

SCALING METHODS AND INVARIANTS ON COMPLEX DOMAINS

BY

HUDA ALZAKI

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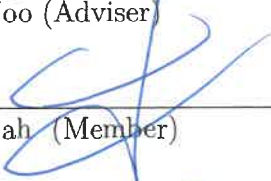
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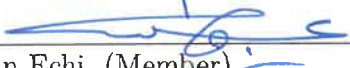
DEANSHIP OF GRADUATE STUDIES

This dissertation, written by **HUDA ALZAKI** under the direction of her dissertation adviser and approved by her dissertation committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS**.

Dissertation Committee

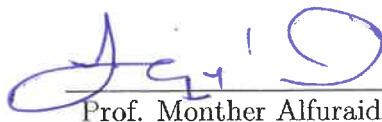

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Prof. Othman Echi (Member)


Dr. Rachid Ait Haddou (Member)


Prof. Hervé Gaussier (Member)


Prof. Monther Alfuraidan
Department Chairman


Prof. Suliman Al-Humidan
Dean of Graduate Studies

Date



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*This dissertation is dedicated to my loving family, my friends and my
advisor*

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LIST OF SYMBOLS

$\partial\Omega$	Boundary of a Domain
$d\varphi_j(p)$	Complex Jacobian Matrix at p
$H_p(\partial\Omega)$	Complex Tangent Space at p
Δ	The Unit Disc
$F(z, v)$	Finsler Metric
$B_\Omega^k(z, r)$	Kobayashi Ball of radius r
$d_\Omega^k(z, w)$	Kobayashi Distance
$K_\Omega(z, v)$	Kobayashi Metric
$c_\Omega(x)$	Maximal Circularity Function
$\text{Aut}(\Omega)$	The Automorphism group of a Domain
$\text{Vol}(G)$	The Euclidean Volume of a Region
$I_{\Omega, F}(z)$	The Indicatrix of a Metric F at z
$C(z, w)$	The Space of all Piecewise C Curves joining z and w
$\text{Hol}(\Delta, \Omega)$	The Space of Holomorphic Mappings From The Unit Disc to a Domain
$s_\Omega(x)$	The Squeezing Function
Δ^n	The Unit Polydisc

LIST OF ABBREVIATIONS

GAP	Generic Analytic Polyhedron
psh	plurisubharmonic
SCV	Several Complex Variables

DISSERTATION ABSTRACT

NAME: Huda Alzaki
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In this study, we investigate the application of the indicatrix within an invariant metric framework for rescaling biholomorphic maps sequence to guarantee the convergence of this rescaled sequence. In addition, to examine the circularity of the domain, we define the maximal circularity function derived from the indicatrix of the Kobayashi-Royden metric. We also study Kobayashi isometries over bounded generic analytic polyhedra.

ملخص الرسالة

الاسم: عنوان الدراسة:

التخصص: تاريخ الدرجة العلمية:

في هذه الدراسة نبحث عن كيفية استخدام المؤشر ضمن دراسة الفراغ المتري غير المتغير لضبط قياس متتابعة من الدوال التحليلية لضمان تقارب هذه المتتابعات. بالإضافة إلى ذلك فإنه لاختبار مدى استدارة أي مجال من المجالات نقوم بتعريف دالة و نسميها بدالة الاستدارة العظمى والتي قمنا باشتقاقها من مقياس متري يعرف باسم كوباياشي. نقوم أيضا بدراسة التقايس المعروف باسم تقايس كوباياشي في المجسم متعدد السطوح التحليلي و المحدود.

CHAPTER 1

INTRODUCTION

1.1 Historical Background

Several Complex Variables (SCV) is a branch of pure mathematics that is about complex structures of higher dimensional complex spaces and holomorphic mappings among them. Historically, this subject has been motivated by several phenomena which are not observed in 1-variable complex analysis. One of them is the failure of the Riemann mapping theorem. Recall that this celebrated theorem states that any proper simply connected domain in $\mathbb{C}$ is biholomorphic to the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

In contrast, it was observed by H. Poincaré that the bidisc

$$\Delta^2 = \{z = (z_1, z_2) : |z_1| < 1, |z_2| < 1\}$$

is not biholomorphic to the ball

$$B^2 = \{z = (z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\}$$

even though both domains are not just simply connected but convex. This observation implies that characterizing conditions for equivalence of high-dimensional complex spaces is not as simple as 1-dimensional cases.

From a geometric point of view, it is necessary to study invariants which means quantities or properties that are preserved by biholomorphic maps to figure out conditions for the equivalence of complex spaces. More specifically, the equivalence problems of SCV consist of

- Finding invariants and relations among invariants.
- Characterizing and classifying complex spaces under given conditions on invariants.

The main focus of this dissertation is on the study of the role of automorphism groups of complex domains and invariant metrics which are important invariants for describing complex domains. The automorphism groups serve as powerful tools in characterizing and classifying geometric structures and studying symmetries. It is natural to consider the case of a non-compact automorphism group which implies that the manifold has a large enough automorphism group action to cover other manifolds and to indicate a high degree of symmetry.

There are some former results on characterizing domains with non-compact automorphism group action. For example, it is proved by B. Wong and J.-P. Rosay ([37], [34]) that any bounded strongly pseudoconvex domain with a non-compact automorphism group is biholomorphic to the unit ball. Also, partial results for the finite type case were obtained by E. Bedford, S. Pinchuk ([4]), and F. Berteloot ([5]). Bounded

generic analytic polyhedra were characterized by K.-T. Kim, S.G. Krantz, A. F. Spiro ([21]) and A. Zimmer ([39]).

One of the main techniques when dealing with automorphism groups to characterize domains is the scaling method. This method involves some steps starting with localization which means contracting compact subsets of the domain to a boundary point, followed by centering, stretching the domain by composing a linear map (called the stretching factor) with the maps in the sequence, analysis of the limit of the sequence and synthesis [12].

The main idea of scaling in the above results was first introduced by S. Pinchuk ([33]). The stretching factor of the Pinchuck scaling sequence is determined by the boundary geometry of the orbit accumulating point. However, it is not completely known in general cases how to construct it.

Another scaling sequence was introduced by S. Frankel ([10]). Frankel constructed the sequence

$$\tau_j = [d\varphi_j(p)]^{-1} \cdot (\varphi_j(\cdot) - \varphi_j(p)) \quad (1.1)$$

where $\{\varphi_j\}$ is a sequence of holomorphic automorphisms and p is a point in the convex domain Ω . The stretching factor here is $[d\varphi_j(p)]^{-1}$ which depends on the map φ_j . Examining the convergence of both scaling sequences becomes challenging when the domain is not convex. The equivalence of the two scalings has been studied ([15],[19]) and proved in some cases. However, it is not known in general cases.

1.2 Justification of The Problem and Main Results

Frankel proved that a hyperbolic convex domain that covers a compact complex manifold is biholomorphic to a bounded symmetric domain ([10]). To show that, he introduced the sequence (1.1) and proved that it is equicontinuous by showing that the derivative for the automorphism is locally uniformly bounded, so the scaled sequence is subsequentially convergent and then proved that the image of the subsequential limit is symmetric.

Although the sequence (1.1) is formally well-defined, the convergence is unclear as mentioned previously, since the geometric information of the domain is veiled in the stretching factor. In this point of view, it is natural to ask the following.

Question (1): Can we replace the stretching factor of the scaled sequence (1.1) with another factor depending only on the geometry of the domain?

Question (2): Can we describe a geometric condition to guarantee the convergence of the scaled sequence (1.1)?

We answer these two questions in the following theorems.

Theorem 1.2.1:

Let Ω be a domain in $\mathbb{C}^n$, $p \in \Omega$ and $\{\varphi_j\}$ be a sequence of holomorphic automorphisms. Then the sequence (1.1) converges subsequentially if and only if

$$\sigma_j := L_j \circ (\varphi_j(\cdot) - p_j)$$

converges subsequentially, where $L_j = L_{p_j}$ is a stretching factor induced by any convex

invariant metric (see section 3.1).

Theorem 1.2.2:

Let Ω be a hyperbolic convex domain in $\mathbb{C}^n$. Then Ω is uniformly dominated by the Kobayashi-Royden metric K_Ω .

In his paper ([10]), Frankel chose a sequence of automorphism that translates an interior point to a regular boundary point. Recall that for a bounded symmetric domain that is not biholomorphic to the ball, any regular boundary point is non-minimal, that is, there is a nontrivial analytic variety in the boundary passing through the point. On the other hand, as seen in many examples ([18],[21],[22],[39]), the sequence of automorphisms moved an interior point to a minimal boundary point and the resulting image of the limit of the scaled sequence is more canonical than the non-minimal boundary point case.

Question (3): What happens if the accumulation point in the convex domain Ω is a minimal boundary point?

To determine an answer to this question, we define the maximal circularity function which is an invariant function that has the property $0 < c_\Omega(x) \leq 1$ for any $x \in \Omega$ and $c_\Omega(x) = 1$, if Ω is biholomorphic to the indicatrix of the Kobayashi Royden metric at $x \in \Omega$, (see section 4.1).

We identify a relation between $c_\Omega(x)$ and the squeezing function and we ask this general question

Question (4): If Ω is a bounded convex domain in $\mathbb{C}^n$, $\{\varphi_j\}$ holomorphic automorphisms of Ω and $x \in \Omega$ is an interior point such that $\varphi_j(x)$ tends to a minimal boundary point, then does $c_\Omega(x)$ tend to 1 ?

To develop an idea to prove this general statement, we consider some special cases, which are the strongly pseudoconvex case and the convex analytic polyhedral case, and demonstrate that the answer to the question is positive.

The study on scaling domains strongly relies on the invariance property of the Kobayashi metric. Therefore, it is natural to ask

- whether or not the scaling theory still works for Kobayashi isometries,
- whether or not a Kobayashi isometry is either holomorphic or anti-holomorphic.

There are several recent progress about the questions above for strongly pseudoconvex domains, ([11],[20],[36]).

In the rest of this dissertation, we will discuss about Kobayashi isometries on another class of domains, called generic analytic polyhedra (GAP).

1.3 Organization of the Rest of the Dissertation

In Chapter (2), entitled (Preliminaries), we present a brief review of some basic materials, terminology, and definitions of holomorphic functions and invariant metrics.

In Chapter (3), entitled (Construction and Convergence of a Scaling Sequence), we construct a new scaling sequence by replacing the linear transform in the Frankel sequence with another stretching factor that is derived from the indicatrix of an invariant metric and prove that the convergence of the new sequence is equivalent to that of Frankel's. (Theorem 3.2.1). A geometric condition is described to guarantee the convergence of the Frankel sequence and we call it the uniform domination of

the domain by a metric (Definition 3.2.1). We also prove that the uniform domination condition is satisfied in any convex hyperbolic domain for the case of Kobayashi Royden metric (Theorem 3.1.5).

In Chapter (4), entitled (Maximal Circularity Function), we define the maximal circularity function $c_{\Omega}(x)$ (Definition 4.1.3) and we prove that $c_{\Omega}(x) \rightarrow 1$ for a point x in Ω if x tends to the boundary of the domain in two specific cases, namely, the strongly pseudoconvex domains (Theorem 4.2.1) and convex polyhedral domains (Theorem 4.3.2).

In Chapter (5), entitled (Holomorphicity of The Kobayashi Isometry), we present some recent progress in scaling theory concerning Kobayashi isometry.

CHAPTER 2

PRELIMINARIES

In these preliminaries, we review some essential definitions and properties regarding mappings and sets that are necessary for the remainder of this dissertation. We will use the notation Δ to denote the unit ball in $\mathbb{C}$.

2.1 Some Fundamentals

Definitions 2.1.1.

