

ON SINGULARLY PERTURBED GENERALIZED PHASE-FIELD DYNAMICAL SYSTEMS

BY

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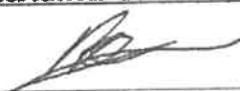
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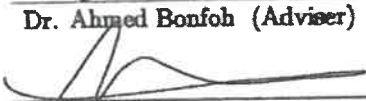
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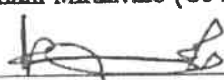
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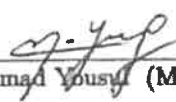
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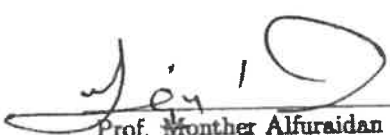

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*This dissertation is dedicated to
My Family*

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DISSERTATION ABSTRACT

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TITLE OF STUDY: On Singularly Perturbed Generalized Phase-field Dynamical Systems

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This dissertation concentrates on models of Phase-field systems that take into account the long-range interactions in phase separation. These models comprise a system of two parabolic equations that involve two unknowns, namely the absolute temperature and the order parameter. We investigate the well-posedness issues and the existence of the global attractor, exponential attractors and inertial manifolds. These objects describe the large-time behavior of dynamical systems. Moreover, considering a singular perturbation parameter in the equations, we examine continuity properties of the global and exponential attractors, as this parameter tends to zero. These results show the convergence of the dynamics to those of the corresponding generalized Cahn-Hilliard equations.

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تتناول في هذه الأطروحة دراسة ثلاثة نماذج من أنظمة مجال الطور للأطوار الانتقالية التي تأخذ في الاعتبار التفاعلات على المدى الطويل في فصل الأطوار. تتألف هذه الأنظمة من معادلتين تتضمنان متغيرين هما درجة الحرارة لمادة قد تظهر في مرحلتين مختلفتين (على سبيل المثال سائل-صلب) بالإضافة إلى المعامل الترتيبي الذي يصف المرحلة الحالية. سوف ندرس وجود الجاذب الشامل والجاذب الأسيّة ومشعبات القصور الذاتي التي تصف السلوك طويل المدى للأنظمة الديناميكية. علاوة على ذلك، نفحص خصائص الاستمرارية لجاذبين وتقارب الديناميكيات للأنظمة المضطربة التي تحتوي المعلمة $\epsilon > 0$ مع نظيرتها الغير مضطربة عندما $\epsilon = 0$ والتي تمثل بعض من معادلات كاهن-هيليارد المعممة.

CHAPTER 1

INTRODUCTION

Phase-field equations describe phase transition processes such as melting or solidification. In this dissertation, we analyse a generalized phase-field system that takes into account the long-range interactions in the phase separation process

$$\begin{aligned}\phi_t - \Delta(-\Delta\phi + g(\phi) - u) + r(\phi) &= 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t - \Delta u + h(u) &= 0, \quad \text{in } \Omega \times \mathbb{R}^+, \end{aligned} \tag{1.0.1}$$

where ϵ is the heat capacity, u is the absolute temperature and ϕ is the order parameter. When $u = 0$, we have the ordinary planar equilibrium melting temperature while ϕ near -1 is related to the solid phase and ϕ near +1 the liquid phase. The set of points at which ϕ vanishes is known as the interface.

Simulation for such a phenomena (e.g., figure [1.1](#)): $\phi = y_B - y_A \in [-1, 1]$; $\phi = -1$: pure comp. A, $\phi = 1$: pure comp. B, $\phi = 0$: interface.

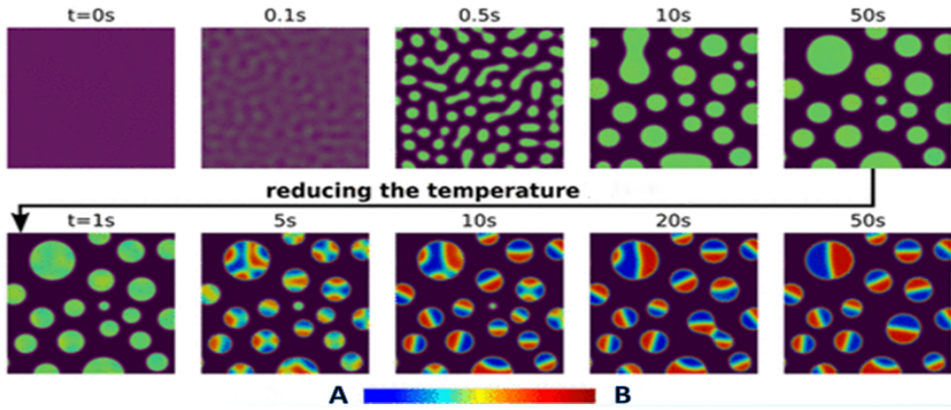


Figure 1.1: Phase separation of two components A & B in a material science (c.f. [44])

The long time behavior of solutions of dissipative systems is described by a certain compact invariant set lying on the phase space which attracts uniformly the trajectories starting from bounded sets when time goes to infinity. This set is called the global attractor.

Let E an infinite-dimension phase space, for any $R > 0$ and $u_0 \in E$ such that $|u_0|_E \leq R$, $\exists t_0(R)$ such that $u(t) \in \mathcal{A}$, $\forall t \geq t_0(R)$:

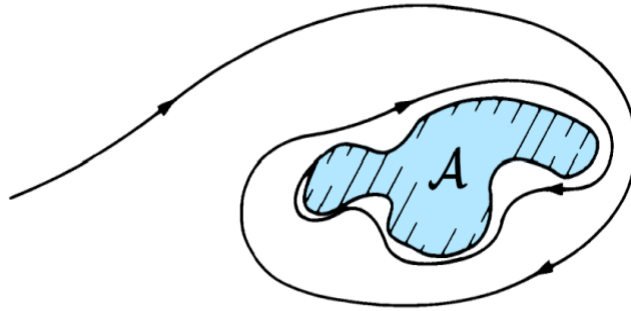


Figure 1.2: Attractor ($\mathcal{A}$) (c.f. [30, Milani and Kokschi])

In many cases, the global attractor has a finite fractal dimension. However, we do not know in general how trajectories approach the global attractor. It is possible

for this set to have a complicated structure like that of a fractal set and the rate of attraction of the trajectories may be small and very sensitive to perturbations and approximations, including numerical simulations. Moreover, the known results on the continuity of global attractors with respect to perturbations make use of restrictive assumptions on the stationary solutions and their unstable sets (see, e.g., [27], [38], [40], [41]). These drawbacks led C. Foias et al. to introduce a new object called exponential attractor. That is a compact and positively invariant set with finite fractal dimension which attracts all the trajectories starting from bounded sets at a uniform exponential rate (see [1]). Clearly, an exponential attractor contains the global attractor and the existence of exponential attractors guarantees the existence of a finite fractal dimensional global attractor.

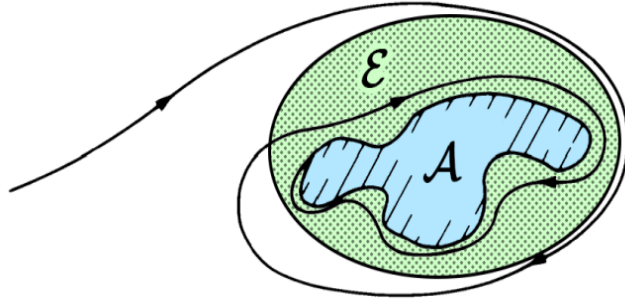


Figure 1.3: Exponential Attractor ($\mathcal{E}$) (c.f. [41, R. Temam])

In addition, exponential attractors are more robust than global attractors with respect to perturbations and approximations. Indeed, to prove lower semi-continuity of the global attractor we need restrictive conditions on the equilibrium solution while these restrictions are not needed for construction of exponential attractors that are

lower semi-continuous. (see [37] and its references).

The finite fractal dimension of the global attractor implies there exists a continuous parameterization of $\mathcal{A}$ using k coordinates where k is large enough. More precisely, for any solution $u(t) \in \mathcal{A}$, there exist continuous functions $\{\alpha_j(t)\}$ such that $u(t) = L^{-1}(\alpha_1(t) e_1 + \alpha_2(t) e_2 + \cdots + \alpha_k(t) e_k)$, where $\{e_j\}_j$ is the canonical basis of $\mathbb{R}^k$ and L is bounded linear map $L : E \rightarrow \mathbb{R}^k$ injective on $\mathcal{A}$ and its inverse is Hölder continuous on $L\mathcal{A}$ (see [40]). However, the description of the functions $\alpha_j(t)$, $j = 1 \dots k$ is currently unknown.

The existence of an inertial manifold provides a way to describe these functions $\alpha_j(t)$ which are solutions of an ODE. An inertial manifold is a positively invariant smooth finite-dimensional manifold that contains the global attractor and that attracts the trajectories at a uniform exponential rate. It follows that the apriori infinite-dimensional dynamical system reduces, on the inertial manifold, to a finite system of ordinary differential equations. See (e.g., [40], [41], [6]).

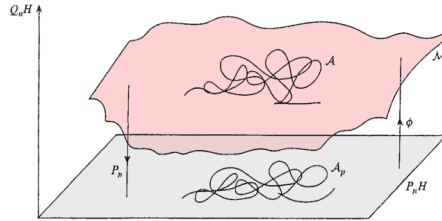


Figure 1.4: Inertial Manifold ($\mathcal{M}$); $P_n H \simeq \mathbb{R}^n$ (c.f. [40, Robinson])

Another interesting issue is how the global attractors, exponential attractors and inertial manifolds are stable under small perturbations of a parameter in the equation in view of upper and lower semi-continuity. This helps to understand the impact

of minor perturbations on the dynamics and also in numerical approximations and simulations. In general, the lower semi-continuity property does not hold, as shown by the following ODE

$$\frac{dx}{dt} = (1 - x)(x^2 - \lambda), \quad \lambda \in [-1, 1].$$

Here $\mathcal{A}_0 = [0, 1]$, $\mathcal{A}_\lambda = [-\sqrt{\lambda}, 1]$ for $\lambda > 0$ and $\mathcal{A}_\lambda = \{1\}$ for $\lambda < 0$. The global attractors change drastically under small perturbations.

The purpose of this dissertation is to analyse the model (1.0.1). Our results complete, improve and generalize previous results. We investigate the well-posedness of the initial boundary value problems. Then, we show that the solutions semigroups admit a global attractor which is upper semi-continuous at $\epsilon = 0$. Next, we construct robust families of exponential attractors that are Hölder continuous at $\epsilon = 0$. Lastly, we also prove the existence of inertial manifolds for the viscous phase-field system.

1.1 Main results

Firstly, in Chapter 2, we consider the Phase-field System of Type Oono

$$\begin{aligned} \phi_t - \Delta(-\Delta\phi + g(\phi) - u) + \sigma\phi &= 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t - \Delta u + \gamma u &= 0, & \text{in } \Omega \times \mathbb{R}^+, \end{aligned}$$

with $\sigma, \gamma > 0$, $\epsilon \in (0, \epsilon_0]$ with $\epsilon_0 = \min\{1, \frac{\gamma}{2\sigma}\}$ is the heat capacity and $g \in \mathcal{C}^3(\mathbb{R})$ is a nonlinear function satisfies the conditions

$$G(s) \geq -C_1, \quad C_1 \geq 0, \quad \forall s \in \mathbb{R}, \quad (1.1.1)$$

$\forall \gamma \in \mathbb{R}$, $\alpha \geq 0$ and $|\gamma| \leq \alpha$, $\exists C_2(\alpha) > 0, C_3(\alpha) \geq 0$ such that

$$(s - \gamma)g(s) - C_2G(s) \geq -C_3, \quad \forall s \in \mathbb{R}, \quad (1.1.2)$$

$$g'(s) \geq -C_4, \quad C_4 \geq 0, \quad \forall s \in \mathbb{R}, \quad (1.1.3)$$

$$\forall \mu > 0, \exists C_\mu > 0 \text{ such that } |g(s)| \leq \mu G(s) + C_\mu, \quad \forall s \in \mathbb{R}. \quad (1.1.4)$$

The problem is subject to the boundary conditions either of Neumann or periodic type

$$\partial_n \phi|_{\partial\Omega} = \partial_n \Delta \phi|_{\partial\Omega} = \partial_n u|_{\partial\Omega} = 0, \quad (1.1.5)$$

(the symbol ∂_n denotes the outward normal derivative) if Ω is a bounded domain of $\mathbb{R}^d$, with smooth boundary $\partial\Omega$, or

$$\begin{cases} D^\beta \phi|_{x_i=0} = D^\beta \phi|_{x_i=L_i}, & |\beta| \leq 3, \quad i = 1, \dots, d, \\ D^\beta u|_{x_i=0} = D^\beta u|_{x_i=L_i}, & |\beta| \leq 1, \quad i = 1, \dots, d, \end{cases} \quad (1.1.6)$$

(where $D^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \dots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}}$, $\beta_i \in \mathbb{N}$, $|\beta| = \beta_1 + \dots + \beta_d$) if $\Omega = \Pi_{i=1}^d(0, L_i)$, for the space dimensions $d = 1, 2$ or 3 .

Our aim in this chapter is to extend the analysis carried out in [10] to our generalized phase-field system. We prove a well-posedness result and the existence of the global attractor which is upper semicontinuity at $\epsilon = 0$. Then we construct a robust family

of exponential attractors that is Hölder continuous at $\epsilon = 0$. We refer the readers to Theorem 2.4.1, Theorem 2.5.1, Theorem 2.6.1 and Theorem 2.5.2.

Secondly, in Chapter 3, we consider the Conserved Viscous Phase-field System

$$\begin{aligned}\phi_t - \Delta(\phi_t - \Delta\phi + g(\phi) - u) &= 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t - \Delta u &= 0, \quad \text{in } \Omega \times \mathbb{R}^+, \end{aligned} \tag{1.1.7}$$

where $\epsilon \in (0, \frac{1}{\lambda_{n+1}}]$ and $g(\phi)$ is (same as in Chapter 2) a nonlinear function satisfies the conditions (1.1.1)-(1.1.4). The problem is associated with Neumann or Periodic boundary conditions (1.1.5) and (1.1.6).

We prove the existence of inertial manifolds for any $\epsilon \in (0, \frac{1}{\lambda_{n+1}}]$. We refer the readers to Theorem 3.0.2.

Lastly, in Chapter 4, we consider the Generalized Phase-field System

$$\begin{aligned}\phi_t - \Delta(-\Delta\phi + g(\phi) - u) + r(\phi) &= 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t - \Delta u + h(u) &= 0, \quad \text{in } \Omega \times \mathbb{R}^+, \end{aligned}$$

where $\epsilon \in (0, 1]$, the nonlinear functions $g(\phi)$, $r(\phi)$ and $h(u)$ are the following polynomials:

$$g(\phi) = \sum_{j=1}^{2m-1} a_j \phi^j, \quad a_{2m-1} > 0, \quad a_2 = 0, \quad m \in \mathbb{N}, \quad m \geq 2, \tag{1.1.8}$$

$$r(\phi) = \sum_{j=1}^m b_j \phi^j, \quad b_m > 0, \tag{1.1.9}$$

and

$$h(u) = \sum_{j=1}^{2n-1} c_j u^j, \quad n \in \mathbb{N}, c_n > 0, \quad H(s) = \int_0^s h(\sigma) d\sigma, \quad (1.1.10)$$

where $n \geq 2$ in $d = 1, 2$ and $n = 3$ in $d = 3$; satisfies the condition

$$h'(s) \geq C_6, \quad C_6 \geq 0, \quad \forall s \in \mathbb{R}. \quad (1.1.11)$$

This problem associated with Dirichlet boundary conditions

$$\phi|_{\partial\Omega} = \Delta\phi|_{\partial\Omega} = u|_{\partial\Omega} = 0, \quad (1.1.12)$$

if $\Omega \subset \mathcal{C}^4$ be a bounded set of $\mathbb{R}^d$ for space dimensions $d = 1, 2$ or 3 .

We prove a well-posedness result, the existence of the global attractor and its upper semi-continuity at $\epsilon = 0$. The construction of a robust family of the exponential attractors at $\epsilon = 0$ in the sense of upper and lower semi-continuity is achieved. We refer the readers to Theorem [4.4.1](#), Theorem [4.5.1](#), Theorem [4.5.2](#) and Theorem [4.6.2](#).

1.2 Models description

The modeling of the equations

$$\phi_t - \Delta(\delta\phi_t - \Delta\phi + g(\phi) - u) + r(\phi) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.2.1)$$

$$\epsilon u_t + \phi_t - \Delta u + h(u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.2.2)$$

where $g(\phi)$, $r(\phi)$ and $h(u)$ are nonlinear functions:

The constitutive law (c.f., e.g., [14])

$$\phi_t = \Delta \left(\frac{\delta \mathcal{F}}{\delta \phi} + \delta \phi_t \right), \quad (1.2.3)$$

where $\delta \geq 0$ is viscosity parameter and $\frac{\delta \mathcal{F}}{\delta \phi}$ denotes the variational derivative of the modified Landau-Ginzburg free energy

$$\mathcal{F}(\phi, u) = \int_{\Omega} \left(\frac{1}{2} |\nabla \phi|^2 + G(\phi) - \phi u + \int_{\Omega} r(\phi(y)) K(x, y) \phi(x) dy \right) dx, \quad (1.2.4)$$

that takes into account of long-range (i.e., nonlocal) interactions in phase separation. $g(\phi)$ is a nonlinear function where G is a double-well potential and $G' = g$ with global minimum 0 at $\phi = \pm 1$. The potential $F(\phi) = G(\phi) - \phi u$ is temperature dependent so that the relative depth of the two wells, representing pure solid and pure liquid phases, changes with temperature. At $u = 0$, $\phi = -1$ and $\phi = +1$ give the pure solid and liquid phases, respectively. (See Figure 1.5)

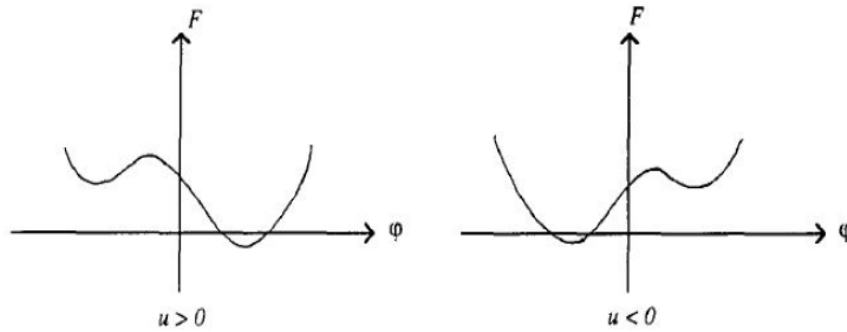


Figure 1.5: Double-well potential (c.f. [2, Bates and Zheng])

When the kernel function K is chosen to be fundamental solution of Laplace equation

$$k(x, y) = \begin{cases} -\frac{1}{2}|x - y|, & \text{when } d = 1, \\ -\frac{1}{2\pi} \ln |x - y|, & \text{when } d = 2, \\ \frac{1}{4\pi|x - y|}, & \text{when } d = 3, \end{cases} \quad (1.2.5)$$

then we have

$$-\Delta_y k(x, y) = \delta(x - y),$$

where δ is the Dirac distribution. The variational derivative of the free energy $\mathcal{F}$ is given by

$$\frac{\delta \mathcal{F}}{\delta \phi} = -\Delta \phi + g(\phi) - u + \int_{\Omega} k(x, y) r(\phi(y)) dy,$$

and since

$$-\int_{\Omega} \Delta_y k(x, y) r(\phi(y)) dy = r(\phi(x)),$$

we deduce Equation (1.2.1) by substituting $\frac{\delta \mathcal{F}}{\delta \phi}$ into (1.2.3).

Now, Equation (1.2.2) is derived from the constitutive law of energy

$$H_t = -h(u) - \nabla \cdot q, \quad (1.2.6)$$

where $H = \epsilon u + \phi$ is the enthalpy, $\epsilon > 0$ is heat capacity, q is the thermal flux vector

that is assumed to obey the classical Fourier law

$$q = -\nabla u. \quad (1.2.7)$$

From (1.2.3)-(1.2.6), we have the system (1.2.1)-(1.2.2).

We consider a particular cases of the this generalized model (1.2.1)-(1.2.2):

1. A Generalized Phase-field Model:

When $\delta = 0$, we have a Generalized Phase-field Model

$$\phi_t - \Delta(-\Delta\phi + g(\phi) - u) + r(\phi) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.2.8)$$

$$\epsilon u_t + \phi_t - \Delta u + h(u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+. \quad (1.2.9)$$

If $\epsilon = 0$, the corresponding unperturbed problem is the following:

$$\begin{cases} \phi_t - \Delta(-\Delta\phi + g(\phi) - u) + r(\phi) = 0, \\ \phi_t - \Delta u + h(u) = 0. \end{cases} \quad (1.2.10)$$

We will show in Section 4.3 that there exists a nonlinear operator $\mathcal{T}$ such that

$u = \mathcal{T}(\phi)$ and the system (1.2.10) can be rewritten as

$$\phi_t - \Delta(-\Delta\phi + g(\phi)) + \Delta\mathcal{T}(\phi(t)) + r(\phi) = 0. \quad (1.2.11)$$

This equation is known as a Cahn-Hilliard equation with a proliferation term.

2. Phase-field of Type Oono Model:

We set $\delta = 0$, $r(\phi) = \sigma\phi$, $\sigma > 0$ in (1.2.1) and $h(u) = \gamma u$, $\gamma > 0$ in (1.2.2). This model reads

$$\phi_t - \Delta(-\Delta\phi + g(\phi) - u) + \sigma\phi = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.2.12)$$

$$\epsilon u_t + \phi_t - \Delta u + \gamma u = 0, \quad \text{in } \Omega \times \mathbb{R}^+. \quad (1.2.13)$$

If $\epsilon = 0$, the corresponding unperturbed problem is the following System

$$\begin{cases} \phi_t - \Delta(-\Delta\phi + g(\phi) - u) + \sigma\phi = 0, \\ \phi_t - \Delta u + \gamma u = 0. \end{cases} \quad (1.2.14)$$

The system can be decoupled by resolving the initial and boundary value problem for ϕ alone first, then by deducing u from ϕ via a nonlinear mapping. Indeed, we deduce from the two equations of (1.2.14), by eliminating ϕ_t , the following equality

$$-2\Delta u + \gamma u = -\Delta(-\Delta\phi + g(\phi)) + \sigma\phi. \quad (1.2.15)$$

Equation (1.2.15) is a Poisson boundary value problem. For any $\gamma > 0$, and for each ϕ , this problem has a unique solution u and we can write

$$u = L_\gamma \phi, \quad (1.2.16)$$

where

$$\mathbf{L}_\gamma \phi = (\gamma I - 2\Delta)^{-1}[-\Delta(-\Delta\phi + g(\phi)) + \sigma\phi].$$

Now, we can deduce the relation $u = -(-\Delta + \gamma I)^{-1}\phi_t$ from (1.2.14)₂, then replace it in (1.2.14)₁ to get the following equation for ϕ alone:

$$B_\gamma \phi_t - \Delta(-\Delta\phi + g(\phi)) + \sigma\phi = 0, \quad (1.2.17)$$

where

$$B_\gamma = I - \Delta(-\Delta + \gamma I)^{-1}.$$

Alternatively, we can sum the equations (1.2.14)₁ and (1.2.14)₂, then replace u by the relation (1.2.16) to find the following equivalent formulation of (1.2.17):

$$2\phi_t - \Delta(-\Delta\phi + g(\phi)) + \gamma \mathbf{L}_\gamma \phi + \sigma\phi = 0. \quad (1.2.18)$$

This equation (1.2.18) is a Cahn-Hilliard equation with a proliferation term.

3. Conserved Viscous Phase-field Model:

When $\delta = 1$, $r(\phi) = 0$ in (1.2.1) and $h(u) = 0$ in (1.2.2), we have the following

Conserved Viscous Phase-field system

$$\begin{aligned} \phi_t - \Delta(\phi_t - \Delta\phi + g(\phi) - u) &= 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t - \Delta u &= 0, & \text{in } \Omega \times \mathbb{R}^+. \end{aligned} \quad (1.2.19)$$

1.3 Literature review

The existence of the global and exponential attractors for the conserved phase-field system

$$\begin{cases} \phi_t - \Delta(-\Delta\phi + g(\phi) - u) = 0, \\ \epsilon u_t + \phi_t - \Delta u = 0, \end{cases} \quad (1.3.1)$$

subject to Neumann boundary conditions was considered with $\epsilon = 1$ in [12] and [13] for

$$g(s) = \sum_{j=1}^{2p-1} b_j s^j, \quad b_{2p-1} > 0, \quad p \geq 2, \quad (1.3.2)$$

if $d = 1, 2$ and $p = 2$ if $d = 3$. In [33], A. Miranville proved the existence of exponential attractors for System (1.3.1), subject to Dirichlet boundary conditions, for $d = 2, 3$ and for a large class of nonlinear functions $g \in \mathcal{C}^2(\mathbb{R})$ containing the polynomials (1.3.2) without any restriction on p if $d = 3$. Continuity properties of the global and exponential attractor for Problem (1.3.1) were investigated in [4], [17], [18], [19], [21], [20], [24], [36]. The stability at $\epsilon = 0$ of the global, exponential attractors and inertial manifolds for System (1.3.1), subject to Neumann boundary conditions and for a class of nonlinearities $g \in \mathcal{C}^8(\mathbb{R})$ with the growth restriction

$$|g''(s)| \leq C_5(1 + |s|^p), \quad C_5 \geq 0, \quad \forall s \in \mathbb{R}, \quad (1.3.3)$$

where $p > 0$ is arbitrary when $d = 1, 2$ and $p \in [0, 3]$ when $d = 3$, were initiated by A. Bonfioh in [5]. The author proved the existence and the upper semi-continuity of the global attractor at $\epsilon = 0$ and then constructed a robust family of exponential attractors

(that is, a family of exponential attractors that is Holder continuous at $\epsilon = 0$). These results were then improved in [8], by weakening the nonlinearity regularity requirement to $g \in \mathcal{C}^4(\mathbb{R})$, and by obtaining stability properties at $(\epsilon, \delta) = (0, 0)$, where $\delta \in [0, 1]$ is the viscosity parameter, for the problem

$$\begin{cases} \phi_t - \Delta(\delta\phi_t - \Delta\phi + g(\phi) - u) = 0, \\ \epsilon u_t + \phi_t - \Delta u = 0, \end{cases} \quad (1.3.4)$$

that takes into account the viscosity effects of the material, and subject to either Neumann boundary conditions or periodic boundary conditions. More recently, in [10], A. Bonfoh and I.A. Suleman studied the upper semi-continuity of the global attractor at $\epsilon = 0$ and constructed a robust family of exponential attractors for System (1.3.4) where $\delta > 0$ is fixed, $g \in \mathcal{C}^2(\mathbb{R})$, if $d = 1$ and $g \in \mathcal{C}^3(\mathbb{R})$ if $d = 2, 3$. They did not require the growth condition (1.3.3), thus improving previous results of [7] by A. Bonfoh and C. Enyi where this growth condition was imposed.

Let us now consider the following problem

$$\begin{cases} \phi_t - \Delta\phi + g(\phi) = u + g_1, & \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t - \Delta u + h(u) = g_2, & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \quad (1.3.5)$$

where $\epsilon \in [0, 1]$, h a nonlinear function, and $g_i(x) \in L^2(\Omega)$ are given external forces, $i = 1, 2$. In [36], the authors considered Problem (1.3.5) in three space dimension, subject to either Dirichlet or Neumann boundary conditions for g , $h \in \mathcal{C}^3(\mathbb{R})$ satisfying some conditions. They constructed exponential attractors which are lower and upper

semi-continuous at $\epsilon = 0$.

Lastly, we consider the following generalized Cahn-Hilliard equation

$$\phi_t - \Delta(-\Delta\phi + g(\phi)) + r(\phi) = 0. \quad (1.3.6)$$

This equation has various applications in biology and image inpainting. In [31], the author considered (1.3.6) with the following functions

$$r(\phi) = \epsilon\phi, \quad \epsilon > 0 \quad \text{and} \quad g(\phi) = \phi^3 - \phi,$$

associated with Neumann boundary conditions in which case is known as the Cahn-Hilliard Oono equation that accounts for long-ranged (nonlocal) interactions in the phase separation process and it was proposed to simplify numerical simulations. The author proved the existence of a robust family of exponential attractor as $\epsilon \rightarrow 0$. In [15], the authors considered

$$r(\phi) = \phi(\phi-1) \quad \text{and} \quad g(\phi) = \phi^3 - \phi,$$

with Neumann boundary conditions in that case, (1.3.6) has applications in biology, and more precisely, it models wound healing and tumor growth. The authors showed that either the average of the local density of cells is bounded, in which case we have a global in time solution, or the solution blows up in finite time. They further proved that the relevant, from a biological point of view, solutions converge to 1 as time goes to infinity. In [32], the author considered (1.3.6) subject to Dirichlet boundary

conditions in which case $r(\phi)$ and $g(\phi)$ are the polynomials

$$g(s) = \sum_{j=1}^{2p-1} a_j s^j, \quad a_{2p-1} > 0, \quad p \geq 2 \quad \text{and} \quad r(\phi) = \sum_{j=1}^q b_j \phi^j, \quad b_q > 0, \quad q \leq p;$$

he constructed the exponential attractors and proved the existence of global attractor.

In [29], the authors considered (1.3.6) subject to Neumann boundary conditions where

$$r(\phi) = \beta\phi + \frac{k\phi}{k' + \phi}, \quad k, k' > 0, \quad \beta \geq 0$$

with logarithmic nonlinear

$$g(\phi) = -c_0\left(\phi - \frac{1}{2}\right) + \theta \ln \frac{\phi}{1-\phi}, \quad c_0, \theta > 0, \quad \phi \in (0, 1).$$

This equation is proposed to model some energy mechanisms in glial cells. The authors prove the existence of a local in time biologically relevant solution that is global under conditions. See also [34] and [16], [3]. In [28], the authors considered

$$\phi_t - \Delta(-\Delta\phi + g(\phi) - \phi) + g(\phi) = 0, \tag{1.3.7}$$

subject to Dirichlet boundary conditions where g is the polynomial function

$$g(s) = \sum_{j=1}^{2p-1} a_j s^j, \quad a_{2p-1} > 0, \quad p \geq 2.$$

This equation is associated with the effect of multiple microscopic mechanisms such as surface and adsorption. The author proved the existence and uniqueness of the solution and the existence of global and exponential attractors.

1.4 The Laplace Operator

1.4.1 Neumann or periodic boundary conditions

Denote

$$m(\varphi) = \frac{1}{|\Omega|} \int_{\Omega} \varphi(x) dx,$$

and

$$\bar{\varphi} = \varphi - m(\varphi).$$

If W is sobolev-type space, then we set

$$\dot{W} = \{\varphi \in W, m(\varphi) = 0\}.$$

We denote by W' the dual space of W .

We define the linear unbounded operator

$$N = -\Delta : \mathcal{D}(N) \rightarrow \dot{L}^2(\Omega),$$

where

$$\begin{cases} \mathcal{D}(N) = \{\varphi \in H^2(\Omega), \partial_n \varphi|_{\partial\Omega} = 0\}, & \text{(N)} \\ \mathcal{D}(N) = H_{per}^2(\Omega), & \text{(P)} \end{cases}$$

which is self-adjoint and nonnegative. If N is restricted to $\mathcal{D}(N) \cap \dot{L}^2(\Omega)$, then it turns to be positive with compact inverse N^{-1} . From the spectral theory, there is a complete orthonormal basis $\{e_j(x)\}$ of $L^2(\Omega)$ made of eigenvectors of N , and corresponding eigenvalues $\{\lambda_j\}$, with

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_j \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Moreover, one can define the powers N^r of N for $r \in \mathbb{R}$ (cf. [41]). The spaces $V_r = \mathcal{D}(N^{r/2})$ are Hilbert spaces. In particular, $V_{-1} = (H^1(\Omega))'$ or $(H_{per}^1(\Omega))'$, $V_0 = L^2(\Omega)$, $V_1 = H^1(\Omega)$ or $H_{per}^1(\Omega)$. The space $\overline{X}^Y$ denotes the closure of a metric space $X \subset Y$ in the topology of the complete metric space Y . For every $r \geq 0$ we endow the Hilbert spaces

$$V_r = \mathcal{D}(N^{\frac{r}{2}}),$$

where V_r is a subspace of $H^r(\Omega)$ and

$$\|\varphi\|_r = \left(\|N^{r/2}\varphi\|^2 + |\Omega| m(\varphi)^2 \right)^{1/2}$$

is a norm on V_r which is equivalent to the usual $H^r(\Omega)$ -norm; we endow V_r' with the norm

$$\|\varphi\|_{-r} = \left(\|N^{-r/2}\bar{\varphi}\|^2 + |\Omega| m(\varphi)^2 \right)^{1/2}.$$

For every $r \geq 0$, $\|(I + N)^{r/2} \cdot\|^2$ is a norm on V_r which is equivalent to $\|\cdot\|_r$.

There hold

$$N^{\frac{r}{2}} \bar{\varphi} = N^{\frac{r}{2}} \varphi, \quad \forall r > 0, \quad \forall \varphi \in V_r,$$

and

$$\|N^{\frac{r_2}{2}} \bar{\psi}\| \leq \lambda_1^{\frac{r_2 - r_1}{2}} \|N^{\frac{r_1}{2}} \bar{\psi}\|, \quad \forall r_1, r_2 \in \mathbb{R}, \quad r_1 > r_2, \quad \forall \psi \in V_{r_1}, \quad (1.4.1)$$

in particular, the case $r_1 = 1$ and $r_2 = 0$ is the Poincaré inequality on V_r .

The injection $V_{r_1} \hookrightarrow V_{r_2}$ is compact and dense, $\forall r_1, r_2 \in \mathbb{R}$, $r_1 > r_2$, and there exists a constant $\mathbf{c}_0 > 0$ such that

$$\|\psi\|_{r_2} \leq \mathbf{c}_0 \|\psi\|_{r_1}, \quad \forall \psi \in V_{r_1}. \quad (1.4.2)$$

In addition, we have

$$N^{\frac{r_1}{2}} N^{\frac{r_2}{2}} = N^{\frac{r_1 + r_2}{2}} = N^{\frac{r_2}{2}} N^{\frac{r_1}{2}} \quad \text{on } V_r, \quad \text{where } r = \max\{r_1, r_2, r_1 + r_2\},$$

$$(\bar{\varphi}, \psi) = (\varphi, \bar{\psi}) = (\bar{\varphi}, \bar{\psi}), \quad \forall \varphi, \psi \in L^2(\Omega),$$

$$\int_{\Omega} N \varphi \psi \, dx = \int_{\Omega} \nabla \varphi \cdot \nabla \psi \, dx, \quad \forall \varphi, \psi \in V_1$$

(Green's identity),

$$\|\varphi\|^2 \leq \|\varphi\|_r \|\varphi\|_{-r}, \quad \forall r \geq 0, \quad \forall \varphi \in V_r, \quad (1.4.3)$$

and

$$\int_{\Omega} \varphi \psi \, dx = (N^{-\frac{r}{2}} \bar{\varphi}, N^{\frac{r}{2}} \psi) + |\Omega| m(\varphi) m(\psi), \quad \forall r \geq 0, \quad \forall \psi \in V_r, \quad \forall \varphi \in V_{-r}. \quad (1.4.4)$$

In case of Neumann or periodic boundary conditions, the eigenvalues $\{\lambda_k\}$ and corresponding eigenvectors $\{e_k\}$, which are orthogonal basis of $L^2(\Omega)$ of the operator $N : \{\varphi \in H^2(\Omega), \partial_n \varphi|_{\partial\Omega} = 0\} \rightarrow \dot{L}^2(\Omega)$, are known and they given respectively by the form:

$$\lambda_k = \pi^2 \frac{k_1^2}{L_1^2}, \quad \text{and} \quad e_k(x) = \sqrt{\frac{2}{L_1}} \cos \frac{\pi k_1 x_1}{L_1} \quad \text{if} \quad d = 1, \quad (1.4.5)$$

and for $N : H_{per}^2(\Omega) \rightarrow \dot{L}^2(\Omega)$, $\lambda_k = 4\pi^2 \sum_{j=1}^d \frac{k_j^2}{L_j^2}$, $d = 1, 2, 3$ with

$$e_k(x) = \begin{cases} \frac{1}{\sqrt{L_1}}, \sqrt{\frac{2}{L_1}} \cos \frac{2\pi k_1 x_1}{L_1}, \sqrt{\frac{2}{L_1}} \sin \frac{2\pi k_1 x_1}{L_1}, & \text{if} \quad d = 1, \\ \frac{1}{\sqrt{L_1 L_2}}, \sqrt{\frac{2}{L_1 L_2}} \cos \frac{2\pi k x}{L}, \sqrt{\frac{2}{L_1 L_2}} \sin \frac{2\pi k x}{L}, & \text{if} \quad d = 2, \\ \frac{1}{\sqrt{L_1 L_2 L_3}}, \sqrt{\frac{2}{L_1 L_2 L_3}} \cos \frac{2\pi k x}{L}, \sqrt{\frac{2}{L_1 L_2 L_3}} \sin \frac{2\pi k x}{L}, & \text{if} \quad d = 3, \end{cases} \quad (1.4.6)$$

with $\frac{kx}{L} = \frac{k_1 x_1}{L_1} + \frac{k_2 x_2}{L_2} + \dots + \frac{k_d x_d}{L_d}$, for $k_1, k_2 = 0, 1, 2, \dots$

1.4.2 Dirichlet boundary conditions

We define the linear unbounded operator

$$A = -\Delta : \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega),$$

which is self-adjoint and nonnegative with compact inverse A^{-1} . From the spectral theory, there is a complete orthonormal basis $\{e_j(x)\}$ of $L^2(\Omega)$ made of eigenvectors of A , and corresponding eigenvalues $\{\lambda_j\}$, with

$$0 < \lambda_1 < \dots < \lambda_j \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Moreover, one can define the powers A^r of A for $r \in \mathbb{R}$ (cf. [41]). The spaces $V_r = \mathcal{D}(A^{r/2})$ are Hilbert spaces. In particular, $V_{-1} = H^{-1}(\Omega)$, $V_0 = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$. For every $r \in \mathbb{R}$, V_r is a subspace of $H^r(\Omega)$ and

$$\|\varphi\|_r = \|A^{r/2}\varphi\|$$

is a norm on V_r which is equivalent to the usual $H^r(\Omega)$ -norm. The relations from (1.4.1) through (1.4.4) are holding with $\bar{\varphi} = \varphi$ (that is, $m(\varphi) = 0$).

In case of Dirichlet boundary conditions, the eigenvalues $\{\lambda_k\}$ and corresponding eigenvectors $\{e_k\}$, which are orthogonal basis of $L^2(\Omega)$ of the operator $A : H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$, are known and they are given respectively by the form:

$$\lambda_k = \pi^2 \frac{k_1^2}{L_1^2}, \text{ and } e_k(x) = \sqrt{\frac{2}{L_1}} \sin \frac{\pi k_1 x_1}{L_1} \quad \text{if } d = 1. \quad (1.4.7)$$

1.5 Useful lemmas and definitions

Let E Banach space.

Definition 1.5.1 (Continuous semigroup) *The family of operators $\{S(t)\}_{t \geq 0}$ de-*

defines a semigroup on E if the following conditions hold:

- $S(t + s) = S(t) \circ S(s), \forall t, s \geq 0,$
- $S(0) = I, I$ being the identity operator,
- $S(t): E \rightarrow E$ is a continuous, $\forall t \geq 0,$
- $S(\cdot)u_0 : \mathbb{R}^+ \rightarrow E, \quad \forall u_0 \in E.$

Definition 1.5.2 (Absorbing set) A set $\mathcal{B} \subset E$ is an absorbing set for the semigroup $S(t) : E \rightarrow E$ if given any bounded set $\mathcal{B} \subset E$ there exists a time $t_0(\mathcal{B})$ such that $S(t)\mathcal{B} \subset \mathcal{B}$, for every $t \geq t_0(\mathcal{B})$.

Definition 1.5.3 (Uniformly compact semigroup) The semigroup $\{S(t)\}_{t \geq 0}$ is uniformly compact in E , if for any bounded set $B \subset E$, there exists $t(B)$ such that $\overline{\bigcup_{t \geq t(B)} S(t)B}$ is compact in E .

Definition 1.5.4 (ω -limit set) The ω -limit of a bounded set $B \subset E$ is defined by $\omega(B) = \bigcap_{n > 0} \overline{\bigcup_{t \geq n} S(t)B}^E$.

The Hausdorff semi-distance with respect to the metric of E is defined as:

$$\text{dist}_E(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_E,$$

where as the symmetric Hausdorff distance between A and B is

$$\text{dist}_E^{\text{sym}}(A, B) = \max\{\text{dist}_E(A, B), \text{dist}_E(B, A)\}.$$

We have the following formal definition of the global attractor.

Definition 1.5.5 (The global attractor) *The global attractor for $\{S(t)\}_{t \geq 0}$ is a compact set $\mathcal{A} \subset E$ such that*

- $\mathcal{A}$ is invariant, that is, $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$,
- $\mathcal{A}$ attracts all bounded sets, that is, for all B bounded in E ,

$$\lim_{t \rightarrow \infty} \text{dist}_E(S(t)B, \mathcal{A}) = 0.$$

The following theorem guarantees the existence of the global attractor.

Theorem 1.5.1 (c.f. [35]) *Let the continuous semigroup $\{S(t)\}$ be compact or uniformly compact on E . Assume that there exists a bounded absorbing set $\mathcal{B}$ in E . Then, $\mathcal{A} = \omega(\mathcal{B})$ is the compact global attractor.*

Theorem 1.5.2 (cf. [42]) *Consider the initial value problem*

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0$$

with $x \in D \subset \mathbb{R}^n$, $|t - t_0| \leq a$, $D = \{x \mid \|x - x_0\| \leq d\}$, a and d are positive constants.

The vector function $f(t, x)$ satisfies the following conditions:

- $f(t, x)$ is continuous in $G = [t_0 - a, t_0 + a] \times D$,
- $f(t, x)$ is Lipschitz continuous in x .

Then the initial value problem has a unique solution for $|t - t_0| \leq \inf(a, \frac{d}{M})$ with $M = \sup_G \|f\|$.

Definition 1.5.6 (Fractal Dimension) Let $X \subset E$ be a (relatively) compact set.

The fractal dimension of X is defined by

$$\dim_F X = \lim_{s \rightarrow 0^+} \sup \frac{\ln N(s, X)}{\ln(1/s)}, \quad (1.5.1)$$

where $s > 0$ and $N(s, X)$ is the minimal number of balls in E of radius s which are necessary to cover X .

Definition 1.5.7 (Exponential attractor) Let $X \subset E$ be compact, $\mathcal{E}$ is an exponential attractor for $S(t)$ if $\mathcal{A} \subseteq \mathcal{E} \subseteq X$ and:

- $\mathcal{E}$ is compact with $S(t)\mathcal{E} \subset \mathcal{E}$, $\forall t \geq 0$,
- $\mathcal{E}$ has finite fractal dimension $d_F(\mathcal{E})$,
- there exists a constant $c_0, c_1 > 0$ such that, $\forall B \subset X$ bounded,

$$\text{dist}_E(S(t)B, \mathcal{E}) \leq c_0 e^{-c_1 t}, \quad \forall t \geq 0.$$

Definition 1.5.8 (Inertial manifolds attractor) A set $\mathcal{M}$ is called an inertial manifold for $S(t)$ if:

- $\mathcal{M}$ is a finite dimensional Lipschitz manifold in E ,
- $\mathcal{M}$ is smooth, i.e., $\mathcal{M}$ is of class $\mathcal{C}^1$,

- $S(t)\mathcal{M} \subset \mathcal{M}, \forall t \geq 0,$
- $\exists c_0 > 0$ s.t, $\forall u_0 \in E, \exists c_1(u_0)$ such that

$$\text{dist}_E(S(t)u_0, \mathcal{M}) \leq c_1 e^{-c_0 t}, \quad \forall t \geq 0.$$

Lemma 1.5.1 (c.f. [11]) *Let A be a maximal monotone operator $A : D(A) \subset H \rightarrow H$. Then*

- $D(A)$ is dense in H ,
- A is a closed operator,
- For every $\lambda > 0$, $(I + \lambda A)$ is bijective from $D(A)$ onto H , $(I + \lambda A)^{-1}$ is a bounded operator and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.

Lemma 1.5.2 (Characteristic of the Attractor) *Let S be a semiflow on a Banach space X , and assume that S admits a global attractor $\mathcal{A}$. Let $x \in X$. Then $x \in \mathcal{A}$ if and only if there exists a complete orbit through x , contained in $\mathcal{A}$.*

Let us give some lemmas that will be helpful in the well-posedness section of this work.

Lemma 1.5.3 (c.f. [40]) *Let X, Y and Z be Banach spaces such that the embedding $X \hookrightarrow Y$ is compact and $Y \subset Z$ is continuous. Assume that X is reflexive. Also, let u_n be a sequence that is uniformly bounded in $L^2(0, T; X)$ and $\frac{du_n}{dt}$ is uniformly bounded in $L^p(0, T; Z)$, for some $p > 1$. Then there exists a subsequence of u_n that converges strongly in $L^2(0, T; Y)$ and a.e in $(0, T) \times \Omega$.*

Lemma 1.5.4 (c.f. [40]) *Let Ω be a bounded open set in $\mathbb{R}^d$, and let g_j be a sequence of functions in $L^p(\Omega)$ where $1 < p < +\infty$, with*

$$\|g_j\|_{L^p(\Omega)} \leq C \text{ for all } j \in \mathbb{Z}^+.$$

If $g \in L^p(\Omega)$ and $g_j \rightarrow g$ almost everywhere then $g_j \rightharpoonup g$ in $L^p(\Omega)$.

Lemma 1.5.5 (c.f. [40]) *for some $k \geq 0$, suppose that $u \in L^2(0, T; H^{k+1}(\Omega))$ and $\frac{d}{dt}u \in L^2(0, T; H^{k-1}(\Omega))$. Then u is continuous from $[0, T]$ into $H^k(\Omega)$.*

Lemma 1.5.6 (Boundedness of weakly converging sequences, [40]) *Suppose $1 \leq p < \infty$ and $u_n \rightharpoonup u$ in $L^p(\Omega)$ ($\overset{*}{\rightharpoonup}$ in $L^\infty(\Omega)$ if $p = \infty$). Then*

$$\|u\|_{L^p(\Omega)} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^p(\Omega)}.$$

The following inequalities will be repeatedly used in this work.

Lemma 1.5.7 (Generalized Hölder's inequality, [41]) *If $p_1, p_2, \dots, p_n$ are such that*

$$\frac{1}{p_1} + \dots + \frac{1}{p_n} = 1 \tag{1.5.2}$$

and $f_j \in L^{p_j}(\Omega)$, then their product $f_1 f_2 \dots f_n \in L^1(\Omega)$ and

$$\int_{\Omega} |f_1(x) f_2(x) \dots f_n(x)| dx \leq \|f_1\|_{L^{p_1}(\Omega)} \dots \|f_n\|_{L^{p_n}(\Omega)}. \tag{1.5.3}$$

Lemma 1.5.8 (Young's inequality, [35]) *Let a and b be positive real numbers. For any $\epsilon > 0$, there holds $ab \leq \epsilon a^p + C_\epsilon b^q$, $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $C_\epsilon = \frac{(\epsilon p)^{-\frac{q}{p}}}{q}$.*

Lemma 1.5.9 (Gronwall's Lemma, cf., e.g., [41]) *Let Θ be an absolutely continuous function on $[t_0, \infty)$, which satisfies the differential inequality*

$$\frac{d\Theta(t)}{dt} \leq f(t)\Theta(t) + h(t)$$

for almost every $t \in [t_0, \infty)$, where h is a positive function and f is locally integrable on $[t_0, \infty)$. Then

$$\Theta(t) \leq \Theta(t_0)e^{\int_{t_0}^t f(s)ds} + \int_{t_0}^t h(r)e^{\int_r^t f(s)ds}dr, \quad \forall t \geq t_0.$$

In particular, if Θ is positive, $f(t) = -\nu t$ and $h(t) = c$, for some $\nu > 0$ and $c \geq 0$, then we find

$$\Theta(t) \leq \Theta(t_0)e^{-\nu(t-t_0)} + \frac{c}{\nu}, \quad \forall t \geq t_0.$$

Lemma 1.5.10 (cf., e.g., [10]) *Let Θ be an absolutely continuous positive function on $[0, \infty)$, which satisfies for some $\nu > 0$ the differential inequality*

$$\frac{d\Theta(t)}{dt} + \nu\Theta(t) \leq h(t)$$

for almost every $t \in [t_0, \infty)$, where f and h are positive functions on $[t_0, \infty)$ such that

$$\int_t^{t+1} h(s)ds \leq Ae^{-\beta(t-t_0)} + B, \quad \forall t \geq t_0,$$

for some $\beta, A, B \geq 0$ such that $\beta < \nu$. Then

$$\Theta(t) \leq \left(\Theta(t_0) + \frac{Ae^\beta}{1 - e^{-(\nu-\beta)}} \right) e^{-\beta(t-t_0)} + \frac{B}{1 - e^{-\nu}}, \quad \forall t \geq t_0.$$

Lemma 1.5.11 (Uniform Gronwall's Lemma, [35]) *Let Θ be an absolutely continuous positive function on $[t_0, \infty)$, which satisfies the differential inequality*

$$\frac{d\Theta(t)}{dt} \leq f(t)\Theta(t) + h(t)$$

for almost every $t \in [t_0, \infty)$, where f and h are positive functions on $[t_0, \infty)$ such that

$$\begin{aligned} \int_t^{t+r} \Theta(s) ds &\leq A_1 e^{-c_1(t-t_0)} + B_1, \quad \forall t \geq t_0, \\ \int_t^{t+r} f(s) ds &\leq A_2, \quad \forall t \geq t_0, \\ \text{and } \int_t^{t+r} h(s) ds &\leq A_3 e^{-c_3(t-t_0)} + B_3, \quad \forall t \geq t_0. \end{aligned}$$

Then

$$\Theta(t) \leq A(r) e^{-c(t-r-t_0)} + B(r), \quad \forall r > 0, \quad \forall t \geq t_0 + r,$$

where

$$A(r) = 2 \max \left\{ \frac{A_1}{r}, A_3 \right\} e^{A_2}, \quad c = \min\{c_1, c_3\} \quad \text{and} \quad B(r) = \left(\frac{B_1}{r} + B_3 \right) e^{A_2}.$$

Lemma 1.5.12 (Agmon's inequality, [35]) *Let $\Omega \subset \mathbb{R}^n$ be an open set of class*

$\mathcal{C}^n$. There exists a constant $c > 0$ which depends on Ω such that:

$$\|u\|_{L^\infty(\Omega)} \leq \begin{cases} c\|u\|_{H^{(n-2)/2}(\Omega)}^{\frac{1}{2}} \|u\|_{H^{(n+2)/2}(\Omega)}^{\frac{1}{2}}, \forall u \in H^{(n+2)/2}(\Omega) & \text{if } n \text{ is even,} \\ c\|u\|_{H^{(n-1)/2}(\Omega)}^{\frac{1}{2}} \|u\|_{H^{(n+1)/2}(\Omega)}^{\frac{1}{2}}, \forall u \in H^{(n+1)/2}(\Omega) & \text{if } n \text{ is odd.} \end{cases} \quad (1.5.4)$$

Lemma 1.5.13 (Interpolation inequalities, [35]) *Let $s_1 \leq s \leq s_2$ such that $s = \theta s_1 + (1 - \theta)s_2, \theta \in [0, 1]$. There exists $c > 0$ such that*

$$\|u\|_{H^s(\Omega)} \leq \|u\|_{H^{s_1}(\Omega)}^{1-\theta} \|u\|_{H^{s_2}(\Omega)}^\theta, \forall u \in H^{s_2}(\Omega). \quad (1.5.5)$$

More generally, let $p \in [1, \infty), m \geq 1$ and $\varphi \in W^{m,p}(\Omega)$. Then, there exists a constant $C > 0$ and $\mu = \frac{n}{m}(\frac{1}{p} - \frac{1}{r})$ such that

$$\|u\|_{L^r(\Omega)} \leq C\|u\|_{L^p(\Omega)}^{1-\mu} \|u\|_{W^{m,p}(\Omega)}^\mu, \quad r \in \begin{cases} [p, \infty] & \text{if } m - \frac{n}{p} > 0, \\ [p, \infty) & \text{if } m - \frac{n}{p} = 0, \\ [p, -\frac{n}{m - (n/p)}] & \text{if } m - \frac{n}{p} < 0. \end{cases} \quad (1.5.6)$$

Throughout this work, we denote by $\|\cdot\|$ and $(\cdot, \cdot)$ the usual norm and scalar product in $L^2(\Omega)$ (and also in $L^2(\Omega)^d$). Moreover, the same letter c, C and c' (and sometimes $c_i, i = 0, 1, 2, \dots$) denote positive constants that may change from line to line, but are always independent of ϵ and on time (unless explicitly specified). The letter $Q : \mathbb{R}^+ \rightarrow (0, \infty)$ denotes a monotonically increasing function such that, for any

$\alpha > 0$ and $A, B \geq 0$, there is a $c > 0$ such that

$$Q(Ae^{-\alpha t} + B) \leq Q(A)e^{-c\alpha t} + Q(B). \quad (1.5.7)$$

For any $m > 0$ and $c > 0$, we will still denote $c + Q$ and cQ^m by Q . Let

$$j = \begin{cases} 1, & \text{if } d = 1, \\ 2, & \text{if } d = 2 \text{ or } 3. \end{cases} \quad (1.5.8)$$

Based on the continuous embedding $H^j(\Omega) \hookrightarrow L^\infty(\Omega)$, we have $\|\phi\|_{L^\infty(\Omega)} \leq c\|\phi\|_j$ and

$$\begin{aligned} \|g(\phi)\|_{L^\infty(\Omega)} &\leq Q(\|\phi\|_1), \text{ for } d = 1, \\ \|g(\phi)\|_{L^\infty(\Omega)} &\leq Q(\|\phi\|_2), \text{ for } d = 2, 3. \end{aligned} \quad (1.5.9)$$

for all $\phi \in H^j(\Omega)$.

