

**ROBUST OPTIMALITY
CONDITIONS AND DUALITY IN
MATHEMATICAL PROGRAMMING**

BY

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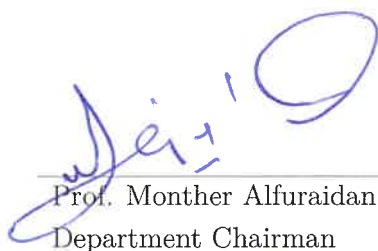
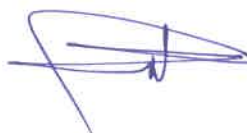


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Dedication

This dissertation is dedicated to my family, who have provided an immense amount of love and support.

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DISSERTATION ABSTRACT

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TITLE OF STUDY: Robust Optimality Conditions and Duality in Mathematical Programming

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The purpose of this dissertation is to analyze several mathematical programming scenarios in the presence of data uncertainty. Many optimization problems in real-world applications require imprecise data during their formulation and solution. By employing the robust approach to study such kinds of problems, the robust semi-infinite multi-objective fractional programming problem is introduced and its parametric robust semi-infinite multi-objective problem is constructed using the Dinkelbach parametrization approach. For these problems, the robust optimality conditions are investigated and the different types of duality theorems are proved. The non-smooth case of previous problems is also examined by using generalized directional derivative tools such as the Clarke subdifferential. The robust conditions for locally isolated and locally proper efficiency are explored and illustrated with some numerical examples. In addition,

the formulation of the robust dual problems in both Wolfe-type and Mond-Weir-type robust duals is demonstrated. Further, the robust vector optimization problem with vanishing constraints is considered, and based on the vectorial concept of robust strong stationary points, the robust necessary and sufficient conditions for optimality of first-order are studied. A strong auxiliary linearized vector problem is constructed using the linearization approach, and the connections between the two problems being examined are clarified. Finally, we consider uncertainty in objective and constraint functions of non-differentiable fractional programming within complex space. The robust second-order Mangasarian type dual model is examined as well as the equivalence of one of the robust non-fractional parametric minimax complex programming problems and the robust complex fractional problem.

ملخص الرسالة

الاسم: فاطمه ابراهيم النباط

عنوان الدراسة: الشروط المثالية القوية والازدواجية في البرمجة الرياضية

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الهدف من هذه الرسالة هو تحليل العديد من سيناريوهات البرمجة الرياضية في ظل عدم دقة البيانات. تتطلب الكثير من مشكلات التحسين في تطبيقات العالم الحقيقي بيانات غير دقيقة أثناء صياغتها وحلها. من خلال استخدام النهج القوي لدراسة مثل هذا النوع من المشاكل، تم تقديم مشكلة البرمجة الجزئية القوية شبه اللانهائية ومتعددة الأهداف وتم إنشاء مسألتها البارامترية القوية شبه اللانهائية متعددة الأهداف باستخدام منهج دينكلباخ. تم دراسة الشروط المثلى القوية الضرورية والكافية لهذه المشاكل وتم إثبات الأنواع المختلفة من نظريات الازدواجية. أيضًا تم دراسة البرمجة الرياضية عندما تكون مشتقات الدوال المتضمنة في المسألة غير متصلة وذلك باستخدام أدوات مشتقة اتجاهية معمة مثل تفاضل كلارك الفرعي، وقد تم استكشاف وتوضيح الشروط القوية لتحقيق الكفاءة المعزولة محليًا والمناسبة محليًا من خلال بعض الأمثلة العددية. بالإضافة إلى ذلك، تم توضيح المسائل المزدوجة القوية التي تمت صياغتها في كل من النوع الثنائي القوي من نوع Wolfe والثنائي القوي من نوع Mond-Weir علاوة على ذلك، تم النظر في مشكلة تحسين المتجهات القوية مع تلاشي القيود، واستنادًا إلى المفهوم المتجه للنقاط الثابتة القوية القوية، تم دراسة الظروف المثلى القوية الضرورية والكافية من الدرجة الأولى. من خلال استخدام المنهج الخطي، تم إنشاء مسألة مساندة قوية ذات متجهات خطية وتم توضيح الروابط والعلاقات بين المسألتين بعين الاعتبار. وأخيرًا، تم الأخذ بعين الاعتبار عدم دقة البيانات في هدف المسألة الكسرية الغير قابلة للاشتقاق وشروطها معا داخل الفضاء المركب. تم دراسة النموذج المزدوج القوي من النوع Mangasarian من الدرجة الثانية و التكافؤ بين إحدى مشكلات البرمجة المركبة البسيطة غير الكسرية القوية والمشكلة الكسرية المركبة القوية.

CHAPTER 1

INTRODUCTION FOR ROBUSTNESS

Deterministic mathematical programming, characterized by the precise knowledge of input data during problem construction and resolution, has been extensively documented in academic literature and implemented across various disciplines, including operations research, management, engineering design, and economics. Scalar mathematical optimization in mathematical programming refers to a problem that involves a single objective function. A problem involving multiple objective functions is known as a multi-objective optimization problem (minimization or maximization), and the optimal solution must consider the objectives' inherent conflicts. For example, achieving a solution to a transportation problem that simultaneously minimizes transport cost and time required to meet demand, or reaching a decision that simultaneously maximizes profit and minimizes product cost, may be unattainable. Therefore, achieving a single optimal solution that fulfills

all objectives is challenging. As a result, researchers have developed the notion of optimality in the context of multi-objective mathematical programming over recent decades. When it is impossible to increase the value of any objective function without compromising some of the other objective values, then the decision is optimal. This type of solution is referred to as a Pareto optimal solution, an efficient solution, a non-inferior solution, or a non-dominated solution, among others, consult ([36], [49], [64], [75]).

Practical optimization issues in the real world have been effectively addressed by mathematical programming models incorporating multi-objective programming problems. Due to a variety of factors, including incomplete information, estimation or measurement errors, and unanticipated future demands, the data utilized by these models may contain imprecise values. In this case, the robust optimization approach has become a vigorous method for studying optimization problems with uncertain data by associating a robust counterpart to this kind of problem. In 1973, Soyster [93] was the first researcher to study robust optimization and apply it to inexact linear programming. Then in the following decade, Falk [39] and Singh [91] developed this method and extended the results obtained by Soyster to be more applicable. Since the beginning of the third millennium, the concept of robustness has experienced extensive study by Ben-Tal and Nemirovski [16], and Ben-Tal et al. ([14], [15]). This approach has been developed over the years and applied in many scientific fields under data uncertainty as in robust engineering designs ([31], [63]), application of internet routing [34], problem of distributionally

hub location [102], timetable information in public transport [48] and applications to portfolio optimization [38]. The method aims to identify a suitable solution by examining all potential values of the parameters within their specified range of uncertainty, including the worst-case scenario among all the available input data choices.

The fundamental analysis, methodology, and application of robust optimization technique under the worst-case scenario approach can be consulted in the work of Beck and Ben-Tal [12], Ben-Tal et al. ([15], [16]) and Jeyakumar et al. ([59], [60], [61]). For more information regarding the characterization of the robust optimal solutions for a robust multi-objective optimization problem, the reader is referred to ([10], [20], [24], [25], [47], [55], [96]).

In the literature, Numerous scholars have introduced diverse notions of robust solutions for mathematical programming problems involving uncertain data. For instance, Branke [18] has discussed the minimax robustness approach, in which the mean function is utilized in place of the single objective function because its minimizer is more robust than the original function's minimizer. Deb and Gupta [31], expanded the definition to include multi-objective optimization and examined the sensitivity of robust solutions to minor perturbations in the neighboring variables. Furthermore, several notions of robust optimal solutions in multi-objective programming have been presented in academic publications, For instance, robust approximate efficiency as in Antczak et al. [10] and Lee and Lee [71], stable robust duality under the assumptions of closedness and convexity in the work of Dinh et al. [32], and efficiency concepts

related to the set order relations as in Ide and Kobis [58]. For more concepts and approaches consult Bertsimas et al. [17], Ehrgott et al. [37], Goberna et al ([44], [45], [47]), Chuong ([24], [25]), Kuroiwa and Lee [65], and for new concepts and developments one can refer to ([31], [34], [86], [103]).

Because of the importance of uncertain parameters in real life, the duality for robust programs has gained the interest of several authors. To gain a comprehensive understanding of the evolution of duality theory in robust programs, one can consult the research conducted by Beck and Ben-Tal [12], Chen et al. [20], Choung [24], Jeyakumar and Li [60], Jeyakumar et al. [61], Lee and Lee [70] and Kim [63].

The literature has presented several concepts of generalized convex functions intending to weaken the convexity condition in optimality and duality of mathematical programming problems as in Suneja et al. [99] who have employed Clarke's generalized gradients of locally Lipschitz functions to introduce different types of generalized convex functions for a non-smooth multi-objective optimization problem. Fakhar et al. [38] studied optimality and duality for non-smooth multi-objective optimization and portfolio optimization problems by presenting a new concept of generalized convexity, namely, pseudo quasi-generalized convex function. Chen et al. [20] investigated the optimality conditions and duality results under uncertainty by introducing the definition of generalized type-I convex functions over cones. Recently, Lee and Lee ([70], [71]) have studied robust semi-infinite multi-objective programs under convexity assumptions. The related functions were considered to be

convex, which is a generalization of the robust linear programs discussed in ([44], [45]).

Further, employing the conjugate analysis and robust approach, necessary and sufficient conditions, and duality results are derived for robust approximate optimal solutions of a robust fractional semi-infinite optimization problem. In [73], Li et al. presented optimality, Wolfe type duality, and Mond-Wier type duality results for a robust minimax nondifferentiable fractional problem under subdifferentiable and convexity. Zhao and Bin [104] established Karush-Kuhn-Tucker optimality conditions for a non-smooth robust optimization problem with data uncertainty on Banach space. Recently, Choung [25] examined the robust optimality conditions and robust duality of an uncertain nondifferentiable multi-objective programming problem by employing the advanced techniques of generalized differentiability and variational analysis. Finally, Many practical optimization models involve multi-objective fractional functions. Limited research has been dedicated to the study of multi-objective fractional functions with uncertain data, such as the work of Sun et al. ([94], [97]) wherein the characterization of robust optimal solutions was investigated..

The objective of this dissertation is to examine a range of mathematical programming situations and explore the robust optimality conditions and duality theorems that apply to these problems providing some illustrative numerical examples. The following describes the remaining structure of this research:

Chapter 2 introduces a semi-infinite multi-objective fractional programming

problem confined to uncertain data in constraint functions and establishes robust Karush–Kuhn–Tucker necessary optimality conditions subject to a generalized robust Slater constraint qualification as well as robust sufficient optimality conditions of the associated robust parametric semi-infinite non-fractional multi-objective program of our considered problem under some assumptions of generalized convexity. Then, based on the parametrization method, a robust duality model of the Mond-Weir type is developed, and its duality theorems are proved.

Chapter 3 is interested in a non-smooth semi-infinite multi-objective fractional programming problem. Under an extended form of a robust type constraint qualification, the non-smooth robust necessary optimality conditions for a local isolated efficient solution and local properly efficient solution of the considered problem and its non-smooth parametric problem, formulated in generalized Dinkelbach approach, are investigated. The non-smooth robust sufficient optimality conditions are studied. Finally, The robust dual problem, based on the Wolfe-type and Mond-Weir-type approaches is proposed.

Chapter 4 focuses on the research of robust vector optimization problems with vanishing constraints. Derived from the notion of robust strong stationary points in vector optimization problems, the robust first-order necessary and sufficient optimality conditions are introduced. Further, using a linearization approach, we construct a robust linearized vector problem and examine its relation with the problems under consideration. As an illustration of our fundamental results, we

present two numerical examples.

Chapter 5 deals with a non-differentiable fractional programming problem under data uncertainty in complex space. The robust-Karush-Kuhn-Tucker necessary optimality conditions are introduced. By employing these conditions, a robust second-order Mangasarian-type dual model of the considered problem is constructed. The theorems of robust weak, strong, and strict converse duality are established and demonstrated.

Chapter 6 concludes our results and gives a brief outlined summary of the main points in this work. Lastly, we propose possible future research in complex spaces including a multi-objective fractional optimization problem in the presence of data uncertainty.

CHAPTER 2

ROBUST SEMI-INFINITE MULTI-OBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS

A specific case of a multi-objective programming challenge is a semi-infinite multi-objective programming problem, which involves minimizing a finite number of objective functions while being subject to an infinite number of constraint functions. Goberna et al. ([44], [45]) conducted research on multi-objective linear semi-infinite algorithms with constraints affected by data uncertainty by providing first a radius of robust feasibility of the parameterized robust counterpart problem. Lee and Lee [70] derived duality theorems for robust multi-objective nonlinear semi-infinite programming problems and established the necessary and sufficient

optimality conditions for a weakly robust efficient solution. In addition, in 2022, Antczak [9] formulated the optimality criteria and proved numerous duality results for differential semi-infinite multi-objective optimization problems with vanishing constraints, which involve an unlimited number of equality and inequality constraints.

Moreover, studying semi-infinite multi-objective optimization problems under data uncertainty with fractional objectives has been of much interest in recent years. For example, Sun et al. [94] employed the robust type constraint qualification to derive the robust approximate necessary and sufficient optimality conditions of robust ϵ -weakly efficient solutions for the uncertain multi-objective fractional semi-infinite optimization problem. Considering the uncertainty in both objective and constraint functions, Antczak [8] utilized a parametric methodology to obtain the optimality conditions for the approximate efficient solution, namely, robust ϵ -efficient solution for the uncertain multi-objective fractional programming problem with a finite number of constraints. In 2022, Ahmad et al. [4] dealt with semi-infinite multi-objective programs with uncertain data and established robust sufficient optimality conditions and duality models.

In this chapter, we investigate a semi-infinite multi-objective fractional programming problem contaminated with uncertainty in constraints. The sections here are arranged in the following manner. Firstly, section 2.1 holds some basic concepts and important definitions which will be used later. In section 2.2, we state a generalized robust Slater constraint qualification and use it to establish the robust Karush–Kuhn–

Tucker necessary optimality conditions for a weakly efficient (efficient) solution of the parametric robust semi-infinite multi-objective non-fractional problem constructed in Dinkelbach parametrization approach. Then, we study the robust sufficient optimality conditions under the assumption of some generalized convexity. In section 2.3, based on the parametrization method, we introduce the Mond-Weir type robust duality theorems for the robust semi-infinite multi-objective programming problems. Finally, section 2.4 illustrates the results obtained in this chapter through a numerical example.

2.1 Preliminaries and Basic Definitions

During this work, we consider the n -dimensional Euclidean space $\mathbb{R}^n$. The partial order relation between any two vectors $\mathbb{A} = (\alpha_1, \dots, \alpha_n)$ and $\mathbb{B} = (\beta_1, \dots, \beta_n)$ in $\mathbb{R}^n$ can be presented as follows:

1. $\mathbb{A} \geq \mathbb{B}$ iff $\alpha_i \geq \beta_i$ for all $i = 1, \dots, n$.
2. $\mathbb{A} > \mathbb{B}$ iff $\mathbb{A} \geq \mathbb{B}$ with $\mathbb{A} \neq \mathbb{B}$.
3. $\mathbb{A} > \mathbb{B}$ iff $\alpha_i > \beta_i$ for all $i = 1, \dots, n$.

The non-negative orthant $\mathbb{R}_+^n$ of $\mathbb{R}^n$ is defined by

$$\mathbb{R}_+^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}.$$

Consider T as a non-empty index set with an infinite number of elements. Define $\mathbb{R}^{(T)}$ to be the linear space of mappings $\mu = (\mu_t)_{t \in T}$ as follows:

$$\mathbb{R}^{(T)} := \{\mu = (\mu_t)_{t \in T} : \mu_t = 0, \forall t \in T, \text{ except for finitely many } t \in T, \mu_t \neq 0\}.$$

Denote the non-negative cone of $\mathbb{R}^{(T)}$ by $\mathbb{R}_+^{(T)}$ and define it by the following:

$$\mathbb{R}_+^{(T)} := \{\mu \in \mathbb{R}^{(T)} : \mu_t \geq 0, \forall t \in T\}.$$

For the general form of semi-infinite multi-objective fractional optimization problem under data uncertainty confined only to the constraint functions, we consider the following problem

$$\min_{x \in \mathbb{R}^n} f(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_k(x)}{g_k(x)} \right) \quad (\text{USIMFP})$$

$$\text{Subject to } h = (-h_t(x, v_t))_{t \in T} \in S, \quad \forall t \in T, x \in \mathbb{R}^n,$$

where $f_i, g_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, k$ and $h_t : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}, t \in T$ are real valued functions. The uncertain parameters v_t belong to some convex and compact uncertainty set $V_t \subset \mathbb{R}^q, t \in T$. Define the set valued mapping $V : T \rightrightarrows \mathbb{R}^q$ as $V(t) := V_t, \forall t \in T$. The graph of the function V_t in which $v_t \in V_t, t \in T$ is denoted and defined by the following:

$$\text{gph}V = \{(t, v_t) \in T \times \mathbb{R}^q, t \in T\}.$$

Assume that the mapping $v(t) := V_t$ is a compact valued and upper semi-continuous

on the metric space T . The subset $S \subset \mathbb{R}_+^{(T)}$ is a non-empty, proper, closed, and convex cone. By applying the robust approach to solve problems with uncertain data in constraints, we formulate the robust counterpart of the problem (USIMFP) to be as follows:

$$\min_{x \in \mathbb{R}^n} f(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_k(x)}{g_k(x)} \right) \quad (\text{RSIMFP})$$

$$\text{Subject to } h = (-h_t(x, v_t))_{t \in T} \in S, \quad \forall v_t \in V_t, \quad t \in T, \quad x \in \mathbb{R}^n,$$

where the set of all feasible solutions of the problem (RSIMFP) is defined by

$$F = \{x \in \mathbb{R}^n : (-h_t(x, v_t))_{t \in T} \in S, \quad \forall v_t \in V_t, \quad t \in T\}.$$

In this model, we assume that f_i, g_i and h_t for $i = 1, \dots, k$ and $t \in T$ are continuously differentiable functions and that f_i is a convex function with $f_i(x) \geq 0$ for all $x \in F$ and each $i = 1, \dots, k$, while g_i is a concave function with $g_i(x) > 0$ for all $x \in F$ and for each $i = 1, \dots, k$.

Let $K \subset \mathbb{R}^k$ be a proper closed convex cone and $\text{int } K$ denotes its interior. Assume that $\text{int } K$ is a non-empty subset in $\mathbb{R}^k$. The positive dual cone of K is denoted by K^* and defined as

$$K^* = \{y \in \mathbb{R}^k : y^T x \geq 0, \quad \forall x \in K\}.$$

A feasible point $x \in F \subset \mathbb{R}^n$ is considered a robust feasible solution of the (USIMFP)

problem if and only if it is a feasible solution of the (RSIMFP) problem.

The following basic concepts are important definitions in the literature that will be utilized in the subsequent part.

Definition 2.1 *A feasible point $x^* \in F$ is considered to be a robust weakly efficient (robust efficient, robust properly efficient) solution to the problem (USIMFP) if and only if it is a weakly efficient (efficient, properly efficient) solution to the problem (RSIMFP).*

Definition 2.2 *A feasible point $x^* \in F$ is considered to be a robust weakly efficient solution to the problem (USIMFP) if there does not exist any $x \in F$ such that $f(x) < f(x^*)$, i.e.,*

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)}, \quad \text{for all } i = 1, \dots, k.$$

Equivalently, if for all $x \in F$, we have

$$f(x^*) - f(x) \notin \text{int } K.$$

Definition 2.3 *A feasible point $x^* \in F$ is considered to be a robust efficient solution to the problem (USIMFP) if there does not exist any $x \in F$ such that $f(x) \leq f(x^*)$, that is*

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(x^*)}{g_i(x^*)}, \quad \text{and } f(x) \neq f(x^*).$$

Equivalently, if for all $x \in F$, we have

$$f(x^*) - f(x) \notin K - \{0\}.$$

Clearly, every robust efficient solution to the problem (USIMFP) is also a robust weakly efficient solution but the opposite does not hold.

Definition 2.4 For the problem (USIMFP), let $\Phi_i(x) = \frac{f_i(x)}{g_i(x)}$, $i = 1, \dots, k$. A feasible point $x^* \in F$ is considered to be a robust properly efficient solution to the problem (USIMFP) if it is a robust efficient solution to (USIMFP) and if there exists a positive scalar $M > 0$, such that for each $x \in F$ and $i = 1, \dots, k$ satisfying $\Phi_i(x) < \Phi_i(x^*)$, we have

$$\frac{\Phi_i(x^*) - \Phi_i(x)}{\Phi_j(x) - \Phi_j(x^*)} \leq M,$$

for some $j = 1, \dots, k$ satisfying $\Phi_j(x) > \Phi_j(x^*)$.

Equivalently, for $N = \frac{1}{M} > 0$, we have

$$\Phi_j(x) - \Phi_j(x^*) \geq N(\Phi_i(x^*) - \Phi_i(x)).$$

For certain results, it can be beneficial to relax the requirement of convexity on the functions used in the optimization problems in order to accomplish the objectives. The subsequent notions of generalized convex functions will be employed subsequently.

Definition 2.5 Consider C , a non-empty convex subset of $\mathbb{R}^n$, and ψ , a differentiable function that maps C to $\mathbb{R}^k$. Then the function ψ is referred to as

(i) *pseudo-convex at $w \in C$ if for all $z \in C$, the following property holds:*

$$\nabla\psi(w)(z - w) \geq 0 \implies \psi(z) \geq \psi(w).$$

or equivalently $\psi(z) < \psi(w) \implies \nabla\psi(w)(z - w) < 0.$

(ii) *quasi-convex at $w \in C$ if for all $z \in C$, the following property holds:*

$$\psi(z) \leq \psi(w) \implies \nabla\psi(w)(z - w) \leq 0.$$

(iii) *strictly pseudo-convex at $w \in C$ if for all $z \in C$ with $z \neq w$, the next property fulfills:*

$$\psi(z) \leq \psi(w) \implies \nabla\psi(w)(z - w) < 0.$$

The generalizations of these definitions over cones can be defined as the following:

Definition 2.6 *Take $\psi : C \rightarrow \mathbb{R}^k$ to be a vector-valued differentiable function where $C \subset \mathbb{R}^n$ is a non-empty convex subset. Let $K \subset \mathbb{R}^k$ be a proper closed convex cone with a non-empty interior. Then*

(i) *The function ψ is called K -pseudo-convex at $w \in C$ if for all $z \in C$, we have*

$$\nabla\psi(w)(z - w) \in K \implies \psi(z) - \psi(w) \notin -\text{int } K.$$

or equivalently $\psi(z) - \psi(w) \in -\text{int } K \implies \nabla\psi(w)(z - w) \notin K.$

(ii) The function ψ is called K -quasi-convex at $w \in C$ if for all $z \in C$, we have

$$\psi(w) - \psi(z) \notin -\text{int } K \implies -\nabla\psi(w)(z - w) \in K.$$

(iii) The function ψ is called K - strictly pseudo-convex at $w \in C$ if for all $z \in C$ with $z \neq w$, we have

$$\psi(w) - \psi(z) \notin -\text{int } K \implies -\nabla\psi(w)(z - w) \in \text{int } K.$$

2.2 Robust Optimality Conditions of (RSIMFP)

In this section, we provide the robust optimality criteria of the robust weakly efficient (robust efficient) solutions of the semi-infinite multi-objective fractional programming problem subject to uncertainty (USIMFP). To get the robust necessary optimality conditions, it is vital to take into account a suitable type of constraint qualification in a robust version, particularly, the Slater constraint qualification. Next, we state a generalized robust Slater constraint qualification ([20], [65]) when dealing with an unlimited number of constraints as follows.

Definition 2.7 The generalized robust Slater constraint qualification (GRSCQ) is considered to be satisfied if there exists $x_0 \in \mathbb{R}^n$ such that

$$(-h_t(x_0, v_t))_{t \in T} \in \text{int } \mathbb{R}_+^{(T)}, \quad \forall t \in T, \quad v_t \in V_t.$$

One way of solving robust semi-infinite multi-objective fractional problems is by using the Dinkelbach parametrization approach [33] in a generalized form for vectorial fractional problem [29], in which the problem (RSIMFP) is associating with a parameter vector $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$. Given that our assumptions regarding the components of the objective functions consist of continuously differentiable non-negative convex functions $f_i(x), i = 1, \dots, k$ and continuously differentiable positive concave functions $g_i(x), i = 1, \dots, k$ for all $x \in F$, then the function $-g_i(x)$ is a convex function for all $i = 1, \dots, k$, and hence for $\lambda_i \geq 0$, each function of the form $f_i(x) - \lambda_i g_i(x), i = 1, \dots, k$ is a convex function.

Now, construct a parametric robust semi-infinite multi-objective non-fractional problem with the parameter vector $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$ as an association to the problem (RSIMFP) as follows:

$$\min_{x \in \mathbb{R}^n} \quad \psi(x) = \left(f_1(x) - \lambda_1 g_1(x), \dots, f_k(x) - \lambda_k g_k(x) \right) \quad (\text{RSIMFP})_\lambda$$

$$\text{Subject to } (-h_t(x, v_t))_{t \in T} \in S, \quad \forall v_t \in V_t, \quad t \in T, \quad x \in \mathbb{R}^n.$$

For this constructed parametric problem, we present the definitions of efficiency as coming next.

Definition 2.8 *Take $x^* \in F$ to be a feasible solution. Then*

(i) *x^* is considered to be a weakly efficient solution to the problem $(\text{RSIMFP})_\lambda$ if*

there does not exist any $x \in F$ such that $\psi(x) < \psi(x^*)$, i.e.,

$$f_i(x) - \lambda_i g_i(x) < f_i(x^*) - \lambda_i g_i(x^*), \text{ for all } i = 1, \dots, k.$$

Equivalently, if for all $x \in F$, we have

$$\psi(x^*) - \psi(x) \notin \text{int } K.$$

(ii) x^* is considered to be an efficient solution to the problem $(\text{RSIMFP})_\lambda$ if there does not exist any $x \in F$ such that $\psi(x) \leq \psi(x^*)$, i.e.,

$$f_i(x) - \lambda_i g_i(x) \leq f_i(x^*) - \lambda_i g_i(x^*), \text{ } i = 1, \dots, k.$$

$$f_j(x) - \lambda_j g_j(x) < f_j(x^*) - \lambda_j g_j(x^*), \text{ for some } j.$$

Equivalently, if for all $x \in F$, we have

$$\psi(x^*) - \psi(x) \notin K - \{0\}.$$

The weakly efficient (efficient) solution of the problem (RSIMFP) constructed in the robust approach can be obtained in an equivalent relation with the weakly efficient (efficient) solution of the parametric problem $(\text{RSIMFP})_\lambda$ constructed in the generalized Dinkelbach approach, where the vector $\lambda = (\lambda_1, \dots, \lambda_k)$ is the parameter vector in $\mathbb{R}_+^k$. The next lemma clarified the method of finding efficient solutions to

both problems.

Lemma 2.1 *Take $x^* \in F$ and consider $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $i = 1, \dots, k$. Then x^* is considered to be a robust weakly efficient solution of the (USIMFP) problem (hence, a weakly efficient solution of the (RSIMFP) problem) if and only if x^* is a weakly efficient solution of the problem $(\text{RSIMFP})_{\lambda^*}$, with $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$.*

Proof. $(\Rightarrow)$ Take $x^* \in F$ to be a robust weakly efficient solution of the problem (USIMFP), to get a contrary, assume that $x^* \in F$ is not a weakly efficient solution of the problem $(\text{RSIMFP})_{\lambda^*}$. Hence, there is $x \in F$ such that

$$f_i(x) - \lambda_i^* g_i(x) < f_i(x^*) - \lambda_i^* g_i(x^*), \text{ for all } i = 1, \dots, k.$$

Since $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, for all $i = 1, \dots, k$, then by substituting λ_i^* in the above inequality, we get the following:

$$f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) < 0, \text{ for all } i = 1, \dots, k.$$

As for any $x \in F$ and for each $i = 1, \dots, k$, $g_i(x) > 0$, then the following holds

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)}, \text{ for all } i = 1, \dots, k.$$

This means that x^* is not a robust weakly efficient solution to the problem (USIMFP) which goes against the assumption. Hence, x^* is a weakly efficient solution of the problem $(\text{RSIMFP})_{\lambda^*}$, with $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$.

($\Leftarrow$) Take $x^* \in F$ to be a weakly efficient solution of the problem $(\text{RSIMFP})_{\lambda^*}$, where $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, for all $i = 1, \dots, k$. We want to proceed with a contradiction, so assume that x^* is not a robust weakly efficient solution to the (USIMFP) problem. Thus, there exists $x \in F$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(x^*)}{g_i(x^*)}, \quad \text{for all } i = 1, \dots, k.$$

By the assumption, replace the term $\frac{f_i(x^*)}{g_i(x^*)}$ with its value λ_i^* , so we obtain

$$\frac{f_i(x)}{g_i(x)} < \lambda_i^*, \quad \text{for all } i = 1, \dots, k,$$

which leads to

$$f_i(x) - \lambda_i^* g_i(x) < 0, \quad i = 1, \dots, k. \quad (2.1)$$

However, using the assumption $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, we also obtain

$$f_i(x^*) - \lambda_i^* g_i(x^*) = 0, \quad i = 1, \dots, k. \quad (2.2)$$

By combining (2.1) and (2.2), the following inequalities hold,

$$f_i(x) - \lambda_i^* g_i(x) < f_i(x^*) - \lambda_i^* g_i(x^*).$$

This contradicts the assumption that x^* is a weakly efficient solution to the problem $(\text{RSIMFP})_{\lambda^*}$. Therefore, x^* is a robust weakly efficient solution to the (USIMFP) problem and the proof is completed. ■

Remark 1 (1) *In the above proof, we considered the case of a weakly efficient solution. Likewise, we can demonstrate the equivalence relation for the efficient solution case by replacing $<$ in the proof with $\leq$ for all $i = 1, \dots, k$ except for some $j \neq i$, we keep the strict inequality considering the concepts of robust efficiency for both problems $(USIMFP)$ and $(RSIMFP)_{\lambda^*}$ respectively.*

(2) *In the case of searching for an approximate efficient solution for the robust multi-objective fractional programming problems, the above Lemma 2.1 is reduced to Lemma 3.1 in [8].*

Next, we study the necessary optimality conditions, known as the Karush-Kuhn-Tucker conditions, for a robust feasible solution to be considered a weakly efficient (efficient) solution for the parametric robust semi-infinite multi-objective non-fractional problem $(RSIMFP)_{\lambda^*}$. These conditions are used to solve the robust semi-infinite multi-objective fractional problem $(RSIMFP)$, under the assumption that the generalized robust Slater constraint qualification (GRSCQ) is met.

Theorem 2.1 (Robust Karush–Kuhn–Tucker necessary optimality conditions)

Take $x^ \in F$ to be a robust weakly efficient solution of the considered problem $(USIMFP)$ with $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ and $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $i = 1, \dots, k$. Consider the function $h_t(\cdot, v_t) : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, $t \in T$ to be a continuous convex function with uncertain parameter $v_t \in V_t$. Assume the fulfillment of the generalized robust Slater constraint qualification (GRSCQ). Then there exist $\alpha \in \mathbb{R}_+^k - \{0\}$, $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, and $v_t \in V_t$, $\forall t \in T$ in a manner that*

$$\sum_{i=1}^k \alpha_i \nabla (f_i(x^*) - \lambda_i^* g_i(x^*)) + \sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t) = 0. \quad (2.3)$$

$$\sum_{t \in T} \mu_t h_t(x^*, v_t) = 0. \quad (2.4)$$

Proof. Take $x^* \in F$ to be a robust weakly efficient solution of the considered problem (USIMFP), hence, it is a weakly efficient solution of the robust problem (RSIMFP). Taking into account Lemma 2.1, x^* is also a weakly efficient solution of the parametric robust semi-infinite multi-objective non-fractional problem $(\text{RSIMFP})_{\lambda^*}$, with $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$. Since the generalized robust Slater constraint qualification (GRSCQ) fulfills, then by applying Theorem 2.1 in [7] and taking into consideration that the involved functions are differentiable convex functions, hence, there exist Lagrange multipliers $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k - \{0\}$, $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, for all $t \in T$ and $v_t \in V_t$ such that the above conditions (2.3) - (2.4) are fulfilled.

We only need to show that α is not the zero vector in $\mathbb{R}_+^k$ (i.e., $\alpha \neq 0$). In fact, suppose to the contrary that $\alpha = 0$, then from the condition (2.3), we obtain

$$\sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t) = 0. \quad (2.5)$$

On the other side, define the index set $T_1(x)$ as follows

$$T_1(x) = \{t \in T : \exists v_t \in V_t \text{ s.t. } h_t(x, v_t) = 0\}, \forall x \in F.$$

Thus, condition (2.4) guarantees that $\mu_t = 0$ for $t \notin T_1(x)$, $x \in F$, in which it follows that for some $t \in T_1(x)$, $\mu_t \neq 0$ and $h_t(x, v_t) = 0$, $v_t \in V_t$. Also, since (GRSCQ) is

satisfied, then there exists a robust feasible solution $x_0 \in F \subset \mathbb{R}^n$ such that

$$(-h_t(x_0, v_t))_{t \in T} \in \text{int } \mathbb{R}_+^{(T)}, \quad \forall t \in T, \quad v_t \in V_t.$$

Which is equivalent to the inequality $h_t(x_0, v_t) < 0$ and hence for $t \in T_1(x)$, we have $h_t(x_0, v_t) < h_t(x^*, v_t)$. By the convexity of $h_t(\cdot, v_t)$ for any uncertain parameter $v_t \in V_t$, it follows also that h_t is a pseudo-convex function, so we get

$$\nabla_x h_t(x^*, v_t)(x_0 - x^*) < 0.$$

Since $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, $t \in T$ are not all zeros, then multiplying the above inequality by μ_t and taking the sum over $t \in T$, obtains

$$\sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t)(x_0 - x^*) < 0.$$

This contradicts condition (2.5). Hence, $\alpha \neq 0$. The proof has been concluded. ■

Here, we provide the robust sufficient optimality conditions that make a robust feasible solution to be a robust weakly efficient (efficient) solution for the problem (USIMFP) under some generalized convexity assumptions.

Theorem 2.2 (Robust sufficient optimality conditions - Weak Efficiency)

Take $x^ \in F$ to be a robust feasible solution for the considered problem (USIMFP).*

Consider $\psi(x)$ to be the next vector-valued non-fractional function:

$$\psi(x) = \left(f_1(x) - \lambda_1 g_1(x), \dots, f_k(x) - \lambda_k g_k(x) \right).$$

Let $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, for all $i = 1, \dots, k$ and suppose the existence of $\alpha \in \mathbb{R}_+^k - \{0\}$, $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, and for all $t \in T$, $v_t \in V_t$ such that the conditions (2.3) - (2.4) have been met at x^* .

Assume also that the function $\psi(x)$ is a pseudo-convex function for all $x \in F$ and the function $\sum_{t \in T} \mu_t h_t(\cdot, v_t)$ is a quasi-convex at x^* . Then the point x^* is a robust weakly efficient solution for the problem (USIMFP).

Proof. For the sake of contradiction, assume that $x^* \in F$ is not a robust weakly efficient solution of the considered problem (USIMFP), hence, it is not a weakly efficient solution of the robust problem (RSIMFP).

By applying Lemma 2.1, we get that x^* is also not a weakly efficient solution of the robust semi-infinite multi-objective non-fractional problem (RSIMFP) $_{\lambda^*}$. Hence, by the definition, there exist $\bar{x} \in F$ such that $\psi(\bar{x}) < \psi(x^*)$. Since $\psi(x)$ is a pseudo-convex function, then by its pseudo-convexity at x^* , we obtain

$$\nabla \psi(x^*)(\bar{x} - x^*) < 0.$$

As $\alpha \in \mathbb{R}_+^k - \{0\}$, then by multiplying the above inequality by α and expanding the

result in the sum form, we get

$$\sum_{i=1}^k \alpha_i \nabla (f_i(x^*) - \lambda_i^* g_i(x^*)) (\bar{x} - x^*) < 0. \quad (2.6)$$

On the other hand, the feasibility condition of the problem $-h_t(x, v_t) \in S \subset \mathbb{R}_+^{(T)}$ implies that $h_t(\bar{x}, v_t) \leq 0$, $\bar{x} \in F$, $t \in T$, $v_t \in V_t$. Since we have $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, then we get $\mu_t h_t(\bar{x}, v_t) \leq 0$, particularly, we can have $\sum_{t \in T} \mu_t h_t(\bar{x}, v_t) \leq 0$.

Now by the assumed condition (2.4), one can have the following inequality

$$\sum_{t \in T} \mu_t h_t(\bar{x}, v_t) \leq \sum_{t \in T} \mu_t h_t(x^*, v_t). \quad (2.7)$$

The quasi-convexity of the function $\sum_{t \in T} \mu_t h_t(\cdot, v_t)$ at x^* implies that

$$\sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t) (\bar{x} - x^*) \leq 0. \quad (2.8)$$

By condition (2.3), we have

$$\sum_{i=1}^k \alpha_i \nabla (f_i(x^*) - \lambda_i^* g_i(x^*)) = - \sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t).$$

Then combining the above equality with (2.8), it comes the following

$$\sum_{i=1}^k \alpha_i \nabla (f_i(x^*) - \lambda_i^* g_i(x^*)) (\bar{x} - x^*) = - \sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t) (\bar{x} - x^*) \geq 0.$$

The last result contradicts (2.6). Therefore, x^* is a robust weakly efficient solution to

the problem (USIMFP), and the proof is completed. ■

Remark 2 *With regard to semi-infinite multi-objective programming with data uncertainty, the above theorem of the robust sufficient optimality conditions of the problem (USIMFP) is reduced to Theorem 3.1 in [4].*

To establish the robust sufficient optimality conditions for getting a robust efficient solution for the problem (USIMFP), we need to set extra strict conditions on the involved functions. The following theorem illustrates these conditions.

Theorem 2.3 (Robust sufficient optimality conditions - Efficiency) *Take $x^* \in F$ be a robust feasible solution for the considered problem (USIMFP) and consider the function $\psi(x)$, the following vector-valued non-fractional function:*

$$\psi(x) = \left(f_1(x) - \lambda_1 g_1(x), \dots, f_k(x) - \lambda_k g_k(x) \right).$$

Let $\lambda^ = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, for all $i = 1, \dots, k$ and assume the existence of $\alpha \in \mathbb{R}_+^k - \{0\}$, $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, and for all $t \in T$, $v_t \in V_t$ such that the conditions (2.3) - (2.4) are met at x^* .*

Moreover, assume that the function $\sum_{t \in T} \mu_t h_t(\cdot, v_t)$ is a quasi-convex at x^ and any condition on the function $\psi(x)$ of the following is fulfilled:*

- (i) $\psi(x)$ is a strictly pseudo-convex function at $x^* \in F$ or
- (ii) $\psi(x)$ is a quasi-convex function at $x^* \in F$ and $\alpha \in \text{int } \mathbb{R}_+^k$.

Then the point x^ is a robust efficient solution for the problem (USIMFP).*

Proof. To prove this theorem, assume on the contrary, that $x^* \in F$ is not a robust efficient solution to the considered problem (USIMFP). So, x^* is not an efficient solution of the robust problem (RSIMFP) and hence it is not an efficient solution of the robust semi-infinite multi-objective non-fractional problem (RSIMFP) $_{\lambda^*}$. Thus, there exists $\bar{x} \in F$ such that $\psi(\bar{x}) \leq \psi(x^*)$.