- (1) A *domain* in $\mathbb{C}^n$ is a connected open set in $\mathbb{C}^n$.
- (2) A real-valued function r on Ω is said to be *plurisubharmonic* (*psh*) if it is subharmonic on the intersection of Ω and any complex line.
- (3) A domain Ω is *pseudoconvex* if it admits a psh exhaustion function. i.e, there exists a function $r : \Omega \rightarrow [-\infty, \infty)$ such that
 - (i) r is psh.
 - (ii) $\forall t \in \mathbb{R}$, the set $\Omega_t = \{z \in \Omega, r(z) \leq t\}$ is relatively compact in Ω .
- (4) In case Ω has a C^1 -smooth boundary, then there exists a function $\rho : \mathbb{C}^n \rightarrow \mathbb{R}$

satisfies the conditions

- (i) $\Omega = \{z \in \mathbb{C}^n, p(z) < 0\}$,
- (ii) $\rho = 0$ on $\partial\Omega$,
- (iii) $\forall p \in \partial\Omega, \nabla\rho(p) \neq 0$.

and is called a *defining function*.

Let $H_p(\partial\Omega)$ be the complex tangent space of $\partial\Omega$ at p , i.e.,

$$H_p(\partial\Omega) = \left\{ v \in \mathbb{C}^n, \sum_{j=1}^n \frac{\partial\rho}{\partial z_j}(p) v_j = 0 \right\}.$$

(5) The *Levi form* of the defining function ρ for Ω is given by

$$L_p(v) = \sum_{i,j=1}^n \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j}(p) v_i \bar{v}_j, \text{ for } v \in H_p(\partial\Omega),$$

for all $p \in \Omega$.

Remark 2.1.2

If Ω has a C^2 -smooth boundary, then

$$\Omega \text{ is pseudoconvex} \iff L_p(v) \geq 0, \forall v \in H_p(\partial\Omega)$$

and Ω is *strongly pseudoconvex* if $L_p(v) > 0$ for any nonzero $v \in H_p(\partial\Omega)$.

Strongly pseudoconvex domains play a crucial role in SCV when studying invariant metrics and equivalence of domains.

Definition 2.1.3.

The upper semicontinuous function $F(z, v)$ defined on $\Omega \times \mathbb{C}^n$ is called a *complex Finsler metric* if:

- $F(z, v) \geq 0$, for all $z \in \Omega$ and $v \in T_z\Omega$ and $F(z, v) = 0$ if and only if $v = 0$,
- $F(z, \lambda v) = |\lambda|f(z, v), \forall (z, v) \in \Omega \times \mathbb{C}^n$.

In this dissertation, we will call a complex Finsler metric simply a metric.

Definition 2.1.4.

The *indicatrix* of the metric F at $z \in \Omega$ is defined by the set

$$I_{\Omega, F}(z) = \{v \in \mathbb{C}^n, F(z, v) < 1\}.$$

Remarks 2.1.5

- F is assumed to satisfy the triangular inequality for each $v \in \mathbb{C}^n$.
- $I_{\Omega, F}(z)$ is not necessarily convex in general.
- If $I_{\Omega, F}(z)$ is a convex domain in $\mathbb{C}^n$, then the metric F is called a *convex metric*.

Definition 2.1.6.

The induced *distance function* is the function defined as

$$d_{\Omega, F}(z, w) = \inf \left\{ \int_0^1 F(\gamma(t), \gamma'(t)) dt, \gamma \in C(z, w) \right\}$$

where $C(z, w)$ denotes the space of all piecewise C^1 -curves that join any two points z and w in Ω .

2.2 Kobayashi-Royden Metric and its Properties

Definitions 2.2.1.

(1) The *infinitesimal Kobayashi pseudo-metric* on a domain Ω is defined by

$$K_{\Omega}(z, v) = \inf \{ \lambda > 0, \exists f \in \text{Hol}(\Delta, \Omega), f(0) = z, \lambda f'(0) = v \},$$

where $z \in \Omega$, $v \in T_z\Omega$, and $\text{Hol}(\Delta, \Omega)$ denotes the space of holomorphic mappings from the unit disc into Ω .

(2) The *Kobayashi pseudo-distance* is

$$d_{\Omega}^k(z, w) = \inf \left\{ \int_0^1 K_{\Omega}(\gamma(t), \gamma'(t)) dt, \gamma \in C(z, w), \gamma(0) = z, \gamma(1) = w \right\}$$

(3) The *Kobayashi-ball*, usually denoted by $B_{\Omega}^k(z, r)$ is the ball

$$\{w \in \Omega, d_{\Omega}^k(z, w) < r\}$$

There are some basic notable remarks regarding Kobayashi metric and Kobayashi distance:

(1) $K_{\Omega}(z, v)$ is not necessarily a metric, since it can be zero for a nonzero vector $v \in T_z\Omega$. However, if Ω is a bounded domain, then $K_{\Omega}(z, v)$ is a metric, and $d_{\Omega}^k(z, w)$ is a distance for any z and w in Ω . Indeed, if Ω is bounded in $\mathbb{C}^n$, $z \in \Omega$ and $v \in \mathbb{C}^n \setminus \{0\}$, then the set

$$\mathcal{F} = \{f : \Delta \longrightarrow \Omega, f(0) = z, f'(0) = \lambda v\}$$

is a normal family. Assume that $K(z, v) = 0$, then there exists $f_j \in \mathcal{F}$ such that $f_j'(0) \rightarrow \infty$ as $j \rightarrow \infty$. This is a contradiction since if $\{f_j\}$ is convergent then $\{f_j'\}$ is also convergent. This means $K(z, v) > 0, \forall v \neq 0$. which implies that $d_\Omega^k(z, w) > 0$, if $z \neq w, \forall z$ and $w \in \Omega$.

The domain Ω is said to be *hyperbolic* if $K_\Omega(z, v)$ is a metric.

(2) One of the most significant properties of Kobayashi metric and distance is the decreasing property. That is, If Ω_1 and Ω_2 are two domains in $\mathbb{C}^n$, and if $f : \Omega_1 \rightarrow \Omega_2$ is a holomorphic map, then

$$K_{\Omega_2}(f(z), df_z(v)) \leq K_{\Omega_1}(z, v),$$

$$\text{and} \quad d_{\Omega_2}(f(z), f(w)) \leq d_{\Omega_1}(z, w),$$

for all $z, w \in \Omega_1$ and $v \in T_z\Omega_1$.

One could prove the first inequality as follows:

Let $\varphi : \Delta \rightarrow \Omega_1$ be a holomorphic map, $z \in \Omega_1$ and $v \in \mathbb{C}^n$, such that $\varphi(0) = z$. Then $f \circ \varphi : \Delta \rightarrow \Omega_2$ is a holomorphic map too, and $f \circ \varphi(0) = f(z)$. Assume $\varphi'(0) = \lambda v, \lambda > 0$. Then $(f \circ \varphi)'(0) = \lambda df_z(v)$.

$$\therefore K_{\Omega_2}(f(z), df_z(v)) \leq \frac{1}{|\lambda|}$$

Taking the infimum on the right-hand side for all φ , we obtain

$$K_{\Omega_2}(f(z), df_z(u)) \leq K_{\Omega_1}(z, v)$$

which proves the desired property.

(3) If f is a biholomorphic mapping then

$$K_{\Omega_2}(f(z), df_z(v)) = K_{\Omega_1}(z, v),$$

$$d_{\Omega_2}(f(z), f(w)) = d_{\Omega_1}(z, w),$$

which means that the map f is an isometry. The metric and distance satisfying these conditions respectively are *invariant metric* and *invariant distance*.

(4) By applying the decreasing property to the inclusion map $i : \Omega_1 \longrightarrow \Omega_2$, $i(z) = z$, we get the following property:

Let $\Omega_1 \subseteq \Omega_2 \subset \mathbb{C}^n$, then for any $z, \omega \in \Omega_1$ and any $v \in \mathbb{C}^a$

$$d_{\Omega_2}^k(z, w) \leq d_{\Omega_1}^k(z, \omega),$$

$$K_{\Omega_2}(z, v) \leq K_{\Omega_1}(z, w).$$

(5) If Ω is a product of domains, i.e. $\Omega = \Omega_1 \times \dots \times \Omega_k$ where $\Omega_i \subseteq \mathbb{C}^{n_i}$, then

$$K_{\Omega}(z, v) = \max_{i=1, \dots, k} K_{\Omega_i}(z_i, v_i)$$

for any $z = (z_1, \dots, z_k) \in \Omega$ and $v = (v_1, \dots, v_k) \in \mathbb{C}^{n_1 + \dots + n_k}$.

Examples 2.2.3.

(1) On the unit disc, the Kobayashi metric (which agrees with the Poincaré metric), is given by

$$K_{\Delta}(z, v) = \frac{|v|}{1 - |z|^2}, \forall z \in \Delta, v \in \mathbb{C},$$

where $|\cdot|$ denotes the usual Euclidean norm. The Kobayashi distance is

$$d_{\Delta}^k(z, \omega) = \frac{1}{2} \log \frac{1 + |\varphi_z(\omega)|}{1 - |\varphi_z(\omega)|}$$

where $\varphi_z(\omega) = \frac{\omega - z}{1 - \bar{z}\omega}$, for any $z, \omega \in \Delta$.

(2) On the disk $\Delta_R = \{z \in \mathbb{C}, |z| < R\}$, the form of the Kobayashi-metric is

$$K_{\Delta_R}(z, v) = \frac{R|v|}{R^2 - |z|^2}, \forall z \in \Delta_R \text{ and } v \in \mathbb{C}.$$

(3) Let $\Delta^2 = \{z = (z_1, z_2) \in \mathbb{C}^2, |z_1| < 1, |z_2| < 1\}$. Then using property (5) we have

$$K_{\Delta^2}(z, v) = \max \{K_{\Delta}(z_1, v_1), K_{\Delta}(z_2, v_2)\}$$

$$\forall z \in \Delta^2, v \in \mathbb{C}^2.$$

The map $\varphi_z(\omega)$ belongs to the automorphism group of the unit disc $\text{Aut}(\Delta)$.

2.3 Automorphism Groups

Definition 2.3.1.

For any domain Ω in $\mathbb{C}^n$, the *automorphism group* of Ω is a collection of holomorphic mappings $f : \Omega \longrightarrow \Omega$ with existing holomorphic inverse, equipped with the operation of function composition, and it is represented by $\text{Aut}(\Omega)$.

Definition 2.3.2.

The set $\{\varphi_j(q), \varphi_j \in \text{Aut}(\Omega)\}$, for $q \in \Omega$ is called the *orbit* of q .

The automorphism group is an interesting invariant that has some properties such as transitivity and non-compactness. This property is used in the following proposition.

Remark 2.3.3.

$\text{Aut}(\Omega)$ turns out to be a Lie group if $\Omega \subset \mathbb{C}^n$ is bounded. In this case, it also turns out that $\text{Aut}(\Omega)$ is non-compact if and only if there exists an interior point $q \in \Omega$, a boundary point $p \in \partial\Omega$ and an automorphism $\varphi_j \in \text{Aut}(\Omega)$ such that

$$\lim_{j \rightarrow \infty} \varphi_j(q) = p.$$

The point p is called a *boundary orbit accumulation point*.

The above proposition allows to apply the localization, (first step of the scaling method), by shrinking the image of any compact subset K of Ω under the automorphism φ_j to a neighborhood of the boundary point p .

2.4 Normal Families and Convergence of Sequences

To prove the convergence of a sequence, one can use the following:

Definition 2.4.1

Let $\mathcal{F} \in \text{Hol}(\Omega_1, \Omega_2)$ be a family, $\mathcal{F}$ is said to be *normal* if every sequence $\{f_j\} \subseteq \mathcal{F}$ either

- (i) has a subsequence that converges uniformly on compact subsets of Ω .

or (ii) has a subsequence, $\{f_{ik}\}$, such that for every compact subset K of Ω and every compact subset L of $\mathbb{C}$, there exists large Number N , such that whenever $k \geq N$, $f_{jk}(K) \cap L = \emptyset$.