1.6 Publications

[1] A. Bonföh, R. Alzahrani, A. Miranville, Stability of exponential attractors for singularly perturbed phase-field systems of Oono type, *Evolution Equations and Control Theory*, 2024, doi:10.3934/eect.2024008.

<https://www.aims sciences.org/article/id/65d463563cb6602db48edf2e>

CHAPTER 2

PHASE-FIELD SYSTEMS OF TYPE OONO

We consider the singular perturbation problem

$$\begin{cases} \phi_t + N(N\phi + g(\phi) - u) + \sigma\phi = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t + Nu + \gamma u = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0, \end{cases} \quad (2.0.1)$$

where the operator N is defined in Subsection 1.4.1, $\sigma, \gamma > 0$ is fixed, $0 < \epsilon \leq \epsilon_0$ with

$$\epsilon_0 = \min\{1, \frac{\gamma}{2\sigma}\},$$

and $g(\phi)$ is a nonlinear function . We set

$$G(s) = \int_0^s g(r)dr,$$

we assume that $g \in \mathcal{C}^3(\mathbb{R})$ satisfies the following conditions when $d = 1, 2, 3$:

$$G(s) \geq -C_1, \quad C_1 \geq 0, \quad \forall s \in \mathbb{R}, \quad (2.0.2)$$

$\forall \gamma \in \mathbb{R}, \alpha \geq 0$ and $|\gamma| \leq \alpha, \exists C_2(\alpha) > 0, C_3(\alpha) \geq 0$ such that

$$(s - \gamma)g(s) - C_2G(s) \geq -C_3, \quad \forall s \in \mathbb{R}, \quad (2.0.3)$$

$$g'(s) \geq -C_4, \quad C_4 \geq 0, \quad \forall s \in \mathbb{R}, \quad (2.0.4)$$

$$\forall \mu > 0, \exists C_\mu > 0 \text{ such that } |g(s)| \leq \mu G(s) + C_\mu, \quad \forall s \in \mathbb{R}. \quad (2.0.5)$$

For instance, the polynomial

$$g(\phi) = \sum_{j=1}^{2p-1} a_j \phi^j, \quad (2.0.6)$$

$p \in \mathbb{N}, p \geq 2, a_{2p-1} > 0$, satisfies Conditions (2.0.2)-(2.0.5).

If $\epsilon = 0$, then the system reduces to

$$\begin{cases} B_\gamma \phi_t + N(N\phi + g(\phi)) + \sigma \phi = 0, \\ \phi|_{t=0} = \phi_0, \end{cases} \quad (2.0.7)$$

where

$$B_\gamma = I + N(\gamma I + N)^{-1} : L^2 \rightarrow L^2.$$

2.1 Functional setting

For every j , (c.f. (1.5.8)), we endow the Hilbert spaces

$$\mathcal{V}_{j,\epsilon} = \mathcal{D}(N^{\frac{j}{2}}) \times \mathcal{D}(N^{\frac{j-1}{2}}),$$

$$\mathcal{V}_{j,0} = \mathcal{D}(N^{\frac{j}{2}}) \times \mathcal{D}(N^{\frac{j-2}{2}}),$$

respectively equipped with the norms

$$\|(\varphi, \psi)\|_{\mathcal{V}_{j,\epsilon}} = (\|\varphi\|_j^2 + \epsilon\|\psi\|_{j-1}^2 + \|\psi\|_{j-2}^2)^{\frac{1}{2}}$$

where

$$\|(\varphi, \psi)\|_{\mathcal{V}_{j,0}} = (\|\varphi\|_j^2 + \|\psi\|_{j-2}^2)^{\frac{1}{2}}.$$

Throughout this chapter, we set

$$\mathcal{V}_j^\alpha = \{\varphi \in V_j, |m(\varphi)| \leq \alpha\},$$

$$\mathcal{V}_{j,\epsilon}^\alpha = \{(\varphi, \psi) \in \mathcal{V}_{j,\epsilon}, |m(\varphi)| + \epsilon|m(\psi)| \leq \alpha\},$$

for any $\alpha \geq 0$. Integrating (2.0.1)₁ and (2.0.1)₂ over Ω , we find

$$\frac{d}{dt}m(\phi) + \sigma m(\phi) = 0$$

and

$$\epsilon \frac{d}{dt}m(u) + m(\phi_t) + \gamma m(u) = 0$$

or

$$\frac{d}{dt}m(u) + \frac{\gamma}{\epsilon}m(u) = \frac{\sigma}{\epsilon}m(\phi),$$

respectively, so that

$$m(\phi(t)) = e^{-\sigma t} m(\phi_0), \quad (2.1.1)$$

hence

$$m(\phi_t(t)) = -\sigma m(\phi_0) e^{-\sigma t}, \forall t \geq 0, \quad (2.1.2)$$

$$m(\phi_{tt}(t)) = \sigma^2 m(\phi_0) e^{-\sigma t}, \forall t \geq 0; \quad (2.1.3)$$

and

$$m(u(t)) = m(u_0) e^{-\frac{\gamma}{\epsilon} t} + \frac{\sigma}{\gamma - \epsilon \sigma} m(\phi_0) (e^{-\sigma t} - e^{-\frac{\gamma}{\epsilon} t}), \quad \forall t \geq 0. \quad (2.1.4)$$

Now, if $|m(\phi_0)| + \epsilon |m(u_0)| \leq \alpha$, then

$$\begin{aligned} & |m(\phi(t))| + \epsilon |m(u(t))| \\ & \leq |m(\phi_0)| e^{-\sigma t} + \epsilon |m(u_0)| e^{-\frac{\gamma}{\epsilon} t} + \frac{\epsilon \sigma}{\gamma - \epsilon \sigma} |m(\phi_0)| (e^{-\sigma t} - e^{-\frac{\gamma}{\epsilon} t}) \\ & \leq \epsilon |m(u_0)| e^{-\frac{\gamma}{\epsilon} t} + |m(\phi_0)| \left(e^{-\sigma t} + \frac{\epsilon \sigma}{\gamma - \epsilon \sigma} (e^{-\sigma t} - e^{-\frac{\gamma}{\epsilon} t}) \right) \\ & \leq |m(\phi_0)| + \epsilon |m(u_0)|, \quad \forall t \geq 0. \end{aligned}$$

Indeed, let the function $f(t) = e^{-\sigma t} + \frac{\epsilon\sigma}{\gamma - \epsilon\sigma}(e^{-\sigma t} - e^{-\frac{\gamma}{\epsilon}t}) - 1, \forall t \geq 0, f(0) = 0$.

Since $\epsilon < \frac{\gamma}{\sigma}$, we have

$$\begin{aligned} f'(t) &= -\sigma e^{-\sigma t} + \frac{\epsilon\sigma}{\gamma - \epsilon\sigma}(-\sigma e^{-\sigma t} + \frac{\gamma}{\epsilon}e^{-\frac{\gamma}{\epsilon}t}), \forall t \geq 0 \\ &< -\sigma e^{-\sigma t} + \frac{\epsilon\sigma}{\gamma - \epsilon\sigma}(-\sigma e^{-\sigma t} + \frac{\gamma}{\epsilon}e^{-\sigma t}) \\ &< -\sigma e^{-\sigma t} + \sigma e^{-\sigma t} = 0, \forall t \geq 0, \end{aligned}$$

then, $f(t) \leq 0, \forall t \geq 0 \implies e^{-\sigma t} + \frac{\epsilon\sigma}{\gamma - \epsilon\sigma}(e^{-\sigma t} - e^{-\frac{\gamma}{\epsilon}t}) \leq 1, \forall t \geq 0$.

Hence, we deduce that

$$|m(\phi(t))| + \epsilon|m(u(t))| \leq \alpha, \quad \forall t \geq 0.$$

2.2 A priori estimates

From now on, let $\alpha \geq 0$ arbitrary, and let us consider a trajectory $(\phi(t), u(t))_{t \geq 0}$ of System (2.0.1) starting from the point (ϕ_0, u_0) and such that $|m(\phi_0)| + \varepsilon|m(u_0)| \leq \alpha$, for every $0 < \varepsilon \leq \varepsilon_0$. Thus, there hold

$$|m(\phi(t))| + \varepsilon|m(u(t))| \leq \alpha, \quad \forall t \geq 0, \tag{2.2.1}$$

$$|m(u(t))| \leq |m(u_0)|e^{-\frac{\gamma}{\varepsilon}t} + \frac{2\sigma\alpha}{\gamma}e^{-\sigma t}, \quad \forall t \geq 0, \tag{2.2.2}$$

$$|m(\phi_t(t))| \leq \sigma \alpha e^{-\sigma t}, \quad \forall t \geq 0 \quad (2.2.3)$$

and

$$|m(\phi_{tt}(t))| \leq \sigma^2 \alpha e^{-\sigma t}, \quad \forall t \geq 0. \quad (2.2.4)$$

In the next subsections, we will derive formal a priori estimates. These estimates, and also the passage to the limit, can be justified by making Galerkin approximations (c.f., e.g., [41]).

2.2.1 Estimates in $\mathcal{V}_{1,\varepsilon}$

We multiply (2.0.1)₁ and (2.0.1)₂ by $N^{-1}\bar{\phi}_t$ and u , respectively, integrate over Ω , use the identity $(u, \bar{\phi}_t) = (u, \phi_t) - |\Omega|m(u)m(\phi_t)$ and sum the resulting equations and we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\varepsilon \|u\|^2 + \|\bar{\phi}\|_1^2 + \sigma \|\bar{\phi}\|_{-1}^2 + 2 \int_{\Omega} G(\phi) dx \right) + \|\bar{u}\|_1^2 + \gamma \|u\|^2 + \|\bar{\phi}_t\|_{-1}^2 \\ &= m(\phi_t) \int_{\Omega} g(\phi) dx - |\Omega|m(\phi_t)m(u). \end{aligned} \quad (2.2.5)$$

There exists a constant $\varrho_1 > 0$ such that

$$\|\bar{u}\|_1^2 + \gamma \|u\|^2 \geq \varrho_1 \|u\|_1^2. \quad (2.2.6)$$

We have $\|\bar{\phi}_t\|_{-1}^2 = \|\phi_t\|_{-1}^2 - |\Omega|m(\phi_t)^2$, $m(\phi_t)^2 \leq \sigma^2\alpha^2$, $|m(\phi_t)m(u)| \leq \sigma\alpha|m(u)|$ and

$$\begin{aligned} |m(\phi_t) \int_{\Omega} g(\phi) dx| &\leq |m(\phi_t)| \int_{\Omega} |g(\phi)| dx \\ &\leq \mu|m(\phi_t)| \int_{\Omega} G(\phi) dx + c_{\mu}|m(\phi_t)| \\ &\leq \mu\alpha\sigma \int_{\Omega} G(\phi) dx + c_{\mu}\alpha\sigma, \quad \forall \mu > 0, \end{aligned} \quad (2.2.7)$$

on account of (2.0.5) and (2.2.3). It follows from (2.2.5), (2.2.6) and (2.2.7) that

$$\begin{aligned} \frac{d}{dt} \left(\varepsilon\|u\|^2 + \|\bar{\phi}\|_1^2 + \sigma\|\bar{\phi}\|_{-1}^2 + 2 \int_{\Omega} G(\phi) dx \right) &+ 2\varrho_1\|u\|_1^2 + 2\|\phi_t\|_{-1}^2 \\ &\leq 2\mu\alpha\sigma \int_{\Omega} G(\phi) dx + 2c_{\mu}\alpha\sigma + 2|\Omega|\sigma^2\alpha^2 + 2|\Omega|\sigma\alpha|m(u)|, \quad \forall \mu > 0. \end{aligned} \quad (2.2.8)$$

We multiply (2.0.1)₁ by $N^{-1}\bar{\phi}$ and we integrate over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|\bar{\phi}\|_{-1}^2 + \|\bar{\phi}\|_1^2 + \int_{\Omega} g(\phi)\bar{\phi} dx - (\bar{u}, \bar{\phi}) + \sigma\|\bar{\phi}\|_{-1}^2 = 0, \quad (2.2.9)$$

and, owing to Assumption (2.0.3), the equality $\|\bar{\phi}\|_1^2 = \|\phi\|_1^2 - |\Omega|m(\phi)^2$, estimates (2.2.1) and

$$\begin{aligned} |(\bar{u}, \bar{\phi})| &\leq \frac{1}{\lambda_1} \|u\|_1 \|\phi\|_1 \\ &\leq \frac{1}{2} \|\phi\|_1^2 + \frac{1}{2\lambda_1^2} \|u\|_1^2 \end{aligned}$$

(got by using Cauchy-Schwarz, Poincaré and Young's inequalities), it follows from (2.2.9) that

$$\frac{d}{dt} \|\bar{\phi}\|_{-1}^2 + \|\phi\|_1^2 + c_1 \int_{\Omega} G(\phi) dx \leq c(\|u\|_1^2 + 1). \quad (2.2.10)$$

We multiply (2.0.1)₁ by ϕ and we integrate over Ω , and we obtain

$$\frac{1}{2} \frac{d}{dt} \|\phi\|^2 + \|\bar{\phi}\|_2^2 + \int_{\Omega} g'(\phi) |\nabla \phi|^2 dx + \sigma \|\phi\|^2 - (N^{\frac{1}{2}} u, N^{\frac{1}{2}} \phi) = 0. \quad (2.2.11)$$

There exists a constant $\varrho_2 > 0$ such that

$$\|\bar{\phi}\|_2^2 + \sigma \|\phi\|^2 \geq \varrho_2 \|\phi\|_2^2.$$

Assumption (2.0.4) yields

$$\int_{\Omega} g'(\phi) |\nabla \phi|^2 dx \geq -C_4 \|\phi\|_1^2 \quad (2.2.12)$$

and we have the estimate

$$\begin{aligned} |(N^{\frac{1}{2}} u, N^{\frac{1}{2}} \phi)| &\leq \|u\|_1 \|\phi\|_1 \\ &\leq \frac{1}{2} \|\phi\|_1^2 + \frac{1}{2} \|u\|_1^2. \end{aligned}$$

We deduce from (2.2.11) that

$$\frac{d}{dt}\|\phi\|^2 + 2\varrho_2\|\phi\|_2^2 \leq c(\|u\|_1^2 + \|\phi\|_1^2). \quad (2.2.13)$$

Summing (2.2.8), ϑ_1 (2.2.10) and ϑ_2 (2.2.13), with $\vartheta_1, \vartheta_2 > 0$, we find

$$\begin{aligned} \frac{dE_1}{dt} + 2\varrho_2\|\phi\|_2^2 + (2\varrho_1 - \vartheta_1c - \vartheta_2c)\|u\|_1^2 + 2\|\phi_t\|_{-1}^2 + (\vartheta_1 - \vartheta_2c)\|\phi\|_1^2 \\ + (c_1\vartheta_1 - 2\mu\alpha\sigma) \int_{\Omega} G(\phi) dx \\ \leq c(\mu, \vartheta_1, \vartheta_2)(1 + |m(u)|), \end{aligned} \quad (2.2.14)$$

where

$$E_1(t) = \varepsilon\|u\|^2 + \|\bar{\phi}\|_1^2 + \sigma\|\bar{\phi}\|_{-1}^2 + 2 \int_{\Omega} G(\phi) dx + \vartheta_1\|\bar{\phi}\|_{-1}^2 + \vartheta_2\|\phi\|^2.$$

We first choose $\vartheta_1 = \frac{\varrho_1}{2c}$ and set $\mu = \frac{c_1\vartheta_1}{4\alpha\sigma}$, then we choose $\vartheta_2 = \min\{\frac{\vartheta_1}{2c}, \frac{\varrho_1}{2c}\}$. We infer from (2.2.14) that

$$\frac{dE_1}{dt} + c \left(\|\phi\|_2^2 + \|u\|_1^2 + \|\phi_t\|_{-1}^2 + \int_{\Omega} G(\phi) dx \right) \leq c(1 + |m(u)|). \quad (2.2.15)$$

Now, we notice that there exist $\tilde{c}_0, c_1, c_2, c_3 \geq 0$ that may depend on α but are all independent of ε such that

$$\|\phi\|_1^2 + \|u\|_1^2 + \int_{\Omega} [G(\phi) + C_1] dx \geq \tilde{c}_0 E_1(t) \quad (2.2.16)$$

(since $G(\phi) + C_1 \geq 0$, by (2.0.2)) and

$$\begin{aligned} c_1 \left(\|\phi(t)\|_1^2 + \varepsilon \|u(t)\|^2 + \int_{\Omega} G(\phi(t)) dx \right) - c_2 \\ \leq E_1(t) \leq c_3 \left(\varepsilon \|u(t)\|^2 + \|\phi(t)\|_1^2 + Q(\|\phi(t)\|_j^2) \right), \end{aligned} \quad (2.2.17)$$

owing to (2.0.2) again, $\|\bar{\phi}\|_1^2 = \|\phi\|_1^2 - |\Omega|m(\phi)^2$, (2.2.1), the continuous embeddings $V_1 \hookrightarrow L^2(\Omega) \hookrightarrow V_1'$ and the estimate $\|G(\phi)\|_{L^\infty} \leq Q(\|\phi\|_j^2)$. We then deduce from (2.2.15), (2.2.16) and (2.2.17) the following estimate

$$\frac{dE_1}{dt} + c_0 E_1 \leq c' (1 + |m(u)|). \quad (2.2.18)$$

Next, we find that

$$\begin{aligned} \int_0^t |m(u(s))| ds &\leq \int_0^t \left(|m(u_0)| e^{-\frac{\gamma}{\varepsilon} s} + \frac{2\sigma\alpha}{\gamma} e^{-\sigma s} \right) ds \\ &\leq \frac{\varepsilon}{\gamma} |m(u_0)| + \frac{2\alpha}{\gamma} \\ &\leq \frac{3\alpha}{\gamma}, \quad \forall t \geq 0, \end{aligned} \quad (2.2.19)$$

on account of (2.2.1) and (2.2.2). Applying Lemma 1.5.9 to (2.2.18), due to (2.2.19), we get

$$\begin{aligned} E_1(t) &\leq E_1(0) e^{-c_0 t} + c \int_0^t (1 + |m(u(r))|) e^{-c_0(t-r)} dr \\ &\leq Q(\|\phi_0\|_j^2 + \varepsilon \|u_0\|^2) e^{-c_0 t} + c, \quad \forall t \geq 0, \end{aligned} \quad (2.2.20)$$

hence

$$\|\phi(t)\|_1^2 + \varepsilon \|u(t)\|^2 + \int_{\Omega} G(\phi(t)) dx \leq Q (\|\phi_0\|_j^2 + \varepsilon \|u_0\|^2) e^{-c_0 t} + c, \quad \forall t \geq 0, \quad (2.2.21)$$

as consequence of (2.2.17). Estimate (2.2.21) implies

$$\int_{\Omega} G(\phi(t)) dx \leq Q (\|\phi_0\|_j^2 + \varepsilon \|u_0\|^2) e^{-c_0 t} + c, \quad \forall t \geq 0, \quad (2.2.22)$$

and

$$\|\phi(t)\|_1^2 + \varepsilon \|u(t)\|^2 \leq Q (\|\phi_0\|_j^2 + \varepsilon \|u_0\|^2) e^{-c_0 t} + c, \quad \forall t \geq 0, \quad (2.2.23)$$

due to Assumption (2.0.2). Similarly to (2.2.19), here we have

$$\begin{aligned} \int_t^{t+r} |m(u(s))| ds &\leq \int_t^{t+r} \left(|m(u_0)| e^{-\frac{\gamma}{\varepsilon} s} + \frac{2\sigma\alpha}{\gamma} e^{-\sigma s} \right) ds \\ &\leq \frac{\varepsilon}{\gamma} |m(u_0)| + \frac{2\alpha}{\gamma} \\ &\leq \frac{3\alpha}{\gamma}, \quad \forall t \geq 0, \quad \forall r > 0. \end{aligned} \quad (2.2.24)$$

Then, integrating (2.2.15) between t and $t+r$, and using again (2.0.2) and (2.2.24),

we find

$$\begin{aligned} &\int_t^{t+r} (\|u(s)\|_1^2 + \|\phi(s)\|_2^2 + \|\phi_t(s)\|_{-1}^2) ds \\ &\leq Q (\|\phi_0\|_j^2 + \varepsilon \|u_0\|^2) e^{-c_0 t} + c, \quad \forall t \geq 0, \quad \forall r > 0. \end{aligned} \quad (2.2.25)$$

Integrating (2.2.15) between 0 and t , we also find

$$\int_0^t (\|u(s)\|_1^2 + \|\phi(s)\|_2^2 + \|\phi_t(s)\|_{-1}^2) ds \leq (1+t)Q(\|\phi_0\|_j^2 + \varepsilon\|u_0\|^2), \quad \forall t \geq 0. \quad (2.2.26)$$

Next, we differentiate (2.0.1)₁ with respect to t to obtain

$$\phi_{tt} + N(N\phi_t + g'(\phi)\phi_t - u_t) + \sigma\phi_t = 0. \quad (2.2.27)$$

We multiply (2.2.27) and (2.0.1)₂ by $N^{-3}\bar{\phi}_t$ and $N^{-2}\bar{u}_t$, respectively, integrate over Ω , sum the resulting equations and obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{\phi}_t\|_{-3}^2 + \|\bar{u}\|_{-1}^2 + \gamma\|\bar{u}\|_{-2}^2) + \varepsilon\|\bar{u}_t\|_{-2}^2 + \|\bar{\phi}_t\|_{-1}^2 + \sigma\|\bar{\phi}_t\|_{-3}^2 \\ & + \int_{\Omega} \nabla N^{-1}\bar{\phi}_t \cdot \nabla [g'(\phi)N^{-2}\bar{\phi}_t] dx + m(\phi_t) \int_{\Omega} g'(\phi)N^{-2}\bar{\phi}_t dx = 0. \end{aligned} \quad (2.2.28)$$

We now exploit estimates (2.2.23) and (1.5.9) in the case of one space dimension to derive some more estimates. To do so, we write

$$\nabla[g'(\phi)N^{-2}\bar{\phi}_t] = g''(\phi)\nabla\phi N^{-2}\bar{\phi}_t + g'(\phi)\nabla N^{-2}\bar{\phi}_t,$$

and, by using (1.5.9) for $j = 1$, we have

$$\begin{aligned}
& \int_{\Omega} |\nabla[g'(\phi)N^{-2}\bar{\phi}_t]|^2 dx \\
& \leq 2\|g''(\phi)\|_{L^\infty(\Omega)}^2 \|\phi\|_1^2 \|N^{-2}\bar{\phi}_t\|_{L^\infty(\Omega)}^2 + 2\|g'(\phi)\|_{L^\infty(\Omega)}^2 \|\phi_t\|_{-3}^2 \\
& \leq Q(\|\phi\|_1^2) \|\phi_t\|_{-3}^2, \quad \text{if } d = 1,
\end{aligned} \tag{2.2.29}$$

so that

$$\left| \int_{\Omega} \nabla N^{-1}\bar{\phi}_t \cdot \nabla[g'(\phi)N^{-2}\phi_t] dx \right| \leq Q(\|\phi\|_1^2) \|\phi_t\|_{-1}^2, \quad \text{if } d = 1. \tag{2.2.30}$$

We also have

$$\begin{aligned}
|m(\phi_t) \int_{\Omega} g'(\phi)N^{-2}\bar{\phi}_t dx| & \leq |m(\phi_t)| \|g(\phi)\|_{L^\infty(\Omega)} (1 + \|\phi_t\|_{-4}^2) \\
& \leq Q(\|\phi\|_1^2) (1 + \|\phi_t\|_{-1}^2), \quad \text{if } d = 1.
\end{aligned} \tag{2.2.31}$$

It follows from (2.2.28), (2.2.30) and (2.2.31) that

$$\frac{dE_2}{dt} + cE_2 + \varepsilon \|\bar{u}_t\|_{-2}^2 \leq Q(\|\phi\|_1^2) (1 + \|\phi_t\|_{-1}^2), \quad \text{if } d = 1, \tag{2.2.32}$$

where

$$E_2(t) = \|\bar{\phi}_t\|_{-3}^2 + \|\bar{u}\|_{-1}^2 + \gamma \|\bar{u}\|_{-2}^2.$$

On account of (2.2.23) and (2.2.25), we get

$$\begin{aligned}
& \int_t^{t+r} Q(\|\phi(s)\|_1^2) (1 + \|\phi_t(s)\|_{-1}^2) ds \\
& \leq [Q(\|\phi_0\|_1^2 + \varepsilon\|u_0\|^2) e^{-ct} + c] \int_t^{t+r} (1 + \|\phi_t(s)\|_{-1}^2) ds \\
& \leq Q(\|\phi_0\|_1^2 + \varepsilon\|u_0\|^2) e^{-ct} + c, \quad \forall t \geq 0.
\end{aligned} \tag{2.2.33}$$

Also, we infer from (2.2.2) that

$$\begin{aligned}
m(u(t))^2 & \leq 2 \left(m(u_0)^2 e^{-2\frac{\gamma}{\varepsilon}t} + \frac{4\sigma^2\alpha^2}{\gamma^2} \right) \\
& \leq 2m(u_0)^2 e^{-4\sigma t} + \frac{8\sigma^2\alpha^2}{\gamma^2}, \quad \forall t \geq 0,
\end{aligned} \tag{2.2.34}$$

as $\varepsilon \leq \frac{\gamma}{2\sigma}$.

Multiplying (2.0.1)₁ by $N^{-3}\bar{\phi}_t$ and integrating over Ω , we use the identity $\|\bar{\phi}_t\|_{-3}^2 = \|\phi_t\|_{-3}^2 - |\Omega|m(\phi_t)^2$, and we obtain

$$\begin{aligned}
\|\phi_t\|_{-3}^2 & = -(N^{\frac{1}{2}}\phi, N^{-\frac{3}{2}}\bar{\phi}_t) - \int_{\Omega} g(\phi)N^{-2}\bar{\phi}_t dx + (N^{-\frac{1}{2}}u, N^{-\frac{3}{2}}\bar{\phi}_t) - \sigma(\phi, N^{-2}\bar{\phi}_t) \\
& \quad + |\Omega|m(\phi_t)^2.
\end{aligned} \tag{2.2.35}$$

We have, by Schwarz and Young's inequalities,

$$\begin{aligned}
|(N^{\frac{1}{2}}\phi, N^{-\frac{3}{2}}\bar{\phi}_t)| + \sigma|(\phi, N^{-2}\bar{\phi}_t)| &\leq \|\phi\|_1\|\phi_t\|_{-3} + \sigma\|\phi\|\|\phi_t\|_{-4} \\
&\leq \frac{1}{12}\|\phi_t\|_{-3}^2 + 3\|\phi\|_1^2 + \frac{1}{12}\|\phi_t\|_{-3}^2 + 3\sigma^2\mathfrak{c}_0^2\|\phi\|^2 \\
&\leq \frac{1}{6}\|\phi_t\|_{-3}^2 + 3\varrho_3^2\|\phi\|_1^2,
\end{aligned}$$

where we used estimates $\|\phi_t\|_{-4} \leq \mathfrak{c}_0\|\phi_t\|_{-3}$ (cf. (1.4.2)) and $\|\phi\|_1^2 + \sigma^2\mathfrak{c}_0^2\|\phi\|^2 \leq \varrho_3^2\|\phi\|_1^2$

for a constant $\varrho_3 > 0$,

$$\begin{aligned}
|(N^{-\frac{1}{2}}\bar{u}, N^{-\frac{3}{2}}\bar{\phi}_t)| &\leq \|u\|_{-1}\|\phi_t\|_{-3} \\
&\leq \frac{1}{6}\|\phi_t\|_{-3}^2 + 3\|u\|_{-1}^2,
\end{aligned}$$

and, by (1.5.9), the continuous embedding $L^2(\Omega) \hookrightarrow L^1(\Omega)$, and (1.4.2) again,

$$\begin{aligned}
\int_{\Omega} |g(\phi)| |N^{-2}\bar{\phi}_t| \, dx &\leq \|g(\phi)\|_{L^\infty(\Omega)} \|N^{-2}\bar{\phi}_t\|_{L^1(\Omega)} \\
&\leq Q(\|\phi\|_1)\|\phi_t\|_{-4} \\
&\leq \frac{1}{6}\|\phi_t\|_{-3}^2 + Q(\|\phi\|_1^2), \quad \text{if } d = 1.
\end{aligned}$$

We deduce from (2.2.35) that

$$\|\phi_t\|_{-3}^2 \leq Q(\|\phi\|_1^2) + c\|u\|_{-1}^2, \quad \text{if } d = 1. \quad (2.2.36)$$

We first apply Lemma 1.5.10 to (2.2.32), thanks to (2.2.33), then we use estimates

(2.2.3), (2.2.34) and (2.2.36) to deduce that

$$\begin{aligned} \|\phi_t(t)\|_{-3}^2 + \|u(t)\|_{-1}^2 &\leq E_2(0)e^{-c_1 t} + c \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{1,\varepsilon}}^2)e^{-c_1 t} + c, \quad \forall t \geq 0, \quad \text{if } d = 1. \end{aligned} \quad (2.2.37)$$

Estimates (2.2.23) and (2.2.37) entail

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{1,\varepsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{1,\varepsilon}}^2)e^{-c_1 t} + c, \quad \forall t \geq 0, \quad \text{if } d = 1. \quad (2.2.38)$$

2.2.2 Estimates in $\mathcal{V}_{2,\varepsilon}$

We multiply (2.2.27) and (2.0.1)₂ by $N^{-1}\bar{\phi}_t$ and $\bar{u}_t$, respectively, integrate over Ω , sum the resulting equations, and we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\bar{\phi}_t\|_{-1}^2 + \|\bar{u}_t\|_1^2 + \gamma \|\bar{u}_t\|^2) + \varepsilon \|\bar{u}_t\|^2 + \|\bar{\phi}_t\|_1^2 + \sigma \|\bar{\phi}_t\|_{-1}^2 + \int_{\Omega} g'(\phi) \phi_t^2 dx \\ = m(\phi_t) \int_{\Omega} g'(\phi) \phi_t dx. \end{aligned} \quad (2.2.39)$$

We have

$$\frac{d}{dt} \left[m(\phi_t) \int_{\Omega} g(\phi) dx \right] = m(\phi_{tt}) \int_{\Omega} g(\phi) dx + m(\phi_t) \int_{\Omega} g'(\phi) \phi_t dx.$$

Therefore equality (2.2.39) can be rewrite as

$$\frac{1}{2} \frac{dE_3}{dt} + \varepsilon \|\bar{u}_t\|^2 + \|\bar{\phi}_t\|_1^2 + \sigma \|\bar{\phi}_t\|_{-1}^2 + \int_{\Omega} g'(\phi) \phi_t^2 dx = -m(\phi_{tt}) \int_{\Omega} g(\phi) dx, \quad (2.2.40)$$

where

$$E_3(t) = \|\bar{\phi}_t\|_{-1}^2 + \|\bar{u}\|_1^2 + \gamma\|\bar{u}\|^2 - m(\phi_t) \int_{\Omega} g(\phi) dx. \quad (2.2.41)$$

We can derive from (2.2.3), (2.2.7) and (2.2.41) that

$$\begin{aligned} c_1 \left(\|\phi_t\|_{-1}^2 + \|u\|_1^2 - \int_{\Omega} G(\phi) dx \right) - c_2 (1 + m(u)^2) \\ \leq E_3(t) \leq c_3 \left(\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \int_{\Omega} G(\phi) dx + 1 \right). \end{aligned} \quad (2.2.42)$$

Similarly to (2.2.7), we have, in view of (2.0.5) and (2.2.4),

$$|m(\phi_{tt}) \int_{\Omega} g(\phi) dx| \leq c \left(\int_{\Omega} G(\phi) dx + 1 \right). \quad (2.2.43)$$

By virtue of (2.0.4), estimate $\|\phi_t\|^2 \leq \|\phi_t\|_1 \|\phi_t\|_{-1}$ and Young's inequality, we also have

$$\begin{aligned} \int_{\Omega} g'(\phi) \phi_t^2 dx &\geq -C_4 \|\phi_t\|^2 \\ &\geq -\mu \|\phi_t\|_1^2 - c_{\mu} \|\phi_t\|_{-1}^2, \quad \forall \mu > 0. \end{aligned} \quad (2.2.44)$$

We now deduce from (2.2.40) that

$$\frac{dE_3}{dt} + \varepsilon \|\bar{u}_t\|^2 + \|\phi_t\|_1^2 \leq c \left(\|\phi_t\|_{-1}^2 + \int_{\Omega} G(\phi) dx + 1 \right). \quad (2.2.45)$$

We first apply Lemma 1.5.11 to (2.2.45), on account of (2.2.22) and (2.2.25), then we use (2.2.3) and (2.2.34) to get

$$\|\phi_t(t)\|_{-1}^2 + \|u(t)\|_1^2 \leq Q(\|(\phi_0, u_0)\|_{V_{j,\varepsilon}}^2) e^{-c_1 t} + c, \quad \forall t \geq r. \quad (2.2.46)$$

Now, we multiply (2.0.1)₁ by ϕ , integrate over Ω and we obtain

$$\|\bar{\phi}\|_2^2 + \sigma\|\phi\|^2 + \int_{\Omega} g'(\phi)|\nabla\phi|^2 dx = (N^{\frac{1}{2}}u, N^{\frac{1}{2}}\phi) - \int_{\Omega} \phi_t\phi dx.$$

Treating the equation above exactly like we did for (2.2.11) and using the estimate

$|\int_{\Omega} \phi_t\phi dx| \leq \|\phi_t\|_{-1}\|\phi\|_1$, it follows that

$$\|\phi\|_2^2 \leq c(\|\phi\|_1^2 + \|u\|_1^2 + \|\phi_t\|_{-1}^2). \quad (2.2.47)$$

Next, we multiply (2.0.1)₁ by $N\phi$, we use the equality $\|\bar{\phi}\|_3^2 = \|\phi\|_3^2 - |\Omega|m(\phi)^2$ and we obtain

$$\|\phi\|_3^2 + \sigma\|\bar{\phi}\|_1^2 + \int_{\Omega} \nabla g(\phi) \cdot \nabla N\phi dx = (N^{\frac{1}{2}}u, N^{\frac{3}{2}}\phi) - (N^{-\frac{1}{2}}\bar{\phi}_t, N^{\frac{3}{2}}\phi) + |\Omega|m(\phi)^2.$$

We have, by Schwarz and Young's inequalities,

$$\begin{aligned} |(N^{\frac{1}{2}}u, N^{\frac{3}{2}}\phi)| &\leq \|u\|_1\|\phi\|_3 \\ &\leq \frac{1}{6}\|\phi\|_3^2 + \frac{3}{2}\|u\|_1^2, \end{aligned}$$

$$\begin{aligned}
|(N^{-\frac{1}{2}}\bar{\phi}_t, N^{\frac{3}{2}}\phi)| &\leq \|\phi_t\|_{-1}\|\phi\|_3 \\
&\leq \frac{1}{6}\|\phi\|_3^2 + \frac{3}{2}\|\phi_t\|_{-1}^2
\end{aligned}$$

and, by the equality $\nabla g(\phi) = g'(\phi)\nabla\phi$ and (1.5.9),

$$\begin{aligned}
\int_{\Omega} |\nabla g(\phi)| |\nabla N\phi| dx &\leq \|g'(\phi)\|_{L^\infty(\Omega)} \|\phi\|_1 \|\phi\|_3 \\
&\leq \frac{1}{6}\|\phi\|_3^2 + Q(\|\phi\|_j^2),
\end{aligned}$$

and we deduce from the above equation that

$$\|\phi\|_3^2 \leq Q(\|\phi\|_j^2) + c(\|u\|_1^2 + \|\phi_t\|_{-1}^2). \quad (2.2.48)$$

We infer from (2.2.46), (2.2.47) and (2.2.48) that

$$\|\phi(t)\|_3^2 + \|\phi_t(t)\|_{-1}^2 + \|u(t)\|_1^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r, \quad \forall r > 0, \quad (2.2.49)$$

that is,

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{2,\varepsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r, \quad \forall r > 0. \quad (2.2.50)$$

Now we want to prove that

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{2,\varepsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2) e^{-ct} + c, \quad \forall t \in [0, 1]. \quad (2.2.51)$$

To deduce (2.2.51) it is enough to prove that

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{2,\varepsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \in [0, 1], \quad (2.2.52)$$

as $1 = e^c e^{-c} \leq e^c e^{-ct}$, $\forall t \in [0, 1]$.

In order to prove Estimate (2.2.52), we multiply (2.0.1)₁ and (2.0.1)₂ by ϕ_t and Nu , respectively, integrate over Ω , sum the resulting equations and we find

$$\frac{1}{2} \frac{d}{dt} (\|\bar{\phi}\|_2^2 + \sigma \|\bar{\phi}\|^2 + \varepsilon \|\bar{u}\|_1^2) + \|\phi_t\|^2 + \|\bar{u}\|_2^2 + \gamma \|\bar{u}\|_1^2 + \int_{\Omega} Ng(\phi) \phi_t dx = 0. \quad (2.2.53)$$

We have $\Delta g(\phi) = g'(\phi) \Delta \phi + g''(\phi) |\nabla \phi|^2$ and the interpolation inequality

$$\|\nabla \phi\|_{L^4(\Omega)^d} \leq c \|\phi\|_1^{1-\frac{d}{4}} \|\phi\|_2^{\frac{d}{4}},$$

hence

$$\begin{aligned} \int_{\Omega} |Ng(\phi)| |\phi_t| dx &\leq \|g'(\phi)\|_{L^\infty(\Omega)} \|\phi\|_2 \|\phi_t\| + \|g''(\phi)\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^4(\Omega)^d}^2 \|\phi_t\| \\ &\leq \frac{1}{2} \|\phi_t\|^2 + Q(\|\phi\|_2^2). \end{aligned} \quad (2.2.54)$$

We deduce from (2.2.53) and (2.2.54) that

$$\frac{d}{dt} (\|\bar{\phi}\|_2^2 + \sigma \|\bar{\phi}\|^2 + \varepsilon \|\bar{u}\|_1^2) + \|\phi_t\|^2 + \|\bar{u}\|_2^2 \leq Q(\|\phi\|_2^2). \quad (2.2.55)$$

We multiply (2.2.27) and (2.0.1)₂ by $N^{-2} \bar{\phi}_t$ and $N^{-1} \bar{u}_t$, respectively, integrate over Ω ,

sum the resulting equations and obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\bar{\phi}_t\|_{-2}^2 + \|\bar{u}\|^2 + \gamma \|\bar{u}\|_{-1}^2) + \varepsilon \|\bar{u}_t\|_{-1}^2 + \|\bar{\phi}_t\|^2 + \sigma \|\bar{\phi}_t\|_{-2}^2 \\ = - \int_{\Omega} g'(\phi) \phi_t N^{-1} \bar{\phi}_t dx. \end{aligned} \quad (2.2.56)$$

We write

$$\begin{aligned} \left| \int_{\Omega} g'(\phi) \phi_t N^{-1} \bar{\phi}_t dx \right| &\leq \|g'(\phi)\|_{L^\infty} \|\phi_t\| \|\phi_t\|_{-2} \\ &\leq \frac{1}{2} \|\phi_t\|^2 + Q(\|\phi\|_2^2) \|\phi_t\|_{-2}^2. \end{aligned} \quad (2.2.57)$$

It follows from (2.2.56) and (2.2.57) that

$$\frac{d}{dt} (\|\bar{\phi}_t\|_{-2}^2 + \|\bar{u}\|^2 + \gamma \|\bar{u}\|_{-1}^2) + \varepsilon \|\bar{u}_t\|_{-1}^2 + \|\phi_t\|^2 \leq Q(\|\phi\|_2^2 + \|\phi_t\|_{-2}^2). \quad (2.2.58)$$

Adding (2.2.55) to (2.2.58), we end up with the inequality

$$\frac{dV}{dt} \leq Q(V), \quad (2.2.59)$$

where

$$V(t) = \|\bar{\phi}\|_2^2 + \sigma \|\bar{\phi}\|^2 + \varepsilon \|\bar{u}\|_1^2 + \|\bar{\phi}_t\|_{-2}^2 + \|\bar{u}\|^2 + \gamma \|\bar{u}\|_{-1}^2.$$

Since $\varepsilon \leq 1$ and

$$m(\phi(t))^2 + 2m(u(t))^2 + m(\phi_t(t))^2 \leq c(1 + m(u_0)^2), \quad c > 0, \quad \forall t \geq 0,$$

thanks to (2.2.1), (2.2.2) and (2.2.3), we deduce that

$$V(t) \geq \|\phi\|_2^2 + \varepsilon \|u\|_1^2 + \|u\|^2 + \|\phi_t\|_{-2}^2 - c(1 + m(u_0)^2). \quad (2.2.60)$$

Multiplying (2.0.1)₁ by $N^{-2}\bar{\phi}_t$ and integrating over Ω , we use the identity $\|\bar{\phi}_t\|_{-2}^2 = \|\phi_t\|_{-2}^2 - |\Omega|m(\phi_t)^2$, and we obtain

$$\begin{aligned} \|\phi_t\|_{-2}^2 &= -(N\phi, N^{-1}\bar{\phi}_t) - \int_{\Omega} g(\phi)N^{-1}\bar{\phi}_t dx + (u, N^{-1}\bar{\phi}_t) - \sigma(\phi, N^{-2}\bar{\phi}_t) \\ &\quad + |\Omega|m(\phi_t)^2. \end{aligned} \quad (2.2.61)$$

Like we proceeded from (2.2.35) through (2.2.36), here we have

$$\begin{aligned} |(N\phi, N^{-1}\bar{\phi}_t)| + \sigma|(\phi, N^{-2}\bar{\phi}_t)| &\leq \|\phi\|_2 \|\phi_t\|_{-2} + \sigma \|\phi\| \|\phi_t\|_{-4} \\ &\leq \frac{1}{12} \|\phi_t\|_{-2}^2 + 3\|\phi\|_2^2 + \frac{1}{12} \|\phi_t\|_{-2}^2 + 3\sigma^2 \mathfrak{c}_0^2 \|\phi\|^2 \\ &\leq \frac{1}{6} \|\phi_t\|_{-2}^2 + 3\varrho_4^2 \|\phi\|_2^2, \end{aligned}$$

for a constant $\varrho_4 > 0$,

$$\begin{aligned} |(u, N^{-1}\bar{\phi}_t)| &\leq \|u\| \|\phi_t\|_{-2} \\ &\leq \frac{1}{6} \|\phi_t\|_{-2}^2 + 3\|u\|^2 \end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega} |g(\phi)| |N^{-1} \bar{\phi}_t| \, dx &\leq \|g(\phi)\|_{L^\infty(\Omega)} \|N^{-1} \bar{\phi}_t\|_{L^1(\Omega)} \\
&\leq Q(\|\phi\|_j) \|\phi_t\|_{-2} \\
&\leq \frac{1}{6} \|\phi_t\|_{-2}^2 + Q(\|\phi\|_j^2).
\end{aligned}$$

We deduce from (2.2.61) that

$$\|\phi_t\|_{-2}^2 \leq Q(\|\phi\|_j^2) + c(\|\phi\|_2^2 + \|u\|^2). \quad (2.2.62)$$

By comparison principle of the solutions of Problem (2.2.59) with the ODE $\frac{dV}{dt} = Q(V)$, there exists a time $t_1 = t_1(V(0)) > 0$ such that

$$V(t) \leq Q(V(0)), \quad \forall t \in [0, t_1]. \quad (2.2.63)$$

We thus infer from (2.2.60), (2.2.62) and (2.2.63) that

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{2,\varepsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \in [0, t_1]. \quad (2.2.64)$$

If $t_1 \geq 1$, then the proof of (2.2.52) is finished. Now assume $t_1 \in (0, 1)$. First, we

integrate (2.2.18) between $\frac{t_1}{2}$ and t , $t \leq 1$, and we obtain

$$\int_{\frac{t_1}{2}}^t \left(\|\phi_t(s)\|_{-1}^2 + \|u(s)\|_1^2 + \int_{\Omega} G(\phi(s)) dx \right) ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \in [\frac{t_1}{2}, 1], \quad (2.2.65)$$

as consequence of (2.2.17) and the fact that $E_1(\frac{t_1}{2}) \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2)$.

Next, we change the variable t by s in (2.2.45), then we multiply by $s - \frac{t_1}{2}$ to obtain

$$\frac{d}{ds} \left[\left(s - \frac{t_1}{2} \right) E_3(s) \right] \leq E_3(s) + c \left(s - \frac{t_1}{2} \right) \left(\|\phi_t\|_{-1}^2 + \int_{\Omega} G(\phi) dx + 1 \right). \quad (2.2.66)$$

Integrating (2.2.66) on s between $\frac{t_1}{2}$ and t , $t \leq 1$, we deduce

$$\left(t - \frac{t_1}{2} \right) E_3(t) \leq c \int_{\frac{t_1}{2}}^t \left(E_3(s) + \|\phi_t\|_{-1}^2 + \int_{\Omega} G(\phi) dx + 1 \right) ds, \quad (2.2.67)$$

hence

$$\begin{aligned} & \left(t - \frac{t_1}{2} \right) (\|\phi_t(t)\|_{-1}^2 + \|u(t)\|_1^2) \\ & \leq c \int_{\frac{t_1}{2}}^t \left(\|\phi_t(s)\|_{-1}^2 + \|u(s)\|_1^2 + \int_{\Omega} G(\phi(s)) dx \right) ds \\ & \quad + c \left(m(u(t))^2 + \int_{\Omega} G(\phi(t)) dx + 1 \right) \\ & \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \in [\frac{t_1}{2}, 1], \end{aligned} \quad (2.2.68)$$

on account of (2.2.2), (2.2.22), (2.2.42) and (2.2.65). It follows from (2.2.47) and

(2.2.68) that

$$\begin{aligned}\|\phi(t)\|_2^2 + \|u(t)\|_1^2 &\leq \frac{1}{t_1} Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2) \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \in [t_1, 1].\end{aligned}\quad (2.2.69)$$

Estimates (2.2.64) and (2.2.69) entail (2.2.52). Eventually, Estimates (2.2.50) and (2.2.51) imply

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{2,\varepsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2) e^{-ct} + c, \quad \forall t \geq 0. \quad (2.2.70)$$

Now, we wish to derive an estimate that is uniform with respect to time.

Owing to (1.5.9), (2.2.3) and (2.2.6), we infer from (2.2.5) that

$$\begin{aligned}\frac{d}{dt} \left(\varepsilon \|u\|^2 + \|\bar{\phi}\|_1^2 + \sigma \|\bar{\phi}\|^2 + 2 \int_{\Omega} G(\phi) dx \right) + 2\varrho_1 \|u\|_1^2 + 2\|\phi_t\|_{-1}^2 \\ \leq Q(\|\phi\|_j^2) e^{-\sigma t} + c|m(u)|.\end{aligned}\quad (2.2.71)$$

By (2.0.2), we have

$$\varepsilon \|u(t)\|^2 + \|\bar{\phi}(t)\|_1^2 + \sigma \|\bar{\phi}(t)\|^2 + 2 \int_{\Omega} G(\phi(t)) dx \geq -c, \quad \forall t \geq 0, \quad (2.2.72)$$

and, by virtue of (2.2.22), there holds

$$\varepsilon \|u(t)\|^2 + \|\bar{\phi}(t)\|_1^2 + \sigma \|\bar{\phi}(t)\|^2 + 2 \int_{\Omega} G(\phi(t)) dx \leq Q(\|(\phi(t), u(t))\|_{\mathcal{V}_{j,\varepsilon}}^2), \quad \forall t \geq 0. \quad (2.2.73)$$

Integrating (2.2.71) from 0 to t and using (2.2.72) and (2.2.73), we reach the estimate

$$\begin{aligned}
& \int_0^t (\|u(s)\|_1^2 + \|\phi_t(s)\|_{-1}^2) ds \\
& \leq \int_0^t Q(\|\phi(s)\|_j^2) e^{-\sigma s} ds + c \int_0^t |m(u(s))| ds + Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2) \\
& \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2),
\end{aligned} \tag{2.2.74}$$

on account of (2.2.19), (2.2.38), (2.2.70). Hence

$$\int_0^\infty (\|u(s)\|_1^2 + \|\phi_t(s)\|_{-1}^2) ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2). \tag{2.2.75}$$

We also infer from (2.0.1)₂ and (2.2.75) that

$$\begin{aligned}
\int_0^\infty \varepsilon^2 \|u_t(s)\|_{-1}^2 ds & \leq c \int_0^\infty (\|\phi_t(s)\|_{-1}^2 + \|u(s)\|_1^2) ds \\
& \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2), \quad \forall t \geq 0.
\end{aligned} \tag{2.2.76}$$

To finish this section, we also derive some more estimates that are useful in the proof of Theorem 2.4.1.

We integrate (2.2.48) between 0 and t , we find

$$\int_0^t \|\phi(s)\|_3^2 ds \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2), \quad \forall t \geq 0, \tag{2.2.77}$$

on account of (2.2.26), (2.2.38) and (2.2.70).

Next, we integrate (2.2.55) between 0 and t to get

$$\int_0^t (\|\phi_t(s)\|^2 + \|u(s)\|_2^2) ds \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \geq 0, \quad (2.2.78)$$

due to (2.2.70). Similarly to (2.2.76), we also deduce from (2.0.1)₂ and (2.2.78) that

$$\begin{aligned} \int_0^t \varepsilon^2 \|u_t(s)\|^2 ds &\leq c \int_0^t (\|\phi_t(s)\|^2 + \|u(s)\|_2^2) ds \\ &\leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \geq 0. \end{aligned} \quad (2.2.79)$$

Then, we multiply (2.0.1)₁ by $N^2\phi$ and obtain

$$\|\bar{\phi}\|_4^2 + \sigma \|\bar{\phi}\|_2^2 + \int_{\Omega} Ng(\phi)N^2\phi dx = (Nu, N^2\phi) - (\phi_t, N^2\phi),$$

from which we deduce, similarly to (2.2.48), that

$$\|\phi\|_4^2 \leq Q(\|\phi\|_2^2) + c(\|u\|_2^2 + \|\phi_t\|^2), \quad (2.2.80)$$

since, by (1.5.9) with reference to (2.2.54), there holds

$$\int_{\Omega} |\Delta g(\phi)|^2 dx \leq Q(\|\phi\|_2^2).$$

Hence

$$\int_0^t \|\phi(s)\|_4^2 ds \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{2,\varepsilon}}^2), \quad \forall t \geq 0, \quad (2.2.81)$$

on account of (2.2.70) and (2.2.78).

2.2.3 Estimates in $\mathcal{V}_{3,\varepsilon}$

Integrating (2.2.48) between t and $t + r$, we find

$$\int_t^{t+r} \|\phi(s)\|_3^2 ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2) e^{-ct} + c, \quad \forall t \geq 0, \quad (2.2.82)$$

on account of (2.2.25), (2.2.38) and (2.2.70).

Next we multiply (2.0.1)₁ and (2.0.1)₂ by $N\phi_t$ and N^2u , respectively, integrate over Ω , sum the resulting equation, and we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{\phi}\|_3^2 + \sigma \|\bar{\phi}\|_1^2 + \varepsilon \|\bar{u}\|_2^2) + \|\bar{\phi}_t\|_1^2 + \|\bar{u}\|_3^2 + \gamma \|\bar{u}\|_2^2 + \int_{\Omega} \nabla N g(\phi) \cdot \nabla \phi_t dx \\ & = 0. \end{aligned} \quad (2.2.83)$$

We have

$$\nabla \Delta g(\phi) = g'(\phi) \nabla \Delta \phi + 3g''(\phi) \Delta \phi \nabla \phi + g'''(\phi) |\nabla \phi|^2 \nabla \phi,$$

and by Sobolev embeddings and Agmon's inequality, we find that

$$\begin{aligned} & \left| \int_{\Omega} \nabla N g(\phi) \cdot \nabla \phi_t dx \right| \\ & \leq (\|g'(\phi)\|_{L^\infty(\Omega)} \|\phi\|_3 + 3\|g''(\phi)\|_{L^\infty(\Omega)} \|\Delta \phi\|_{L^\infty(\Omega)} \|\phi\|_1 \\ & \quad + \|g'''(\phi)\|_{L^\infty(\Omega)} \|\phi\|_1 \|\nabla \phi\|_{L^\infty(\Omega)^d}^2) \|\phi_t\|_1 \\ & \leq Q(\|\phi\|_1^2) \|\phi\|_3 \|\phi_t\|_1, \quad \text{if } d = 1, \end{aligned} \quad (2.2.84)$$

and

$$\begin{aligned}
& \left| \int_{\Omega} \nabla N g(\phi) \cdot \nabla \phi_t \, dx \right| \\
& \leq (\|g'(\phi)\|_{L^\infty(\Omega)} \|\phi\|_3 + 3\|g''(\phi)\|_{L^\infty(\Omega)} \|\Delta \phi\|_{L^4(\Omega)} \|\nabla \phi\|_{L^4(\Omega)^d} \\
& \quad + \|g'''(\phi)\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^6(\Omega)^d}^3) \|\phi_t\|_1 \\
& \leq Q(\|\phi\|_2^2) (1 + \|\phi\|_3) \|\phi_t\|_1, \text{ if } d = 2, 3.
\end{aligned} \tag{2.2.85}$$

We infer from (2.2.83), (2.2.84) and (2.2.85) that

$$\frac{dE_4}{dt} + c_0 E_4 + \|\phi_t\|_1^2 + \|\bar{u}\|_3^2 \leq Q(\|\phi\|_j^2) (1 + \|\phi\|_3^2), \tag{2.2.86}$$

where

$$E_4(t) = \|\bar{\phi}\|_3^2 + \varepsilon \|\bar{u}\|_2^2.$$

Applying Lemma 1.5.10 to (2.2.86), we find

$$\|\phi(t)\|_3^2 + \varepsilon \|u(t)\|_2^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2) e^{-ct} + c, \quad \forall t \geq r, \quad \forall r > 0, \tag{2.2.87}$$

on account of (2.2.1), (2.2.34) and (2.2.82). Putting (2.2.49) and (2.2.87) together we end up with the desired estimate

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{3,\varepsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2) e^{-ct} + c, \quad \forall t \geq r, \quad \forall r > 0. \tag{2.2.88}$$

From (2.0.1)₁, and $\int_{\Omega} |\nabla N g(\phi)|^2 dx \leq Q(\|\phi\|_3^2)$, we deduce that

$$\|\phi\|_5^2 \leq Q(\|\phi\|_3^2) + c(\|u\|_3^2 + \|\phi_t\|_1^2), \quad (2.2.89)$$

and there holds

$$\int_r^t \|\phi(s)\|_5^2 ds \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2), \quad \forall t \geq r, \quad (2.2.90)$$

on account of (2.2.88) and (2.2.86).

