

**AUTOMATED RITZ METHOD FOR THE ANALYSIS OF
THICK LAMINATED COMPOSITE PLATES**

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Dedicated

to

My parents, brother and sisters

for their help and support

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TABLE OF CONTENTS

ACKNOWLEDGMENTS	IV
TABLE OF CONTENTS	V
LIST OF TABLES	VIII
LIST OF FIGURES	X
LIST OF ABBREVIATIONS	XIII
ABSTRACT	XVI
ملخص الرسالة	XVIII
CHAPTER 1 INTRODUCTION	1
1.1 General.....	1
1.2 Practical Applications.....	5
1.3 Problem Statement.....	6
1.4 Need of Research.....	7
1.5 Research Objectives and Scope.....	8
1.6 Research Methodology	9
1.7 Thesis Outlines	10
CHAPTER 2 LITERATURE REVIEW	12
2.1 General.....	12
2.2 Classical lamination plate theory (CLPT)	17
2.3 First-order shear deformation theory (FSDT)	19
2.4 Higher-order shear deformation theory (HSDT)	22
2.5 Assessments of various methods to model laminated composite plates	24
2.5.1 Navier and Levy solutions.....	26
2.5.2 Elasticity solutions	28

2.5.3	Power, trigonometric & Fourier series approaches.....	28
2.5.4	Rayleigh-Ritz and Galerkin methods.....	29
2.5.5	Variational methods.....	31
2.5.6	Differential quadrature methods (DQM)	31
2.5.7	Some recent analytical methods	32
2.5.8	Finite element analysis	33
2.5.9	Meshless technique.....	37
2.5.10	Boundary element method.....	38
2.5.11	Some recent numerical methods.....	38
2.5.12	Artificial Intelligence based on artificial neural network approaches	39

CHAPTER 3 CLPT FOR THE ANALYSIS OF THIN LAMINATED COMPOSITE

PLATES 42

3.1	General	42
3.2	CLPT for the Behavior of Thin Laminated Composite Plates	44
3.2.1	Constitutive relations	45
3.2.2	Determination of transformed stiffness matrix and rigidities	49
3.3	Finite Element Modeling (FEM) in ABAQUS	55
3.3.1	Introduction	55
3.3.2	Modeling of laminated composite plates in ABAQUS.....	56
3.4	Ritz Method Formulation Based on CLPT	59
3.4.1	Strain and potential energy	59
3.4.2	Selection of the approximation functions F_{iu} , F_{jv} and F_{kw}	62
3.4.3	Ritz method in matrix form	66
3.5	Numerical Examples and Discussions.....	70
3.5.1	Convergence Study.....	74
3.5.2	Cross-Ply Laminates.....	75
3.5.3	Angle-Ply Laminates	77
3.5.4	Impact of varied stacking sequence on deflection	80

CHAPTER 4 FSDT FOR THE ANALYSIS OF THICK LAMINATED COMPOSITE

PLATES 82

4.1	General	82
4.2	FSDT for the Behavior of Thick Laminated Composite Plates.....	85
4.2.1	Strain Displacement Relations.....	86
4.2.2	Constitutive Equations.....	86
4.2.3	Rigidities of Thick Laminated Composite Plates (FSDT)	91
4.3	Finite Element Modeling (FEM) in COMSOL	97
4.3.1	Introduction	97
4.3.2	Modeling of laminated composite plates in COMSOL	98

4.4	Ritz Method Formulation Based on FSDT	102
4.4.1	Potential Energy Components	102
4.4.2	Ritz Method in a Matrix Form	108
4.5	Numerical Examples and Discussions	114
4.5.1	Cross-ply $[0^\circ/90^\circ/0^\circ]$	114
4.5.2	Angle-ply $[-45^\circ/45^\circ]$	126
CHAPTER 5 HSDT FOR THE ANALYSIS OF THICK LAMINATED PLATES		134
5.1	General.....	134
5.2	HSDT for the Behavior of Thick Laminated Composite Plates	135
5.2.1	Constitutive Equations	135
5.2.2	Potential Energy Components	141
5.3	Ritz Method Formulation Based on HSDT.....	151
5.4	Numerical Examples and Discussions	165
5.4.1	Cross ply Laminates	168
5.4.2	Angle ply Laminates	171
5.4.3	Impact of varied boundaries and number of layers on deflection and stresses	172
CHAPTER 6 SUMMARY, CONCLUSIONS AND RECOMMENDATIONS		174
6.1	Summary.....	174
6.2	Conclusion	177
6.3	Recommendations for Future Work	178
REFERENCES.....		180
APPENDIX A: DERIVATION OF RITZ METHOD IN A MATRIX FORM FOR THIN LAMINATED COMPOSITE PLATE		200
APPENDIX B: DERIVATION OF RITZ METHOD IN A MATRIX FORM FOR THICK LAMINATED COMPOSITE BASED ON FSDT PLATES		206
APPENDIX C: DERIVATION OF RITZ METHOD IN A MATRIX FORM FOR THICK RECTANGULAR LAMINATED COMPOSITE BASED ON HSDT PLATES		215
VITAE.....		226

LIST OF TABLES

Table 1 : Plate theories classification.....	3
Table 2 : Historical review and advancement analysis for the mentioned theories	12
Table 3 : Equations of displacements for ESL theories	16
Table 4 : Historical Progress for shear correction factor	22
Table 5: Convergence study for different stacking sequence at the center of the plate	74
Table 6 : Normalized deflection and stresses for cross ply corner supported square plate ($0^\circ/90^\circ$)	77
Table 7 : Dimensionless maximum deflection for symmetric angle ply corner supported plate	79
Table 8 : Dimensionless stress components for symmetric and anti-symmetric angle ply corner supported plate.....	80
Table 9 : The comparison of the three solutions for SSSS case ($0^\circ/90^\circ/0^\circ$)	117
Table 10: The comparison of the three solutions CCCC ($0^\circ/90^\circ/0^\circ$)	122
Table 11: The comparison of the three solutions for SSSS ($-45^\circ/45^\circ$)	128
Table 12 : The comparison of the two solutions for CCCC ($-45^\circ/45^\circ$).....	131
Table 13 : Normalized deflection and stresses for cross ply simply supported square plate [($0^\circ/90^\circ/90^\circ/0^\circ$)]	169
Table 14 : Normalized deflection and in-plane stress for cross ply simply supported square plate [$0^\circ/90^\circ/0^\circ$]	169

Table 15 : Normalized Deflection for Different layers 2,6 ($0^\circ/90^\circ/0^\circ/90^\circ \dots$) at (0, 0) for SSSS.....	170
Table 16 : Normalized deflection for angle ply simply supported square plate [$(-\theta)/\theta^\circ \dots (-\theta^\circ)/\theta^\circ \dots$].....	171
Table 17 : Normalized deflection and stresses for cross ply square plate ($0^\circ/90^\circ$) (2 layers) and $a/h=5$	172
Table 18 : Normalized deflection and stresses for cross ply square plate ($0^\circ/90^\circ$) (10 Layers) and $a/h=5$	173

LIST OF FIGURES

Figure 1: Schematic of a laminate.	2
Figure 2 : Deformation of plate element based on different theories	14
Figure 3 : Progress on FEM for laminated composite plates.....	26
Figure 4 : Plate's element before and after bending	45
Figure 5 : (a) Laminated composite plate; (b) Coordinate locations of plies in the laminated plate	50
Figure 6 : Composite laminated shell section; (a) Defining material, thickness, and orientation of each layer; (b) Plies stacking sequence of the composite laminated plate.	58
Figure 7 : Rectangular plate with sides of $(2a \times 2b)$ and centered at $(0, 0)$	65
Figure 8 : Plate restrained at the corners.....	71
Figure 9 : Stacking configurations of cross-ply and angle-ply composite plate.....	72
Figure 10 : Deflection vs θ for square plate.....	81
Figure 11 : Deflection vs θ for rectangle plate.....	81
Figure 12: (a) Thick Laminated composite plate; (b) Coordinate locations of plies in the laminated plate	83
Figure 13 : (a) choose the model (b) Defining the parameters (c) Defining material (d)thickness, and orientation of each layer; (e) Defining the variables (f) Defining the shear correction factor.	102
Figure 14 : Square laminated plate centered at the origin with different B.C	115
Figure 15 : lateral displacement $\bar{w}$ vs (a/h)	118
Figure 16 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM.....	119

Figure 17 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM.....	119
Figure 18 : The contours of the in-plane stress $\bar{\sigma}_y$; (a) Ritz method and (b) FEM.....	120
Figure 19 : The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM	120
Figure 20 : The contours of the out-plane shear stress $\bar{\tau}_{xz}$; (a) Ritz method and (b) FEM	121
Figure 21 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM	121
Figure 22 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM..	123
Figure 23 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM.....	123
Figure 24: The contours of the in-plane stress $\bar{\sigma}_y$; (a) Ritz method and (b) FEM.....	124
Figure 25: The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM	124
Figure 26 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM	125
Figure 27 : The contours of the out-plane shear stress $\bar{\tau}_{xz}$; (a) Ritz method and (b) FEM	125
Figure 28 : Plate with two antisymmetric layers oriented at $(-45^\circ/45^\circ)$	126
Figure 29 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM.	128
Figure 30 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM.....	129
Figure 31 : The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM	129
Figure 32 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM	130

Figure 33 : The contours of the out-plane shear stress $\bar{\tau}_{xz}$; (a) Ritz method and (b) FEM	130
Figure 34 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM.	132
Figure 35 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM.....	132
Figure 36 : The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM	132
Figure 37 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM	133
Figure 38 : The contours of the out-plane shear stress $\bar{\tau}_{xz}$; (a) Ritz method and (b) FEM	133
Figure 39 : Deformed Shape for Thick Laminated Composite Plate based on HSDT	135
Figure 40 : Normalized Deflection for Different layers 2,6 (0°/90°/0°/90° ...) for SSSS	170
Figure 41 : Normalized Deflection Vs Different Aspects Ratios	176

LIST OF ABBREVIATIONS

The following is the list of abbreviations used in this dissertation. Other abbreviations are defined within the text.

a	:	The plate's dimension along x -axis
b	:	The plate's dimension along y -axis
t	:	The plate's thickness.
E	:	The elastic modulus.
ν	:	Poisson's ratio.
G	:	The shear modulus.
$[Q]$	:	The reduced stiffness matrix.
$[A]$	:	The extensional stiffness matrix.
$[B]$	:	The coupling stiffness matrix.
$[D]$	:	The bending stiffness matrix.
U^m	:	The membrane strain energy.
U^b	:	The bending strain energy.
U^c	:	The strain energy due to coupling.

U^{ts} : The strain energy due to transverse shear.

Π : The total potential energy.

W : The work done by the external force.

q : The applied lateral pressure.

u, v, w : The plate's displacements in x, y and z -direction, respectively.

ϕ_x, ϕ_y : The plate's rotations in y, x direction, respectively.

$N_u, N_v, N_w, N_{\phi_x}, N_{\phi_y}$: The numbers of terms used in the trial functions of u, v, w, ϕ_x and ϕ_y .

$F_i^u, F_j^v, F_k^w, F_n^{\phi_x}, F_m^{\phi_y}$: Trial shape functions of the displacements u, v, w, ϕ_x and ϕ_y respectively.

$C_i^u, C_j^v, C_k^w, C_n^{\phi_x}, C_m^{\phi_y}$: The unknown coefficients of the trial functions $F_i^u, F_j^v, F_k^w, F_n^{\phi_x}, F_m^{\phi_y}$.

ABSTRACT

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Due to their relatively high thickness to in-plane dimension ratio, the classical lamination theory is not sufficiently precise in predicting the behavior of thick laminated composite plates. For such plates, more refined and high-order shear deformation theories become essential. The flexural and shear behavior of thick laminated plates is described by five coupled partial differential equations. These equations have been analytically solved by many researchers for specially stacked laminated plates with simple boundary conditions and geometries. Particularly for the case of arbitrary stacked thick laminated plates, the form of governing equations is too difficult to be solved by classical methods. The difficulty of the solution is further increased if boundary conditions involve free edges. Therefore, this research is destined to develop a versatile method to handle such a complicated problem. The proposed method is based on an automated Ritz method capable of yielding accurate solutions for thick laminated plates with arbitrary stacking sequences and different boundary conditions. Indicinal notations are utilized to cast Ritz method in a matrix form which can be implemented in a symbolic software like Mathematica. The matrix form is implemented in a Mathematica code and automated in a way to enable the addition of as many polynomial terms of the approximation functions as required for the accuracy and convergence of the solution. The validity of the proposed method is

established through comparing the its solutions against findings from earlier research as well as against finite element method (FEM) results generated using the commercial software ABAQUS and COMSOL. The functional form solutions are utilized to study the influence of different geometric and material parameters on the critical values of deflection and stresses in the studied thick laminated plates.

ملخص الرسالة

الاسم الكامل: علي محمد محمود علي عودة

عنوان الرسالة: طريقة ريتز الآلية لتحليل الألواح المركبة السميكة

التخصص: الهندسة المدنية

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نظرًا لسمكها المرتفع نسبيًا إلى نسبة البعد داخل المستوى، فإن نظرية التصفيح الكلاسيكية ليست دقيقة بدرجة كافية في التنبؤ بسلوك الصفائح المركبة السميكة. بالنسبة لمثل هذه الصفائح، تصبح نظريات تشوه القص الأكثر دقة وعالية الترتيب ضرورية. تم وصف سلوك الانحناء والقص للصفائح السميكة من خلال خمس معادلات تفاضلية جزئية مقترنة. تم حل هذه المعادلات تحليليًا من قبل العديد من الباحثين من أجل صفائح مغلقة مكسدة خصيصًا ذات شروط حدودية وهندسية بسيطة. خاصة بالنسبة لحالة الصفائح السميكة والمكسدة بشكل عشوائي، فإن شكل المعادلات الحاكمة يصعب حلها بالطرق الكلاسيكية. وتزداد صعوبة الحل إذا كانت الشروط الحدودية تنطوي على حواف حرة. ولذلك، يهدف هذا البحث إلى تطوير طريقة متعددة الاستخدامات للتعامل مع مثل هذه المشكلة المعقدة. تعتمد الطريقة المقترحة على طريقة ريتز الآلية القادرة على تقديم حلول دقيقة للألواح المصفحة السميكة ذات تسلسلات تراص عشوائية وشروط حدودية مختلفة. تُستخدم الرموز الإرشادية لصياغة طريقة ريتز في شكل مصفوفة يمكن تنفيذها في برنامج رمزي مثل Mathematica. تم تطوير نموذج المصفوفة ليتم ترميزه وأتمتته بطريقة تمكن من إضافة أكبر عدد ممكن من مصطلحات متعددة الحدود لوظائف التقريب كما هو مطلوب لضمان دقة الحل وتقاربه. يتم إثبات صحة الطريقة المقترحة من خلال مقارنة النتائج التي تم الحصول عليها مع نتائج الأبحاث السابقة وكذلك مع نتائج طريقة العناصر المحدودة (FEM) التي تم إنشاؤها باستخدام البرنامج التجاري ABAQUS و COMSOL. تم استخدام حلول الشكل الوظيفي لدراسة تأثير العوامل الهندسية والمادية المختلفة على القيم الحرجة للانحراف والإجهادات في الصفائح السميكة المدروسة.

CHAPTER 1

INTRODUCTION

1.1 General

A composite material is formed by combining various materials to achieve specific engineering properties such as bending, buckling, and stiffness. This production can result in superior properties compared to the individual components. A lamina is the essential unit of modern composite laminated structures, which is also referred to as a layer or a ply. Multiple layers of laminae form a laminate [1]. Composite materials provide a broad spectrum of advantages compared to traditional materials. Their favorable characteristics encompass superior stiffness to weight ratios and strength, extended fatigue life, lightweight construction, resilience against environmental deterioration, and the capacity to achieve structures that are nearly in their final shape during manufacturing processes [2].

Anisotropic plates made of composite laminated layers consist of a group of single layers bonded to each other. Each layer can be identified by its material's properties, its location in the laminate, and its angle of fiber orientation with respect to a reference axis as shown in Figure 1.

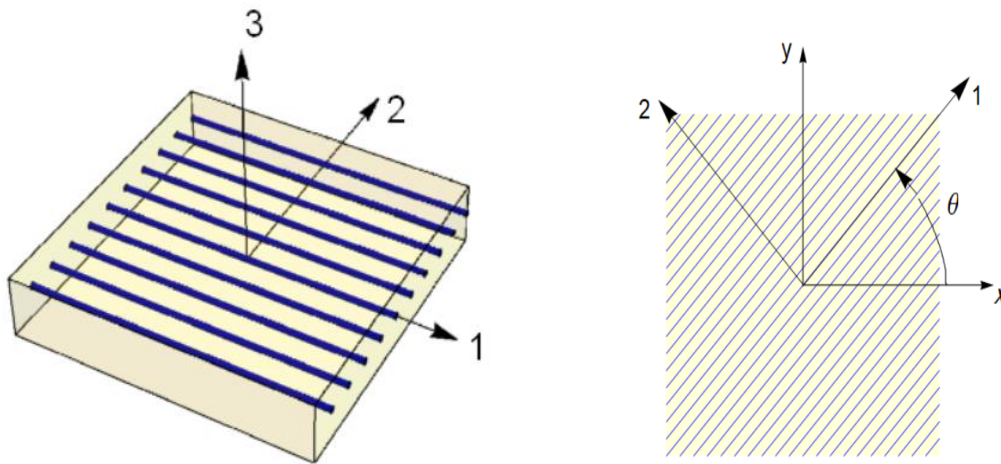


Figure 1: Schematic of a laminate.

Absolutely, as the utilization of composite materials in structural applications continues to grow, it becomes imperative to thoroughly analyze laminated anisotropic plates. However, due to the complexity arising from numerous variables, achieving an exact analysis of the structure-property relationship is challenging. To tackle this, some simplifying assumptions have been introduced, focusing on specific structural details and simplifications regarding the state of stress within the composite. These simplifications aim to facilitate a more manageable analysis process while still providing valuable insights into the behavior of laminated anisotropic plates.

Laminated composite plates are gaining increasing significance as structural components due to their remarkable combination of low specific density and high specific strength. As structures are being redesigned to prioritize weight reduction and enhanced performance, these composites offer a compelling solution to meet these objectives. Their unique properties make them indispensable in modern engineering applications including bridge decks, pressure vessels and tanks, pipes, aircraft structures, solar panels, wind turbines,

helicopters, space station structures, marine constructions, gliders, fighters and vehicles to mention a few. One of their significant benefits is their capability to bear loads effectively in multiple directions. Among these plates, anisotropic ones stand out for their exceptional efficiency, primarily attributed to their remarkable strength to weight ratio which makes them crucial in modern engineering applications [3,4]. To analyze the behavior of such plates, there should be mathematical representations that can describe their mechanics which can be classified into different theories based on Table 1. During the last 7 decades, the equivalent single layer (ESL) theories [3–12], which are the concern of this study, and the layer-wise theories [13,14] have drawn the attention of researchers.

Table 1 : Plate theories classification [15]

Classification of Plate Theories		
Based on elimination of thickness coordinate Z	Based upon field variable	Based upon choice of variable description
- Continuum - Asymptotic - Axiomatic	- Stress - Displacement - Mixed	- ESL - MLT

Despite their performance, the anisotropy (presence of highly directional properties) of composite laminates leads to the complexity of designing and analyzing structures made of such materials as compared to other types of structures, even under simple loadings. This is attributed to the occurrence of bending-extension coupling which leads to a very complex failure process with no possibility, except in special cases, to find analytical solutions [16].

In the analysis of composite plates, several Equivalent Single Layer (ESL) theories are employed to model the behavior of the laminated structure. Classical Laminate Plate Theory (CLPT) is a basic approach that assumes no transverse shear deformation, making it suitable for thin laminates. First-Order Shear Deformation Theory (FSDT) improves upon CLPT by including transverse shear deformation effects, providing better accuracy for moderately thick laminates. Higher-Order Shear Deformation Theory (HSDT) further refines the model by accounting for non-linear distribution of transverse shear strains through the laminate thickness, offering enhanced precision for thick laminates.

The laminated plate is modeled as a single layer in these theories, known as the equivalent single layer (ESL). The ESL's stiffness matrices are considered by the summation of stiffness for all layers with respect to the reference axis. The ESL stiffness-matrices consist of three main components for all theories: The extensional stiffness matrix $[A]$, the coupling stiffness matrix $[B]$, and the bending stiffness matrix $[D]$. Each serves distinct functions concerning laminated composite plates. The extensional stiffness matrix $[A]$ establishes a connection between the resultant in-plane forces and the corresponding mid-plane strains, enabling the characterization of how forces applied within the plane of the laminate influence its deformations. On the other hand, the bending stiffness matrix $[D]$ establishes a relationship between the bending moments and the associated curvatures of the plate, providing insights into how bending moments affect the shape and deflections of the laminate. Finally, the coupling stiffness matrix $[B]$ correlates the in-plane forces and bending moments with the mid-plane strains and plate curvatures, facilitating a comprehensive understanding of the complex interactions between different types of loads and deformations within the laminated composite structure [17,18]. However, for the other

theories such as FSDT and HSDT, additional stiffness components are introduced to account for transverse shear deformation and higher-order effects. These additional stiffnesses will be clarified in later sections of the manuscript.

In engineering applications, composite laminated structures are susceptible to failure when exposed to a variety of static loading conditions (out-of-plane and in-plane loads). One of the important failure modes is bending under transverse loading where bending–stretching effects need to be considered. Due to this effect, obtaining solutions in closed-form is not possible, and thus many studies neglect the coupling effect by using symmetrically laminated angle-ply configurations [19].

1.2 Practical Applications

Using laminated composite materials in structural design improves performance and dependability, including specific modulus, stiffness-to-weight ratio, strength, impact resistance, total weight, and design flexibility [20,21]. Their physical structure and mechanical behavior are determined by fiber orientation, layer size, form, and thickness of lamina [22]. Additionally, adding piezoelectric layers to laminated composites transforms them into smart structures. Their unique properties make them indispensable in modern engineering applications including laminated composite plates, bridge decks, pressure vessels and tanks, pipes, aircraft structures, solar panels, wind turbines, helicopters, space station structures, marine constructions, gliders, fighters, and vehicles to mention a few [23,24]. The most traditional structural elements in engineering applications that are made of composite laminated materials are plates and shells [25]. For further enhancement of the structural behavior of composite laminates, advanced composites are a promising material

to be structurally used in numerous engineering fields. Recently, Carbon nanotubes (CNTs) are gaining popularity among researchers for their electrical, thermal, mechanical, and chemical capabilities [26,27]. Also, functionally graded materials (FGM) are inhomogeneous composites utilized in engineering applications where their characteristics vary smoothly in a specified direction [28,29]. Additionally, plates with different geometries such as circular, annular, and sector annular are universal configurations in a wide range of industries and applications. They are widely utilized in aerospace, shipbuilding, petrochemical containers, aircraft manufacturing, civil construction, railway transportation, and other engineering industries [30]. Addition to their high load carrying capacity, composite materials have the advantages of being durable and resistant to corrosion, particularly, when utilized in marine and aerospace engineering [31]. Moreover, building-up structural elements using composite materials offers flexibility in design by allowing for different stacking sequences of fiber and matrix [32]. Consequently, all these applications have led to increased interest in optimizing laminate design techniques [33].

1.3 Problem Statement

There are analytical solutions available for specific shapes of plates, with specific boundary conditions. However, plates in reality are designed to fit different applications, and therefore complex and general shapes of plates may be encountered, along with general patterns of boundary conditions. By reviewing the literature, one can realize that analytical solutions are not available for such generality of boundary conditions and plate configurations. Due to the complexities of handling the generality in shape and boundary conditions of thick plates, it becomes inevitable to resort to a more general approach

capable of generating the solution in functional form. Energy-based methods such as Galerkin or Ritz methods are among the well-known approaches that have been employed for solving many boundary value problems in the area of engineering mechanics. Ritz method relies on minimizing the total potential energy within the system which is a thick laminated plate in our case. It reformulates the problem and transforms it to a weak form in which several integrals are to be calculated over the whole domain.

The aim of this research is to formulate Ritz method in matrix form and apply it to solve the thick plate with general stacking sequence and mixed boundary conditions. Furthermore, this approach enables the incorporation of different types of approximation functions to characterize the displacement shapes (u , v , w , ϕ_x and ϕ_y). The developed matrix equations are coded using the symbolic features of Mathematica software, facilitating the derivation of solutions in a functional manner through straightforward polynomial approximation functions. the code is built up to automate the Ritz method so as to enable accommodating more trial functions and hence more accurate solutions.

1.4 Need of Research

This research is significant in terms of theory and practicality. Recently, plates and beams made of laminated composite materials have attracted the attention of many engineers and designers due to their favorable properties of high strength to weight ratio and stiffness. However, their analysis and hence design is not easy as compared to structures made of traditional materials such as steel and concrete. The task of analysis/design becomes more complicated for thick laminated plates with general stacking sequences due to their high anisotropy and involved formulation. The conventional and classical solution procedures

are limited to cases involving symmetric stacking sequences and simple boundary conditions. This difficult task is usually overcome by using numerical methods. However, these methods yield discrete solution at the nodal points of the plate boundary and domain. Having the solution in a functional form will serve the following purposes:

- a) The ability to get the deflection and its derivatives in functional forms enables the design engineer to perform parametric analysis to investigate the impact of different geometric and material parameters on the behavior of the plate.
- b) It can be utilized for validation of other developed analytical/ numerical techniques.
- c) It allows direct and accurate computation of the secondary variables such as shear forces, stresses and bending moments.
- d) It can be utilized in carrying out design optimization.

1.5 Research Objectives and Scope

The major goal of this study is to develop a functional form solution for displacements and stresses in thick laminated composite plates with general boundary conditions and stacking sequences. The specific objectives are:

1. Derive the equilibrium equations and minimum potential energy equation representing generally stacked composite laminated plates based on FSDT and HSDT
2. Derive Ritz method formulations in matrix forms.
3. Implement the developed Ritz matrix form in a Mathematica code capable of incorporating as many trial functions as required for solution convergence and

accuracy.

4. Perform parametric analysis to investigate the impact of geometric and material parameters on the behavior of thick composite laminated plates.
5. Verify the accuracy of the proposed solution against the obtained solutions by the FEM software ABAQUS.

Throughout this research, the material is considered to be linear elastic. For cases of anisotropic laminate composite plates, it is assumed that bond between the plies is perfect with no possibility of debonding.

1.6 Research Methodology

The following tasks will be completed in order to attain the objectives listed above:

Task 1: Literature Review

A comprehensive literature survey needs to be conducted to document the historical progress of plate deformation theories (formulations and solutions) with more emphasis on thick laminated plates. This task has already initiated at the beginning of the work and continuously updated in parallel with other tasks.

Task 2: Mathematical Formulation

The equilibrium equations and the force/moments-strains/curvature relations corresponding to composite laminated plates will be derived based on the assumption of FSDT and HSDT.

Task 3: Formulation of Ritz Method

Utilizing the concept of minimum potential energy, the standard Ritz method will be cast in matrix forms for generally stacked thick laminated plates.

Task 4: Development of Computer Codes

The obtained formulations in Tasks 2 and 3 will be implemented in computer codes using Mathematica software to obtain the required functional form solutions based on Ritz method. Different numerical examples will be analyzed for the cases of thick laminated plates with different boundary conditions and stacking sequences.

Task 5: Parametric Analysis

Perform parametric analysis to investigate the impact of geometric and material parameters on the behavior of thick composite laminated plates.

Task 6: Verification of Formulations and Computer Codes.

The accuracy of the developed computer codes will be validated through a comparison of their results with existing literature and FEM results using ABAQUS.

1.7 Thesis Outlines

This thesis consists of 6 chapters. Chapter 1 contains the introduction and general descriptions of the problem and stating the objectives. Chapter 2 gives a comprehensive literature review with more focus on the solution of thick laminated composite plate analytically and numerically.

Chapter 3 extensively examines aspects related to thin plates, encompassing the evaluation of stiffness rigidities for various plate types in the sense of Classical Laminated Plate Theory (CLPT). It further includes derivations of potential energy and the Ritz method in matrix form for thin rectangular plates, supported by numerical examples. Chapters 4

extend this exploration to thick plates, utilizing the First-order Shear Deformation Theory (FSDT). Building upon this foundation, Chapter 5 continues the analysis of thick plates, employing the Higher-order Shear Deformation Theory (HSDT). Ultimately, Chapter 6 serves as the peak of results and subsequent discussions, combining insights from both thin and thick plate analyses to provide a comprehensive conclusion to the study.

CHAPTER 2

LITERATURE REVIEW

2.1 General

Several researchers have put impressive efforts into solving boundary value problems, analytically and numerically, of different plates including laminated plates. The preceding discussion includes a concise overview of prior studies conducted on composite laminated plates. However, the literature search encompasses all relevant studies, offering a historical review and advancement analysis for the mentioned theories, as illustrated in Table 2.

Table 2 : Historical review and advancement analysis for the mentioned theories

Theory	CLPT	FSDT	HSDT	LLT
Historical Period	Approximately 1850-1950	Approximately 1950-1980	Approximately 1980 - present	Had evolved over time to the present
Model Name	Called Kirchhoff- Love plate theory	Called Ressiner- Mindlin plate theory	Extension of Ressiner- Mindlin plate theory	Different researchers may have proposed their own versions. (e.g. zigzag and 3D elasticity models)
Suitable usage	Suitable for thin plates	Suitable for thick plates	This theory is utilized for	More accurate for predicting the global and local response of

Theory	CLPT	FSDT	HSDT	LLT
	with thickness to width ratio less than (1/10) [34,35].	with thickness to width ratio more than (1/10) [34,35].	analyzing thicker laminated anisotropic composite plates where FSDT is not accurate enough [34,35].	laminated composite thick plates than the ESL theories.
Shear Effect Consideration	Neglects shear effects	Include shear effects	Include shear effects	Include shear effects

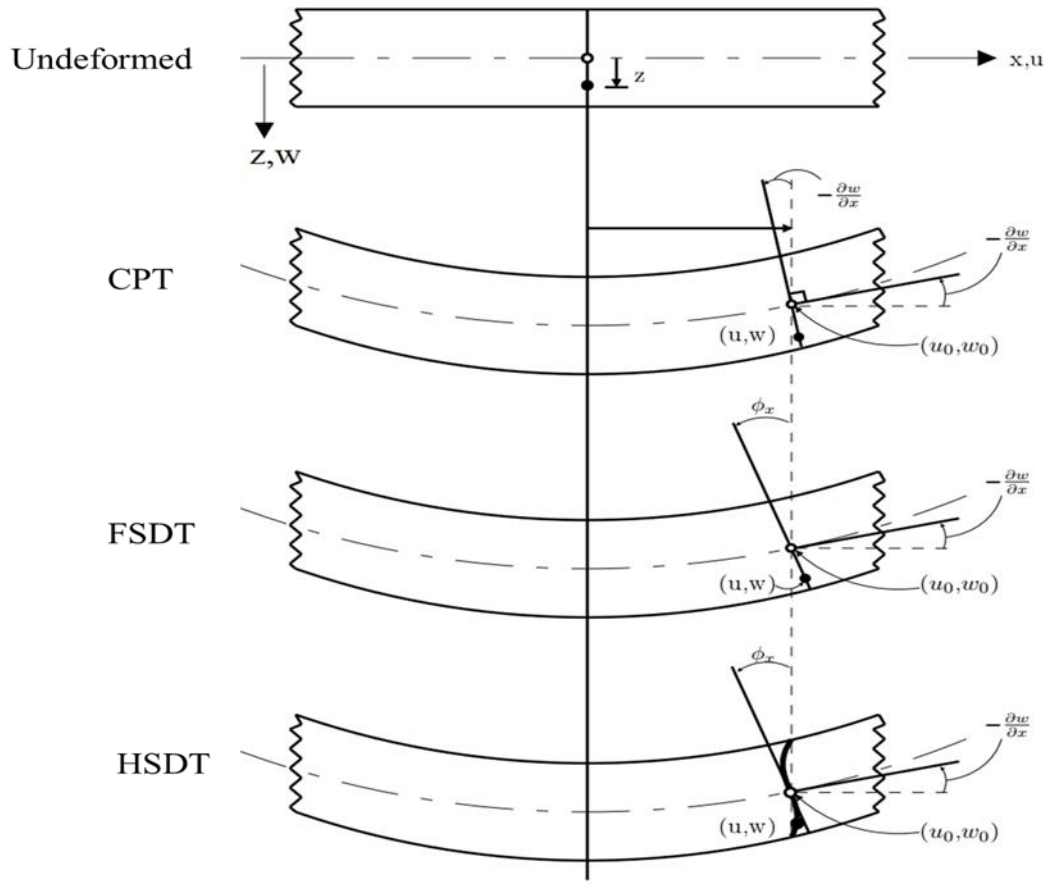


Figure 2 : Deformation of plate element based on different theories [36]

The key distinction among the theories is clarified in Table 2 and Figure 2, which illustrates the orientation of the transverse plane after deformation. In the scenario of the Classical Plate Theory (CPT), the transverse plane maintains its perpendicular and straight alignment to the midplane surface both before and after deformation, resulting in the elimination of shearing strains (γ_{yz} and γ_{xz}) .

In general, the theories used to analyze laminated plates may be divided into different major categories, namely: (i) classical lamination theory (CLPT) [37], (ii) first-order shear deformation theory (FSDT) [38], (iii) higher-order shear deformation theory (HSDT) [39], (iv) layer-wise lamination theory (LWT) [40] , (v) Zig-Zag theories [41] ,(vi) 3D

elasticity theories [42] and (vii) discrete-layer theories[43]. Each theoretical approach offers a unique perspective on comprehending the behavior of laminated composites. These theories cater to various levels of complexity and accuracy in their analyses, providing researchers with a range of options to study and understand the intricacies of laminated composites [44–46]. It should be noted that FSDT, HSDT's are theoretical models describing the kinematics of displacements whereas ESL, LWT are approaches of applying these theories (FSDT HSDTs) to laminated composites. A FSDT can be an ESL and at the same time can also be a layer-wise model if written for each layer in a laminate. Each theory has evolved over different historical periods and is tailored for specific applications based on plate thickness.

All the ESL theories are based on the concept of perfectly bonded layers, the laminate is under plane stress, and that each single lamina is orthotropic and linearly elastic. Due to their non-uniform behavior across the thickness and overall anisotropic nature, normal and transverse shear stresses are fundamentally present in composite plates [47]. The transverse and normal shear stress distributions in composite materials hold significant importance for several reasons. Firstly, when the in-plane stress exceeds the fiber materials' yield strength, it can lead to the failure of interlaminar shear stress. This failure is caused by tangential slips occurring between the layers, the components are arranged in a specific relationship to each other. Secondly, a substantial increase in transverse normal stress can result in severe bonding failure. This occurs when two layers of the composite separate from each other, experiencing opposing forces that pull them apart [48].

Equations of displacements for CLPT, FSDT and HSDT provided in Table 3, these equations are crucial components in the analysis of laminated composite plates. These displacement equations, as outlined in Table 3, play a pivotal role in describing the deformations and behaviors of such plates under various loading conditions. CLPT provides a simplified but effective representation, assuming no shear deformation through the thickness. FSDT introduces shear deformation effects, offering a more realistic depiction of plate behavior. On the other hand, HSDT, with its higher-order consideration of shear deformations, further refines the accuracy of predictions. The comparison and understanding of these displacement equations contribute significantly to the selection of appropriate theories based on the specific requirements and precision needed for a given laminated composite plate analysis.

Table 3 : Equations of displacements for ESL theories

CLPT	FSDT	HSDT
$u_1 = u_0(x, y) - z \frac{\partial w}{\partial x}$	$u_1 = u_0(x, y) + z \phi_x(x, y)$	$u_1 = u_0(x, y) - z \left(\frac{\partial w(x, y)}{\partial x} + f(z)(\phi_x(x, y) + \frac{\partial w(x, y)}{\partial x}) \right)$
$u_2 = v_0(x, y) - z \frac{\partial w}{\partial y}$	$u_2 = v_0(x, y) + z \phi_y(x, y)$	$u_2 = v_0(x, y) - z \left(\frac{\partial w(x, y)}{\partial y} + f(z)(\phi_y(x, y) + \frac{\partial w(x, y)}{\partial y}) \right)$
$u_3 = w(x, y)$	$u_3 = w(x, y)$	$u_3 = w(x, y)$

Where u_0 , v_0 , and w denote the equivalent deformation of midplane in the x , y , and z directions and u_1 , u_2 , and u_3 represent the coordinate point (x, y, z) displacements in the plate element. Additionally, ϕ_y and ϕ_x are cross sectional rotations and $f(z)$ takes different expressions for different methods.

2.2 Classical lamination plate theory (CLPT)

CLPT is a development of classical plate theory (CPT) and its adaptation for analyzing composite laminated plates can be attributed to the invention of Reissner and Stavsky in 1961 [49]. CLPT is the most fundamental approach initially developed for analyzing homogeneous plates and then extended to laminated structures. It considers the laminate as a single layer in two dimensions, ignoring the three transverse normal stresses and strains components according to Kirchhoff [50], Love [51], and Rayleigh [52]. As a result, it is also referred to as the equivalent single layer (ESL) theory, assuming that the layers in laminated composites behave as one equivalent single layer. Although this theory has been applied extensively, it is most accurate for relatively thin plates, offering reasonably accurate predictions for such cases [44,53].

Volokh [54] presented an improved version of CLPT which assumed that shear forces are statically equivalent to twisting and bending moments. This consideration contrasts with CPT, where the shear forces are unrelated to the Saint-Venant principle. In this proposed version of the theory, the treatment of shear forces serves as the foundation for describing the behavior of laminated plates. By incorporating this enhanced approach, the theory aims to get the exact deformation of laminated composite plates. As per Reissner's theory [38], neglecting shear stresses in the CLPT results in a decrease or elimination of the three essential boundary conditions that must be met along the free edges. These boundary conditions encompass the bending moment, twisting, and normal force. Due to the exclusion of shear stresses in the theory, certain complexities are simplified.

Wang [55], Pagano [56] and Reddy [57] demonstrated the limitations of CLPT according to linear displacement when analyzing thick laminated plates throughout the entire composite laminate. This inadequacy results from the neglect of shear deformation in CLPT. Khandan et al. [58] assured that CLPT remains a widely employed method for obtaining rapid and straightforward predictions, particularly concerning the behavior of thin plated structures. The key simplification involves treating shells or 3D thick structural plates as shells positioned at their mid thickness or 2D plates. This simplification significantly reduces the number of variables and equations involved, leading to substantial savings in computational time and effort. Li and Lee [59] conducted an analysis of a circular plate under uniformly distributed and concentrated transverse loads to calculate the rotation, stresses, and deflection at the center of the plate. The study utilized explicit expressions and Bessel function for this purpose. The model was found to converge to the classical Mindlin assumption for circular plate. Additionally, Mahato et al [60] extended these investigations to square and rectangular plates, expanding the scope of the study.

Based on the information provided, it can be inferred that CLPT is most suitable for thin plate structures composed of balanced composite laminates and symmetric experiencing bending and pure tension. In such cases, CLPT can provide reasonably accurate results. However, its accuracy deteriorates slightly for unsymmetrically stacked thin laminated plates and the deterioration of the solution becomes even larger as the plate thickness increased due to the neglect of transverse shear stresses.

2.3 First-order shear deformation theory (FSDT)

FSDT is the simplest theory that considers the transverse normal and transverse shear deformations. Reddy [61] introduced a two-dimensional general shear deformation theory for laminated composite plates that encompassed CLPT, FSDT (Reissner-Mindlin). Reissner [38] considered the transverse shear deformability of the plate, taking into consideration impact of the analysis on shear deformation. Consequently, the system of equations derived from this theory is such that three boundary conditions (simply supported, fixed and free) need to be provided along the edge of the plate. Mindlin [62] incorporated a similar approach for including shear effects and rotatory inertia in the flexural motions of isotropic elastic plates.

Roufaeil and Cong [63] proposed a FSDT formulation under the hypothesis that w is constant along the plate's thickness, while u and v varies linearly over each layer's thickness. Additionally, the cross-sectional rotations ϕ_x and ϕ_y are considered constant across the plate. Chandrashekhara [64] introduced a theory for shear-deformable laminated composite plates by utilizing Reissner's principle to compute their in-plane response. Reddy and Arciniega [39] discussed the analyzed shell and shear deformation plate theories proposed by Stavsky, which utilize straight forward fundamental equations and exclude transverse shear strain. Shimpi and Ghugal [65] conducted a revision of advanced shear deformation theories that aim to improve the accuracy of anisotropic and isotropic laminated plates, taking into account stress and displacement considerations. Reddy and Wang [66] provided a comprehensive analysis of the exact connections between solutions (such as buckling loads, natural frequencies, and deflection) obtained by employing CLPT

and FSDT theories. Their focus was on sandwich plates and isotropic with various boundary conditions and shapes. Their study demonstrated that FSDT solutions could be readily derived from known CLPT solutions for isotropic plates. Frederick [67] employed FSDT to investigate bending behavior of thick circular plates supported by elastic foundations. Wu and Chen [68] assumed constant transverse shear stress, which is a simplification often used in plate theories.

However, in order to satisfy the plate upper and lower surfaces boundary conditions, a shear correction factor (K) was developed to account for the transverse shear stiffness, thus affecting the validity of the outcomes utilizing FSDT. Maji and Mahato [48] proposed different shear correction factors for modifying the transverse shear stress in FSDT formulation. The selection of appropriate shear correction factor is essential in achieving better agreement between the theoretical predictions and real behavior of the thick laminated plates. Pai [69] presented a novel approach to derive shear correction factor via aligning the shear strain energy and exact shear stress results to be similar to that of FSDT. This formulation allows for obtaining shear correction factors without the need to solve a plate problem with particular boundary conditions and loadings. Furthermore, K for both symmetric angle-ply laminates and asymmetric are provided. Auricchio and Sacco [70] created novel mixed variational formulations for FSDT without using K . By this innovative approach, out of plane shear stresses can be estimated directly without the use of any post-processing. By eliminating the reliance on shear correction factors, this formulation streamlines the analysis and ensures more accurate transverse shear effects predictions in laminated composite structures.

Reddy and Nguyen [71,72] derived the expression of transverse stresses and membrane stresses from the equilibrium equations. Qi and Knight [73] presented refined FSDT-based solution in both symmetric cross-ply laminated and homogeneous plates for the plane strain bending problem. This theory introduces effective transverse shear stiffness based on constant shear stress distribution through thickness. The refined theory improves deflection predictions and includes variable transverse shear strain distribution. This assumption is supported and satisfied by exact elasticity solution. Dobyns [74] , Nelson [75] and Medwadowski [76] investigated laminated orthotropic composites plates behavior based on refined FSDT. This study involved an investigation of various aspects, including transverse normal and transverse shear effect, and quadratic displacement effects. Averill [77] put forward an accurate solution for symmetrical laminated composite plates utilizing FSDT and considering the rotational inertia. Thai [78] introduced a new approach based on FSDT instead of the traditional original method by reducing the number of unknown displacement functions from five to four. The approach assumes that the angular displacement, theta (θ), replaces two rotations ϕ_x and ϕ_y , resulting in the displacement field represented by (u, v, θ, w) . This approach enables the derivation of analytical solutions for vibration, buckling, and bending for rectangular plate analysis using different boundary conditions, all without requiring the use of shear correction factors.

Consequently, as shown in Table 4, K is chosen considering various aspects — the energy of shear deformation, material constants, the assembling pattern of laminate, geometry, configuration, loading, and end conditions [38,62] .

Table 4 : Historical Progress for shear correction factor

Authors Name	Contribution
Reissner [38] and Mindlin's [62]	K for laminate transverse shear in FSDT are an extension of Reissner and Mindlin's theories for isotropic homogeneous plates.
Whitney [34,40] and Kaneko [79]	Discussed how to introduce K
Stephen and Hutchinson [80]	Was the first to include a dependence on cross-sectional aspect ratio.
Isaksson [81]	Derived K for a corrugated board sandwich panel based on equilibrium stress.
Tanov and Tabiei [82]	Utilized FEM to determine K
Puchegger [83]	Obtained K experimentally for validation
Hadavinia and Gordnian [84]	Dervide shear correction factor based using the energy equivalence method

Following the preceding analysis, it can be deduced that the utilization of FSDT, which neglects cross-sectional deformation effects, leads to a transverse shear stress that uniformly varies across the thickness of the laminate. Furthermore, the precision of the analysis is notably influenced by the shear correction factor employed. This inference prompts exploration into novel approaches that may eliminate the need for shear correction factors, potentially enhancing the accuracy of the analytical methods employed.

2.4 Higher-order shear deformation theory (HSDT)

HSDT utilized for analyzing thicker laminated anisotropic composite plates where FSDT is not accurate enough for predicting buckling loads, natural frequencies, stresses, and

deflections for such plates. It considers the transverse shear stresses to follow a parabolic distribution, and notably, it does not necessitate any shear correction factors [85].

Reddy [86] proposed a HSDT for laminated composite plates which offers improved accuracy in predicting stresses and deflections as compared to FSDT. Aydogdu [87] achieved a comparative analysis among various shear deformation theories (SDT) for cross-ply rectangular plates having simply supported edges, concerning the buckling, vibration, and bending behavior. Law and Feng [88] demonstrated that the newly proposed and straightforward HSDT, based on the principle of virtual work, offers improved reliability, accuracy, and computational efficiency compared to both FSDT and other HSDT theories. Matsunaga [89] provided two-dimensional theoretical equations, considering both transverse shear and bending deformations for evaluating displacement distributions and stress within a thick elastic plate subjected to fixed boundaries. Mantari [90] introduced a novel HSDT in the case of composite laminated and sandwich plates. It assured that the plate's boundary surface is free of tangential stress and ensured a precise transverse shear strain distribution throughout the thickness of the plate without using K . Dhuria [91] provided an innovative HSDT for analyzing the mechanical behavior of anti-symmetric and symmetric composite laminates in both angle-ply and cross-ply arrangements. This theory is suitable for buckling and static bending analysis. It incorporated a secant hyperbolic function within the displacement field, capturing non-linear displacement distributions. Shi et al. [92] introduced a new hyperbolic tangent SDT for the derivation of weak forms without K to analyze the static buckling, static behavior, and free vibration of laminated composite plates. These derived forms are then solved numerically using the iso-geometric analysis approach. Li and Yan [93] created a

comprehensive framework, unifying all HSDT theories. This framework is designed to model and analyze functionally graded as well as laminated plates. Shishehsaz [94] employed layer-wise theory combined with third order shear deformation theory (TSDT) and HSDT to examine the adhesive behavior within a uniformly loaded circular plate. Both analytical and FEM solutions are used to analyze and understand the adhesive behavior under these conditions.

Based on the above, it is concluded that 3D equilibrium equations need to be applied to achieve more satisfactory results. However, despite the progress made, researchers still need to investigate the integration of HSDT with other advanced theories or numerical methods to enhance its capabilities and applicability to a wider range of practical engineering problems.

2.5 Assessments of various methods to model laminated composite plates

Several researchers have put remarkable efforts into solving boundary value problems, analytically and numerically of different laminated plates [95]. In this section, a brief summary of previously reported studies in the literature on the analytical and numerical analysis of laminated plates is presented. Prior to delving into the analytical and numerical solutions, various mathematical models exist that are based on shear deformation theory as listed below:

The expansion technique (based on thickness) [96]:

1. The mixed variational technique.

2. The successive approximations method.
3. An expansion method based on asymptotic analysis.
4. Symbolic integrations technique.
5. Initial functions technique.

The Method of hypotheses' (2D problem) [48] :

1. The semi-inverse technique.
2. The technique of trigonometric functions.
3. The expansion based on Fourier series.

Analytical solutions are typically limited to cases involving simple geometry and boundary conditions in engineering and mathematics. This section is a review of some research related to analytical solutions of thin and thick laminated composite plates. Generally, analytical solutions are primarily applicable to situations involving specific boundary conditions, simplified theories, and particular cases of loadings. These solutions are typically applicable to specific kinds of boundary conditions.

Numerical techniques have been innovatively developed to efficiently find solutions for diverse structural components under various loadings and boundary conditions as per Figure 3. These techniques offer the advantage of significantly reducing design time and associated costs for engineers and researchers. The fundamental idea behind FEM is to represent the solution region analytically or through approximation by dividing it into discrete elements (discretization). By assembling these elements in various configurations, it becomes possible to represent highly complicated shapes and structures [97].

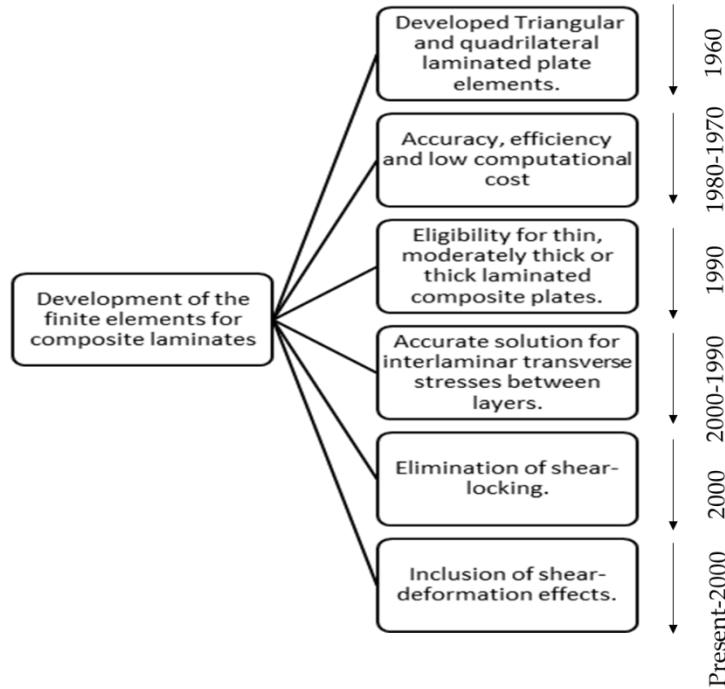


Figure 3 : Progress on FEM for laminated composite plates [4]

2.5.1 Navier and Levy solutions

Leissa and Whitney [98] formulated the Navier solution for simply supported rectangular plates, addressing aspects of buckling, and bending. Bshaskar and Varadan [99] investigated the equations of dynamic equilibrium by a combined approach involving Laplace transform technique and Navier's method based on FSDT. By applying this approach, the dynamic magnification factor is calculated for both stresses and deflections. The study used various loading conditions and considered different geometric parameters. Mantari and soares [100] provided numerical results for static and free vibration assessments of cross-ply simply supported plates using Navier's solution. Adim [101] used refined HSDT which has five unknown variables to analyze the free vibration and static bending of simply supported laminated composite plates. He considered the parabolic

evolution of transverse shear strains in the thickness direction. The theory comprises four unknown displacement components, which were enlarged using trigonometric functions and Navier's technique. Dan [102] applied HSDT to study free vibration response in simply supported beams with solid cross-sections. The Carrera unified formulation (CUF) is taken into account for displacement fields. The resultant differential equations are solved analytically using Navier's method.