The following theorem is related to Normality.

Theorem 2.4.2 (Arzela-Ascoli).

Let $\mathcal{F} \in \text{Hol}(\Omega_1, \Omega_2)$ be a family, then $\mathcal{F}$ is normal $\Leftrightarrow$

- (i) $\mathcal{F}$ is equicontinuous.
- (ii) $\forall x \in \Omega_1, \mathcal{F}(x) = \{f(x) \in \Omega_2, f \in \mathcal{F}\}$ is bounded.

As previously stated in the introduction, S. Frankel used the above theorem to prove the convergence of the sequence he constructed, and the resulting image is a bounded symmetric domain.

We mean by *bounded symmetric domains*, domains $\Omega \subset \mathbb{C}^n$ with an automorphism group satisfying these two conditions:

- (i) $\text{Aut}(\Omega)$ is transitive.
- (ii) $\forall z \in \Omega, \exists \sigma_z \in \text{Aut}(\Omega)$, such that $\sigma_z \circ \sigma_z = Id$.

For future information on the general theory on domains and invariant metrics, see e.g., ([12],[14],[29]).

CHAPTER 3

CONSTRUCTION AND CONVERGENCE OF A SCALING SEQUENCE

3.1 Constructing a New Scaling Sequence

Suppose Ω is a bounded domain in the complex Euclidean space $\mathbb{C}^n$, $z \in \Omega$ and F is a metric on Ω . Since F is upper semicontinuous, we may find $u_1 = u_1(z) \in \mathbb{C}^n$ with $|u_1| = 1$ such that $\mu_1(z) := F_\Omega(z, u_1) = \max \{F(z, v) : v \in \mathbb{C}^n, |v| = 1\}$.

By induction, choose unit vectors $u_1, \dots, u_\alpha$ and $\mu_\beta = F(z, u_\beta)$ for $1 \leq \beta, \alpha \leq n - 1$. If V_α denotes the orthogonal complement of $\{u_1, \dots, u_\alpha\}$ in $\mathbb{C}^n$, then

$$\mu_{\alpha+1}(z) := F(z, u_{\alpha+1}) = \max \{F(z, v) : v \in V_\alpha, |v| = 1\}$$

Let L_z be a linear transform on $\mathbb{C}^n$ given as

$$L_z(u_\alpha) = \mu_\alpha(z)e_\alpha, \quad \alpha = 1, \dots, n \quad (3.1)$$

where $\{e_1, \dots, e_n\}$ is the standard Euclidean basis for $\mathbb{C}^n$.

Before we proceed, we recall that S. Frankel ([6]) has defined a sequence

$$\tau_j = [d\varphi_j(p)]^{-1} \cdot (\varphi_j(\cdot) - p_j) \quad (3.2)$$

where $\{\varphi_j\}$ is a sequence of holomorphic automorphisms on Ω , $p_j = \varphi_j(p)$ and $d\varphi_j(p)$ is the complex Jacobian matrix of φ_j at p . Frankel proved that if Ω is convex and hyperbolic, the $\{\tau_j\}$ has a convergent subsequence. In the following theorem, we prove that the linear transform $[d\varphi_j(p)]^{-1}$ can be replaced by the linear map L_{p_j} that we constructed and we call it a stretching factor of Ω at p_j induced by the metric F .

Theorem 3.1.1.

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $p \in \Omega$. Suppose that there is a non-compact sequence of automorphisms $\{\varphi_j\}$. Let $L_j = L_{p_j}$ be the stretching factor induced by any convex invariant metric. Then the scaling sequence σ_j defined by

$$\sigma_j := L_j \circ (\varphi_j(\cdot) - p_j) \quad (3.3)$$

converges subsequentially if and only if Frankel's sequence (3.2) converges subsequentially.

Proof.

Suppose F is a convex invariant metric on the domain Ω . Choose u_α^j and μ_α^j to

construct L_j by (3.1) at $p_j = \varphi_j(p)$ and let

$$\tilde{u}_\alpha^j = d\varphi_j(p)^{-1} (u_\alpha^j), \quad \alpha = 1, \dots, n.$$

The automorphism φ_j preserves the metric F , and this implies $F(p; \tilde{u}_\alpha^j) = \mu_\alpha^j$.

Hence, the vectors $\tilde{u}_\alpha^j / \mu_\alpha^j \in \partial I_{\Omega, F}(p)$, where

$$I_{\Omega, F}(s) = \{v \in \mathbb{C}^n : F(s; v) < 1\},$$

is the indicatrix of F at $s \in \Omega$. Notice that the indicatrix is always convex for any $s \in \Omega$. Define the linear map A_j on $\mathbb{C}^n$ as

$$A_j \left(\frac{\tilde{u}_\alpha^j}{\mu_\alpha^j} \right) = e_\alpha, \quad \alpha = 1, \dots, n.$$

Then

$$L_j = A_j \circ d\varphi_j(p)^{-1}$$

by definition. This means that we only need to show that

$$C_1|v| \leq |A_j v| \leq C_2|v|, \quad \forall v \in \mathbb{C}^n \tag{3.4}$$

where C_1 and C_2 are two positive constants that do not depend on j , to prove that the convergence of $\{\tau_j\}$ is equivalent to the convergence of $\{\sigma_j\}$.

The lower estimate is equivalent to the upper bound of A_j^{-1} . This is clear because

$$|A_j^{-1}v| = \left| A_j^{-1} \left(\sum_{\alpha} a_{\alpha} e_{\alpha} \right) \right| = \left| \sum_{\alpha} a_{\alpha} \left(\frac{\tilde{u}_{\alpha}^j}{\mu_{\alpha}^j} \right) \right| \leq nM|v|$$

for every $v = \sum_{\alpha} a_{\alpha} e_{\alpha} \in \mathbb{C}^n$, where $M = \sup \{|w| : w \in \partial I_{\Omega, F}(p)\}$. Therefore, if we prove

$$|\det A_j| \leq C \quad (3.5)$$

where C is a positive constant that doesn't depend on j , we can find the desired upper estimate. By the definition of L_j , we have

$$|\det L_j| = \prod_{\alpha} \mu_{\alpha}^j. \quad (3.6)$$

If we denote the determinant of the real linear operator on $\mathbb{R}^{2n} = \mathbb{C}^n$ by $\det_{\mathbb{R}}$ and the Euclidean volume of a region G by $\text{Vol}(G)$, then since $d\varphi_j(p)$ maps $I_{\Omega}(p)$ onto $I_{\Omega}(p_j)$, we have

$$\det_{\mathbb{R}} d\varphi_j(p) = |\det d\varphi_j(p)|^2 = \frac{\text{Vol}(I_{\Omega, F}(p_j))}{\text{Vol}(I_{\Omega, F}(p))} = C \text{Vol}(I_{\Omega, F}(p_j)) \quad (3.7)$$

Where $\text{Vol}(I_{\Omega, F}(p)) = \frac{1}{C}$. Regarding $\mathbb{C}^n = \mathbb{R}^n + \sqrt{-1}\mathbb{R}^n$, let

$$R_j = I_{\Omega, F}(p_j) \cap \mathbb{R}^n = \{(a_1, \dots, a_n) \in I_{\Omega, F}(p_j) : a_{\alpha} \text{'s are real}\}.$$

Since $I_{\Omega, F}(p_j)$ is completely circular, it is a circular region generated by R_j and

R_j is symmetric with respect to the origin. Moreover, the region R_j is convex in $\mathbb{R}^n$ since $I_{\Omega,F}(p_j)$ is convex.

The following lemma is needed to establish the proof of the estimation (3.5).

Lemma 3.1.2.

Let R be a bounded convex domain in $\mathbb{R}^n$ which is symmetric with respect to the origin. Let

$$r_1 = \min\{|x| : x \in \partial R\}.$$

Suppose that $(r_1, 0, \dots, 0) \in \partial R$ (this means the minimum distance to the boundary is attained at the x_1 -intercept of ∂R). Let

$$\tilde{R} = \{(x_2, \dots, x_n) \in \mathbb{R}^{n-1} : (0, x_2, \dots, x_n) \in R\}.$$

Suppose

$$\tilde{R} \subset \tilde{B} := [-r_2, r_2] \times \dots \times [-r_n, r_n]$$

for some $r_2, \dots, r_n > 0$. Then

$$R \subset B := [-r_1, r_1] \times 2\tilde{B} = [-r_1, r_1] \times [-2r_2, 2r_2] \times \dots \times [-2r_n, 2r_n].$$

Proof of Lemma 3.1.2.

We notice first that the supporting hyperplanes at $(\pm r_1, 0, \dots, 0)$ are

$$x_1 = \pm r_1,$$

since r_1 is the minimum distance from the origin to ∂R . Therefore

$$x = (x_1, \dots, x_n) \in R \implies |x_1| \leq r_1.$$

We may assume $\tilde{R} = \tilde{B}$. Since $\partial\tilde{R} = \partial\tilde{B}$ contains boxes $\{\pm r_2\} \times [-r_3, r_3] \times \dots \times [-r_n, r_n]$, the supporting hyperplanes of R at $(0, \pm r_2, 0, \dots, 0) \in \partial R$ also contains those boxes. This means if

$$x_2 = \alpha_1 x_1 + \alpha_3 x_3 + \dots + \alpha_n x_n \pm r_2$$

are the equation of the supporting hyperplanes, then $\alpha_3 = \dots = \alpha_n = 0$. Then from the condition that the distance from the origin to the supporting hyperplanes is greater than or equal to r_1 , we have

$$\frac{r_2}{\sqrt{\alpha_1^2 + 1}} \geq r_1.$$

Therefore,

$$|\alpha_1| \leq \sqrt{\left(\frac{r_2}{r_1}\right)^2 - 1}$$

Since $x = (x_1, \dots, x_n)$ lies between two support hyperplanes $x_2 = \alpha_1 x_1 \pm r_2$ and $|x_1| \leq r_1$ if $x \in R$, we have

$$|x_2| \leq \sqrt{r_2^2 - r_1^2} + r_2 \leq 2r_2.$$

Similarly, we can also show that $|x_\alpha| \leq 2r_\alpha$ for any $\alpha = 2, \dots, n$ whenever

$x = (x_1, \dots, x_n) \in R$. Thus the desired conclusion follows.

Resuming the proof of Theorem 3.1.1, we let

$$R_{j,1} = \{(x_{n-1}, x_n) \in \mathbb{R}^2 : (0, \dots, 0, x_{n-1}, x_n) \in R_j\}.$$

Applying Lemma 3.1.2, we see that

$$R_{j,1} \subset B_{j,1} := [-1/\mu_{n-1}^j, 1/\mu_{n-1}^j] \times [-2/\mu_n^j, 2/\mu_n^j],$$

since $\tilde{R}_{j,1} = [-1/\mu_n^j, 1/\mu_n^j]$. Applying the lemma to

$$R_{j,2} = \{(x_{n-2}, x_{n-1}, x_n) \in \mathbb{R}^3 : (0, \dots, 0, x_{n-2}, x_{n-1}, x_n) \in R_j\},$$

we see that

$$R_{j,1} \subset B_{j,2} := [-1/\mu_{n-2}^j, 1/\mu_{n-2}^j] \times [-2/\mu_{n-1}^j, 2/\mu_{n-1}^j] \times [-2^2/\mu_n^j, 2^2/\mu_n^j],$$

since $\tilde{R}_{j,2} = R_{j,1} \subset B_{j,1}$. After repeating the same procedure, we get

$$R_j \subset B_j := \prod_{\alpha=1}^n [-2^{\alpha-1}/\mu_\alpha^j, 2^{\alpha-1}/\mu_\alpha^j].$$

Therefore, the indicatrix $I_{\Omega,F}(p_j)$ is a subset of the circular region in $\mathbb{C}^n$ generated

by B_j which is the polydisc

$$P_j := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_\alpha| \leq 2^{\alpha-1}/\mu_\alpha^j, \alpha = 1, \dots, n\}.$$

This yields

$$\text{Vol}(I_{\Omega, F}(p_j)) \leq \text{Vol}(P_j) \leq C \prod_{\alpha} \left(\frac{1}{\mu_\alpha^j} \right)^2$$

for a positive constant C doesn't depend on j . Therefore, by using (3.6) and (3.7) we get the estimate (3.5).