To end this subsection, we derive few more estimates that will be needed in the proof of Theorem 4.6.2.

We integrate (2.2.86) between 0 and t , after using the identity $\|\bar{u}\|_3^2 = \|u\|_3^2 - |\Omega|m(u)^2$, and we find

$$\begin{aligned} & \int_0^t (\|\phi_t(s)\|_1^2 + \|u(s)\|_3^2) ds \\ & \leq E_4(0) + \int_0^t Q(\|\phi(s)\|_j^2) (1 + \|\phi(s)\|_3^2) ds + |\Omega| \int_0^t m(u(s))^2 ds \\ & \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{3,\varepsilon}}^2), \quad \forall t \geq 0, \end{aligned} \quad (2.2.91)$$

due to (2.2.34), (2.2.70) and (2.2.77). Like (2.2.76) and (2.2.79), we deduce from (2.0.1)₂ and (2.2.91) that

$$\begin{aligned} & \int_0^t \varepsilon^2 \|u_t(s)\|_1^2 ds \leq c \int_0^t (\|\phi_t(s)\|_1^2 + \|u(s)\|_3^2) ds \\ & \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{3,\varepsilon}}^2), \quad \forall t \geq 0, \end{aligned} \quad (2.2.92)$$

and

$$\int_r^t \varepsilon^2 \|u_t(s)\|_1^2 ds \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\varepsilon}}^2), \quad \forall t \geq r. \quad (2.2.93)$$

2.2.4 $\mathcal{V}_{1,\varepsilon}$ - $\mathcal{V}_{1,\varepsilon}$ Estimate of the difference of two solutions

We consider the difference (ϕ, u) of two solutions (ϕ_1, u_1) and (ϕ_2, u_2) of Problem (2.0.1) starting at the points (ϕ_{10}, u_{10}) and (ϕ_{20}, u_{20}) in $\mathcal{V}_j$ at time $t = 0$, respectively,

that is,

$$\begin{cases} \phi_t + N(N\phi + l(t)\phi - u) + \sigma\phi = 0, \\ \varepsilon u_t + \phi_t + Nu + \gamma u = 0, \\ \phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0, \end{cases} \quad (2.2.94)$$

where, by Taylor's formula with integral reminder, $g(\phi_1) - g(\phi_2) = l(t)\phi$, with

$$l(t) = \int_0^1 g'(s\phi_1(t) + (1-s)\phi_2(t)) ds.$$

We observe that ϕ and u have the same expressions (2.1.1), (2.1.2) and (2.1.4), and

we have the estimates

$$m(\phi(t))^2 \leq m(\phi_0)^2, \quad \forall t \geq 0, \quad (2.2.95)$$

$$m(\phi_t(t))^2 \leq \sigma^2 m(\phi_0)^2, \quad \forall t \geq 0 \quad (2.2.96)$$

and

$$m(u(t))^2 \leq 2 \left(m(u_0)^2 + \frac{4\sigma^2}{\gamma^2} m(\phi_0)^2 \right), \quad \forall t \geq 0. \quad (2.2.97)$$

We multiply (2.2.94)₁ and (2.2.94)₂ by $N^{-1}\bar{\phi}_t$ and u , respectively, integrate over Ω , sum the resulting equations, and we obtain (cf. (2.2.5))

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{\phi}\|_1^2 + \sigma \|\bar{\phi}\|_{-1}^2 + \varepsilon \|u\|^2) + \|\bar{\phi}_t\|_{-1}^2 + \|\bar{u}\|_1^2 + \gamma \|u\|^2 \\ &= - \int_{\Omega} \nabla[l(t)\phi] \cdot \nabla N^{-1}\bar{\phi}_t \, dx - |\Omega| m(u)m(\phi_t). \end{aligned} \quad (2.2.98)$$

We multiply (2.2.94)₁ by ϕ , integrate over Ω , and we obtain

$$\frac{1}{2} \frac{d}{dt} \|\phi\|^2 + \|\bar{\phi}\|_2^2 + \sigma \|\phi\|^2 + \int_{\Omega} l(t)\phi N\phi \, dx - (u, N\phi) = 0. \quad (2.2.99)$$

We have $\nabla[l(t)\phi] = l(t)\nabla\phi + \phi\nabla l(t)$,

$$\nabla l(t) = \int_0^1 g''(s\phi_1(t) + (1-s)\phi_2(t)) (s\nabla\phi_1(t) + (1-s)\nabla\phi_2(t)) \, ds,$$

$$\begin{aligned} \int_{\Omega} |\nabla l(t)|^2 \, dx &\leq \sup_{s \in [0,1]} \|g''(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty(\Omega)}^2 (\|\phi_1(t)\|_1^2 + \|\phi_2(t)\|_1^2) \\ &\leq \tilde{c}_j, \quad \forall t \geq 0, \end{aligned}$$

where

$$\tilde{c}_j = Q \left(\|(\phi_{10}, u_{10})\|_{\mathcal{V}_{j,\varepsilon}}^2, \|(\phi_{20}, u_{20})\|_{\mathcal{V}_{j,\varepsilon}}^2 \right),$$

hence

$$\begin{aligned} \int_{\Omega} |\nabla[l(t)\phi]|^2 dx &\leq \|l(t)\|_{L^\infty(\Omega)}^2 \|\phi\|_1^2 + \|\phi\|_{L^\infty(\Omega)}^2 \|\nabla l(t)\|^2 \\ &\leq \tilde{c}_j \|\phi\|_1 \|\phi\|_2, \quad \forall t \geq 0, \end{aligned} \quad (2.2.100)$$

where we used the Agmon's inequality $\|\phi\|_{L^\infty}^2 \leq c \|\phi\|_1 \|\phi\|_2$ (valid for all three space dimensions $d = 1, 2, 3$).

Adding (2.2.98) to (2.2.99) and by using Cauchy-Schwarz and Young's inequalities, together with (2.2.96), (2.2.97) and (2.2.100), we deduce that

$$\frac{dE_5}{dt} + c (\|\phi_t\|_{-1}^2 + \|\phi\|_2^2 + \|u\|_1^2) \leq \tilde{c}_j \|\phi\|_1^2 + c (m(u_0)^2 + m(\phi_0)^2), \quad (2.2.101)$$

where

$$E_5(t) = \|\phi\|^2 + \sigma \|\bar{\phi}\|_{-1}^2 + \|\bar{\phi}\|_1^2 + \varepsilon \|u\|^2.$$

There exist two constants $c_1, c_2 > 0$, independent of ε , such that

$$c_1 (\|\phi(t)\|_1^2 + \varepsilon \|u(t)\|^2) \leq E_5(t) \leq c_2 (\|\phi(t)\|_1^2 + \varepsilon \|u(t)\|^2). \quad (2.2.102)$$

We apply the Gronwall's lemma to (2.2.101) and we deduce

$$\|\phi(t)\|_1^2 + \varepsilon \|u(t)\|^2 \leq c (\|\phi_0\|_1^2 + \varepsilon \|u_0\|^2) e^{\tilde{c}t}, \quad \forall t \geq 0, \quad (2.2.103)$$

by virtue of (2.2.102).

Then, integrating (2.2.101) between 0 and t , we find

$$\int_0^t (\|\phi_t(s)\|_{-1}^2 + \|\phi(s)\|_2^2 + \|u(s)\|_1^2) ds \leq c (\|\phi_0\|_1^2 + \varepsilon \|u_0\|^2) e^{\tilde{c}t}, \quad \forall t \geq 0. \quad (2.2.104)$$

Next we differentiate (2.2.94)₁ with respect to t to obtain

$$\phi_{tt} + N(N\phi_t + l(t)\phi_t + l'(t)\phi - u_t) + \sigma\phi_t = 0, \quad (2.2.105)$$

where

$$l'(t) = \int_0^1 g''(s\phi_1(t) + (1-s)\phi_2(t))(s\phi_{1t}(t) + (1-s)\phi_{2t}(t)) ds.$$

We multiply (2.2.105) and (2.2.94)₂ by $N^{-3}\bar{\phi}_t$ and $N^{-2}\bar{u}_t$, respectively, integrate over Ω , sum the resulting equations, and we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{\phi}_t\|_{-3}^2 + \|\bar{u}\|_{-1}^2 + \gamma \|\bar{u}\|_{-2}^2) + \|\bar{\phi}_t\|_{-1}^2 + \varepsilon \|\bar{u}_t\|_{-2}^2 + \sigma \|\bar{\phi}_t\|_{-3}^2 \\ &= - \int_{\Omega} \phi_t l(t) N^{-2} \bar{\phi}_t dx - \int_{\Omega} l'(t) \phi N^{-2} \bar{\phi}_t dx. \end{aligned} \quad (2.2.106)$$

We use the identity $\phi_t = \bar{\phi}_t + m(\phi_t) = NN^{-1}\bar{\phi}_t + m(\phi_t)$ to write

$$\int_{\Omega} \phi_t l(t) N^{-2} \bar{\phi}_t dx = \int_{\Omega} (NN^{-1}\bar{\phi}_t) l(t) N^{-2} \bar{\phi}_t dx + m(\phi_t) \int_{\Omega} l(t) N^{-2} \bar{\phi}_t dx, \quad (2.2.107)$$

and, by using Green's identity, we have

$$\int_{\Omega} (NN^{-1}\bar{\phi}_t)(l(t)N^{-2}\bar{\phi}_t) dx = \int_{\Omega} \nabla N^{-1}\bar{\phi}_t \cdot \nabla [l(t)N^{-2}\bar{\phi}_t] dx. \quad (2.2.108)$$

On the one hand, we have

$$\nabla [l(t)N^{-2}\bar{\phi}_t] = \nabla l(t)N^{-2}\bar{\phi}_t + l(t)\nabla N^{-2}\bar{\phi}_t,$$

$$\begin{aligned} \int_{\Omega} |\nabla [l(t)N^{-2}\bar{\phi}_t]|^2 dx &\leq 2\|\nabla l(t)\|^2 \|N^{-2}\bar{\phi}_t\|_{L^\infty(\Omega)}^2 + 2\|l(t)\|_{L^\infty(\Omega)}^2 \|\phi_t\|_{-3}^2 \\ &\leq \begin{cases} \tilde{c}_1 \|\phi_t\|_{-3}^2, & \text{if } d = 1, \\ \tilde{c}_2 \|\phi_t\|_{-2}^2, & \text{if } d = 2, 3, \end{cases} \end{aligned}$$

so that

$$|\int_{\Omega} \nabla N^{-1}\bar{\phi}_t \cdot \nabla [l(t)N^{-2}\bar{\phi}_t] dx| \leq \frac{1}{4} \|\phi_t\|_{-1}^2 + \tilde{c}_j \|\phi_t\|_{-3}^2, \quad (2.2.109)$$

where we used the interpolation estimate $\|\varphi\|_{-2} \leq c\|\varphi\|_{-3}^{\frac{1}{2}}\|\varphi\|_{-1}^{\frac{1}{2}}$, when $d = 2, 3$. On

the other hand, similarly to (2.2.107) and (2.2.108), we have

$$\begin{aligned}
& \int_{\Omega} l'(t) \phi N^{-2} \bar{\phi}_t dx \\
&= \int_{\Omega} \int_0^1 \nabla [g''(s\phi_1 + (1-s)\phi_2) \phi N^{-2} \bar{\phi}_t] \cdot \nabla N^{-1} [s\bar{\phi}_{1t} + (1-s)\bar{\phi}_{2t}] ds dx \\
&+ \int_{\Omega} \int_0^1 [g''(s\phi_1 + (1-s)\phi_2) \phi N^{-2} \bar{\phi}_t] [sm(\phi_{1t}) + (1-s)m(\phi_{2t})] ds dx \\
&\quad \text{if } d = 1,
\end{aligned}$$

$$\begin{aligned}
& \nabla [g''(s\phi_1 + (1-s)\phi_2) \phi N^{-2} \bar{\phi}_t] \\
&= g''(s\phi_1 + (1-s)\phi_2) (N^{-2} \bar{\phi}_t \nabla \phi + \phi \nabla N^{-2} \bar{\phi}_t) \\
&+ g'''(s\phi_1 + (1-s)\phi_2) (s\nabla \phi_1 + (1-s)\nabla \phi_2) \phi N^{-2} \bar{\phi}_t,
\end{aligned}$$

$$\begin{aligned}
& \int_{\Omega} |\nabla [g''(s\phi_1 + (1-s)\phi_2) \phi N^{-2} \bar{\phi}_t]|^2 dx \\
&\leq 2 \sup_{s \in [0,1]} \|g''(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty(\Omega)}^2 \\
&\quad \times (\|\phi\|_{L^\infty}^2 \|\phi_t\|_{-3}^2 + \|\phi\|_1^2 \|N^{-2} \bar{\phi}_t\|_{L^\infty}^2) \\
&+ 2 \sup_{s \in [0,1]} \|g'''(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty}^2 \\
&\quad \times (\|\phi_1\|_1^2 + \|\phi_2\|_1^2) \|\phi\|_{L^\infty}^2 \|N^{-2} \bar{\phi}_t\|_{L^\infty(\Omega)}^2 \\
&\leq \tilde{c}_1 \|\phi\|_1^2 \|\phi_t\|_{-3}^2, \text{ if } d = 1,
\end{aligned}$$

so that

$$|\int_{\Omega} l'(t) \phi N^{-2} \bar{\phi}_t dx| \leq \frac{1}{6} \|\phi_t\|_{-1}^2 + \dot{c}_1(t) \|\phi\|_1^2, \text{ if } d = 1, \quad (2.2.110)$$

since $\|\phi_t\|_{-3} \leq \mathfrak{c}_0 \|\phi_t\|_{-1}$ (see (1.4.2)), where

$$\dot{c}_j(t) = Q \left(\|(\phi_{10}, u_{10})\|_{\mathcal{V}_{j,\varepsilon}}^2, \|(\phi_{20}, u_{20})\|_{\mathcal{V}_{j,\varepsilon}}^2 \right) (1 + \|\phi_{1t}(t)\|_{j-2}^2 + \|\phi_{2t}(t)\|_{j-2}^2).$$

We also have

$$\begin{aligned} |\int_{\Omega} l'(t) \phi N^{-2} \bar{\phi}_t dx| &\leq \sup_{s \in [0,1]} \|g''(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty(\Omega)}^2 \\ &\quad \times \|\phi\|_{L^4(\Omega)} \|N^{-2}\phi_t\|_{L^4(\Omega)} (\|\phi_{1t}(t)\| + \|\phi_{2t}(t)\|) \\ &\leq \frac{1}{6} \|\phi_t\|_{-1}^2 + \dot{c}_2(t) \|\phi\|_1^2, \text{ if } d = 2, 3, \end{aligned} \quad (2.2.111)$$

and

$$\begin{aligned} |m(\phi_t) \int_{\Omega} l(t) N^{-2} \bar{\phi}_t dx| &\leq c |m(\phi_t)| \sup_{s \in [0,1]} \|g'(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty} \|\phi_t\|_{-1} \\ &\leq \frac{1}{6} \|\phi_t\|_{-1}^2 + \tilde{c}_j |m(\phi_0)|^2. \end{aligned} \quad (2.2.112)$$

It follows from (2.2.106) through (2.2.112) that

$$\frac{dE_6}{dt} \leq \tilde{c}_j E_6 + \dot{c}_j(t) \|\phi\|_1^2 + \tilde{c}_j |m(\phi_0)|^2, \quad (2.2.113)$$

where

$$E_6(t) = \|\bar{\phi}_t\|_{-3}^2 + \|\bar{u}\|_{-1}^2 + \gamma \|\bar{u}\|_{-2}^2,$$

and by (2.2.97), we have

$$\begin{aligned} E_6(t) &\geq \|u(t)\|_{-1}^2 - |\Omega| m(u(t))^2 \\ &\geq \|u(t)\|_{-1}^2 - c \left(m(\phi_0)^2 + m(u_0)^2 \right). \end{aligned}$$

We apply the Gronwall lemma to (2.2.113), as $\dot{c}_j(s) \in L^1(0, t)$, $\forall t \geq 0$, due to (2.2.26) and (2.2.78), and we conclude that

$$\|u(t)\|_{-1}^2 \leq c \|(\phi_0, u_0)\|_{\mathcal{V}_{1,\varepsilon}}^2 e^{ct}, \quad \forall t \geq 0, \quad (2.2.114)$$

since

$$\begin{aligned} \|\phi_t(0)\|_{-3}^2 &\leq c(\|\phi_0\|_1^2 + \|l(0)\phi_0\|_{-1}^2 + \|u_0\|_{-1}^2) \\ &\leq \tilde{c}(\|\phi_0\|_1^2 + \|u_0\|_{-1}^2). \end{aligned} \quad (2.2.115)$$

Combining (2.2.103) and (2.2.114) we end up with the estimate

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{1,\varepsilon}}^2 \leq c \|(\phi_0, u_0)\|_{\mathcal{V}_{1,\varepsilon}}^2 e^{ct}, \quad \forall t \geq 0. \quad (2.2.116)$$

2.2.5 $\mathcal{V}_{1,\varepsilon}$ - $\mathcal{V}_{2,\varepsilon}$ Estimate of the difference of two solutions

We multiply (2.2.94)₁ and (2.2.94)₂ by ϕ_t and Nu , respectively, integrate over Ω , sum the resulting equations, and we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\bar{\phi}\|_2^2 + \sigma \|\bar{\phi}\|^2 + \varepsilon \|\bar{u}\|_1^2) + \|\phi_t\|^2 + \|\bar{u}\|_2^2 + \gamma \|\bar{u}\|_1^2 = - \int_{\Omega} N[l(t)\phi] \phi_t dx. \quad (2.2.117)$$

We have $\Delta[l(t)\phi] = l(t)\Delta\phi + 2\nabla l(t) \cdot \nabla\phi + \phi\Delta l(t)$,

$$\begin{aligned} \Delta l(t) = & \int_0^1 [g''(s\phi_1(t) + (1-s)\phi_2(t)) (s\Delta\phi_1(t) + (1-s)\Delta\phi_2(t)) \\ & + g'''(s\phi_1(t) + (1-s)\phi_2(t)) |s\nabla\phi_1(t) + (1-s)\nabla\phi_2(t)|^2] ds, \end{aligned}$$

$$\begin{aligned} & \int_{\Omega} |\Delta l(t)|^2 dx \\ & \leq c \sup_{s \in [0,1]} \|g''(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty(\Omega)}^2 (\|\phi_1\|_2^2 + \|\phi_2\|_2^2) \\ & \quad + c \sup_{s \in [0,1]} \|g'''(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty(\Omega)}^2 \left(\|\nabla\phi_1\|_{L^4(\Omega)^d}^4 + \|\nabla\phi_2\|_{L^4(\Omega)^d}^4 \right), \end{aligned}$$

so that

$$\begin{aligned} \int_{\Omega} |\Delta l(t)|^2 dx & \leq Q (\|\phi_1\|_1^2, \|\phi_2\|_1^2) (1 + \|\phi_1\|_2^2 + \|\phi_2\|_2^2) \\ & \leq \dot{c}_1(t), \quad \forall t \geq 0, \text{ if } d = 1, \end{aligned}$$

where

$$\dot{c}_1(t) = Q \left(\|(\phi_{10}, u_{10})\|_{\mathcal{V}_{1,\varepsilon}}^2, \|(\phi_{20}, u_{20})\|_{\mathcal{V}_{1,\varepsilon}}^2 \right) (1 + \|\phi_1(t)\|_2^2 + \|\phi_2(t)\|_2^2),$$

and

$$\begin{aligned} \int_{\Omega} |\Delta l(t)|^2 dx &\leq Q (\|\phi_1\|_2^2, \|\phi_2\|_2^2) \\ &\leq \tilde{c}_2, \quad \forall t \geq 0, \text{ if } d = 2, 3, \end{aligned}$$

hence

$$\begin{aligned} \int_{\Omega} |\Delta[l(t)\phi]|^2 dx &\leq 3\|l(t)\|_{L^\infty(\Omega)}^2 \|\phi\|_2^2 + 6\|\nabla l(t)\|_{L^\infty(\Omega)^d}^2 \|\phi\|_1^2 + 3\|\phi\|_{L^\infty(\Omega)}^2 \|\Delta l(t)\|^2 \\ &\leq \dot{c}_1(t) \|\phi\|_1^2 + \tilde{c}_1 \|\phi\|_2^2, \quad \forall t \geq 0, \text{ if } d = 1, \end{aligned} \tag{2.2.118}$$

and

$$\begin{aligned} \int_{\Omega} |\Delta[l(t)\phi]|^2 dx &\leq 3\|l(t)\|_{L^\infty(\Omega)}^2 \|\phi\|_2^2 + 6\|\nabla l(t)\|_{L^4(\Omega)^d}^2 \|\nabla \phi\|_{L^4(\Omega)^d}^2 + 3\|\phi\|_{L^\infty(\Omega)}^2 \|\Delta l(t)\|^2 \\ &\leq \tilde{c}_2 \|\phi\|_2^2, \quad \forall t \geq 0, \text{ if } d = 2, 3. \end{aligned} \tag{2.2.119}$$

We derive from (2.2.118) and (2.2.119) that

$$\begin{aligned} \int_{\Omega} |N[l(t)\phi]| |\phi_t| dx &\leq \|N[l(t)\phi]\| \|\phi_t\| \\ &\leq \frac{1}{2} \|\phi_t\|^2 + \tilde{c}_j \|\phi\|_2^2 + \dot{c}_1(t) \|\phi\|_1^2, \end{aligned} \quad (2.2.120)$$

and we infer from (2.2.117) and (2.2.120) that

$$\frac{d}{dt} (\|\bar{\phi}\|_2^2 + \sigma \|\bar{\phi}\|^2 + \varepsilon \|\bar{u}\|_1^2) + \|\phi_t\|^2 + \|\bar{u}\|_2^2 \leq \tilde{c}_j \|\phi\|_2^2 + \dot{c}_1(t) \|\phi\|_1^2. \quad (2.2.121)$$

To finish, we multiply (2.2.105) and (2.2.94)₂ by $N^{-2}\bar{\phi}_t$ and $N^{-1}\bar{u}_t$, respectively, integrate over Ω , sum the resulting equations, and we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\bar{\phi}_t\|_{-2}^2 + \|\bar{u}\|^2 + \gamma \|\bar{u}\|_{-1}^2) + \|\bar{\phi}_t\|^2 + \sigma \|\bar{\phi}_t\|_{-1}^2 + \varepsilon \|\bar{u}_t\|_{-1}^2 \\ = - \int_{\Omega} l(t) \phi_t N^{-1} \bar{\phi}_t dx - \int_{\Omega} l'(t) \phi N^{-1} \bar{\phi}_t dx. \end{aligned} \quad (2.2.122)$$

But we have

$$\begin{aligned} \left| \int_{\Omega} \phi_t l(t) N^{-1} \bar{\phi}_t dx \right| &\leq \sup_{s \in [0,1]} \|g'(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty(\Omega)} \|\phi_t\| \|\phi_t\|_{-2} \\ &\leq \frac{1}{4} \|\phi_t\|^2 + \tilde{c}_j \|\phi_t\|_{-2}^2. \end{aligned} \quad (2.2.123)$$

We also have (cf. (2.2.110))

$$\int_{\Omega} |l'(t) \phi N^{-1} \bar{\phi}_t|^2 dx \leq \dot{c}_1(t) \|\phi\|_1^2 \|\phi_t\|^2, \text{ if } d = 1,$$

so that

$$|\int_{\Omega} l'(t) \phi N^{-1} \bar{\phi}_t dx| \leq \frac{1}{4} \|\phi_t\|^2 + \acute{c}_1(t) \|\phi\|_1^2, \text{ if } d = 1. \quad (2.2.124)$$

Finally, we have (cf. (2.2.111))

$$|\int_{\Omega} l'(t) \phi N^{-1} \bar{\phi}_t dx| \leq \frac{1}{4} \|\phi_t\|^2 + \acute{c}_2(t) \|\phi\|_1^2, \text{ if } d = 2, 3. \quad (2.2.125)$$

We infer from (2.2.122), (2.2.123), (2.2.124) and (2.2.125) that

$$\frac{d}{dt} (\|\bar{\phi}_t\|_{-2}^2 + \|\bar{u}\|^2 + \gamma \|\bar{u}\|_{-1}^2) + \|\bar{\phi}_t\|^2 + \varepsilon \|\bar{u}_t\|_{-1}^2 \leq \tilde{c}_j \|\phi_t\|_{-2}^2 + \acute{c}_j(t) \|\phi\|_1^2. \quad (2.2.126)$$

Adding (2.2.121) and (2.2.126), we reach the estimate

$$\frac{dE_7}{dt} \leq \tilde{c}_j E_7 + (\dot{c}_1(t) + \acute{c}_j(t)) \|\phi\|_1^2 + \tilde{c}_j |m(\phi_0)|^2, \quad (2.2.127)$$

where

$$E_7(t) = \|\bar{\phi}\|_2^2 + \sigma \|\bar{\phi}\|^2 + \varepsilon \|\bar{u}\|_1^2 + \|\bar{\phi}_t\|_{-2}^2 + \|\bar{u}\|^2 + \gamma \|\bar{u}\|_{-1}^2.$$

There exists a constant $c > 0$, independent of ε , such that

$$\begin{aligned} E_7(t) &\geq \|(\phi(t), u(t))\|_{\mathbb{V}_{2,\varepsilon}}^2 - |\Omega| (m(\phi(t))^2 + \varepsilon m(u(t))^2 + m(u(t))^2) \\ &\geq \|(\phi(t), u(t))\|_{\mathbb{V}_{2,\varepsilon}}^2 - c (m(\phi_0)^2 + \varepsilon m(u_0)^2), \end{aligned} \quad (2.2.128)$$

due to (2.2.95) and (2.2.97), and we have

$$E_7(t) \leq c \left(\|\phi(t)\|_2^2 + \varepsilon \|u(t)\|^2 + \|u(t)\|^2 + \|\phi_t(t)\|_{-2}^2 \right). \quad (2.2.129)$$

We change the variable t by s in (2.2.127), then multiply by s , $0 \leq s \leq t$, to get

$$\frac{d}{ds} [sE_7(s)] \leq E_7(s) + t \left[\tilde{c}_j E_7(s) + (\dot{c}_1(s) + \dot{c}_j(s)) \|\phi(s)\|_1^2 + \tilde{c}_j |m(\phi_0)|^2 \right]. \quad (2.2.130)$$

Now, we integrate (2.2.130) between 0 and t , and we deduce that

$$E_7(t) \leq \tilde{c}_j \left(1 + \frac{1}{t} \right) \int_0^t (E_7(s) + (\dot{c}_1(s) + \dot{c}_j(s)) \|\phi(s)\|_1^2 + |m(\phi_0)|^2) ds, \quad \forall t > 0, \quad (2.2.131)$$

and, by using Sobolev embeddings and (2.2.103), (2.2.104), (2.2.128) and (2.2.129), we find

$$\begin{aligned} & \|(\phi(t), u(t))\|_{\mathcal{V}_{2,\varepsilon}}^2 \\ & \leq \tilde{c}_j \left(1 + \frac{1}{t} \right) \int_0^t (\|\phi_t(s)\|_{-1}^2 + \|\phi(s)\|_2^2 + \|u(s)\|_1^2 + (\dot{c}_1(s) + \dot{c}_j(s)) \|\phi(s)\|_1^2) ds \\ & \quad + c (m(\phi_0)^2 + \varepsilon m(u_0)^2) \\ & \leq c(\omega) \|(\phi_0, u_0)\|_{\mathcal{V}_{1,\varepsilon}}^2 e^{ct}, \quad \forall t \geq \omega, \quad \forall \omega > 0. \end{aligned} \quad (2.2.132)$$

Estimate (2.2.132) implies

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{2,\varepsilon}}^2 \leq c \|(\phi_0, u_0)\|_{\mathcal{V}_{1,\varepsilon}}^2 e^{ct}, \quad \forall t \geq \omega, \quad \forall \omega > 0. \quad (2.2.133)$$

2.2.6 $\mathcal{V}_{2,\epsilon}$ - $\mathcal{V}_{2,\epsilon}$ Estimate of the difference of two solutions

Like (2.2.115), we also deduce from (2.2.94)₁ that

$$\begin{aligned} \|\phi_t(0)\|_{-2}^2 &\leq c(\|\phi_0\|_2^2 + \|l(0)\phi_0\|^2 + \|u_0\|^2) \\ &\leq \tilde{c}(\|\phi_0\|_2^2 + \|u_0\|^2). \end{aligned} \quad (2.2.134)$$

We apply the Gronwall lemma to (2.2.127) to obtain

$$\begin{aligned} E_7(t) &\leq \left(E_7(0) + \int_0^t (\dot{c}_1(s) + \dot{c}_j(s)) \|\phi(s)\|_1^2 + \tilde{c}_j |m(\phi_0)|^2 ds \right) e^{\tilde{c}_j t} \\ &\leq c \|(\phi_0, u_0)\|_{\mathcal{V}_{2,\epsilon}}^2 e^{ct}, \quad \forall t \geq 0, \end{aligned}$$

on account of (2.2.26), (2.2.103) and (2.2.134). Hence

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{2,\epsilon}}^2 \leq c \|(\phi_0, u_0)\|_{\mathcal{V}_{2,\epsilon}}^2 e^{ct}, \quad \forall t \geq 0. \quad (2.2.135)$$

2.3 The unperturbed problem

Recall that, if $\epsilon = 0$, then the system reduces to

$$\begin{cases} B_\gamma \phi_t + N(N\phi + g(\phi)) + \sigma\phi = 0, \\ \phi|_{t=0} = \phi_0, \end{cases} \quad (2.3.1)$$

where

$$B_\gamma = I + N(\gamma I + N)^{-1}.$$

To obtain a priori estimates for the unperturbed problem (2.3.1), we introduce the following lemma.

We set, for any $\gamma > 0$,

$$A_\gamma = \gamma I + N : \mathcal{D}(N) \rightarrow L^2(\Omega),$$

$$B_\gamma = I + NA_\gamma^{-1} : L^2(\Omega) \rightarrow L^2(\Omega),$$

$$C_\gamma = \gamma I + 2N (= A_\gamma + N) : \mathcal{D}(N) \rightarrow L^2(\Omega).$$

Lemma 2.3.1 *The following estimates are hold*

$$\frac{1}{\sqrt{2}}\|\psi\| \leq \|B_\gamma^{-\frac{1}{2}}\psi\| \leq \|\psi\|, \quad \forall \psi \in L^2(\Omega), \quad (2.3.2)$$

$$\frac{1}{2}\|\psi\| \leq \|B_\gamma^{-1}\psi\| \leq \|\psi\|, \quad \forall \psi \in L^2(\Omega), \quad (2.3.3)$$

$$\|\psi\| \leq \|B_\gamma^{\frac{1}{2}}\psi\| \leq \sqrt{2}\|\psi\|, \quad \forall \psi \in L^2(\Omega), \quad (2.3.4)$$

$$\|C_\gamma^{-1}N\psi\| \leq \frac{1}{2}\|\psi\|, \quad \forall \psi \in V_2, \quad (2.3.5)$$

$$\|C_\gamma^{-1}\psi\| \leq \frac{1}{\gamma}\|\psi\|, \quad \forall \psi \in L^2(\Omega). \quad (2.3.6)$$

Proof: Let $\varphi(x) = \sum_{j=0}^{\infty} c_j(t)e_j(x)$. We have

$$\|B_\gamma^{-\frac{1}{2}}\psi\|^2 = \sum_{j=0}^{\infty} \frac{\gamma + \lambda_j}{\gamma + 2\lambda_j} c_j^2(t),$$

$$\|B_\gamma^{-1}\psi\|^2 = \sum_{j=0}^{\infty} \frac{(\gamma + \lambda_j)^2}{(\gamma + 2\lambda_j)^2} c_j^2(t),$$

$$\|B_\gamma^{\frac{1}{2}}\psi\|^2 = \sum_{j=0}^{\infty} \frac{\gamma + 2\lambda_j}{\gamma + \lambda_j} c_j^2(t),$$

$$\|C_\gamma^{-1}N\psi\|^2 = \sum_{j=0}^{\infty} \frac{\lambda_j^2}{(\gamma + 2\lambda_j)^2} c_j^2(t)$$

and

$$\|C_\gamma^{-1}\psi\|^2 = \sum_{j=0}^{\infty} \frac{1}{(\gamma + 2\lambda_j)^2} c_j^2(t).$$

The function $h(x) = \frac{\gamma+x}{\gamma+2x}$ is strictly decreasing and we have $1 \leq h(x) \leq \frac{1}{2}$, $\forall x \in \mathbb{R}^+$, so that (2.3.2) and (2.3.3) follow. The function $s(x) = \frac{1}{h(x)}$ is then strictly increasing and satisfies $1 \leq s(x) \leq 2$, $\forall x \in \mathbb{R}^+$, and we deduce (2.3.4). Finally, we note that the functions $v(x) = \frac{x}{\gamma+2x}$ and $l(x) = \frac{1}{\gamma+2x}$ are strictly increasing and decreasing, respectively, and we have $v(x) < \frac{1}{2}$ and $l(x) \leq \frac{1}{\gamma}$ both on $\mathbb{R}^+$, so we get Estimates (2.3.5) and (2.3.6).

Now due to the equivalence of norms, a priori estimates of the unperturbed problem (2.3.1) can be obtained in a similar way as in the perturbed problem in section (2.2), hence we can deduce the following estimates

$$\|\phi(t)\|_j^2 \leq Q(\|\phi_0\|_j^2)e^{-c_0 t} + c, \quad \forall t \geq 0, \quad (2.3.7)$$

$$\|\phi(t)\|_2^2 \leq Q(\|\phi_0\|_1^2)e^{-c_0 t} + c, \quad \forall t \geq r, \quad (2.3.8)$$

$$\|\phi(t)\|_3^2 \leq Q(\|\phi_0\|_j^2)e^{-c_0 t} + c, \quad \forall t \geq r, \quad (2.3.9)$$

$$\int_0^t \|\phi_t(s)\|_{-1}^2 ds \leq Q(\|\phi_0\|_j^2), \quad \forall t \geq 0. \quad (2.3.10)$$

Moreover, the following are the estimates of the difference ϕ of two solutions ϕ_1 and ϕ_2 of Problem (2.3.1) starting at the points ϕ_{10} and ϕ_{20} in V_j at time $t = 0$, respectively:

$$\|\phi(t)\|_j^2 \leq \|\phi_0\|_j^2 e^{\tilde{c}_1 t}, \quad \forall t \geq 0, \quad (2.3.11)$$

$$\|\phi(t)\|_2^2 \leq \tilde{c} \|\phi_0\|_1^2 e^{c_1 t}, \quad \forall t \geq \omega. \quad (2.3.12)$$

2.4 Well-posedness

Relying on a priori estimate obtain in section 2.2, we will prove a well-posedness result. Let consider the following weak formulation of Problem (2.0.1).

For any $T > 0$, find

$$(\phi, u) : [0, T] \rightarrow \mathcal{V}_{j+1, \epsilon}$$

such that

$$\phi(0, x) = \phi_0(x), \quad u(0, x) = u_0(x),$$

and for almost any $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt}(\phi, q) + (N\phi, Nq) + (\nabla g(\phi), \nabla q) - (\nabla u, \nabla q) + \sigma(\phi, q) &= 0, \quad q \in V_2, \\ \frac{d}{dt}[\epsilon(u, v) + (\phi, v)] + (\nabla u, \nabla v) + \gamma(u, v) &= 0, \quad v \in V_1. \end{aligned} \quad (2.4.1)$$

Theorem 2.4.1 *Let $g \in \mathcal{C}^3(\mathbb{R})$. We assume Conditions (2.0.2)-(2.0.5) hold. Let $j = 1$ if $d = 1$ and $j = 2$ if $d = 2$ or 3 . If $(\phi_0, u_0) \in \mathcal{V}_{j, \epsilon}$, then Problem (2.0.1)*

possesses a unique solution (ϕ, u) such that

$$(\phi, u) \in \mathcal{C}([0, \infty); \mathcal{V}_{j,\epsilon}) \cap \mathcal{C}([\tau, \infty); \mathcal{V}_{3,\epsilon}) \cap L^2(0, T; \mathcal{V}_{j+1,\epsilon}) \cap L^2(0, T; \mathcal{V}_{j+2,0}),$$

$\forall T > 0, \forall \tau > 0$. Moreover, there holds

$$|m(\phi(t))| + \epsilon |m(u(t))| \leq |m(\phi_0)| + \epsilon |m(u_0)|, \quad \forall t \geq 0.$$

Proof. This theorem will be proven in three parts.

Part1: Existence of the solution

We implement the Faedo-Galerkin method. There is a complete orthonormal basis ω_j on $L^2(\Omega)$ made up of eigenvectors of $N = -\Delta$ such that

$$N\omega_j = \lambda_j \omega_j, \quad \omega_j \in V_2, \quad j = 0, 1, 2, 3, \dots$$

We set $E_m = \text{span}\{\omega_1, \omega_2, \dots, \omega_m\}$ and P_m be the orthogonal projection on E_m . We consider the approximate problem:

For each m , find an approximate solution $(\phi_m, u_m) : [0, T] \rightarrow E_m$, such that

$$\begin{aligned} \phi_m(t) &= \sum_0^m a_j(t) \omega_j, \\ u_m(t) &= \sum_0^m b_j(t) \omega_j, \end{aligned}$$

which satisfies

$$\begin{aligned}
\frac{d}{dt}(\phi_m, \omega_j) + (\Delta\phi_m, \Delta\omega_j) + (g(\phi_m), -\Delta\omega_j) - (\nabla u_m, \nabla\omega_j) + \sigma(\phi_m, \omega_j) &= 0, \\
\frac{d}{dt}[\epsilon(u_m, \omega_j) + (\phi_m, \omega_j)] + (\nabla u_m, \nabla\omega_j) + \gamma(u_m, \omega_j) &= 0, \quad \forall j = 0, 1, 2, \dots, m, \\
\phi_m(0) = P_m\phi_0, \quad u_m(0) = P_mu_0. &
\end{aligned} \tag{2.4.2}$$

This leads to the following equations for each $j \in \{0, 1, \dots, m\}$

$$\begin{aligned}
a'_j(t) + \lambda_j^2 a_j(t) + (g(\phi_m), -\Delta\omega_j) - \lambda_j b_j(t) + \sigma a_j(t) &= 0, \\
\epsilon b'_j(t) + a'_j(t) + \lambda_j b_j(t) + \gamma b_j &= 0.
\end{aligned} \tag{2.4.3}$$

Let set $Y = (a_0, a_1, \dots, a_m)$, $Z = (b_0, b_1, \dots, b_m)$ and $M = (\lambda_j \delta_{ij})_{i,j=0,1,\dots,m}$, then (2.4.3) produces the system

$$\begin{aligned}
Y' + (M^2 + \sigma I)Y + G(Y) - MZ &= 0, \\
\epsilon Z' - (M^2 - \sigma I)Y - G(Y) + (2M + \gamma I)Z &= 0.
\end{aligned} \tag{2.4.4}$$

The matrix M is positive definite and

$$G(Y) = \begin{pmatrix} (g(\sum_{j=0}^m a_j \omega_j), \lambda_0 \omega_0) \\ \vdots \\ (g(\sum_{j=0}^m a_j \omega_j), \lambda_m \omega_m) \end{pmatrix} \tag{2.4.5}$$

depends continuously on Y . Hence (2.4.2) has a unique solution on some finite interval

$[0, T_m)$ by an ODE theorem on the existence of a solution (c.f., e.g., Theorem 1.5.2).

Now we show that the solutions are bounded in time and uniformly bounded in m .

We rewrite (2.4.2) in the equivalent form

$$\begin{aligned} \frac{d}{dt}\phi_m - \Delta(-\Delta\phi_m + P_m g(\phi_m) - u_m) + \sigma\phi_m &= 0, \\ \epsilon \frac{d}{dt}u_m + \frac{d}{dt}\phi_m - \Delta u_m + \gamma u_m &= 0, \\ \phi_m(0) = P_m\phi_0, \quad u_m(0) &= P_m u_0. \end{aligned} \tag{2.4.6}$$

Keeping in mind that in the computations ahead, we will use the facts that

$\|\phi_m(0)\| \leq \|\phi_0\|$, $\|u_m(0)\| \leq \|u_0\|$, $P_m\phi_0 \rightarrow \phi_0$ and $P_mu_0 \rightarrow u_0$, we also note that

ϕ_m, u_m satisfy a priori estimates in section (2.2).

Therefore, due to (2.2.38), (2.2.70), (2.2.37) and (2.2.62), we have

$$\sup_{t \in [0, T]} \|\phi_m(t)\|_j^2 \leq C, \tag{2.4.7}$$

$$\sup_{t \in [0, T]} \epsilon \|u_m(t)\|_{j-1}^2 \leq C, \tag{2.4.8}$$

$$\sup_{t \in [0, T]} \|\phi_{mt}(t)\|_{j-4}^2 \leq C, \tag{2.4.9}$$

with C independent of t and m .

Thus the sequences ϕ_m , u_m and ϕ_{mt} remains uniformly bounded in $L^\infty(0, T; V_j)$, $L^\infty(0, T; V_{j-1})$ and $L^\infty(0, T; V_{j-4})$, respectively.

Moreover, by utilize the estimates (2.2.81), (2.2.77), (2.2.75), (2.2.78), (2.2.76) and (2.2.79), where the right hand sides are independent of m ,

we have the sequences ϕ_m, u_m, ϕ_{mt} and u_{mt} remains uniformly bounded in $L^2(0, T; V_{j+2}), L^2(0, T; V_j), L^2(0, T; V_{j-2})$ and $L^2(0, T; V_{j-2})$, respectively.

Now we have subsequences still denoted ϕ_m and u_m such that

$$\phi_m \xrightarrow{*} \phi \quad \text{in} \quad L^\infty(0, T; V_j), \quad (2.4.10)$$

$$u_m \xrightarrow{*} u \quad \text{in} \quad L^\infty(0, T; V_{j-1}),$$

$$\phi_m \rightharpoonup \phi \quad \text{in} \quad L^2(0, T; V_{j+2}), \quad (2.4.11)$$

$$u_m \rightharpoonup u \quad \text{in} \quad L^2(0, T; V_j),$$

$$\phi_{mt} \rightharpoonup \phi_t \quad \text{in} \quad L^2(0, T; V_{j-2}), \quad (2.4.12)$$

$$u_{mt} \rightharpoonup u_t \quad \text{in} \quad L^2(0, T; V_{j-2}),$$

$$\phi_{mt} \xrightarrow{*} \phi_t \quad \text{in} \quad L^\infty(0, T; V_{j-4}). \quad (2.4.13)$$

From Lemma 1.5.3, due to (2.4.10) and (2.4.12) we deduce that

$$\phi_m \rightarrow \phi \quad \text{in} \quad L^2(0, T; V_{j-1}(\Omega)) \quad \text{strongly and a.e in } (0, T) \times \Omega. \quad (2.4.14)$$

Therefore, by using the integral form of Main Value Theorem, we have

$$g(\phi_m(t)) - g(\phi(t)) = (\phi_m(t) - \phi(t)) \int_0^t g'(s\phi_m(t) + (1-s)\phi(t)) ds,$$

so that

$$\int_0^T \|g(\phi_m(s)) - g(\phi(s))\| dt \leq \int_0^T Q(\|\phi_m(s)\|_j, \|\phi(s)\|_j) \|\phi_m(s) - \phi(s)\| dt.$$

On account of (2.4.7), (2.4.10); and by using Lemma 1.5.6, we have that

If $\phi_m \xrightarrow{*} \phi$ in $L^\infty(0, T, V_j)$, then $\|\phi\|_{L^\infty(0, T, V_j)} \leq \lim_{n \rightarrow \infty} \inf \|\phi_m\|_{L^\infty(0, T, V_j)} \leq C$.

As a result, we have

$$\int_0^T \|g(\phi_m(s)) - g(\phi(s))\| ds \leq \sup_{t \in [0, T]} Q(\|\phi_m(s)\|_j, \|\phi(s)\|_j) \int_0^T \|\phi_m(s) - \phi(s)\| ds.$$

Hence

$$\int_0^T \|g(\phi_m(s)) - g(\phi(s))\| ds \leq C \int_0^T \|\phi_m(s) - \phi(s)\| ds \rightarrow 0, \quad d = 1,$$

$$\int_0^T \|g(\phi_m(s)) - g(\phi(s))\| ds \leq C \int_0^T \|\phi_m(s) - \phi(s)\|_1 ds \rightarrow 0, \quad d = 2, 3.$$

Thus,

$$g(\phi_m) \rightarrow g(\phi) \quad \text{in} \quad L^2(0, T; L^2(\Omega)). \quad (2.4.15)$$

Next, to show $P_m g(\phi_m) \rightarrow g(\phi)$ in $L^2(0, T; L^2)$, we can write

$$\begin{aligned} & \int_0^T \int_\Omega (P_m g(\phi_m) - g(\phi)) \Phi \, dx \, dt \\ &= \int_0^T \int_\Omega (g(\phi_m) - g(\phi)) \Phi \, dx \, dt - \int_0^T \int_\Omega Q_m g(\phi_m) \Phi \, dx \, dt, \end{aligned} \quad (2.4.16)$$

(since $Q_m = I - P_m$), for all $\Phi \in (L^2(0, T; L^2))'$.

Clearly, the first term on the right hand side tends to zero. For the second term, we

note that the functions of the form

$$\Phi = \sum_{j=1}^m \alpha_j(t) \Phi_j,$$

and we have

$$\int_0^T \int_{\Omega} Q_m g(\phi_m) \left(\sum_{j=1}^m \alpha_j(t) \Phi_j \right) dx dt = \int_0^T \int_{\Omega} g(\phi_m) \left(\sum_{j=1}^m \alpha_j(t) Q_m \Phi_j \right) dx dt, \quad (2.4.17)$$

since $Q_m \Phi_j \rightarrow 0$ for each j we have the required convergence of $P_m g(\phi_m)$.

Now, to pass to the limit in the approximate Problem (2.4.2), we multiply the first equation in (2.4.2) by ψ and the second by ϑ and integrate by parts with respect to t , where $\psi, \vartheta \in \mathcal{D}(0, T)$, we get

$$\begin{aligned} & - \int_0^T (\phi_m, \omega_j) \frac{d\psi}{dt}(t) dt + \int_0^T (\Delta \phi_m, \Delta \omega_j) \psi(t) dt + \int_0^T (g(\phi_m), -\Delta \omega_j) \psi(t) dt \\ & - \int_0^T (\nabla u_m, \nabla \omega_j) \psi(t) dt + \sigma \int_0^T (\phi_m, \omega_j) \psi(t) dt = 0, \end{aligned} \quad (2.4.18)$$

$$\begin{aligned} & - \epsilon \int_0^T (u_m, \omega_j) \frac{d\vartheta}{dt}(t) dt - \int_0^T (\phi_m, \omega_j) \frac{d\vartheta}{dt}(t) dt + \int_0^T (\nabla u_m, \nabla \omega_j) \vartheta(t) dt \\ & + \gamma \int_0^T (u_m, \omega_j) \vartheta(t) dt = 0. \end{aligned} \quad (2.4.19)$$

We notice that, on account of (2.4.11), (2.4.10), (2.4.13) and (2.4.15), we have

$$\begin{aligned}
& - \int_0^T (\phi_m, \omega_j) \frac{d\psi}{dt}(t) dt \rightarrow - \int_0^T (\phi, \omega_j) \frac{d\psi}{dt}(t) dt \\
& \int_0^T (\Delta \phi_m, \Delta \omega_j) \psi(t) dt \rightarrow \int_0^T (\Delta \phi, \Delta \omega_j) \psi(t) dt \\
& \int_0^T (g(\phi_m), -\Delta \omega_j) \psi(t) dt \rightarrow \int_0^T (g(\phi), -\Delta \omega_j) \psi(t) dt \\
& - \int_0^T (\nabla u_m, \nabla \omega_j) \psi(t) dt \rightarrow - \int_0^T (\nabla u, \nabla \omega_j) \psi(t) dt \\
& \sigma \int_0^T (\phi_m, \omega_j) \psi(t) dt \rightarrow \sigma \int_0^T (\phi, \omega_j) \psi(t) dt,
\end{aligned}$$

along with

$$\begin{aligned}
& - \epsilon \int_0^T (u_m, \omega_j) \frac{d\vartheta}{dt}(t) dt \rightarrow - \epsilon \int_0^T (u, \omega_j) \frac{d\vartheta}{dt}(t) dt \\
& - \int_0^T (\phi_m, \omega_j) \frac{d\vartheta}{dt}(t) dt \rightarrow - \int_0^T (\phi, \omega_j) \frac{d\vartheta}{dt}(t) dt \\
& \int_0^T (\nabla u_m, \nabla \omega_j) \vartheta(t) dt \rightarrow \int_0^T (\nabla u, \nabla \omega_j) \vartheta(t) dt \\
& \gamma \int_0^T (u_m, \omega_j) \vartheta(t) dt \rightarrow \gamma \int_0^T (u, \omega_j) \vartheta(t) dt.
\end{aligned}$$

The subspaces E_m are dense in V_2 and V_1 , that is,

$$V_2 = H^2 = \overline{\bigcup_{m \in \mathbb{N}^*} E_m}^{H^2}, \quad V_1 = H^1 = \overline{\bigcup_{m \in \mathbb{N}^*} E_m}^{H^1}.$$

Observing that the approximate problem is equivalent to

$$\begin{aligned} & - \int_0^T (\phi_m, q) \frac{d\psi}{dt}(t) dt + \int_0^T (\Delta \phi_m, \Delta q) \psi(t) dt + \int_0^T (g(\phi_m), -\Delta q) \psi(t) dt \\ & - \int_0^T (\nabla u_m, \nabla q) \psi(t) dt + \sigma \int_0^T (\phi_m, q) \psi(t) dt = 0, \quad \forall q \in E_m, \end{aligned} \quad (2.4.20)$$

$$\begin{aligned} & - \epsilon \int_0^T (u_m, v) \frac{d\vartheta}{dt}(t) dt - \int_0^T (\phi_m, v) \frac{d\vartheta}{dt}(t) dt + \int_0^T (\nabla u_m, \nabla v) \vartheta(t) dt \\ & + \gamma \int_0^T (u_m, v) \vartheta(t) dt = 0, \quad \forall v \in E_m. \end{aligned} \quad (2.4.21)$$

Passing to the limit as $m \rightarrow \infty$, we deduce that (2.4.20) and (2.4.21) are satisfied for ϕ, u in place of ϕ_m, u_m . Therefore,

$$\begin{aligned} & - \int_0^T (\phi, q) \frac{d\psi}{dt}(t) dt + \int_0^T (\Delta \phi, \Delta q) \psi(t) dt + \int_0^T (g(\phi), -\Delta q) \psi(t) dt \\ & - \int_0^T (\nabla u, \nabla q) \psi(t) dt + \sigma \int_0^T (\phi, q) \psi(t) dt = 0, \quad \forall q \in V_2, \end{aligned} \quad (2.4.22)$$

$$\begin{aligned} & - \epsilon \int_0^T (u, v) \frac{d\vartheta}{dt}(t) dt - \int_0^T (\phi, v) \frac{d\vartheta}{dt}(t) dt + \int_0^T (\nabla u, \nabla v) \vartheta(t) dt \\ & + \gamma \int_0^T (u, v) \vartheta(t) dt = 0, \quad \forall v \in V_1. \end{aligned} \quad (2.4.23)$$

It follows that, we have in the sense of distribution on $(0, T)$:

$$\begin{aligned} & \frac{d}{dt}(\phi, q) + (\Delta \phi, \Delta q) + (g(\phi), -\Delta q) - (\nabla u, \nabla q) + \sigma(\phi, q) = 0, \quad \forall q \in V_2, \\ & \frac{d}{dt}[\epsilon(u, v) + (\phi, v)] + (\nabla u, \nabla v) + \gamma(u, v) = 0, \quad \forall v \in V_1, \end{aligned} \quad (2.4.24)$$

which is exactly the weak formulation. Thus, we have

$$\begin{aligned}\phi_t + \Delta^2 \phi - \Delta g(\phi) + \Delta u + \sigma \phi &= 0, \quad \text{in } L^2(0, T, V_{-2}), \\ \epsilon u_t + \phi_t - \Delta u + \gamma u &= 0, \quad \text{in } L^2(0, T, V_{-1}).\end{aligned}\tag{2.4.25}$$

Next, showing $\phi_0 = \phi(0)$ and $u_0 = u(0)$:

To do this, multiply the first equation in (2.4.25) by ρ and the second one by ω both continuously differentiable on $[0, T]$ satisfying $\rho(T) = \omega(T) = 0$ and $\rho(0) = \omega(0) = 1$ to get

$$\begin{aligned}- \int_0^T (\phi, q) \frac{d\rho}{dt}(t) dt + \int_0^T (\Delta \phi, \Delta q) \rho(t) dt + (\nabla g(\phi), \nabla q) \rho(t) dt \\ - \int_0^T (\nabla u, \nabla q) \rho(t) dt + \sigma \int_0^T (\phi, q) \rho(t) dt &= (\phi(0), q),\end{aligned}\tag{2.4.26}$$

$$\begin{aligned}- \epsilon \int_0^T (u, v) \frac{d\omega}{dt}(t) dt - \int_0^T (\phi, v) \frac{d\omega}{dt}(t) dt + \int_0^T (\nabla u, \nabla v) \omega(t) dt \\ + \gamma \int_0^T (u, v) \omega(t) dt &= \epsilon(u(0), v) + (\phi(0), v).\end{aligned}\tag{2.4.27}$$

Replacing ψ, ϑ in (2.4.20) and (2.4.21) with ρ, ω and allowing $m \rightarrow \infty$, we see that

for every $q \in V_2$ and $v \in V_1$

$$\begin{aligned} & - \int_0^T (\phi, q) \frac{d\rho}{dt}(t) dt + \int_0^T (\Delta\phi, \Delta q) \rho(t) dt + (\nabla g(\phi), \nabla q) \rho(t) dt \\ & - \int_0^T (\nabla u, \nabla q) \rho(t) dt + \sigma \int_0^T (\phi, q) \rho(t) dt = (\phi_0, q), \end{aligned} \quad (2.4.28)$$

$$\begin{aligned} & - \epsilon \int_0^T (u, v) \frac{d\omega}{dt}(t) dt - \int_0^T (\phi, v) \frac{d\omega}{dt}(t) dt + \int_0^T (\nabla u, \nabla v) \omega(t) dt \\ & + \gamma \int_0^T (u, v) \omega(t) dt = \epsilon(u_0, v) + (\phi_0, v). \end{aligned} \quad (2.4.29)$$

Using these last two systems of equations, we have $\phi(0) = \phi_0$ and $u(0) = u_0$.