Reddy and Khdeir [103] introduced Levy's solution for symmetrically laminated rectangular plates, encompassing various boundary conditions such as free, clamped and simply supported. These solutions were derived using FSDT. Reddy and Xiang [104] investigated rectangular plate's natural frequencies by employing state-space technique and Levy's solution, in conjunction with FSDT. Mohammadi [105] presented studied buckling stresses in thin FG plates using CLPT with supported opposing edges and arbitrary boundary conditions along remaining edges by Levy's solution. The theory posits that in-plane displacement components vary linearly with thickness, while material characteristics vary as a power function of thickness coordinate. Naderi and Saidi [106] examined the buckling reaction of relatively thick FG sector plates using FSDT. The linked stability equations are decoupled and solved analytically for SS conditions by Levy's solution on straight edges and free boundaries on circumferential edges. Atashipour [107] used TSDT to analyze the bending of thick FG circular plates. Closed form solutions for transverse plate deflection are found by utilizing boundary layer phenomena.

2.5.2 Elasticity solutions

Hussainy, Srinivas [108] and Rao [109] derived solutions to the bending behavior of moderately thick rectangular laminates plates under simply supported conditions based on 3D elasticity solutions. They explored the flexural response of two-sided composite plates with different edge conditions while being supported on the remaining sides considering the transverse shear effects. Pagano [110] used the same elasticity solution to develop a solution for rectangular laminates with pinned edges. The study included the solution of specific cases, including a sandwich plate, which were then compared with corresponding solutions obtained in the sense of CLPT. Ray [111] derived 3D elasticity solutions for static analysis of rectangular antisymmetric angle-ply plates. Resulting formulas for stresses and displacements are versatile and can be readily applied to calculate numerical results for plates with various fiber orientation angles. However, it's important to note that the exact solutions obtained for square antisymmetric angle-ply plates having a fiber orientation angle of 45° are not directly applicable to plates that have varying fiber orientation angles. Singh and Karathanasopoulos [112] developed a 3D elasticity solution to analyze the mechanical behavior of composite plates having arbitrary support conditions, both in cross-ply and angle-ply, when subjected to patch loads.

2.5.3 Power, trigonometric & Fourier series approaches

Alipour [113] used the application of a power series method based on FSDT to analyze the stress and bending distribution in sandwich and angle-ply laminated composite plates subjected to non-uniform loads. This method offered analytical solutions over a broad spectrum of boundary conditions and diverse load distributions. The outcomes

demonstrated its applicability to sandwich plates and moderately thick cross/angle-ply laminated composite, accounting for composite plate orientation, edge conditions, variable elastic foundation conditions and non-uniform transversely distributed loads. Sayyad and Shinde [114] proposed a straightforward double trigonometric series solution based FSDT to analyze the buckling and bending for cross-ply laminated composite plates. The field of in-plane displacements were represented using sinusoidal functions in terms of the thickness coordinate, effectively incorporating shear deformation effects. The transverse displacement encompassed both shear and bending components without using K. Stresses, displacements, and critical buckling loads have been obtained. Kabir [115] introduced an analytical solution using an innovative approach involving a boundary-continuous double Fourier series solution based on (FSDT), considering the shear flexibility of arbitrary rectangular laminates.

2.5.4 Rayleigh-Ritz and Galerkin methods

Ritz method is an approximate energy-based technique utilized in mechanics problems. In this approach, the displacement function is estimated by composing it as a linear combination of trial functions. It is essential that this displacement function satisfies the geometric or fundamental boundary conditions [116]. It is a classical approach extensively employed to examine the dynamic, buckling, and static responses of structures. It is employed in the determination of moments, stresses, natural frequencies, and critical buckling in vibrating systems. Additionally, it can be applied to describe and solve various boundary value problems for shells, beams, and plates [117]. Ritz method is among the well-known approaches that has been employed for solving many boundary value

problems, which proved its capability and efficiency in predicting the bending behavior of composite laminated plate with boundary conditions including free edges [118–125].

Song [126] employed Rayleigh-Ritz approach to determine the natural frequency of laminated composite cylindrical shells under arbitrary boundary conditions. Nallim and Grossi [127] offered the application of orthogonal polynomials within Ritz method, particularly concerning the examination of rectangular anisotropic plates to determine free vibration and bending deflection. Vaseghi [128] used the hierarchical Rayleigh-Ritz approach to investigate the unilateral buckling of restricted rectangular plates. The study examined composite plate behavior under compressive in-plane stresses using a nonlinear elastic constraint on one side. Kharghani and Soares [129] derived an efficient semi-analytical solution to assess composite plates' load bearing capacity and flexural response. This solution employed the Layerwise HSDT theory in conjunction with the Ritz approximation technique. Dinis and Belinha [130] developed the Element-Free Galerkin approach, to analyze anisotropic laminated plates using FSDT. Fazzolari and Carrera developed a trigonometric Ritz formulation for thermal and mechanical buckled anisotropic laminated plates, based on enhanced ESL, LW, and zig-zag plate theories, to determine non-dimensional natural frequencies [131]. Wang and Wu [132] studied the free vibration of an FG porous cylindrical shell using higher order sinusoidal theory. The Rayleigh-Ritz approach was used to determine the governing equations of plates under different boundary conditions.

2.5.5 Variational methods

The variational asymptotic approach, created in a mathematically rigorous method and group to analyze laminated structures, guarantees accurate results [133]. Hodges and Lee [134] used the variational-asymptotical method to split the solution of deformations in laminated plates into two distinct parts. The first part involves a linear, one-dimensional analysis through the plate's thickness, while the second part focuses on a nonlinear, two-dimensional plate analysis to investigate the linear cylindrical bending of laminated plates with varying stacking sequences. Madenci and Özütok [135] introduced a variational approximate method with four nodes, each node having 11-DOF, using HSDT for laminated composite plates.

2.5.6 Differential quadrature methods (DQM)

Chaudhuri and Seide [136] introduced a semi- analytical solution to estimate interlaminar shear stress distribution in thick laminated plates. This method employs FEM with a quadratic displacement potential energy assumption for the analysis. Sherbourne and Pandey [137] examined the precision and convergence of differential quadrature method in relation to addressing various differential equations having changing coefficients related to the buckling of plates. Rectangular plates with different material properties, including anisotropic and composite laminates with orthotropic symmetric angle-ply subjected to linearly changing uniaxial compression were considered. Jahromi [138] introduced the generalized DQ approach to analyze vibrations of relatively thick plates on a Pasternak basis, where the plate's governing equations are derived using FSDT. Ferreira and Carrera [139] employed DQM to forecast both free vibration and static deformations characteristics

of cross-ply and variable thickness laminated composite plates, spanning from thin to thick. The study encompassed a range of boundary conditions, including fully clamped or complete simply supported boundaries, as well as mixed external conditions, involving combinations of free, simply support, and clamped. Khalid and Ojo [140] introduced a novel method known as the two-dimensional inverse DQM for approximating solutions to higher-order systems of differential equations concerning laminated composite plates, employing FSDT. Through numerical assessments focusing on buckling and bending for plates with varying boundary conditions and loadings. The proposed method's accuracy is found to be comparable to FEM and Navier's solution. Szekrényes [141] utilized DQM to analyze composite plates based on FSDT. The investigated plates were subjected to concentrated transverse forces at their midpoints. The study verified comparing the convergence of deflection obtained through the DQM with deflection values derived from spatial FEM and analytical methods. Civalek and Baltacıoglu [142] analyzed the vibration of laminated and FGM composite annular sector plates using the harmonic DQM, applying both theories FSDT and CLPT.

2.5.7 Some recent analytical methods

Dorduncu [143] presented a new approach using a nonlocal Peridynamic Differential Operator to perform static analyses of simply supported laminated composite plates based on FSDT. In his formulation, the peridynamic differential operators were used instead of the traditional local spatial derivatives. Pathan and Singh [144] presented a solution for the static analysis of functionally graded and smart laminated composite plates. The deformation kinetics within smart structures were represented using five newly introduced non-polynomial HSDT theories. Shukla and Singh [145] investigated angle-ply laminated

plates using a method that employed multiquadric radial basis functions in a mesh-free approach for flexure analysis based on HSDT. It enabled the calculation of deflection and stress results for both cross-ply and angle-ply laminated plates. Naik and Sayyad [146] introduced a unique analytical solution to examine the bending behavior of mechanically loaded simply supported sandwich and laminated composite plates. The analysis employed HSDT, which utilized polynomial-type form functions up to the fifth order in the displacement field. Additionally, they presented the same theory formulated for the comprehensive analysis of the same plates undergoing bi-directional bending under transverse loads. In the displacement field, this theory involved nine unknowns, without using shear correction factor [147]. Li [148,149] analyzed thin rectangular cantilever plates using the superposition method in symplectic space (SSM) to achieve a reasonable analytical solution for free vibration characteristics. The SSM technique provides a new analytical solution for buckling response of thin rectangular plates.

2.5.8 Finite element analysis

Roufaeil and Cing [150] introduced a novel method for laminated plates by analyzing finite strip solution, incorporating FSDT. The study predicted that displacements u and v varying linearly over the thickness of individual layers and maintain continuity at interfaces adjacent to each other. Additionally, throughout the thickness, constant displacement w was assumed. As a significant outcome of their work, they developed 2 and 3-noded finite strip elements. Roufacil and Tran-Cong [151] and Pryor and Barker [152] applied CLPT and formulated FEM using the principles of Mindlin and Reissner theories. In their approach, they assumed that the transverse displacement of the composite plate is constant across its thickness. Additionally, they assumed that the two normal direction

displacements vary linearly within each lamina. Turvey [153] introduced both approximate and exact methods to examine the flexural behavior of simply supported antisymmetric rectangular moderately thick laminated plates. He examined how transverse shear deformations impact the plates' response to different lateral load distributions. Reddy and Chao [154] examined how element type (linear or quadratic), mesh size, and the application of reduced integration impacted the accuracy of FEM with governing anisotropic, laminated, thick composite plates. Owen and Li [155] introduced a finite element local model that builds upon enhanced approximate theory for anisotropic thick laminated plates. They demonstrated the practicality and cost-effectiveness of this model by successfully predicting plates' in-plane displacements, transverse shearing stresses, and static bending stresses.

Rolfes and Rohwer [156,157] developed a novel FEM technique to compute enhanced transverse shear stresses in composite laminated plates, applying FSDT. The core concept involved directly estimating transverse shear stresses based on transverse shear forces, without taking membrane forces into account. The method eliminated the need for shear correction factors. Additionally, they developed a method in layered composite plate systems, to compute the transverse normal stress in the direction of thickness. Examples of antisymmetric angle-ply and symmetric cross-ply laminates in numerical form demonstrated that the proposed approach closely approximates the exact 3D elasticity solution. This holds true, applicable to thin and relatively thick plates, with a slenderness ratio as low as five. Touratier [158] conducted a FEM to examine plates with large side to thickness ratios and to obtain precise shear constraints without using shear correction factors. The proposed theory was then assessed for accuracy by investigating various

significant problems, including buckling and bending of a three layered (laminated and sandwich), simply supported edges, rectangular or square symmetric cross-ply plates. Kim and Zhang [159] obtained LDT18 which is a basic displacement-based 3-node, 18-DOF operator. This element was designed for linear FEM of both thick and thin laminated composite plates using FSDT. Timoshenko's laminated composite beam functions were used to calculate the rotation and deflection along the element's boundary, to prevent shear-locking problems. Sze and He [160] presented a predictor-corrector approach for the FEM analysis of laminated composite plate. In this approach, Mindlin finite element models were used to compute predictive strain, deflection, and stress. A least-squares fit was used to estimate the corrector transverse shear stress, predictor in-plane stress, and condition of homogenous stress equilibrium. Subsequently, the corrector deflection was calculated, based on FSDT. Ge [161] developed improved FEM discrete 15-DOF triangular laminated composite plate element based on FSDT. The proposed element incorporates the shear strain, which is suitable to analyze both thin and moderately thick anisotropic laminated composite plates. Singh et al [162] proposed a new rectangular material finite element based on FSDT, free from any locking issues, and has 5-DOF at each node. The shape functions, derived from the equilibrium equations of moment-shear, incorporated the element coordinates, material properties and thickness. Gim [163] created a 2-D finite element based on FSDT to model delamination's in laminated plates, allowing for computation of the plate element's strain energy. Kosmatka [164] introduced a novel 6-node triangular plate element for analyzing laminated composite structures. It was based on Hamilton's concept and incorporated the FSDT, without shear locking and exhibited the correct rank. Through numerical validation, the new element demonstrated remarkable

convergence performance compared to available triangular elements. Kabir [165] developed FEM for fiber-reinforced laminated simply supported plate with cross-ply laminations under uniform transverse loads with various thicknesses, using isoparametric 3-node triangular finite element. The element's strain-displacement relations were based on FSDT, which accounted for transverse shear deformations. Also, in the context of FSDT, Wang and To [166] introduced straightforward 3-node, 6-DOF per node, triangular shell finite elements designed for laminated composites. The outcomes demonstrated that the suggested simplified plate elements provided exceptionally precise results for length-to-thickness ratios higher than or equal to 20. Erdem and Gunay [167] introduced 9-node quadrilateral element, capable of shear deformation and heterosis. To create this element, Kirchhoff constraints have been adjusted using FSDT assumptions. Sadek [168] proposed 3 rectangular 8-nodes elements and implemented them to analyze cross-ply square composite plates and having different thickness ratios. These elements were designed to include transverse shear deformations by accounting for FSDT-HSDT terms in the thickness coordinate. Load, stiffness, and stress matrices were derived for these elements. Sacco and Auricchio [169] developed a 4-node finite for analyzing laminated composite plates using FSDT, considering the element's advantage of its locking-free. Additionally, this element also includes an algorithmic iteration for evaluating the shear correction factors. Alfano [170] developed a novel 9 and 4-node elements, for analyzing laminated composite plates using FSDT. These elements are based on model of tensorial components developed by Bathe and Dvorkin [171,172], which also have the advantages of being locking-free. This method produces very good results for normal and transverse stresses. Rao and Meyer-Piening [173] used Toledano-Murakami higher-order bending theory to

construct a unique FEM formulation to examine thick laminated plates comprising layers of arbitrary orientations. The investigation involved changing the fiber orientation angle inside a rectangular laminate's bottom and top layers to assess in-plane stresses and central deflection. Vinh and Belarbi [174] introduced an innovative approach for static bending analysis of functionally graded (FG) plates using a 4-node quadrilateral plate element using HSDT. It is achieved by incorporating higher order terms that have quadratic shape functions. Mathew [175] employed least square finite element solution to obtain displacements and stresses through a state-space model, for anisotropic, linear elastic, multilayered rectangular plate subjected to arbitrary boundary conditions on one edge while being simply supported on the other edge.

2.5.9 Meshless technique

Meshless methods have been shown to be capable of handling a wide range of engineering issues, and they outperform traditional numerical methods, particularly when dealing with discontinuities and large deformation problems [176,177]. Wang and Liew [178] introduced meshless technique utilizing kernel particle approach to investigate the buckling and flexural behavior of rectangular laminated composite plates. This method adopted FSDT and constructed displacement shape functions through reproducing kernel approximation that ensured adherence to consistency conditions. By embracing this meshless methodology, an efficient solution approach for analyzing laminated composite plates was achieved. Singh [179] employed a meshless approach to analyze laminated composite plates, and deflections were measured using HSDT. Various RBFs were used to discretize differential equations and approximate solutions. Sator [180] analyzed the bending of thin and thick elastic FG plates utilizing CLPT, FSDT, and HSDT. They used

a meshless approach with the moving least squares approximation to govern field variables. Khosravifard and Zarei [181] developed a mesh-free method of radial point interpolation for conducting buckling and static analyses of thick laminated plates. This method takes into consideration the zigzag displacement field changes with plate thickness and ensures consistency of interlaminar transverse shearing stresses. Van Do and Lee investigated the bending response using HSDT [182] and buckling response using n^{th} -order shear deformation theory [183] of multilayered composite FGM plates using an enhanced meshless moving kriging approach.

2.5.10 Boundary element method

Hsu and Hwu [184] developed boundary element method to address a diverse set of complicated problems linked to thick laminated plates including deformations based on FSDT. This method is versatile and can handle the coupling problems, which often occur in unsymmetric laminated composite plates. Garg [185,186] examined the analysis of poorly bonded laminated composite plates under cylindrical bending. Also, for thick laminated composite plates subjected to bi-axial bending using the scaled boundary finite element method (SBFEM), a relatively recent approach. Daros [187] uses a pure BEM to analyze the static behavior of 2D and 3D structures made out of thin, unsymmetric laminated composite plates.

2.5.11 Some recent numerical methods

Ferreira conducted several studies on the collocation method for the analysis of isotropic laminated composite plates and doubly-curved composite shells. He utilized Deslaurier–Dubuc interpolating wavelets to evaluate bending, static, and free vibration analyses[188].

FEM was employed to discretize the plates, assuming constant transverse shear deformation based on FSDT. The meshless approach based on wavelet collocation was used for discretization of boundary conditions and equilibrium equations. The research covered topics such as elastic buckling loads [189], bending response [190], and natural frequencies [191], employing the collocation technique with interpolating wavelets for structural analysis. The studies demonstrated the applicability of this method for various analyses in composite materials. Kien [192] developed a radial basis function-based finite element approach to construct shape functions for analyzing laminated composite plates buckling, bending, and vibration based on FSDT. Compared to the traditional FEM, it simplifies the construction of shape functions as the number of nodes in the element increases. Shukla and Vishwakarma [193] employed radial basis function-based mesh-free approach for analyzing the buckling of both antisymmetric and symmetric laminated plates based on HSDT.

2.5.12 Artificial Intelligence based on artificial neural network approaches

Within the engineering discipline, artificial neural networks appear from publications statistics to be one of the great successes of structural engineering computing [194]. Artificial neural networks are widely used in engineering due to their efficacy in handling strong non-linear relationships [195]. Especially valuable when conventional methods demand significant computational resources or time, these networks find applications in diverse areas, including building material studies, structural identification (e.g., analysis of laminated composite structures), geotechnical engineering (e.g., earthquake-induced liquefaction potential), civil engineering heat transfer problems, transportation engineering (e.g., traffic problem identification), construction technology

and management (e.g., estimating building costs), and building services (e.g., analyzing water distribution networks) [196]. Al Rjoub and Alshatnawi [197] used an ANN model to forecast the frequency of porous FGM plates with a side crack. Third-order plate theory is used to examine how porosity and cracking affect dynamic behavior. Saadatmorad [198] used convolutional neural networks (CNN) and two-dimensional wavelet transforms to detect damage in rectangular laminated plates. Martinez [199] analyzed the reliability of a smart laminated plate with a piezoelectric fiber-reinforced composite actuator layer using an artificial neural network trained model. Atilla [200] studied the stability and vibration of composite plates including circular cutouts.

The previously mentioned studies primarily focus on solutions tailored to specific stacking sequences or address particular boundary conditions, showcasing a limitation in applicability. Additionally, these investigations provide an extensive survey of diverse methodologies utilized for the analysis of laminated composite plates. The spectrum of approaches discussed spans classical analytical methods, finite element methods (FEM), meshless techniques, boundary element methods (BEM), and artificial intelligence-based approaches, with a specific emphasis on artificial neural networks (ANN). The discourse encompasses a comprehensive exploration of each method's relevance to the study of laminated composite plates, encompassing aspects like bending, buckling, and free vibration. Furthermore, it offers a meticulous examination of various methodologies employed in the analysis of laminated composite and functionally graded (FG) plates, presenting valuable insights into the strengths and implications associated with each approach. The conventional and classical solution procedures are limited to case involving symmetric stacking sequences and simple boundary conditions. This difficult task is

usually overcome by using numerical methods. However, these methods yield discrete solution at the nodal points of the plate boundary and domain. The development of Ritz formulation in matrix form for thick plates governed by high order shear deformation theory is an original work and worth publishing. The achievement of the solution in functional form is another strength of the work.

CHAPTER 3

CLPT FOR THE ANALYSIS OF THIN LAMINATED COMPOSITE PLATES

3.1 General

Depending on their thickness to smaller span length ratio, plates can be classified into two groups: thin plates and thick plates. A plate is considered as a thin when the ratio of the thickness to the smaller span length is less than $(1/20)$ [201]. Furthermore, thin plates can be classified into thin plate with small deflection for cases where the deflection is less than the plate thickness ($w < t$). This chapter provides the fundamental principles of anisotropic thin plate theory developed based on the CLPT. The presentation on this principle is based on the works of Lekhnitskii [202], Jones [203], Whitney [204] and Reddy [39]. In an anisotropic continuum, the development of a thin plate theory which includes in-plane force effects that requires to relate stresses on the deformed body to the original configuration. CLPT assumed all points through thickness to have the same deflection. Thus, the displacements u , v and w any point are expressed as:

$$u = u_0(x, y) - z \frac{\partial w}{\partial x} \quad (3.1a)$$

$$v = v_0(x, y) - z \frac{\partial w}{\partial y} \quad (3.1b)$$

$$w = w(x, y) \quad (3.1c)$$

Where u , v , and w are the three displacement components in the x , y , and z directions, respectively. The mid-surface has the displacement components; u_0 , v_0 , and w . Now, using the definitions of the three strains for small displacements in the xy -plane yields:

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \quad (3.2a)$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2} \quad (3.2b)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \quad (3.2c)$$

Also, the generalized Hook's law can be expressed in cartesian coordinates by the following matrix form:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} & Q_{45} & Q_{46} \\ Q_{15} & Q_{25} & Q_{35} & Q_{45} & Q_{55} & Q_{56} \\ Q_{16} & Q_{26} & Q_{36} & Q_{46} & Q_{56} & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \quad (3.3)$$

Where, Q_{ij} is the stiffness matrix which includes 21 independent elastic constants for general case.

This chapter focuses on the behavior of thin anisotropic plates where the CLPT can be applied and gives reasonably accurate results for many practical problems, as stated by Reissner and Stavsky [49]. Therefore, the CLPT is detailed more in the following section.

3.2 CLPT for the Behavior of Thin Laminated Composite Plates

The classical plate theory (CLPT) which is known as Kirchhoff's plate theory. The plate deformation according to this theory is illustrated by Figure 4. The fundamental assumptions of classical plate theory of elastic may be stated as follows:

- The plane normal (nm) to the mid-surface before bending remains straight and normal to that surface after bending.
- The stress normal to the midplane, σ_z , is small compared with the other stress components and may be neglected.
- The transverse shearing strains are assumed to be zero. i.e., shear strains γ_{xz} and γ_{yz} will be zero. Thus, thickness of the plate does not change during bending. It is assumed, therefore, that the normal strain ε_z resulting from transverse loading may also be ignored.

To spot a light on plate bending behavior, consider a load free flat rectangular plate (Figure 4), in which the xy -plane is coincide with the mid-surface of the plate. When a lateral load is exerted on the plate, deformation takes place leading to three displacement components u , v , and w in the x , y , and z directions, respectively. The mid-surface has the displacement components: u_0 , v_0 , and w .

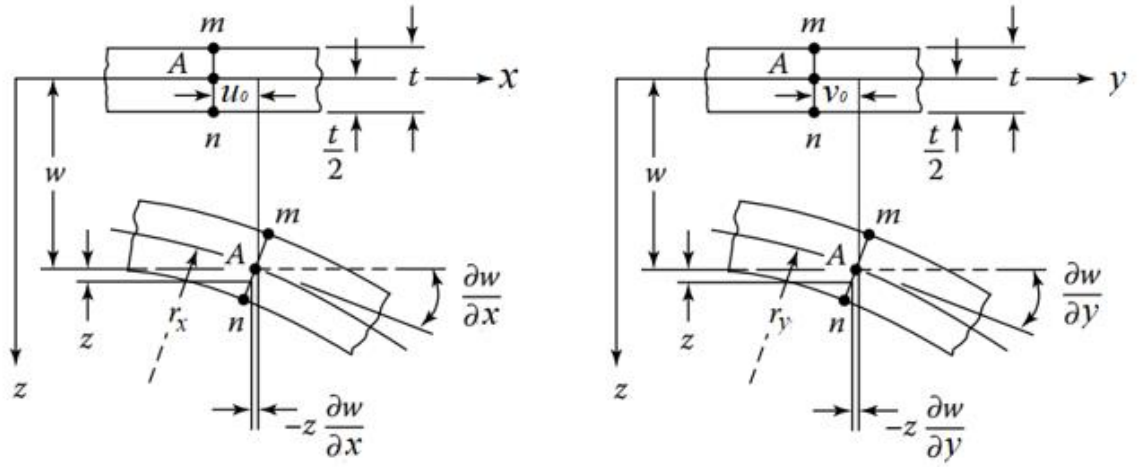
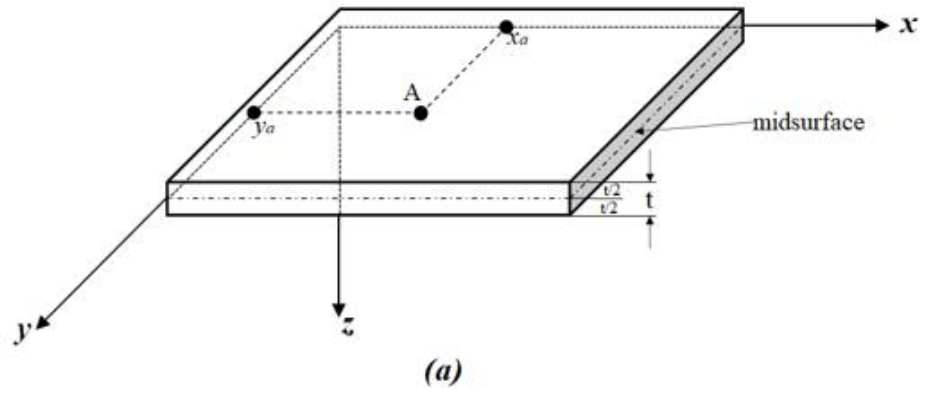


Figure 4 : Plate's element before and after bending

3.2.1 Constitutive relations

Considering the geometry of the deflected plate (Figure 4), strains vary linearly through the thickness of the plate and depend on the midspan strains ($\epsilon_x^{(0)}, \epsilon_y^{(0)}, \gamma_{xy}^{(0)}$) and curvatures ($\kappa_x, \kappa_y, \kappa_{xy}$). They may be expressed as:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.4)$$

where:

$$\begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{bmatrix} \quad (3.5)$$

and

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (3.6)$$

Considering the assumptions of CLPT, the generalized Hook's law (Eq. 3.3) is reduced to:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (3.7)$$

The transformed reduced stiffness matrix $[\bar{Q}]$ in Eq. (3.7) relates the stresses to the corresponding strains considering the coupling responses. The components of the matrix $[\bar{Q}]$ are defined as follows; $\bar{Q}_{11}$ and $\bar{Q}_{22}$ are the extension components relate normal stresses to normal stains, $\bar{Q}_{12}$ is the extension-extension coupling component that relates the normal stress in one direction to the normal strain in the perpendicular direction due to Poisson's effect, $\bar{Q}_{66}$ is the shear coupling component that relates the shear stress to the shear strain, and $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are the shear-extension coupling components that relate the

normal stresses to the shear strains [205]. However, this will be explained, in detail, in Section 3.2.2.

After the combination of Eqs. (3.4) and (3.7), the stress components for the laminate in global coordinates are expressed in Eq. (3.8).

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.8)$$

From Eq. (3.8), it is obvious that for each lamina the stresses will vary linearly through the thickness. Also, the stresses change as the transformed reduced stiffness matrix $[\bar{Q}]$ changes (depends on the material and angle orientation of the lamina θ_k).

The unknowns of the problem are reduced to the midplane strains and plate curvatures. However, because of the stresses are not continuous through the laminate, it is more convenient to deal with resultant forces and moments. This can be done by integrating the stresses in each lamina through the thickness of the laminate. Thus, the resultant normal forces and moments are defined as follows:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \quad (3.9a)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z dz \quad (3.9b)$$

Using Eqs. (3.9) in conjunction with (3.8) and (3.7) with considering that the transformed reduced stiffness matrix $[\bar{Q}]$ is constant and the midplane strains and curvatures are independent of the z -coordinate. The result equations can be written as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} dz \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z dz \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.10a)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z dz \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z^2 dz \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.1b)$$

As a result of integrating Eq. (3.10), the membrane forces and bending moments are expressed as follows:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.11a)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.2b)$$

The expressions above can be combined into one general expression that relates in plane forces (N_x, N_y, N_{xy}) and moments (M_x, M_y, M_{xy}) per unit width to the midspan strains $(\epsilon_x^{(0)}, \epsilon_y^{(0)}, \gamma_{xy}^{(0)})$ and curvatures $(\kappa_x, \kappa_y, \kappa_{xy})$, resulting in the following constitutive relations for the anisotropic thin plate:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.12)$$

where

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-t/2}^{t/2} \bar{Q}_{ij} (1, z, z^2) dz, \quad i = 1, 2, 6; \quad j = 1, 2, 6 \quad (3.13)$$

The component A_{ij} , B_{ij} and D_{ij} are of the matrices representing the extensional stiffness matrix [A], the coupling stiffness matrix [B], and the bending stiffness matrix [D], respectively. The extensional stiffness matrix [A] relates the resultant in-plane forces to the corresponding mid-plane strains. The matrix [D] relates the bending moments to the associated plate curvatures. The matrix [B] relates the in-plane forces and bending moments to the mid-plane strains and plate curvatures. The components A_{ij} , B_{ij} and D_{ij} are, evaluated using Eq. (3.13), resulting in different categories of the plates (e.g. isotropic, orthotropic, etc.) depending on the degree of anisotropy.

3.2.2 Determination of transformed stiffness matrix and rigidities

In the composite structures, a composite laminated plate is an assembly of layers made of fiber reinforced materials which can be joined to provide the required engineering properties, including in-plane stiffness, bending stiffness, and strength. The individual layers are generally orthotropic or transversely isotropic but the whole laminated plate exhibits anisotropic properties. To employ the CLPT in the investigation of laminate plate

behavior, the bond between the lamina (layers) of the laminated plate are assumed to be perfect and the laminated composite plate is modeled as an equivalent single layer plate (ESL). The properties of the resultant ESL of the laminated composite plate are evaluated by approximating the resultant stiffness matrix considering the stacking sequence of the individual layers.

For example, consider a laminated composite plate having a thickness (t) and each layer or ply have its engineering constants in its local coordinate axes making an angle θ with the positive x -axis in the global coordinate system as shown in Figure 5. The reference datum plane is the mid-surface plane of the plate at $z = 0$ and the location of the top and bottom surfaces of the k^{th} layer (ply) is defined by z_{k-1} and z_k from the reference surface, respectively. Consequently, the thickness (t_k) of the k^{th} ply is evaluated by subtracting the location of the surface (z_{k-1}) from the location of the bottom one (z_k), where, $k = 1, 2, 3, \dots, n$.

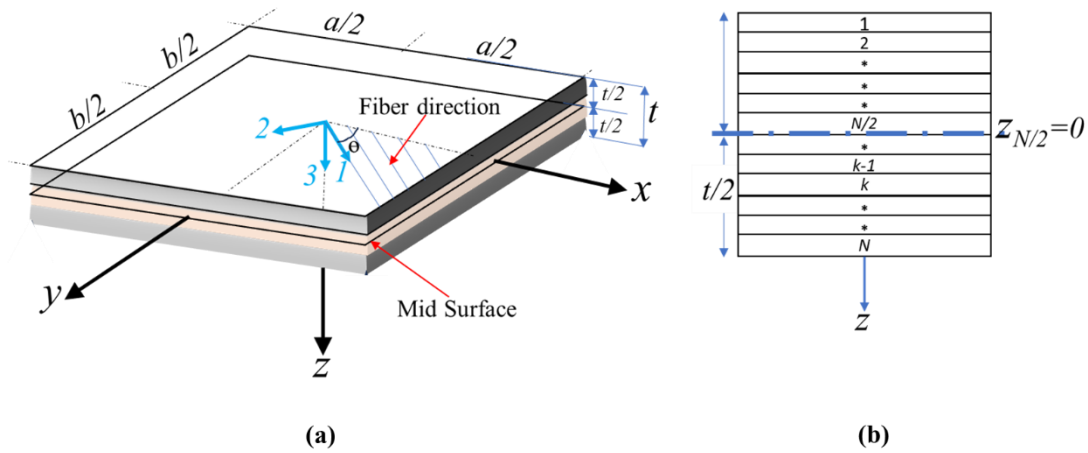


Figure 5 : (a) Laminated composite plate; (b) Coordinate locations of plies in the laminated plate

Let x, y and z be the coordinate directions (global coordinate system) of the laminated plate which are geometrically natural to the solution of the problem. However, the principal directions of each individual ply do not coincide with the coordinate directions of the plate. As shown in Figure 5(b), the local coordinate system of the plies is defined as $1, 2$ and 3 where the 1 and 2 axes are coincided with the longitudinal and transverse directions of the fiber, respectively. The ply's thickness is along the direction of 3 -axis which coincides with z -axis. In the 1 - 2 plane, the fiber's orientation in the longitudinal direction (1 -axis) of a ply makes an angle θ with the positive x -axis.

The stress-strain relations of an element for k^{th} ply in its local coordinates system is expressed by Hooke's law as:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}_k = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}_k \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k \quad (3.14)$$

Inverting Eq. (3.14) to get the stress components in terms of the strain components of the element in its local coordinate system.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = \frac{1}{1 - \nu_{12}\nu_{21}} \begin{bmatrix} E_1 & \nu_{12}E_2 & 0 \\ \nu_{21}E_1 & E_2 & 0 \\ 0 & 0 & G_{12}(1 - \nu_{12}\nu_{21}) \end{bmatrix}_k \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}_k \quad (3.15)$$

where the matrix in the right-side of Eq. (3.15) is the reduced stiffness matrix $[\bar{Q}]_k$ for each ply i.e.:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}_k = \frac{1}{1 - \nu_{12}\nu_{21}} \begin{bmatrix} E_1 & \nu_{12}E_2 & 0 \\ \nu_{21}E_1 & E_2 & 0 \\ 0 & 0 & G_{12}(1 - \nu_{12}\nu_{21}) \end{bmatrix}_k \quad (3.16)$$

In order to get the stresses and strains of the laminated plates in the directions of the global coordinates, the stresses and strains of the ply's element is transformed from its local coordinate system to the global coordinate system based on its orientation's angle θ_k . As discussed previously, Figure 5, the fiber direction of the ply makes an angle θ_k from the positive x -axis to 1 -axis and the stress and strain components in 12 -system are expressed in the xy -system as following:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{1}{2}\gamma_{12} \end{bmatrix}_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}_k \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{bmatrix}_k \quad (3.17a)$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}_k \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k \quad (3.17b)$$

where the matrix in the right-side of Eq. (3.17) is called the transformation matrix $[T]_k$.

The engineering strains can be introduced using Reuter Matrix $[R]$ along with Eq. (3.17a) to get:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}_k = [R][T]_k[R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k \quad (3.18a)$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = [T]_k \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k \quad (3.18b)$$

where:

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } [R]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

where the value of $[R][T]_k[R]^{-1}$ is equal to $[T]_k^{-T}$ (the transpose of the inverse of the transformation matrix $[T]_k$).

Substituting Eq. (3.18) in Eq. (3.15) yields:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = [\bar{Q}]_k \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}_k \quad (3.19)$$

where the $[\bar{Q}]_k$ is the transformed stiffness matrix of each ply i.e.:

$$[\bar{Q}]_k = [T]_k^{-1} [Q]_k [T]_k^{-T} \quad (3.20)$$

Where:

$$[T]_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$$

$$[T]_k^{-1} = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & -2\sin(\theta)\cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & -\sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$$

$$[T]_k^{-T} = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -\sin(\theta)\cos(\theta) \\ -2\sin(\theta)\cos(\theta) & 2\sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}$$

Substituting the above matrices and Eq. (16) in Eq. (3.20) gives:

$$[\bar{Q}]_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \quad (3.21)$$

Where the components of Eq. (3.21) are evaluated for the k^{th} ply as:

$$\bar{Q}_{11} = Q_{11} \cos^4(\theta) + Q_{22} \sin^4(\theta) + 2(Q_{12} + 2Q_{66}) \sin^2(\theta) \cos^2(\theta),$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2(\theta) \cos^2(\theta) + Q_{12}(\cos^4(\theta) + \sin^4(\theta)),$$

$$\bar{Q}_{22} = Q_{11} \sin^4(\theta) + Q_{22} \cos^4(\theta) + 2(Q_{12} + 2Q_{66}) \sin^2(\theta) \cos^2(\theta),$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin(\theta) \cos^3(\theta) - (Q_{22} - Q_{12} - 2Q_{66}) \sin^3(\theta) \cos(\theta),$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3(\theta) \cos(\theta) - (Q_{22} - Q_{12} - 2Q_{66}) \sin(\theta) \cos^3(\theta),$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2(\theta) \cos^2(\theta) + Q_{66}(\cos^4(\theta) + \sin^4(\theta))$$

After evaluating the transformed reduced stiffness matrix $[\bar{Q}]_k$ for each ply (layer), Eq. (3.13) is used to find the rigidity components (A_{ij}, B_{ij}, D_{ij}) of the laminated composite plate as an ESL plate with reference to the middle surface plane. The integrals in Eq. (3.13) must be performed over each ply across its own thickness and then summation over the number of plies to be carried out. Using the layers layout of the laminated plate as shown in Figure 5b, Eq. (3.13) can be written as:

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} [\bar{Q}_{ij}]_k (1, z, z^2) dz \quad (3.22)$$

Knowing that

$$\int_{z_{k-1}}^{z_k} dz = (z_k - z_{k-1}), \int_{z_{k-1}}^{z_k} z dz = \frac{1}{2}(z_k^2 - z_{k-1}^2) \text{ and } \int_{z_{k-1}}^{z_k} z^2 dz = \frac{1}{3}(z_k^3 - z_{k-1}^3)$$

results in re-writing Eq. (3.22) as:

$$A_{ij} = \sum_{k=1}^n [\bar{Q}_{ij}]_k (z_k - z_{k-1}) \quad (3.23a)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^n [\bar{Q}_{ij}]_k (z_k^2 - z_{k-1}^2) \quad (3.23b)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^n [\bar{Q}_{ij}]_k (z_k^3 - z_{k-1}^3) \quad (3.23c)$$

Now the extensional [A], coupling [B], and bending [D] stiffness matrices are evaluated and can be used in Eq. (3.12). The components of these stiffness matrices are affected by the stacking sequence and orientation of the individual layers.

For symmetric laminates (laminates that layered in the same sequence above and below the mid-plane reference), the mid-surface plane is also the neutral plane of the plate. The similarities in the properties and geometry above and below the mid-surface also implies that the elements of the coupling matrix [B] become zeros. The same cannot be said for case of unsymmetrical arrangement of the laminated plate where the [B] matrix will have some nonzero elements.

3.3 Finite Element Modeling (FEM) in ABAQUS

3.3.1 Introduction

Throughout this study, the FEM analysis is performed for the sake of comparison and verification of the proposed method's validity. Therefore, a brief description on FEM modeling of thin anisotropic plates is presented in this section.

Abaqus has the feature to model all types of plates; isotropic, orthotropic, and anisotropic plates. The laminated composite plates can be modeled in ABAQUS environment by either the composite laminated section or general shell stiffness section. The composite laminated section method allows for inserting the properties of the lamina and constructing the composite plate based on the stacking sequence and orientation. The general shell stiffness section models the plate by defining the resultant rigidity matrices [A], [B] and [D]. The output using the composite section method can be shown in terms of displacements, forces, moments, strains and stresses while in case of the general shell stiffness section model only displacements, forces and moments are provided. ABAQUS has different element types developed based on different theories describing the plate bending behavior. In the foregoing investigations, it is important to use element type that suitable for the plate behavior described in the above sections. Thin shell element STRI3 is available in ABAQUS /Standard which is a triangular small membrane strain thin shell element. As in the ABAQUS documentation [206], the element STRI3 is based on the classical (Kirchhoff) shell theory and involves no transverse shear strain energy calculation. Consequently, this element is used in all the numerical examples of this study, an extensively fine unstructured mesh with a maximum element size of $0.005 a$, resulting in a total of 80000 elements which accurately predicts the behavior of thin plates based on CLPT.

3.3.2 Modeling of laminated composite plates in ABAQUS

Analysis of anisotropic plates in the ABAQUS environment requires, as mentioned above, consideration of the convenient model including meshing, section type and element type. Hence, the following example highlights the important steps of using ABAQUS to perform a proper FEM model of anisotropic thin plates in the sense of CLPT.

A composite plate can be modeled as a laminated (layered) shell made of one or more materials where each layer can be defined as a single isotropic, orthotropic, or anisotropic material. In the composite laminated shell section, you specify the thickness, the material, and the orientation (either as a reference to an orientation definition or as an angle measured relative to the overall orientation definition) for each layer of the shell (Figure 6). The order of the laminated shell (stacking sequence) layers with respect to the positive direction of the shell normal is defined by the order in which the layers are specified.

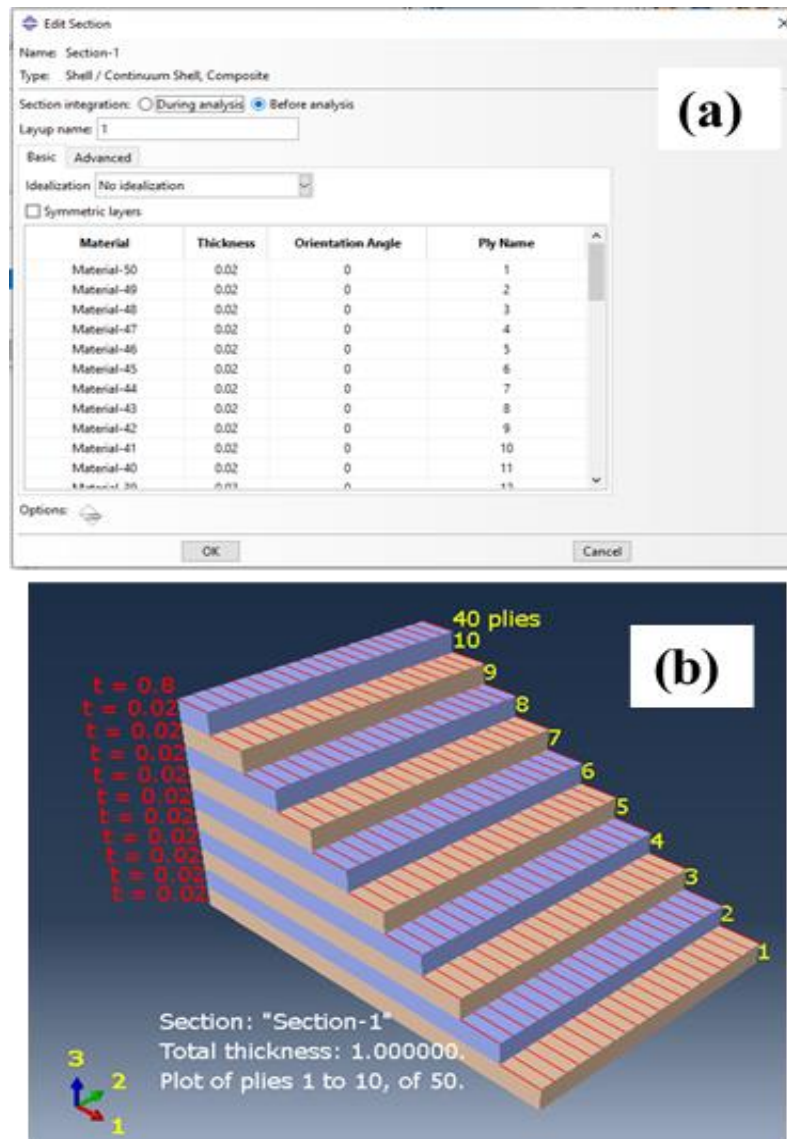


Figure 6 : Composite laminated shell section; (a) Defining material, thickness, and orientation of each layer; (b) Plies stacking sequence of the composite laminated plate.

3.4 Ritz Method Formulation Based on CLPT

Ritz method is one of the energy methods used extensively for solving problems in engineering mechanics. It is basically based on minimizing the potential energy in a deformed body. Ritz method can be a capable candidate to obtain semi-analytical solutions with remarkable improvement in the accuracy due to its simple and modest requirement of satisfying only the essential boundary conditions. To adopt Ritz method in solving the considered problem, the generalized stiffness matrix should be employed in the derivations of the membrane strain energy, bending strain energy and external work due to the applied lateral load $q(x, y)$. The potential energy of a plate subjected to a lateral load q is defined as the difference between the strain energy stored in the plate, during bending and stretching of the middle surface, and the work done by the applied load. In the following sections, the strain and potential energies corresponding to small deflection of the plate are derived first. Then, the equations resulting from the minimization of the potential energy are cast in matrix form in order to automate the method and make it easily coded for an arbitrary number of Ritz trial functions.

3.4.1 Strain and potential energy

Consider the previous laterally loaded thin rectangular laminated composite plate in Figure 5 represented by an equivalent single layer (ESL). Implementing the discussed assumptions of plate bending behavior along with constitutive equations in Section 3.2, the strain energy stored in the plate during bending with consideration of the small deflection strains has mainly three components: the membrane strain energy, the bending strain energy and the

coupling strain energy. For laminate composite thin plate, the strain energy stored in the system is evaluated using the following expression

$$U = \frac{1}{2} \iint_A (N_x \varepsilon_{x0} + N_y \varepsilon_{y0} + N_{xy} \gamma_{xy0} + M_x \kappa_x + M_y \kappa_y + M_{xy} \kappa_{xy}) dA \quad (3.24)$$

Recalling Eq. (3.5), Eq. (3.6), and Eq. (3.11) and substituting them in Eq. (3.24) and rearrangement, results in expressing the strain energy as:

$$\begin{aligned} U = \frac{1}{2} \iint_A & \left(A_{11} \left(\frac{\partial u_0}{\partial x} \right)^2 + A_{22} \left(\frac{\partial v_0}{\partial y} \right)^2 + 2A_{12} \left(\frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial y} \right) + A_{66} \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right)^2 \right. \\ & + 2A_{16} \left(\frac{\partial u_0}{\partial x} \frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial x} \right) + 2A_{26} \left(\frac{\partial v_0}{\partial y} \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial v_0}{\partial x} \right) \\ & + D_{11} \left(\frac{\partial w}{\partial x} \right)^2 + D_{22} \left(\frac{\partial w}{\partial y} \right)^2 + 2D_{12} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial w}{\partial y} \right)^2 \\ & + 4D_{66} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right)^2 + 4D_{16} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \\ & + 4D_{26} \left(\frac{\partial w}{\partial y} \right)^2 \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{11} \left(\frac{\partial u_0}{\partial x} + \left(\frac{\partial w}{\partial x} \right)^2 \right) \\ & + B_{22} \left(\frac{\partial v_0}{\partial y} \left(\frac{\partial w}{\partial y} \right)^2 \right) + B_{12} \left(\frac{\partial v_0}{\partial y} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial u_0}{\partial x} \left(\frac{\partial w}{\partial y} \right)^2 \right) \\ & + 2B_{66} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \right) + B_{16} \left(\left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \right) \\ & + 2 \left(\frac{\partial u_0}{\partial x} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right) + B_{26} \left(\left(\frac{\partial w}{\partial y} \right)^2 \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \right) \\ & \left. + 2 \left(\frac{\partial v_0}{\partial y} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right) \right) dA \end{aligned} \quad (3.25)$$

Eq. (3.25) can be splitted into three terms: membrane strain energy (U^m), bending strain energy (U^b), and coupling strain energy (U^c).

$$U = U^m + U^b + U^c \quad (3.26)$$

where, U^m is the membrane strain energy due to membrane stresses (membrane stress to midplane strains) and is defined as:

$$\begin{aligned} U^m = \frac{1}{2} \iint_A & \left(A_{11} \left(\frac{\partial u_0}{\partial x} \right)^2 + A_{22} \left(\frac{\partial v_0}{\partial y} \right)^2 + 2A_{12} \left(\frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial y} \right) \right. \\ & + A_{66} \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right)^2 + 2A_{16} \left(\frac{\partial u_0}{\partial x} \frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial x} \right) \\ & \left. + 2A_{26} \left(\frac{\partial v_0}{\partial y} \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial v_0}{\partial x} \right) \right) dA \end{aligned} \quad (3.27)$$

U^b is the bending strain energy due to effect of bending moments to curvatures which expressed as follows:

$$\begin{aligned} U^b = \frac{1}{2} \iint_A & \left(D_{11} \left(\frac{\partial w}{\partial x} \right)^2 + D_{22} \left(\frac{\partial w}{\partial y} \right)^2 + 2D_{12} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial w}{\partial y} \right)^2 \right. \\ & + 4D_{66} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right)^2 + 4D_{16} \left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \\ & \left. + 4D_{26} \left(\frac{\partial w}{\partial y} \right)^2 \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right) dA \end{aligned} \quad (3.28)$$

U^c is the strain energy due to coupling effect of bending moments to midplane strains and the membrane forces to curvatures which expressed as follows:

$$\begin{aligned}
U^c = \frac{1}{2} \iint_A & \left(B_{11} \left(\frac{\partial u_0}{\partial x} + \left(\frac{\partial w}{\partial x} \right)^2 \right) + B_{22} \left(\frac{\partial v_0}{\partial y} + \left(\frac{\partial w}{\partial y} \right)^2 \right) \right. \\
& + B_{12} \left(\frac{\partial v_0}{\partial y} + \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial u_0}{\partial x} + \left(\frac{\partial w}{\partial y} \right)^2 \right) + 2B_{66} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \right) \\
& + B_{16} \left(\left(\frac{\partial w}{\partial x} \right)^2 \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \right) + 2 \left(\frac{\partial u_0}{\partial x} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right) \\
& \left. + B_{26} \left(\left(\frac{\partial w}{\partial y} \right)^2 \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \right) + 2 \left(\frac{\partial v_0}{\partial y} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right) \right) dA
\end{aligned} \tag{3.29}$$

The potential energy due to the work done by the applied load is given by:

$$V = \iint_A q w dA \tag{3.30}$$

The total potential energy Π is defined as the difference between the strain energy stored in the system and the external work by the applied load:

$$\Pi = U - V \tag{3.31}$$

3.4.2 Selection of the approximation functions F_i^u, F_j^v and F_k^w

Ritz method requires the approximation functions F_i^u, F_j^v and F_k^w for u , v and w , respectively, to satisfy only the essential boundary conditions, i.e. deflection and slope boundary conditions, which makes it advantageous for plates with mixed boundary conditions. To explain how to derive the approximating functions, let us assume that the equation of the line representing the k^{th} edge of the plate is given by $f_k(x, y) = 0$. Then, the following polynomial product satisfies the essential boundary conditions of the plate:

$$F_{ij}^u(x, y) = \left(\prod_{k=1}^4 f_k(x, y)^{n_k} \right) x^i y^j \quad (3.32a)$$

$$F_{ij}^v(x, y) = \left(\prod_{k=1}^4 f_k(x, y)^{n_k} \right) x^i y^j \quad (3.32b)$$

$$F_{ij}^w(x, y) = \left(\prod_{k=1}^4 f_k(x, y)^{n_k} \right) x^i y^j \quad (3.32c)$$

where Π is the product operator.