3.2 Convergence of Frankel Sequence

In this section, we are seeking a condition of metric such that Frankel sequence converges. Let Ω be a given domain in $\mathbb{C}^n$ and let F be a metric on Ω . Let $B_{\Omega, F}(x; r)$ represents the metric ball of center x and radius r For $r > 0$, i.e.

$$B_{\Omega, F}(x; r) = \{y \in \Omega : d_{\Omega, F}(x, y) < r\}.$$

Definition 3.2.1.

Let Ω be a domain in $\mathbb{C}^n$. We say that Ω is *uniformly dominated by a metric F* on Ω if

- F is complete (i.e. $d_{\Omega, F}$ is Cauchy complete),
- For each $r > 0$, there is $\lambda(r) > 0$ such that

$$B_{\Omega,F}(x;r) - x \subset \lambda(r)I_{\Omega,F}(x) \quad (3.8)$$

for any $x \in \Omega$, where $B_{\Omega,F}(x;r) - x = \{y - x : y \in B_{\Omega,F}(x;r)\}$, is the Euclidean translation.

Proposition 3.2.2.

Let Ω be a domain and $p \in \Omega$ be fixed. Assume that Ω is uniformly dominated by an invariant metric F . Then

$$\mathcal{A} = \{\tau_\varphi : \Omega \rightarrow \mathbb{C}^n : \varphi \in \text{Aut}(\Omega)\}$$

is a normal family, where $\tau_\varphi = [d\varphi(p)]^{-1} \circ (\varphi(\cdot) - \varphi(p))$,

Proof.

Let $r > 0$ and let φ be an automorphism of Ω . Then

$$\begin{aligned} \tau_\varphi(B_{\Omega,F}(p;r)) &= [d\varphi(p)]^{-1}(B_{\Omega,F}(\varphi(p);r) - \varphi(p)) \\ &\subset \lambda(r)[d\varphi(p)]^{-1}(I_{\Omega,F}(\varphi(p))) \\ &= \lambda(r)I_{\Omega,F}(p) \end{aligned}$$

by (3.8), since F is invariant. Therefore, the family $\mathcal{A}$ is uniformly bounded on $B_{\Omega,F}(p;r)$. This means that $\mathcal{A}$ is uniformly bounded on every compact subset of Ω . Apply Montel's theorem to obtain the conclusion.

Example 3.2.3.

This example shows that the upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$ is uniformly

dominated by the Poincaré metric which agrees with the Kobayashi-Royden metric on $\mathcal{H}$.

For $v \in \mathbb{C}$ and $p, z \in \mathcal{H}$, The forms of the metric and the distance are given respectively by

$$F(p; v) = \frac{|v|}{2 \operatorname{Im} p}, \quad d_{\mathcal{H}, F}(z, p) = 2 \tanh^{-1} \left| \frac{z - p}{z - \bar{p}} \right|. \quad (3.9)$$

By translation, we may assume $p = ib$, where $i = \sqrt{-1}$, $b > 0$. Therefore,

$$I_{\mathcal{H}, F}(p) = \{\zeta \in \mathbb{C} : |\zeta| < 2b\}$$

Assume that $z \in B_{\mathcal{H}}(p, r)$, then

$$\frac{|z - ib|}{|z + ib|} < \tanh \frac{r}{2}$$

After some calculations we have $|z - ib(r)|^2 < [b(r)]^2 - b^2$

where

$$b(r) = \left(\frac{1 + \tanh^2(r/2)}{1 - \tanh^2(r/2)} \right) b$$

This means that the ball $B_{\mathcal{H}, F}(p, r)$ is centred at $ib(r)$ and of radius $\sqrt{b(r)^2 - b^2}$.

In order to satisfy the condition (3.8), we define $\lambda_{\mathcal{H}}(r)$ as

$$\begin{aligned}\lambda_{\mathcal{H}}(r) &= \frac{1}{2b} \left(b(r) - b + \sqrt{b(r)^2 - b^2} \right) \\ &= \frac{\tanh(r/2)}{1 - \tanh(r/2)}\end{aligned}\tag{3.10}$$

Remark 3.2.4.

The uniform domination condition of a metric is not holomorphic invariant, because holomorphic maps do not preserve Euclidean distance. However, it is an affine transform invariant.

Now we consider conditions of invariant metrics that guarantee uniform domination. For the rest of this section, we will denote the Kobayashi indicatrix and the Kobayashi distance ball by $I_{\Omega}(x)$ and $B_{\Omega}(x; r)$, respectively.

Theorem 3.2.5.

Let Ω be a hyperbolic convex domain in $\mathbb{C}^n$. Then Ω is uniformly dominated by the Kobayashi-Royden metric K_{Ω} . Indeed,

$$B_{\Omega}(x; r) - x \subset 2\lambda_{\mathcal{H}}(r)I_{\Omega}(x)$$

for each $r > 0$ and $x \in \Omega$, where $\lambda_{\mathcal{H}}$ is the function defined as (3.10)

Proof.

The proof is based on reducing the dimension to dimension one and then applying the upper half plane case result (Example 3.2.3). Let $x \in \Omega$ be a point and let $v \in \mathbb{C}^n \setminus \{0\}$. Define the complex lines L_v and $L_{x,v}$ respectively as

$$L_v = \{\xi v : \xi \in \mathbb{C}\}, \quad L_{x,v} := L_v + x = \{\xi v + x : \xi \in \mathbb{C}\}$$

Let $\Omega_{x,v} = \Omega \cap l_{x,v}$. Notice that the intersection is a bounded hyperbolic and convex domain. Suppose $p = p(x, v)$ is a point on $\partial\Omega_{x,v}$ such that

$$d := |x - p| = \text{dis}(x, \partial\Omega_{x,v})$$

where dis denotes the Euclidean distance. Then the disk

$$\Delta_{x,v}(d) = \{z \in L_{x,v} : |z - x| < d\}$$

is contained in $\Omega_{x,v}$. Hence,

$$K_{\Omega}(x, v) \leq K_{\Omega_{x,v}}(x, v) \leq \frac{|v|}{d} = 2K_H(\sqrt{-1}d, v) \quad (3.11)$$

By applying that maps $(x - p)/|x - p|$ to e_1 , a translation $z \rightarrow z - p$ and a linear map

$$e_1 \longrightarrow e_1, v_2 \longrightarrow e_2, \dots, v_n \longrightarrow e_n,$$

where $v_2, \dots, v_n$ are linearly independent vectors over $\mathbb{C}$ and $e_1, \dots, e_n$ are the standard orthonormal unit vectors, we may assume

- (i) $p = 0$,
- (ii) $l_{x,v}$ is the z_1 plane,
- (iii) $\Omega_{x,v}$ lies on $\mathcal{H}$ in z_1 -plane and the real axis is the supporting line of $\Omega_{x,v}$,
- (iv) $x = \sqrt{-1}d$, and
- (v) the supporting hyperplane of Ω at $p = 0$ is $\Pi = \{\text{Im } z_1 = 0\}$ and hence $\Omega \subset$

$\mathcal{H} \times \mathbb{C}^{n-1}$.

Let $y \in \Omega_{x,v} \subset \mathcal{H} \times \{O\}$, then by using the Kobayashi distance decreasing property and the property for the distance on a product of domains we have

$$d_{\Omega}(x, y) \geq d_{\mathcal{H} \times \mathbb{C}^{n-1}}(x, y) = d_{\mathcal{H}}(x, y). \quad (3.12)$$

Finally, combining (3.11) with (3.12) would imply

$$(B_{\Omega}(x, r) - x) \cap l_v \subset 2\lambda_{\mathcal{H}}(I_{\Omega}(x) \cap l_v),$$

which conclude the proof since $x \in \Omega$ and $v \in \mathbb{C}^n \setminus \{0\}$ are arbitrary.

CHAPTER 4

MAXIMAL CIRCULARITY FUNCTION

4.1 Maximal Circularity Function in Bounded Domains

Definition 4.1.1.

Let Ω be a bounded domain in $\mathbb{C}^n$. Ω is said to be a *completely circular domain* if $\forall c \in \bar{\Delta}$ and $z \in \Omega, cz \in \Omega$. where $\bar{\Delta}$ represents the closure of the unit disc.

Remark 4.1.2.

The indicatrix $I_\Omega(z) = \{v \in \mathbb{C}^n : K_\Omega(z, v) < 1\}$ is a completely circular domain by homogeneity property of the Kobayashi metric.

Definition 4.1.3.

Let Ω be a bounded domain in $\mathbb{C}^n$, and x is a point in Ω . Let $B_\Omega(x) = \{\varphi : \Omega \longrightarrow \mathbb{C}^n, \varphi \text{ is a bounded holomorphic embedding, } \varphi(x) = 0\}$. Let $I_\Omega(x)$ be

the Kobayashi indicatrix of Ω at the point x . we define the function $c_\Omega(x)$ as follows

$$c_\Omega(x) := \sup \left\{ \frac{r}{R} : \exists \varphi \in B_\Omega(x), R > 0, r > 0, \text{ such that } rI_{\varphi(\Omega)}(0) \subset \varphi(\Omega) \subset RI_{\varphi(\Omega)}(0) \right\} \quad (4.1)$$

We call $c_\Omega(x)$ the *maximal circularity function* of Ω .

Remarks 4.1.4.

- (i) $0 < c_\Omega(x) \leq 1, \forall x \in \Omega$,
- (ii) If Ω is bounded completely circular domain, then Ω is biholomorphic to $I_\Omega(0)$, ([3]), and then $c_\Omega(0) = 1$.

In the next proposition, we prove that the converse of (ii) is valid too.

proposition 4.1.5.

Let Ω be a bounded domain in $\mathbb{C}^n$. Suppose that $c_\Omega(x) = 1$ at a point $x \in \Omega$. Then Ω is biholomorphic to a complete circular domain.

proof.

Suppose we choose $\varphi_j \in B_\Omega(x)$ such that

$$r_j I_j \subset \varphi_j(\Omega) \subset R_j I_j \quad (4.2)$$

for some $R_j \geq r_j > 0$ and $r_j/R_j \rightarrow 1$ as $j \rightarrow \infty$, where I_j is the indicatrix of $\varphi_j(\Omega)$ at the origin

$$I_j := I_{\varphi_j(\Omega)}(0) = d\varphi_j(x) (I_\Omega(x)),$$

and $d\varphi_j(x) : T_x\Omega \rightarrow T_0\varphi_j(\Omega)$ is a linear map.

Let

$$\psi_j = [d\varphi_j(x)]^{-1} \cdot \varphi_j$$

Then $\psi_j \in B_\Omega(x)$. Applying $[d\varphi_j(x)]^{-1}$ to (4.2), we get

$$r_j I_\Omega(x) \subset \psi_j(\Omega) \subset R_j I_\Omega(x)$$

To prove R_j is bounded, assume by contradiction that $R_j \rightarrow \infty$ as $j \rightarrow \infty$. Then $r_j \rightarrow \infty$ as well. Choose a small $\varepsilon > 0$ such that the Euclidean ball $B(0, \varepsilon)$ is contained in $I_\Omega(x)$, and let $v \in \mathbb{C}^n \setminus \{0\}$. Then since $d\psi_j(x) = Id$ and Ω is hyperbolic,

$$0 < K_\Omega(x, v) = K_{\psi_j(\Omega)}(0, v) \leq K_{r_j B(0, \varepsilon)}(0, v) = \frac{|v|}{r_j \varepsilon} \rightarrow 0$$

as $j \rightarrow \infty$, which yields a contradiction.