Part2: Uniqueness of the solution

Let (ϕ_1, u_1) and (ϕ_2, u_2) be two solutions of (2.0.1). Setting $\phi = \phi_1 - \phi_2$ and $u = u_1 - u_2$, we have $\phi(0) = 0$, $u(0) = 0$, $m(\phi(t)) = 0$, $m(u(t)) = 0$, $\forall t \geq 0$ and (ϕ, u) satisfies the equations

$$\phi_t - \Delta(-\Delta\phi + g(\phi_1) - g(\phi_2) - u) + \sigma\phi = 0, \quad (2.4.30)$$

$$\epsilon u_t + \phi_t - \Delta u + \gamma u = 0. \quad (2.4.31)$$

Observe that in (2.2.116) and (2.2.135), we found

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{j,\epsilon}}^2 \leq c \|(\phi_0, u_0)\|_{\mathcal{V}_{j,\epsilon}}^2 e^{ct}, \quad \forall t \in [0, T], \text{ for any } T > 0, \quad (2.4.32)$$

and since $\phi_0 = 0$ and $u_0 = 0$, hence the result.

Part3: Continuity of the solution

From our discussion above and according to Lemma 1.5.5, we can deduce the continuity of ϕ and u .

We start by showing the continuity of ϕ and u on $[0, T]$, we have

$$\phi \in L^2(0, T; V_{j+2}) \text{ and } \phi_t \in L^2(0, T; V_{j-2}) \implies \phi \in \mathcal{C}([0, T], V_j),$$

$$u \in L^2(0, T; V_j) \text{ and } u_t \in L^2(0, T; V_{j-2}) \implies u \in \mathcal{C}([0, T], V_{j-1}).$$

We can deduce that $(\phi, u) \in \mathcal{C}([0, T], \mathcal{V}_{j,\epsilon})$, and since $\forall (\phi_0, u_0) \in \mathcal{V}_j$, $\|(\phi(t), u(t))\|_{\mathcal{V}_{j,\epsilon}}^2 \leq c$, $\forall t \geq 0$, (the solution is uniformly bounded independently of time in $\mathcal{V}_{j,\epsilon}$), then

$$(\phi, u) \in \mathcal{C}([0, \infty], \mathcal{V}_{j,\epsilon}).$$

Moreover, on account of (2.2.90), (2.2.93) and (2.2.86), we have

$$\phi \in L^2([\tau, T]; V_5) \text{ and } \phi_t \in L^2([\tau, T]; V_1) \implies \phi \in \mathcal{C}([\tau, T]; V_3),$$

$$u \in L^2([\tau, T]; V_3) \text{ and } u_t \in L^2([\tau, T]; V_1) \implies u \in \mathcal{C}([\tau, T]; V_2).$$

Thus, we can conclude that

$$(\phi, u) \in \mathcal{C}([\tau, T], \mathcal{V}_{3,\epsilon}),$$

and since $\forall (\phi_0, u_0) \in \mathcal{V}_j$, $\|(\phi(t), u(t))\|_{\mathcal{V}_{3,\epsilon}}^2 \leq c$, $\forall t \geq \tau$ (the solution is uniformly

bounded independently of time in $\mathcal{V}_{3,\epsilon}$. Thus, we deduce that

$$(\phi, u) \in \mathcal{C}([\tau, \infty], \mathcal{V}_{3,\epsilon}).$$

I

2.5 The global attractor

Thanks to Theorem 2.4.1, we define the semigroup

$$S_\epsilon(t) : \mathcal{V}_{j,\epsilon}^\alpha \rightarrow \mathcal{V}_{j,\epsilon}^\alpha, \quad (\phi_0, u_0) \mapsto (\phi(t), u(t)), \quad t \geq 0,$$

where $(\phi(t), u(t))$ is the solution to (2.0.1) at time t and for every $\epsilon \in (0, \epsilon_0]$.

Due to Theorem 2.4.1, the operators $S_\epsilon(t)$ are continuous with respect to time $t \rightarrow S_\epsilon(t)(\phi_0, u_0) \in \mathcal{C}([0, \infty], \mathcal{V}_{j,\epsilon})$ for almost every $x \in \Omega$, and the continuity with respect to initial data which is given by

$$\|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t)(\phi_{02}, u_{02})\|_{\mathcal{V}_{j,\epsilon}} \leq c(t) \|(\phi_{01}, u_{01}) - (\phi_{02}, u_{02})\|_{\mathcal{V}_{j,\epsilon}}, \quad \forall t \geq 0$$

is proved in (2.2.135) and (2.2.116), hence we can deduce that $S_\epsilon(t)$ are continuous.

Owing to the uniqueness of the solution, there hold

$$S_\epsilon(t+s) = S_\epsilon(t) \circ S_\epsilon(s),$$

$$S_\epsilon(0) = I.$$

Thus, the operator $S_\epsilon(t)$ defines a semigroup on $\mathcal{V}_{j,\epsilon}^\alpha$.

Note that, if $(\phi, u) \in \mathcal{V}_{j,\epsilon}^\alpha$, then all the constants in Section 2.2 are independent of $m(\phi_0)$ and $m(u_0)$.

We then deduce from the estimates (2.2.88), (2.2.38), (2.2.50) and (2.2.70) the existence of absorbing sets $\mathcal{B}_{k,\epsilon} \subset \mathcal{V}_{k,\epsilon}^\alpha$ for $S_\epsilon(t)$ of the form

$$\mathcal{B}_{1,\epsilon} = \{(\varphi, \psi) \in \mathcal{V}_{1,\epsilon}^\alpha, \|(\varphi, \psi)\|_{\mathcal{V}_{1,\epsilon}} \leq r_1\}, \quad d = 1, \quad (2.5.1)$$

$$\mathcal{B}_{2,\epsilon} = \{(\varphi, \psi) \in \mathcal{V}_{2,\epsilon}^\alpha, \|(\varphi, \psi)\|_{\mathcal{V}_{2,\epsilon}} \leq r_2\}, \quad d = 1, 2, 3, \quad (2.5.2)$$

$$\mathcal{B}_{3,\epsilon} = \{(\varphi, \psi) \in \mathcal{V}_{3,\epsilon}^\alpha, \|(\varphi, \psi)\|_{\mathcal{V}_{3,\epsilon}} \leq r_3\}, \quad d = 1, 2, 3, \quad (2.5.3)$$

where r_1 , r_2 and r_3 are independent of ϵ .

Proposition 2.5.1 *The semigroup $S_\epsilon(t)$ is uniformly compact for every $\epsilon \in (0, \epsilon_0]$.*

Proof. Let $B \subset \mathcal{V}_{j,\epsilon}^\alpha$ be a bounded set. Then there exists $t(B) \geq 0$ independent of ϵ , such that

$$\bigcup_{t \geq t(B)} S(t)B \subset \mathcal{B}_{j,\epsilon} \subset \mathcal{V}_{j,\epsilon} \quad \text{and} \quad \bigcup_{t \geq t(B)+1} S(t)B \subset \mathcal{B}_{3,\epsilon} \subset \mathcal{V}_{3,\epsilon},$$

hence

$$\bigcup_{t \geq t(B)+1} S(t)B \text{ is bounded in } \mathcal{V}_{j,\epsilon}^\alpha \text{ and in } \mathcal{V}_{3,\epsilon}^\alpha.$$

Since $\mathcal{V}_{3,\epsilon}^\alpha$ is compact in $\mathcal{V}_{j,\epsilon}^\alpha$, we have that $\bigcup_{t \geq t(B)+1} S(t)B$ is precompact in $\mathcal{V}_{j,\epsilon}^\alpha$, i.e., $\overline{\bigcup_{t \geq t(B)+1} S(t)B}$ is compact in $\mathcal{V}_{j,\epsilon}^\alpha$. ■

Now, we state the following result.

Theorem 2.5.1 *Let the assumptions of Theorem 2.4.1 hold. Then, for any $\alpha \geq 0$ and every $\epsilon \in (0, \epsilon_0]$, the semigroup $S_\epsilon(t)$ has the global attractor $\mathcal{A}_\epsilon^\alpha$ which is bounded in $\mathcal{V}_{3,\epsilon}$, that is,*

(i) $\mathcal{A}_\epsilon^\alpha$ is compact in $\mathcal{V}_{j,\epsilon}^\alpha$ and invariant, that is, $S_\epsilon(t)\mathcal{A}_\epsilon^\alpha = \mathcal{A}_\epsilon^\alpha$, $\forall t \geq 0$,

(ii) for any bounded subset $B \subset \mathcal{V}_{j,\epsilon}^\alpha$ there holds

$$\lim_{t \rightarrow \infty} \text{dist}_{\mathcal{V}_{j,\epsilon}}(S_\epsilon(t)B, \mathcal{A}_\epsilon^\alpha) = 0.$$

From now on, we will assume that

$$\mathcal{B}_{0,1} = \{\varphi \in \mathcal{V}_k^\alpha, \|\varphi\|_1 \leq r_1\}, \quad d = 1, \quad (2.5.4)$$

$$\mathcal{B}_{0,2} = \{\varphi \in \mathcal{V}_2^\alpha, \|\varphi\|_2 \leq r_2\}, \quad d = 1, 2, 3, \quad (2.5.5)$$

$$\mathcal{B}_{0,3} = \{\varphi \in \mathcal{V}_3^\alpha, \|\varphi\|_3 \leq r_3\}, \quad d = 1, 2, 3, \quad (2.5.6)$$

where $r_1, r_2, r_3 > 0$ are the same as in $\mathcal{B}_{1,\epsilon}, \mathcal{B}_{2,\epsilon}$ and $\mathcal{B}_{3,\epsilon}$, are absorbing sets for the semigroup

$$S(t) : \mathcal{V}_j^\alpha \rightarrow \mathcal{V}_j^\alpha, \quad \phi_0 \mapsto \phi(t), \quad t \geq 0,$$

generated by the corresponding unperturbed problem (for the variable ϕ alone). The semigroup $S(t)$ possesses the global attractor $\mathcal{A}^\alpha$. Observe that, a solution of the unperturbed problem for both variables ϕ and u (at time t) is given by

$$\phi(t) = S(t)\phi_0 \quad \text{and} \quad u(t) = L_\gamma \phi(t), \quad (2.5.7)$$

where (cf. (1.2.16))

$$L_\gamma \varphi = C_\gamma^{-1} N[N\varphi + g(\varphi)] + \sigma C_\gamma^{-1} \varphi \quad (2.5.8)$$

(we recall that $C_\gamma = \gamma I + 2N$). We define

$$(\mathcal{A}^\alpha)_0 = \{(\varphi, L_\gamma \varphi), \varphi \in \mathcal{A}^\alpha\}.$$

We now wish to compare the global attractor $\mathcal{A}_\epsilon^\alpha$ with the set $(\mathcal{A}^\alpha)_0$ when ϵ goes to zero. Indeed, we need first the following lemma.

Lemma 2.5.1 *There exists $C > 0$ independent of ϵ such that any complete trajectory $(\phi(t), u(t))_{t \in \mathbb{R}} = S_\epsilon(t)(\phi_0, u_0)$ satisfies*

$$\|\phi(t)\|_3^2 + \|u(t)\|_1^2 + \|\phi_t(t)\|_{-1}^2 + \epsilon \|u_t(t)\|_{-2}^2 \leq C, \quad \forall t \in \mathbb{R}, \quad (2.5.9)$$

for any $(\phi_0, u_0) \in \mathcal{A}_\epsilon^\alpha$ and any $\epsilon \in (0, \epsilon_0]$.

Proof. Consider a complete trajectory $(\phi(t), u(t))_{t \in \mathbb{R}} \in \mathcal{A}_\epsilon^\alpha \subset \mathcal{B}_{3,\epsilon}$ which is a solution of (2.0.1), and $S_\epsilon(t)\mathcal{A}_{\epsilon_n}^\alpha = \mathcal{A}_{\epsilon_n}^\alpha$ (by exploiting the invariance property of the attractor).

Hence it follows that there exists $c > 0$ independent of ϵ such that

$$\|(\phi(t), u(t))\|_{\mathcal{V}_{3,\epsilon}}^2 \leq C, \quad \forall t \in \mathbb{R}. \quad (2.5.10)$$

Moreover, there holds

$$m(\phi(t)) = m(u(t)) = 0, \quad \forall t \in \mathbb{R}. \quad (2.5.11)$$

Indeed, like at the beginning of Section 2.2, we have

$$m(\phi(t)) = m(\phi(t_0))e^{-\sigma(t-t_0)}, \quad \forall t_0 \leq t, \quad t \in \mathbb{R}, \quad (2.5.12)$$

and

$$\begin{aligned} m(u(t)) &= m(u(t_0))e^{-\frac{\gamma}{\epsilon}(t-t_0)} + \frac{\sigma}{\gamma - \epsilon\sigma} m(\phi(t_0))(e^{-\sigma(t-t_0)} - e^{-\frac{\gamma}{\epsilon}(t-t_0)}), \\ &\forall t_0 \leq t, \quad t \in \mathbb{R}, \end{aligned} \quad (2.5.13)$$

we observe that, since $\frac{\sigma}{\gamma - \epsilon\sigma} > 0$, $(e^{-\sigma(t-t_0)} - e^{-\frac{\gamma}{\epsilon}(t-t_0)}) > 0$ and $(\phi(t_0), u(t_0)) \in \mathcal{A}_\epsilon^\alpha$, so that $|m(\phi(t_0))| \leq \alpha$ and

$$\begin{aligned} |m(u(t_0))| &\leq \frac{1}{|\Omega|} \int_{\Omega} |u(t_0)| \, dx \\ &\leq c \|u(t_0)\| \leq c. \end{aligned}$$

Then, we can deduce that

$$|m(\phi(t))| \leq ce^{-\sigma(t-t_0)}, \quad (2.5.14)$$

$$\begin{aligned} |m(u(t))| &\leq ce^{-\frac{\gamma}{\epsilon}(t-t_0)} + ce^{-\sigma(t-t_0)} \\ &\leq ce^{-\sigma(t-t_0)}. \end{aligned} \quad (2.5.15)$$

We have

$$e^{-\sigma(t-t_0)} \rightarrow 0 \quad \text{as} \quad t_0 \rightarrow -\infty,$$

thus, by taking the limit in (2.5.14) and (2.5.15) as $t_0 \rightarrow -\infty$, we entail (2.5.11).

In addition, it follows from (2.0.1)₁ that $\phi_t = -N(N\phi + g(\phi) - u) - \sigma\phi$, and we deduce, on account of (2.5.10), that

$$\begin{aligned} \|\phi_t(t)\|_{-1}^2 &\leq Q(\|\phi(t)\|_j^2) + c(\|\phi(t)\|_3^2 + \|u(t)\|_1^2) \\ &\leq c, \quad \forall t \in \mathbb{R}. \end{aligned} \quad (2.5.16)$$

We deduce from (2.2.39) that

$$\frac{d}{dt} (\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \gamma\|u\|^2) + \varepsilon\|u_t\|^2 + \|\phi_t\|_1^2 \leq c. \quad (2.5.17)$$

Now, we differentiate (2.0.1)₂ with respect to t to obtain

$$\epsilon u_{tt} + \phi_{tt} + Nu_t + \gamma u_t = 0. \quad (2.5.18)$$

We multiply (2.5.18) and (2.0.1)₁ by $N^{-2}u_t$ and $N^{-3}\phi_{tt}$, respectively, integrate over Ω , we get

$$\epsilon \frac{1}{2} \frac{d}{dt} \|u_t\|_{-2}^2 + \|u_t\|_{-1}^2 + (\phi_{tt}, N^{-2}u_t) + \gamma \|u_t\|_{-2}^2 = 0, \quad (2.5.19)$$

and

$$\frac{1}{2} \frac{d}{dt} (\|\phi_t\|_{-1}^2 + \sigma\|\phi_t\|_{-3}^2) + \|\phi_{tt}\|_{-3}^2 - (u_t, N^{-2}\phi_{tt}) + \int_{\Omega} g'(\phi)\phi_t N^{-2}\phi_{tt} dx = 0. \quad (2.5.20)$$

Summing the equations (2.5.19) and (2.5.20), then using $\|\phi_t g'(\phi)\| \leq c\|\phi_t\|$, $\|\phi_t\|^2 \leq \mu\|\phi_t\|_1^2 + c_\mu\|\phi_t\|_{-1}^2$ and (2.5.16), we obtain

$$\frac{d}{dt} (\|\phi_t\|_{-1}^2 + \sigma\|\phi_t\|_{-3}^2 + \varepsilon\|u_t\|_{-2}^2) + \|\phi_{tt}\|_{-3}^2 + \|u_t\|_{-1}^2 \leq \mu\|\phi_t\|_1^2 + c_\mu, \quad \forall \mu > 0. \quad (2.5.21)$$

Summing (2.5.17) and (2.5.21), we deduce that

$$\begin{aligned} \frac{d}{dt} (2\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \epsilon\|u_t\|_{-2}^2 + \gamma\|u\|^2 + \sigma\|\phi_t\|_{-3}^2) + \|\phi_t\|_1^2 + \epsilon\|u_t\|^2 \\ + \|\phi_{tt}\|_{-3}^2 + \|u_t\|_{-1}^2 \leq c. \end{aligned} \quad (2.5.22)$$

Thus, we deduce that

$$\frac{d}{dt} E_8 + c_0 E_8 \leq C, \quad (2.5.23)$$

where

$$E_8(t) = 2\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \epsilon\|u_t\|_{-2}^2 + \gamma\|u\|^2 + \sigma\|\phi_t\|_{-3}^2.$$

Now, multiply (2.0.1)₂ by $\epsilon N^{-1}u_t$, we get

$$\begin{aligned} \epsilon^2 \|u_t(t)\|_{-2}^2 &\leq 2(\|\phi_t(t)\|_{-2}^2 + \|u\|^2) \\ &\leq c, \quad \forall t \in \mathbb{R}, \end{aligned} \quad (2.5.24)$$

hence

$$\epsilon \|u_t(t)\|_{-2}^2 \leq \frac{c}{\epsilon}, \quad \forall t \in \mathbb{R}.$$

Applying the Gronwall's Lemma to (2.5.23) between t_0 to t , for any $t_0, t \in \mathbb{R}$, we obtain

$$\begin{aligned} E_8(t) &\leq e^{-c_0(t-t_0)} E_8(t_0) + c \\ &\leq c(\|\phi_t(t_0)\|_{-1}^2 + \|u(t_0)\|_1^2 + \epsilon \|u_t(t_0)\|_{-2}^2 + \|\phi(t_0)\|_j^2) e^{-c_0(t-t_0)} + c \\ &\leq \frac{c}{\epsilon} e^{-c_0(t-t_0)} + c, \quad \forall t \geq t_0, t \in \mathbb{R}, \end{aligned} \tag{2.5.25}$$

due to (2.5.16); hence

$$\|\phi_t(t)\|_{-1}^2 + \|u(t)\|_1^2 + \epsilon \|u_t(t)\|_{-2}^2 \leq \frac{c}{\epsilon} e^{-c_0(t-t_0)} + C, \quad \forall t \in \mathbb{R}. \tag{2.5.26}$$

Now, taking the limit in (2.5.26) as $t_0 \rightarrow -\infty$, we deduce that

$$\|\phi_t(t)\|_{-1}^2 + \|u(t)\|_1^2 + \epsilon \|u_t(t)\|_{-2}^2 \leq c, \quad \forall t \in \mathbb{R}. \tag{2.5.27}$$

Finally, the estimates (2.5.27) and (2.5.10) entail the desired estimate. ■

We are now in position to prove the following stability property of the global attractor.

Theorem 2.5.2 *Let the assumptions of Theorem 4.5.1 hold. Then, for any $\eta > 0$ the*

global attractor $\mathcal{A}_\epsilon^\alpha$ is upper semicontinuous at $\epsilon = 0$, that is,

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{V}_{3-\eta,0}}(\mathcal{A}_\epsilon^\alpha, (\mathcal{A}^\alpha)_0) = 0. \quad (2.5.28)$$

Proof. According to [25], [27], [26], it suffices to show that the following property holds:

Let ϵ_n be a sequence of positive numbers with ϵ_n converging to 0 when n goes to infinity, and let (ϕ_n, u_n) be a corresponding sequence of solutions of (2.0.1) such that, for any $t \in \mathbb{R}$, $(\phi_n(t), u_n(t)) \in \mathcal{A}_{\epsilon_n}^\alpha$ then there is a subsequence ϵ_{jn} of ϵ_n , such that $(\phi^{jn}(0), u^{jn}(0))$ converges to $(\tilde{\phi}, \tilde{u})$ in $\mathcal{V}_{3-\eta,0}$ and $(\tilde{\phi}, \tilde{u})$ belongs to $(\mathcal{A}^\alpha)_0$.

We now prove this property by a contradiction argument. Assume there exists $\rho > 0$, sequences $\epsilon_n \in (0, 1]$, $\epsilon_n \rightarrow 0$ and a corresponding sequence $z_{0n} = (\phi^n(0), u^n(0)) \in \mathcal{A}_{\epsilon_n}^\alpha$ such that

$$\text{dist}_{\mathcal{V}_{3-\eta,0}}(z_{0n}, (\mathcal{A}^\alpha)_0) \geq \rho. \quad (2.5.29)$$

We set

$$\mathcal{A}^\alpha = \bigcup_{\epsilon \in (0, \epsilon_0]} \mathcal{A}_\epsilon^\alpha. \quad (2.5.30)$$

we proved that $\mathcal{A}^\alpha$ is uniformly bounded in $\mathcal{V}_{3,0}$ with respect to ϵ (cf. lemma 2.5.1), hence it is relatively compact in $\mathcal{V}_{3-\eta,0}$, for any $\eta > 0$.

Let the complete trajectory $(\phi^n(t), u^n(t))_{t \in \mathbb{R}}$ such that $z_n(t) = (\phi^n(t), u^n(t)) = S_\epsilon(t)z_{0n}$. Due to the invariance property of the global attractor, it follows that

$$\bigcup_{t \in \mathbb{R}} \bigcup_{n \in \mathbb{N}} z_n(t) \subset \mathcal{A}^\alpha$$

is a relatively compact set in $\mathcal{V}_{3-\eta,0}$. Hence, $\bigcup_{t \in \mathbb{R}} \bigcup_{n \in \mathbb{N}} z_n(t)$ is a precompact set in $\mathcal{V}_{3-\eta,0}$, and the family of mappings $\{z_n(t) = (\phi^n(t), u^n(t)) \in \mathcal{C}(\mathbb{R}; \mathcal{V}_{3-\eta,0}), n \geq 0\}$ is equicontinuous from $\mathbb{R}$ into $\mathcal{V}_{3-\eta,0}$. By Ascoli's theorem and a classical diagonalization method, it follows the existence of an element $(\tilde{\phi}, \tilde{u}) \in \mathcal{C}(\mathbb{R}, \mathcal{V}_{3-\eta,0})$ such that (up to a subsequence) $\phi^n \rightarrow \tilde{\phi}$ in $\mathcal{C}([-T, T], V_{3-\eta})$ and $u^n \rightarrow \tilde{u}$ in $\mathcal{C}([-T, T], V_{1-\eta})$ that is,

$$z_{j_n} = (\phi^n, u^n) \rightarrow (\tilde{\phi}, \tilde{u}) \quad \text{in } \mathcal{C}([-T, T], \mathcal{V}_{3-\eta,0}) \quad (2.5.31)$$

for any $T > 0$, and

$$\sup_{t \in \mathbb{R}} \|(\tilde{\phi}(t), \tilde{u}(t))\|_{\mathcal{V}_{3-\eta,0}} < \infty, \quad (2.5.32)$$

on account of (2.5.9).

The next step is to show $(\tilde{\phi}(t), \tilde{u}(t))$ is a complete trajectory of $S(t)$ (weak solution of unperturbed problem). Observe that

$$\begin{cases} \phi_t^n = -N(N\phi^n + g(\phi^n) - u^n) - \sigma\phi^n, \\ \phi_t^n = -\epsilon u_t^n - Nu^n - \gamma u^n. \end{cases} \quad (2.5.33)$$

Exploiting (2.5.9) again, we learn that (up to a subsequence)

$$\phi_t^n \xrightarrow{*} \tilde{\phi}_t \text{ in } L^\infty([-T, T]; V_{-1}),$$

$$\epsilon_n u_t^n \xrightarrow{*} 0 \text{ in } L^\infty([-T, T]; V_{-2}).$$

Since $\phi^n \rightarrow \tilde{\phi} \in \mathcal{C}([-T, T], V_{3-\eta})$ and $g : V_2 \rightarrow V_2$ is continuous, it follows that

$$g(\phi^n) \rightharpoonup g(\tilde{\phi}) \text{ in } \mathcal{C}([-T, T], V_2).$$

Hence,

$$\phi_t^n \rightarrow \tilde{\phi}_t \text{ in } \mathcal{C}([-T, T]; V_{-(1+\eta)}),$$

$$\phi_t^n \rightarrow -N(N\tilde{\phi} + g(\tilde{\phi}) - \tilde{u}) - \sigma\tilde{\phi} \quad \text{in} \quad \mathcal{C}([-T, T], V_{-(1+\eta)}),$$

and

$$\phi_t^n \rightarrow -N\tilde{u} - \gamma\tilde{u} \quad \text{in} \quad \mathcal{C}([-T, T], V_{-(2+\eta)}), \quad (2.5.34)$$

for any $T > 0$. Passing to the limit $n \rightarrow \infty$ in (2.5.33), and by uniqueness of the limit, we find that

$$\begin{cases} \tilde{\phi}_t + N(N\tilde{\phi} + g(\tilde{\phi}) - \tilde{u}) + \sigma\tilde{\phi} = 0 & \text{in } \mathcal{C}([-T, T], V_{-(1+\eta)}), \\ \tilde{\phi}_t + N\tilde{u} + \gamma\tilde{u} = 0 & \text{in } \mathcal{C}([-T, T], V_{-(2+\eta)}), \end{cases} \quad (2.5.35)$$

or

$$B_\gamma \tilde{\phi}_t + N(N\tilde{\phi} + g(\tilde{\phi})) + \sigma\tilde{\phi} = 0 \text{ in } \mathcal{C}([-T, T], V_{-(1+\eta)}), \quad (2.5.36)$$

for any $T > 0$. It follows that $(\tilde{\phi}, \tilde{u})$ is a bounded complete trajectory of the unperturbed problem (2.5.36).

Therefore, by characterization of the attractor (cf. lemma 1.5.2), that is

$$u \in \mathcal{A} \iff \exists \{\psi(t)\}_{t \in \mathbb{R}} \in \mathcal{A} \text{ such that } \psi(0) = u \text{ and } \|\psi(t)\|_{3-\eta} \leq C, \forall t \in \mathbb{R},$$

we have $\tilde{\phi}(t)$ belong to $\mathcal{A}^\alpha$ for any time t . As $\phi^n(t)$ converge to $\tilde{\phi}(t)$ in $\mathcal{C}([-T, T], V_{3-\eta})$, for any compact interval $[-T, T]$ of $\mathbb{R}$, $\phi^n(0)$, in particular, converges to $\tilde{\phi}(0) \in \mathcal{A}^\alpha$ in $V_{3-\eta}$, and by definition $(\tilde{\phi}(0), L_\gamma \tilde{\phi}(0)) \in (\mathcal{A}^\alpha)_0$. Thus the convergence

$$z_{0n} \rightarrow (\tilde{\phi}(0), L_\gamma \tilde{\phi}(0)) \text{ in } \mathcal{V}_{3-\eta, 0},$$

implies that

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{V}_{3-\eta, 0}}(z_{0n}, (\mathcal{A}^\alpha)_0) = 0, \quad (2.5.37)$$

against the initial assumption. █

2.6 Exponential attractors

We now recall that the fractal dimension of a compact subset $\mathcal{C} \subset \mathcal{V}_{j, \epsilon}$ is given by

$$\dim_{\mathcal{V}_{j, \epsilon}}(\mathcal{C}) = \limsup_{\theta \rightarrow 0} \frac{\ln N_\theta(\mathcal{C})}{\log \frac{1}{\theta}},$$

where $N_\theta(\mathcal{C})$ is the minimum number of θ -balls of $\mathcal{V}_{j, \epsilon}$ necessary to cover $\mathcal{C}$.

To prove the existence of exponential attractors the following lemma and proposition are useful.

Lemma 2.6.1 *Let the assumptions of Theorem 2.4.1 hold. If $\phi_0 \in \mathcal{B}_{0,3}$, then we have*

the following estimate

$$\int_0^t \|u_t(s)\|_{-1}^2 ds \leq c(1+t), \quad \forall t \geq 0. \quad (2.6.1)$$

Proof. As $\phi_0 \in \mathcal{B}_{0,3}$, there holds

$$\|\phi(t)\|_3^2 \leq c, \quad \forall t \geq 0. \quad (2.6.2)$$

First, we differentiate (2.3.1) with respect to t to obtain

$$\phi_{tt} + N(N\phi_t + g'(\phi)\phi_t + (\gamma I + N)^{-1}\phi_{tt}) + \sigma\phi_t = 0, \quad (2.6.3)$$

then

$$B_\gamma \phi_{tt} + N(N\phi_t + g'(\phi)\phi_t) + \sigma\phi_t = 0,$$

where

$$B_\gamma = I + N(\gamma I + N)^{-1}.$$

From which we get that

$$u_t(t) = C_\gamma^{-1}[N(N\phi_t(t) + g'(\phi(t))\phi_t(t)) + \sigma\phi_t], \quad \forall t \geq 0, \quad (2.6.4)$$

where

$$m(u_t(t)) = -\frac{\sigma^2}{\gamma}m(\phi_0)e^{-\sigma t}.$$

We multiply (2.6.3) by $N^{-1}\bar{\phi}_t$, and integrate over Ω , we find

$$\frac{1}{2} \frac{d}{dt} \|B_\gamma^{\frac{1}{2}} N^{-\frac{1}{2}} \bar{\phi}_t\|^2 + \|\bar{\phi}_t\|_1^2 + \int_\Omega \nabla[g'(\phi)\phi_t] \cdot \nabla N^{-1} \bar{\phi}_t dx + \sigma \|\bar{\phi}_t\|_{-1}^2 = 0. \quad (2.6.5)$$

We have

$$\begin{aligned} \int_\Omega |\nabla[g'(\phi)\phi_t]|^2 dx &\leq 2\|g''(\phi)\|_{L^\infty(\Omega)}^2 \|\nabla\phi\|_{L^\infty(\Omega)}^2 \|\phi_t\|^2 + 2\|g'(\phi)\|_{L^\infty(\Omega)}^2 \|\nabla\phi_t\|^2 \\ &\leq Q(\|\phi\|_j^2) \|\phi_t\|_1^2, \end{aligned}$$

and by virtue (2.6.2), we deduce

$$\begin{aligned} \left| \int_\Omega \nabla[g'(\phi)\phi_t] \cdot \nabla N^{-1} \bar{\phi}_t dx \right| &\leq c \|\phi_t\|_1 \|\phi_t\|_{-1} \\ &\leq \frac{1}{2} \|\phi_t\|_1 + c \|\phi_t\|_{-1}^2. \end{aligned} \quad (2.6.6)$$

On account of (2.6.2) again, the estimate (2.3.3) and

$$\phi_t(x) = -B_\gamma^{-1} N(N\phi(x) + g(\phi(x))) - B_\gamma^{-1} \sigma \phi(x),$$

we can deduce that

$$\|\phi_t\|_{-1}^2 \leq c \|\phi\|_3^2 \leq c, \quad \forall t \geq 0.$$

We infer from (2.6.5) that

$$\frac{d}{dt} \|B_\gamma^{\frac{1}{2}} N^{-\frac{1}{2}} \bar{\phi}_t\|^2 + \frac{1}{2} \|\phi_t\|_1^2 + \sigma \|\bar{\phi}_t\|_{-1}^2 \leq c + |m(\phi_t)|^2, \quad (2.6.7)$$

then

$$\frac{d}{dt} \|B_\gamma^{\frac{1}{2}} N^{-\frac{1}{2}} \bar{\phi}_t\|^2 + \|\phi_t\|_1^2 + \sigma \|\bar{\phi}_t\|_{-1}^2 \leq c + ce^{-\sigma t}. \quad (2.6.8)$$

It follows from (2.3.4) that

$$\|\bar{\phi}_t\|_{-1}^2 \leq \|B_\gamma^{\frac{1}{2}} N^{-\frac{1}{2}} \bar{\phi}_t\|^2 \leq 2\|\bar{\phi}_t\|_{-1}^2.$$

Hence, by integrating from 0 to t , we get

$$\|\phi_t\|_{-1}^2 + \int_0^t \|\phi_t\|_1^2 \leq c(1+t), \quad \forall t \geq 0. \quad (2.6.9)$$

Next, from (2.6.4), we can obtain

$$N^{\frac{-1}{2}} \bar{u}_t(t) = C_\gamma^{-1} [N^{\frac{1}{2}} (N\phi_t(t) + g'(\phi(t))\phi_t(t)) + \sigma N^{\frac{-1}{2}} \bar{\phi}_t], \quad \forall t \geq 0,$$

and, by utilizing (2.3.5) and (2.3.6), we infer from the following estimates

$$\begin{aligned} \|C_\gamma^{-1} \nabla[g'(\phi(t))\phi_t(t)]\|^2 &\leq \frac{1}{\gamma^2} \|\nabla[g'(\phi)\phi_t]\|^2 \\ &\leq c\|\phi_t(t)\|_1, \quad \forall t \geq 0, \end{aligned} \quad (2.6.10)$$

$$\begin{aligned} \|g'(\phi)\phi_t\|^2 &\leq \|g'(\phi)\|_{L^\infty}^2 \|\phi_t\|^2 \\ &\leq c\|\phi_t\|^2 \end{aligned} \quad (2.6.11)$$

and

$$\|C_\gamma^{-1} N N^{\frac{1}{2}} \phi_t(t)\|^2 \leq \frac{1}{4} \|\phi_t(t)\|_1^2, \quad \forall t \geq 0, \quad (2.6.12)$$

so that, we find

$$\|\bar{u}_t(t)\|_{-1}^2 \leq c \|\phi_t(t)\|_1^2, \quad (2.6.13)$$

and

$$\|u_t(t)\|_{-1}^2 \leq c \|\phi_t(t)\|_1^2 + |m(u_t)|^2. \quad (2.6.14)$$

On account of (2.6.9), and by taking the integral of (2.6.14) from 0 to t , we derive

$$\int_0^t \|u_t(s)\|_{-1}^2 ds \leq c(1+t), \quad \forall t \geq 0. \quad (2.6.15)$$

■

Proposition 2.6.1 *Let the assumptions of Theorem 2.4.1 hold. Then, there exists $c > 0$ and $c' > 0$, all independent of ϵ , such that*

$$\|S_\epsilon(t)(\phi_0, u_0) - (S(t)\phi_0, \mathbf{L} S(t)\phi_0)\|_{\mathbf{V}_{2,\epsilon}}^2 \leq c\epsilon e^{c't}, \quad \forall t \geq \frac{1}{\sigma}, \quad (2.6.16)$$

for any $(\phi_0, u_0) \in \mathcal{B}_{3,\epsilon}$ and any $\epsilon \in (0, \epsilon_0]$.

Proof. Let us take $(\phi_0, u_0) \in \mathcal{B}_{3,\epsilon}$, we set

$$(\phi^\epsilon(t), u^\epsilon(t)) = S_\epsilon(t)(\phi_0, u_0), \quad (\phi(t), u(t)) = (S(t)\phi_0, L_\gamma S(t)\phi_0).$$

The pair of function $(\phi^\epsilon(t), u^\epsilon(t))$ and $\phi(t)$ satisfy the estimates

$$\|\phi^\epsilon(t)\|_3^2 + \epsilon \|u^\epsilon(t)\|_2^2 + \|u^\epsilon(t)\|_1^2 \leq c, \quad \forall t \geq 0, \quad (2.6.17)$$

and

$$\|\phi(t)\|_3^2 \leq c, \quad \forall t \geq 0,$$

also, from (2.5.8), we have

$$\|u\|_1^2 \leq c \|\phi\|_3^2 \leq c, \quad \forall t \geq 0. \quad (2.6.18)$$

Also, on account of (2.6.9), and by integrating (2.2.86) from 0 to t we obtain

$$\int_0^t (\|\phi_t^\epsilon(s)\|_1^2 + \|\phi_t(s)\|_1^2) ds \leq C, \quad \forall t \geq 0. \quad (2.6.19)$$

Note that,

$$m(u_t(t)) = -\frac{\sigma^2}{\gamma} m(\phi_0) e^{-\sigma t}.$$

We now set $P = \phi^\epsilon - \phi$ and $R = u^\epsilon - u$ and they satisfy the following problem:

$$\begin{cases} P_t + N(NP + h(t)P - R) + \sigma P = 0, \\ \epsilon R_t + P_t + NR + \gamma R = -\epsilon u_t, \\ P|_{t=0} = 0, \quad R|_{t=0} = u_0 - L_\gamma \phi_0, \end{cases} \quad (2.6.20)$$

where

$$h(t) = \int_0^1 g'(s\phi^\epsilon(t) + (1-s)\phi(t))ds.$$

We first observe that

$$m(P(t)) = m(\phi^\epsilon(t)) - m(\phi(t)) = 0, \quad \forall t \geq 0,$$

and

$$\begin{aligned} m(R(t)) &= m(u^\epsilon(t)) - m(u(t)) \\ &= m(u_0)e^{-\frac{\gamma}{\epsilon}t} + m(\phi_0)\frac{\sigma\epsilon}{\gamma(\gamma - \epsilon\sigma)}\left[\sigma e^{-\sigma t} - \frac{\gamma}{\epsilon}e^{-\frac{\gamma}{\epsilon}t}\right], \quad \forall t \geq 0. \end{aligned} \quad (2.6.21)$$

Note that ($xe^{-x} \leq \frac{1}{e}$, $\forall x \geq 0$): Indeed, let $f(x) = xe^{-x}$, we have

$$f'(x) = e^{-x} - xe^{-x} = 0 \iff x = 1,$$

then the function has its maximum value $\frac{1}{e}$ at $x = 1$, hence

$$f(x) \leq \frac{1}{e} \implies xe^{-x} \leq \frac{1}{e}, \quad \forall x \geq 0.$$

Now, we have

$$e^{-\frac{\gamma t}{\epsilon}} \leq \frac{\epsilon}{\gamma t e}, \quad \forall t \geq 0, \implies e^{-\frac{\gamma}{\epsilon} t} \leq \frac{\epsilon \sigma}{\gamma e}, \quad \forall t \geq \frac{1}{\sigma}.$$

Next we prove that $\sigma e^{-\sigma t} - \frac{\gamma}{\epsilon} e^{-\frac{\gamma}{\epsilon} t} > 0$: Let $f(x) = x e^{-x t}$, then

$$f'(x) = e^{-x t}(1 - x t) < 0 \text{ (} f \text{ is decreasing)} \iff x > \frac{1}{t},$$

since $\sigma < \frac{\gamma}{\epsilon} \implies \sigma e^{-\sigma t} > \frac{\gamma}{\epsilon} e^{-\frac{\gamma}{\epsilon} t}$, $\forall t > \frac{1}{\sigma}$.

Also, since $\epsilon \leq \frac{\gamma}{2\sigma}$ then $-\epsilon \geq -\frac{\gamma}{2\sigma} \implies \gamma - \epsilon \sigma \geq \frac{\gamma}{2}$, therefore, we can deduce that

$$\frac{1}{\gamma - \epsilon \sigma} < \frac{2}{\gamma}.$$

Recalling (2.6.21), it follows that

$$\begin{aligned} |m(R(t))| &\leq |m(u_0)| e^{-\frac{\gamma}{\epsilon} t} + |m(\phi_0)| \frac{\sigma \epsilon}{\gamma(\gamma - \epsilon \sigma)} [\sigma e^{-\sigma t} - \frac{\gamma}{\epsilon} e^{-\frac{\gamma}{\epsilon} t}], \quad \forall t \geq 0 \\ &\leq |m(u_0)| e^{-\frac{\gamma}{\epsilon} t} + \frac{2\epsilon \sigma^2}{\gamma^2} |m(\phi_0)| e^{-\sigma t}, \quad \forall t > \frac{1}{\sigma} \\ &\leq c\epsilon, \quad \forall t > \frac{1}{\sigma}. \end{aligned} \tag{2.6.22}$$

We multiply (2.6.20)₁ and (2.6.20)₂ by $N^{-1}P_t$ and R , respectively, integrate over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} (\|P\|_1^2 + \sigma \|P\|_{-1}^2) + \|P_t\|_{-1}^2 - (R, P_t) + \int_{\Omega} \nabla[h(t)P] \cdot \nabla N^{-1}P_t \, dx = 0, \tag{2.6.23}$$

and

$$\frac{\epsilon}{2} \frac{d}{dt} \|R\|^2 + \|\bar{R}\|_1^2 + (R, P_t) + \gamma \|R\|^2 = -\epsilon(N^{-1/2}\bar{u}_t, N^{1/2}R) - \epsilon m(u_t) |\Omega| m(R). \quad (2.6.24)$$

Summing the equations (2.6.23) and (2.6.24), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|P\|_1^2 + \epsilon \|R\|^2 + \sigma \|P\|_{-1}^2) + \|P_t\|_{-1}^2 + \|\bar{R}\|_1^2 \\ & + \int_{\Omega} \nabla[h(t)P] \cdot \nabla N^{-1}P_t + \gamma \|R\|^2 = -\epsilon(N^{-\frac{1}{2}}\bar{u}_t, N^{\frac{1}{2}}\bar{R}) - \epsilon m(u_t) |\Omega| m(R). \end{aligned} \quad (2.6.25)$$

We have

$$\begin{aligned} |\epsilon(N^{-1/2}\bar{u}_t, N^{1/2}\bar{R})| & \leq \epsilon \|\bar{u}_t\|_{-1} \|\bar{R}\|_1 \\ & \leq \frac{1}{2} \epsilon^2 \|\bar{u}_t\|_{-1}^2 + \frac{1}{2} \|\bar{R}\|_1^2. \end{aligned}$$

Moreover, we have

$$\nabla h(t) = \int_0^1 g''(s\phi^\epsilon(t) + (1-s)\phi(t))(s\nabla\phi^\epsilon(t) + (1-s)\nabla\phi(t))ds,$$

and $\nabla[h(t)P] = P\nabla h(t) + h(t)\nabla P$, so that

$$\begin{aligned}
& \int_{\Omega} |\nabla[h(t)P]|^2 dx \\
& \leq 2 \int_{\Omega} |P|^2 \int_0^1 |g''(s\phi^\epsilon(t) + (1-s)\phi(t))|^2 |s\nabla\phi^\epsilon(t) + (1-s)\nabla\phi(t)|^2 ds \\
& \quad + \int_0^1 |g'(s\phi^\epsilon(t) + (1-s)\phi(t))|^2 ds |\nabla P|^2 dx \\
& \leq \sup_{s \in [0,1]} \|g''(s\phi^\epsilon(t) + (1-s)\phi(t))\|_{L^\infty} \|P\|_{L^4}^2 (\|\nabla\phi^\epsilon(t)\|_{L^4}^2 + \|\nabla\phi(t)\|_{L^4}^2) \\
& \quad + \sup_{s \in [0,1]} \|g'(s\phi^\epsilon(t) + (1-s)\phi(t))\|_{L^\infty} \|\nabla P\|^2 \\
& \leq \|P\|_1^2 Q(\|\phi^\epsilon\|_j^2 + \|\phi\|_j^2)(\|\phi^\epsilon(t)\|_2^2 + \|\phi(t)\|_2^2) + \|P\|_1^2 Q(\|\phi^\epsilon\|_j^2 + \|\phi\|_j^2) \\
& \leq c\|P\|_1^2, \quad \forall t \geq 0,
\end{aligned} \tag{2.6.26}$$

then

$$\begin{aligned}
& \left| \int_{\Omega} \nabla[h(t)P] \cdot \nabla N^{-1}P_t dx \right| \leq \|\nabla[h(t)P]\| \|P_t\|_{-1} \\
& \leq \frac{c}{2} \|P\|_1^2 + \frac{1}{2} \|P_t\|_{-1}^2.
\end{aligned}$$

It follows from (2.6.25) that

$$\begin{aligned}
& \frac{d}{dt} (\|P\|_1^2 + \epsilon \|R\|^2 + \sigma \|P\|_{-1}^2) \\
& + \|P_t\|_{-1}^2 + \|\bar{R}\|_1^2 + \gamma \|R\|^2 \leq c' \|P\|_1^2 + \epsilon^2 \|\bar{u}_t\|_{-1}^2 + c\epsilon |(m(u_t))| |m(R)|,
\end{aligned} \tag{2.6.27}$$

by using Young's inequality and the estimate $\|\bar{R}\|_1^2 + \gamma\|R\|^2 \geq \|R\|_1^2$, we deduce that

$$\frac{d}{dt}E_9 + \|P_t\|_{-1}^2 + \|R\|_1^2 \leq c'E_9 + \epsilon^2\|\bar{u}_t\|_{-1}^2 + c\epsilon^2|m(u_t)|^2 + \frac{1}{2}|m(R)|^2, \quad (2.6.28)$$

where

$$E_9 = \|P\|_1^2 + \epsilon\|R\|^2 + \sigma\|P\|_{-1}^2.$$

Moreover, we have

$$\begin{aligned} |m(R(t))| &\leq |m(u_0)|e^{-\frac{\gamma}{\epsilon}t} + |m(\phi_0)|\frac{\sigma^2\epsilon}{\gamma(\gamma - \epsilon\sigma)}e^{-\sigma t} + |m(\phi_0)|\frac{\sigma}{(\gamma - \epsilon\sigma)}e^{-\frac{\gamma}{\epsilon}t} \\ &\leq (|m(u_0)| + |m(\phi_0)|\frac{2\sigma}{\gamma})e^{-\frac{\gamma}{\epsilon}t} + |m(\phi_0)|\frac{2\sigma^2\epsilon}{\gamma^2}e^{-\sigma t} \\ &\leq ce^{-\frac{\gamma}{\epsilon}t} + c\epsilon e^{-\sigma t}, \quad \forall t \geq 0, \end{aligned}$$

hence

$$\begin{aligned} |m(R(t))|^2 &\leq 2(c^2e^{-\frac{2\gamma}{\epsilon}t} + c^2\epsilon^2e^{-2\sigma t}) \\ &\leq ce^{-\frac{2\gamma}{\epsilon}t} + c\epsilon^2e^{-2\sigma t}, \quad \forall t \geq 0, \end{aligned}$$

and

$$\begin{aligned} \int_0^t |m(R(s))|^2 ds &\leq c \int_0^t e^{-\frac{2\gamma}{\epsilon}s} ds + c \int_0^t \epsilon^2 e^{-2\sigma s} ds \\ &\leq \frac{c\epsilon}{2\gamma}(1 - e^{-\frac{2\gamma}{\epsilon}t}) + \frac{c\epsilon^2}{2\sigma}(1 - e^{-2\sigma t}) \\ &\leq c\epsilon, \quad \forall t \geq 0. \end{aligned}$$

Now, by applying the Gronwall Lemma to (2.6.28), we find

$$\begin{aligned}
E_9(t) &\leq [E_9(0) + \int_0^t (c\epsilon^2 \|\bar{u}_t\|_{-1}^2 + c\epsilon^2 |m(u_t)|^2 + \frac{1}{2} |m(R)|^2) ds] e^{c't} \\
&\leq [\epsilon \|u_0 - L_\gamma \phi_0\|^2 + \int_0^t (c\epsilon^2 \|\bar{u}_t\|_{-1}^2 + c\epsilon^2 |m(u_t)|^2 + \frac{1}{2} |m(R)|^2) ds] e^{c't} \\
&\leq c\epsilon e^{c't}, \quad \forall t \geq 0,
\end{aligned} \tag{2.6.29}$$

due to (2.6.1), the fact that $\|u_0\|_1 \leq c$ (since $(\phi_0, u_0) \in \mathcal{B}_{3,\epsilon}$), and

$$\begin{aligned}
\|L_\gamma \phi_0\| &\leq \|C_\gamma^{-1} [N(N\phi_0 + g(\phi_0)) + \sigma\phi_0]\| \\
&\leq c(\|\phi_0\|_2 + \|g(\phi_0)\|) \\
&\leq c,
\end{aligned}$$

then

$$\|u_0 - L_\gamma \phi_0\|^2 \leq c.$$

Hence, it follows from (2.6.29) that

$$\|P(t)\|_1^2 + \epsilon \|R(t)\|^2 \leq c\epsilon e^{c't}, \quad \forall t \geq 0. \tag{2.6.30}$$

Integrating the equation (2.6.28) between 0 and t , we deduce that

$$\int_0^t (\|P_t(s)\|_{-1}^2 + \|R(s)\|_1^2) ds \leq c\epsilon e^{c't}, \quad \forall t \geq 0. \tag{2.6.31}$$

Next, we differentiate (2.6.20)₁ with respect to t to obtain

$$P_{tt} + N(NP_t + h(t)P_t + h'(t)P - R_t) + \sigma P_t = 0, \quad (2.6.32)$$

where

$$h'(t) = \int_0^1 g''(s\phi^\epsilon(t) + (1-s)\phi(t))(s\phi_t^\epsilon(t) + (1-s)\phi_t(t)) ds.$$

We multiply (2.6.32) and (2.6.20)₂ by $N^{-2}P_t$ and $N^{-1}\bar{R}_t$, respectively, integrate over Ω , then sum the resulting equations, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|P_t\|_{-2}^2 + \|\bar{R}\|^2 + \gamma \|\bar{R}\|_{-1}^2) + \|P_t\|^2 + \epsilon \|\bar{R}_t\|_{-1}^2 + \sigma \|P_t\|_{-2}^2 + \int_{\Omega} h(t) P_t N^{-1} P_t dx \\ & + \int_{\Omega} h'(t) P N^{-1} P_t dx = -\epsilon (N^{-\frac{1}{2}} \bar{u}_t, N^{-\frac{1}{2}} \bar{R}_t). \end{aligned} \quad (2.6.33)$$

Since

$$\begin{aligned} |\epsilon (N^{-\frac{1}{2}} \bar{u}_t, N^{-\frac{1}{2}} \bar{R}_t)| & \leq \frac{\epsilon}{2} \|\bar{u}_t\|_{-1}^2 + \frac{\epsilon}{2} \|\bar{R}_t\|_{-1}^2 \\ & \leq \frac{\epsilon}{2} \|\bar{u}_t\|_{-1}^2 + \frac{\epsilon}{2} \|\bar{R}_t\|_{-1}^2, \end{aligned}$$

then it follows from (2.6.33) that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\bar{R}\|^2 + \|P_t\|_{-2}^2 + \gamma \|\bar{R}\|_{-1}^2) + \|P_t\|^2 + \frac{\epsilon}{2} \|\bar{R}_t\|_{-1}^2 + \sigma \|P_t\|_{-2}^2 + \int_{\Omega} h(t) P_t N^{-1} P_t dx \\ & + \int_{\Omega} h'(t) P N^{-1} P_t dx \leq \epsilon \|\bar{u}_t\|_{-1}^2. \end{aligned} \quad (2.6.34)$$

We have

$$\begin{aligned}
& \int_{\Omega} |h(t)N^{-1}P_t|^2 dx \\
& \leq \sup_{s \in [0,1]} \|g'(s\phi^\epsilon(t) + (1-s)\phi(t))\|_{L^\infty(\Omega)}^2 \|N^{-1}P_t\|^2 \\
& \leq \tilde{Q}(\|\phi^\epsilon\|_j^2, \|\phi\|_j^2) \|P_t\|_{-2}^2 \\
& \leq C \|P_t\|_{-1}^2, \quad \forall t \geq 0,
\end{aligned} \tag{2.6.35}$$

then

$$\begin{aligned}
| \int_{\Omega} P_t h(t) N^{-1} P_t dx | & \leq \|P_t\| \|h(t)N^{-1}P_t\| \\
& \leq C \|P_t\|_{-1}^2 + \frac{1}{4} \|P_t\|^2.
\end{aligned} \tag{2.6.36}$$

Moreover, we have

$$\begin{aligned}
& | \int_{\Omega} h'(t) P N^{-1} P_t dx | \\
& \leq \int_{\Omega} \int_0^1 |g''(s\phi^\epsilon(t) + (1-s)\phi(t))| |(s\phi_t^\epsilon(t) + (1-s)\phi_t(t))| |P| |N^{-1}P_t| ds dx \\
& \leq c \|N^{-1}P_t\|_{L^4} \|P\|_{L^4} (\|\phi_t^\epsilon\| + \|\phi_t\|) \\
& \leq c \|P_t\|_{-1} \|P\|_1 (\|\phi_t^\epsilon\| + \|\phi_t\|) \\
& \leq \frac{1}{4} \|P_t\|^2 + c \|P\|_1^2 (\|\phi_t^\epsilon\|^2 + \|\phi_t\|^2), \quad \forall t \geq 0.
\end{aligned} \tag{2.6.37}$$

Owing to (2.6.36) and (2.6.37), we get

$$\begin{aligned} \frac{d}{dt}(\|\bar{R}\|^2 + \|P_t\|_{-2}^2 + \gamma\|\bar{R}\|_{-1}^2) + \|P_t\|^2 + \epsilon\|\bar{R}_t\|_{-1}^2 + \sigma\|P_t\|_{-2}^2 \\ \leq c[\epsilon\|u_t\|_{-1}^2 + \|P\|_1^2(\|\phi_t^\epsilon\|^2 + \|\phi_t\|^2) + \|P_t\|_{-1}^2], \quad \forall t \geq 0. \end{aligned} \quad (2.6.38)$$

Hence

$$\begin{aligned} \frac{dE_{10}}{dt} + c(\|P_t\|^2 + \epsilon\|\bar{R}_t\|_{-1}^2 + \sigma\|P_t\|_{-2}^2) \\ \leq cE_{10} + \epsilon c\|u_t\|_{-1}^2 + c\|P\|_1^2(\|\phi_t^\epsilon\|^2 + \|\phi_t\|^2), \end{aligned} \quad (2.6.39)$$

where

$$E_{10}(t) = \|\bar{R}\|^2 + \|P_t\|_{-2}^2 + \gamma\|\bar{R}\|_{-1}^2.$$

We multiply (2.6.39) by t , then we integrate between 0 and t , we find

$$\begin{aligned} t(\|\bar{R}(t)\|^2 + \|P_t(t)\|_{-2}^2 + \gamma\|\bar{R}(t)\|_{-1}^2) \\ \leq c(1+t) \int_0^t [\|R(s)\|_1^2 + \|P_t(s)\|_{-1}^2 + (\|\phi_t^\epsilon(s)\|^2 + \|\phi_t(s)\|^2)\|P(s)\|_1^2 + \epsilon\|u_t(s)\|_{-1}^2] ds \\ \leq c(1+t)\epsilon e^{c't}, \quad \forall t > 0, \end{aligned}$$

due to (2.6.1), (2.6.19), (2.6.30) and (2.6.31). Hence

$$\|\bar{R}(t)\|^2 + \|P_t(t)\|_{-2}^2 + \gamma\|\bar{R}(t)\|_{-1}^2 \leq \frac{(1+t)}{t} c\epsilon e^{c't}, \quad \forall t > 0,$$

therefore,

$$\|\bar{R}(t)\|^2 + \|P_t(t)\|_{-2}^2 \leq c\epsilon e^{c't}, \quad \forall t \geq \frac{1}{\sigma}, \quad (2.6.40)$$

and

$$\begin{aligned} \|R(t)\|^2 + \|P_t(t)\|_{-2}^2 &\leq c\epsilon e^{c't} + |m(R)|^2 \\ &\leq c_1\epsilon e^{c't}, \quad \forall t \geq \frac{1}{\sigma}. \end{aligned} \quad (2.6.41)$$

Collecting (2.6.30) and (2.6.41), we deduce that

$$\|P(t)\|_1^2 + \epsilon\|R(t)\|^2 + \|R(t)\|_{-1}^2 \leq c\epsilon e^{c't}, \quad \forall t \geq \frac{1}{\sigma}. \quad (2.6.42)$$

In addition, on account of (2.6.41) and (2.6.30), we have

$$\begin{aligned} \|P(t)\|_2^2 &\leq c(\|P_t(t)\|_{-2}^2 + \|P(t)\|^2 + \|R(t)\|^2) \\ &\leq c(\|P_t(t)\|_{-2}^2 + \|P(t)\|_1^2 + \|R(t)\|^2) \\ &\leq c\epsilon e^{c't} \quad \forall t \geq \frac{1}{\sigma}. \end{aligned} \quad (2.6.43)$$

Now, since

$$R(t) = u^\epsilon(t) - u(t)$$

then, by virtue (2.6.17) and (2.6.18), we obtain

$$\begin{aligned}\epsilon \|R(t)\|_1^2 &\leq 2\epsilon(\|u^\epsilon(t)\|_1^2 + \|u(t)\|_1^2) \\ &\leq c\epsilon, \quad \forall t \geq \frac{1}{\sigma}.\end{aligned}\tag{2.6.44}$$

Hence,

$$\|P(t)\|_2^2 + \epsilon \|R(t)\|_1^2 + \|R(t)\|^2 \leq c\epsilon e^{c't}, \quad \forall t \geq \frac{1}{\sigma},\tag{2.6.45}$$

as required. ■

We recall the following transitivity of the exponential attraction property.