If the plate geometry, boundary conditions, stacking sequence and loads are symmetric about x and y coordinates, the i and j powers of x and y in the above approximating polynomials can be replaced by $2i$ and $2j$, respectively, in Eq. (3.32c) and by $2i+1$ and $2j+1$ in Eqs. (3.32a & b).

The approximation functions for the in-plane displacements u and v are expressed in Eqs. (3.32a) and (3.32b), respectively. The edges of the plate either allow the movement in the in-plane direction (movable) or prevent the in-plane movement (immovable). Accordingly, the value of n_k can be taken as 0 and 1 for movable and immovable edges, respectively. In the case of lateral movement w , the value of n_k in Eq. (3.32c) is defined as 0, 1 and 2 for free, simply supported and clamped edges, respectively. The displacement components u , v and w of the plate can be approximated by a linear combination of the above product polynomials, i.e.

$$u(x, y) = \sum_{i=0}^{n_u} \sum_{j=0}^{l_u} C_{ij}^u F_{ij}^u \quad (3.33a)$$

$$v(x, y) = \sum_{i=0}^{n_v} \sum_{j=0}^{l_v} C_{ij}^v F_{ij}^v \quad (3.33b)$$

$$w(x, y) = \sum_{i=0}^{n_w} \sum_{j=0}^{l_w} C_{ij}^w F_{ij}^w \quad (3.33c)$$

Setting $F_{00}^u = F_1^u$, $F_{01}^u = F_2^u \dots$, $F_{n_u l_u}^u = F_{N_u}^u$ and, similarly, setting $C_{00}^u = C_1^u$, $C_{01}^u = C_2^u \dots$, $C_{n_u l_u}^u = C_{N_u}^u$, where $N_u = (n_u + 1)(l_u + 1)$; and applying the same for Eqs. (3.33b) and (3.33c) yields:

$$u(x, y) = \sum_{i=1}^{N_u} C_i^u F_i^u \quad (3.34a)$$

$$v(x, y) = \sum_{j=1}^{N_v} C_j^v F_j^v \quad (3.34b)$$

$$w(x, y) = \sum_{k=1}^{N_w} C_k^w F_k^w \quad (3.34c)$$

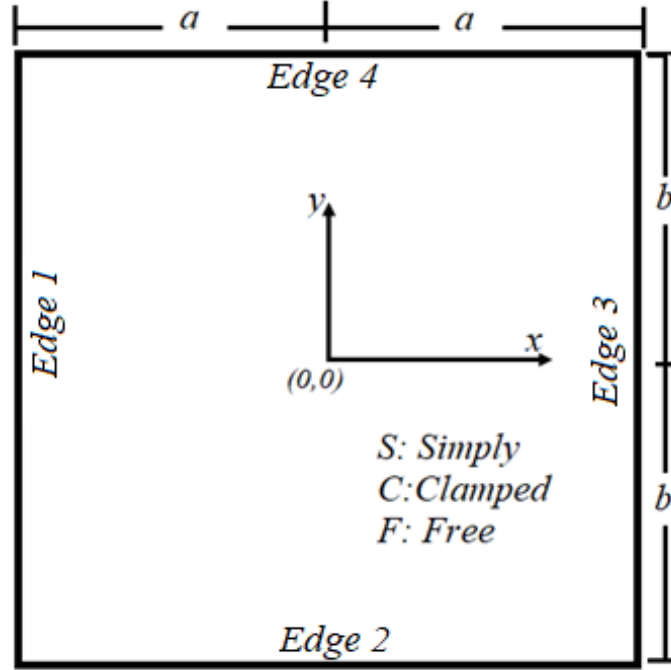


Figure 7 : Rectangular plate with sides of $(2a \times 2b)$ and centered at $(0, 0)$

For clarity, consider a rectangular plate, Figure 7 , with mixed boundary conditions detailed as following:

- Edge 1 is fully simply supported, $k = 1$ and $f_1(x, y) = (a + x)$
- Edge 2 is fully clamped, $k = 2$, $f_2(x, y) = (b + y)$
- Edge 3 is simply supported and allowed to move in the x -direction, $k = 3$,
 $f_3(x, y) = (a - x)$
- Edge 4 is free to move in all directions, $k = 4$, $f_4(x, y) = (b - y)$

Using the above details in Eq. (3.33) yields:

$$\begin{aligned}
 F_{ij}^u(x, y) &= ((a + x)^{n_1} (b + y)^{n_2} (a - x)^{n_3} (b - y)^{n_4}) x^i y^j \\
 &= ((a + x)^1 (b + y)^1 (a - x)^0 (b - y)^0) x^i y^j = ((a + x)(b + y)) x^i y^j
 \end{aligned}$$

$$\begin{aligned}
F_{ij}^v(x, y) &= ((a+x)^{n_1}(b+y)^{n_2}(a-x)^{n_3}(b-y)^{n_4})x^i y^j \\
&= ((a+x)^1(b+y)^1(a-x)^1(b-y)^0)x^i y^j = ((a^2-x^2)(b+y))x^i y^j
\end{aligned}$$

$$\begin{aligned}
F_{ij}^w(x, y) &= ((a+x)^{n_1}(b+y)^{n_2}(a-x)^{n_3}(b-y)^{n_4})x^i y^j \\
&= ((a+x)^1(b+y)^2(a-x)^1(b-y)^0)x^i y^j = ((a^2-x^2)(b+y)^2)x^i y^j
\end{aligned}$$

Substituting the values of F_{ij}^u , F_{ij}^v and F_{ij}^k in Eq. (3.34) to get the approximation functions for u , v and w , respectively i.e.:

$$u(x, y) = \sum_{i=0}^{n_u} \sum_{j=0}^{l_u} C_{ij}^u (a+x)(b+y)x^i y^j \quad (3.35a)$$

$$v(x, y) = \sum_{i=0}^{n_v} \sum_{j=0}^{l_v} C_{ij}^v (a^2-x^2)(b+y)x^i y^j \quad (3.35b)$$

$$w(x, y) = \sum_{i=0}^{n_w} \sum_{j=0}^{l_w} C_{ij}^w (a^2-x^2)(b+y)^2 x^i y^j \quad (3.35c)$$

The whole procedures are automated in a developed Mathematica code shown in Appendix A and the code is adopted to solve the numerical examples of the following section.

3.4.3 Ritz method in matrix form

The derivation of the equations is performed by the use of indicial notation. the first step in formulating Ritz method is to approximate the solutions for u , v , and w by the following:

$$\begin{aligned}
u(x, y) &= C_i^u F_i^u(x, y), \quad i = 1, N_u \\
v(x, y) &= C_j^v F_j^v(x, y), \quad j = 1, N_v
\end{aligned} \tag{3.36}$$

$$w(x, y) = C_k^w F_k^w(x, y), \quad k = 1, N_w$$

in which the trial functions F_i^u, F_j^v and F_k^w in Eq. (3.36) are selected to satisfy only the geometric boundary conditions; N_u, N_v , and N_w are the number of terms for each of the three trial functions, respectively; and the coefficients C_i^u, C_j^v and C_k^w are to be determined based on the principle of minimum potential energy where the strain energy is derived with respect to each coefficients C_r^u, C_s^v and C_t^w as follows:

$$\begin{aligned}
\Pi_{,C_r^u} &= U_{,C_r^u}^m + U_{,C_r^u}^b + U_{,C_r^u}^c - W_{,C_r^u} = 0, \quad r = 1, N_u \\
\Pi_{,C_s^v} &= U_{,C_s^v}^m + U_{,C_s^v}^b + U_{,C_s^v}^c - W_{,C_s^v} = 0, \quad s = 1, N_v \\
\Pi_{,C_t^w} &= U_{,C_t^w}^m + U_{,C_t^w}^b + U_{,C_t^w}^c - W_{,C_t^w} = 0, \quad t = 1, N_w
\end{aligned} \tag{3.37}$$

Inspecting the dependency of the energy quantities, U^m, U^b and W , on the constants C_r^u, C_s^v and C_t^w reveals that

$$U_{,C_r^u}^b = U_{,C_s^v}^b = W_{,C_r^u} = W_{,C_s^v} = 0 \tag{3.38}$$

Furthermore, all the remaining non-zero energy differential terms produce linear functions of the constants, yielding:

$$\begin{aligned}
U_{,C_r^u}^m + U_{,C_r^u}^c &= 0, \quad r = 1, N_u \\
U_{,C_s^v}^m + U_{,C_s^v}^c &= 0, \quad s = 1, N_v
\end{aligned} \tag{3.39}$$

$$U_{,C_t^w}^b + U_{,C_t^w}^c = W_{,C_t^w}, \quad t = 1, N_w$$

Carrying out the lengthy task of differentiations in the system of equations shown in Eqs.

(3.39) results in the following equations written in a matrix form:

$$\begin{bmatrix} K_{11}^{ri} & K_{12}^{rj} & K_{13}^{rk} \\ K_{21}^{si} & K_{22}^{sj} & K_{23}^{sk} \\ K_{31}^{ti} & K_{32}^{tj} & K_{33}^{tk} \end{bmatrix} \begin{bmatrix} C_i^u \\ C_j^v \\ C_k^w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \iint_A q \psi_t dA \end{bmatrix}, \quad (3.40)$$

where:

$$K_{11}^{ri} = \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_r^u}{\partial x} A_{11} + \frac{\partial F_r^u}{\partial y} A_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_r^u}{\partial x} A_{16} + \frac{\partial F_r^u}{\partial y} A_{66} \right) \right] dA \quad (3.41a)$$

$$K_{12}^{rj} = \iint_A \left[\frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_r^u}{\partial x} A_{12} + \frac{\partial F_r^u}{\partial y} A_{26} \right) + \frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_r^u}{\partial x} A_{16} + \frac{\partial F_r^u}{\partial y} A_{66} \right) \right] dA \quad (3.41b)$$

$$\begin{aligned} K_{13}^{rk} = & - \iint_A \left(B_{11} \frac{\partial F_r^u}{\partial x} \left(\frac{\partial F_k^w}{\partial x} \right)^2 + 2 B_{12} \frac{\partial F_r^u}{\partial x} \left(\frac{\partial F_k^w}{\partial y} \right)^2 \right. \\ & + 2 B_{66} \frac{\partial F_r^u}{\partial y} \left(\frac{\partial F_k^w}{\partial x} \frac{\partial F_k^w}{\partial y} \right) \\ & + B_{16} \left[\frac{\partial F_r^u}{\partial y} \left(\frac{\partial F_k^w}{\partial x} \right)^2 + 2 \frac{\partial F_r^u}{\partial x} \left(\frac{\partial F_k^w}{\partial x} \frac{\partial F_k^w}{\partial y} \right) \right] \\ & \left. + B_{26} \frac{\partial F_r^u}{\partial y} \left(\frac{\partial F_k^w}{\partial y} \right)^2 \right) dA \end{aligned} \quad (3.41c)$$

$$K_{21}^{si} = \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_s^v}{\partial y} A_{12} + \frac{\partial F_s^v}{\partial x} A_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_s^v}{\partial y} A_{26} + \frac{\partial F_s^v}{\partial x} A_{66} \right) \right] dA \quad (3.41d)$$

$$K_{22}^{sj} = \iint_A \left[\frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_s^v}{\partial y} A_{26} + \frac{\partial F_s^v}{\partial x} A_{66} \right) + \frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_s^v}{\partial x} A_{26} + \frac{\partial F_s^v}{\partial y} A_{22} \right) \right] dA \quad (3.41e)$$

$$\begin{aligned}
K_{23}^{sk} = & - \iint_A \left(B_{22} \frac{\partial F_s^v}{\partial y} \left(\frac{\partial F_k^w}{\partial y} \right)^2 + 2B_{12} \frac{\partial F_s^v}{\partial y} \left(\frac{\partial F_k^w}{\partial x} \right)^2 \right. \\
& + 2B_{66} \frac{\partial F_s^v}{\partial x} \left(\frac{\partial F_k^w}{\partial x} \frac{\partial F_k^w}{\partial y} \right) + B_{16} \frac{\partial F_s^v}{\partial x} \left(\frac{\partial F_k^w}{\partial x} \right)^2 \\
& \left. + B_{26} \left[\frac{\partial F_s^v}{\partial x} \left(\frac{\partial F_k^w}{\partial y} \right)^2 + 2 \frac{\partial F_s^v}{\partial y} \left(\frac{\partial F_k^w}{\partial x} \frac{\partial F_k^w}{\partial y} \right) \right] \right) dA \quad (3.41f)
\end{aligned}$$

$$\begin{aligned}
K_{31}^{tk} = & - \iint_A \left(B_{11} \frac{\partial F_k^u}{\partial x} \left(\frac{\partial F_t^w}{\partial x} \right)^2 + 2B_{12} \frac{\partial F_k^u}{\partial x} \left(\frac{\partial F_t^w}{\partial y} \right)^2 \right. \\
& + 2B_{66} \frac{\partial F_k^u}{\partial y} \left(\frac{\partial F_t^w}{\partial x} \frac{\partial F_t^w}{\partial y} \right) \\
& + B_{16} \left[\frac{\partial F_k^u}{\partial y} \left(\frac{\partial F_t^w}{\partial x} \right)^2 + 2 \frac{\partial F_k^u}{\partial x} \left(\frac{\partial F_t^w}{\partial x} \frac{\partial F_t^w}{\partial y} \right) \right] \\
& \left. + B_{26} \frac{\partial F_k^u}{\partial y} \left(\frac{\partial F_t^w}{\partial y} \right)^2 \right) dA \quad (3.41g)
\end{aligned}$$

$$\begin{aligned}
K_{32}^{tk} = & - \iint_A \left(B_{22} \frac{\partial F_k^v}{\partial y} \left(\frac{\partial F_t^w}{\partial y} \right)^2 + 2B_{12} \frac{\partial F_k^v}{\partial y} \left(\frac{\partial F_t^w}{\partial x} \right)^2 \right. \\
& + 2B_{66} \frac{\partial F_k^v}{\partial x} \left(\frac{\partial F_t^w}{\partial x} \frac{\partial F_t^w}{\partial y} \right) + B_{16} \frac{\partial F_k^v}{\partial x} \left(\frac{\partial F_t^w}{\partial x} \right)^2 \\
& \left. + B_{26} \left[\frac{\partial F_k^v}{\partial x} \left(\frac{\partial F_t^w}{\partial y} \right)^2 + 2 \frac{\partial F_k^v}{\partial y} \left(\frac{\partial F_t^w}{\partial x} \frac{\partial F_t^w}{\partial y} \right) \right] \right) dA \quad (3.41h)
\end{aligned}$$

$$\begin{aligned}
K_{33}^{tk} = & \iint_A \left(D_{11} \left(\frac{\partial F_k^w}{\partial x} \right)^2 \left(\frac{\partial F_t^w}{\partial x} \right)^2 + D_{22} \left(\frac{\partial F_k^w}{\partial y} \right)^2 \left(\frac{\partial F_t^w}{\partial y} \right)^2 \right. \\
& + D_{12} \left[\left(\frac{\partial F_k^w}{\partial y} \right)^2 \left(\frac{\partial F_t^w}{\partial x} \right)^2 + \left(\frac{\partial F_k^w}{\partial x} \right)^2 \left(\frac{\partial F_t^w}{\partial y} \right)^2 \right] \\
& + 4D_{66} \frac{\partial F_k^w}{\partial x} \frac{\partial F_k^w}{\partial y} \frac{\partial F_t^w}{\partial x} \frac{\partial F_t^w}{\partial y} \\
& + 2D_{16} \left[\left(\frac{\partial F_k^w}{\partial x} \right)^2 \frac{\partial F_t^w}{\partial x} \frac{\partial F_t^w}{\partial y} + \left(\frac{\partial F_t^w}{\partial x} \right)^2 \frac{\partial F_k^w}{\partial x} \frac{\partial F_k^w}{\partial y} \right] \\
& \left. + 2D_{26} \left[\left(\frac{\partial F_k^w}{\partial y} \right)^2 \frac{\partial F_t^w}{\partial x} \frac{\partial F_t^w}{\partial y} + \left(\frac{\partial F_t^w}{\partial y} \right)^2 \frac{\partial F_k^w}{\partial x} \frac{\partial F_k^w}{\partial y} \right] \right) dA \quad (3.41i)
\end{aligned}$$

The indices in Eq. (3.40) and Eqs. (3.41) are according to $(r, i = 1, N_u)$, $(s, j = 1, N_v)$ and $(k, t = 1, N_w)$. The matrix equation (Eq. (3.41)) consists of a total of $(N_u + N_v +$

N_w) equations, all of which have an equal number of unknown coefficients. The integrals given in Eqs. (3.41a to i) can be computed, and the solution for the unknown constants C^u , C^v , and C^w in Eq. (3.40) can be obtained using any suitable software such as Wolfram Mathematica. Once the unknown coefficients are determined, the plate problem is fully solved, and the displacements u , v , and w can be obtained in functional forms (Eq. (3.36)). Consequently, the secondary variables of the plate, such as stresses, can be obtained by directly differentiating these displacements.

3.5 Numerical Examples and Discussions

Different sets of numerical examples are presented in this section to verify the capability of the proposed method and its accuracy in predicting the bending behavior of uniformly loaded rectangular laminated composite plate supported at the corners. The results of the proposed method are compared with those obtained by FEM software ABAQUS due to the lack of data availability in the literature for the case of corner supported laminate composite plate.

In the presented examples, the proposed method is applied to analyze a rectangular laminated composite plate as depicted in Figure 8. The plate is restrained at the corner points to prevent any movement in the x , y , and z directions, resulting in zero displacements (u , v , and w) at those locations.

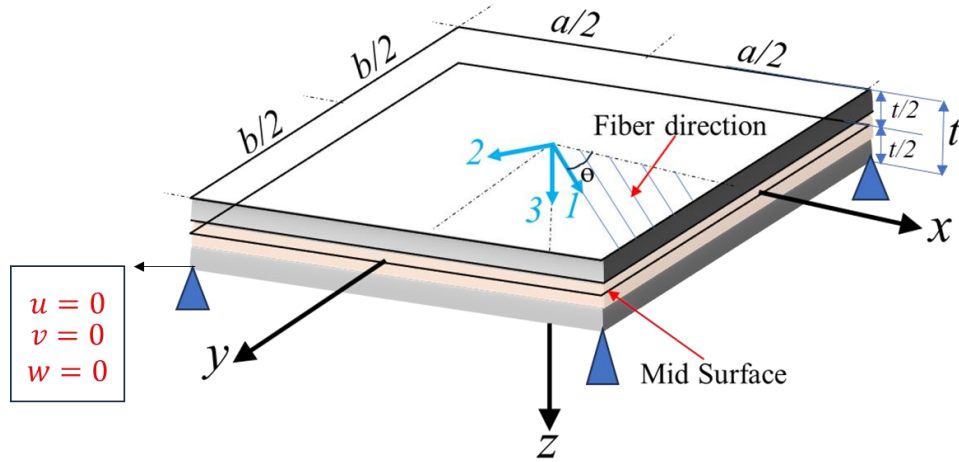


Figure 8 : Plate restrained at the corners.

In accordance with the devised formulation of the Ritz Method, the approximation functions only need to fulfill the geometric boundary conditions. Consequently, the polynomial trial functions expressed in Eq. (3.41) are employed in the preceding analysis. These trial functions satisfy the zero-displacement at the corners and accurately represent the deformation shape of uniformly loaded composite plates. Four categories of stacking patterns are considered in this study as shown in Figure 9.

Figure 9: symmetric/antisymmetric cross-ply and symmetric/antisymmetric angle-ply laminates.

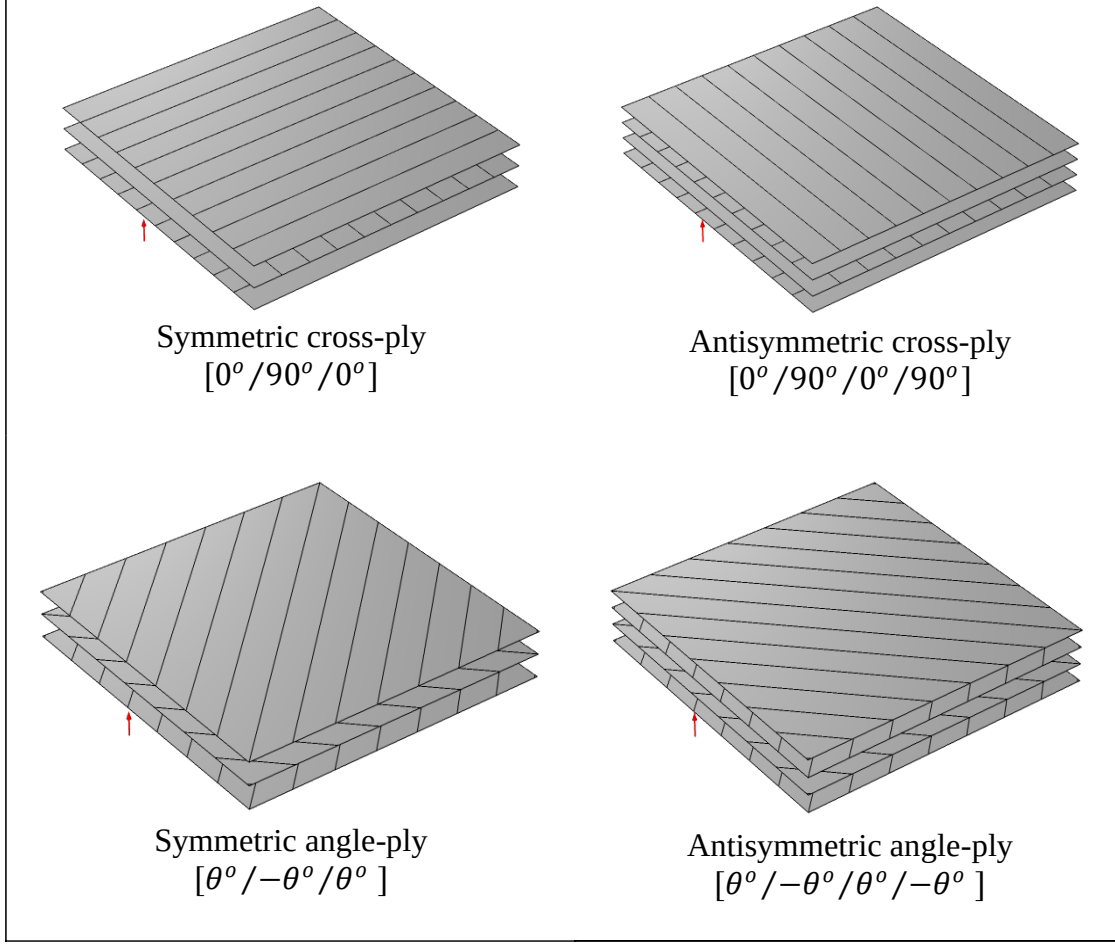


Figure 9 : Stacking configurations of cross-ply and angle-ply composite plate

The trial functions for symmetrically stacked laminates are given by:

$$\begin{aligned}
 u &= 0 \\
 v &= 0 \\
 w &= C_1^w \left(\left(\frac{a}{2} \right)^2 - x^2 \right) + C_2^w \left(\left(\frac{b}{2} \right)^2 - y^2 \right) + \sum_{j=0}^i \sum_{i=0}^n C_{ij}^w \left(\left(\frac{a}{2} \right)^2 \left(\frac{b}{2} \right)^2 - x^2 y^2 \right) x^i y^{j-i} \quad (3.42a)
 \end{aligned}$$

The trial functions for antisymmetric cross-ply and angle-ply laminates are given by:

$$\begin{aligned}
u &= \sum_{i=0}^n \sum_{j=0}^m C_{ij}^u \left(\left(\frac{a}{2} \right)^{2i} \left(\frac{b}{2} \right)^{2j} - x^{2i} y^{2j} \right) x^\alpha y^\beta \\
v &= \sum_{i=0}^n \sum_{j=0}^m C_{ij}^v \left(\left(\frac{a}{2} \right)^{2i} \left(\frac{b}{2} \right)^{2j} - x^{2i} y^{2j} \right) x^\beta y^\alpha
\end{aligned} \tag{3.42b}$$

$$w = C_1^w \left(\left(\frac{a}{2} \right)^2 - x^2 \right) + C_2^w \left(\left(\frac{b}{2} \right)^2 - y^2 \right) + \sum_{j=0}^i \sum_{i=0}^n C_{ij}^w \left(\left(\frac{a}{2} \right)^2 \left(\frac{b}{2} \right)^2 - x^2 y^2 \right) x^i y^{j-i}$$

where

$$\alpha = \begin{cases} 0 & \text{for antisymmetric angle - ply laminates} \\ 1 & \text{for antisymmetric cross - ply laminates} \end{cases}$$

$$\beta = \begin{cases} 0 & \text{for antisymmetric cross - ply laminates} \\ 1 & \text{for antisymmetric angle - ply laminates} \end{cases}$$

In each of the following examples, numerical results for both the proposed method and FEM solutions are obtained for plates having side dimensions of $a = 1 \text{ m}$ and $b = 1 \text{ m}$ & 0.5 m with side to thickness ratio of $a/h = 100$. The elastic material properties are Poison's ratio $\nu = 0.25$, the elastic modulus $E_1 = 25 \text{ GPa}$, $E_2 = 1 \text{ GPa}$, and the modulus of rigidity $G_{12} = 0.2 \text{ GPa}$.

For each numerical example, a computer code is generated in Mathematica software [207] to compute all integrals and solve the system of equations (Eqs. (3.41)) while the FEM analysis is performed using ABAQUS [206]. In the FEM simulation, the Laminated Composite Shell Section available in ABAQUS is employed to analyze the plate with the shell element STRI3, which is based on the classical (Kirchhoff) shell. An extensively fine unstructured mesh with a maximum element size of $0.005a$, resulting in a total of 80000 elements.

The values reported in the forgoing tables and figures are for the normalized deflection and stresses evaluated, respectively, at $(0, 0)$ and $(0, 0, -h/2)$ using the following expressions:

$$\left. \begin{aligned} \bar{w} &= w \times \frac{E_2 h^3}{a^4 q_0} \\ \bar{\sigma}_x &= \sigma_x \times \frac{h^2}{a^2 q_0} \\ \bar{\sigma}_y &= \sigma_y \times \frac{h^2}{a^2 q_0} \\ \bar{\tau}_{xy} &= \tau_{xy} \times \frac{h^2}{a^2 q_0} \end{aligned} \right\} \quad (3.43)$$

3.5.1 Convergence Study

Convergence analysis is performed for a corner supported square laminated composite plates in order to determine the number of terms required to reach accurate values for the central deflection and stresses in reference to FEM results. The results of the convergence study are shown in Table 5 for symmetric and anti-symmetric cross-ply and angle-ply laminated composite plate. The results reveal that the deflection converges rapidly reaching a good agreement with FEM solutions at $m = n = 12$. Based on the convergence results, it is sufficient to truncate the polynomial series at $m = n = 12$ for the foregoing analysis.

Table 5: Convergence study for different stacking sequence at the center of the plate

Stacking Angles $\{\theta_1, \theta_2\}$	Anti-Symmetric $[\{\theta_1, \theta_2\}_8]$					Symmetric $[\{\theta_1, \theta_2\}_7, \theta_1]$			
	m , n	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
{0/90}	9	3.335	0.0171	0.312	0	3.398	0.011	0.733	0
	10	3.341	0.0191	0.347	0	3.405	0.011	0.733	0
	11	3.341	0.0191	0.347	0	3.405	0.010	0.720	0
	12	3.342	0.0179	0.326	0	3.406	0.010	0.721	0
	FEM	3.346	0.0183	0.334	0	3.410	0.010	0.727	0
{-15/15}	9	15.190	0.512	0.732	0.141	15.194	0.561	0.736	0.161
	10	15.231	0.555	0.734	0.152	15.236	0.605	0.738	0.172
	11	15.231	0.555	0.734	0.152	15.236	0.605	0.738	0.172
	12	15.235	0.560	0.739	0.153	15.241	0.609	0.742	0.173
	FEM	15.240	0.560	0.744	0.152	15.245	0.6011	0.742	0.169
{-30/30}	9	8.460	1.072	0.811	0.640	8.454	1.130	0.828	0.672
	10	8.472	1.093	0.820	0.651	8.468	1.141	0.836	0.683

	11	8.472	1.092	0.820	0.651	8.468	1.141	0.836	0.683
	12	8.473	1.098	0.821	0.654	8.470	1.147	0.839	0.686
	FEM	8.474	1.098	0.821	0.654	8.470	1.143	0.838	0.684
{-45/45}	9	3.423	0.945	0.945	0.851	3.411	0.971	0.971	0.879
	10	3.425	0.952	0.952	0.857	3.414	0.975	0.975	0.883
	11	3.425	0.952	0.952	0.857	3.414	0.975	0.975	0.883
	12	3.426	0.954	0.954	0.859	3.415	0.979	0.979	0.886
	FEM	3.427	0.949	0.949	0.852	3.415	0.979	0.979	0.886

3.5.2 Cross-Ply Laminates

A cross-ply $[0^\circ/90^\circ]$ laminated composite square plate supported at the corners is investigated using the proposed method by employing the trial polynomial functions in Eqs. (3.42) and Eq. (3.43), respectively, for symmetric and antisymmetric stacking. The analysis is performed for plates composed of different number of layers; 4, 8, 12, and 16 layers in the case of anti-symmetric stacking and 3, 7, 11, and 15 layers in the case of symmetric stacking.

Table 6 shows the results for normalized deflection at the center of the plates $(0, 0)$ and stresses at point coordinates of $(0, 0, -h/2)$. The results clearly illustrate full agreement between the proposed method results and those obtained by the FEM analysis for all the investigated cases within a percentage difference less than 3.48%.

Table 6 : Normalized deflection and stresses for cross ply corner supported square plate [0/90]

Method	Anti-Symmetric					Symmetric				
	Number of Layers	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	Number of Layers	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
Present	4	3.768	0.027	0.360	0	3	10.01 1	0.08 7	9.630	0
FEM		3.794	0.028	0.373	0		10.03 1	0.08 7	9.596	0
Difference %		0.68 %	0.35 %	3.48 %	0		0.19 %	0	0.35 %	0
Present	8	3.418	0.019	0.323	0	7	3.777	0.01 0	1.421	0
FEM		3.421	0.020	0.332	0		3.777	0.01 0	1.426	0
Difference %		0.8%	0.5%	2.7%	0		0	0	0.35 %	0
Present	12	3.361	0.018 4	0.323	0	11	3.487	0.00 9	0.911	0
FEM		3.366	0.018 8	0.332	0		3.491	0.00 9	0.911	0
Difference %		0.15 %	2.1%	2.7%	0		0.11 %	0	0	0
Present	16	3.342	0.017 9	0.326	0	15	3.406	0.01 0	0.721	0
FEM		3.346	0.018 3	0.334	0		3.410	0.01 0	0.727	0
Difference %		0.12 %	2.1%	2.4%	0		0.11 %	0	0.82 %	0

3.5.3 Angle-Ply Laminates

In the case of angle-ply stacking sequence, square plates with various lamina's fiber orientation (θ) and different number of layers are analyzed using both the proposed method and the FEM method. The plies are stacked in symmetric $[-\theta/\theta \dots / \theta / \dots \theta / -\theta]$ and anti-symmetric $[-\theta/\theta \dots -\theta/\theta]$ layups, where θ takes the values of 15°, 30°, and 45°. For the stacking of each orientation angle, the plates are composed of 4, 8, 12, and 16 layers

in case of anti-symmetric sequence and 3, 7, 11, and 15 layers in the case of symmetric sequence.

The results in Table 7 show that, regardless of the number of layers, the deflection is found to be maximum at the mid-edges for all orientation angle except for $\theta = 45^\circ$ where the deflection is maximum at the center of the plate. For the anti-symmetric case with an orientation angle of 15° and 30° , the solution by the present method shows excellent agreement with FEM results at all locations, with differences ranging from 0% to 2.6%. For the 45° case, the solution of the present method consistently aligns well with FEM results for all orientations and locations with a maximum difference of 1.44%. Overall, the results suggest that the present method is effective in predicting the maximum deflection for both symmetric and anti-symmetric angle-ply corner-supported plates across various orientations.

The results of the central normalized stresses are shown in Table 8. Based on the percentage difference, all the results demonstrate excellent agreement between the current solution and the FEM solution for both deflection and stresses. The negligible differences observed indicate the efficiency of the proposed solutions in predicting stresses, particularly the shear stress, where the maximum difference is only 2.3%.

Table 7: Dimensionless maximum deflection for symmetric angle ply corner supported plate

		Anti-Symmetric					Symmetric				
Method	θ	Location	$n = 4$	$n = 8$	$n = 12$	$n = 16$	Location	$n = 3$	$n = 7$	$n = 11$	$n = 15$
Present	15	(0,0)	15.428	15.272	15.245	15.235	(0,0)	15.910	15.305	15.255	15.241
FEM			15.439	15.276	15.250	15.240		15.942	15.314	15.261	15.245
Difference%			0.07%	0.02%	0.03%	0.03%		0.2%	0.05%	0.06%	0.02%
Present		(a/2,0)	16.232	16.090	16.067	16.059	(a/2,0)	17.234	16.202	16.109	16.081
FEM			16.235	16.094	16.070	16.063		17.140	16.208	16.111	16.084
Difference%			0.01%	0.02%	0.01%	0.02%		0.54%	0.04%	0.1%	0.01%
Present		(0,b/2)	1.519	1.316	1.282	1.272	(0,b/2)	2.401	1.385	1.306	1.283
FEM			1.524	1.319	1.274	1.274		2.418	1.389	1.309	1.286
Difference%			0.33%	0.23%	0.62%	0.15%		0.7%	0.28%	0.01%	0.01%
Present	30	(0,0)	8.877	8.547	8.492	8.473	(0,0)	9.264	8.548	8.488	8.470
FEM			8.887	8.549	8.493	8.474		9.498	8.569	8.496	8.470
Difference%			0.11%	0.02%	0.01%	0.01%		2.46%	0.24%	0.09%	0%
Present		(a/2,0)	10.531	10.297	10.258	10.245	(a/2,0)	11.760	10.374	10.283	10.235
FEM			10.536	10.298	10.258	10.244		12.076	10.403	10.294	10.224
Difference%			0.05%	0.01%	0%	0.01%		2.6%	0.28%	0.1%	0.1%
Present		(0,b/2)	0.208	0.446	0.483	0.495	(0,b/2)	0.478	0.432	0.481	0.496
FEM			0.208	0.443	0.494	0.495		0.483	0.433	0.482	0.496
Difference%			0%	0.01%	2.2%	0%		1.03%	0.01%	0.01%	0%
Present	45	(0,0)	3.868	3.504	3.443	3.426	(0,0)	3.967	3.471	3.428	3.415
FEM			3.898	3.511	3.448	3.427		4.025	3.473	3.428	3.415
Difference%			0.77%	0.2%	0.14%	0.01%		1.44%	0.05%	0%	0%
Present		(a/2,0)	2.474	2.249	2.208	2.197	(a/2,0)	3.010	2.262	2.213	2.199
FEM			2.490	2.253	2.210	2.197		3.024	2.264	2.214	2.200
Difference%			0.64%	0.18%	0.09%	0%		0.46%	0.05%	0.01%	0.01%
Present		(0,b/2)	2.474	2.249	2.208	2.197	(0,b/2)	3.010	2.262	2.213	2.199
FEM			2.495	2.253	2.210	2.197		3.024	2.264	2.214	2.200
Difference%			0.64%	0.18%	0.09%	0%		0.46%	0.05%	0.01%	0.01%

Table 8 : Dimensionless stress components for symmetric and anti-symmetric angle ply corner supported plate

		Anti-Symmetric $n=16$			Symmetric $n=15$		
Method	θ	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
Present	15	0.560	0.739	0.153	0.609	0.742	0.173
FEM		0.560	0.744	0.152	0.601	0.742	0.169
Difference%		0%	0.67%	0.65%	1.33%	0%	2.3%
Present	30	1.098	0.821	0.654	1.147	0.839	0.686
FEM		1.098	0.821	0.654	1.143	0.838	0.684
Difference%		0%	0%	0	0.35%	0.11%	0.29%
Present	45	0.954	0.954	0.859	0.979	0.979	0.886
FEM		0.949	0.949	0.852	0.979	0.979	0.886
Difference%		0.53%	0.53%	0.82%	0%	0%	0%

3.5.4 Impact of varied stacking sequence on deflection

Figure 10 and Figure 11 show the influence of the fiber orientation and number of layers on plate's deflection, respectively, for square and rectangular ($b/a = 1/2$) plate. Notably, it is evident that there exists a particular angle that minimizes the plate's deflection, signifying an optimal configuration for enhanced structural performance. The graphical representation provides a clear visual understanding of how different angles impact the deflection of the plates. In the case of square plates, the minimum deflection is observed at a fiber orientation angle of 45° for both symmetric and anti-symmetric angle-ply stacking, while for rectangular plates, the orientation angle of 25° yields the minimum deflection. Also, a consistent convergence towards the orientation angle that minimizes deflection is observed as the number of layers increases.

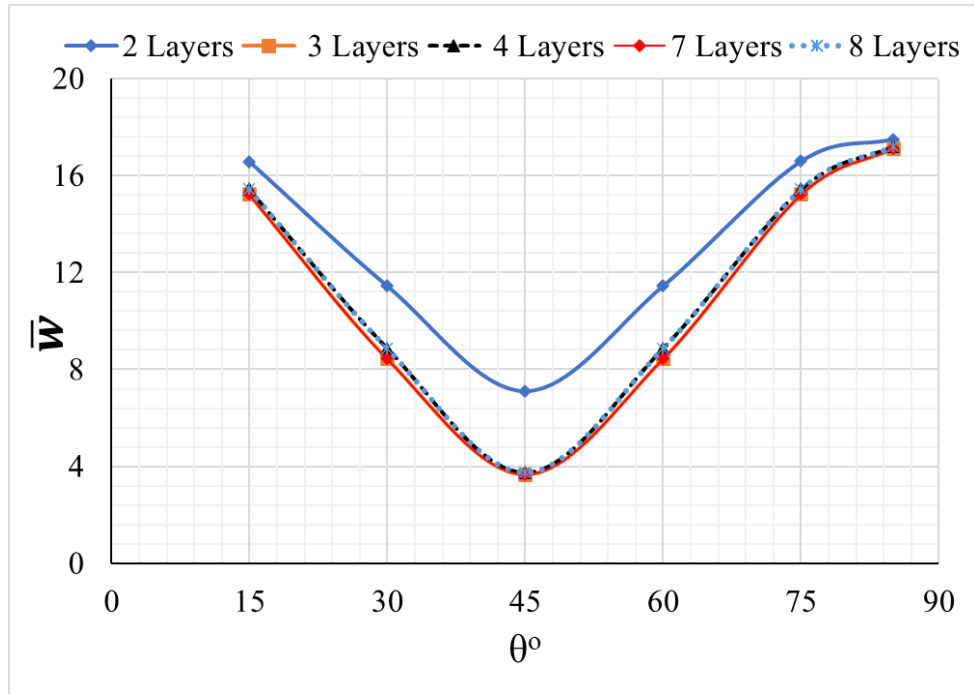


Figure 10 : Deflection vs θ for square plate

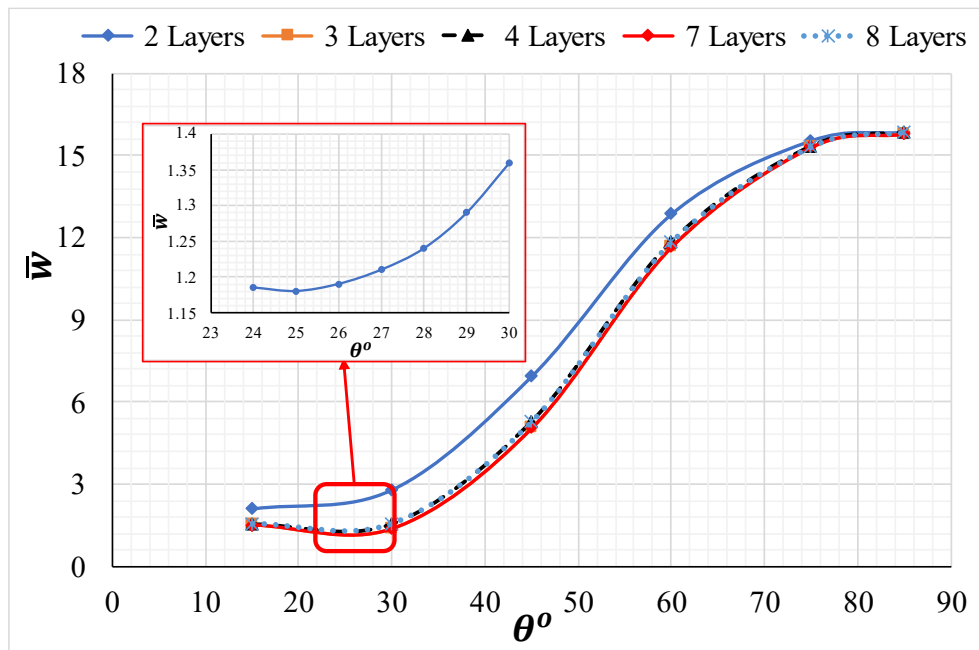


Figure 11 : Deflection vs θ for rectangle plate

CHAPTER 4

FSDT FOR THE ANALYSIS OF THICK LAMINATED COMPOSITE PLATES

4.1 General

The flexural and shear behavior of thick laminated plates is described by five coupled partial differential equations, derived based on the “Rissener-Mindlin plate theory”. Thick laminated plates have a relatively large thickness compared to their overall length. A plate is classified as a thick plate when its thickness to smaller span length ratio is more than $(1/20)$ [34,35].

First-order shear deformation theory (FSDT) is the simplest theory that takes into account for transverse normal and transverse shear deformations [61]. To illustrate the behavior of a thick plate, consider a rectangular plate (Figure 12) subjected to a transverse lateral uniform load $q(x, y)$, in which the xy -plane coincides with the mid-surface of the plate. The plate has a total thickness t where the top and bottom surfaces are located at $z = -t/2$ and $t/2$, respectively, length a , width b and N -plies with angles $(\theta_1, \theta_2, \dots, \theta_N)$, as shown in Figure 12. The plies of the plate have the same thickness t and assumed to be homogeneous and perfectly bonded. When a lateral load is exerted on the plate, deformation takes place leading to three in-plane displacement components u , v , and w in

the x , y , and z directions, respectively, where w is constant along the plate's thickness, while u and v varies linearly over each layer's thickness. Also, two out of plane rotations: ϕ_y that represents the rotation of a transverse normal about x -axis and ϕ_x that represents the rotation of a transverse normal about y -axis are both considered constant across the plate thickness. The mid-surface has the displacement components: u_0 , v_0 , and w .

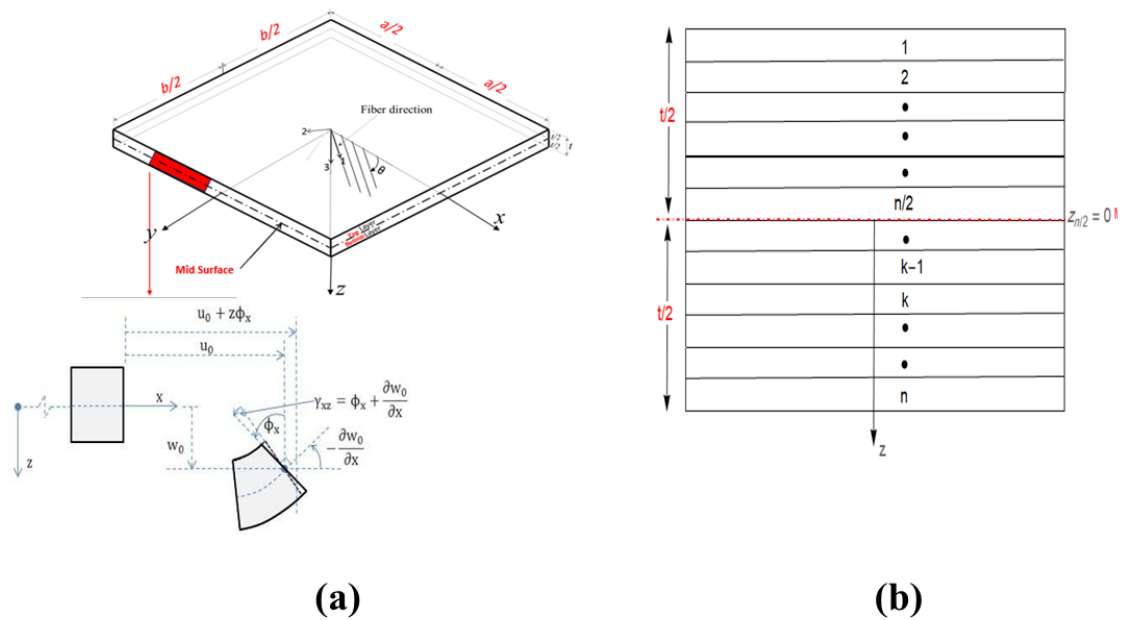


Figure 12: (a) Thick Laminated composite plate; (b) Coordinate locations of plies in the laminated plate

In an anisotropic continuum, the development of a thick plate theory which includes (out-of-plane and in-plane loads) effects requires relating the stresses on the deformed body to the original configuration. The strain may be related to the displacement components by the following relations:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (4.1a)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad (4.1b)$$

$$\varepsilon_z = 0 \quad (4.1c)$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (4.1d)$$

$$\gamma_{xz} = \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad (4.1e)$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad (4.1f)$$

Also, the generalized Hook's law can be expressed in cartesian coordinates by the following matrix form:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ Q_{12} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\ Q_{13} & Q_{23} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ Q_{14} & Q_{24} & Q_{34} & Q_{44} & Q_{45} & Q_{46} \\ Q_{15} & Q_{25} & Q_{35} & Q_{45} & Q_{55} & Q_{56} \\ Q_{16} & Q_{26} & Q_{36} & Q_{46} & Q_{56} & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \quad (4.2)$$

Where, Q_{ij} is the stiffness matrix which includes 21 independent elastic constants for the general case.

Considering the geometry of the deflected plate, Figure **12a**, the integration of Equations (4.1c), (4.1e) and (4.1f) is taken over the thickness, resulting in expressions defining the displacement components at any point as following:

$$w = w(x, y) \quad (4.3a)$$

$$u = u_0(x, y) + z \phi_x \quad (4.3b)$$

$$v = v_0(x, y) + z \phi_y \quad (4.3c)$$

4.2 FSDT for the Behavior of Thick Laminated Composite Plates

In investigating the thick laminated composite plate's behavior, Figure 12, (FSDT) stands out as a simple yet robust method. It effectively incorporates transverse normal and shear deformations, making it a practical choice for a comprehensive analysis of thick plates. The fundamental assumptions of FSDT's may be stated as follows:

- The transverse normal in Figure **12a** doesn't remain perpendicular to the mid-surface after bending.
- The first-order shear deformation theory also assumes transverse inextensibility, which produces shear strains $(\gamma_{xz}, \gamma_{yz})$.
- The transverse shear strain is assumed to remain constant through the thickness
- The stress normal to the midplane, σ_z , is small compared with the other stress components and may be neglected.
- The first-order shear deformation theory requires shear correction factor K which depend on the lamination, geometric parameters, loading and boundary conditions.