Now, in case of assumption (i), by the strict pseudo-convexity of the function $\psi(x)$ at $x^* \in F$, we obtain $\nabla\psi(x^*)(\bar{x} - x^*) < 0$. For $\alpha \in \mathbb{R}_+^k - \{0\}$, the following inequality holds:

$$\sum_{i=1}^k \alpha_i \nabla(f_i(x^*) - \lambda_i^* g_i(x^*))(\bar{x} - x^*) < 0. \quad (2.9)$$

Furthermore, on the other case of assumption (ii), the quasi-convexity of the function $\psi(x)$ at $x^* \in F$ implies that $\nabla\psi(x^*)(\bar{x} - x^*) \leq 0$. That is for each $i = 1, \dots, k$, we have

$$\sum_{i=1}^k \nabla(f_i(x^*) - \lambda_i^* g_i(x^*))(\bar{x} - x^*) \leq 0.$$

But along with the assumption that $\alpha \in \text{int } \mathbb{R}_+^k$, one can get also the following

$$\sum_{i=1}^k \alpha_i \nabla(f_i(x^*) - \lambda_i^* g_i(x^*))(\bar{x} - x^*) < 0. \quad (2.10)$$

We can proceed with the rest of the proof in the same way as in the proof of Theorem 2.2, and then for the uncertain parameter $v_t \in V_t, t \in T$, we get the next result

$$\sum_{i=1}^k \alpha_i \nabla(f_i(x^*) - \lambda_i^* g_i(x^*))(\bar{x} - x^*) = - \sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t)(\bar{x} - x^*) \geq 0.$$

Which is a contradiction for both inequalities (2.9) and (2.10) of both cases (i) and (ii) respectively. Hence, x^* is a robust efficient solution to the problem (USIMFP) and the proof is completed. ■

Next, we present the robust sufficient optimality conditions that let a robust feasible solution be a robust properly efficient solution of the considered problem (USIMFP). At the start, we provide an interesting relation of the multi-objective programming problem to its associated scalar optimization problem which is introduced first by Geoffrion [41] and characterizes the proper efficient solution of the vector optimization problem. To this result, by applying the scalarization method with a strictly positive scalar $\alpha \in \mathbb{R}_+^k - \{0\}$ for the corresponding robust semi-infinite multi-objective non-fractional problem (RSIMFP) $_\lambda$, with $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\lambda_i = \frac{f_i(x)}{g_i(x)}$, $i = 1, \dots, k$, we define the associated scalar optimization problem as follows.

$$\min \quad \alpha^T \psi(x) = \sum_{i=1}^k \alpha_i (f_i(x) - \lambda_i g_i(x)) \quad (\text{SRSIMFP})_\lambda$$

$$\text{Subject to } (-h_t(x, v_t))_{t \in T} \in S, \quad \forall v_t \in V_t, \quad t \in T, \quad x \in \mathbb{R}^n.$$

A feasible point $x^* \in F$ of the problem (SRSIMFP) $_\lambda$ is said to be an optimal solution of the problem if for all $x \in F$, $\alpha^T \psi(x^*) \leq \alpha^T \psi(x)$ fulfills. That is

$$\sum_{i=1}^k \alpha_i (f_i(x^*) - \lambda_i^* g_i(x^*)) \leq \sum_{i=1}^k \alpha_i (f_i(x) - \lambda_i g_i(x)),$$

where $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, $i = 1, \dots, k$.

The following Lemma [41] states the above-mentioned relation to characterizing the robust properly efficient solution of the problem $(RSIMFP)_\lambda$.

Lemma 2.2 *Let $\alpha \in \mathbb{R}_+^k - \{0\}$ be fixed and x^* be an optimal solution of the scalar minimization problem $(SRSIMFP)_\lambda$. Then x^* is a properly efficient solution to the problem $(RSIMFP)_\lambda$.*

Theorem 2.4 (Robust sufficient optimality conditions - Proper Efficiency)

Take x^ to be a robust feasible solution for the considered problem $(USIMFP)$ and consider $\psi(x)$ to be the following vector-valued non-fractional function:*

$$\psi(x) = \left(f_1(x) - \lambda_1 g_1(x), \dots, f_k(x) - \lambda_k g_k(x) \right).$$

Let $\lambda^ = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, for all $i = 1, \dots, k$ and assume the existence of $\alpha \in \mathbb{R}_+^k - \{0\}$, $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, and for all $t \in T$, $v_t \in V_t$ such that the conditions (2.3) - (2.4) are fulfilled at x^* .*

Moreover, suppose that the function $\sum_{t \in T} \mu_t h_t(\cdot, v_t)$ is a quasi-convex at x^ and the function $\alpha^T \psi$ is pseudo-convex at x^* . Then the point x^* is a robust properly efficient solution to the problem $(USIMFP)$.*

Proof. To prove this theorem, we follow the same steps given in the proof of Theorem (2.2) since $x^* \in F$ is a robust feasible solution satisfying the feasibility condition of the problem that $h_t(x^*, v_t) \leq 0$, $t \in T, v_t \in V_t$, and the function $\sum_{t \in T} \mu_t h_t(\cdot, v_t)$ is a

quasi-convex at x^* . Hence, we obtain

$$\sum_{i=1}^k \alpha_i \nabla(f_i(x^*) - \lambda_i^* g_i(x^*))(x - x^*) = - \sum_{t \in T} \mu_t \nabla_x h_t(x^*, v_t)(x - x^*) \geq 0.$$

The above inequality can be written in the following equivalent form

$$\nabla(\alpha^T \psi(x^*))(x - x^*) \geq 0.$$

Now, by assumption, $\alpha^T \psi$ is a pseudo-convex function at x^* , so for all $x \in F$, the next result holds

$$\alpha^T \psi(x) \geq \alpha^T \psi(x^*).$$

This implies that x^* is an optimal solution to the problem $(\text{SRSIMFP})_{\lambda^*}$. Thus by Lemma (2.2), x^* is a properly efficient solution to the problem $(\text{RSIMFP})_{\lambda}$, and by Lemma 2.1, it is a properly efficient solution to the problem (RSIMFP) . Therefore, Definition 2.1 gives that x^* is a robust properly efficient solution to the problem (USIMFP) , which completes the proof. ■

2.3 Robust Duality of (RSIMFP)

This section is dedicated to the robust dual model of the considered semi-infinite multi-objective fractional programming problem (USIMFP) under uncertainty with an infinite number of constraints. Here, we present the Mond-Weir type robust dual

for the (RSIMFP) problem in the sense of the parametrization approach as follows:

$$\max_{u \in \mathbb{R}^n} \psi(u) = \left(f_1(u) - \lambda_1 g_1(u), \dots, f_k(u) - \lambda_k g_k(u) \right) \quad (\text{MWRDSIMFP})_\lambda$$

$$\begin{aligned} \text{Subject to } & \sum_{i=1}^k \alpha_i \nabla(f_i(u) - \lambda_i g_i(u)) + \sum_{t \in T} \mu_t \nabla_u h_t(u, v_t) = 0, \\ & \sum_{t \in T} \mu_t h_t(u, v_t) \geq 0, \quad v_t \in V_t, \quad t \in T, \\ & \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k - \{0\}, \quad \mu_t \in \mathbb{R}_+^{(T)}, \quad t \in T. \end{aligned}$$

Let $F_D = \{(u, \alpha, \lambda, \mu_t, v_t) \in \mathbb{R}^n \times \mathbb{R}_+^k - \{0\} \times \mathbb{R}_+^k \times \mathbb{R}_+^{(T)} \times \mathbb{R}^q\}$ be the set of all feasible solutions of the Mond-Weir robust dual model $(\text{MWRDSIMFP})_\lambda$ with $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$ under the uncertain parameter $v_t \in V_t$, $t \in T$.

For this robust dual problem, the notions of weak efficiency and efficiency can be introduced as follows:

Definition 2.9 *Let $(u^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*) \in F_D$ be a feasible solution. Then*

- (i) *$(u^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is called a weakly efficient solution to the maximization dual problem $(\text{MWRDSIMFP})_\lambda$, if there does not exist any $(u, \alpha, \lambda, \mu_t, v_t) \in F_D$ such that*

$$\psi(u) > \psi(u^*).$$

- (ii) *$(u^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is called an efficient solution to the maximization dual problem*

$(MWRDSIMFP)_\lambda$, if there does not exist any $(u, \alpha, \lambda, \mu_t, v_t) \in F_D$ such that

$$\psi(u) \geq \psi(u^*).$$

Next, motivated by the approach of [4], we introduce the robust duality theorems for the robust semi-infinite multi-objective fractional problem (RSIMFP) including robust weak duality, robust strong duality and robust strict converse duality theorems under the conditions of extended convexity, namely, quasi-convex functions and pseudo-convex functions.

Theorem 2.5 (Robust Weak Duality Theorem) *Let x and $(u, \alpha, \lambda, \mu_t, v_t)$ be feasible solutions of the problems (RSIMFP) and $(MWRDSIMFP)_\lambda$ respectively. For any $x \in \mathbb{R}^n$ and a parameter $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$, consider the vector-valued non-fractional function $\psi(x)$ as follows:*

$$\psi(x) = \left(f_1(x) - \lambda_1 g_1(x), \dots, f_k(x) - \lambda_k g_k(x) \right).$$

Assume that the function $\sum_{t \in T} \mu_t h_t(\cdot, v_t)$ is a quasi-convex at u with v_t is the uncertain parameter for $t \in T$. Then we have the following cases:

(i) *if the function ψ is pseudo-convex at u , then the following is not true*

$$\psi(x) < \psi(u).$$

(ii) if the function ψ is strictly-pseudo-convex at u , then the following is not true

$$\psi(x) \leq \psi(u).$$

(iii) if the function ψ is quasi-convex at u and $\alpha \in \text{int } \mathbb{R}_+^k$, then the following statement is not true

$$\psi(x) \leq \psi(u).$$

Proof. As x is assumed to be a feasible solution to the problem (RSIMFP), then it satisfies the robust feasibility condition that $h_t(x, v_t) \leq 0$. Hence, for the multiplier $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, we obtain $\sum_{t \in T} \mu_t h_t(x, v_t) \leq 0$. Also, as $(u, \alpha, \lambda, \mu_t, v_t)$ is a feasible solution of the dual problem (MWRDSIMFP) $_\lambda$, then it fulfills the robust duality condition that $\sum_{t \in T} \mu_t h_t(u, v_t) \geq 0$.

Thus, by getting together both results, we get

$$\sum_{t \in T} \mu_t h_t(x, v_t) - \sum_{t \in T} \mu_t h_t(u, v_t) \leq 0 \iff \sum_{t \in T} \mu_t h_t(x, v_t) \leq \sum_{t \in T} \mu_t h_t(u, v_t).$$

By the quasi-convexity of the function $\sum_{t \in T} \mu_t h_t(\cdot, v_t)$, it follows:

$$\sum_{t \in T} \mu_t \nabla_x h_t(u, v_t)(x - u) \leq 0. \quad (2.11)$$

Furthermore, we have the following robust Mond-Weir dual feasibility condition

$$\sum_{i=1}^k \alpha_i \nabla(f_i(u) - \lambda_i g_i(u)) + \sum_{t \in T} \mu_t \nabla_u h_t(u, v_t) = 0.$$

Then, we get

$$\sum_{i=1}^k \alpha_i \nabla(f_i(u) - \lambda_i g_i(u)) = - \sum_{t \in T} \mu_t \nabla_u h_t(u, v_t).$$

Therefore, by the above equality and (2.11), we have

$$\sum_{i=1}^k \alpha_i \nabla(f_i(u) - \lambda_i g_i(u))(x - u) = - \sum_{t \in T} \mu_t \nabla_u h_t(u, v_t)(x - u) \geq 0. \quad (2.12)$$

Now, we proceed to prove the three cases (i) – (iii):

- (i) To prove (i), using the contradiction, suppose that the statement $\psi(x) < \psi(u)$ is true, then by the pseudo-convexity of ψ at u , we get $\nabla\psi(u)(x - u) < 0$, which implies $\sum_{i=1}^k \nabla(f_i(u) - \lambda_i g_i(u))(x - u) < 0$. Thus, for $\alpha \in \mathbb{R}_+^k - \{0\}$, we also get

$$\sum_{i=1}^k \alpha_i \nabla(f_i(u) - \lambda_i g_i(u))(x - u) < 0. \quad (2.13)$$

Clearly, the inequality (2.13) contradicts (2.12). Therefore, the statement $\psi(x) < \psi(u)$ does not hold.

- (ii) By contradiction, we can prove (ii) following the same steps as in (i).

- (iii) To get a contrary, suppose that the statement $\psi(x) \leq \psi(u)$ holds. Then as the function ψ is quasi-convex at u , we have $\nabla\psi(u)(x - u) \leq 0$. Now, for $\alpha \in \text{int } \mathbb{R}_+^k$, the following holds

$$\sum_{i=1}^k \alpha_i \nabla(f_i(u) - \lambda_i g_i(u))(x - u) < 0.$$

Which again contradicts (2.12). Therefore, $\psi(x) \leq \psi(u)$ does not hold. ■

Theorem 2.6 (Robust Strong Duality Theorem) *Take $x^* \in F$ to be a weakly efficient (efficient) solution of the problem (RSIMFP) and $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $i = 1, \dots, k$. Consider $h_t(\cdot, v_t)$, $t \in T$ to be a continuous convex function with uncertain parameter $v_t \in V_t$. Suppose that the generalized robust Slater constraint qualification (GRSCQ) holds. Then there exist $\alpha^* \in \mathbb{R}_+^k - \{0\}$, $(\mu_t^*)_{t \in T} \in \mathbb{R}_+^{(T)}$, and $v_t^* \in V_t$, $\forall t \in T$ such that $(x^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is a feasible solution of the dual problem $(MWRDSIMFP)_{\lambda^*}$. Moreover, if all the hypotheses of the robust weak duality Theorem (2.5) are satisfied at every feasible point of the Mond-Weir robust dual problem, then $(x^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is also a weakly efficient (efficient) solution of the dual problem $(MWRDSIMFP)_{\lambda^*}$, and both problems (RSIMFP) and $(MWRDSIMFP)_{\lambda^*}$ have the same corresponding optimal value.*

Proof. Let $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $i = 1, \dots, k$. By assumption, x^* is a weakly efficient (efficient) solution of the problem (RSIMFP) and hence Lemma (2.1) implies that x^* is also a weakly efficient (efficient) solution of the problem $(RSIMFP)_{\lambda^*}$. Assume that the generalized robust Slater constraint qualification (GRSCQ) is met at x^* . Then by the robust Karush–Kuhn–Tucker necessary optimality conditions, there exist multipliers $\alpha^* = (\alpha_1^*, \dots, \alpha_k^*) \in \mathbb{R}_+^k - \{0\}$, $(\mu_t^*)_{t \in T} \in \mathbb{R}_+^{(T)}$, and $v_t^* \in V_t$, $\forall t \in T$ such that

$$\sum_{i=1}^k \alpha_i^* \nabla(f_i(x^*) - \lambda_i^* g_i(x^*)) + \sum_{t \in T} \mu_t^* \nabla_x h_t(x^*, v_t^*) = 0. \quad (2.14)$$

$$\sum_{t \in T} \mu_t^* h_t(x^*, v_t^*) = 0. \quad (2.15)$$

The above equations imply that $(x^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is a feasible solution of the dual problem $(\text{MWRDSIMFP})_{\lambda^*}$. Now, we need to show that $(x^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is also a weakly efficient (efficient) solution of the dual problem $(\text{MWRDSIMFP})_{\lambda^*}$. Thus, define first the vector-valued non-fractional function $\psi(x)$ as follows:

$$\psi(x) = \left(f_1(x) - \lambda_1 g_1(x), \dots, f_k(x) - \lambda_k g_k(x) \right).$$

Suppose that the hypotheses of the robust weak duality Theorem (2.5) are fulfilled for all possible solutions to the dual problem $(\text{MWRDSIMFP})_{\lambda^*}$. To reach a contradiction, suppose that $(x^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is not a weakly efficient (efficient) solution of $(\text{MWRDSIMFP})_{\lambda^*}$. This implies the existence of a feasible solution $(x, \alpha, \lambda, \mu_t, v_t)$ of the dual maximization problem $(\text{MWRDSIMFP})_{\lambda^*}$ satisfying that

$$\psi(x) > \psi(x^*).$$

The above inequality holds for x^* which is assumed to be a weakly efficient (efficient) solution of the problem (RSIMFP) , but this contradicts the first case of Theorem (2.5). Hence, $(x^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ is a weakly efficient (efficient) solution of the dual problem $(\text{MWRDSIMFP})_{\lambda^*}$ and clearly both problems (RSIMFP) and $(\text{MWRDSIMFP})_{\lambda^*}$ have the same corresponding optimal value. ■

Theorem 2.7 (Robust Strict Converse Duality Theorem) *Let x^* and*

$(u^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ be feasible solutions of the problems (RSIMFP) and $(MWRDSIMFP)_\lambda$ respectively. For a parameter $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*) \in \mathbb{R}_+^k$, consider the following multi-objective non-fractional function $\psi(\cdot)$

$$\psi(\cdot) = \left(f_1(\cdot) - \lambda_1^* g_1(\cdot), \dots, f_k(\cdot) - \lambda_k^* g_k(\cdot) \right).$$

For $\alpha^* \in \mathbb{R}_+^k - \{0\}$, let the function $\alpha^{*T} \psi(\cdot)$ be strictly pseudo-convex at u^* , and assume that $\sum_{t \in T} \mu_t^* h_t(\cdot, v_t^*)$ is a quasi-convex function at u^* with the uncertain parameter v_t^* for all $t \in T$. If $\alpha^{*T} \psi(x^*) = \alpha^{*T} \psi(u^*)$ holds. That is

$$\sum_{i=1}^k \alpha_i^* (f_i(x^*) - \lambda_i^* g_i(x^*)) = \sum_{i=1}^k \alpha_i^* (f_i(u^*) - \lambda_i^* g_i(u^*)). \quad (2.16)$$

Then $x^* = u^*$.

Proof. To prove this theorem, Assume, on the contrary, that $x^* \neq u^*$. Then by the feasibility of x^* to the problem (RSIMFP) and for $(\mu_t^*)_{t \in T} \in \mathbb{R}_+^{(T)}$, $t \in T$, we obtain $\sum_{t \in T} \mu_t^* h_t(x^*, v_t^*) \leq 0$. In addition, the feasibility of $(u^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ to the robust Mond-Weir dual problem $(MWRDSIMFP)_\lambda^*$ implies that $\sum_{t \in T} \mu_t^* h_t(u^*, v_t^*) \geq 0$. Thus, we get

$$\sum_{t \in T} \mu_t^* h_t(x^*, v_t^*) \leq \sum_{t \in T} \mu_t^* h_t(u^*, v_t^*).$$

Now, by assumption, the function $\sum_{t \in T} \mu_t^* h_t(\cdot, v_t^*)$ is a quasi-convex function at u^* , hence, for $t \in T$, the next result is satisfied

$$\sum_{t \in T} \mu_t^* \nabla_x h_t(u^*, v_t^*)(x^* - u^*) \leq 0. \quad (2.17)$$

Moreover, since $(u^*, \alpha^*, \lambda^*, \mu_t^*, v_t^*)$ satisfies the first robust Mond-Weir dual feasibility criteria, that is

$$\sum_{i=1}^k \alpha_i^* \nabla(f_i(u^*) - \lambda_i^* g_i(u^*)) = - \sum_{t \in T} \mu_t^* \nabla_u h_t(u^*, v_t^*).$$

Then, along with (2.17), we get the following

$$\sum_{i=1}^k \alpha_i^* \nabla(f_i(u^*) - \lambda_i^* g_i(u^*))(x^* - u^*) = - \sum_{t \in T} \mu_t^* \nabla_u h_t(u^*, v_t^*)(x - u) \geq 0, \quad (2.18)$$

which gives that

$$\alpha^{*T} \nabla \psi(u^*)(x^* - u^*) \geq 0. \quad (2.19)$$

As the function $\alpha^{*T} \psi(\cdot)$ is assumed to be strictly pseudo-convex at u^* , then the inequality (2.19) yields that

$$\alpha^{*T} \psi(x^*) > \alpha^{*T} \psi(u^*).$$

That is

$$\sum_{i=1}^k \alpha_i^* (f_i(x^*) - \lambda_i^* g_i(x^*)) > \sum_{i=1}^k \alpha_i^* (f_i(u^*) - \lambda_i^* g_i(u^*)).$$

Obviously, this last inequality contradicts the assumption (2.16). Therefore, $x^* = u^*$, which completes the proof. ■

2.4 Numerical Example

This section is devoted to illustrating the obtained main results through the following numerical example:

$$\min_{x \in \mathbb{R}} \left(\frac{x^2 + 3}{x + 1}, \frac{3x^2 + x + 1}{x + 2} \right) \quad (\text{USIMFP})$$

$$\text{Subject to } 3xv_t - 1 \leq 0,$$

where $x \in \mathbb{R}$ and $v_t \in V_t = [-t, t]$ with $t \in T = [0, 1]$ is an uncertain parameter.

Applying the robust approach to this considered problem, the robust counterpart is formulated as follows:

$$\min_{x \in \mathbb{R}} \left(\frac{x^2 + 3}{x + 1}, \frac{3x^2 + x + 1}{x + 2} \right) \quad (\text{RSIMFP})$$

$$\text{Subject to } 3xv_t - 1 \leq 0, \quad \forall v_t \in V_t = [-t, t], \quad t \in T = [0, 1], \quad x \in \mathbb{R}.$$

Clearly, we have $f_1(x) = x^2 + 3$, $g_1(x) = x + 1$, $f_2(x) = 3x^2 + x + 1$, $g_2(x) = x + 2$ and $h_t(x, v_t) = 3xv_t - 1$, $\forall t \in [0, 1]$, $x \in \mathbb{R}$.

The set of all feasible solutions is defined by

$$F = \{x \in \mathbb{R} : 3xv_t - 1 \leq 0, \quad \forall v_t \in [-t, t], \quad t \in [0, 1]\}$$

Hence, the feasible region of the problem (RSIMFP) is given by $F = [\frac{-1}{3}, \frac{1}{3}]$. In this problem, it is obvious that $f_1(x)$ and $f_2(x)$ are continuously differentiable and convex

functions with $f_i(x) \geq 0$, $i = 1, 2$, $\forall x \in F = [-\frac{1}{3}, \frac{1}{3}]$. Also, $g_1(x)$ and $g_2(x)$ are continuously differentiable and concave functions with $g_i(x) > 0$, $i = 1, 2$, $\forall x \in F$.

Now, we associate with the robust counterpart problem the following parametric semi-infinite multi-objective non-fractional programming problem based on the Dinkelbach parametrization approach with parameter $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$.

$$\min_{x \in \mathbb{R}} \quad \psi(x) = \left(x^2 + 3 - \lambda_1(x + 1), 3x^2 + x + 1 - \lambda_2(x + 2) \right) \quad (\text{RSIMFP})_\lambda$$

$$\text{Subject to } h_t(x, v_t) = 3xv_t - 1 \leq 0, \quad \forall v_t \in [-t, t], \quad t \in [0, 1], \quad x \in \mathbb{R}.$$

Consider the feasible point $x^* = 0 \in F = [-\frac{1}{3}, \frac{1}{3}]$, we need to show that $x^* = 0$ is a robust efficient solution to the problem (USIMFP). To obtain this result, first, by Lemma 2.1 and Remark 1, we shall show that $x^* = 0$ is an efficient solution to the problem (RSIMFP) $_{\lambda^*}$ where $\lambda^* = (\lambda_1^*, \lambda_2^*)$ such that

$$\lambda_1^* = \frac{f_1(x^*)}{g_1(x^*)} = \frac{f_1(0)}{g_1(0)} = 3 > 0, \quad \lambda_2^* = \frac{f_2(x^*)}{g_2(x^*)} = \frac{f_2(0)}{g_2(0)} = \frac{1}{2} > 0.$$

In addition, It should be noted that the generalized robust Slater constraint qualification (GRSCQ) is fulfilled when $x^* = 0$. In fact, $h_t(0, v_t) = -1 < 0$, $\forall v_t \in [-t, t]$, $t \in [0, 1]$. Thus, for $x^* = 0$, there exists $\alpha = (\alpha_1, \alpha_2) = (\frac{1}{7}, \frac{6}{7}) \in \mathbb{R}_+^2 - \{0\}$ and with $t \in [0, 1]$, $\mu_t = 0$, $v_t = 1$, so the conditions (2.3) - (2.4) are satisfied at $x^* = 0$.

Furthermore, for $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, the function $\psi(x)$ is a strictly pseudo-convex function for all $x \in F$, and the function $\sum_{t \in [0, 1]} \mu_t h_t(x, v_t)$ is a quasi-convex at $x^* = 0$.

Indeed, we have the following:

$$\begin{aligned} \sum_{t \in [0,1]} \mu_t h_t(x, v_t) - \sum_{t \in [0,1]} \mu_t h_t(0, v_t) &= \sum_{t \in [0,1]} \mu_t ((3xv_t - 1) - (3(0)v_t - 1)) \\ &= \sum_{t \in [0,1]} 3\mu_t v_t x. \end{aligned}$$

Now as $x \in [-\frac{1}{3}, \frac{1}{3}]$, we get that $\sum_{t \in [0,1]} 3\mu_t v_t x \leq 0$ for $x \in [-\frac{1}{3}, 0]$. It follows:

$$\begin{aligned} \nabla_x \left(\sum_{t \in [0,1]} \mu_t h_t(x, v_t) \right) (x - x^*) &= \nabla_x \left(\sum_{t \in [0,1]} \mu_t (3xv_t - 1) \right) (x - 0) \\ &= \sum_{t \in [0,1]} 3\mu_t v_t x \leq 0, \quad x \in [-\frac{1}{3}, 0]. \end{aligned}$$

Hence, at $x^* = 0$, the function $\sum_{t \in [0,1]} \mu_t h_t(x, v_t)$ is a quasi-convex function. Therefore, by Theorem 2.3, $x^* = 0$ is a robust efficient solution to the problem (USIMFP), and hence, it is also a robust weakly efficient solution.

CHAPTER 3

NON-SMOOTH ROBUST SEMI-INFINITE MULTI-OBJECTIVE FRACTIONAL PROGRAMMING PROBLEM

Non-smooth optimization has been interestingly applied in real-life practical problems such as engineering problems, economics, optimal control, and data analysis. In this kind of application, mostly, the nature of the objective functions or constraint functions appears to be non-differentiable or discontinuously differentiable functions, and hence adopting the non-smooth optimization approach to modeling and solving such problems has been of highly effective technique in recent years.

By taking into consideration some contributions from generalized differentiation and variational analysis such as Mordukhovich subdifferential [83] of a function that takes values from the extended real number system, Choung [26] provided some new necessary and sufficient optimal conditions in a fuzzy form for an approximate weak Pareto solution of a cone constrained non-smooth non-convex vector optimization problem. Li and Yu [74] employed another tool of variational analysis, Clarke subdifferential, and established necessary optimality conditions for an approximate quasi-weakly efficient solution of a non-smooth vector optimization problem. The authors introduced a novel kind of generalized convexity called approximate pseudo quasi type-I functions and utilized it to study the sufficient optimality condition to the problem with respect to the concept of approximate quasi weakly efficiency.

One important progress in non-smooth optimization problems is dealing with data infected with uncertainty. Several researchers have employed the robust approach to formulate and solve uncertain non-smooth optimization problems as in the research done by Sun et al. [95] in which they introduced the approximate optimality conditions for solving non-smooth semi-infinite optimization problems with unknown data. These conditions are based on a robust type of constraint qualifying condition and in terms of the Legendre-Fenchel conjugate function. Fakhar et al [38] provided non-smooth necessary and sufficient optimality conditions of a robust weakly efficient solution for a non-smooth robust multi-objective optimization for the so-called strictly pseudo-quasi generalized convexity and applied the obtained

results to the portfolio optimization. By employing Clarke's generalized subgradient concept and introducing two types of generalized convex functions, Chen et al. [20] investigated the robust optimality conditions for a weakly robust efficient solution and properly robust efficient solution of non-smooth/non-convex multi-objective optimization problems with constraint data uncertainty. Moreover, Antczak et al. [10] applied the worst-case approach for finding an approximate efficient solution for a non-smooth multi-objective programming problem with the uncertainty in both the objective and constraint functions and used a scalarizing method in proving the optimality conditions for such problem. Then, Ahmad et al. [3] proposed a category of generalized convex functions and studied a non-smooth non-convex vector mathematical program over a cone with data uncertainty in constraints as well as constructed its robust dual problem.

Moreover, modeling the fractional programs in which the objective function of an optimization problem is expressed as the ratio of two functions has been a very active field in recent years as it arises from many real-world applications. Many researchers have investigated mathematical programming with multi-objective fractional programming problems confined to uncertain data. Sun et al. [94] derived the robust approximate optimality conditions of robust ϵ -weakly efficient solutions for the uncertain multi-objective fractional semi-infinite optimization problem under a type of constraint qualification. Antczak [8] established the optimality conditions for the robust ϵ -efficient solution for multi-objective fractional programming problems under uncertainty using a parametric approach. In [4], Ahmad et al. studied the

duality models as well as the robust sufficient optimality conditions in semi-infinite multi-objective programs.

On the other hand, there were few studies on proper and isolated efficient solutions for various vector optimization problems, the reader can consult ([27], [87], [88], [92]). In 2023, based on the robustness methodology, Pham [85] presented a paper on studying the optimality conditions and duality results for local isolated and local properly efficient solutions for non-smooth semi-infinite multi-objective programming problems under uncertainty.

In this chapter, we take into consideration a semi-infinite multi-objective fractional programming problem endowed with an infinite number of constraints and infected with data uncertainty. In addition, we assume that this programming problem involves non-smooth locally Lipschitz functions. The layout of this chapter is designed as follows: Section 3.1 enriches the background with the most crucial concepts and notations that will be useful throughout the chapter. In section 3.2, under an extension of a robust type constraint qualification (RCQ), we investigate non-smooth robust necessary optimality conditions for local isolated efficient solutions as well as local properly efficient solutions of both the non-smooth robust semi-infinite multi-objective fractional problems and the non-smooth robust semi-infinite parametric problem constructed by the generalized Dinkelbach parametrization approach. In section 3.3, we establish the non-smooth robust sufficient optimality conditions for the non-smooth robust semi-infinite multi-objective fractional problems subject to some generalized

convex functions. Section 3.4 consists of robust dual problems which are formulated in Wolfe-type robust dual problems and Mond-Weir-type robust dual problems for local properly efficient solutions. We demonstrate the relation between the robust primal problem and the robust dual problem through the different types of robust duality theorems.

3.1 Primary Concepts

For a positive integer n , consider the two vectors $x, y \in \mathbb{R}^n$, the notation $\langle x, y \rangle$ defines the standard inner product in $\mathbb{R}^n$, while $\|x\|$ denotes the norm of x .

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued function. Then, ϕ is locally Lipschitz function (a function of order L) at a point $\bar{u} \in \mathbb{R}^n$, if there is a positive scalar $L > 0$ so that for any u_1, u_2 in a neighborhood N of $\bar{u}$, we have

$$\|\phi(u_1) - \phi(u_2)\| \leq L\|u_1 - u_2\|, \quad \forall u_1, u_2 \in N.$$

The function ϕ is considered to be a locally Lipschitz function on $\mathbb{R}^n$, if it is a locally Lipschitz function at every point in $\mathbb{R}^n$. For a vector-valued function $\Phi = (\phi_1, \dots, \phi_k)^T : \mathbb{R}^n \rightarrow \mathbb{R}^k$, we say that Φ is a locally Lipschitz function on $\mathbb{R}^n$, if and only if each ϕ_i is locally Lipschitz function on $\mathbb{R}^n$, for all $i = 1, \dots, k$.

In the case of optimization problems with non-smooth functions, it is a crucial situation to recall a definition of generalized directional derivative such as the definition

in the sense of Clarke [28], as follows.

Definition 3.1 [28] *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function. The Clarke generalized directional derivative of ϕ at $\bar{u} \in \mathbb{R}^n$ in the direction of a vector $d \in \mathbb{R}^n$ is defined as*

$$\phi^\circ(\bar{u}; d) := \limsup_{u \rightarrow \bar{u}} \sup_{t \rightarrow 0} \frac{\phi(u + td) - \phi(u)}{t},$$

where t is a positive scalar.

The Clarke subdifferential of ϕ at $\bar{u} \in \mathbb{R}^n$ is defined to be as follows.

$$\partial^C \phi(\bar{u}) := \{\xi \in \mathbb{R}^n : \phi^\circ(\bar{u}; d) \geq \langle \xi, d \rangle, \forall d \in \mathbb{R}^n\}.$$

The subset $\partial^C \phi(\bar{u})$ is a non-empty compact subset of $\mathbb{R}^n$ along with the assumption that ϕ is a locally Lipschitz function, and for each $d \in \mathbb{R}^n$, the following fact holds

$$\phi^\circ(\bar{u}; d) = \max\{\langle \xi, d \rangle : \xi \in \partial^C \phi(\bar{u})\}, \quad \forall d \in \mathbb{R}^n.$$

In addition, if the function ϕ is assumed to be a differentiable function, then clearly we have $\partial^C \phi(\bar{u}) = \{\nabla \phi(\bar{u})\}$. Furthermore, if ϕ is a convex function, then the Clarke subdifferential $\partial^C \phi(\bar{u})$ matches the original concept of subdifferential defined in the convex analysis, i.e.,

$$\partial \phi(\bar{u}) := \{\xi \in \mathbb{R}^n : \phi(u) - \phi(\bar{u}) \geq \langle \xi, u - \bar{u} \rangle, \forall u \in \mathbb{R}^n\}.$$

Let $C \subseteq \mathbb{R}^n$ be a non-empty closed subset of $\mathbb{R}^n$. The distance function related to

the set C , denoted by d_C , which is a non-smooth Lipschitz function can be defined as follows.

$$d_C(x; C) = \min\{\|x - c\| : c \in C\}.$$

Denote the Clarke tangent cone to C at $\bar{u} \in C$ by $T^C(\bar{u}; C)$ and it is defined by

$$T^C(\bar{u}; C) := \{w \in \mathbb{R}^n : d_C^\circ(\bar{u}; w) = 0\}.$$

The Clarke normal cone to C at $\bar{u} \in C$ is defined by

$$N^C(\bar{u}; C) := \{\xi \in \mathbb{R}^n : \langle \xi, w \rangle \leq 0, \forall w \in T^C(\bar{u}; C)\}.$$

In case that the non-empty closed subset $C \subset \mathbb{R}^n$ is also a convex set, that is for all $c_1, c_2 \in C$ and $\lambda \in [0, 1]$, $\lambda c_1 + (1 - \lambda)c_2 \in C$, then the Clarke normal cone $N^C(\bar{u}; C)$ matches the concept of normal cone at $\bar{u}$ to C defined in the convex analysis, i.e.,

$$N(\bar{u}; C) := \{\xi \in \mathbb{R}^n : \langle \xi, c - \bar{u} \rangle \leq 0, \forall c \in C\}.$$

For any subset $C \subseteq \mathbb{R}^n$, its closure and interior are denoted by clC and $intC$ respectively. The polar cone of the C is defined by $C^\circ = \{\hat{u} \in \mathbb{R}^n : \langle \hat{u}, u \rangle \leq 0, \forall u \in C\}$. The set C is said to be locally closed if and only if for each $\bar{u} \in C$, there exists a neighborhood N of $\bar{u}$ such that $C \cap clN$ is closed. The closed unit ball in $\mathbb{R}^n$ is denoted by $\mathbb{B}$.

Next, we recall various important properties and results related to the Clarke subdif-

ferential in [28] which will be used throughout this chapter.

Proposition 3.1 [28] *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\phi_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, k$ be real-valued locally Lipschitz functions in a neighborhood of a point $\bar{u} \in \mathbb{R}^n$. Then*

(i) *For any real scalar $\lambda \in \mathbb{R}$, $\partial^C(\lambda\phi)(\bar{u}) = \lambda \partial^C\phi(\bar{u})$. (Scalar Multiples).*

(ii) *If $\bar{u} \in \mathbb{R}^n$ is a local minimum or maximum of ϕ , then $0 \in \partial^C\phi(\bar{u})$.*

(iii) *$\partial^C(\phi_1 + \dots + \phi_k)(\bar{u}) \subseteq \partial^C\phi_1(\bar{u}) + \dots + \partial^C\phi_k(\bar{u})$. (Finite Sum).*

For any real scalars $\lambda_i \in \mathbb{R}$ with $i = 1, \dots, k$, we have

$$\partial^C(\lambda_1\phi_1 + \dots + \lambda_k\phi_k)(\bar{u}) \subset \lambda_1\partial^C\phi_1(\bar{u}) + \dots + \lambda_k\partial^C\phi_k(\bar{u}).$$

(iv) *$\partial^C(\max\{\phi_i, i = 1, \dots, k\})(\bar{u}) \subseteq \text{conv}(\partial^C\phi_1(\bar{u}) \cup \dots \cup \partial^C\phi_k(\bar{u}))$.*

Where "conv" denotes the convex hull of the considered set.

Consider the indicator function $\delta(\cdot; C)$ of a set $C \subseteq \mathbb{R}^n$ which is an extended real-valued function defined by

$$\delta(u; C) = \begin{cases} 0, & u \in C \\ \infty, & \text{otherwise,} \end{cases}$$

The relation between the Clarke subdifferential of the indicator function and the Clarke normal cone to C at $\bar{u} \in C$ is obtained by the following Proposition.

Proposition 3.2 [28] *Let $\bar{u} \in C$. Then we have*

$$\partial^C \delta(\bar{u}; C) = N(\bar{u}; C), \quad \forall \bar{u} \in C.$$

Lemma 3.1 [28] *Let $\phi_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, k$ be real valued locally Lipschitz functions near a point $\bar{u} \in \mathbb{R}^n$. Consider the function $\Psi(\cdot) := \max\{\phi_i, i = 1, \dots, k\}$. Then, the function $\Psi(\cdot)$ is also a locally Lipschitz function near $\bar{u}$ and the following fulfills.*

$$\partial^C \Psi(\bar{u}) \subset \bigcup \left\{ \sum_{i=1}^k \gamma_i \partial^C \phi_i(\bar{u}) : (\gamma_1, \dots, \gamma_k) \in \mathbb{R}_+^k, \sum_{i=1}^k \gamma_i = 1, \gamma_i [\phi_i(\bar{u}) - \Psi(\bar{u})] = 0 \right\}.$$

Lemma 3.2 [28] *Suppose that $C \subseteq \mathbb{R}^n$ is a non-empty subset of $\mathbb{R}^n$ and $\bar{u} \in C$. Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a real-valued locally Lipschitz function around $\bar{u}$ and attain a minimum over C at $\bar{u}$. Then, we have*

$$0 \in \partial^C \phi(\bar{u}) + N^C(\bar{u}; C).$$

Lemma 3.3 [28] *Let $\phi_1, \phi_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be real valued locally Lipschitz functions near $\bar{u} \in \mathbb{R}^n$. Suppose that $\phi_2(\bar{u}) \neq 0$. Then the function $\frac{\phi_1}{\phi_2}$ is locally Lipschitz function near $\bar{u}$ and we have*

$$\partial^C \left(\frac{\phi_1}{\phi_2} \right)(\bar{u}) \subset \frac{\phi_2(\bar{u}) \partial^C \phi_1(\bar{u}) - \phi_1(\bar{u}) \partial^C \phi_2(\bar{u})}{[\phi_2(\bar{u})]^2}.$$

Another useful tool in generalized differentiation is the Fréchet subdifferential of an extended real-valued function $\phi : \mathbb{X} \subseteq \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ at a point $\bar{u}$ in a Banach space $\mathbb{X}$

with $|\phi(\bar{u})| < \infty$, which is defined by the following:

$$\widehat{\partial}\phi(\bar{u}) := \left\{ u^* \in \mathbb{X}^* : \liminf_{u \rightarrow \bar{u}} \frac{\phi(u) - \phi(\bar{u}) - \langle u^*, u - \bar{u} \rangle}{\|u - \bar{u}\|} \geq 0 \right\}.$$

Where $\mathbb{X}^*$ is the dual space of $\mathbb{X}$. The upper counterpart of $\widehat{\partial}\phi(\bar{u})$ is called the Fréchet upper subdifferential and denoted by $\widehat{\partial}^+\phi(\bar{u})$.

The following Lemma enriches our background with the Fréchet upper subdifferential that will be useful later for the proofs of the theorems.

