_{y2}^i \xi^i) \\
& + e^{S^i l^i (2+\xi^i)} (w_1^i + w_2^i - l^i \theta_{y1}^i + l^i \theta_{y1}^i \xi^i + l^i \theta_{y2}^i \xi^i) - 2e^{S^i l^i (1+\xi^i)} (w_1^i + w_2^i - l^i \theta_{y2}^i + l^i \theta_{y1}^i \xi^i + l^i \theta_{y2}^i \xi^i)) \\
& 24S^{i4} e^{S^i l^i \xi^i} (-1 + e^{2S^i l^i}) l^{i3} (w_1^i (-1 + \xi^i) - w_2^i \xi^i) \Phi - 12S^{i3} l^{i2} (e^{2S^i l^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) + e^{2S^i l^i \xi^i} (w_1^i - w_2^i + l^i \theta_{y1}^i) \\
& - e^{S^i l^i} (w_1^i - w_2^i + l^i \theta_{y2}^i) - e^{S^i l^i (1+2\xi^i)} (w_1^i - w_2^i + l^i \theta_{y2}^i) + e^{S^i l^i \xi^i} (w_1^i + w_2^i - l^i \theta_{y1}^i + l^i \theta_{y1}^i \xi^i + l^i \theta_{y2}^i \xi^i) \\
& + e^{S^i l^i (2+\xi^i)} (w_1^i + w_2^i - l^i \theta_{y1}^i + l^i \theta_{y1}^i \xi^i + l^i \theta_{y2}^i \xi^i) - 2e^{S^i l^i (1+\xi^i)} (w_1^i + w_2^i - l^i \theta_{y2}^i + l^i \theta_{y1}^i \xi^i + l^i \theta_{y2}^i \xi^i)) \Phi \\
& \left. + S^{i6} (e^{S^i l^i \xi^i} - e^{S^i l^i (2+\xi^i)}) l^{i5} (w_1^i (-1 + \xi^i) - w_2^i \xi^i) \Phi^2 \right) \\
W^i = & \left(\frac{(S^i (-1 + e^{S^i l^i}) (12 + S^{i2} l^{i2} \Phi) (24 + 12S^i l^i + S^{i3} l^{i3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i3} l^{i3} \Phi)))}{(S^i (-1 + e^{S^i l^i}) (12 + S^{i2} l^{i2} \Phi) (24 + 12S^i l^i + S^{i3} l^{i3} \Phi + e^{S^i l^i} (-24 + 12S^i l^i + S^{i3} l^{i3} \Phi)))} \right)
\end{aligned} \tag{3.61c}$$

In the z -direction, dimensionless coordinates are represented by $\zeta = \frac{z}{l^i}$.

This can be written in matrix form as

$$\{d^i\}_{3 \times 1} = \begin{bmatrix} U^i & V^i & W^i \end{bmatrix}^T = [\mathcal{N}^i]_{3 \times 4} \{e^i\}_{4 \times 1} \quad (3.62)$$

For non-rotating beam, F_x^i 3.31 will be zero. Further more, S^i will be zero. As it can be observed in equations 3.60d and 3.61c, S^i is in the denominator. When S^i is set to zero for non rotating beam, this will result in undefined shape functions for the beam element. This is why the violin shape function can be used only for rotating beam element.

3.6 Stiffness Matrices

3.6.1 Elastic Stiffness Matrix

The equation (2.48) shows the composite elemental elastic stiffness matrix of a non-rotating tapered Timoshenko beam element that has length l^i .

$$[k_{ee}^i] = [k_b^i] + [k_s^i]$$

where each of the previous matrices is given by

$$[k_b^i] = \int_0^{l^i} [B_\theta^i]^T EI_{yy}^i [B_\theta^i] dx^i \quad (3.63)$$

$$[k_s^i] = \int_0^{l^i} \left([B_w^i] - [\mathcal{N}_\theta^i] \right)^T \kappa GA^i \left([B_w^i] - [\mathcal{N}_\theta^i] \right) dx^i \quad (3.64)$$

3.6.2 Stiffness Matrices due to Rotation

Increasing strain energy from the i^{th} beam element as the hub rotates introduces the elemental rotating stiffness matrix. In the (xz) plane, the elemental stiffness matrix represents centrifugal forces:

$$[k_c^i] = \int_0^{l^i} [B_w^i]^T F_P^i [B_w^i] dx^i \quad (3.65)$$

The variable F_P^i indicates the centrifugal force applied to the i^{th} element during hub rotation. Assuming small displacements, higher-order components in the axial direction for the i^{th} element may be ignored. The equation below shows the tensile force F_P^i on a beam element connected to a differential element at position P^i of the i^{th} element:

$$dF_P^i = \rho A^i \Omega^2 (\mathcal{R}_0 + r_P^i) dr_P^i \quad (3.66)$$

where r_P^i represents the position vector of point P^i , and $\mathcal{R}_0$ represents the hub radius.

Minor deformations can be written as

$$r_P^i = (L^i + x^i) = (i-1)l^i + x^i \quad (3.67)$$

Hence, the formulation of the force expression F_P^i can be obtained by performing integration of equation (3.66) throughout the interval extending from point P^i to the unfixed extremity of the beam, as depicted in Figures 3.1-3.2.

$$F_P^i = \rho\Omega^2 \left(\int_{x^i}^{l^i} (\mathcal{R}_0 + ((i-1)l^i + x^i)) A^i dx^i + \int_{il^i}^{nl^i} (\mathcal{R}_0 + x^i) A^i dx^i \right) \quad (3.68)$$

3.7 Mass Matrices

Equation (2.85) represents the mass matrix for the i^{th} beam element with a length of l^i .

$$[m_{ee}^i] = [m_t^i] + [m_r^i]$$

where

$$[m_t^i] = \int_0^{l^i} [\mathcal{N}_w^i]^T \rho A^i [\mathcal{N}_w^i] dx^i \quad (3.69)$$

$$[m_r^i] = \int_0^{l^i} [\mathcal{N}_\theta^i]^T \rho I_{yy}^i [\mathcal{N}_\theta^i] dx^i \quad (3.70)$$

CHAPTER 4

RESULTS AND DISCUSSIONS

The finite element model has been implemented successfully using Mathematica code. A comparison investigation was undertaken to examine the impact of various factors on the dynamical characteristics of the model. The code consists of nine modules that can be modified independently, which are:

- Shape function module: This module can adopt any shape function to solve the finite element model.
- Geometrical properties module: This module represents the dimensionless geometrical properties of the beam, where it can handle rectangular or circular cross sections.
- Parameters definitions module: This module is used as a support module to define the experienced Parameters on the beam during the analysis (i.e. tension due to rotation, spin Parameters, shear deformation factor and non-dimensional parameters used to simplify the equation of motion)
- Stiffness and mass matrices module: As a function of element number i and non-

dimensional distance ζ in the x direction, this module creates mass matrices and establishes the parameters for the dimensionless elemental stiffness matrix.

- Boundary conditions module: This module defines the different boundary conditions for both Euler and Timoshenko beams. The module can handle all the classical boundary conditions (i.e., free, hinged, or clamped) based on the user input.
- Applying boundary conditions module: This module applies the boundary conditions to both ends of each element of the beam.
- Generate the boundary conditions rule: This gives the solution of the first and last element based on the selected boundary conditions.
- The numerical integration module: This is where the power of Mathematica is demonstrated and the mass and stiffness matrices are generated by applying the numerical integration of equations in the chapter. The `NIntegrate` function in Mathematica gives very accurate results.
- Assembly module: This module assembles the elemental mass and stiffness matrices to form the global matrices.

After running all modules, the eigenvalues can be found directly by the built-in function in Mathematica "Eigenvalues". The developed model is suitable for linear and exponential beam profiles, any doubly symmetric cross-section, and various combinations of boundary conditions, such as clamped-free and hinged-free. Additionally, if feasible, the results have been contrasted with the results of existing literature.

A flow chart of the developed scheme is presented in Figure 4.1

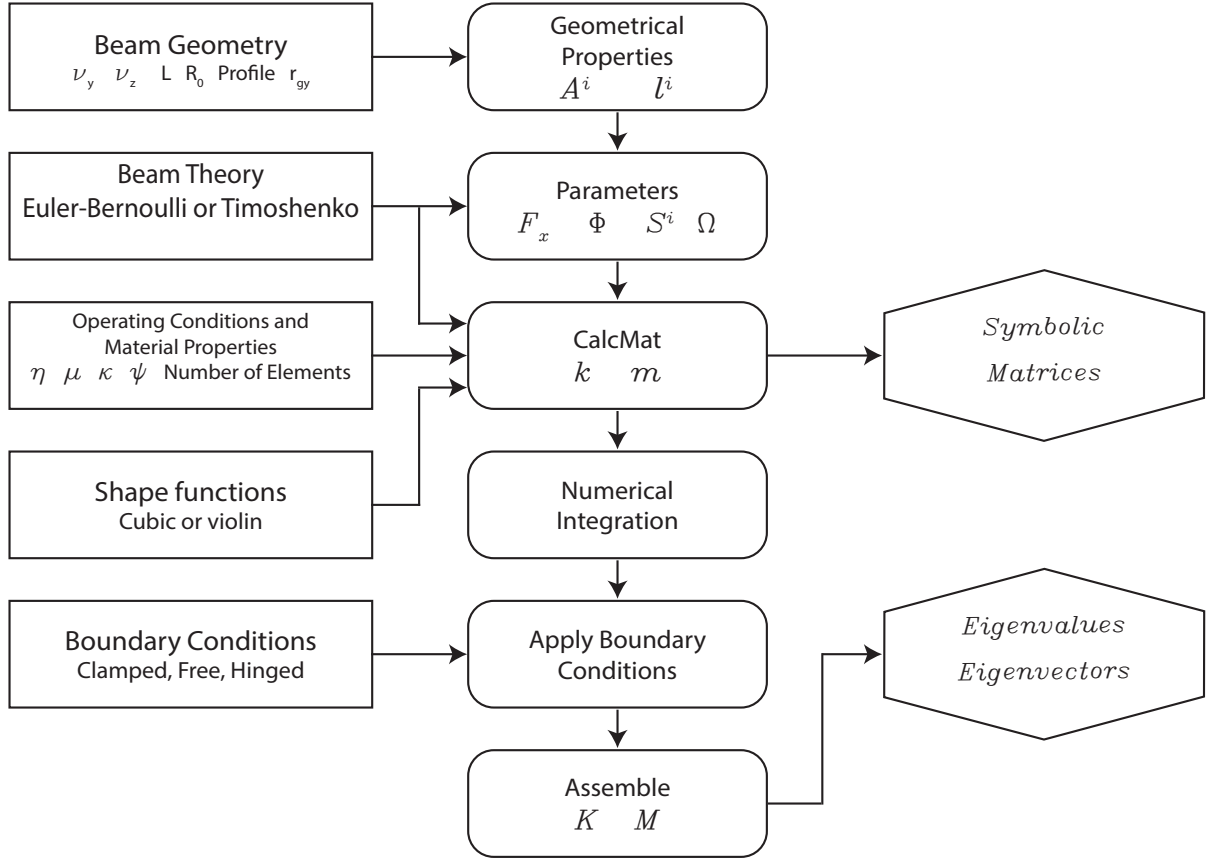


Figure 4.1: The flow chart for the computer scheme.

4.1 Numerical Scheme

The provided Mathematica code executes the following operations:

- Generates and assembles the system matrices.
- Evaluates the modal characteristics.

4.1.1 Input Data

Several finite elements are discretized from the beam, and each element's material and geometrical attributes need to be specified.

- **Element Properties:** For every element, there are two kinds of properties that must be specified:

- **The geometrical properties:** Include the hub radius R_0 , the shear correction factor, the element length, width, and depth, as well as the slenderness ratios and the taper ratios. The beam profile—whether it is linearly tapered, exponentially tapered, or linear—is another factor. Zero taper ratios mean a uniform beam. There is a pyramid when the taper ratios are equal to one. It's crucial to notice that the highest taper ratio that may be utilized for a linear beam profile is 1, but for an exponential taper beam, greater than 1 can be used as it will result in an appropriate exponential beam.
- **The material properties:** The code and equations are done for non-dimensional calculations to reduce the material-related variables as much as possible. The only needed input for the analysis is μ which is Poisson's ratio. Another material variable that was found to be used in various literary works was the ratio of Young's modulus of elasticity E to shear modulus G [20]. By ignoring the shear factor $kappa$, it is possible to derive the Rayleigh beam case. This suggests the omission of the shear stiffness matrix $[k_s^i]$. When disregarding the shear factor κ and the shear stiffness

matrix $[k_s^i]$ as well as the rotary mass inertia $[m_r^i]$ is excluded, the model will be reduced to the Euler-Bernoulli beam case.

- **Type of Analysis:** The code can use two different sets of shape functions: the conventional cubical shape function and the violin string shape function. After selecting which shape function can be used, beam theory as well as the number of beam elements must be specified.

4.1.2 System Matrices

To utilize the symbolic capabilities of Mathematica, the elemental stiffness and mass matrices are derived in Mathematica utilizing the symbolic shape functions. The symbolic calculation of Mathematica provides an advantage where the exact expressions of these matrices are calculated at the beginning. This eliminates the round-off errors. Then, the boundary conditions are applied.

4.1.3 Eigenvalue Solution

The eigenvalue problem that corresponds to this formulation is then specified and solved using the LAPACK procedures, which are a commonly used set of routines in Mathematica. To address the eigenvalue problem, standard built-in Mathematica functions were used. More insight on the basic procedure of Mathematica is given by Cecílio and dos Santos [36]. The procedure given in [36] was improved to handle symbolic and dimensionless calculations.

4.1.4 Numerical Integration

The system mass and stiffness matrices are numerically integrated using the built-in Mathematica function `NIntegrate` to reduce the time and get accurate results.

4.2 Modal Characteristics

The analysis used a Full Order Model (FOM) to provide the natural frequencies in graphical and tabular representations, including a broad spectrum of parameter variations. A thorough comparison with analytical solutions and other numerical findings obtained from reliable sources in the body of literature will be part of a rigorous validation process that will evaluate the accuracy of numerical solutions.

4.2.1 Model convergence

To model convergence by increasing the number of beam elements, figures 4.3 - 4.4 are presented to show the convergence of the model. Both Euler-Bernoulli and Timoshenko are going to be investigated.

Euler-Bernoulli Beam

The convergence of the present Euler-Bernoulli beam model is shown in Figure 4.3 in comparison to Wright et al. [16]'s exact solution. The exact values for the first three frequency parameters, as determined by Wright et al. [16], are: 7.3603730, 26.809082, and 66.683914. The model shows good stability and approaches to the exact solution.

To identify the deviation in percentage between the present finite element model and

exact solution of Wright et al. [16], figure 4.2 presents the percentage difference between them. Fig 4.2 shows that the accuracy will be excellent at 12 elements for the Euler-Bernoulli beam.

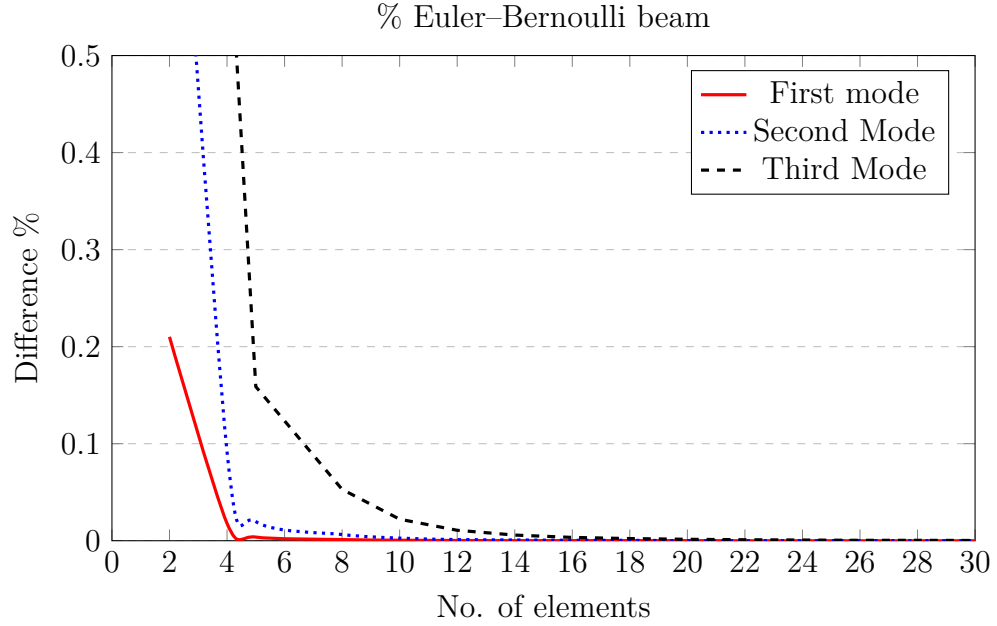


Figure 4.2: Percentage difference between present finite element model and exact solution of [16] of Euler-Bernoulli beam

Timoshenko Beam

Similarly, figure 4.5 illustrates how the results of a rotating Timoshenko beam spinning at $\bar{\Omega} = 6$ converges. The first four frequencies of the current model are compared with Nagaraj's exact solution [88] in figure 4.5. Nagaraj [88] first four frequency parameters are: 7.3319, 26.2488, 63.2386 and 114.5177. The current model has a high level of concordance with the exact solution. The slight overestimation of the first frequency parameter due to the consistent mass formulations [19]. To evaluate the exact difference between the exact solution and the present model of the finite element

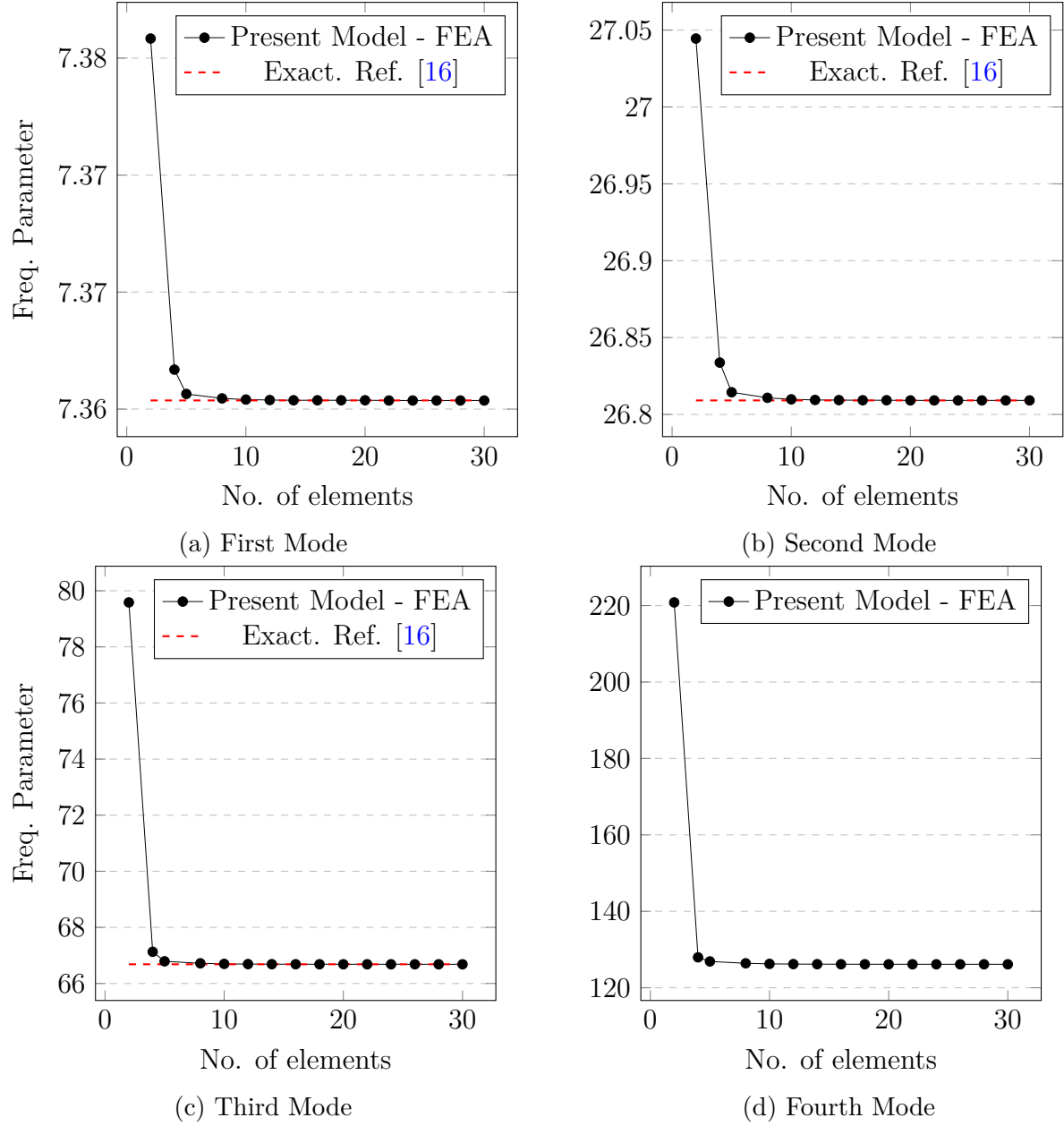


Figure 4.3: Rotating uniform clamped-free Euler-Bernoulli beam convergence of the present finite element model ($R = 0$, $\bar{\Omega} = 6$, $L = 1$)

mode, figure 4.4. The 16 elements give good accuracy and higher beam elements would reduce the difference even below 0.2%.