4.2.1 Strain Displacement Relations

Equation (4.3a) indicates that the lateral deflection w does not change over the plate thickness. In contrast to the lateral displacements, the in-plane displacements (u and v), Equations (4.3b and c), are changing linearly over the thickness. Substituting Equations (4.3a-c) in Equations (4.1a), (4.1b) and (4.1d), the strains can be expressed at any point of the plate as following:

$$\varepsilon_x = \frac{\partial u}{\partial x} + z \frac{\phi_x}{\partial x} \quad (4.4a)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + z \frac{\phi_y}{\partial y} \quad (4.4b)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \quad (4.4c)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z \left(\frac{\phi_x}{\partial y} + \frac{\phi_y}{\partial x} \right) \quad (4.4d)$$

$$\gamma_{xz} = \phi_x + \frac{\partial w}{\partial x} \quad (4.4e)$$

$$\gamma_{yz} = \phi_y + \frac{\partial w}{\partial y} \quad (4.4f)$$

4.2.2 Constitutive Equations

Considering the assumptions of FSDT, the generalized Hook's law (Equation 4.2) is reduced to be expressed for thick anisotropic plates as following:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{14} & \bar{Q}_{15} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{24} & \bar{Q}_{25} & \bar{Q}_{26} \\ \bar{Q}_{14} & \bar{Q}_{24} & \bar{Q}_{44} & \bar{Q}_{45} & \bar{Q}_{46} \\ \bar{Q}_{15} & \bar{Q}_{25} & \bar{Q}_{45} & \bar{Q}_{55} & \bar{Q}_{56} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{46} & \bar{Q}_{56} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \quad (4.5)$$

The transformed reduced stiffness matrix $[\bar{Q}]$ in Equation (4.5) relates the stresses to the corresponding strains considering the coupling responses [208]. The components of the matrix $[\bar{Q}]$ $\bar{Q}_{11}$, $\bar{Q}_{22}$, $\bar{Q}_{12}$, $\bar{Q}_{66}$, $\bar{Q}_{16}$ and $\bar{Q}_{26}$ are defined earlier in chapter 3. While $\bar{Q}_{44}$ relates the shear stress τ_{yz} to the shear strain γ_{yz} , $\bar{Q}_{45}$ & $\bar{Q}_{45}$ represents the coupling shear modulus, relating the shear stress τ_{yz} to the shear strain γ_{xz} . It indicates the interaction between shear stresses and strains in different planes. Finally, $\bar{Q}_{55}$ relates the shear stress τ_{xz} to the shear strain γ_{xz} .

After the combination of Eqs. (4.4) and (4.5), the stress components for the laminate in global coordinates are expressed in Eq. (4.6).

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix} \quad (4.6a)$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \quad (4.6b)$$

From Eq. (4.6), it is obvious that for each lamina the stresses will vary linearly through the thickness. Also, the stresses changes as the transformed reduced stiffness matrix $[\bar{Q}]$ changes (depends on the material and angle orientation of the lamina θ_k).

The only unknowns for finding the lamina strains and stresses are the midplane strains and plate rotations. However, because the stresses are not continuous through the laminate, it is more convenient to deal with resultant forces and moments. This can be done by integrating the stresses in each lamina through the thickness of the laminate. Thus, the resultant normal forces and moments are defined as follows:

The resultant normal forces, moments and shear forces are defined as follows:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \quad (4.7a)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z dz \quad (4.7b)$$

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} dz \quad (4.7c)$$

The stresses can be divided into three components to simplify the derivation of the potential energy as follows:

(a) Membrane stresses (σ_0)

$$\begin{Bmatrix} \sigma_x^{(0)} \\ \sigma_y^{(0)} \\ \tau_{xy}^{(0)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} =$$

$$\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \quad (4.8a)$$

(b) Bending stresses (σ_1)

$$\begin{Bmatrix} \sigma_x^{(1)} \\ \sigma_y^{(1)} \\ \tau_{xy}^{(1)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi x}{\partial x} \\ \frac{\partial \phi y}{\partial y} \\ \frac{\partial \phi x}{\partial y} + \frac{\partial \phi y}{\partial x} \end{Bmatrix} \quad (4.8b)$$

(c) Transverse shear stresses (τ_z)

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (4.8c)$$

Using Eqs. (4.7) in conjunction with Eqs.(4.6) and (4.5) with considering that the transformed reduced stiffness matrix $[\bar{Q}]$ is constant as well as that midplane strains and curvatures are independent of the z-coordinate. The result equations can be written as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} dz \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z dz \begin{bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix} \quad (4.9a)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z dz \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z^2 dz \begin{bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix} \quad (4.9b)$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} dz \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \quad (4.9c)$$

As a result of integrating the expressions above, we get:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (4.10 \text{ a})$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \quad (4.10 \text{ b})$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (4.10 \text{ c})$$

They can be combined into one general expression that relates in plane force (N_x, N_y, N_{xy}), moments (M_x, M_y, M_{xy}) and shear forces (Q_x, Q_y) per unit width to the midspan strains ($\epsilon_x^{(0)}, \epsilon_y^{(0)}, \gamma_{xy}^{(0)}$), bending curvatures ($\epsilon_x^{(1)}, \epsilon_y^{(1)}, \gamma_{xy}^{(1)}$) and shear strains (γ_{yz}, γ_{xz}), resulting in the following constitutive relations for the anisotropic thin plate:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (4.11)$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad (4.12)$$

Where

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{t}{2}}^{\frac{t}{2}} Q_{ij} (1, z, z^2) dz, \quad i = 1, 2, 6; \quad j = 1, 2, 6 \quad (4.13)$$

$$(A_{ij}) = \int_{-\frac{t}{2}}^{\frac{t}{2}} Q_{ij} dz, \quad i = 4, 5; \quad j = 4, 5 \quad (4.14)$$

A_{ij}, B_{ij}, D_{ij} are defined earlier in chapter 3 while in Eq. (4.14) A_{ij} are the transverse stiffnesses that relate in-plane loads to transverse shear strains, where i and j are indices ranging from 1 to 6 and 4 to 5 for the transverse stiffness. The solution of laminated plates based on the above differential equations can be obtained exactly using Navier or Levy method, only for specially orthotropic rectangular plates ($B_{11} = B_{12} = B_{22} = A_{16} = A_{26} = B_{66} = D_{16} = D_{26} = 0$).

4.2.3 Rigidities of Thick Laminated Composite Plates (FSDT)

Consider same laminated composite plate in Figure 12 having a thickness (t) and each layer or ply have its engineering constants in its local coordinate axes making an angle θ with the positive x -axis in the global coordinate system. The reference datum plane is the mid-surface plane of the plate at $z = 0$ and the location of the top and bottom surfaces of the k^{th} layer (ply) is defined by z_{k-1} and z_k from the reference surface, respectively. Consequently, the thickness (t_k) of the k^{th} ply is evaluated by subtracting the location of the surface (z_{k-1}) from the location of the bottom one (z_k), where, $k = 1, 2, 3, \dots, n$.

Let x, y and z be the coordinate directions (global coordinate system) of the laminated plate which are geometrically natural to the solution of the problem. However, the principal

directions of each individual ply do not coincide with the coordinate directions of the plate.

As shown in **Error! Reference source not found.b**, the local coordinate system of the ply is defined as 1, 2 and 3 where the 1 and 2 axes are coinciding with the longitudinal and transverse directions of the fiber, respectively. The ply's thickness is along the direction of 3-axis which coincides with z-axis. The fiber's orientation in the 1-2 plane makes an angle θ from the positive x-axis to 1-axis.

The stress-strain relations of a ply element in its local coordinates system are expressed by Hooke's law as:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix}_k = \begin{bmatrix} \frac{1 - \nu_{12}\nu_{21}}{E_1} & \frac{1 - \nu_{12}\nu_{21}}{E_1\nu_{12}} & 0 & 0 & 0 \\ \frac{1 - \nu_{12}\nu_{21}}{E_2\nu_{12}} & \frac{1 - \nu_{12}\nu_{21}}{E_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{G_{12}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix}_k \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}_k \quad (4.15)$$

Inverting Eq. (4.15) to get the stress components in terms of the strain components of the element in its local coordinate system.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix}_k = \frac{1}{1 - \nu_{12}\nu_{21}} \begin{bmatrix} E_1 & E_1\nu_{12} & 0 & 0 & 0 \\ E_2\nu_{12} & E_2 & 0 & 0 & 0 \\ 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 \\ 0 & 0 & 0 & 0 & G_{13} \end{bmatrix}_k \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix}_k \quad (4.16)$$

where the matrix in the right-side of Eq. (4.16) is the reduced stiffness matrix $[Q]$ for each ply i.e.:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{54} & Q_{55} \end{bmatrix}_k = \frac{1}{1 - \nu_{12}\nu_{21}} \begin{bmatrix} E_1 & E_1\nu_{12} & 0 & 0 & 0 \\ E_2\nu_{12} & E_2 & 0 & 0 & 0 \\ 0 & 0 & G_{12} & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 \\ 0 & 0 & 0 & 0 & G_{13} \end{bmatrix}_k \quad (4.17)$$

In order to get the stresses and strains of the laminated plates in the directions of the global stiffness matrix, the stresses and strains of the ply's element is transformed from its local coordinate system to the global coordinate system based on its orientation's angle θ . As discussed previously, Figure 12, the fiber direction of the ply makes an angle θ from x-axis to 1-axis and the stress and strain components in 21-system are expressed the xy-system as following:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \frac{1}{2}\gamma_{12} \\ \frac{1}{2}\gamma_{23} \\ \frac{1}{2}\gamma_{13} \end{bmatrix}_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) & 0 & 0 \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) & 0 & 0 \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}_k \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} \end{bmatrix}_k \quad (4.18a)$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix}_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) & 0 & 0 \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) & 0 & 0 \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}_k \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix}_k \quad (4.18b)$$

where the matrix in the right-side of Eq. (4.18) is called the transformation matrix $[T]$.

The engineering strains can be introduced using Reuter Matrix $[R]$ along with Eq. (4.18a)

to get:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{bmatrix}_k = [R][T]_k[R]^{-1} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}_k \quad (4.19a)$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{bmatrix}_k = [T]_k \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}_k \quad (4.19b)$$

where:

$$[R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } [R]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

where the value of $[R][T]_k[R]^{-1}$ is equal to $[T]^{-T}$ (the transpose of the inverse of the transformation matrix $[T]_k$).

Substituting Eq. (4.17) in Eq. (4.14) yields:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = [\bar{Q}] \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \quad (4.20)$$

where the $[\bar{Q}]$ is the transformed stiffness matrix of each ply i.e.:

$$[\bar{Q}] = [T]^{-1}[Q][T]^{-T} \quad (4.21)$$

where:

$$[T]_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) & 0 & 0 \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) & 0 & 0 \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}_k$$

$$[T]^{-1}_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & -2\sin(\theta)\cos(\theta) & 0 & 0 \\ \sin^2(\theta) & \cos^2(\theta) & 2\sin(\theta)\cos(\theta) & 0 & 0 \\ \sin(\theta)\cos(\theta) & -\sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}_k$$

$$[T]^{-T}_k = \begin{bmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) & 0 & 0 \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) & 0 & 0 \\ -\sin(\theta)\cos(\theta) & \sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}_k$$

Substituting the above matrices in Eq.(4.21) gives:

$$[\bar{Q}]_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{14} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{24} & 0 & 0 \\ \bar{Q}_{14} & \bar{Q}_{24} & \bar{Q}_{44} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & \bar{Q}_{56} \\ 0 & 0 & 0 & \bar{Q}_{56} & \bar{Q}_{66} \end{bmatrix}_k \quad (4.22)$$

Where the components of Eq. (4.22) are evaluated for the k^{th} ply as:

$$\bar{Q}_{11} = Q_{11} \cos^4(\theta) + Q_{22} \sin^4(\theta) + 2(Q_{12} + 2 Q_{66}) \sin^2(\theta) \cos^2(\theta),$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2(\theta) \cos^2(\theta) + Q_{12}(\cos^4(\theta) + \sin^2(\theta)),$$

$$\bar{Q}_{22} = Q_{11} \sin^4(\theta) + Q_{22} \cos^4(\theta) + 2(Q_{12} + 2 Q_{66}) \sin^2(\theta) \cos^2(\theta),$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin(\theta) \cos^3(\theta) - (Q_{22} - Q_{12} - 2Q_{66}) \sin^3(\theta) \cos(\theta),$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3(\theta) \cos(\theta) - (Q_{22} - Q_{12} - 2Q_{66}) \sin(\theta) \cos^3(\theta),$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2(\theta) \cos^2(\theta) + Q_{66}(\cos^4(\theta) + \sin^4(\theta)),$$

$$\bar{Q}_{44} = Q_{44} \cos^2(\theta) + Q_{55} \sin^2(\theta),$$

$$\bar{Q}_{45} = (Q_{55} - Q_{44}) \sin(\theta) \cos(\theta),$$

$$\bar{Q}_{55} = Q_{44} \sin^2(\theta) + Q_{55} \cos^2(\theta),$$

After evaluating the transformed reduced stiffness matrix $[\bar{Q}]_k$ for each ply (layer), Eq. (4.15) is used to find the rigidity components (A_{ij}, B_{ij}, D_{ij}) of the laminated composite plate as an ESL plate with reference to the middle surface plane. The integrals in Eq.(4.15) must be performed over each ply across its own thickness and then summation over the number of plies to be carried out. Using the layers layout of the laminated plate as shown in Figure **12b**, Eq. (4.15) can be written as:

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} [\bar{Q}_{ij}]_k (1, z, z^2) dz \quad (4.23)$$

$$A_{ij} = \sum_{k=1}^n \int_{z_{k-1}}^{z_k} [\bar{Q}_{ij}]_k dz \quad (4.24)$$

Knowing that

$$\int_{z_{k-1}}^{z_k} dz = (z_k - z_{k-1}), \quad \int_{z_{k-1}}^{z_k} z dz = \frac{1}{2}(z_k^2 - z_{k-1}^2) \text{ and } \int_{z_{k-1}}^{z_k} z^2 dz = \frac{1}{3}(z_k^3 - z_{k-1}^3)$$

results in re-writing Eq. (4.13) as:

$$A_{ij} = \sum_{k=1}^n [\bar{Q}_{ij}]_k (z_k - z_{k-1}) \quad (4.25a)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^n [\bar{Q}_{ij}]_k (z_k^2 - z_{k-1}^2) \quad (4.25b)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^n [\bar{Q}_{ij}]_k (z_k^3 - z_{k-1}^3) \quad (4.25c)$$

4.3 Finite Element Modeling (FEM) in COMSOL

4.3.1 Introduction

A brief description on the FEM modeling of thick laminated composite plates is presented in this section using the commercial software COMSOL.

COMSOL Multiphysics is highly capable of modeling composite plates based on various theoretical frameworks, such as Classical Laminated Plate Theory (CLPT) and First-Order Shear Deformation Theory (FSDT). The software offers specialized technical features for the analysis of laminated composite plates. It supports the definition of anisotropic and orthotropic material properties, enabling accurate modeling of the complex behavior of laminated composites. The software's advanced meshing techniques, such as adaptive meshing and boundary layer refinement, ensure precise representation of geometrical and material discontinuities. Additionally, COMSOL allows for the implementation of custom constitutive equations and user-defined functions, facilitating the incorporation of specific composite material models and different shear deformation theories. Robust post-processing tools in COMSOL facilitate detailed visualization of stress distributions, deformation patterns, and failure modes under various loading conditions and boundary

constraints making it an invaluable tool for engineers and researchers working on laminated composite plates.

COMSOL has the feature to model all types of plates; isotropic, orthotropic, and anisotropic plates. The laminated composite plates can be modeled in COMSOL environment by the shell section for FSDT. The shell section allows for inserting the properties of the lamina and constructing the composite plate based on the stacking sequence and orientation. The output using the composite section method can be shown in terms of displacements, forces, moments, strains, and stresses. The selection of element type in the FEM analysis is very important to get the proper and fair comparison. Consequently, the mesh element (mapped) is used in all the numerical examples of this study, with an extremely fine mesh having a maximum element size of 0.001 which accurately predicts the behavior of thick plates based on FSDT.

4.3.2 Modeling of laminated composite plates in COMSOL

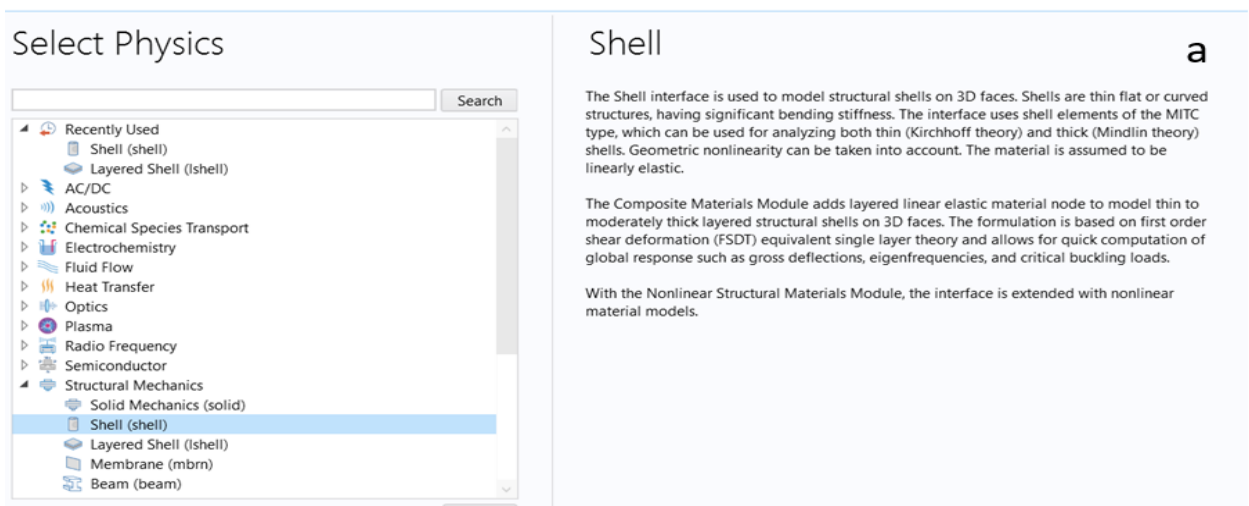
Analysis of laminated composite plates in the COMSOL environment requires, as mentioned above, consideration of the convenient model including meshing, section type and element type. Hence, the following example highlights the important steps of using COMSOL to perform a proper FEM model of anisotropic thick plates in the sense of FSDT.

A composite plate can be modeled as a laminated (layered) shell made of one or more materials where each layer can be defined as a single isotropic, orthotropic, or anisotropic material. For the inputs, COMSOL allows users to define the composite laminated shell model, you specify the thickness, the material, and the orientation (either as a reference to an orientation definition or as an angle measured relative to the overall orientation

definition) for each layer of the shell (**Error! Reference source not found.a-c**). The order of the laminated shell (stacking sequence) layers with respect to the positive direction of the shell normal is defined by the order in which the layers are specified [209].

The software supports the inclusion of shear correction factors, such as the commonly used $K=5/6$ that can be easily incorporated into the material property definitions to improve the accuracy of the simulation. It can be entered manually as in Figure 13f or automatically based in 3D elasticity theory to account for transverse shear deformation in First-Order Shear Deformation Theory (FSDT).

For the outputs, COMSOL provides robust visualization tools that allow for detailed analysis of stress distributions, deformation patterns, and failure modes under various loading conditions and boundary constraints as shown in Figure 13d. Users can generate comprehensive plots and animations to visualize the mechanical behavior of laminated composite plates.



Settings

Parameters

Label: Parameters 1

Parameters

Name	Expression	Value	Description
a	200[mm]	0.2 m	Side length
th	2*a/10	0.04 m	Laminate thickness
thl	th/3	0.013333 m	Lamina thickness
E1	25e6[psi]	1.7237E11 Pa	Lamina Young's modulus,...
E2	1e6[psi]	6.8948E9 Pa	Lamina Young's modulus,...
E3	E2	6.8948E9 Pa	Lamina Young's modulus,...
G12	0.5e6[psi]	3.4474E9 Pa	Lamina shear modulus, 1...
G23	0.2e6[psi]	1.379E9 Pa	Lamina shear modulus, 2...
G13	G12	3.4474E9 Pa	Lamina shear modulus, 1...
nu12	0.25	0.25	Lamina Poisson's ratio, 12...
nu23	nu12	0.25	Lamina Poisson's ratio, 23...
nu13	nu12	0.25	Lamina Poisson's ratio, 13...
q0	1[N/m^2]	1 N/m ²	Transverse load amplitude

Settings

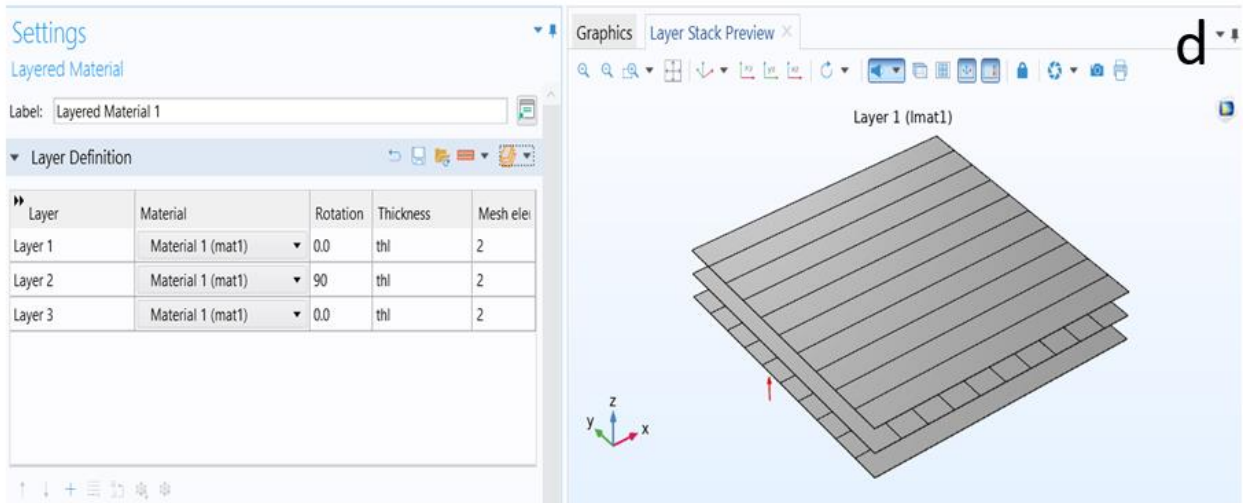
Material

Label: Material 1

Material Properties

Material Contents

Property	Variable	Value	Unit
☒ Density	rho	1	kg/m ³
☒ Young's modulus	{Evec...	{E1, E2, E3}	Pa
☒ Poisson's ratio	{nuvec...	{nu12, nu23, nu13}	1
☒ Shear modulus	{Gvec...	{G12, G23, G13}	N/m ²
☒ Elasticity matrix	{D11,...	{Evector1*(1-nuvector2...	Pa
Loss factor for orthotropic You...	{eta_E...	{0, 0, 0}	1
Loss factor for orthotropic she...	{eta_G...	{0, 0, 0}	1



Settings
Variables

Label: Variables 1

Geometric Entity Selection

Geometric entity level: Entire model

Variables

Name	Expression	Unit	Description
sdl	$q0 \cdot \sin(0.5 \cdot \pi \cdot X/a) \cdot \sin(0.5 \cdot \pi \cdot Y/a)$	N/m ²	Sinusoidal distributed tra...
SXX_esl	$shell.SXX/q0 \cdot (th/(2 \cdot a))^2$		In-plane normal stress (S...
SZZ_esl	$shell.SZZ/q0$		Transverse normal stress (...)
SYZ_esl	$shell.SYZ/q0 \cdot (th/(2 \cdot a))$		Transverse shear stress (S...
SXZ_esl	$shell.SXZ/q0 \cdot (th/(2 \cdot a))$		Transverse shear stress (S...
A	$1.105 \cdot a/2$	m	Cut point coordinate
B	$1.894 \cdot a/2$	m	Cut point coordinate
W	$shell.w/q0 \cdot (th^3/a^4) \cdot E2$		Deflection
C	0		Cut point at center coord...
SYX_esl	$shell.SYX/q0 \cdot (th/(2 \cdot a))^2$		
SXY_esl	$shell.SXY/q0 \cdot (th/(2 \cdot a))^2$		shear stress (SXY), normal...

The image shows the 'Settings' window for 'Variables'. The 'Geometric Entity Selection' section shows 'Entire model' selected. Below this, a 'Variables' table is displayed. The table has four columns: Name, Expression, Unit, and Description. It lists various variables used in the model, including stress components (sdl, SXX_esl, SZZ_esl, SYZ_esl, SXZ_esl, SYX_esl, SXY_esl), coordinates (A, B, C), and deflection (W). The expressions for these variables are given in mathematical notation. The units are specified where applicable (N/m² for stress, m for coordinates).



Figure 13 : (a) choose the model (b) Defining the parameters (c) Defining material (d) thickness, and orientation of each layer; (e) Defining the variables (f) Defining the shear correction factor.

4.4 Ritz Method Formulation Based on FSDT

4.4.1 Potential Energy Components

The potential energy is defined as the difference between the total strain energy stored in the plate and the work done by the applied load, i.e.

$$\Pi = U - W \quad (4.25)$$

where:

$$U = \frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} (\sigma \varepsilon) dz \right) dx dy \quad (4.26)$$

and the work done by the external force can be expressed by:

$$W = \iint_A q w dA \quad (4.27)$$

The strain energy in Eq.(4.26) can be expressed using Eqs.(4.8a-c) by :

$$U = \frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} (\sigma^{(0)} \epsilon^{(0)} + \sigma^{(1)} \epsilon^{(0)} + \sigma^{(0)} \epsilon^{(1)} + \sigma^{(1)} \epsilon^{(1)} + \tau_z \gamma_z) dz \right) dx dy \quad (4.28)$$

For facilitating the lengthy task of derivation of strain energy, the above expression can be splitted into the following Eqs.(4.29-4.32):

U^m is the membrane strain energy due to normal strains to normal forces effect.

$$U^m = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma^{(0)} \epsilon^{(0)}) dz \right) dx dy \quad (4.29)$$

U^b is the bending strain energy due to effect of bending moments to curvatures.

$$U^b = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma^{(1)} \epsilon^{(1)}) dz \right) dx dy \quad (4.30)$$

U^c is the strain energy due to coupling effect.

$$U^c = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\boldsymbol{\sigma}^{(1)} \boldsymbol{\epsilon}^{(0)} + \boldsymbol{\sigma}^{(0)} \boldsymbol{\epsilon}^{(1)}) dz \right) dx dy \quad (4.31)$$

U^{ts} is the transverse shear strain energy.

$$U^{ts} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\boldsymbol{\tau}_z \gamma_z) dz \right) dx dy \quad (4.32)$$

Recalling the membrane strain energy Eq.(4.29):

$$U^m = \frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} dz \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \right) dxdy \quad (4.33)$$

where, the integration of $[\bar{Q}_{ij}]_k$ over the thickness represents the extensional stiffness

matrix of the laminate whereas defined in Section 3.23. Using Eq.(4.23) in Eq.(4.33) yields:

$$U^m = \frac{1}{2} \iint_A \left(\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \right) \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} dxdy \quad (4.34)$$

respectively, results in expressing the membrane energy as:

$$\begin{aligned}
 U^m = \frac{1}{2} \iint_A & \left(\left(\frac{\partial u_0}{\partial x} \right)^2 A_{11} + 2 \frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial y} A_{12} + 2 \left(\frac{\partial u_0}{\partial x} \frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial x} \right) A_{16} \right. \\
 & + 2 \left(\frac{\partial v_0}{\partial y} \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial v_0}{\partial x} \right) A_{26} + \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right)^2 A_{66} \\
 & \left. + \left(\frac{\partial v_0}{\partial y} \right)^2 A_{22} \right) dA
 \end{aligned} \tag{4.35}$$

Similarly, recalling of Equation (4.30) and substitution of the expressions of curvatures, results in expressing the bending strain energy in terms of displacements by:

$$U^b = \frac{1}{2} \iint_A \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z^2 dz \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \right) \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} dxdy \tag{4.36}$$

Using Eq.(4.23) in Eq.(4.36) yields:

$$U^b = \frac{1}{2} \iint_A \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \right) \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} dxdy \tag{4.37}$$

respectively, results in expressing the bending energy as:

$$\begin{aligned}
U^b = \frac{1}{2} \iint_A & \left(\left(\frac{\partial \phi_x}{\partial x} \right)^2 D_{11} + 2 \frac{\partial \phi_x}{\partial x} \frac{\partial \phi_y}{\partial y} D_{12} + 2 \left(\frac{\partial \phi_x}{\partial x} \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_x}{\partial x} \frac{\partial \phi_y}{\partial x} \right) D_{16} \right. \\
& + 2 \left(\frac{\partial \phi_y}{\partial y} \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial y} \frac{\partial \phi_y}{\partial x} \right) D_{26} + \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right)^2 D_{66} \\
& \left. + \left(\frac{\partial \phi_y}{\partial y} \right)^2 D_{22} \right) dA
\end{aligned} \tag{4.38}$$

Obviously, the strain energy expressions due to the coupling in Eq. (4.31) are identical.

Thus, adopting this approach, the coupling strain energy is given by:

$$\begin{aligned}
U^c = \frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} & \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} dz dxdy + \\
\frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} & \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} dz dxdy
\end{aligned} \tag{4.39}$$

Simplifying Eq.(4.39) yields:

$$U^c = \frac{1}{2} \iint_A \left(2 \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z dz \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \right) dxdy \tag{4.40a}$$

Using Eq.(4.23) in Eqs.(4.40a) yields :

$$U^c = \iint_A \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} dz \right) dxdy \quad (4.40b)$$

Results in expressing the coupling energy as:

$$\begin{aligned} U^c = \iint_A & \left(\frac{\partial u_0}{\partial x} \frac{\partial \phi_x}{\partial x} B_{11} + \left(\frac{\partial v_0}{\partial y} \frac{\partial \phi_x}{\partial x} + \frac{\partial u_0}{\partial x} \frac{\partial \phi_y}{\partial y} \right) B_{12} \right. \\ & + \left(\frac{\partial u_0}{\partial y} \frac{\partial \phi_x}{\partial x} + \frac{\partial v_0}{\partial x} \frac{\partial \phi_x}{\partial x} + \frac{\partial u_0}{\partial x} \frac{\partial \phi_x}{\partial y} + \frac{\partial u_0}{\partial x} \frac{\partial \phi_y}{\partial x} \right) B_{16} \\ & + \left(\frac{\partial u_0}{\partial y} \frac{\partial \phi_y}{\partial y} + \frac{\partial v_0}{\partial x} \frac{\partial \phi_y}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial \phi_x}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial \phi_y}{\partial x} \right) B_{26} \\ & + \left(\frac{\partial u_0}{\partial y} \frac{\partial \phi_x}{\partial y} + \frac{\partial u_0}{\partial y} \frac{\partial \phi_y}{\partial x} + \frac{\partial v_0}{\partial x} \frac{\partial \phi_x}{\partial y} + \frac{\partial v_0}{\partial x} \frac{\partial \phi_y}{\partial x} \right) B_{66} \\ & \left. + \frac{\partial v_0}{\partial y} \frac{\partial \phi_y}{\partial y} B_{22} \right) dA \end{aligned} \quad (4.41)$$

Similarly, recalling of Equation (4.32) and substitution of the expressions of shear strain's results in expressing the shear strain energy in terms of displacements by:

$$U^{ts} = \frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} K \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{Bmatrix} dz \begin{Bmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{Bmatrix} dxdy \quad (4.42)$$

Using Eq.(4.3) in Eqs.(4.42) yields :

$$U^{ts} = \frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} K \begin{bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{Bmatrix} \begin{Bmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{Bmatrix} dx dy \quad (4.43)$$

respectively, results in expressing the transverse shear strain energy as:

$$U^{ts} = \frac{1}{2} \int_y \int_x \left[A_{44} \left(\phi_y + \frac{\partial w}{\partial y} \right)^2 + 2A_{45} \left(\phi_x + \frac{\partial w}{\partial x} \right) \left(\phi_y + \frac{\partial w}{\partial y} \right) + A_{55} \left(\phi_x + \frac{\partial w}{\partial x} \right)^2 \right] dx dy \quad (4.44)$$

4.4.2 Ritz Method in a Matrix Form

The derivation of the equations is performed by the use of indicial notation. The first step in formulating Ritz method is to approximate the solutions for u , v , w , ϕ_x and ϕ_y . The proposed Ritz method is based on the following approximation trial function:

$$\begin{aligned} u_0(x, y) &= C_i^u F_i^u(x, y), & i &= 1, N_u \\ v_0(x, y) &= C_j^v F_j^v(x, y), & j &= 1, N_v \\ w(x, y) &= C_k^w F_k^w(x, y), & k &= 1, N_w \\ \phi_x(x, y) &= C_n^{\phi_x} F_n^{\phi_x}(x, y), & n &= 1, N_{\phi_x} \\ \phi_y(x, y) &= C_m^{\phi_y} F_m^{\phi_y}(x, y), & m &= 1, N_{\phi_y} \end{aligned} \quad (4.45)$$

where $N_u, N_v, N_w, N_{\phi_x}$ and N_{ϕ_y} are the number of terms for each of the five trial functions, the trial functions $F_i^u, F_j^v, F_k^w, F_m^{\phi_x}$ and $F_n^{\phi_y}$ are selected in order to satisfy the geometric boundary conditions and the coefficients $C_i^u, C_j^v, C_k^w, C_n^{\phi_x}$ and $C_m^{\phi_y}$ need to be determined using the principle of minimum potential energy results in:

$$\begin{aligned}
\Pi_{,C_r^u} &= \frac{\partial U^m}{\partial C_r^u} + \frac{\partial U^b}{\partial C_r^u} + \frac{\partial U^c}{\partial C_r^u} + \frac{\partial U^{ts}}{\partial C_r^u} - \frac{\partial W}{\partial C_r^u} = 0, & r &= 1, N_u \\
\Pi_{,C_s^v} &= \frac{\partial U^m}{\partial C_s^v} + \frac{\partial U^b}{\partial C_s^v} + \frac{\partial U^c}{\partial C_s^v} + \frac{\partial U^{ts}}{\partial C_s^v} - \frac{\partial W}{\partial C_s^v} = 0, & s &= 1, N_v \\
\Pi_{,C_t^w} &= \frac{\partial U^m}{\partial C_t^w} + \frac{\partial U^b}{\partial C_t^w} + \frac{\partial U^c}{\partial C_t^w} + \frac{\partial U^{ts}}{\partial C_t^w} - \frac{\partial W}{\partial C_t^w} = 0, & t &= 1, N_w \\
\Pi_{,C_p^{\phi_x}} &= \frac{\partial U^m}{\partial C_p^{\phi_x}} + \frac{\partial U^b}{\partial C_p^{\phi_x}} + \frac{\partial U^c}{\partial C_p^{\phi_x}} + \frac{\partial U^{ts}}{\partial C_p^{\phi_x}} - \frac{\partial W}{\partial C_p^{\phi_x}} = 0, & p &= 1, N_{\phi_x} \\
\Pi_{,C_q^{\phi_y}} &= \frac{\partial U^m}{\partial C_q^{\phi_y}} + \frac{\partial U^b}{\partial C_q^{\phi_y}} + \frac{\partial U^c}{\partial C_q^{\phi_y}} + \frac{\partial U^{ts}}{\partial C_q^{\phi_y}} - \frac{\partial W}{\partial C_q^{\phi_y}} = 0, & q &= 1, N_{\phi_y}
\end{aligned} \tag{4.46}$$

Inspecting the dependency of the energy quantities, U^m, U^b, U^{ts}, U^c and W , on the constants $C_i^u, C_j^v, C_k^w, C_n^{\phi_x}$ and $C_m^{\phi_y}$ reveals that

$$\begin{aligned}
\frac{\partial U^m}{\partial C_t^w} &= \frac{\partial U^m}{\partial C_p^{\phi_x}} = \frac{\partial U^m}{\partial C_q^{\phi_y}} = \frac{\partial U^b}{\partial C_r^u} = \frac{\partial U^b}{\partial C_s^v} = \frac{\partial U^b}{\partial C_t^w} = \frac{\partial U^c}{\partial C_t^w} = \frac{\partial U^{ts}}{\partial C_r^u} = \frac{\partial U^{ts}}{\partial C_s^v} = \frac{\partial U^{ts}}{\partial C_t^w} = \frac{\partial W}{\partial C_r^u} \\
&= \frac{\partial W}{\partial C_s^v} = \frac{\partial W}{\partial C_p^{\phi_x}} = \frac{\partial W}{\partial C_q^{\phi_y}} = 0
\end{aligned} \tag{4.47}$$

Furthermore, all the remaining non-zero energy differential terms produce linear functions of the constants resulting in the following system of equations:

$$\frac{\partial U^m}{\partial C_r^u} + \frac{\partial U^c}{\partial C_r^u} = 0, \quad r = 1, N_u$$

$$\frac{\partial U^m}{\partial C_s^v} + \frac{\partial U^c}{\partial C_s^v} = 0, \quad s = 1, N_v$$

$$\frac{\partial U^{ts}}{\partial C_t^w} = \frac{\partial W}{\partial C_t^w} = 0, \quad t = 1, N_w$$

$$\frac{\partial U^b}{\partial C_p^{\phi_x}} + \frac{\partial U^c}{\partial C_p^{\phi_x}} + \frac{\partial U^{ts}}{\partial C_p^{\phi_x}} = 0, \quad p = 1, N_{\phi_x}$$

$$\frac{\partial U^b}{\partial C_q^{\phi_y}} + \frac{\partial U^c}{\partial C_q^{\phi_y}} + \frac{\partial U^{ts}}{\partial C_q^{\phi_y}} = 0, \quad q = 1, N_{\phi_y}$$

With the help of indicial notation, Ritz method can be cast in the following equations and carrying out the lengthy task of differentiations in the system of equations shown in Equations results in the following equations written in a matrix form:

$$\begin{bmatrix} K_{11}^{ri} & K_{12}^{rj} & K_{13}^{rk} & K_{14}^{rn} & K_{15}^{rm} \\ K_{21}^{si} & K_{22}^{sj} & K_{23}^{sk} & K_{24}^{sn} & K_{25}^{sm} \\ K_{31}^{ti} & K_{32}^{tj} & K_{33}^{tk} & K_{34}^{tn} & K_{35}^{tm} \\ K_{41}^{pi} & K_{42}^{pj} & K_{43}^{pk} & K_{44}^{pn} & K_{45}^{pm} \\ K_{51}^{qi} & K_{52}^{qj} & K_{53}^{qk} & K_{54}^{qn} & K_{55}^{qm} \end{bmatrix} \begin{bmatrix} C_i^u \\ C_j^v \\ C_k^w \\ C_n^{\phi_x} \\ C_m^{\phi_y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \iint_A q \psi_t dA \\ 0 \\ 0 \end{bmatrix} \quad (4.48)$$

where

$$K_{11}^{ri} = \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_r^u}{\partial x} A_{11} + \frac{\partial F_r^u}{\partial y} A_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_r^u}{\partial x} A_{16} + \frac{\partial F_r^u}{\partial y} A_{66} \right) \right] dA \quad (4.49 \text{ a})$$

$$K_{12}^{rj} = \iint_A \left[\frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_r^u}{\partial x} A_{12} + \frac{\partial F_r^u}{\partial y} A_{26} \right) + \frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_r^u}{\partial x} A_{16} + \frac{\partial F_r^u}{\partial y} A_{66} \right) \right] dA \quad (4.49 \text{ b})$$

$$K_{13}^{rk} = 0 \quad (4.49 \text{ c})$$

$$K_{14}^{rn} = \iint_A \left[\frac{\partial F_n^{\phi x}}{\partial x} \left(\frac{\partial F_r^u}{\partial x} B_{11} + \frac{\partial F_r^u}{\partial y} B_{16} \right) + \frac{\partial F_n^{\phi x}}{\partial y} \left(\frac{\partial F_r^u}{\partial x} B_{16} + \frac{\partial F_r^u}{\partial y} B_{66} \right) \right] dA \quad (4.49 \text{ d})$$

$$K_{15}^{rm} = \iint_A \left[\frac{\partial F_m^{\phi y}}{\partial y} \left(\frac{\partial F_r^u}{\partial x} B_{12} + \frac{\partial F_r^u}{\partial y} B_{26} \right) + \frac{\partial F_m^{\phi y}}{\partial x} \left(\frac{\partial F_r^u}{\partial x} B_{16} + \frac{\partial F_r^u}{\partial y} B_{66} \right) \right] dA \quad (4.49 \text{ e})$$

$$K_{21}^{si} = \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_s^v}{\partial y} A_{12} + \frac{\partial F_s^v}{\partial x} A_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_s^v}{\partial x} A_{26} + \frac{\partial F_s^v}{\partial y} A_{66} \right) \right] dA \quad (4.49 \text{ f})$$

$$K_{22}^{sj} = \iint_A \left[\frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_s^v}{\partial y} A_{26} + \frac{\partial F_s^v}{\partial x} A_{66} \right) + \frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_s^v}{\partial x} A_{26} + \frac{\partial F_s^v}{\partial y} A_{22} \right) \right] dA \quad (4.49 \text{ g})$$

$$K_{23}^{sk} = 0 \quad (4.49 \text{ h})$$

$$K_{24}^{sn} = \iint_A \left[\frac{\partial F_n^{\phi x}}{\partial x} \left(\frac{\partial F_s^v}{\partial y} B_{12} + \frac{\partial F_s^v}{\partial x} B_{16} \right) + \frac{\partial F_n^{\phi x}}{\partial y} \left(\frac{\partial F_s^v}{\partial y} B_{26} + \frac{\partial F_s^v}{\partial x} B_{66} \right) \right] dA \quad (4.49 \text{ i})$$

$$K_{25}^{sm} = \iint_A \left[\frac{\partial F_m^{\phi y}}{\partial y} \left(\frac{\partial F_s^v}{\partial x} B_{26} + \frac{\partial F_s^v}{\partial y} B_{22} \right) + \frac{\partial F_m^{\phi y}}{\partial x} \left(\frac{\partial F_s^v}{\partial y} B_{26} + \frac{\partial F_s^v}{\partial x} B_{66} \right) \right] dA \quad (4.49 \text{ j})$$

$$K_{31}^{ti} = 0 \quad (4.49 \text{ k})$$

$$K_{32}^{tj} = 0 \quad (4.49 \text{ l})$$

$$K_{33}^{tk}$$

$$= K \iint_A \left[\frac{\partial F_t^w}{\partial y} \left(\frac{\partial F_k^w}{\partial y} A_{44} + \frac{\partial F_k^w}{\partial x} A_{45} \right) + \frac{\partial F_t^w}{\partial x} \left(\frac{\partial F_k^w}{\partial y} A_{45} + \frac{\partial F_k^w}{\partial x} A_{55} \right) \right] dA \quad (4.49 \text{ m})$$

$$K_{34}^{tn} = K \iint_A F_n^{\phi_x} \left(\frac{\partial F_t^w}{\partial y} A_{45} + \frac{\partial F_t^w}{\partial x} A_{55} \right) dA \quad (4.49n)$$

$$K_{35}^{tm} = K \iint_A F_m^{\phi_y} \left(\frac{\partial F_t^w}{\partial y} A_{44} + \frac{\partial F_t^w}{\partial x} A_{45} \right) dA \quad (4.49o)$$

$$K_{41}^{pi}$$

$$= \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} B_{11} + \frac{\partial F_p^{\phi_x}}{\partial y} B_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} B_{16} + \frac{\partial F_p^{\phi_x}}{\partial y} B_{66} \right) \right] dA \quad (4.49p)$$

$$K_{42}^{pj}$$

$$= \iint_A \left[\frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} B_{12} + \frac{\partial F_p^{\phi_x}}{\partial y} B_{26} \right) + \frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} B_{16} + \frac{\partial F_p^{\phi_x}}{\partial y} B_{66} \right) \right] dA \quad (4.49q)$$

$$K_{43}^{pk} = K \iint_A F_p^{\phi_x} \left(\frac{\partial F_k^w}{\partial y} A_{45} + \frac{\partial F_k^w}{\partial x} A_{55} \right) dA \quad (4.49r)$$

$$K_{44}^{pn} = \iint_A \left[\frac{\partial F_n^{\phi_x}}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} D_{11} + \frac{\partial F_p^{\phi_x}}{\partial y} D_{16} \right) + \frac{\partial F_n^{\phi_x}}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} D_{16} + \frac{\partial F_p^{\phi_x}}{\partial y} D_{66} \right) + K F_p^{\phi_x} F_n^{\phi_x} A_{55} \right] dA \quad (4.49s)$$

$$K_{45}^{pm} = \iint_A \left[\frac{\partial F_m^{\phi_y}}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} D_{12} + \frac{\partial F_p^{\phi_x}}{\partial y} D_{26} \right) + \frac{\partial F_m^{\phi_y}}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} D_{16} + \frac{\partial F_p^{\phi_x}}{\partial y} D_{66} \right) + K F_p^{\phi_x} F_m^{\phi_y} A_{45} \right] dA \quad (4.49t)$$

$$K_{51}^{qi}$$

$$= \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} B_{12} + \frac{\partial F_q^{\phi_y}}{\partial x} B_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial y} B_{26} + \frac{\partial F_q^{\phi_y}}{\partial x} B_{66} \right) \right] dA \quad (4.49u)$$

$$K_{52}^{qj} = \iint_A \left[\frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} B_{26} + \frac{\partial F_q^{\phi_y}}{\partial x} B_{66} \right) + \frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial x} B_{26} + \frac{\partial F_q^{\phi_y}}{\partial y} B_{22} \right) \right] dA \quad (4.49v)$$

$$K_{53}^{qk} = K \iint_A F_q^{\phi_y} \left(\frac{\partial F_k^w}{\partial y} A_{44} + \frac{\partial F_k^w}{\partial x} A_{45} \right) dA \quad (4.49w)$$

$$K_{54}^{qn} = \iint_A \left[\frac{\partial F_n^{\phi_x}}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} D_{12} + \frac{\partial F_q^{\phi_y}}{\partial x} D_{16} \right) + \frac{\partial F_n^{\phi_x}}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial y} D_{26} + \frac{\partial F_q^{\phi_y}}{\partial x} D_{66} \right) + K F_q^{\phi_y} F_n^{\phi_x} A_{45} \right] dA \quad (4.49x)$$

$$K_{55}^{qm} = \iint_A \left[\frac{\partial F_m^{\phi_y}}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} D_{26} + \frac{\partial F_q^{\phi_y}}{\partial x} D_{66} \right) + \frac{\partial F_m^{\phi_y}}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial x} D_{26} + \frac{\partial F_q^{\phi_y}}{\partial y} D_{22} \right) + K F_q^{\phi_y} F_m^{\phi_y} A_{44} \right] dA \quad (4.49y)$$

where the indices take their values according to:

$$r, i = 1, N_u \quad s, j = 1, N_v \quad t, k = 1, N_w \quad p, n = 1, N_{\phi_x} \text{ and } q, m = 1, N_{\phi_y}$$

The derivations provided above account for the coupling effect and it can be employed to study the behavior of laterally loaded thick plates with general anisotropic properties and different boundary conditions. The derivation and computations of all integrals and the solution of the resulting linear equations are carried out using Mathematica software as discussed in detail in Appendix B.

4.5 Numerical Examples and Discussions

Different sets of numerical examples are presented in this section to verify the capability of the proposed method and its accuracy in solving deflection, stresses and moments problems of thick laminated composite plates. The developed code is employed to investigate the behavior of thick composite laminated plates through different examples of uniformly loaded square laminated plates with cross-ply and angle-ply stacking.

4.5.1 Cross-ply [0°/90°/0°]

Consider a square laminated plate centered at the origin having the dimensions of $(a \times a)$ and thickness (t) . The plate is layered in symmetrical sequence and different orientation to represent general-ply laminates with anisotropy. The investigated plate is under SSSS and CCCC boundary conditions and subjected to uniform lateral load as depicted in Figure 14.

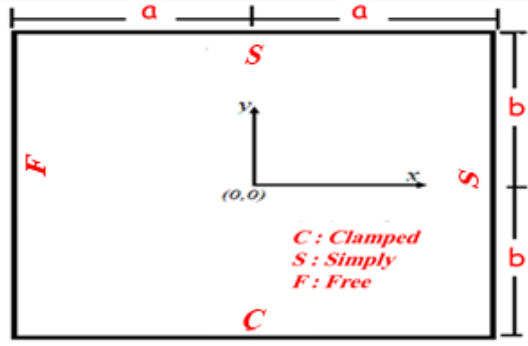


Figure 14 : Square laminated plate centered at the origin with different B.C

The plates are composed of three symtrical layers oriented at $(0^\circ/90^\circ/0^\circ)$ and having elastic material properties for each lamina as in Reddy book [208]:

$$E_1 = 1GPa, E_2 = 25GPa, G_{12} = G_{13} = 0.5 E_2 GPa, G_{23} = 0.2 E_2 GPa, \nu_{12} = 0.25, K = 5/6$$

The shear correction factor $K=5/6$ is commonly selected in FSDT to account for the transverse shear deformation. This factor adjusts the shear stiffness to more accurately represent the distribution of shear stresses across the plate thickness. The value $5/6$ was derived from analytical studies and empirical data, showing that it provides a good approximation for the transverse shear deformation of plates as mentioned by Reddy [208]. Using this correction factor helps to balance the accuracy and simplicity in the mathematical modeling of shear effects, ensuring that the theoretical predictions closely match experimental observations and finite element analysis results.