Lemma 3.4 [84] *Let $\bar{u} \in \mathbb{R}^n$. Consider ϕ_1, ϕ_2 be two extended real-valued functions with $|\phi_i(\bar{u})| < \infty$, $i = 1, 2$. Then*

$$(i) \quad \widehat{\partial}^+\phi_1(\bar{u}) = -\widehat{\partial}(-\phi_1)(\bar{u}).$$

(ii) *If $\widehat{\partial}^+\phi_2(\bar{u}) \neq \emptyset$. Then we have*

$$\widehat{\partial}(\phi_1 + \phi_2)(\bar{u}) \subset \bigcap_{y \in \widehat{\partial}^+\phi_2(\bar{u})} [\widehat{\partial}\phi_1(\bar{u}) + y].$$

In this chapter, consider the following non-smooth semi-infinite multi-objective fractional optimization problem confined with uncertain data in constraint functions:

$$\min_{x \in \mathbb{R}^n} \quad F(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_k(x)}{g_k(x)} \right) \quad (\text{UNSIMFP})$$

$$\text{Subject to } (-h_t(x, v_t))_{t \in T} \in S, \quad \forall t \in T, \quad x \in \Delta \subseteq \mathbb{R}^n,$$

where $f_i, g_i : \mathbb{R}^n \rightarrow \mathbb{R}, i = 1, \dots, k$ are locally Lipschitz functions. The constraint

functions $h_t : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, $t \in T$ are infected with the uncertain parameters v_t that belong to some convex and compact uncertainty set $V_t \subseteq \mathbb{R}^q, t \in T$, the functions $h_t(x, v_t)$ are locally Lipschitz functions with respect to the first argument, that is for any $x \in \mathbb{R}^n$, there exists a positive scalar $L > 0$ such that for any x_1, x_2 in a neighborhood N of x , and $v_t \in V_t$, $t \in T$, the following holds:

$$\|h_t(x_1, v_t) - h_t(x_2, v_t)\| \leq L\|x_1 - x_2\|, \quad \forall x_1, x_2 \in N, \forall v_t \in V_t, t \in T.$$

The subset $S \subseteq \mathbb{R}_+^{(T)}$ is a non-empty, proper, closed, and convex cone. The subset Δ is a non-empty locally closed subset of $\mathbb{R}^n$.

By using the technique of robust optimization, the robust counterpart of the problem (UNSIMFP) under uncertainty is formulated as follows:

$$\min_{x \in \mathbb{R}^n} F(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_k(x)}{g_k(x)} \right) \quad (\text{NRSIMFP})$$

$$\text{Subject to } (-h_t(x, v_t))_{t \in T} \in S, \quad \forall v_t \in V_t, t \in T, x \in \Delta \subseteq \mathbb{R}^n.$$

The feasible solution set for the aforementioned problem (NRSIMFP) is determined by the following:

$$G = \{x \in \Delta \subseteq \mathbb{R}^n : (-h_t(x, v_t))_{t \in T} \in S, \quad \forall v_t \in V_t, t \in T\}.$$

Any feasible point $x \in G$ is called a robust feasible solution to the problem (UN-

SIMFP) or a feasible solution to the (NRSIMFP) problem. Also, the feasible point $x \in G$ is called a robust efficient solution to the problem (UNSIMFP) if and only if it is an efficient solution to the problem (NRSIMFP). Moreover, for this model, It is assumed that the functions $f_i(x) \geq 0$ and $g_i(x) > 0$ for all $x \in G$ and for each $i = 1, \dots, k$, and the functions $h_t(x, v_t)$ with $x \in G, v_t \in V_t, t \in T$ are upper semi-continuous with respect to its second argument.

In actual real-life applications, it is sometimes better, based on the available data, to deal with a local efficient solution rather than a global efficient one. Furthermore, it makes sense to strengthen the notion of the local efficiency of a vector optimization problem to the notion of local isolated efficiency in which there exists a neighborhood of the efficient solution so that it is the only locally efficient in that neighborhood.

Below, are different concepts of local efficiency for a multi-objective optimization problem such as local efficient, local properly efficient solution (Chuong and Yao [27]), and the local isolated efficient solution (Ginchev et al. [43]) are extended to the situation of multi-objective fractional optimization problem in the literature. The following extensions will be considered in this chapter.

Definition 3.2 [23] *Take $x^* \in G$ to be a feasible solution. Then x^* is considered to be a local efficient solution of the problem (NRSIMFP) if and only if there exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* so that*

$$F(x^*) - F(x) \notin \mathbb{R}_+^k - \{0\}, \forall x \in G \cap N.$$

Definition 3.3 [92] Take $x^* \in G$ to be a feasible solution. Then x^* is considered to be a local isolated efficient solution of the problem (NRSIMFP) if and only if there exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* and a positive constant $\omega > 0$ so that

$$\max_{1 \leq i \leq k} \{F_i(x) - F_i(x^*)\} \geq \omega \|x - x^*\|, \quad \forall x \in G \cap N.$$

Where $F_i(\cdot) = \frac{f_i(\cdot)}{g_i(\cdot)}$, $i = 1, \dots, k$.

Definition 3.4 [92] Take $x^* \in G$ to be a feasible solution. Then x^* is considered to be a local properly efficient solution of the problem (NRSIMFP) if and only if there exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* and $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ so that

$$\langle \alpha, F(x) \rangle \geq \langle \alpha, F(x^*) \rangle, \quad \forall x \in G \cap N.$$

Remark 3 In case we have $N = \mathbb{R}^n$, the above concepts of local efficiency become the concepts of global efficiency, that is the efficient solution, isolated efficient solution, and properly efficient solution of the problem (NRSIMFP) respectively.

3.2 Non-smooth Robust Necessary Optimality Conditions

This section is focused on investigating a non-smooth robust necessary optimality condition for both local isolated and local properly efficient solutions of the non-smooth robust semi-infinite multi-objective fractional problems (NRSIMFP). Sun et

al. in [98] defined a robust type constraint qualification (Definition 3.2) to study uncertain semi-infinite programming problems. An extension of this definition is presented by Pham [85] and is considered here in the next lines.

Let $\Delta \subset \mathbb{R}^n$ be a locally closed subset of $\mathbb{R}^n$. For a non-negative multiplier $\mu = (\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ and uncertain parameters $v_t \in V_t$, $t \in T$, denote the set of active constraint multipliers at $x^* \in \Delta$ by $J(x^*)$ in which it is defined as follows:

$$J(x^*) = \{\mu \in \mathbb{R}_+^{(T)} : \mu_t h_t(x^*, v_t) = 0, \forall v_t \in V_t, t \in T\}.$$

Definition 3.5 [85] *Let $x^* \in G$. The robust type constraint qualification (RCQ) is satisfied at x^* if and only if the following holds:*

$$N^C(x^*; G) \subseteq \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta).$$

Where $J(x^*)$ is the set of active constraint multipliers at $x^* \in \Delta$ and $v_t \in V_t$, $t \in T$ are the uncertain parameters..

3.2.1 Robust Necessary Conditions for Locally Isolated Efficiency and Locally Properly Efficiency

Upcoming, under the satisfying of the robust type constraint qualification (RCQ), we first start with the non-smooth robust necessary optimality condition for local isolated efficient solutions of the problem (NRSIMFP).

Theorem 3.1 Take $x^* \in G$ to be a local isolated efficient solution of the (NRSIMFP) problem for some positive constant $\omega > 0$. Assume that the robust type constraint qualification (RCQ) holds at x^* , then there exist $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ with $\sum_{i=1}^k \alpha_i = 1$ and $\mu \in J(x^*)$ with uncertain parameters $v_t \in V_t, t \in T$ such that

$$\omega \mathbb{B} \subset \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta). \quad (3.1)$$

Proof. Consider x^* to be a local isolated efficient solution to the problem (NRSIMFP). Take the next scalar optimization problem:

$$\min_{x \in G} \Phi(x), \quad (3.2)$$

where $\Phi(x)$ is defined as follows:

$$\Phi(x) = \max_{1 \leq i \leq k} \left\{ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right\} - \omega \|x - x^*\|, \quad \forall x \in \mathbb{R}^n.$$

As x^* is a local isolated efficient solution to (NRSIMFP), then there exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* in such a way that

$$\Phi(x) \geq \Phi(x^*) = 0, \quad \forall x \in G \cap N.$$

Hence, x^* is a local minimizer of the problem (3.2). Therefore, x^* can be identified as

a local minimizer of the given unconstrained minimization problem.

$$\min_{x \in \mathbb{R}^n} \Phi(x) + \delta(x; G).$$

Applying Fermat's rule in its non-smooth version ([82] - Proposition 1.114) to the previous problem yields that

$$0 \in \widehat{\partial}(\Phi + \delta(\cdot; G))(x^*). \quad (3.3)$$

Define $\Phi_1(x) := \max_{1 \leq i \leq k} \left\{ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right\} + \delta(x; G)$ and $\Phi_2(x) := -\omega \|x - x^*\|$. Then the function $\Phi + \delta(\cdot; G)$ can be written as $\Phi(x) + \delta(\cdot; G) = \Phi_1(x) + \Phi_2(x)$ and then (3.3) becomes as follows:

$$0 \in \widehat{\partial}(\Phi_1 + \Phi_2)(x^*). \quad (3.4)$$

Note that the function $-\Phi_2$ is a convex function. Then it gives

$$\widehat{\partial}^+ \Phi_2(x^*) = -\widehat{\partial}(-\Phi_2)(x^*) = -\widehat{\partial}(\omega \|x - x^*\|)(x^*) = \omega \mathbb{B} \neq \emptyset.$$

Now, as $0 \in \widehat{\partial}(\Phi_1 + \Phi_2)(x^*)$, by utilizing Lemma 3.4 with $\widehat{\partial}^+ \Phi_2(x^*) = \omega \mathbb{B} \neq \emptyset$, the next result is satisfied:

$$0 \in \bigcap_{y \in \omega \mathbb{B}} [\widehat{\partial} \Phi_1(x^*) + y].$$

Which gives that $\omega \mathbb{B} \subset \widehat{\partial} \Phi_1(x^*) \subset \partial^C \Phi_1(x^*)$. Thus

$$\omega \mathbb{B} \subset \partial^C \left(\max_{1 \leq i \leq k} \left\{ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right\} + \delta(\cdot; G) \right)(x^*). \quad (3.5)$$

Since the function $\Psi(x) = \max_{1 \leq i \leq k} \left\{ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right\}$ is Locally Lipschitz function near x^* and the indicator function $\delta(\cdot; G)$ is lower semi-continuous near the same point x^* , then applying the sum rule in [82] to (3.5), we get that

$$\omega\mathbb{B} \subset \partial^C \Psi(x^*) + \partial^C \delta(x^*; G). \quad (3.6)$$

Then along with the result in Proposition 3.2, we obtain

$$\omega\mathbb{B} \subset \partial^C \Psi(x^*) + N^C(x^*; G). \quad (3.7)$$

This implies the following:

$$\omega\mathbb{B} \subset \partial^C \left(\max_{1 \leq i \leq k} \left\{ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right\} \right) (x^*) + N^C(x^*; G). \quad (3.8)$$

Further, employing Lemma 3.1 and the finite sum of Proposition 3.1, we get

$$\partial^C \left(\max_{1 \leq i \leq k} \left\{ \frac{f_i(\cdot)}{g_i(\cdot)} - \frac{f_i(x^*)}{g_i(x^*)} \right\} \right) (x^*) \subset \left\{ \sum_{i=1}^k \alpha_i \partial^C \left(\frac{f_i}{g_i} \right) (x^*) : \alpha_i \geq 0, i = 1, \dots, k, \sum_{i=1}^k \alpha_i = 1 \right\}.$$

By applying Lemma 3.3, we have

$$\begin{aligned} & \partial^C \left(\max_{1 \leq i \leq k} \left\{ \frac{f_i(\cdot)}{g_i(\cdot)} - \frac{f_i(x^*)}{g_i(x^*)} \right\} \right) (x^*) \\ & \subset \left\{ \sum_{i=1}^k \alpha_i \frac{g_i(x^*) \partial^C f_i(x^*) - f_i(x^*) \partial^C g_i(x^*)}{[g_i(x^*)]^2} : \alpha_i \geq 0, i = 1, \dots, k, \sum_{i=1}^k \alpha_i = 1 \right\} \\ & = \left\{ \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) : \alpha_i \geq 0, i = 1, \dots, k, \sum_{i=1}^k \alpha_i = 1 \right\}. \end{aligned} \quad (3.9)$$

Now, since the robust type constraint qualification (RCQ) is satisfied at x^* , then for uncertain parameters $v_t \in V_t$, $t \in T$, it leads to

$$N^C(x^*; G) \subseteq \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta). \quad (3.10)$$

Where $J(x^*)$ is the set of active constraint multipliers at $x^* \in \Delta$, i.e.,

$$J(x^*) = \{\mu = (\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)} : \mu_t h_t(x^*, v_t) = 0, \forall v_t \in V_t, t \in T\}.$$

Hence, for $\alpha \geq 0$, the relation (3.8) follows that

$$\begin{aligned} \omega \mathbb{B} \subset & \left\{ \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) \right\} \\ & + \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta). \end{aligned}$$

Thus, there exist $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ with $\sum_{i=1}^k \alpha_i = 1$ and $\mu \in J(x^*)$ with uncertain parameters $v_t \in V_t, t \in T$ such that

$$\omega \mathbb{B} \subset \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta). \quad (3.11)$$

Which brings out the completion of the proof. ■

The fulfilling of the robust type constraint qualification (RCQ) at the considered point is essential to obtain the result of Theorem 3.1. If this kind of constraint qualification is not satisfied, the above conclusion may fail to be held. The next example of a non-smooth robust semi-infinite bi-objective fractional programming

problem clarifies this significance.

Example 1

$$\min_{x \in \mathbb{R}} \left(\frac{x^3 - x^2}{|x| + 1}, \frac{2x^2}{|x| + 2} \right), \quad (UNSIMFP)$$

$$\text{Subject to } 2v_t x^2 \leq 0.$$

Consider the uncertain parameters $v_t \in V_t = [2 - t, 2 + t]$ with $t \in T = [0, 1]$.

Formulating the robust counterpart of the aforementioned problem as follows:

$$\min_{x \in \mathbb{R}} \left(\frac{x^3 - x^2}{|x| + 1}, \frac{2x^2}{|x| + 2} \right) \quad (NRSIMFP)$$

$$\text{Subject to } 2v_t x^2 \leq 0, \quad \forall v_t \in V_t = [2 - t, 2 + t], \quad t \in T = [0, 1], \quad x \in \mathbb{R}.$$

Let $f_1(x) = x^3 - x^2$, $g_1(x) = |x| + 1$, $f_2(x) = 2x^2$, $g_2(x) = |x| + 2$, $h_t(x, v_t) = 2v_t x^2$, $\forall t \in [0, 1]$, $x \in \mathbb{R}$, and $\Delta = (-\infty, 0] \subset \mathbb{R}$. Hence, it is clear that $G = \{0\}$ is the feasible set of the problem (NRSIMFP).

Considering the point $x^* = 0 \in G$, it is obvious to notice that $x^* = 0$ is a global isolated efficient solution to the problem (NRSIMFP) as well as the conclusion of Theorem 3.1 fails to hold.

In fact, for all $\omega > 0$ and $\forall x \in G$, we have

$$\max \left\{ \frac{x^3 - x^2}{|x| + 1} - 0 \right\} = \frac{x^3 - x^2}{|x| + 1} \geq \omega|x| = \omega\|x - x^*\|,$$

$$\max \left\{ \frac{2x^2}{|x| + 2} - 0 \right\} = \frac{2x^2}{|x| + 2} \geq \omega|x| = \omega\|x - x^*\|.$$

In addition, for $x^* = 0 \in G$, let $\omega = 1 > 0$ and $\mathbb{B} = [-1, 1]$. Consider the positive parameter $\alpha = (\alpha_1, \alpha_2) \in \text{int } \mathbb{R}_+^2$ such that $\alpha_1 + \alpha_2 = 1$. Then, for $i = 1, 2$, we get that $\partial^C f_i(x^*) = \partial^C f_i(0) = 0$ and $\partial_x^C h_t(x^*, v_t) = 0$, for any uncertain parameter $v_t \in V_t = [2 - t, 2 + t]$, $t \in T = [0, 1]$. Furthermore, we have that $N^C(x^*; \Delta) = N^C(0; (-\infty, 0]) = [0, \infty)$, and simply at $x^* = 0$, one has $\omega\mathbb{B} = [-1, 1]$ and

$$\sum_{i=1}^2 \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta) = [0, \infty).$$

Which gives the following:

$$\omega\mathbb{B} \not\subset \sum_{i=1}^2 \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta).$$

This failure is because that robust type constraint qualification (RCQ) at $x^* = 0$ does not hold. Indeed, for any $\mu \in J(x^*)$ and $v_t \in V_t$, $t \in T$ we have

$$N^C(x^*; G) = \mathbb{R} \not\subset [0, \infty) = \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta).$$

Next, we propose a non-smooth robust necessary optimality condition for local properly efficient solutions of the non-smooth robust semi-infinite multi-objective fractional problems (NRSIMFP) subject to the conditions of robust type constraint qualification (RCQ).

Theorem 3.2 Consider $x^* \in G$ to be a local properly efficient solution to the problem (NRSIMFP). Assume that the robust type constraint qualification (RCQ) is satisfied

at x^* , then there are positive multipliers $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ and uncertain parameters $v_t \in V_t$, $t \in T$ with $\mu = (\mu_t)_{t \in T} \in J(x^*)$ such that

$$0 \in \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta). \quad (3.12)$$

Proof. Consider $x^* \in G$ to be a local properly efficient solution to the problem (NRSIMFP). By the definition, there exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* and positive multipliers $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ such that

$$\langle \alpha, F(x) \rangle \geq \langle \alpha, F(x^*) \rangle, \quad \forall x \in G \cap N.$$

Which implies that $\forall x \in G \cap N$,

$$\sum_{i=1}^k \alpha_i \left(\frac{f_i(x)}{g_i(x)} \right) \geq \sum_{i=1}^k \alpha_i \left(\frac{f_i(x^*)}{g_i(x^*)} \right).$$

Hence,

$$\sum_{i=1}^k \alpha_i \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right] \geq 0, \quad \forall x \in G \cap N.$$

Setting $\Phi(x) = \sum_{i=1}^k \alpha_i \left(\frac{f_i(x)}{g_i(x)} \right)$, then x^* is evidently a local minimizer of the scalar minimization problem

$$\min_{x \in G} \Phi(x).$$

Since we have $\Phi(x)$ is locally Lipschitz at x^* , then by Lemma 3.2, we obtain

$$0 \in \partial^C \Phi(x^*) + N^C(x^*; G).$$

By employing the finite sum rule involved with positive real scalars $\alpha_i \geq 0$, $i = 1, \dots, k$ of Proposition 3.1, we get

$$\partial^C \Phi(x^*) = \partial^C \left(\sum_{i=1}^k \alpha_i \frac{f_i(\cdot)}{g_i} \right)(x^*) \subset \sum_{i=1}^k \alpha_i \partial^C \left(\frac{f_i}{g_i} \right)(x^*).$$

By Lemma 3.3, we get the next implication

$$\partial^C \Phi(x^*) \subset \sum_{i=1}^k \alpha_i \frac{g_i(x^*) \partial^C f_i(x^*) - f_i(x^*) \partial^C g_i(x^*)}{[g_i(x^*)]^2}. \quad (3.13)$$

Since the robust type constraint qualification (RCQ) fulfills at x^* . Then for uncertain parameters $v_t \in V_t$, $t \in T$ and $\mu = (\mu_t)_{t \in T} \in J(x^*)$, it yields

$$N^C(x^*; G) \subseteq \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta). \quad (3.14)$$

Therefore, combining the above results in (3.13) and (3.22), we have

$$0 \in \sum_{i=1}^k \alpha_i \frac{g_i(x^*) \partial^C f_i(x^*) - f_i(x^*) \partial^C g_i(x^*)}{[g_i(x^*)]^2} + \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta).$$

Thus, there exist positive scalars $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ and uncertain parameters $v_t \in V_t$, $t \in T$ with $\mu = (\mu_t)_{t \in T} \in J(x^*)$ such that

$$0 \in \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta).$$

Which concludes the proof. |

The given example further demonstrates the requirement for the robust type constraint qualification (RCQ) to be fulfilled at the considered point is crucial to obtain the results of Theorem 3.2.

Example 2 *Take into account the below non-smooth semi-infinite multi-objective fractional programming problem with $k = 2$ under uncertainty.*

$$\min_{x \in \mathbb{R}} \left(\frac{x}{|x| + 1}, \frac{x}{|x| + 1} \right), \quad (UNSIMFP)$$

$$\text{Subject to } v_t x^2 \leq 0.$$

Choose $v_t \in V_t = [2 - t, 2 + t]$ with $t \in T = [0, 1]$ to be uncertain parameters. Then the formulation of the robust counterpart of the problem (UNSIMFP) is as described next:

$$\min_{x \in \mathbb{R}} \left(\frac{x}{|x| + 1}, \frac{x}{|x| + 1} \right) \quad (NRSIMFP)$$

$$\text{Subject to } v_t x^2 \leq 0, \quad \forall v_t \in V_t = [2 - t, 2 + t], \quad t \in T = [0, 1], \quad x \in \mathbb{R}.$$

Clearly, we have $f_1(x) = f_2(x) = x$, $g_1(x) = g_2(x) = |x| + 1$ and $h_t(x, v_t) = v_t x^2$, $\forall t \in [0, 1]$, $x \in \mathbb{R}$ with $\Delta = (-\infty, 0] \subset \mathbb{R}$. Hence, it is simple to show that the feasible region of the problem (NRSIMFP) is given by the set $G = \{0\}$.

Take the point $x^* = 0 \in G = \{0\}$, then evidently we get that $x^* = 0$ is a global properly efficient solution to the problem (NRSIMFP) as for positive real scalars $\alpha_1, \alpha_2 \geq 0$

with $\alpha_1 + \alpha_2 = 1$, the following is satisfied at $x^* = 0$.

$$\sum_{i=1}^2 \alpha_i \frac{f_i(x)}{g_i(x)} \geq \sum_{i=1}^2 \alpha_i \frac{f_i(x^*)}{g_i(x^*)}.$$

Further, for $x^* = 0 \in G$, we have $\partial^C f_i(x^*) = \partial^C f_i(0) = 1$, $i = 1, 2$, and for any uncertain parameter $v_t \in V_t$, $t \in T$, $\partial_x^C h_t(x^*, v_t) = \partial_x^C h_t(0, v_t) = 0$. Now, since $N^C(x^*; \Delta) = N^C(0; (-\infty, 0]) = [0, \infty)$, we obtain the next calculation:

$$0 \notin \sum_{i=1}^2 \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta).$$

In fact,

$$\begin{aligned} & \sum_{i=1}^2 \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta) \\ &= \sum_{i=1}^2 \frac{\alpha_i}{g_i(0)} \left(\partial^C f_i(0) - \frac{f_i(0)}{g_i(0)} \partial^C g_i(0) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(0; (-\infty, 0]) \\ &= 1 + [0, \infty) = [1, \infty). \end{aligned}$$

Which yields that $0 \notin [1, \infty)$.

The reason behind this unsatisfied outcome is that the robust type constraint qualification (RCQ) at $x^* = 0$ is not satisfied as for any $\mu \in J(x^*)$ and $v_t \in V_t$, $t \in T$ we have

$$N^C(x^*; G) = \mathbb{R} \not\supseteq [0, \infty) = \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta).$$

Remark 4 *Contrary to the above example is not necessary to hold that a feasible point satisfying condition (3.12) is not necessarily a global properly efficient solution to the problem (NRSIMFP) although the robust type constraint qualification (RCQ) holds at the point under consideration. This will be shown in the next example.*

Example 3 *Take the following non-smooth robust semi-infinite bi-objective fractional programming problem.*

$$\min_{x \in \mathbb{R}} \left(\frac{x^3}{|x| + 1}, \frac{x^3}{|x| + 1} \right), \quad (NRSIMFP)$$

$$\text{Subject to } -v_t x^2 \leq 0, \quad \forall v_t \in V_t = [2 - t, 2 + t], \quad t \in T = [0, 1], \quad x \in \mathbb{R}.$$

Here we have $f_1(x) = f_2(x) = x^3$, $g_1(x) = g_2(x) = |x| + 1$ and $h_t(x, v_t) = -v_t x^2$, $\forall t \in [0, 1]$, $x \in \mathbb{R}$. Let $\Delta = (-\infty, 0] \subset \mathbb{R}$. Then, the feasible region $G = \mathbb{R}$.

Now, consider the feasible point $x^* = 0 \in G$, then clearly we have $\partial^C f_i(x^*) = \partial^C f_i(0) = 0$, $i = 1, 2$, and $\partial_x^C h_t(x^*, v_t) = \partial_x^C h_t(0, v_t) = 0$, $\forall v_t \in V_t$, $t \in T$. and $N^C(x^*; \Delta) = N^C(0; (-\infty, 0]) = [0, \infty)$, So, for all $(\mu_t)_{t \in T} \in J(x^*)$ and positive $\alpha = (\alpha_1, \alpha_2) \in \text{int } \mathbb{R}_+^2$ with uncertain parameters $v_t \in V_t$, $t \in T$, the condition (3.12) is satisfied as follows:

$$0 \in [0, \infty) = \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta).$$

Further, as $0 \in \text{int } \mathbb{R}$, we get that $N^C(x^*; G) = N^C(0; \mathbb{R}) = \{0\}$. Hence the robust

type constraint qualification (RCQ) fulfills at $x^* = 0$ in this way:

$$N^C(x^*; G) = \{0\} \subseteq [0, \infty) = \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta).$$

Nevertheless, $x^* = 0$ does not qualify as a global, properly efficient solution to the problem (NRSIMFP). To clarify, put $x = -1 \in G = \mathbb{R}$ and positive vector $(\alpha_1, \alpha_2) \geq 0$.

Hence, we get

$$\sum_{i=1}^2 \alpha_i \frac{f_i(x)}{g_i(x)} = -\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) < 0 = \sum_{i=1}^2 \alpha_i \frac{f_i(x^*)}{g_i(x^*)}.$$

This implies that x^* is not a global properly efficient solution to the problem (NRSIMFP).

3.2.2 Robust Necessary Conditions of Non-smooth Robust Semi-infinite Parametric Problem of (NRSIMFP)

This section will consider the convex case of the non-smooth semi-infinite multi-objective programming problem confined with uncertain data (UNSIMFP), that is when the functions $f_i, i = 1, \dots, k$ are convex functions with $f_i(x) \geq 0$ and the functions $g_i, i = 1, \dots, k$ are concave functions with $g_i(x) > 0$ for all $x \in G$. One method of dealing with this kind of problem is by employing the generalized Dinkelbach parametrization approach [29], associating the problem with a non-negative parameter vector $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}_+^k$. Since for $\lambda_i \geq 0$, each function of the form $f_i(x) - \lambda_i g_i(x)$, $i = 1, \dots, k$ is a convex function. Hence, consider the following uncertain non-smooth semi-infinite parametric convex programming problem in its robust

counterpart:

$$\min_{x \in \mathbb{R}^n} P(x) = \left(f_1(x) - \lambda_1 g_1(x), \dots, f_k(x) - \lambda_k g_k(x) \right) \quad (\text{NRSIMFP})_\lambda$$

$$\text{Subject to } (-h_t(x, v_t))_{t \in T} \in S, \quad \forall v_t \in V_t, \quad t \in T, \quad x \in \Delta.$$

Next, we present the definitions of local isolated efficient solution and local properly efficient solution for this considered problem,

Definition 3.6 Take $x^* \in G$ to be a feasible point of the problem $(\text{NRSIMFP})_\lambda$. Then

- (i) x^* is considered to be a local isolated efficient solution to the problem $(\text{NRSIMFP})_\lambda$ if and only if there exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* and a positive constant $\omega > 0$ such that

$$\max_{1 \leq i \leq k} \{ [f_i(x) - \lambda_i g_i(x)] - [f_i(x^*) - \lambda_i g_i(x^*)] \} \geq \omega \|x - x^*\|, \quad \forall x \in G \cap N.$$

- (ii) x^* is considered to be a local properly efficient solution of the problem (NRSIMFP) if and only if there exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* and $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ such that

$$\langle \alpha, P(x) \rangle \geq \langle \alpha, P(x^*) \rangle, \quad \forall x \in G \cap N.$$

Equivalently, we have

$$\sum_{i=1}^k \alpha_i (f_i(x) - \lambda_i g_i(x)) \geq \sum_{i=1}^k \alpha_i (f_i(x^*) - \lambda_i g_i(x^*)), \quad \forall x \in G \cap N.$$

Below, we propose an equivalent technique for determining the local isolated and local properly efficient solutions to the non-smooth robust semi-infinite multi-objective programming problem and its non-smooth robust parametric formulation constructed in the generalized Dinkelbach approach.

Lemma 3.5 *Let $x^* \in G$ and consider $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $i = 1, \dots, k$. Then x^* is referred to a local isolated (resp. local properly) efficient solution of (NRSIMFP) if and only if x^* is a local isolated (resp. local properly) efficient solution of the problem $(NRSIMFP)_{\lambda^*}$, with $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$.*

Proof. Firstly, we prove the local isolated efficient solution case as follows:

($\Rightarrow$) Take $x^* \in G$ to be a local isolated efficient solution of the problem (NRSIMFP) and consider $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, $i = 1, \dots, k$. Assume that $x^* \in G$ is not a local isolated efficient solution of the problem $(NRSIMFP)_{\lambda^*}$. Then, for any neighborhood N of x^* and $\omega > 0$, we have

$$\max_{1 \leq i \leq k} \{ [f_i(x) - \lambda_i^* g_i(x)] - [f_i(x^*) - \lambda_i^* g_i(x^*)] \} < \omega \|x - x^*\|, \quad \forall x \in G \cap N.$$

Since $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, for all $i = 1, \dots, k$, then exchanging λ_i^* with its value in the above inequality gives the following:

$$\max_{1 \leq i \leq k} \left\{ f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right\} < \omega \|x - x^*\|, \quad \forall x \in G \cap N.$$

Hence, for all $x \in G \cap N$, we get

$$\max_{1 \leq i \leq k} \left\{ g_i(x) \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right) \right\} < \omega \|x - x^*\|.$$

Since for all $x \in G$, $g_i(x) > 0$, $i = 1, \dots, k$, it follows that

$$\max_{1 \leq i \leq k} \left\{ \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right) \right\} < \omega^* \|x - x^*\|, \quad \forall x \in G \cap N.$$

Where $\omega^* = \frac{\omega}{\max\{g_i(x): 1 \leq i \leq k\}} > 0$.

Hence, we deduce that x^* is not a local isolated efficient solution to the problem (NRSIMFP) but this contradicts the assumption. Therefore, x^* is a local isolated efficient solution of the problem (NRSIMFP) $_{\lambda^*}$ with $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$.

($\Leftarrow$) Take $x^* \in G$ to be a local isolated efficient solution of the problem (NRSIMFP) $_{\lambda^*}$ where $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, $i = 1, \dots, k$. By contradiction, suppose that x^* is not a local isolated efficient solution to the problem (NRSIMFP). Thus, for any neighborhood N of x^* and $\omega > 0$, we have

$$\max_{1 \leq i \leq k} \left\{ \frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right\} < \omega \|x - x^*\|, \quad \forall x \in G \cap N.$$

Since, $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, for all $i = 1, \dots, k$, then the above inequality becomes

$$\max_{1 \leq i \leq k} \left\{ \frac{f_i(x)}{g_i(x)} - \lambda_i^* \right\} < \omega \|x - x^*\|, \quad \forall x \in G \cap N.$$

On the other hand, the equality $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$ also implies that $f_i(x^*) - \lambda_i^* g_i(x^*) = 0$, for

all $i = 1, \dots, k$. Then it yields

$$\max_{1 \leq i \leq k} \left\{ \frac{1}{g_i(x)} \left(f_i(x) - \lambda_i^* g_i(x) \right) - 0 \right\} < \omega \|x - x^*\|, \quad \forall x \in G \cap N.$$

It follows

$$\max_{1 \leq i \leq k} \left\{ [f_i(x) - \lambda_i^* g_i(x)] - [f_i(x^*) - \lambda_i^* g_i(x^*)] \right\} < \omega^* \|x - x^*\|, \quad \forall x \in G \cap N.$$

Where $\omega^* = \frac{\omega}{\tau} > 0$, with $\tau = \max\{\frac{1}{g_i(x)} : g_i(x) > 0, 1 \leq i \leq k\}$. This contradicts the assumption that x^* is a local isolated efficient solution of the problem $(\text{NRSIMFP})_{\lambda^*}$. Therefore, x^* is a local isolated efficient solution to the problem (NRSIMFP) which completes the first part of the proof.

Secondly, to prove the equivalence relation in the case of a local properly efficient solution, we proceed with the same procedure as in the local isolated efficient solution.

($\Rightarrow$) Let $x^* \in G$ be a local properly efficient solution to the problem (NRSIMFP) . To get a contradiction, assume that $x^* \in G$ is not a local properly efficient solution of the problem $(\text{NRSIMFP})_{\lambda^*}$. Then, for any neighborhood N of x^* and $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$, we have

$$\sum_{i=1}^k \alpha_i (f_i(x) - \lambda_i^* g_i(x)) < \sum_{i=1}^k \alpha_i (f_i(x^*) - \lambda_i^* g_i(x^*)), \quad \forall x \in G \cap N.$$

Since $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)} \geq 0$, for all $i = 1, \dots, k$, then it implies

$$\sum_{i=1}^k \alpha_i \left(f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right) < 0, \quad \forall x \in G \cap N.$$

Now, as for all $x \in G$, $g_i(x) > 0$, $i = 1, \dots, k$, the following holds.

$$\sum_{i=1}^k \alpha_i \frac{f_i(x)}{g_i(x)} < \sum_{i=1}^k \alpha_i \frac{f_i(x^*)}{g_i(x^*)}, \quad \forall x \in G \cap N.$$

This shows that x^* is not a local properly efficient solution to the problem (NRSIMFP) which contradicts the assumption. Therefore, x^* is a local properly efficient solution of the problem (NRSIMFP) $_{\lambda^*}$.

($\Leftarrow$) Let $x^* \in G$ be a local properly efficient solution of the problem (NRSIMFP) $_{\lambda^*}$ where $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ as well as $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, $i = 1, \dots, k$. To the contrary, suppose that x^* is not a local properly efficient solution to the problem (NRSIMFP). Thus, for any neighborhood N of x^* and any positive vector $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$, we have

$$\sum_{i=1}^k \alpha_i \frac{f_i(x)}{g_i(x)} < \sum_{i=1}^k \alpha_i \frac{f_i(x^*)}{g_i(x^*)}, \quad \forall x \in G \cap N. \quad (3.15)$$

But $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$ so we get that $f_i(x^*) - \lambda_i^* g_i(x^*) = 0$, for all $i = 1, \dots, k$.

On the other hand, the inequality (3.15) implies that

$$\sum_{i=1}^k \alpha_i \frac{f_i(x)}{g_i(x)} - \sum_{i=1}^k \alpha_i \lambda_i^* < 0, \quad \forall x \in G \cap N.$$

That is with $g_i(x) > 0$, $i = 1, \dots, k$,

$$\sum_{i=1}^k \alpha_i \left(f_i(x) - \lambda_i^* g_i(x) \right) < 0 = \sum_{i=1}^k \alpha_i \left(f_i(x^*) - \lambda_i^* g_i(x^*) \right), \quad \forall x \in G \cap N.$$

This contradicts the assumption that x^* is a local properly efficient solution of the

problem $(\text{NRSIMFP})_{\lambda^*}$. Thus, x^* is a local properly efficient solution to the problem (NRSIMFP) and the proof is completed. $\blacksquare$

The non-smooth robust necessary optimality conditions for obtaining a local isolated or local properly efficient solution to the problem (NRSIMFP) can be studied through its non-smooth robust semi-infinite parametric problem formulated in the generalized Dinkelbach approach subject to some conditions as it can be seen in the next theorems.

Theorem 3.3 *Take $x^* \in G$ to be a local isolated efficient solution of the problem (NRSIMFP) for some positive constant $\omega > 0$. Let $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, $i = 1, \dots, k$. Assume that the robust type constraint qualification (RCQ) is satisfied at x^* , and suppose that the functions $(f_i(x) - \lambda g_i(x))$, $i = 1, \dots, k$ are convex functions for all $x \in G$. Then there exist $\alpha := (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k$ with $\sum_{i=1}^k \alpha_i = 1$, $\mu = (\mu_t)_{t \in T} \in J(x^*)$ and uncertain parameters $v_t \in V_t, t \in T$ such that*

$$\omega \mathbb{B} \subset \sum_{i=1}^k \alpha_i \left(\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta).$$

Proof. Take x^* to be a local isolated efficient solution to the problem (NRSIMFP) and $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$. By Lemma 3.5, x^* is also a local isolated efficient solution of the parametric problem $(\text{NRSIMFP})_{\lambda^*}$. Define the function $\Phi(x)$ as follows:

$$\Phi(x) = \max_{1 \leq i \leq k} \left\{ \left(f_i(x) - \lambda g_i(x) \right) - \left(f_i(x^*) - \lambda g_i(x^*) \right) \right\} - \omega \|x - x^*\|, \quad \forall x \in \mathbb{R}^n.$$

As x^* is a local isolated efficient solution to the problem $(\text{NRSIMFP})_{\lambda^*}$, then there

exists a neighborhood $N \subseteq \mathbb{R}^n$ of x^* such that

$$\Phi(x) \geq \Phi(x^*) = 0, \quad \forall x \in G \cap N.$$

This means that x^* is a local minimizer of the scalar optimization problem

$$\min_{x \in G} \Phi(x).$$

In addition, Let $\Phi_1(x) = \max_{1 \leq i \leq k} \{ (f_i(x) - \lambda g_i(x)) - (f_i(x^*) - \lambda g_i(x^*)) \}$ and $\Phi_2(x) = \omega \|x - x^*\|$. Then clearly both functions Φ_1 and Φ_2 are convex functions and the function $\Phi = \Phi_1 - \Phi_2$ represents the difference of two convex functions. Thus by employing Corollary 3.1 in [6], we obtain the following.

$$\partial^C(\omega \|\cdot - x^*\|)(x^*) \subset \partial^C \left(\max_{1 \leq i \leq k} \{ (f_i(\cdot) - \lambda g_i(\cdot)) - (f_i(x^*) - \lambda g_i(x^*)) \} \right)(x^*) + N^C(x^*; G).$$

Since $\partial^C(\omega \|\cdot - x^*\|)(x^*) = \omega \mathbb{B}$ for all $\omega > 0$, then the above inclusion yields

$$\omega \mathbb{B} \subset \partial^C \left(\max_{1 \leq i \leq k} \{ (f_i(\cdot) - \lambda g_i(\cdot)) - (f_i(x^*) - \lambda g_i(x^*)) \} \right)(x^*) + N^C(x^*; G). \quad (3.16)$$

Furthermore, employing Lemma 3.1 and the finite sum of Proposition 3.1, we get

$$\begin{aligned} & \partial^C \left(\max_{1 \leq i \leq k} \{ (f_i(\cdot) - \lambda g_i(\cdot)) - (f_i(x^*) - \lambda g_i(x^*)) \} \right)(x^*) \\ & \subset \left\{ \sum_{i=1}^k \alpha_i (\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*)) : \alpha_i \geq 0, i = 1, \dots, k, \sum_{i=1}^k \alpha_i = 1 \right\}. \end{aligned} \quad (3.17)$$

By assumption, the robust type constraint qualification (RCQ) is satisfied at x^* and hence for uncertain parameters $v_t \in V_t$, $t \in T$, we have

$$N^C(x^*; G) \subseteq \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta). \quad (3.18)$$

Hence, utilizing results (3.17) and (3.18) in 3.16 implies that

$$\begin{aligned} \omega \mathbb{B} \subset & \left\{ \sum_{i=1}^k \alpha_i (\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*)) : \alpha_i \geq 0, i = 1, \dots, k, \sum_{i=1}^k \alpha_i = 1 \right\} \\ & + \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta). \end{aligned} \quad (3.19)$$

Thus, there exist $\alpha := (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k$ with $\sum_{i=1}^k \alpha_i = 1$ and $\mu \in J(x^*)$ with uncertain parameters $v_t \in V_t$, $t \in T$ such that

$$\omega \mathbb{B} \subset \sum_{i=1}^k \alpha_i \left(\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta). \quad (3.20)$$

This completes the proof. ■

Theorem 3.4 *Let $x^* \in G$ be a local properly efficient solution to the problem (NR-SIMFP). Let $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, $i = 1, \dots, k$. Assume that the robust type constraint qualification (RCQ) is satisfied at x^* , then there exist positive multipliers $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ and uncertain parameters $v_t \in V_t$, $t \in T$ with $\mu = (\mu_t)_{t \in T} \in J(x^*)$ such that*

$$0 \in \sum_{i=1}^k \alpha_i \left(\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta). \quad (3.21)$$

Proof. Consider $x^* \in G$ to be a local properly efficient solution to the problem (NR-SIMFP) and $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*)$ with $\lambda_i^* = \frac{f_i(x^*)}{g_i(x^*)}$, $i = 1, \dots, k$. According to Lemma 3.5, x^* is also a properly isolated efficient solution of the parametric problem $(\text{NRSIMFP})_{\lambda^*}$. Hence, there exists a neighborhood N of x^* and $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ such that $\forall x \in G \cap N$, we have

$$\sum_{i=1}^k \alpha_i (f_i(x) - \lambda_i g_i(x)) \geq \sum_{i=1}^k \alpha_i (f_i(x^*) - \lambda_i g_i(x^*)).$$

Let $\Phi(x) = \sum_{i=1}^k \alpha_i (f_i(x) - \lambda_i g_i(x))$. Then clearly the point x^* is a local minimizer of the scalar minimization problem

$$\min_{x \in G} \Phi(x).$$

Now, since $\Phi(x)$ is locally Lipschitz at x^* , then by Lemma 3.2, we obtain

$$0 \in \partial^C \Phi(x^*) + N^C(x^*; G).$$

Using the finite sum rule of Proposition 3.1, we get

$$\partial^C \Phi(x^*) = \partial^C \left(\sum_{i=1}^k \alpha_i (f_i(x^*) - \lambda_i g_i(x^*)) \right) \subset \sum_{i=1}^k \alpha_i \left(\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*) \right)$$

Besides, the fulfillment of the robust type constraint qualification (RCQ) at x^* guarantees that for uncertain parameters $v_t \in V_t$, $t \in T$ and $\mu = (\mu_t)_{t \in T} \in J(x^*)$, we

have

$$N^C(x^*; G) \subseteq \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta). \quad (3.22)$$

Therefore, the following holds.

$$0 \in \sum_{i=1}^k \alpha_i \left(\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*) \right) + \bigcup_{\mu \in J(x^*), v_t \in V_t} \left[\sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) \right] + N^C(x^*; \Delta).$$

So, there exist positive scalars $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ and uncertain parameters $v_t \in V_t$, $t \in T$ with $\mu = (\mu_t)_{t \in T} \in J(x^*)$ such that

$$0 \in \sum_{i=1}^k \alpha_i \left(\partial^C f_i(x^*) - \lambda^* \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta).$$

Which completes the proof. |

3.3 Non-smooth Robust Sufficient Optimality Conditions

In the present section, we establish the conditions of non-smooth robust sufficient optimality for a robust feasible point to represent a local isolated efficient solution or a local properly efficient solution of the non-smooth robust semi-infinite multi-objective fractional problem (NRSIMFP). To formulate these conditions, inspired by analogous concepts in [23], first, we present some definitions related to generalized convex functions as shown below.