Lemma 2.6.2 (cf. Lemma 4.1, [23]) *Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \subset \mathcal{V}_{j,\epsilon}$ be such that*

$$\text{dist}_{\mathcal{V}_{j,\epsilon}}(S(t)\mathcal{B}_1, \mathcal{B}_2) \leq M_1 e^{-c_1 t}, \text{dist}_{\mathcal{V}_{j,\epsilon}}(S(t)\mathcal{B}_2, \mathcal{B}_3) \leq M_2 e^{-c_2 t}, \quad \forall t \geq 0,$$

for some $c_1, c_2 > 0$ and $M_1, M_2 \geq 0$. Assume also that for all $z_1, z_2 \in \mathcal{V}_{j,\epsilon}$ there holds

$$\|S(t)z_1 - S(t)z_2\|_{\mathcal{V}_{1,\epsilon}} \leq M_0 e^{c_0 t} \|z_1 - z_2\|_{\mathcal{V}_{1,\epsilon}}, \quad \forall t \geq 0,$$

for some $c_0 \geq 0$ and some $M_0 \geq 0$. Then it follows that

$$\text{dist}_{\mathcal{V}_{j,\epsilon}}(S(t)\mathcal{B}_1, \mathcal{B}_3) \leq M e^{-ct}, \quad \forall t \geq 0,$$

where $c = \frac{c_1 c_2}{c_0 + c_1 + c_2}$ and $M = M_0 M_1 + M_2$.

Now, theorems ensuring the existence of a family of exponential attractors which are Hölder continuous with respect to some perturbation parameters are available in the literature. Here we report the following result.

Lemma 2.6.3 ([22], [37]) *Assume that there exist $\varkappa_j \in (0, 1]$ and $\Upsilon_j \geq 0$ (all independent of ϵ), and such that the following conditions hold:*

1. *The map L_γ is continuous in the metric topology of $L^2(\Omega)$.*
2. *There exists $t^* > 0$, independent of ϵ , such that*

$$S_\epsilon(t)\mathcal{B}_{3,\epsilon} \subset \mathcal{B}_{3,\epsilon}, \quad \forall t \geq t^*.$$

Moreover, setting $S_\epsilon(t^) = S_\epsilon$, the map S_ϵ satisfies, for every $z_1, z_2 \in \mathcal{B}_{3,\epsilon}$,*

$$\|S_\epsilon z_1 - S_\epsilon z_2\|_{\mathcal{V}_{2,\epsilon}} \leq \Upsilon_1 \|z_1 - z_2\|_{\mathcal{V}_{1,\epsilon}}. \quad (2.6.46)$$

3. *For any $z = (\phi_0, \phi_1) \in \mathcal{B}_{3,\epsilon}$, there hold*

$$\|S_\epsilon^m(\phi_0, \phi_1) - S^m(\phi_0, L_\gamma \phi_0)\|_{\mathcal{V}_{1,\epsilon}} \leq \Upsilon_2^m \epsilon^{\varkappa_1}, \quad \forall m \in \mathbb{N}, \quad (2.6.47)$$

$$\|S_\epsilon(t)(\phi_0, \phi_1) - S(t)(\phi_0, L_\gamma \phi_0)\|_{\mathcal{V}_{1,\epsilon}} \leq \Upsilon_3 \epsilon^{\varkappa_2}, \quad \forall t \in [t^*, 2t^*]. \quad (2.6.48)$$

4. *The map $z \mapsto S_\epsilon(t)z$ is Lipschitz continuous on $\mathcal{B}_{3,\epsilon}$ endowed with the metric topology of $\mathcal{V}_{1,\epsilon}$, with a Lipschitz constant independent of ϵ and $t \in [t^*, 2t^*]$.*

5. *The map*

$$(t, z) \mapsto S_\epsilon(t)z : [t^*, 2t^*] \times \mathcal{B}_{3,\epsilon} \rightarrow \mathcal{B}_{3,\epsilon}$$

is continuous, where $\mathcal{B}_{3,\epsilon}$ is endowed with the metric topology of $\mathcal{V}_{1,\epsilon}$.

Then there exists a family of exponential attractors $\mathcal{E}_\epsilon$ on $\mathcal{B}_3 = \overline{\mathcal{B}}_{3,\epsilon}^{\mathcal{V}_{1,\epsilon}}$ with the following properties:

(i) $\mathcal{E}_\epsilon$ attracts $\mathcal{B}_3$ with an exponential rate which is uniform with respect to ϵ , that is,

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)\mathcal{B}_3, \mathcal{E}_\epsilon) \leq \mathcal{M}_1 e^{-\omega t}, \quad \forall t \geq 0,$$

for some $\mathcal{M}_1 > 0$ and some $\omega > 0$.

(ii) The fractal dimension of $\mathcal{E}_\epsilon$ is uniformly bounded with respect to ϵ , that is,

$$\dim_{\mathcal{V}_{1,\epsilon}}[\mathcal{E}_\epsilon] \leq \mathcal{M}_2,$$

for some $\mathcal{M}_2 > 0$.

(iii) There exist positive constants $\mathcal{M}_3, \varpi_1 \in (0, \frac{1}{2}]$ such that

$$\text{dist}_{\mathcal{V}_{1,0}}^{\text{sym}}(\mathcal{E}_\epsilon, (\mathcal{E})_0) \leq \mathcal{M}_3 \epsilon^{\varpi_1}, \quad (2.6.49)$$

for all $\epsilon \in (0, 1]$, where $\mathcal{E}$ is an exponential attractor for $S(t)$ on $\mathcal{B}_3 = \overline{\mathcal{B}}_3^{V_1}$ given that Conditions 2, 4 & 5 are satisfied in the metric topology of V_1 . Here ω, ϖ_1 and $\mathcal{M}_j$ are independent of ϵ , and they can be computed explicitly.

Remark 2.6.1 Estimate (2.6.49) is a consequence of the estimate

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}^{\text{sym}}(\mathcal{E}_\epsilon, (\mathcal{E})_0) \leq \mathcal{M}_3 \epsilon^{\varpi_1}$$

guaranteed by Lemma 2.6.3 and the fact that $\|z\|_{\mathcal{V}_{1,\epsilon}} \geq \|z\|_{\mathcal{V}_{1,0}}, \forall z \in \mathcal{V}_{1,\epsilon}$.

We prove now the main theorem of this chapter.

Theorem 2.6.1 *Let the assumptions of Theorem 2.4.1 hold. Then, for any $\alpha \geq 0$ and every $\epsilon \in (0, \epsilon_0]$, the semigroup $S_\epsilon(t)$ possesses an exponential attractor $\mathcal{E}_\epsilon^\alpha$ on $\mathcal{V}_{j,\epsilon}^\alpha$, that is,*

- (i) $\mathcal{E}_\epsilon^\alpha$ is compact and positively invariant, that is, $S_\epsilon(t)\mathcal{E}_\epsilon^\alpha \subset \mathcal{E}_\epsilon^\alpha, \forall t \geq 0$,
- (ii) the fractal dimension of $\mathcal{E}_\epsilon^\alpha$ is uniformly bounded with respect to ϵ ,
- (iii) there exists a constant $\mathfrak{z}_0 > 0$ such that, for every bounded subset $B \subset \mathcal{V}_{j,\epsilon}^\alpha$, there exists a constant $\mathfrak{z}_1(B) > 0$ such that $\mathfrak{z}_0$ and $\mathfrak{z}_1$ are independent of ϵ)

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{E}_\epsilon^\alpha) \leq \mathfrak{z}_1 e^{-\mathfrak{z}_0 t}, \quad \forall t \geq 0.$$

Moreover, there exist $0 < \varpi_2, \varpi_3 \leq \frac{1}{2}$ and $\mathcal{M}_5, \mathcal{M}_6 > 0$, all independent of ϵ , and a family of exponential attractors $\mathcal{E}_\epsilon^\alpha$ that satisfies the following property

$$\text{dist}_{\mathcal{V}_{2,\epsilon}}^{\text{sym}}(\mathcal{E}_\epsilon^\alpha, (\mathcal{E}^\alpha)_0) \leq \mathcal{M}_5 \epsilon^{\varpi_2}, \text{ and therefore } \text{dist}_{\mathcal{V}_{2,0}}^{\text{sym}}(\mathcal{E}_\epsilon^\alpha, (\mathcal{E}^\alpha)_0) \leq \mathcal{M}_6 \epsilon^{\varpi_3}$$

for all $\epsilon \in (0, \epsilon_0]$.

Proof. On account of Lemma 2.6.3, we check all the assumptions 1-5. This will prove the existence of exponential attractors $\mathcal{E}_\epsilon^\alpha$ on $\overline{\mathcal{B}_{3,\epsilon}}^{\mathcal{V}_{1,\epsilon}}$ satisfying the estimate

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)\overline{\mathcal{B}_{3,\epsilon}}^{\mathcal{V}_{1,\epsilon}}, \mathcal{E}_\epsilon^\alpha) \leq c(r_3)e^{-c_0 t}, \quad \forall t \geq 0, \quad (2.6.50)$$

where $c_0 > 0$ only depends of α , and $c(r_3) > 0$ depends on the radius of the absorbing set $\mathcal{B}_{3,\varepsilon}$ (where both c_0 and $c(r_3)$ are independent of ε). Then, we can extend the basin of attraction of each exponential attractor from $\overline{\mathcal{B}_{3,\varepsilon}}^{\mathcal{V}_{1,\varepsilon}}$ to the whole phase-space $\mathcal{V}_{j,\varepsilon}^\alpha$ by using Lemma 2.6.2, on account of Lemma 2.6.4 and Estimate (2.6.50).

Observe first that there exists $t^* > \frac{1}{\sigma}$ such that $S_\varepsilon(t)\mathcal{B}_{3,\varepsilon} \subset \mathcal{B}_{3,\varepsilon}$ for all $t \geq t^*$. Let $(\phi_0, u_0) \in \mathcal{B}_{3,\varepsilon}$. Recalling (2.5.8), to verify Assumption 1, we will prove that L_γ is continuous in the metric topology of V_1 . Indeed, we have

$$\begin{aligned} \|L_\gamma \phi_1 - L_\gamma \phi_2\| &\leq c(\|\phi_1 - \phi_2\|_2 + \|g(\phi_1) - g(\phi_2)\|) \\ &\leq c(\|\phi_1 - \phi_2\|_2 + \|\phi_1 - \phi_2\|) \\ &\leq c(\|\phi_1 - \phi_2\|_3^{\frac{1}{2}} + \|\phi_1 - \phi_2\|^{\frac{1}{2}})\|\phi_1 - \phi_2\|_1^{\frac{1}{2}} \\ &\leq c\|\phi_1 - \phi_2\|_1^{\frac{1}{2}}, \end{aligned}$$

owing to interpolation estimate $\|\phi\|_2 \leq \|\phi\|_1^{\frac{1}{2}} \|\phi\|_3^{\frac{1}{2}}$, the boundedness $\|\phi\|_3^{\frac{1}{2}} \leq c$ and $g(\phi_1) - g(\phi_2) = \int_0^s g'(s\phi_1 + (1-s)\phi_2)(\phi_1 - \phi_2) ds$. Hence, L_γ satisfies Assumption 1. Estimate (2.2.133) entails Assumption 2. As $t^* > \frac{1}{\sigma}$, Estimate (2.6.16) holds, so that Assumptions 3 follows.

Thus we are left to prove Assumptions 4 and 5. Indeed, defining

$$(\phi(t), u(t)) = S_\varepsilon(t)(\phi_{01}, u_{01}), \quad (\phi(t'), u(t')) = S_\varepsilon(t')(\phi_{02}, u_{02}),$$

with $(\phi_{01}, u_{01}), (\phi_{02}, u_{02}) \in \mathcal{B}_{3,\epsilon}$, and $t, t' \in [t^*, 2t^*]$, we obtain

$$\begin{aligned}
& \|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{V}_{1,\epsilon}} \\
& \leq \|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{01}, u_{01})\|_{\mathcal{V}_{1,\epsilon}} \\
& \quad + \|S_\epsilon(t')(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{V}_{1,\epsilon}}. \tag{2.6.51}
\end{aligned}$$

On the one hand, by utilizing (2.2.91) and (2.2.92), we have

$$\begin{aligned}
& \|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{01}, u_{01})\|_{\mathcal{V}_{1,\epsilon}} \\
& \leq c(\|\phi(t) - \phi(t')\|_1 + \sqrt{\epsilon}\|u(t) - u(t')\| + \|u(t) - u(t')\|_{-1}) \\
& \leq c(|\int_t^{t'} \|\phi_t(s)\|_1 ds| + \sqrt{\epsilon}|\int_t^{t'} \|u_t(s)\| ds| + |\int_t^{t'} \|u_t(s)\|_{-1} ds|) \\
& \leq c(t^*, \epsilon)|t' - t|^{\frac{1}{2}}. \tag{2.6.52}
\end{aligned}$$

On the other hand, it follows from (2.2.116) that

$$\begin{aligned}
& \|S_\epsilon(t')(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{V}_{1,\epsilon}} \\
& \leq c(t^*)\|(\phi_{01}, u_{01}) - (\phi_{02}, u_{02})\|_{\mathcal{V}_{1,\epsilon}}, \quad \forall t' > 0. \tag{2.6.53}
\end{aligned}$$

Hence, we conclude with

$$\begin{aligned}
& \|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{V}_{1,\epsilon}} \\
& \leq c(t^*, \epsilon)(|t' - t|^{\frac{1}{2}} + \|(\phi_{01}, u_{01}) - (\phi_{02}, u_{02})\|_{\mathcal{V}_{1,\epsilon}}). \tag{2.6.54}
\end{aligned}$$

Lastly, we notice that the semigroup $S(t)$ of the unperturbed problem (2.3.1) possesses an exponential attractor $\mathcal{E}^\alpha$ on $\mathcal{B}_3$ in the metric topology of V_1 (cf. also, e.g., [35], [31]) and the basin of attraction of $\mathcal{E}^\alpha$ can be extended to the whole space $\mathcal{V}_j^\alpha$ as well. This completes the proof of the theorem. ■

In Theorem 2.6.1, the basin of attraction of each exponential attractor was extended from $\overline{\mathcal{B}_{3,\epsilon}}^{\mathcal{V}_{1,\epsilon}}$ to the whole phase-space $\mathcal{V}_{j,\epsilon}$. To prove this result, we exploit Lemma 2.6.2 and the next Lemma.

Lemma 2.6.4 *Let $g \in \mathcal{C}^3$ and $\mathcal{B}_{3,\epsilon}$ is a bounded absorbing set in $\mathcal{V}_{3,\epsilon}$. For any bounded set $B \subset \mathcal{V}_{j,\epsilon}$ with $R = \sup_{\xi \in B} \|\xi\|_{\mathcal{V}_{j,\epsilon}}$, there exist $c > 0$ independent of R , and $K(R) > 0$ that depends on R (all independent of ϵ) such that*

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{B}_{3,\epsilon}) \leq K(R)e^{-ct}, \quad \forall t \geq 0.$$

Proof. Since $\mathcal{B}_{3,\epsilon}$ is absorbing set, then for any $(\phi_0, u_0) \in B$ bounded set in $\mathcal{V}_{j,\epsilon}$, there exists $t_1(B)$ such that

$$S_\epsilon(t)B \subset \mathcal{B}_{3,\epsilon}, \quad \forall t \geq t_1(B),$$

hence

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{B}_{3,\epsilon}) = 0, \quad \forall t \geq t_1(B). \quad (2.6.55)$$

Now, we need to show it for $0 \leq t \leq t_1(B)$:

On account of (2.2.38) and (2.2.70), we obtained

$$\begin{aligned} \|(\phi(t), u(t))\|_{\mathcal{V}_{j,\epsilon}} &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{V}_{j,\epsilon}})e^{-ct} + c, \\ &\leq C(R), \quad \forall t \geq 0, \end{aligned}$$

where $(\phi(t), u(t)) = S_\epsilon(t)(\phi_0, u_0)$. Since $\mathcal{B}_{3,\epsilon}$ is a bounded set in the metric topology of $\mathcal{V}_{1,\epsilon}$, then for any $(\psi, \varphi) \in \mathcal{B}_{3,\epsilon}$ we have $\|(\psi, \varphi)\|_{\mathcal{V}_{1,\epsilon}} \leq r_3$.

Hence, the difference is also bounded in the metric topology of $\mathcal{V}_{1,\epsilon}$. Indeed,

$$\begin{aligned} \|(\phi, u) - (\psi, \varphi)\|_{\mathcal{V}_{1,\epsilon}} &\leq \|(\phi, u)\|_{\mathcal{V}_{1,\epsilon}} + \|(\psi, \varphi)\|_{\mathcal{V}_{1,\epsilon}} \\ &\leq C(R), \quad \forall t \geq 0, \end{aligned} \tag{2.6.56}$$

therefore, we can deduce that

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{B}_{3,\epsilon}) = \sup_{(\phi, u) \in S_\epsilon(t)B} \inf_{(\psi, \varphi) \in \mathcal{B}_{3,\epsilon}} \|(\phi, u) - (\psi, \varphi)\|_{\mathcal{W}_{1,\epsilon}} \leq C(R), \quad \forall t \geq 0,$$

so that

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{B}_{3,\epsilon}) \leq C(R)e^{-ct_1(B)}, \quad \forall t \geq 0, \quad c > 0,$$

then when we take $0 \leq t \leq t_1(B)$, we have

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{B}_{3,\epsilon}) \leq C(R)e^{-ct}, \quad \forall t \in [0, t_1(B)], \quad \forall c > 0. \tag{2.6.57}$$

Thus, on account of (2.6.55) and (2.6.57) we obtain

$$\text{dist}_{\mathcal{V}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{B}_{3,\epsilon}) \leq C(R)e^{-ct}, \quad \forall t \geq 0.$$

■

CHAPTER 3

INERTIAL MANIFOLD FOR CONSERVED VISCOUS PHASE-FIELD SYSTEM

We considered the conserved viscous phase-field system

$$\begin{cases} \phi_t + N(\phi_t + N\phi + g(\phi) - u) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t + Nu = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \phi|_{t=0} = \phi_0, \ u|_{t=0} = u_0, \end{cases} \quad (3.0.1)$$

where $\epsilon \in (0, 1]$, the operator N is same as in the previous chapter and the nonlinear function $g(\phi)$ satisfies the Conditions (2.0.2)-(2.0.5).

For every $r \geq 0$, we define the Hilbert space

$$\mathcal{Z}_{r,\epsilon} = V_r \times V_{r-1}$$

normed by

$$\|(u_1, u_2)\|_{\mathcal{Z}_{r,\epsilon}} = (2\|u_1\|_{r-1}^2 + \|N^{\frac{r}{2}}u_1\|^2 + 2\epsilon(N^{\frac{r-1}{2}}\bar{u}_1, N^{\frac{r-1}{2}}\bar{u}_2) + \epsilon^2\|u_2\|_{r-1}^2)^{\frac{1}{2}},$$

with the scalar product

$$\begin{aligned} (U, V)_{\mathcal{Z}_{r,\epsilon}} &= 2(N^{\frac{r-1}{2}}\bar{u}_1, N^{\frac{r-1}{2}}\bar{v}_1) + 2|\Omega|m(u_1)m(v_1) + (N^{\frac{r}{2}}u_1, N^{\frac{r}{2}}v_1) \\ &\quad + \epsilon(N^{\frac{r-1}{2}}\bar{u}_1, N^{\frac{r-1}{2}}\bar{v}_2) + \epsilon(N^{\frac{r-1}{2}}\bar{v}_1, N^{\frac{r-1}{2}}\bar{u}_2) \\ &\quad + \epsilon^2[(N^{\frac{r-1}{2}}\bar{u}_2, N^{\frac{r-1}{2}}\bar{v}_2) + |\Omega|m(u_2)m(v_2)], \end{aligned} \quad (3.0.2)$$

$$\forall U = (u_1, u_2), V = (v_1, v_2) \in \mathcal{Z}_{r,\epsilon}.$$

We set

$$\mathcal{X}_{r,\epsilon}^\alpha = \{U \in \mathcal{Z}_{r,\epsilon}, |m(U)| \leq \alpha\}, \quad |U| = (|u_1|^2 + |u_2|^2)^{\frac{1}{2}},$$

$$V_r^\alpha = \{u \in V_r, |m(u)| \leq \alpha\}.$$

For every (ϕ_0, u_0) in the phase space $\mathcal{X}_{j,\epsilon}^\alpha$, the problem (3.0.1) has a solution in $\mathcal{X}_{j,\epsilon}^\alpha$:

We multiply (3.0.1)₁ and (3.0.1)₂ by ϕ and ϵu , respectively, integrate over Ω , then sum the resulting equations, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\phi\|^2 + \|\bar{\phi}\|_1^2 + \epsilon^2\|u\|^2) + \|\bar{\phi}\|_2^2 + \epsilon\|\bar{u}\|_1^2 + (\epsilon u, \phi_t) - (Nu, \phi) \leq C_4\|\bar{\phi}\|_1^2. \quad (3.0.3)$$

On account of (3.0.1)₂, we have

$$\begin{aligned}
(\epsilon u, \phi_t) &= \frac{d}{dt}(\epsilon u, \phi) - (\epsilon u_t, \phi) \\
&= \frac{d}{dt}(\epsilon u, \phi) + (\phi_t, \phi) + (Nu, \phi) \\
&= \frac{d}{dt}((\epsilon u, \phi) + \|\phi\|^2) + (Nu, \phi),
\end{aligned} \tag{3.0.4}$$

hence

$$\frac{1}{2} \frac{d}{dt} (\|\bar{\phi}\|_1^2 + \epsilon^2 \|u\|^2 + 2\|\phi\|^2 + (\epsilon u, \phi)) + \|\bar{\phi}\|_2^2 + \epsilon \|\bar{u}\|_1^2 \leq C_4 \|\bar{\phi}\|_1^2. \tag{3.0.5}$$

We multiply (3.0.1)₁ and (3.0.1)₂ by $N^{-1}\bar{\phi}$ and $\epsilon N^{-1}\bar{u}$, respectively, integrate over Ω , then summing the resulting equations, and by proceeding as we did in (3.0.4), we obtain

$$\frac{1}{2} \frac{d}{dt} (2\|\bar{\phi}\|_{-1}^2 + \|\bar{\phi}\|^2 + \epsilon^2 \|\bar{u}\|_{-1}^2 + (\epsilon N^{-1}\bar{u}, \bar{\phi})) + \|\bar{\phi}\|_1^2 + C_2 \int_{\Omega} G(\phi) dx + \epsilon \|\bar{u}\|^2 \leq c. \tag{3.0.6}$$

Summing (3.0.6) and $\vartheta_1(3.0.5)$, with $\vartheta_1 > 0$ small enough, we deduce

$$\frac{d}{dt} H(t) + cH(t) + c(\|\phi\|_2^2 + \epsilon \|u\|_1^2) + C_2 \int_{\Omega} G(\phi) dx \leq c, \tag{3.0.7}$$

where

$$H(t) = \vartheta_1(\|\bar{\phi}\|_1^2 + \epsilon^2 \|u\|^2 + 2\|\phi\|^2 + (\epsilon u, \phi)) + 2\|\bar{\phi}\|_{-1}^2 + \|\bar{\phi}\|^2 + \epsilon^2 \|\bar{u}\|_{-1}^2 + (N^{-1}\epsilon \bar{u}, \bar{\phi}).$$

Apply Gronwall's Lemma to (3.0.7), then integrate it from t to $t + r$, $r > 0$, we find

$$\|(\phi, u)\|_{\mathcal{Z}_{1,\epsilon}}^2 + \int_t^{t+r} \left(\|\phi\|_2^2 + \epsilon \|u\|_1^2 + \int_{\Omega} G(\phi) dx \right) ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{Z}_{j,\epsilon}}^2) e^{-ct} + c. \quad (3.0.8)$$

Next, multiply (3.0.1)₁ by $N^{-1}\phi_t$ and (3.0.1)₂ by u , we get

$$\frac{1}{2} \frac{d}{dt} \left(\|\bar{\phi}\|_1^2 + \epsilon \|u\|^2 + 2 \int_{\Omega} G(\phi) dx \right) + \|\phi_t\|^2 + \|\phi_t\|_{-1}^2 + 2\|\bar{u}\|_1^2 = 0,$$

by using Uniform Gronwall's Lemma, on account of (3.0.8), then integrating it from t to $t + r$, we find

$$\begin{aligned} \epsilon \|u(t)\|^2 + \int_{\Omega} G(\phi) dx + c \int_t^{t+1} (\|\phi_t\|^2 + \|u\|_1^2) ds \\ \leq Q(\|(\phi_0, u_0)\|_{\mathcal{Z}_{j,\epsilon}}^2) e^{-ct} + c, \quad \forall t \geq r. \end{aligned} \quad (3.0.9)$$

We differentiate (3.0.1)₁ with respect to t to obtain

$$\phi_{tt} + N(\phi_{tt} + N\phi_t + g'(\phi)\phi_t - u_t) = 0. \quad (3.0.10)$$

We multiply (3.0.10) and (3.0.1)₂ by $N^{-1}\phi_t$ and u_t , respectively, integrate over Ω , summing the resulting equations, then apply Uniform Gronwall's Lemma, on account of (3.0.9), we find

$$\|\phi_t(t)\|_{-1}^2 + \|\phi_t(t)\|^2 + \|u(t)\|_1^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{Z}_{j,\epsilon}}^2) e^{-ct} + c, \quad \forall t \geq 2r.$$

From (3.0.1)₁, we have

$$\begin{aligned}\|\phi\|_2 &\leq c(\|\phi_t\|^2 + \|u\|^2 + \|\phi\|_1^2) \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{Z}_{j,\epsilon}}^2)e^{-ct} + c, \quad \forall t \geq 2r,\end{aligned}\tag{3.0.11}$$

so that

$$\|(\phi(t), u(t))\|_{\mathcal{Z}_{2,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{Z}_{j,\epsilon}}^2)e^{-ct} + c, \quad \forall t \geq 2r.$$

Next, we multiply (3.0.1)₁ and (3.0.1)₂ by $N\phi$ and ϵNu , respectively, integrate over Ω , then sum the resulting equations, and by proceed similar to (3.0.4), we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\phi\|_2^2 + \epsilon^2 \|u\|_1^2 + 2\|\phi\|_1^2 + (\epsilon Nu, \phi)) + \frac{1}{2} \|\bar{\phi}\|_3^2 + \epsilon \|\bar{u}\|_2^2 \leq Q(\|\phi\|_j^2) \|\phi\|_1^2, \tag{3.0.12}$$

hence

$$\frac{d}{dt} V(t) \leq Q(V(t)) \quad \text{where} \quad V(t) = \|\phi\|_2^2 + \epsilon^2 \|u\|^2 + (\epsilon Nu, \phi).$$

Next, we proceed as we did in Chapter 2 (section 2.2.2), by using comparison principle, we can obtain

$$\|(\phi(t), u(t))\|_{\mathcal{Z}_{2,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{Z}_{2,\epsilon}}^2)e^{-ct} + c, \quad \forall t \geq 0.$$

Thus, we can deduce that there exists absorbing sets $C_{j,\epsilon} \subset \mathcal{Z}_{j,\epsilon}^\alpha$ of the form

$$C_{j,\epsilon} = \{(\varphi, \psi) \in \mathcal{Z}_{j,\epsilon}^\alpha, \|\varphi, \psi\|_{\mathcal{Z}_{j,\epsilon}} \leq r_j\}. \tag{3.0.13}$$

Now, we will report the following results that were proved in [6, Theorem 2.1].

We consider the problem

$$U_t + A_\epsilon U + G_\epsilon(U) = 0, \quad (3.0.14)$$

$$U|_{t=0} = U_0,$$

in $\mathcal{Z}_{j,\epsilon}^\alpha$, for every $\epsilon \in (0, 1]$.

Assumption (H1). The operator $A_\epsilon : \mathcal{D}(A_\epsilon) \rightarrow \mathcal{Z}_{j,\epsilon}^\alpha$ is linear and satisfies the following conditions

- A_ϵ is maximal accretive, that is, $(A_\epsilon U, U) \geq 0$, $\forall u \in \mathcal{D}(A_\epsilon)$, and the range of $I + A_\epsilon$ is equal to $\mathcal{Z}_{j,\epsilon}^\alpha$,
- A_ϵ is closed and $\mathcal{D}(A_\epsilon)$ is dense in $\mathcal{Z}_{j,\epsilon}^\alpha$.

There is an orthogonal projection $\mathcal{P}$ of finite rank n such that $\mathcal{Z}_{j,\epsilon}^\alpha = \mathcal{P}\mathcal{Z}_{j,\epsilon}^\alpha \oplus \mathcal{Q}\mathcal{Z}_{j,\epsilon}^\alpha$ where $\mathcal{Q} = I - \mathcal{P}$. There is an increasing family $\{\gamma_{\epsilon,k}\}_{k \in \mathbb{N}}$ of eigenvalues of A_ϵ .

Assumption (H2). The nonlinear function $G_\epsilon : \mathcal{Z}_{j,\epsilon}^\alpha \rightarrow \mathcal{Z}_{j,\epsilon}^\alpha$ satisfies the following conditions, for all U and V in $\mathcal{Z}_{j,\epsilon}^\alpha$

$$\|G_\epsilon(U)\|_{\mathcal{Z}_{j,\epsilon}} \leq M_0 \quad (3.0.15)$$

$$\|G_\epsilon(U) - G_\epsilon(V)\|_{\mathcal{Z}_{j,\epsilon}} \leq M_1 \|U - V\|_{\mathcal{Z}_{j,\epsilon}} \quad (3.0.16)$$

where $M_0, M_1 > 0$ all independent of ϵ .

Assumption (H3). There hold

$$\gamma_{\epsilon, n+1} - \gamma_{\epsilon, n} > 6M_1, \quad (3.0.17)$$

$$\gamma_{\epsilon, n} > 3M_1, \quad (3.0.18)$$

where M_1 is the same as in Assumption (H2).

Assumption (H4). Problem (3.0.14) generates a continuous semigroup

$$\tilde{S}_\epsilon(t) : \mathcal{Z}_{j,\epsilon}^\alpha \rightarrow \mathcal{Z}_{j,\epsilon}^\alpha, \quad U_0 \mapsto U(t), \quad t \geq 0,$$

where $U(t)$ is the solution to (3.0.14) at time $t \geq 0$.

We now state the following result.

Theorem 3.0.1 [6, c.f. Theorem 2.1] *Let Assumptions from (H1) through (H4) hold.*

Then, there exists a positive integer n independent of ϵ such that, for every $\epsilon \in (0, 1]$,

Equation (3.0.14) has an inertial manifold $\mathcal{M}_\epsilon$ in $\mathcal{Z}_{j,\epsilon}^\alpha$, that is,

- $\mathcal{M}_\epsilon$ is a finite-dimensional Lipschitz manifold in $\mathcal{Z}_{j,\epsilon}^\alpha$ and $\dim \mathcal{M}_\epsilon = n$,
- $\mathcal{M}_\epsilon$ is invariant under $\tilde{S}_\epsilon(t)$, that is, $\tilde{S}_\epsilon(t)\mathcal{M}_\epsilon = \mathcal{M}_\epsilon$, $\forall t \geq 0$,
- $\mathcal{M}_\epsilon$ is exponentially attracting, that is, there exists a constant k such that, for every $U_0 \in \mathcal{Z}_{j,\epsilon}^\alpha$, there exists a constant $C = C(\|U_0\|_{\mathcal{Z}_{j,\epsilon}}) > 0$ such that (k and C are independent of ϵ)

$$\text{dist}_{\mathcal{Z}_{j,\epsilon}}(\tilde{S}_\epsilon(t)U_0, \mathcal{M}_\epsilon) \leq Ce^{-kt}, \quad \forall t \geq 0.$$

Now, in order to apply Theorem 3.0.1 for Problem (3.0.1), we will check that the Assumptions **(H1)**, **(H2)**, **(H3)** and **(H4)** are satisfied.

First, we need to introduce the "prepared problem":

$$\begin{cases} \phi_t + N(\phi_t + N\phi + \mathbf{g}(\phi) - u) = 0, \\ \epsilon u_t + \phi_t + Nu = 0, \end{cases} \quad (3.0.19)$$

where

$$\mathbf{g}(\phi) = \theta \left(\frac{\|\phi\|_j}{r_j} \right) g(\phi),$$

r_j is the radius of the absorbing set $C_{j,\epsilon} \subset \mathcal{X}_{j,\epsilon}^\alpha$ for a semigroup $S_\epsilon(t)|_{\mathcal{X}_{j,\epsilon}^\alpha}$, and $\theta : \mathbb{R}^+ \rightarrow [0, 1]$ is a $\mathcal{C}^\infty$ function such that $\theta(s)$ is equal to 1 when $0 \leq s \leq 1$, and is equal to 0 when $s > 2$, and $|\theta'(s)| \leq 2$, $\forall s \geq 0$.

We note that $\mathbf{g} \in \mathcal{C}^1(V_j^\alpha, V_j^\alpha)$ and its derivatives are globally lipschitz on V_j^α ; and there exists a constant $\mathbf{c}_0 > 0$ such that

$$\|\mathbf{g}(\phi)\|_j \leq \mathbf{c}_0, \quad \forall \phi \in V_j^\alpha, \quad (3.0.20)$$

$$\|\mathbf{g}(\phi_1) - \mathbf{g}(\phi_2)\|_j \leq \mathbf{c}_0 \|\phi_1 - \phi_2\|_j, \quad \forall \phi_1, \phi_2 \in V_j^\alpha, \quad (3.0.21)$$

$$\|\mathbf{g}'(\phi)v\|_j \leq \mathbf{c}_0 \|v\|_j, \quad \forall \phi, v \in V_j^\alpha, \quad (3.0.22)$$

see [41]. Now, we rewrite system (3.0.19) in the following form:

$$U_t + A_\epsilon U + G_\epsilon(U) = 0, \quad (3.0.23)$$

where $U = (\phi, u)$. Let $\wedge$ be the operator

$$\wedge = (I + N)^{-1}$$

so that

$$G_\epsilon(U) = (\wedge N \mathfrak{g}(\phi), -\frac{1}{\epsilon} \wedge N \mathfrak{g}(\phi))$$

and

$$A_\epsilon = \begin{pmatrix} \wedge N^2 & -\wedge N \\ -\frac{1}{\epsilon} \wedge N^2 & \frac{1}{\epsilon} N(\wedge + I) \end{pmatrix}. \quad (3.0.24)$$

The operator

$$A_\epsilon : \mathcal{D}(A_\epsilon) = \mathcal{Z}_{j+2,\epsilon} = V_{j+2} \times V_{j+1} \rightarrow \mathcal{Z}_{j,\epsilon},$$

satisfies the monotonicity property:

$$\begin{aligned} (A_\epsilon U, U)_{\mathcal{Z}_{j,\epsilon}} &= \|\wedge^{\frac{1}{2}} N^{\frac{j+1}{2}} u_1\|^2 + \|\wedge^{\frac{1}{2}} N^{\frac{j+2}{2}} u_1\|^2 + \epsilon \|N^{\frac{j}{2}} u_2\|^2 \\ &\quad - (\wedge N^{\frac{j}{2}} \bar{u}_2, N^{\frac{j}{2}} \bar{u}_1) - (\wedge N^{\frac{j+2}{2}} \bar{u}_2, N^{\frac{j}{2}} \bar{u}_1) + (N^{\frac{j}{2}} u_1, N^{\frac{j}{2}} u_2), \\ &= \|\wedge^{\frac{1}{2}} N^{\frac{j+1}{2}} u_1\|^2 + \|\wedge^{\frac{1}{2}} N^{\frac{j+2}{2}} u_1\|^2 + \epsilon \|N^{\frac{j}{2}} u_2\|^2, \\ &\geq 0, \end{aligned} \quad (3.0.25)$$

for $U = (u_1, u_2) \in \mathcal{D}(A_\epsilon)$; this result is owing to the following

$$\begin{aligned}
& -(\wedge N^{\frac{j}{2}} \bar{u}_2, N^{\frac{j}{2}} \bar{u}_1) - (\wedge N^{\frac{j+2}{2}} \bar{u}_2, N^{\frac{j}{2}} \bar{u}_1) + (N^{\frac{j}{2}} u_1, N^{\frac{j}{2}} u_2) \\
& = -([\wedge N^{\frac{j}{2}} + \wedge N^{\frac{j+2}{2}}] \bar{u}_2, N^{\frac{j}{2}} \bar{u}_1) + (N^{\frac{j}{2}} u_1, N^{\frac{j}{2}} u_2) \\
& = -([N^{\frac{j}{2}} \wedge (I + N)] \bar{u}_2, N^{\frac{j}{2}} \bar{u}_1) + (N^{\frac{j}{2}} u_1, N^{\frac{j}{2}} u_2), \\
& = -(N^{\frac{j}{2}} \bar{u}_2, N^{\frac{j}{2}} \bar{u}_1) + (N^{\frac{j}{2}} u_1, N^{\frac{j}{2}} u_2), \quad (\text{due to } \wedge (I + N) = I) \\
& = 0.
\end{aligned}$$

The operator A_ϵ has a positive eigenvalues

$$\mu_k^\pm = \frac{\lambda_k}{2\epsilon(1 + \lambda_k)} (2 + (\epsilon + 1)\lambda_k \pm \sqrt{[(\epsilon + 1)\lambda_k + 2]^2 - 4\epsilon(1 + \lambda_k)\lambda_k}),$$

for $k = 0, 1, 2, \dots$ and corresponding eigenfunctions

$$U_k^\pm = (e_k, -\tilde{\mu}_k^\pm e_k),$$

where

$$\tilde{\mu}_k^\pm = \frac{1}{2\epsilon} \left(2 + (1 - \epsilon)\lambda_k \pm \sqrt{(2 + (1 + \epsilon)\lambda_k)^2 - 4\epsilon\lambda_k(1 + \lambda_k)} \right),$$

$\{\lambda_k\}$ are eigenvalues of N ordered in an increasing sequence and $\{e_k\}$ are the corresponding eigenfunctions which form an orthogonal basis $L^2(\Omega)$.

From (3.0.23), by integrating over Ω , we get

$$\frac{d}{dt}m(U) = 0,$$

so that

$$m(U(t)) = m(U_0), \quad \forall t \in \mathbb{R}.$$

We notice that

$$\begin{aligned} \|U(t)\|_{\mathcal{Z}_{j,\epsilon}} &= (2\|N^{\frac{j-1}{2}}\bar{\phi}\|^2 + \|N^{\frac{j}{2}}\phi\|^2 + 2\epsilon(N^{\frac{j-1}{2}}\bar{\phi}, N^{\frac{j-1}{2}}\bar{u}) + \epsilon^2\|N^{\frac{j-1}{2}}\bar{u}\|^2 \\ &\quad + 2|\Omega|m(\phi)^2 + \epsilon^2|\Omega|m(u)^2)^{\frac{1}{2}}, \end{aligned} \quad (3.0.26)$$

$$\|\bar{U}(t)\|_{\mathcal{Z}_{j,\epsilon}} = (2\|N^{\frac{j-1}{2}}\bar{\phi}\|^2 + \|N^{\frac{j}{2}}\phi\|^2 + 2\epsilon(N^{\frac{j-1}{2}}\bar{\phi}, N^{\frac{j-1}{2}}\bar{u}) + \epsilon^2\|N^{\frac{j-1}{2}}\bar{u}\|^2)^{\frac{1}{2}}, \quad (3.0.27)$$

so that

$$\begin{aligned} \|U(t)\|_{\mathcal{Z}_{j,\epsilon}} &= (\|\bar{U}(t)\|_{\mathcal{Z}_{j,\epsilon}}^2 + 2|\Omega|m(\phi)^2 + \epsilon^2|\Omega|m(u)^2)^{\frac{1}{2}} \\ &\leq \left(\|\bar{U}(t)\|_{\mathcal{Z}_{j,\epsilon}}^2 + 2|\Omega|(m(\phi)^2 + m(u)^2) \right)^{\frac{1}{2}} \\ &\leq \left(\|\bar{U}(t)\|_{\mathcal{Z}_{j,\epsilon}}^2 + 2|\Omega||m(U)|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (3.0.28)$$

Now, if $|m(U_0)| \leq \alpha$ then

$$\begin{aligned} |m(U)| &= |m(U_0)| \\ &\leq \alpha, \quad \forall t \geq 0. \end{aligned} \quad (3.0.29)$$

Proposition 3.0.1 *For every $\epsilon \in (0, 1]$, if $U_0 \in \mathcal{Z}_{j,\epsilon}^\alpha$, then Problem (3.0.23) possesses a unique solution $U(t)$ such that $U(t) \in \mathcal{C}([0, T]; \mathcal{Z}_{j,\epsilon}^\alpha)$, for any $T > 0$.*

Proof. First, we notice that

$$\begin{aligned}
\|G(U)\|_{\mathcal{Z}_{j,\epsilon}}^2 &= \|N^{\frac{j-1}{2}}(I+N)^{-1}N\mathfrak{g}(\phi)\|^2 + \|N^{\frac{j}{2}}(I+N)^{-1}N\mathfrak{g}(\phi)\|^2 \\
&\leq \|\mathfrak{g}(\phi)\|_{j-1}^2 + \|\mathfrak{g}(\phi)\|_j^2 \\
&\leq c\|\mathfrak{g}(\phi)\|_j^2 \\
&\leq \mathfrak{c}, \quad \forall U = (\phi, u) \in \mathcal{Z}_{j,\epsilon}^\alpha,
\end{aligned} \tag{3.0.30}$$

due to (3.0.20). Moreover, on account of Poincaré inequality and the Estimate (1.4.1), there holds

$$\|\wedge^{\frac{1}{2}} N^{\frac{\sigma+2}{2}} \varphi\| \geq \frac{\lambda_1}{(1+\lambda_1)^{\frac{1}{2}}} \|N^{\frac{\sigma}{2}} \varphi\|, \quad \forall \sigma \in \mathbb{R}, \quad \forall \varphi \in \dot{V}_\sigma. \tag{3.0.31}$$

Owing to (3.0.25), (3.0.31) and $\epsilon \geq \epsilon^2$, we have

$$\begin{aligned}
(AU, U)_{\mathcal{Z}_{j,\epsilon}} &= \|\wedge^{\frac{1}{2}} N^{\frac{j+1}{2}} \bar{u}_1\|^2 + \|\wedge^{\frac{1}{2}} N^{\frac{j+2}{2}} \bar{u}_1\|^2 + \epsilon \|N^{\frac{j}{2}} \bar{u}_2\|^2, \\
&\geq \frac{\lambda_1^2}{1+\lambda_1} \|N^{\frac{j-1}{2}} \bar{u}_1\|^2 + \frac{\lambda_1^2}{1+\lambda_1} \|N^{\frac{j}{2}} \bar{u}_1\|^2 + \epsilon^2 \lambda_1 \|N^{\frac{j-1}{2}} \bar{u}_2\|^2 \\
&\quad + \frac{\epsilon \lambda_1^2}{4(1+\lambda_1)} (N^{\frac{j-1}{2}} \bar{u}_1, N^{\frac{j-1}{2}} \bar{u}_2) - \frac{\epsilon \lambda_1^2}{4(1+\lambda_1)} (N^{\frac{j-1}{2}} \bar{u}_1, N^{\frac{j-1}{2}} \bar{u}_2),
\end{aligned} \tag{3.0.32}$$

and by using Young's inequality, we notice that

$$-\frac{\epsilon \lambda_1^2}{4(1+\lambda_1)} (N^{\frac{j-1}{2}} \bar{u}_1, N^{\frac{j-1}{2}} \bar{u}_2) \geq \frac{\lambda_1^2}{4(1+\lambda_1)} \left(-\|N^{\frac{j-1}{2}} \bar{u}_1\|^2 - \frac{\epsilon^2}{4} \|N^{\frac{j-1}{2}} \bar{u}_2\|^2 \right),$$

it follows from (3.0.32) that

$$\begin{aligned}
& (AU, U)_{\mathcal{Z}_{j,\epsilon}} \\
& \geq \frac{3\lambda_1^2}{8(1+\lambda_1)} 2\|N^{\frac{i-1}{2}}\bar{u}_1\|^2 + \frac{\lambda_1^2}{1+\lambda_1}\|N^{\frac{i}{2}}\bar{u}_1\|^2 + \frac{16\lambda_1+15\lambda_1^2}{16(1+\lambda_1)}\epsilon^2\|N^{\frac{i-1}{2}}\bar{u}_2\|^2 \\
& \quad + \frac{\lambda_1^2}{8(1+\lambda_1)}2\epsilon(N^{\frac{i-1}{2}}\bar{u}_1, N^{\frac{i-1}{2}}\bar{u}_2) \\
& \geq \frac{\lambda_1^2}{8(1+\lambda_1)} \left(2\|N^{\frac{i-1}{2}}\bar{u}_1\|^2 + \|N^{\frac{i}{2}}\bar{u}_1\|^2 + \epsilon^2\|N^{\frac{i-1}{2}}\bar{u}_2\|^2 + 2\epsilon(N^{\frac{i-1}{2}}\bar{u}_1, N^{\frac{i-1}{2}}\bar{u}_2) \right) \\
& \geq \frac{\lambda_1^2}{8(1+\lambda_1)} \|\bar{U}\|_{\mathcal{Z}_{j,\epsilon}}^2, \tag{3.0.33}
\end{aligned}$$

due to (3.0.27).

We take the $(\cdot, \cdot)_{\mathcal{Z}_{j,\epsilon}}$ - scalar product of (3.0.23) with U to obtain

$$\frac{1}{2} \frac{d}{dt} \|U\|_{\mathcal{Z}_{j,\epsilon}}^2 + (A_\epsilon U, U)_{\mathcal{Z}_{j,\epsilon}} + (G(U), U)_{\mathcal{Z}_{j,\epsilon}} = 0, \tag{3.0.34}$$

by utilizing (3.0.33), we get

$$\frac{1}{2} \frac{d}{dt} \|U\|_{\mathcal{Z}_{j,\epsilon}}^2 + \frac{\lambda_1^2}{8(1+\lambda_1)} \|\bar{U}\|_{\mathcal{Z}_{j,\epsilon}}^2 \leq 2\|G(U)\|_{\mathcal{Z}_{j,\epsilon}} \|U\|_{\mathcal{Z}_{j,\epsilon}}, \tag{3.0.35}$$

once $(\frac{\lambda_1^2}{8(1+\lambda_1)} 2|\Omega| |m(U)|^2)$ is added to each side, we can use the relation (3.0.28) to obtain

$$\frac{1}{2} \frac{d}{dt} \|U\|_{\mathcal{Z}_{j,\epsilon}}^2 + \frac{\lambda_1^2}{8(1+\lambda_1)} \|U\|_{\mathcal{Z}_{j,\epsilon}}^2 \leq 2\|G(U)\|_{\mathcal{Z}_{j,\epsilon}} \|U\|_{\mathcal{Z}_{j,\epsilon}} + c'|m(U)|^2. \tag{3.0.36}$$

Hence, by virtue (3.0.29), (3.0.30) and using Young's inequality, we get

$$\begin{aligned} \frac{d}{dt} \|U\|_{\mathcal{Z}_{j,\epsilon}}^2 + c_1 \|U\|_{\mathcal{Z}_{j,\epsilon}}^2 &\leq \frac{1}{c_1} \|G(U)\|_{\mathcal{Z}_{j,\epsilon}}^2 + c' \alpha^2 \\ &\leq \mathfrak{c}', \quad c_1 = \frac{\lambda_1^2}{16(1 + \lambda_1)}. \end{aligned} \quad (3.0.37)$$

Integrating the equation from 0 to t , we deduce that

$$\|U(t)\|_{\mathcal{Z}_{j,\epsilon}}^2 \leq \|U_0\|_{\mathcal{Z}_{j,\epsilon}}^2 e^{-c_1 t} + \mathfrak{c}, \quad \forall t \geq 0. \quad (3.0.38)$$

We also notice that

$$\begin{aligned} &\|G(U_1) - G(U_2)\|_{\mathcal{Z}_{j,\epsilon}}^2 \\ &= \|N^{\frac{j-1}{2}}(I + N)^{-1} N[\mathfrak{g}(\phi_1) - \mathfrak{g}(\phi_2)]\|^2 + \|N^{\frac{j}{2}}(I + N)^{-1} N[\mathfrak{g}(\phi_1) - \mathfrak{g}(\phi_2)]\|^2 \\ &\leq \|\mathfrak{g}(\phi_1) - \mathfrak{g}(\phi_2)\|_{j-1}^2 + \|\mathfrak{g}(\phi_1) - \mathfrak{g}(\phi_2)\|_j^2 \\ &\leq \mathfrak{c} \|\phi_1 - \phi_2\|_j^2 \\ &\leq \mathfrak{c} \|U_1 - U_2\|_{\mathcal{Z}_{j,\epsilon}}, \quad \forall U_1 = (\phi_1, u_1), U_2 = (\phi_2, u_2) \in \mathcal{Z}_{j,\epsilon}, \end{aligned} \quad (3.0.39)$$

due to (3.0.21).

To show the uniqueness of the solution, we take the $(\cdot, \cdot)_{\mathcal{Z}_{j,\epsilon}}$ - scalar product of the difference $W = U_1 - U_2$ of two solution U_1 and U_2 of propblem (3.0.23) with W , we obtain

$$\frac{1}{2} \frac{d}{dt} \|W\|_{\mathcal{Z}_{j,\epsilon}}^2 + (A_\epsilon W, W)_{\mathcal{Z}_{j,\epsilon}} + (G(U_1) - G(U_2), W)_{\mathcal{Z}_{j,\epsilon}} = 0, \quad (3.0.40)$$

hence

$$\begin{aligned} \frac{d}{dt} \|W\|_{\mathcal{Z}_{j,\epsilon}}^2 &\leq 2\|G(U_1) - G(U_2)\|_{\mathcal{Z}_{j,\epsilon}} \|W\|_{\mathcal{Z}_{j,\epsilon}} \\ &\leq c\|W\|_{\mathcal{Z}_{j,\epsilon}}^2. \end{aligned} \quad (3.0.41)$$

Integrating this equation we deduce that

$$\|W(t)\|_{\mathcal{Z}_{j,\epsilon}}^2 \leq \|W_0\|_{\mathcal{Z}_{j,\epsilon}}^2 e^{ct}, \quad \forall t \in [0, T]. \quad (3.0.42)$$

This finishes the proof of the proposition. ■

Thanks to Proposition 3.0.1, we can define the continuous semigroup

$$\tilde{S}_\epsilon(t) : \mathcal{X}_{j,\epsilon}^\alpha \rightarrow \mathcal{X}_{j,\epsilon}^\alpha, \quad U_0 \mapsto U(t), \quad t \geq 0,$$

where $U(t)$ is the solution to Problem (3.0.23) at time t .