The geometrical properties of the laminated plate are:

$$a = b = 1 \text{ m, and } a/h = 4, 10, 20, 100$$

As per Ritz method requirements, the trial functions have been selected to satisfy the essential boundary conditions, and the solution has been obtained using the developed Mathematica code. The following trial polynomials were employed in the analysis:

$$\left. \begin{aligned} u &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^u (a^2 - x^2) x^i y^j \\ v &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^v (b^2 - y^2) x^i y^j \\ w &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^w (a^2 - x^2)^h (b^2 - y^2)^h x^i y^j \\ \phi_x &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^{\phi_x} (b^2 - y^2) x^i y^j \\ \phi_y &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^{\phi_y} (a^2 - x^2) x^i y^j \end{aligned} \right\} \quad (4.50)$$

The in-plane movement is prevented at all edges as shown in the above expression and the value of h is equal to 1 for simply supported edges and for clamped edges.

The results of the lateral deflection and stresses are compared with those obtained by Reddy based on Navier solution and FEM analysis using the commercial software COMSOL.

4.5.1.1 SSSS-Simply supported plate at four edges

Table 9 compares the results obtained from Reddy's (Navier) solution, Ritz solution, and Finite Element Method (FEM) solution for a cross-ply $[0^\circ/90^\circ/0^\circ]$ square thick SSSS-plate with various values of the aspect ratio (length to thickness ratio) (a/t): 4, 10, 20, and

100. The deflection and stress components are reported for the investigated four aspect ratios: 4, 10, 20, and 100.

For all aspects ratio of 4 ,10, 20 and 100, all three methods show close agreement in terms of deflection and stress components. As the aspect ratio increases, the deflection values decrease as shown in

Figure 15, indicating the stiffening effect of the plate. The stress components also follow these criteria, with reductions in both normal and shear stresses.

Comparing the Reddy's solution, Ritz solution, and FEM solution, it is observed that the results are in perfect agreement for all aspect ratios, demonstrating the reliability of the proposed Ritz method. The deflection, as well as normal and shear stresses, shows consistent patterns, affirming the accuracy of the numerical solutions.

In conclusion, the table illustrates that the proposed Ritz method aligns well with established Reddy's solutions and FEM results for thick anisotropic plates across different aspect ratios. The agreement between the methods supports the validity and applicability of the Ritz method in efficiently predicting the bending behavior and stress distribution in thick anisotropic plates, providing a valuable tool for engineering analysis. Also, Ritz method provides a computationally efficient alternative to the FEM solution as shown from Table 9 and contours Figures (16-21) , delivering accurate results for the considered thick plate.

Table 9 : The comparison of the three solutions for SSSS case ($0^\circ/90^\circ/0^\circ$)

a/h	Reddy (Navier) Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$

4	2.659	0.654	0.659	0.073	0.691	0.419
10	1.021	0.771	0.307	0.051	0.764	0.323
20	0.757	0.798	0.222	0.045	0.779	0.302
100	0.669	0.807	0.192	0.042	0.784	0.296
a/h	Ritz Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	2.659	0.654	0.658	0.074	0.695	0.425
10	1.021	0.771	0.306	0.052	0.768	0.330
20	0.757	0.798	0.221	0.045	0.781	0.309
100	0.669	0.807	0.190	0.042	0.787	0.301
a/h	FEM Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	2.659	0.654	0.654	0.073	0.680	0.416
10	1.021	0.772	0.307	0.051	0.762	0.323
20	0.757	0.798	0.223	0.045	0.776	0.291
100	0.669	0.807	0.193	0.042	0.770	0.289

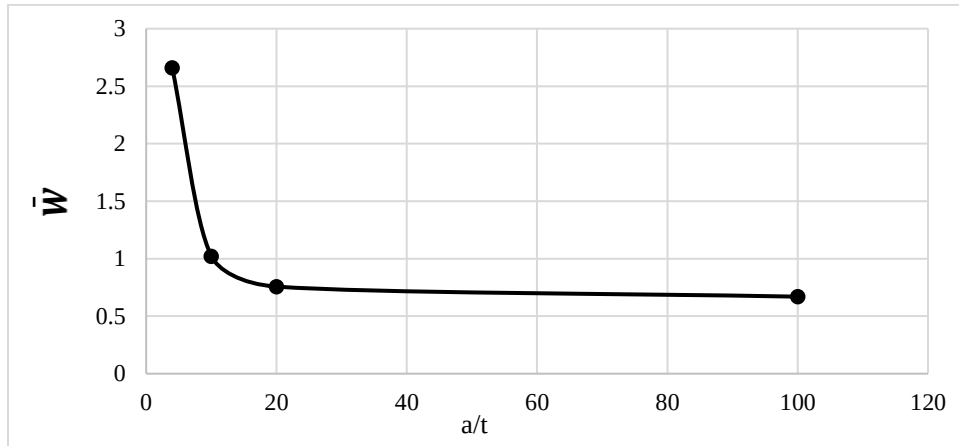
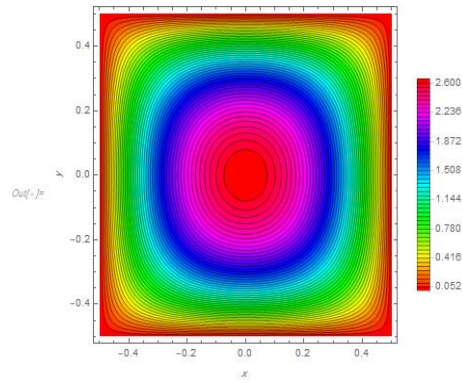
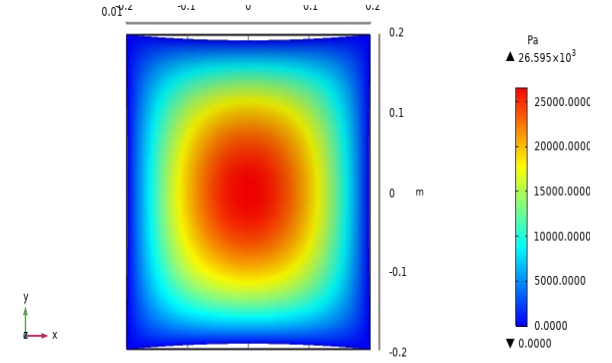


Figure 15 : lateral displacement $\bar{w}$ vs (a/h)

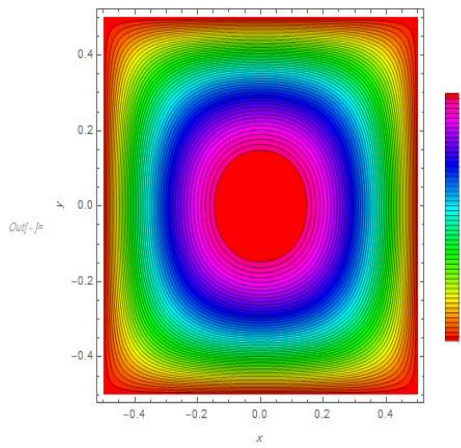


RITZ

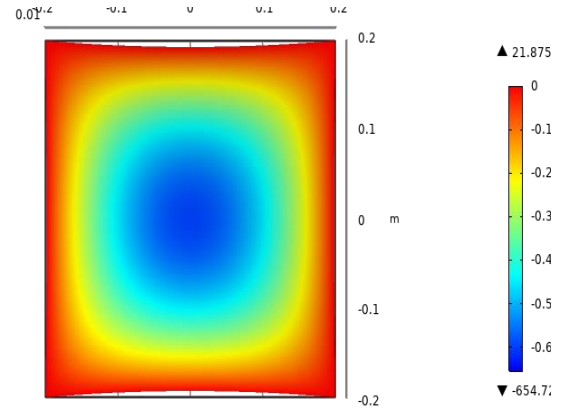


FEM

Figure 16 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM



RITZ



FEM

Figure 17 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM

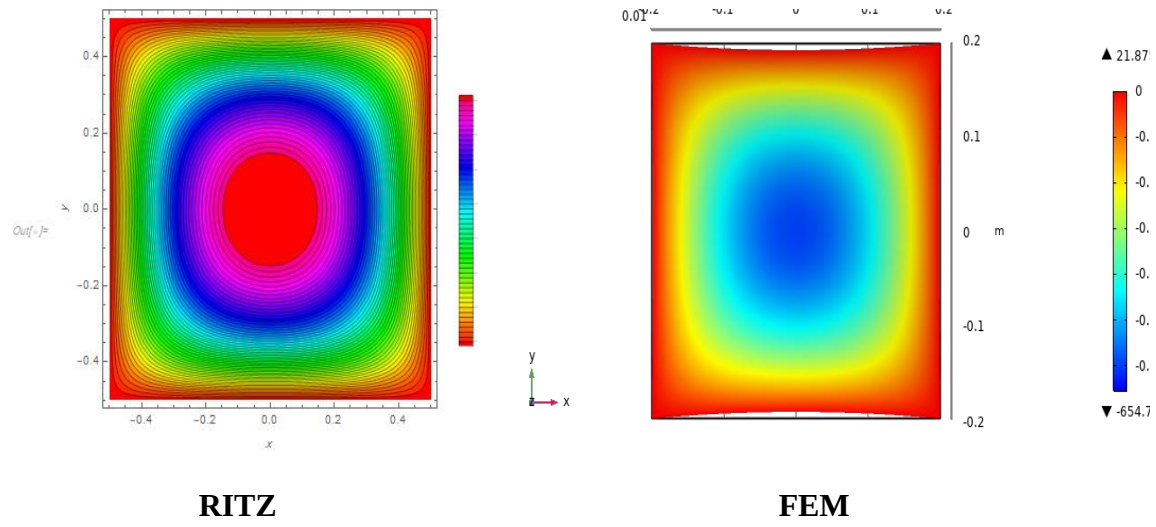


Figure 18 : The contours of the in-plane stress $\bar{\sigma}_y$; (a) Ritz method and (b) FEM

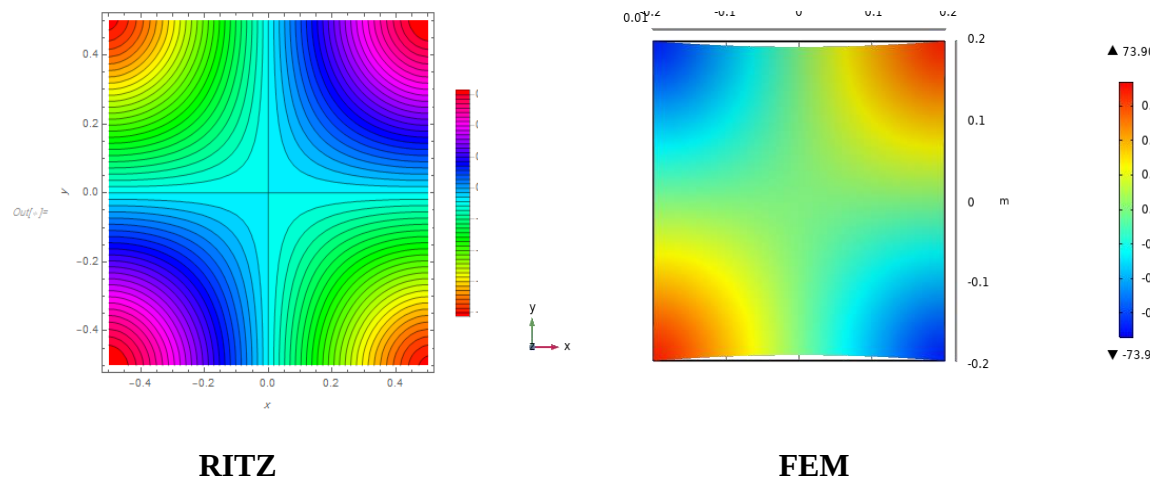


Figure 19 : The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM

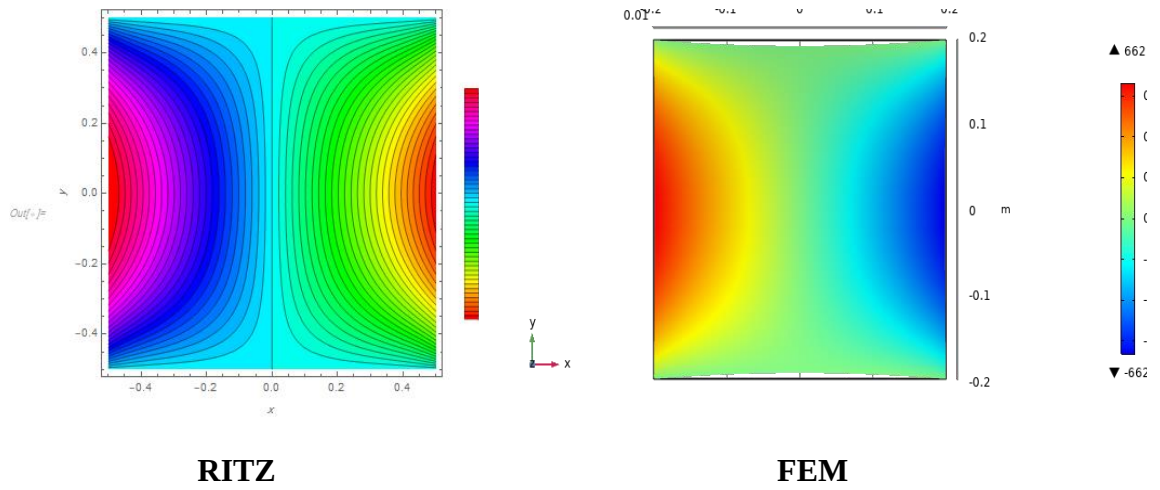


Figure 20 : The contours of the out-plane shear stress $\bar{\tau}_{xz}$; (a) Ritz method and (b) FEM

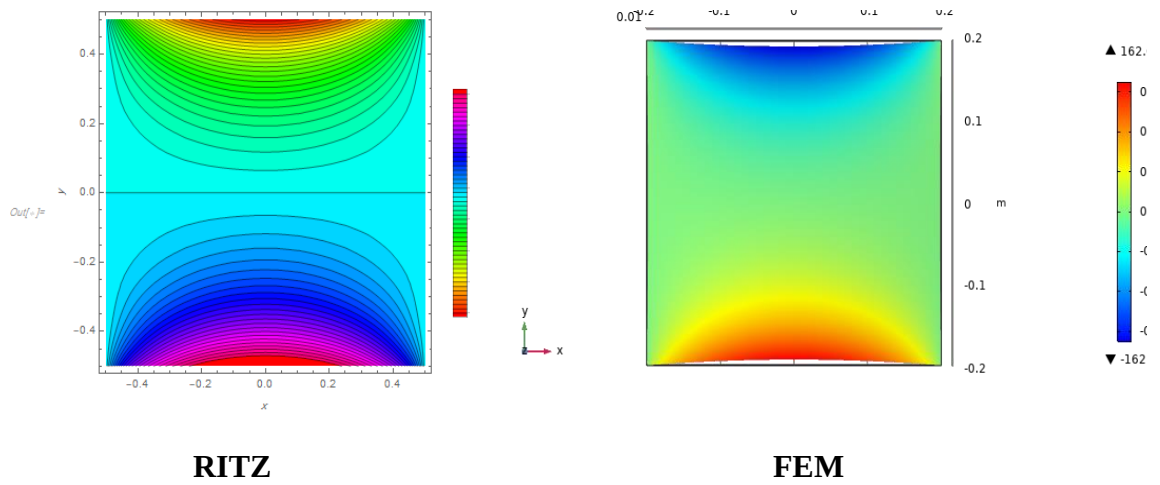


Figure 21 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM

4.5.1.2 CCCC-Clamped plate at four edges

Table 10 presents a comparative analysis of the Ritz method and the Finite Element Method (FEM) solutions for a cross-ply $[0^\circ/90^\circ/0^\circ]$ square thick CCCC-plate, examining various aspect ratios (length to thickness ratio) (a/t): 4, 10, 20, and 100. The deflection and stress components are detailed for the four investigated aspect ratios: 4, 10, 20, and 100.

Examining the results for all aspect ratios reveals a reasonably close match between the Ritz and FEM solutions in terms of both deflection and stress components. As the aspect ratio increases, there is a noticeable decrease in deflection, also reflected in the stress components, with reductions in both normal and shear stresses as the aspect ratio grows.

A key observation is the consistent agreement between the Ritz and FEM solutions across all aspect ratios and contours in Figures (22-27), emphasizing the reliability of the proposed Ritz method for CCCC case.

Table 10: The comparison of the three solutions CCCC ($0^\circ/90^\circ/0^\circ$)

a/h	Ritz Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	1.900	0.197	0.046	0.001	0.600	0.201
10	4.794	0.248	0.024	0.009	0.720	0.156
20	0.230	0.270	0.0124	0.007	0.760	0.132
100	0.141	0.277	0.007	0.007	0.762	0.120
a/h	FEM Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	1.922	0.195	0.046	0.001	0.591	0.210
10	4.828	0.246	0.024	0.001	0.698	0.174
20	0.231	0.269	0.014	0.008	0.742	0.155
100	0.141	0.277	0.007	0.007	0.750	0.372

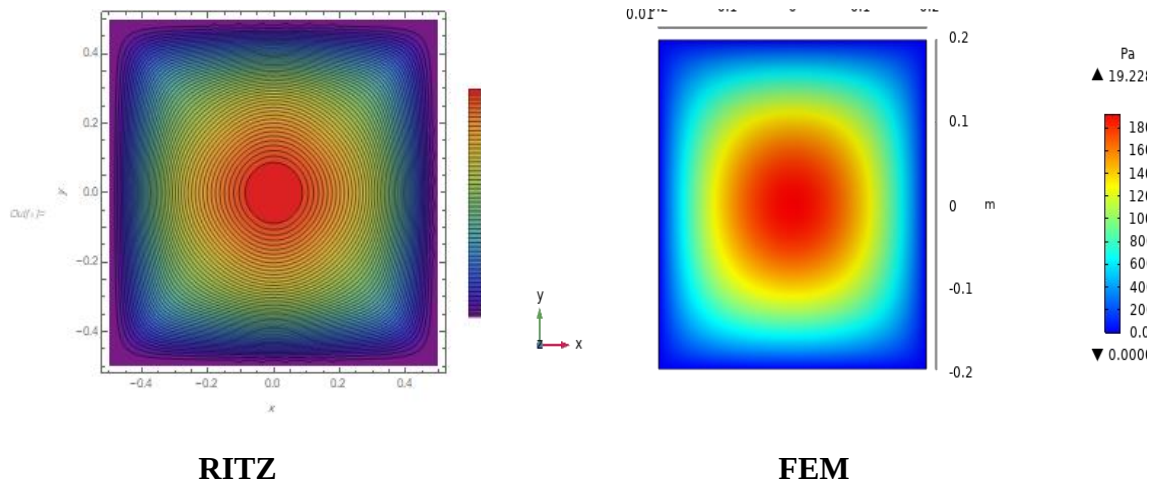


Figure 22 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM

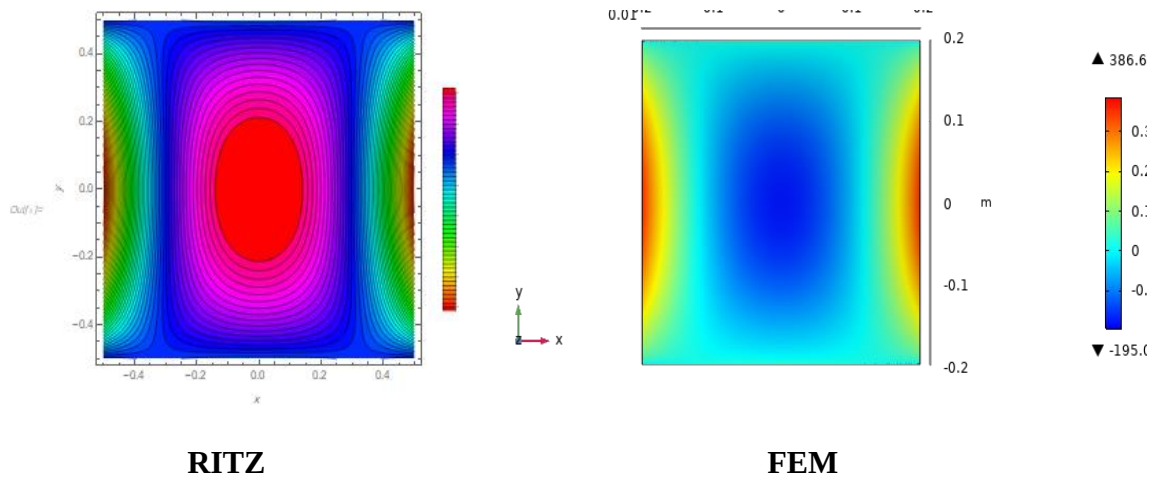


Figure 23 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM

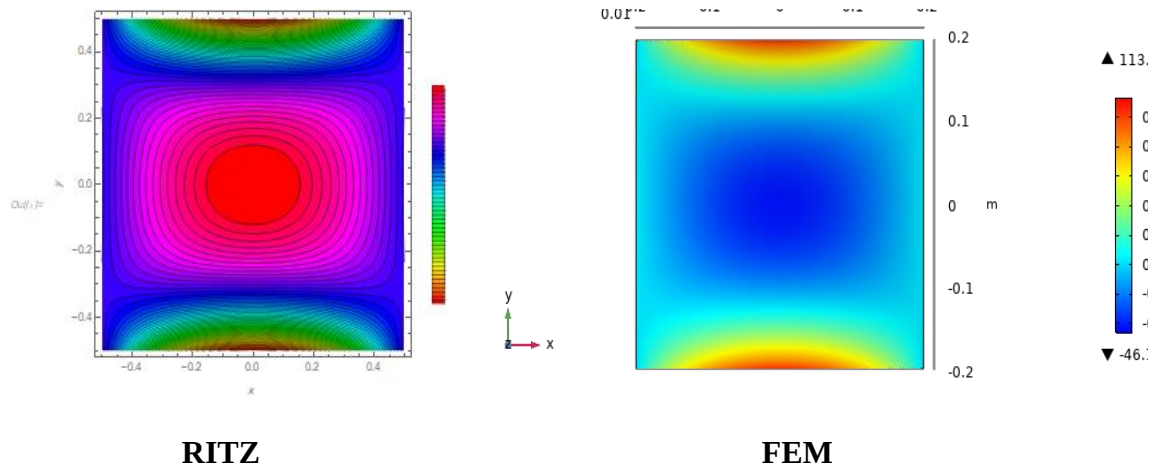


Figure 24: The contours of the in-plane stress $\bar{\sigma}_y$; (a) Ritz method and (b) FEM

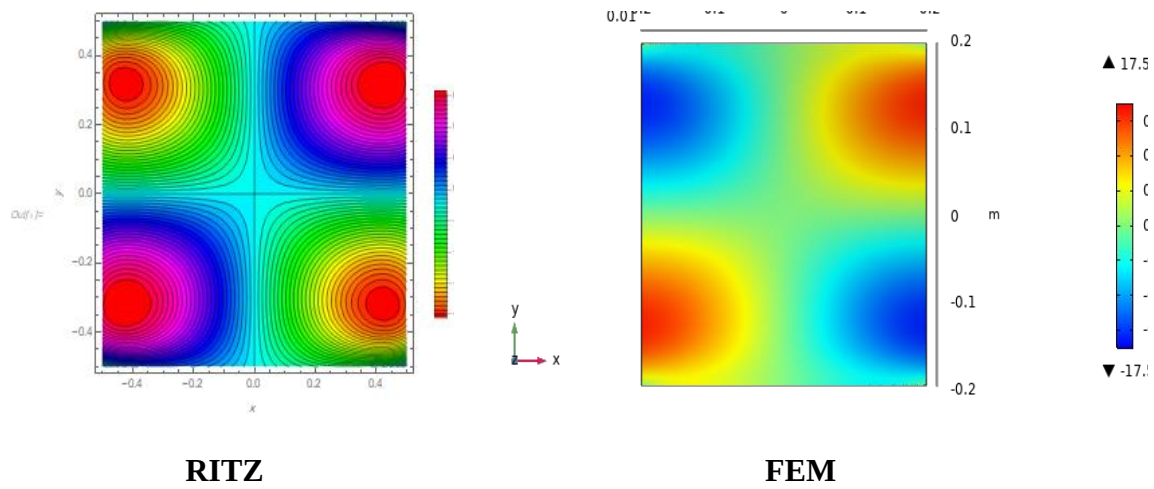


Figure 25: The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM

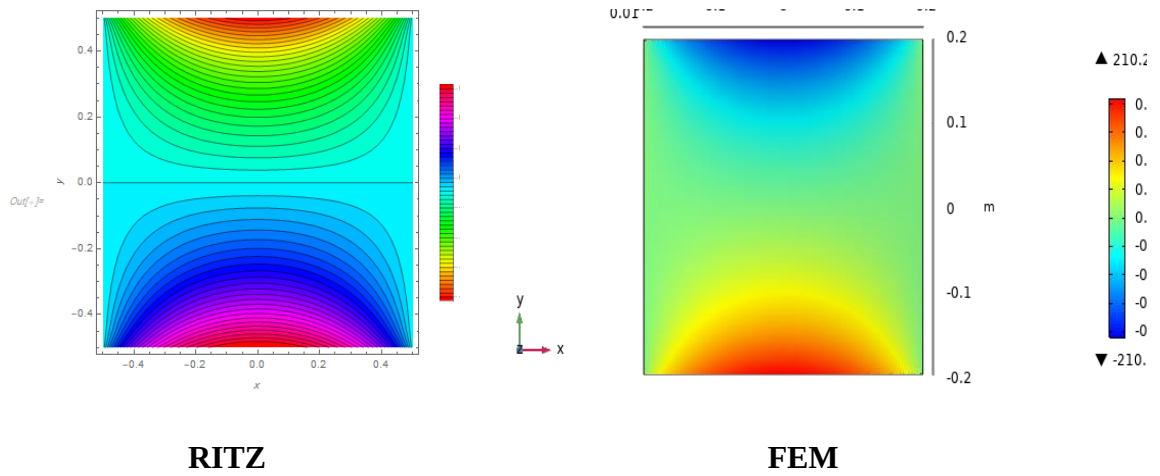


Figure 26 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM

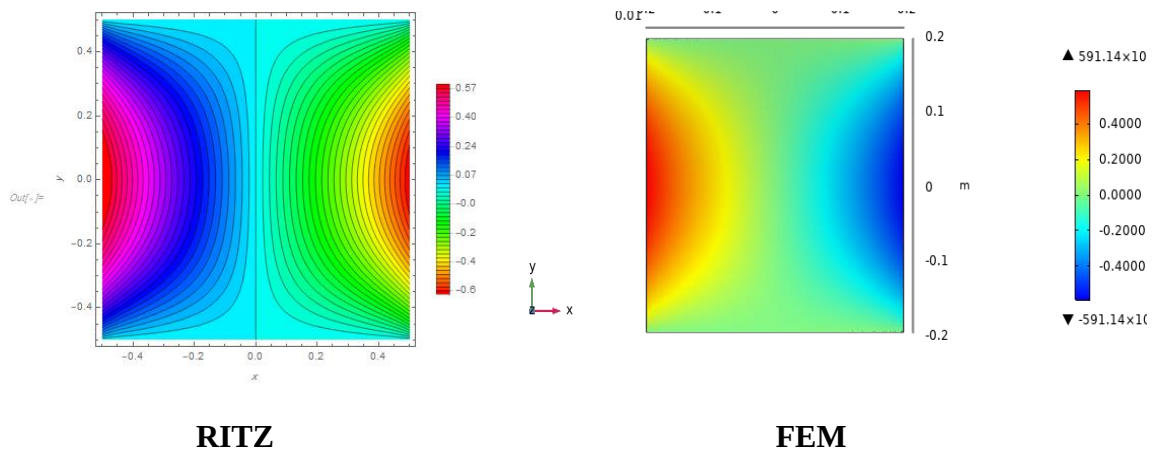


Figure 27 : The contours of the out-plane shear stress $\bar{\tau}_{xz}$; (a) Ritz method and (b) FEM

4.5.2 Angle-ply [-45°/45°]

Fiber orientation angle (45°) is investigated for antisymmetric laminated composite square plate under simply supported and clamped edges. The plates are composed of two antisymmetric layers oriented at $(-45^\circ/45^\circ)$ as shown in Figure 28.

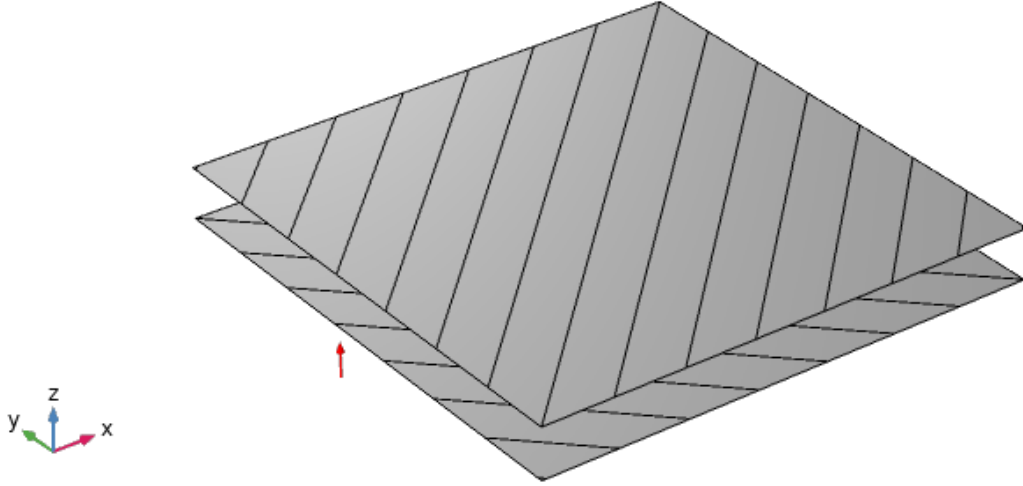


Figure 28 : Plate with two antisymmetric layers oriented at $(-45^\circ/45^\circ)$

The plate is centered at the origin $(0, 0)$ and dimensioned as $(a \times a)$ with thickness of (t) .

The details of the plate and elastic material properties for each lamina are:

$$E_1 = 1GPa, E_2 = 25GPa, G_{12} = G_{13} = 0.5 E_2 GPa, G_{23} = 0.2 E_2 GPa, \nu_{12} = 0.25, K = 5/6$$

The geometrical properties of the laminated plate are:

$$a = b = 1, \text{ and } a/h = 4, 10, 20, 100$$

The obtained results are illustrated in tables, graphically and compared with the FEM results. The approximation polynomial functions in this example:

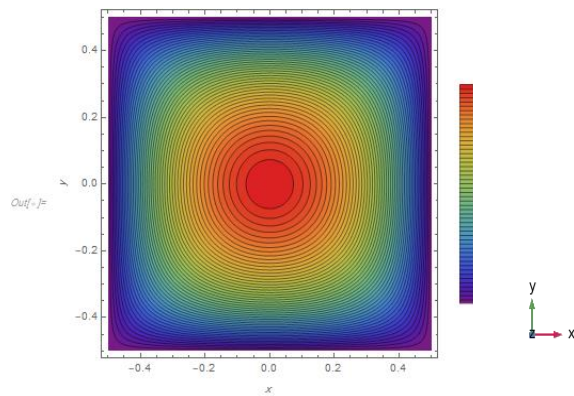
$$\left. \begin{aligned}
u &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^u (a^2 - x^2)(b^2 - y^2) x^i y^j \\
v &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^v (a^2 - x^2)(b^2 - y^2) x^i y^j \\
w &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^w (a^2 - x^2)^h (b^2 - y^2)^h x^i y^j \\
\phi_x &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^{\phi_x} (a^2 - x^2)(b^2 - y^2) x^i y^j \\
\phi_y &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^{\phi_y} (a^2 - x^2)(b^2 - y^2) x^i y^j
\end{aligned} \right\} \quad (4.51)$$

4.5.2.1 Simply supported at four edges (SSSS)

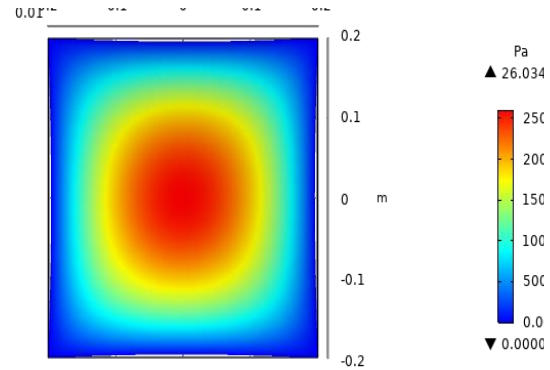
Table **11** provides a detailed comparison of the Navier solution, Ritz solution, and Finite Element Method (FEM) solution for simply supported angle-ply laminates with angles of $(-45^\circ/45^\circ)$. It examines various parameters, including the aspect ratio (a/h) and mechanical responses such as deflection $(\bar{w})$ and stress components. The results show a consistent trend across all three methods, with deflection decreasing and stresses in the laminate diminishing as the aspect ratio increases. This trend is also reflected in the contour plots in Figures **(29-33)**, demonstrating the convergence and agreement of the different analytical and numerical methods.

Table 11: The comparison of the three solutions for SSSS (-45°/45°)

a/h	Navier Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	2.603	0.343	0.343	0.414	0.388	0.388
10	1.279	0.347	0.347	0.427	0.381	0.381
20	1.090	0.349	0.349	0.435	0.377	0.377
100	1.030	0.350	0.350	0.441	0.376	0.376
a/h	Ritz Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	2.603	0.343	0.343	0.416	0.389	0.389
10	1.278	0.346	0.346	0.428	0.381	0.381
20	1.090	0.348	0.348	0.436	0.377	0.377
100	1.030	0.349	0.349	0.444	0.375	0.375
a/h	FEM Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	2.603	0.338	0.338	0.416	0.380	0.380
10	1.279	0.349	0.349	0.428	0.373	0.373
20	1.090	0.356	0.356	0.437	0.370	0.370
100	1.030	0.374	0.374	0.443	0.369	0.369



RITZ



FEM

Figure 29 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM

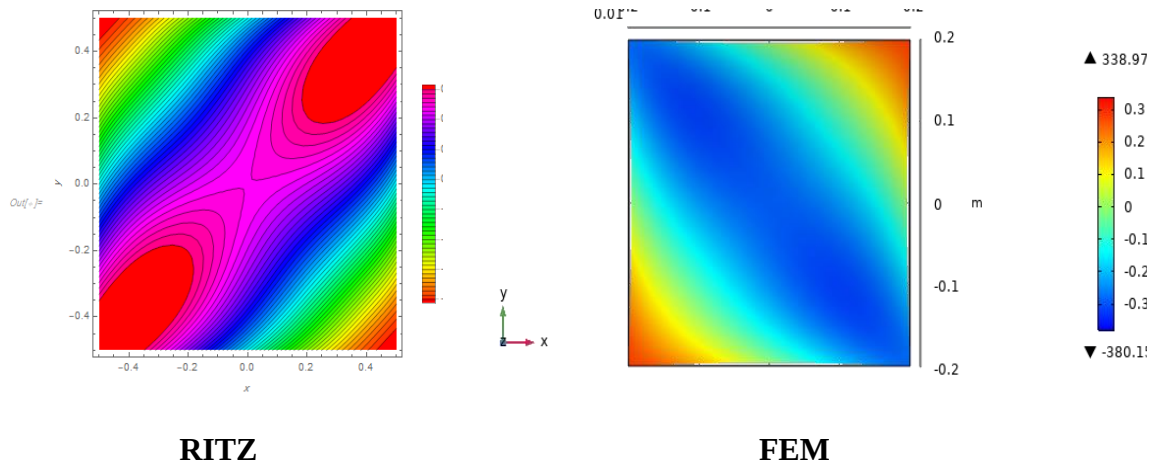


Figure 30 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM

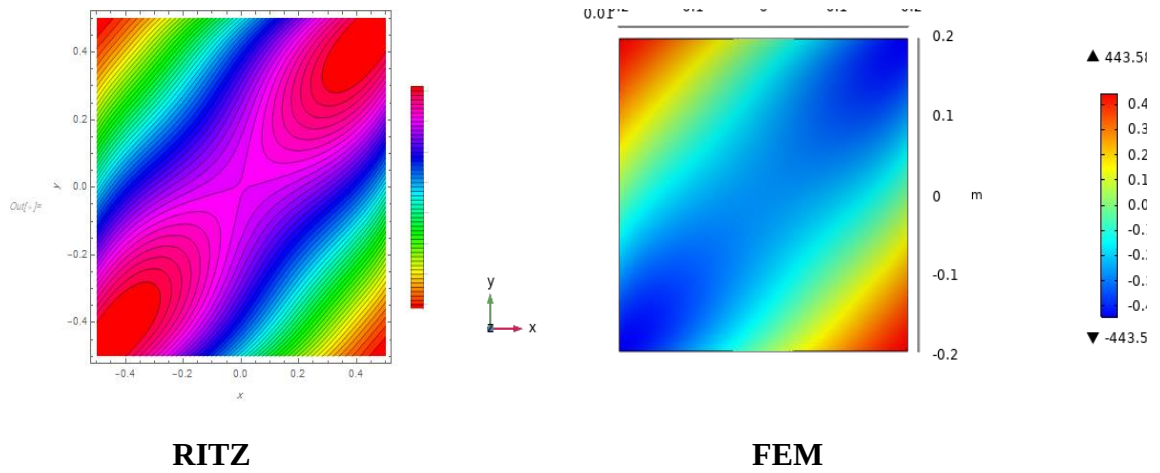


Figure 31 : The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM

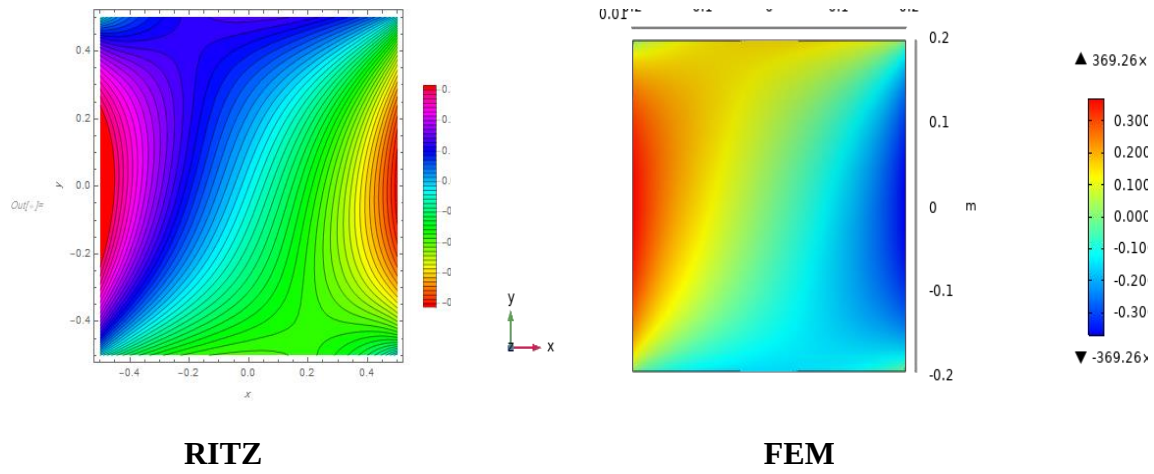


Figure 32 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM

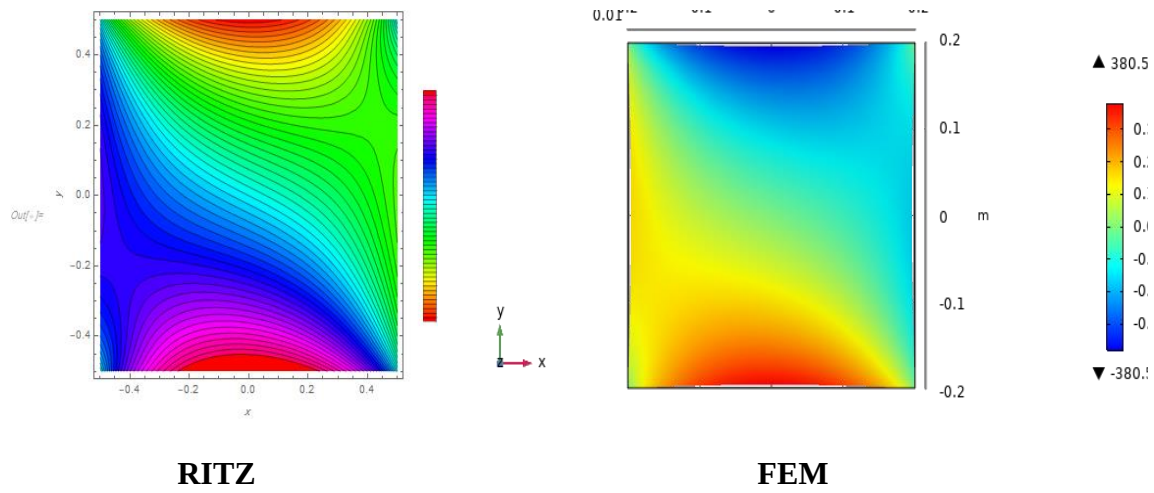


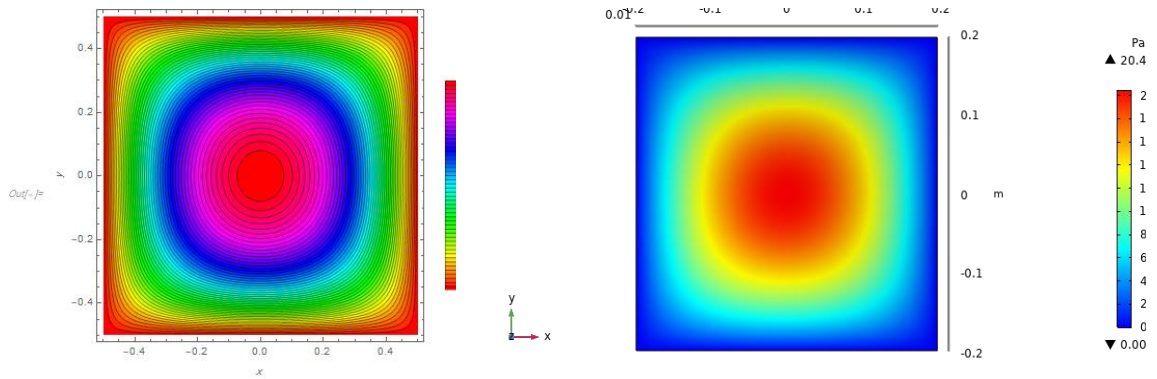
Figure 33 : The contours of the out-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM

4.5.2.2 Clamped at four edges (CCCC)

Table 12 presents a comparative analysis of the Ritz solution and Finite Element Method (FEM) solution for angle-ply laminates with varying aspect ratios (a/h). The data show that for different aspect ratios, both methods yield similar results in terms of deflection and stress components. Figures (34-38) illustrate the corresponding deflection ($\bar{w}$) and stress contours, specifically highlighting the values for an aspect ratio of 4. This close agreement between the two solutions underscores the reliability of the methods for analyzing angle-ply laminates.

Table 12 : The comparison of the two solutions for CCCC (-45°/45°)

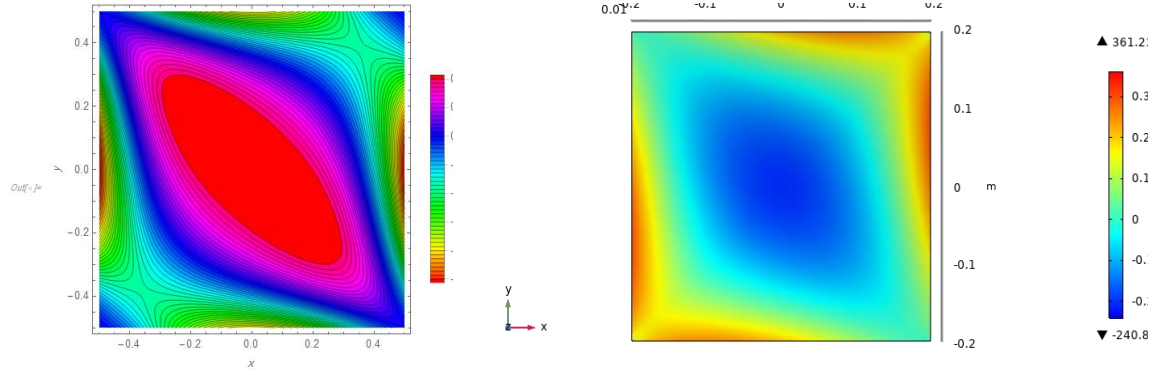
a/h	Ritz Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	2.012	0.237	0.237	0.191	0.400	0.400
10	0.6911	0.223	0.223	0.177	0.438	0.438
20	0.485	0.201	0.201	0.167	0.460	0.460
100	0.409	0.197	0.197	0.156	0.462	0.462
a/h	FEM Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	2.047	0.240	0.240	0.189	0.401	0.401
10	0.6893	0.222	0.222	0.177	0.444	0.444
20	0.488	0.210	0.210	0.176	0.486	0.486
100	0.411	0.197	0.197	0.156	0.460	0.460



RITZ

FEM

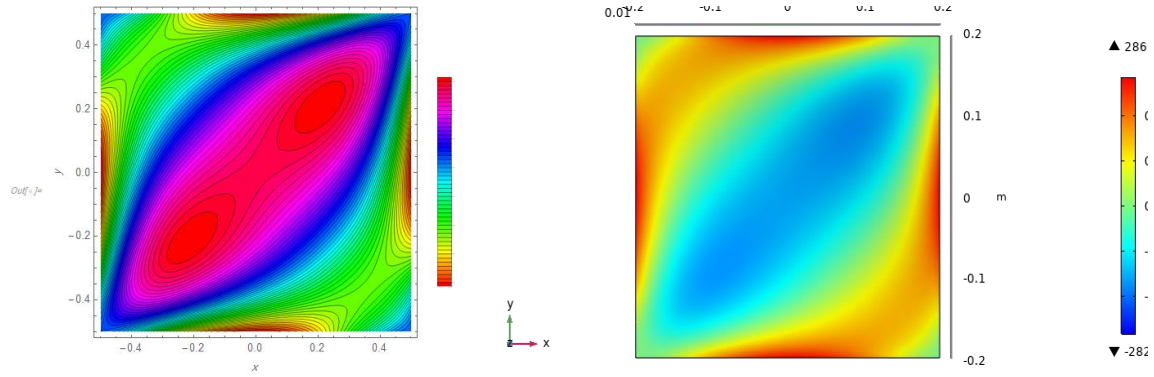
Figure 34 : The contours of the lateral displacement $\bar{w}$; (a) Ritz method and (b) FEM



RITZ

FEM

Figure 35 : The contours of the in-plane stress $\bar{\sigma}_x$; (a) Ritz method and (b) FEM



RITZ

FEM

Figure 36 : The contours of the in-plane shear stress $\bar{\tau}_{xy}$; (a) Ritz method and (b) FEM

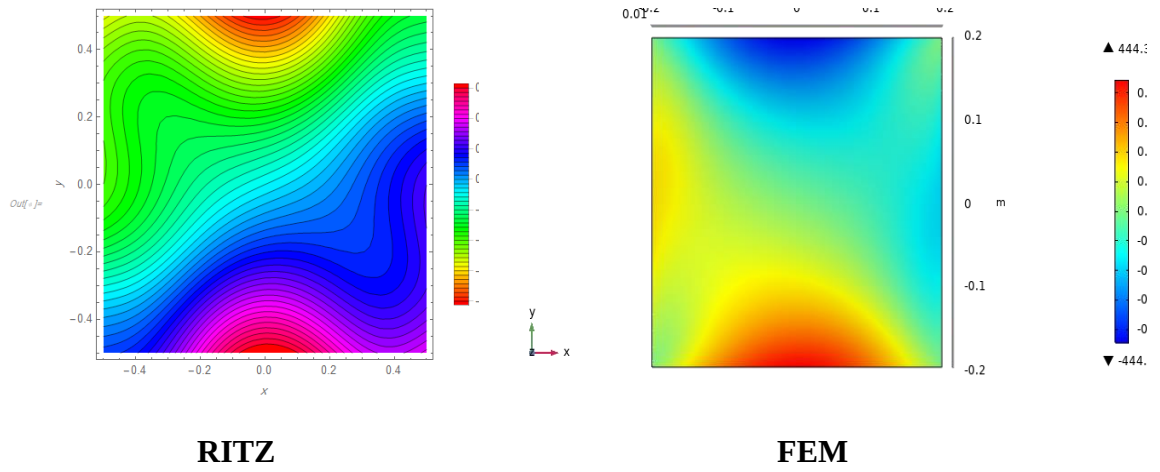


Figure 37 : The contours of the out-plane shear stress $\bar{\tau}_{yz}$; (a) Ritz method and (b) FEM

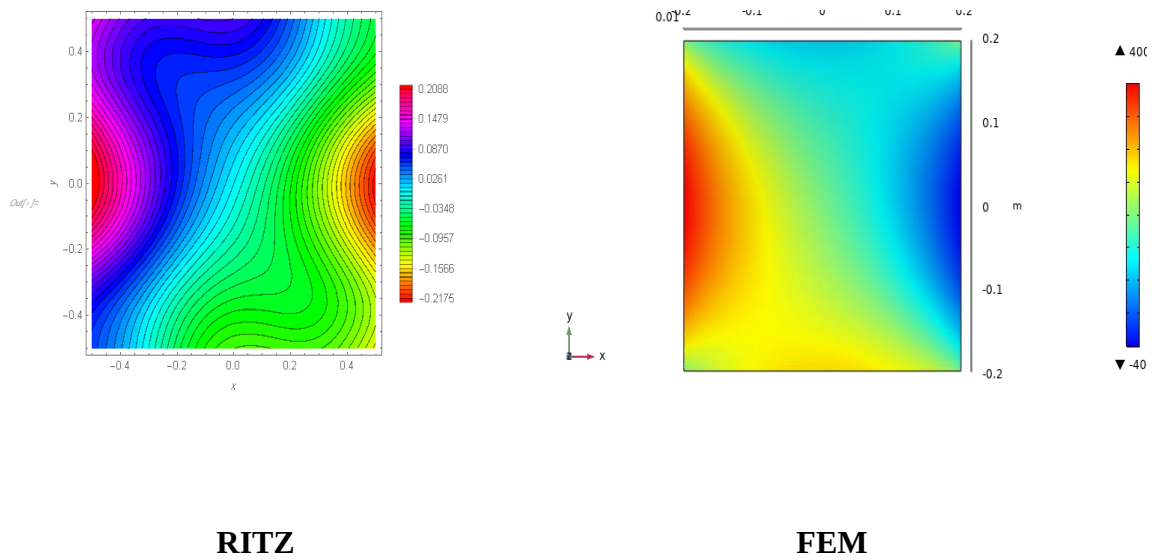


Figure 38 : The contours of the out-plane shear stress $\bar{\tau}_{xz}$; (a) Ritz method and (b) FEM

CHAPTER 5

HSDT FOR THE ANALYSIS OF THICK LAMINATED PLATES

5.1 General

This theory is utilized for analyzing thicker laminated anisotropic composite plates where FSDT is not accurate enough for predicting stresses, and deflections for such plates. It considers the transverse shear stresses to follow a parabolic distribution, and notably, it does not necessitate any shear correction coefficients [85]. As per HSDT, the displacement field can be described by Reddy [86] :

$$u = u_0(x, y) - z \frac{\partial w(x, y)}{\partial x} + f(z) \left(\phi_x(x, y) + \frac{\partial w(x, y)}{\partial x} \right) \quad (5.1a)$$

$$v = v_0(x, y) - z \frac{\partial w(x, y)}{\partial y} + f(z) \left(\phi_y(x, y) + \frac{\partial w(x, y)}{\partial y} \right) \quad (5.1b)$$

$$w = w(x, y) \quad (5.1c)$$

Where $f(z)$ takes different expressions for different methods.