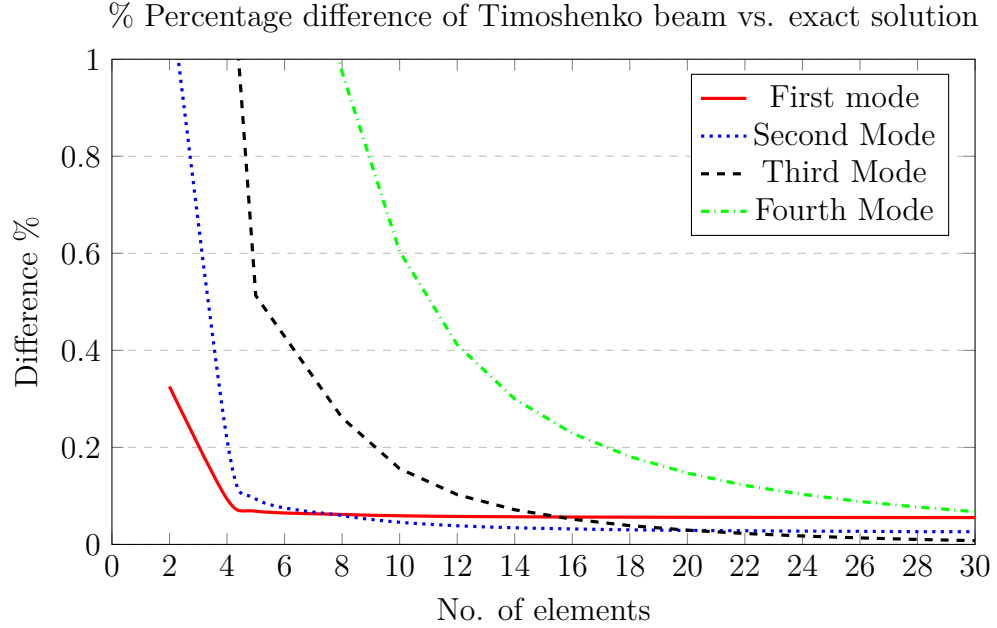


Figure 4.4: Percentage difference between present finite element model and exact solution of [88] of Timoshenko beam

Violin shape function

Greater model convergence was identified as a key advantage of the violin string shape functions by Vindo Kumar and Ganguli [93]. This section will investigate the convergence of violin shape function in scenarios other than those previously explored by Vindo Kumar and Ganguli [93]. In their study, Vindo Kumar and Ganguli [93] conducted a comparison between the outcomes obtained by violin string and cubic shape functions. This comparison was carried out for uniform Timoshenko beams, considering various spin parameters and slenderness ratios. However, Vindo Kumar and Ganguli [93] did not take into account other beam properties, such as hub radius,

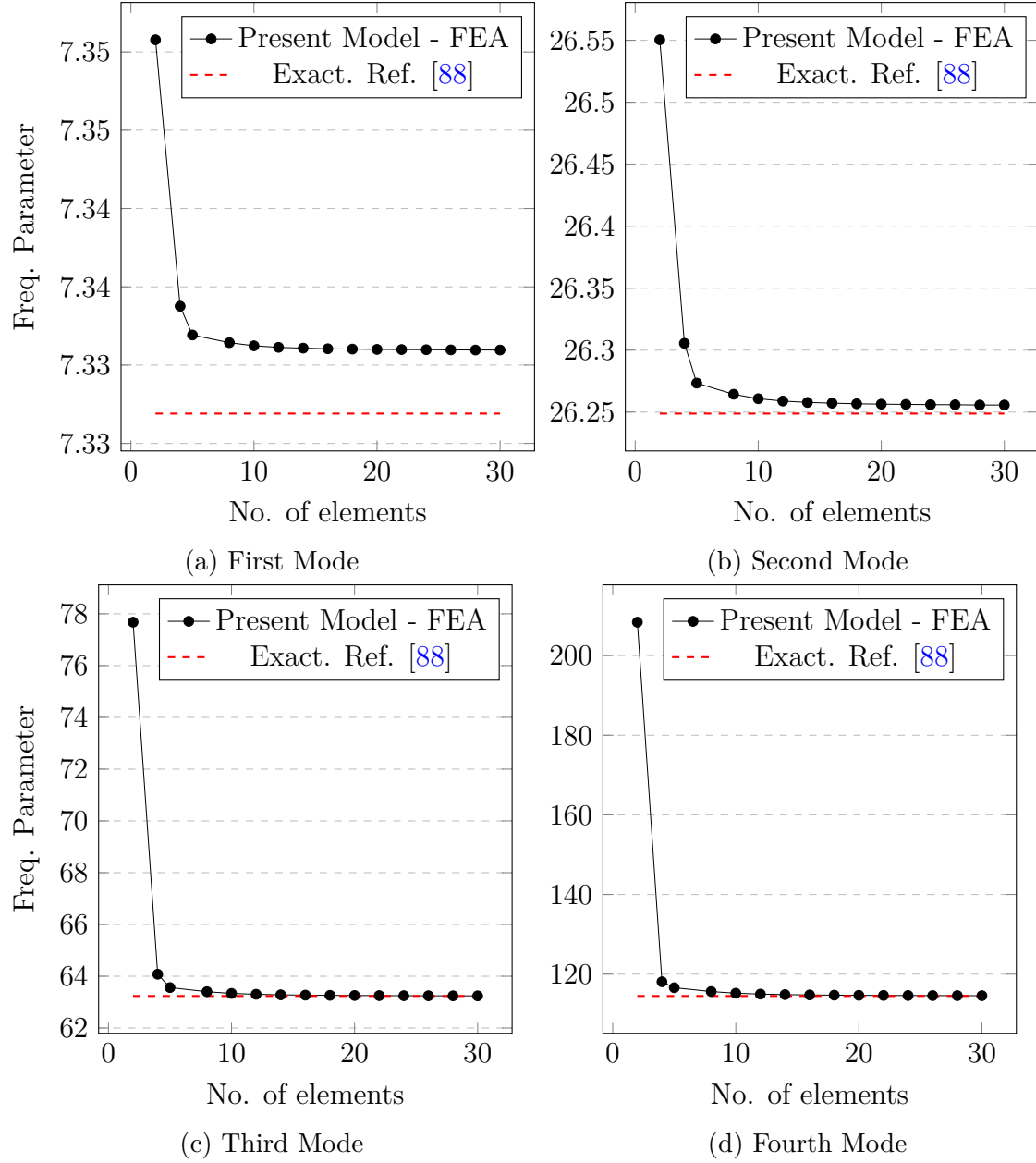


Figure 4.5: Rotating uniform clamped-free Timoshenko beam convergence of the present finite element model ($R = 0$, $\bar{\Omega} = 6$, $L = 0.72$, $r_{gy}/L = 0.02$, $\kappa = 5/6$)

linear taper, exponential taper, and setting angle. Tables 4.1 - 4.3 duplicate the results of a uniform Timoshenko rotating beam with a spin parameter of 4, 12 and 100 at different slenderness ratios. The number of elements in the present model in tables 4.1 - 4.3 is 20 for both cubic and violin shape functions. As it can be observed from tables 4.1 - 4.3, the results of the present model are in excellent agreement with the results of Vindo Kumar and Ganguli [93] for a spin parameter of 4 and 12. Also, the agreement is excellent for fundamental frequency at $\bar{\Omega} = 100$. However, the deviation between the present study and the work of Kumar and Ganguli [93] observed at frequency parameters 2 to 5. It is not clear from the paper of Vindo Kumar and Ganguli [93] what was the number of elements used to get the results. To investigate more the discrepancies, the model convergence at $\bar{\Omega} = 100$ is plotted in figures 4.6.

As it can be observed in Figure 4.6, the convergence of the rotating element at higher speed requires higher number of elements. This explain the discrepancies between the tabulated results of Vindo Kumar and Ganguli [93] and the present model. If, for instance, the present model was used with 10 elements for a uniform rotating Timoshenko clamped-free beam with $L = 1$, $\bar{\Omega} = 100$, $\kappa = 2/3$ and slenderness ratio of 0.045, then model frequencies will be 346.328 and 414.381 for the fourth and fifth frequencies respectively. When these frequencies compared with results of Vindo Kumar and Ganguli [93] which are 347.11095 and 402.69847 for for the fourth and fifth frequencies respectively which has a difference of 0.22.6% and 2.82%. This shows that the model

Figure 4.6 shows that both the violin string and the cubic shape functions have similar convergence. The cubic shape function exhibits a slight improvement in per-

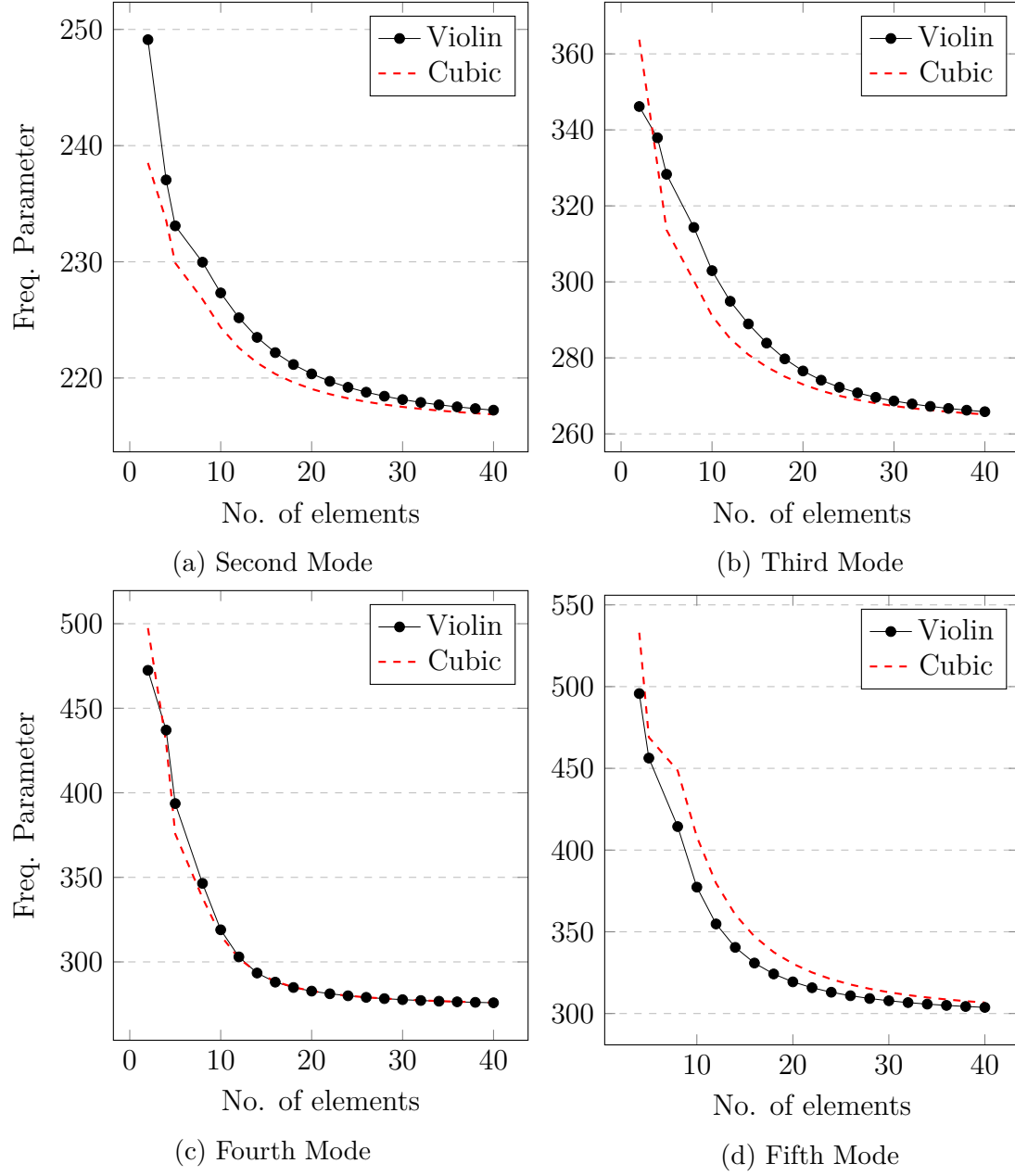


Figure 4.6: Convergence of comparison of cubic and violin shape functions for a rotating uniform clamped-free Timoshenko beam ($R = 0$, $\bar{\Omega} = 100$, $L = 1$, $r_{gy}/L = 0.045$, $\kappa = 2/3$)

Table 4.1: Gyration ratio effect on frequency parameters using cubic and violin shape functions for a clamped-free rotating Timoshenko beam, $R = 0$, $\Omega = 4$, $\kappa = 2/3$, and $L = 1$

	$(r_{gy})/L \longrightarrow$		0.09	0.05	0.045
λ_{T1}	Present work	Cubic	5.24975	5.46307	5.48471
		Violin	5.24951	5.46296	5.48461
	Ref. [93]		5.24943	5.46293	5.48459
λ_{T2}	Present work	Cubic	17.2647	20.9967	21.4913
		Violin	17.2682	20.9987	21.4931
	Ref. [93]		17.25698	20.99124	21.48647
λ_{T3}	Present work	Cubic	35.435	47.5031	49.4908
		Violin	35.4481	47.5147	49.5017
	Ref. [93]		35.34860	47.42615	49.41892
λ_{T4}	Present work	Cubic	53.173	78.3101	82.7398
		Violin	53.2046	78.3412	82.7698
	Ref. [93]		52.89319	77.95112	82.38964
λ_{T5}	Present work	Cubic	68.1059	111.491	118.978
		Violin	68.1469	111.549	119.035
	Ref. [93]		67.59790	110.44737	117.92903

formance for the second and third modes. The form functions have almost identical values for the fourth mode. In the context of the Fifth mode, it can be seen that the form function of the violin string exhibits a slightly superior level of convergence. Figure 4.6 is for a slenderness ratio of 0.045. Figure 4.7 is presented to examine the slenderness ratio effect on the convergence of both shape functions where the slenderness ratio was set to 0.09.

Figure 4.6 illustrates that both the violin string and cubic form functions exhibit comparable convergence. The cubic shape function exhibits a minor improvement in performance for the second and third modes. The form functions exhibit a near

Table 4.2: Gyration ratio effect on frequency parameters using cubic and violin shape functions for a clamped-free rotating Timoshenko beam, $R = 0$, $\Omega = 12$, $\kappa = 2/3$, and $L = 1$

	$(r_{gy})/L \longrightarrow$		0.09	0.05	0.045
λ_{T1}	Present work	Cubic	12.4907	12.8301	12.8776
		Violin	12.4883	12.8281	12.8757
	Ref. [93]		12.48715	12.82721	12.87493
λ_{T2}	Present work	Cubic	30.7003	34.3755	34.8296
		Violin	30.7151	34.3852	34.8384
	Ref. [93]		30.69199	34.36649	34.82098
λ_{T3}	Present work	Cubic	53.6085	64.9446	66.6226
		Violin	53.665	65.0104	66.687
	Ref. [93]		53.50637	64.87082	66.55376
λ_{T4}	Present work	Cubic	66.8566	100.422	104.201
		Violin	67.0886	100.605	104.386
	Ref. [93]		66.5894	100.07833	103.8689
λ_{T5}	Present work	Cubic	83.9479	138.082	144.69
		Violin	84.0892	138.428	145.05
	Ref. [93]		83.31214	137.05581	143.66688

equivalence in the case of the fourth mode. In the context of the fifth mode, it can be observed that the violin string shape function has a marginally superior level of convergence. The data presented in Figure 4.6 represents a slenderness ratio of 0.045. The offered figure 4.7 is utilized to investigate the impact of the slenderness ratio on the convergence of form functions. In this study, the slenderness ratio was set at a value of 0.09. As illustrated in figure 4.7, the violin shape function exhibits superior convergence at elevated slenderness ratios. In order to get identical outcomes for the frequency parameter's second mode of 85, it is needed to utilize 18 elements employing the violin form function, while 32 elements are required for the cubic shape function.

Table 4.3: Gyration ratio effect on frequency parameters using cubic and violin shape functions for a clamped-free rotating Timoshenko beam, $R = 0$, $\Omega = 100$, $\kappa = 2/3$, and $L = 1$

	$(r_{gy})/L \longrightarrow$		0.09	0.05	0.045
λ_{T1}	Present work	Cubic	65.4113	99.8209	99.939
		Violin	64.4649	99.7602	99.8797
	Ref. [93]		63.06658	99.72437	100.09367
λ_{T2}	Present work	Cubic	91.3506	192.153	218.153
		Violin	84.2595	194.236	219.498
	Ref. [93]		80.27038	188.10892	226.9068
λ_{T3}	Present work	Cubic	100.804	227.842	270.348
		Violin	100.756	225.606	273.345
	Ref. [93]		100.68832	214.91474	340.20408
λ_{T4}	Present work	Cubic	128.753	261.588	281.335
		Violin	114.178	258.476	281.231
	Ref. [93]		106.53987	246.341	347.11095
λ_{T5}	Present work	Cubic	170.684	282.939	328.255
		Violin	148.621	271.557	316.604
	Ref. [93]		136.70214	262.9316	402.69847

The third mode in figure 4.7 demonstrates improved convergence of the violin form function while maintaining the near results of the cubic shape function. In order to get a third frequency parameter of 100.9, the violin string shape function requires 12 elements, but the cubic shape function requires 18 elements. In order to achieve comparable outcomes, it is necessary to employ a violin form function of 20 elements for the fourth and fifth modes, as opposed to the cubic shape function, which requires 36 elements.

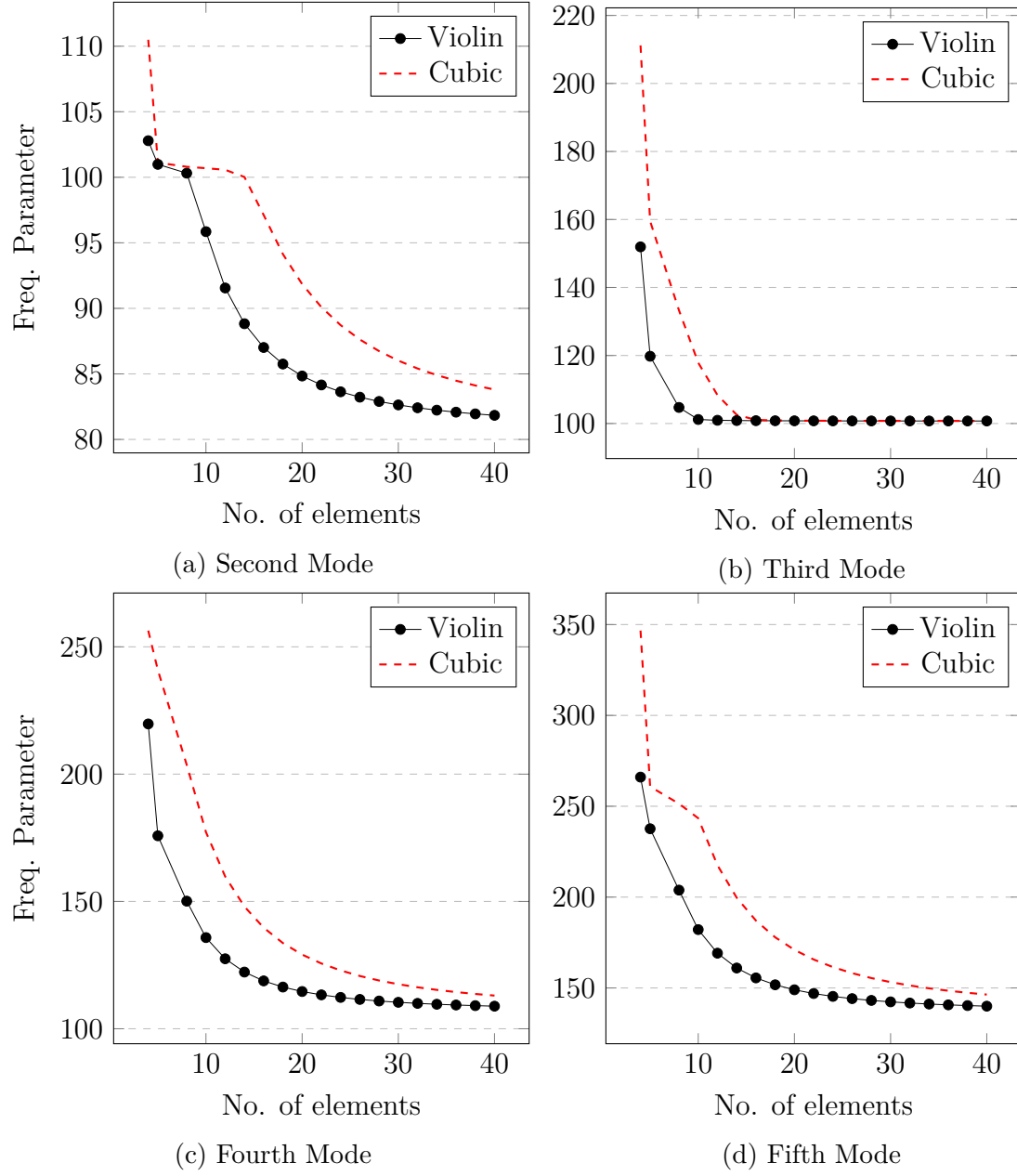


Figure 4.7: Convergence of comparison of cubic and violin shape functions for a rotating uniform clamped-free Timoshenko beam ($R = 0$, $\bar{\Omega} = 100$, $L = 1$, $r_{gy}/L = 0.09$, $\kappa = 2/3$)

Uniform rotating Timoshenko beam with hub radius It is clear that with a higher slenderness ratio, the violin string shape function has better convergence at a higher spin parameter for the higher modes. Figure 4.8 is presented to examine the effect of the hub radius where the hub radius R was set at 3.

Figure 4.9 presents the convergence at the lower spin parameter of $\bar{\Omega} = 10$ and $R = 3$ to examine the effect of the hub radius at lower speeds. Both shape functions are in excellent agreement, which puts emphasis on the convergence of the violin string shape function at higher speeds.

Linearly taper rotating Timoshenko beam Figures 4.10 and 4.11 present the convergence of a rotating linearly tapered Timoshenko beam with taper parameters $\nu_y = \nu_z = 0.5$ with spin parameters of 10 and 100, respectively.

4.2.2 Validation of the Model

This section will discuss numerous cases to demonstrate the present mathematical model's capabilities and stability. The developed code can handle both polynomial (cubic) and violin string shape functions. Due to the undefined shape functions of the violin string shape function that are described at the end of 3.5 when the spin parameter is set to zero (non-rotating beam), the non-rotating beam will be verified by polynomial shape functions. The validation will cover different cases, which include different combinations of uniform, tapered, non-rotating, rotating, Euler, and Timoshenko beams.