Applying Mantel's Theorem, ψ_j converges subsequentially to a map ψ , and then $r_j \rightarrow R_0, R_j \rightarrow R_0$ for some $R_0 \geq 0$. Since $d\psi_j(x) = Id$, the limit ψ of the sequence ψ_j is also a 1-1 holomorphic mapping and $\psi(\Omega) = R_0 I_\Omega(x)$. Hence, $R_0 > 0$ and Ω is biholomorphic to $R_0 I_\Omega(x)$.

Definition 4.1.6.

Let D be a bounded complete circular domain in $\mathbb{C}^n$. Then the *squeezing function* $s_\Omega(x)$ of Ω at $x \in \Omega$ modelled on D is defined by

$$s_\Omega^D(x) := \sup \left\{ \frac{r}{R} : \exists \varphi \in \mathcal{B}_\Omega(x), R > 0, r > 0 \text{ such that } rD \subset \varphi(\Omega) \subset RD \right\}.$$

Since the definition of the squeezing function is similar to the maximal circularity function, we could identify a correlation between the two functions.

Proposition 4.1.7.

Let Ω be a bounded domain in $\mathbb{C}^n$ and $x \in \Omega$. Then for any bounded complete circular domain $D \subset \mathbb{C}^n$,

$$c_\Omega(x) \geq (s_\Omega^D(x))^2$$

Proof.

Suppose $\varphi \in \mathcal{B}_\Omega(x)$ such that $\varphi(\Omega) \subset D$. Choose $r > 0$ such that $rD \subset \varphi(\Omega)$. Since the bounded complete circular domain is the same with its Kobayashi indicatrix at the origin ([3]), considering that both domains D and rD are completely circular, we have

$$I_D(0) = D, \quad I_{rD}(0) = rD$$

Therefore,

$$rD = I_{rD}(0) \subset I_{\varphi(\Omega)}(0) \subset I_D(0) = D$$

Using the decreasing property of the Kobayashi metric. Hence,

$$rI_{\varphi(\Omega)}(0) \subset rD \subset \varphi(\Omega) \subset D \subset \frac{1}{r}I_{\varphi(\Omega)}(0).$$

This means $c_\Omega(x) \geq r^2$. Taking the supremum for r , the desired result is obtained.

4.2 Maximal Circularity Function in Strongly Pseudoconvex Domains

Theorem 4.2.1.

If Ω is a bounded domain in $\mathbb{C}^n$, with C^2 - smooth strongly pseudoconvex boundary, then $c_\Omega(x) \rightarrow 1$ as $x \rightarrow p \in \partial\Omega$.

Proof.

This is a consequence of the work ([9], [23]), since in strongly pseudo-convex domains, for every point $p \in \partial\Omega$, and a sequence $\{x_j\}$ in Ω , if $x_j \rightarrow p$, then $s_\Omega(x_j) = s_\Omega^{B^n}(x_j) \rightarrow 1$ as $j \rightarrow \infty$, where B^n represents the unit ball in $\mathbb{C}^n$. Using Proposition 4.1.7, the conclusion follows.

4.3 Maximal Circularity Function in Convex Polyhedral Domains

Definitions 4.3.1

(1) Let Ω be a bounded domain in $\mathbb{C}^n$. Ω is called a *polyhedral domain* if there are smooth real-valued functions $\rho_1, \dots, \rho_N$ defined on a neighborhood of $\bar{\Omega}$ such that

- $\Omega = \{z \in \mathbb{C}^n : \rho_1(z) < 0, \dots, \rho_N(z) < 0\}$
- on $\partial_\alpha\Omega := \{z \in \partial\Omega : \rho_\alpha(z) = 0\}$ for $\alpha = 1, \dots, N$, $\nabla\rho_\alpha$ never vanishes,
- each $\partial_\alpha\Omega$ is Levi flat.

(2) The polyhedral domain Ω is said to be *normal* if it satisfies in addition

- on each $\partial_{\alpha_1 \dots \alpha_k} \Omega := \{z \in \partial\Omega : \rho_{\alpha_1}(z) = \dots = \rho_{\alpha_k}(z) = 0\}$, $\nabla \rho_{\alpha_1}, \dots, \nabla \rho_{\alpha_k}$ are linearly independent over $\mathbb{C}$.

Theorem 4.3.2.

Let $\Omega = \{z \in \mathbb{C}^n : \rho_1(z) < 0, \dots, \rho_N(z) < 0\}$ be a convex normal polyhedral domain in $\mathbb{C}^n$. Let $q \in \partial_{\alpha_1 \dots \alpha_n} \Omega$ for some distinct indices $\alpha_1, \dots, \alpha_n$. Then $c_\Omega(x) \rightarrow 1$ as $x \rightarrow q$.

Proof.

Assume without loss of generality that $\{\alpha_1, \dots, \alpha_n\} = \{1, \dots, n\}$ and $|\nabla \rho_\alpha| \equiv 1$ on $\partial_\alpha \Omega$ for each $\alpha = 1, \dots, n$. We need to show that $s_\Omega^{\Delta^n}(x) \rightarrow 1$ when $x \rightarrow q$, where Δ^n denotes the unit polydisc in $\mathbb{C}^n$.

Since $\partial_\alpha \Omega$ is Levi flat, there exist an analytic variety $V_\alpha \subset \partial_\alpha \Omega$ for every α , and since Ω is convex, V_α is a part of affine hyperplanes H_α such that

- $\cap H_\alpha = \{q\}$
- $V_\alpha := H_\alpha \cap \partial\Omega$ is the maximal variety that contains q in $\partial_\alpha \Omega$.

After a change of coordinates, we may assume

- $q = 0$
- $H_\alpha = \{z_\alpha = 0\}$
- $\{\operatorname{Re} z_\alpha = 0\}$ are the supporting hyperplanes of Ω .
- Ω is a subset of $\mathcal{H}^n$, where $\mathcal{H} = \{z \in \mathbb{C}, \operatorname{Re} z < 0\}$ is the left half plane in $\mathbb{C}$.

Now suppose that x is a point in Ω that is very close to the origin, then there exists an affine hyperplane $h_\alpha = h_\alpha(x)$ that is tangent to $\partial_\alpha \Omega$ for each $\alpha = 1, \dots, n$, unique and is closest to x . Let $p = p(x) \in \partial_{12\dots n} \Omega$, such that $\cap h_\alpha = \{p\}$, and let $v_\alpha = v_\alpha(x) = \nabla \rho_\alpha(p)$.

Define an affine transform A_x as follows

- $A_x(p) = 0$
- $dA_x(v_\alpha) = e_\alpha$, for each α .

Then $A_x(\Omega) \subset \mathcal{H}^n$, and $A_x(x) = (x, \dots, x_n)$, where $x_\alpha < 0$ and $x_\alpha \rightarrow 0$ as $x \rightarrow 0$.

Now, Let $\Phi : \mathcal{H}^n \rightarrow \Delta^n$ be the Cayley transform such that $\Phi(0) = (1, \dots, 1)$, then $\Phi(A_x(x)) = (1 - r_1, \dots, 1 - r_n)$, where $r_\alpha > 0$, and $r_\alpha \rightarrow 0$ as $x \rightarrow 0$ for each α .

Let $\pi_\alpha : \mathbb{C}^n \rightarrow \mathbb{C}$, be the natural projection, i.e.,

$$\pi_\alpha(z_1, \dots, z_n) = z_\alpha, \quad \forall \alpha = 1, \dots, n.$$

Since Ω is normal, $\partial_{1\dots n} \Omega$ is a smooth real n -dimensional, and hence $\pi_\alpha(\Phi(A_x(\Omega)))$ has a smooth boundary near 1 .

Therefore, $\forall x \in \Omega$ that is close enough to 0, $\exists \varepsilon > 0$, such that $\Delta_\varepsilon^n \subset \Phi(A_x(\Omega))$, where $\Delta_\varepsilon = \{z \in \mathbb{C}, |z - (1 - \varepsilon)| < \varepsilon\}$.

Now, suppose $\Psi_x \in \text{Aut}(\Delta^n)$, such that

$$\Psi_x(1, 1, \dots, 1) = (1, \dots, 1) \text{ and } \Psi_x(1 - r_1, \dots, 1 - r_n) = (0, \dots, 0)$$

Then, for an arbitrary r , $0 < r < 1$, and any $x \in \Omega$ close enough to 0

$$r\Delta^n \subset \Psi_x(\Delta_\varepsilon^n) \subset \Psi_x \circ \Phi \circ A_x(\Omega)$$

Taking supremum for r , we get $s_\Omega^{\Delta^n}(x) \longrightarrow 1$ when $x \longrightarrow q$.

CHAPTER 5

HOLOMORPHICITY OF THE KOBAYASHI ISOMETRY

This part of the dissertation is included in the second paper we prepare for publication ([2]). Our future goal is to prove that Kobayashi isometry between two generic analytic polyhedra (GAP) is either holomorphic or antiholomorphic if it is irreducible. In this chapter, we introduce scaling theory regarding Kobayashi isometry between analytic polyhedra for this purpose. Although all the demonstrations in this chapter work for any dimensional case, we consider the case $n = 2$ for simplicity. In section 5.1, we consider a special case of GAP, which is the bidisc. We prove that Kobayashi isometry from a bidisc into itself is either holomorphic or anti-holomorphic componentwisely. In section 5.2, we consider the general case.

5.1 Kobayashi Isometry on Bidiscs

Theorem 5.1.1

Let $f : \Delta^2 \longrightarrow \Delta^2$ be a Kobayashi isometry, where Δ^2 denotes the bidisc. Then by composing with an automorphism on Δ^2 , we have

$$f(z_1, z_2) = (\Phi(z_1), \Phi(z_2))$$

where $\Phi(z)$ is either z or $\bar{z}$.

Remarks 5.1.2

(1) Although we consider the case of Δ^2 for simplicity, the same statement is true for arbitrary Δ^n .

(2) The result has been already proved in ([14],[25]) under the assumption of C^1 -regularity of the Kobayashi isometry. Our proof is a modification of that in ([25]) to fit to C^0 -assumption.

In preparation for presenting the proof of the theorem, we review some known results in the following remarks:

Remarks 5.1.3

- (1) The Kobayashi distance on the bidisc is complete.
- (2) $\forall p, q \in \Delta^2$, there exists a geodesic $\sigma : \mathbb{R} \longrightarrow \Delta^2$ joining p and q .
- (3) If $\sigma : [a, b] \longrightarrow \Delta^2$ is a Kobayashi geodesic, then $\forall [c, d] \subset [a, b]$, $\sigma|_{[c, d]}$ is also a geodesic.
- (4) A Kobayashi geodesic passing through the origin is a straight line. In particular, $\gamma(s) = \tanh s : (-\infty, \infty) \longrightarrow \Delta$ is the unit speed parametrization of the geodesic. Therefore, $\forall \lambda = e^{i\theta} \in S^1$, $\lambda\gamma(s)$ is the geodesic through 0 to λ , where S^1 is the unit circle.

(5) Circular arc in Δ intersecting $\partial\Delta$ orthogonally are Kobayashi geodesics on Δ .