However, despite the progress made, researchers still need to investigate the integration of HSDT with other advanced theories or numerical methods to enhance its capabilities and applicability to a wider range of practical engineering problems. Therefore, this chapter aims to propose a semi-analytical solution for thick laminated composite plates with different boundary conditions using minimization of potential energy in the sense of HSDT.

5.2 HSDT for the Behavior of Thick Laminated Composite Plates

5.2.1 Constitutive Equations

In the following, a thick rectangular laminated composite plate located in the x, y, z plane. The plate has a total thickness t where upper surface is located at $z = t/2$ and lower surfaces at $-t/2$, length a , width b and N -layer with angles $(\theta_1, \theta_2, \dots, \theta_N)$, as shown in Figure 39. The plies of the plate have the same thickness t and assumed to be perfectly bonded.

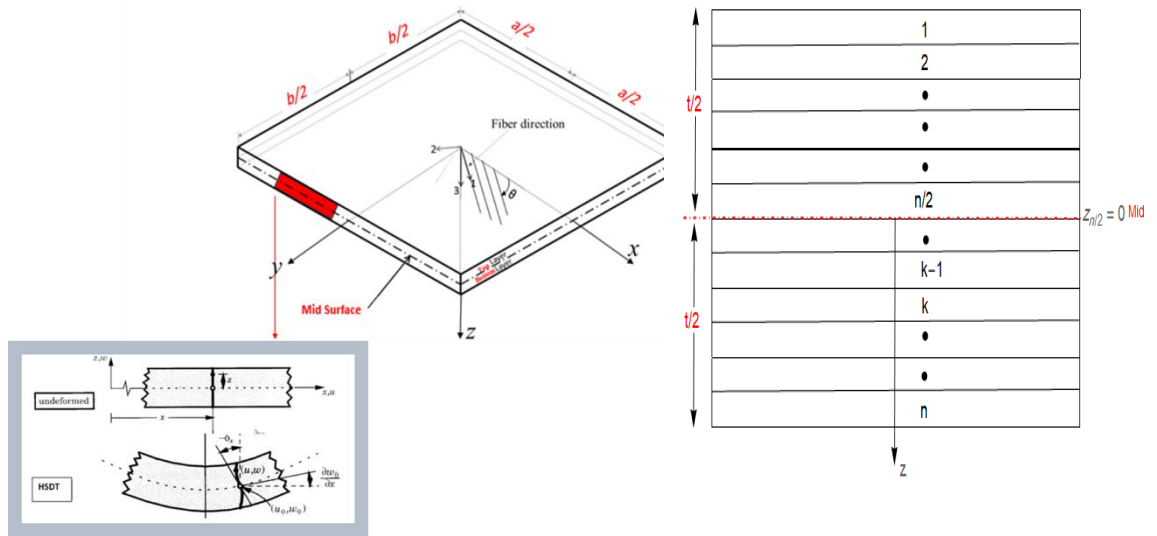


Figure 39 : Deformed Shape for Thick Laminated Composite Plate based on HSDT

The fundamental assumptions of HSDT may be stated as follows:

- HSDT can represent the kinematics better and also may not require shear correction factors.
- HSDT can yield more accurate interlaminar stress distributions.
- HSDT involves high order stresses resultants that are difficult to interpret physically and require considerably more computational efforts.
- HSDT expand the displacement up to cubic term in the thickness coordinate to have quadratic variation of the transverse shear strains and transverse shear stresses through each layer as shown in Figure 39.

Equation (5.1a) indicates that the lateral deflection w does not change over the plate thickness. In contrast to the lateral displacements, the in-plane displacements (u and v), Equations (5.1b and c), are changing over the thickness.

Recalling Eqs.(4.1) and Substituting them in Eqs. (5.1), the in-plane strains are given by:

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{bmatrix} + \begin{bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{bmatrix} + \begin{bmatrix} \epsilon_x^{(3)} \\ \epsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{bmatrix} \quad (5.2)$$

where : $C_1 = 4/3 t^2$ and $C_2 = 3C_1$

the out-plane strains are given by:

$$\begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{bmatrix} + \begin{bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{bmatrix} \quad (5.4)$$

Where $(\epsilon_x^{(0)}, \epsilon_y^{(0)}, \gamma_{xy}^{(0)})$ are the midspan strains which are given by:

$$\begin{Bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \quad (5.5)$$

Where $(\epsilon_x^{(1)}, \epsilon_y^{(1)}, \gamma_{xy}^{(1)})$ are given by:

$$\begin{Bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = z \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \quad (5.6)$$

Where $(\epsilon_x^{(3)}, \epsilon_y^{(3)}, \gamma_{xy}^{(3)})$ are given by:

$$\begin{Bmatrix} \epsilon_x^{(3)} \\ \epsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = z^3 \begin{Bmatrix} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix} \quad (5.7)$$

Where $(\gamma_{yz}^{(0)}, \gamma_{xz}^{(0)})$ are given by:

$$\begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{Bmatrix} \quad (5.8)$$

Where $(\gamma_{yz}^{(2)}, \gamma_{xz}^{(2)})$ are given by:

$$\begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} = z^2 \begin{Bmatrix} -C_2 \left(\phi_y + \frac{\partial w}{\partial y} \right) \\ -C_2 \left(\phi_x + \frac{\partial w}{\partial x} \right) \end{Bmatrix} \quad (5.9)$$

The stress-strain relations, in global coordinates, for an individual layer k (located between $z = z_{k-1}$ and $z = z_k$ in **Figure 39** are given by:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{14} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{24} & 0 & 0 \\ \bar{Q}_{14} & \bar{Q}_{24} & \bar{Q}_{44} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & \bar{Q}_{56} \\ 0 & 0 & 0 & \bar{Q}_{56} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}_k \quad (5.10)$$

where $\bar{Q}_{ij}$ are the components of the off axis reduced stiffness matrix, which have been defined earlier in Chapter 3.

After the combination of Eqs. (5.6 - 5.9) with 5.10, the stress components for the laminate in global

coordinates are expressed as:

$$(N_i, M_i, P_i) = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_i (1, z, z^3) dz, \quad i = 1, 2, 6 \quad (5.11a)$$

$$(Q_i, R_i) = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_i (1, z^2) dz, \quad i = 4, 5 \quad (5.11b))$$

From Eqs.(5.11) the resultants forces , bending moments , higher order stresses and shear forces per unit width can be found by the integration of stress components through plate thicknesses, i.e. as follows :

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz \quad (5.12a)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k z dz \quad (5.12b)$$

$$\begin{bmatrix} P_x \\ P_y \\ P_{xy} \end{bmatrix} = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k z^3 dz \quad (5.12c)$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix}_k dz \quad (5.12d)$$

$$\begin{bmatrix} R_y \\ R_x \end{bmatrix} = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix}_k z^2 dz \quad (5.12f)$$

As a result of integrating the expressions above yields the following matrix-form equation that relates the in-plane forces (N_x, N_y, N_{xy}) , moments (M_x, M_y, M_{xy}) , high order in-plane forces (P_x, P_y, P_{xy}) , out-plane shear forces (Q_x, Q_y) and high order out-plane shear forces

high (R_x, R_y) to the midspan strains $(\epsilon_x^{(0)}, \epsilon_y^{(0)}, \gamma_{xy}^{(0)})$, $(\epsilon_x^{(1)}, \epsilon_y^{(1)}, \gamma_{xy}^{(1)})$, $(\epsilon_x^{(3)}, \epsilon_y^{(3)}, \gamma_{xy}^{(3)})$ and shear strains $(\gamma_{yz}^{(0)}, \gamma_{xz}^{(0)})$, $(\gamma_{yz}^{(2)}, \gamma_{xz}^{(2)})$:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ P_x \\ P_y \\ P_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & E_{12} & E_{22} & E_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & E_{16} & E_{26} & E_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & F_{11} & F_{12} & F_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & F_{12} & F_{22} & F_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & F_{16} & F_{26} & F_{66} \\ E_{11} & E_{12} & E_{16} & F_{11} & F_{12} & F_{16} & H_{11} & H_{12} & H_{16} \\ E_{12} & E_{22} & E_{26} & F_{12} & F_{22} & F_{26} & H_{12} & H_{22} & H_{26} \\ E_{16} & E_{26} & E_{66} & F_{16} & F_{26} & F_{66} & H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^{(0)} \\ \epsilon_y^{(0)} \\ \gamma_{xy}^{(0)} \\ \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \gamma_{xy}^{(1)} \\ \epsilon_x^{(3)} \\ \epsilon_y^{(3)} \\ \epsilon_{xy}^{(3)} \end{bmatrix} \quad (5.13)$$

$$\begin{bmatrix} Q_Y \\ Q_X \\ R_y \\ R_x \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} & D_{44} & D_{45} \\ A_{45} & A_{55} & D_{45} & D_{55} \\ D_{44} & D_{45} & F_{44} & F_{45} \\ D_{45} & D_{55} & F_{45} & F_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{bmatrix} \quad (5.14)$$

where A_{ij} , B_{ij} and D_{ij} are defined in chapter 3 while E_{ij} , F_{ij} , H_{ij} are the stiffness that accounts for higher order stresses and $(A_{44}, A_{45}$ and $A_{55} \dots F_{45}, D_{45})$ are the transverse stiffnesses that relate out-plane loads to transverse shear strains, these stiffnesses elements are evaluated as follows:

$$A_{ij} = \sum_{k=1}^N \bar{Q}_{ij}(z_k - z_{k-1}) \quad (5.15)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N \bar{Q}_{ij}(z_k^2 - z_{k-1}^2) \quad (5.16)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N \bar{Q}_{ij} (z_k^3 - z_{k-1}^3) \quad (5.17)$$

$$E_{ij} = \frac{1}{4} \sum_{k=1}^N \bar{Q}_{ij} (z_k^4 - z_{k-1}^4) \quad (5.18)$$

$$F_{ij} = \frac{1}{5} \sum_{k=1}^N \bar{Q}_{ij} (z_k^5 - z_{k-1}^5) \quad (5.19)$$

$$H_{ij} = \frac{1}{7} \sum_{k=1}^N \bar{Q}_{ij} (z_k^7 - z_{k-1}^7) \quad (5.20)$$

where i and j take the values 1, 2, 6 from equ (5.15-5.20) and 4, 5 for equ (5.15, 5.17 and 5.19).

5.2.2 Potential Energy Components

For laminate composite thick plate, the potential energy is defined as the difference between the total strain energy stored in the plate and the work done by the applied load, i.e.

$$\Pi = U - W \quad (5.21)$$

where:

$$U = \frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} (\sigma \varepsilon) dz \right) dx dy \quad (5.22)$$

and the work done by the external force can be expressed by:

$$W = \iint_A q w dA \quad (5.23)$$

using Eqs.(4.8a-c) in addition to higher order stresses (σ_3) :

$$\begin{Bmatrix} \sigma_x^{(3)} \\ \sigma_y^{(3)} \\ \tau_{xy}^{(3)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^{(3)} \\ \epsilon_y^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix} \quad (5.24a)$$

and high order Transverse shear stresses (τ_z^2):

$$\begin{Bmatrix} \tau_{yz}^2 \\ \tau_{xz}^2 \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{xz}^{(2)} \end{Bmatrix} \quad (5.24b)$$

Strain energy in Eq.(5.22) can be expressed using Eqs.(4.8a-c) and (5.24):

$$\begin{aligned}
U = \frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} (\sigma^{(0)} \epsilon^{(0)} + \sigma^{(1)} \epsilon^{(0)} + \sigma^{(0)} \epsilon^{(1)} + \sigma^{(1)} \epsilon^{(1)} + \sigma^{(3)} \epsilon^{(3)} + \sigma^{(0)} \epsilon^{(3)} + \right. \\
\left. \sigma^{(3)} \epsilon^{(0)} + \sigma^{(1)} \epsilon^{(3)} + \sigma^{(3)} \epsilon^{(1)} + \tau_z \gamma_z + \tau_z \gamma_z^2 + \tau_z^2 \gamma_z + \tau_z^2 \gamma_z^2) dz \right) dx dy
\end{aligned}
\tag{5.25}$$

For facilitating the lengthy task of derivation of strain energy, the above expression can be splitted into the following Eqs.(5.26-5.30):

U^{00} is the membrane strain energy due to normal strains to normal forces effect.

$$U^{00} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma^{(0)} \epsilon^{(0)}) dz \right) dx dy \tag{5.26}$$

U^{11} is the bending strain energy due to effect of bending moments to curvatures.

$$U^{11} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma^{(1)} \epsilon^{(1)}) dz \right) dx dy \tag{5.27}$$

U^{33} is the inextensible strain energy due to effect of high order forces to high order strains.

$$U^{33} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma^{(3)} \epsilon^{(3)}) dz \right) dx dy \tag{5.28}$$

$U^{c01}, U^{c03}, U^{c13}$ is the strain energy due to coupling effect.

$$U^{c01} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\sigma^{(1)} \epsilon^{(0)} + \sigma^{(0)} \epsilon^{(1)}) dz \right) dx dy$$

(5.29a)

$$U^{c03} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\boldsymbol{\sigma}^{(3)} \epsilon^{(0)} + \boldsymbol{\sigma}^{(0)} \epsilon^{(3)}) dz \right) dx dy$$

(5.29b)

$$U^{c13} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\boldsymbol{\sigma}^{(1)} \epsilon^{(3)} + \boldsymbol{\sigma}^{(3)} \epsilon^{(1)}) dz \right) dx dy$$

(5.29c)

U^{SH00} is the transverse shear strain energy due to coupling and higher orders transverse forces respectively:

$$U^{SH00} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\boldsymbol{\tau}_z \gamma_z) dz \right) dx dy \quad (5.30a)$$

$$U^{SH33} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\boldsymbol{\tau}_z^{(2)} \gamma_z^{(2)}) dz \right) dx dy$$

(5.30b)

$$U^{SH30} = \frac{1}{2} \int_y \int_x \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} (\boldsymbol{\tau}_z^{(2)} \gamma_z) dz \right) dx dy$$

(5.30c)

Finally, the strain energy (U) stored in the plate can expressed into the following terms:

$$U = U^{00} + U^{11} + U^{33} + U^{01} + U^{03} + U^{13} + U^{SH00} + U^{SH33} + U^{SH30}$$

Recalling the membrane strain energy in chapter 4 Eq.(4.35):

$$U^{00} = \frac{1}{2} \int_y \int_x \left[\left(\frac{\partial u_0}{\partial x} \right)^2 A_{11} + 2 \frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial y} A_{12} + 2 \left(\frac{\partial u_0}{\partial x} \frac{\partial u_0}{\partial y} + \frac{\partial u_0}{\partial x} \frac{\partial v_0}{\partial x} \right) A_{16} + 2 \left(\frac{\partial v_0}{\partial y} \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial v_0}{\partial x} \right) A_{26} + \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right)^2 A_{66} + \left(\frac{\partial v_0}{\partial y} \right)^2 A_{22} \right] dx dy \quad (5.31)$$

Recalling the bending strain energy chapter 4 Eq.(4.38):

$$U^{11} = \frac{1}{2} \int_y \int_x \left[\left(\frac{\partial \phi_x}{\partial x} \right)^2 D_{11} + 2 \frac{\partial \phi_x}{\partial x} \frac{\partial \phi_y}{\partial y} D_{12} + 2 \left(\frac{\partial \phi_x}{\partial x} \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_x}{\partial x} \frac{\partial \phi_y}{\partial x} \right) D_{16} + 2 \left(\frac{\partial \phi_y}{\partial y} \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial y} \frac{\partial \phi_y}{\partial x} \right) D_{26} + \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right)^2 D_{66} + \left(\frac{\partial \phi_y}{\partial y} \right)^2 D_{22} \right] dx dy \quad (5.32)$$

Recalling the inextensible strain energy Eq.(5.28):

$$U^{33} = \frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} Z^6 dz \right) \left\{ \begin{array}{l} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{array} \right\} \left\{ \begin{array}{l} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{array} \right\} dx dy \quad (5.33)$$

where, the integration of $[\bar{Q}_{ij}]_k$ over the thickness represents the higher order stiffness

matrix of the laminate. Using Eq.(5.20) in Eq.(5.33) yields:

$$U^{33} =$$

$$\frac{1}{2} \iint_A \left(\begin{bmatrix} H_{11} & H_{12} & H_{16} \\ H_{12} & H_{22} & H_{26} \\ H_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix} \right) \begin{Bmatrix} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix} dx dy$$

(5.34)

respectively, results in expressing the inextensible energy as:

$$\begin{aligned} U^{33} = \frac{c_1^2}{2} \int_y \int_x & \left[\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right)^2 H_{11} + 2 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) H_{12} + 2 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + \right. \right. \\ & \left. \left. 2 \frac{\partial^2 w}{\partial x \partial y} \right) \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) H_{16} + 2 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) H_{26} + \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + \right. \right. \\ & \left. \left. 2 \frac{\partial^2 w}{\partial x \partial y} \right)^2 H_{66} + \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right)^2 H_{22} \right] dx dy \end{aligned} \quad (5.35)$$

Recalling the coupling strain energy in chapter 4 Eq.(4.41):

$$\begin{aligned} U^{10} = \int_y \int_x & \left[\frac{\partial u_0}{\partial x} \frac{\partial \phi_x}{\partial x} B_{11} + \left(\frac{\partial v_0}{\partial y} \frac{\partial \phi_x}{\partial x} + \frac{\partial u_0}{\partial x} \frac{\partial \phi_y}{\partial y} \right) B_{12} + \left(\frac{\partial u_0}{\partial y} \frac{\partial \phi_x}{\partial x} + \frac{\partial v_0}{\partial x} \frac{\partial \phi_x}{\partial x} + \frac{\partial u_0}{\partial x} \frac{\partial \phi_x}{\partial y} + \right. \right. \\ & \left. \frac{\partial u_0}{\partial x} \frac{\partial \phi_y}{\partial x} \right) B_{16} + \left(\frac{\partial u_0}{\partial y} \frac{\partial \phi_y}{\partial y} + \frac{\partial v_0}{\partial x} \frac{\partial \phi_y}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial \phi_x}{\partial y} + \frac{\partial v_0}{\partial y} \frac{\partial \phi_y}{\partial x} \right) B_{26} + \left(\frac{\partial u_0}{\partial y} \frac{\partial \phi_x}{\partial y} + \frac{\partial u_0}{\partial y} \frac{\partial \phi_y}{\partial x} + \right. \\ & \left. \frac{\partial v_0}{\partial x} \frac{\partial \phi_x}{\partial y} + \frac{\partial v_0}{\partial x} \frac{\partial \phi_y}{\partial x} \right) B_{66} + \frac{\partial v_0}{\partial y} \frac{\partial \phi_y}{\partial y} B_{22} \right] dx dy \end{aligned} \quad (5.36)$$

Recalling the coupling strain energy (membrane- inextensible) Eq.(5.29 b):

$$U^{03} =$$

$$\frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z^3 dz \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \begin{Bmatrix} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix} \right) dx dy$$

(5.37)

where, the integration of $[\bar{Q}_{ij}]_k$ over the thickness represents the higher order stiffness matrix of the laminate. Using Eq.(5.18) in Eq.(5.37) yields:

$$U^{03} = \frac{1}{2} \iint_A \left(\begin{bmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & H_{26} \\ E_{16} & H_{26} & H_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix} \right) \begin{Bmatrix} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix} dx dy$$

(5.38)

respectively, results in expressing the coupling (membrane-inextensible) energy as:

$$\begin{aligned} U^{c30} = & -C_1 \int_y \int_x \left[\frac{\partial u_0}{\partial x} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) E_{11} + \left(\frac{\partial v_0}{\partial y} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + \frac{\partial u_0}{\partial x} \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \right) E_{12} + \right. \\ & \left(\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + \frac{\partial u_0}{\partial x} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \right) E_{16} + \left(\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + \right. \\ & \left. \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial v_0}{\partial y} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \left. \right) E_{26} + \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \right) \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) E_{66} + \\ & \left. \frac{\partial v_0}{\partial y} \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) E_{22} \right] dx dy \end{aligned}$$

(5.39)

Recalling the coupling strain energy (bending- inextensible) Eq.(5.29 c):

$$U^{13} =$$

$$\frac{1}{2} \iint_A \left(\int_{-t/2}^{t/2} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} z^4 dz \right) \left\{ \begin{array}{c} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{array} \right\} \left\{ \begin{array}{c} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{array} \right\} dxdy$$

(5.40)

where, the integration of $[\bar{Q}_{ij}]_k$ over the thickness represents the higher order stiffness matrix of the laminate. Using Eq.(5.19) in Eq.(5.38) yields:

$$U^{13} = \frac{1}{2} \iint_A \left(\begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{12} & F_{22} & F_{26} \\ F_{16} & F_{26} & F_{66} \end{bmatrix} \left\{ \begin{array}{c} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{array} \right\} \right) \left\{ \begin{array}{c} -C_1 \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -C_1 \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -C_1 \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{array} \right\} dxdy$$

(5.41)

respectively, results in expressing the coupling (bending-inextensible energy) as:

$$\begin{aligned}
U^{c13} = & -C_1 \int_y \int_x \left[\left(\frac{\partial \phi_x}{\partial x} \right) \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) F_{11} + \left(\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_x}{\partial x} \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \right) F_{12} + \right. \\
& \left(\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) + \frac{\partial \phi_x}{\partial x} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \right) F_{16} + \left(\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \left(\frac{\partial \phi_x}{\partial y} + \right. \right. \\
& \left. \left. \frac{\partial \phi_y}{\partial x} \right) + \frac{\partial \phi_y}{\partial y} \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \right) F_{26} + \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right) \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) F_{66} + \\
& \left. \left(\frac{\partial \phi_y}{\partial y} \right) \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) F_{22} \right] dx dy
\end{aligned}
\tag{5.42}$$

Recalling the transverse shear strain energy in chapter 4 Eq.(4.44):

$$\begin{aligned}
U^{SH00} = & \frac{1}{2} \int_y \int_x \left[A_{44} \left(\phi_y + \frac{\partial w}{\partial y} \right)^2 + 2A_{45} \left(\phi_x + \frac{\partial w}{\partial x} \right) \left(\phi_y + \frac{\partial w}{\partial y} \right) + A_{55} \left(\phi_x + \right. \right. \\
& \left. \left. \frac{\partial w}{\partial x} \right)^2 \right] dx dy
\end{aligned}
\tag{5.43}$$

Similarly, recalling of Equation (5.30 b) and substitution of the expressions of shear strain's results in expressing the shear strain energy in terms of displacements by:

$$U^{SH33} = \frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} K \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} -C_2 \left(\phi_y + \frac{\partial w}{\partial y} \right) \\ -C_2 \left(\phi_x + \frac{\partial w}{\partial x} \right) \end{Bmatrix} z^4 dz \begin{Bmatrix} -C_2 \left(\phi_y + \frac{\partial w}{\partial y} \right) \\ -C_2 \left(\phi_x + \frac{\partial w}{\partial x} \right) \end{Bmatrix} dx dy \quad (5.44)$$

Using Eq.(5.15) in Eqs.(4.44) yields :

$$U^{SH33} = \frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} K \begin{bmatrix} F_{44} & F_{45} \\ F_{54} & F_{55} \end{bmatrix} \begin{Bmatrix} -C_2 \left(\phi_y + \frac{\partial w}{\partial y} \right) \\ -C_2 \left(\phi_x + \frac{\partial w}{\partial x} \right) \end{Bmatrix} \begin{Bmatrix} -C_2 \left(\phi_y + \frac{\partial w}{\partial y} \right) \\ -C_2 \left(\phi_x + \frac{\partial w}{\partial x} \right) \end{Bmatrix} dx dy \quad (5.45)$$

respectively, results in expressing the high order transverse shear strain energy as:

$$U^{SH33} = \frac{c_2^2}{2} \int \int_x \left[F_{44} \left(\phi_y + \frac{\partial w}{\partial y} \right)^2 + 2F_{45} \left(\phi_x + \frac{\partial w}{\partial x} \right) \left(\phi_y + \frac{\partial w}{\partial y} \right) + F_{55} \left(\phi_x + \frac{\partial w}{\partial x} \right)^2 \right] dx dy \quad (5.46)$$

Similarly, recalling of Equation (5.30 c) and substitution of the expressions of shear strain's results in expressing the shear strain energy in terms of displacements by:

$$U^{SH03} = \frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} K \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{Bmatrix} z^2 dz \begin{Bmatrix} -C_2 \left(\phi_y + \frac{\partial w}{\partial y} \right) \\ -C_2 \left(\phi_x + \frac{\partial w}{\partial x} \right) \end{Bmatrix} dxdy \quad (5.47)$$

Using Eq.(5.15) in Eqs.(4.44) yields :

$$U^{SH03} = \frac{1}{2} \iint_A \int_{-\frac{t}{2}}^{\frac{t}{2}} K \begin{bmatrix} D_{44} & D_{45} \\ D_{54} & D_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \frac{\partial w}{\partial y} \\ \phi_x + \frac{\partial w}{\partial x} \end{Bmatrix} \begin{Bmatrix} -C_2 \left(\phi_y + \frac{\partial w}{\partial y} \right) \\ -C_2 \left(\phi_x + \frac{\partial w}{\partial x} \right) \end{Bmatrix} dxdy \quad (5.48)$$

respectively, results in expressing the coupling transverse shear strain energy as:

$$U^{SH03} = -C_2 \int \int_x \left[D_{44} \left(\phi_y + \frac{\partial w}{\partial y} \right)^2 + 2D_{45} \left(\phi_x + \frac{\partial w}{\partial x} \right) \left(\phi_y + \frac{\partial w}{\partial y} \right) + D_{55} \left(\phi_x + \frac{\partial w}{\partial x} \right)^2 \right] \quad (5.49)$$

5.3 Ritz Method Formulation Based on HSDT

In order to facilitate the formulation of Ritz method, the indicial notation was employed during the extensive derivation of the equations lead to symbolic function used for integrating the polynomials, helped in speeding up the computation process. The use of indicial notation helped in simplify and streamline the mathematical expressions, making

the derivation process more manageable and efficient [118–120]. As per requirement of Ritz method, the initial phase in formulating the matrix form is to approximate the solutions for u , v , and w as follows:

$$\left. \begin{aligned} u &= \sum_{i=0}^{N_u} C_i^u F_i^u(x, y) \\ v &= \sum_{j=0}^{N_v} C_j^v F_j^v(x, y) \\ w &= \sum_{k=0}^{N_w} C_k^w F_k^w(x, y) \\ \phi_x &= \sum_{n=0}^{N_{\phi_x}} C_n^{\phi_x} F_n^{\phi_x}(x, y) \\ \phi_y &= \sum_{m=0}^{N_{\phi_y}} C_m^{\phi_y} F_m^{\phi_y}(x, y) \end{aligned} \right\} \quad (5.50)$$

where $F_i^u, F_j^v, F_k^w, F_n^{\phi_x}$ and $F_m^{\phi_y}$ denotes the approximation trial functions that satisfy the geometric boundary conditions, $N_u, N_v, N_w, N_{\phi_x}$ and N_{ϕ_y} are the number of terms for each of the three trial functions, and $C_i^u, C_j^v, C_k^w, C_n^{\phi_x}$ and $C_m^{\phi_y}$ are unknown coefficients need to be determined.

Using the trial functions in Eq. (5.50) and then applying the minimization of potential energy by taking the derivative of Π with respect to each coefficient, where $\partial\Pi/\partial C_i^u = 0$, $\partial\Pi/\partial C_j^v = 0$, and $\partial\Pi/\partial C_k^w = 0$. After tackling the lengthy derivations, the resultant expressions of Ritz method can be cast in the following matrix form:

$$\begin{bmatrix} K_{11}^{ri} & K_{12}^{rj} & K_{13}^{rk} & K_{14}^{rn} & K_{15}^{rm} \\ K_{21}^{si} & K_{22}^{sj} & K_{23}^{sk} & K_{24}^{sn} & K_{25}^{sm} \\ K_{31}^{ti} & K_{32}^{tj} & K_{33}^{tk} & K_{34}^{tn} & K_{35}^{tm} \\ K_{41}^{pi} & K_{42}^{pj} & K_{43}^{pk} & K_{44}^{pn} & K_{45}^{pm} \\ K_{51}^{qi} & K_{52}^{qj} & K_{53}^{qk} & K_{54}^{qn} & K_{55}^{qm} \end{bmatrix} \begin{bmatrix} C_i^u \\ C_j^v \\ C_k^w \\ C_n^{\phi_x} \\ C_m^{\phi_y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \iint_A q \psi_t dA \\ 0 \\ 0 \end{bmatrix} \quad (5.51)$$

where :

$$K_{11}^{ri} = \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_r^u}{\partial x} A_{11} + \frac{\partial F_r^u}{\partial y} A_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_r^u}{\partial x} A_{16} + \frac{\partial F_r^u}{\partial y} A_{66} \right) \right] dA \quad (5.52a)$$

$$K_{12}^{rj} = \iint_A \left[\frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_r^u}{\partial x} A_{12} + \frac{\partial F_r^u}{\partial y} A_{26} \right) + \frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_r^u}{\partial x} A_{16} + \frac{\partial F_r^u}{\partial y} A_{66} \right) \right] dA \quad (5.52b)$$

$$\begin{aligned} K_{13}^{rk} = -C_1 \int_y \int_x & \left[\frac{\partial^2 F_k^w}{\partial x^2} \left(\frac{\partial F_r^u}{\partial x} E_{11} + \frac{\partial F_r^u}{\partial y} E_{16} \right) \right. \\ & + \frac{\partial^2 F_k^w}{\partial y^2} \left(\frac{\partial F_r^u}{\partial x} E_{12} + \frac{\partial F_r^u}{\partial y} E_{26} \right) \\ & \left. + 2 \frac{\partial^2 F_k^w}{\partial x \partial y} \left(\frac{\partial F_r^u}{\partial x} E_{16} + \frac{\partial F_r^u}{\partial y} E_{66} \right) \right] dA \end{aligned} \quad (5.52c)$$

$$\begin{aligned} K_{14}^{rn} = \int_y \int_x & \left[\frac{\partial F_n^{\phi_x}}{\partial x} \left(\frac{\partial F_r^u}{\partial x} (B_{11} - C_1 E_{11}) + \frac{\partial F_r^u}{\partial y} (B_{16} - C_1 E_{16}) \right) \right. \\ & \left. + \frac{\partial F_n^{\phi_x}}{\partial y} \left(\frac{\partial F_r^u}{\partial x} (B_{16} - C_1 E_{16}) + \frac{\partial F_r^u}{\partial y} (B_{66} - C_1 E_{66}) \right) \right] dA \end{aligned} \quad (5.52d)$$

$$\begin{aligned}
K_{15}^{rm} = & \int_y \int_x \left[\frac{\partial F_m^{\phi_y}}{\partial y} \left(\frac{\partial F_r^u}{\partial x} (B_{12} - C_1 E_{12}) + \frac{\partial F_r^u}{\partial y} (B_{26} - C_1 E_{26}) \right) \right. \\
& \left. + \frac{\partial F_m^{\phi_y}}{\partial x} \left(\frac{\partial F_r^u}{\partial x} (B_{16} - C_1 E_{16}) + \frac{\partial F_r^u}{\partial y} (B_{66} - C_1 E_{66}) \right) \right] dA \quad (5.52e)
\end{aligned}$$

$$K_{21}^{si} = \iint_A \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_s^v}{\partial y} A_{12} + \frac{\partial F_s^v}{\partial x} A_{16} \right) + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_s^v}{\partial y} A_{26} + \frac{\partial F_s^v}{\partial x} A_{66} \right) \right] dA \quad (5.52f)$$

$$K_{22}^{sj} = \iint_A \left[\frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_s^v}{\partial y} A_{26} + \frac{\partial F_s^v}{\partial x} A_{66} \right) + \frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_s^v}{\partial x} A_{26} + \frac{\partial F_s^v}{\partial y} A_{22} \right) \right] dA \quad (5.52g)$$

$$\begin{aligned}
K_{23}^{sk} = & -C_1 \int_y \int_x \left[\frac{\partial^2 F_k^w}{\partial x^2} \left(\frac{\partial F_s^v}{\partial x} E_{16} + \frac{\partial F_s^v}{\partial y} E_{12} \right) \right. \\
& + \frac{\partial^2 F_k^w}{\partial y^2} \left(\frac{\partial F_s^v}{\partial x} E_{26} + \frac{\partial F_s^v}{\partial y} E_{22} \right) \\
& \left. + 2 \frac{\partial^2 F_k^w}{\partial x \partial y} \left(\frac{\partial F_s^v}{\partial x} E_{66} + \frac{\partial F_s^v}{\partial y} E_{26} \right) \right] dA \quad (5.52h)
\end{aligned}$$

$$\begin{aligned}
K_{24}^{sn} = & \int_y \int_x \left[\frac{\partial F_n^{\phi_x}}{\partial x} \left(\frac{\partial F_s^v}{\partial y} (B_{12} - C_1 E_{12}) + \frac{\partial F_s^v}{\partial x} (B_{16} - C_1 E_{16}) \right) \right. \\
& \left. + \frac{\partial F_n^{\phi_x}}{\partial y} \left(\frac{\partial F_s^v}{\partial y} (B_{26} - C_1 E_{26}) + \frac{\partial F_s^v}{\partial x} (B_{66} - C_1 E_{66}) \right) \right] dA \quad (5.52i)
\end{aligned}$$

$$\begin{aligned}
K_{25}^{sm} = \int_y \int_x & \left[\frac{\partial F_m^{\phi_y}}{\partial y} \left(\frac{\partial F_s^v}{\partial x} (B_{26} - C_1 E_{26}) + \frac{\partial F_s^v}{\partial y} (B_{22} - C_1 E_{22}) \right) \right. \\
& \left. + \frac{\partial F_m^{\phi_y}}{\partial x} \left(\frac{\partial F_s^v}{\partial y} (B_{26} - C_1 E_{26}) + \frac{\partial F_s^v}{\partial x} (B_{66} - C_1 E_{66}) \right) \right] dA
\end{aligned} \tag{5.52j}$$

$$\begin{aligned}
K_{31}^{ti} = -C_1 \int_y \int_x & \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial^2 F_t^w}{\partial x^2} E_{11} + \frac{\partial^2 F_t^w}{\partial y^2} E_{12} + 2 \frac{\partial^2 F_t^w}{\partial x \partial y} E_{16} \right) \right. \\
& \left. + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial^2 F_t^w}{\partial x^2} E_{16} + \frac{\partial^2 F_t^w}{\partial y^2} E_{26} + 2 \frac{\partial^2 F_t^w}{\partial x \partial y} E_{66} \right) \right] dA
\end{aligned} \tag{5.52k}$$

$$\begin{aligned}
K_{32}^{tj} = -C_1 \int_y \int_x & \left[\frac{\partial F_j^v}{\partial x} \left(2 \frac{\partial^2 F_t^w}{\partial x \partial y} E_{66} + \frac{\partial^2 F_t^w}{\partial x^2} E_{16} + \frac{\partial^2 F_t^w}{\partial y^2} E_{26} \right) \right. \\
& \left. + \frac{\partial F_j^v}{\partial y} \left(2 \frac{\partial^2 F_t^w}{\partial x \partial y} E_{26} + \frac{\partial^2 F_t^w}{\partial x^2} E_{12} + \frac{\partial^2 F_t^w}{\partial y^2} E_{22} \right) \right] dA
\end{aligned} \tag{5.52l}$$

$$\begin{aligned}
K_{33}^{tk} = & \int_y \int_x \left[\frac{\partial F_k^w}{\partial x} \left(\frac{\partial F_t^w}{\partial x} (A_{55} + C_2^2 F_{55} - 2C_2 D_{55}) \right. \right. \\
& + \frac{\partial F_t^w}{\partial y} (A_{45} + C_2^2 F_{45} - 2C_2 D_{45}) \Big) \\
& + \frac{\partial F_k^w}{\partial y} \left(\frac{\partial F_t^w}{\partial x} (A_{45} + C_2^2 F_{45} - 2C_2 D_{45}) \right. \\
& + \frac{\partial F_t^w}{\partial y} (A_{44} + C_2^2 F_{44} - 2C_2 D_{44}) \Big) \\
& + C_1^2 \left(\frac{\partial^2 F_k^w}{\partial x^2} \left(\frac{\partial^2 F_t^w}{\partial x^2} H_{11} + \frac{\partial^2 F_t^w}{\partial y^2} H_{12} + 2 \frac{\partial^2 F_t^w}{\partial x \partial y} H_{16} \right) \right. \\
& + \frac{\partial^2 F_k^w}{\partial y^2} \left(\frac{\partial^2 F_t^w}{\partial x^2} H_{12} + \frac{\partial^2 F_t^w}{\partial y^2} H_{22} + 2 \frac{\partial^2 F_t^w}{\partial x \partial y} H_{26} \right) \\
& \left. \left. + 2 \frac{\partial^2 F_k^w}{\partial x \partial y} \left(\frac{\partial^2 F_t^w}{\partial x^2} H_{16} + \frac{\partial^2 F_k^w}{\partial x^2} H_{16} + 2 \frac{\partial^2 F_t^w}{\partial x \partial y} H_{66} \right) \right) \right] dx dy
\end{aligned}
\tag{5.52m}$$

$$\begin{aligned}
K_{34}^{tn} = & \int_y \int_x \left[F_n^{\phi_x} \left(\frac{\partial F_t^w}{\partial y} (A_{45} + (C_2)^2 F_{45} - 2C_2 D_{45}) \right. \right. \\
& + \frac{\partial F_t^w}{\partial x} (A_{55} + (C_2)^2 F_{55} - 2C_2 D_{55}) \left. \right) \\
& + \frac{\partial F_n^{\phi_x}}{\partial x} \left(\frac{\partial^2 F_t^w}{\partial x^2} (C_1^2 H_{11} - C_2 F_{11}) + \frac{\partial^2 F_t^w}{\partial y^2} (C_2^2 H_{12} - C_1 F_{12}) \right. \\
& + 2 \frac{\partial^2 F_t^w}{\partial x \partial y} (C_1^2 H_{16} - C_1 F_{16}) \left. \right) \\
& + \frac{\partial F_n^{\phi_x}}{\partial y} \left(\frac{\partial^2 F_t^w}{\partial x^2} (C_1^2 H_{16} - C_1 F_{16}) + \frac{\partial^2 F_t^w}{\partial y^2} (C_1^2 H_{26} - C_1 F_{26}) \right. \\
& + 2 \frac{\partial^2 F_t^w}{\partial x \partial y} (C_1^2 H_{66} - C_1 F_{66}) \left. \right) \left. \right] dx dy
\end{aligned} \tag{5.52n}$$

$$\begin{aligned}
K_{35}^{tm} = & \int_y \int_x \left[F_m^{\phi_y} \left(\frac{\partial F_t^w}{\partial y} (A_{44} + (C_2)^2 F_{44} - 2C_2 D_{44}) \right. \right. \\
& + \left. \frac{\partial F_t^w}{\partial x} (A_{45} + (C_2)^2 F_{45} - 2C_2 D_{45}) \right) \\
& + \frac{\partial F_m^{\phi_y}}{\partial y} \left(\frac{\partial^2 F_t^w}{\partial x^2} (C_1^2 H_{12} - C_1 F_{12}) + \frac{\partial^2 F_t^w}{\partial y^2} (C_1^2 H_{22} - C_1 F_{22}) \right. \\
& + \left. 2 \frac{\partial^2 F_t^w}{\partial x \partial y} (C_1^2 H_{26} - C_1 F_{26}) \right) \\
& + \frac{\partial F_m^{\phi_y}}{\partial x} \left(\frac{\partial^2 F_t^w}{\partial x^2} (C_1^2 H_{16} - C_1 F_{16}) + \frac{\partial^2 F_t^w}{\partial y^2} (C_1^2 H_{16} - C_1 F_{26}) \right. \\
& + \left. \left. 2 \frac{\partial^2 F_t^w}{\partial x \partial y} (C_1^2 H_{66} - C_1 F_{66}) \right) \right] dA \tag{5.52o}
\end{aligned}$$

$$\begin{aligned}
K_{41}^{pi} = & \int_y \int_x \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (B_{11} - C_1 E_{11}) + \frac{\partial F_p^{\phi_x}}{\partial y} (B_{16} - C_1 E_{16}) \right) \right. \\
& + \left. \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (B_{16} - C_1 E_{16}) + \frac{\partial F_p^{\phi_x}}{\partial y} (B_{66} - C_1 E_{66}) \right) \right] dA \tag{5.52p}
\end{aligned}$$

$$\begin{aligned}
K_{42}^{pj} = & \int_y \int_x \left[\frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (B_{16} - C_1 E_{16}) + \frac{\partial F_p^{\phi_x}}{\partial y} (B_{66} - C_1 E_{66}) \right) \right. \\
& + \left. \frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (B_{12} - C_1 E_{12}) + \frac{\partial F_p^{\phi_x}}{\partial y} (B_{26} - C_1 E_{26}) \right) \right] dA \tag{5.52q}
\end{aligned}$$

$$\begin{aligned}
K_{43}^{pk} = & \int_y \int_x \left[F_p^{\phi_x} \left(\frac{\partial F_k^w}{\partial y} (A_{45} + (C_2)^2 F_{45} - 2C_2 D_{45}) \right. \right. \\
& + \frac{\partial F_k^w}{\partial x} (A_{55} + (C_2)^2 F_{55} - 2C_2 D_{55}) \Big) \\
& + \frac{\partial F_p^{\phi_x}}{\partial x} \left(\frac{\partial^2 F_k^w}{\partial x^2} (H_{11} C_1^2 - C_1 F_{11}) + \frac{\partial^2 F_k^w}{\partial y^2} (C_1^2 H_{12} - C_1 F_{12}) \right. \\
& + 2 \frac{\partial^2 F_k^w}{\partial x \partial y} (H_{16} C_1^2 - C_1 F_{16}) \Big) \\
& + \frac{\partial F_p^{\phi_x}}{\partial y} \left(\frac{\partial^2 F_k^w}{\partial x^2} (C_1^2 H_{16} - C_1 F_{16}) + \frac{\partial^2 F_k^w}{\partial y^2} (C_1^2 H_{26} - C_1 F_{26}) \right. \\
& + 2 \frac{\partial^2 F_k^w}{\partial x \partial y} (C_1^2 H_{66} - C_1 F_{66}) \Big) \Big] dA
\end{aligned} \tag{5.52r}$$

$$\begin{aligned}
K_{44}^{pn} = & \int_y \int_x \left[\frac{\partial F_n^{\phi_x}}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (D_{11} + C_1^2 H_{11} - 2C_1 F_{11}) \right. \right. \\
& + \frac{\partial F_p^{\phi_x}}{\partial y} (D_{16} + C_1^2 H_{16} - 2C_1 F_{16}) \Big) \\
& + \frac{\partial F_n^{\phi_x}}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (D_{16} + C_1^2 H_{16} - 2C_1 F_{16}) \right. \\
& + \frac{\partial F_p^{\phi_x}}{\partial y} (D_{66} + C_1^2 H_{66} - 2C_1 F_{66}) \Big) \\
& \left. + F_p^{\phi_x} F_n^{\phi_x} (A_{55} + (C_2)^2 F_{55} - 2C_2 D_{55}) \right] dA
\end{aligned} \tag{5.52s}$$

$$\begin{aligned}
K_{45}^{pm} = \int_y \int_x & \left[\frac{\partial F_m^{\phi_y}}{\partial y} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (D_{12} + C_1^2 H_{12} - 2C_1 F_{12}) \right. \right. \\
& + \frac{\partial F_p^{\phi_x}}{\partial y} (D_{26} + C_1^2 H_{26} - 2C_1 F_{26}) \Big) \\
& + \frac{\partial F_m^{\phi_y}}{\partial x} \left(\frac{\partial F_p^{\phi_x}}{\partial x} (D_{16} + C_1^2 H_{16} - 2C_1 F_{16}) \right. \\
& + \frac{\partial F_p^{\phi_x}}{\partial y} (D_{66} + C_1^2 H_{66} - 2C_1 F_{66}) \Big) \\
& \left. + F_p^{\phi_x} F_m^{\phi_y} (A_{45} + (C_2)^2 F_{45} - 2C_2 D_{45}) \right] dA \tag{5.52t}
\end{aligned}$$

$$\begin{aligned}
K_{51}^{qi} = \int_y \int_x & \left[\frac{\partial F_i^u}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} (B_{12} - C_1 E_{12}) + \frac{\partial F_q^{\phi_y}}{\partial x} (B_{16} - C_1 E_{16}) \right) \right. \\
& \left. + \frac{\partial F_i^u}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial y} (B_{26} - C_1 E_{26}) + \frac{\partial F_q^{\phi_y}}{\partial x} (B_{66} - C_1 E_{66}) \right) \right] dA \tag{5.52u}
\end{aligned}$$

$$\begin{aligned}
K_{52}^{qj} = \int_y \int_x & \left[\frac{\partial F_j^v}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} (B_{26} - C_1 E_{26}) + \frac{\partial F_q^{\phi_y}}{\partial x} (B_{66} - C_1 E_{66}) \right) \right. \\
& \left. + \frac{\partial F_j^v}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial x} (B_{26} - C_1 E_{26}) + \frac{\partial F_q^{\phi_y}}{\partial y} (B_{22} - C_1 E_{22}) \right) \right] dA
\end{aligned}
\tag{5.52v}$$

$$\begin{aligned}
K_{53}^{qk} = \int_y \int_x & \left[F_q^{\phi_y} \left(\frac{\partial F_k^w}{\partial y} (A_{44} + (C_2)^2 F_{44} - 2C_2 D_{44}) \right. \right. \\
& \left. + \frac{\partial F_k^w}{\partial x} (A_{45} + (C_2)^2 F_{45} - 2C_2 D_{45}) \right) \\
& + \frac{\partial F_q^{\phi_y}}{\partial x} \left(\frac{\partial^2 F_k^w}{\partial x^2} (C_1^2 H_{16} - C_1 F_{16}) + \frac{\partial^2 F_k^w}{\partial y^2} (C_1^2 H_{26} - C_1 F_{26}) \right. \\
& \left. + 2 \frac{\partial^2 F_k^w}{\partial x \partial y} (C_1^2 H_{66} - C_1 F_{66}) \right) \\
& + \frac{\partial F_q^{\phi_y}}{\partial y} \left(\frac{\partial^2 F_k^w}{\partial x^2} (C_1^2 H_{12} - C_1 F_{12}) + \frac{\partial^2 F_k^w}{\partial y^2} (C_1^2 H_{22} - C_1 F_{22}) \right. \\
& \left. \left. + 2 \frac{\partial^2 F_k^w}{\partial x \partial y} (C_1^2 H_{26} - C_1 F_{26}) \right) \right] dA
\end{aligned}
\tag{5.52w}$$

$$\begin{aligned}
K_{54}^{qn} = \int_y \int_x & \left[\frac{\partial F_n^{\phi_x}}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} (D_{12} + C_1^2 H_{12} - 2C_1 F_{12}) \right. \right. \\
& \left. \left. + \frac{\partial F_q^{\phi_y}}{\partial x} (D_{16} + C_1^2 H_{16} - 2C_1 F_{16}) \right) \right. \\
& \left. + \frac{\partial F_n^{\phi_x}}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial y} (D_{26} + H_{26} C_1^2 - 2C_1 F_{26}) \right. \right. \\
& \left. \left. + \frac{\partial F_q^{\phi_y}}{\partial x} (D_{66} + H_{66} C_1^2 - 2C_1 F_{66}) \right) \right. \\
& \left. + F_q^{\phi_y} F_n^{\phi_x} (A_{45} + (C_2)^2 F_{45} - 2C_2 D_{45}) \right] dA
\end{aligned}
\tag{5.52y}$$

$$\begin{aligned}
K_{55}^{qm} = \int_y \int_x \left[\frac{\partial F_m^{\phi_y}}{\partial x} \left(\frac{\partial F_q^{\phi_y}}{\partial y} (D_{26} + C_1^2 H_{26} - 2C_1 F_{26}) \right. \right. \\
+ \frac{\partial F_q^{\phi_y}}{\partial x} (D_{66} + C_1^2 H_{66} - 2C_1 F_{66}) \Big) \\
+ \frac{\partial F_m^{\phi_y}}{\partial y} \left(\frac{\partial F_q^{\phi_y}}{\partial x} (D_{26} + H_{26} C_1^2 - 2C_1 F_{26}) \right. \\
+ \frac{\partial F_q^{\phi_y}}{\partial y} (D_{22} + H_{22} C_1^2 - 2C_1 F_{22}) \Big) \\
\left. + F_q^{\phi_y} F_m^{\phi_y} (A_{44} + (C_2)^2 F_{44} - 2C_2 D_{44}) \right] dA
\end{aligned}
\tag{5.52x}$$

where the indices take their values according to:

$$r, i = 1, N_u \quad s, j = 1, N_v \quad t, k = 1, N_w \quad p, n = 1, N_{\phi_x} \text{ and } q, m = 1, N_{\phi_y}$$

The derivations provided above accounts for the coupling effect and it can be employed to study the behavior of sinusoidally loaded thick plates with general anisotropic properties and different boundary conditions. The derivation and computations of all integrals and the solution of the resulting linear equations are carried out using Mathematica software as discussed in detail in Appendix C.