The beams under consideration for this research possess rectangular cross-sectional

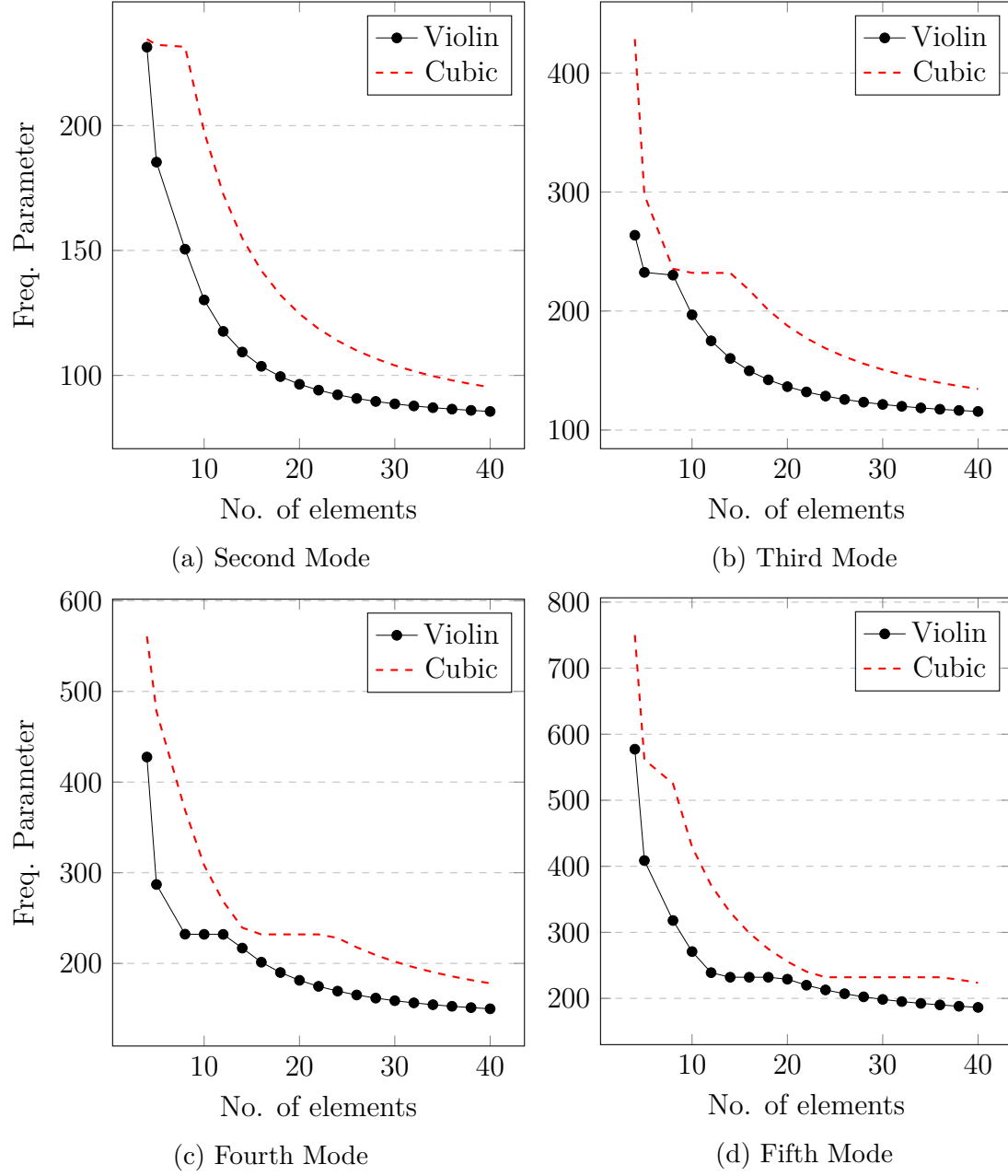


Figure 4.8: Convergence of comparison of cubic and violin shape functions for a rotating uniform clamped-free Timoshenko beam ($R = 3$, $\bar{\Omega} = 100$, $L = 1$, $r_{gy}/L = 0.09$, $\kappa = 2/3$)

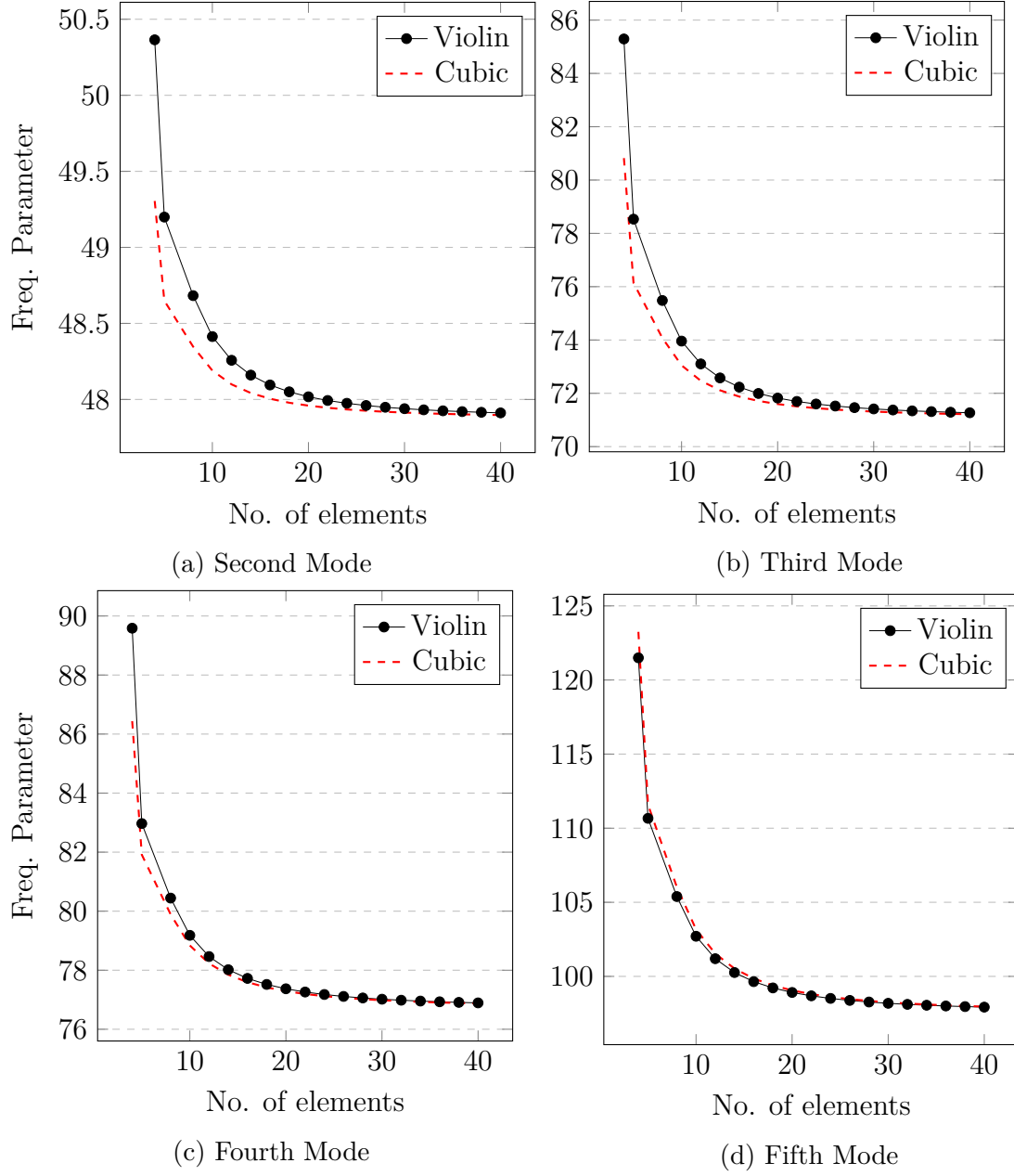


Figure 4.9: Convergence of comparison of cubic and violin shape functions for a rotating uniform clamped-free Timoshenko beam ($R = 3$, $\bar{\Omega} = 10$, $L = 1$, $r_{gy}/L = 0.09$, $\kappa = 2/3$)

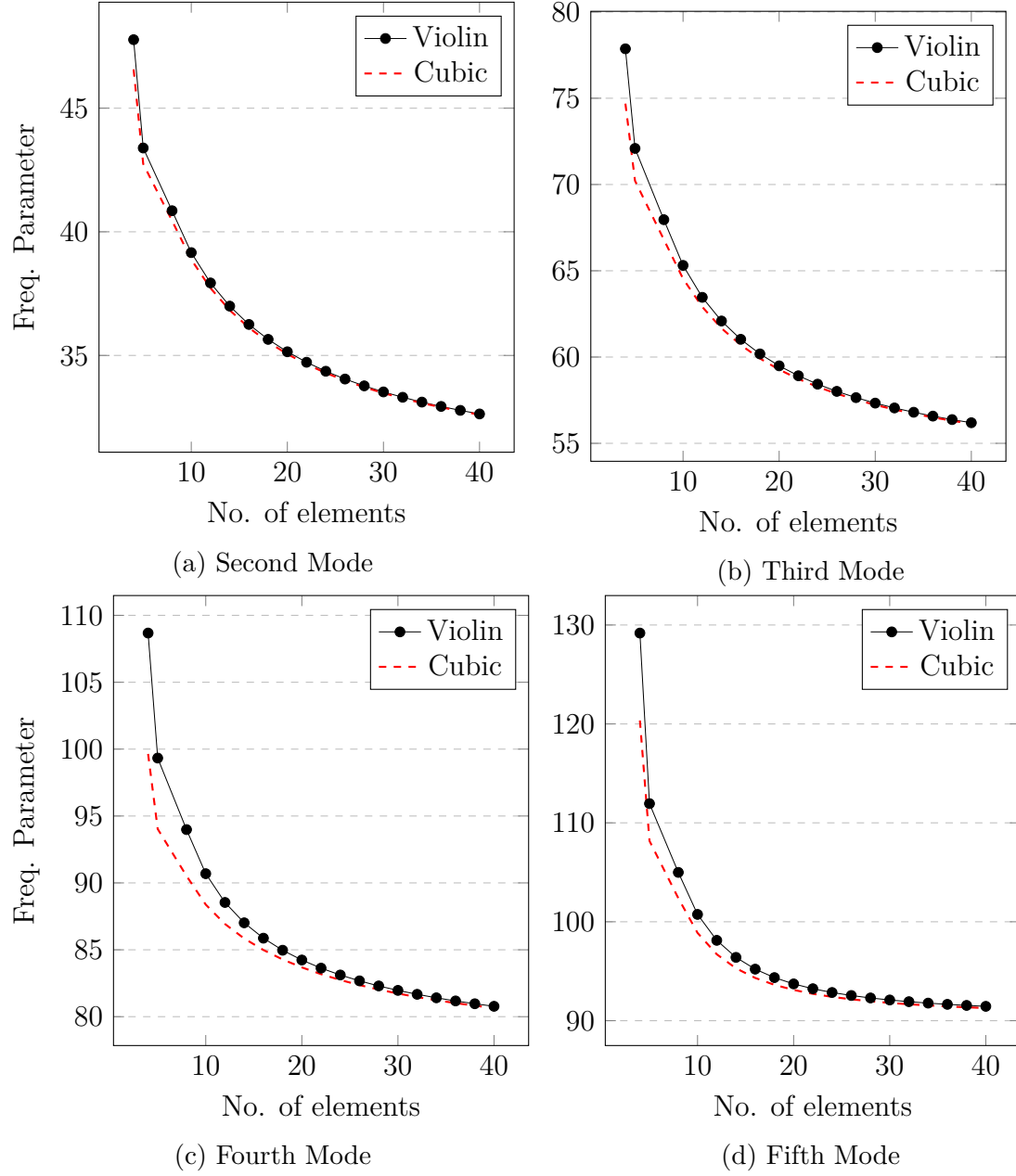


Figure 4.10: Convergence of comparison of cubic and violin shape functions for a rotating linearly tapered clamped-free Timoshenko beam ($R = 3$, $\bar{\Omega} = 10$, $L = 1$, $r_{yy}/L = 0.09$, $\kappa = 2/3$, $\nu_y = \nu_z = 0.5$)

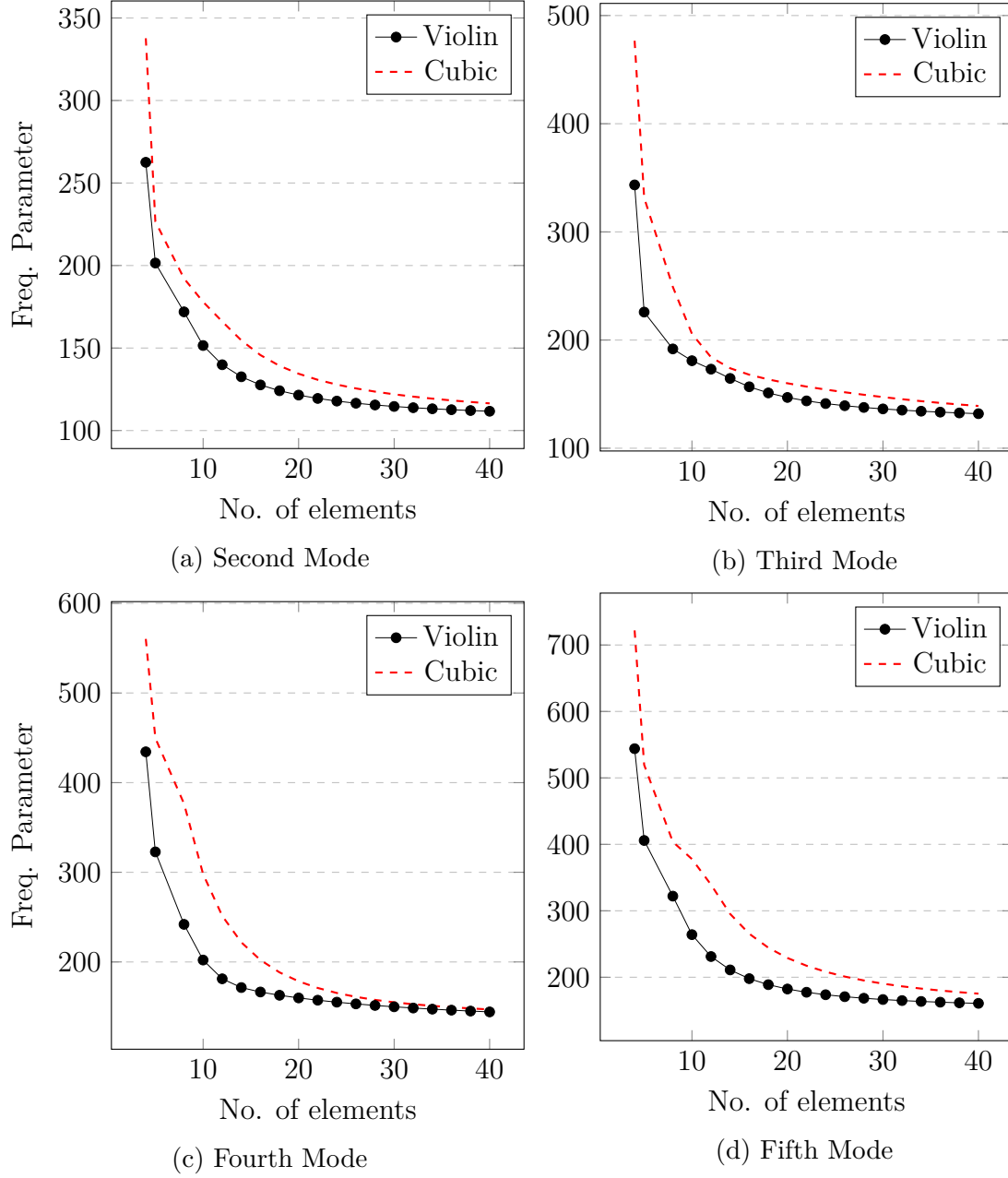


Figure 4.11: Convergence of comparison of cubic and violin shape functions for a rotating linearly tapered clamped-free Timoshenko beam ($R = 3$, $\bar{\Omega} = 100$, $L = 1$, $r_{gy}/L = 0.09$, $\kappa = 2/3$, $\nu_y = \nu_z = 0.5$)

shapes and are constructed from steel. The material and geometric characteristics of the beams used in this study are shown in Table 4.4.

Table 4.4: Geometric and material properties for Timoshenko and Euler-Bernoulli beams

Property	Value
Density, ρ	7850 kg/m ³
Shear modulus, G	77 GPa
Modulus of elasticity, E	200 GPa
Shear factors, $\kappa_y = \kappa_z = \kappa$	5/6
Ratio (r_{gy}/L)	0.02
Number of elements	12 for Euler-Bernoulli beam 16 for Timoshenko beam
Total length of the beam, L	1.00 m for Euler-Bernoulli beam 0.72 m for Timoshenko beam

Natural frequencies

Euler-Bernoulli Beam

Example 4.1 (Non-rotating clamped-free beam). The first example in the model validation is a symmetrical cross-section non-rotating clamped-free Euler-Bernoulli beam. Both homogeneous and evenly tapered beams were taken into consideration. The non-rotating frequencies for the uniform and doubly tapered beams are shown in Table 4.5. The data used in this study was sourced from the research conducted by Khulief and Bazoune [26]. The obtained findings closely align with those of their studies. Moreover, the findings suggest that with increasing taper ratios, all frequencies, except for the fundamental frequency, exhibit a reduction. In contrast, the

fundamental frequency exhibits a contrasting pattern.

Table 4.5: Taper effect on frequency parameters using cubic shape functions for a clamped-free non-rotating Euler-Bernoulli beam

Mode No.	$\nu_y = \nu_z \longrightarrow$	0.0	0.2	0.5	0.7	0.9
1	Present, work	3.51602	3.85512	4.62515	5.50927	7.20488
	Ref. [26]	3.51602	3.85512	4.62515	5.50927	7.20487
2	Present work	22.0348	21.0571	19.548	18.6416	18.6808
	Ref. [26]	22.0348	21.0571	19.548	18.6416	18.6808
3	Present work	61.7049	56.6372	48.5854	42.817	37.1326
	Ref. [26]	61.7049	56.6372	48.5853	42.8169	37.1326

Example 4.2 (Rotating uniform clamped-free beam with hub radius parameters). Table 4.6 shows numerical predictions for the first three lowest frequency features of a uniform clamped-free beam with different values for spin and hub radius. The current values shown in the table have been determined by the use of a cubic polynomial form function. The present Finite Element Method (FEM) solution achieved a high level of agreement when compared to the findings presented by Bazoune Khulief in [27]. Based on the obtained data, it is observed that the frequency parameters exhibit an increase when there is a rise in either the rotation parameter or the hub radius parameter. As these factors grow, the beam's stiffness also increases, resulting in a stiffer beam. The aforementioned phenomena is often denoted as the rotational stiffening effect.

Example 4.3 (Rotating uniform hinged-free beam with hub radius parameters). Table 4.7 shows the result of solving the beam discussed in example 4.3, using hinged-free end boundary conditions. The conclusions shown in this study are very consistent with the results reported by Wright et al. [16], who used the Frobenius approach to get a power series solution. Under the specified boundary condition where the hub radius is reduced to zero, a rigid body flapping mode is seen. This mode exhibits a frequency that is equivalent to the rotational speed of the hub, as shown in Table 4.7. Furthermore, it should be noted that there is a specific mode known as the rigid body mode that is applicable to beams that do not undergo rotation. When the values of both the hub radius and spin parameters are not equal to zero, the spin parameter is precisely equivalent to the fundamental frequency parameter.

Table 4.6: Spin parameter and hub radius effect on frequency parameters using cubic shape functions for a clamped-free Euler-Bernoulli beam, $\psi = 0^\circ$

$\bar{\Omega}$		λ_{E1}		λ_{E2}		λ_{E3}	
		$R = 0.0$	$R = 1.0$	$R = 0.0$	$R = 1.0$	$R = 0.0$	$R = 1.0$
0.0	Present work	3.51598	3.51598	22.0334	22.0334	61.6954	61.6954
	Ref. [27]	3.51602	3.51602	22.0348	22.0348	61.7049	61.7049
2.0	Present work	4.13729	4.83366	22.6138	23.365	62.2713	63.0657
	Ref. [27]	4.13733	4.83371	22.6153	23.3664	62.2808	63.0751
4.0	Present work	5.58496	7.47501	24.2723	26.9562	63.9648	66.9848
	Ref. [27]	5.58503	7.47511	24.2737	26.9577	63.9742	66.9941
6.0	Present work	7.36033	10.4439	26.808	32.0264	66.6819	72.98461
	Ref. [27]	7.36043	10.4440	26.8095	32.0280	66.6912	72.9939
8.0	Present work	9.2568	13.5075	29.9944	37.9534	70.2909	80.5288
	Ref. [27]	9.25694	13.5077	29.9959	37.9552	70.3002	80.5382
10.0	Present work	11.2023	16.6067	33.6395	44.3687	74.6474	89.1576
	Ref. [27]	11.2025	16.6070	33.6411	44.3707	74.5670	89.1673

In line with the aforementioned example, it can be noticed that the flapping frequency characteristics demonstrate a phenomenon of increased rigidity when subjected to hinged-free end boundary conditions.

Table 4.7: Spin parameter and hub radius effect on frequency parameters using cubic shape functions for a hinged-free Euler-Bernoulli beam, $\psi = 0^\circ$

$\bar{\Omega}$		λ_{E1}		λ_{E2}		λ_{E3}	
		$R = 0.0$	$R = 1.0$	$R = 0.0$	$R = 1.0$	$R = 0.0$	$R = 1.0$
0.0	Present work	0.0	0.0	15.4178	15.4178	49.9640	49.9640
	Exact, Ref. [16]	0.0	0.0	15.4182	15.4182	49.9649	–
2.0	Present work	2.0	3.1586	16.2257	17.3175	50.6751	51.7059
	Exact, Ref. [16]	2.0	3.1586	16.2261	17.3179	50.6760	–
4.0	Present work	4.0	6.3056	18.4309	21.9893	52.7454	56.565
	Exact, Ref. [16]	4.0	6.3056	18.4313	21.9897	52.7463	–
8.0	Present work	8.0	12.5864	25.3431	34.5367	60.2500	72.3649
	Exact, Ref. [16]	8.0	12.5865	25.3436	34.5373	60.2513	–
12.0	Present work	12.0	18.8664	33.7598	48.3564	70.8357	92.0336
	Exact, Ref. [16]	12.0	18.8665	33.7603	48.3572	70.8373	–

Example 4.4 (Tapered rotating Euler-Bernoulli beam with). The work of Khulief and Bazoune [26] shows the first 3 frequency parameters of rotating beam with different combinations that includes the rotating speed, taper ratios and boundary conditions. Figures 4.12-4.21 show the comparison between present work and [26]. Figures 4.12-4.16 are for clamped-free rotating Euler-Bernoulli beam. Figures 4.17-4.21 are for hinged rotating Euler-Bernoulli beam. Based on the data presented, it can be inferred that the current findings exhibit a high level of agreement with the results documented in the cited reference [26].