(6) If σ_1 and σ_2 are two geodesics on Δ such that

$$\lim_{s \rightarrow \pm\infty} \sigma_1(s) = \lim_{s \rightarrow \pm\infty} \sigma_2(s)$$

then $\sigma_1(s) = \sigma_2(s + s_0)$, for some $s_0 \in \mathbb{R}$.

(7) If $\sigma : \mathbb{R} \rightarrow \Delta^2$ is a Kobayashi geodesic and $f : \Delta^2 \rightarrow \Delta^2$ is a Kobayashi isometry, then $f \circ \sigma : \mathbb{R} \rightarrow \Delta^2$ is also a Kobayashi geodesic.

(8) Let $\sigma : \mathbb{R} \rightarrow \Delta$ be a geodesic, then $\forall y \in \Delta, \tilde{\sigma}(s) = (\sigma(s), y)$ is a geodesic in Δ^2 . Similarly, $(x, \sigma(s))$ is also a geodesic in $\Delta^2, \forall x \in \Delta$.

Lemma 5.1.4

Let $f : \Delta^2 \rightarrow \Delta^2$ be a Kobayashi isometry and $\sigma : \mathbb{R} \rightarrow \Delta^2$ is any geodesic. Let $f(x, y) = (f_1(x, y), f_2(x, y)) \in \Delta^2$, then either $f_1 \circ \sigma$ or $f_2 \circ \sigma$ is a Kobayashi geodesic in Δ .

Proof

for each $s > 0$,

$$\begin{aligned} d_{\Delta^2}(\sigma(-s), \sigma(s)) &= d_{\Delta^2}(f(\sigma(-s)), f(\sigma(s))) \\ &= \max \{d_{\Delta}(f_1 \circ \sigma(-s), f_1 \circ \sigma(s)), d_{\Delta}(f_2 \circ \sigma(-s), f_2 \circ \sigma(s))\} \end{aligned}$$

Therefore, for each $n > 0$,

$$\begin{aligned}
d_{\Delta}(f_1 \circ \sigma(-n), f_1 \circ \sigma(n)) &= d_{\Delta^2}(\sigma(-n), \sigma(n)) \\
&= l_{\Delta^2}(\sigma|_{[-n, n]}) \\
&= l_{\Delta^2}(f \circ \sigma|_{[-n, n]}) \\
&\geq l_{\Delta}(f_1 \circ \sigma|_{[-n, n]})
\end{aligned}$$

Hence $f_1 \circ \sigma$ is a geodesic.

Proof of theorem 5.1.1.

Let $\gamma(s) = \tanh s$ and let

$$K_1 = \{(\lambda, y) \in S^1 \times \Delta, f_1(\lambda\gamma(s), y) \text{ is a geodesic} \}$$

$$K_2 = \{(\lambda, y) \in S^1 \times \Delta, f_2(\lambda\gamma(s), y) \text{ is a geodesic} \}.$$

Since $f(\lambda\gamma(s), y)$ is a geodesic $\forall (\lambda, y) \in S^1 \times \Delta$, we have $K_1 \cup K_2 = S^1 \times \Delta$ by lemma 5.1.3.

Also, the sets K_1 and K_2 are closed. Indeed, if $(\lambda_j, y_j) \in K_1$ and $(\lambda_j, y_j) \rightarrow (\lambda, y)$, then since $(f_1(\lambda_j\gamma(\cdot), y_j))$ is a geodesic, we have $\forall s, t \in \mathbb{R}, d_{\Delta}(f_1(\lambda_j\gamma(s), y_j), f_1(\lambda_j\gamma(t), y_j)) = |s - t|$, and as $j \rightarrow \infty$, $d_{\Delta}(f_1(\lambda\gamma(s), y), f_1(\lambda\gamma(t), y)) = |s - t|$. Hence, $f_1(\lambda\gamma(\cdot), y)$ is a geodesic and $(\lambda, y) \in K_1$. Similarly, we can prove that K_2 is closed.

Now, suppose that either K_1 or K_2 is $S^1 \times \Delta$. We may assume that $K_1 = S^1 \times \Delta$. Or if we assume that neither K_1 or K_2 are $S^1 \times \Delta$, then $\exists (\lambda_0, y_0) \notin K_2$, and since K_2

is closed, there exists an open neighborhood I of λ_0 in S^1 and U of y_0 in Δ such that

$$(\lambda_0, y_0) \in I \times U \subset (S^1 \times \Delta) \setminus K_2$$

Since $K_1 \cup K_2 = S^1 \times \Delta$, $I \times U \subset K_1$. Therefore, we may assume that, $\exists (\lambda_0, y_0) \in S^1 \times \Delta$ and a neighborhood $I \times U$ of (λ_0, y_0) in $S^1 \times \Delta$ such that $I \times U \subset K_1$. That is, $\forall (\lambda, y) \in I \times U$, $f_1(\lambda\gamma(\cdot), y) : \mathbb{R} \longrightarrow \Delta$ is a geodesic on Δ .

Let

$$\varphi(x, y) = \left(\lambda_0 x, \frac{y + y_0}{1 - \bar{y}_0 y} \right) \in \text{Aut}(\Delta^2)$$

then $\varphi(0, 0) = (0, y_0)$, and $\varphi(\gamma(s), 0) = (\lambda_0 \gamma(s), y_0)$.

Replacing f by $f \circ \varphi$, we may assume $(\lambda_0, y_0) = (1, 0) \in S^1 \times \Delta$.

Let $(\alpha, \beta) = f \circ \varphi(0, 0)$, and let

$$\psi(z, \omega) = \left(\frac{z - \alpha}{1 - \bar{\alpha}z}, \frac{\omega - \beta}{1 - \bar{\beta}\omega} \right)$$

$f_1(\lambda\gamma(s), y)$ is a geodesic on Δ , and $(\psi \circ f)_1(\lambda\gamma(s), y)$ is a geodesic on Δ .

Replacing f by $\psi \circ f \circ \varphi$, we may assume:

- $f(0, 0) = (0, 0)$
- $(\lambda_0, y_0) = (1, 0)$
- $f_1(\lambda\gamma(s), y)$ is a geodesic, $\forall (\lambda, y) \in I \times U$.

Since $f_1(\gamma(s), 0)$ is a geodesic passing through the origin, it satisfies

$$f_1(\gamma(s), 0) = \mu\gamma(s)$$

for some $\mu = e^{i\theta} \in S^1$. Composing with the rotation $r(z, \omega) = (\mu^{-1}z, \omega)$, we may assume

$$f_1(\gamma(s), 0) = \gamma(s).$$

We claim the following.

Claim 1. Either $f_1(\lambda\gamma(s), 0) = \lambda\gamma(s)$, or $f_1(\lambda\gamma(s), 0) = \bar{\lambda}\gamma(s)$.

Proof of the claim.

Since $f_1(\lambda\gamma(s), 0)$ is a geodesic passing through the origin, we have

$$\begin{aligned} d_\Delta(\gamma(s), \lambda\gamma(s)) &= d_{\Delta^2}((\gamma(s), 0), (\lambda\gamma(s), 0)) \\ &= d_{\Delta^2}(f(\gamma(s), 0), f(\lambda\gamma(s), 0)) \\ &\geq d_\Delta(f_1(\gamma(s), 0), f_1(\lambda\gamma(s), 0)) \\ &= d_\Delta(\gamma(s), \mu\gamma(s)) \end{aligned}$$

Similarly, $d_\Delta(\gamma(s), \lambda\gamma(-s)) \geq d_\Delta(\gamma(s), \mu\gamma(-s))$. Hence, $\mu = \lambda$ or $\mu = \bar{\lambda}$, which concludes the proof of the claim.

We may assume that $f_1(\lambda\gamma(s), 0) = \lambda\gamma(s), \forall \lambda \in I$. Let

$$C = \bigcup \{\lambda\gamma(s), \lambda \in I, s \in \mathbb{R}\}.$$

If $x \in C$, then $x = \lambda\gamma(s)$ for some $\lambda \in I$ and $s \in \mathbb{R}$. Therefore,

$$f(x, 0) = f_1(\lambda\gamma(s), 0) = \lambda\gamma(s) = x.$$

Let $\hat{C}$ denotes the geodesic hull of the set C , and let $H = \hat{C}$.

Claim 2. $\forall x \in H, f_1(x, 0) = x$

proof of the claim.

Let λ_1 and λ_2 belongs to I , and let a and b be points on $\lambda_1\gamma$ and $\lambda_2\gamma$ respectively.

Let $d = d_\Delta(a, b)$ and let σ be a geodesic in Δ joining a and b . We want to show that

$$f_1(\sigma(u), 0) = \sigma(u), \forall u \in [a, b].$$

Since

$$\begin{aligned} l_k(f_1(\sigma(u), 0)) &\leq l_k(f(\sigma(u), 0)) \\ &= l_k(\sigma(u), 0) \\ &= l_k(\sigma(u)) \\ &= d, \end{aligned}$$

$f_1(\sigma(u), 0)$ is a geodesic in Δ , and hence

$$f_1(\sigma(u), 0) = \sigma(u)$$

Claim 3. For each $\lambda \in I$ and $y \in U, f_1(\lambda\gamma(s), y) = \lambda\gamma(s)$.

proof of the claim.

Since $d_{\Delta^2}((\lambda\gamma(s), y), (\lambda\gamma(s), 0)) = d_{\Delta}(y, 0) < \infty$, and

$$\begin{aligned} & d_{\Delta^2}(f(\lambda\gamma(s), y), f(\lambda\gamma(s), 0)) \\ & \geq d_{\Delta}(f_1(\lambda\gamma(s), y), f_1(\lambda\gamma(s), 0)) \\ & = d_{\Delta}(f_1(\lambda\gamma(s), y), \lambda\gamma(s)) \end{aligned}$$

We have $d_{\Delta}(f_1(\lambda\gamma(s), y), \lambda\gamma(s)) < \infty$, and

$$\lim_{s \rightarrow \pm\infty} f_1(\lambda\gamma(s), y) = \lim_{s \rightarrow \pm\infty} \lambda\gamma(s) = \lambda.$$

Since $f_1(\lambda\gamma(s), y)$ is also a geodesic in Δ ,

$$f_1(\lambda\gamma(s), y) = \lambda\gamma(s + s_0), \text{ for some } s_0 = s_0(\lambda, y).$$

Suppose $s = 0$, then $f_1(\lambda\gamma(s), y) = f_1(0, y)$. Since $0 = \lambda\gamma(0)$ belongs to all geodesics, $f_1(\lambda\gamma(0), y) = f_1(0, y)$ belongs to all geodesics, which implies that $f_1(\lambda\gamma(0), y) = 0$. Hence, $\lambda\gamma(s_0) = 0$ and then $s_0 = 0$. This proves the claim.

This means, $f_1(\lambda\gamma(s), y) = \lambda\gamma(s), \forall \lambda \in I$ and $y \in U$. In other words, if $x \in C$ and $y \in U$, then $f_1(x, y) = x$.

Claim 4. Following the steps of claim 2, we also see that $f_1(x, y) = x$, for all $x \in H$ and $y \in U$.

Now, let $x \in H$ and consider $(x, \lambda\gamma(s))$, a geodesic in Δ^2 , then $f(x, \lambda\gamma(s))$ is a geodesic in Δ^2 .

Claim 5. $f_2(x, \lambda\gamma(s))$ is a geodesic in Δ .

proof of the claim.