5.4 Numerical Examples and Discussions

Different sets of numerical examples are presented in this section to verify the capability of the proposed method and its accuracy in predicting the bending behavior of sinusoidally loaded rectangular thick laminated composite plate. The results of the proposed method are compared with those obtained by Reddy.

In the presented examples, the proposed method is applied to analyze a rectangular laminated composite plate as depicted in Figure 39. The plate has different boundary conditions.

In accordance with the devised formulation of the Ritz Method, the approximation functions only need to fulfill the geometric boundary conditions. Consequently, the following polynomial trial functions are employed in the preceding analysis. These trial functions satisfy and accurately represent the deformation shape of sinusoidally loaded composite plates. three categories of stacking patterns are considered in this study: symmetric/antisymmetric cross-ply and antisymmetric angle-ply laminates. The trial functions are given by:

$$\left. \begin{aligned} u &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^u (a^2 - x^2)(b^2 - y^2) x^i y^j \\ v &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^v (a^2 - x^2)(b^2 - y^2) x^i y^j \\ w &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^w (a^2 - x^2)^h (b^2 - y^2)^h x^i y^j \\ \phi x &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^{\phi x} (a^2 - x^2)(b^2 - y^2) x^i y^j \\ \phi y &= \sum_{j=0}^n \sum_{i=0}^m C_{ij}^{\phi y} (a^2 - x^2)(b^2 - y^2) x^i y^j \end{aligned} \right\} \quad (5.53)$$

- Where :
- Edges fully simply supported, $h = 1$
- Edges fully clamped, $h = 2$
- Edges free to move in all directions, $h = 0$

In each of the following examples, numerical results for both the proposed method and Reddy solution are obtained for plates having side dimensions of $a = 1m$ and $b = 1m$ with side to thickness ratio of $a/h = 4, 5, 10, 20, 50$ and 100 . The elastic material properties are Poison's ratio $\nu = 0.25$, the elastic modulus $E_1 = 25GPa$, $E_2 = 1GPa$, and the modulus of rigidity $G_{12} = 0.2GPa$.

The values reported in the forgoing tables and figures are for the normalized deflection and stresses evaluated, respectively, at $(0, 0)$ and $(0, 0, h/2)$ while $(0, 0, h/4)$ for $\bar{\sigma}_y$ using the following expressions:

$$\left. \begin{aligned}
\bar{w} &= w \times \frac{E_2 h^3}{a^4 q_0} \\
\bar{\sigma}_x &= \sigma_x \times \frac{h^2}{a^2 q_0} \\
\bar{\sigma}_y &= \sigma_y \times \frac{h^2}{a^2 q_0} \\
\bar{\tau}_{xy} &= \tau_{xy} \times \frac{h^2}{a^2 q_0} \\
\bar{\tau}_{xz} &= \tau_{xz} \times \frac{h}{a q_0} \\
\bar{\tau}_{yz} &= \tau_{yz} \times \frac{h}{a q_0}
\end{aligned} \right\} \quad (5.54)$$

5.4.1 Cross ply laminates

A cross-ply laminated composite square plate with simply supported boundary conditions is investigated using the proposed method by employing the trial polynomial function, respectively, for symmetric and antisymmetric stacking. The analysis is performed for plates composed of different number of layers; 3, 4, and 6 layers in the case of antisymmetric stacking and symmetric stacking.

Table 13, 14 and 15 shows the results for normalized deflection at the center of the plates $(0, 0)$ and stresses at point coordinates of $(0, 0, h/2)$. The results clearly illustrate full agreement between the proposed method results and those obtained by Reddy analysis for all the investigated cases.

When considering the influence of the number of layers on the deflection values in Table 15 and Figure 40, it is evident that both the Navier Solution and the Ritz Solution exhibit consistent behavior as the number of layers increases. For both solutions, the deflection values decrease as the number of layers increases. This behavior aligns with the expected behaviour in composite laminates, where a higher number of layers enhances stiffness and reduces deflection.

Table 13 : Normalized deflection and stresses for cross ply simply supported square plate
[(0°/90°/90°/0°)]

a/h	Reddy Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	1.894	0.665	0.632	0.0440	0.216	0.220
10	0.715	0.546	0.389	0.0268	0.260	0.160
20	0.506	0.539	0.304	0.0228	0.282	0.113
100	0.434	0.539	0.271	0.0213	0.290	0.112
a/h	Ritz Solution					
	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	$\bar{\tau}_{xz}$	$\bar{\tau}_{yz}$
4	1.895	0.665	0.632	0.0450	0.206	0.239
10	0.715	0.546	0.389	0.0273	0.264	0.153
20	0.506	0.539	0.304	0.0230	0.282	0.120
100	0.434	0.539	0.271	0.0216	0.290	0.112

Table 14 : Normalized deflection and in-plane stress for cross ply simply supported square plate [0°/90°/0°]

a/h	Reddy Solution	
	$\bar{w}$	$\bar{\sigma}_x$
4	1.9218	0.7345
10	0.7125	0.5684
100	0.4342	0.5390
a/h	Ritz Solution	
	w	$\bar{\sigma}_x$
4	1.9218	0.7342
10	0.7125	0.5682
100	0.4342	0.539

Table 15 : Normalized Deflection for Different layers 2,6 ($0^\circ/90^\circ/0^\circ/90^\circ \dots$) at (0, 0) for SSSS

a/h	Reddy Solution	
	$\bar{w}$	$\bar{w}$
N. of Layers	2	6
4	1.9985	1.5411
10	1.2161	0.6382
20	1.1018	0.5060
50	1.0697	0.4688
100	1.0651	0.4635
a/h	Ritz Solution	
	$\bar{w}$	$\bar{w}$
4	1.9985	1.5411
10	1.2161	0.6382
20	1.1018	0.5060
50	1.0697	0.4688
100	1.6501	0.4635

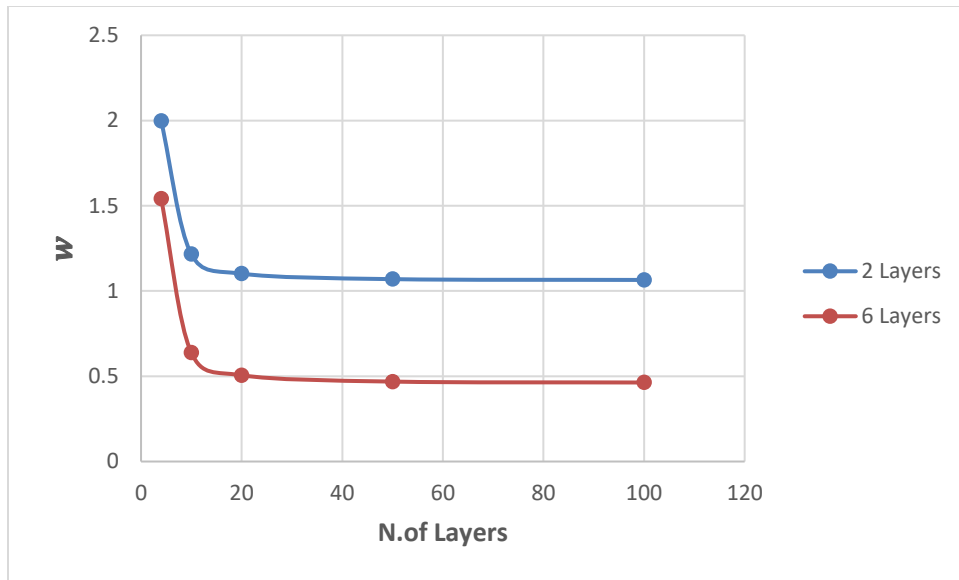


Figure 40 : Normalized Deflection for Different layers 2,6 ($0^\circ/90^\circ/0^\circ/90^\circ \dots$) for SSSS

5.4.2 Angle ply laminates

In the case of angle-ply stacking sequence, square plates with various lamina's fiber orientation (θ) and different number of layers are analyzed. The plies are stacked in symmetric anti-symmetric $[-\theta/\theta \dots -\theta/\theta]$ sequence, where θ takes the values of 5° , 30° , and 45° . For the stacking of each orientation angle, the plates are composed of 2 and 6 layers.

The obtained results in Table 16 present that the maximum normalized deflection occurs either at the center of the plate. Regardless of the number of layers, the deflection is found to be maximum at $\theta = 5^\circ$. All the results demonstrate excellent agreement between the current solution and Reddy solution for both deflection values.

Table 16 : Normalized deflection for angle ply simply supported square plate $[(-\theta^\circ)/\theta^\circ \dots (-\theta^\circ)/\theta^\circ]$

a/h	Reddy Solution		Reddy Solution		Reddy Solution	
	$\bar{w}$	$\bar{w}$	$\bar{w}$	$\bar{w}$	$\bar{w}$	$\bar{w}$
$-\theta/\theta$	$\theta = 5$		$\theta = 30$		$\theta = 45$	
N. of Layers	2	6	2	6	2	6
4	1.2625	1.2282	1.0838	0.8851	1.0203	0.8375
10	0.4848	0.4485	0.5916	0.3007	0.5581	0.2745
20	0.3579	0.3209	0.5180	0.2127	0.4897	0.1905
50	0.3215	0.2842	0.4972	0.1878	0.4704	0.1668
100	0.3162	0.2789	0.4942	0.1842	0.4676	0.1634
a/h	Ritz Solution		Ritz Solution		Ritz Solution	
	$\bar{w}$	$\bar{w}$	$\bar{w}$	$\bar{w}$	$\bar{w}$	$\bar{w}$
4	1.2625	1.2281	1.0837	0.8851	1.0203	0.8375
10	0.4848	0.4485	0.5916	0.3007	0.5581	0.2745
20	0.3579	0.3209	0.5180	0.2127	0.4896	0.1904
50	0.3215	0.2842	0.4972	0.1878	0.4704	0.1668
100	0.3162	0.2789	0.4942	0.1842	0.4676	0.1634

5.4.3 Impact of varied boundaries and number of layers on deflection and stresses

Table 17 and Table 18 show the influence of different boundaries and number of layers on plate's with ($a/h = 5$) deflection and stresses. Notably, it is evident that the deflection signifying an optimal value for **SSFF** case, while it's fluctuating for stresses. Visual understanding of how different layers and boundaries impact the deflection and stresses performance of the plates. The minimum deflection was observed at **SSCC**. Also, a consistent decreasing in deflection was observed as the number of layers increase.

Table 17 : Normalized deflection and stresses for cross ply square plate ($0^\circ/90^\circ$) (2 layers) and $a/h=5$

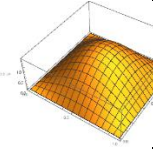
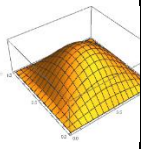
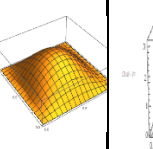
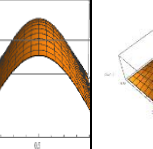
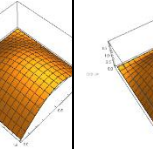
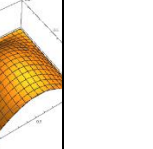
<i>Variables</i>	Reddy Solution					
	SSSS	SSSC	SSCC	SSFF	SSFS	SSFC
Deformed Shape						
$\bar{w}$	1.667	1.333	1.088	2.624	2.211	1.733
$\bar{\sigma}_x$	8.385	6.816	5.679	3.171	5.349	3.727
$\bar{\sigma}_y$	8.385	6.725	5.505	13.551	11.324	8.919
$\bar{\tau}_{yz}$	3.155	2.543	2.095	4.457	3.893	3.048
<i>Variables</i>	Ritz Solution					
	SSSS	SSSC	SSCC	SSFF	SSFS	SSFC
$\bar{w}$	1.667	1.320	1.088	2.624	2.2109	1.755
$\bar{\sigma}_x$	8.382	6.933	5.600	3.133	5.358	3.713
$\bar{\sigma}_y$	8.382	6.623	5.440	13.540	11.316	8.964
$\bar{\tau}_{yz}$	3.157	2.523	2.168	4.430	3.840	3.084

Table 18 : Normalized deflection and stresses for cross ply square plate ($0^\circ/90^\circ$) (10 Layers) and $a/h=5$

<i>Variables</i>	Reddy Solution					
	<i>SSSS</i>	<i>SSSC</i>	<i>SSCC</i>	<i>SSFF</i>	<i>SSFS</i>	<i>SSFC</i>
$\bar{w}$	1.129	1.001	0.879	1.651	1.450	1.214
$\bar{\sigma}_x$	6.340	5.196	4.025	2.482	3.946	2.608
$\bar{\sigma}_y$	6.340	5.635	4.963	9.545	8.525	6.934
$\bar{\tau}_{yz}$	3.362	2.974	2.601	4.784	4.234	3.535
<i>Variables</i>	Ritz Solution					
	<i>SSSS</i>	<i>SSSC</i>	<i>SSCC</i>	<i>SSFF</i>	<i>SSFS</i>	<i>SSFC</i>
$\bar{w}$	1.129	0.988	0.880	1.634	1.449	1.193
$\bar{\sigma}_x$	6.340	5.185	4.340	2.400	3.960	2.554
$\bar{\sigma}_y$	6.340	5.593	4.863	9.423	8.458	6.899
$\bar{\tau}_{yz}$	3.365	2.950	2.561	4.720	4.183	3.532

CHAPTER 6

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The aim of this research is to study the behavior of anisotropic rectangular thin & thick laminated composite plate subjected to uniform lateral, sinusoidally loaded and having different boundary conditions and stacking sequences. The classical plate theory (CLPT) has been employed to describe the behavior of thin anisotropic laminated composite plate, The first order shear deformation theory (FSDT) and the high order shear deformation theory has been employed to describe the behavior of thick anisotropic laminated composite plate and the analysis was performed using Ritz method. This chapter summarizes the main accomplishments of the conducted research and outlines the conclusions drawn from the obtained results. It also provides some directions of future research.

6.1 Summary

The potential energy of anisotropic thin and thick laminated composite plates has been derived in $u-v-w$ and $u-v-w-\phi_x-\phi_y$ formulation. Then, the principle of minimization potential energy using Ritz method was cast in matrix forms utilizing the advantage of indicial notations. This results in system of equations in terms of the unknown coefficients to be determined. The developed matrix forms of Ritz method are generalized and can be employed for different lateral, sinusoidally loads and complicated boundary conditions

including free edges. Additionally, they allow for implementing of as many approximation polynomials as required for convergence of the solutions for the problem main variables $(u, v, w, \phi_x, \phi_y)$. Utilizing the symbolic capabilities of Mathematica software, computer codes have been developed to evaluate all the required derivatives and integrals of the polynomial approximation functions used in this investigation. The codes were also used to solve the system of equations using the direct built-in functions (FindRoots) available in Mathematica software and, then, getting the solutions for u, v, w, ϕ_x and ϕ_y in functional forms.

The developed codes have been tested through different numerical examples for thin and thick plates. Figure **41** highlights a particular example, providing a visual representation of the testing process and demonstrating the robustness of the developed codes. Normalized deflection values for a SSSS rectangular plate with varying aspect ratios (a/h) and three layers (0/90/0) under sinusoidally load and different plate theories, including HSDT, FSDT with shear correction factors $K=1$, $K=5/6$ and $K=3/4$, as well as (CLPT). The results indicate that as the aspect ratio increases, the normalized deflection generally decreases. Comparative analysis reveals that HSDT and FSDT ($K=1$) closely approximate each other, underscoring their effectiveness in capturing the plate's bending behavior. Differences become more apparent among FSDT theories with varying shear correction factors for higher aspect ratios, emphasizing the significant influence of the chosen K on predicted deflections. Additionally, CLPT consistently yields higher deflection values, suggesting potential oversimplification by neglecting shear deformations. Practical considerations highlight the accuracy between the theories, with HSDT offering high accuracy for ($a/h=4$) and FSDT ($K=5/6$) presenting a balanced solution for varying aspect ratios. For the aspect

ratio ($a/h=10$), which serves as a critical point in differentiating the theories. Moreover, when we consider the aspect ratio ($a/h=100$), all theories produce similar results. This emphasizes the reliability and accuracy of the developed codes. Consequently, the study highlights how important it is to choose the right plate theory depending on the (a/h) ratio. This ratio helps determine the most suitable approach for accurate results.

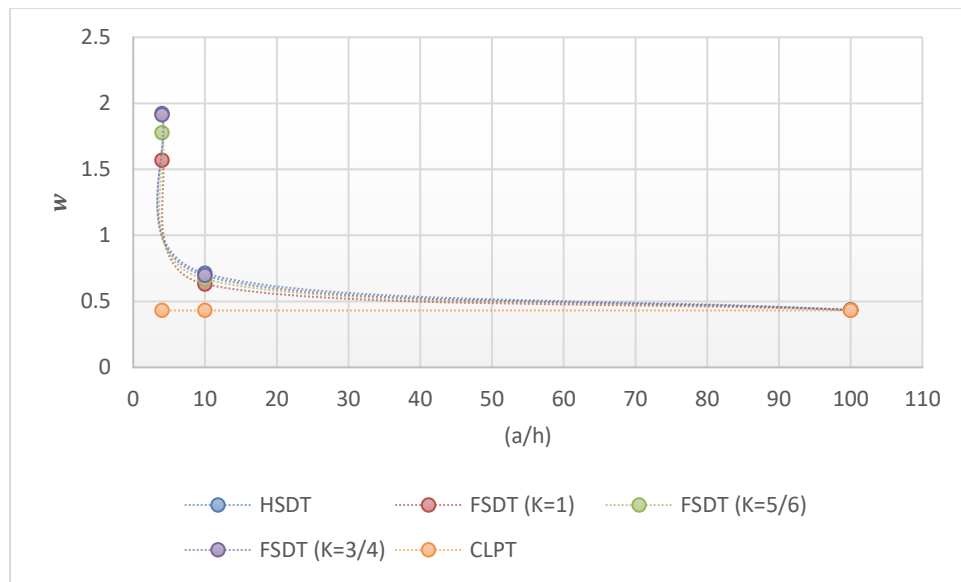


Figure 41 : Normalized Deflection Vs Different Aspects Ratios

Also, these codes tested different boundary conditions including free edges. The results of the investigated cases were compared with the published results from Reddy and FEM prepared models using the commercial software ABAQUS and COMSOL. The comparison showed excellent accuracy in terms of the deflection and stresses and verified the versatility of the proposed method in thin and thick rectangular plates with complicated boundary conditions.

Furthermore, getting the solution in a functional form allowed to emphasize more on the behavior of composite laminated by studying the variation of the plate variables across the thickness and the surface of the plate.

6.2 Conclusion

Based on the developed automation of Ritz method and the obtained results in this study, the following has been concluded:

1. Ritz method has been formulated in a matrix form for solving the problem of for bending behavior of thin and thick laminated composite plates in the sense of CLPT, FSDT and HSDT with different boundary conditions.
2. Casting Ritz method in a matrix form automates and speeds up the solution process and allows the accommodation of as many polynomial terms as required for solution convergence and accuracy.
3. Another important advantage of the automated Ritz method is its capability to be programmed easily using Mathematica and producing the solution of the approximation functions in terms of the displacements in functional symbolic form. This provides more accurate solutions over the plate dimensions when compared to other numerical methods, where interpolation would be required considering their built-in discrete nature.
4. The order of the polynomial terms and the number of terms are affected by the boundary conditions, degree of anisotropy and aspect ratio of the rectangular thin and thick plate.
5. Different combinations of numerical examples involving cross-ply, angle-ply are stacked in symmetric and anti-symmetric sequences to test the accuracy and the

versatility of the proposed method. The comparison of the present solution with FEM and the literature confirms the competency of the proposed automated Ritz method solution. Also, the effect of the fiber's orientation angles on the deflection has been investigated. The results show that the plates attain minimal deflection at an orientation angle of 45° , and 30° for square and rectangular ($b/a = 1/2$) thin plates, respectively.

6. For FSDT to work well, a search for the correct correction factor needs to be performed for each case with different boundary conditions and stacking sequence. This factor is crucial to ensure accurate and reliable results when applying FSDT in our analyses
7. The study highlights how important it is to choose the right plate theory depending on the plate dimension/thickness ratio (a/h) ratio. This ratio helps determine the most suitable approach for accurate results.
8. In general, the results conducted on composite laminated rectangular plates showed excellent convergence and accuracy in terms of deflection and stresses for different boundary conditions involving free edges.

6.3 Recommendations for Future Work

Finally, as the solution of anisotropic thin rectangular plate undergoing large deflection has been presented in this study, there is a scope for further application of the proposed method. Below are some of the recommendations that can be made:

1. Derivations of Ritz method in matrix form in polar coordinates allow extending the present method to plates with curved boundaries such as circular, semi-circular, triangular and polygonal sectorial plates.

2. The proposed method can be extended to plates subjected to buckling and dynamic loads.
3. Using the same approach, Ritz method may be developed to study the plate under combined lateral and axial loads.
4. Explore the integration of thermal effects into the Ritz method for a more comprehensive analysis.
5. Explore the potential of the Ritz method in predicting failure mechanisms, such as delamination or fiber breakage, and develop methodologies for assessing the structural integrity.
6. Anisotropic plate resting on elastic foundation is also a scope of research to be considered.

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Appendix A:

Derivation of Ritz Method in a Matrix Form for Thin

Laminated Composite Plate

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ClearAll["Global`*"]
(* This code "Intrec" is to evaluate the exact integral of \
polynomials quickly*)
Intrec[f_, a_, b_] :=
Module[{coeff, ii, jj, i, j},
coeff = CoefficientList[f, {x, y}]; {ii, jj} = Dimensions[coeff];
Sum[coeff[[i, j]] ((a/2)^i - (-a/2)^i)/i ((b/2)^j - (-b/2)^j)/
j, {i, 1, ii}, {j, 1, jj}];
(* Material prpoerties of composite antisymmetric laminated plate, \
Reddy Page 331*)

```

```

\[Theta]1 = {-30, 30, -30} Degree;
\[Nu] = 0.25; \[Nu]12 = \[Nu];
tt = 0.01; E1 = 25; E2 = 1; G12 = 0.5 E2 ;
a = 1; bb = {1.};
\[Nu]21 = (\[Nu]12 E2)/E1;
nL = Length[\[Theta]1];
tL = tt/nL;
\[Nu]21 = E2/E1 \[Nu]12;
Q11 = E1/(1 - \[Nu]12 \[Nu]21); Q22 = E2/(
1 - \[Nu]12 \[Nu]21); Q12 = (\[Nu]12 E2)/(
1 - \[Nu]12 \[Nu]21); Q66 = G12;

```

(* Calculation of rigidities of the resulted equivalent single layer*)

```

Q = {{Q11, Q12, 0}, {Q12, Q22, 0}, {0, 0, Q66}};
T = {{Cos[\[Theta]]^2, Sin[\[Theta]]^2,
2 Sin[\[Theta]] Cos[\[Theta]]}, {Sin[\[Theta]]^2,
Cos[\[Theta]]^2, -2 Sin[\[Theta]] Cos[\[Theta]]}, \
{-Sin[\[Theta]] Cos[\[Theta]], Sin[\[Theta]] Cos[\[Theta]],
Cos[\[Theta]]^2 - Sin[\[Theta]]^2}} ;
Qb = Inverse[T].Q.Transpose[Inverse[T]];
Do[z[k] = -(tt/2) + tL k, {k, 0, Length[\[Theta]1]}];
DD = AA = BB = Table[0, {i, 1, 3}, {j, 1, 3}];
Do[DD[[i, j]] =
1/3 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^3 -
z[k - 1]^3), {k, 1, Length[\[Theta]1]}], {i, 1, 3}, {j, 1, 3}];
Do[Qbb[k] = Qb /. \[Theta] -> \[Theta]1[[k]], {k, 1,
Length[\[Theta]1]}];

Do[AA[[i, j]] =
Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k] -
z[k - 1]), {k, 1, Length[\[Theta]1]}], {i, 1, 3}, {j, 1, 3}];
Do[BB[[i, j]] =
1/2 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^2 -

```

```

z[k - 1]^2), {k, 1, Length[\Theta1]}, {i, 1, 3}, {j, 1, 3}];
A11 = AA[[1, 1]]; A12 = AA[[1, 2]]; A16 = AA[[1, 3]]; A22 =
AA[[2, 2]]; A26 = AA[[2, 3]]; A66 = AA[[3, 3]];
D11 = DD[[1, 1]]; D12 = DD[[1, 2]]; D16 = DD[[1, 3]]; D22 =
DD[[2, 2]]; D26 = DD[[2, 3]]; D66 = DD[[3, 3]];
B11 = BB[[1, 1]]; B12 = BB[[1, 2]]; B16 = BB[[1, 3]]; B22 =
BB[[2, 2]]; B26 = BB[[2, 3]]; B66 = BB[[3, 3]];
p = {1};

```

```

Do[Do[q = p[[1]] /. b -> bb[[bn]];
(* Approximation function for simply supported plate (SSSS) \
centered at (0, 0)*)
u = Simplify[
0 Sum[Cu[i, j] ((a/2)^(2 i) (b/2)^(2 j) - x^(2 i) y^(2 j)) y, {i,
0, 5}, {j, 0, 5}] /. b -> bb[[bn]]];
v = Simplify[
0 Sum[Cv[i, j] ((a/2)^(2 i) (b/2)^(2 j) - x^(2 i) y^(2 j)) x, {i,
0, 5}, {j, 0, 5}] /. b -> bb[[bn]]];
w = Simplify[
Cw[0, 20] ((b/2)^2 - y^2) + Cw[20, 0] ((a/2)^2 - x^2) +
Sum[Cw[i, j] ((a/2)^2 (b/2)^2 - x^2 y^2) x^i y^j, {j, 0,
pn}, {i, 0, j}] /. b -> bb[[bn]]];
sx = D[w, x]; sy = D[w, y]; wxy = D[D[w, x], y];
coefu = Cases[Cases[Variables[u], Except[x]], Except[y]];
Nu = Length[coefu];
coefv = Cases[Cases[Variables[v], Except[x]], Except[y]];
Nv = Length[coefv];
coefw = Cases[Cases[Variables[w], Except[x]], Except[y]];
Nw = Length[coefw];
nn = Nu + Nv + Nw;

```

```

(* Extracting number of coefficients and corresponding \
approximation polynomial functions and their derivatives*)
u = Simplify[u /. Table[coefu[[i]] -> Cu[i], {i, 1, Nu}]];
v = Simplify[v /. Table[coefv[[j]] -> Cv[j], {j, 1, Nv}]];
w = Simplify[w /. Table[coefw[[k]] -> Cw[k], {k, 1, Nw}]];
Do[\Phi[i] = Coefficient[u, Cu[i]], {i, 1, Nu}];
Do[\Phi x[i] = Simplify[D[\Phi[i], x]], {i, 1, Nu}];
Do[\Phi y[i] = Simplify[D[\Phi[i], y]], {i, 1, Nu}];
Do[\Chi[j] = Coefficient[v, Cv[j]], {j, 1, Nv}];
Do[\Chi x[j] = Simplify[D[\Chi[j], x]], {j, 1, Nv}];
Do[\Chi y[j] = Simplify[D[\Chi[j], y]], {j, 1, Nv}];
Do[\Psi[k] = Coefficient[w, Cw[k]], {k, 1, Nw}];
Do[\Psi x[k] = Simplify[D[\Psi[k], x]], {k, 1, Nw}];
Do[\Psi y[k] = Simplify[D[\Psi[k], y]], {k, 1, Nw}];

```

```

Do[\Psi]xx[k] = Simplify[D[\Psi][k], {x, 2}], {k, 1, Nw}];
Do[\Psi]yy[k] = Simplify[D[\Psi][k], {y, 2}], {k, 1, Nw}];
Do[\Psi]xy[k] = Simplify[D[D[\Psi][k], {x, 1}], {y, 1}], {k, 1,
  Nw}];
Do[Lap[k] =
  Simplify[D[\Psi][k], {x, 2}] + D[\Psi][k], {y, 2}], {k, 1, Nw}];
K11 = Table[0, {r, 1, Nu}, {i, 1, Nu}];
K12 = Table[0, {r, 1, Nu}, {j, 1, Nv}];
K13 = Table[0, {r, 1, Nu}, {k, 1, Nw}];
K21 = Table[0, {s, 1, Nv}, {i, 1, Nu}];
K22 = Table[0, {s, 1, Nv}, {j, 1, Nv}];
K23 = Table[0, {s, 1, Nv}, {k, 1, Nw}];
K31 = Table[0, {t, 1, Nw}, {i, 1, Nu}];
K32 = Table[0, {t, 1, Nw}, {j, 1, Nv}];
K33 = Table[0, {t, 1, Nw}, {k, 1, Nw}];
K = Table[0, {i, 1, nn}, {j, 1, nn}];
FAB = NLFAB = Table[0, {i, 1, Nu + Nv}];
FC = NLFC = Table[0, {t, 1, Nw}];
F = NLF = Table[0, {i, 1, nn}];

```

(* Terms of Equation (C-18) LHS*)

```

Do[K11[[r, i]] =
  K[[r, i]] =
    Intrec[(\[Phi]x[i] (A11 \[Phi]x[r] + A16 \[Phi]y[r]) + \[Phi]y[
      i] (A66 \[Phi]y[r] + A16 \[Phi]x[r])), a, bb[[bn]]], {r, 1,
    Nu}, {i, 1, Nu}];
Do[K12[[r, j]] =
  K[[r, j + Nu]] =
    Intrec[(\[Chi]x[j] (A66 \[Phi]y[r] + A16 \[Phi]x[r]) + \[Chi]y[
      j] (A12 \[Phi]x[r] + A26 \[Phi]y[r])), a, bb[[bn]]], {r, 1,
    Nu}, {j, 1, Nv}];
Do[K13[[r, k]] =
  K[[r, k + Nu +
    Nv]] = -Intrec[(\Psi)xx[
      k] (B11 \[Phi]x[r] + B16 \[Phi]y[r]) + \Psi]yy[
      k] (B12 \[Phi]x[r] + B26 \[Phi]y[r]) +
      2 \Psi]xy[k] (B16 \[Phi]x[r] + B66 \[Phi]y[r])), a,
    bb[[bn]]], {r, 1, Nu}, {k, 1, Nw}];
Do[K21[[s, i]] =
  K[[s + Nu, i]] =
    Intrec[(\[Phi]y[i] (A66 \[Chi]x[s] + A26 \[Chi]y[s]) + \[Phi]x[
      i] (A12 \[Chi]y[s] + A16 \[Chi]x[s])), a, bb[[bn]]], {s, 1,
    Nv}, {i, 1, Nu}];
Do[K22[[s, j]] =
  K[[s + Nu, j + Nu]] =
    Intrec[(\[Chi]x[j] (A66 \[Chi]x[s] + A26 \[Chi]y[s]) + \[Chi]y[

```

j] (A22 \[Chi]y[s] + A26 \[Chi]x[s])), a, bb[[bn]]], {s, 1, Nv}, {j, 1, Nv}}];

Do[K23[[s, k]] =

K[[s + Nu,
k + Nu +
Nv]] = -Intrec[(\[Psi]xx[
k] (B12 \[Chi]y[s] + B16 \[Chi]x[s]) + \[Psi]yy[
k] (B22 \[Chi]y[s] + B26 \[Chi]x[s]) +
2 \[Psi]xy[k] (B26 \[Chi]y[s] + B66 \[Chi]x[s])), a,
bb[[bn]]], {s, 1, Nv}, {k, 1, Nw}}];

Do[K31[[t, i]] =

K[[t + Nu + Nv,
i]] = -Intrec[(\[Phi]x[
i] (B11 \[Psi]xx[t] + 2 B16 \[Psi]xy[t] +
B12 \[Psi]yy[t]) + \[Phi]y[
i] (B16 \[Psi]xx[t] + 2 B66 \[Psi]xy[t] +
B26 \[Psi]yy[t])), a, bb[[bn]]], {t, 1, Nw}, {i, 1, Nu}}];

Do[K32[[t, j]] =

K[[t + Nu + Nv,
j + Nu]] = -Intrec[(\[Chi]x[
j] (B16 \[Psi]xx[t] + 2 B66 \[Psi]xy[t] +
B26 \[Psi]yy[t]) + \[Chi]y[
j] (B22 \[Psi]yy[t] + 2 B26 \[Psi]xy[t] +
B12 \[Psi]xx[t])), a, bb[[bn]]], {t, 1, Nw}, {j, 1, Nv}}];

Do[K33[[t, k]] =

K[[t + Nu + Nv, k + Nu + Nv]] =
Intrec[(\[Psi]xx[
k] (D11 \[Psi]xx[t] + D12 \[Psi]yy[t] +
2 D16 \[Psi]xy[t]) + \[Psi]yy[
k] (D22 \[Psi]yy[t] + D12 \[Psi]xx[t] +
2 D26 \[Psi]xy[t]) + \[Psi]xy[
k] (4 D66 \[Psi]xy[t] + 2 D16 \[Psi]xx[t] +
2 D26 \[Psi]yy[t])), a, bb[[bn]]], {t, 1, Nw}, {k, 1, Nw}}];

Do[FC[[t]] = Intrec[q \[Psi][t], a, bb[[bn]]], {t, 1, Nw}}];

Do[F[[t + Nu + Nv]] = FC[[t]], {t, 1, Nw}}];

(* Direct solution of the coefficients Subscript[Cu, i], Subscript[Cv, j] and Subscript[Cw, k] using FindRoot*)

cCu = Table[Cu[r], {r, 1, Nu}];

cCv = Table[Cv[s], {s, 1, Nv}];

cCw = Table[Cw[t], {t, 1, Nw}];

Cuvw[pn, bn] = Join[cCu, cCv, cCw];

(*solCuvw[0,bn]={Table[Cuvw[[i]] \[Rule] 0.00i,{i,1,nn}}];

Cuvw[i[pn-1]=Table[{Cuvw[[i]] ,Cuvw[[i]]/. solCuvw[pn-1,bn]][[1, i]],{i,1,nn}};*)

```

solCuvw[pn, bn] = Solve[K.Cuvw[pn, bn] == F, Cuvw[pn, bn]];

(* Soltion of u, v and w and evaluating forces,
moments and stresses in functional form*)
w = w /. solCuvw[pn, bn][[1]];
u = u /. solCuvw[pn, bn][[1]];
v = v /. solCuvw[pn, bn][[1]];
wx = D[w, x]; wy = D[w, y]; wxx = D[wx, x]; wyy = D[wy, y];
wxy = D[wx, y];
ux = D[u, x]; vx = D[v, x]; uy = D[u, y]; vy = D[v, y];
(* Save the solution for each aspect ratio and loading increment*)
W[pn, bn] = w; u1[pn, bn] = u; v1[pn, bn] = v;
\[Kappa][pn, bn] = {wxx, wyy, 2 wxy}; \[Epsilon]0[pn, bn] = {ux, vy,
  uy + vx};
{Mx1[pn, bn], My1[pn, bn], Mxy[pn, bn]} = -DD.\[Kappa][pn, bn] +
  BB.\[Epsilon]0[pn, bn];
{Nx1[pn, bn], Ny1[pn, bn], Nxy[pn, bn]} = -BB.\[Kappa][pn, bn] +
  AA.\[Epsilon]0[pn, bn];
{\[Sigma]x[pn, bn], \[Sigma]y[pn, bn], \[Sigma]xy[pn,
  bn]} = -z Qb.\[Kappa][pn, bn] + Qb.\[Epsilon]0[pn, bn], {pn, 1,
  10}}, {bn, 1, Length[bb]}}

```

Appendix B:

Derivation of Ritz Method in a Matrix Form for Thick

Laminated Composite Based On FSDT Plates

```

ClearAll["Global`*"]
(* This code "Intrec" is to evaluate the exact integral of \
polynomials quickly*)
Intrec[f_, a_, b_] :=
Module[{coeff, ii, jj, i, j},
coeff = CoefficientList[f, {x, y}]; {ii, jj} = Dimensions[coeff];
Sum[coeff[[i, j]] (a^i - (-a)^i)/i (b^j - (-b)^j)/j, {i, 1,
ii}, {j, 1, jj}]];

(*L=-15 Degree;
U=15Degree;
nL=5;
\[Theta]1=Table[L+(U-L)/(nL-1)i,{i,0,nL-1}]*)

\[Theta]1 = {0, 90, 0} Degree;

a = 0.5;
\[Nu] = 0.25; \[Nu]12 = \[Nu];
tt = 0.25; E2 = 1 ; E1 = 25 E2;
G12 = 0.5 E2 ;
G13 = G12;
G23 = 0.2 E2;
bb = {0.5};
\[Nu]21 = (\[Nu]12 E2)/E1;
nL = Length[\[Theta]1];
tL = tt/nL;
\[Nu]21 = E2/E1 \[Nu]12;
\[Psi] = 5/6;(* Shear Correction Factor*)
Q11 = E1/(1 - \[Nu]12 \[Nu]21); Q22 = E2/(
1 - \[Nu]12 \[Nu]21); Q12 = (\[Nu]21 E1)/(
1 - \[Nu]12 \[Nu]21); Q21 = (\[Nu]12 E2)/(
1 - \[Nu]12 \[Nu]21); Q66 = G12; Q44 = G23; Q55 = G13;

(* Calculation of rigidities of the resulted equivalent single layer*)

Q = {{Q11, Q12, 0, 0, 0}, {Q21, Q22, 0, 0, 0}, {0, 0, Q66, 0, 0}, {0,
0, 0, Q44, 0}, {0, 0, 0, 0, Q55}};
T = {{Cos[\[Theta]]^2, Sin[\[Theta]]^2,
2 Sin[\[Theta]] Cos[\[Theta]], 0, 0}, {Sin[\[Theta]]^2,
Cos[\[Theta]]^2, -2 Sin[\[Theta]] Cos[\[Theta]], 0,
0}, {-Sin[\[Theta]] Cos[\[Theta]], Sin[\[Theta]] Cos[\[Theta]],
Cos[\[Theta]]^2 - Sin[\[Theta]]^2, 0, 0}, {0, 0, 0,
Cos[\[Theta]], -Sin[\[Theta]]}, {0, 0, 0, Sin[\[Theta]],
Cos[\[Theta]]}} ;
Qb = Inverse[T].Q.Transpose[Inverse[T]];

```

```

Do[z[k] = -(tt/2) + tL k, {k, 0, Length[\[Theta]1]};
DD = AA = BB = Table[0, {i, 1, 3}, {j, 1, 3}];
aa = Table[0, {i, 1, 2}, {j, 1, 2}];
Do[DD[[i, j]] =
  1/3 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^3 -
    z[k - 1]^3), {k, 1, Length[\[Theta]1]}, {i, 1, 3}, {j, 1, 3}];
Do[AA[[i, j]] =
  Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k] -
    z[k - 1]), {k, 1, Length[\[Theta]1]}, {i, 1, 3}, {j, 1, 3}];
Do[BB[[i, j]] =
  1/2 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^2 -
    z[k - 1]^2), {k, 1, Length[\[Theta]1]}, {i, 1, 3}, {j, 1, 3}];
Do[aa[[i, j]] =
  Sum[(Qb[[i + 3, j + 3]] /. \[Theta] -> \[Theta]1[[k]]) (z[k] -
    z[k - 1]), {k, 1, Length[\[Theta]1]}, {i, 1, 2}, {j, 1, 2}];

A11 = AA[[1, 1]]; A12 = AA[[1, 2]]; A16 = AA[[1, 3]]; A22 =
AA[[2, 2]]; A26 = AA[[2, 3]]; A66 = AA[[3, 3]];
D11 = DD[[1, 1]]; D12 = DD[[1, 2]]; D16 = DD[[1, 3]]; D22 =
DD[[2, 2]]; D26 = DD[[2, 3]]; D66 = DD[[3, 3]];
B11 = BB[[1, 1]]; B12 = BB[[1, 2]]; B16 = BB[[1, 3]]; B22 =
BB[[2, 2]]; B26 = BB[[2, 3]]; B66 = BB[[3, 3]];
A44 = aa[[1, 1]]; A45 = aa[[1, 2]]; A55 = aa[[2, 2]];
P = {1};

```

```

Do[Do[q1 = P[[Pn]] /. b -> bb[[bn]];
(* Approximation function for simply supported plate (SSSS) \
centered at (0, 0)*)
u = Simplify[
  Sum[Cu[i, j] (b^2 - y^2) x^i y^j, {i, 0, 6}, {j, 0, 6}] /.
  b -> bb[[bn]]];
v = Simplify[
  Sum[Cv[i, j] (a^2 - x^2) x^i y^j, {i, 0, 6}, {j, 0, 6}] /.
  b -> bb[[bn]]];
w = Simplify[
  Sum[Cw[i, j] (a^2 - x^2) (b^2 - y^2) x^i y^j, {i, 0, 6}, {j, 0,
    6}] /. b -> bb[[bn]]];
\[Phi]x =
Simplify[
  Sum[C\[Phi]x[i, j] (b^2 - y^2) x^i y^j, {i, 0, 6}, {j, 0, 6}] /.
  b -> bb[[bn]]];
\[Phi]y =
Simplify[
  Sum[C\[Phi]y[i, j] (a^2 - x^2) x^i y^j, {i, 0, 6}, {j, 0, 6}] /.
  b -> bb[[bn]]];

```

```

coefu = Cases[Cases[Variables[u], Except[x]], Except[y]];
Nu = Length[coefu];
coefv = Cases[Cases[Variables[v], Except[x]], Except[y]];
Nv = Length[coefv];
coefw = Cases[Cases[Variables[w], Except[x]], Except[y]];
Nw = Length[coefw];
coef\[Phi]x = Cases[Cases[Variables[\[Phi]x], Except[x]], Except[y]];
N\[Phi]x = Length[coef\[Phi]x];
coef\[Phi]y = Cases[Cases[Variables[\[Phi]y], Except[x]], Except[y]];
N\[Phi]y = Length[coef\[Phi]y];
nn = Nu + Nv + Nw + N\[Phi]x + N\[Phi]y;

```

```

(* Extracting number of coefficients and corresponding \
approximation polynomial functions and their derivatives*)
u = Simplify[u /. Table[coefu[[i]] -> Cu[i], {i, 1, Nu}]];
v = Simplify[v /. Table[coefv[[j]] -> Cv[j], {j, 1, Nv}]];
w = Simplify[w /. Table[coefw[[k]] -> Cw[k], {k, 1, Nw}]];
\[Phi]x =
Simplify[\[Phi]x /.
Table[coef\[Phi]x[[n]] -> C\[Phi]x[n], {n, 1, N\[Phi]x}]];
\[Phi]y =
Simplify[\[Phi]y /.
Table[coef\[Phi]y[[m]] -> C\[Phi]y[m], {m, 1, N\[Phi]y}]];

```

```

Do[Fu[i] = Coefficient[u, Cu[i]], {i, 1, Nu}]
Do[Fux[i] = Simplify[D[Fu[i], x]], {i, 1, Nu}]
Do[Fuy[i] = Simplify[D[Fu[i], y]], {i, 1, Nu}]
Do[Fv[j] = Coefficient[v, Cv[j]], {j, 1, Nv}]
Do[Fvx[j] = Simplify[D[Fv[j], x]], {j, 1, Nv}]
Do[Fvy[j] = Simplify[D[Fv[j], y]], {j, 1, Nv}]
Do[Fw[k] = Coefficient[w, Cw[k]], {k, 1, Nw}]
Do[Fwx[k] = Simplify[D[Fw[k], x]], {k, 1, Nw}]
Do[Fwy[k] = Simplify[D[Fw[k], y]], {k, 1, Nw}]
Do[F\[Phi]x[n] = Coefficient[\[Phi]x, C\[Phi]x[n]], {n, 1, Nw}]
Do[Fx\[Phi]x[n] = Simplify[D[F\[Phi]x[n], x]], {n, 1, N\[Phi]x}]
Do[Fy\[Phi]x[n] = Simplify[D[F\[Phi]x[n], y]], {n, 1, N\[Phi]x}]
Do[F\[Phi]y[m] = Coefficient[\[Phi]y, C\[Phi]y[m]], {m, 1,
N\[Phi]y}]
Do[Fx\[Phi]y[m] = Simplify[D[F\[Phi]y[m], x]], {m, 1, N\[Phi]y}]
Do[Fy\[Phi]y[m] = Simplify[D[F\[Phi]y[m], y]], {m, 1, N\[Phi]y}];

```