Moreover, from (3.0.38) we deduced the existence of an absorbing set for $\tilde{S}_\epsilon(t)$ on $\mathcal{X}_{j,\epsilon}^\alpha$ of the form

$$\mathfrak{B}_{j,\epsilon} = \{U \in \mathcal{X}_{j,\epsilon}^\alpha, \|U(t)\|_{\mathcal{Z}_{j,\epsilon}} \leq r_j\}. \quad (3.0.43)$$

Now, we will show that Assumptions **(H1)**, **(H2)**, **(H3)** and **(H4)** are all satisfied.

Assumption (H1):

We will start with the following proposition.

Proposition 3.0.2 *For every $\epsilon \in (0, 1]$, the operator A_ϵ is defined on $\mathcal{Z}_{j,\epsilon}$, and*

satisfies the following conditions

- A_ϵ is a maximal accretive operator, that is, $(A_\epsilon U, U)_{\mathcal{Z}_{j,\epsilon}} \geq 0$, $\forall U \in \mathcal{D}(A_\epsilon)$; and the range of $I + A_\epsilon$ is equal to $\mathcal{Z}_{j,\epsilon}$
- A_ϵ is closed linear operator and $\mathcal{D}(A_\epsilon)$ is dense in $\mathcal{Z}_{j,\epsilon}$.

Proof. First, the estimate (3.0.25) gives $(A_\epsilon U, U)_{\mathcal{Z}_{j,\epsilon}} \geq 0$, $\forall U \in \mathcal{Z}_{j+2,\epsilon}$. To complete second part the range of $I + A_\epsilon$ is equal to $\mathcal{Z}_{j,\epsilon}$, we need to prove for every $U^* \in \mathcal{Z}_{j,\epsilon}$, there exists $U \in \mathcal{Z}_{j+2,\epsilon}$ such that $(I + A_\epsilon)U = U^*$. Indeed, for every $U^* = (u^*, v^*) \in \mathcal{Z}_{j,\epsilon}$, there exists U such that

$$(I + A_\epsilon)U = U^*$$

that is

$$(I + \wedge N^2)u - \wedge Nv = u^*, \quad (3.0.44)$$

$$(I + \frac{1}{\epsilon}N(\wedge + I))v - \frac{1}{\epsilon}\wedge N^2u = v^*, \quad (3.0.45)$$

by finding $v = (\wedge^{-1}N^{-1} + N)u - \wedge^{-1}N^{-1}u^*$ from (3.0.44), then substituting v into (3.0.45), we obtain

$$\left(N^{-1} + \frac{2+\epsilon}{\epsilon}I + \frac{1+\epsilon}{\epsilon}N + \frac{1}{\epsilon}N^2\right)u = v^* + \left(N^{-1} + \frac{2+\epsilon}{\epsilon}I + \frac{1}{\epsilon}N\right)u^* \in V_{j-2},$$

due to $U^* \in \mathcal{Z}_{j,\epsilon}$. Consequently, there exists a solution u such that

$$u \in V_{j+2}. \quad (3.0.46)$$

Now, from (3.0.45), we deduce that

$$\left(I + \frac{1}{\epsilon} \wedge N + \frac{1}{\epsilon} N \right) v = \frac{1}{\epsilon} \wedge N^2 u + v^* \in V_{j-1},$$

due to (3.0.46) and $U^* \in \mathcal{Z}_{j,\epsilon}$. Similarly, there exists a solution v such that $v \in V_{j+1}$.

Thus,

$$U = (u, v) \in \mathcal{Z}_{j+2,\epsilon}.$$

From first and second parts, we deduce the operator A_ϵ is maximal monotone.

Now, on account of Lemma 1.5.1, since A_ϵ is a maximal monotone operator then

$\mathcal{D}(A_\epsilon) = \mathcal{Z}_{j+2,\epsilon}$ is dense in $\mathcal{Z}_{j,\epsilon}$ and A_ϵ is a closed operator and for every $\lambda > 0$,

$(I + \lambda A_\epsilon)$ is bijective from $\mathcal{D}(A_\epsilon)$ onto $\mathcal{Z}_{j,\epsilon}$, $(I + \lambda A_\epsilon)^{-1}$ is a bounded operator.

Hence, the result. ■

Recall that we set

$$X_n = \text{span}\{U_k^\pm, k = 1, \dots, n\}, \quad Y_n = \overline{\text{span}\{U_k^\pm, k = n+1, n+2, \dots\}}^{\mathcal{Z}_{j,\epsilon}^\alpha},$$

and

$$\sigma_1 = \{\mu_l^-, \mu_m^+, \max\{\mu_l^-, \mu_m^+\} \leq \mu_n^-\}, \quad (3.0.47)$$

$$\sigma_2 = \{\mu_l^+, \mu_m^\pm, \mu_l^- \leq \mu_n^- < \min\{\mu_l^+, \mu_m^\pm\}\}, \quad (3.0.48)$$

$$X_{n1} = \text{span}\{U_l^-, U_m^+, \mu_l^-, \mu_m^+ \in \sigma_1\}, \quad (3.0.49)$$

$$X_{n2} = \text{span}\{U_l^+, \mu_l^- \leq \mu_n^- < \mu_l^+\}. \quad (3.0.50)$$

We have

$$\tilde{\mu}_l^+ + \tilde{\mu}_l^- = \frac{1}{\epsilon}(2 + (1 - \epsilon)\lambda_l), \quad (3.0.51)$$

$$\tilde{\mu}_l^+ \tilde{\mu}_l^- = -\frac{\lambda_l}{\epsilon}. \quad (3.0.52)$$

When $\mu_0^\pm = 0$, and for $U_0^- = (e_0, 0) \in X_{n1}$, $U_0^+ = (e_0, -\frac{2}{\epsilon}e_0) \in X_{n2}$, we deduce

$$(U_0^-, U_0^+) = 2 + \lambda_0 - 2 = 0,$$

and for $U_l^- \in X_{n1}$ and $U_l^+ \in X_{n2}$ there holds

$$\begin{aligned} (U_l^-, U_l^+) &= 2(N^{\frac{j-1}{2}} \bar{e}_l, N^{\frac{j-1}{2}} \bar{e}_l) + (N^{\frac{j}{2}} e_l, N^{\frac{j}{2}} e_l) + 2|\Omega|m(e_1)^2 - \epsilon(N^{\frac{j-1}{2}} e_l, N^{\frac{j-1}{2}} \tilde{\mu}_l^+ e_l) \\ &\quad - \epsilon(N^{\frac{j-1}{2}} e_l, N^{\frac{j-1}{2}} \tilde{\mu}_l^- e_l) + \epsilon^2 \tilde{\mu}_l^+ \tilde{\mu}_l^- (N^{\frac{j-1}{2}} \bar{e}_l, N^{\frac{j-1}{2}} \bar{e}_l) + \epsilon^2 \tilde{\mu}_l^+ \tilde{\mu}_l^- |\Omega|m(e_1)^2 \\ &= 2\lambda_l^{j-1} + \lambda_l^j - \epsilon(\tilde{\mu}_l^- + \tilde{\mu}_l^+) \lambda_l^{j-1} + \epsilon^2 \lambda_l^{j-1} \tilde{\mu}_l^- \tilde{\mu}_l^+ \\ &= 2\lambda_l^{j-1} + \lambda_l^j - (2 + (1 - \epsilon)\lambda_l) \lambda_l^{j-1} - \epsilon \lambda_l^j \\ &= 0, \end{aligned} \quad (3.0.53)$$

due to (3.0.51) and (3.0.52).

As a consequence, X_{n1} is orthogonal to X_{n2} and to Y_n , and we have $\mathcal{X}_{j,\epsilon}^\alpha = X_{n1} \oplus X_{n2} \oplus Y_n$ with respect to the scalar product $(\cdot, \cdot)_{\mathcal{X}_{j,\epsilon}^\alpha}$ and we set

$$(\mathcal{X}_{j,\epsilon}^\alpha)_1 = X_{n1} \quad \text{and} \quad (\mathcal{X}_{j,\epsilon}^\alpha)_2 = X_{n2} \oplus Y_n.$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be the unique orthogonal projections onto $(\mathcal{X}_{j,\epsilon}^\alpha)_1$ and $(\mathcal{X}_{j,\epsilon}^\alpha)_2$, respectively, where $\mathcal{Q} = I - \mathcal{P}$.

Assumption (H2):

It follows from Estimates (3.0.30) and (3.0.39).

Assumption (H3):

Let

$$M = 6M_1 = 6\mathfrak{c} > 0, \tag{3.0.54}$$

where $M_1 = \mathfrak{c}$ is given by (3.0.39). There hold

$$\mu_{n+1}^- - \mu_n^- > M, \tag{3.0.55}$$

provided there exists a positive integer n such that $\lambda_n \geq 1$ and the following condition is satisfied

$$\lambda_{n+1} - \lambda_n > 8M. \tag{3.0.56}$$

Indeed, if $d = 1$, then this is immediate, and if $d = 2$, then the result is well-known on a rectangular domain $\Omega = [0, L_1] \times [0, L_2]$, where L_1/L_2 is rational, due to I. Richards (see [39]). When $d=3$, it holds true on the unit sphere (c.f., e.g., [43]).

Of course, the latter estimate (3.0.56) implies that

$$\lambda_{n+1}^2 - \lambda_n^2 > 8M(\lambda_n + \lambda_{n+1}). \quad (3.0.57)$$

Now, we have

$$\mu_k^- = \frac{\lambda_k}{2\epsilon(1 + \lambda_k)} \left(2 + (\epsilon + 1)\lambda_k - \sqrt{(\lambda_k + 2)^2 - 2\epsilon\lambda_k^2 + \epsilon^2\lambda_k^2} \right),$$

so that

$$\begin{aligned} \mu_{n+1}^- - \mu_n^- &= \frac{\lambda_{n+1}}{2\epsilon(1 + \lambda_{n+1})} \left(2 + \lambda_{n+1} - \sqrt{(\lambda_{n+1} + 2)^2 - 2\epsilon\lambda_{n+1}^2 + \epsilon^2\lambda_{n+1}^2} \right) \\ &\quad - \frac{\lambda_n}{2\epsilon(1 + \lambda_n)} \left(2 + \lambda_n - \sqrt{(\lambda_n + 2)^2 - 2\epsilon\lambda_n^2 + \epsilon^2\lambda_n^2} \right) \\ &\quad + \frac{\lambda_{n+1}^2}{2(1 + \lambda_{n+1})} - \frac{\lambda_n^2}{2(1 + \lambda_n)} - M + M, \end{aligned}$$

we rewrite it as

$$\mu_{n+1}^- - \mu_n^- = h_\epsilon(n) + M,$$

where

$$\begin{aligned}
h_\epsilon(n) &= \frac{\lambda_{n+1}}{2\epsilon(1+\lambda_{n+1})} \left(2 + \lambda_{n+1} - \sqrt{W_{n+1}} \right) - \frac{\lambda_n}{2\epsilon(1+\lambda_n)} \left(2 + \lambda_n - \sqrt{W_n} \right) \\
&\quad + \frac{\lambda_{n+1}^2}{2(1+\lambda_{n+1})} - \frac{\lambda_n^2}{2(1+\lambda_n)} - M,
\end{aligned} \tag{3.0.58}$$

and

$$W_k = (\lambda_k + 2)^2 - 2\epsilon\lambda_k^2 + \epsilon^2\lambda_k^2, \quad k = n, n+1.$$

On account of Lemma 3.0.1, we have

$$\begin{aligned}
&\frac{\lambda_{n+1}}{(1+\lambda_{n+1})} \left(\sqrt{W_{n+1}} - (2 + \lambda_{n+1}) \right) - \frac{\lambda_n}{(1+\lambda_n)} \left(\sqrt{W_n} - (2 + \lambda_n) \right) \\
&\leq \frac{\epsilon^2\lambda_{n+1}^3}{4(1+\lambda_{n+1})} - \frac{\epsilon^2\lambda_n^3}{4(1+\lambda_n)},
\end{aligned} \tag{3.0.59}$$

it follows from (3.0.58) and (3.0.59) that

$$\begin{aligned}
&\frac{1}{2\epsilon} \left| \frac{\lambda_{n+1}}{(1+\lambda_{n+1})} \left((2 + \lambda_{n+1}) - \sqrt{W_{n+1}} \right) - \frac{\lambda_n}{(1+\lambda_n)} \left((2 + \lambda_n) - \sqrt{W_n} \right) \right| \\
&\leq \frac{\epsilon\lambda_{n+1}^3}{8(1+\lambda_{n+1})} - \frac{\epsilon\lambda_n^3}{8(1+\lambda_n)}.
\end{aligned} \tag{3.0.60}$$

Thus

$$\begin{aligned}
h_\epsilon(n) &> \frac{\lambda_{n+1}^2}{2(1+\lambda_{n+1})} - \frac{\lambda_n^2}{2(1+\lambda_n)} - M - \epsilon \left(\frac{\lambda_{n+1}^3}{8(1+\lambda_{n+1})} - \frac{\lambda_n^3}{8(1+\lambda_n)} \right) \\
&> \frac{\lambda_{n+1}^2}{4(1+\lambda_{n+1})} - \frac{\lambda_n^2}{4(1+\lambda_n)} - M \\
&\quad + \frac{\lambda_{n+1}^2}{4(1+\lambda_{n+1})} - \frac{\lambda_n^2}{4(1+\lambda_n)} - \epsilon \left(\frac{\lambda_{n+1}^3}{8(1+\lambda_{n+1})} - \frac{\lambda_n^3}{8(1+\lambda_n)} \right).
\end{aligned} \tag{3.0.61}$$

There holds $\frac{\lambda_{n+1}^2}{4(1+\lambda_{n+1})} - \frac{\lambda_n^2}{4(1+\lambda_n)} - M > 0$. Indeed,

$$\begin{aligned}
& \frac{\lambda_{n+1}^2}{4(1+\lambda_{n+1})} - \frac{\lambda_n^2}{4(1+\lambda_n)} - M \\
&= \frac{1}{8(1+\lambda_{n+1})(1+\lambda_n)} ([\lambda_{n+1}^2 - \lambda_n^2 - 8M] + [\lambda_{n+1}^2 - \lambda_n^2 - 8M(\lambda_{n+1} + \lambda_n)]) \\
&\quad + \frac{\lambda_n \lambda_{n+1}}{4(1+\lambda_{n+1})(1+\lambda_n)} (\lambda_{n+1} - \lambda_n - 4M) \\
&> 0,
\end{aligned} \tag{3.0.62}$$

due to (3.0.56) and (3.0.57).

There also holds

$$\frac{\lambda_{n+1}^2}{4(1+\lambda_{n+1})} - \frac{\lambda_n^2}{4(1+\lambda_n)} - \epsilon \left(\frac{\lambda_{n+1}^3}{8(1+\lambda_{n+1})} - \frac{\lambda_n^3}{8(1+\lambda_n)} \right) > 0, \quad \forall \epsilon \in (0, \frac{1}{\lambda_{n+1}}), \tag{3.0.63}$$

whenever (3.0.56) is satisfied. Indeed, on account of (3.0.56) and (3.0.57), we have

$$\begin{aligned}
& \frac{2\lambda_{n+1}^2(1+\lambda_n) - 2\lambda_n^2(1+\lambda_{n+1})}{\lambda_{n+1}^3(1+\lambda_n) - \lambda_n^3(1+\lambda_{n+1})} - \frac{1}{\lambda_{n+1}} \\
&= \frac{2\lambda_{n+1}^3(1+\lambda_n) - 2\lambda_n^2\lambda_{n+1}(1+\lambda_{n+1}) - [\lambda_{n+1}^3(1+\lambda_n) - \lambda_n^3(1+\lambda_{n+1})]}{[\lambda_{n+1}^3(1+\lambda_n) - \lambda_n^3(1+\lambda_{n+1})] \lambda_{n+1}} \\
&= \frac{\lambda_{n+1}^3(1+\lambda_n) - \lambda_n^2\lambda_{n+1}(1+\lambda_{n+1}) + \lambda_n^3(1+\lambda_{n+1}) - \lambda_n^2\lambda_{n+1}(1+\lambda_{n+1})}{[\lambda_{n+1}^3(1+\lambda_n) - \lambda_n^3(1+\lambda_{n+1})] \lambda_{n+1}} \\
&= \frac{\lambda_{n+1}[(\lambda_{n+1}^2 - \lambda_n^2) + \lambda_{n+1}\lambda_n(\lambda_{n+1} - \lambda_n)] - \lambda_n^2(1+\lambda_{n+1})(\lambda_{n+1} - \lambda_n)}{[\lambda_{n+1}^3(1+\lambda_n) - \lambda_n^3(1+\lambda_{n+1})] \lambda_{n+1}} \\
&= \frac{(\lambda_{n+1} - \lambda_n)[(\lambda_{n+1}^2 - \lambda_n^2) + \lambda_{n+1}\lambda_n(\lambda_{n+1} - \lambda_n) + \lambda_n\lambda_{n+1}]}{[\lambda_{n+1}^3(1+\lambda_n) - \lambda_n^3(1+\lambda_{n+1})] \lambda_{n+1}} \\
&> 0,
\end{aligned} \tag{3.0.64}$$

hence

$$0 < \epsilon < \frac{1}{\lambda_{n+1}} < \frac{2[\lambda_{n+1}^2(1 + \lambda_n) - \lambda_n^2(1 + \lambda_{n+1})]}{\lambda_{n+1}^3(1 + \lambda_n) - \lambda_n^3(1 + \lambda_{n+1})}. \quad (3.0.65)$$

Thus, we entail (3.0.63).

Now, on account of (3.0.63) and (3.0.62), we can deduce that

$$h_\epsilon(n) > 0,$$

hence (3.0.55) holds true, for all $0 < \epsilon < \frac{1}{\lambda_{n+1}}$, provided (3.0.56) and (3.0.57) are satisfied. Thus, **Assumption (H3)** holds.

Assumption (H4). It holds true on account of Proposition 3.0.1.

The following lemma is used to show Assumption(H3).

Lemma 3.0.1 *For any $\epsilon \in (0, 1]$, the function*

$$f(x) = \frac{x}{1+x} \left(\sqrt{(2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2} - (2+x) - \frac{1}{4}\epsilon^2 x^2 \right)$$

is decreasing function on $\mathbb{R}^+$.

Proof. First, there holds

$$\sqrt{(2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2} - (2+x) - \frac{1}{4}\epsilon^2 x^2 < 0, \quad (3.0.66)$$

Indeed, we observe that $(2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2 = (\epsilon - 1)^2 x^2 + 4x + 4 \geq 0$, hence the

sign of

$$\sqrt{(2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2} - [(2+x) + \frac{1}{4}\epsilon^2 x^2], \quad (3.0.67)$$

is same as that of

$$(2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2 - [(2+x) + \frac{1}{4}\epsilon^2 x^2]^2, \quad (3.0.68)$$

and

$$\begin{aligned} & (2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2 - [(2+x) + \frac{1}{4}\epsilon^2 x^2]^2 \\ &= (2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2 - (2+x)^2 - \frac{1}{16}\epsilon^4 x^4 - \frac{1}{2}\epsilon^2 x^3 - \epsilon^2 x^2 \\ &= -2\epsilon x^2 - \frac{1}{2}\epsilon^2 x^3 - \frac{1}{16}\epsilon^4 x^4 \\ &< 0. \end{aligned} \quad (3.0.69)$$

Second, there also holds

$$\frac{2+x-2\epsilon x+\epsilon^2 x}{\sqrt{(2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2}} - 1 - \frac{1}{2}\epsilon^2 x < 0. \quad (3.0.70)$$

Indeed, the sign of

$$\frac{2+x-2\epsilon x+\epsilon^2 x}{\sqrt{(2+x)^2 - 2\epsilon x^2 + \epsilon^2 x^2}} - 1 - \frac{1}{2}\epsilon^2 x, \quad (3.0.71)$$

is same as that of

$$\begin{aligned} & [2 + x - 2\epsilon x + \epsilon^2 x]^2 - (1 + \frac{1}{2}\epsilon^2 x)^2[(2 + x)^2 - 2\epsilon x^2 + \epsilon^2 x^2] \\ &= [(\epsilon - 1)^2 x + 2]^2 - (1 + \frac{1}{2}\epsilon^2 x)^2[(\epsilon - 1)^2 x^2 + 4x + 4], \end{aligned}$$

and

$$\begin{aligned} & [(\epsilon - 1)^2 x + 2]^2 - (1 + \frac{1}{2}\epsilon^2 x)^2[(\epsilon - 1)^2 x^2 + 4x + 4] \\ &= (\epsilon - 1)^4 x^2 + 4(\epsilon - 1)^2 x + 4 - (\epsilon - 1)^2 x^2 - 4x - 4 \\ &\quad - (\epsilon^2 x + \frac{1}{4}\epsilon^4 x^2)[(\epsilon - 1)^2 x^2 + 4x + 4] \\ &= -(\epsilon - 1)^2 x^2[1 - (\epsilon - 1)^2] - 4x[1 - (\epsilon - 1)^2] \\ &\quad - (\epsilon^2 x + \frac{1}{4}\epsilon^4 x^2)[(\epsilon - 1)^2 x^2 + 4x + 4] \\ &= -\epsilon(\epsilon - 1)^2 x^2[2 - \epsilon] - 4\epsilon x[2 - \epsilon] - (\epsilon^2 x + \frac{1}{4}\epsilon^4 x^2)[(\epsilon - 1)^2 x^2 + 4x + 4] \\ &< 0, \end{aligned} \tag{3.0.72}$$

due to $(2 - \epsilon) > 0$.

Thus, on account of (3.0.66) and (3.0.70), the sign of $f'(x)$ is negative. Hence, the result. ■

Now, we can state the following result.

Theorem 3.0.2 *Let Assumptions from (H1) through (H4) hold. Then, there exists a positive integer n independent of ϵ such that, for every $\epsilon \in (0, \frac{1}{\lambda_{n+1}}]$, Equation (3.0.23) has an inertial manifold $\mathcal{M}_\epsilon^\alpha$ in $\mathcal{Z}_{j,\epsilon}^\alpha$, that is,*

- $\mathcal{M}_\epsilon^\alpha$ is a finite-dimensional Lipschitz manifold in $\mathcal{Z}_{j,\epsilon}^\alpha$ and $\dim \mathcal{M}_\epsilon^\alpha = n$;
- $\mathcal{M}_\epsilon^\alpha$ is invariant under $\tilde{S}_\epsilon(t)$, that is, $\tilde{S}_\epsilon(t)\mathcal{M}_\epsilon^\alpha = \mathcal{M}_\epsilon^\alpha$, $\forall t \geq 0$;
- $\mathcal{M}_\epsilon^\alpha$ is exponentially attracting, that is, there exists a constant k such that, for every $U_0 \in \mathcal{Z}_{j,\epsilon}^\alpha$, there exists a constant $C = C(\|U_0\|_{\mathcal{Z}_{j,\epsilon}}) > 0$ such that (k and C are independent of ϵ)

$$\text{dist}_{\mathcal{Z}_{j,\epsilon}}(\tilde{S}_\epsilon(t)U_0, \mathcal{M}_\epsilon^\alpha) \leq Ce^{-kt}, \quad \forall t \geq 0.$$

It follows from the existence of inertial manifolds that the semigroup $\tilde{S}_\epsilon(t)$ generated by Equation (3.0.23) admits an inertial manifold $\mathcal{M}_\epsilon^\alpha$. More precisely, there exists a Lipschitz mapping $\Phi_\epsilon^\alpha : (\mathcal{Z}_{j,\epsilon}^\alpha)_1 \rightarrow (\mathcal{Z}_{j,\epsilon}^\alpha)_2$ such that the graph of Φ_ϵ^α defined an inertial manifold

$$\mathcal{M}_\epsilon^\alpha = \{p + \Phi_\epsilon^\alpha(p), p \in (\mathcal{Z}_{j,\epsilon}^\alpha)_1\},$$

for $\tilde{S}_\epsilon(t)$ of dimension n independent of ϵ in the metric indicating by the norm $\|\cdot\|_{\mathcal{Z}_{j,\epsilon}}$.

CHAPTER 4

A GENERALIZED PHASE-FIELD SYSTEM

We consider the Generalized Phase-field model

$$\begin{cases} \phi_t + A(A\phi + g(\phi) - u) + r(\phi) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \epsilon u_t + \phi_t + Au + h(u) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0, \end{cases} \quad (4.0.1)$$

where the operator A is defined in the Subsection 1.4.2, $\epsilon \in (0, 1]$ and $g(\phi)$, $r(\phi)$ and $h(\phi)$ are the following polynomials:

$$g(\phi) = \sum_{j=1}^{2m-1} a_j \phi^j, \quad a_{2m-1} > 0, \quad a_2 = 0, \quad m \in \mathbb{N}, \quad m \geq 2, \quad (4.0.2)$$

$$r(\phi) = \sum_{j=1}^m b_j \phi^j, \quad b_m > 0, \quad (4.0.3)$$

where

$$G(\phi) = \sum_{j=2}^{2m} c_j \phi^j, \quad c_j = \frac{a_{j-1}}{j}, \quad c_{2m} > 0, \quad (4.0.4)$$

and

$$h(u) = \sum_{j=1}^{2n-1} c_j u^j, \quad n \in \mathbb{N}, c_n > 0, \quad H(s) = \int_0^s h(\sigma) d\sigma, \quad (4.0.5)$$

where $n \geq 2$ in $d = 1, 2$ and $n = 3$ in $d = 3$, satisfies the condition

$$h'(s) \geq C_6, \quad C_6 \geq 0, \quad \forall s \in \mathbb{R}. \quad (4.0.6)$$

If $\epsilon = 0$, then the system reduces to

$$\begin{cases} \phi_t + A(A\phi + g(\phi) - u) + r(\phi) = 0, \\ \phi_t + Au + h(u) = 0, \\ \phi|_{t=0} = \phi_0. \end{cases} \quad (4.0.7)$$

4.1 Functional setting

For every j , (c.f. (1.5.8)), we endow the Hilbert spaces

$$\mathcal{W}_{j,\epsilon} = \mathcal{D}(A^{\frac{j}{2}}) \times \mathcal{D}(A^{\frac{j-1}{2}})$$

and

$$\mathcal{U}_j = \mathcal{D}(A^{\frac{j}{2}}) \times \mathcal{D}(A^{\frac{j-2}{2}}),$$

respectively equipped with the norms

$$\|(\varphi, \psi)\|_{\mathcal{W}_{j,\epsilon}} = (\|\varphi\|_j^2 + \epsilon\|\psi\|_{j-1}^2)^{\frac{1}{2}}$$

and

$$\|(\varphi, \psi)\|_{\mathcal{U}_j} = (\|\varphi\|_j^2 + \|\psi\|_{j-2}^2)^{\frac{1}{2}}.$$

Remark 4.1.1 *Since h is the polynomial (4.0.5) and by virtue (4.0.6), it is easy to conclude the following (c.f. [41])*

$$|h(s)| \leq \mathcal{C}(1 + |s|^{p+4}), \quad \mathcal{C} > 0, \quad \forall s \in \mathbb{R}, \quad (4.1.1)$$

$$h(s)s \geq \mathcal{C}_2(|s|^{p+5} - 1), \quad \mathcal{C}_2 > 0, \quad (4.1.2)$$

$$|h'''(s)| \leq \mathcal{C}_5(|s|^{p+1} + 1), \quad \mathcal{C}_5 > 0, \quad \forall s \in \mathbb{R}, \quad (4.1.3)$$

where $p > 0$ is arbitrary if $d = 1, 2$ and $p = 1$ if $d = 3$,

Moreover, since g is polynomial of order $2m - 1$, and $a_{2m-1} > 0$, we conclude that

$$g'(s) \geq -C_4, \quad C_4 \geq 0, \quad \forall s \in \mathbb{R}. \quad (4.1.4)$$

Also, by using Young's inequality where $2 < 2m$ and (4.0.4), we obtain

$$\begin{aligned} s^2 &\leq \frac{\mu}{c} s^{2m} + c_\mu \\ &\leq \mu G(s) + c_\mu, \quad c_\mu > 0, \quad \forall \mu > 0 \quad \forall s \in \mathbb{R}, \end{aligned} \quad (4.1.5)$$

Let $M(s) = sg(s) - G(s)$, then by the integral form of the Main Value Theorem we get

$$M(s) = s^2 \int_0^1 \sigma g'(\sigma s) d\sigma \geq -\frac{C_4}{2} s^2, \quad (4.1.6)$$

that is

$$sg(s) \geq \frac{3}{4}G(s) - C_2, \quad \forall s \in \mathbb{R}, \text{ with } C_2 > 0. \quad (4.1.7)$$

Similarly, setting $z(s) = sh(s) - H(s)$, we have

$$z(s) = s^2 \int_0^1 rh'(rs) dr \geq \frac{C_6}{2} s^2,$$

that is, $sh(s) \geq H(s) + cs^2$. On account of (4.1.1) and by using Young's inequality, we obtain

$$\begin{aligned} H(s) &\leq sh(s) - cs^2 \\ &\leq \mathcal{C}|s|(1 + |s|^{p+4}) - cs^2 \\ &\leq C(1 + |s|^{p+5}), \quad \forall s \in \mathbb{R}. \end{aligned} \quad (4.1.8)$$

From (4.1.5), (4.0.2) and (4.0.3), we can deduce the following lemma.

Lemma 4.1.1 *From (4.0.2), (4.0.3) and (4.1.5) we can deduce the following estimates*

$$\|r(s)\|^2 \leq \mathcal{C}_7 \left(\int_{\Omega} [G(s) + 1] dx \right), \quad \mathcal{C}_7 > 0, \quad (4.1.9)$$

$$\|r'(s)\|^2 \leq \mathcal{C} \left(\int_{\Omega} [G(s) + 1] dx \right) \quad \mathcal{C} > 0, \quad (4.1.10)$$

$$\forall \varsigma > 0, \exists c_{\varsigma} > 0 \text{ such that } \|s\| \|r(s)\| \leq \varsigma \int_{\Omega} G(\phi) dx + c_{\varsigma}, \quad (4.1.11)$$

$$\forall \mu > 0, \exists c > 0 \text{ such that } \|s\|^2 \leq \mu \int_{\Omega} G(\phi) dx + c_{\mu}. \quad (4.1.12)$$

Proof. We have

$$\|r'(\phi)\|^2 \leq c \int_{\Omega} |r'(\phi)|^2 dx.$$

On account of (4.0.3) and by Young's inequality with the fact that $2(m-1) \leq 2m$,

we can find

$$\begin{aligned} \|r'(\phi)\|^2 &\leq c \int_{\Omega} \left| \sum_{j=1}^m j b_j \phi^{j-1} \right|^2 dx \\ &\leq c \int_{\Omega} (1 + |\phi|^{2m-2}) dx \\ &\leq C \int_{\Omega} [G(\phi) + 1] dx. \end{aligned}$$

Similarly, on account of (4.0.3) and by using Young's inequality, we have

$$\begin{aligned}
\|r(\phi)\|^2 &\leq c \int_{\Omega} |r(\phi)|^2 dx \\
&\leq c \int_{\Omega} \left| \sum_{j=1}^m b_j \phi^j \right|^2 dx \\
&\leq C \int_{\Omega} (|\phi|^{2m} + 1) dx \\
&\leq C' \int_{\Omega} [G(\phi) + 1] dx.
\end{aligned} \tag{4.1.13}$$

From Assumption (4.1.5), we obtain

$$\begin{aligned}
\|\phi\|^2 &= \int_{\Omega} |\phi|^2 dx \\
&\leq \mu \int_{\Omega} G(\phi) dx + c.
\end{aligned} \tag{4.1.14}$$

Moreover, on account of (4.1.13), (4.1.5) and by using Young's inequality, we have

$$\begin{aligned}
\|r(\phi)\|^2 \|\phi\|^2 &\leq c \left(\int_{\Omega} (G(\phi) + 1) dx \right) \left(\frac{\mu'}{c} \int_{\Omega} G(\phi) dx + c_{\mu'} \right), \quad \forall \mu' > 0 \\
&\leq \mu' \left(\int_{\Omega} G(\phi) dx \right)^2 + (C + \mu') \int_{\Omega} G(\phi) dx + c_{\mu'} \\
&\leq \mu' \left(\int_{\Omega} G(\phi) dx \right)^2 + \mu' \left(\int_{\Omega} G(\phi) dx \right)^2 + c_{\mu'} \\
&\leq 2\mu' \left(\int_{\Omega} G(\phi) dx \right)^2 + c_{\mu'} \\
&\leq \varsigma^2 \left(\int_{\Omega} G(\phi) dx \right)^2 + c_{\varsigma}, \quad \varsigma^2 = 2\mu', \quad \forall \varsigma > 0.
\end{aligned}$$

I

4.2 A priori estimates

In the following subsections, we will derive formal a priori estimates. These estimates and the passage to the limit can be justified by making Galerkin approximations.

4.2.1 Estimates in $\mathcal{W}_{1,\epsilon}$

We multiply (4.0.1)₁ and (4.0.1)₂ by $A^{-1}\phi_t$ and u , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\begin{aligned} \frac{d}{dt}(\epsilon\|u\|^2 + \|\phi\|_1^2 + 2 \int_{\Omega} G(\phi) dx) \\ + 2\|u\|_1^2 + 2\|\phi_t\|_{-1}^2 + 2 \int_{\Omega} r(\phi) A^{-1}\phi_t dx + 2 \int_{\Omega} u h(u) dx = 0. \end{aligned} \quad (4.2.1)$$

We have

$$\begin{aligned} \left| \int_{\Omega} \nabla A^{-1}r(\phi) \cdot \nabla A^{-1}\phi_t dx \right| &\leq \frac{1}{2}\|r(\phi)\|_{-1}^2 + \frac{1}{2}\|\phi_t\|_{-1}^2 \\ &\leq c \left(\int_{\Omega} [G(s) + 1] dx \right) + \frac{1}{2}\|\phi_t\|_{-1}^2, \end{aligned} \quad (4.2.2)$$

due to (4.1.9). Hence on account of (4.1.2) and (4.2.2), it follows that

$$\begin{aligned} \frac{d}{dt}(\epsilon\|u\|^2 + \|\phi\|_1^2 + 2 \int_{\Omega} G(\phi) dx) + \|u\|_1^2 + \|\phi_t\|_{-1}^2 + 2\mathcal{C}_2\|u\|_{L^{p+5}}^{p+5} &\leq c \int_{\Omega} G(s) dx + c'. \end{aligned} \quad (4.2.3)$$

We multiply (4.0.1)₁ by $A^{-1}\phi$ and we integrate over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|\phi\|_{-1}^2 + \|\phi\|_1^2 + \int_{\Omega} g(\phi) \phi \, dx - (u, \phi) + \int_{\Omega} \nabla A^{-1} r(\phi) \cdot \nabla A^{-1} \phi \, dx = 0. \quad (4.2.4)$$

Now using (4.1.7), we deduce

$$\frac{1}{2} \frac{d}{dt} \|\phi\|_{-1}^2 + \|\phi\|_1^2 + \frac{3}{4} \int_{\Omega} G(\phi) \, dx - (u, \phi) + \int_{\Omega} \nabla A^{-1} r(\phi) \cdot \nabla A^{-1} \phi \, dx \leq C. \quad (4.2.5)$$

On account of Cauchy-Schwarz, Young's inequalities and (4.1.12) by choosing $\mu = \frac{\eta}{8c}$, we have

$$\begin{aligned} |(u, \phi)| &\leq \|u\| \|\phi\| \\ &\leq \frac{\eta}{4} \|u\|_1^2 + \frac{c}{\eta} \|\phi\|^2 \\ &\leq \frac{\eta}{4} \|u\|_1^2 + \frac{1}{8} \int_{\Omega} G(\phi) \, dx + c_{\eta}, \end{aligned}$$

and, by virtue (4.1.11) where we choose $\varsigma = \frac{1}{8c_1}$, we obtain

$$\begin{aligned} \left| \int_{\Omega} \nabla A^{-1} r(\phi) \cdot \nabla A^{-1} \phi \, dx \right| &\leq \|r(\phi)\|_{-1} \|\phi\|_{-1} \\ &\leq c_1 \|r(\phi)\| \|\phi\| \\ &\leq \frac{1}{8} \int_{\Omega} G(\phi) \, dx + c. \end{aligned}$$

It follows from (4.2.5) that

$$\frac{d}{dt}\|\phi\|_{-1}^2 + 2\|\phi\|_1^2 + \int_{\Omega} G(\phi) dx \leq \frac{\eta}{2}\|u\|_1^2 + C, \quad \forall \eta > 0. \quad (4.2.6)$$

We multiply (4.0.1)₁ by ϕ and we integrate over Ω , and we obtain

$$\frac{1}{2} \frac{d}{dt} \|\phi\|^2 + \|\phi\|_2^2 + \int_{\Omega} g'(\phi) |\nabla \phi|^2 dx - (A^{1/2}u, A^{1/2}\phi) + \int_{\Omega} r(\phi) \phi dx = 0. \quad (4.2.7)$$

Owing to estimate (2.2.12) and the fact that

$$\begin{aligned} \left| \int_{\Omega} r(\phi) \phi dx \right| &\leq \|r(\phi)\| \|\phi\| \\ &\leq \frac{1}{8} \int_{\Omega} G(\phi) dx + c, \end{aligned} \quad (4.2.8)$$

due to (4.1.11); along with the estimate $|(A^{1/2}u, A^{1/2}\phi)| \leq \frac{1}{4}\|u\|_1^2 + \|\phi\|_1^2$, we can deduce from (4.2.7) that

$$\frac{d}{dt} \|\phi\|^2 + 2\|\phi\|_2^2 \leq \frac{1}{2}\|u\|_1^2 + c_1\|\phi\|_1^2 + \frac{1}{4} \int_{\Omega} G(\phi) dx. \quad (4.2.9)$$

Summing (4.2.6), μ_1 (4.2.3) and μ_2 (4.2.9), with $\mu_1, \mu_2 > 0$, we have

$$\begin{aligned} &\frac{d}{dt} E_1(t) + (2 - c_1\mu_2) \|\phi\|_1^2 + \left(1 - \frac{\mu_2}{4} - c\mu_1\right) \int_{\Omega} G(\phi) dx \\ &+ \left(\mu_1 - \left[\frac{\eta}{2} + \frac{\mu_2}{2}\right]\right) \|u\|_1^2 + \mu_1 \|\phi_t\|_{-1}^2 + 2\mu_2 \|\phi\|_2^2 + 2\mu_1 \mathcal{C}_2 \|u\|_{L^{p+5}}^{p+5} \leq c_{\eta}, \end{aligned} \quad (4.2.10)$$

where

$$E_1(t) = \mu_1(\epsilon \|u\|^2 + \|\phi\|_1^2 + 2 \int_{\Omega} G(\phi) dx) + \|\phi\|_{-1}^2 + \mu_2 \|\phi\|^2.$$

First we choose $\eta = \mu_2$, and we can take $\mu_1 = \frac{4}{1+8c}$ and $\mu_2 = \frac{1}{2} \inf\{\frac{1}{c_1}, \mu_1\}$, we deduce that there exist $c, c_2 \geq 0$ such that

$$\frac{d}{dt} E_1(t) + c \left(\|u\|_1^2 + \|\phi\|_2^2 + \|\phi_t\|_{-1}^2 + \int_{\Omega} G(\phi) dx \right) + 2\mu_1 \mathcal{C}_2 \|u\|_{L^{p+5}}^{p+5} \leq C. \quad (4.2.11)$$

Now, we noticed that there exist $c_0, c'_1 c_2, c_3 \geq 0$ that are all independent of ϵ , such that

$$\|\phi\|_1^2 + \|u\|^2 + \int_{\Omega} [G(\phi) + C_1] dx \geq c_0 E_1(t),$$

and

$$\begin{aligned} E_1(t) &\leq c_2 (\epsilon \|u(t)\|^2 + \|\phi(t)\|_1^2 + \|G(\phi(t))\|_{L^\infty(\Omega)}) \\ &\leq Q(\epsilon \|u(t)\|^2 + \|\phi(t)\|_j^2), \end{aligned}$$

then

$$c_1 \left(\epsilon \|u\|^2 + \|\phi\|_1^2 + \int_{\Omega} [G(\phi) + C_1] dx \right) - c_2 \leq E_1(t) \leq Q(\epsilon \|u(t)\|^2 + \|\phi(t)\|_j^2).$$

Thus, from (4.2.11), we obtain that

$$\frac{d}{dt}E_1(t) + c_0E_1(t) + c(\|u\|_1^2 + \|\phi\|_2^2 + \|\phi_t\|_{-1}^2 + \|u\|_{L^{p+5}}^{p+5}) \leq c_2. \quad (4.2.12)$$

Applying the Gronwall's Lemma 1.5.9 to (4.2.12), we obtain that

$$E_1(t) \leq E_1(0)e^{-c_1t} + c, \quad \forall t \geq 0,$$

hence

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{1,\epsilon}}^2 + \int_{\Omega} G(\phi(t)) dx \leq Q(\|\phi_0\|_j^2 + \epsilon\|u_0\|^2)e^{-c_1t} + c, \quad \forall t \geq 0. \quad (4.2.13)$$

Then, integrating (4.2.12) between t and $t+r$, by utilize (4.2.13), we find

$$\begin{aligned} & \int_t^{t+r} \left(\|u(s)\|_1^2 + \|\phi(s)\|_2^2 + \|\phi_t(s)\|_{-1}^2 + \int_{\Omega} G(\phi) dx + \|u\|_{L^{p+5}}^{p+5} \right) ds \\ & \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2)e^{-c_1t} + c, \quad \forall t \geq 0. \end{aligned} \quad (4.2.14)$$

Furthermore, integrating (4.2.12) between 0 and t , we obtain

$$\begin{aligned} & \int_0^t \left(\|u(s)\|_1^2 + \|\phi(s)\|_2^2 + \|\phi_t(s)\|_{-1}^2 + \int_{\Omega} G(\phi) dx + \|u\|_{L^{p+5}}^{p+5} \right) ds \\ & \leq (1+t)Q\left(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2\right), \quad \forall t \geq 0. \end{aligned} \quad (4.2.15)$$

4.2.2 Estimates in $\mathcal{W}_{2,\epsilon}$

We differentiate (4.0.1)₁ with respect to t to obtain

$$\phi_{tt} + A(A\phi_t + g'(\phi)\phi_t - u_t) + r'(\phi)\phi_t = 0. \quad (4.2.16)$$

We multiply (4.2.16) and (4.0.1)₂ by $A^{-1}\phi_t$ and u_t , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\phi_t\|_{-1}^2 + \|u\|_1^2) \\ & + \|\phi_t\|_1^2 + \int_{\Omega} g'(\phi) |\phi_t|^2 dx + \epsilon \|u_t\|^2 + \int_{\Omega} r'(\phi) \phi_t A^{-1} \phi_t dx + \int_{\Omega} h(u) u_t dx = 0. \end{aligned} \quad (4.2.17)$$

We have

$$\begin{aligned} \left| \int_{\Omega} r'(\phi) \phi_t A^{-1} \phi_t dx \right| & \leq \|r'(\phi)\| \|\phi_t\| \|A^{-1} \phi_t\|_{L^\infty(\Omega)} \\ & \leq c \|r'(\phi)\| \|\phi_t\|^2 \\ & \leq c' \|r'(\phi)\|^2 \|\phi_t\|_{-1}^2 + \frac{1}{2} \|\phi_t\|_1^2 \\ & \leq c \left(\int_{\Omega} G(\phi) dx + 1 \right) \|\phi_t\|_{-1}^2 + \frac{1}{2} \|\phi_t\|_1^2, \end{aligned} \quad (4.2.18)$$

due to (4.1.10), and, by using the estimates (2.2.12), $\|\phi_t\|^2 \leq \|\phi_t\|_1 \|\phi_t\|_{-1}$ and Young's inequality, we can obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \int_{\Omega} H(u) dx \right) + \frac{1}{2} \|\phi_t\|_1^2 + \epsilon \|u_t\|^2 \leq c \left(\int_{\Omega} [G(\phi) dx + 1] \right) \|\phi_t\|_{-1}^2, \quad (4.2.19)$$

hence

$$\frac{d}{dt} \left(\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \int_{\Omega} H(u) dx \right) + \|\phi_t\|_1^2 + \epsilon \|u_t\|^2 \leq c \left(\int_{\Omega} [G(\phi) dx + C_1] \right) \|\phi_t\|_{-1}^2. \quad (4.2.20)$$

On account of (4.1.8) and (4.2.14), we notice that

$$\begin{aligned} \int_t^{t+r} \int_{\Omega} |H(u)| dx ds &\leq c \int_t^{t+r} (1 + \|u\|_{L^{p+5}}^{p+5}) ds \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \forall t \geq 0. \end{aligned} \quad (4.2.21)$$

Apply Lemma 1.5.11 to (4.2.20), on account of (4.2.13), (4.2.21) and (4.2.14), we find

$$\|\phi_t(t)\|_{-1}^2 + \|u(t)\|_1^2 + \int_{\Omega} H(u) dx \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r. \quad (4.2.22)$$

Integrating (4.2.20) between t and $t+r$, we get

$$\int_t^{t+r} (\|\phi_t(s)\|_1^2 + \epsilon \|u_t(s)\|^2) ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r. \quad (4.2.23)$$

Now, we multiply (4.0.1)₁ by ϕ and we integrate over Ω , and we obtain

$$\|\phi\|_2^2 + (\phi_t, \phi) + \int_{\Omega} g'(\phi) |\nabla \phi|^2 dx - (u, A\phi) + \int_{\Omega} r(\phi) \phi dx = 0, \quad (4.2.24)$$

it follows that

$$\begin{aligned} \|\phi(t)\|_2^2 &\leq c (\|u\|^2 + \|\phi\|_1^2 + \|\phi_t\|_{-2}^2 + \|r(\phi)\|_{-2}^2) \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r, \quad \forall r > 0. \end{aligned} \quad (4.2.25)$$

Thus,

$$\|\phi(t)\|_2^2 + \|\phi_t(t)\|_{-1}^2 + \|u(t)\|_1^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r. \quad (4.2.26)$$

Next, we will proceed as we did in Section 2.2.2 (c.f. from (2.2.51) through (2.2.55)),

it is enough to prove that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{2,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}}^2), \quad \forall t \in [0, r].$$

We multiply (4.0.1)₁ and (4.0.1)₂ by ϕ_t and Au , respectively, integrate over Ω , summing the resulting equations, then using the estimate (4.0.6), (2.2.54) and

$$\begin{aligned} \left| \int_{\Omega} r(\phi) \phi_t dx \right| &\leq \|r(\phi)\| \|\phi_t\| \\ &\leq c \|r(\phi)\|_{L^\infty}^2 + \frac{1}{4} \|\phi_t\|^2 \\ &\leq Q(\|\phi\|_j^2) + \frac{1}{4} \|\phi_t\|^2, \end{aligned} \quad (4.2.27)$$

we obtain that

$$\frac{d}{dt} (\|\phi\|_2^2 + \epsilon \|u\|_1^2) + \|\phi_t\|^2 + \|u\|_2^2 \leq Q(\|\phi\|_2^2 + \epsilon \|u\|_1^2). \quad (4.2.28)$$

Hence, we end up with the inequality

$$\frac{dV}{dt} \leq \tilde{Q}(V), \quad (4.2.29)$$

where

$$V(t) = \|\phi\|_2^2 + \epsilon \|u\|_1^2.$$

By comparison principle of the solutions of Problem (4.2.29) with the ODE $\frac{dV}{dt} = Q(V)$, there exists a time $t_1 = t_1(V(0)) > 0$ such that

$$V(t) \leq Q(\|\phi_0\|_2^2 + \epsilon \|u_0\|_1^2), \quad \forall t \in [0, t_1].$$

We have

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{2,\epsilon}}^2 \leq V(t) \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}}^2), \quad \forall t \in [0, t_1]. \quad (4.2.30)$$

Then

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{2,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}}^2), \quad \forall t \in [0, t_1]. \quad (4.2.31)$$

If $t_1 \geq 1$, then the proof of is finished. If not, we use the same technique as we proceed from (2.2.65) to (2.2.69) to conclude that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{2,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}}^2)e^{-c_0 t} + c, \quad \forall t \in [t_1, 1]. \quad (4.2.32)$$

From estimates (4.2.31), (4.2.32) and (4.2.26), we entail that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{2,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}}^2)e^{-c_0 t} + c, \quad \forall t \geq 0. \quad (4.2.33)$$

The following estimates are useful in the proof of Theorem 4.4.1.

We integrate (4.2.28) between 0 and t , on account of (4.2.33), to get

$$\int_0^t (\|\phi_t(s)\|^2 + \|u(s)\|_2^2) ds \leq (1+t) Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}}^2), \quad \forall t \geq 0. \quad (4.2.34)$$

Owing to (4.2.15), we can obtain the following estimate

$$\begin{aligned}
\|h(u_m)\|_{L^{q^*}(0,t;L^{q^*})} &\leq c \int_0^t \int_{\Omega} (1 + |u_m|^{p+4})^{\frac{p+5}{p+4}} dx ds, \quad q^* = \frac{p+5}{p+4} \\
&\leq C \int_0^t (1 + \|u_m\|_{L^{p+5}(\Omega)}^{p+5}) ds \\
&\leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \forall t \geq 0.
\end{aligned} \tag{4.2.35}$$

We infer from (4.0.1)₂, (4.2.35) and (4.2.15) that

$$\begin{aligned}
\int_0^t \varepsilon^2 \|u_t(s)\|_{-1}^2 ds &\leq c \int_0^t (\|\phi_t(s)\|_{-1}^2 + \|u(s)\|_1^2 + \|h(u)\|_{-1}) ds \\
&\leq c \int_0^t (\|\phi_t(s)\|_{-1}^2 + \|u(s)\|_1^2 + \|h(u)\|_{L^{q^*}}) ds \\
&\leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\varepsilon}}^2), \quad \forall t \geq 0.
\end{aligned} \tag{4.2.36}$$

We also deduce from (4.0.1)₂, (4.2.35) and (4.2.34) that

$$\begin{aligned}
\int_0^t \varepsilon^2 \|u_t(s)\|^2 ds &\leq c \int_0^t (\|\phi_t(s)\|^2 + \|u(s)\|_2^2 + \|h(u)\|^2) ds \\
&\leq c \int_0^t (\|\phi_t(s)\|^2 + \|u(s)\|_2^2) ds + \int_0^t \|h(u)\|_{L^{q^*}}^2 ds \\
&\leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\varepsilon}}^2), \quad \forall t \geq 0.
\end{aligned} \tag{4.2.37}$$

Then, we multiply (4.0.1)₁ by $N^2\phi$ and obtain

$$\|\phi\|_4^2 + \int_{\Omega} r(\phi) N^2 \phi dx + \int_{\Omega} Ng(\phi) N^2 \phi dx = (Nu, N^2 \phi) - (\phi_t, N^2 \phi),$$

and we deduce that

$$\|\phi\|_4^2 \leq Q(\|\phi\|_2^2) + c(\|u\|_2^2 + \|\phi_t\|^2), \quad (4.2.38)$$

hence

$$\int_0^t \|\phi(s)\|_4^2 ds \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{2,\epsilon}}^2), \quad \forall t \geq 0, \quad (4.2.39)$$

on account of (4.2.34) and (4.2.33).