Definition 3.7 Take $\Delta \subset \mathbb{R}^n$ and consider the function $F(x) = \left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_k(x)}{g_k(x)} \right)$ with $f_i, g_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, k$ be locally Lipschitz functions. Then The function $F(x)$ is referred to as a generalized convex function at $x^* \in \Delta$ if for each $x \in \Delta$, $\hat{\xi}_i \in \partial^C f_i(x^*)$ and $\hat{\eta}_i \in \partial^C g_i(x^*)$, $i = 1, \dots, k$, there is $u \in N^C(x^*; \Delta)^\circ$ so that

$$f_i(x) - f_i(x^*) \geq \langle \hat{\xi}_i, u \rangle, \quad i = 1, \dots, k.$$

$$g_i(x) - g_i(x^*) \geq \langle \hat{\eta}_i, u \rangle, \quad i = 1, \dots, k.$$

The function F is called a generalized locally convex function at $x^* \in \Delta$ if and only if there is a neighborhood N of x^* such that the set $N \cap \Delta$ is a convex set and F is a generalized convex function on $N \cap \Delta$.

Definition 3.8 Let $\Delta \subset \mathbb{R}^n$ and consider the function $h_t(x, v_t) : \mathbb{R}^n \times \mathbb{R}^q \rightarrow \mathbb{R}$, $t \in T$ be locally Lipschitz functions with respect to the first argument. Then the function $h_t(x, v_t), t \in T$ is called a generalized convex function at $x^* \in \Delta$ if for each $x \in \Delta$ and $\hat{\zeta}_t \in \partial^C h_t(x^*, v_t)$, $v_t \in V_t \subset \mathbb{R}^q, t \in T$, there exists $u \in N^C(x^*; \Delta)^\circ$ such that

$$h_t(x, v_t) - h_t(x^*, v_t) \geq \langle \hat{\zeta}_t, u \rangle, \quad t \in T.$$

The function $h_t(x, v_t), t \in T$ is called a generalized locally convex function at $x^* \in \Delta$ if and only if there is a neighborhood N of x^* so that $N \cap \Delta$ is a convex set and $h_t(x, v_t), t \in T$ is a generalized convex function on $N \cap \Delta$.

Remark 5 In the above definition, if the constraint function $h_t(x, v_t), t \in T$ is

uncertain-free with respect to the uncertain parameter $v_t \in V_t$, $t \in T$, that is $h_t(x, v_t) = h_t(x)$, $t \in T$, then Definition 3.8 reduces to the Definition 3.3 in [27].

It is clear to see that in case Δ is a convex set and each of f_i, g_i , $i = 1, \dots, k$ and $h_t(x, v_t)$, $t \in T$ are convex functions, then for each $x \in \Delta$, both functions $F(x)$ and $h_t(x, v_t)$, $t \in T$ are generalized convex functions at $x^* \in \Delta$ with $u = x - x^*$.

Next, we present the non-smooth robust sufficient optimality conditions for a local isolated efficient solution to the problem (NRSIMFP).

Theorem 3.5 *Consider $x^* \in G$ to be a robust feasible solution to the problem (NRSIMFP). For given positive constant $\omega > 0$, assume that there exist $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ such that $\sum_{i=1}^k \alpha_i = 1$ and $\mu \in J(x^*)$ as well as uncertain parameters $v_t \in V_t, t \in T$ so that (3.1) holds. In addition, suppose that the functions $F(x) = \left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_k(x)}{g_k(x)}\right)$ and $h_t(x, v_t), t \in T$ are generalized locally convex function at x^* . Then the point x^* is a local isolated efficient solution to the problem (NRSIMFP).*

Proof. Consider $x^* \in G$ to be a robust feasible solution to the problem (NRSIMFP) satisfying the hypothesis in the theorem for some $\omega > 0$. Note that the below condition is met for any $\beta \in \mathbb{R}^n$.

$$\|\beta\| = \max_{\hat{\beta} \in \mathbb{B}} \langle \hat{\beta}, \beta \rangle.$$

Hence, clearly, there exists $\hat{\beta} \in \mathbb{B}$ such that $\|\beta\| = \langle \hat{\beta}, \beta \rangle$.

On the other side, since $F(x)$ and $h_t(x, v_t), t \in T$ are assumed to be generalized locally convex function at x^* , then there exists a neighborhood N of x^* such that $N \cap \Delta$ is a

convex set and $F(x)$ and $h_t(x, v_t), t \in T$ are generalized convex functions on $N \cap \Delta$.

Take any $x \in N \cap G$, then there exists $\hat{x} \in \mathbb{B}$ such that

$$\|x - x^*\| = \langle \hat{x}, x - x^* \rangle. \quad (3.23)$$

Using the assumption, for $i = 1, \dots, k$, there exist $\alpha_i \geq 0$ with $\sum_{i=1}^k \alpha_i = 1$, $\mu \in J(x^*)$ and uncertain parameters $v_t \in V_t, t \in T$ such that the feasible point $x^* \in G$ satisfies (3.1). Hence, there exist $\hat{\xi}_i \in \partial^C f_i(x^*), \hat{\eta}_i \in \partial^C g_i(x^*), i = 1, \dots, k$ and $\hat{\zeta}_t \in \partial^C h_t(x^*, v_t), v_t \in V_t, t \in T$ such that we have the follows

$$\omega \hat{x} - \left[\sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\hat{\xi}_i - \frac{f_i(x^*)}{g_i(x^*)} \hat{\eta}_i \right) + \sum_{t \in T} \mu_t \hat{\zeta}_t \right] \in N^C(x^*; \Delta).$$

Utilizing the definition of Clarke normal cone to Δ at x^* with Δ is being locally closed set and $N \cap \Delta$ is a convex set, we obtain

$$\omega \langle \hat{x}, x - x^* \rangle - \left[\sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\langle \hat{\xi}_i, x - x^* \rangle - \frac{f_i(x^*)}{g_i(x^*)} \langle \hat{\eta}_i, x - x^* \rangle \right) + \sum_{t \in T} \mu_t \langle \hat{\zeta}_t, x - x^* \rangle \right] \leq 0.$$

Now by generalized locally convexity of the functions $F(x)$ and $h_t(x, v_t), t \in T$, and the equality in (3.23), we obtain the following result

$$\begin{aligned} \omega \|x - x^*\| &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} (g_i(x) - g_i(x^*)) \right] + \sum_{t \in T} \mu_t (h_t(x) - h_t(x^*)). \\ &= \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right] + \sum_{t \in T} \mu_t (h_t(x) - h_t(x^*)). \\ &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right]. \end{aligned}$$

The reason behind the above inequality is that for all uncertain parameters $v_t \in V_t$, $t \in T$, we have that $\mu_t h_t(x, v_t) \leq 0$ and $\mu_t h_t(x^*, v_t) = 0$.

Thus, the next lines are fulfilled.

$$\begin{aligned}
\omega \|x - x^*\| &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right]. \\
&\leq \sum_{i=1}^k \alpha_i \max_{1 \leq i \leq k} \left[\frac{1}{g_i(x^*)} \left(f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right) \right]. \\
&= \sum_{i=1}^k \alpha_i \max_{1 \leq i \leq k} \left[\frac{g_i(x)}{g_i(x^*)} \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right) \right]. \\
&= \max_{1 \leq i \leq k} \left[\frac{g_i(x)}{g_i(x^*)} \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right) \right].
\end{aligned}$$

Set $\omega^* = \frac{\omega}{m}$ where $m = \max_{1 \leq i \leq k} \left[\frac{g_i(x)}{g_i(x^*)} : x \in N \cap G \right] > 0$. Then it follows that for any $x \in N \cap G$ there exists a positive constant $\omega^* > 0$ such that

$$\omega^* \|x - x^*\| \leq \max_{1 \leq i \leq k} \left[\frac{f_i(x)}{g_i(x)} - \frac{f_i(x^*)}{g_i(x^*)} \right].$$

This implies that x^* is a local isolated efficient solution to the problem (NRSIMFP) and the proof is completed. ■

In the coming theorem, the conditions for the non-smooth robust sufficient optimality for a local properly efficient solution to the problem (NRSIMFP) will be discussed under the fulfillment of generalized local convexity of functions under consideration.

Theorem 3.6 *Consider $x^* \in G$ to be a robust feasible solution to the problem (NRSIMFP). Suppose that there exist $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ and uncertain parameters*

$v_t \in V_t, t \in T$ with $\mu = (\mu_t)_{t \in T} \in J(x^*)$ such that (3.12) holds. Assume that the functions $F(x) = \left(\frac{f_1(x)}{g_1(x)}, \dots, \frac{f_k(x)}{g_k(x)}\right)$ and $h_t(x, v_t), t \in T$ are generalized locally convex function at x^* . Then the feasible point x^* is a local properly efficient solution to the problem (NRSIMFP).

Proof. Take $x^* \in G$ to be a robust feasible solution to the problem (NRSIMFP).

Consider the existence of positive multipliers $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k, \mu \in J(x^*)$ and uncertain parameters $v_t \in V_t, t \in T$ so that (3.12) is satisfied at $x^* \in G$. Hence, there exist $\hat{\xi}_i \in \partial^C f_i(x^*), \hat{\eta}_i \in \partial^C g_i(x^*), i = 1, \dots, k$ and $\hat{\zeta}_t \in \partial^C h_t(x^*, v_t), v_t \in V_t, t \in T$ such that

$$-\left[\sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\hat{\xi}_i - \frac{f_i(x^*)}{g_i(x^*)} \hat{\eta}_i\right) + \sum_{t \in T} \mu_t \hat{\zeta}_t\right] \in N^C(x^*; \Delta).$$

By the definition of Clarke normal cone to Δ at x^* and the assumption of functions $F(x)$ and $h_t(x, v_t), t \in T$ are being generalized locally convex function at x^* , there exists a neighborhood N of x^* such that the set $N \cap \Delta$ is a convex set and $u \in N^C(x^*; \Delta)^\circ$ such that the following is fulfilled.

$$\begin{aligned} 0 &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\langle \hat{\xi}_i, u \rangle - \frac{f_i(x^*)}{g_i(x^*)} \langle \hat{\eta}_i, u \rangle\right) + \sum_{t \in T} \mu_t \langle \hat{\zeta}_t, u \rangle. \\ &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} (g_i(x) - g_i(x^*))\right] + \sum_{t \in T} \mu_t (h_t(x) - h_t(x^*)). \\ &= \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x)\right] + \sum_{t \in T} \mu_t (h_t(x) - h_t(x^*)). \\ &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x)\right]. \end{aligned}$$

Where $\mu_t h_t(x, v_t) \leq 0$ and $\mu_t h_t(x^*, v_t) = 0$ for all uncertain parameters $v_t \in V_t$, $t \in T$. This implies that

$$0 \leq \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left[f_i(x) - \frac{f_i(x^*)}{g_i(x^*)} g_i(x) \right].$$

Since for each $i = 1, \dots, k$, $g_i(x) > 0$ for all $x \in G$, we obtain

$$\sum_{i=1}^k \alpha_i \frac{f_i(x^*)}{g_i(x^*)} \leq \sum_{i=1}^k \alpha_i \frac{f_i(x)}{g_i(x)}.$$

Thus, we have the existence of a neighborhood $N \subseteq \mathbb{R}^n$ of x^* and a positive vector $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ such that $\forall x \in G \cap N$, we get

$$\langle \alpha, F(x^*) \rangle \leq \langle \alpha, F(x) \rangle.$$

This implies that x^* is a local properly efficient solution to the non-smooth robust semi-infinite multi-objective fractional problem (NRSIMFP).

I

Remark 6 *Note that if we have $N = \mathbb{R}^n$, then the concept of generalized locally convex functions coincides with the concepts of global generalized convex functions that can be used to investigate the non-smooth robust sufficient optimality conditions for a robust feasible point to be a global isolated efficient or a global properly efficient solution to the problem (NRSIMFP).*

Next, motivated by Example 3.3 in [92], we give an extension to the case when un-

certain data is involved within the constraints functions and illustrate the essentiality of generalized convexity assumption in Theorem 3.6 for a feasible point of the problem (NRSIMFP) to be a global properly efficient solution even if this point under consideration is fulfilling condition (3.12). The following example shows this result.

Example 4 *Consider the robust counterpart of the following non-smooth uncertain semi-infinite multi-objective fractional programming problem with $k = 2$.*

$$\min_{x \in \mathbb{R}} \quad F(x) = \left(\frac{-\arctan|x|}{xe^x + 1}, \frac{-\arctan|x|}{x^2 + 1} \right), \quad (NRSIMFP)$$

$$\text{Subject to } v_t x^3 \leq 0, \quad \forall v_t \in V_t = [2t, t], \quad t \in T = (-\infty, 0), \quad x \in \Delta = \mathbb{R}.$$

Clearly we have $f_1(x) = f_2(x) = -\arctan|x|$, $g_1(x) = xe^x + 1$, $g_2(x) = x^2 + 1$ and $h_t(x, v_t) = v_t x^3$, $\forall v_t \in [2t, t]$, $t \in (-\infty, 0)$, $x \in \mathbb{R}$. Then the feasible set is $G = [0, \infty)$.

Consider the feasible point $x^* = 0 \in G$. Thus, we have

$$\partial^C f_i(x^*) = \partial^C f_i(0) = [-1, 1], \quad i = 1, 2.$$

$$\partial^C g_1(x^*) = \partial^C g_1(0) = \{1\}, \quad \partial^C g_2(x^*) = \partial^C g_2(0) = \{0\}.$$

$$\partial_x^C h_t(x^*, v_t) = \partial_x^C h_t(0, v_t) = \{0\}, \quad \forall v_t \in V_t, \quad t \in T.$$

Moreover, we have $N^C(x^*; \Delta) = N^C(0; \mathbb{R}) = \{0\}$ so that its polar cone $N^C(0; \mathbb{R})^\circ = \{0\}^\circ = \mathbb{R}$. Hence, for all $(\mu_t)_{t \in T} \in J(x^*)$ and positive $\alpha = (\alpha_1, \alpha_2) \in \text{int } \mathbb{R}_+^2$ with uncertain parameters $v_t \in V_t$, $t \in T$, the condition (3.12) is satisfied at $x^* = 0$.

However, $x^* = 0$ is not a global properly efficient solution to the problem (NRSIMFP).

In fact, choose $x = 1 \in G = [0, \infty)$ and $(\alpha_1, \alpha_2) \in \text{int } \mathbb{R}_+^2$. Hence, the following is satisfied

$$\sum_{i=1}^2 \alpha_i \frac{f_i(x)}{g_i(x)} = -\beta(\alpha_1 + \alpha_2) < 0 = \sum_{i=1}^2 \alpha_i \frac{f_i(x^*)}{g_i(x^*)}.$$

Where $\beta = \frac{f_1(1)}{g_1(1)} + \frac{f_2(1)}{g_2(1)} = \frac{0.785}{e+1} + \frac{0.785}{2} > 0$. This is a consequence of that the functions $F(x)$ and $h_t(x, v_t)$, $v_t \in V_t$, $t \in T$ are not being generalized convex functions at $x^* = 0$. Indeed, there exists $x = 1 \in \mathbb{R}$, $\hat{\xi}_i = 0 \in \partial^C f_i(0) = [-1, 1]$, $i = 1, 2$, and $\hat{\zeta}_t = 0 \in \partial^C h_t(x^*, v_t) = \{0\}$, $v_t \in V_t, t \in T$ such that for all $u \in N^C(0; \mathbb{R})^\circ = \mathbb{R}$, and all $v_t \in [2t, t]$, $t \in (-\infty, 0)$ we have

$$f_i(x) - f_i(x^*) = f_i(1) - f_i(0) = -0.785 < 0 = \langle 0, u \rangle = \langle \hat{\xi}_i, u \rangle, \quad i = 1, \dots, k.$$

$$h_t(x, v_t) - h_t(x^*, v_t) = h_t(1, v_t) - h_t(0, v_t) = v_t < 0 = \langle 0, u \rangle = \langle \hat{\zeta}_t, u \rangle.$$

Remark 7 Observe that if $g_i(x) \equiv 1$, for all $i = 1, \dots, k$, then the non-smooth robust semi-infinite multi-objective fractional problem (NRSIMFP) reduces to the non-smooth robust semi-infinite multi-objective problem which is considered by Pham in [85]. In this paper, the author employed different types of generalized convexity functions, namely, pseudoconvex and quasiconvex functions, to establish a sufficient optimality condition for a global properly efficient solution to its studied problem.

3.4 Robust Duality Theorems of (NRSIMFP)

This section will formulate the robust dual problem of the non-smooth robust semi-infinite multi-objective fractional problem (NRSIMFP) using both Wolfe-type robust

duality and Mond-Weir-type robust duality. The study will examine the robust duality theorems including robust weak, robust strong, and robust strict converse theorems in relation to generalized convexity.

3.4.1 Wolfe-type Robust Dual Formulation

To begin with, we explore below the formulation of a Wolfe-type robust duality for robust local properly efficient solutions only as we can deal with local isolated efficient solutions in the same manner.

Let $z \in \mathbb{R}^n$ and $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ with $\sum_{i=1}^k \alpha_i = 1$. Consider we have the uncertain parameters $v_t \in V_t$, $t \in T$ with $\mu = (\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$. The generalized Lagrange function of the objective function and the uncertain constraint functions of the problem (NRSIMFP) are constructed as follows:

$$\bar{F}(z, \alpha, \mu, v_t) = F(z) + \sum_{t \in T} \mu_t h_t(z, v_t) e,$$

where $e = (1, \dots, 1) \in \mathbb{R}^k$ is the vector with all components being one.

Connecting to the primal non-smooth robust semi-infinite multi-objective fractional problem (NRSIMFP), the formulation of its Wolfe-type robust dual problem is in the next method.

$$\max_{z \in \mathbb{R}^n} \quad \bar{F}(z, \alpha, \mu, v_t) \quad (\text{WDNRSIMFP})$$

$$\begin{aligned}
\text{Subject to } \quad & 0 \in \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(\partial^C f_i(z) - \frac{f_i(z)}{g_i(z)} \partial^C g_i(z) \right) + \sum_{t \in T} \mu_t \partial_z^C h_t(z, v_t) + N^C(z; \Delta), \\
& \alpha = (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k, \quad \sum_{i=1}^k \alpha_i = 1, \quad z \in \Delta, \\
& \mu = (\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}, v_t \in V_t \subset \mathbb{R}^q, \quad t \in T.
\end{aligned}$$

The Wolfe-type robust dual problem feasible set is defined by

$$\begin{aligned}
G_{WD} = \Big\{ (z, \alpha, \mu, v_t) \in \Delta \times \text{int } \mathbb{R}_+^k \times \mathbb{R}_+^{(T)} \times \mathbb{R}^q : & 0 \in \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(\partial^C f_i(z) - \frac{f_i(z)}{g_i(z)} \partial^C g_i(z) \right) \\
& + \sum_{t \in T} \mu_t \partial_z^C h_t(z, v_t) + N^C(z; \Delta), \sum_{i=1}^k \alpha_i = 1, v_t \in V_t, t \in T \Big\}. \quad (3.24)
\end{aligned}$$

In the following, analogous to the primal case, we present the concepts of robust efficiency and robust local proper efficiency for this robust dual problem.

Definition 3.9 *A feasible solution $(z^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$ is considered to be an efficient solution of the problem (WDNRSIMFP) if and only if $\forall (z, \alpha, \mu, v_t) \in G_{WD}$ we have*

$$\bar{F}(z^*, \alpha^*, \mu^*, v_t^*) - \bar{F}(z, \alpha, \mu, v_t) \notin -\mathbb{R}_+^k - \{0\}.$$

Definition 3.10 *A feasible solution $(z^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$ is considered to be a properly efficient solution to the problem (WDNRSIMFP) if and only if there is $\alpha := (\alpha_1, \dots, \alpha_k) \in -\text{int } \mathbb{R}_+^k$ such that*

$$\langle \alpha, \bar{F}(z, \alpha, \mu, v_t) \rangle \geq \langle \alpha, \bar{F}(z^*, \alpha^*, \mu^*, v_t^*) \rangle, \quad \forall (z, \alpha, \mu, v_t) \in G_{WD}.$$

Turning now to the robust weak duality between the primal problem (NRSIMFP) and its Wolfe dual problem (WDNRSIMFP), this relation can be examined in the next theorem.

Theorem 3.7 (Robust Weak Wolfe-type Duality) *Consider $x \in G$ to be a feasible solution to the primal problem (NRSIMFP) and $(z, \alpha, \mu, v_t) \in G_{WD}$ be a feasible solution to the dual problem (WDNRSIMFP). Suppose that $F(\cdot) = (\frac{f_1(\cdot)}{g_1(\cdot)}, \dots, \frac{f_k(\cdot)}{g_k(\cdot)})$ and $h_t(\cdot, v_t), t \in T$ are generalized convex functions at z on Δ with uncertain parameters $v_t, t \in T$. Then the next statement is not valid.*

$$F(x) \leq \bar{F}(z, \alpha, \mu, v_t).$$

Proof. Let $(z, \alpha, \mu, v_t) \in G_{WD}$ be a feasible solution to the dual problem (WDNRSIMFP), then as it satisfies the robust Wolfe duality conditions, there exist positive scalars $\alpha_i > 0, i = 1, \dots, k$ with $\sum_{i=1}^k \alpha_i = 1$ and $\hat{\xi}_i \in \partial^C f_i(z), \hat{\eta}_i \in \partial^C g_i(z), i = 1, \dots, k$ as well as $\hat{\zeta}_t \in \partial^C h_t(z, v_t)$ with $v_t \in V_t, t \in T$ and $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ such that

$$-\left[\sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(\hat{\xi}_i - \frac{f_i(z)}{g_i(z)} \hat{\eta}_i \right) + \sum_{t \in T} \mu_t \hat{\zeta}_t \right] \in N^C(z; \Delta). \quad (3.25)$$

On the sake of the contrary, assume that the next inequality is true.

$$F(x) \leq \bar{F}(z, \alpha, \mu, v_t).$$

Which means $F(x) - \bar{F}(z, \alpha, \mu, v_t) \leq 0$. Now, since $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$, then it

implies that $\langle \alpha, F(x) - \bar{F}(z, \alpha, \mu, v_t) \rangle < 0$. The construction of $\bar{F}$ gives that

$$\langle \alpha, F(x) - F(z) - \sum_{t \in T} \mu_t h_t(z, v_t) e \rangle < 0$$

As $\langle \alpha, e \rangle = \sum_{i=1}^k \alpha_i = 1$, this implies

$$\sum_{i=1}^k \alpha_i \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(z)}{g_i(z)} \right) - \sum_{t \in T} \mu_t h_t(z, v_t) < 0. \quad (3.26)$$

On the other hand, using the definition of the polar cone and the assumption of $F(\cdot)$ and $h_t(\cdot, v_t), t \in T$ being generalized convex function at z on Δ , the relation (3.25) yields the existence of $u \in N^C(z; \Delta)^\circ$ such that for $x \in G$, we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(\langle \hat{\xi}_i, u \rangle - \frac{f_i(z)}{g_i(z)} \langle \hat{\eta}_i, u \rangle \right) + \sum_{t \in T} \mu_t \langle \hat{\zeta}_t, u \rangle \\ &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(f_i(x) - f_i(z) - \frac{f_i(z)}{g_i(z)} (g_i(x) - g_i(z)) \right) + \sum_{t \in T} \mu_t (h_t(x, v_t) - h_t(z, v_t)) \\ &\leq \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(f_i(x) - \frac{f_i(z)}{g_i(z)} g_i(x) \right) - \sum_{t \in T} \mu_t h_t(z, v_t). \end{aligned} \quad (3.27)$$

Where the last inequality is obtained due to the fact that a feasible point $x \in G$ fulfills the robust feasibility conditions, i.e., $\sum_{t \in T} \mu_t h_t(x, v_t) \leq 0$.

Thus, by simplification, we get this next inequality

$$0 \leq \sum_{i=1}^k \alpha_i \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(z)}{g_i(z)} \right) - \sum_{t \in T} \mu_t h_t(z, v_t).$$

Clearly, this contradicts (3.26), so the proof is completed. |

Next, in the sense of Wolfe-type formulation, we will explore the robust strong duality theorem which examines the relationship between the (NRSIMFP) primal problem and its (WDNRSIMFP) Wolfe-type dual problem considering the absence of a duality gap and supposing the fulfillment of (RCQ), the robust type constraint qualification.

Theorem 3.8 (Robust Strong Wolfe-type Duality) *Consider $x^* \in G$ to be a local properly efficient solution to the primal (NRSIMFP). Assuming that the robust type constraint qualification (RCQ) is satisfied at x^* , then there are $\alpha^* := (\alpha_1^*, \dots, \alpha_k^*) \in \text{int } \mathbb{R}_+^k$ and $\mu^* \in \mathbb{R}_+^{(T)}$ with uncertain parameters $v_t^* \in V_t, t \in T$ so that $(x^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$ is considered as a feasible point of the robust Wolfe dual problem (WDNRSIMFP) and $F(x^*) = \bar{F}(x^*, \alpha^*, \mu^*, v_t^*)$. Moreover, if the functions $F(\cdot)$ and $h_t(\cdot, v_t), t \in T$ are generalized convex functions at any z on Δ . Then the solution $(x^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$ is concluded to be an efficient solution to the Wolfe-type dual problem (WDNRSIMFP).*

Proof. Since by assumption, x^* is a local properly efficient solution satisfying the robust type constraint qualification (RCQ). Then there is positive multipliers $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$ and uncertain parameters $v_t \in V_t, t \in T$ with $\mu = (\mu_t)_{t \in T} \in J(x^*) = \{\mu \in \mathbb{R}_+^{(T)} : \mu_t h_t(x^*, v_t) = 0, \forall v_t \in V_t, t \in T\}$. such that

$$0 \in \sum_{i=1}^k \frac{\alpha_i}{g_i(x^*)} \left(\partial^C f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \partial^C g_i(x^*) \right) + \sum_{t \in T} \mu_t \partial_x^C h_t(x^*, v_t) + N^C(x^*; \Delta). \quad (3.28)$$

Now consider the following:

$$\alpha_i^* = \frac{\alpha_i}{\sum_{i=1}^k \alpha_i}, i = 1, \dots, k, \quad \mu_t^* = \frac{\mu_t}{\sum_{i=1}^k \alpha_i}, t \in T,$$

Hence, one can observe that $\alpha^* = (\alpha_1^*, \dots, \alpha_k^*) \in \text{int } \mathbb{R}_+^k$ with $\sum_{i=1}^k \alpha_i^* = \frac{\sum_{i=1}^k \alpha_i}{\sum_{i=1}^k \alpha_i} = 1$, and $\mu^* = (\mu_t^*)_{t \in T} \in \mathbb{R}_+^{(T)}$. By inserting α_i^* 's and μ_t^* 's in (3.28) instead of α_i 's and μ_t 's, respectively, the relation still holds.

Further, as $\mu = (\mu_t)_{t \in T} \in J(x^*)$, we have that $\mu_t h_t(x^*, v_t) = 0, \forall v_t \in V_t, t \in T$, and hence for uncertain parameter $v_t^* \in V_t, t \in T$, $\sum_{t \in T} \mu_t h_t(x^*, v_t^*) = 0$. Thus, clearly, the point $(x^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$ is a feasible solution of the robust Wolfe dual problem (WDNRSIMFP), as well as the following, is true.

$$\bar{F}(x^*, \alpha^*, \mu^*, v_t^*) = F(x^*) + \sum_{t \in T} \mu_t^* h_t(x^*, v_t^*) e = F(x^*). \quad (3.29)$$

Moreover, since $F(\cdot)$ and $h_t(\cdot, v_t), t \in T$ are generalized convex functions at any z on Δ . Then by the result of the robust weak Wolfe-type duality (Theorem 3.7), the following does not hold.

$$F(x^*) \leq \bar{F}(z, \alpha, \mu, v_t).$$

Therefore, by (3.29), we obtain also that the following does not hold.

$$\bar{F}(x^*, \alpha^*, \mu^*, v_t^*) \leq \bar{F}(z, \alpha, \mu, v_t), \quad \forall (z, \alpha, \mu, v_t) \in G_{WD}.$$

This concludes that the feasible solution $(x^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$ is an efficient solution of the (WDNRSIMFP) Wolfe-type dual problem. This completes the proof. ■

Remark 8 *As we have seen in Theorem 3.8, the optimal solution of the robust Wolfe-type dual problem is an efficient solution in spite of that the solution of the primal*

problem is a properly efficient one. Mainly, having a properly efficient solution for the (NRSIMFP) primal problem does not guarantee to gain a properly efficient solution for the (WDNRSIMFP) dual problem. This is still true even in the case of a convex situation and having a singleton uncertain parameter set V_t with finite index set T , (see [22], Example 4.6).

Another robust duality theorem that describes the connection between (NRSIMFP) problem and its Wolfe-type dual problem (WDNRSIMFP) will be given below.

Theorem 3.9 (Robust Converse Wolfe-type Duality) *Consider*

$(x^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$ to be a feasible point to the dual problem (WDNRSIMFP). If $x^* \in G$ and the functions $F(\cdot)$ and $h_t(\cdot, v_t)$, $t \in T$ are generalized convex functions on Δ with uncertain parameters $v_t \in V_t$, $t \in T$. Then x^* is considered to be a properly efficient solution to the primal (NRSIMFP).

Proof. Let $(x^*, \alpha^*, \mu^*, v_t^*) \in G_{WD}$. Then, there exist $\alpha^* := (\alpha_1^*, \dots, \alpha_k^*) \in \text{int } \mathbb{R}_+^k$ with $\sum_{i=1}^k \alpha_i^* = 1$ and $\mu^* = (\mu_t^*)_{t \in T} \in \mathbb{R}_+^{(T)}$ with uncertain parameters $v_t^* \in V_t$, $t \in T$ as well as $\hat{\xi}_i \in \partial^C f_i(x^*)$, $\hat{\eta}_i \in \partial^C g_i(x^*)$, $i = 1, \dots, k$ together with $\hat{\zeta}_t \in \partial^C h_t(x^*, v_t^*)$ such that

$$-\left[\sum_{i=1}^k \frac{\alpha_i^*}{g_i(x^*)} \left(\hat{\xi}_i - \frac{f_i(x^*)}{g_i(x^*)} \hat{\eta}_i \right) + \sum_{t \in T} \mu_t^* \hat{\zeta}_t \right] \in N^C(x^*; \Delta). \quad (3.30)$$

Let $x^* \in G$ and assume for the contrary that x^* is not a properly efficient solution to (NRSIMFP). Then for such existing α^* , it implies that there is $y \in G$, another

feasible solution in such way that

$$\langle \alpha^*, F(y) \rangle < \langle \alpha^*, F(x^*) \rangle.$$

Which is equivalent to

$$\sum_{i=1}^k \alpha_i^* \left(\frac{f_i(y)}{g_i(y)} \right) < \sum_{i=1}^k \alpha_i^* \left(\frac{f_i(x^*)}{g_i(x^*)} \right).$$

Thus, we obtain

$$\sum_{i=1}^k \alpha_i^* \left(\frac{f_i(y)}{g_i(y)} - \frac{f_i(x^*)}{g_i(x^*)} \right) < 0. \quad (3.31)$$

Further, according to the definition of Clarke normal cone to the set Δ at x^* , there exists $u \in N^C(x^*; \Delta)^\circ$ so that (3.30) fulfills the following:

$$\sum_{i=1}^k \frac{\alpha_i^*}{g_i(x^*)} \left(\langle \hat{\xi}_i, u \rangle - \frac{f_i(x^*)}{g_i(x^*)} \langle \hat{\eta}_i, u \rangle \right) + \sum_{t \in T} \mu_t \langle \hat{\zeta}_t, u \rangle \geq 0$$

Now, since the functions $F(\cdot)$ and $h_t(\cdot, v_t), t \in T$ are generalized convex functions at any x^* on Δ , then for such $y \in G \subseteq \Delta$, $\hat{\xi}_i \in \partial^C f_i(x^*)$, $\hat{\eta}_i \in \partial^C g_i(x^*)$ for $i = 1, \dots, k$, $\hat{\zeta}_t \in \partial^C h_t(x^*, v_t^*)$ with $v_t^* \in V_t, t \in T$ and $u \in N^C(x^*; \Delta)^\circ$, we get

$$\begin{aligned} 0 &\leq \sum_{i=1}^k \frac{\alpha_i^*}{g_i(x^*)} \left(\langle \hat{\xi}_i, u \rangle - \frac{f_i(x^*)}{g_i(x^*)} \langle \hat{\eta}_i, u \rangle \right) + \sum_{t \in T} \mu_t \langle \hat{\zeta}_t, u \rangle. \\ &\leq \sum_{i=1}^k \frac{\alpha_i^*}{g_i(x^*)} \left[f_i(y) - f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} (g_i(y) - g_i(x^*)) \right] + \sum_{t \in T} \mu_t (h_t(y) - h_t(x^*)). \\ &= \sum_{i=1}^k \frac{\alpha_i^*}{g_i(x^*)} \left[f_i(y) - \frac{f_i(x^*)}{g_i(x^*)} g_i(y) \right] + \sum_{t \in T} \mu_t (h_t(y) - h_t(x^*)). \end{aligned}$$

$$\leq \sum_{i=1}^k \frac{\alpha_i^*}{g_i(x^*)} \left[f_i(y) - \frac{f_i(x^*)}{g_i(x^*)} g_i(y) \right].$$

The last inequality is a consequence of that for $y \in G$, we have by the primal feasibility condition that $\mu_t h_t(y, v_t) \leq 0$ and $\mu_t h_t(x^*, v_t) = 0$ for all uncertain parameters $v_t \in V_t$, $t \in T$. Hence, after some simplification along the assumption that $g_i(y) > 0$ for all $y \in G$, we get

$$\sum_{i=1}^k \alpha_i^* \frac{f_i(x^*)}{g_i(x^*)} \leq \sum_{i=1}^k \alpha_i^* \frac{f_i(y)}{g_i(y)}.$$

Which implies

$$\sum_{i=1}^k \alpha_i^* \left(\frac{f_i(y)}{g_i(y)} - \frac{f_i(x^*)}{g_i(x^*)} \right) \geq 0. \quad (3.32)$$

Clearly, (3.32) contradicts (3.31). Thus, x^* is a properly efficient solution to the problem (NRSIMFP). ■

3.4.2 Mond-Weir-type Robust Dual Formulation

Moving on to the Mond-Weir-type robust dual formulation of the non-smooth robust semi-infinite multi-objective fractional problem (NRSIMFP) for robust local properly efficient solutions, we start with the next construction:

Let $z \in \mathbb{R}^n$ and $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$. Take uncertain parameters $v_t \in V_t$, $t \in T$ and $\mu = (\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$. Define the following multi-objective fractional function:

$$\bar{F}(z, \alpha, \mu, v_t) := F(z).$$

Hence, the formulation of a Mond-Weir-type robust dual problem of (NRSIMFP) is conducted as follows:

$$\max_{z \in \mathbb{R}^n} \quad \bar{F}(z, \alpha, \mu, v_t) = \left(\frac{f_1(z)}{g_1(z)}, \dots, \frac{f_k(z)}{g_k(z)} \right) \quad (\text{MWDNRSIMFP})$$

$$\begin{aligned} \text{Subject to} \quad & 0 \in \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(\partial^C f_i(z) - \frac{f_i(z)}{g_i(z)} \partial^C g_i(z) \right) + \sum_{t \in T} \mu_t \partial_z^C h_t(z, v_t) + N^C(z; \Delta), \\ & \sum_{t \in T} \mu_t h_t(z, v_t) \geq 0, \quad z \in \Delta, \quad v_t \in V_t \subset \mathbb{R}^q, \quad t \in T, \\ & \alpha = (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k, \quad \mu = (\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}. \end{aligned}$$

The Mond-Weir-type robust dual problem feasible set is defined by

$$\begin{aligned} G_{MWD} = \left\{ (z, \alpha, \mu, v_t) \in \Delta \times \text{int } \mathbb{R}_+^k \times \mathbb{R}_+^{(T)} \times \mathbb{R}^q : 0 \in \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(\partial^C f_i(z) - \frac{f_i(z)}{g_i(z)} \partial^C g_i(z) \right) \right. \\ \left. + \sum_{t \in T} \mu_t \partial_z^C h_t(z, v_t) + N^C(z; \Delta), \quad \sum_{t \in T} \mu_t h_t(z, v_t) \geq 0, \quad v_t \in V_t, \quad t \in T \right\}. \quad (3.33) \end{aligned}$$

Observe that the concepts of robust efficiency and robust proper efficiency for this robust dual problem constructed in Mond-Weir-type are defined exactly as indicated earlier for the case of Wolfe-type robust dual problem.

In the following, we present the robust duality theorems, including robust weak, robust strong, and robust converse Mond-Weir duality theorems. These theorems are used to examine the relationship between the primal problem (NRSIMFP) and its

dual problem (MWDNRSIMFP).