```

K11 = Table[0, {r, 1, Nu}, {i, 1, Nu}];
K12 = Table[0, {r, 1, Nu}, {j, 1, Nv}];
K13 = Table[0, {r, 1, Nu}, {k, 1, Nw}];
K14 = Table[0, {r, 1, Nu}, {n, 1, N\[Phi]x}];

```

K15 = Table[0, {r, 1, Nu}, {m, 1, N[Phi]y}];
 K21 = Table[0, {s, 1, Nv}, {i, 1, Nu}];
 K22 = Table[0, {s, 1, Nv}, {j, 1, Nv}];
 K23 = Table[0, {s, 1, Nv}, {k, 1, Nw}];
 K24 = Table[0, {s, 1, Nv}, {n, 1, N[Phi]x}];
 K25 = Table[0, {s, 1, Nv}, {m, 1, N[Phi]y}];
 K31 = Table[0, {t, 1, Nw}, {i, 1, Nu}];
 K32 = Table[0, {t, 1, Nw}, {j, 1, Nv}];
 K33 = Table[0, {t, 1, Nw}, {k, 1, Nw}];
 K34 = Table[0, {t, 1, Nw}, {n, 1, N[Phi]x}];
 K35 = Table[0, {t, 1, Nw}, {m, 1, N[Phi]y}];
 K41 = Table[0, {p, 1, N[Phi]x}, {i, 1, Nu}];
 K42 = Table[0, {p, 1, N[Phi]x}, {j, 1, Nv}];
 K43 = Table[0, {p, 1, N[Phi]x}, {k, 1, Nw}];
 K44 = Table[0, {p, 1, N[Phi]x}, {n, 1, N[Phi]x}];
 K45 = Table[0, {p, 1, N[Phi]x}, {m, 1, N[Phi]y}];
 K51 = Table[0, {q, 1, N[Phi]y}, {i, 1, Nu}];
 K52 = Table[0, {q, 1, N[Phi]y}, {j, 1, Nv}];
 K53 = Table[0, {q, 1, N[Phi]y}, {k, 1, Nw}];
 K54 = Table[0, {q, 1, N[Phi]y}, {n, 1, N[Phi]x}];
 K55 = Table[0, {q, 1, N[Phi]y}, {m, 1, N[Phi]y}];

K = Table[0, {i, 1, nn}, {j, 1, nn}];
 FAB = NLFAB = Table[0, {i, 1, Nu + Nv}];
 FBC = NLFBC = Table[0, {i, 1, N[Phi]x + N[Phi]y}];
 FC = NLFC = Table[0, {t, 1, Nw}];
 F = NLF = Table[0, {i, 1, nn}];

(* Terms of Equation (C-18) LHS*)

Do[K11[[r, i]] =
 K[[r, i]] =
 Intrec[(Fux[i] (A11 Fux[r] + A16 Fuy[r]) +
 Fuy[i] (A66 Fuy[r] + A16 Fux[r])), a, bb[[bn]]], {r, 1,
 Nu}, {i, 1, Nu}];
 Do[K12[[r, j]] =
 K[[r, j + Nu]] =
 Intrec[(Fvx[j] (A66 Fuy[r] + A16 Fux[r]) +
 Fvy[j] (A12 Fux[r] + A26 Fuy[r])), a, bb[[bn]]], {r, 1,
 Nu}, {j, 1, Nv}];
 Do[K14[[r, n]] =
 K[[r, n + Nu + Nv + Nw]] =
 Intrec[(Fv[Phi]x[n] (B11 Fux[r] + B16 Fuy[r]) +
 Fy[Phi]x[n] (B16 Fux[r] + B66 Fuy[r])), a, bb[[bn]]], {r, 1,
 Nu}, {n, 1, N[Phi]x}];
 Do[K15[[r, m]] =
 K[[r, m + Nu + Nv + Nw + N[Phi]x]] =

$\text{Intrec}[(Fy\backslash[\Phi]y[m] (B12 Fux[r] + B26 Fuy[r]) +$
 $Fx\backslash[\Phi]y[m] (B16 Fux[r] + B66 Fuy[r])), a, bb[[bn]]], \{r, 1,$
 $Nu\}, \{m, 1, N\backslash[\Phi]y\}];$
 $\text{Do}[K21[[s, i]] =$
 $K[[s + Nu, i]] =$
 $\text{Intrec}[(Fuy[i] (A66 Fvx[s] + A26 Fvy[s]) +$
 $Fux[i] (A12 Fvy[s] + A16 Fvx[s])), a, bb[[bn]]], \{s, 1,$
 $Nv\}, \{i, 1, Nu\}];$
 $\text{Do}[K22[[s, j]] =$
 $K[[s + Nu, j + Nu]] =$
 $\text{Intrec}[(Fvx[j] (A66 Fvx[s] + A26 Fvy[s]) +$
 $Fvy[j] (A22 Fvy[s] + A26 Fvx[s])), a, bb[[bn]]], \{s, 1,$
 $Nv\}, \{j, 1, Nv\}];$
 $\text{Do}[K24[[s, n]] =$
 $K[[s + Nu, n + Nu + Nv + Nw]] =$
 $\text{Intrec}[(Fx\backslash[\Phi]x[n] (B12 Fvy[s] + B16 Fvx[s]) +$
 $Fy\backslash[\Phi]x[n] (B26 Fvy[s] + B66 Fvx[s])), a, bb[[bn]]], \{s, 1,$
 $Nv\}, \{n, 1, N\backslash[\Phi]x\}];$
 $\text{Do}[K25[[s, m]] =$
 $K[[s + Nu, m + Nu + Nv + Nw + N\backslash[\Phi]x]] =$
 $\text{Intrec}[(Fy\backslash[\Phi]y[m] (B26 Fvx[s] + B22 Fvy[s]) +$
 $Fx\backslash[\Phi]y[m] (B26 Fvy[s] + B66 Fvx[s])), a, bb[[bn]]], \{s, 1,$
 $Nv\}, \{m, 1, N\backslash[\Phi]y\}];$
 $\text{Do}[K33[[t, k]] =$
 $K[[t + Nu + Nv, k + Nu + Nv]] = \backslash[\Psi]^*$
 $\text{Intrec}[(Fwy[t] (A44 Fwy[k] + A45 Fwx[k]) +$
 $Fwx[t] (A45 Fwy[k] + A55 Fwx[k])), a, bb[[bn]]], \{t, 1,$
 $Nw\}, \{k, 1, Nw\}];$
 $\text{Do}[K34[[t, n]] =$
 $K[[t + Nu + Nv, n + Nu + Nv + Nw]] = \backslash[\Psi]^*$
 $\text{Intrec}[(F\backslash[\Phi]x[n] (A45 Fwy[t] + A55 Fwx[t])), a,$
 $bb[[bn]]], \{t, 1, Nw\}, \{n, 1, N\backslash[\Phi]x\}];$
 $\text{Do}[K35[[t, m]] =$
 $K[[t + Nu + Nv, m + Nu + Nv + Nw + N\backslash[\Phi]x]] = \backslash[\Psi]^*$
 $\text{Intrec}[(F\backslash[\Phi]y[m] (A44 Fwy[t] + A45 Fwx[t])), a,$
 $bb[[bn]]], \{t, 1, Nw\}, \{m, 1, N\backslash[\Phi]y\}];$
 $\text{Do}[K41[[p, i]] =$
 $K[[p + Nu + Nv + Nw, i]] =$
 $\text{Intrec}[(Fux[i] (B11 Fx\backslash[\Phi]x[p] + B16 Fy\backslash[\Phi]x[p]) +$
 $Fuy[i] (B16 Fx\backslash[\Phi]x[p] + B66 Fy\backslash[\Phi]x[p])), a,$
 $bb[[bn]]], \{p, 1, N\backslash[\Phi]x\}, \{i, 1, Nu\}];$
 $\text{Do}[K42[[p, j]] =$
 $K[[p + Nu + Nv + Nw, j + Nu]] =$
 $\text{Intrec}[(Fvy[j] (B12 Fx\backslash[\Phi]x[p] + B26 Fy\backslash[\Phi]x[p]) +$
 $Fvx[j] (B16 Fx\backslash[\Phi]x[p] + B66 Fy\backslash[\Phi]x[p])), a,$
 $bb[[bn]]], \{p, 1, N\backslash[\Phi]x\}, \{j, 1, Nv\}];$

$\text{Do}[K43][p, k] =$
 $K[[p + \text{Nu} + \text{Nv} + \text{Nw}, k + \text{Nu} + \text{Nv}]] = \backslash[\text{Psi}]^*$
 $\text{Intrec}[(F\backslash[\text{Phi}]x[p] (A45 \text{Fwy}[k] + A55 \text{Fwx} [k])), a,$
 $\text{bb}[[\text{bn}]]], \{p, 1, N\backslash[\text{Phi}]x\}, \{k, 1, \text{Nw}\};$
 $\text{Do}[K44][p, n] =$
 $K[[p + \text{Nu} + \text{Nv} + \text{Nw}, n + \text{Nu} + \text{Nv} + \text{Nw}]] =$
 $\text{Intrec}[(F\backslash[\text{Phi}]x[n] (D11 \text{Fx}\backslash[\text{Phi}]x[p] + D16 \text{Fy}\backslash[\text{Phi}]x[p]) +$
 $\text{Fy}\backslash[\text{Phi}]x[$
 $n] (D16 \text{Fx}\backslash[\text{Phi}]x[p] +$
 $D66 \text{Fy}\backslash[\text{Phi}]x[p]) + (A55 \backslash[\text{Psi}] F\backslash[\text{Phi}]x[p] F\backslash[\text{Phi}]x[n])),$
 $a, \text{bb}[[\text{bn}]]], \{p, 1, N\backslash[\text{Phi}]x\}, \{n, 1, N\backslash[\text{Phi}]x\};$
 $\text{Do}[K45][p, m] =$
 $K[[p + \text{Nu} + \text{Nv} + \text{Nw}, m + \text{Nu} + \text{Nv} + \text{Nw} + N\backslash[\text{Phi}]x]] =$
 $\text{Intrec}[(F\backslash[\text{Phi}]y[m] (D12 \text{Fx}\backslash[\text{Phi}]x[p] + D26 \text{Fy}\backslash[\text{Phi}]x[p]) +$
 $\text{Fx}\backslash[\text{Phi}]y[$
 $m] (D16 \text{Fx}\backslash[\text{Phi}]x[p] +$
 $D66 \text{Fy}\backslash[\text{Phi}]x[p]) + (A45 \backslash[\text{Psi}] F\backslash[\text{Phi}]x[p] F\backslash[\text{Phi}]y[m])),$
 $a, \text{bb}[[\text{bn}]]], \{p, 1, N\backslash[\text{Phi}]x\}, \{m, 1, N\backslash[\text{Phi}]y\};$
 $\text{Do}[K51][q, i] =$
 $K[[q + \text{Nu} + \text{Nv} + \text{Nw} + N\backslash[\text{Phi}]x, i]] =$
 $\text{Intrec}[(F\backslash[\text{Phi}]y[i] (B12 \text{Fy}\backslash[\text{Phi}]y[q] + B16 \text{Fx}\backslash[\text{Phi}]y[q]) +$
 $\text{Fuy}[i] (B26 \text{Fy}\backslash[\text{Phi}]y[q] + B66 \text{Fx}\backslash[\text{Phi}]y[q])), a,$
 $\text{bb}[[\text{bn}]]], \{q, 1, N\backslash[\text{Phi}]y\}, \{i, 1, \text{Nu}\};$
 $\text{Do}[K52][q, j] =$
 $K[[q + \text{Nu} + \text{Nv} + \text{Nw} + N\backslash[\text{Phi}]x, j + \text{Nu}]] =$
 $\text{Intrec}[(F\backslash[\text{Phi}]y[j] (B26 \text{Fy}\backslash[\text{Phi}]y[q] + B66 \text{Fx}\backslash[\text{Phi}]y[q]) +$
 $\text{Fvy}[j] (B26 \text{Fx}\backslash[\text{Phi}]y[q] + B22 \text{Fy}\backslash[\text{Phi}]y[q])), a,$
 $\text{bb}[[\text{bn}]]], \{q, 1, N\backslash[\text{Phi}]y\}, \{j, 1, \text{Nv}\};$
 $\text{Do}[K53][q, k] =$
 $K[[q + \text{Nu} + \text{Nv} + \text{Nw} + N\backslash[\text{Phi}]x, k + \text{Nu} + \text{Nv}]] = \backslash[\text{Psi}]^*$
 $\text{Intrec}[(F\backslash[\text{Phi}]y[q] (A44 \text{Fwy}[k] + A45 \text{Fwx} [k])), a,$
 $\text{bb}[[\text{bn}]]], \{q, 1, N\backslash[\text{Phi}]y\}, \{k, 1, \text{Nw}\};$
 $\text{Do}[K54][q, n] =$
 $K[[q + \text{Nu} + \text{Nv} + \text{Nw} + N\backslash[\text{Phi}]x, n + \text{Nu} + \text{Nv} + \text{Nw}]] =$
 $\text{Intrec}[(F\backslash[\text{Phi}]x[n] (D12 \text{Fy}\backslash[\text{Phi}]y[q] + D16 \text{Fx}\backslash[\text{Phi}]y[q]) +$
 $\text{Fy}\backslash[\text{Phi}]x[$
 $n] (D26 \text{Fy}\backslash[\text{Phi}]y[q] +$
 $D66 \text{Fx}\backslash[\text{Phi}]y[q]) + (A45 \backslash[\text{Psi}] F\backslash[\text{Phi}]y[q] F\backslash[\text{Phi}]x[n])),$
 $a, \text{bb}[[\text{bn}]]], \{q, 1, N\backslash[\text{Phi}]y\}, \{n, 1, N\backslash[\text{Phi}]x\};$
 $\text{Do}[K55][q, m] =$
 $K[[q + \text{Nu} + \text{Nv} + \text{Nw} + N\backslash[\text{Phi}]x, m + \text{Nu} + \text{Nv} + \text{Nw} + N\backslash[\text{Phi}]x]] =$
 $\text{Intrec}[(F\backslash[\text{Phi}]y[m] (D26 \text{Fy}\backslash[\text{Phi}]y[q] + D66 \text{Fx}\backslash[\text{Phi}]y[q]) +$
 $\text{Fy}\backslash[\text{Phi}]y[$
 $m] (D26 \text{Fx}\backslash[\text{Phi}]y[q] +$
 $D22 \text{Fy}\backslash[\text{Phi}]y[q]) + (A44 \backslash[\text{Psi}] F\backslash[\text{Phi}]y[q] F\backslash[\text{Phi}]y[m])),$
 $a, \text{bb}[[\text{bn}]]], \{q, 1, N\backslash[\text{Phi}]y\}, \{m, 1, N\backslash[\text{Phi}]y\};$

```

Do[FC[[t]] = Intrec[q1 Fw[t], a, bb[[bn]]], {t, 1, Nw}];
Do[F[[t + Nu + Nv]] = FC[[t]], {t, 1, Nw}];
Do[F[[t + N[Phi]y + N[Phi]x]] = FC[[t]], {t, 1, Nw}];

```

```

(* Direct solution of the coefficients Subscript[Cu, i], Subscript[
Cv, j] , Subscript[Cw, k] , C\[Phi]x and Subscript[C\[Phi], y ]
using FindRoot*)

```

```

cCu = Table[Cu[r], {r, 1, Nu}];
cCv = Table[Cv[s], {s, 1, Nv}];
cCw = Table[Cw[t], {t, 1, Nw}];
cC\[Phi]x = Table[C\[Phi]x[p], {p, 1, N[Phi]x}];
cC\[Phi]y = Table[C\[Phi]y[q], {q, 1, N[Phi]y}];
Cuvw\[Phi]x\[Phi]y[Pn, bn] =
Join[cCu, cCv, cCw, cC\[Phi]x, cC\[Phi]y];

```

```

(*solCuvw[0,bn]={Table[Cuvw[[i]] \[Rule] 0.00i,{i,1,nn}}];
Cuvwi[Pn-1]=Table[{Cuvw[[i]] ,Cuvw[[i]]/. solCuvw[Pn-1,bn][[1,
i]]},{i,1,nn}];*)
solCuvw\[Phi]x\[Phi]y[Pn, bn] =
Solve[K.Cuvw\[Phi]x\[Phi]y[Pn, bn] == F,
Cuvw\[Phi]x\[Phi]y[Pn, bn]];

```

```

(* Soltion of u, v and w and evaluating forces,
moments and stresses in functional form*)

```

```

w = w /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
u = u /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
v = v /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
\[Phi]x = \[Phi]x /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
\[Phi]y = \[Phi]y /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
wx = D[w, x]; wy = D[w, y]; wxx = D[w, x]; wyy = D[w, y];
wxy = D[w, y];
ux = D[u, x]; vx = D[v, x]; uy = D[u, y];
vy = D[v, y]; \[Phi]xx = D[\[Phi]x, x]; \[Phi]yy =
D[\[Phi]y, y]; \[Phi]xy = D[\[Phi]x, y]; \[Phi]yx = D[\[Phi]y, x];
(* Save the solution for each aspect ratio and loading increment*)
W[Pn, bn] = w; u1[Pn, bn] = u;
v1[Pn, bn] =
v; \[Phi]x1[Pn, bn] = \[Phi]x; \[Phi]y1[Pn, bn] = \[Phi]y;
\[Kappa][Pn,
bn] = {\[Phi]xx, \[Phi]yy, \[Phi]xy + \[Phi]yx}; \[Epsilon]0[Pn,
bn] = {ux, vy, uy + vx}; \[Kappa]2[Pn,
bn] = {\[Phi]y + wy, \[Phi]x + wx};
{Mx1[Pn, bn], My1[Pn, bn], Mxy[Pn, bn]} =
DD.\[Kappa][Pn, bn] + BB.\[Epsilon]0[Pn, bn];
{Nx1[Pn, bn], Ny1[Pn, bn], Nxy[Pn, bn]} =

```

```

BB.\[Kappa][Pn, bn] + AA.\[Epsilon]0[Pn, bn];
{Qy[Pn, bn], Qx[Pn, bn]} = aa.\[Kappa]2[Pn, bn];
{\[Sigma]x[Pn, bn], \[Sigma]y[Pn, bn], \[Sigma]xy[Pn, bn]} =
Qb[[1 ;; 3, 1 ;; 3]].\[Epsilon]0[Pn, bn] +
z Qb[[1 ;; 3, 1 ;; 3]].\[Kappa][Pn, bn]; {\[Tau]yz[Pn,
bn], \[Tau]xz[Pn, bn]} =
Qb[[4 ;; 5, 4 ;; 5]].\[Kappa]2[Pn, bn];, {Pn, 1, Length[P]}}, {bn,
1, Length[bb]]}

```

Appendix C:

Derivation of Ritz Method in a Matrix Form for Thick

Rectangular Laminated Composite Based on HSDT Plates

```

ClearAll["Global`*"]
(* This code "Intrec" is to evaluate the exact integral of \
polynomials quickly*)
Intrec[f_, a_, b_] :=
Module[{coeff, ii, jj, i, j},
coeff = CoefficientList[f, {x, y}]; {ii, jj} = Dimensions[coeff];
Sum[coeff[[i, j]] (a^i)/i (b^j)/j, {i, 1, ii}, {j, 1, jj}]];

\[Theta]1 = {0, 90, 90, 0} Degree;

\[Nu] = 0.25; \[Nu]12 = \[Nu];
tt = a/4; E2 = 1 ; E1 = 25 E2;
G12 = 0.5 E2 ;
G13 = G12;
G23 = 0.2 E2;
a = 1; bb = {1};
\[Nu]21 = (\[Nu]12 E2)/E1;
nL = Length[\[Theta]1];
tL = tt/nL;
\[Nu]21 = E2/E1 \[Nu]12;
C1 = 4/(3 tt^2);
C2 = 3 C1;
Q11 = E1/(1 - \[Nu]12 \[Nu]21); Q22 = E2/(
1 - \[Nu]12 \[Nu]21); Q12 = (\[Nu]21 E1)/(
1 - \[Nu]12 \[Nu]21); Q21 = (\[Nu]12 E2)/(
1 - \[Nu]12 \[Nu]21); Q66 = G12; Q44 = G23; Q55 = G13;

(* Calculation of rigidities of the resulted equivalent single layer*)

Q = {{Q11, Q12, 0, 0, 0}, {Q21, Q22, 0, 0, 0}, {0, 0, Q66, 0, 0}, {0,
0, 0, Q44, 0}, {0, 0, 0, 0, Q55}};
T = {{Cos[\[Theta]]^2, Sin[\[Theta]]^2,
2 Sin[\[Theta]] Cos[\[Theta]], 0, 0}, {Sin[\[Theta]]^2,
Cos[\[Theta]]^2, -2 Sin[\[Theta]] Cos[\[Theta]], 0,
0}, {-Sin[\[Theta]] Cos[\[Theta]], Sin[\[Theta]] Cos[\[Theta]],
Cos[\[Theta]]^2 - Sin[\[Theta]]^2, 0, 0}, {0, 0, 0,
Cos[\[Theta]], -Sin[\[Theta]]}, {0, 0, 0, Sin[\[Theta]],
Cos[\[Theta]]}};
Qb = Inverse[T].Q.Transpose[Inverse[T]];
Do[z[k] = -(tt/2) + tL k, {k, 0, Length[\[Theta]1]}];
DD = AA = BB = EE = FF = HH = Table[0, {i, 1, 3}, {j, 1, 3}];
aa = dd = gg = Table[0, {i, 1, 2}, {j, 1, 2}];

```

```

Do[DD[[i, j]] =
  1/3 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^3 -
    z[k - 1]^3), {k, 1, Length[\[Theta]1}}, {i, 1, 3}, {j, 1, 3}];
Do[AA[[i, j]] =
  Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k] -
    z[k - 1]), {k, 1, Length[\[Theta]1}}, {i, 1, 3}, {j, 1, 3}];
Do[BB[[i, j]] =
  1/2 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^2 -
    z[k - 1]^2), {k, 1, Length[\[Theta]1}}, {i, 1, 3}, {j, 1, 3}];
Do[EE[[i, j]] =
  1/4 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^4 -
    z[k - 1]^4), {k, 1, Length[\[Theta]1}}, {i, 1, 3}, {j, 1, 3}];
Do[FF[[i, j]] =
  1/5 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^5 -
    z[k - 1]^5), {k, 1, Length[\[Theta]1}}, {i, 1, 3}, {j, 1, 3}];
Do[HH[[i, j]] =
  1/7 Sum[(Qb[[i, j]] /. \[Theta] -> \[Theta]1[[k]]) (z[k]^7 -
    z[k - 1]^7), {k, 1, Length[\[Theta]1}}, {i, 1, 3}, {j, 1, 3}];
Do[aa[[i, j]] =
  Sum[(Qb[[i + 3, j + 3]] /. \[Theta] -> \[Theta]1[[k]]) (z[k] -
    z[k - 1]), {k, 1, Length[\[Theta]1}}, {i, 1, 2}, {j, 1, 2}];
Do[dd[[i, j]] =
  1/3 Sum[(Qb[[i + 3, j + 3]] /. \[Theta] -> \[Theta]1[[k]]) (z[
    k]^3 - z[k - 1]^3), {k, 1, Length[\[Theta]1}}, {i, 1,
    2}, {j, 1, 2}];
Do[gg[[i, j]] =
  1/5 Sum[(Qb[[i + 3, j + 3]] /. \[Theta] -> \[Theta]1[[k]]) (z[
    k]^5 - z[k - 1]^5), {k, 1, Length[\[Theta]1}}, {i, 1,
    2}, {j, 1, 2}];

A11 = AA[[1, 1]]; A12 = AA[[1, 2]]; A16 = AA[[1, 3]]; A22 =
  AA[[2, 2]]; A26 = AA[[2, 3]]; A66 = AA[[3, 3]];
D11 = DD[[1, 1]]; D12 = DD[[1, 2]]; D16 = DD[[1, 3]]; D22 =
  DD[[2, 2]]; D26 = DD[[2, 3]]; D66 = DD[[3, 3]];
B11 = BB[[1, 1]]; B12 = BB[[1, 2]]; B16 = BB[[1, 3]]; B22 =
  BB[[2, 2]]; B26 = BB[[2, 3]]; B66 = BB[[3, 3]];
E11 = EE[[1, 1]]; E12 = EE[[1, 2]]; E16 = EE[[1, 3]]; E22 =
  EE[[2, 2]]; E26 = EE[[2, 3]]; E66 = EE[[3, 3]];
F11 = FF[[1, 1]]; F12 = FF[[1, 2]]; F16 = FF[[1, 3]]; F22 =
  FF[[2, 2]]; F26 = FF[[2, 3]]; F66 = FF[[3, 3]];
H11 = HH[[1, 1]]; H12 = HH[[1, 2]]; H16 = HH[[1, 3]]; H22 =
  HH[[2, 2]]; H26 = HH[[2, 3]]; H66 = HH[[3, 3]];
A44 = aa[[1, 1]]; A45 = aa[[1, 2]]; A55 = aa[[2, 2]];
D44 = dd[[1, 1]]; D45 = dd[[1, 2]]; D55 = dd[[2, 2]];
F44 = gg[[1, 1]]; F45 = gg[[1, 2]]; F55 = gg[[2, 2]];
P = {1};

```

```

Do[Do[q1 = P[[Pn]] Sin[ \[Pi]/a x] Sin[ \[Pi]/b y] /. b -> bb[[bn]];
(* Approximation function for simply supported plate (SSSS) \
centered at (0, 0)*)
u = Simplify[
  Sum[Cu[i, j] (b - y) y x^i y^j, {i, 0, 5}, {j, 0, 5}] /.
  b -> bb[[bn]]];
v = Simplify[
  Sum[Cv[i, j] (a - x) x x^i y^j, {i, 0, 5}, {j, 0, 5}] /.
  b -> bb[[bn]]];
w = Simplify[
  Sum[Cw[i, j] (a - x) (b - y) x y x^i y^j, {i, 0, 5}, {j, 0,
  5}] /. b -> bb[[bn]]];
\[Phi]x =
Simplify[
  Sum[C\[Phi]x[i, j] (b - y) y x^i y^j, {i, 0, 5}, {j, 0, 5}] /.
  b -> bb[[bn]]];
\[Phi]y =
Simplify[
  Sum[C\[Phi]y[i, j] (a - x) x x^i y^j, {i, 0, 5}, {j, 0, 5}] /.
  b -> bb[[bn]]];

coefu = Cases[Cases[Variables[u], Except[x]], Except[y]];
Nu = Length[coefu];
coefv = Cases[Cases[Variables[v], Except[x]], Except[y]];
Nv = Length[coefv];
coefw = Cases[Cases[Variables[w], Except[x]], Except[y]];
Nw = Length[coefw];
coef\[Phi]x = Cases[Cases[Variables[\[Phi]x], Except[x]], Except[y]];
N\[Phi]x = Length[coef\[Phi]x];
coef\[Phi]y = Cases[Cases[Variables[\[Phi]y], Except[x]], Except[y]];
N\[Phi]y = Length[coef\[Phi]y];
nn = Nu + Nv + Nw + N\[Phi]x + N\[Phi]y;

(* Extracting number of coefficients and corresponding \
approximation polynomial functions and their derivatives*)
u = Simplify[u /. Table[coefu[[i]] -> Cu[i], {i, 1, Nu}]];
v = Simplify[v /. Table[coefv[[j]] -> Cv[j], {j, 1, Nv}]];
w = Simplify[w /. Table[coefw[[k]] -> Cw[k], {k, 1, Nw}]];
\[Phi]x =
Simplify[\[Phi]x /.
  Table[coef\[Phi]x[[n]] -> C\[Phi]x[n], {n, 1, N\[Phi]x}]];
\[Phi]y =
Simplify[\[Phi]y /.
  Table[coef\[Phi]y[[m]] -> C\[Phi]y[m], {m, 1, N\[Phi]y}]];

```

```

Do[Fu[i] = Coefficient[u, Cu[i]], {i, 1, Nu}];
Do[Fux[i] = Simplify[D[Fu[i], x]], {i, 1, Nu}];
Do[Fuy[i] = Simplify[D[Fu[i], y]], {i, 1, Nu}];
Do[Fv[j] = Coefficient[v, Cv[j]], {j, 1, Nv}];
Do[Fvx[j] = Simplify[D[Fv[j], x]], {j, 1, Nv}];
Do[Fvy[j] = Simplify[D[Fv[j], y]], {j, 1, Nv}];
Do[Fw[k] = Coefficient[w, Cw[k]], {k, 1, Nw}];
Do[Fwx[k] = Simplify[D[Fw[k], x]], {k, 1, Nw}];
Do[Fwy[k] = Simplify[D[Fw[k], y]], {k, 1, Nw}];
Do[Fwxx[k] = Simplify[D[Fw[k], {x, 2}]], {k, 1, Nw}];
Do[Fwyy[k] = Simplify[D[Fw[k], {y, 2}]], {k, 1, Nw}];
Do[Fwxy[k] = Simplify[D[D[Fw[k], {x, 1}], {y, 1}], {k, 1, Nw}];
(*Do[Lap[k]=Simplify[D[Fw[k],{x,2}]+D[Fw[k],{y,2}],{k,1,Nw}]*);
Do[F\[Phi]x[n] = Coefficient[\[Phi]x, C\[Phi]x[n]], {n, 1,
  N\[Phi]x}];
Do[Fx\[Phi]x[n] = Simplify[D[F\[Phi]x[n], x]], {n, 1, N\[Phi]x}];
Do[Fy\[Phi]x[n] = Simplify[D[F\[Phi]x[n], y]], {n, 1, N\[Phi]x}];
Do[F\[Phi]y[m] = Coefficient[\[Phi]y, C\[Phi]y[m]], {m, 1,
  N\[Phi]y}];
Do[Fx\[Phi]y[m] = Simplify[D[F\[Phi]y[m], x]], {m, 1, N\[Phi]y}];
Do[Fy\[Phi]y[m] = Simplify[D[F\[Phi]y[m], y]], {m, 1, N\[Phi]y}];

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K11 = Table[0, {r, 1, Nu}, {i, 1, Nu}];
K12 = Table[0, {r, 1, Nu}, {j, 1, Nv}];
K13 = Table[0, {r, 1, Nu}, {k, 1, Nw}];
K14 = Table[0, {r, 1, Nu}, {n, 1, N\[Phi]x}];
K15 = Table[0, {r, 1, Nu}, {m, 1, N\[Phi]y}];
K21 = Table[0, {s, 1, Nv}, {i, 1, Nu}];
K22 = Table[0, {s, 1, Nv}, {j, 1, Nv}];
K23 = Table[0, {s, 1, Nv}, {k, 1, Nw}];
K24 = Table[0, {s, 1, Nv}, {n, 1, N\[Phi]x}];
K25 = Table[0, {s, 1, Nv}, {m, 1, N\[Phi]y}];
K31 = Table[0, {t, 1, Nw}, {i, 1, Nu}];
K32 = Table[0, {t, 1, Nw}, {j, 1, Nv}];
K33 = Table[0, {t, 1, Nw}, {k, 1, Nw}];
K34 = Table[0, {t, 1, Nw}, {n, 1, N\[Phi]x}];
K35 = Table[0, {t, 1, Nw}, {m, 1, N\[Phi]y}];
K41 = Table[0, {p, 1, N\[Phi]x}, {i, 1, Nu}];
K42 = Table[0, {p, 1, N\[Phi]x}, {j, 1, Nv}];
K43 = Table[0, {p, 1, N\[Phi]x}, {k, 1, Nw}];
K44 = Table[0, {p, 1, N\[Phi]x}, {n, 1, N\[Phi]x}];
K45 = Table[0, {p, 1, N\[Phi]x}, {m, 1, N\[Phi]y}];
K51 = Table[0, {q, 1, N\[Phi]y}, {i, 1, Nu}];
K52 = Table[0, {q, 1, N\[Phi]y}, {j, 1, Nv}];
K53 = Table[0, {q, 1, N\[Phi]y}, {k, 1, Nw}];

```

K54 = Table[0, {q, 1, N\[Phi]y}, {n, 1, N\[Phi]x}];
K55 = Table[0, {q, 1, N\[Phi]y}, {m, 1, N\[Phi]y}];

K = Table[0, {i, 1, nn}, {j, 1, nn}];
FAB = NLFAB = Table[0, {i, 1, Nu + Nv}];
FBC = NLFBC = Table[0, {i, 1, N\[Phi]x + N\[Phi]y}];
FC = NLFC = Table[0, {t, 1, Nw}];
F = NLF = Table[0, {i, 1, nn}];

(* Terms of Equation casted *)

Do[K11[[r, i]] =
K[[r, i]] =
Intrec[(Fux[i] (A11 Fux[r] + A16 Fuy[r]) +
Fuy[i] (A66 Fuy[r] + A16 Fux[r])), a, bb[[bn]]], {r, 1,
Nu}, {i, 1, Nu}];
Do[K12[[r, j]] =
K[[r, j + Nu]] =
Intrec[(Fvx[j] (A66 Fuy[r] + A16 Fux[r]) +
Fvy[j] (A12 Fux[r] + A26 Fuy[r])), a, bb[[bn]]], {r, 1,
Nu}, {j, 1, Nv}];
Do[K13[[r, k]] =
K[[r, k + Nu +
Nv]] = -C1 Intrec[(Fwxx[k] (E11 Fux[r] + E16 Fuy[r]) +
Fwyy[k] (E12 Fux[r] + E26 Fuy[r]) +
2 Fwxy[k] (E16 Fux[r] + E66 Fuy[r])), a, bb[[bn]]], {r, 1,
Nu}, {k, 1, Nw}];
Do[K14[[r, n]] =
K[[r, n + Nu + Nv + Nw]] =
Intrec[(Fv\[Phi]x[
n] ((B11 - C1 E11) Fux[r] + (B16 - C1 E16) Fuy[r]) +
Fy\[Phi]x[n] ((B16 - C1 E16) Fux[r] + (B66 - C1 E66) Fuy[r])),
a, bb[[bn]]], {r, 1, Nu}, {n, 1, N\[Phi]x}];
Do[K15[[r, m]] =
K[[r, m + Nu + Nv + Nw + N\[Phi]x]] =
Intrec[(Fy\[Phi]y[
m] ((B12 - C1 E12) Fux[r] + (B26 - C1 E26) Fuy[r]) +
Fx\[Phi]y[
m] ((B16 - C1 E16) Fux[r] + (B66 - C1 E66) Fuy[r])), a,
bb[[bn]]], {r, 1, Nu}, {m, 1, N\[Phi]y}];
Do[K21[[s, i]] =
K[[s + Nu, i]] =
Intrec[(Fuy[i] (A66 Fvx[s] + A26 Fvy[s]) +
Fux[i] (A12 Fvy[s] + A16 Fvx[s])), a, bb[[bn]]], {s, 1,
Nv}, {i, 1, Nu}];
Do[K22[[s, j]] =
K[[s + Nu, j + Nu]] =

$\text{Intrec}[(Fvx[j] (A66 Fvx[s] + A26 Fvy[s]) +$
 $Fvy[j] (A22 Fvy[s] + A26 Fvx[s])), a, bb[[bn]]], \{s, 1,$
 $Nv\}, \{j, 1, Nv\}];$
 $\text{Do}[K23[[s, k]] =$
 $K[[s + Nu,$
 $k + Nu +$
 $Nv]] = -C1 \text{Intrec}[(Fwxx[k] (E16 Fvx[s] + E12 Fvy[s]) +$
 $Fwyy[k] (E22 Fvy[s] + E26 Fvx[s]) +$
 $2 Fwxy[k] (E26 Fvy[s] + E66 Fvx[s])), a, bb[[bn]]], \{s, 1,$
 $Nv\}, \{k, 1, Nw\}];$
 $\text{Do}[K24[[s, n]] =$
 $K[[s + Nu, n + Nu + Nv + Nw]] =$
 $\text{Intrec}[(Fx\backslash[\text{Phi}]x[$
 $n] ((B12 - C1 E12) Fvy[s] + (B16 - C1 E16) Fvx[s]) +$
 $Fy\backslash[\text{Phi}]x[n] ((B26 - C1 E26) Fvy[s] + (B66 - C1 E66) Fvx[s])),$
 $a, bb[[bn]]], \{s, 1, Nv\}, \{n, 1, N\backslash[\text{Phi}]x\}];$
 $\text{Do}[K25[[s, m]] =$
 $K[[s + Nu, m + Nu + Nv + Nw + N\backslash[\text{Phi}]x]] =$
 $\text{Intrec}[(Fy\backslash[\text{Phi}]y[$
 $m] ((B26 - C1 E26) Fvx[s] + (B22 - C1 E22) Fvy[s]) +$
 $Fx\backslash[\text{Phi}]y[m] ((B26 - C1 E26) Fvy[s] + (B66 - C1 E66) Fvx[s])),$
 $a, bb[[bn]]], \{s, 1, Nv\}, \{m, 1, N\backslash[\text{Phi}]y\}];$
 $\text{Do}[K31[[t, i]] =$
 $K[[t + Nu + Nv,$
 $i]] = -C1 \text{Intrec}[(Fux[$
 $i] (E11 Fwxx[t] + E12 Fwyy[t] + 2 E16 Fwxy[t]) +$
 $Fuy[i] (E16 Fwxx[t] + E26 Fwyy[t] + 2 E66 Fwxy[t])), a,$
 $bb[[bn]]], \{t, 1, Nw\}, \{i, 1, Nu\}];$
 $\text{Do}[K32[[t, j]] =$
 $K[[t + Nu + Nv,$
 $j + Nu]] = -C1 \text{Intrec}[(Fvx[$
 $j] (2 E66 Fwxy[t] + E16 Fwxx[t] + E26 Fwyy[t]) +$
 $Fvy[j] (2 E26 Fwxy[t] + E12 Fwxx[t] + E22 Fwyy[t])), a,$
 $bb[[bn]]], \{t, 1, Nw\}, \{j, 1, Nv\}];$
 $\text{Do}[K33[[t, k]] =$
 $K[[t + Nu + Nv, k + Nu + Nv]] =$
 $\text{Intrec}[(Fwx[$
 $k] ((A55 + C2^2 F55 - 2 C2 D55) Fwx[$
 $t] + (A45 + C2^2 F45 - 2 C2 D45) Fwy[t]) +$
 $Fwy[k] ((A45 + C2^2 F45 - 2 C2 D45) Fwx[$
 $t] + (A44 + C2^2 F44 - 2 C2 D44) Fwy[t]) +$
 $Fwxx[k] C1^2 (H11 Fwxx[t] + H12 Fwyy[t] + 2 H16 Fwxy[t]) +$
 $Fwyy[k] C1^2 (H12 Fwxx[t] + H22 Fwyy[t] + 2 H26 Fwxy[t]) +$
 $2 Fwxy[k] C1^2 (H16 Fwxx[t] + H26 Fwyy[t] + 2 H66 Fwxy[t])),$
 $a, bb[[bn]]], \{t, 1, Nw\}, \{k, 1, Nw\}];$
 $\text{Do}[K34[[t, n]] =$

$K[[t + Nu + Nv, n + Nu + Nv + Nw]] =$
 $\text{Intrec}[(F[\Phi]x[$
 $n] ((A45 + C2^2 F45 - 2 C2 D45) Fwy[$
 $t] + (A55 + C2^2 F55 - 2 C2 D55) Fwx[t]) +$
 $Fx\backslash[\Phi]x[$
 $n] ((C1^2 H11 - C1 F11) Fwxx[t] + (C1^2 H12 - C1 F12) Fwyy[$
 $t] + (C1^2 H16 - C1 F16) 2 Fwxy[t]) +$
 $Fy\backslash[\Phi]x[$
 $n] ((C1^2 H16 - C1 F16) Fwxx[t] + (C1^2 H26 - C1 F26) Fwyy[$
 $t] + (C1^2 H66 - C1 F66) 2 Fwxy[t])), a, bb[[bn]]], \{t,$
 $1, Nw\}, \{n, 1, N\backslash[\Phi]x\}];$
 $\text{Do}[K35[[t, m]] =$
 $K[[t + Nu + Nv, m + Nu + Nv + Nw + N\backslash[\Phi]x]] =$
 $\text{Intrec}[(F[\Phi]y[$
 $m] ((A44 + C2^2 F44 - 2 C2 D44) Fwy[$
 $t] + (A45 + C2^2 F45 - 2 C2 D45) Fwx[t]) +$
 $Fy\backslash[\Phi]y[$
 $m] ((C1^2 H12 - C1 F12) Fwxx[t] + (C1^2 H22 - C1 F22) Fwyy[$
 $t] + (C1^2 H26 - C1 F26) 2 Fwxy[t]) +$
 $Fx\backslash[\Phi]y[$
 $m] ((C1^2 H16 - C1 F16) Fwxx[t] + (C1^2 H26 - C1 F26) Fwyy[$
 $t] + (C1^2 H66 - C1 F66) 2 Fwxy[t])), a, bb[[bn]]], \{t,$
 $1, Nw\}, \{m, 1, N\backslash[\Phi]y\}];$
 $\text{Do}[K41[[p, i]] =$
 $K[[p + Nu + Nv + Nw, i]] =$
 $\text{Intrec}[(Fux[$
 $i] ((B11 - C1 E11) Fx\backslash[\Phi]x[p] + (B16 - C1 E16) Fy\backslash[\Phi]x[$
 $p]) + Fuy[$
 $i] ((B16 - C1 E16) Fx\backslash[\Phi]x[p] + (B66 - C1 E66) Fy\backslash[\Phi]x[$
 $p])), a, bb[[bn]]], \{p, 1, N\backslash[\Phi]x\}, \{i, 1, Nu\}];$
 $\text{Do}[K42[[p, j]] =$
 $K[[p + Nu + Nv + Nw, j + Nu]] =$
 $\text{Intrec}[(Fvx[$
 $j] ((B16 - C1 E16) Fx\backslash[\Phi]x[p] + (B66 - C1 E66) Fy\backslash[\Phi]x[$
 $p]) + Fvy[$
 $j] ((B12 - C1 E12) Fx\backslash[\Phi]x[p] + (B26 - C1 E26) Fy\backslash[\Phi]x[$
 $p])), a, bb[[bn]]], \{p, 1, N\backslash[\Phi]x\}, \{j, 1, Nv\}];$
 $\text{Do}[K43[[p, k]] =$
 $K[[p + Nu + Nv + Nw, k + Nu + Nv]] =$
 $\text{Intrec}[(F[\Phi]x[$
 $p] ((A45 + C2^2 F45 - 2 C2 D45) Fwy[$
 $k] + (A55 + C2^2 F55 - 2 C2 D55) Fwx[k]) +$
 $Fx\backslash[\Phi]x[$
 $p] ((C1^2 H11 - C1 F11) Fwxx[k] + (C1^2 H12 - C1 F12) Fwyy[$
 $k] + (C1^2 H16 - C1 F16) 2 Fwxy[k]) +$
 $Fy\backslash[\Phi]x[$

$$p] ((C1^2 H16 - C1 F16) Fwxx[k] + (C1^2 H26 - C1 F26) Fwyy [$$

$$k] + (C1^2 H66 - C1 F66) 2 Fwxy [k])), a, bb[[bn]]], \{p,$$

$$1, N[\Phi]x\}, \{k, 1, Nw\};$$

$$Do[K44[[p, n]] =$$

$$K[[p + Nu + Nv + Nw, n + Nu + Nv + Nw]] =$$

$$Intrec[(Fx[\Phi]x[$$

$$n] ((D11 + C1^2 H11 - 2 C1 F11) Fx[\Phi]x[$$

$$p] + (D16 + C1^2 H16 - 2 C1 F16) Fy[\Phi]x[p]) +$$

$$Fy[\Phi]x[$$

$$n] ((D16 + C1^2 H16 - 2 C1 F16) Fx[\Phi]x[$$

$$p] + (D66 + C1^2 H66 - 2 C1 F66) Fy[\Phi]x[p]) + ((A55 +$$

$$C2^2 F55 - 2 C2 D55) F[\Phi]x[p] F[\Phi]x[n])), a,$$

$$bb[[bn]]], \{p, 1, N[\Phi]x\}, \{n, 1, N[\Phi]x\};$$

$$Do[K45[[p, m]] =$$

$$K[[p + Nu + Nv + Nw, m + Nu + Nv + Nw + N[\Phi]x]] =$$

$$Intrec[(Fy[\Phi]y[$$

$$m] ((D12 + C1^2 H12 - 2 C1 F12) Fx[\Phi]x[$$

$$p] + (D26 + C1^2 H26 - 2 C1 F26) Fy[\Phi]x[p]) +$$

$$Fx[\Phi]y[$$

$$m] ((D16 + C1^2 H16 - 2 C1 F16) Fx[\Phi]x[$$

$$p] + (D66 + C1^2 H66 - 2 C1 F66) Fy[\Phi]x[p]) + ((A45 +$$

$$C2^2 F45 - 2 C2 D45) F[\Phi]x[p] F[\Phi]y[m])), a,$$

$$bb[[bn]]], \{p, 1, N[\Phi]x\}, \{m, 1, N[\Phi]y\};$$

$$Do[K51[[q, i]] =$$

$$K[[q + Nu + Nv + Nw + N[\Phi]x, i]] =$$

$$Intrec[(Fux[$$

$$i] ((B12 - C1 E12) Fy[\Phi]y[q] + (B16 - C1 E16) Fx[\Phi]y[$$

$$q]) + Fuy[$$

$$i] ((B26 - C1 E26) Fy[\Phi]y[q] + (B66 - C1 E66) Fx[\Phi]y[$$

$$q])), a, bb[[bn]]], \{q, 1, N[\Phi]y\}, \{i, 1, Nu\};$$

$$Do[K52[[q, j]] =$$

$$K[[q + Nu + Nv + Nw + N[\Phi]x, j + Nu]] =$$

$$Intrec[(Fvx[$$

$$j] ((B26 - C1 E26) Fy[\Phi]y[q] + (B66 - C1 E66) Fx[\Phi]y[$$

$$q]) + Fvy[$$

$$j] ((B26 - C1 E26) Fx[\Phi]y[q] + (B22 - C1 E22) Fy[\Phi]y[$$

$$q])), a, bb[[bn]]], \{q, 1, N[\Phi]y\}, \{j, 1, Nv\};$$

$$Do[K53[[q, k]] =$$

$$K[[q + Nu + Nv + Nw + N[\Phi]x, k + Nu + Nv]] =$$

$$Intrec[(F[\Phi]y[$$

$$q] ((A44 + C2^2 F44 - 2 C2 D44) Fwy[$$

$$k] + (A45 + C2^2 F45 - 2 C2 D45) Fwx [k]) +$$

$$Fx[\Phi]y[$$

$$q] ((C1^2 H16 - C1 F16) Fwxx[k] + (C1^2 H26 - C1 F26) Fwyy [$$

$$k] + (C1^2 H66 - C1 F66) 2 Fwxy [k]) +$$

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Fy\[Phi]y[
q] ((C1^2 H12 - C1 F12) Fwxx[
k] + (C1^2 H22 - C1 F22) Fwyy [
k] + (C1^2 H26 - C1 F26) 2 Fwxy [k])), a, bb[[bn]]], {q,
1, N\[Phi]y}, {k, 1, Nw}}];
Do[K54[[q, n]] =
K[[q + Nu + Nv + Nw + N\[Phi]x, n + Nu + Nv + Nw]] =
Intrec[(Fx\[Phi]x[
n] ((D12 + C1^2 H12 - 2 C1 F12) Fy\[Phi]y[
q] + (D16 + C1^2 H16 - 2 C1 F16) Fx\[Phi]y[q]) +
Fy\[Phi]x[
n] ((D26 + C1^2 H26 - 2 C1 F26) Fy\[Phi]y[
q] + (D66 + C1^2 H66 - 2 C1 F66) Fx\[Phi]y[q]) + ((A45 +
C2^2 F45 - 2 C2 D45) F\[Phi]y[q] F\[Phi]x[n])), a,
bb[[bn]]], {q, 1, N\[Phi]y}, {n, 1, N\[Phi]x}}];
Do[K55[[q, m]] =
K[[q + Nu + Nv + Nw + N\[Phi]x, m + Nu + Nv + Nw + N\[Phi]x]] =
Intrec[(Fx\[Phi]y[
m] ((D26 + C1^2 H26 - 2 C1 F26) Fy\[Phi]y[
q] + (D66 + C1^2 H66 - 2 C1 F66) Fx\[Phi]y[q]) +
Fy\[Phi]y[
m] ((D26 + C1^2 H26 - 2 C1 F26) Fx\[Phi]y[
q] + (D22 + C1^2 H22 - 2 C1 F22) Fy\[Phi]y[q]) + ((A44 +
C2^2 F44 - 2 C2 D44) F\[Phi]y[q] F\[Phi]y[m])), a,
bb[[bn]]], {q, 1, N\[Phi]y}, {m, 1, N\[Phi]y}}];

Do[FC[[t]] = NIntegrate[q1 Fw[t], {x, 0, a}, {y, 0, bb[[bn]]}], {t,
1, Nw}}];
Do[F[[t + Nu + Nv]] = FC[[t]], {t, 1, Nw}}];
(*Do[F[[t+N\[Phi]y+N\[Phi]x]]=FC[[t]],{t,1,Nw}}]*);

(* Direct solution of the coefficients Subscript[Cu, i], Subscript[
Cv, j] , Subscript[Cw, k] , C\[Phi]x and Subscript[C\[Phi], y ]
using FindRoot*)
cCu = Table[Cu[r], {r, 1, Nu}];
cCv = Table[Cv[s], {s, 1, Nv}];
cCw = Table[Cw[t], {t, 1, Nw}];
cC\[Phi]x = Table[C\[Phi]x[p], {p, 1, N\[Phi]x}];
cC\[Phi]y = Table[C\[Phi]y[q], {q, 1, N\[Phi]y}];
Cuvw\[Phi]x\[Phi]y[Pn, bn] =
Join[cCu, cCv, cCw, cC\[Phi]x, cC\[Phi]y];

(*solCuvw[0,bn]={Table[Cuvw[[i]] \[Rule] 0.00i,{i,1,nn}}];
Cuvw[i[Pn-1]]=Table[{Cuvw[[i]] ,Cuvw[[i]]/. solCuvw[Pn-1,bn]][[1,
i]]},{i,1,nn}}];*)
solCuvw\[Phi]x\[Phi]y[Pn, bn] =

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Solve[K.Cuvw\[Phi]x\[Phi]y[Pn, bn] == F,
Cuvw\[Phi]x\[Phi]y[Pn, bn]];

(* Soltion of u, v and w and evaluating forces,
moments and stresses in functional form*)
w = w /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
u = u /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
v = v /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
\[Phi]x = \[Phi]x /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
\[Phi]y = \[Phi]y /. solCuvw\[Phi]x\[Phi]y[Pn, bn][[1]];
wx = D[w, x]; wy = D[w, y]; wxx = D[w, x]; wyy = D[w, y];
wxy = D[w, x, y];
ux = D[u, x]; vx = D[v, x]; uy = D[u, y];
vy = D[v, y]; \[Phi]xx = D[\[Phi]x, x]; \[Phi]yy =
D[\[Phi]y, y]; \[Phi]xy = D[\[Phi]x, y]; \[Phi]yx = D[\[Phi]y, x];
(* Save the solution for each aspect ratio and loading increment*)
W[Pn, bn] = w; u1[Pn, bn] = u;
v1[Pn, bn] =
v; \[Phi]x1[Pn, bn] = \[Phi]x; \[Phi]y1[Pn, bn] = \[Phi]y;
\[Kappa][Pn,
bn] = {\[Phi]xx, \[Phi]yy, \[Phi]xy + \[Phi]yx}; \[Epsilon]0[Pn,
bn] = {ux, vy, uy + vx}; \[Kappa]2[Pn,
bn] = {\[Phi]y + wy, \[Phi]x + wx}; \[Kappa]3[Pn,
bn] = {-C1 (\[Phi]xx + wxx), -C1 (\[Phi]yy +
wyy), -C1 (\[Phi]xy + \[Phi]yx + 2 wxy)};
\[Kappa]4[Pn, bn] = {-C2 (\[Phi]y + wy), -C2 (\[Phi]x + wx)};
{Mx1[Pn, bn], My1[Pn, bn], Mxy[Pn, bn]} =
BB.\[Epsilon]0[Pn, bn] + DD.\[Kappa][Pn, bn] + FF.\[Kappa]3[Pn, bn];
{Nx1[Pn, bn], Ny1[Pn, bn], Nxy[Pn, bn]} =
BB.\[Kappa][Pn, bn] + AA.\[Epsilon]0[Pn, bn];
{Qy[Pn, bn], Qx[Pn, bn]} = aa.\[Kappa]2[Pn, bn];
{\[Sigma]x[Pn, bn], \[Sigma]y[Pn, bn], \[Sigma]xy[Pn, bn]} =
z^3 Qb[[1 ;; 3, 1 ;; 3]].\[Kappa]3[Pn, bn] +
z Qb[[1 ;; 3, 1 ;; 3]].\[Kappa][Pn, bn] +
Qb[[1 ;; 3, 1 ;; 3]].\[Epsilon]0[Pn, bn]; {\[Tau]yz[Pn,
bn], \[Tau]xz[Pn, bn]} =
z^2 Qb[[4 ;; 5, 4 ;; 5]].\[Kappa]4[Pn, bn] +
Qb[[4 ;; 5, 4 ;; 5]].\[Kappa]2[Pn, bn];, {Pn, 1, Length[P]}, {bn,
1, Length[bb]}]

```

Vitae

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Publications :

Journals:

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