4.2.3 Estimates in $\mathcal{W}_{3,\epsilon}$

On account of (4.0.1)₂, we obtain

$$\|u\|_2^2 \leq c(\|\phi_t\|^2 + \epsilon\|u_t\|^2 + \|u\|_1^2).$$

Differentiate (4.0.1)₂ we get

$$\epsilon u_{tt} + \phi_{tt} + Au_t + h'(u)u_t = 0. \quad (4.2.40)$$

We multiply (4.2.40) and (4.2.16) by u_t and $A^{-1}\phi_{tt}$, respectively, integrate over Ω , we get

$$\epsilon \frac{1}{2} \frac{d}{dt} \|u_t\|^2 + \|u_t\|_1^2 + (\phi_{tt}, u_t) + \int_{\Omega} h'(u) |u_t|^2 dx = 0, \quad (4.2.41)$$

and

$$\frac{1}{2} \frac{d}{dt} \|\phi_t\|_1^2 + \|\phi_{tt}\|_{-1}^2 - (u_t, \phi_{tt}) + \int_{\Omega} \nabla[g'(\phi)\phi_t] \cdot \nabla A^{-1} \phi_{tt} dx + \int_{\Omega} r'(\phi)\phi_t A^{-1} \phi_{tt} dx = 0. \quad (4.2.42)$$

Summing the equations (4.2.41) and (4.2.42) we can deduce

$$\frac{d}{dt} (\epsilon \|u_t\|^2 + \|\phi_t\|_1^2) + \|\phi_{tt}\|_{-1}^2 + \|u_t\|_1^2 \leq c \|\phi_t\|_1^2. \quad (4.2.43)$$

By applying Uniform Gronwall's lemma, we obtain

$$\epsilon \|u_t\|^2 + \|\phi_t\|_1^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 2r, \quad (4.2.44)$$

due to (4.2.59). Thus, we can conclude that

$$\begin{aligned} \|u\|_2^2 &\leq c(\|\phi_t\|^2 + \epsilon \|u_t\|^2 + \|u\|_1^2) \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 2r. \end{aligned} \quad (4.2.45)$$

We multiply (4.0.1)₁ by $A\phi$, integrate over Ω , and we obtain

$$\begin{aligned} \|\phi\|_3^2 + \int_{\Omega} \nabla g(\phi) \cdot \nabla A\phi dx + \int_{\Omega} \nabla A^{-1} r(\phi) \cdot \nabla A\phi dx \\ = \int_{\Omega} \nabla Au \cdot \nabla A\phi dx - \int_{\Omega} \nabla A^{-1} \phi_t \cdot \nabla A\phi dx. \end{aligned} \quad (4.2.46)$$

Since we have the estimates $\int_{\Omega} |\nabla g(\phi)|^2 dx \leq Q(\|\phi\|_j^2)$ and

$$\begin{aligned} \int_{\Omega} |\nabla A^{-1}r(\phi)|^2 dx &\leq \|\nabla A^{-1}r(\phi)\|^2 \\ &\leq Q(\|\phi\|_j^2), \end{aligned} \quad (4.2.47)$$

then we deduce from (4.2.46) that

$$\begin{aligned} \|\phi\|_3^2 &\leq Q(\|\phi\|_j^2) + c(\|u\|_1^2 + \|\phi_t\|_{-1}^2), \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r. \end{aligned} \quad (4.2.48)$$

Therefore, by using the estimate (4.2.15), we have

$$\int_0^t \|\phi\|_3^2 ds \leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 0. \quad (4.2.49)$$

As a consequence of (4.2.45) and (4.2.48), we have

$$\|\phi\|_3^2 + \|u\|_2^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 2r, \quad (4.2.50)$$

so that

$$\|\phi\|_3^2 + \epsilon \|u\|_2^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 2r. \quad (4.2.51)$$

We multiply (4.0.1)₁ and (4.0.1)₂ by $A\phi_t$ and A^2u , respectively, integrate over Ω , then

Summing the resulting equations, we obtain

$$\begin{aligned} & \frac{d}{dt}(\|\phi\|_3^2 + \epsilon\|u\|_2^2) + \|\phi_t\|_1^2 + \|u\|_3^2 \\ & + \int_{\Omega} \nabla\phi_t \cdot \nabla Ag(\phi) dx + \int_{\Omega} \nabla r(\phi) \cdot \nabla\phi_t dx + \int_{\Omega} \nabla h(u) \cdot \nabla Au dx = 0. \end{aligned} \quad (4.2.52)$$

We have

$$\begin{aligned} \int_{\Omega} |r'(\phi) \nabla\phi|^2 dx & \leq \|r'(\phi)\|_{L^\infty}^2 \|\nabla\phi\|^2 \\ & \leq Q(\|\phi\|_j^2) \|\phi\|_1^2 \\ & \leq Q(\|\phi\|_2^2), \end{aligned} \quad (4.2.53)$$

then

$$\begin{aligned} \left| \int_{\Omega} \nabla r(\phi) \cdot \nabla\phi_t dx \right| & \leq \|r'(\phi)\nabla\phi\| \|\nabla\phi_t\| \\ & \leq Q(\|\phi\|_2^2) + \frac{1}{4}\|\phi_t\|_1^2. \end{aligned} \quad (4.2.54)$$

By Taylor's formula with integral remainder and (4.1.3), we can deduce that there exists a constant $c > 0$ such that

$$|h'(u)| \leq c(1 + |u|^{p+3}).$$

We notice that

$$\begin{aligned}
\left| \int_{\Omega} h'(u) \nabla u \nabla Au \, dx \right| &\leq \int_{\Omega} |h'(u)| |\nabla u| |\nabla Au| \, dx \\
&\leq \|h'(u)\| \|\nabla u\|_{L^\infty} \|\nabla Au\| \\
&\leq C(1 + \|u\|_1^{p+3}) \|u\|_1^{\frac{1}{2}} \|u\|_3^{\frac{3}{2}} \\
&\leq Q(\|u\|_1^2) + \frac{1}{2} \|u\|_3^2, \quad d = 1, 2,
\end{aligned} \tag{4.2.55}$$

and

$$\begin{aligned}
\left| \int_{\Omega} h'(u) \nabla u \nabla Au \, dx \right| &\leq \int_{\Omega} (1 + |u|^4) |\nabla u| |\nabla Au| \, dx, \\
&\leq \|\nabla u\| \|\nabla Au\| + \|u\|_{L^6}^3 \|u\|_{L^\infty} \|\nabla u\|_{L^\infty} \|\nabla Au\| \\
&\leq \|\nabla u\| \|\nabla Au\| + \|u\|_1^3 \|u\|_2 \|u\|_2^{\frac{1}{2}} \|u\|_3^{\frac{1}{2}} \|u\|_3 \\
&\leq \|\nabla u\| \|\nabla Au\| + \|u\|_1^3 \|u\|_2^{\frac{3}{2}} \|u\|_3^{\frac{3}{2}} \\
&\leq Q(\|u\|_2^2) + \frac{1}{2} \|u\|_3^2, \quad d = 3,
\end{aligned} \tag{4.2.56}$$

and owing to (2.2.84) and (2.2.85), we can deduce from (4.2.52) that

$$\frac{d}{dt} (\|\phi\|_3^2 + \epsilon \|u\|_2^2) + \|\phi_t\|_1^2 + \|u\|_3^2 \leq Q(\|\phi\|_2^2)(1 + \|\phi\|_3^2) + Q(\|u\|_2^2). \tag{4.2.57}$$

Then, apply Lemma 1.5.11 to (4.2.57), and we obtain

$$\|\phi(t)\|_3^2 + \epsilon \|u(t)\|_2^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-ct} + c_1, \quad \forall t \geq 2r, \quad (4.2.58)$$

due to (4.2.49) and (4.2.23) (since $\int_t^{t+1} \|u\|_2^2 ds \leq \int_t^{t+r} (\epsilon \|u_t\|^2 + \|\phi_t\|^2) ds$).

Then, integrating (4.2.57) between t and $t+r$ we get

$$\int_t^{t+r} (\|\phi_t(s)\|_1^2 + \|u(s)\|_3^2) ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + C, \quad \forall t \geq 2r, \quad (4.2.59)$$

due to (4.2.51).

4.2.4 Estimates in $\mathcal{W}_{4,\epsilon}$

We multiply (4.0.1)₁ and (4.0.1)₂ by $A^2\phi_t$ and ∇Au , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\begin{aligned} \frac{d}{dt} (\|\phi\|_4^2 + \epsilon \|u\|_3^2) + \|\phi_t\|_2^2 + \|u\|_4^2 + \int_{\Omega} A\phi_t A^2 g(\phi) dx \\ + \int_{\Omega} A r(\phi) A\phi_t dx + \int_{\Omega} A h(u) A^2 u dx = 0. \end{aligned} \quad (4.2.60)$$

We have

$$\begin{aligned} A^2 g(\phi) &= 4g''(\phi) \nabla \phi \nabla \Delta \phi + g'(\phi) A^2 \phi \\ &\quad + 6g'''(\phi) |\nabla \phi|^2 \Delta \phi + 3g''(\phi) |\Delta \phi|^2 + g^{(4)}(\phi) |\nabla \phi|^4, \end{aligned} \quad (4.2.61)$$

then , we deduce the following estimate

$$\begin{aligned}
\int_{\Omega} |A^2 g(\phi)|^2 dx &\leq 4 \|g''(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^4}^2 \|\nabla \Delta \phi\|_{L^4}^2 + \|g'(\phi)\|_{L^\infty}^2 \|A^2 \phi\|^2 \\
&\quad + 6 \|g'''(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^4}^4 \|\Delta \phi\|_{L^\infty}^2 + 3 \|g''(\phi)\|_{L^\infty}^2 \|\Delta \phi\|_{L^4}^4 \\
&\quad + \|g^{(4)}(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^\infty}^2 \|\nabla \phi\|_{L^6}^6. \\
&\leq Q(\|\phi\|_3^2) \|\phi\|_4^2,
\end{aligned} \tag{4.2.62}$$

hence

$$\begin{aligned}
\left| \int_{\Omega} A^2 g(\phi) A \phi_t dx \right| &\leq \frac{1}{4} \|\phi_t\|_2^2 + c \|A^2 g(\phi)\|^2 \\
&\leq \frac{1}{4} \|\phi_t\|_2^2 + Q(\|\phi\|_3^2) \|\phi\|_4^2.
\end{aligned} \tag{4.2.63}$$

Furthermore, since we have

$$Ar(\phi) = r''(\phi)(\nabla \phi)^2 + r'(\phi)\Delta \phi,$$

then

$$\begin{aligned}
\int_{\Omega} |Ar(\phi)|^2 dx &\leq 2 \int_{\Omega} (|r''(\phi)|^2 |\nabla \phi|^4 + |r'(\phi)|^2 |\Delta \phi|^2) dx \\
&\leq C(\|r''(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^4}^4 + \|r'(\phi)\|_{L^\infty}^2 \|\Delta \phi\|^2) \\
&\leq Q(\|\phi\|_2^2),
\end{aligned}$$

hence, the following estimate holds

$$\begin{aligned}
\left| \int_{\Omega} Ar(\phi) A\phi_t dx \right| &\leq \|Ar(\phi)\| \|A\phi_t\| \\
&\leq Q(\|\phi\|_2^2) + \frac{1}{4}\|\phi_t\|_2^2.
\end{aligned} \tag{4.2.64}$$

By using Taylor's formula with integral remainder and (4.1.3), we can deduce that there exists a constant $c > 0$ such that

$$|h'(u)| \leq c(1 + |u|^{p+3}) \quad \text{and} \quad |h''(u)| \leq c(1 + |u|^{p+2}).$$

Now, by using Agmon's inequality and interpolation, we obtain

$$\begin{aligned}
\left| \int_{\Omega} h'(u) \Delta u A^2 u dx \right| &\leq \|h'(u)\| \|\Delta u\|_{L^\infty} \|A^2 u\| \\
&\leq c(1 + \|u\|_1^{p+3}) \|u\|_2^{\frac{1}{2}} \|u\|_4^{\frac{3}{2}} \\
&\leq Q(\|u\|_2^2) + \frac{1}{4}\|u\|_4^2, \quad d = 1, 2,
\end{aligned} \tag{4.2.65}$$

and

$$\begin{aligned}
& \left| \int_{\Omega} h'(u) \Delta u A^2 u \, dx \right| \\
& \leq c \|\Delta u\| \|A^2 u\| + c \|u\|_{L^6}^3 \|u\|_{L^\infty} \|\Delta u\|_{L^\infty} \|A^2 u\| \\
& \leq c \|u\|_2 \|u\|_4 + c \|u\|_1^3 \|u\|_2 (\|u\|_3^{\frac{1}{2}} \|u\|_4^{\frac{1}{2}}) \|u\|_4 \\
& \leq c \|u\|_2 \|u\|_4 + Q(\|u\|_2) \|u\|_3^{\frac{1}{2}} \|u\|_4^{\frac{3}{2}} \\
& \leq Q(\|u\|_2^2) (1 + \|u\|_3^2) + \frac{1}{4} \|u\|_4^2, \quad d = 3,
\end{aligned} \tag{4.2.66}$$

along with

$$\begin{aligned}
& \left| \int_{\Omega} h''(u) |\nabla u|^2 A^2 u \, dx \right| \leq \|h''(u)\|_{L^3} \|\nabla u\|_{L^6} \|\nabla u\|_{L^\infty} \|A^2 u\| \\
& \leq C(1 + \|u\|_1^{p+2}) \|u\|_2 \|u\|_3 \|u\|_4 \\
& \leq Q(\|u\|_2^2) \|u\|_3^2 + \frac{1}{4} \|u\|_4^2, \quad d = 1, 2,
\end{aligned} \tag{4.2.67}$$

and

$$\begin{aligned}
& \left| \int_{\Omega} h''(u) |\nabla u|^2 A^2 u \, dx \right| \leq c \|\nabla u\|_{L^\infty} \|\nabla u\| \|A^2 u\| + c \|u\|_{L^6}^3 \|\nabla u\|_{L^\infty}^2 \|A^2 u\| \\
& \leq c \|u\|_1 \|u\|_3 \|u\|_4 + c \|u\|_1^3 \|u\|_2 \|u\|_3 \|u\|_4, \\
& \leq Q(\|u\|_2^2) \|u\|_3^2 + \frac{1}{4} \|u\|_4^2 \quad d = 3.
\end{aligned} \tag{4.2.68}$$

Then, on account of (4.2.66) and (4.2.68), we deduce

$$\begin{aligned} \left| \int_{\Omega} Ah(u) A^2 u \, dx \right| &\leq \int_{\Omega} (|h''(u)| |\nabla u|^2 + |h'(u)| |\Delta u|) |A^2 u| \, dx \\ &\leq Q(\|u\|_2^2)(1 + \|u\|_3^2) + \frac{1}{2} \|u\|_4^2. \end{aligned} \quad (4.2.69)$$

Owing to the Equation (4.2.60), we find

$$\frac{d}{dt} (\|\phi\|_4^2 + \epsilon \|u\|_3^2) + \|\phi_t\|_2^2 + \|u\|_4^2 \leq Q(\|\phi\|_3^2)(1 + \|\phi\|_4^2) + Q(\|u\|_2^2)(1 + \|u\|_3^2). \quad (4.2.70)$$

Now, since we have the following estimates

$$\begin{aligned} \int_t^{t+r} \|\phi(s)\|_4^2 \, ds &\leq c \int_t^{t+r} (\|\phi_t\|^2 + \|u\|_2^2 + Q(\|\phi\|_j^2)) \, ds \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq r, \end{aligned} \quad (4.2.71)$$

along with

$$\int_t^{t+r} Q(\|u\|_2^2)(1 + \|u\|_3^2) \, ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 2r, \quad (4.2.72)$$

due to (4.2.59) and (4.2.45).

Then, applying Lemma 1.5.11 to (4.2.70), on account of (4.2.50), (4.2.71) and (4.2.72),

we deduce that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{4,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-ct} + c_1, \quad \forall t \geq 3r. \quad (4.2.73)$$

Integrating (4.2.70) between t and $t + r$ we get

$$\int_t^{t+r} (\|\phi_t(s)\|_2^2 + \|u(s)\|_4^2) ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + C, \quad \forall t \geq 3r. \quad (4.2.74)$$

4.2.5 Estimates in $\mathcal{W}_{5,\epsilon}$

We multiply (4.0.1)₁ and (4.0.1)₂ by $A^3\phi_t$ and N^4u , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\begin{aligned} & \frac{d}{dt} (\|\phi\|_5^2 + \epsilon \|u\|_4^2) + \|\phi_t\|_3^2 + \|u\|_5^2 \\ & + \int_{\Omega} \nabla A\phi_t \cdot \nabla A^2g(\phi) dx + \int_{\Omega} \nabla Ar(\phi) \cdot \nabla A\phi_t dx + \int_{\Omega} \nabla Ah(u) \cdot \nabla A^2u dx = 0. \end{aligned} \quad (4.2.75)$$

We have

$$\begin{aligned} \nabla A^2g(\phi) &= 10 g'''(\phi) |\nabla\phi|^2 \nabla\Delta\phi + 15 g'''(\phi) \nabla\phi |\Delta\phi|^2 \\ &+ 5 g''(\phi) \nabla\phi A^2\phi + 10 g''(\phi) \nabla\Delta\phi \Delta\phi \\ &+ g'(\phi) \nabla A^2\phi + 10 g^{(4)}(\phi) (\nabla\phi)^3 \Delta\phi + g^{(5)}(\phi) |\nabla\phi|^5, \end{aligned} \quad (4.2.76)$$

and, we find that

$$\begin{aligned}
& \int_{\Omega} |\nabla A^2 g(\phi)|^2 dx \\
& \leq c(10 \|g'''(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^6}^4 \|\nabla \Delta \phi\|_{L^6}^2 + 15 \|g'''(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^6}^2 \|\Delta \phi\|_{L^6}^4 \\
& \quad + 5 \|g''(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^\infty}^2 \|A^2 \phi\|^2 + 10 \|g''(\phi)\|_{L^\infty}^2 \|\nabla \Delta \phi\|_{L^4}^2 \|\Delta \phi\|_{L^4}^2 \\
& \quad + \|g'(\phi)\|_{L^\infty}^2 \|\nabla A^2 \phi\|^2 + 10 \|g^{(4)}(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^6}^6 \|\Delta \phi\|_{L^\infty}^2 \\
& \quad + \|g^{(5)}(\phi)\|_{L^\infty}^2 \|\nabla \phi\|_{L^6}^6 \|\nabla \phi\|_{L^\infty}^4). \\
& \leq Q(\|\phi\|_3^2) \|\phi\|_5^2,
\end{aligned} \tag{4.2.77}$$

hence, we deduce the following estimate

$$\begin{aligned}
| \int_{\Omega} \nabla A^2 g(\phi) \cdot \nabla A \phi_t dx | & \leq \frac{1}{4} \|\phi_t\|_3^2 + c \|\nabla A^2 g(\phi)\|^2 \\
& \leq \frac{1}{4} \|\phi_t\|_3^2 + Q(\|\phi\|_3^2) \|\phi\|_5^2.
\end{aligned} \tag{4.2.78}$$

In addition, since we also have

$$\nabla A r(\phi) = r'''(\phi) \nabla \phi |\nabla \phi|^2 + 3r''(\phi) \nabla \phi \Delta \phi + r'(\phi) \nabla \Delta \phi,$$

then

$$\begin{aligned}
\int_{\Omega} |\nabla Ar(\phi)|^2 dx &\leq 3\|r'''(\phi)\|_{L^\infty}^2 \|\nabla\phi\|_{L^\infty}^2 \|\nabla\phi\|_{L^4}^4 \\
&\quad + 9\|r''(\phi)\|_{L^\infty}^2 \|\nabla\phi\|_{L^4}^2 \|\Delta\phi\|_{L^4}^2 + 3\|r'(\phi)\|_{L^\infty}^2 \|\nabla\Delta\phi\|^2 \\
&\leq Q(\|\phi\|_2^2) \|\phi\|_3^2,
\end{aligned}$$

hence, we deduce the following estimate

$$\left| \int_{\Omega} \nabla Ar(\phi) \cdot \nabla A\phi_t dx \right| \leq Q(\|\phi\|_2^2) \|\phi\|_3^2 + \frac{1}{4} \|\phi_t\|_3^2. \quad (4.2.79)$$

We have

$$\nabla Ah(u) = h'''(u) |\nabla u|^2 + 3h''(u) \nabla u \cdot \Delta u + h'(u) \nabla \Delta u,$$

then, by using Agmon and interpolation inequalities, we obtain

$$\begin{aligned}
\left| \int_{\Omega} h'(u) \nabla \Delta u \cdot \nabla A^2 u dx \right| &\leq \|h'(u)\| \|\nabla \Delta u\|_{L^\infty} \|\nabla A^2 u\| \\
&\leq C(1 + \|u\|_1^{p+3}) \|u\|_3^{\frac{1}{2}} \|u\|_5^{\frac{3}{2}} \\
&\leq Q(\|u\|_1^2) \|u\|_3^2 + \frac{1}{6} \|u\|_5^2, \quad d = 1, 2, \quad (4.2.80)
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_{\Omega} h'(u) \nabla \Delta u \nabla A^2 u \, dx \right| &\leq c \|\nabla \Delta u\|_{L^\infty} \|\nabla A^2 u\| + c \|u\|_{L^6}^3 \|u\|_{L^\infty} \|\nabla \Delta u\|_{L^\infty} \|\nabla A^2 u\| \\
&\leq c \|u\|_4^{\frac{1}{2}} \|u\|_5^{\frac{3}{2}} + c \|u\|_1^3 \|u\|_2 \|u\|_4^{\frac{1}{2}} \|u\|_5^{\frac{3}{2}} \\
&\leq Q(\|u\|_2^2) \|u\|_4^2 + \frac{1}{6} \|u\|_5^2, \quad d = 3,
\end{aligned} \tag{4.2.81}$$

along with

$$\begin{aligned}
\left| \int_{\Omega} h''(u) \nabla u \Delta u \nabla A^2 u \, dx \right| &\leq \|h''(u)\|_{L^3} \|\nabla u\|_{L^6} \|\Delta u\|_{L^\infty} \|\nabla A^2 u\| \\
&\leq c(1 + \|u\|_1^{p+2}) \|u\|_2 \|u\|_4 \|u\|_5 \\
&\leq Q(\|u\|_2^2) \|u\|_4^2 + \frac{1}{6} \|u\|_5^2, \quad d = 1, 2,
\end{aligned} \tag{4.2.82}$$

and

$$\begin{aligned}
\left| \int_{\Omega} h''(u) \nabla u \Delta u \nabla A^2 u \, dx \right| &\leq c \int_{\Omega} (1 + |u|^3) |\nabla u| |\Delta u| |\nabla A^2 u| \, dx \\
&\leq c \|\nabla u\| \|\Delta u\|_{L^\infty} \|\nabla A^2 u\| + c \|u\|_{L^6}^2 \|u\|_{L^\infty} \|\nabla u\|_{L^6} \|\Delta u\|_{L^\infty} \|\nabla A^2 u\| \\
&\leq c \|u\|_1 \|u\|_4 \|u\|_5 + Q(\|u\|_2^2) \|u\|_4 \|u\|_5 \\
&\leq Q(\|u\|_2^2) \|u\|_4^2 + \frac{1}{4} \|u\|_5^2 \quad d = 3,
\end{aligned} \tag{4.2.83}$$

and with

$$\begin{aligned}
\left| \int_{\Omega} h'''(u) \nabla u |\nabla u|^2 \nabla A^2 u \, dx \right| &\leq \|h'''(u)\|_{L^6} \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^6}^2 \|\nabla A^2 u\| \\
&\leq C(1 + \|u\|_1^{p+1}) \|u\|_3 \|u\|_2^2 \|u\|_5 \\
&\leq Q(\|u\|_2^2) \|u\|_3^2 + \frac{1}{6} \|u\|_5^2, \quad d = 1, 2, \quad (4.2.84)
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_{\Omega} h'''(u) \nabla u |\nabla u|^2 \nabla A^2 u \, dx \right| &\leq c \|\nabla u\| \|\nabla u\|_{L^\infty}^2 \|\nabla A^2 u\| + c \|u\|_{L^6} \|u\|_{L^\infty} \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^6}^2 \|\nabla A^2 u\| \\
&\leq \|u\|_1 \|u\|_2 \|u\|_3 \|u\|_5 + c \|u\|_1 \|u\|_2^3 \|u\|_3 \|u\|_5 \\
&\leq Q(\|u\|_2^2) \|u\|_3^2 + \frac{1}{6} \|u\|_5^2, \quad d = 3. \quad (4.2.85)
\end{aligned}$$

Now, thanks to (4.2.80), (4.2.81), (4.2.82), (4.2.83), (4.2.84) and (4.2.85), we deduce

$$\left| \int_{\Omega} \nabla A h(u) \cdot \nabla A^2 u \, dx \right| \leq Q(\|u\|_2^2) \|u\|_4^2 + \frac{1}{2} \|u\|_5^2. \quad (4.2.86)$$

Owing to the Equation (4.2.75), we deduce the following estimate

$$\frac{d}{dt} (\|\phi\|_5^2 + \epsilon \|u\|_4^2) + \|\phi_t\|_3^2 + \|u\|_5^2 \leq Q(\|\phi\|_3^2) \|\phi\|_5^2 + Q(\|u\|_2^2) \|u\|_4^2. \quad (4.2.87)$$

Now, since we have (4.2.50) and

$$\begin{aligned} \int_t^{t+r} \|\phi(s)\|_5^2 ds &\leq c \int_t^{t+r} (\|\phi_t\|_1^2 + Q(\|\phi\|_2^2) \|\phi\|_3^2 + \|u\|_3^2 + Q(\|\phi\|_2^2)) ds, \\ &\leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 2r, \end{aligned} \quad (4.2.88)$$

and

$$\int_t^{t+r} Q(\|u\|_2^2) \|u\|_4^2 ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + c, \quad \forall t \geq 3r, \quad (4.2.89)$$

due to (4.2.74). Applying Lemma 1.5.11 to (4.2.87), we deduce that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{5,\epsilon}}^2 \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c t} + c_1, \quad \forall t \geq 4r. \quad (4.2.90)$$

Integrating (4.2.87) between t and $t + r$, we get

$$\int_t^{t+r} (\|\phi_t(s)\|_3^2 + \|u(s)\|_5^2) ds \leq Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2) e^{-c_0 t} + C, \quad \forall t \geq 4r. \quad (4.2.91)$$

Moreover, we can find

$$\int_{4r}^t (\|\phi_t(s)\|_3^2 + \|u(s)\|_5^2) ds \leq (1+t) Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2), \quad \forall t \geq 4r,$$

so that, from (4.0.1)₁ and (4.0.1)₂, we deduce that

$$\begin{aligned} \int_{4r}^t \|\phi\|_7^2 ds &\leq \int_{4r}^t [Q(\|\phi\|_5^2) + c(\|u\|_5^2 + \|\phi_t\|_3^2)] ds \\ &\leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2), \quad \forall t \geq 4r, \end{aligned} \quad (4.2.92)$$

and

$$\begin{aligned} \int_{4r}^t \varepsilon^2 \|u_t(s)\|_3^2 ds &\leq c \int_{4r}^t (\|\phi_t(s)\|_3^2 + \|u(s)\|_5^2) ds \\ &\leq (1+t)Q(\|(\phi_0, u_0)\|_{\mathcal{W}_{j,\epsilon}}^2), \quad \forall t \geq 4r. \end{aligned} \quad (4.2.93)$$

4.2.6 $\mathcal{W}_{1,\epsilon}$ - $\mathcal{W}_{1,\epsilon}$ Estimate of the difference of two solutions

We consider the difference (ϕ, u) of two solutions (ϕ_1, u_1) and (ϕ_2, u_2) of Problem (4.0.1) starting at the points (ϕ_{10}, u_{10}) and (ϕ_{20}, u_{20}) in $\mathcal{W}_j$ at time $t = 0$, respectively, that is,

$$\begin{cases} \phi_t + A(A\phi + l_1(t)\phi - u) + l_2(t)\phi = 0, \\ \epsilon u_t + \phi_t + Au + l_3(t)u = 0, \\ \phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0, \end{cases} \quad (4.2.94)$$

where

$$\begin{aligned} l(t) &= \int_0^1 g'(s\phi_1(t) + (1-s)\phi_2(t)) ds, \\ l_2(t) &= \int_0^1 r'(s\phi_1(t) + (1-s)\phi_2(t)) ds, \\ l_3(t) &= \int_0^1 h'(su_1(t) + (1-s)u_2(t)) ds. \end{aligned}$$

We multiply (4.2.94)₁ and (4.2.94)₂ by $A^{-1}\phi_t$ and u , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\phi\|_1^2 + \epsilon \|u\|^2) + \|\phi_t\|_{-1}^2 + \|u\|_1^2 \\ & + \int_{\Omega} \nabla[l(t)\phi] \cdot \nabla A^{-1}\phi_t \, dx + \int_{\Omega} l_2(t)\phi A^{-1}\phi_t \, dx + \int_{\Omega} l_3(t)|u|^2 \, dx = 0. \end{aligned} \quad (4.2.95)$$

On account of (2.2.100), we have

$$| \int_{\Omega} \nabla[l(t)\phi] \cdot \nabla A^{-1}\phi_t \, dx | \leq \frac{1}{4} \|\phi_t\|_{-1}^2 + \tilde{c}_j \|\phi\|_1^2 + \frac{1}{2} \|\phi\|_2^2, \quad (4.2.96)$$

where

$$\tilde{c}_j = Q(\|\phi_{10}\|_j^2 + \epsilon \|u_{10}\|_{j-1}^2, \|\phi_{20}\|_j^2 + \epsilon \|u_{20}\|_{j-1}^2).$$

Furthermore, since we have the following estimate

$$\begin{aligned} \|l_2(t)\phi\|^2 &= \int_{\Omega} |l_2(t)|^2 |\phi|^2 \, dx \\ &\leq \sup_{s \in [0,1]} \|r'(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty(\Omega)}^2 \|\phi\|^2 \\ &\leq \tilde{c}_j \|\phi\|_1^2, \end{aligned} \quad (4.2.97)$$

then, we deduce

$$\begin{aligned}
\left| \int_{\Omega} l_2(t) \phi A^{-1} \phi_t dx \right| &\leq \|l_2(t) \phi\| \|A^{-1} \phi_t\| \\
&\leq \tilde{c}_j \|\phi\|_1^2 + \frac{1}{4} \|\phi_t\|_{-1}^2.
\end{aligned} \tag{4.2.98}$$

By using Assumption (4.0.6), we get

$$\int_{\Omega} l_3(t) |u|^2 dx = \int_{\Omega} \int_0^1 h'(su_1(t) + (1-s)u_2(t)) ds |u|^2 dx \geq C_6 \|u\|^2. \tag{4.2.99}$$

Hence, it follows from (4.2.95) that

$$\frac{d}{dt} (\|\phi\|_1^2 + \epsilon \|u\|^2) + \|\phi_t\|_{-1}^2 + \|u\|_1^2 \leq \tilde{c}_j \|\phi\|_1^2 + \frac{1}{2} \|\phi\|_2^2. \tag{4.2.100}$$

Then, we multiply (4.2.94)₁ by ϕ , integrate over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|\phi\|^2 + \|\phi\|_2^2 + \int_{\Omega} l_1(t) \phi A \phi dx - \int_{\Omega} \nabla u \cdot \nabla \phi dx + \int_{\Omega} l_2(t) \phi \phi dx = 0. \tag{4.2.101}$$

We have

$$\begin{aligned}
\int_{\Omega} |l_2(t) \phi|^2 dx &\leq \int_{\Omega} |l_2(t)|^2 |\phi|^2 dx \\
&\leq \sup_{s \in [0,1]} \|r'(s\phi_1(t) + (1-s)\phi_2(t))\|_{L^\infty}^2 \|\phi\|^2 \\
&\leq \tilde{c}_j \|\phi\|^2,
\end{aligned} \tag{4.2.102}$$

and due to (2.2.100) and (4.2.102), we obtain

$$\frac{d}{dt}\|\phi\|^2 + \|\phi\|_2^2 \leq \tilde{c}_j\|\phi\|_1^2 + \frac{1}{2}\|u\|_1^2. \quad (4.2.103)$$

Summing the estimates (4.2.100) to (4.2.103), we deduce

$$\frac{dE_8}{dt} + c(\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \|\phi\|_2^2) \leq \tilde{c}_j E_8, \quad (4.2.104)$$

where

$$E_8(t) = \|\phi(t)\|_1^2 + \epsilon\|u(t)\|^2 + \vartheta_5\|\phi(t)\|^2.$$

Then, we apply the Gronwall's Lemma to (4.2.104), we find

$$E_8(t) \leq E_8(0)e^{ct},$$

so that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{1,\epsilon}}^2 \leq c\|(\phi_0, u_0)\|_{\mathcal{W}_{1,\epsilon}}^2 e^{c_1 t}, \quad \forall t \geq 0. \quad (4.2.105)$$

We integrate (4.2.104) between 0 and t , we deduce that

$$\int_0^t (\|\phi(s)\|_2^2 + \|u(s)\|_1^2 + \|\phi_t(s)\|_{-1}^2) ds \leq c\|(\phi_0, u_0)\|_{\mathcal{W}_{1,\epsilon}}^2 e^{c_1 t}, \quad \forall t \geq 0. \quad (4.2.106)$$

4.2.7 $\mathcal{W}_{1,\epsilon}$ - $\mathcal{W}_{2,\epsilon}$ Estimate of the difference of two solutions

We multiply (2.2.94)₁ and (2.2.94)₂ by ϕ_t and Au , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\frac{d}{dt}(\|\phi\|_2^2 + \epsilon\|u\|_1^2) + 2\|\phi_t\|^2 + 2\|u\|_2^2 + 2 \int_{\Omega} \phi_t A[l(t)\phi] dx + \int_{\Omega} l_2(t)\phi\phi_t dx + \int_{\Omega} l_3(t)uAu dx = 0. \quad (4.2.107)$$

We have

$$\begin{aligned} \left| \int_{\Omega} l_2(t)\phi\phi_t dx \right| &\leq \|l_2(t)\phi\| \|\phi_t\| \\ &\leq \tilde{c}_j \|\phi\|_1^2 + \frac{1}{4} \|\phi_t\|^2, \end{aligned} \quad (4.2.108)$$

due to (4.2.97). Moreover, we have

$$\begin{aligned} \left| \int_{\Omega} l_3(t)uAu dx \right| &\leq \int_{\Omega} |l_3(t)| |u| |Au| dx \\ &\leq \sup_{s \in [0,1]} \|h'(su_1(t) + (1-s)u_2(t))\| \|u\|_{L^\infty} \|Au\| \\ &\leq (1 + \|u_1\|_{L^{2p+4}}^{p+2} + \|u_2\|_{L^{2p+4}}^{p+2}) \|u\|_1^{\frac{1}{2}} \|u\|_2^{\frac{1}{2}} \|u\|_2 \\ &\leq (1 + \|u_1\|_1^{p+2} + \|u_2\|_1^{p+2}) \|u\|_1^{\frac{1}{2}} \|u\|_2^{\frac{3}{2}} \\ &\leq \tilde{c}_j \|u\|_1^2 + \frac{1}{2} \|u\|_2^2, \quad \forall t \geq r. \end{aligned} \quad (4.2.109)$$

Also by using the Estimate (2.2.120); we deduce from (4.2.107) that

$$\frac{d}{dt}(\|\phi\|_2^2 + \epsilon\|u\|_1^2) + \|\phi_t\|^2 + \|u\|_2^2 \leq \tilde{c}_j \|\phi\|_2^2 + \tilde{c}_1(t) \|\phi\|_1^2 + \tilde{c}_j \|u\|_1^2, \quad (4.2.110)$$

so that

$$\frac{d}{dt}E_{10} + \|\phi_t\|^2 + \|u\|_2^2 \leq \tilde{c}_j E_{10} + \tilde{c}_1(t)\|\phi\|_1^2 + \tilde{c}_j \|u\|_1^2, \quad \forall t \geq r, \quad (4.2.111)$$

where

$$E_{10}(t) = \|\phi(t)\|_2^2 + \epsilon \|u(t)\|_1^2.$$

Now, we multiply (4.2.111) by $(t - r)$, we get

$$\frac{d}{dt}[(t - r)E_{10}] \leq E_{10}(t) + \tilde{c}_j(t - r)E_{10}(t) + (t - r)(\tilde{c}_j \|u\|_1^2 + \tilde{c}_1(t)\|\phi\|_1^2), \quad (4.2.112)$$

then we integrate between r and t , we find

$$(t - r)E_{10}(t) \leq (1 + \tilde{c}_j(t - r)) \int_r^t E_{10}(s)ds + (t - r) \int_r^t (\tilde{c}_j \|u\|_1^2 + \tilde{c}_1(s)\|\phi\|_1^2)ds,$$

hence

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{2,\epsilon}}^2 \leq \tilde{c}_j \|(\phi_0, u_0)\|_{\mathcal{W}_{1,\epsilon}}^2 e^{c_1 t}, \quad \forall t \geq 2r. \quad (4.2.113)$$

4.3 The unperturbed problem

Let us now consider the limit equation of (4.0.1) (i.e. $\epsilon = 0$ in (4.0.1)):

$$\begin{cases} \phi_t + A(A\phi + g(\phi) - u) + r(\phi) = 0, \\ \phi_t + Au + h(u) = 0, \\ \phi|_{t=0} = \phi_0. \end{cases} \quad (4.3.1)$$

We notice that, in contrast to the case $\epsilon > 0$, the values of $(\phi(t), u(t))$ are not independent in that case. Indeed, it follows from (4.3.1), by eliminating ϕ_t , the following equality

$$2Au + h(u) = A(A\phi + g(\phi)) + r(\phi). \quad (4.3.2)$$

Therefore, the value of $u(t)$ is uniquely defined by (4.3.2), if the value $\phi(t)$ is known.

Next, we will show there exists a nonlinear operator $\mathcal{T}$ such that (4.3.2) is equivalent to

$$u(t) = \mathcal{T}(\phi(t)).$$

From (4.3.2), and by setting $F(\phi) = A(A\phi + g(\phi)) + r(\phi)$, we have

$$\begin{cases} 2Au + h(u) = F(\phi), \\ u|_{\partial u} = 0. \end{cases} \quad (4.3.3)$$

From a well known result, Assume $h'(u) \geq 0$, if $F(\phi) \in H^{-1}(\Omega)$, then there exists a unique solution $u(t) \in H_0^1(\Omega)$.

Indeed, we multiply (4.3.3) by u and integrate over Ω , and by using (4.0.6), we get

$$\|u(t)\|_1^2 + \mathcal{C}\|u\|_{L^{p+5}}^{p+5} \leq \|F(\phi(t))\|_{-1}^2 + c \implies \|u(t)\|_1^2 \leq C.$$

Moreover, if $F(\phi) \in L^2(\Omega)$, then there exists unique solution $u(t) \in H^2(\Omega)$.

Indeed, we multiply (4.3.3) by Au and integrate over Ω , we get

$$\|u\|_2^2 + \int_{\Omega} h'(u)|\nabla u|^2 dx \leq c\|F(\phi)\|^2 \implies \|u\|_2^2 \leq C.$$

Thus, we can deduce there is exist a unique u , such that

$$u(t) = \mathcal{T}(\phi(t)),$$

and

$$\|\mathcal{T}(\phi)\|_1^2 = \|u\|_1^2 \leq c \quad \text{and} \quad \|\mathcal{T}(\phi)\|_2^2 = \|u\|_2^2 \leq c.$$

Therefore, we can rewrite the limit equation of (4.0.1) as

$$\begin{cases} \phi_t + A(A\phi + g(\phi) - \mathcal{T}(\phi(t))) + r(\phi) = 0, \\ \phi|_{t=0} = \phi_0. \end{cases} \quad (4.3.4)$$

A priori estimates for the unperturbed problem can be obtained by following the same method used for the perturbed problem in Section 4.2 but with $\epsilon = 0$. We deduced

the following estimates

$$\|\phi(t)\|_j^2 \leq Q(\|\phi_0\|_j^2)e^{-c_0t} + c, \quad \forall t \geq 0, \quad (4.3.5)$$

$$\int_{\Omega} H(s) dx \leq Q(\|\phi_0\|_j^2)e^{-c_0t} + c, \quad \forall t \geq 0, \quad (4.3.6)$$

$$\|\phi(t)\|_2^2 \leq Q(\|\phi_0\|_1^2)e^{-ct} + c_1, \quad \forall t \geq r, \quad (4.3.7)$$

$$\|\phi(t)\|_3^2 \leq Q(\|\phi_0\|_j^2)e^{-ct} + c_1, \quad \forall t \geq 2r, \quad (4.3.8)$$

$$\|\phi(t)\|_4^2 \leq Q(\|\phi_0\|_j^2)e^{-ct} + c_1, \quad \forall t \geq 3r, \quad (4.3.9)$$

$$\|\phi(t)\|_5^2 \leq Q(\|\phi_0\|_j^2)e^{-ct} + c_1, \quad \forall t \geq 4r. \quad (4.3.10)$$

Moreover, the estimates of the difference ϕ of two solutions ϕ_1, ϕ_2 of problem (4.3.1) starting at the points ϕ_{10} and ϕ_{20} in V_j at time $t = 0$ are

$$\|\phi(t)\|_1^2 \leq \|\phi_0\|_1^2 e^{c_1t}, \quad \forall t \geq 0, \quad (4.3.11)$$

$$\|\phi(t)\|_2^2 \leq \tilde{c}_j \|\phi_0\|_1^2 e^{c_1t}, \quad \forall t \geq 2r. \quad (4.3.12)$$

4.4 Well-posedness

We start the proof by giving the weak formulation of the problem (4.0.1).

Weak formulation: For any $T > 0$, find

$$(\phi, u) : [0, T] \rightarrow \mathcal{W}_{2,\epsilon},$$

such that

$$\phi(0, x) = \phi_0(x), \quad u(0, x) = u_0(x),$$

and for almost any $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt}(\phi, q) + (A\phi, Aq) + (\nabla g(\phi), \nabla q) - (\nabla u, \nabla q) + (r(\phi), q) &= 0, \quad q \in V_2, \\ \frac{d}{dt}[\epsilon(u, v) + (\phi, v)] + (\nabla u, \nabla v) + (h(u), v) &= 0, \quad v \in V_1. \end{aligned} \quad (4.4.1)$$

Theorem 4.4.1 *We assume (4.0.2)-(4.1.3) hold. If $(\phi_0, u_0) \in \mathcal{W}_{j,\epsilon}$, then Problem (4.0.1) possesses a unique solution (ϕ, u) such that*

$$(\phi, u) \in \mathcal{C}([0, \infty]; \mathcal{W}_{j,\epsilon}) \cap \mathcal{C}([\tau, \infty]; \mathcal{W}_{5,\epsilon}) \cap L^2(0, T; \mathcal{W}_{j+1,\epsilon}) \cap L^2(0, T; \mathcal{W}_{j+2,0}),$$

$$\forall T > 0, \quad \forall \tau > 0.$$

Proof. Part1: Existence of the solution

We proceed Like in the proof of Theorem 2.4.1. We first notice that, according to a priori estimates derived in the Section 4.2, we have

$$\sup_{t \in [0, T]} \|\phi_m(t)\|_j^2 \leq C, \quad (4.4.2)$$

$$\sup_{t \in [0, T]} \epsilon \|u_m(t)\|_{j-1}^2 \leq C, \quad (4.4.3)$$

$$\int_0^T (\|\phi_m(s)\|_3^2 + \|\phi_{mt}(s)\|_{-1}^2 + \|u_{mt}(s)\|_{-1}^2 + \|u_m(s)\|_1^2) ds \leq C(T), \quad d = 1, \quad (4.4.4)$$

$$\int_0^T (\|\phi_m(s)\|_4^2 + \|\phi_{mt}(s)\|^2 + \|u_{mt}(s)\|^2 + \|u_m(s)\|_2^2) ds \leq C(T), \quad d = 2, 3. \quad (4.4.5)$$

The right hand side of the inequalities (4.4.2) to (4.4.5) are independent of m . Thus the sequences ϕ_m , u_m and ϕ_{mt} remain uniformly bounded in $L^\infty(0, T; V_j)$, $L^\infty(0, T; V_{j-1})$ and $L^\infty(0, T; V_{j-1})$, respectively. Moreover, the sequences ϕ_m , u_m , ϕ_{mt} and u_{mt} remains uniformly bounded in $L^2(0, T; V_{j+2})$, $L^2(0, T; V_j)$, $L^2(0, T; V_{j-2})$ and $L^2(0, T; V_{j-2})$, respectively.

We now use the bounds on ϕ_m, u_m to obtain bounds on the nonlinear terms $g(\phi_m)$, $r(\phi_m)$ and $h(u_m)$. Since we have

$$\phi_m \rightarrow \phi \text{ in } L^2(0, T; V_{j-1}(\Omega)) \text{ strongly and a.e in } (0, T) \times \Omega \quad (4.4.6)$$

then, by proceeding as we did from (2.4.14) through (2.4.16), we deduce that

$$P_m g(\phi_m) \rightarrow g(\phi) \quad \text{in } L^2(0, T; L^2), \quad (4.4.7)$$

$$P_m r(\phi_m) \rightarrow r(\phi) \quad \text{in } L^2(0, T; L^2). \quad (4.4.8)$$

Moreover, from (4.2.35), we have $\|h(u)\|_{L^{q^*}(0, T; L^{q^*})} \leq C$ where $q^* = \frac{p+5}{p+4}$.

Therefore, we have subsequences still denoted ϕ_m and u_m such that

$$\phi_m \xrightarrow{*} \phi \quad \text{in} \quad L^\infty(0, T; V_j) \quad (4.4.9)$$

$$u_m \xrightarrow{*} u \quad \text{in} \quad L^\infty(0, T; V_{j-1}) \quad (4.4.10)$$

$$\phi_m \rightharpoonup \phi \quad \text{in} \quad L^2(0, T; V_{j+2}) \quad (4.4.11)$$

$$u_m \rightharpoonup u \quad \text{in} \quad L^2(0, T; V_j) \quad (4.4.12)$$

$$\phi_{mt} \rightharpoonup \phi_t \quad \text{in} \quad L^2(0, T; V_{j-2}) \quad (4.4.13)$$

$$u_{mt} \rightharpoonup u_t \quad \text{in} \quad L^2(0, T; V_{j-2}) \quad (4.4.14)$$

$$h(u_m) \rightharpoonup \chi_3 \quad \text{in} \quad L^q(0, T; L^q), \quad q = \frac{p+5}{p+4}. \quad (4.4.15)$$

It remains to show that $\chi_3 = h(u)$. Thanks to (4.4.10) and (4.4.14), we deduce

$$u_m \rightarrow u \quad \text{in} \quad L^2(0, T; V_{j-1}(\Omega)) \quad \text{strongly and a.e in } (0, T) \times \Omega. \quad (4.4.16)$$

Therefore there is a subsequence u_{mj} such that

$$u_{mj}(t, x) \rightarrow u(t, x) \quad \text{for almost every } (t, x) \text{ in } (0, T) \times \Omega.$$

It follows from the continuity of h that $h(u_{mj}(t, x)) \rightarrow h(u(t, x))$ for almost every $(t, x) \in (0, T) \times \Omega$.

This together with (4.2.35) by applying Lemma 1.5.4 will gives the following weak

convergence

$$h(u_m) \rightharpoonup h(u) \text{ in } L^{q^*}(0, T; L^{q^*}), \quad q^* = \frac{p+5}{p+4}. \quad (4.4.17)$$

By the uniqueness of limits $\chi_3 = h(u)$. Hence, $P_m h(u_m) \rightharpoonup h(u)$ in $L^{q^*}(0, T; L^{q^*})$.

Therefore, if $(\phi_0, u_0) \in \mathcal{W}_{j,\epsilon}$, then the approximate solution (ϕ_m, u_m) are bounded independently of m and there exist the pair of functions (ϕ, u) such that $(\phi_m, u_m) \rightarrow (\phi, u)$ and $\phi \in L^\infty(0, T; V_j) \cap L^2(0, T; V_{j+2})$, $u \in L^\infty(0, T; V_{j-1}) \cap L^2(0, T; V_j)$. Moreover, $\phi_t \in L^2(0, T; V_{j-2})$ and $u_t \in L^2(0, T; V_{j-2})$. Again Like in the proof of Theorem 2.4.1, we can pass to the limit as $m \rightarrow \infty$ in the approximate problem, where (ϕ, u) is solution to (4.0.1); with $\phi_0 = \phi(0)$ and $u_0 = u(0)$.

Part2: Uniqueness of the solution

Let (ϕ_1, u_1) and (ϕ_2, u_2) be two solutions of (4.0.1). Setting $\phi = \phi_1 - \phi_2$ and $u = u_1 - u_2$, we have $\phi(0) = 0$, $u(0) = 0$ and (ϕ, u) satisfies the equations

$$\phi_t + A(A\phi + g(\phi_1) - g(\phi_2) - u) + r(\phi_1) - r(\phi_2) = 0, \quad (4.4.18)$$

$$\epsilon u_t + \phi_t + Au + h(u_1) - h(u_2) = 0. \quad (4.4.19)$$

Proceeding like in the proof of uniqueness in Theorem 2.4.1, we obtain

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{1,\epsilon}}^2 \leq \tilde{c} \|(\phi_0, u_0)\|_{\mathcal{W}_{1,\epsilon}}^2 e^{c_1 t}, \quad \forall t \in [0, T], \text{ for any } T > 0, \quad (4.4.20)$$

and since $\phi_0 = 0$ and $u_0 = 0$, hence the result.

Part3: Regularity of the solution

Like in the proof of Theorem 2.4.1, from classical compactness theorems, on account of $\phi \in L^2(0, T; V_{j+2})$, $\phi_t \in L^2(0, T; V_{j-2})$ and $u \in L^2(0, T; V_j)$, $u_t \in L^2(0, T; V_{j-2})$; also $\phi \in L^2(\tau, T; V_7)$, $\phi_t \in L^2(\tau, T; V_3)$ and $u \in L^2(\tau, T; V_5)$, $u_t \in L^2(\tau, T; V_3)$ we can deduce

$$(\phi, u) \in \mathcal{C}([0, \infty], \mathcal{W}_{j,\epsilon}) \text{ and } (\phi, u) \in \mathcal{C}([\tau, \infty]; \mathcal{W}_{5,\epsilon}).$$

I

4.5 The global attractor

Thanks to Theorem 4.4.1, we define the semigroup

$$S_\epsilon(t) : \mathcal{W}_{j,\epsilon} \rightarrow \mathcal{W}_{j,\epsilon}, \quad (\phi_0, u_0) \mapsto (\phi(t), u(t)), \quad t \geq 0,$$

where $(\phi(t), u(t))$ is the solution to (4.3.1) at time t and for every $\epsilon \in (0, 1]$.

Due to Theorem 4.4.1, the semigroup $S_\epsilon(t)$ is continuous with respect to time $t \rightarrow S_\epsilon(t)(\phi_0, u_0) \in \mathcal{C}([0, \infty], \mathcal{W}_{j,\epsilon})$ for almost every $x \in \Omega$, and the continuity with respect to initial data which is given by

$$\|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t)(\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}} \leq c\|(\phi_{01}, u_{01}) - (\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}}$$

is proved in (4.2.105), hence we can deduce that the semigroup $S_\epsilon(t)$ is continuous.

Note that, from Section 4.2, we can deduce the existence of absorbing sets $\mathcal{B}_{k,\epsilon} \subset \mathcal{W}_{k,\epsilon}$

for $S_\epsilon(t)$ of the form

$$\mathcal{B}_{k,\epsilon} = \{(\varphi, \psi) \in \mathcal{W}_{k,\epsilon}, \|(\varphi, \psi)\|_{\mathcal{W}_{k,\epsilon}} \leq r_k\}, \quad k = j, \dots, 5 \quad (4.5.1)$$

where r_k is independent of ϵ . Hence, as in Proposition 2.5.1, we can deduce that the semigroup $S_\epsilon(t)$ is uniformly compact for every $\epsilon \in (0, 1]$.

Now, we state the following result.

Theorem 4.5.1 *Let the assumptions of Theorem 4.4.1 hold. Then, for every $\epsilon \in (0, 1]$, the semigroup $S_\epsilon(t)$ has the global attractor $\mathcal{A}_\epsilon$, in the metric topology of $\mathcal{W}_{1,\epsilon}$, which is bounded in $\mathcal{W}_{5,\epsilon}$; there holds*

$$\lim_{t \rightarrow \infty} \text{dist}_{\mathcal{W}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{A}_\epsilon) = 0,$$

for any bounded subset $B \subset \mathcal{W}_{j,\epsilon}$.

From now on, we will assume that

$$\mathcal{B}_{0,k} = \{\varphi \in V_k, \|\varphi\|_{V_k} \leq r_k\}, \quad k = 1, 2, 3, 4, 5 \quad (4.5.2)$$

(where $r_k > 0$ are the same as in $\mathcal{B}_{k,\epsilon}$) are absorbing sets for the semigroup

$$S(t) : V_j \rightarrow V_j, \quad \phi_0 \mapsto \phi(t), \quad t \geq 0,$$

generated by the corresponding unperturbed problem (for the variable ϕ alone). A priori estimates for the unperturbed problem are obtained in Section 4.3 by following

the same method used for the perturbed problem in Section 4.2. The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$. Observe that, a solution of the unperturbed problem for both variables ϕ and u (at time t) is given by

$$\phi(t) = S(t)\phi_0 \quad \text{and} \quad u(t) = \mathcal{T}(\phi(t)). \quad (4.5.3)$$

We define

$$(\mathcal{A})_0 = \{(\varphi, \mathcal{T}(\varphi)), \varphi \in \mathcal{A}\}.$$

In order to compare the global attractor $\mathcal{A}_\epsilon$ with the set $(\mathcal{A})_0$ for ϵ in a neighborhood of $\epsilon = 0$, we need the following lemma.

Lemma 4.5.1 *There exists a constant $C > 0$ independent of ϵ such that for any complete trajectory $(\phi(t), u(t))_{t \in \mathbb{R}} = (S_\epsilon(t)(\phi_0, u_0))_{t \in \mathbb{R}} \subset \mathcal{A}_\epsilon$ we have*

$$\|\phi(t)\|_5^2 + \|u(t)\|_3^2 + \|\phi_t(t)\|_1^2 + \epsilon \|u_t(t)\|^2 \leq C, \quad \forall t \in \mathbb{R}. \quad (4.5.4)$$

Proof. Let $(\phi(t), u(t))_{t \in \mathbb{R}}$ be complete trajectory $\mathcal{A}_\epsilon$ which is a solution of (4.0.1).

As $\mathcal{A}_\epsilon \subset \mathcal{B}_{5,\epsilon}$, it follows that there exists $c > 0$ independent of ϵ such that

$$\|(\phi(t), u(t))\|_{\mathcal{W}_{5,\epsilon}}^2 \leq c, \quad \forall t \in \mathbb{R}. \quad (4.5.5)$$

Now, we multiply (4.2.40) and (4.2.16) by u_t and $A^{-1}\phi_{tt}$, respectively, integrate over Ω , we get

$$\frac{d}{dt}(\epsilon \|u_t\|^2 + \|\phi_t\|_1^2) + \|\phi_{tt}\|_{-1}^2 + \frac{1}{2} \|u_t\|_1^2 \leq c \|\phi_t\|_1^2. \quad (4.5.6)$$

From (4.2.20), we have

$$\frac{d}{dt} \left(\|\phi_t\|_{-1}^2 + \|u\|_1^2 + \int_{\Omega} H(u) dx \right) + \|\phi_t\|_1^2 + \epsilon \|u_t\|^2 \leq c \|\phi_t\|_{-1}^2. \quad (4.5.7)$$

In addition, from (4.2.3), we have

$$\frac{d}{dt} \left(\epsilon \|u\|^2 + \|\phi\|_1^2 + 2 \int_{\Omega} G(\phi) dx \right) + \|u\|_1^2 + \|\phi_t\|_{-1}^2 + \int_{\Omega} H(u) dx \leq C. \quad (4.5.8)$$

Summing (4.5.8), ϑ_1 (4.5.6) and ϑ_2 (4.5.7), we obtain

$$\begin{aligned} \frac{d}{dt} \left(\epsilon \|u_t\|^2 + \|\phi_t\|_1^2 + \|\phi_t\|_{-1}^2 + \|u\|_1^2 + \epsilon \|u\|^2 + \|\phi\|_1^2 + 2 \int_{\Omega} G(\phi) dx + \int_{\Omega} H(u) dx \right) \\ + \|\phi_t\|_1^2 + \epsilon \|u_t\|^2 + \|\phi_{tt}\|_{-1}^2 + \frac{1}{2} \|u_t\|_1^2 + c \|u\|_1^2 + \int_{\Omega} H(u) dx \leq C, \end{aligned} \quad (4.5.9)$$

then

$$\frac{d}{dt} E_6(t) + c E_6(t) \leq C, \quad (4.5.10)$$

where

$$E_6 = \epsilon \|u_t\|^2 + \|\phi_t\|_1^2 + \|\phi_t\|_{-1}^2 + \|u\|_1^2 + \epsilon \|u\|^2 + \|\phi\|_1^2 + 2 \int_{\Omega} G(\phi) dx + \int_{\Omega} H(u) dx.$$

Now, we multiply (4.0.1)₂ by ϵAu_t , on account of (4.5.5), we get

$$\begin{aligned}
\epsilon^2 \|u_t(t)\|^2 &\leq c(\|\phi_t\|^2 + \|u\|_2^2) \\
&\leq c(\|\phi_t\|^2 + \frac{1}{\epsilon} \epsilon \|u\|_2^2) \\
&\leq \frac{c}{\epsilon}, \quad \forall t \in \mathbb{R}.
\end{aligned} \tag{4.5.11}$$

Applying the Gronwall's Lemma to (4.5.10) between t_0 to t , for any $t_0, t \in \mathbb{R}$ to obtain

$$\begin{aligned}
E_6(t) &\leq e^{-c_0(t-t_0)} E_6(t_0) + c \\
&\leq Q(\epsilon \|u_t(t_0)\|^2 + \|\phi_t(t_0)\|_1^2 + \|\phi(t_0)\|_j^2 + \|u(t_0)\|_1^2) e^{-c_0(t-t_0)} + c \\
&\leq \frac{c}{\epsilon^2} e^{-c_0(t-t_0)} + c, \quad \forall t \geq t_0, \quad t_0 \in \mathbb{R},
\end{aligned} \tag{4.5.12}$$

hence

$$\epsilon \|u_t\|^2 + \|\phi_t\|_1^2 + \|u\|_1^2 \leq \frac{c}{\epsilon^2} e^{-c_0(t-t_0)} + c, \quad \forall t \geq t_0.$$

Since we have $\|u\|_3^2 \leq c(\|\phi_t\|_1^2 + \|\phi\|_5^2)$, then we obtain

$$\|\phi_t(t)\|_1^2 + \|u(t)\|_3^2 + \epsilon \|u_t(t)\|^2 \leq \frac{c}{\epsilon^2} e^{-c_0(t-t_0)} + C, \quad \forall t \geq t_0, \quad t_0, t \in \mathbb{R}. \tag{4.5.13}$$

Taking the limit in (4.5.13) as $t_0 \rightarrow -\infty$, we deduce that

$$\|\phi(t)\|_5^2 + \|u(t)\|_3^2 + \|\phi_t(t)\|_1^2 + \epsilon \|u_t(t)\|^2 \leq c, \quad \forall t \in \mathbb{R}.$$

I

We show now the stability property of the global attractor.