Theorem 3.10 (Robust Weak Mond-Weir-type Duality) *Consider $x \in G$ to be a feasible point to the primal problem (NRSIMFP) and $(z, \alpha, \mu, v_t) \in G_{MWD}$ be a feasible point to the dual problem (MWDNRSIMFP). Assume that $F(\cdot) = (\frac{f_1(\cdot)}{g_1(\cdot)}, \dots, \frac{f_k(\cdot)}{g_k(\cdot)})$ and $h_t(\cdot, v_t), t \in T$ are generalized convex functions at z on Δ with uncertain parameters $v_t, t \in T$. Then the next statement is not valid.*

$$F(x) \leq \bar{F}(z, \alpha, \mu, v_t).$$

Proof. Let $(z, \alpha, \mu, v_t) \in G_{MWD}$, then it satisfies the robust Mond-Weir duality conditions and hence there are positive constants $\alpha_i > 0, i = 1, \dots, k$ with $\hat{\xi}_i \in \partial^C f_i(z), \hat{\eta}_i \in \partial^C g_i(z), i = 1, \dots, k$ and $\hat{\zeta}_t \in \partial^C h_t(z, v_t)$ with $v_t \in V_t, t \in T$ and $(\mu_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ in a manner that

$$-\left[\sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(\hat{\xi}_i - \frac{f_i(z)}{g_i(z)} \hat{\eta}_i \right) + \sum_{t \in T} \mu_t \hat{\zeta}_t \right] \in N^C(z; \Delta). \quad (3.34)$$

$$\sum_{t \in T} \mu_t h_t(z, v_t) \geq 0. \quad (3.35)$$

To prove by contradiction, assume that for $x \in G$, the following holds.

$$F(x) \leq \bar{F}(z, \alpha, \mu, v_t).$$

Since $\alpha := (\alpha_1, \dots, \alpha_k) \in \text{int } \mathbb{R}_+^k$, and $F(x) - \bar{F}(z, \alpha, \mu, v_t) \leq 0$, one can get that

$\langle \alpha, F(x) - \bar{F}(z, \alpha, \mu, v_t) \rangle < 0$. which yields

$$\sum_{i=1}^k \alpha_i \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(z)}{g_i(z)} \right) < 0. \quad (3.36)$$

Now, by the concepts of the normal and polar cone and the assumption of $F(\cdot)$ and $h_t(\cdot, v_t), t \in T$ being generalized convex function at z on Δ , we proceed similar to the argument of Theorem 3.7, hence, we obtain

$$0 \leq \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(f_i(x) - \frac{f_i(z)}{g_i(z)} g_i(x) \right) - \sum_{t \in T} \mu_t (h_t(x, v_t) - h_t(z, v_t)).$$

By the feasibility of $x \in G$ and the fulfillment of (3.35), the above inequality implies

$$0 \leq \sum_{i=1}^k \frac{\alpha_i}{g_i(z)} \left(f_i(x) - \frac{f_i(z)}{g_i(z)} g_i(x) \right).$$

Which simplified to

$$0 \leq \sum_{i=1}^k \alpha_i \left(\frac{f_i(x)}{g_i(x)} - \frac{f_i(z)}{g_i(z)} \right).$$

Clearly, we get a contradiction to (3.36); hence, the proof is completed. ■

Theorem 3.11 (Robust Strong Mond-Weir-type Duality) *Consider $x^* \in G$ a local properly efficient point to the primal problem (NRSIMFP). Assume that the robust type constraint qualification (RCQ) is satisfied at x^* . Then there exist $\alpha^* := (\alpha_1^*, \dots, \alpha_k^*) \in \text{int } \mathbb{R}_+^k$ and $\mu^* = (\mu_t^*)_{t \in T} \in \mathbb{R}_+^{(T)}$ with uncertain parameters $v_t^* \in V_t, t \in T$ so that $(x^*, \alpha^*, \mu^*, v_t^*) \in G_{MWD}$ and $F(x^*) = \bar{F}(x^*, \alpha^*, \mu^*, v_t^*)$. Furthermore, if the functions $F(\cdot)$ and $h_t(\cdot, v_t), t \in T$ are generalized convex functions at any z on Δ .*

Then $(x^*, \alpha^*, \mu^*, v_t^*)$ is an efficient solution to dual problem of the Mond-Weir-type (MWDNRSIMFP).

Proof. It is similar to the demonstration of Theorem 3.8. |

Theorem 3.12 (Robust Converse Mond-Weir-type Duality) *Consider*

$(x^*, \alpha^*, \mu^*, v_t^*) \in G_{MWD}$ to be a feasible point to the dual problem (MWDNRSIMFP). If $x^* \in G$ and the functions $F(\cdot)$ and $h_t(\cdot, v_t), t \in T$ are generalized convex functions at any x^* on Δ with uncertain parameters $v_t \in V_t, t \in T$. Then the solution x^* is considered to be a properly efficient solution to the primal problem (NRSIMFP).

Proof. It is similar to the demonstration of Theorem 3.9. |

CHAPTER 4

ROBUST VECTOR OPTIMIZATION PROBLEMS WITH VANISHING CONSTRAINTS

An optimization problem with vanishing constraints is a form of mathematical programming problem that has been interesting to many researchers for more than a decade. Optimization of truss topology and other structural topological problems can be reconstructed using mathematical programming with vanishing constraints. This type of optimization was a foundation established by Achtziger and Kanzow [2], and Hoheisel and Kanzow ([51], [52], [53]). Dorsch et al. [35] presented the concept of a T-stationary point which is a suitable stationarity concept for topological aspects of mathematical programming. In the literature, many researchers have

approved that this difficult form of mathematical programming under some moderate assumptions violates most of the standard constraint qualifications, for instance, the linear independence constraint qualification, Mangasarian-Fromovitz constraint qualification, and the Slater constraint qualification. For more details, the reader can consult ([2], [53], [78]). The notion of duality has a significant role in optimization in order to search the lower bound and find out the optimal solution of the mathematical programming with vanishing constraints, see Mishra et al. [79] and Ahmad et al. [5]. Recently, some works of minimization programming have been continued to the case of vector optimization problems with equality, inequality, and vanishing constraints in which the objectives are conflicting under some set of constraints, the reader refers to Mishra et al. [78] where some suitable modifications of constraint qualification were introduced and the relationships among them were studied for multi-objective optimization problem with vanishing constraints. Shaker et al. [90] studied the necessary optimality condition for properly efficient solutions in convex multi-objective optimization with non-differentiable vanishing constraints. Employing semi-infinite mathematical programming with vanishing constraints has been of interest to many researchers such as Guu et al. [50] in which the concept of strong KKT optimality conditions has been extended to the multi-objective semi-infinite programming problems with vanishing constraints without involving any constraint qualification. Tung [101] investigated the Karush–Kuhn–Tucker optimality conditions and explored the weak and strong duality results for the multiobjective semi-infinite programming with vanishing constraints. Antczak [9] extended the definition of S-stationarity to the case of smooth semi-infinite multiobjective optimization problems with vanishing

constraints. Further, the necessary optimality conditions for such differentiable vector optimization problems with respect to such stationary points were formulated and the sufficient optimality conditions were proved under the invexity assumptions. For further related information, consult [62].

In [2], Achtziger and Kanzow established the KKT necessary optimality conditions for the scalar mathematical programming with vanishing constraints under the standard Abadie constraint qualification. The Guignard constraint qualification is considered to be the weakest constraint qualification among all the other constraint qualifications ensuring that the KKT conditions are first-order necessary conditions for efficiency, that is at a local (global) efficient solution there exist Lagrange multipliers such that the standard KKT necessary conditions hold. This has been proved for the vectorial case of mathematical programming with vanishing constraints in Theorem 6.2 of [78]. In this work, we study the robust first-order necessary and sufficient optimality conditions based on the concept of robust strong stationary points in vector optimization problems under uncertain data. Further, we construct a robust auxiliary linearized vector problem using a linearization approach and investigate the relation between the problems under consideration. This chapter is organized by introducing first the essential notations that will be useful throughout the chapter in Section 4.1. Then Section 4.2 explores the linearization approach that could be utilized to describe and solve the considered problem. Sections 4.3 and 4.4 establish the robust first-order stationarity type necessary and sufficient optimality conditions respectively. Finally, two numerical examples are introduced in Section 4.5 to illustrate our basic results.

4.1 Preliminaries

Let $x \in \mathbb{R}^n$ and take into account the following vector optimization problem involving conditions with uncertain data and vanishing constraints:

$$\min_{x \in \mathbb{R}^n} F(x) = \left(f_1(x), f_2(x), \dots, f_k(x) \right) \quad (\text{VP})$$

$$\begin{aligned} \text{Subject to} \quad & g_j(x, v_j) \leq 0, & j \in J = \{1, 2, \dots, m\}, \\ & h_i(x, w_i) = 0, & i \in I = \{1, 2, \dots, r\}, \\ & H_p(x) \geq 0, & p \in P = \{1, 2, \dots, q\}, \\ & G_p(x)H_p(x) \leq 0, & p \in P = \{1, 2, \dots, q\}, \end{aligned}$$

Where the functions $f_t : \mathbb{R}^n \rightarrow \mathbb{R}$, $t \in T = \{1, \dots, k\}$, $H_p, G_p : \mathbb{R}^n \rightarrow \mathbb{R}$, $p \in P = \{1, 2, \dots, q\}$ are continuously differentiable functions. The inequality and equality constraint functions $g_j, h_i : \mathbb{R}^n \times \mathbb{R}^s \rightarrow \mathbb{R}$, $j \in J = \{1, 2, \dots, m\}$ and $i \in I = \{1, 2, \dots, r\}$ are continuously differentiable functions and confined with uncertain parameters $v_j \in V_j \subseteq \mathbb{R}^s$ and $w_i \in W_i \subseteq \mathbb{R}^s$, respectively. The subsets V_j and W_i of $\mathbb{R}^s$ are called the uncertainty sets which are assumed to be convex and compact.

Due to the uncertainty in constraint functions, we utilize the robust approach to considering the worst-cases of constraints and formulate the following robust counter-

part of the problem (VP):

$$\min_{x \in \mathbb{R}^n} F(x) = \left(f_1(x), f_2(x), \dots, f_k(x) \right) \quad (\text{RVP})$$

$$\text{Subject to} \quad g_j(x, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, 2, \dots, m,$$

$$h_i(x, w_i) = 0, \quad \forall w_i \in W_i, \quad i = 1, 2, \dots, r,$$

$$H_p(x) \geq 0, \quad p = 1, 2, \dots, q,$$

$$G_p(x)H_p(x) \leq 0, \quad p = 1, 2, \dots, q,$$

The following set can define all the feasible solutions to the problem (RVP):

$$Q = \{x \in \mathbb{R}^n : g_j(x, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, \dots, m,$$

$$h_i(x, w_i) = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r,$$

$$H_p(x) \geq 0, \quad p = 1, 2, \dots, q,$$

$$G_p(x)H_p(x) \leq 0, \quad p = 1, 2, \dots, q\}.$$

Any feasible solution for the problem (RVP) is a robust feasible solution for (VP).

The concepts of efficiency and weak efficiency in solving the considered problems are important tools to determine the optimal solutions in vector optimization. A feasible solution $x^* \in Q$ is called an efficient [a weakly efficient] solution to (RVP) or (a robust efficient [a robust weakly efficient] solution to (VP)) if there does not exist any $x \in Q$ such that $F(x) \leq F(x^*)$ [$F(x) < F(x^*)$].

Turning to some essential tools we need to employ in studying mathematical programming with vanishing constraints, we define specific index sets for any feasible solution $x^* \in Q$. The following index sets concern the indices related to the sign of function H_p at the point x^* .

$$\Upsilon_+(x^*) = \{p \in P : H_p(x^*) > 0\},$$

$$\Upsilon_0(x^*) = \{p \in P : H_p(x^*) = 0\}.$$

Further, the following index subsets of both $\Upsilon_+(x^*)$ and $\Upsilon_0(x^*)$, are related to the signs of the functions H_p and G_p for $p = 1, 2, \dots, q$ simultaneously.

$$\Upsilon_{+0}(x^*) = \{p \in P : H_p(x^*) > 0, G_p(x^*) = 0\},$$

$$\Upsilon_{+-}(x^*) = \{p \in P : H_p(x^*) > 0, G_p(x^*) < 0\},$$

$$\Upsilon_{00}(x^*) = \{p \in P : H_p(x^*) = 0, G_p(x^*) = 0\},$$

$$\Upsilon_{0+}(x^*) = \{p \in P : H_p(x^*) = 0, G_p(x^*) > 0\},$$

$$\Upsilon_{0-}(x^*) = \{p \in P : H_p(x^*) = 0, G_p(x^*) < 0\}.$$

Let $\Omega \in \mathbb{R}^n$ be any subset in $\mathbb{R}^n$. Then, the notations *span* Ω and *conv* Ω are denoted the linear hull and convex hull of Ω , while *cone* Ω and *convex* Ω stand for cone and convex cone generated by Ω that containing the origin, respectively. Further,

For any subsets Ω_1 and Ω_2 in $\mathbb{R}^n$, we have

$$\text{span} (\Omega_1 \cup \Omega_2) = \text{span} (\Omega_1) + \text{span} (\Omega_2), \quad (4.1)$$

$$\text{convex} (\Omega_1 \cup \Omega_2) = \text{convex} (\Omega_1) + \text{convex} (\Omega_2), \quad (4.2)$$

The positive dual cone of $\Omega \in \mathbb{R}^n$ is defined by $\Omega^* = \{a \in \mathbb{R}^n : \langle a, b \rangle \geq 0, \forall b \in \Omega\}$.

Similarly, both negative Ω° and strictly negative Ω° polar cones of $\Omega \in \mathbb{R}^n$ can be defined as follows:

$$\Omega^\circ = \{a \in \mathbb{R}^n : \langle a, b \rangle \leq 0, \forall b \in \Omega\}.$$

$$\Omega^\circ = \{a \in \mathbb{R}^n : \langle a, b \rangle < 0, \forall b \in \Omega - \{0\}\}.$$

The perpendicular complement of Ω is defined by

$$\Omega^\perp = \{a \in \mathbb{R}^n : \langle a, b \rangle = 0, \forall b \in \Omega\}.$$

Next, we recall some useful Lemmas that will be used in the proofs later.

Lemma 4.1 [89] *Consider the non-empty convex subsets S_i of $\mathbb{R}^n$ for some i and let $S = \text{convex}\left(\bigcup S_i\right)$. Then any non-zero element in S is expressed as a non-negative linear combination of elements of different S_i in which the multipliers are all non-negative.*

Lemma 4.2 [77] *Let M, N , and Z be arbitrary finite index sets. consider the mappings*

$$\gamma_m : M \rightarrow \mathbb{R}^n, \quad \gamma_n : N \rightarrow \mathbb{R}^n, \quad \gamma_z : Z \rightarrow \mathbb{R}^n.$$

Assume that the set $\text{conv} \{\gamma_m : m \in M\} + \text{convex} \{\gamma_n : n \in N\} + \text{span} \{\gamma_z : z \in Z\}$ is closed. Then the next statements are equivalent

(i) The system

$$\begin{cases} \langle \gamma_m, y \rangle < 0, & m \in M, \quad M \neq \emptyset. \\ \langle \gamma_n, y \rangle \leq 0, & n \in N. \\ \langle \gamma_z, y \rangle = 0, & z \in Z. \end{cases}$$

has no solution $y \in \mathbb{R}^n$.

(ii) $0 \in \text{conv} \{\gamma_m : m \in M\} + \text{convex} \{\gamma_n : n \in N\} + \text{span} \{\gamma_z : z \in Z\}$.

4.2 Linearization Approach for (RVP)

One way of solving such problems is by using the linearization approach which can be described as the following:

Take $\bar{x} \in Q$ to be an arbitrary solution of (RVP). Then we constructed its robust auxiliary vector problem as follows:

$$\min_{x \in \mathbb{R}^n} L(x) = \left(f_1(\bar{x}) + (x - \bar{x})^T \nabla f_1(\bar{x}), \dots, f_k(\bar{x}) + (x - \bar{x})^T \nabla f_k(\bar{x}) \right) \quad (\text{RVP}(\bar{x}))$$

$$\text{Subject to} \quad g_j(\bar{x}, v_j) + (x - \bar{x})^T \nabla_x g_j(\bar{x}, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, 2, \dots, m,$$

$$h_i(\bar{x}, w_i) + (x - \bar{x})^T \nabla_x h_i(\bar{x}, w_i) = 0, \quad \forall w_i \in W_i, \quad i = 1, 2, \dots, r,$$

$$H_p(\bar{x}) + (x - \bar{x})^T \nabla H_p(\bar{x}) \geq 0, \quad p = 1, 2, \dots, q,$$

$$\begin{aligned}
& G_p(\bar{x})H_p(\bar{x}) + (x - \bar{x})^T G_p(\bar{x})\nabla H_p(\bar{x}) \\
& + (x - \bar{x})^T \nabla G_p(\bar{x})H_p(\bar{x}) \leq 0, \quad p = 1, 2, \dots, q.
\end{aligned}$$

Clearly, $L(x)$ is a convex function as it is a vector linear function in which each component is a linear function. Denote the feasible set consisting of elements satisfying all the above constraints of the problem $(RVP(\bar{x}))$ by $Q(\bar{x})$ which is defined by

$$\begin{aligned}
Q(\bar{x}) = \{x \in \mathbb{R}^n : & g_j(\bar{x}, v_j) + (x - \bar{x})^T \nabla_x g_j(\bar{x}, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, \dots, m, \\
& h_i(\bar{x}, w_i) + (x - \bar{x})^T \nabla_x h_i(\bar{x}, w_i) = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r, \\
& H_p(\bar{x}) + (x - \bar{x})^T \nabla H_p(\bar{x}) \geq 0, \quad p = 1, 2, \dots, q, \\
& G_p(\bar{x})H_p(\bar{x}) + (x - \bar{x})^T G_p(\bar{x})\nabla H_p(\bar{x}) \\
& + (x - \bar{x})^T \nabla G_p(\bar{x})H_p(\bar{x}) \leq 0, \quad p = 1, 2, \dots, q\}.
\end{aligned}$$

The concepts of efficiency for the above vector linear (convex) programming problem can be presented in the coming definitions:

Definition 4.1 Take x^* to be a feasible solution to the problem $(RVP(\bar{x}))$. Then

1. A feasible solution x^* is considered to be a local weakly efficient solution to the problem $(RVP(\bar{x}))$ if there is a neighborhood $\mathbb{N} \subseteq \mathbb{R}^n$ such that there does not exist any $x \in Q(\bar{x}) \cap \mathbb{N}$ satisfying $L(x) < L(x^*)$. That is

$$f_t(\bar{x}) + (x - \bar{x})^T \nabla f_t(\bar{x}) < f_t(\bar{x}) + (x^* - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\}.$$

This is equivalent to

$$(x - \bar{x})^T \nabla f_t(\bar{x}) < (x^* - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\}.$$

It is called a weakly efficient solution to the problem $(RVP(\bar{x}))$ if $\mathbb{N} = \mathbb{R}^n$.

2. *A feasible solution x^* is considered to be a local efficient solution to the problem $(RVP(\bar{x}))$ if there is a neighborhood $\mathbb{N} \subseteq \mathbb{R}^n$ such that there does not exist any $x \in Q(\bar{x}) \cap \mathbb{N}$ satisfying $L(x) \leq L(x^*)$. That is*

$$f_t(\bar{x}) + (x - \bar{x})^T \nabla f_t(\bar{x}) \leq f_t(\bar{x}) + (x^* - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\},$$

$$f_{\hat{t}}(\bar{x}) + (x - \bar{x})^T \nabla f_{\hat{t}}(\bar{x}) < f_{\hat{t}}(\bar{x}) + (x^* - \bar{x})^T \nabla f_{\hat{t}}(\bar{x}), \quad \text{for some } \hat{t} \in T \text{ with } \hat{t} \neq t.$$

This is equivalent to

$$(x - \bar{x})^T \nabla f_t(\bar{x}) \leq (x^* - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\},$$

$$(x - \bar{x})^T \nabla f_{\hat{t}}(\bar{x}) < (x^* - \bar{x})^T \nabla f_{\hat{t}}(\bar{x}), \quad \text{for some } \hat{t} \in T \text{ with } \hat{t} \neq t.$$

It is called an efficient solution to the problem $(RVP(\bar{x}))$ if $\mathbb{N} = \mathbb{R}^n$.

The next theorem is a generalization form of Theorem 3.4.3 in [11] to the vector case which will be useful subsequently.

Theorem 4.1 *Let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a differentiable convex vector-valued function.*

Consider a minimization problem of $\gamma(x) = (\gamma_1(x), \dots, \gamma_k(x))$ subject to $x \in \mathbb{X}$ where

$\mathbb{X}$ is a nonempty convex subset of $\mathbb{R}^n$. The solution $y \in \mathbb{X}$ is a weakly efficient solution to the problem if and only if the following fulfills:

$$\nabla\gamma(y)(x - y) \geq 0, \quad \forall x \in \mathbb{X}.$$

Next, we present the relation between the problem (RVP) and its robust linearized form (RVP($\bar{x}$)) around the feasible point $\bar{x} \in Q$.

Proposition 4.1 *Consider $\bar{x} \in Q$ to be a feasible solution to the problem (RVP). Suppose that the functions $g_j(\cdot, v_j), v_j \in V_j, j = 1, \dots, m$ and $h_i(\cdot, w_i), w_i \in W_i, i = 1, \dots, r$ are convex functions confined with uncertain parameters $v_j \in V_j$ and $w_i \in W_i$ respectively. Moreover, suppose that the functions $-H_p(\cdot), G_p(\cdot)$ and $G_p(\cdot)H_p(\cdot), p = 1, \dots, q$ are also convex functions on $\mathbb{R}^n$. Then, each feasible solution $y \in Q$ of the problem (RVP) is a feasible solution of the problem (RVP($\bar{x}$)). That is*

$$y \in Q \Rightarrow y \in Q(\bar{x}), \quad \bar{x} \in Q$$

Proof. First, consider the problem (RVP) and take $\bar{x} \in Q$ be an arbitrary feasible solution in (RVP). Now, take $y \in Q \subseteq \mathbb{R}^n$ to be any feasible solution to the problem (RVP). Due to the assumption that $g_j(\cdot, v_j), h_i(\cdot, w_i), -H_p(\cdot), G_p(\cdot)$ and $G_p(\cdot)H_p(\cdot)$ with uncertain parameters $v_j \in V_j, j = 1, \dots, m$ and $w_i \in W_i, i = 1, \dots, r, p = 1, \dots, q$ are all convex functions on $\mathbb{R}^n$, the following inequalities are satisfied.

$$g_j(y, v_j) \geq g_j(\bar{x}, v_j) + (y - \bar{x})^T \nabla_y g_j(\bar{x}, v_j), \quad \forall v_j \in V_j, j = 1, \dots, m,$$

$$h_i(y, w_i) \geq h_i(\bar{x}, w_i) + (y - \bar{x})^T \nabla_y h_i(\bar{x}, w_i), \quad \forall w_i \in W_i, \quad i = 1, \dots, r,$$

$$-H_p(y) \geq -H_p(\bar{x}) - (y - \bar{x})^T \nabla H_p(\bar{x}), \quad p = 1, \dots, q,$$

$$G_p(y)H_p(y) \geq G_p(\bar{x})H_p(\bar{x}) + (y - \bar{x})^T G_p(\bar{x})\nabla H_p(\bar{x}) + (y - \bar{x})^T \nabla G_p(\bar{x})H_p(\bar{x}), \quad p = 1, \dots, q.$$

Since $y \in Q$ is a feasible solution of (RVP), then we have

$$g_j(y, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, \dots, m,$$

$$h_i(y, w_i) = 0 \Rightarrow h_i(y, w_i) \leq 0 \text{ and } -h_i(y, w_i) \leq 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r,$$

$$H_p(y) \geq 0 \Rightarrow -H_p(y) \leq 0, \quad p = 1, \dots, q,$$

$$G_p(y)H_p(y) \leq 0, \quad p = 1, \dots, q.$$

Thus, we get

$$g_j(\bar{x}, v_j) + (y - \bar{x})^T \nabla_y g_j(\bar{x}, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, \dots, m,$$

$$h_i(\bar{x}, w_i) + (y - \bar{x})^T \nabla_y h_i(\bar{x}, w_i) = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r,$$

$$H_p(\bar{x}) + (y - \bar{x})^T \nabla H_p(\bar{x}) \geq 0, \quad p = 1, \dots, q,$$

$$G_p(\bar{x})H_p(\bar{x}) + (y - \bar{x})^T G_p(\bar{x})\nabla H_p(\bar{x}) + (y - \bar{x})^T \nabla G_p(\bar{x})H_p(\bar{x}) \leq 0, \quad p = 1, \dots, q.$$

This yields that y is also a feasible solution to the problem (RVP($\bar{x}$)). Hence $y \in Q(\bar{x})$,

and the proof is completed. I

Now, we show that the (local) weak efficiency of the point $\bar{x}$ in the problem $(RVP(\bar{x}))$ leads to the same (local) weak efficiency of $\bar{x}$ in the considered robust vector optimization problem with vanishing constraints (RVP) .

Theorem 4.2 *Consider $\bar{x} \in Q$ to be a weakly efficient solution to the problem $(RVP(\bar{x}))$. Suppose that the functions $f_t(\cdot), t = 1, \dots, k, g_j(\cdot, v_j), v_j \in V_j, j = 1, \dots, m, h_i(\cdot, w_i), w_i \in W_i, i = 1, \dots, r, -H_p(\cdot), G_p(\cdot)$ and $G_p(\cdot)H_p(\cdot), p = 1, \dots, q$ are all convex functions on $\mathbb{R}^n$ where $v_j \in V_j$ and $w_i \in W_i$ are uncertain parameters. Hence, $\bar{x}$ is also a weakly efficient solution to the (RVP) problem.*

Proof. Take $\bar{x} \in Q$ to be a weakly efficient solution to the problem $(RVP(\bar{x}))$. Then, by the definition, there is no $x \in Q(\bar{x})$ fulfilling that

$$f_t(\bar{x}) + (x - \bar{x})^T \nabla f_t(\bar{x}) < f_t(\bar{x}) + (\bar{x} - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\}.$$

Which is equivalent to

$$(x - \bar{x})^T \nabla f_t(\bar{x}) < 0, \quad \forall t \in T = \{1, \dots, k\}.$$

Now as the functions $g_j(\cdot, v_j), h_i(\cdot, w_i), -H_p(\cdot), G_p(\cdot)$ and $G_p(\cdot)H_p(\cdot)$ are convex functions on $\mathbb{R}^n$, Then by Proposition 4.1, each feasible solution in Q of the problem (RVP) is a feasible solution in $Q(\bar{x})$ of the problem $(RVP(\bar{x}))$. Thus, we can deduce that there is no feasible solution $x \in Q$ such that

$$(x - \bar{x})^T \nabla f_t(\bar{x}) < 0, \quad \forall t \in T = \{1, \dots, k\}. \quad (4.3)$$

To prove our result, on the other hand, suppose by contradiction that $\bar{x}$ is not a weakly efficient solution to the problem (RVP). Hence, there is a feasible point $x \in Q$ such that

$$f_t(x) < f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\}. \quad (4.4)$$

By assumption, the functions $f_t(\cdot)$ are convex on $\mathbb{R}^n$ for all $t = 1, \dots, k$. This implies that for $x \in Q \subseteq \mathbb{R}^n$, the following is fulfilled

$$f_t(x) \geq f_t(\bar{x}) + (x - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\}. \quad (4.5)$$

By utilizing relation (4.4) in (4.5), we yield

$$f_t(\bar{x}) > f_t(\bar{x}) + (x - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t \in T = \{1, \dots, k\}.$$

Hence, from the above inequality, we deduce that there is a feasible solution $x \in Q$ such that

$$(x - \bar{x})^T \nabla f_t(\bar{x}) < 0, \quad \forall t \in T = \{1, \dots, k\}. \quad (4.6)$$

Clearly, relation (4.6) contradicts (4.3). Thus, we deduce that $\bar{x}$ is a weakly efficient solution to the problem (RVP). ■

Remark 9 *A similar outcome to the above theorem could be gained to the efficient solution case of the point $\bar{x}$ in the same way.*

The next theorem shows that in general under some generalized convexity hypothesis on the objective function, any weakly efficient solution to the problem (RVP($\bar{x}$)) is

also a weakly efficient solution to the problem (RVP).

Theorem 4.3 *Consider $\bar{x} \in Q$ to be a feasible point in the problem (RVP). Assume that the vector-valued function F is a pseudoconvex function. If $\bar{x}$ is a weakly efficient solution to the problem $(RVP(\bar{x}))$, then $\bar{x}$ is also a weakly efficient solution to the problem (RVP).*

Proof. Consider $\bar{x} \in Q$ to be a feasible solution to the problem (RVP) assuming that $\bar{x}$ is a weakly efficient solution to the problem $(RVP(\bar{x}))$. Proceeding the proof with a contradiction, assume that $\bar{x}$ is not a weakly efficient solution to the problem (RVP), hence, there is another feasible solution $x \in Q$ such that $F(x) < F(\bar{x})$. Since F is a pseudoconvex function, then we get

$$\nabla F(\bar{x})(x - \bar{x}) < 0 \quad (4.7)$$

Further, the function L is a convex vector function satisfying the following

$$\nabla L(\bar{x}) = \left(\nabla f_1(\bar{x}), \dots, \nabla f_k(\bar{x}) \right) = \nabla F(\bar{x})$$

Thus the inequality (4.7) yields

$$\nabla L(\bar{x})(x - \bar{x}) < 0$$

By Theorem 4.1, we obtain that $\bar{x}$ is not a weakly efficient solution to the problem $(RVP(\bar{x}))$, and hence, the assumption is contradicted. This completes the proof. ■

4.3 Robust First-Order Stationarity Type Necessary Optimality Conditions

This section is dedicated to extending the definition of strong stationary points introduced by Achtziger and Kanzow [2], and Hoheisel and Kanzow ([51], [52]) for the differentiable scalar mathematical programming problem with vanishing constraints with regard to the vector optimization problems with vanishing constraints under uncertainty. It has been shown in ([2], [51]) that the standard KKT conditions of an optimization problem with vanishing constraints are equivalent to strong-stationary conditions, and hence strong stationarity is a necessary optimality condition under all constraint qualifications that imply a special type of constraint condition say the Guignard constraint qualification, for more details, consult ([52], [53]).

Now, motivated by the strong-stationarity notion defined in the above references, we first consider our problem(RVP) and introduce the concepts of strong-stationarity. Then, we apply the results to the problem (RVP($\bar{x}$)) that linearized around a feasible point $\bar{x} \in Q$ of the original robust counterpart problem (RVP). This could be done in the next definitions:

Definition 4.2 *A feasible point $x^* \in Q$ is considered to be a robust strong-stationary point of the problem (RVP) if and only if there exist uncertain parameters $v_j \in V_j \subseteq \mathbb{R}^s$ and $w_i \in W_i \subseteq \mathbb{R}^s$, and Lagrange multipliers $(\alpha, \beta^g, \beta^h, \beta^H, \beta^G) \in \mathbb{R}_+^k \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^q$ satisfying that $\sum_{t=1}^k \alpha_t = 1, \beta_p^H = 0, p \in \Upsilon_+(x^*), \beta_p^H \geq 0, p \in \Upsilon_{00}(x^*) \cup$*

$\Upsilon_{0-}(x^*), \beta_p^G = 0, p \in \Upsilon_{+-}(x^*) \cup \Upsilon_{0+}(x^*) \cup \Upsilon_{00}(x^*) \cup \Upsilon_{0-}(x^*)$ and $\beta_p^G \geq 0, p \in \Upsilon_{+0}(x^*)$

such that

$$\begin{aligned} & \sum_{t=1}^k \alpha_t \nabla f_t(x^*) + \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \\ & - \sum_{p=1}^q \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*) = 0. \end{aligned}$$

Definition 4.3 A feasible point $x^* \in Q$ is considered to be a robust VP-strong-stationary point of the problem (RVP) if and only if there exist uncertain parameters $v_j \in V_j \subseteq \mathbb{R}^s$ and $w_i \in W_i \subseteq \mathbb{R}^s$, and Lagrange multipliers $(\alpha, \beta^g, \beta^h, \beta^H, \beta^G) \in \mathbb{R}_+^k \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^q$ satisfying that $\sum_{t=1}^k \alpha_t = 1, \beta_p^H = 0, p \in \Upsilon_+(x^*), \beta_p^H \geq 0, p \in \Upsilon_{00}(x^*) \cup \Upsilon_{0-}(x^*), \beta_p^G = 0, p \in \Upsilon_{+-}(x^*) \cup \Upsilon_{0+}(x^*) \cup \Upsilon_{0-}(x^*)$ and $\beta_p^G \geq 0, p \in \Upsilon_{+0}(x^*) \cup \Upsilon_{00}(x^*)$ such that

$$\begin{aligned} & \sum_{t=1}^k \alpha_t \nabla f_t(x^*) + \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \\ & - \sum_{p=1}^q \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*) = 0. \end{aligned}$$

Note that the difference between the above concepts of robust strong-stationarity is related to the multipliers with indices related to the index set $\Upsilon_{00}(x^*)$.

Remark 10 In case our considered problem is constructed without uncertainty and characterized by an infinite number of inequality constraints, then the problem matches the differentiable semi-infinite multi-objective programming problems involving vanishing constraints established in 2022 by Tung [101] and Antczak [9].

To establish the first-order stationarity necessary optimality conditions for the problem (RVP), we need first to introduce a robust suitable constraint qualification. The Abadie constraint qualification defined in [2] for the scalar case of mathematical programming with vanishing constraints is based on the relation between two significant cones: the standard tangent cone (see, e.g., [28], [89]) and the linearized cone introduced in [2]. For a feasible point $x^* \in Q$, the standard tangent cone to the feasible set Q of (RVP) at x^* is defined as follows:

$$\begin{aligned} \mathbb{T}(Q; x^*) &= \left\{ \tau \in \mathbb{R}^n : \exists \{x_n\} \subseteq Q, \{\sigma_n\} \downarrow 0 : x_n \rightarrow x^*, \frac{x_n - x^*}{\sigma_n} \rightarrow \tau \right\} \\ &= \left\{ \tau \in \mathbb{R}^n : \exists \tau_n \rightarrow \tau, \sigma_n \downarrow 0 : x^* + \sigma_n \tau_n \in Q, \forall n \in \mathbb{Z}^+ \right\}. \end{aligned}$$

The linearized cone and its modification, the VP-linearized cone, were introduced by Achtziger and Kanzow in [2] for the scalar objective optimization problems and in [78] for the multiobjective optimization problems with vanishing constraints. The robust representation of these cones at a feasible point of our problem (RVP) confined with uncertain parameters $v_j \in V_j$, $j = 1, \dots, m$ and $w_i \in W_i$, $i = 1, \dots, r$ can be shown below:

Definition 4.4 *Let $x^* \in Q$. Then*

- (i) *The linearized cone to the set Q of (RVP) at $x^* \in Q$ is denoted by $\mathbb{L}(Q; x^*)$ and defined as follows*

$$\mathbb{L}(Q; x^*) = \left\{ \tau \in \mathbb{R}^n : \langle \nabla_x g_j(x^*, v_j), \tau \rangle \leq 0, \quad \forall v_j \in V_j, \quad j \in J(x^*), \right\}$$

$$\begin{aligned}
\langle \nabla_x h_i(x^*, w_i), \tau \rangle &= 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r, \\
\langle \nabla H_p(x^*), \tau \rangle &= 0, \quad \forall p \in \Upsilon_{0+}(x^*), \\
\langle \nabla H_p(x^*), \tau \rangle &\geq 0, \quad \forall p \in \Upsilon_{00}(x^*) \cup \Upsilon_{0-}(x^*), \\
\langle \nabla G_p(x^*), \tau \rangle &\leq 0, \quad \forall p \in \Upsilon_{+0}(x^*) \}.
\end{aligned}$$

Where $J(x^*) = \{j \in J = \{1, \dots, m\} : g_j(x^*, v_j) = 0, \quad \forall v_j \in V_j, \quad j = 1, \dots, m\}$ is the index set of all active constraints $g_j(x^*, v_j)$, $\forall v_j \in V_j, \quad j = 1, \dots, m$.

(ii) The RVP-linearized cone to the set Q of (RVP) at $x^* \in Q$ is denoted by $\mathbb{L}_{RVP}(Q; x^*)$ and defined as follows

$$\begin{aligned}
\mathbb{L}_{RVP}(Q; x^*) &= \left\{ \tau \in \mathbb{R}^n : \langle \nabla_x g_j(x^*, v_j), \tau \rangle \leq 0, \quad \forall v_j \in V_j, \quad j \in J(x^*), \right. \\
&\quad \langle \nabla_x h_i(x^*, w_i), \tau \rangle = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r, \\
&\quad \langle \nabla H_p(x^*), \tau \rangle = 0, \quad \forall p \in \Upsilon_{0+}(x^*), \\
&\quad \langle \nabla H_p(x^*), \tau \rangle \geq 0, \quad \forall p \in \Upsilon_{00}(x^*) \cup \Upsilon_{0-}(x^*), \\
&\quad \left. \langle \nabla G_p(x^*), \tau \rangle \leq 0, \quad \forall p \in \Upsilon_{+0}(x^*) \cup \Upsilon_{00}(x^*) \right\}.
\end{aligned}$$

Remark 11 Utilizing the concepts of the negative polar cone, strictly negative polar cone, and the perpendicular complement of a subset Q of $\mathbb{R}^n$, we can construct the linearized cone $\mathbb{L}(Q; x^*)$ and its modification, the RVP-linearized cone $\mathbb{L}_{RVP}(Q; x^*)$ to

the set Q of (RVP) at $x^* \in Q$ as follows:

$$\begin{aligned} \mathbb{L}(Q; x^*) = & \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \right)^\circ \cap \left(\bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \right)^\perp \cap \left(\bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) \right)^\perp \\ & \cap \left(\bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) \right)^\circ \cap \left(\bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*) \right)^\circ. \end{aligned}$$

$$\begin{aligned} \mathbb{L}_{RVP}(Q; x^*) = & \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \right)^\circ \cap \left(\bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \right)^\perp \cap \left(\bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) \right)^\perp \\ & \cap \left(\bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) \right)^\circ \cap \left(\bigcup_{p \in \Upsilon_{+0} \cup \Upsilon_{00}} \nabla G_p(x^*) \right)^\circ. \end{aligned}$$

Now, under the differentiability of the functions involved in the considered problem, we can present the robust version of the Abadie constraint qualification and its modification, the RVP-Abadie constraint qualification along with the next definitions:

Definition 4.5 *Let $x^* \in Q$. Then*

- (i) *The robust Abadie constraint qualification (ACQ) for the problem (RVP) holds at the feasible point $x^* \in Q$ if we have*

$$\mathbb{L}(Q; x^*) = \mathbb{T}(Q; x^*).$$

- (ii) *The modified robust Abadie constraint qualification (RVP-ACQ) for the problem (RVP) holds at the feasible point $x^* \in Q$ if we have*

$$\mathbb{L}_{RVP}(Q; x^*) \subseteq \mathbb{T}(Q; x^*).$$

Note that in (i), it is sufficient to prove that $\mathbb{L}(Q; x^*) \subseteq \mathbb{T}(Q; x^*)$ as the other inclusion $\mathbb{T}(Q; x^*) \subseteq \mathbb{L}(Q; x^*)$ is always satisfied. In general, $\mathbb{T}(Q; x^*)$ is always closed but not necessarily convex, whereas $\mathbb{L}(Q; x^*)$ is clearly a polyhedral cone so it is closed and convex one.

By applying the previous concepts of robust strong-stationary points, we formulate the robust first-order stationarity necessary optimality conditions for the problem (RVP) under the robust Abadie constraint qualification and its modification. Throughout the next content, we refer to the index set $\Upsilon_{00}(x^*)$ by Υ_{00} only for simplicity, similarly for the other index sets.

Theorem 4.4 *Consider $x^* \in Q$ to be a local weakly efficient solution to the problem (RVP). Suppose that the robust Abadie constraint qualification (ACQ) holds at $x^* \in Q$ and the following set defined with uncertain parameters $v_j \in V_j, j \in J(x^*)$ and $w_i \in W_i, i \in I = 1, \dots, r$,*

$$\begin{aligned} \Theta = & \text{convex} \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \cup \bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) \cup \bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*) \right) \\ & + \text{span} \left(\bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \cup \bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) \right) \end{aligned}$$

is closed. Hence, x^ is a robust strong-stationary point of the problem (RVP).*

Proof. Consider $x^* \in Q$ to be a local weakly efficient solution to the problem (RVP).