Since $f(x, \lambda\gamma(s))$ is a geodesic in Δ^2 , either $f_1(x, \lambda\gamma(s))$ or $f_2(x, \lambda\gamma(s))$ is a geodesic. Since $\lambda\gamma(s) \in U$ if s is small, $f_1(x, \lambda\gamma(s)) = x$ if s is small enough. Hence $f_2(x, \lambda\gamma(s))$ is a geodesic. Since $f_2(0, \lambda\gamma(s))$ is a geodesic in Δ and $f_2(0, \lambda\gamma(0)) = 0$, $f_2(0, \lambda\gamma(s))$ is a geodesic passing through the origin. We may then assume that $f_2(0, \gamma(s)) = \gamma(s)$ and by following the claims we may assume that $f_2(0, \lambda\gamma(s)) = \lambda\gamma(s), \forall \lambda \in S^1$. Moreover, following claims 3 and 4, we can assume,

$$f_2(x, \lambda\gamma(s)) = \lambda\gamma(s).$$

which implies the desired result.

Now, consider $f(\lambda\gamma(s), y)$, which is a geodesic in Δ^2 , for each $y \in \Delta$ and $\lambda \in S^1$.

Then either $f_1(\lambda\gamma(s), y)$ or $f_2(\lambda\gamma(s), y)$ is a geodesic. Since if s is small, we have $\lambda\gamma(s) \in H$ and in that case $f_2(\lambda\gamma(s), y) = y$. Hence, $f_1(\lambda\gamma(s), y)$ is a geodesic, $\forall \lambda \in S^1$ and $\forall y \in \Delta$. Following the same argument in claims 3 and 4, we get

$$f_1(\lambda\gamma(s), y) = \lambda\gamma(s), \forall \lambda \in S^1, \text{ and } y \in \Delta.$$

Hence,

$$f_1(x, y) = x, \forall (x, y) \in \Delta^2$$

.

Similarly, we can see that $f_2(x, \lambda\gamma(s))$ is a geodesic, $\forall x \in \Delta, \lambda \in S^1$ and that

$f_2(x, \lambda\gamma(s)) = \lambda\gamma(s)$, which implies that

$$f_2(x, y) = y, \quad \forall (x, y) \in \Delta^2.$$

Finally, we conclude that by taking a composition with automorphisms or conjugations

$$f(x, y) = (f_1(x, y), f_2(x, y)) = (x, y) \text{ on } \Delta^2$$

which completes the proof of the theorem.

5.2 Kobayashi Isometry and Bounded Generic Analytic Polyhedron

Definitions 5.2.1.

(1) An *analytic polyhedron* is a domain Ω in $\mathbb{C}^n$ such that

- There exist holomorphic functions $f_1, \dots, f_N : U \rightarrow \mathbb{C}$, where U is a neighborhood of $\bar{\Omega}$.
- $\Omega = \{z \in U, |f_1(z)| < 1, \dots, |f_N(z)| < 1\}$.

(2) Ω is said to be *generic* if we can choose the functions $f_1, \dots, f_N$ such that if $|f_{i_1}(p)| = \dots = |f_{i_r}(p)| = 1$, for a collection $\{i_1, \dots, i_r\}$ of un-repeated indices and $p \in \partial\Omega$, then $\nabla f_{i_1}(p), \dots, \nabla f_{i_r}(p)$ are linearly independent.

(3) Let $p \in \partial\Omega$, and

$$\mathcal{I}(p) := \{j \in \{1, \dots, N\}, |f_j(p)| = 1\}, r(p) := \text{number of indices in } \mathcal{I}(p).$$

We denote the analytic variety in $\partial\Omega$ passing through p by $\mathcal{F}(p)$, which represents the connected component of

$$\{z \in \partial\Omega, |f_i(z)| = |f_i(p)| = 1 \text{ if } i \in \mathcal{I}(p) \text{ and } |f_i(z)| < 1, i \notin \mathcal{I}(p)\}$$

For the rest of this chapter, we assume having a Kobayashi isometry φ between two bounded GAPs as follows,

Let

$$\Omega = \{z = (z_1, z_2) \in \mathbb{C}^2, |f_1(z)| < 1, \dots, |f_n(z)| < 1\}$$

$$\tilde{\Omega} = \{w = (w_1, w_2) \in \mathbb{C}^2, |g_1(w)| < 1, \dots, |g_m(w)| < 1\}$$

be bounded GAP in $\mathbb{C}^2$. Suppose $\exists \varphi : \Omega \rightarrow \tilde{\Omega}$ a Kobayashi isometry which is C^1 smooth and both φ and φ^{-1} extend continuously to the Silov boundary. Let p be a point on the Silov boundary of Ω . Recall that the Silov boundary is defined as a closed set $S \subset \bar{\Omega}$ such that:

(i) every analytic function f on Ω and continuous on $\bar{\Omega}$ satisfies

$$\max_{z \in \bar{\Omega}} |f(z)| = \max_{z \in S} |f(z)|,$$

(ii) every other closed set $H \subset \bar{\Omega}$ with the property (i) implies that $S \subset H$.

In the generic analytic polyhedron case, the Silov boundary is

$$S = \{p \in \partial\Omega, r(p) = 2\}$$

Lemma 5.2.2.

Assuming an isometry as described above, then $\varphi(p)$ lies on the Silov boundary of $\tilde{\Omega}$.

Proof.

Let S_1 and S_2 denote the Silov boundaries on Ω and $\tilde{\Omega}$ respectively and $p \in \Omega$. We will prove that $\varphi(p) \in S_2$ by contradiction. Suppose $\varphi(p)$ is a flat point. Let $g_1, \dots, g_m$ be holomorphic functions on a neighborhood of $\tilde{\Omega}$ to $\mathbb{C}$ and without loss of generality, assume

$$\mathcal{F}(\varphi(p)) = \left\{ q \in \partial\tilde{\Omega}, g_1(\varphi(p)) = 1, |g_i(\varphi(p))| < 1, i = 2, \dots, m \right\}.$$

choose two sequences q_j and $\tilde{q}_j$ in $\tilde{\Omega}$ converge to $\varphi(p)$ and q_1 respectively, where $q_1 \neq \varphi(p)$ lie on $\mathcal{F}(\varphi(p))$, and such that $g_1(q_j) = g_1(\tilde{q}_j)$.

By estimating the Kobayashi distance in [39], we have

$$d_{\tilde{\Omega}}(q_j, \tilde{q}_j) \leq c [\|q_j - \tilde{q}_j\| + d_{\Delta}(g_1(q_j), g_1(\tilde{q}_j))].$$

By assumption, $d_{\Delta}(g_1(q_j), g_1(\tilde{q}_j)) = 0$ so $d_{\tilde{\Omega}}(q_j, \tilde{q}_j) < \infty$. Since φ is an isometry and $q_j \rightarrow \varphi(p)$, we have, $p_j := \varphi^{-1}(q_j) \rightarrow p$. Let $\varphi^{-1}(\tilde{q}_j) = \tilde{p}_j$, then $d_{\Omega}(p_j, \tilde{p}_j) < \infty$, (see[39]). Therefore, both p_j and $\tilde{p}_j$ converge to the same point $p \in S_1$, but the images by the isometry φ converge to different points, a contradiction, which ends the proof.

Let's assume without losing the generality that,

$$f_1(p) = f_2(p) = 1, \quad |f_j(p)| < 1, \quad j = 3, \dots, n.$$

$$g_1(\tilde{p}) = g_2(\tilde{p}) = 1, \quad |g_j(\tilde{p})| < 1, \quad j = 3, \dots, m.$$

where $\tilde{p} = \varphi(p)$.

Lemma 5.2.3.(Localization Lemma).

For any neighborhood U of p and any $\varepsilon > 0$, there exists a neighborhood $V \subset U$ of p such that

$$(1 - \varepsilon)K_{\Omega \cap U}(q, v) \leq K_{\Omega}(q, v) \leq K_{\Omega \cap U}(q, v)$$

for any $q \in V$ and $v \in \mathbb{C}^2$. (see [32]).

Let U and $\tilde{U}$ be chosen such that

$$F = (f_1, f_2) : U \rightarrow O := F(U) \subset \mathbb{C}^2$$

$$G = (g_1, g_2) : \tilde{U} \rightarrow \tilde{O} := G(\tilde{U}) \subset \mathbb{C}^2$$

are biholomorphisms and.

$$\tilde{U} \cap \tilde{\Omega} = \varphi(U \cap \Omega)$$

Let p_j be a sequence in Ω such that $p_j \rightarrow p$ as $j \rightarrow \infty$ Assume $p_j \in U$ for all j .

Let

$$\cdot \tilde{p}_j := \varphi(p_j) \rightarrow \tilde{p}$$

$$\cdot q_j := F(p_j) \rightarrow (1, 1)$$

$$\cdot \tilde{q}_j := G(\tilde{p}_j) \rightarrow (1.1)$$

as $j \rightarrow \infty$

Let $\varepsilon > 0$, be given, and let V be a neighborhood of p chosen as lemma 5.2.3. Let $W = F(V)$, and let $\tilde{V}$ be a neighborhood of $\tilde{p}$ such that

$$\varphi(\Omega \cap V) = \tilde{\Omega} \cap \tilde{V} \quad \text{and} \quad \tilde{W} = G(\tilde{V}).$$

We may shrink V if necessary and we assume the localization lemma holds for $(\tilde{\Omega}, \tilde{U}, \tilde{V})$, (Δ^2, O, W) and $(\Delta^2, \tilde{O}, \tilde{W})$ that is

$$(1 - \varepsilon)K_{\tilde{\Omega} \cap \tilde{U}}(w; v) \leq K_{\tilde{\Omega}}(w; v) \leq K_{\tilde{\Omega} \cap \tilde{U}}(w; v)$$

$$\forall w \in \tilde{V} \cap \tilde{\Omega}, v \in \mathbb{C}^2.$$

Suppose $\varphi_j, \tilde{\varphi}_j \in \text{Aut}(\Delta^2)$ such that $\varphi_j(q_j) = (0, 0), \tilde{\varphi}_j(\tilde{q}_j) = (0, 0)$. Let K be a compact subset of Δ^2 . Then $\exists N_1 = N_1(K, \varepsilon)$ such that $\varphi_j^{-1}(K) \subset W \cap \Delta^2$ if $j > N_1$.

For each $j > N_1$. choose $r_j \rightarrow 1$ such that

$$\varphi_j^{-1}(\Delta_{r_j}^2) \subset \Delta^2 \cap O$$

where $\Delta_{r_j}^2 = \{\zeta = (\zeta_1, \zeta_2) \in \Delta^2 : |\zeta_1| < r_j, |\zeta_2| < r_j\}$

Construct the sequence of mapping

$$\Lambda_j = \tilde{\varphi}_j \circ G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1}|_{\Delta_{r_j}^2} : \Delta_{r_j}^2 \longrightarrow \Delta^2 \quad (5.1)$$

Assume that K is a compact subset in $\Delta_{r_j}^2$.

Claim.

If $j > N_1$, then

$$K_{\Delta^2}(z, v) \leq (1 - \varepsilon)^{-2} K_{\Delta^2}(\Lambda_j(z), d\Lambda_j|_z(v))$$

$\forall z \in K$ and $v \in \mathbb{C}^2$.

Proof If $z \in K$

$$\begin{aligned} K_{\Delta^2}(z, v) &= K_{\Delta^2}(\varphi_j^{-1}(z), d\varphi_j^{-1}(v)) \\ &\leq K_{\Delta^2 \cap O}(\varphi_j^{-1}(z), d\varphi_j^{-1}(v)) \\ &= K_{\Omega \cap U}(F^{-1} \circ \varphi_j^{-1}(z), d\varphi F^{-1} \circ \varphi_j^{-1}(v)) \\ &\leq (1 - \varepsilon)^{-1} K_{\Omega}(F^{-1} \circ \varphi_j^{-1}(z), dF^{-1} \circ \varphi_j^{-1}(v)) \\ &= (1 - \varepsilon)^{-1} K_{\tilde{\Omega}}(\varphi \circ F^{-1} \circ \varphi_j^{-1}(z), d(\varphi \circ F^{-1} \circ \varphi_j^{-1})(v)) \\ &\leq (1 - \varepsilon)^{-1} K_{\Delta^2 \cap \tilde{O}}(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1}(z), d(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1})(v)) \\ &\leq (1 - \varepsilon)^{-2} K_{\Delta^2}(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1}(z), d(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1})(v)) \\ &= (1 - \varepsilon)^{-2} K_{\Delta^2}(\Lambda_j(z), d\Lambda_j(v)) \end{aligned}$$

Claim .