Theorem 4.5.2 *Let the assumptions of Theorem 4.4.1 hold. Then, for any $\eta > 0$ the global attractor $\mathcal{A}_\epsilon$ is upper semicontinuous at $\epsilon = 0$, that is,*

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{U}_{5-\eta}}(\mathcal{A}_\epsilon, (\mathcal{A})_0) = 0. \quad (4.5.14)$$

Proof. Similar to the proof of Theorem 2.5.2. We assume there exists $\rho > 0$, sequences $\epsilon_n \in (0, 1]$, $\epsilon_n \rightarrow 0$, and a corresponding sequence $z_{0n} \in \mathcal{A}_{\epsilon_n}$ such that

$$\text{dist}_{\mathcal{U}_{5-\eta}}(z_{0n}, (\mathcal{A})_0) \geq \rho. \quad (4.5.15)$$

We set

$$\mathcal{A}^* = \bigcup_{\epsilon \in (0, 1]} \mathcal{A}_\epsilon. \quad (4.5.16)$$

Owing to Lemma 4.5.1, $\mathcal{A}^*$ is relatively compact in $\mathcal{U}_{5-\eta}$, for any $\eta > 0$. Let the complete trajectory $(\phi^n(t), u^n(t))_{t \in \mathbb{R}}$ such that $z_n(t) = (\phi^n(t), u^n(t)) = S_\epsilon(t)z_{0n}$. Due to the invariance property of the global attractor, it follows that

$$\bigcup_{t \in \mathbb{R}} \bigcup_{n \in \mathbb{N}} z_n(t) \subset \mathcal{A}^*$$

is a relatively compact set in $\mathcal{U}_{5-\eta}$. Hence, $\bigcup_{t \in \mathbb{R}} \bigcup_{n \in \mathbb{N}} z_n(t)$ is a precompact set in $\mathcal{U}_{5-\eta}$, and the family of mappings $\{(\phi^n(t), u^n(t)) \in \mathcal{C}(\mathbb{R}; \mathcal{U}_{5-\eta}), n \geq 0\}$ is equicontinu-

ous from $\mathbb{R}$ into $\mathcal{U}_{5-\eta}$. By Ascoli's theorem and a classical diagonalization method,

It follows the existence of an element $(\tilde{\phi}, \tilde{u}) \in \mathcal{C}(\mathbb{R}; \mathcal{U}_{5-\eta})$ such that $\phi^n \rightarrow \tilde{\phi}$ in $\mathcal{C}([-T, T]; V_{5-\eta})$ and $u^n \rightarrow \tilde{u}$ in $\mathcal{C}([-T, T]; V_{3-\eta})$ that is,

$$(\phi^n, u^n) \rightarrow (\tilde{\phi}, \tilde{u}) \quad \text{in } \mathcal{C}([-T, T]; \mathcal{U}_{5-\eta}), \quad (4.5.17)$$

for any $T > 0$, and

$$\sup_{t \in \mathbb{R}} \|(\tilde{\phi}(t), \tilde{u}(t))\|_{\mathcal{U}_{5-\eta}} < \infty, \quad (4.5.18)$$

on account of (4.5.4). Observe that

$$\begin{cases} \phi_t^n = -A(A\phi^n + g(\phi^n) - u^n) - r(\phi^n), \\ \phi_t^n = -\epsilon u_t^n - Au^n - h(u^n). \end{cases} \quad (4.5.19)$$

Exploiting (4.5.4) , we deduce that

$$\phi_t^n \xrightarrow{*} \tilde{\phi}_t \text{ in } L^\infty([-T, T]; V_1),$$

$$\epsilon_n u_t^n \xrightarrow{*} 0 \text{ in } L^\infty([-T, T]; L^2).$$

Since $\phi^n \rightarrow \tilde{\phi} \in \mathcal{C}([-T, T]; V_{5-\eta})$ and the functions g and r are continuous, it follows that

$$g(\phi^n) \rightarrow g(\tilde{\phi}) \text{ in } \mathcal{C}([-T, T]; V_4),$$

$$r(\phi^n) \rightarrow r(\tilde{\phi}) \text{ in } \mathcal{C}([-T, T]; V_2).$$

Similarly, since $u^n \rightarrow \tilde{u} \in \mathcal{C}([-T, T]; V_{3-\eta})$ and h is continuous, it follows that

$$h(u^n) \rightarrow h(\tilde{u}) \text{ in } \mathcal{C}([-T, T]; V_2).$$

Hence,

$$\begin{aligned} \phi_t^n &\rightarrow \tilde{\phi}_t && \text{in } \mathcal{C}([-T, T]; V_{1-\eta}), \\ \phi_t^n &\rightarrow -A(A\tilde{\phi} + g(\tilde{\phi}) - \tilde{u}) - r(\tilde{\phi}) && \text{in } \mathcal{C}([-T, T]; V_{1-\eta}), \end{aligned}$$

and

$$\phi_t^n \rightarrow -A\tilde{u} - h(\tilde{u}) \quad \text{in } \mathcal{C}([-T, T]; V_{-\eta}), \quad (4.5.20)$$

for any $T > 0$. Passing to the limit $n \rightarrow \infty$ in (4.5.19), and by uniqueness of the limit, we find that

$$\begin{cases} \tilde{\phi}_t + A(A\tilde{\phi} + g(\tilde{\phi}) - \tilde{u}) + r(\tilde{\phi}) = 0 & \text{in } \mathcal{C}([-T, T]; V_{1-\eta}), \\ \tilde{\phi}_t + A\tilde{u} + h(\tilde{u}) = 0 & \text{in } \mathcal{C}([-T, T]; V_{-\eta}), \end{cases} \quad (4.5.21)$$

or

$$\phi_t + A(A\phi + g(\phi) - \mathcal{T}(\phi(t))) + r(\phi) = 0, \quad \text{in } \mathcal{C}([-T, T]; V_{1-\eta}) \quad (4.5.22)$$

for any $T > 0$. It follows that $(\tilde{\phi}(t), \tilde{u}(t))_{t \in \mathbb{R}}$ is a bounded complete trajectory of the unperturbed problem (4.5.21). By characterization of the attractor it is clear that $\tilde{\phi}(0) \in \mathcal{A}$, and by definition $(\tilde{\phi}(0), \mathcal{T}(\tilde{\phi}(0))) \in (\mathcal{A})_0$. Thus the convergence

$$z_{0n} \rightarrow (\tilde{\phi}(0), \mathcal{T}(\tilde{\phi}(0))) \text{ in } \mathcal{U}_{5-\eta},$$

implies that

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{U}_{5-\eta}}(z_{0n}, (\mathcal{A}^\alpha)_0) = 0, \quad (4.5.23)$$

against the initial assumption. I

4.6 Exponential attractor

We show the following estimate that will be utilized later in Proposition 4.6.1.

Lemma 4.6.1 *Let the assumptions of Theorem 2.4.1 hold. If $\phi_0 \in \mathcal{B}_{0,5}$, then*

$$\int_0^t \|u_t(s)\|_1^2 ds \leq c(1+t), \quad \forall t \geq 0. \quad (4.6.1)$$

Proof. Let $\phi_0 \in \mathcal{B}_{0,5}$. There holds

$$\|\phi(t)\|_5^2 \leq c, \quad \forall t \geq 0. \quad (4.6.2)$$

Recall the following equality for the unperturbed problem

$$2Au + h(u) = A(A\phi + g(\phi)) + r(\phi). \quad (4.6.3)$$

Multiplying (4.6.3) by Au , then integrate over Ω , we get

$$2\|u\|_2^2 + \int_{\Omega} h'(u)|\nabla u|^2 dx = \int_{\Omega} A^2\phi Au dx + \int_{\Omega} Ag(\phi) Au dx + \int_{\Omega} r(\phi) Au dx, \quad (4.6.4)$$

on account of the Assumption (4.0.6) and the estimate (4.6.2), we deduce

$$\begin{aligned} \|u(t)\|_2^2 &\leq c\|\phi(t)\|_4^2 + Q(\|\phi(t)\|_2^2) \\ &\leq c, \quad \forall t \geq 0. \end{aligned} \tag{4.6.5}$$

Multiplying (4.6.3) by A^2u , then integrate over Ω , we deduce

$$\begin{aligned} 2\|u\|_3^2 &= - \int_{\Omega} \nabla h(u) \cdot \nabla Au \, dx + \int_{\Omega} \nabla A^2\phi \cdot \nabla Au \, dx + \int_{\Omega} \nabla g(\phi) \cdot \nabla Au \, dx \\ &\quad + \int_{\Omega} \nabla r(\phi) \cdot \nabla Au \, dx, \end{aligned} \tag{4.6.6}$$

we notice that

$$\begin{aligned} \left| \int_{\Omega} h'(u) \nabla u \cdot \nabla Au \, dx \right| &\leq \|h'(u)\| \|\nabla u\|_{L^\infty} \|\nabla Au\| \\ &\leq C(1 + \|u\|_1^{p+2}) \|u\|_2^{\frac{1}{2}} \|u\|_3^{\frac{3}{2}} \\ &\leq Q(\|u\|_2^2) + \frac{1}{2}\|u\|_3^2, \end{aligned}$$

and by virtue (4.6.2) and (4.6.5), it follows from (4.6.6) that

$$\begin{aligned} \|u\|_3^2 &\leq c \|\phi\|_5^2 + Q(\|u\|_2^2) \\ &\leq c, \quad \forall t \geq 0. \end{aligned} \tag{4.6.7}$$

Differentiate (4.6.3) with respect to t , we obtain

$$2Au_t + h'(u)u_t = A(A\phi_t + g'(\phi)\phi_t) + r'(\phi)\phi_t. \quad (4.6.8)$$

Multiplying (4.6.8) by u_t and then integrate over Ω to get

$$\|u_t(t)\|_1^2 \leq c\|\phi_t\|_3^2. \quad (4.6.9)$$

We multiply (4.3.1)₁ and (4.3.1)₂ by $\nabla A\phi_t$ and A^4u , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\begin{aligned} \frac{d}{dt}\|\phi\|_5^2 + \|\phi_t\|_3^2 + \|u\|_5^2 + \int_{\Omega} \nabla A\phi_t \cdot \nabla A^2g(\phi) dx + \int_{\Omega} \nabla Ar(\phi) \cdot \nabla A\phi_t dx \\ + \int_{\Omega} \nabla Ah(u) \cdot \nabla A^2u dx = 0. \end{aligned} \quad (4.6.10)$$

On account of the estimates from (4.2.77) through (4.2.87), we deduce

$$\frac{d}{dt}\|\phi\|_5^2 + \|\phi_t\|_3^2 + \|u\|_5^2 \leq Q(\|\phi\|_3^2)\|\phi\|_5^2 + Q(\|u\|_2^2)\|u\|_4^2, \quad (4.6.11)$$

by using interpolation and Young's inequality, we have

$$\begin{aligned} Q(\|u\|_2^2)\|u\|_4^2 &\leq Q(\|u\|_2^2)\|u\|_3\|u\|_5 \\ &\leq Q(\|u\|_2^4)\|u\|_3^2 + \frac{1}{4}\|u\|_5^2, \end{aligned} \quad (4.6.12)$$

hence, it follows from (4.6.12) and (4.6.11) that

$$\frac{d}{dt}\|\phi\|_5^2 + \|\phi_t\|_3^2 + \frac{1}{2}\|u\|_5^2 \leq Q(\|\phi\|_5^2) + Q(\|u\|_3^2). \quad (4.6.13)$$

On account of (4.6.7) and (4.6.2), we deduce

$$\frac{d}{dt}\|\phi\|_5^2 + \|\phi_t\|_3^2 + \frac{1}{2}\|u\|_5^2 \leq c. \quad (4.6.14)$$

Integrating the equation (4.6.14) from 0 to t to get

$$\int_0^t \|\phi_t(s)\|_3^2 ds \leq c(1+t), \quad \forall t \geq 0. \quad (4.6.15)$$

Thus, on account of (4.6.9) and (4.6.15), we deduce that

$$\begin{aligned} \int_0^t \|u_t(s)\|_1^2 ds &\leq c \int_0^t \|\phi_t(s)\|_3^2 ds \\ &\leq c(1+t), \quad \forall t \geq 0, \end{aligned} \quad (4.6.16)$$

hence the result. ■

Now we assume that

$$\mathcal{B}_{0,5} = \{\phi \in V_5, \|\phi\|_{V_5} \leq r_5\},$$

(where r_5 is the same as in (4.5.1)) is a bounded absorbing set for $\{S(t)\}_{t \geq 0}$ in V_5 . That is, there exists $t^* > 1$, independent of ϵ , such that $S_\epsilon(t)\mathcal{B}_{5,\epsilon} \subset \mathcal{B}_{5,\epsilon}$ and $S(t)\mathcal{B}_{0,5} \subset \mathcal{B}_{0,5}$,

for all $t \geq t^*$. We set

$$\tilde{\mathcal{B}}_{5,\epsilon} = S_\epsilon(t^*)\mathcal{B}_{5,\epsilon},$$

$$\tilde{\mathcal{B}}_{0,5} = S(t^*)\mathcal{B}_{0,5}.$$

Then we have

$$\tilde{\mathcal{B}}_{5,\epsilon} \subset \{(\phi, u) \in \mathcal{W}_{5,\epsilon}, \|(\phi, u)\|_{\mathcal{W}_{5,\epsilon}} \leq r_5\},$$

and

$$\tilde{\mathcal{B}}_{0,5} \subset \{\phi \in V_5, \|\phi\|_5 \leq r_5\}.$$

Notice that $\tilde{\mathcal{B}}_{5,\epsilon}$ and $\tilde{\mathcal{B}}_{0,5}$ are a bounded absorbing sets for $S_\epsilon(t)$ and $S(t)$ as well, respectively.

Proposition 4.6.1 *Let the assumptions of Theorem 4.4.1 hold. There exists $c > 0$ and $c' > 0$ all independent of ϵ such that*

$$\|S_\epsilon(t)(\phi_0, u_0) - (S(t)\phi_0, \mathcal{T}(S(t)\phi_0))\|_{\mathcal{W}_{1,\epsilon}}^2 \leq c\epsilon^{\frac{1}{4}}e^{c't}, \quad \forall t \geq 1, \quad (4.6.17)$$

for any $(\phi_0, u_0) \in \mathcal{B}_{5,\epsilon}$, and

$$\|S_\epsilon(t)(\phi_0, u_0) - (S(t)\phi_0, \mathcal{T}(S(t)\phi_0))\|_{\mathcal{W}_{1,1}}^2 \leq c\epsilon^{\frac{1}{4}}e^{c't}, \quad \forall t \geq 1, \quad (4.6.18)$$

for any $(\phi_0, u_0) \in S_\epsilon(t^*)\mathcal{B}_{5,\epsilon}$ and any $\epsilon \in (0, 1]$.

Proof. Let us take $(\phi_0, u_0) \in \mathcal{B}_{5,\epsilon}$, We set

$$(\phi^\epsilon(t), u^\epsilon(t)) = S_\epsilon(t)(\phi_0, u_0), \quad (\phi(t), u(t)) = (S(t)\phi_0, \mathcal{T}(S(t)\phi_0)).$$

There holds that

$$\|\phi^\epsilon(t)\|_5^2 + \epsilon \|u^\epsilon(t)\|_4^2 \leq c, \quad \forall t \geq 0, \quad (4.6.19)$$

$$\|u^\epsilon(t)\|_1^2 \leq c, \quad \forall t \geq r, \quad (4.6.20)$$

$$\|u^\epsilon(t)\|_2^2 \leq c, \quad \forall t \geq 2r, \quad \text{for any } r > 0, \quad (4.6.21)$$

and

$$\|\phi(t)\|_5^2 \leq c, \quad \forall t \geq 0, \quad (4.6.22)$$

also, from (4.6.7) we have

$$\|u(t)\|_3^2 \leq c, \quad \forall t \geq 0. \quad (4.6.23)$$

We now set $P = \phi^\epsilon - \phi$ and $R = u^\epsilon - u$ and they satisfy the following problem:

$$\begin{cases} P_t + A(AP + V_1(t)P - R) + V_2(t)P = 0, \\ \epsilon R_t + P_t + AR + V_3(t)R = -\epsilon u_t, \\ P|_{t=0} = 0, \quad R|_{t=0} = u_0 - \mathcal{T}(\phi_0), \end{cases} \quad (4.6.24)$$

where

$$V_1(t) = \int_0^1 g'(s\phi^\epsilon(t) + (1-s)\phi(t))ds,$$

$$V_2(t) = \int_0^1 r'(s\phi^\epsilon(t) + (1-s)\phi(t))ds,$$

$$V_3(t) = \int_0^1 h'(su^\epsilon(t) + (1-s)u(t))ds.$$

We multiply (4.6.24)₁ and (4.6.24)₂ by $A^{-1}P_t$ and R , respectively, integrate over Ω , we obtain

$$\frac{1}{2} \frac{d}{dt} \|P\|_1^2 + \|P_t\|_{-1}^2 - (R, P_t) + \int_{\Omega} \nabla[V_1(t)P] \cdot \nabla A^{-1}P_t dx + \int_{\Omega} V_2(t)P A^{-1}P_t dx = 0, \quad (4.6.25)$$

and

$$\frac{\epsilon}{2} \frac{d}{dt} \|R\|^2 + \|R\|_1^2 + (R, P_t) + \int_{\Omega} V_3(t)|R|^2 dx = -\epsilon(A^{-1/2}u_t, A^{1/2}R). \quad (4.6.26)$$

Summing the equations (4.6.25) and (4.6.26) to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|P\|_1^2 + \epsilon \|R\|^2) + \|P_t\|_{-1}^2 + \|R\|_1^2 + \int_{\Omega} \nabla[V_1(t)P] \cdot \nabla A^{-1}P_t dx \\ + \int_{\Omega} V_3(t)|R|^2 dx + \int_{\Omega} V_2(t)P A^{-1}P_t dx = -\epsilon(A^{\frac{-1}{2}}u_t, A^{\frac{1}{2}}R). \end{aligned} \quad (4.6.27)$$

We have

$$\begin{aligned} |\epsilon(A^{-1/2}u_t, A^{1/2}R)| &\leq \epsilon\|u_t\|_{-1}\|R\|_1 \\ &\leq \frac{1}{2}\epsilon^2\|u_t\|_{-1}^2 + \frac{1}{2}\|R\|_1^2, \end{aligned}$$

and we proceed as in (4.2.96), we obtain

$$|\int_{\Omega} \nabla[V_1(t)P] \cdot \nabla A^{-1}P_t dx| \leq \frac{c}{2}\|P\|_1^2 + \frac{1}{2}\|P_t\|_{-1}^2 \quad (4.6.28)$$

and

$$\int_{\Omega} |V_2(t) P|^2 dx \leq \tilde{c}\|P\|^2. \quad (4.6.29)$$

Also, by virtue (4.0.6), we have

$$\int_{\Omega} V_3(t)|R|^2 dx = \int_{\Omega} \int_0^1 h'(su^\epsilon(t) + (1-s)u) ds |R|^2 dx \geq C_6\|R\|^2.$$

Hence, it follows from (4.6.27) that

$$\frac{d}{dt}(\|P\|_1^2 + \epsilon\|R\|^2) + \|P_t\|_{-1}^2 + \|R\|_1^2 \leq c'\|P\|_1^2 + \epsilon^2\|u_t\|_{-1}^2. \quad (4.6.30)$$

We Apply the Gronwall's Lemma to (4.6.30), on account of (4.6.1), we find

$$\begin{aligned}
\|P(t)\|_1^2 + \epsilon \|R(t)\|^2 &\leq [E_{21}(0) + \int_0^t c\epsilon^2 \|u_t\|_{-1}^2 ds] e^{c't} \\
&\leq [\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \int_0^t c\epsilon^2 \|u_t\|_{-1}^2 ds] e^{c't} \\
&\leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{c't}, \quad \forall t \geq 0.
\end{aligned} \tag{4.6.31}$$

Integrating the equation (4.6.30) between 0 and t , we deduce that

$$\int_0^t (\|P_t(s)\|_{-1}^2 + \|R(s)\|_1^2) ds \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{c't}, \quad \forall t \geq 0. \tag{4.6.32}$$

Next, we multiply (4.6.24)₁ and (4.6.24)₂ by P_t and AR , respectively, integrate over Ω , then summing the resulting equations, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} (\|P\|_2^2 + \epsilon \|R\|_1^2) + \|P_t\|^2 + \|R\|_2^2 + \int_{\Omega} A[V_1(t)P] P_t dx \\
+ \int_{\Omega} V_2(t)P P_t dx + \int_{\Omega} V_3(t)R AR dx = -\epsilon(u_t, AR).
\end{aligned} \tag{4.6.33}$$

We have

$$\begin{aligned}
|\epsilon(u_t, AR)| &\leq \epsilon \|u_t\| \|R\|_2 \\
&\leq \frac{1}{2} \epsilon^2 \|u_t\|^2 + \frac{1}{2} \|R\|_2^2.
\end{aligned}$$

Also, we proceed as in the estimates (2.2.118) and (2.2.119)) (4.2.108) and (4.2.109),

we obtain

$$|\int_{\Omega} A[V_1(t)P] P_t dx| \leq \frac{1}{4}\|P_t\|^2 + \tilde{c}\|P\|_2^2, \quad (4.6.34)$$

$$|\int_{\Omega} V_2(t)P P_t dx| \leq \tilde{c}\|P\|^2 + \frac{1}{4}\|P_t\|^2, \quad (4.6.35)$$

$$\begin{aligned} |\int_{\Omega} V_3(t)R AR dx| &\leq (1 + \|u^\epsilon\|_1^{p+2} + \|u\|_1^{p+2}) \|R\|_1^{\frac{1}{2}} \|R\|_2^{\frac{3}{2}} \\ &\leq c \|R\|_1^2 + \frac{1}{2}\|R\|_2^2, \quad \forall t \geq r. \end{aligned} \quad (4.6.36)$$

It follows from (4.6.33) that

$$\frac{d}{dt}(\|P\|_2^2 + \epsilon\|R\|_1^2) + \|P_t\|^2 + \|R\|_2^2 \leq c'\|P\|_2^2 + c\|R\|_1^2 + \epsilon^2\|u_t\|^2, \quad \forall t \geq r, \quad (4.6.37)$$

hence

$$\frac{d}{dt}E_{21} + \|P_t\|^2 + \|R\|_2^2 \leq c'E_{21} + c\|R\|_1^2 + \epsilon^2\|u_t\|^2, \quad \forall t \geq r, \quad (4.6.38)$$

where

$$E_{21} = \|P\|_2^2 + \epsilon\|R\|_1^2.$$

Since we have (4.6.32) and

$$\begin{aligned}
\int_r^t \|P(t)\|_2^2 ds &\leq c \int_r^t (\|P_t(t)\|_{-2}^2 + \|P(t)\|^2 + \|R(t)\|^2) ds \\
&\leq c \int_r^t (\|P_t(t)\|_{-1}^2 + \|P(t)\|_1^2 + \|R(t)\|_1^2) ds \\
&\leq c(1+t)(\epsilon\|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{ct}, \tag{4.6.39}
\end{aligned}$$

due to (4.6.31) and (4.6.32).

Then, we multiply (4.6.38) by $(t-r)$, then integrate it between r and t , we get

$$\begin{aligned}
(t-r)E_{21}(t) &\leq \int_r^t E_{21}(s) ds + c \int_r^t \|R\|_1^2 ds + \epsilon^2 \int_r^t \|u_t\|^2 ds \\
&\leq c(1+t)(\epsilon\|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon)e^{ct}. \tag{4.6.40}
\end{aligned}$$

Hence

$$\|P(t)\|_2^2 + \epsilon\|R(t)\|_1^2 \leq c(\epsilon\|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon)e^{ct}, \quad \forall t \geq 2r. \tag{4.6.41}$$

Integrating the equation (4.6.38) between $2r$ and t , we deduce that

$$\int_{2r}^t (\|P_t(s)\|^2 + \|R(s)\|_2^2) ds \leq c(\epsilon\|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon)e^{ct}, \quad \forall t \geq 2r. \tag{4.6.42}$$

We also multiply (4.6.24)₁ and (4.6.24)₂ by AP_t and A^2R , respectively, integrate over

Ω , summing the resulting equations , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|P\|_3^2 + \epsilon \|R\|_2^2) + \|P_t\|_1^2 + \|R\|_3^2 + \int_{\Omega} \nabla A[V_1(t)P] \cdot \nabla P_t \, dx \\ + \int_{\Omega} \nabla[V_2(t)P] \cdot \nabla P_t \, dx + \int_{\Omega} \nabla[V_3(t)R] \cdot \nabla AR \, dx = -\epsilon \int_{\Omega} \nabla u_t \cdot \nabla AR \, dx. \end{aligned} \quad (4.6.43)$$

We have

$$\begin{aligned} \left| \epsilon \int_{\Omega} \nabla u_t \cdot \nabla AR \, dx \right| &\leq \epsilon \|u_t\|_1 \|R\|_3 \\ &\leq \frac{1}{2} \epsilon^2 \|u_t\|_1^2 + \frac{1}{2} \|R\|_3^2, \end{aligned}$$

and $\int_{\Omega} |\nabla \Delta[V_1(t)P]|^2 \, dx \leq \tilde{c} \|P\|_3^2$, then

$$\left| \int_{\Omega} \nabla A[V_1(t)P] \cdot \nabla P_t \, dx \right| \leq \frac{1}{4} \|P_t\|_1^2 + \tilde{c} \|P\|_3^2. \quad (4.6.44)$$

Moreover, we have $\int_{\Omega} |\nabla[V_2(t)P]|^2 \, dx \leq \tilde{c} \|P\|_1^2$, then

$$\left| \int_{\Omega} \nabla[V_2(t)P] \cdot \nabla P_t \, dx \right| \leq \tilde{c} \|P\|_1^2 + \frac{1}{4} \|P_t\|_1^2; \quad (4.6.45)$$

and

$$\begin{aligned}
& \left| \int_{\Omega} \nabla[V_3(t)R] \cdot \nabla AR \, dx \right| \\
& \leq \sup_{s \in [0,1]} \|h''(su^\epsilon(t) + (1-s)u(t))\|_{L^3} (\|\nabla u^\epsilon\|_{L^6} + \|\nabla u\|_{L^6}) \|R\|_{L^\infty} \|\nabla AR\| \\
& \quad + \sup_{s \in [0,1]} \|h'(su^\epsilon(t) + (1-s)u(t))\| \|\nabla R\|_{L^\infty} \|\nabla AR\| \\
& \leq c \|R\|_2^2 + \frac{1}{2} \|R\|_3^2, \quad \forall t \geq 2r.
\end{aligned} \tag{4.6.46}$$

It follows from (4.6.43) that

$$\frac{d}{dt} (\|P\|_3^2 + \epsilon \|R\|_2^2) + \|P_t\|_1^2 + \|R\|_3^2 \leq c' \|P\|_3^2 + c \|R\|_2^2 + \epsilon^2 \|u_t\|_1^2, \quad \forall t \geq 2r, \tag{4.6.47}$$

then

$$\frac{d}{dt} E(t) + \|P_t\|_1^2 + \|R\|_3^2 \leq c' E(t) + c \|R\|_2^2 + \epsilon^2 \|u_t\|_1^2, \quad \forall t \geq 2r, \tag{4.6.48}$$

where

$$E(t) = \|P\|_3^2 + \epsilon \|R\|_2^2.$$

Since

$$\begin{aligned}
\int_{2r}^t \|P(t)\|_3^2 \, ds & \leq c \int_{2r}^t [\|P_t(t)\|_{-1}^2 + \|P(t)\|_1^2 + \|R(t)\|_1^2] \, ds \\
& \leq c(1+t)(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{ct},
\end{aligned} \tag{4.6.49}$$

due to (4.6.31) and (4.6.32); and

$$\int_{2r}^t \|R(s)\|_2^2 ds \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{c't}, \quad \forall t \geq 2r. \quad (4.6.50)$$

Then, we multiply (4.6.48) by $(t - 2r)$, integrate between $2r$ and t to get

$$\begin{aligned} (t - 2r)E(t) &\leq \int_{2r}^t E(t) ds + c \int_{2r}^t \|R\|_2^2 ds + \epsilon^2 \int_{2r}^t \|u_t\|^2 ds, \\ &\leq (1 + t)[c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{ct} + \epsilon^2 c], \\ &\leq c(1 + t)(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{ct}, \quad \forall t \geq 2r, \end{aligned} \quad (4.6.51)$$

hence

$$\|P(t)\|_3^2 + \epsilon \|R(t)\|_2^2 \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{c't}, \quad \forall t \geq 3r. \quad (4.6.52)$$

Integrating the equation (4.6.48) between $3r$ and t , we deduce that

$$\int_{3r}^t (\|P_t(s)\|_1^2 + \|R(s)\|_3^2) ds \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{c't}, \quad \forall t \geq 3r. \quad (4.6.53)$$

Next, from (4.6.26), we have

$$\frac{\epsilon}{2} \frac{d}{dt} \|R\|^2 + \|R\|_1^2 + (R, P_t) \leq c\epsilon^2 \|u_t\|_{-1}^2, \quad (4.6.54)$$

that is

$$\frac{1}{2} \frac{d}{dt} \epsilon \|R\|^2 + \frac{c}{\epsilon} \epsilon \|R\|^2 + (R, P_t) \leq c \epsilon^2 \|u_t\|_{-1}^2. \quad (4.6.55)$$

To control the term (R, P_t) , we obtain from equation (4.6.24)₁ the following relation

$$P_t = -A(AP + V_1(t)P - R) - V_2(t)P. \quad (4.6.56)$$

Subsisting (4.6.56) into (4.6.55), it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \epsilon \|R\|^2 + \frac{c}{\epsilon} \epsilon \|R\|^2 + \|R\|_1^2 \\ & \leq c \epsilon^2 \|u_t\|_{-1}^2 + \int_{\Omega} \nabla R \cdot \nabla A P \, dx + \int_{\Omega} \nabla R \cdot \nabla [V_1(t)P] \, dx + \int_{\Omega} R V_2(t) P \, dx, \end{aligned} \quad (4.6.57)$$

since

$$\begin{aligned} \left| \int_{\Omega} \nabla R \cdot \nabla A P \, dx \right| & \leq \frac{1}{6} \|R\|_1^2 + c \|P\|_3^2 \\ & \leq \frac{1}{6} \|R\|_1^2 + c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{c't}, \forall t \geq 3r, \end{aligned} \quad (4.6.58)$$

$$\left| \int_{\Omega} \nabla R \cdot \nabla [V_1(t)P] \, dx \right| \leq \frac{1}{6} \|R\|_1^2 + c \|P\|_1^2, \quad (4.6.59)$$

$$\left| \int_{\Omega} R V_2(t) P \, dx \right| \leq \frac{1}{6} \|R\|_1^2 + \tilde{c} \|P\|^2, \quad (4.6.60)$$

then, it follows from (4.6.57) that

$$\frac{d}{dt} \epsilon \|R(t)\|^2 + \frac{c}{\epsilon} \epsilon \|R(t)\|^2 \leq c_1 (\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{ct} + c_2 \epsilon^2 \|u_t\|_{-1}^2, \quad \forall t \geq 3r. \quad (4.6.61)$$

We multiply the equation (4.6.61) by $e^{\frac{ct}{\epsilon}}$ then multiply it by $(t - 3r)$ we obtain

$$\begin{aligned} & \frac{d}{dt} ((t - 3r) \epsilon \|R(t)\|^2 e^{\frac{ct}{\epsilon}}) \\ & \leq \epsilon \|R(t)\|^2 e^{\frac{ct}{\epsilon}} + c_1 (t - 3r) e^{\frac{ct}{\epsilon}} (\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{ct} + c_2 (t - 3r) e^{\frac{ct}{\epsilon}} \epsilon^2 \|u_t\|_{-1}^2. \end{aligned} \quad (4.6.62)$$

Integrating the equation (4.6.62) between $3r$ and t , we find

$$\begin{aligned} (t - 3r) \epsilon \|R(t)\|^2 & \leq \epsilon \int_{3r}^t \|R(s)\|^2 ds + c_1 (t - 3r) \epsilon (\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{ct} \\ & \quad + c_2 (t - 3r) \int_{3r}^t \epsilon^2 \|u_t\|_{-1}^2 ds. \end{aligned} \quad (4.6.63)$$

On account of (4.6.1) and (4.6.32), we deduce

$$\begin{aligned} & \epsilon \|R(t)\|^2 \\ & \leq \epsilon c_1 (t - 3r)^{-1} (\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{c't} + c_2 [\epsilon (\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) + \epsilon^2] e^{ct}, \end{aligned} \quad (4.6.64)$$

hence, by taking $t = 4r$ for any $r > 0$, we obtain

$$\begin{aligned} & \epsilon \|R(4r)\|^2 \\ & \leq \epsilon c_1 r^{-1} (\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) e^{4c'r} + c_2 [\epsilon (\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) + \epsilon^2] e^{4cr}. \end{aligned} \quad (4.6.65)$$

In addition, we have

$$\begin{aligned} \|P(t)\|_1^2 & \leq \|P(t)\|_{-1} \|P(t)\|_3 \\ & \leq \left(\int_0^t \|P_t(t)\|_{-1}^2 dx \right)^{\frac{1}{2}} t^{\frac{1}{2}} \|P(t)\|_3 \\ & \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) t^{\frac{1}{2}} e^{ct}, \quad \forall t \geq 3r, \end{aligned} \quad (4.6.66)$$

so that

$$\|P(4r)\|_1^2 \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) (4r)^{\frac{1}{2}} e^{4cr}, \quad \forall r > 0. \quad (4.6.67)$$

Since $r > 0$ is arbitrary, we choose $r = \frac{\sqrt{\epsilon}}{4}$ to have

$$\|P(\sqrt{\epsilon})\|_1^2 \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) \epsilon^{\frac{1}{4}}, \quad (4.6.68)$$

and

$$\epsilon \|R(\sqrt{\epsilon})\|^2 \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) \sqrt{\epsilon}. \quad (4.6.69)$$

Now, back to the equation (4.6.30), that is

$$\frac{d}{dt}(\|P\|_1^2 + \epsilon\|R\|^2) + \|P_t\|_{-1}^2 + \|R\|_1^2 \leq c'\|P\|_1^2 + \epsilon^2\|u_t\|_{-1}^2. \quad (4.6.70)$$

Applying the Gronwall Lemma to (4.6.70) between $\sqrt{\epsilon}$ and $t' + \sqrt{\epsilon}$, we find

$$\begin{aligned} & \|P(t' + \sqrt{\epsilon})\|_1^2 + \epsilon\|R(t' + \sqrt{\epsilon})\|^2 \\ & \leq c[\|P(\sqrt{\epsilon})\|_1^2 + \epsilon\|R(\sqrt{\epsilon})\|^2] + \epsilon^2 e^{ct'} \\ & \leq c(\epsilon\|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) \epsilon^{\frac{1}{4}} e^{ct'}, \quad \forall t' \geq 0, \end{aligned} \quad (4.6.71)$$

due to (4.6.68) and (4.6.69).

By using the simple substitution $t = t' + \sqrt{\epsilon}$, we can easily achieve that

$$\|P(t)\|_1^2 + \epsilon\|R(t)\|^2 \leq c(\epsilon\|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) \epsilon^{\frac{1}{4}} e^{ct}, \quad \forall t > \sqrt{\epsilon}. \quad (4.6.72)$$

Since $(\phi_0, u_0) \in \mathcal{B}_{5,\epsilon}$ and $(\|\phi_0\|_5^2 + \epsilon\|u_0\|_4^2 \leq c)$, then

$$\begin{aligned} \epsilon\|u_0 - \mathcal{T}(\phi_0)\|^2 & \leq \epsilon\|u_0\|^2 + \epsilon\|\mathcal{T}(\phi_0)\|^2 \\ & \leq c, \end{aligned}$$

so that

$$\|P(t)\|_1^2 + \epsilon\|R(t)\|^2 \leq c\epsilon^{\frac{1}{4}} e^{ct}, \quad \forall t > \sqrt{\epsilon}. \quad (4.6.73)$$

To make t independent of ϵ , we consider it for $t > 1$, and we deduce

$$\|P(t)\|_1^2 + \epsilon \|R(t)\|^2 \leq c\epsilon^{\frac{1}{4}} e^{c't}, \quad \forall t > 1, \quad (4.6.74)$$

as required in (4.6.17).

Finally, to deduce (4.6.18), we back to (4.6.72), we proved

$$\epsilon \|R(t)\|^2 \leq c(\epsilon \|u_0 - \mathcal{T}(\phi_0)\|^2 + \epsilon) \epsilon^{\frac{1}{4}} e^{c't}, \quad \forall t > 1. \quad (4.6.75)$$

If we take $(\phi_0, u_0) \in S_\epsilon(1)\mathcal{B}_{5,\epsilon}$, then we obtain

$$\epsilon \|R(t)\|^2 \leq c(\epsilon \|S_\epsilon(1)(u_0 - \mathcal{T}(\phi_0))\|^2 + \epsilon) \epsilon^{\frac{1}{4}} e^{c't}, \quad \forall t > 1, \quad (4.6.76)$$

hence, on account of $(\|u^\epsilon(t)\|^2 \leq c, \forall t \geq 1)$ and $(\|u\|^2 \leq c, \forall t \geq 0)$, we deduce

$$\begin{aligned} \|R(t)\|^2 &\leq c(\|S_\epsilon(1)(u_0 - \mathcal{T}(\phi_0))\|^2 + 1) \epsilon^{\frac{1}{4}} e^{c't}, \\ &\leq c\epsilon^{\frac{1}{4}} e^{c't}, \quad \forall t > 1. \end{aligned} \quad (4.6.77)$$

Thus, we have

$$\|P(t)\|_1^2 + \|R(t)\|^2 \leq \epsilon^{\frac{1}{4}} e^{c't}, \quad \forall t > 1, \quad (4.6.78)$$

as required in (4.6.18). I

4.6.1 A robust family of exponential attractors

The following theorems give sufficient conditions to ensure the existence of a robust family of exponential attractors.

Theorem 4.6.1 ([9], [4]) *Assume that there exist $\alpha, \vartheta \in (0, 1]$, $\varkappa_j \in (0, 1]$ and $\Upsilon_j \geq 0$ (all independent of ϵ), let $C_\epsilon(r)$ denoted a closed ball in $\mathcal{W}_{2,\epsilon}$ of radius $r > 0$ and concentrated at zero such that following conditions hold:*

1. *There exists a map $\mathcal{T} : C_0 \rightarrow L^2$ which is Hölder continuous of exponent α . Here C_0 is endowed with the metric topology of $\mathcal{W}_{1,\epsilon}$. Indeed, there exists a constant c such that*

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\| \leq c \|\phi_1 - \phi_2\|_1^\alpha,$$

for any ϕ_1 and ϕ_2 in C_0 .

2. *There exists $t^* > 0$, independent of ϵ , such that*

$$S_\epsilon(t)C_\epsilon \subset C_\epsilon, \quad \forall t \geq t^*,$$

and C_ϵ is uniformly bounded (with respect to ϵ) in the $\mathcal{W}_{2,1}$ - norm. Moreover, setting

$S_\epsilon(t^) = S_\epsilon$, the map S_ϵ satisfies, for every $z_1, z_2 \in C_\epsilon$,*

$$\|S_\epsilon z_1 - S_\epsilon z_2\|_{\mathcal{W}_{2,\epsilon}} \leq \Upsilon_1 \|z_1 - z_2\|_{\mathcal{W}_{1,\epsilon}}. \quad (4.6.79)$$

3. For any $z = (\phi_0, \phi_1) \in C_\epsilon$, there hold

$$\|S_\epsilon^m(\phi_0, \phi_1) - S^m(\phi_0, \mathcal{T}(\phi_0))\|_{\mathcal{W}_{1,1}} \leq \Upsilon_2^m \epsilon^{\varkappa_1}, \quad \forall m \in \mathbb{A}, \quad (4.6.80)$$

$$\|S_\epsilon(t)(\phi_0, \phi_1) - S(t)(\phi_0, \mathcal{T}(\phi_0))\|_{\mathcal{W}_{1,1}} \leq \Upsilon_3 \epsilon^{\varkappa_2}, \quad \forall t \in [t^*, 2t^*]. \quad (4.6.81)$$

4. The map $z \mapsto S_\epsilon(t)z$ is Lipschitz continuous on C_ϵ endowed with the metric topology of $\mathcal{W}_{1,\epsilon}$, with a Lipschitz constant independent of ϵ and $t \in [t^*, 2t^*]$.

5. The map

$$(t, z) \mapsto S_\epsilon(t)z : [t^*, 2t^*] \times C_\epsilon \rightarrow C_\epsilon$$

is Hölder continuous of exponent ϑ , where C_ϵ is endowed with the metric topology of $\mathcal{W}_{1,\epsilon}$.

Then there exists a family of exponential attractors $\mathcal{E}_\epsilon$ on $\mathcal{B} = \overline{C}_\epsilon^{\mathcal{W}_{1,\epsilon}}$ with the following properties:

(i) $\mathcal{E}_\epsilon$ attracts $\mathcal{B}$ with an exponential rate which is uniform with respect to ϵ , that is,

$$\text{dist}_{\mathcal{W}_{1,\epsilon}}(S_\epsilon(t)\mathcal{B}, \mathcal{E}_\epsilon) \leq \mathcal{M}_1 e^{-\omega t}, \quad \forall t \geq 0,$$

for some $\mathcal{M}_1 > 0$ and some $\omega > 0$.

(ii) The fractal dimension of $\mathcal{E}_\epsilon$ is uniformly bounded with respect to ϵ , that is,

$$\dim_{\mathcal{W}_{1,\epsilon}}[\mathcal{E}_\epsilon] \leq \mathcal{M}_2,$$

for some $\mathcal{M}_2 > 0$.

(iii) There exist a positive constants $\mathcal{M}_3, \mathcal{M}_4, \varpi_1, \varpi_2 \in (0, \frac{1}{2}]$ such that

$$\text{dist}_{\mathcal{W}_{1,\epsilon}}^{\text{sym}}(\mathcal{E}_\epsilon, (\mathcal{E})_0) \leq \mathcal{M}_3 \epsilon^{\varpi_1}, \quad (4.6.82)$$

$$\text{dist}_{\mathcal{W}_{1,1}}(\mathcal{E}_\epsilon, (\mathcal{E})_0) \leq \mathcal{M}_4 \epsilon^{\varpi_2}, \quad (4.6.83)$$

and

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{W}_{1,1}}((\mathcal{E})_0, \mathcal{E}_\epsilon) = 0, \quad (4.6.84)$$

for all $\epsilon \in (0, 1]$. Here ω, ϖ_j and $\mathcal{M}_j$ are independent of ϵ , and they can be computed explicitly.

Observe that there exists $t^* \geq 1$ such that $S_\epsilon(t)\tilde{\mathcal{B}}_{5,\epsilon} \subset \tilde{\mathcal{B}}_{5,\epsilon}$ for all $t \geq t^*$; $\tilde{\mathcal{B}}_{5,\epsilon}$ and $\tilde{\mathcal{B}}_{0,5}$ are a bounded absorbing set for $S_\epsilon(t)$ and $S(t)$ as well, respectively. Thus we can apply Theorem 4.6.1 and the following theorem with $C_\epsilon = \tilde{\mathcal{B}}_{5,\epsilon}$.

Theorem 4.6.2 *Let the assumptions of Theorem 4.4.1 hold. Then, for every $\epsilon \in (0, 1]$, the semigroup $S_\epsilon(t)$ possesses an exponential attractor $\mathcal{E}_\epsilon$ on $\mathcal{W}_{j,\epsilon}$ in the metric topology of $\mathcal{W}_{1,\epsilon}$, that is, there exists a constant $\mathfrak{z}_0 > 0$ such that, for every bounded subset $B \subset \mathcal{W}_{j,\epsilon}$, there exists a constant $\mathfrak{z}_1(B) > 0$ such that ($\mathfrak{z}_0$ and $\mathfrak{z}_1$ are independent of ϵ)*

$$\text{dist}_{\mathcal{W}_{1,\epsilon}}(S_\epsilon(t)B, \mathcal{E}_\epsilon) \leq \mathfrak{z}_1 e^{-\mathfrak{z}_0 t}, \quad \forall t \geq 0.$$

Moreover, there exist $0 < \varpi_2, \varpi_3 \leq \frac{1}{2}$ and $\mathcal{M}_5, \mathcal{M}_6 > 0$, all independent of ϵ , and a

family of exponential attractors $\mathcal{E}_\epsilon$ that satisfies the following property

$$\text{dist}_{\mathcal{W}_{1,\epsilon}}^{\text{sym}}(\mathcal{E}_\epsilon, (\mathcal{E})_0) \leq \mathcal{M}_5 \epsilon^{\varpi_2},$$

and

$$\text{dist}_{\mathcal{W}_{1,1}}(\mathcal{E}_\epsilon, (\mathcal{E})_0) \leq \mathcal{M}_6 \epsilon^{\varpi_3},$$

for all $\epsilon \in (0, 1]$. Also,

$$\lim_{\epsilon \rightarrow 0} \text{dist}_{\mathcal{W}_{1,1}}((\mathcal{E})_0, \mathcal{E}_\epsilon) = 0.$$

Proof. Let us now prove the theorem. On account of Theorem 4.6.1, we check all the assumptions 1-5.

Let $(\phi_0, u_0) \in \tilde{\mathcal{B}}_{5,\epsilon}$. To verify Assumption 1, we will prove that $\mathcal{T} : \tilde{\mathcal{B}}_{5,0} \rightarrow L^2$ is continuous in the metric topology of V_1 . Indeed, we consider the difference of two solutions (ϕ_1, u_1) and (ϕ_2, u_2) of Problem (4.3.2) where $u_i = \mathcal{T}(\phi_i)$, we obtain

$$2A(u_1 - u_2) + h(u_1) - h(u_2) = A[A(\phi_1 - \phi_2) + g(\phi_1) - g(\phi_2)] + r(\phi_1) - r(\phi_2), \quad (4.6.85)$$

and since

$$h(u_1) - h(u_2) = \int_0^1 h'(su_1 + (1-s)u_2) ds (u_1 - u_2),$$

then

$$\begin{aligned}
& 2A(u_1 - u_2) + \int_0^1 h'(su_1 + (1-s)u_2)(u_1 - u_2)ds \\
& = A[A(\phi_1 - \phi_2) + g(\phi_1) - g(\phi_2)] + r(\phi_1) - r(\phi_2).
\end{aligned} \tag{4.6.86}$$

We multiply (4.6.86) by $u_1 - u_2$, integrate over Ω , we obtain

$$\begin{aligned}
\|u_1 - u_2\|_1^2 & \leq \|\phi_1 - \phi_2\|_3^2 + \|g(\phi_1) - g(\phi_2)\|_1^2 + c\|r(\phi_1) - r(\phi_2)\|^2 \\
& \leq c(\|\phi_1 - \phi_2\|_5^{\frac{1}{2}} + \|\phi_1 - \phi_2\|_1^{\frac{1}{2}})\|\phi_1 - \phi_2\|_1^{\frac{1}{2}} \\
& \leq c\|\phi_1 - \phi_2\|_1^{\frac{1}{2}},
\end{aligned} \tag{4.6.87}$$

where we used the integral form of the Main Value Theorem and the interpolation estimate $\|\phi\|_3 \leq \|\phi\|_1^{\frac{1}{2}} \|\phi\|_5^{\frac{1}{2}}$ and the bounded estimate $\|\phi\|_5^{\frac{1}{2}} \leq c$.

Thus, we can deduce that

$$\|\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)\|_1 = \|u_1 - u_2\|_1 \leq c\|\phi_1 - \phi_2\|_1^{\frac{1}{2}}, \tag{4.6.88}$$

hence $\mathcal{T}$ satisfies Assumption 1.

As $t^* \geq 1$, and $r > 0$ arbitrary the estimates (4.2.25) and (4.2.22) with $t \geq r \geq 1$ show that $\tilde{\mathcal{B}}_{5,\epsilon}$ is uniformly bounded (with respect to ϵ) in the $\mathcal{W}_{2,1}$ - norm.

Moreover, from estimate (4.2.113) we can entail Assumption 2. Estimate (4.6.18) will obtain Assumption 3.

Thus we are left to prove Assumptions 4 and 5. Indeed, defining

$$(\phi(t), u(t)) = S_\epsilon(t)(\phi_{01}, u_{01}), \quad (\phi(t'), u(t')) = S_\epsilon(t')(\phi_{02}, u_{02}),$$

with $(\phi_{01}, u_{01}), (\phi_{02}, u_{02}) \in \tilde{\mathcal{B}}_{5,\epsilon}$, and $t, t' \in [t^*, 2t^*]$, we obtain

$$\begin{aligned} \|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}} &\leq \|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{01}, u_{01})\|_{\mathcal{W}_{1,\epsilon}} \\ &\quad + \|S_\epsilon(t')(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}}. \end{aligned} \tag{4.6.89}$$

On one hand, on account of (4.2.20), we have

$$\begin{aligned} &\|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{01}, u_{01})\|_{\mathcal{W}_{1,\epsilon}} \\ &\leq c (\|\phi(t) - \phi(t')\|_1 + \sqrt{\epsilon}\|u(t) - u(t')\|) \\ &\leq c (|\int_t^{t'} \|\phi_t(s)\|_1 ds| + |\int_t^{t'} \sqrt{\epsilon}\|u_t(s)\| ds|) \\ &\leq c(t^*, \epsilon)|t' - t|^{\frac{1}{2}}. \end{aligned} \tag{4.6.90}$$

On the another hand, it follows from (4.2.105) that

$$\begin{aligned} &\|S_\epsilon(t')(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}} \\ &\leq c(t^*)\|(\phi_{01}, u_{01}) - (\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}}, \quad \forall t' > 0. \end{aligned} \tag{4.6.91}$$

Hence, we conclude with

$$\begin{aligned} & \|S_\epsilon(t)(\phi_{01}, u_{01}) - S_\epsilon(t')(\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}} \\ & \leq c(t^*, \epsilon)(|t' - t|^{\frac{1}{2}} + \|(\phi_{01}, u_{01}) - (\phi_{02}, u_{02})\|_{\mathcal{W}_{1,\epsilon}}). \end{aligned} \quad (4.6.92)$$

Theorem 4.6.1 shows the existence of exponential attractors on $\overline{\mathcal{B}_{5,\epsilon}}^{\mathcal{W}_{1,\epsilon}}$, we have

$$\text{dist}_{\mathcal{W}_{1,\epsilon}}(S_\epsilon(t)\tilde{\mathcal{B}}_5, \mathcal{E}_\epsilon) \leq \mathcal{M}_1 e^{-\omega t}, \quad \forall t \geq 0.$$

Next by the following Lemma 4.6.2, we can use the transitivity property of the exponential attraction to extend the basin of attraction of each exponential attractor from $\overline{\tilde{\mathcal{B}}_{5,\epsilon}}^{\mathcal{W}_{1,\epsilon}}$ to the whole phase-space $\mathcal{W}_{1,\epsilon}$.

This completes the proof of the theorem. ■

Lemma 4.6.2 *Let $\mathcal{B}_{5,\epsilon}$ is abounded absorbing set in $\mathcal{W}_{5,\epsilon}$. For any bounded set $B \subset \mathcal{W}_{j,\epsilon}$ with $R = \sup_{\xi \in B} \|\xi\|_{\mathcal{W}_{j,\epsilon}}$, there exist $c > 0$ independent of R , and $C(R) > 0$ that depends on R (all independent of ϵ) such that*

$$\text{dist}_{\mathcal{W}_{1,\epsilon}}(S_\epsilon(t)B, \tilde{\mathcal{B}}_{5,\epsilon}) \leq C(R)e^{-ct}, \quad \forall t \geq 0.$$

Proof. we can prove this Lemma by using similar method as in the proof of Lemma 2.6.4 but with $\tilde{\mathcal{B}}_{5,\epsilon}$ and $\mathcal{W}_{1,\epsilon}$. ■

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