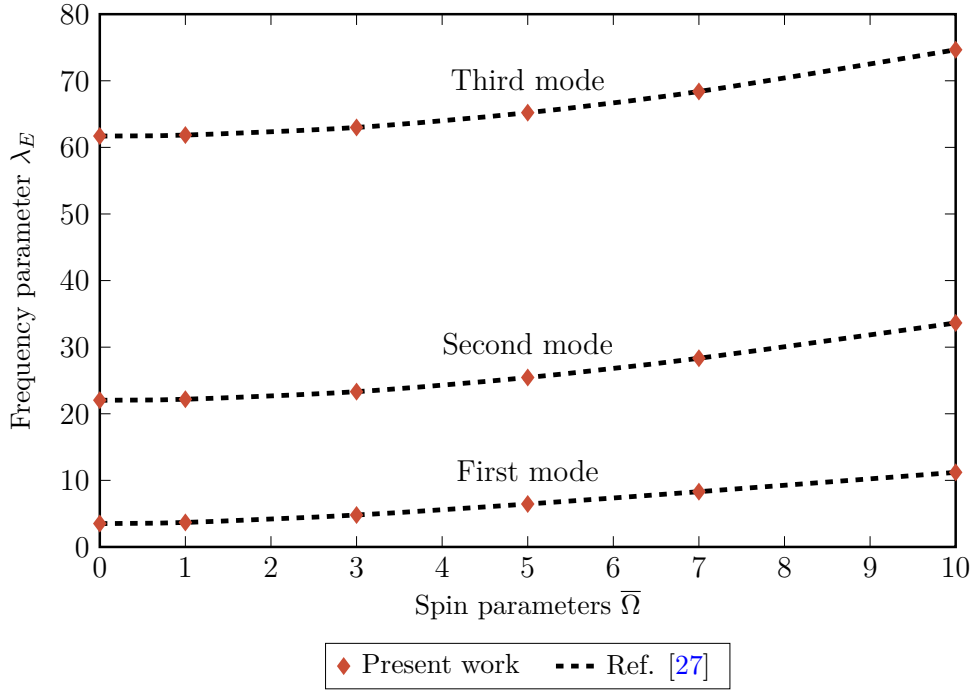


Figure 4.12: Spin parameter effect on the first three frequency parameters using cubic shape functions for a uniform clamped-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.0$

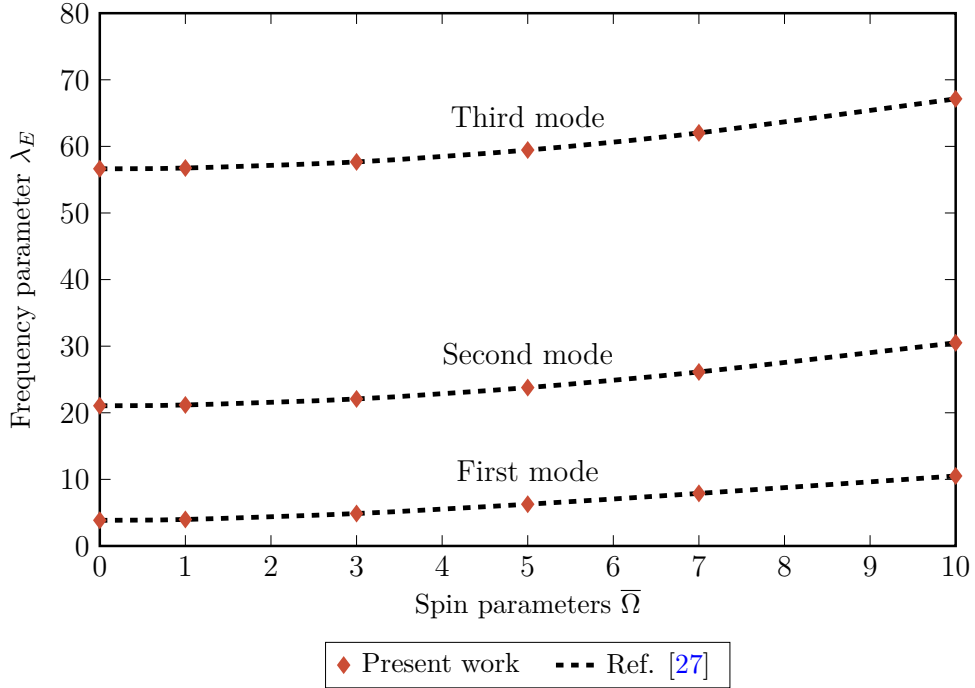


Figure 4.13: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered clamped-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.2$

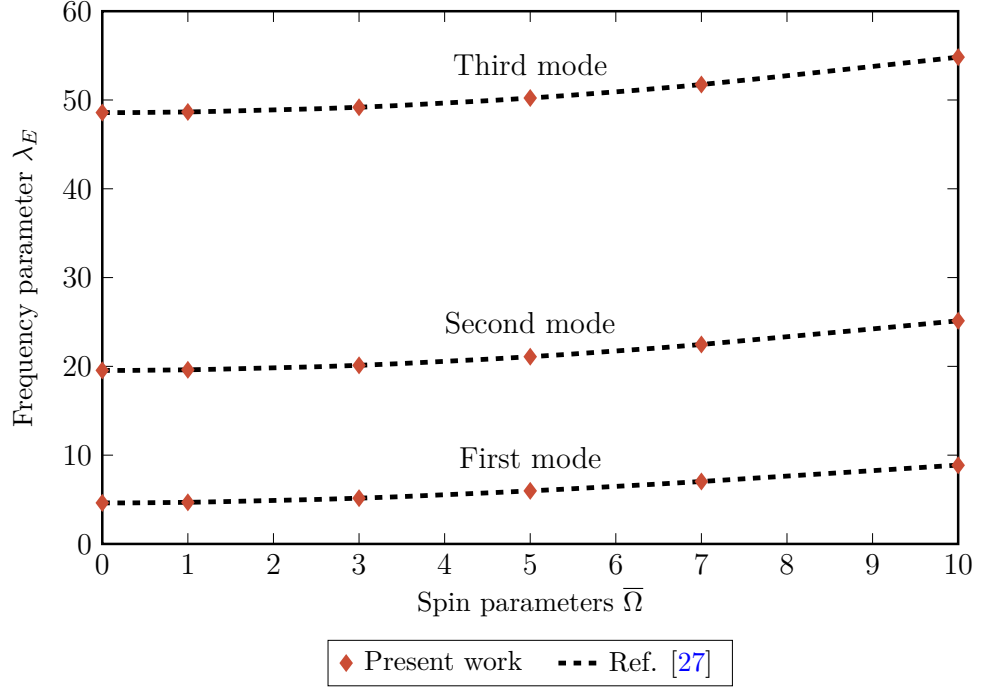


Figure 4.14: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered clamped-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.5$

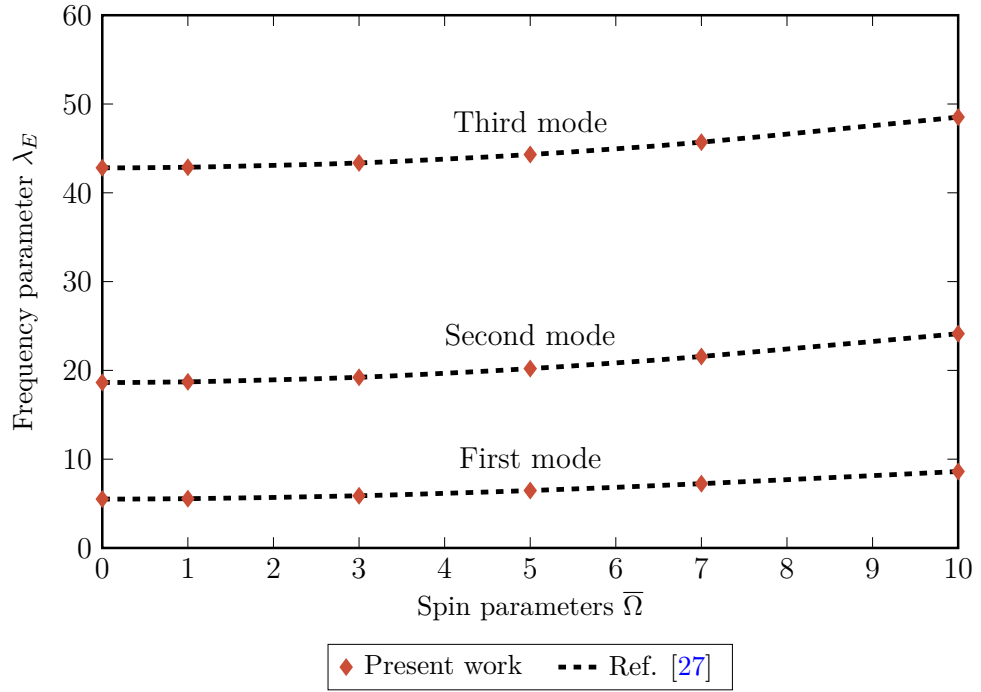


Figure 4.15: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered clamped-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.7$

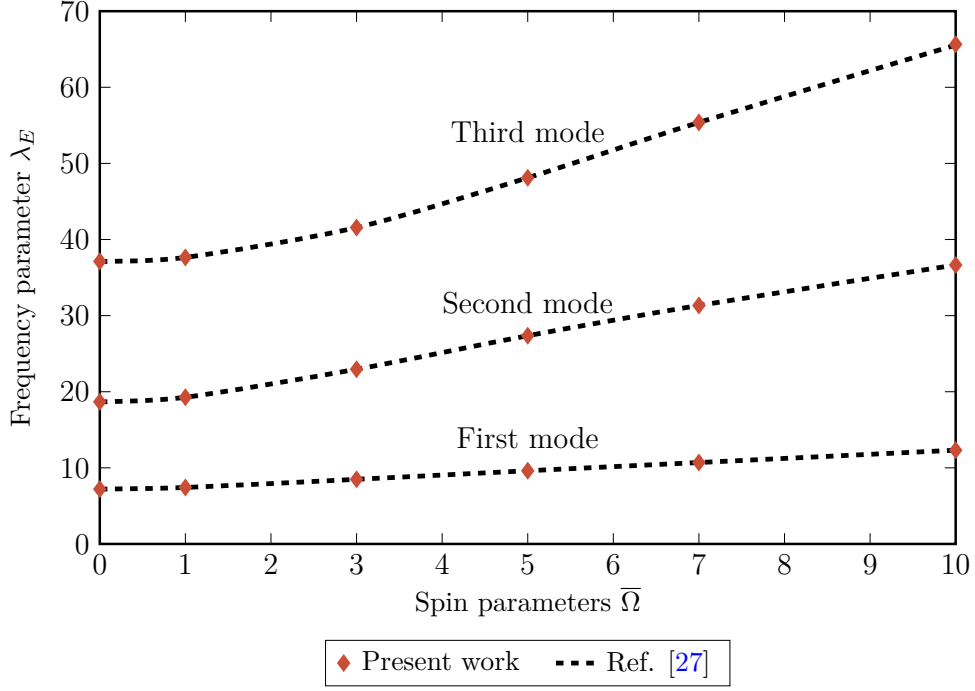


Figure 4.16: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered clamped-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.9$

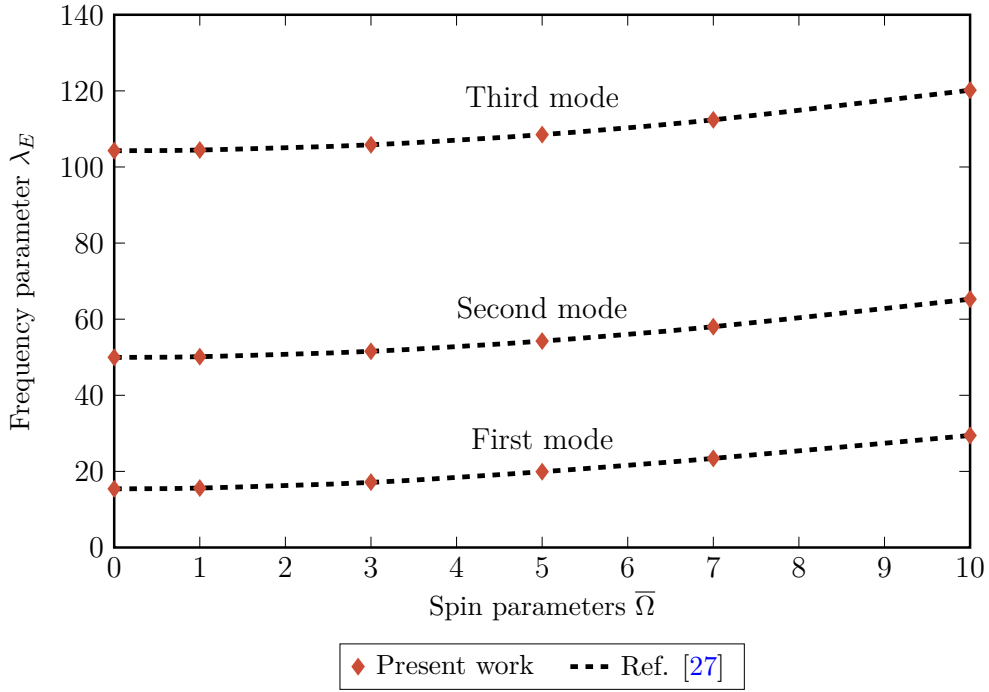


Figure 4.17: Spin parameter effect on the first three frequency parameters using cubic shape functions for a uniform hinged-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.0$

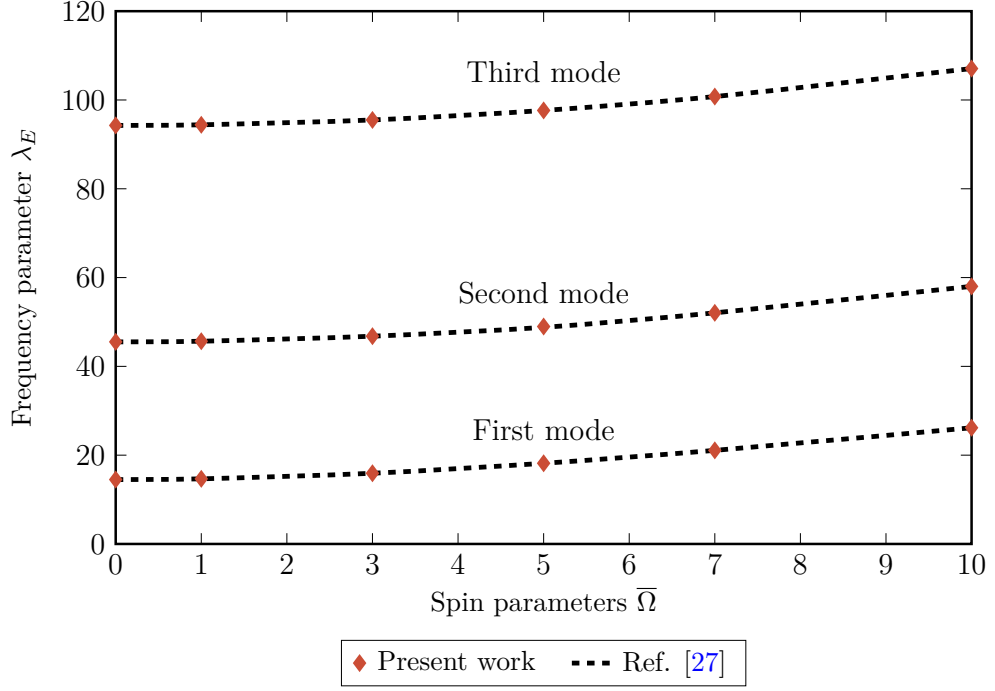


Figure 4.18: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered hinged-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.2$

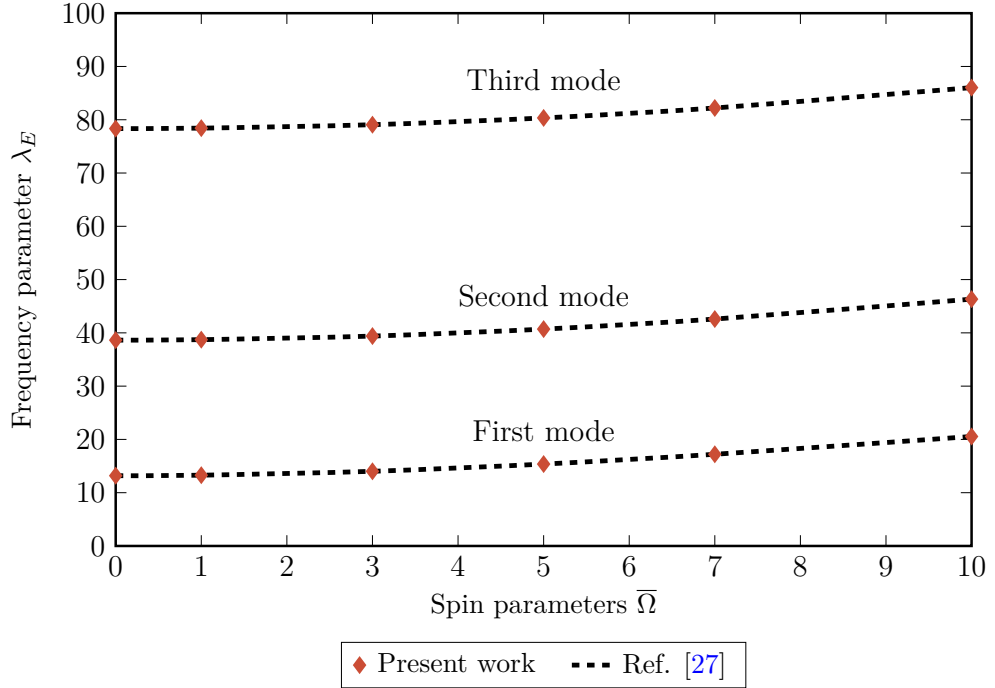


Figure 4.19: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered hinged-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.5$

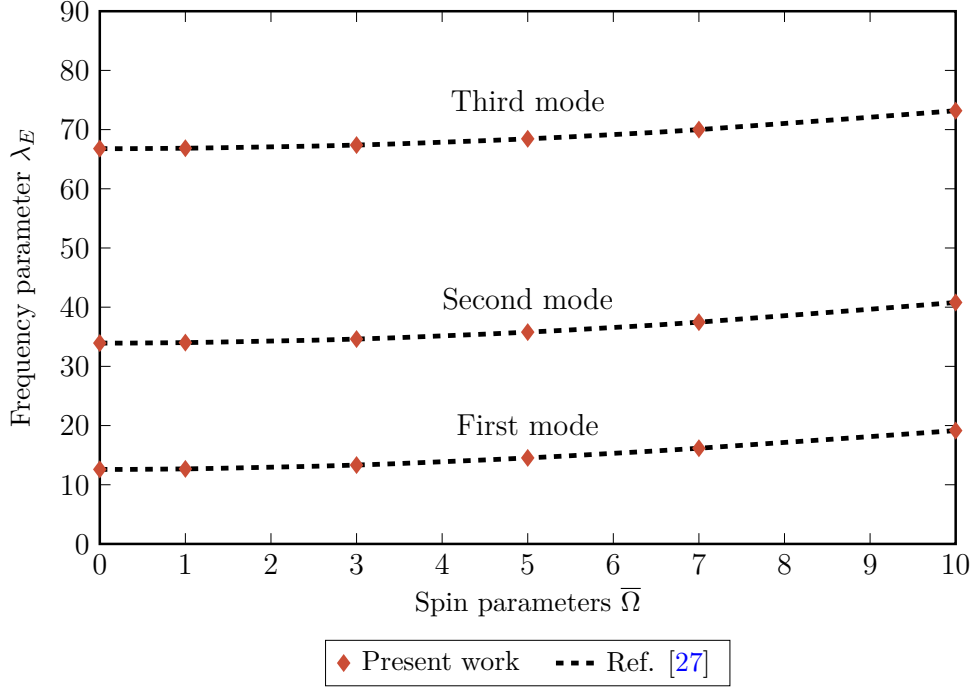


Figure 4.20: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered hinged-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.7$

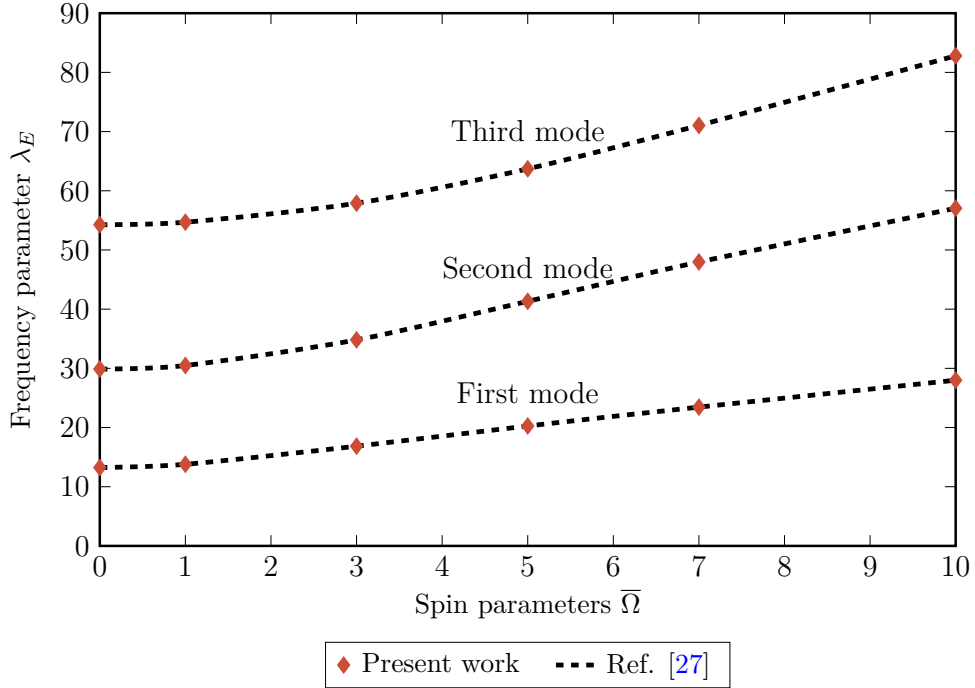


Figure 4.21: Spin parameter effect on the first three frequency parameters using cubic shape functions for a linearly tapered hinged-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.9$

Example 4.5 (Exponentially tapered rotating Euler-Bernoulli beam). Except for the report by Nourifar et al. [97], there is no literature demonstrating the impact of exponential tapering on the frequency parameters of rotating Euler-Bernoulli beams. The analysis was done on exponentially circular tapered beam. From figures 4.22 and 4.23 It may be inferred that the current findings are consistent with those reported in the reference [97]. The deviation observed at the higher speeds is mainly due to the difference in the beam element cross-section. In the present model it is rectangular while it is circular in the work of Nourifar et al. [97] which effects the mass and stiffness matrices.

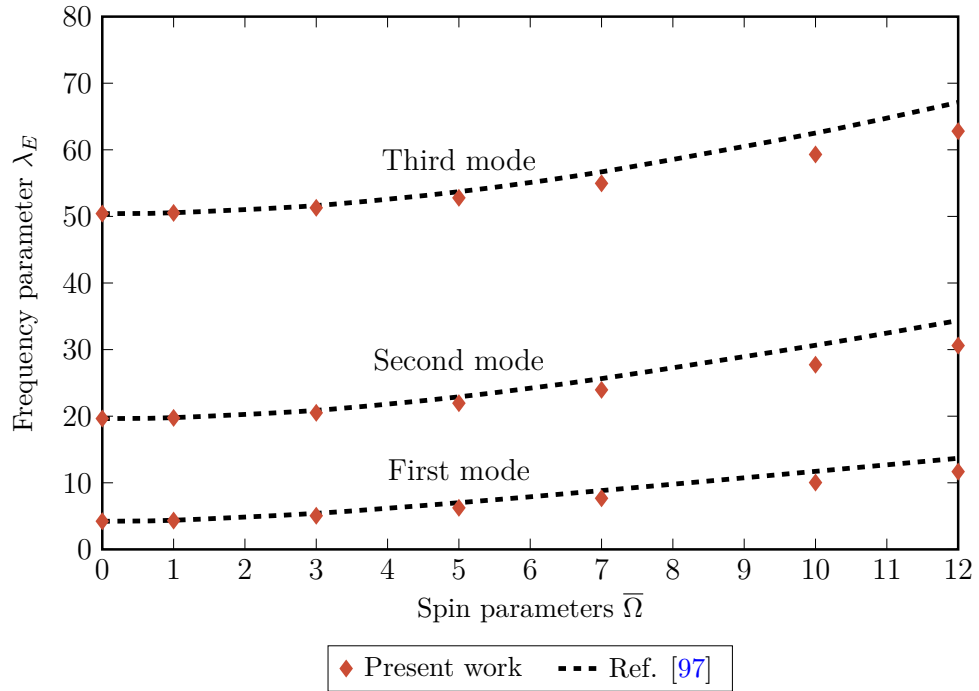


Figure 4.22: Spin parameter effect on the first three frequency parameters using cubic shape functions for an exponentially tapered clamped-free Euler-Bernoulli beam, $\nu_y = \nu_z = 0.5$

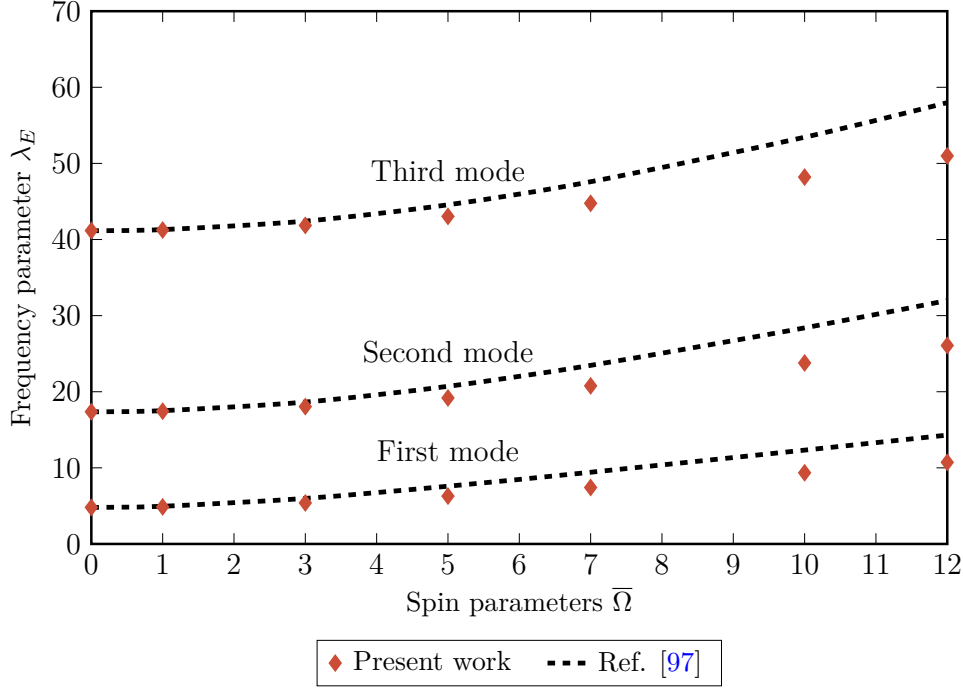


Figure 4.23: Spin parameter effect on the first three frequency parameters using cubic shape functions for an exponentially tapered clamped-free Euler-Bernoulli beam, $\nu_y = \nu_z = 1.0$

Timoshenko Beam The significance of rotational inertia and shear deformation increases notably when examining stubby¹ beams that need greater bending modes.