If $j > N_1$, then

$$K_{\Delta^2}(z, v) \geq (1 - \varepsilon)^2 K_{\Delta^2}(\Lambda_j(z), d\Lambda_j(v))$$

$\forall z \in K, v \in \mathbb{C}^2$

Proof.

$$\begin{aligned}
K_{\Delta^2}(z, v) &= K_{\Delta^2}(\varphi_j^{-1}(z), d\varphi_j^{-1}(v)) \\
&\geq (1 - \varepsilon)K_{\Delta^2 \cap O}(\varphi_j^{-1}(z), d\varphi_j^{-1}(v)) \\
&= (1 - \varepsilon)K_{\Omega \cap U}(F^{-1} \circ \varphi_j^{-1}(z), d(F^{-1} \circ \varphi_j^{-1})(v)) \\
&\geq (1 - \varepsilon)K_{\Omega}(F^{-1} \circ \varphi_j^{-1}(z), d(F^{-1} \circ \varphi_j^{-1})(v)) \\
&= (1 - \varepsilon)K_{\tilde{\Omega}}(\varphi \circ F^{-1} \circ \varphi_j^{-1}(z), d(\varphi \circ F^{-1} \circ \varphi_j^{-1})(v)) \\
&\geq (1 - \varepsilon)^2 K_{\tilde{\Omega} \cap \tilde{U}}(\varphi \circ F^{-1} \circ \varphi_j^{-1}(z), d(\varphi \circ F^{-1} \circ \varphi_j^{-1})(v)) \\
&= (1 - \varepsilon)^2 K_{\Delta^2 \cap D}(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1}(z), d(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1})(v)) \\
&\geq (1 - \varepsilon)^2 K_{\Delta^2}(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1}(z), d(G \circ \varphi \circ F^{-1} \circ \varphi_j^{-1})(v)) \\
&= (1 - \varepsilon)^2 K_{\Delta^2}(\Lambda_j(z), d\Lambda_j(v))
\end{aligned}$$

Combining the results from the above claims we get the following proposition

Proposition 5.2.4

Assuming Λ_j as constructed in (5.1) and if $j > N_1$, then

$$((1 - \varepsilon)^2 K_{\Delta^2}(\Lambda_j(z), d\Lambda_j(v)) \leq K_{\Delta^2}(z, v) \leq (1 - \varepsilon)^{-2} K_{\Delta^2}(\Lambda_j(z), d\Lambda_j(v))$$

$$\forall z \in K, v \in \mathbb{C}^2.$$

Claim .

Let $K \subset \Delta^2$ be a compact subset. Then there exists $N = N(K, \varepsilon)$, such that

$$d_{\Delta^2}(\Lambda_j(q_1), \Lambda_j(q_2)) \leq (1 - \varepsilon)^{-2} d_{\Delta^2}(q_1, q_2)$$

$\forall q_1, q_2 \in K, j > N$.

Proof.

Let $r = \max \{d_{\Delta^2}(x, y) : x, y \in K\} < \infty$. Since Δ^2 is complete hyperbolic.

$$\tilde{K} := \overline{\bigcup_{x \in K} B(x; r)}$$

is compact in Δ^2 , where $B(x, r)$ denotes the Kobayashi ball. Let $N_1 = N_1(\tilde{K}, \varepsilon)$ and let N_2 be chosen such that $\tilde{K} \subset \Delta_{r_j}^2, \forall j > N_2$.

Let $N = \max \{N_1, N_2\}$. If $j > N$, then for $q_1, q_2 \in K$, the Kobayashi geodesic $\gamma(t)$ joining q_1 to q_2 lies on $\tilde{K}$.

Therefore,

$$\begin{aligned} d_{\Delta^2}(q_1, q_2) &= \int_0^1 K_{\Delta^2}(\gamma(t), \gamma'(t)) dt \\ &\geq (1 - \varepsilon)^2 \int_0^1 K_{\Delta^2}(\Lambda_j(\gamma(t)), d\Lambda_j(\gamma'(t))) dt \\ &= (1 - \varepsilon)^2 l_k(\Lambda_j \circ \gamma), \text{ where } l_k \text{ is Kobayashi length} \\ &\geq (1 - \varepsilon)^2 d_{\Delta^2}(\Lambda_j(q_1), \Lambda_j(q_2)) \end{aligned}$$

if $j > N$.

Since both φ and φ^{-1} are assumed to be extended continuously up the Silov boundaries of Ω and $\tilde{\Omega}$, respectively, we can apply the same procedure to φ^{-1} to conclude that:

For each compact subset $K \subset \Delta^2$, $\varepsilon > 0$, $\exists N = N(K, \varepsilon)$ such that

$$d_{\Delta^2}(\Lambda_j^{-1}(q_1), \Lambda_j^{-1}(q_2)) \leq (1 - \varepsilon)^{-2} d_{\Delta^2}(q_1, q_2)$$

for any q_1 and $q_2 \in K$, whenever $j > N$. Now, we claim the following:

Claim

If K is compact in Δ^2 , then $\exists N = N(K, \varepsilon)$, such that

$$(1 - \varepsilon)^2 d_{\Delta^2}(q_1, q_2) \leq d_{\Delta^2}(\Lambda_j(q_1), \Lambda_j(q_2))$$

$\forall q_1, q_2 \in K$, if $j > N$.

Let $\varepsilon_k \rightarrow 0$ be a decreasing sequence. Assume that

$$(1 - \varepsilon_k)^{-2} < 2, \forall k.$$

Let $r_k \rightarrow \infty$ be an increasing sequence and

$$K_k = B(0, r_k) = \{z \in \Delta^2 : d_{\Delta^2}(0, z) \leq r_k\}$$

be the closed Kobayashi ball, where $r_1 = 1$. Choose m_k such that

$$d_{\Delta^2}(\Lambda_j(q_1), \Lambda_j(q_2)) \leq (1 - \varepsilon_k)^{-2} d_{\Delta^2}(q_1, q_2)$$

$\forall q_1, q_2 \in K_k, j > m_k$. Since $(1 - \varepsilon_1)^{-2} \leq 2$, in particular,

$$d_{\Delta^2}(0, \Lambda_j(z)) \leq 2d_{\Delta^2}(0, z) \leq 2, \quad \forall z \in K_k$$

Then there exists $0 < \delta_k < 2r_k$ such that

$$\Lambda_j(K_k) \subset B(0, s_k) =: L_k$$

if $j > m_k$. Choose n_k such that

$$d_{\Delta^2}(\Lambda_j^{-1}(q_1), \Lambda_j^{-1}(q_2)) \leq (1 - \varepsilon_k)^{-2} d_{\Delta^2}(q_1, q_2)$$

$\forall q_1, q_2 \in L_k$ and $j > n_k$. Let $N_k = \max\{n_k, m_k\}$. Notice that if $j > N_k$, then

$$\Lambda_j^{-1}(L_k) \supset K_k$$

For any $q_1, q_2 \in K_k$, if $j > N_k$, then $\tilde{q}_1 := \Lambda_j(q_1), \tilde{q}_2 := \Lambda_j(q_2) \in L_k$ and

$$\begin{aligned} d_{\Delta^2}(q_1, q_2) &= d_{\Delta^2}(\Lambda_j^{-1}(\tilde{q}_1), \Lambda_j^{-1}(\tilde{q}_2)) \\ &\leq (1 - \varepsilon_k)^{-2} d_{\Delta^2}(\tilde{q}_1, \tilde{q}_2) \\ &= (1 - \varepsilon_k)^{-2} d_{\Delta^2}(\Lambda_j(q_1), \Lambda_j(q_2)) \end{aligned}$$

Therefore, we get the following estimate

Proposition 5.2.5.

For a sequence $\varepsilon_k \rightarrow 0, r_k \rightarrow \infty, \exists N_k$ such that if $j > N_k$, then Λ_j is well-defined

on $B^k(0, r_k)$ and

$$(1 - \varepsilon_k)^2 d_{\Delta^2}(q_1, q_2) \leq d_{\Delta^2}(\Lambda_j(q_1), \Lambda_j(q_2)) \leq (1 - \varepsilon_k)^{-2} d_{\Delta^2}(q_1, q_2) \quad (5.2)$$

Using (5.2), we can prove that Λ_j has a subsequence converging to a Kobayashi isometry Λ from Δ^2 into Δ^2 .

Theorem 5.2.6.

Let $\Omega, \tilde{\Omega}$ be two bounded generic analytic polyhedra and the map $\varphi : \Omega \longrightarrow \tilde{\Omega}$ be a C^1 - Kobayashi isometry. Suppose that both φ and φ^{-1} extend continuously up to the Silov boundaries of Ω and $\tilde{\Omega}$, respectively. Let p be a point on the Silov boundary $S(\Omega)$ of Ω and let $\tilde{p} = \varphi(p) \in S(\tilde{\Omega})$. Let $p_j \in \Omega$ be a sequence such that $p_j \rightarrow p$ and $\tilde{p}_j = \varphi(p_j) \rightarrow \tilde{p}$. suppose $\{\Lambda_j\}$ is a sequence of mappings defined by (5.1). Then Λ_j has a subsequence converging to a C^0 -Kobayashi isometry Λ from Δ^2 into Δ^2 .

Proof .

Since Λ_j is locally uniformly Lipschitz, any subsequence of Λ_j has a convergent subsequence by the Arzela-Ascoli theorem. Let ε_k, r_k and N_k , be as in last claim and let

(i) Λ_{1j} be a convergent subsequence of Λ_j which satisfies (5.2) $\forall j$ in case $k = 1$.

(ii) Λ_{2j} be a convergent subsequence of Λ_{1j} which satisfies (5.2) $\forall j$ in case $k = 2$.

Inductively, we construct Λ_{kj} , a convergent subsequence of $\Lambda_{(k-1)j}$ for k . Then by the standard argument of the diagonal process, Λ_{kk} converges locally uniformly on Δ^2 and if satisfies (5.2) for all k . which implies the limit Λ is a Kobayashi isometry of Δ^2 .

Finally, we highlight some important remarks.

Remarks 5.2.7.

(1) Although φ is assumed to be C^1 , Λ is only guaranteed to be C^0 , since the Arzela-Ascoli's theorem is about a continuous limit.

(2) The limit map is surjective since a Kobayashi isometry on the bidisc must be a composition of automorphism of Δ^2 and a coordinate-wise conjugation, which is surjective.

(3) Theorem 5.2.6 gives information on the restriction of the Kobayashi isometry on the Silov boundary. Recall that a holomorphic function on a domain that is continuous up to the boundary is completely determined by its values on the Silov boundary. In this scope, the next problem we have to solve is whether or not a Kobayashi isometry is fully determined by its restriction on the Silov boundary, and in the case of generic analytic polyhedra, a Kobayashi isometry is either holomorphic or anti-holomorphic. This is the main subject of ([2]).

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VITAE

- Name: Huda Ahmed Alzaki
- Nationality: Saudi
- Date of Birth: August 1981
- Email: *hudaalzaki81@gmail.com*
- Permanent Address: P.O. Box 5991, Dhahran 31311, Saudi Arabia.
- Publications
 - R. ALHARBEY, H. ALZAKI, AND S. HASSAN, *Transient Spectrum of $\sin^2$ Pulsed Driven Harmonic Oscillator*, Journal of Nonlinear Optical Physics and Materials, 48(4) (2014).
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