By the definition, there exists a neighborhood $\mathbb{N} \subseteq \mathbb{R}^n$ such that there does not exist

any $x \in Q \cap \mathbb{N}$ satisfying

$$f_t(x) < f_t(x^*), \quad \forall t \in T = \{1, \dots, k\}. \quad (4.8)$$

First, we need to show that the following system has no solution $\tau \in \mathbb{R}^n$.

$$\left\{ \begin{array}{l} \langle \nabla f_t(x^*), \tau \rangle < 0, \quad \forall t \in T = \{1, \dots, k\}. \\ \langle \nabla_x g_j(x^*, v_j), \tau \rangle \leq 0, \quad \forall v_j \in V_j, \quad j \in J(x^*). \\ \langle \nabla_x h_i(x^*, w_i), \tau \rangle = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r. \\ \langle \nabla H_p(x^*), \tau \rangle = 0, \quad \forall p \in \Upsilon_{0+}. \\ \langle \nabla - H_p(x^*), \tau \rangle \leq 0, \quad \forall p \in \Upsilon_{00} \cup \Upsilon_{0-}. \\ \langle \nabla G_p(x^*), \tau \rangle \leq 0, \quad \forall p \in \Upsilon_{+0}. \end{array} \right.$$

Applying the constructions of Remark 11 to the above system, means we need to show that the following intersections are empty.

$$\begin{aligned} & \left(\bigcup_{t \in T} \nabla f_t(x^*) \right)^{\bar{\circ}} \cap \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \right)^{\circ} \cap \left(\bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \right)^{\perp} \cap \left(\bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) \right)^{\perp} \\ & \cap \left(\bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) \right)^{\circ} \cap \left(\bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*) \right)^{\circ} = \emptyset. \end{aligned}$$

That is

$$\left(\bigcup_{t \in T} \nabla f_t(x^*) \right)^{\bar{\circ}} \cap \mathbb{L}(Q; x^*) = \emptyset.$$

Since the robust Abadie constraint qualification (ACQ) holds at $x^* \in Q(\bar{x})$, we have

$$\mathbb{L}(Q; x^*) = \mathbb{T}(Q; x^*).$$

Thus, our aim now is to prove that $\left(\bigcup_{t \in T} \nabla f_t(x^*)\right)^{\bar{\circ}} \cap \mathbb{T}(Q; x^*) = \emptyset$. To illustrate this, suppose to get a contradiction that there exists a vector $\tau \in \left(\bigcup_{t \in T} \nabla f_t(x^*)\right)^{\bar{\circ}} \cap \mathbb{T}(Q; x^*)$. As we have $\tau \in \left(\bigcup_{t \in T} \nabla f_t(x^*)\right)^{\bar{\circ}}$, then the strictly negative polar cone implies that

$$\langle \nabla f_t(x^*), \tau \rangle < 0, \quad \forall t \in T = \{1, \dots, k\}.$$

On the other hand, as $\tau \in \mathbb{T}(Q; x^*)$, there exists $\tau_n \rightarrow \tau, \sigma_n \downarrow 0$ such that

$$x^* + \sigma_n \tau_n \in Q, \quad \forall n \in \mathbb{Z}^+.$$

Now, as each function $f_t, \forall t \in T = \{1, \dots, k\}$ assumed to be continuously differentiable function at x^* , then we have

$$f_t(x^* + \sigma_n \tau_n) = f_t(x^*) + \sigma_n \langle \nabla f_t(x^*), \tau_n \rangle + o(\sigma_n \|\tau_n\|), \quad \forall t \in T = \{1, \dots, k\}.$$

Hence, by arranging the above equality, we get

$$\frac{f_t(x^* + \sigma_n \tau_n) - f_t(x^*)}{\sigma_n} = \langle \nabla f_t(x^*), \tau_n \rangle + \frac{o(\sigma_n \|\tau_n\|)}{\sigma_n \|\tau_n\|} \|\tau_n\|, \quad \forall t \in T. \quad (4.9)$$

Now as $n \rightarrow \infty$, this yields

$$\langle \nabla f_t(x^*), \tau_n \rangle + \frac{o(\sigma_n \|\tau_n\|)}{\sigma_n \|\tau_n\|} \|\tau_n\| \rightarrow \langle \nabla f_t(x^*), \tau \rangle < 0 \quad \forall t \in T. \quad (4.10)$$

Therefore, (4.10) implies that for each index $t \in T = \{1, \dots, k\}$, there is a positive constant $\hat{n}_t > 0$ such that

$$\frac{f_t(x^* + \sigma_n \tau_n) - f_t(x^*)}{\sigma_n} < 0, \quad \forall n > \hat{n}_t.$$

Put $\hat{n} = \max\{\hat{n}_t : t \in T\}$. Hence, there exists sufficiently large constant $n > \hat{n}$ such that the element $x^* + \sigma_n \tau_n \in Q \cap \mathbb{N}$ satisfying

$$f_t(x^* + \sigma_n \tau_n) < f_t(x^*), \quad \forall t \in T = \{1, \dots, k\}.$$

Which contradicts (4.8). Thus, $\left(\bigcup_{t \in T} \nabla f_t(x^*)\right)^{\bar{\circ}} \cap \mathbb{L}(Q; x^*) = \emptyset$ and the previously mentioned system has no solution $\tau \in \mathbb{R}^n$.

Further, the set $\text{conv} \{\bigcup_{t \in T} \nabla f_t(x^*)\}$ is compact and since Θ is closed by assumption, then the set $\text{conv} \{\bigcup_{t \in T} \nabla f_t(x^*)\} + \Theta$ is closed. Based on Lemma 4.2, it follows equivalently that

$$\begin{aligned} 0 \in & \text{conv} \bigcup_{t \in T} \nabla f_t(x^*) + \text{convex} \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \cup \bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) \right. \\ & \left. \cup \bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*) \right) + \text{span} \left(\bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \cup \bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) \right). \end{aligned}$$

Now, applying (4.1) and (4.2) to the above statements, we get

$$\begin{aligned}
0 \in & \text{conv} \bigcup_{t \in T} \nabla f_t(x^*) + \text{convex} \bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) + \text{span} \bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \\
& + \text{span} \bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) + \text{convex} \bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) + \text{convex} \bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*).
\end{aligned}$$

Hence, using Lemma 4.1, there exist Lagrange multipliers $(\alpha, \beta^g, \beta^h, \beta^H, \beta^G) \in \mathbb{R}_+^k \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^q$ satisfying that $\sum_{t=1}^k \alpha_t = 1, \beta_p^H = 0, p \in \Upsilon_+, \beta_p^H \geq 0, p \in \Upsilon_{00} \cup \Upsilon_{0-}, \beta_p^G = 0, p \in \Upsilon_{+-} \cup \Upsilon_{0+} \cup \Upsilon_{00} \cup \Upsilon_{0-}$ and $\beta_p^G \geq 0, p \in \Upsilon_{+0}$ as well as uncertain parameters $v_j \in V_j \subseteq \mathbb{R}^s$ and $w_i \in W_i \subseteq \mathbb{R}^s$ such that

$$\begin{aligned}
& \sum_{t=1}^k \alpha_t \nabla f_t(x^*) + \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \\
& - \sum_{p=1}^q \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*) = 0.
\end{aligned}$$

Thus, x^* is a robust strong-stationary point of the problem (RVP), and the proof is completed. ■

Theorem 4.5 *Consider $x^* \in Q$ to be a local weakly efficient solution to the problem (RVP). Assume that the modified robust Abadie constraint qualification (RVP-ACQ) holds at $x^* \in Q$ and the following set defined with uncertain parameters $v_j \in V_j, j \in J(x^*)$ and $w_i \in W_i, i \in I = 1, \dots, r$,*

$$\begin{aligned}
\bar{\Theta} = & \text{convex} \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \cup \bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) \cup \bigcup_{p \in \Upsilon_{+0} \cup \Upsilon_{00}} \nabla G_p(x^*) \right) \\
& + \text{span} \left(\bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \cup \bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) \right)
\end{aligned}$$

is closed. Then x^* is a robust VP-strong-stationary point of the problem (RVP).

Proof. Similarly to the argument of the Theorem 4.4. ■

Remark 12 In case the neighborhood $N \subseteq \mathbb{R}^n$ in the previous theorems coincides with the whole space $\mathbb{R}^n$, that is $N = \mathbb{R}^n$, then we get the weakly efficient solution to the problem (RVP).

By the equivalence between the standard KKT necessary optimality conditions and the strong-stationarity necessary optimality conditions ([2], [51], [78]), we generalize our results of the robust first-order stationarity necessary optimality conditions for a robust weakly efficient (efficient) solution of the problem (RVP) under the modified robust Abadie constraint qualification (RVP-ACQ) to the robust standard KKT necessary optimality conditions as following.

Theorem 4.6 Let $x^* \in Q$ be a weakly efficient solution to the problem (RVP). Assume that the modified robust Abadie constraint qualification (RVP-ACQ) is satisfied at $x^* \in Q$ and the following set

$$\begin{aligned} \bar{\Theta} = & \text{convex} \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \cup \bigcup_{p \in \Upsilon_{00} \cup \Upsilon_{0-}} (-\nabla H_p(x^*)) \cup \bigcup_{p \in \Upsilon_{+0} \cup \Upsilon_{00}} \nabla G_p(x^*) \right) \\ & + \text{span} \left(\bigcup_{i \in I} \nabla_x h_i(x^*, w_i) \cup \bigcup_{p \in \Upsilon_{0+}} \nabla H_p(x^*) \right) \end{aligned}$$

is closed where $v_j \in V_j, j \in J(x^*)$ and $w_i \in W_i, i \in I = 1, \dots, r$ are uncertain parameters. Then the existence of Lagrange multipliers $\alpha \in \mathbb{R}_+^k, \beta^g \in \mathbb{R}^m, \beta^h \in \mathbb{R}^r$ and

$\beta^H, \beta^G \in \mathbb{R}^q$ is guaranteed such that the next conditions

$$\begin{aligned} \sum_{t=1}^k \alpha_t \nabla f_t(x^*) + \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \\ - \sum_{p=1}^q \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*) = 0, \end{aligned} \quad (4.11)$$

$$\alpha \geq 0, \quad \sum_{t=1}^k \alpha_t = 1, \quad \beta^g \geq 0, \quad (4.12)$$

$$\beta_j^g g_j(x^*, v_j) = 0, \quad \forall v_j \in V_j, \quad j \in J(x^*), \quad (4.13)$$

$$\beta_i^h h_i(x^*, w_i) = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r, \quad (4.14)$$

$$\beta_p^H H_p(x^*) = 0, \quad \forall p = 1, \dots, q, \quad (4.15)$$

$$\beta_p^G G_p(x^*) = 0, \quad \forall p = 1, \dots, q, \quad (4.16)$$

$$\beta_p^H = 0, p \in \Upsilon_+(x^*), \beta_p^H \geq 0, p \in \Upsilon_{00}(x^*) \cup \Upsilon_{0-}(x^*), \beta_p^H \text{ free}, p \in \Upsilon_{0+}(x^*), \quad (4.17)$$

$$\beta_p^G = 0, p \in \Upsilon_{+-}(x^*) \cup \Upsilon_{0+}(x^*) \cup \Upsilon_{0-}(x^*), \beta_p^G \geq 0, p \in \Upsilon_{+0}(x^*) \cup \Upsilon_{00}(x^*), \quad (4.18)$$

are fulfilled.

4.4 Robust First-Order Stationarity Type Sufficient Optimality Conditions

In this section, based on some generalized convexity concepts, we present robust first-order sufficient optimality conditions for our considered vector optimization

problem (RVP) based on robust strong stationarity. In the literature, it is known that in usual nonlinear programming, both strong-stationarity and M-stationarity [52] are first-order necessary optimality conditions. As these concepts are equivalent to the concept of KKT necessary conditions subject to some constraint qualifications, the KKT conditions of the problem, which is assumed to be formulated under convexity, are also sufficient optimality conditions. However, in our problem, this is not satisfied in general as mentioned before, the product constraint does not hold to be convex in general. Here, assuming that the constraint functions g_j, h_i, H_p, G_p for $j = 1, \dots, m, i = 1, \dots, r$, and $p = 1, \dots, q$ are employed with some certain generalized convexity, we show that strong stationarity is also sufficient optimality conditions for our considered problem (RVP).

To achieve our aim, we need to set other important tools to study and analyze robust vector optimization problems with vanishing constraints are the next index sets related to the multipliers involved in constraint function conditions for a given feasible solution to the robust problem (RVP).

Let $x^* \in Q$ and $(\beta^g, \beta^h, \beta^H, \beta^G) \in \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^q$. Consider the index set $I(x^*)$ defined as $I(x^*) = \{i \in I = \{1, \dots, r\} : h_i(x^*, w_i) = 0, \forall w_i \in W_i, i = 1, \dots, r\}$ and the previously indicated index set $J(x^*)$ of all active constraints $g_j(x^*, v_j), \forall v_j \in V_j, j = 1, \dots, m$. Then we have

$$J_g^+(x^*) = \{j \in J(x^*) : \beta_j^g > 0\},$$

$$I_h^+(x^*) = \{i \in I(x^*) : \beta_i^h > 0\}, \quad I_h^-(x^*) = \{i \in I(x^*) : \beta_i^h < 0\},$$

$$\Upsilon_+^+(x^*) = \{p \in \Upsilon_+(x^*) : \beta_p^H > 0\},$$

$$\Upsilon_0^+(x^*) = \{p \in \Upsilon_0(x^*) : \beta_p^H > 0\}, \quad \Upsilon_0^-(x^*) = \{p \in \Upsilon_0(x^*) : \beta_p^H < 0\},$$

$$\Upsilon_{0+}^+(x^*) = \{p \in \Upsilon_{0+}(x^*) : \beta_p^H > 0\}, \quad \Upsilon_{0+}^-(x^*) = \{p \in \Upsilon_{0+}(x^*) : \beta_p^H < 0\},$$

$$\Upsilon_{00}^+(x^*) = \{p \in \Upsilon_{00}(x^*) : \beta_p^H > 0\}, \quad \Upsilon_{00}^-(x^*) = \{p \in \Upsilon_{00}(x^*) : \beta_p^H < 0\},$$

$$\Upsilon_{0-}^+(x^*) = \{p \in \Upsilon_{0-}(x^*) : \beta_p^H > 0\},$$

$$\bar{\Upsilon}_{+0}^+(x^*) = \{p \in \Upsilon_{+0}(x^*) : \beta_p^G > 0\}, \quad \bar{\Upsilon}_{+0}^-(x^*) = \{p \in \Upsilon_{+0}(x^*) : \beta_p^G < 0\},$$

$$\bar{\Upsilon}_{+-}^+(x^*) = \{p \in \Upsilon_{+-}(x^*) : \beta_p^G > 0\},$$

$$\bar{\Upsilon}_{0+}^+(x^*) = \{p \in \Upsilon_{0+}(x^*) : \beta_p^G > 0\}, \quad \bar{\Upsilon}_{0+}^-(x^*) = \{p \in \Upsilon_{0+}(x^*) : \beta_p^G < 0\},$$

$$\bar{\Upsilon}_{00}^+(x^*) = \{p \in \Upsilon_{00}(x^*) : \beta_p^G > 0\}, \quad \bar{\Upsilon}_{00}^-(x^*) = \{p \in \Upsilon_{00}(x^*) : \beta_p^G < 0\},$$

$$\bar{\Upsilon}_{0-}^+(x^*) = \{p \in \Upsilon_{0-}(x^*) : \beta_p^G > 0\}.$$

Hence, utilizing some generalized convexity, we introduce the robust first-order sufficient optimality conditions for a robust strong-stationary point to be a weakly efficient solution (efficient solution) to the problem (RVP) in the next theorems.

Theorem 4.7 *Consider $x^* \in Q$ to be a robust strong-stationary point of the problem (RVP) and assume that the index set $\Upsilon_{0+}^- = \emptyset$. Further, suppose that the following*

functions

$$g_j(\cdot, v_j), v_j \in V_j, \quad j \in J(x^*),$$

$$h_i(\cdot, w_i), w_i \in W_i, \quad i \in I_h^+(x^*), \quad -h_i(\cdot, w_i), w_i \in W_i, \quad i \in I_h^-(x^*),$$

$$-H_p(\cdot), p \in \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*), \quad G_p(\cdot), p \in \tilde{\Upsilon}_{+0}^+(x^*),$$

are all quasiconvex at x^* . If the functions $f_t(\cdot), \forall t \in T$ are pseudoconvex at x^* , then the point x^* is a weakly efficient solution to the (RVP) problem.

Proof. Consider x^* to be a robust strong-stationary point of the problem (RVP).

Then there exist uncertain parameters $v_j \in V_j$ and $w_i \in W_i$, as well as Lagrange multipliers $(\alpha, \beta^g, \beta^h, \beta^H, \beta^G) \in \mathbb{R}_+^k \times \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^q \times \mathbb{R}^q$ such that $\sum_{t=1}^k \alpha_t = 1, \beta_p^H = 0, p \in \Upsilon_+(x^*), \beta_p^H \geq 0, p \in \Upsilon_{00}(x^*) \cup \Upsilon_{0-}(x^*), \beta_p^G = 0, p \in \Upsilon_{+-}(x^*) \cup \Upsilon_{0+}(x^*) \cup \Upsilon_{00}(x^*) \cup \Upsilon_{0-}(x^*), \beta_p^G \geq 0, p \in \Upsilon_{+0}(x^*)$ and the following holds

$$\begin{aligned} & \sum_{t=1}^k \alpha_t \nabla f_t(x^*) + \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \\ & - \sum_{p=1}^q \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*) = 0. \end{aligned} \quad (4.19)$$

Take any feasible point $x \in Q$. Then by the feasibility condition, we have that for any uncertain parameter $v_j \in V_j$, $g_j(x, v_j) \leq 0 = g_j(x^*, v_j)$ for each $j \in J(x^*)$. Since $g_j(\cdot, v_j), v_j \in V_j, j \in J(x^*)$ is quasiconvex at x^* , it yields that

$$\langle \nabla_x g_j(x^*, v_j), x - x^* \rangle \leq 0, \quad \forall v_j \in V_j, \quad j \in J(x^*).$$

Multiply the above inequality by $\beta_j^g \in \mathbb{R}^m$ and take the summation over $j = 1, \dots, m$, we get the following inequality for each $v_j \in V_j$, $j = 1, \dots, m$.

$$\left\langle \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j), x - x^* \right\rangle \leq 0. \quad (4.20)$$

Now, as $x, x^* \in Q$ are feasible points to the problem (RVP), then for $w_i \in W_i$ we have $h_i(x, w_i) = h_i(x^*, w_i) = 0$, $\forall i = 1, \dots, r$. This gives

$$h_i(x, w_i) \leq h_i(x^*, w_i), \quad \forall i \in I_h^+(x^*).$$

$$-h_i(x, w_i) \leq -h_i(x^*, w_i), \quad \forall i \in I_h^-(x^*).$$

By assumption, both functions $h_i(\cdot, w_i)$ and $-h_i(\cdot, w_i)$ are quasiconvex at x^* . Hence, we get

$$\langle \nabla_x h_i(x^*, w_i), x - x^* \rangle \leq 0, \quad \forall w_i \in W_i, \quad i \in I_h^+(x^*).$$

$$\langle -\nabla_x h_i(x^*, w_i), x - x^* \rangle \leq 0, \quad \forall w_i \in W_i, \quad i \in I_h^-(x^*).$$

Since $\beta_i^h > 0$ for $i \in I_h^+(x^*)$ and $\beta_i^h < 0$ for $i \in I_h^-(x^*)$, we get next unified result

$$\left\langle \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i), x - x^* \right\rangle \leq 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r. \quad (4.21)$$

On the other hand, by the feasibility of the point $x \in Q$, one has $H_p(x) \geq 0, \forall p = 1, \dots, q$ which implies that $-H_p(x) \leq 0, \forall p = 1, \dots, q$. Therefore, by the definitions of the index sets $\Upsilon_{0+}^+(x^*), \Upsilon_{00}^+(x^*)$ and $\Upsilon_{0-}^+(x^*)$ where $H_p(x^*) = 0$ and all related

multipliers $\beta_p^H > 0$ are positive, the following inequality still hold.

$$-H_p(x) \leq -H_p(x^*), \quad \forall p \in \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*),$$

Thus, the quasiconvexity of the function $-H_p(\cdot)$, $p \in \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*)$

implies that

$$\langle -\nabla H_p(x^*), x - x^* \rangle \leq 0, \quad \forall p \in \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*). \quad (4.22)$$

From (4.19), it is obvious that for all $x \in Q$ we have

$$\begin{aligned} \left\langle \sum_{t=1}^k \alpha_t \nabla f_t(x^*), x - x^* \right\rangle &= - \left\langle \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \right. \\ &\quad \left. - \sum_{p=1}^q \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*), x - x^* \right\rangle. \end{aligned} \quad (4.23)$$

Since $\Upsilon_{0+}^- = \emptyset$ and the relations (4.20) - (4.22) are satisfied, then $\forall x \in Q$, it yields

$$\begin{aligned} \left\langle \sum_{t=1}^k \alpha_t \nabla f_t(x^*), x - x^* \right\rangle &= - \left\langle \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \right. \\ &\quad \left. - \sum_{p \in \Upsilon_0^{H+}(x^*)} \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*), x - x^* \right\rangle \geq 0. \end{aligned} \quad (4.24)$$

Where $\Upsilon_0^{H+}(x^*) = \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*)$.

Now, seeking a contrary, suppose that the point x^* is not a weakly efficient solution to the problem (RVP). Hence, there exists another feasible point $y \in Q$ such that

$F(y) < F(x^*)$, that is

$$f_t(y) < f_t(x^*), \forall t \in T = \{1, \dots, k\}.$$

As assumed, each function $f_t(\cdot), \forall t \in T$ is a pseudoconvex function at x^* , this implies

$$\langle \nabla f_t(x^*), y - x^* \rangle < 0, \forall t \in T.$$

The existence of Lagrange multipliers $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k$ with $\sum_{t=1}^k \alpha_t = 1$ gurant-
ess that the following strict inequality hold.

$$\left\langle \sum_{t=1}^k \alpha_t \nabla f_t(x^*), y - x^* \right\rangle < 0.$$

The above result contradicts (4.24). Therefore, x^* is a weakly efficient solution to the
problem (RVP). ■

Theorem 4.8 *Consider $x^* \in Q$ to be a robust strong-stationary point of the problem
(RVP) and assume that the set $\Upsilon_{0+}^- = \emptyset$. Further, suppose that the following functions*

$$g_j(\cdot, v_j), v_j \in V_j, j \in J(x^*),$$

$$h_i(\cdot, w_i), w_i \in W_i, i \in I_h^+(x^*), \quad -h_i(\cdot, w_i), w_i \in W_i, i \in I_h^-(x^*),$$

$$-H_p(\cdot), p \in \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*), \quad G_p(\cdot), p \in \tilde{\Upsilon}_{+0}^+(x^*).$$

are all quasiconvex at x^ . If the functions $f_t(\cdot), \forall t \in T$ are strictly pseudoconvex at*

x^* , then the point x^* is an efficient solution to the problem (RVP).

Proof. Following the same procedure of demonstration of Theorem 4.7, we get the

next inequality with $\Upsilon_0^{H+}(x^*) = \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*)$.

$$\begin{aligned} \left\langle \sum_{t=1}^k \alpha_t \nabla f_t(x^*), x - x^* \right\rangle = & - \left\langle \sum_{j=1}^m \beta_j^g \nabla_x g_j(x^*, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(x^*, w_i) \right. \\ & \left. - \sum_{p \in \Upsilon_0^{H+}(x^*)} \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^q \beta_p^G \nabla G_p(x^*), x - x^* \right\rangle \geq 0. \end{aligned} \quad (4.25)$$

Now assume to the contrary that the point x^* is not an efficient solution to the problem

(RVP). Hence, there exists a feasible point $y \in Q$ such that $F(y) \leq F(x^*)$, that is

$$f_t(y) \leq f_t(x^*), \quad \forall t \in T = \{1, \dots, k\},$$

$$f_{t_0}(y) < f_{t_0}(x^*), \quad \text{for some } t_0 \in T \text{ with } t_0 \neq t.$$

By assumption, the functions $f_t(\cdot), \forall t \in T$ are strictly pseudoconvex functions at x^* .

Thus we get

$$\langle \nabla f_t(x^*), y - x^* \rangle < 0, \quad \forall t \in T.$$

This implies together with Lagrange multipliers $\alpha_t \geq 0, t \in T$ with $\sum_{t=1}^k \alpha_t = 1$ that the following strict inequality fulfills.

$$\left\langle \sum_{t=1}^k \alpha_t \nabla f_t(x^*), y - x^* \right\rangle < 0.$$

The above inequality contradicts (4.25) which results in that the point x^* is an efficient

solution to the problem (RVP). |

Similar results will be derived for the case of a robust VP-strong-stationary point of the problem (RVP) under some generalized convexity as can be seen next.

Theorem 4.9 *Let $x^* \in Q$ be a robust VP-strong-stationary point of the problem (RVP) and assume that the index set $\Upsilon_{0+}^- = \emptyset$. Further, suppose that the following functions*

$$g_j(\cdot, v_j), v_j \in V_j, j \in J(x^*),$$

$$h_i(\cdot, w_i), w_i \in W_i, i \in I_h^+(x^*), -h_i(\cdot, w_i), w_i \in W_i, i \in I_h^-(x^*),$$

$$-H_p(\cdot), p \in \Upsilon_{0+}^+(x^*) \cup \Upsilon_{00}^+(x^*) \cup \Upsilon_{0-}^+(x^*), G_p(\cdot), p \in \bar{\Upsilon}_{+0}^+(x^*) \cup \bar{\Upsilon}_{00}^+(x^*).$$

are quasiconvex functions at x^ . Then we have the following cases:*

- (i) *If the functions $f_t(\cdot), \forall t \in T$ are pseudoconvex at x^* , then the point x^* is a weakly efficient solution to the problem (RVP).*
- (ii) *If the functions $f_t(\cdot), \forall t \in T$ are strictly pseudoconvex at x^* , then the point x^* is an efficient solution to the problem (RVP).*

Proof. The proof is analogous to those of Theorem 4.7 and Theorem 4.8. |

Turning now to some applications of the robust strong-stationary points for our considered problem (RVP), we establish the equivalence relation between the weakly efficient solutions of both optimization problems, the vector optimization problem (RVP) and its robust auxiliary form (RVP($\bar{x}$)) where $\bar{x} \in Q$.

Theorem 4.10 *Let $\bar{x} \in Q$ be a robust strong-stationarity point in the problem (RVP).*

Suppose that the index set $\Upsilon_{0+}^-(\bar{x}) = \{p \in \Upsilon_{0+}(\bar{x}) : \beta_p^H < 0\} = \emptyset$. Then the point $\bar{x}$ is a weakly efficient solution to the problem (RVP($\bar{x}$)).

Proof. Assume that $\bar{x} \in Q$ be a robust strong-stationarity point in the problem (RVP). By contradiction, suppose that $\bar{x}$ is not a weakly efficient solution to the problem (RVP($\bar{x}$)). Hence, there is another feasible solution $x \in Q(\bar{x})$ such that

$$f_t(\bar{x}) + (x - \bar{x})^T \nabla f_t(\bar{x}) < f_t(\bar{x}) + (\bar{x} - \bar{x})^T \nabla f_t(\bar{x}), \quad \forall t = 1, \dots, k.$$

That is

$$(x - \bar{x})^T \nabla f_t(\bar{x}) < 0, \quad \forall t = 1, \dots, k.$$

Multiply the above inequality by $\alpha_t \geq 0, t = 1, \dots, k$ with $\sum_{t=1}^k \alpha_t = 1$ and take the summation over $t = 1, \dots, k$, we get the following inequality

$$(x - \bar{x})^T \sum_{t=1}^k \alpha_t \nabla f_t(\bar{x}) < 0, \quad \forall t = 1, \dots, k. \quad (4.26)$$

Furthermore, by the feasibility of the point $x \in Q(\bar{x})$, we have

$$g_j(\bar{x}, v_j) + (x - \bar{x})^T \nabla_x g_j(\bar{x}, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, \dots, m. \quad (4.27)$$

$$h_i(\bar{x}, w_i) + (x - \bar{x})^T \nabla_x h_i(\bar{x}, w_i) = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r. \quad (4.28)$$

$$H_p(\bar{x}) + (x - \bar{x})^T \nabla H_p(\bar{x}) \geq 0, \quad p = 1, \dots, q. \quad (4.29)$$

$$G_p(\bar{x})H_p(\bar{x}) + (x - \bar{x})^T G_p(\bar{x}) \nabla H_p(\bar{x}) + (x - \bar{x})^T \nabla G_p(\bar{x}) H_p(\bar{x}) \leq 0, \quad p = 1, \dots, q. \quad (4.30)$$

Multiply the inequality (4.27) by $\beta_j^g \geq 0, j = 1, \dots, m$. By considering (4.13) and taking the summation over $j = 1, \dots, m$. This gives

$$(x - \bar{x})^T \sum_{j=1}^m \beta_j^g \nabla_x g_j(\bar{x}, v_j) \leq 0, \quad \forall v_j \in V_j, \quad j = 1, \dots, m. \quad (4.31)$$

Now, multiply (4.28) by $\beta_i^h \in \mathbb{R}, i = 1, \dots, r$ and use (4.14). Then, take the summation over $i = 1, \dots, r$. we get

$$(x - \bar{x})^T \sum_{i=1}^r \beta_i^h \nabla_x h_i(\bar{x}, v_j) = 0, \quad \forall w_i \in W_i, \quad i = 1, \dots, r. \quad (4.32)$$

Since $\Upsilon_{0+}^-(\bar{x}) = \emptyset$, we can multiply (4.29) by $\beta_p^H \geq 0, p = 1, \dots, q$, and then consider the relation (4.15). By taking the summation over $p = 1, \dots, q$, It will yield

$$-(x - \bar{x})^T \sum_{p=1}^q \beta_p^H \nabla H_p(\bar{x}) \leq 0, \quad \forall p = 1, \dots, q. \quad (4.33)$$

Similarly, by multiplying (4.30) by $\beta_p^H \geq 0, p = 1, \dots, q$ and considering (4.15), we get the following result

$$(x - \bar{x})^T G_p(\bar{x}) \beta_p^H \nabla H_p(\bar{x}) \leq 0, \quad p = 1, \dots, q. \quad (4.34)$$

We claim that (4.34) holds and can be written as follows

$$-(x - \bar{x})^T \sum_{p=1}^q \beta_p^H \nabla H_p(\bar{x}) \leq 0, \quad p = 1, \dots, q. \quad (4.35)$$

In fact, we consider three cases to get this result:

(i) If $G_p(\bar{x}) = 0$, $p \in \Upsilon_{00}(\bar{x}) \cup \Upsilon_{+0}(\bar{x})$. Then (4.34) is trivial and one has

$$(x - \bar{x})^T \sum_{p \in \Upsilon_{00}(\bar{x}) \cup \Upsilon_{+0}(\bar{x})} \beta_p^H \nabla H_p(\bar{x}) = 0.$$

(ii) If $G_p(\bar{x}) < 0$, $p \in \Upsilon_{0-}(\bar{x}) \cup \Upsilon_{+-}(\bar{x})$. Then (4.34) holds and one has

$$-(x - \bar{x})^T \sum_{p \in \Upsilon_{0-}(\bar{x}) \cup \Upsilon_{+-}(\bar{x})} \beta_p^H \nabla H_p(\bar{x}) \leq 0.$$

(iii) If $G_p(\bar{x}) > 0$, $p \in \Upsilon_{0+}(\bar{x})$. Then (4.34) implies that

$$(x - \bar{x})^T \sum_{p \in \Upsilon_{0+}(\bar{x})} \beta_p^H \nabla H_p(\bar{x}) \leq 0.$$

By combining the above inequality together with the inequality (4.33), we get the following equality

$$(x - \bar{x})^T \sum_{p \in \Upsilon_{0+}(\bar{x})} \beta_p^H \nabla H_p(\bar{x}) = 0, \quad p \in \Upsilon_{0+}(\bar{x}).$$

This proves the fulfillment of the inequality (4.35) for all $p = 1, \dots, q$.

Now, following the same procedure in the previous steps, we multiply (4.30) by $\beta_p^G \geq 0$

for $p = 1, \dots, q$, and consider (4.16), we get the next result

$$(x - \bar{x})^T H_p(\bar{x}) \beta_p^G \nabla G_p(\bar{x}) \leq 0, \quad p = 1, \dots, q. \quad (4.36)$$

We also claim that (4.36) is satisfied and can be written as follows

$$(x - \bar{x})^T \sum_{p=1}^q \beta_p^G \nabla G_p(\bar{x}) \leq 0, \quad p = 1, \dots, q. \quad (4.37)$$

Indeed, the above result (4.37) can be illustrated in two cases as follows:

(i) If $H_p(\bar{x}) = 0$, $p \in \Upsilon_{00}(\bar{x}) \cup \Upsilon_{0+}(\bar{x}) \cup \Upsilon_{0-}(\bar{x})$. Then (4.36) is clear and we have

$$(x - \bar{x})^T \sum_{p \in \Upsilon_{00}(\bar{x}) \cup \Upsilon_{0+}(\bar{x}) \cup \Upsilon_{0-}(\bar{x})} \beta_p^G \nabla G_p(\bar{x}) = 0.$$

(ii) If $H_p(\bar{x}) > 0$, $p \in \Upsilon_{+0}(\bar{x}) \cup \Upsilon_{+-}(\bar{x})$. Then (4.36) holds and we have

$$(x - \bar{x})^T \sum_{p \in \Upsilon_{+0}(\bar{x}) \cup \Upsilon_{+-}(\bar{x})} \beta_p^G \nabla G_p(\bar{x}) \leq 0.$$

Hence, the fulfillment of the inequality (4.37) holds for all $p = 1, \dots, q$.

To finish the proof, combine the results (4.26), (4.31), (4.32), (4.35) and (4.37). Then,

we get

$$(x - \bar{x})^T \left[\sum_{t=1}^k \alpha_t \nabla f_t(\bar{x}) + \sum_{j=1}^m \beta_j^g \nabla_x g_j(\bar{x}, v_j) + \sum_{i=1}^r \beta_i^h \nabla_x h_i(\bar{x}, w_i) - \sum_{p=1}^q \beta_p^H \nabla H_p(\bar{x}) + \sum_{p=1}^q \beta_p^G \nabla G_p(\bar{x}) \right] < 0.$$

This is a contradiction to the assumption that $\bar{x} \in Q$ is a robust strong-stationarity point in the problem (RVP). This completes the proof. ■

Corollary 1 *Let $\bar{x} \in Q$ be a weakly efficient solution to the problem (RVP) satisfying all the hypotheses of the Theorem 4.4. Suppose that $\Upsilon_{0+}^-(\bar{x}) = \emptyset$. Then $\bar{x}$ is a weakly efficient solution to the problem (RVP($\bar{x}$)).*

4.5 Numerical Examples

This section is dedicated to illustrating the results of this chapter in the following two examples.

Example 5 *Consider the next programming of a robust vector optimization problem involving vanishing constraints under uncertain data.*

$$\min_{x \in \mathbb{R}^2} F(x) = \left(2x_1^2 + x_2^2, x_1^2 + 3x_1 + 1 \right) \quad (RVP)$$

$$\begin{aligned} \text{Subject to} \quad & g_1(x, v_1) = -v_1 x_2 \leq 0, \quad v_1 \in V_1 = [1, 2], \\ & g_2(x, v_2) = v_2(x_1^2 - x_2) \leq 0, \quad v_2 \in V_2 = [1, 2], \\ & H_1(x) = 1 - x_2 \geq 0, \\ & G_1(x)H_1(x) = -x_1(1 - x_2) \leq 0. \end{aligned}$$

Then, we have $k = 2$ where $f_1(x) = 2x_1^2 + x_2^2$, $f_2(x) = x_1^2 + 3x_1 + 1$ with $x = (x_1, x_2) \in \mathbb{R}^2$ are continuously differentiable functions. The constraints functions g_1 and g_2 are

affected with uncertain parameters $v_1, v_2 \in [1, 2]$ respectively. The set of feasible solutions to the problem (RVP) is defined by

$$Q = \{x \in \mathbb{R}^2 : x_1 = 0, 1 \geq x_2 \geq 0\} \cup \{x \in \mathbb{R}^2 : x_1 > 0, x_2 \geq x_1^2, 1 \geq x_2 \geq 0\}.$$

Take the feasible point $x^* = (0, 0) \in Q$. Then clearly we have

$$\mathbb{T}(Q; x^*) = \mathbb{T}(Q; (0, 0)) = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \right\}.$$

Note that $J(x^*) = \{1, 2\}$ and $\Upsilon_{+0} = \{1\}$ while $\Upsilon_{00} = \Upsilon_{0+} = \Upsilon_{0-} = \Upsilon_{+-} = \emptyset$.

$$\nabla_x g_1(x^*, v_1) = \left\{ \begin{bmatrix} 0 \\ -v_1 \end{bmatrix}, v_1 \in [1, 2] \right\}, \quad \nabla_x g_2(x^*, v_2) = \left\{ \begin{bmatrix} 0 \\ -v_2 \end{bmatrix}, v_2 \in [1, 2] \right\},$$

$$\nabla H_1(x^*) = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}, \quad \nabla G_1(x^*) = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\},$$

$$\nabla f_1(x) = \left\{ \begin{bmatrix} 4x_1 \\ 2x_2 \end{bmatrix} \right\}, \quad \nabla f_2(x) = \left\{ \begin{bmatrix} 2x_1 + 3 \\ 0 \end{bmatrix} \right\},$$

$$\begin{aligned} \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \right)^\circ &= \{x \in \mathbb{R}^2 : (x_1, x_2) \begin{bmatrix} 0 \\ -v_1 \end{bmatrix} \leq 0\} \cup \{x \in \mathbb{R}^2 : (x_1, x_2) \begin{bmatrix} 0 \\ -v_1 \end{bmatrix} \leq 0\}. \\ &= \{x \in \mathbb{R}^2 : -v_1 x_2 \leq 0\} \cup \{x \in \mathbb{R}^2 : -v_2 x_2 \leq 0\}. \\ &= \{x \in \mathbb{R}^2 : x_2 \geq 0\}. \end{aligned}$$

$$\begin{aligned}
\left(\bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*) \right)^\circ &= \{x \in \mathbb{R}^2 : (x_1, x_2) \begin{bmatrix} -1 \\ 0 \end{bmatrix} \leq 0\}. \\
&= \{x \in \mathbb{R}^2 : -x_1 \leq 0\} = \{x \in \mathbb{R}^2 : x_1 \geq 0\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathbb{L}_{RVP}(Q; x^*) &= \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \right)^\circ \cap \left(\bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*) \right)^\circ. \\
&= \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0, x_1 \geq 0\} \subseteq \mathbb{T}(Q; x^*).
\end{aligned}$$

Similarly, we have that $\mathbb{L}(Q; x^*) = \mathbb{T}(Q; x^*)$.

Thus, both robust Abadie constraint qualification (ACQ) and the modified robust Abadie constraint qualification (RVP-ACQ) for the problem (RVP) at the feasible point $x^* = (0, 0)$ are fulfilled for this example as for the index p we only have $p \in \Upsilon_{+0} = \{1\}$, whereas the other related index sets are empty.

Furthermore, the following set

$$\begin{aligned}
\bar{\Theta} &= \text{convex} \left(\bigcup_{j \in J(x^*)} \nabla_x g_j(x^*, v_j) \cup \bigcup_{p \in \Upsilon_{+0}} \nabla G_p(x^*) \right) \\
&= \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}
\end{aligned}$$

is closed. We need to show that the feasible point $x^* = (0, 0)$ is a robust VP-strong-stationary point of the problem (RVP). Indeed, choose Lagrange multipliers $\alpha_1 = \alpha_2 =$

$\frac{1}{2}, \beta_1^g = \beta_2^g = 0, \beta_1^H = 0, \beta_1^G = \frac{3}{2} > 0$ with $v_1 = v_2 = 1$. Hence, we get

$$\begin{aligned} & \sum_{t=1}^2 \alpha_t \nabla f_t(x^*) + \sum_{j=1}^2 \beta_j^g \nabla_x g_j(x^*, v_j) - \sum_{p=1}^1 \beta_p^H \nabla H_p(x^*) + \sum_{p=1}^1 \beta_p^G \nabla G_p(x^*) \\ &= \frac{1}{2}(0, 0) + \frac{1}{2}(3, 0) + \beta_1^g(0, -1) + \beta_2^g(0, -1) - \beta_1^H(0, -1) + \frac{3}{2}(-1, 0) = 0. \end{aligned}$$

This means that $x^* = (0, 0)$ is a robust VP-strong-stationary point to (RVP). Now, since $\Upsilon_{0+}^- = \emptyset$, and the functions $g_j(x, v_j), v_j \in [1, 2], j \in J(x^*) = \{1, 2\}$ and $G_1(x)$ are convex functions and hence quasiconvex functions, also the functions $f_1(x)$ and $f_2(x)$ are convex functions and hence pseudoconvex functions. Then, the hypotheses of Theorem 4.9 are all fulfilled, so the feasible point $x^* = (0, 0)$ is a weakly efficient solution to the problem (RVP).