Example 4.6 (Rotating clamped-free uniform Timoshenko beam). Table 4.8 displays the comparison between the first four flapping frequencies of a uniform, clamped-free Timoshenko beam, which does not rotate, with the frequencies given by Downs in [84]. The degree of agreement between the current forecasts and the results reported in Downs' reference [84] is quite acceptable. When you look at the Euler-Bernoulli beam results in Table 4.6 without rotation ($\bar{\Omega} = 0$), you can see that the frequencies show a trend for the vibration frequencies to go down. This is because of shear deformation and rotary inertia. The effects indicated above are more

¹The beam's cross-sectional dimensions have a higher magnitude relative to its length.

prominent in beams with larger values of the ratio (r_{gy}/L) and have a stronger impact on higher modes, as expected. It may be inferred that an increase in the ratio of the gyration radius to the length, denoted as (r_{gy}/L) , leads to a drop in the natural frequencies across all modes.

Table 4.8: Gyration ratio effect on frequency parameters using cubic shape functions for a clamped-free non-rotating Euler-Bernoulli beam, $R = 0$, $\psi = 0^\circ$

	(r_{gy}/L)	0.01	0.02	0.05	0.08	0.1
λ_{T1}	Present work	3.51271	3.50285	3.43648	3.32417	3.23109
	Ref. [84]	3.51270	3.50284	3.43643	3.32405	3.23093
λ_{T2}	Present work	21.8917	21.4799	19.1475	16.3018	14.5447
	Ref. [84]	21.8910	21.4805	19.1391	16.2890	14.5309
λ_{T3}	Present work	60.7681	58.2381	46.8704	36.8501	31.8104
	Ref. [84]	60.7548	58.1980	46.7510	36.70780	31.6707
λ_{T4}	Present work	117.66	109.441	79.8087	58.8286	48.6777
	Ref. [84]	117.564	109.180	79.2392	58.2788	48.2281

Figure 4.17 displays the percentage difference between the frequencies of Timoshenko and Euler-Bernoulli beams. The range that was chosen to investigate the impact on the first four frequency parameters was $0.01 \leq (r_{gy}/L) \leq 0.1$. According to expectations, a rise in the ratio (r_{gy}/L) leads to a drop in the frequency values, hence leading to the generation of thicker beams. As an example, the difference in percentage between $(r_{gy}/L) = 0.01$ and $(r_{gy}/L) = 0.1$ is inconsequential. Furthermore, as previously mentioned, the influence of shear deformation and rotating inertia effects becomes more significant at higher modes. A drop of 8.3% was observed in the initial mode for a thick beam (i.e., gyration ratio $(r_{gy}/L) = 1$). Furthermore, a more sig-

nificant decrease of 34.5%, 48.9%, and 60% for the second, third, and fourth modes, respectively.

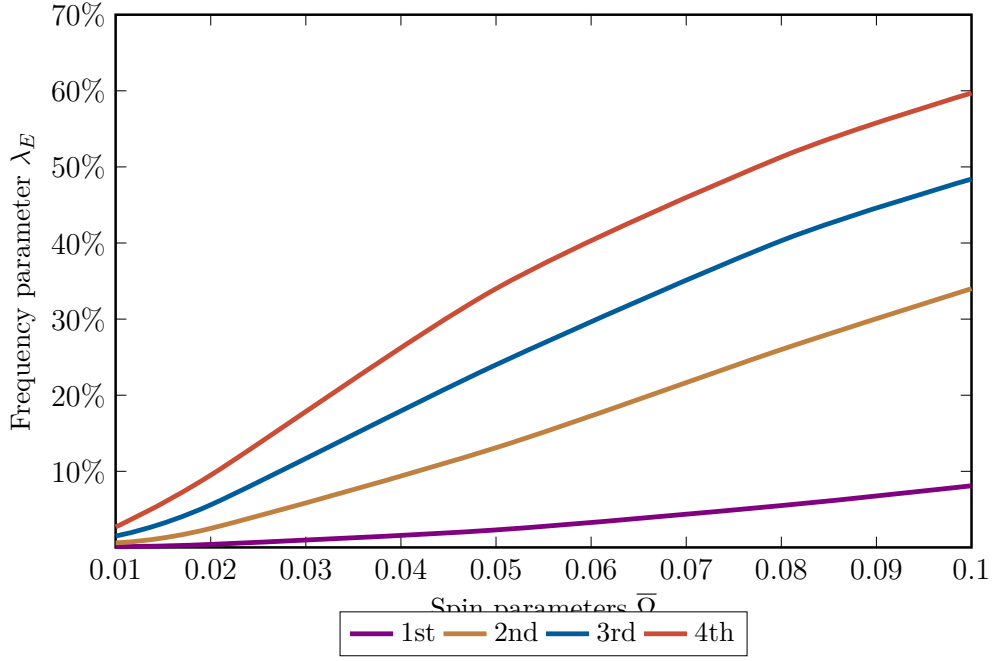


Figure 4.24: Frequency percentage difference between TBT and EBT as a function of (r_{gy}/L) for non-rotating beams.

Example 4.7 (Rotating clamped-free uniform beam with different slenderness ratio). The effect to the frequency parameter λ_T of a uniform Timoshenko beam when the spin rotation and slenderness ratio are changed. The changes in the fundamental frequency parameter for the slenderness ratio (L/r_{gy}) are shown in Table 4.9 for both rotating and non-rotating beams. The rotating beam utilizes a combination of cubic form functions and violin string shape functions. In the equation of Δ , the frequency parameter λ_E is assigned a value of 3.516016 when $\bar{\Omega}$ is equal to zero and a value of 6.449548 when $\bar{\Omega}$ is equal to 5. The findings of this study demonstrate a high level of concurrence with the theoretical forecasts put forward by Lee and Kuo [22] in relation to both non-rotating and rotating situations. The frequency parameters

for a rotating beam with a slenderness ratio of $\bar{\Omega} = 5$ are seen to be greater than those of a non-rotating beam. This discrepancy may be attributed to the stiffening effect, which leads to an overall rise in the frequency values across all slenderness ratios. Also, as slenderness ratios get bigger, the difference between the Euler-Bernoulli beam (EBT) and the Timoshenko beam (TBT) decreases until it almost equals zero. This means that TBT converges to EBT. To clarify, as the ratio of length to radius of gyration (L/r_{gy}) is increased, the frequency parameters for Timoshenko beams also rise. However, as shown in Table 4.9, at very high slenderness ratios, these frequency parameters eventually converge to the Euler-Bernoulli frequencies.

Table 4.9: Timoshenko clamped-free beam planar fundamental frequency parameter $\rho = 7850 \text{ kg/m}^3$, λ_T , $E/\kappa G = 3.059$, $E = 200 \text{ GPa}$, $\psi = 0^\circ$, $R = 0$, $\lambda_{E1} = 3.5160$, $\lambda_{E2} = 6.449548$ and $\Delta = \frac{\lambda_E - \lambda_T}{\lambda_E} \times 100\%$

$\left(\frac{L}{r_{gy}}\right)$	$\bar{\Omega} = 0$			$\bar{\Omega} = 5$				
	Present work λ_T	Ref. [22] λ_T	Δ	Present work λ_T		Ref. [22] λ_T	Δ	
	Cubic			Cubic	Violin		Cubic	Violin
20	3.43648	3.4364	2.26	6.33206	6.33177	6.3241	1.82	1.83
40	3.49552	3.4954	0.58	6.41827	6.41818	6.4179	0.48	0.49
80	3.51085	3.5108	0.15	6.4416	6.44157	6.4418	0.12	0.12
100	3.51271	3.5126	0.09	6.44445	6.44443	6.4446	0.08	0.08
200	3.51519	3.5152	0.02	6.44827	6.44826	6.4485	0.02	0.02
300	3.51565	3.5155	0.01	6.44898	6.44898	6.4493	0.01	0.01
∞	3.51602	3.516	0.00	6.44955	6.44954	6.4497	0.00	0.00

Example 4.8 (Gyration ratio effect on rotating clamped-free beam). To see the impact of gyration ratio on a rotating uniform Timoshenko beam frequency pa-

rameters. Nagaraj [88] studied a Timoshenko beam with different (r_{gy}/L) ratio. Table 4.10 illustrates the effect of low and high gyrations ratios in the first four planar flapping frequency parameters. The study revealed that there was a positive correlation between the spin parameter and the frequency parameters across all modes. Nevertheless, it was observed that the frequency values exhibited a reduction with an increase in the (r_{gy}/L) ratio, suggesting that the presence of rotational inertia had a diminishing influence on the frequency values. The validation of the current model's correctness was conducted by comparing the findings obtained with the frequencies provided by Nagaraj [88], which showed agreement.

Table 4.10: Rotating uniform clamped-free Timoshenko beam planar frequency parameters for different values of (r_{gy}/L) , $R = 0$, and $\psi = 0^\circ$

$\bar{\Omega}$	Model No.	$(r_{gy})/L = 0.01$		$(r_{gy})/L = 0.1$	
		Present work	Ref. [88]	Present work	Ref. [88]
0	1	3.5026	3.5026	3.2273	3.2274
	2	21.4691	21.4698	14.4803	14.4689
	3	58.1765	58.1498	31.5992	31.5025
	4	109.2609	109.0275	48.2100	47.9090
6	1	7.3359	7.3319	6.9311	6.8509
	2	26.2571	26.2488	19.8483	19.6787
	3	63.2656	63.2386	38.7717	38.5785
	4	114.7635	114.5177	55.4906	56.2950
12	1	13.1089	13.1046	12.4657	12.1665
	2	36.9947	36.9790	30.0027	29.3024
	3	76.2875	76.2744	52.0115	51.2741
	4	129.6420	129.3472	80.2560	79.0558

Example 4.9 (Tapered non-rotating clamped-free Timoshenko beam). The influence of taper in Timoshenko beams is shown in Table 4.11, which includes the predictions made by Downs [85]. The research investigation centered on examining the non-rotating Timoshenko beam first three flapping frequency characteristics. The study explored eight distinct combinations of uneven tapering in both the width and depth directions. The outcomes acquired by the current model demonstrate a remarkable concurrence with Downs’s solution. The numerical predictions suggest that an increase in the taper ratio ν_y is associated with a drop in thickness towards the tip of the beam. This decrease in thickness leads to a reduction in both the mass and stiffness of the beam. Consequently, Table 4.11 illustrates the observed increase in the frequency characteristics for every mode. Regarding the effect of tapering, it is observed that the first mode remains unchanged, but the second and third modes exhibit a reversal in their respective patterns.

Table 4.11: Non-rotating clamped-free tapered Timoshenko beam frequency parameters, $\kappa = 0.85$, $R = 0$, $(r_{gy}/L) = 0.08$, and $\psi = 0^\circ$

ν_y		λ_{E1}		λ_{E2}		λ_{E3}	
		$\nu_z = 0.3$	$\nu_z = 0.6$	$\nu_z = 0.3$	$\nu_z = 0.6$	$\nu_z = 0.3$	$\nu_z = 0.6$
0.0	Present work	3.48603	3.76095	15.6488	14.6489	34.778	31.7084
	Ref. [85]	3.4869	3.7623	15.6411	14.6449	34.6625	31.6243
0.3	Present work	3.85286	4.1332	16.1855	15.1363	35.2207	32.1475
	Ref. [85]	3.8535	4.1339	16.1766	15.1297	35.1026	32.0582
0.6	Present work	4.461	4.75022	17.0347	15.9225	35.9595	32.8698
	Ref. [85]	4.4611	4.7498	17.0229	15.9107	35.8346	32.7692
0.9	Present work	5.78465	6.09532	19.1278	17.9524	38.1255	34.9876
	Ref. [85]	5.7836	6.0927	19.1076	17.9264	37.9753	34.8498

Example 4.10 (Effect of setting angle on uniform rotating Timoshenko beam). The study conducted by Vindo Kumar and Ganguli [93] introduced the violin string form functions for a rotating Timoshenko beam. However, their analysis did not account for the hub radius or setting angle. In order to investigate the impact on rotating uniform Timoshenko beams by setting angle on the flapping frequencies, the first four flapping frequency parameters is presented in Table 4.12 for setting angles of $\psi = 0^\circ$ and 90° . The material and geometric characteristics of this beam exhibit similarities to the qualities described by Bazoune and Khulief [27]. The present results for both violin and cubic shape functions are in excellent agreement when compared to [27].

Table 4.12: Rotating uniform clamped-free Timoshenko beam the first four planar frequency parameters of with $R = 3$, $\Omega = 10$, $L = 1$, $\kappa = 0.85$, and $\nu = 0.3$

		$\psi \longrightarrow$	$(r_{gy})/L = 0.05$		$(r_{gy})/L = 0.1$	
			0°	90°	0°	90°
λ_{T1}	Present work	Violin	23.514	21.3026	23.0405	20.8565
		Cubic	23.524	21.3136	23.0503	20.8669
	Ref. [27]		23.522	21.2816	23.006	20.8321
λ_{T2}	Present work	Violin	56.1252	55.3045	45.7041	45.2174
		Cubic	56.105	55.2839	45.598	45.1153
	Ref. [27]		56.0951	55.257	45.8150	45.0809
λ_{T3}	Present work	Violin	97.3534	96.9148	68.0342	67.8449
		Cubic	97.1875	96.7474	67.7163	67.5198
	Ref. [27]		97.1537	96.7106	68.9308	67.3641
λ_{T4}	Present work	Violin	145.005	144.725	73.1892	72.8704
		Cubic	144.49	144.208	73.0756	72.7562
	Ref. [27]		144.395	144.115	73.0185	72.7188

Example 4.11 (Effect of taper on rotating clamped-free beam). Bazoune and Khulief [40] studied a rotating Timoshenko beam with different speed and taper ratios, and their findings were analyzed to see how rotation and taper ratios affect the beam. Tables 4.13 to 4.16 show the results obtained using both the cubical and violin shape functions for different taper ratios and spin parameters. It worth emphasizing that only the rotating beam is considered since the violin string shape function is can not be applied, as the violin shape function produces undefined terms when the speed parameter is zero. Nonetheless, the frequencies reported by Bazoune and Khulief [40] are closely matched, confirming the precision of the present model's results with either shape function.

The predictions that have been made so far, as seen in Tables 4.5 to 4.16 and Figures 4.12 to 4.17, are examples that fall within the scope of the current model. The purpose of their presentation was to cultivate trust in the suggested concept. It is evident that the current method exhibits a high level of concordance with other numerical findings documented in the existing literature for these particular scenarios.

4.2.3 Effect of exponential taper on rotating Timoshenko beam

The literature lacks data on the results of an exponentially tapered rotating Timoshenko beam, with the exception of the study conducted by Nourifar et al. (2018) [97]. In their research, Nourifar et al. utilized the differential transform method (DTM) to investigate the behavior of a rotating beam with a circular cross section. The validation of the present model versus Nourifar et al. [97] results was done previously in 4.5.

Table 4.13: Spin parameter effect on frequency parameters λ using cubic and violin shape functions for a uniform clamped-free Timoshenko beam ($v_y = v_z = 0.0$)

$\bar{\Omega}$		Shape function	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
1	present work	Cubic	3.48898	16.4687	37.057	59.0851	81.8789	96.0541	109.675	118.117	138.902	145.784
		violin	3.48897	16.469	37.0584	59.0884	81.8842	96.0634	109.683	118.127	138.91	145.797
3	Ref [40]	Cubic	3.48952	16.4534	37.0035	58.9728	81.6906	95.6941	109.356	117.763	138.532	145.385
	present work	Cubic	4.5852	17.744	38.6569	61.0696	84.0775	96.9167	111.405	119.782	140.907	148.023
5	Ref [40]	violin	4.58507	17.7465	38.6683	61.0976	84.1221	96.9991	111.471	119.873	140.975	148.132
	present work	Cubic	4.58641	17.731	38.6081	60.9631	83.8921	96.5552	111.09	119.445	140.537	147.645
7	Ref [10]	Cubic	6.18216	20.0371	41.6031	64.7206	87.989	98.3343	114.479	123.077	144.11	152.795
	present work	violin	6.18163	20.0433	41.6312	64.7908	88.0995	98.5542	114.64	123.306	144.279	153.07
10	Ref [40]	Cubic	6.18416	20.0216	41.5617	65.6224	87.8	97.9783	114.161	122.775	143.733	152.462
	present work	Cubic	7.95479	23.02	45.5296	69.5446	92.7099	100.212	118.313	127.741	148.161	159.587
10	Ref [40]	violin	7.95361	23.0302	45.5766	69.6668	92.9089	100.605	118.585	128.13	148.449	160.05
	present work	Cubic	7.95761	23.0142	45.496	69.4524	92.4923	99.8844	117.992	127.475	147.79	159.296
10	Ref [40]	Cubic	10.7335	28.2408	52.5281	77.8175	98.6421	104.761	125.34	135.254	157.095	169.721
	present work	violin	10.731	28.256	52.6011	78.0385	99.0938	105.319	125.768	135.842	157.561	170.384
10	Ref [40]	Cubic	10.7377	28.2394	52.5039	77.7166	98.3338	104.512	125.058	134.985	156.79	169.423

Table 4.14: Spin parameter effect on frequency parameters λ using cubic and violin shape functions for a linearly tapered clamped-free Timoshenko beam ($v_y = v_z = 0.3$)

$\bar{\Omega}$		Shape function	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
1	present work	Cubic	3.95529	16.2919	35.3529	56.9707	80.3066	103.396	115.926	126.062	142.37	152.321
		violin	3.95527	16.2922	35.3546	56.975	80.3142	103.409	115.942	126.079	142.387	152.34
3	Ref [40]	Cubic	3.95744	16.2847	35.314	56.8817	80.1576	103.133	115.427	125.673	141.891	151.867
		Cubic	4.69161	17.1176	36.3888	58.2676	81.8524	104.909	116.267	127.438	143.383	153.711
5	Ref [40]	violin	4.69146	17.1207	36.4032	58.3035	81.9168	105.02	116.407	127.586	143.525	153.873
		Cubic	4.69414	17.1117	36.3525	58.1825	81.7084	104.638	115.782	127.038	142.925	153.251
7	Ref [40]	Cubic	5.88268	18.6528	38.3538	60.7382	84.7976	107.533	117.168	129.765	145.54	156.174
		violin	5.8823	18.6612	38.3906	60.8278	84.9567	107.831	117.508	130.152	145.888	156.59
10	Ref [40]	Cubic	5.8858	18.6489	38.3221	60.66	84.662	107.229	116.725	129.342	145.12	155.704
		Cubic	7.29748	20.7304	41.0789	64.1815	88.8959	110.331	119.253	132.373	149.041	159.254
7	Ref [40]	violin	7.29674	20.7454	41.1426	64.3344	89.1669	110.922	119.773	133.079	149.607	159.993
		Cubic	7.30128	20.7288	41.0528	64.1115	88.7697	109.95	118.896	131.919	148.666	158.779
10	Ref [40]	Cubic	9.60803	24.5347	46.2085	70.6871	96.5755	112.863	125.743	135.852	156.497	164.583
		violin	9.60645	24.5597	46.3116	70.9351	97.0214	113.998	126.46	137.064	157.383	165.795
10	Ref [40]	Cubic	9.61296	24.5366	46.1907	70.6286	96.4592	112.377	125.486	135.378	156.133	164.156

Table 4.15: Spin parameter effect on frequency parameters λ using cubic and violin shape functions for a linearly tapered clamped-free Timoshenko beam ($v_y = v_z = 0.6$)

$\bar{\Omega}$		Shape function	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
1	present work	Cubic	4.79158	15.9739	32.9333	53.1242	75.5397	99.4707	123.997	131.415	152.01	164.796
		violin	4.7914	15.9774	32.9875	53.3281	75.9383	99.9573	124.362	131.508	152.263	165.138
3	Ref [40]	Cubic	4.79747	15.9839	32.9303	53.0917	75.4699	99.3544	123.773	130.796	151.765	163.904
	present work	Cubic	5.10967	16.3788	33.4359	53.7437	76.2741	100.313	124.792	131.645	153.077	164.921
5	Ref [40]	violin	5.10861	16.3817	33.5341	54.0935	76.8904	101.041	125.453	131.962	153.92	165.753
	present work	Cubic	5.11557	16.389	33.434	53.7131	77.2069	100.2	124.553	131.045	152.835	164.026
7	Ref [40]	violin	5.68918	17.1753	34.5611	55.3645	78.3934	102.837	127.093	132.949	156.47	166.709
	present work	Cubic	5.69567	17.1675	34.4126	54.9229	77.6429	101.846	125.923	131.68	154.902	164.274
10	Ref [40]	violin	6.45695	18.2552	35.8138	56.6857	79.7664	104.308	127.825	133.492	158.034	165.583
	present work	Cubic	6.4581	18.2956	36.0127	57.1606	80.538	105.361	128.875	134.708	159.738	167.866
10	Ref [40]	violin	7.82649	20.3743	38.5789	60.1263	83.8567	108.941	129.565	137.48	162.95	167.191
	present work	Cubic	7.83068	20.445	38.8415	60.6899	84.7667	110.235	130.973	139.28	165.192	170.365
10	Ref [40]	violin	7.83303	20.3867	38.5856	60.112	83.8124	108.849	129.015	137.222	162.526	166.495
	present work	Cubic	7.83303	20.3867	38.5856	60.112	83.8124	108.849	129.015	137.222	162.526	166.495

Table 4.16: Spin parameter effect on frequency parameters λ using cubic and violin shape functions for a linearly tapered clamped-free Timoshenko beam ($v_y = v_z = 0.9$)

$\bar{\Omega}$		Shape function	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
1	present work	Cubic	7.00446	17.1292	30.7131	47.3348	66.4954	87.7506	110.734	134.728	146.07	162.558
		violin	7.00991	17.1988	30.701	47.5952	66.5527	88.1812	111.088	135.267	146.229	163.156
3	Ref [40]	Cubic	7.05313	17.2264	30.8415	47.4974	66.7295	88.1178	111.309	135.485	145.693	163.841
		Cubic	7.95105	20.4061	35.0694	51.4601	70.0325	90.9254	113.815	137.606	146.683	166.055
5	Ref [40]	violin	7.95346	20.5342	35.1647	51.7436	70.4158	91.4549	114.611	138.561	147.108	167.193
		Cubic	8.00965	20.5635	35.3639	51.8855	70.5499	91.5462	114.591	138.361	146.431	167.384
7	Ref [40]	Cubic	8.90666	23.7247	40.6185	58.0859	76.0809	96.3233	119.201	141.901	148.697	172.521
		violin	8.90702	23.8433	40.864	58.4007	76.5523	97.0737	120.14	142.994	149.689	174.129
10	Ref [40]	Cubic	8.49536	22.29	38.2053	55.1828	73.5663	94.2432	117.206	140.391	147.349	170.336
		Cubic	9.844	26.6477	45.6718	65.1636	83.3705	102.84	125.892	144.59	154.24	180.95
	present work	violin	9.84581	26.756	45.9765	65.6113	83.8362	103.706	127.012	145.724	155.782	182.999
		Cubic	9.91047	26.8225	46.0153	65.94	84.8615	104.423	127.241	144.346	155.38	182.373
	present work	Cubic	11.2906	30.6838	52.4423	75.0233	94.8212	113.295	136.544	146.78	166.796	192.399
		violin	11.2948	30.7967	52.7728	75.6061	95.342	114.143	137.886	148.392	168.735	196.27
	Ref [40]	Cubic	11.3619	30.8723	52.7854	75.7917	96.8864	115.709	137.942	146.662	168.316	192.366

The breadth and depth equations for exponentially tapered profiles are illustrated in 3.1. The exponentially tapered rotating beam will be studied in this section using Timoshenko and Euler-Bernoulli theories. The research looks at the non-dimensional natural frequencies of a rectangular cross-section spinning an exponentially tapered beam. The width and height of the beam vary exponentially. The study investigates several situations, including rotating, non-rotating, uniform, and exponentially tapering beams. In the study, clamped and hinged end boundary conditions are taken into account. There are two ways to show the results: tabular and graphical. When a linear taper beam is considered, the maximum taper parameter that can be used is 1, which defines the case of a pyramid. However, when an exponential taper beam is considered, the taper parameter can be higher than 1. The higher the taper parameter ν , the more taper will be modeled. Figure 4.25 illustrates the effects of different exponential tapers and compares them with a linear taper beam with $\nu = 0.5$.

clamped-free Timoshenko beam

In tables 4.17 - 4.22, the taper parameter are varied from 0 to 2 for an exponentially tapered clamped-free Timoshenko beam. The first six bending frequencies of clamped-free Euler-Bernoulli and Timoshenko beams are shown in Figures 4.26 to 4.31. These frequencies are calculated by using different spin and taper parameters. The comparison of frequencies between Euler-Bernoulli and Timoshenko models, as illustrated in Figures 4.26 to 4.31, reveals that the Euler-Bernoulli frequencies exhibit greater values. The higher difference between the Euler-Bernoulli and Timoshenko frequencies is most observed in a uniform beam, then it decays until they both almost match

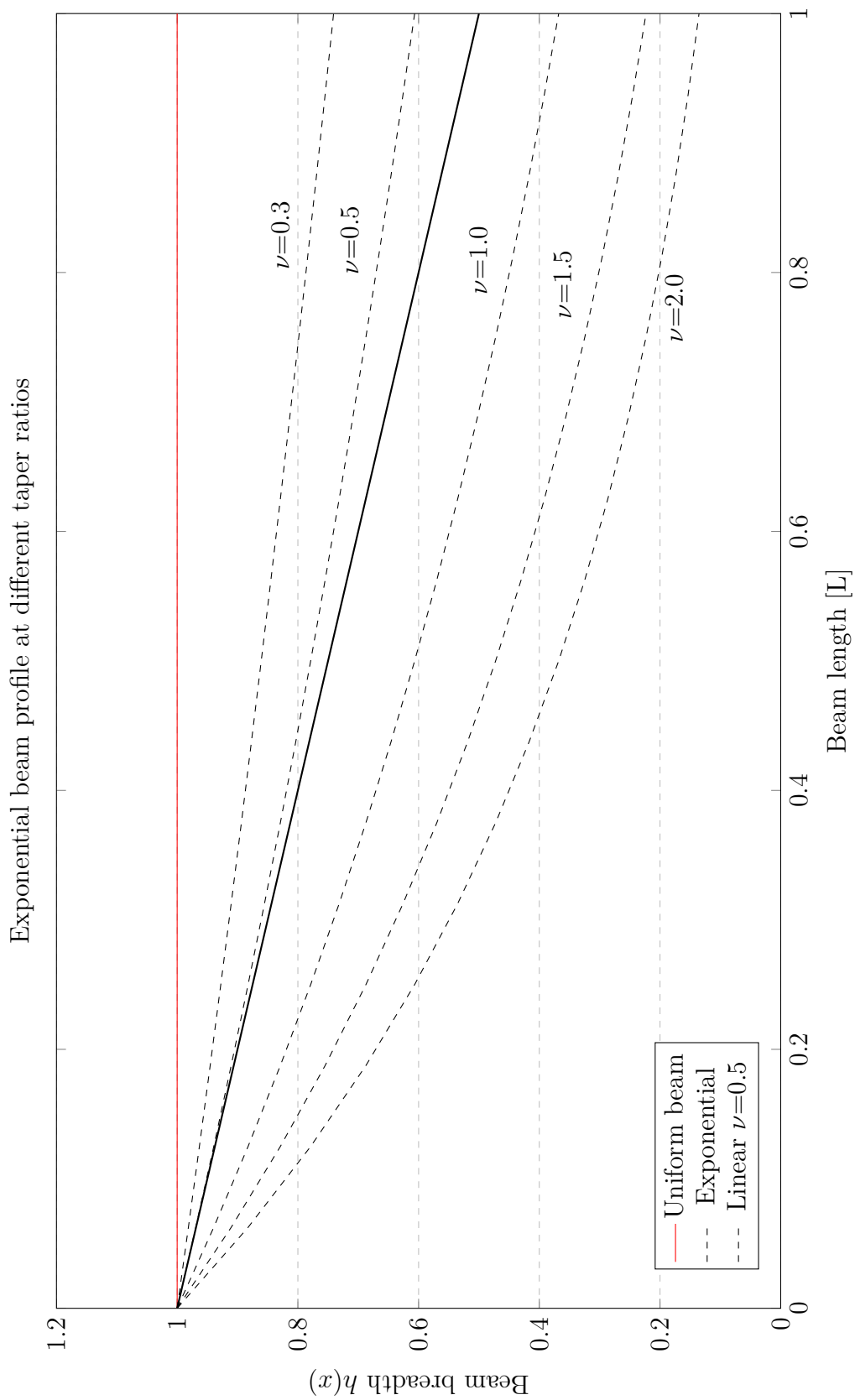


Figure 4.25: Exponential beam profile

for both beam theories. The non-rotating beam frequencies presented in Figures 4.26 and 4.27 exhibit comparable characteristics to the linearly tapered beam previously examined by Bazoune and Khulief [40]. In order to gain a more comprehensive understanding of the relationship between speed and taper ratio, a spin parameter of 10 was subjected to detailed analysis.

Table 4.17: Spin parameter effect on frequency parameters using cubic shape functions for a uniform clamped-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	3.32417	16.3018	36.8501	58.8286	81.592	95.935	109.451	117.907	138.625	145.517
1	3.48898	16.4687	37.057	59.0851	81.8789	96.0541	109.675	118.117	138.902	145.784
3	4.5852	17.744	38.6569	61.0696	84.0775	96.9167	111.405	119.782	140.907	148.023
5	6.18216	20.0371	41.6031	64.7206	87.989	98.3343	114.479	123.077	144.11	152.795
7	7.95479	23.02	45.5296	69.5446	92.7099	100.212	118.313	127.741	148.161	159.587
10	10.7335	28.2408	52.5281	77.8175	98.6421	104.761	125.34	135.254	157.095	169.721

Table 4.18: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered clamped-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 0.3$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	3.74584	16.13	35.3411	56.998	80.301	102.947	115.096	124.887	141.432	151.363
1	3.86441	16.2526	35.4904	57.1833	80.521	103.157	115.15	125.073	141.593	151.551
3	4.70062	17.2013	36.6573	58.635	82.2444	104.74	115.631	126.466	142.903	152.98
5	6.01509	18.9508	38.8604	61.389	85.5125	107.354	116.912	128.788	145.615	155.531
7	7.54387	21.2955	41.896	65.2037	90.0261	109.867	119.721	131.397	149.752	158.85
10	10.0048	25.5416	47.5596	72.3408	98.3656	111.998	127.081	135.248	157.542	165.327

Table 4.19: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered clamped-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 0.5$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	4.0155	15.8746	34.1864	55.3429	78.4177	102.335	122.627	127.178	148.746	153.582
1	4.11238	15.9768	34.3084	55.4926	78.5954	102.535	122.653	127.381	148.824	153.774
3	4.81422	16.7715	35.2666	56.6706	79.9947	104.098	122.838	128.983	149.399	155.318
5	5.96293	18.2542	37.0963	58.932	82.6846	107.061	123.175	132.011	150.434	158.294
7	7.33867	20.2701	39.6566	62.1182	86.4784	111.087	123.757	135.977	152.064	162.219
10	9.597	23.9814	44.5353	68.2362	93.7519	117.525	126.111	141.853	156.701	168.352

Table 4.20: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered clamped-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 1.0$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	4.59689	14.9205	30.9417	50.3292	72.0737	95.5718	120.426	141.629	148.985	174.008
1	4.66059	14.9928	31.0229	50.4238	72.1831	95.6966	120.565	141.67	149.114	174.158
3	5.14019	15.5584	31.6647	51.1736	73.0511	96.6873	121.667	141.964	150.166	175.308
5	5.97848	16.6306	32.9102	52.6377	74.7496	98.6265	123.814	142.424	152.353	177.319
7	7.04045	18.1169	34.693	54.7527	77.2113	101.437	126.896	142.957	155.69	179.453
10	8.86205	20.9167	38.2005	58.9697	82.1423	107.063	132.841	144.022	162.591	181.499

Table 4.21: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered clamped-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 1.5$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	4.94883	13.7246	27.3959	44.5678	64.3218	86.1596	109.778	134.887	158.557	165.414
1	4.9994	13.7865	27.462	44.6396	64.4008	86.2468	109.874	134.991	158.612	165.484
3	5.38369	14.271	27.9849	45.2098	65.0293	86.9406	110.637	135.817	159.022	166.072
5	6.06715	15.1866	29.0014	46.3293	66.268	88.3101	112.144	137.446	159.694	167.377
7	6.94765	16.4497	30.4598	47.9597	68.0826	90.3214	114.359	139.829	160.436	169.557
10	8.48228	18.8113	33.3359	51.2504	71.7814	94.4375	118.893	144.645	161.517	174.552

Table 4.22: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered clamped-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 2.0$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	4.97427	12.4325	23.8017	38.5062	55.8435	75.3788	96.8682	120.134	144.982	170.414
1	5.02676	12.4976	23.8695	38.5754	55.9153	75.4542	96.948	120.219	145.071	170.495
3	5.41842	13.0016	24.4029	39.1248	56.4865	76.055	97.5839	120.893	145.784	171.132
5	6.09027	13.9305	25.4267	40.1981	57.6112	77.2425	98.8432	122.231	147.195	172.343
7	6.92559	15.1693	26.8664	41.7478	59.2557	78.9893	100.701	124.207	149.279	173.966
10	8.34145	17.3904	29.617	44.8277	62.593	82.5723	104.534	128.293	153.575	176.501

Bazoune and Khulief observed a critical taper ratio for the liner taper beam, where the frequency starts increasing after the 0.7 taper ratio for $\bar{\Omega} = 10$.

In their study, Bazoune and Khulief [40] observed a consistent pattern in the first three frequencies for the specified value of $\bar{\Omega} = 10$ for a linearly tapered beam. At first, the frequencies exhibit a decline within the interval of $0.0 \leq \nu \leq 0.7$, followed by an increase within the interval of $0.7 \leq \nu \leq 0.9$. In the present study, as shown in figure 4.30, the frequencies are showing different behavior, where the frequencies are decreasing across the whole examined range of $0.0 \leq \nu \leq 2.0$. The fourth frequency in the Bazoune and Khulief study [40], at the same specified value of $\bar{\Omega} = 10$, is decreasing in $0.0 \leq \nu \leq 0.7$ and increasing in $0.7 \leq \nu \leq 0.9$. In the present study, the fourth frequency at the same investigated speed, at the same specified value of $\bar{\Omega} = 10$, is decreasing throughout the whole range $0.0 \leq \nu \leq 2$ of the taper parameter, as shown in figure 4.31. The fifth frequency in Bazoune and Khulief study [40], at the same specified value of $\bar{\Omega} = 10$, is increasing for $0.0 \leq \nu \leq 0.1$ and $0.75 \leq \nu \leq 0.9$ and decreasing in $0.1 \leq \nu \leq 0.75$. In the present study, the fifth frequency shows similar

behaviour in the range of $0.0 \leq \nu \leq 0.75$, where the frequency slightly increases $0.0 \leq \nu \leq 0.1$ and decreases $0.1 \leq \nu \leq 0.75$. However, after that the exponential taper beam continue to decrease for $0.75 \leq \nu \leq 2.0$ where for the linear taper, the frequency start increasing again in $0.75 \leq \nu \leq 0.9$. Finally, the sixth one in Bazoune and Khulief study [40], at the same specified value of $\bar{\Omega} = 10$, is increasing for $0.0 \leq \nu \leq 0.4$ and $0.7 \leq \nu \leq 0.9$ and decreases in $0.4 \leq \nu \leq 0.7$. In comparison in the present study as presented in figure 4.31, the frequency increase for $0.0 \leq \nu \leq 0.5$ and decreases for $0.5 \leq \nu \leq 2.0$.

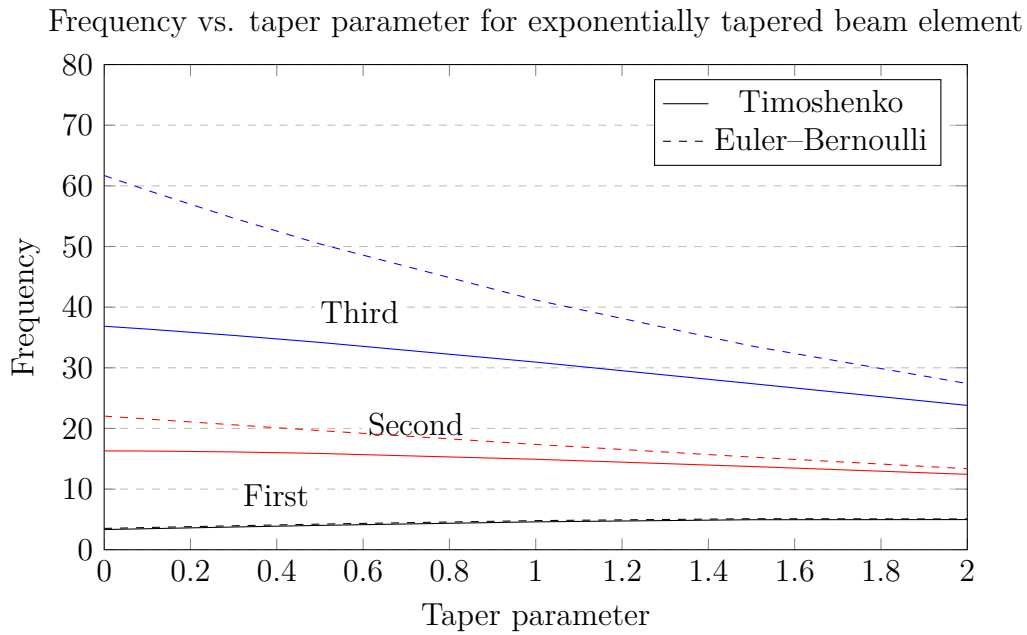


Figure 4.26: First, second and third frequency parameters of non rotating ($\bar{\Omega} = 0$) clamped-free exponentially tapered beam

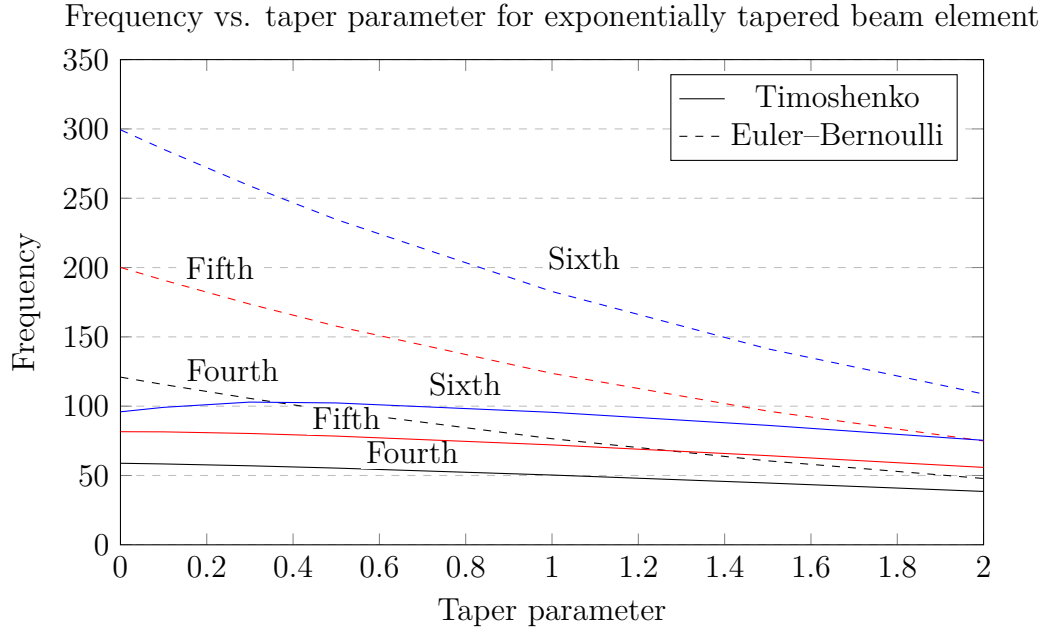


Figure 4.27: Fourth, fifth and sixth frequency parameters of non rotating ($\bar{\Omega} = 0$) clamped-free exponentially tapered beam

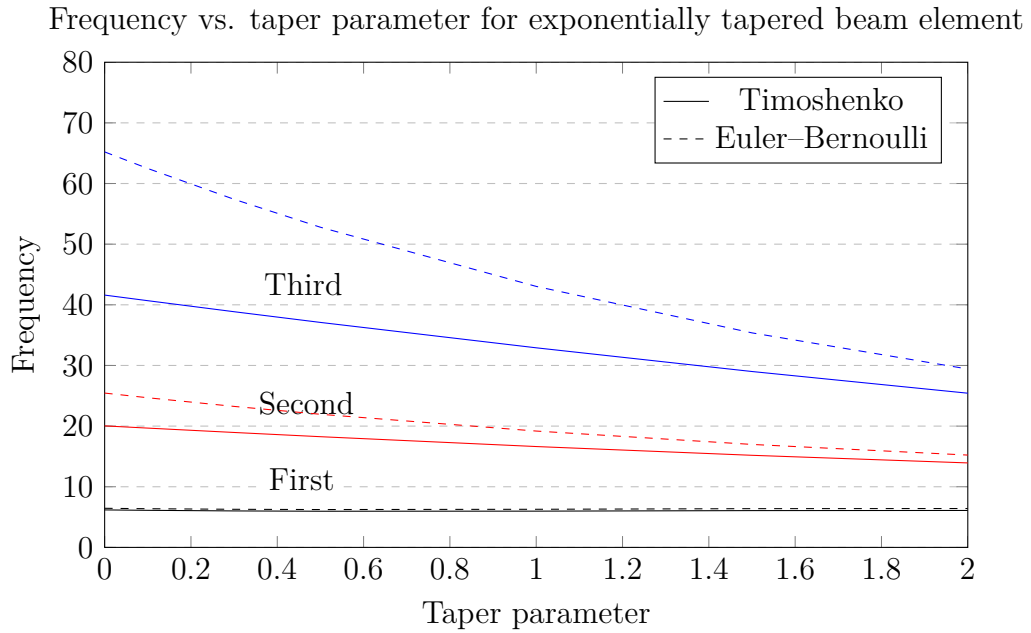


Figure 4.28: First, second and third frequency parameters of clamped-free exponentially tapered beam with spin parameter $\bar{\Omega} = 5$

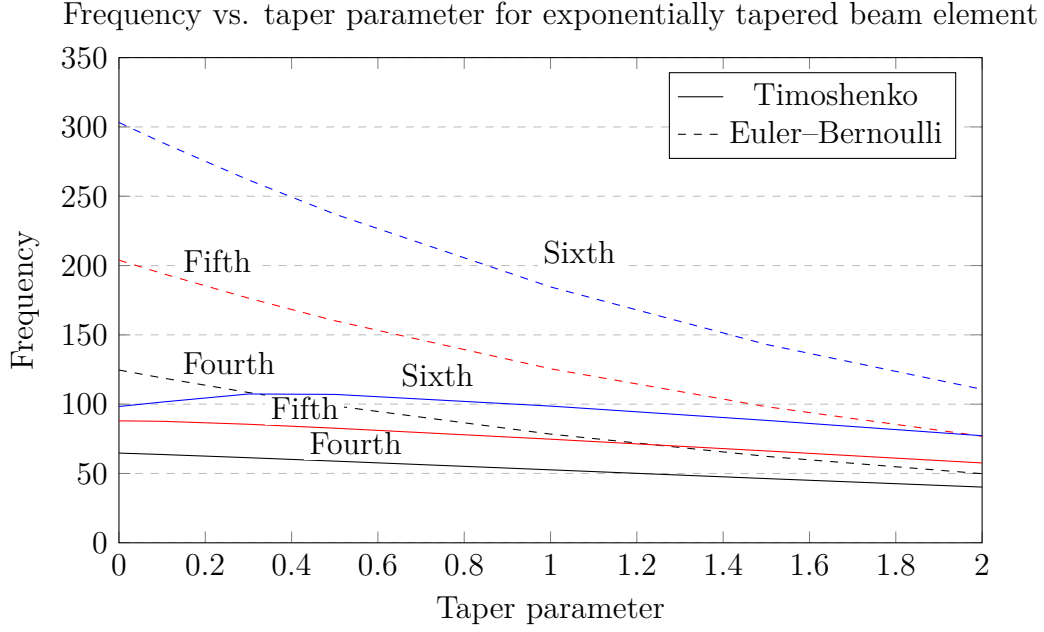


Figure 4.29: Fourth, fifth and sixth frequency parameters of clamped-free exponentially tapered beam with spin parameter $\bar{\Omega} = 5$