Moreover, as the modified robust Abadie constraint qualification (RVP-ACQ) is satisfied at this weakly efficient solution $x^* = (0, 0)$ to the problem (RVP) and the previously mentioned set $\bar{\Theta}$ is closed, we deduce from the above discussion that the existence of Lagrange multipliers $\alpha = (\alpha_1, \alpha_2) \geq 0$ with $\alpha_1 + \alpha_2 = 1, \beta_1^g, \beta_2^g \geq 0, \beta_1^H = 0$ and $\beta_1^G \geq 0$ with uncertain parameters $v_1, v_2 \in [1, 2]$ is guaranteed such that all the conditions of Theorem 4.6 are satisfied.

The next example clarifies the linearization approach to solving a robust vector optimization problem with vanishing constraints.

Example 6 First, consider a robust vector optimization problem involving vanishing constraints as follows.

$$\min_{x \in \mathbb{R}^2} F(x) = \left(e^{x_2+3x_1}, e^{3x_1} \right) \quad (RVP)$$

$$\text{Subject to} \quad g_1(x, v_1) = v_1 x_1^2 - v_1 x_2 \leq 0, \quad v_1 \in V_1 = [1, 2],$$

$$g_2(x, v_2) = v_2 x_1^2 - v_2 x_1 \leq 0, \quad v_2 \in V_2 = [1, 2],$$

$$H_1(x) = 1 - x_2 \geq 0,$$

$$G_1(x)H_1(x) = -x_2(1 - x_2) \leq 0.$$

Then, $f_1(x) = e^{x_2+3x_1}$, $f_2(x) = e^{3x_1}$ are continuously differentiable functions. The set of feasible solutions to the above (RVP) problem is defined by

$$Q = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = 0\} \cup \{x \in \mathbb{R}^2 : 1 \geq x_2 > 0, x_2 \geq x_1^2\}.$$

Clearly the point $\bar{x} = (0, 0) \in Q$ is a feasible point. Then we have

$$\nabla_x g_1(\bar{x}, v_1) = \left\{ \begin{bmatrix} 0 \\ -v_1 \end{bmatrix}, v_1 \in [1, 2] \right\}, \quad \nabla_x g_2(\bar{x}, v_2) = \left\{ \begin{bmatrix} -v_2 \\ 0 \end{bmatrix}, v_2 \in [1, 2] \right\},$$

$$\nabla H_1(\bar{x}) = \nabla G_1(\bar{x}) = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\},$$

$$\nabla f_1(\bar{x}) = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, \quad \nabla f_2(\bar{x}) = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\},$$

Now, using the linearization approach, we construct the robust auxiliary linear vector

problem of the considered problem (RVP) as follows:

$$\min_{x \in \mathbb{R}^2} \quad L(x) = \left(1 + 3x_1 + x_2, 1 + 3x_1\right) \quad (RVP(\bar{x}))$$

$$\text{Subject to} \quad -v_1x_2 \leq 0, \quad v_1 \in [1, 2],$$

$$-v_2x_1 \leq 0, \quad v_2 \in [1, 2],$$

$$1 - x_2 \geq 0,$$

Indeed, the feasible point $\bar{x} = (0, 0)$ is a robust strong-stationary point of the problem (RVP). To show this, take Lagrange multipliers $\alpha_1 = \alpha_2 = \frac{1}{2} > 0$, $\beta_1^g = \frac{1}{4} > 0$, $\beta_2^g = 3 > 0$, $\beta_1^H = 0$, and $\beta_1^G = \frac{1}{4} > 0$ with $v_1 = v_2 = 1$. Then we get

$$\begin{aligned} & \sum_{t=1}^2 \alpha_t \nabla f_t(\bar{x}) + \sum_{j=1}^2 \beta_j^g \nabla_x g_j(\bar{x}, v_j) - \sum_{p=1}^1 \beta_p^H \nabla H_p(\bar{x}) + \sum_{p=1}^1 \beta_p^G \nabla G_p(\bar{x}) \\ &= \frac{1}{2}(3, 1) + \frac{1}{2}(3, 0) + \frac{1}{4}(0, -1) + 3(-1, 0) - \beta_1^H(0, -1) + \frac{1}{4}(0, -1) = (0, 0). \end{aligned}$$

This clarifies that $\bar{x} = (0, 0)$ is a robust strong-stationary point to the problem (RVP).

In addition, we have $\Upsilon_{0+}^- = \emptyset$, thus by Theorem 4.10, the point $\bar{x}$ is a weakly efficient solution to the problem (RVP($\bar{x}$)). Since the functions $f_1(x)$ and $f_2(x)$ are convex functions and hence pseudoconvex functions, then the assumptions of Theorem 4.3 hold and we conclude that $\bar{x}$ is also a weakly efficient solution to the problem (RVP). Furthermore, we can use any method studied in the literature for solving robust multi-objective linear programming problems, for instance, the techniques discussed in the

work of Abbassi [1], Georgiev et al. [42] and Goberna et al. [46]. The problem $(RVP(\bar{x}))$ is confined with constraint parameters which are uncertain and vary in the interval $[1, 2]$. Based on the robust approach introduced by Abbassi in 2023 [1], we formulate the following deterministic mathematical alternative model:

$$\begin{aligned}
& \min_{x \in \mathbb{R}^2} \quad \left(1 + 3x_1 + x_2, \ 1 + 3x_1 \right) \\
& \text{Subject to} \quad -v_1x_2 + w_{11} + \tau_1u_1 \leq 0, \\
& \quad \quad \quad -v_2x_1 + w_{21} + \tau_2u_2 \leq 0, \\
& \quad \quad \quad 1 - x_2 \geq 0, \\
& \quad \quad \quad w_{11} + u_1 \geq \frac{1}{2}x_1, \\
& \quad \quad \quad w_{21} + u_2 \geq \frac{1}{2}x_1, \\
& \quad \quad \quad w_{11}, w_{21}, u_1, u_2 \geq 0, \\
& \quad \quad \quad x_1, x_2 \geq 0.
\end{aligned} \tag{4.38}$$

Where $\tau_1, \tau_2 \in [0, 1]$ are introduced to be other integer parameters as well as $u_1, u_2 \in \{0, 1\}$. By solving the above problem with the weighted sum scalarization method along with parameters $v_1 = v_2 = \tau_1 = \tau_2 = 1$ and $w_{11} = w_{21} = u_1 = u_2 = 0$, we get that the point $\bar{x} = (0, 0)$ is a weakly efficient solution to the problem $(RVP(\bar{x}))$. Hence, it is also a weakly efficient solution to the problem (RVP) .

CHAPTER 5

ROBUST SECOND ORDER DUALITY FOR A NON-DIFFERENTIABLE FRACTIONAL PROGRAMMING WITH COMPLEX VARIABLES

Most of the real-life practical optimization problems need to be formulated into mathematical programming that may involve a fractional objective function with complex variables such as some applications in electrical engineering. In these problems, the alternative currents for the elements of the electrical network could be represented by a complex variable $z \in \mathbb{C}^n$. Complex optimization plays a significant role in constructing and solving these kinds of practical problems. Levinson [72]

in 1966, first extended the linear programming problem to the complex space and formulated the complex mathematical programming problem. In 1970, Swarup and Shama [100] studied linear fractional programming in complex variables. Since then various work in the field of complex optimization has been introduced to the literature, for more details regarding the non-linear fractional and non-fractional complex programming, the reader is referred to ([21], [30], [40], [81]).

It is well-known that minimax problems are optimization problems where the performance of maximization and minimization mechanisms occurs. It is applied to game theory, fixed point theory, and statistics. By the inspiration of [30], Lai et al. [69] proved that in complex space, a complex minimax fractional programming problem is equivalent to a complex minimax non-fractional parametric problem for a given parameter and established the Kuhn-Tucker type necessary optimality conditions as well as duality theorems involving generalized convex complex functions. Then, Lai and Huang ([66], [67]) studied the necessary and sufficient optimality conditions for non-differentiable minimax programming problems with square root terms in complex space. Subsequently, Lai and Huang [68] presented the first-order duality theorems for complex non-differentiable minimax fractional programming with complex variables. Thereafter, Huang ([56], [57]) formulated a second-order dual problem for a complex non-differentiable minimax programming problem as well as a second-order parametric free dual problem for complex minimax fractional programming in complex space.

In fact, by considering the worst-case approach in solving problems of robust optimization, the minimax fractional and non-fractional programming problems can be considered as the robust counterpart of the robust minimization problem based on the worst-case scenario. In 2021, Chen et al. [19] investigated the robust Fritz John type necessary optimality conditions and robust Karush-Kuhn-Tucker type for non-differentiable complex fractional programming problems under data uncertainty in complex space.

This chapter aims to discuss the robust second-order duality theorems for non-differentiable fractional programming with uncertain data in complex space. It is organized as follows. Section 5.1 is dedicated to introducing the basic concepts, notations, and existing lemmas and results that will be useful in the following sections. In section 5.2, the construction of a robust non-fractional problem of a robust complex fractional problem using a parametric approach is presented as well as some results related to the equivalence between the robust complex fractional problem and one of the robust non-fractional parametric minimax complex programming problem such as the robust optimal solution and the robust necessary optimality conditions. In section 5.3, the robust second-order Mangasarian type dual model of a robust complex fractional problem is formulated. Many kinds of robust duality theorems are proved in section 5.4.

5.1 Problem Background and Formulation

Consider the robust complex fractional programming as follows:

$$\min_{\zeta} \frac{Re[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}]}{Re[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}]} \quad (\text{RCFP})$$

$$\text{Subject to } -h(\zeta, v) \in T, \quad \zeta = (z, \bar{z}) \in \mathbb{C}^{2n},$$

where $f(\zeta, \eta_1), g(\zeta, \eta_2) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}$ and $h(\zeta, v) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}^p$ are analytic functions at each $\zeta \in Q \subset \mathbb{C}^{2n}$ such that $Q = \{\zeta = (z, \bar{z}) : z \in \mathbb{C}^n\}$ is a linear manifold over the field of real numbers $\mathbb{R}$ which is also considered to be a convex cone. Since a non-linear analytic function lacks a convex real part (consult Ferrero [40]), we regard all the complex functions to be defined on Q . Assume that the functions f , g and h are continuous functions on the second argument in which $\eta_1 \in Y_1 \subset \mathbb{C}^{2m}, \eta_2 \in Y_2 \subset \mathbb{C}^{2m}$ and $v \in V \subset \mathbb{C}^{2m}$ are uncertain parameters belong to the non-empty compact uncertain subsets Y_1, Y_2 and V of $\mathbb{C}^{2m}$ respectively.

The matrices A and B are complex positive semidefinite hermitian matrices with dimensions of $n \times n$. In case there exists a point $\zeta_0 = (z_0, \bar{z}_0)$ such that either $z_0^H A z_0 = 0$ or $z_0^H B z_0 = 0$, or both of them are zeros, then the corresponding term $(z^H A z)^{\frac{1}{2}}$ or $(z^H B z)^{\frac{1}{2}}$ is non-differentiable near ζ_0 , hence, the problem (RCFP) is considered to be a non-differentiable programming problem.

The cone $T = \{\xi \in \mathbb{C}^p : \text{Re}(D\xi) \geq 0\}$ is a polyhedral cone in $\mathbb{C}^p$ with a matrix $D \in \mathbb{C}^{k \times p}$ and $k \in \mathbb{Z}^+$. The cone $T^* = \{\mu \in \mathbb{C}^p : \text{Re}\langle \xi, \mu \rangle \geq 0, \text{ for } \xi \in T\}$, is the dual cone of T where $\langle \xi, \mu \rangle = \mu^H \xi = \bar{\mu}^T \xi$ is the complex inner product. It is clear that in this case $T = (T^*)^*$. For $\xi_0 \in T$, The set $T(\xi_0)$ is defined to be the intersection of those closed half-spaces identifying the cone T and including ξ_0 in their boundaries. Therefore, if $\xi_0 \in \text{int}(T)$, then $T(\xi_0) = \mathbb{C}^p$.

Let $X = \{\zeta = (z, \bar{z}) \in \mathbb{C}^{2n} : -h(\zeta, v) \in T\}$ be the feasible solution set of (RCFP). For every $(\zeta, \eta_1) \in X \times Y_1, (\zeta, \eta_2) \in X \times Y_2$ and $v \in V$, assume also that $\text{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}] \geq 0$ and $\text{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}] > 0$.

Based on the worst-case approach, the robust counterpart of (RCFP) can be formulated as the following minimax programming:

$$\min_{\zeta} \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{\text{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}]}{\text{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}]} \quad (\text{RCCFP})$$

$$\text{Subject to} \quad -h(\zeta, v) \in T, \quad \zeta = (z, \bar{z}) \in \mathbb{C}^{2n}, \quad \forall v \in V.$$

The robust feasible solution set of (RCCFP) is defined by

$$G = \{\zeta = (z, \bar{z}) \in \mathbb{C}^{2n} : -h(\zeta, v) \in T, \quad \forall v \in V\},$$

A decision point $\zeta_0 = (z_0, \bar{z}_0) \in G \subset \mathbb{C}^{2n}$ is considered a robust optimal solution of

(RCFP) if it is an optimal solution of (RCCFP). That is for each $\zeta \in G$, we have

$$\max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{\operatorname{Re}[f(\zeta_0, \eta_1) + (z_0^H A z_0)^{\frac{1}{2}}]}{\operatorname{Re}[g(\zeta_0, \eta_2) - (z_0^H B z_0)^{\frac{1}{2}}]} \leq \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{\operatorname{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}]}{\operatorname{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}]}.$$

For a twice differentiable complex analytic function $\psi : \mathbb{C}^{2n} \rightarrow \mathbb{C}$, we can express the differentiability of $\psi(\zeta)$ with $\zeta = (z, \bar{z}) \in \mathbb{C}^{2n}$ by the following gradient notations:

$$\nabla \psi(\zeta) = (\nabla_z \psi(\zeta), \nabla_{\bar{z}} \psi(\zeta)) \in \mathbb{C}^{2n},$$

where $\nabla_z \psi(\zeta) = (\psi_{z_1}(\zeta), \dots, \psi_{z_n}(\zeta)) \in \mathbb{C}^n$ and $\nabla_{\bar{z}} \psi(\zeta) = (\psi_{\bar{z}_1}(\zeta), \dots, \psi_{\bar{z}_n}(\zeta)) \in \mathbb{C}^n$.

The expression of the second-order gradient $\nabla^2 \psi(\zeta)$ is defined by the following Hessian matrix

$$\nabla^2 \psi(\zeta) = \begin{pmatrix} \nabla_{zz} \psi(\zeta) & \nabla_{z\bar{z}} \psi(\zeta) \\ \nabla_{\bar{z}z} \psi(\zeta) & \nabla_{\bar{z}\bar{z}} \psi(\zeta) \end{pmatrix} \in \mathbb{C}^{2n \times 2n},$$

where

$$\nabla_{zz} \psi(\zeta) = (\psi_{z_i z_j}(\zeta))_{n \times n}, \quad \text{for } i, j = 1, \dots, n,$$

$$\nabla_{z\bar{z}} \psi(\zeta) = (\psi_{z_i \bar{z}_j}(\zeta))_{n \times n}, \quad \text{for } i, j = 1, \dots, n,$$

$$\nabla_{\bar{z}z} \psi(\zeta) = (\psi_{\bar{z}_i z_j}(\zeta))_{n \times n}, \quad \text{for } i, j = 1, \dots, n,$$

$$\nabla_{\bar{z}\bar{z}} \psi(\zeta) = (\psi_{\bar{z}_i \bar{z}_j}(\zeta))_{n \times n}, \quad \text{for } i, j = 1, \dots, n,$$

To express the differentiability of a complex function perturbed by uncertain parameters using the gradient expressions, we have the next lemma:

Lemma 5.1 [19] *Consider the below function*

$$\psi(\zeta) = f(\zeta, \eta) + z^H A w + \langle h(\zeta, v), \mu \rangle,$$

where $f(\zeta, \eta) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}$ and $h(\zeta, v) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}^p$ are analytic functions at each $\zeta = (z, \bar{z}) \in Q \subset \mathbb{C}^{2n}$, A is $n \times n$ complex positive semidefinite Hermitian matrix, $\eta \in Y$ and $v \in V$ are uncertain parameters. For each vector $w \in \mathbb{C}^n$ and $\mu \in \mathbb{C}^p$, suppose the function $\psi(\zeta)$ is differentiable at $\zeta_0 = (z_0, \bar{z}_0) \in \mathbb{C}^{2n}$. Then

$$\begin{aligned} & Re[\psi'(\zeta_0)(\zeta - \zeta_0)] \\ &= Re[\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \nabla_{\bar{z}} f(\zeta_0, \eta) + A w + \mu^T \overline{\nabla_z h(\zeta_0, v)} + \mu^H \nabla_{\bar{z}} h(\zeta_0, v) \rangle]. \end{aligned}$$

Mond [80], extended the generalized Schwarz inequality to complex space. For a complex positive semidefinite Hermitian matrix $A \in \mathbb{C}^{n \times n}$ and $u, v \in \mathbb{C}^n$, we have

$$Re(u^H A v) \leq (u^H A u)^{\frac{1}{2}} (v^H A v)^{\frac{1}{2}}.$$

The equality holds in case $Au = \lambda Av$ for some $\lambda \geq 0$ with $u = \lambda v$.

5.2 Robust Non-fractional Construction of (RCCFP) using Parametric Approach

The problem (RCCFP) is reconstructed into a non-fractional parametric problem with a suitable parameter. Indeed, let

$$\lambda = \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{\operatorname{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}]}{\operatorname{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}]} \geq 0. \quad (5.1)$$

Hence,

$$\frac{\operatorname{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}]}{\operatorname{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}]} \leq \lambda.$$

Then $\forall \eta_1 \in Y_1, \eta_2 \in Y_2$, we obtain

$$\operatorname{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}] - \lambda \operatorname{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}] \leq 0.$$

The following robust non-fractional parametric minimax complex programming problem is an equivalent problem of (RCCFP) with respect to parameter λ :

$$\min_{\zeta \in G} \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \operatorname{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}] - \lambda \operatorname{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}] \quad (\text{RCCFP})_\lambda$$

Since $f(\zeta, \cdot)$ and $g(\zeta, \cdot)$ are continuous functions on the non-empty compact sets Y_1 and Y_2 respectively, then the existence of $\hat{\eta}_1 \in Y_1$ and $\hat{\eta}_2 \in Y_2$ is guaranteed such that

for each $\zeta \in G$, we have

$$\max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{Re[f(\zeta, \eta_1) + (z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta, \eta_2) - (z^H Bz)^{\frac{1}{2}}]} = \frac{Re[f(\zeta, \hat{\eta}_1) + (z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta, \hat{\eta}_2) - (z^H Bz)^{\frac{1}{2}}]}.$$

The following sets are useful in the sequel.

For every element $\zeta \in Q$, we define the following:

$$Y(\zeta) = \left\{ (\hat{\eta}_1, \hat{\eta}_2) \in Y_1 \times Y_2 : \frac{Re[f(\zeta, \hat{\eta}_1) + (z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta, \hat{\eta}_2) - (z^H Bz)^{\frac{1}{2}}]} = \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{Re[f(\zeta, \eta_1) + (z^H Az)^{\frac{1}{2}}]}{Re[g(\zeta, \eta_2) - (z^H Bz)^{\frac{1}{2}}]} \right\}.$$

Equivalently, we can define the following set concerning parameter λ :

$$\begin{aligned} Y_\lambda(\zeta) &= \left\{ (\hat{\eta}_1, \hat{\eta}_2) \in Y_1 \times Y_2 : Re[f(\zeta, \hat{\eta}_1) + (z^H Az)^{\frac{1}{2}}] - \lambda Re[g(\zeta, \hat{\eta}_2) - (z^H Bz)^{\frac{1}{2}}] \right. \\ &\quad \left. = \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \left[Re[f(\zeta, \eta_1) + (z^H Az)^{\frac{1}{2}}] - \lambda Re[g(\zeta, \eta_2) - (z^H Bz)^{\frac{1}{2}}] \right] \right\}. \end{aligned}$$

Clearly, for $\zeta \in Q$, both sets $Y(\zeta)$ and $Y_\lambda(\zeta)$ are compact subsets of $Y_1 \times Y_2$. Hence, for a robust feasible solution $\zeta = (z, \bar{z}) \in G$ of the problem $(RCCFP)_\lambda$, there exist finite points $(\eta_{1_i}, \eta_{2_i}) \in Y_\lambda(\zeta)$, multipliers $\alpha_i > 0$ for $i = 1, \dots, k$ with $k \in \mathbb{Z}^+$ and $\sum_{i=1}^k \alpha_i = 1$ such that we can write the objective function of the problem $(RCCFP)_\lambda$ as follows:

$$\begin{aligned} &\max_{(\eta_1, \eta_2) \in (Y_1, Y_2)} Re[f(\zeta, \eta_1) + (z^H Az)^{\frac{1}{2}}] - \lambda Re[g(\zeta, \eta_2) - (z^H Bz)^{\frac{1}{2}}] \\ &= \sum_{i=1}^k \alpha_i \left[Re[f(\zeta, \eta_{1_i}) + (z^H Az)^{\frac{1}{2}}] - \lambda Re[g(\zeta, \eta_{2_i}) - (z^H Bz)^{\frac{1}{2}}] \right] \\ &= \sum_{i=1}^k \alpha_i Re[f(\zeta, \eta_{1_i}) + (z^H Az)^{\frac{1}{2}}] - \lambda \sum_{i=1}^k \alpha_i Re[g(\zeta, \eta_{2_i}) - (z^H Bz)^{\frac{1}{2}}]. \end{aligned}$$

Chen et al. [19] established the equivalent relation between the robust optimal solution of the problem (RCFP), hence of (RCCFP), and one of the robust non-fractional parametric minimax complex programming problem $(\text{RCCFP})_\lambda$ as follows:

Theorem 5.1 [19]

1. The feasible point $\zeta_0 = (z_0, \bar{z}_0) \in G$ is a robust optimal solution of (RCFP) with optimal value

$$\lambda^* = \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{\text{Re}[f(\zeta_0, \eta_1) + (z_0^H A z_0)^{\frac{1}{2}}]}{\text{Re}[g(\zeta_0, \eta_2) - (z_0^H B z_0)^{\frac{1}{2}}]} = \min_{\zeta \in G} \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \frac{\text{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}]}{\text{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}]}$$

if and only if

$$\begin{aligned} & \min_{\zeta \in G} \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \left\{ \text{Re}[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}] - \lambda^* \text{Re}[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}] \right\} \\ &= \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \left\{ \text{Re}[f(\zeta_0, \eta_1) + (z_0^H A z_0)^{\frac{1}{2}}] - \lambda^* \text{Re}[g(\zeta_0, \eta_2) - (z_0^H B z_0)^{\frac{1}{2}}] \right\} = 0. \end{aligned}$$

That is ζ_0 is an optimal solution of $(\text{RCCFP})_\lambda$ with $\lambda = \lambda^*$ and optimal value equals 0.

2. If ζ_0 is a robust optimal solution of (RCFP) with optimal value λ^* , then

$$Y(\zeta_0) = Y_{\lambda^*}(\zeta_0).$$

The robust necessary optimality conditions for the robust optimal solution $\zeta_0 = (z_0, \bar{z}_0)$ of the robust non-differentiable fractional programming (RCFP) in which the term $(z^H A z)^{\frac{1}{2}}$ or $(z^H B z)^{\frac{1}{2}}$ is non-differentiable when $\zeta = \zeta_0$, that is either $z_0^H A z_0 = 0$ or $z_0^H B z_0 = 0$, or $z_0^H A z_0 = z_0^H B z_0 = 0$, can be extended to the complex space as the work of [19], [56], [66],

[67]. For these results, we need first to introduce the subset $Z(\zeta_0) \subset Q \subset \mathbb{C}^{2n}$ subject to one of three cases.

Consider ζ_0 to be the robust optimal solution of (RCFP) with optimal value λ^* , both sets $Y(\zeta_0)$ and $Y_{\lambda^*}(\zeta_0)$ are compact subsets of $Y_1 \times Y_2$. Hence, there exist $(\widehat{\eta}_{1_i}, \widehat{\eta}_{2_i}) \in Y_{\lambda^*}(\zeta_0)$, scalars $\alpha_i > 0$ with $\sum_{i=1}^k \alpha_i = 1$, for $i = 1, \dots, k$, and $\hat{v} \in V$. Then the definition of the set $Z(\zeta_0)$ is determined as follows:

$$Z(\zeta_0) = \{\zeta = (z, \bar{z}) \in Q \subset \mathbb{C}^{2n} : -h'_\zeta(\zeta_0, \hat{v})\zeta \in T(-h(\zeta_0, \hat{v}))\}.$$

If any of the cases specified below in (i), (ii), and (iii) is met:

Case (i) : if $z_0^H A z_0 > 0$ and $z_0^H B z_0 = 0$. Then

$$Re \left\{ \sum_{i=1}^k \alpha_i [f'_\zeta(\zeta_0, \widehat{\eta}_{1_i}) - \lambda^* g'_\zeta(\zeta_0, \widehat{\eta}_{2_i})] \zeta + \frac{\langle A z_0, z \rangle}{\langle A z_0, z_0 \rangle^{\frac{1}{2}}} + \langle (\lambda^*)^2 B z, z \rangle^{\frac{1}{2}} \right\} < 0.$$

Case (ii) : if $z_0^H A z_0 = 0$ and $z_0^H B z_0 > 0$. Then

$$Re \left\{ \sum_{i=1}^k \alpha_i [f'_\zeta(\zeta_0, \widehat{\eta}_{1_i}) - \lambda^* g'_\zeta(\zeta_0, \widehat{\eta}_{2_i})] \zeta + \langle A z, z \rangle^{\frac{1}{2}} + \frac{\langle \lambda^* B z_0, z \rangle}{\langle B z_0, z_0 \rangle^{\frac{1}{2}}} \right\} < 0.$$

Case (iii) : if $z_0^H A z_0 = 0$ and $z_0^H B z_0 = 0$. Then

$$Re \left\{ \sum_{i=1}^k \alpha_i [f'_\zeta(\zeta_0, \widehat{\eta}_{1_i}) - \lambda^* g'_\zeta(\zeta_0, \widehat{\eta}_{2_i})] \zeta + \langle [A + (\lambda^*)^2 B] z, z \rangle^{\frac{1}{2}} \right\} < 0.$$

The following definition [19] is the robust version of the constraint qualification property in complex space [66], in short, (RCQ) at a point ζ_0 .

Definition 5.1 (Robust Constraint Qualification) *The robust constraint qualification*

is considered to be satisfied by the problem (RCFP) at ζ_0 , if for any non-zero $\mu \in T^* \subset \mathbb{C}^p$ and for any uncertain parameter $v \in V$, we have

$$\operatorname{Re}\langle h'_\zeta(\zeta_0, v)(\zeta - \zeta_0), \mu \rangle \neq 0, \quad \zeta \neq \zeta_0.$$

Which is equivalent to the following gradient expression:

$$\mu^T \overline{\nabla_z h(\zeta_0, v)} + \mu^H \nabla_{\bar{z}} h(\zeta_0, v) \neq 0, \quad \forall 0 \neq \mu \in T^*, \quad v \in V.$$

For robust complex non-differentiable fractional programming (RCFP), we consider the following Karush-Kuhn-Tucker (KKT)-type robust necessary optimality conditions [19] using the robust constraint qualification RCQ :

Theorem 5.2 (KKT- Robust Necessary Optimality Conditions) Consider $\zeta_0 = (z_0, \bar{z}_0) \in G$ to be a robust optimal solution of (RCFP) with optimal value equals λ^* . Assume that the (RCFP) problem fulfills the robust constraint qualification RCQ at ζ_0 , with either $z_0^H A z_0 = 0$ or $z_0^H B z_0 = 0$, or $z_0^H A z_0 = z_0^H B z_0 = 0$, and $Z(\zeta_0) = \emptyset$. Then there exist $\mu \in T^*, u_1, u_2 \in \mathbb{C}^n, \hat{v} \in V$ and finite points $(\widehat{\eta}_{1_i}, \widehat{\eta}_{2_i}) \in Y_{\lambda^*}(\zeta_0)$, multipliers $\alpha_i > 0$ with $\sum_{i=1}^k \alpha_i = 1$, for $i = 1, \dots, k$, in which the function

$$\sum_{i=1}^k \alpha_i \operatorname{Re}[f(\zeta, \widehat{\eta}_{1_i}) - \lambda^* g(\zeta, \widehat{\eta}_{2_i}) + (z^H A z)^{\frac{1}{2}} + \lambda^* (z^H B z)^{\frac{1}{2}}] + \langle h(\zeta, \hat{v}), \mu \rangle$$

satisfies the following KKT- robust necessary conditions at ζ_0 .

$$\begin{aligned} \sum_{i=1}^k \alpha_i \left[\overline{[\nabla_z f(\zeta_0, \widehat{\eta}_{1_i}) + \nabla_{\bar{z}} f(\zeta_0, \widehat{\eta}_{1_i})]} - \lambda^* [\overline{[\nabla_z g(\zeta_0, \widehat{\eta}_{2_i}) + \nabla_{\bar{z}} g(\zeta_0, \widehat{\eta}_{2_i})]} \right] \\ + A u_1 + \lambda^* B u_2 + \mu^T \overline{\nabla_z h(\zeta_0, \hat{v})} + \mu^H \nabla_{\bar{z}} h(\zeta_0, \hat{v}) = 0, \end{aligned} \quad (5.2)$$

$$\operatorname{Re}\langle h(\zeta_0, \hat{v}), \mu \rangle = 0, \quad (5.3)$$

$$\langle Au_1, u_1 \rangle = u_1^H Au_1 \leq 1, \quad \operatorname{Re}(z_0^H Au_1) = \operatorname{Re}(z_0^H Az_0)^{\frac{1}{2}}, \quad (5.4)$$

$$\langle Bu_2, u_2 \rangle = u_2^H Bu_2 \leq 1, \quad \operatorname{Re}(z_0^H Bu_2) = \operatorname{Re}(z_0^H Bz_0)^{\frac{1}{2}}. \quad (5.5)$$

Remark 13 *If both matrices A and B are assumed to be positive definite Hermitian matrices, that is $z_0^H Az_0 > 0$ and $z_0^H Bz_0 > 0$, then the problem (RCFP) is a differentiable programming problem and the above results still hold without the assumption $Z(\zeta_0) = \emptyset$.*

5.3 Robust Second-Order Duality Models of (RCFP)

A second-order duality is a significant tool in mathematical programming when quadratic approximations of both the twice differentiable analytic objective and constraint functions are considered as this kind of dual provides tighter bounds for the objective value. Moreover, in case first-order duality is not applicable for a given feasible point of the primary problem, then the second-order duality can be used to provide the value of the primary problem with a lower bound. In 1975, Mangasarian [76] first introduced the second-order dual of a nonlinear minimization problem of the form

$$\min_{x \in \mathbb{R}^n} \{f(x) : g(x) \leq 0\},$$

with $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and then formulated the second-order dual associated with a vector $p \in \mathbb{R}^n$ to be as follows:

$$\max_{x \in \mathbb{R}^n, u \in \mathbb{R}^m} f(x) + ug(x) - \frac{1}{2}p[\nabla^2 f(x) + \nabla^2 ug(x)]p$$

$$\text{Subject to } \nabla f(x) + u\nabla g(x) + p(\nabla^2 f(x) + \nabla^2 ug(x)) = 0, \quad u \geq 0.$$

To construct the second-order duality model of the problem (RCFP) based on the Mangasarian dual type, the following lemma is needed :

Lemma 5.2 *Take $f(\zeta, \eta) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}$ and $h(\zeta, v) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}^p$ to be analytic functions at each $\zeta = (z, \bar{z}) \in Q \subset \mathbb{C}^{2n}$ and contaminated with uncertain parameters $\eta \in Y \subset \mathbb{C}^{2m}$, and $v \in V \subset \mathbb{C}^{2m}$ respectively. Given $\zeta_0 = (z_0, \bar{z}_0) \in Q$, consider $s = z - z_0$. Then the twice differentiable complex analytic functions $f(\zeta, \eta)$ and $\langle h(\zeta, v), \mu \rangle$ with non-zero $\mu \in T^* \subset \mathbb{C}^p$ have the following second-order gradient representation at ζ_0 :*

(i)

$$\begin{aligned} (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta) (\zeta - \zeta_0) &= \langle s, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)}] \rangle + \langle s^H [\nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)], s \rangle \\ &+ \langle s, s^T [\overline{\nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)}] \rangle + \langle s^T [\nabla_{zz} f(\zeta_0, \eta)], s \rangle. \end{aligned}$$

In particular, the real part of the above expression is equal to

$$\text{Re}(\langle z - z_0, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)} + \nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)] + s^T [\overline{\nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)} + \nabla_{zz} f(\zeta_0, \eta)] \rangle).$$

(ii)

$$\begin{aligned}
(\zeta - \zeta_0)^T \nabla^2 \langle h(\zeta_0, v), \mu \rangle (\zeta - \zeta_0) &= \langle s, s^H [\mu^T \overline{\nabla_{zz} h(\zeta_0, v)}] \rangle + \langle s^H [\mu^H \nabla_{\bar{z}\bar{z}} h(\zeta_0, v)], s \rangle \\
&+ \langle s, s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}} h(\zeta_0, v)}] \rangle + \langle s^T [\mu^H \nabla_{z\bar{z}} h(\zeta_0, v)], s \rangle.
\end{aligned}$$

In particular, the real part of the above expression is equal to

$$\text{Re}(\langle z - z_0, s^H [\mu^T \overline{\nabla_{zz} h(\zeta_0, v)} + \mu^H \nabla_{\bar{z}\bar{z}} h(\zeta_0, v)] + s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}} h(\zeta_0, v)} + \mu^H \nabla_{z\bar{z}} h(\zeta_0, v)] \rangle).$$

Proof. Let $\zeta = (z, \bar{z}), \zeta_0 = (z_0, \bar{z}_0)$. Then $(\zeta - \zeta_0)^T = (z - z_0, \bar{z} - \bar{z}_0) = (s, \bar{s})$, where $s = (s_1, \dots, s_n) \in \mathbb{C}^n$ and $\bar{s} = (\bar{s}_1, \dots, \bar{s}_n) \in \mathbb{C}^n$. Hence, we have

(i)

$$\begin{aligned}
(\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta) (\zeta - \zeta_0) &= (s, \bar{s}) \begin{pmatrix} \nabla_{zz} f(\zeta_0, \eta) & \nabla_{z\bar{z}} f(\zeta_0, \eta) \\ \nabla_{\bar{z}z} f(\zeta_0, \eta) & \nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta) \end{pmatrix} \begin{pmatrix} s \\ \bar{s} \end{pmatrix} \\
&= (s \nabla_{zz} f(\zeta_0, \eta) + \bar{s} \nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta), s \nabla_{z\bar{z}} f(\zeta_0, \eta) + \bar{s} \nabla_{\bar{z}z} f(\zeta_0, \eta)) \begin{pmatrix} s \\ \bar{s} \end{pmatrix} \\
&= (s \nabla_{zz} f(\zeta_0, \eta) + \bar{s} \nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)) s + (s \nabla_{z\bar{z}} f(\zeta_0, \eta) + \bar{s} \nabla_{\bar{z}z} f(\zeta_0, \eta)) \bar{s} \\
&= s \nabla_{zz} f(\zeta_0, \eta) s + \bar{s} \nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta) s + s \nabla_{z\bar{z}} f(\zeta_0, \eta) \bar{s} + \bar{s} \nabla_{\bar{z}z} f(\zeta_0, \eta) \bar{s} \\
&= \sum_{j=1}^n \left[\sum_{i=1}^n s_i f_{z_i z_j}(\zeta_0, \eta) \right] s_j + \sum_{j=1}^n \left[\sum_{i=1}^n \bar{s}_i f_{\bar{z}_i z_j}(\zeta_0, \eta) \right] s_j \\
&\quad + \sum_{j=1}^n \left[\sum_{i=1}^n s_i f_{z_i \bar{z}_j}(\zeta_0, \eta) \right] \bar{s}_j + \sum_{j=1}^n \left[\sum_{i=1}^n \bar{s}_i f_{\bar{z}_i \bar{z}_j}(\zeta_0, \eta) \right] \bar{s}_j \\
&= \sum_{j=1}^n \left[\sum_{i=1}^n s_i f_{z_i z_j}(\zeta_0, \eta) \right] s_j + \sum_{j=1}^n \left[\sum_{i=1}^n \bar{s}_i f_{\bar{z}_i z_j}(\zeta_0, \eta) \right] \bar{s}_j \\
&\quad + \sum_{j=1}^n \left[\sum_{i=1}^n \bar{s}_i f_{\bar{z}_i z_j}(\zeta_0, \eta) \right] s_j + \sum_{j=1}^n \left[\sum_{i=1}^n s_i f_{z_i \bar{z}_j}(\zeta_0, \eta) \right] \bar{s}_j
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^n s_j \left[\sum_{i=1}^n \overline{s_i f_{z_i z_j}(\zeta_0, \eta)} \right] + \sum_{j=1}^n \left[\sum_{i=1}^n \bar{s}_i f_{\bar{z}_i \bar{z}_j}(\zeta_0, \eta) \right] \bar{s}_j \\
&\quad + \sum_{j=1}^n s_j \left[\sum_{i=1}^n \overline{s_i f_{\bar{z}_i z_j}(\zeta_0, \eta)} \right] + \sum_{j=1}^n \left[\sum_{i=1}^n s_i f_{z_i \bar{z}_j}(\zeta_0, \eta) \right] \bar{s}_j \\
&= \langle s, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)}] \rangle + \langle s^H [\nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)], s \rangle \\
&\quad + \langle s, s^T [\overline{\nabla_{\bar{z}z} f(\zeta_0, \eta)}] \rangle + \langle s^T [\nabla_{z\bar{z}} f(\zeta_0, \eta)], s \rangle,
\end{aligned}$$

where the complex inner product $\langle u, v \rangle = v^H u = \bar{v}^T u = \sum_{i=1}^n u_i \bar{v}_i$ is considered for any $u, v \in \mathbb{C}^n$.