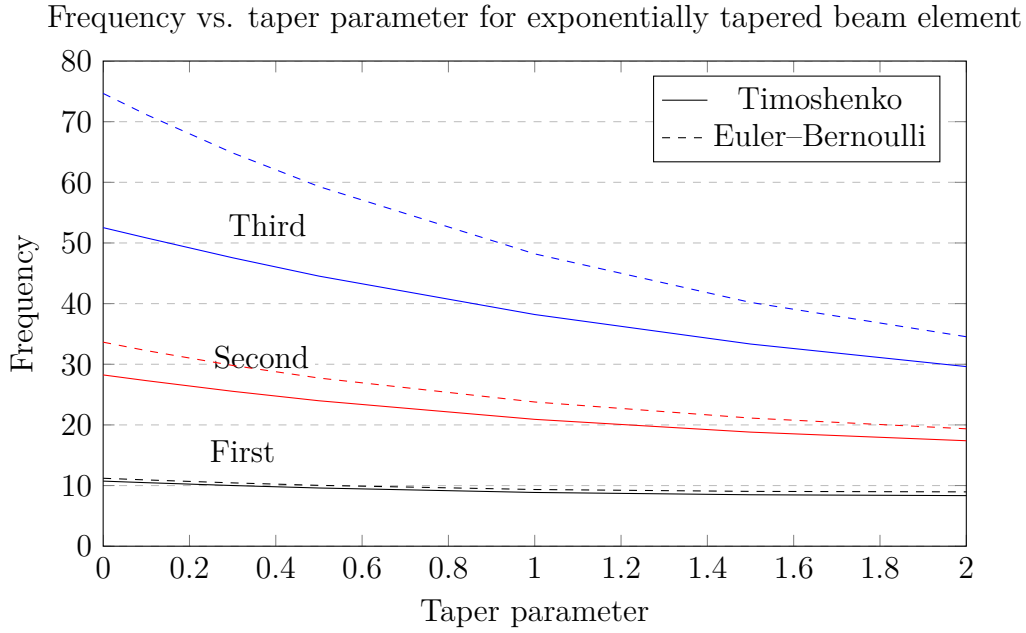


Figure 4.30: First, second and third frequency parameters of clamped-free exponentially tapered beam with spin parameter $\bar{\Omega} = 10$

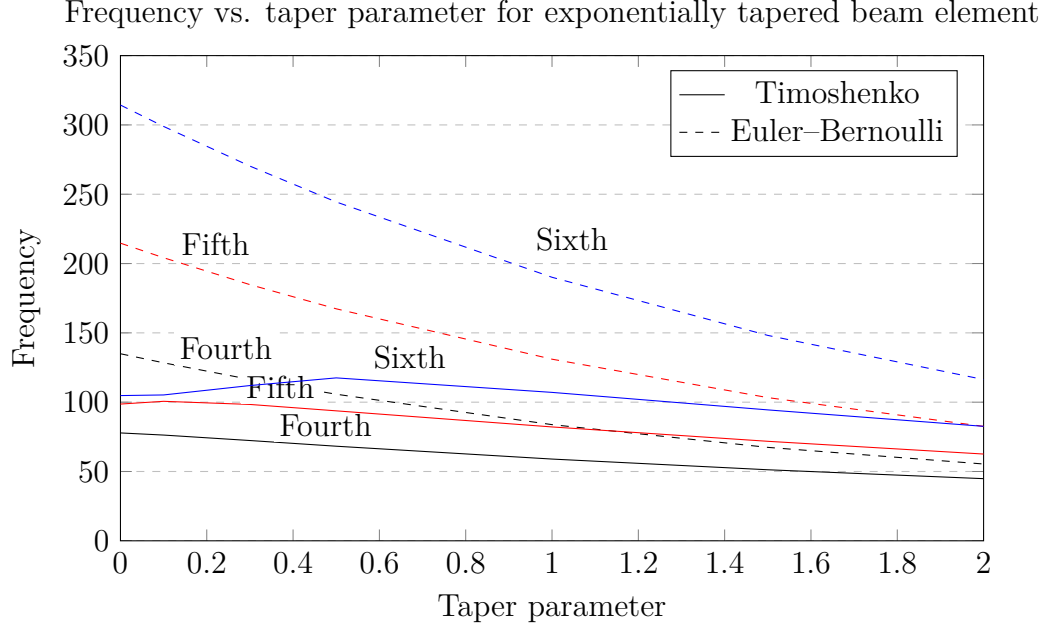


Figure 4.31: Fourth, fifth and sixth frequency parameters of clamped-free exponentially tapered beam with spin parameter $\bar{\Omega} = 10$

The effect of rotation on the frequency ratios (λ_T/λ_E) of the clamped-free beam is shown graphically in figures 4.32 - 4.35. The first mode ratio is observed to be stable across different spin for a taper ratio ≤ 1.5 in figures 4.32 - 4.34. The first mode is decreasing at a taper parameter of 2 as presented in 4.35. For other modes, the ratios are stable or slightly increasing at different taper parameter.

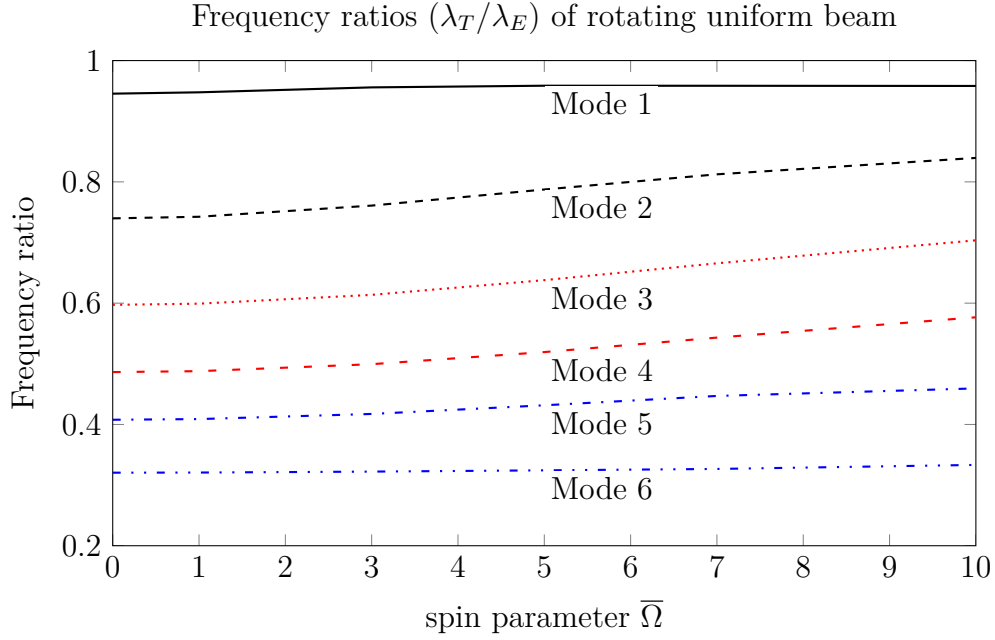


Figure 4.32: Effect of rotation on the frequency ratios of a uniform clamped-free beam

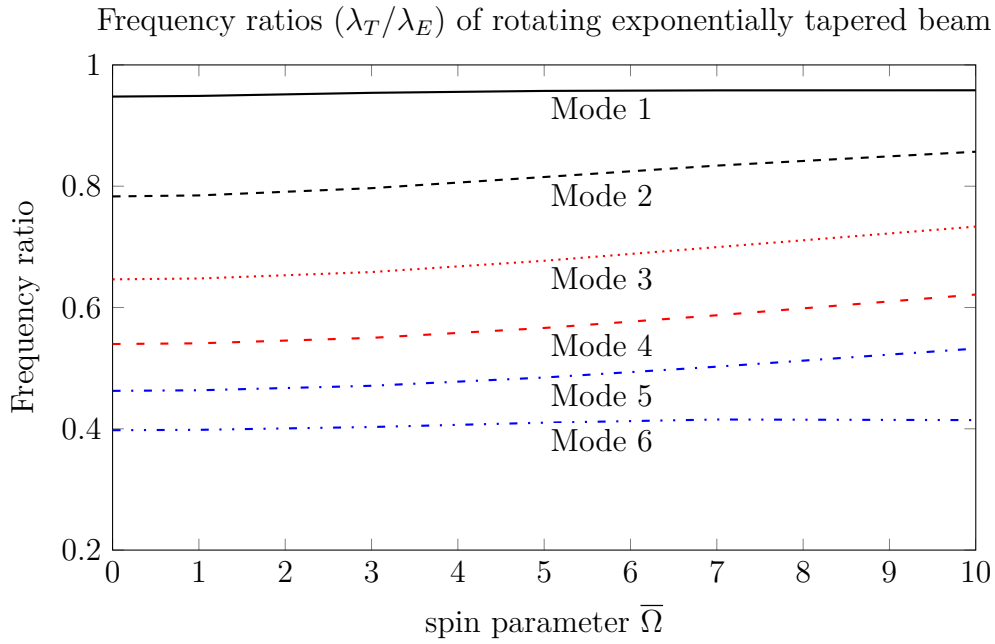


Figure 4.33: Effect of rotation on the frequency ratios of a exponentially tapered clamped-free beam ($\nu = 0.3$)

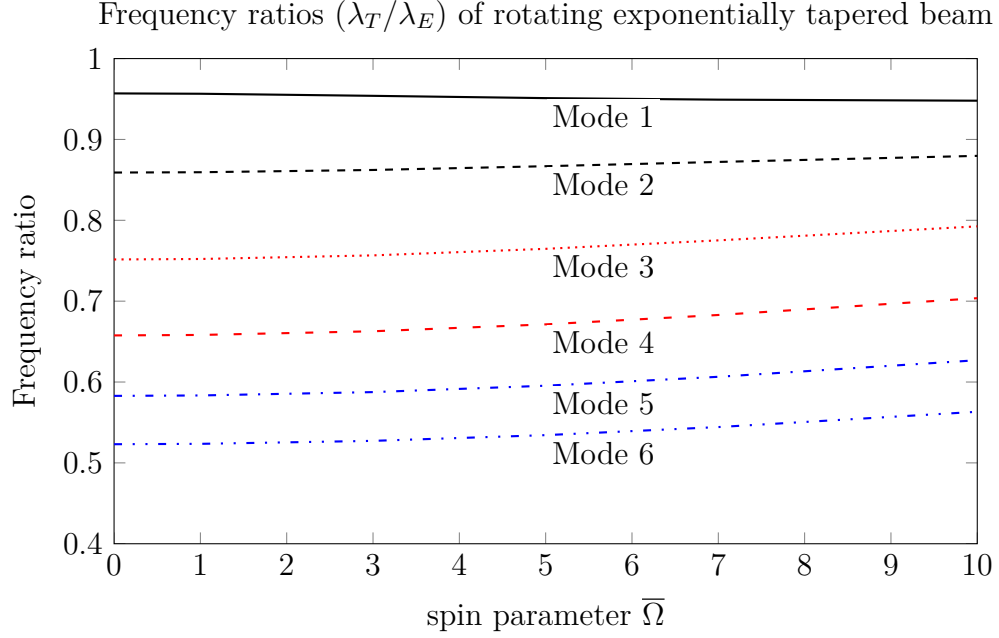


Figure 4.34: Effect of rotation on the frequency ratios of a exponentially tapered clamped-free beam ($\nu = 1.0$)

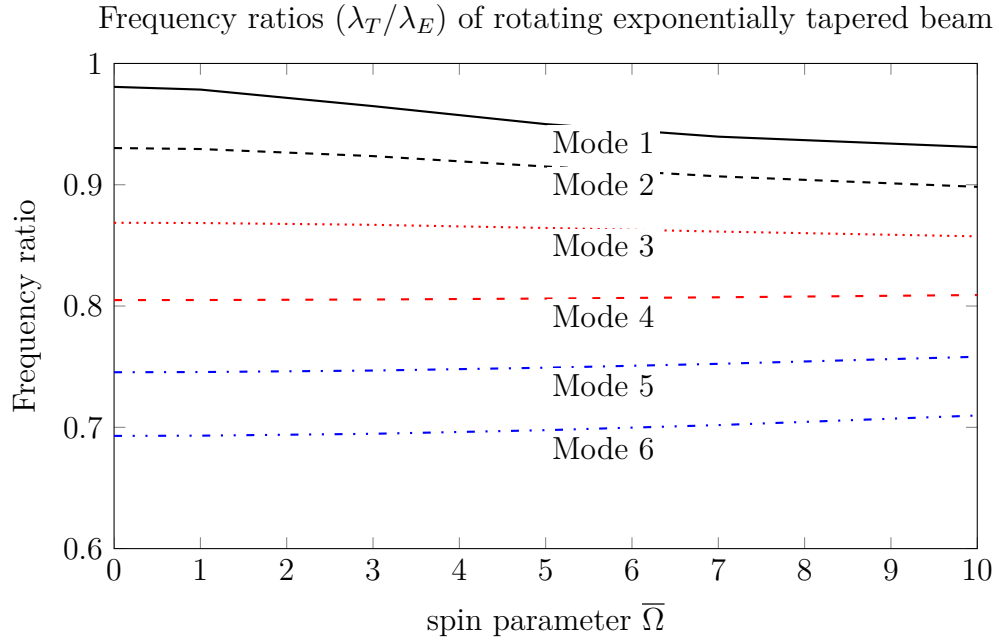


Figure 4.35: Effect of rotation on the frequency ratios of an exponentially tapered clamped-free beam ($\nu = 2.0$)

Figures 4.36 - 4.42 present first three mode shapes of exponentially tapered clamped-

free beam with two different slenderness ratios (0.08 and 0.02) with a taper parameter of $\nu = 2.0$.

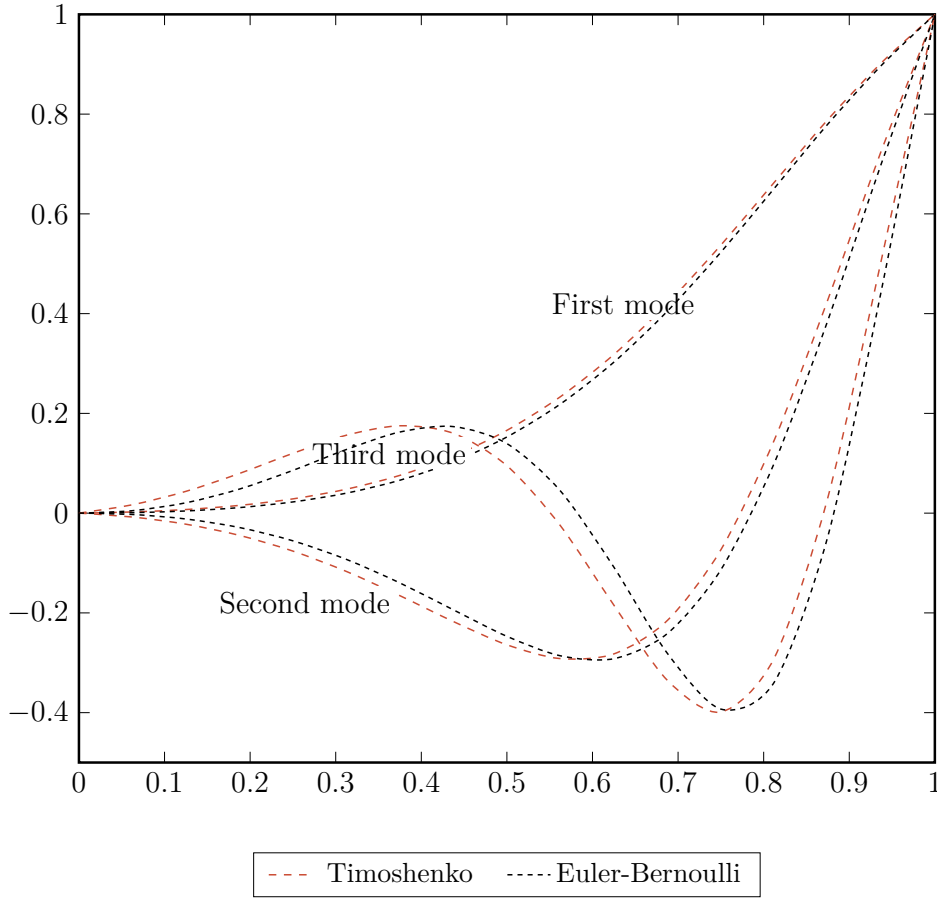


Figure 4.36: First, second and third bending mode shapes using cubic shape functions for rotating clamped-free exponentially tapered beam, $\nu = 2.0$, $R = 0$, $\psi = 0^\circ$, $\bar{\Omega} = 10$, $r_{gy}/L = 0.08$

Hinged-free Timoshenko beam

Up to the knowledge of the author, also as mentioned earlier, there is no available results in the literature of exponentially tapered rotating beam. This thesis investigated the rotating exponentially tapered clamped-free Timoshenko beam in 4.2.3. In this section, the boundary conditions is going to be changed to hinged-free. In tables 4.23 - 4.28, the taper parameter are varied from 0 to 2 for an exponentially tapered

hinged Timoshenko beam. Table 4.23 compares the planar frequency characteristics of the rotating uniform hinged Timoshenko beam to Bazoune and Khulief's results [40]. There is good excellent agreement with the uniform beam. Tables 4.24 - 4.28 shows the effect of exponential taper.

Table 4.23: Spin parameter effect on frequency parameters using cubic shape functions for a uniform hinged-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 0$

$\bar{\Omega}$		λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	Present Work	13.1348	34.1067	57.502	80.6905	92.698	101.893	114.018	127.909	142.338	158.838
	Ref. [40]	13.1312	34.0693	57.4010	80.4776	92.1319	101.293	113.587	126.992	141.89	157.404
1	Present Work	13.3507	34.3326	57.7672	80.9734	92.7416	102.095	114.239	128.116	142.682	159.053
	Ref. [40]	13.3473	34.2957	57.6669	80.7598	92.174	101.494	113.811	127.198	142.237	157.614
3	Present Work	14.9552	36.0698	59.8184	83.1216	93.0784	103.609	115.991	129.719	145.366	160.675
	Ref. [40]	14.9538	36.0638	59.7233	82.8982	92.5054	102.993	115.579	128.793	144.936	159.208
5	Present Work	17.6923	39.232	63.5922	86.7793	93.8226	106.173	119.29	132.81	150.214	163.796
	Ref. [40]	17.6935	39.2047	63.5044	86.5037	93.2765	105.525	118.89	131.889	149.767	162.338
7	Present Work	21.0766	43.387	68.5884	90.3543	95.7108	109.366	123.465	137.563	155.722	169.162
	Ref. [40]	21.0801	43.3652	68.506	89.8926	95.3093	108.687	123.039	136.67	155.143	167.823
10	Present Work	26.7413	50.6719	77.2418	92.627	101.587	115.502	129.776	147.868	162.588	182.608
	Ref. [40]	26.7475	53.2287	80.0831	92.3669	103.224	117.193	131.491	150.584	164.312	186.277

Rigid body mode is not included

Table 4.24: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered hinged-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 0.3$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	0.909683	12.5936	32.2711	55.3173	79.5419	99.5381	103.992	122.176	131.515	146.897
1	2.72568	13.8304	33.5595	56.815	81.2903	99.5744	105.885	122.865	133.104	148.155
3	4.53536	16.0005	35.965	59.6501	84.6027	99.6274	109.412	123.975	136.183	150.556
5	6.34045	18.7585	39.2317	63.566	89.1636	99.7077	113.993	125.463	140.224	154.111
7	9.04426	23.4989	45.215	70.8708	97.2036	100.271	120.437	129.7	146.485	161.682

Table 4.25: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered hinged-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 0.5$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	0.866785	11.9963	30.7393	53.2531	77.3813	102.32	103.098	127.353	135.896	152.078
1	2.59332	13.0538	31.8107	54.4721	78.8024	102.97	104.099	128.985	136.195	153.54
3	4.30593	14.9363	33.8343	56.8059	81.5327	103.038	107.206	131.749	137.109	156.035
5	6.00751	17.3631	36.6248	60.0814	85.3807	103.09	111.624	134.102	139.858	158.855
7	8.54978	21.5899	41.8389	66.3365	92.7409	103.217	119.998	135.594	147.718	162.95

Table 4.26: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered hinged-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 1.0$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	0.789817	10.3588	26.6019	47.1981	70.1551	94.6135	109.758	120.635	147.438	162.612
1	2.35408	11.1339	27.3433	47.9883	71.0383	95.6083	109.798	121.769	148.73	162.641
3	3.88464	12.5464	28.7665	49.5261	72.7644	97.5512	109.878	123.997	151.261	162.7
5	5.38357	14.4115	30.7728	51.736	75.2608	100.35	110.01	127.24	154.935	162.792
7	7.59607	17.7312	34.6344	56.1068	80.2438	105.736	110.507	133.768	162.138	163.165

Table 4.27: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered hinged-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 1.5$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	0.744775	8.71255	22.348	40.4357	61.383	84.2895	108.379	115.908	135.14	162.392
1	2.21426	9.36705	22.9667	41.0547	62.0332	84.9896	109.078	116.014	135.991	163.338
3	3.63693	10.5636	24.1541	42.2643	63.3121	86.37	110.409	116.277	137.675	165.208
5	5.01071	12.148	25.8272	44.0133	65.1798	88.3929	112.159	116.873	140.155	167.962
7	7.00687	14.9719	29.0452	47.5043	68.9664	92.5142	114.364	119.494	145.255	173.623

Table 4.28: Spin parameter effect on frequency parameters using cubic shape functions for an exponentially tapered hinged-free Timoshenko beam, $R = 0$, $\psi = 0^\circ$, $\nu_y = \nu_z = 2.0$

$\bar{\Omega}$	λ_{T1}	λ_{T2}	λ_{T3}	λ_{T4}	λ_{T5}	λ_{T6}	λ_{T7}	λ_{T8}	λ_{T9}	λ_{T10}
0	0.720855	7.19584	18.3023	33.5594	51.8828	72.4907	94.8524	117.114	122.665	145.188
1	2.14103	7.84113	18.9395	34.1738	52.4921	73.1103	95.4907	117.516	122.991	145.914
3	3.50914	8.9883	20.1378	35.3647	53.6874	74.3328	96.752	118.204	123.749	147.357
5	4.81892	10.4659	21.7791	37.0643	55.4256	76.126	98.6066	118.972	125.124	149.494
7	6.69818	13.0382	24.819	40.3845	58.9211	79.7849	102.4	119.962	128.6	153.933

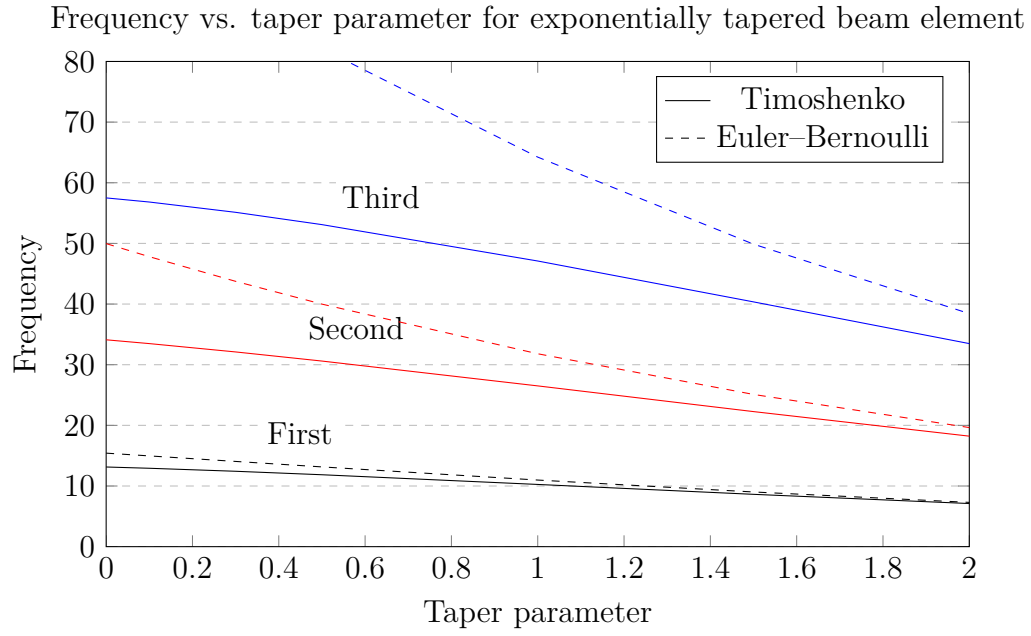


Figure 4.37: First, second and third frequency parameters of non rotating ($\bar{\Omega} = 0$) hinged-free exponentially tapered beam

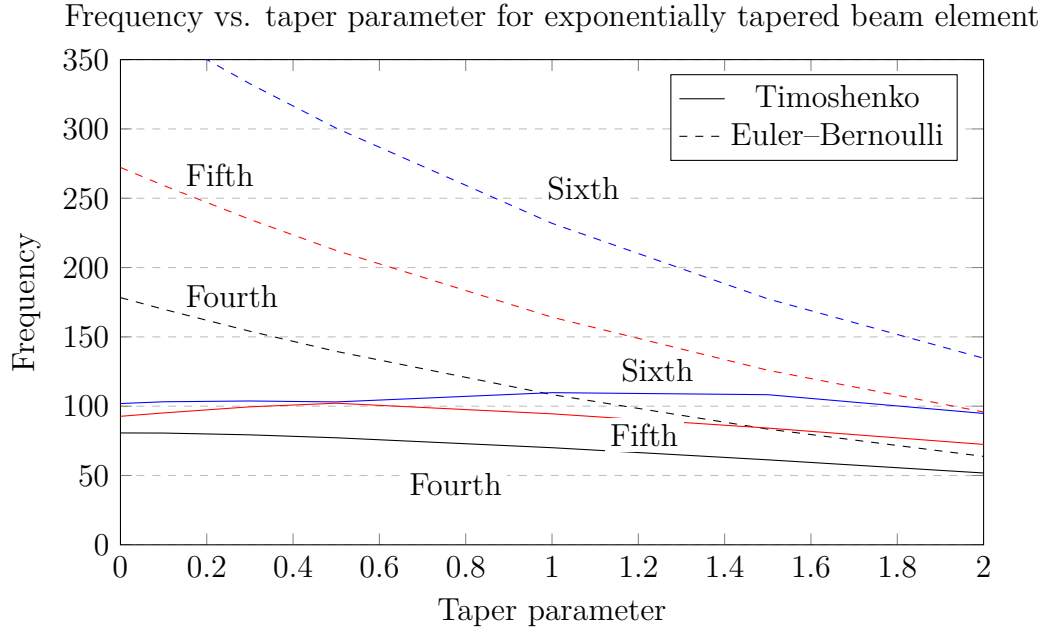


Figure 4.38: Fourth, fifth and sixth frequency parameters of non rotating ($\bar{\Omega} = 0$) hinged-free exponentially tapered beam

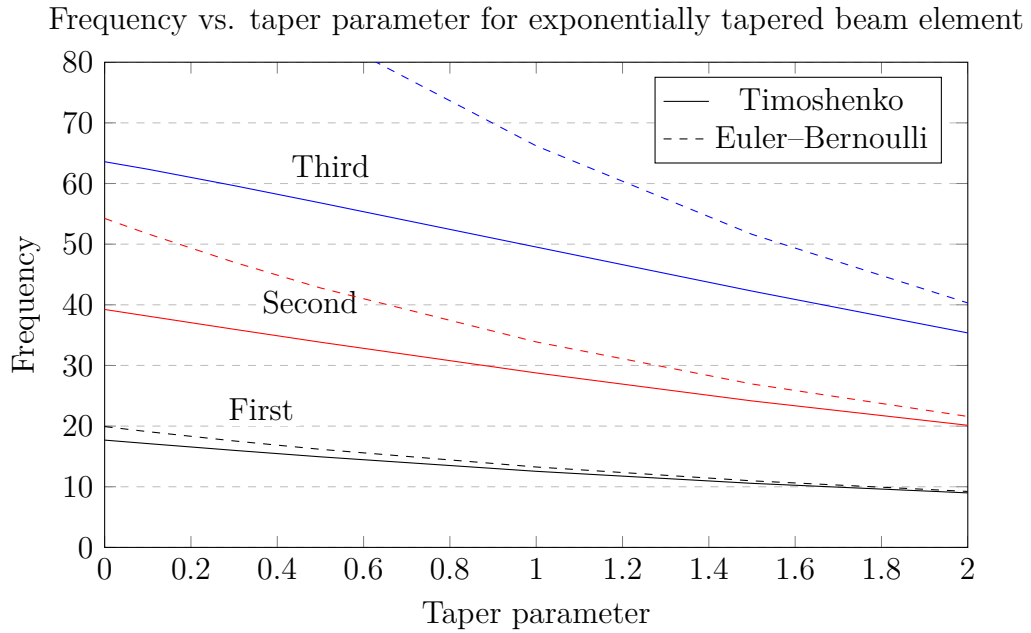


Figure 4.39: First, second and third frequency parameters of hinged-free exponentially tapered beam with spin parameter $\bar{\Omega} = 5$

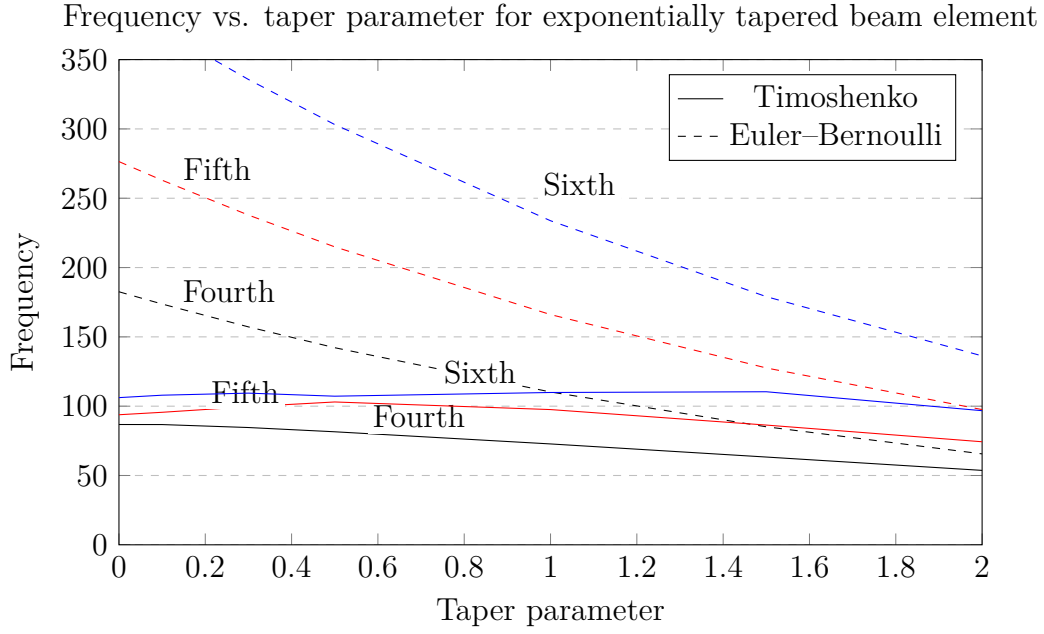


Figure 4.40: Fourth, fifth and sixth frequency parameters of hinged-free exponentially tapered beam with spin parameter $\bar{\Omega} = 5$

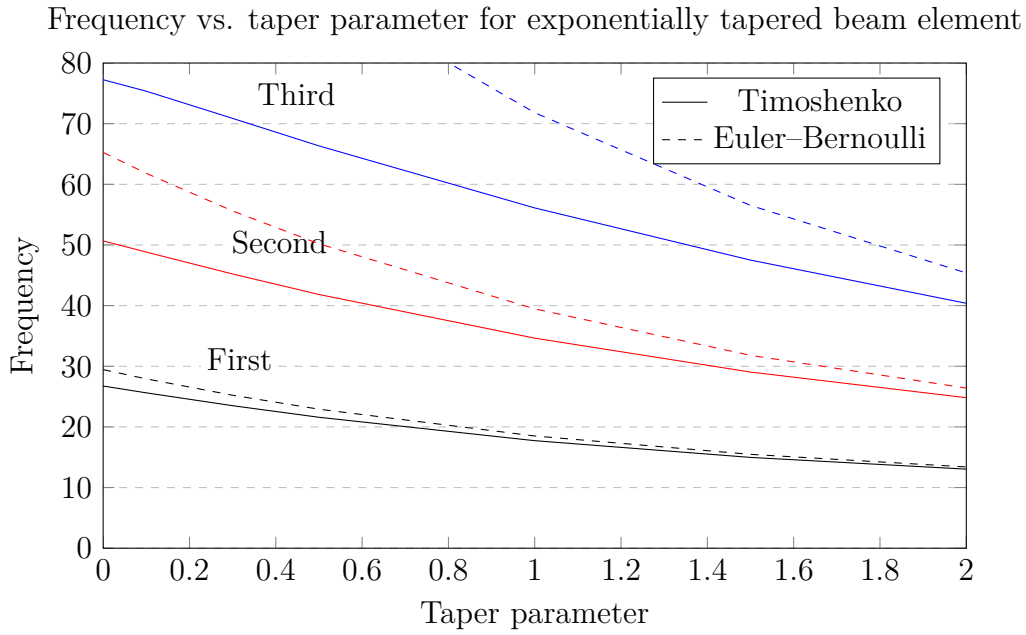


Figure 4.41: First, second and third frequency parameters of hinged-free exponentially tapered beam with spin parameter $\bar{\Omega} = 10$

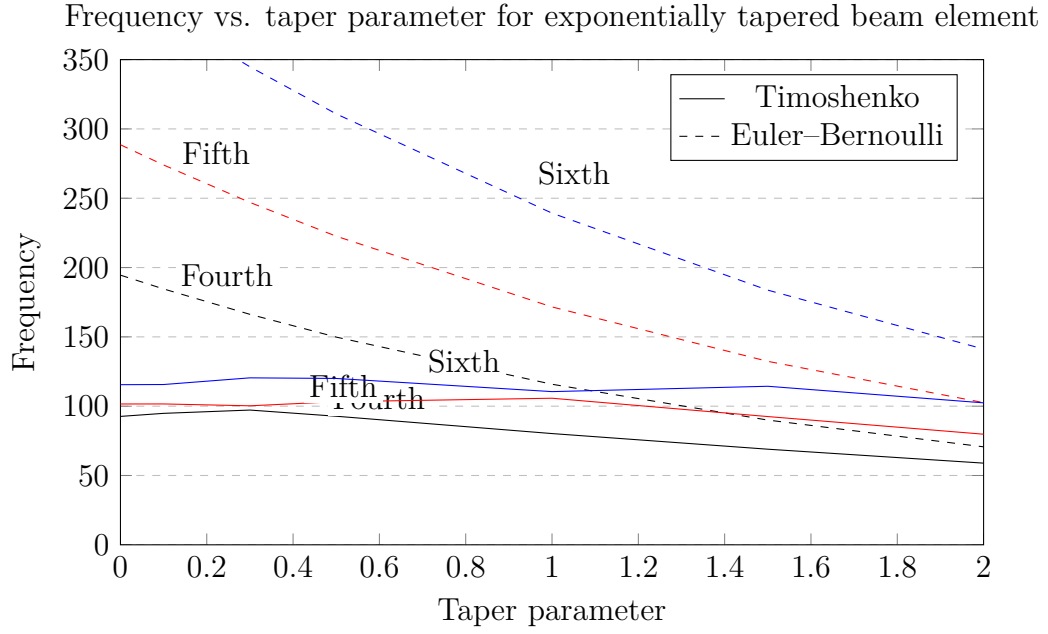


Figure 4.42: Fourth, fifth and sixth frequency parameters of hinged-free exponentially tapered beam with spin parameter $\bar{\Omega} = 10$

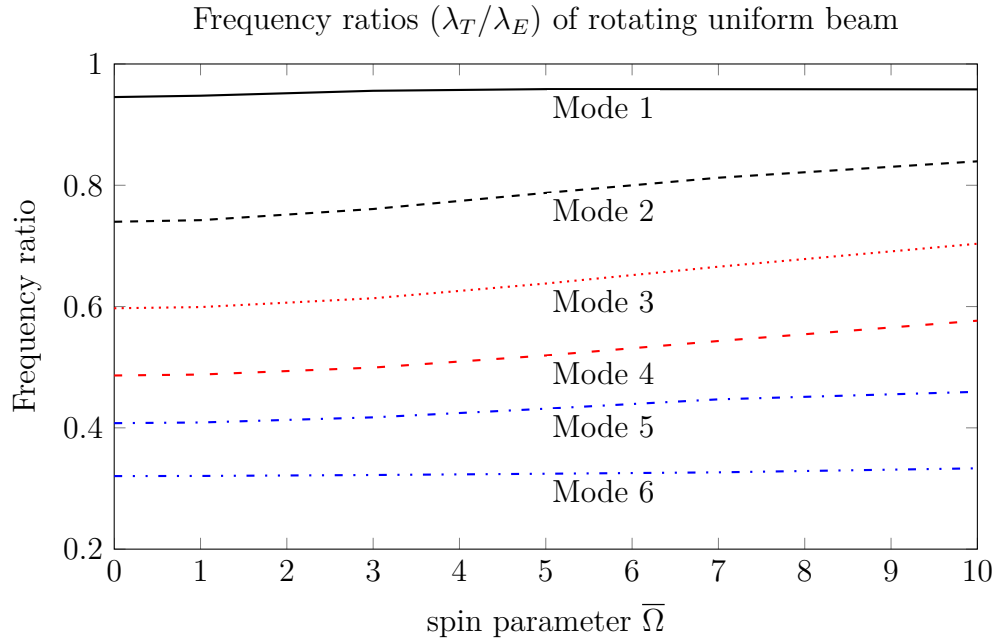


Figure 4.43: Effect of rotation on the frequency ratios of a uniform clamped-free beam

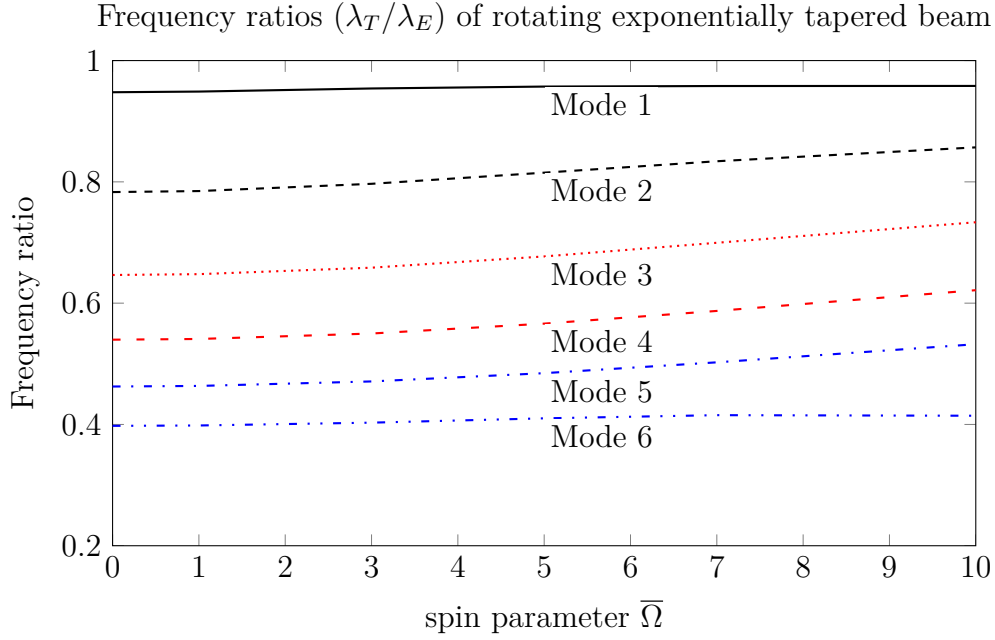


Figure 4.44: Effect of rotation on the frequency ratios of a exponentially tapered clamped-free beam ($\nu = 0.3$)

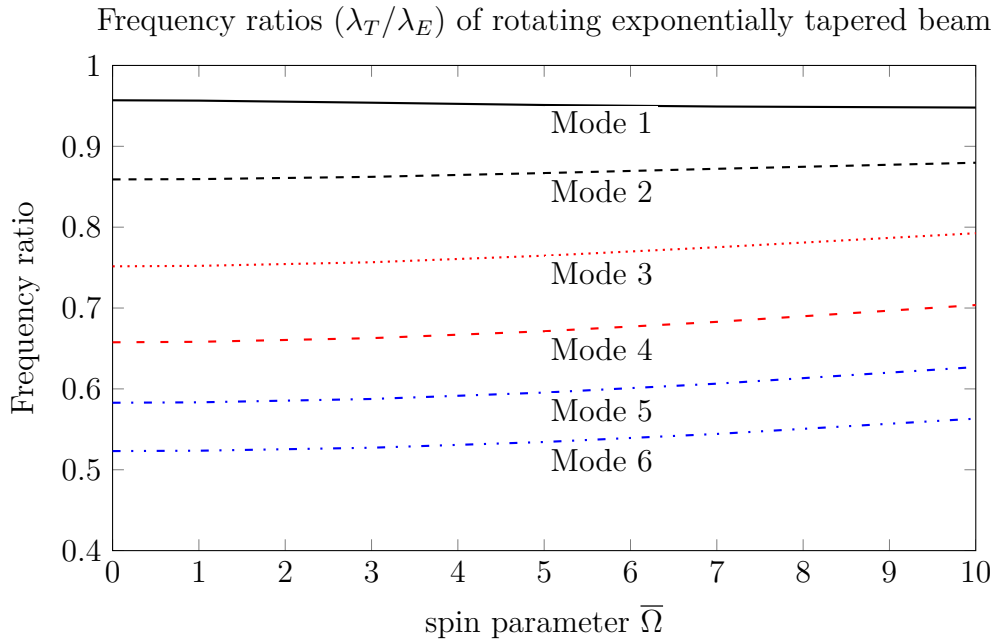


Figure 4.45: Effect of rotation on the frequency ratios of a exponentially tapered clamped-free beam ($\nu = 1.0$)

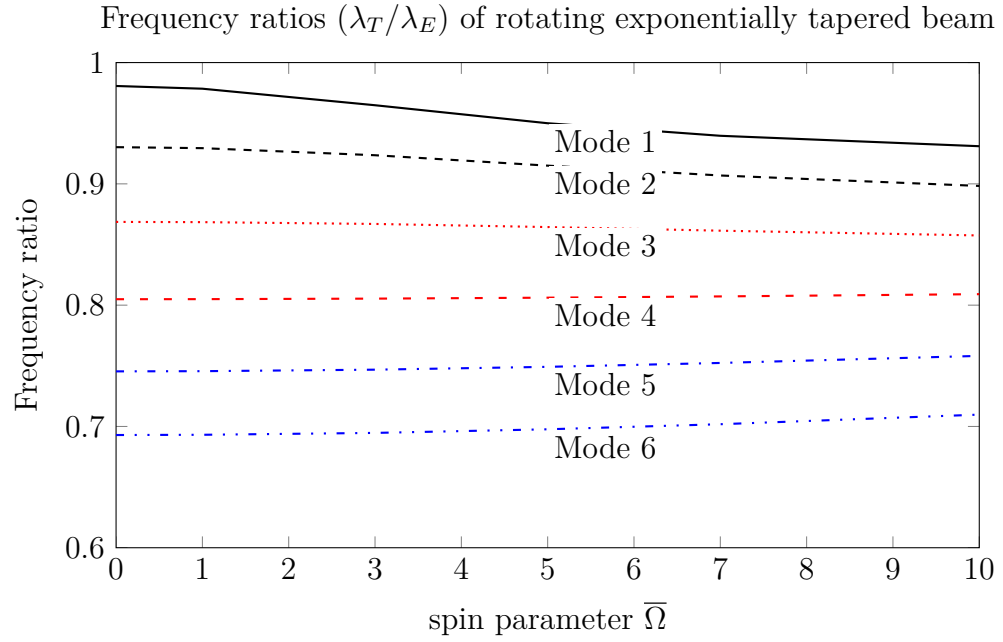


Figure 4.46: Effect of rotation on the frequency ratios of an exponentially tapered clamped-free beam ($\nu = 2.0$)

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

A consistent mass finite element Timoshenko or Euler-Bernoulli beam model was created, with one degree of freedom for each node in terms of translation and one for rotation. The model takes into consideration a number of real-world design characteristics, such as hub radius, setting angle, gyration ratios, boundary conditions, and uniform, linearly, or exponentially tapered profiles.

Two sets of shape functions were utilized. The first set are the widely used cubic Hermitian polynomial shape functions, which were referred to in the present work as "cubic" shape functions. The cubic shape functions are applicable for non-rotating and rotating beams since they don't account for rotation during the derivation of these shape functions. The second set are recently developed shape functions, which were referred to as "violin string" shape functions, which were introduced and referred

to in the present work as "violin" shape functions. The violin shape function was reproduced to overcome some of the mistakes observed in the work of Vinod Kumar and Ganguli [93].

Then, Mathematica's symbolic computation features were used to get the eigenvalues and eigenvectors by solving the full-order model eigenvalue problem and deriving mass and stiffness matrices from shape functions. Symbolic computation in Mathematica lets you use shape functions, cross-sectional area, and forces as input to drive the stiffness and mass matrices. This is different from the old methods, which required manually deriving mass and stiffness matrices in very large matrices. This gives an advantage in the future because inputs can be changed easily and tested. The model was tested against many scenarios in the published literature, and there was great agreement between the present model and the published results.

The investigation of the violin shape function revealed higher accuracy and convergence properties compared to cubic shape functions. Especially for high gyration ratios (0.09) and high spin parameters. This advantage is due to the consideration of tension on each element due to rotation. This consideration of the tension due to rotation is not accounted for in the cubic shape functions. The effect of hub radius was also investigated when using the violin shape functions, which were not addressed previously. The hub radius showed minimal effect compared to the gyration ratios and spin parameters. The investigation of the exponentially tapered beam led to an interesting conclusion when it came to the taper parameter effect on beam natural frequencies. Previous work by Khulief and Bazoune shed light on an interesting feature of the linearly tapered beam, which is called the "critical taper parameter,"

where the frequencies of the rotating beam reverse direction. This is mainly because, as the cross-sectional area decreases, the centrifugal impact becomes greater than the softening effect. In the case of the exponential tapered, the critical taper parameter was not observed, although the taper parameter was tested up to high numbers (i.e., 2). This concludes that in the case of exponential taper, the softening effect of the cross-section area remains the dominant factor, and the natural frequencies keep reducing with increasing taper ratios.

5.2 Recommendations

For further investigation, the following recommendations are made:

- Extending the model to include a functionally graded beam
- Extend the present model to include a pre-twisted beam element
- Extend the present model to include the damping effect
- Extended the present model to include hub spin acceleration instead of the fixed speed hub that is assumed in the present study

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