Now, by taking the real part of the above statement, we obtain

$$\begin{aligned}
&Re((\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta) (\zeta - \zeta_0)) \\
&= Re(\langle s, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)}] \rangle) + Re(\langle s^H [\nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)], s \rangle) \\
&\quad + Re(\langle s, s^T [\overline{\nabla_{\bar{z}z} f(\zeta_0, \eta)}] \rangle) + Re(\langle s^T [\nabla_{z\bar{z}} f(\zeta_0, \eta)], s \rangle) \\
&= Re(\langle s, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)}] \rangle) + Re(\langle s, s^H [\nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)] \rangle) \\
&\quad + Re(\langle s, s^T [\overline{\nabla_{\bar{z}z} f(\zeta_0, \eta)}] \rangle) + Re(\langle s, s^T [\nabla_{z\bar{z}} f(\zeta_0, \eta)] \rangle) \\
&= Re(\langle s, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)}] \rangle + \langle s, s^H [\nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)] \rangle) \\
&\quad + \langle s, s^T [\overline{\nabla_{\bar{z}z} f(\zeta_0, \eta)}] \rangle + \langle s, s^T [\nabla_{z\bar{z}} f(\zeta_0, \eta)] \rangle) \\
&= Re(\langle s, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)} + \nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)] + s^T [\overline{\nabla_{\bar{z}z} f(\zeta_0, \eta)} + \nabla_{z\bar{z}} f(\zeta_0, \eta)] \rangle) \\
&= Re(\langle z - z_0, s^H [\overline{\nabla_{zz} f(\zeta_0, \eta)} + \nabla_{\bar{z}\bar{z}} f(\zeta_0, \eta)] + s^T [\overline{\nabla_{\bar{z}z} f(\zeta_0, \eta)} + \nabla_{z\bar{z}} f(\zeta_0, \eta)] \rangle),
\end{aligned}$$

where these equalities are followed from the facts that for any $x, y \in \mathbb{C}$, we have

$$Re(x + y) = Re(x) + Re(y) \text{ and } Re\langle x, y \rangle = Re\langle y, x \rangle.$$

(ii) By the definition of the complex inner product, we have

$$\begin{aligned}
(\zeta - \zeta_0)^T \nabla^2 \langle h(\zeta_0, v), \mu \rangle (\zeta - \zeta_0) &= \langle (\zeta - \zeta_0)^T \nabla^2 h(\zeta_0, v) (\zeta - \zeta_0), \mu \rangle \\
&= \sum_{i=1}^p (\zeta - \zeta_0)^T \nabla^2 h_i(\zeta_0, v) (\zeta - \zeta_0) \bar{\mu}_i \quad \text{for } i = 1, \dots, p. \\
&= (\zeta - \zeta_0)^T \nabla^2 h_1(\zeta_0, v) (\zeta - \zeta_0) \bar{\mu}_1 + \dots + (\zeta - \zeta_0)^T \nabla^2 h_p(\zeta_0, v) (\zeta - \zeta_0) \bar{\mu}_p.
\end{aligned}$$

Following the same steps in 5.2, we get

$$\begin{aligned}
(\zeta - \zeta_0)^T \nabla^2 h_i(\zeta_0, v) (\zeta - \zeta_0) \bar{\mu}_i &= \langle s, s^H [\mu_i \overline{\nabla_{zz} h_i(\zeta_0, v)}] \rangle + \langle s^H [\bar{\mu}_i \nabla_{\bar{z}\bar{z}} h_i(\zeta_0, v)], s \rangle \\
&\quad + \langle s, s^T [\mu_i \overline{\nabla_{\bar{z}\bar{z}} h_i(\zeta_0, v)}] \rangle + \langle s^T [\bar{\mu}_i \nabla_{zz} h_i(\zeta_0, v)], s \rangle.
\end{aligned}$$

Then, considering the sum of all terms, it follows

$$\begin{aligned}
(\zeta - \zeta_0)^T \nabla^2 \langle h(\zeta_0, v), \mu \rangle (\zeta - \zeta_0) &= \langle s, s^H [\mu^T \overline{\nabla_{zz} h(\zeta_0, v)}] \rangle + \langle s^H [\mu^H \nabla_{\bar{z}\bar{z}} h(\zeta_0, v)], s \rangle \\
&\quad + \langle s, s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}} h(\zeta_0, v)}] \rangle + \langle s^T [\mu^H \nabla_{zz} h(\zeta_0, v)], s \rangle.
\end{aligned}$$

Similarly, the real part of the above expression is equal to

$$\text{Re}(\langle s, s^H [\mu^T \overline{\nabla_{zz} h(\zeta_0, v)}] + \mu^H \nabla_{\bar{z}\bar{z}} h(\zeta_0, v) \rangle + \langle s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}} h(\zeta_0, v)}] + \mu^H \nabla_{zz} h(\zeta_0, v) \rangle).$$

■

Remark 14 In case the functions $f(\zeta, \eta)$ and $h(\zeta, v)$ have no uncertainty with respect to the parameters $\eta \in Y$ and $v \in V$ respectively, then Lemma 5.2 coincides with Lemma 3.1 in [56].

Based on Mangasarian dual type and motivated by Huang [56], the robust second-order duality model $(\text{RCDP})_\lambda$ of the problem (RCFP) with parameter λ satisfying (5.1), is constructed by employing the Karush-Kuhn-Tucker (KKT)-robust necessary optimality conditions and Lemma 5.2, as follows:

$$\begin{aligned}
& \max_{(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}) \in k(\xi)} \max_{(\xi, \mu, u_1, u_2, s) \in X(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v})} \sum_{i=1}^k \alpha_i \text{Re}[f(\xi, \eta_{1_i}) + (u^H A u)^{\frac{1}{2}}] \quad (\text{RCDP})_\lambda \\
& - \lambda \sum_{i=1}^k \alpha_i \text{Re}[g(\xi, \eta_{2_i}) - (u^H B u)^{\frac{1}{2}}] + \text{Re}\langle h(\xi, \hat{v}), \mu \rangle \\
& - \frac{1}{2} \sum_{i=1}^k \alpha_i \text{Re}\langle s, s^H [\overline{\nabla_{zz} f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})} + \nabla_{zz} f(\xi, \eta_{1_i})] \rangle \\
& + \frac{1}{2} \lambda \sum_{i=1}^k \alpha_i \text{Re}\langle s, s^H [\overline{\nabla_{zz} g(\xi, \eta_{2_i})} + \nabla_{\bar{z}\bar{z}} g(\xi, \eta_{2_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} g(\xi, \eta_{2_i})} + \nabla_{zz} g(\xi, \eta_{2_i})] \rangle \\
& - \frac{1}{2} \text{Re}\langle s, s^H [\mu^T \overline{\nabla_{zz} h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}\bar{z}} h(\xi, \hat{v})] + s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}} h(\xi, \hat{v})} + \mu^H \nabla_{zz} h(\xi, \hat{v})] \rangle. \quad (5.6)
\end{aligned}$$

Subject to the following conditions:

1. The set $k(\xi)$ consists of points $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v})$ such that $k \in \mathbb{Z}^+$, $\hat{\alpha} = (\alpha_1, \dots, \alpha_k)$, $\hat{\eta}_1 = (\eta_{1_1}, \dots, \eta_{1_k})$, $\hat{\eta}_2 = (\eta_{2_1}, \dots, \eta_{2_k})$, and $\hat{v} \in V$ satisfying the KKT-robust necessary optimality conditions of (RCFP) with parameter λ for any feasible solution $\xi = (u, \bar{u}) \in Q$. Also, for the non-zero $\mu \in T^*$ together with the feasibility condition $-h(\xi, \hat{v}) \in T$, we obtain $\text{Re}\langle h(\xi, \hat{v}), \mu \rangle \leq 0$.
2. The set $X(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v})$ consists of all feasible solutions of the robust dual $(\text{RCDP})_\lambda$ and is defined by

$$X(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}) = \{(\xi, \mu, u_1, u_2, s) \in \mathbb{C}^{2n} \times \mathbb{C}^p \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n\}$$

such that for $\xi = (u, \bar{u}) \in Q \subset \mathbb{C}^{2n}$, the followings are satisfied:

$$\begin{aligned}
& \sum_{i=1}^k \alpha_i [\overline{\nabla_z f(\xi, \eta_{1_i})} + \nabla_{\bar{z}} f(\xi, \eta_{1_i}) + Au_1] - \lambda \sum_{i=1}^k \alpha_i [\overline{\nabla_z g(\xi, \eta_{2_i})} + \nabla_{\bar{z}} g(\xi, \eta_{2_i}) - Bu_2] \\
& + \mu^T \overline{\nabla_z h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}} h(\xi, \hat{v}) \\
& + \sum_{i=1}^k \alpha_i \{s^H [\overline{\nabla_{zz} f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})} + \nabla_{zz} f(\xi, \eta_{1_i})]\} \\
& - \lambda \sum_{i=1}^k \alpha_i \{s^H [\overline{\nabla_{zz} g(\xi, \eta_{2_i})} + \nabla_{\bar{z}\bar{z}} g(\xi, \eta_{2_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} g(\xi, \eta_{2_i})} + \nabla_{zz} g(\xi, \eta_{2_i})]\} \\
& + s^H [\mu^T \overline{\nabla_{zz} h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}\bar{z}} h(\xi, \hat{v})] + s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}} h(\xi, \hat{v})} + \mu^H \nabla_{zz} h(\xi, \hat{v})] = 0, \quad (5.7)
\end{aligned}$$

$$\mu \in T^* - \{0\}, \quad (5.8)$$

$$u_1^H Au_1 \leq 1, \quad u_2^H Bu_2 \leq 1, \quad (5.9)$$

$$Re(u^H Au_1) = Re(u^H Au)^{\frac{1}{2}}, \quad Re(u^H Bu_2) = Re(u^H Bu)^{\frac{1}{2}}. \quad (5.10)$$

The robust duality theorems in complex space can be established in order to show the relation between the robust primal problem (RCFP) and the robust second-order dual problem (RCDP) $_{\lambda}$ with respect to the parameter λ under some suitable assumptions. To achieve these results, we need first to state some definitions related to the generalized second-order convexity as in the following section.

5.4 Robust Duality Theorems

By the combination of the definitions of generalized η -bonvexity of a twice differentiable function in real space ([13], [54]) and generalized convexity for a real part of a function in complex space [66], Huang [56] introduced the concept of Θ -bonvexities for the real part of

a complex function.

Next, we give the definition of generalized second-order convexity in complex space, say, Θ -bonvexities for the real part of an analytic function perturbed with uncertain parameters.

Definition 5.2 *Let $f(\zeta, \eta) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}$ be a twice differentiable analytic function at each $\zeta = (z, \bar{z}) \in Q \subset \mathbb{C}^{2n}$ and $\eta \in Y$ be an uncertain parameter. The real part of $f(\zeta, \eta)$, that is $\text{Re}f(\zeta, \eta)$ is said to be*

1. Θ -bonvex at $\zeta_0 \in Q$ if there exists a certain mapping $\Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ such that for all $\zeta \in Q$, we have

$$\text{Re} \left[f(\zeta, \eta) - f(\zeta_0, \eta) + \frac{1}{2}(\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta)(\zeta - \zeta_0) \right] \geq$$

$$\text{Re} \left[\{ \nabla f(\zeta_0, \eta) + (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta) \} \Theta(\zeta, \zeta_0) \right].$$

It is called strictly Θ -bonvex if we replace $(\geq)$ with $(>)$.

2. Θ -pseudobonvex at $\zeta_0 \in Q$ if there exists a certain mapping $\Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ such that for all $\zeta \in Q$, we have

$$\text{Re} \left[\{ \nabla f(\zeta_0, \eta) + (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta) \} \Theta(\zeta, \zeta_0) \right] \geq 0$$

$$\Rightarrow \text{Re} \left[f(\zeta, \eta) - f(\zeta_0, \eta) + \frac{1}{2}(\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta)(\zeta - \zeta_0) \right] \geq 0.$$

It is called strictly Θ -pseudobonvex if we replace (≥ 0) with (> 0) .

3. Θ -quasibonvex at $\zeta_0 \in Q$ if there exists a certain mapping $\Theta : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \rightarrow \mathbb{C}^{2n}$ such

that for all $\zeta \in Q$, we have

$$\begin{aligned} \operatorname{Re} \left[f(\zeta, \eta) - f(\zeta_0, \eta) + \frac{1}{2}(\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta)(\zeta - \zeta_0) \right] &\leq 0 \\ \Rightarrow \operatorname{Re} \left[\{ \nabla f(\zeta_0, \eta) + (\zeta - \zeta_0)^T \nabla^2 f(\zeta_0, \eta) \} \Theta(\zeta, \zeta_0) \right] &\leq 0. \end{aligned}$$

Remark 15 If the function f is not associated with uncertain parameter $\eta \in Y$, and if we have $(\zeta - \zeta_0) = p \in \mathbb{R}^n$, with both vectors $\zeta, \zeta_0 \in \mathbb{R}^n$, then the above definition of Θ -bonvex is reduced to the definition of η -bonvex as it was originally defined by Bector and Chandra [13] in 1987.

Now, using the above definitions, we construct the robust duality theorems as follows:

Theorem 5.3 (Robust Weak Duality Theorem) Let $\zeta = (z, \bar{z})$ be the robust primal feasible solution of (RCFP) and $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}, \xi, \mu, u_1, u_2, s)$ be the robust dual feasible solution of (RCDP) $_{\lambda}$. Consider the following function for any $\zeta = (z, \bar{z}) \in Q$ and for any uncertain parameters $\eta_1 \in Y_1, \eta_2 \in Y_2$ with $i = 1, \dots, k$ and $\hat{v} \in V$,

$$\psi(\zeta) = \sum_{i=1}^k \alpha_i f(\zeta, \eta_{1_i}) + z^H A u_1 - \lambda \sum_{i=1}^k \alpha_i g(\zeta, \eta_{2_i}) + z^H B u_2 + \langle h(\zeta, \hat{v}), \mu \rangle.$$

Assume that the real part of ψ is Θ -pseudobonvex in Q . Then we have

$$\begin{aligned} &\max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \operatorname{Re} [f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}] - \lambda \operatorname{Re} [g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}] \\ &\geq \sum_{i=1}^k \alpha_i \operatorname{Re} [f(\xi, \eta_{1_i}) + (u^H A u)^{\frac{1}{2}}] - \lambda \sum_{i=1}^k \alpha_i \operatorname{Re} [g(\xi, \eta_{2_i}) - (u^H B u)^{\frac{1}{2}}] + \operatorname{Re} \langle h(\xi, \hat{v}), \mu \rangle \\ &\quad - \frac{1}{2} \sum_{i=1}^k \alpha_i \operatorname{Re} \langle s, s^H [\overline{\nabla_{zz} f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})] + s^T [\overline{\nabla_{\bar{z}z} f(\xi, \eta_{1_i})} + \nabla_{z\bar{z}} f(\xi, \eta_{1_i})] \rangle \\ &\quad + \frac{1}{2} \lambda \sum_{i=1}^k \alpha_i \operatorname{Re} \langle s, s^H [\overline{\nabla_{zz} g(\xi, \eta_{2_i})} + \nabla_{\bar{z}\bar{z}} g(\xi, \eta_{2_i})] + s^T [\overline{\nabla_{\bar{z}z} g(\xi, \eta_{2_i})} + \nabla_{z\bar{z}} g(\xi, \eta_{2_i})] \rangle \end{aligned}$$

$$-\frac{1}{2}Re\langle s, s^H[\mu^T \overline{\nabla_{zz}h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})] + s^T[\mu^T \overline{\nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})} + \mu^H \nabla_{zz}h(\xi, \hat{v})]\rangle.$$

Proof. We prove this theorem using the contradiction. Suppose the following

$$\begin{aligned} & \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} Re[f(\zeta, \eta_1) + (z^H A z)^{\frac{1}{2}}] - \lambda Re[g(\zeta, \eta_2) - (z^H B z)^{\frac{1}{2}}] \\ & < \sum_{i=1}^k \alpha_i Re[f(\xi, \eta_{1_i}) + (u^H A u)^{\frac{1}{2}}] - \lambda \sum_{i=1}^k \alpha_i Re[g(\xi, \eta_{2_i}) - (u^H B u)^{\frac{1}{2}}] + Re\langle h(\xi, \hat{v}), \mu \rangle \\ & - \frac{1}{2} \sum_{i=1}^k \alpha_i Re\langle s, s^H[\overline{\nabla_{zz}f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}}f(\xi, \eta_{1_i})] + s^T[\overline{\nabla_{\bar{z}\bar{z}}f(\xi, \eta_{1_i})} + \nabla_{zz}f(\xi, \eta_{1_i})]\rangle \\ & + \frac{1}{2} \lambda \sum_{i=1}^k \alpha_i Re\langle s, s^H[\overline{\nabla_{zz}g(\xi, \eta_{2_i})} + \nabla_{\bar{z}\bar{z}}g(\xi, \eta_{2_i})] + s^T[\overline{\nabla_{\bar{z}\bar{z}}g(\xi, \eta_{2_i})} + \nabla_{zz}g(\xi, \eta_{2_i})]\rangle \\ & - \frac{1}{2} Re\langle s, s^H[\mu^T \overline{\nabla_{zz}h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})] + s^T[\mu^T \overline{\nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})} + \mu^H \nabla_{zz}h(\xi, \hat{v})]\rangle. \end{aligned}$$

Hence, for the robust feasible solution ζ of (RCFP), and for each $\eta_1 \in Y_1, \eta_2 \in Y_2$, there exist finite multipliers $\alpha_i > 0$ for $i = 1, \dots, k$ such that $\sum_{i=1}^k \alpha_i = 1$. Thus, we can write the above inequality as follows:

$$\begin{aligned} & \sum_{i=1}^k \alpha_i Re[f(\zeta, \eta_{1_i}) + (z^H A z)^{\frac{1}{2}}] - \lambda \sum_{i=1}^k \alpha_i Re[g(\zeta, \eta_{2_i}) - (z^H B z)^{\frac{1}{2}}] \\ & < \sum_{i=1}^k \alpha_i Re[f(\xi, \eta_{1_i}) + (u^H A u)^{\frac{1}{2}}] - \lambda \sum_{i=1}^k \alpha_i Re[g(\xi, \eta_{2_i}) - (u^H B u)^{\frac{1}{2}}] + Re\langle h(\xi, \hat{v}), \mu \rangle \\ & - \frac{1}{2} \sum_{i=1}^k \alpha_i Re\langle s, s^H[\overline{\nabla_{zz}f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}}f(\xi, \eta_{1_i})] + s^T[\overline{\nabla_{\bar{z}\bar{z}}f(\xi, \eta_{1_i})} + \nabla_{zz}f(\xi, \eta_{1_i})]\rangle \\ & + \frac{1}{2} \lambda \sum_{i=1}^k \alpha_i Re\langle s, s^H[\overline{\nabla_{zz}g(\xi, \eta_{2_i})} + \nabla_{\bar{z}\bar{z}}g(\xi, \eta_{2_i})] + s^T[\overline{\nabla_{\bar{z}\bar{z}}g(\xi, \eta_{2_i})} + \nabla_{zz}g(\xi, \eta_{2_i})]\rangle \\ & - \frac{1}{2} Re\langle s, s^H[\mu^T \overline{\nabla_{zz}h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})] + s^T[\mu^T \overline{\nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})} + \mu^H \nabla_{zz}h(\xi, \hat{v})]\rangle. \end{aligned}$$

Since ζ is a robust primal feasible solution satisfying the feasibility condition $-h(\zeta, \hat{v}) \in T$ that is $h(\zeta, \hat{v}) \leq 0$ for any uncertain parameter $\hat{v}$, and $\mu \in T^*$ is a non-zero vector in $\mathbb{C}^p$, we

get $Re\langle h(\zeta, \hat{v}), \mu \rangle \leq 0$. Thus, by adding the term $Re\langle h(\zeta, \hat{v}), \mu \rangle$ to the left-hand side of the above inequality, the following still hold.

$$\begin{aligned}
& \left[\sum_{i=1}^k \alpha_i Re[f(\zeta, \eta_{1_i}) + (z^H A z)^{\frac{1}{2}}] - \lambda \sum_{i=1}^k \alpha_i Re[g(\zeta, \eta_{2_i}) - (z^H B z)^{\frac{1}{2}}] + Re\langle h(\zeta, \hat{v}), \mu \rangle \right] \\
& - \left[\sum_{i=1}^k \alpha_i Re[f(\xi, \eta_{1_i}) + (u^H A u)^{\frac{1}{2}}] - \lambda \sum_{i=1}^k \alpha_i Re[g(\xi, \eta_{2_i}) - (u^H B u)^{\frac{1}{2}}] + Re\langle h(\xi, \hat{v}), \mu \rangle \right] \\
& + \frac{1}{2} \sum_{i=1}^k \alpha_i Re\langle s, s^H [\overline{\nabla_{zz} f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})} + \nabla_{zz} f(\xi, \eta_{1_i})] \rangle \\
& - \frac{1}{2} \lambda \sum_{i=1}^k \alpha_i Re\langle s, s^H [\overline{\nabla_{zz} g(\xi, \eta_{2_i})} + \nabla_{\bar{z}\bar{z}} g(\xi, \eta_{2_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} g(\xi, \eta_{2_i})} + \nabla_{zz} g(\xi, \eta_{2_i})] \rangle \\
& + \frac{1}{2} Re\langle s, s^H [\mu^T \overline{\nabla_{zz} h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}\bar{z}} h(\xi, \hat{v})] + s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}} h(\xi, \hat{v})} + \mu^H \nabla_{zz} h(\xi, \hat{v})] \rangle < 0.
\end{aligned} \tag{5.11}$$

For vectors $u_1, u_2 \in \mathbb{C}^n$ satisfying the robust dual feasibility that $u_1^H A u_1 \leq 1$ and $u_2^H B u_2 \leq 1$ and by the generalized Schwarz inequality, we get

$$Re(z^H A u_1) \leq (z^H A z)^{\frac{1}{2}} (u_1^H A u_1)^{\frac{1}{2}} \leq (z^H A z)^{\frac{1}{2}} \tag{5.12}$$

$$Re(z^H B u_2) \leq (z^H B z)^{\frac{1}{2}} (u_2^H B u_2)^{\frac{1}{2}} \leq (z^H B z)^{\frac{1}{2}}. \tag{5.13}$$

By employing (5.10) and the inequalities (5.12) and (5.13) in (5.11), the results are still satisfying the following inequality:

$$\begin{aligned}
& Re \left[\sum_{i=1}^k \alpha_i f(\zeta, \eta_{1_i}) + z^H A u_1 - \lambda \sum_{i=1}^k \alpha_i g(\zeta, \eta_{2_i}) + z^H B u_2 + \langle h(\zeta, \hat{v}), \mu \rangle \right] \\
& - Re \left[\sum_{i=1}^k \alpha_i f(\xi, \eta_{1_i}) + u^H A u_1 - \lambda \sum_{i=1}^k \alpha_i g(\xi, \eta_{2_i}) + u^H B u_2 + \langle h(\xi, \hat{v}), \mu \rangle \right] \\
& + \frac{1}{2} \left[\sum_{i=1}^k \alpha_i Re\langle s, s^H [\overline{\nabla_{zz} f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})} + \nabla_{zz} f(\xi, \eta_{1_i})] \rangle \right.
\end{aligned}$$

$$\begin{aligned}
& -\lambda \sum_{i=1}^k \alpha_i \operatorname{Re} \langle s, s^H [\overline{\nabla_{zz}g(\xi, \eta_{2_i})} + \nabla_{\bar{z}\bar{z}}g(\xi, \eta_{2_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}}g(\xi, \eta_{2_i})} + \nabla_{zz}g(\xi, \eta_{2_i})] \rangle \\
& + \operatorname{Re} \langle s, s^H [\mu^T \overline{\nabla_{zz}h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})] + s^T [\mu^T \overline{\nabla_{\bar{z}\bar{z}}h(\xi, \hat{v})} + \mu^H \nabla_{zz}h(\xi, \hat{v})] \rangle \Big] < 0.
\end{aligned}$$

Considering $(s, \bar{s}) = \zeta - \xi \in Q$ and applying the results of Lemma 5.2, we obtain

$$\begin{aligned}
& \operatorname{Re} \left[\sum_{i=1}^k \alpha_i f(\zeta, \eta_{1_i}) + z^H A u_1 - \lambda \sum_{i=1}^k \alpha_i g(\zeta, \eta_{2_i}) + z^H B u_2 + \langle h(\zeta, \hat{v}), \mu \rangle \right] \\
& - \operatorname{Re} \left[\sum_{i=1}^k \alpha_i f(\xi, \eta_{1_i}) + u^H A u_1 - \lambda \sum_{i=1}^k \alpha_i g(\xi, \eta_{2_i}) + u^H B u_2 + \langle h(\xi, \hat{v}), \mu \rangle \right] \\
& + \frac{1}{2} \left[\sum_{i=1}^k \alpha_i \operatorname{Re} \left[(\zeta - \xi)^T \nabla^2 f(\xi, \eta_{1_i}) (\zeta - \xi) \right] - \lambda \sum_{i=1}^k \alpha_i \operatorname{Re} \left[(\zeta - \xi)^T \nabla^2 g(\xi, \eta_{2_i}) (\zeta - \xi) \right] \right. \\
& \left. + \operatorname{Re} \left[(\zeta - \xi)^T \nabla^2 \langle h(\xi, \hat{v}), \mu \rangle (\zeta - \xi) \right] \right] < 0.
\end{aligned}$$

Which can be simplified to the following

$$\operatorname{Re} [\psi(\zeta) - \psi(\xi) + \frac{1}{2} (\zeta - \xi)^T \nabla^2 \psi(\xi) (\zeta - \xi)] < 0.$$

By assumption, for each $\zeta \in Q$, the real part of ψ is Θ -pseudobonvex, hence, we get

$$\operatorname{Re} [\{\nabla \psi(\xi) + (\zeta - \xi)^T \nabla^2 \psi(\xi)\} \Theta(\zeta, \xi)] < 0.$$

Using Lemma 5.1 and Lemma 5.2, the above inequality implies the following

$$\begin{aligned}
& \sum_{i=1}^k \alpha_i [\overline{\nabla_z f(\xi, \eta_{1_i})} + \nabla_{\bar{z}} f(\xi, \eta_{1_i}) + A u_1] - \lambda \sum_{i=1}^k \alpha_i [\overline{\nabla_z g(\xi, \eta_{2_i})} + \nabla_{\bar{z}} g(\xi, \eta_{2_i}) - B u_2] \\
& + \mu^T \overline{\nabla_z h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}} h(\xi, \hat{v}) \\
& + \sum_{i=1}^k \alpha_i \{ s^H [\overline{\nabla_{zz} f(\xi, \eta_{1_i})} + \nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})] + s^T [\overline{\nabla_{\bar{z}\bar{z}} f(\xi, \eta_{1_i})} + \nabla_{zz} f(\xi, \eta_{1_i})] \}
\end{aligned}$$

$$\begin{aligned}
& -\lambda \sum_{i=1}^k \alpha_i \{s^H [\overline{\nabla_{zz}g(\xi, \eta_{2_i})} + \nabla_{\bar{z}z}g(\xi, \eta_{2_i})] + s^T [\overline{\nabla_{\bar{z}z}g(\xi, \eta_{2_i})} + \nabla_{z\bar{z}}g(\xi, \eta_{2_i})]\} \\
& + s^H [\mu^T \overline{\nabla_{zz}h(\xi, \hat{v})} + \mu^H \nabla_{\bar{z}z}h(\xi, \hat{v})] + s^T [\mu^T \overline{\nabla_{\bar{z}z}h(\xi, \hat{v})} + \mu^H \nabla_{z\bar{z}}h(\xi, \hat{v})] < 0. \quad (5.14)
\end{aligned}$$

Clearly (5.14) contradicts the equality (5.7) , hence, the proof is completed. ■

The next theorem shows that for a given robust optimal solution of (RCFP), a robust feasible solution of the robust dual problem (RCDP) $_{\lambda}$ can be constructed, and furthermore, if the assumptions of the robust weak duality theorem are fulfilled, then a robust optimal solution of (RCDP) $_{\lambda}$ can be obtained with the fact that the optimal values of the problems (RCFP) and (RCDP) $_{\lambda}$ are equal as there is no robust duality gap between the robust primary problem and it's a robust duality problem.

Theorem 5.4 (Robust Strong Duality Theorem) *Consider ζ_0 a robust optimal solution of (RCFP) and suppose that the (RCFP) problem satisfies the robust constraint qualification RCQ at ζ_0 . Then there exist $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}) \in k(\zeta_0)$ and $(\zeta_0, \mu, u_1, u_2, s) \in X(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v})$ such that $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}, \zeta_0, \mu, u_1, u_2, s)$ is a robust feasible solution of the robust dual (RCDP) $_{\lambda}$. Moreover, in case the hypotheses of the robust weak duality theorem (Theorem 5.3) are met for all robust feasible solutions of (RCDP) $_{\lambda}$, then $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}, \zeta_0, \mu, u_1, u_2, s)$ is a robust optimal solution of (RCDP) $_{\lambda}$ and both (RCFP) and (RCDP) $_{\lambda}$ problems have identical optimal values.*

Proof. Assume that $\zeta_0 = (z_0, \bar{z}_0)$ is a robust optimal solution of the problem (RCFP) with $s = z_0 - \bar{z}_0 = 0$. By the Karush-Kuhn-Tucker (KKT)- robust necessary optimality conditions, there exist $k \in \mathbb{Z}^+$ with finite points $\eta_{1_i}, \eta_{2_i} \in Y(\zeta_0)$, multipliers $\alpha_i > 0$ so that $\sum_{i=1}^k \alpha_i = 1$, $i = 1, \dots, k$, $\hat{v} \in V$, $u_1, u_2 \in \mathbb{C}^n$, and a non-zero $\mu \in T^*$. For $\hat{\eta}_1 = (\eta_{1_1}, \dots, \eta_{1_k})$, $\hat{\eta}_2 = (\eta_{2_1}, \dots, \eta_{2_k})$ and $\hat{\alpha} = (\alpha_1, \dots, \alpha_k)$, we have $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}) \in k(\zeta_0)$ and

$(\zeta_0, \mu, u_1, u_2, s = 0) \in X(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v})$. Then $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}, \zeta_0, \mu, u_1, u_2, s = 0)$ is a robust feasible solution of the robust dual $(\text{RCDP})_\lambda$. In case the hypotheses of the robust weak duality theorem are fulfilled for all robust feasible solutions of the problem $(\text{RCDP})_\lambda$, then $(k, \hat{\alpha}, \hat{\eta}_1, \hat{\eta}_2, \hat{v}, \zeta_0, \mu, u_1, u_2, s = 0)$ is a robust optimal solution of $(\text{RCDP})_\lambda$. Hence, both (RCFP) and $(\text{RCDP})_\lambda$ problems have identical optimal values. $\blacksquare$

The following theorem states that under the hypotheses of the robust strong duality theorem as well as some extra assumptions, the optimal values of the (RCFP) and $(\text{RCDP})_\lambda$ problems are equal if both the robust optimal solution of (RCFP) and the robust optimal solution of its robust dual $(\text{RCDP})_\lambda$ exist.

Theorem 5.5 (Robust Strict Converse Duality Theorem) *Let ζ^* be the robust optimal solution of the problem (RCFP) and $(k^*, \alpha^*, \eta_1^*, \eta_2^*, v^*, \xi^*, \mu^*, u_1^*, u_2^*, s^*)$ be the robust optimal solution of its robust dual problem $(\text{RCDP})_\lambda$. Assume that the hypotheses of the robust strong duality theorem (Theorem 5.4) are satisfied. For each $\zeta = (z, \bar{z}) \in Q$ and uncertain parameters $\eta_1^* \in Y_1, \eta_2^* \in Y_2$ and $v^* \in V$, let the real part of the following function*

$$\psi(\zeta) = \sum_{i=1}^{k^*} \alpha_i^* f(\zeta, \eta_{1_i}^*) + z^H A u_1^* - \lambda \sum_{i=1}^{k^*} \alpha_i^* g(\zeta, \eta_{2_i}^*) + z^H B u_2^* + \langle h(\zeta, v^*), \mu^* \rangle$$

be strictly Θ -pseudobonvex in Q . Then we have $\zeta^ = \xi^*$, and the optimal values are equal.*

Proof. Let $\zeta^* = (z^*, \bar{z}^*) \in Q$ and $\xi^* = (u^*, \bar{u}^*) \in Q$. Seeking for contradiction, suppose that $\zeta^* \neq \xi^*$. Since the hypotheses of Theorem 5.4 are satisfied, therefore

$$\begin{aligned} & \max_{\eta_1 \in Y_1, \eta_2 \in Y_2} \text{Re}[f(\zeta^*, \eta_1) + (z^{*H} A z^*)^{\frac{1}{2}}] - \lambda \text{Re}[g(\zeta^*, \eta_2) - (z^{*H} B z^*)^{\frac{1}{2}}] \\ &= \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[f(\xi^*, \eta_{1_i}^*) + (u^{*H} A u^*)^{\frac{1}{2}}] - \lambda \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[g(\xi^*, \eta_{2_i}^*) - (u^{*H} B u^*)^{\frac{1}{2}}] + \text{Re}\langle h(\xi^*, v^*), \mu^* \rangle \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \sum_{i=1}^{k^*} \alpha_i^* \text{Re} \langle s^*, s^{*H} [\overline{\nabla_{zz} f(\xi^*, \eta_{1_i}^*)} + \nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)} + \nabla_{zz} f(\xi^*, \eta_{1_i}^*)] \rangle \\
& + \frac{1}{2} \lambda \sum_{i=1}^{k^*} \alpha_i^* \text{Re} \langle s^*, s^{*H} [\overline{\nabla_{zz} g(\xi^*, \eta_{2_i}^*)} + \nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)} + \nabla_{zz} g(\xi^*, \eta_{2_i}^*)] \rangle \\
& - \frac{1}{2} \text{Re} \langle s^*, s^{*H} [\mu^{*T} \overline{\nabla_{zz} h(\xi^*, v^*)} + \mu^{*H} \nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)] + s^{*T} [\mu^{*T} \overline{\nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)} + \mu^{*H} \nabla_{zz} h(\xi^*, v^*)] \rangle.
\end{aligned}$$

Then, for each $\eta_1 \in Y_1$ and $\eta_2 \in Y_2$, we have

$$\begin{aligned}
& \text{Re}[f(\zeta^*, \eta_1) + (z^{*H} A z^*)^{\frac{1}{2}}] - \lambda \text{Re}[g(\zeta^*, \eta_2) - (z^{*H} B z^*)^{\frac{1}{2}}] \\
& \leq \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[f(\xi^*, \eta_{1_i}^*) + (u^{*H} A u^*)^{\frac{1}{2}}] - \lambda \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[g(\xi^*, \eta_{2_i}^*) - (u^{*H} B u^*)^{\frac{1}{2}}] + \text{Re} \langle h(\xi^*, v^*), \mu^* \rangle \\
& - \frac{1}{2} \sum_{i=1}^{k^*} \alpha_i^* \text{Re} \langle s^*, s^{*H} [\overline{\nabla_{zz} f(\xi^*, \eta_{1_i}^*)} + \nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)} + \nabla_{zz} f(\xi^*, \eta_{1_i}^*)] \rangle \\
& + \frac{1}{2} \lambda \sum_{i=1}^{k^*} \alpha_i^* \text{Re} \langle s^*, s^{*H} [\overline{\nabla_{zz} g(\xi^*, \eta_{2_i}^*)} + \nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)} + \nabla_{zz} g(\xi^*, \eta_{2_i}^*)] \rangle \\
& - \frac{1}{2} \text{Re} \langle s^*, s^{*H} [\mu^{*T} \overline{\nabla_{zz} h(\xi^*, v^*)} + \mu^{*H} \nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)] + s^{*T} [\mu^{*T} \overline{\nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)} + \mu^{*H} \nabla_{zz} h(\xi^*, v^*)] \rangle.
\end{aligned}$$

Thus, for the positive integer k^* with finite points $\eta_{1_i}^* \in Y_1, \eta_{2_i}^* \in Y_2$ and scalars $\alpha_i^* > 0$ for

$i = 1, \dots, k^*$ such that $\sum_{i=1}^{k^*} \alpha_i^* = 1$, we obtain

$$\begin{aligned}
& \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[f(\zeta^*, \eta_{1_i}^*) + (z^{*H} A z^*)^{\frac{1}{2}}] - \lambda \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[g(\zeta^*, \eta_{2_i}^*) - (z^{*H} B z^*)^{\frac{1}{2}}] \\
& \leq \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[f(\xi^*, \eta_{1_i}^*) + (u^{*H} A u^*)^{\frac{1}{2}}] - \lambda \sum_{i=1}^{k^*} \alpha_i^* \text{Re}[g(\xi^*, \eta_{2_i}^*) - (u^{*H} B u^*)^{\frac{1}{2}}] + \text{Re} \langle h(\xi^*, v^*), \mu^* \rangle \\
& - \frac{1}{2} \sum_{i=1}^{k^*} \alpha_i^* \text{Re} \langle s^*, s^{*H} [\overline{\nabla_{zz} f(\xi^*, \eta_{1_i}^*)} + \nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)} + \nabla_{zz} f(\xi^*, \eta_{1_i}^*)] \rangle \\
& + \frac{1}{2} \lambda \sum_{i=1}^{k^*} \alpha_i^* \text{Re} \langle s^*, s^{*H} [\overline{\nabla_{zz} g(\xi^*, \eta_{2_i}^*)} + \nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)} + \nabla_{zz} g(\xi^*, \eta_{2_i}^*)] \rangle \\
& - \frac{1}{2} \text{Re} \langle s^*, s^{*H} [\mu^{*T} \overline{\nabla_{zz} h(\xi^*, v^*)} + \mu^{*H} \nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)] + s^{*T} [\mu^{*T} \overline{\nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)} + \mu^{*H} \nabla_{zz} h(\xi^*, v^*)] \rangle.
\end{aligned}$$

Since ζ^* is a robust optimal solution of (RCFP) satisfying the robust necessary condition

$Re\langle h(\zeta^*, v^*), \mu^* \rangle = 0$, then we can write the above inequality as follows

$$\begin{aligned}
& \left[\sum_{i=1}^{k^*} \alpha_i^* Re[f(\zeta^*, \eta_{1_i}^*) + (z^{*H} A z^*)^{\frac{1}{2}}] - \lambda \sum_{i=1}^{k^*} \alpha_i^* Re[g(\zeta^*, \eta_{2_i}^*) - (z^{*H} B z^*)^{\frac{1}{2}}] + Re\langle h(\zeta^*, v^*), \mu^* \rangle \right] \\
& - \left[\sum_{i=1}^{k^*} \alpha_i^* Re[f(\xi^*, \eta_{1_i}^*) + (u^{*H} A u^*)^{\frac{1}{2}}] - \lambda \sum_{i=1}^{k^*} \alpha_i^* Re[g(\xi^*, \eta_{2_i}^*) - (u^{*H} B u^*)^{\frac{1}{2}}] + Re\langle h(\xi^*, v^*), \mu^* \rangle \right] \\
& + \frac{1}{2} \sum_{i=1}^{k^*} \alpha_i^* Re\langle s^*, s^{*H} [\overline{\nabla_{zz} f(\xi^*, \eta_{1_i}^*)} + \nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)} + \nabla_{zz} f(\xi^*, \eta_{1_i}^*)] \rangle \\
& - \frac{1}{2} \lambda \sum_{i=1}^{k^*} \alpha_i^* Re\langle s^*, s^{*H} [\overline{\nabla_{zz} g(\xi^*, \eta_{2_i}^*)} + \nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)} + \nabla_{zz} g(\xi^*, \eta_{2_i}^*)] \rangle \\
& + \frac{1}{2} Re\langle s^*, s^{*H} [\mu^{*T} \overline{\nabla_{zz} h(\xi^*, v^*)} + \mu^{*H} \nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)] + s^{*T} [\mu^{*T} \overline{\nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)} + \mu^{*H} \nabla_{zz} h(\xi^*, v^*)] \rangle \leq 0.
\end{aligned} \tag{5.15}$$

Since the following robust optimal and dual conditions respectively are satisfied for $u_1^*, u_2^* \in$

$\mathbb{C}^n$, that is

$$\begin{aligned}
Re(z^{*H} A u_1^*) & \leq (z^{*H} A z^*)^{\frac{1}{2}}, & Re(z^{*H} B u_2^*) & \leq (z^{*H} B z^*)^{\frac{1}{2}}, \\
Re(u^{*H} A u_1^*) & \leq (u^{*H} A u^*)^{\frac{1}{2}}, & Re(u^{*H} B u_2^*) & \leq (u^{*H} B u^*)^{\frac{1}{2}}.
\end{aligned}$$

Using the definition of the function $\psi(\zeta)$ for each $\zeta \in Q$ together with Lemma 5.2, we

proceed similarly to the proof of Theorem 5.3. Then inequality (5.15) is simplified as

$$Re[\psi(\zeta^*) - \psi(\xi^*) + \frac{1}{2}(\zeta^* - \xi^*)^T \nabla^2 \psi(\xi^*)(\zeta^* - \xi^*)] \leq 0.$$

Since for each $\zeta \in Q$, $Re(\psi(\zeta))$ is strictly Θ -pseudobonvex in Q , we get

$$Re[\{\nabla \psi(\xi^*) + (\zeta^* - \xi^*)^T \nabla^2 \psi(\xi^*)\} \Theta(\zeta^*, \xi^*)] < 0.$$

Thus, we have

$$\begin{aligned}
& \sum_{i=1}^{k^*} \alpha_i^* [\overline{\nabla_z f(\xi^*, \eta_{1_i}^*)} + \nabla_{\bar{z}} f(\xi^*, \eta_{1_i}^*) + Au_1^*] - \lambda \sum_{i=1}^{k^*} \alpha_i^* [\overline{\nabla_z g(\xi^*, \eta_{2_i}^*)} + \nabla_{\bar{z}} g(\xi^*, \eta_{2_i}^*) - Bu_2^*] \\
& + \mu^{*T} \overline{\nabla_z h(\xi^*, v^*)} + \mu^{*H} \nabla_{\bar{z}} h(\xi^*, v^*) \\
& + \sum_{i=1}^{k^*} \alpha_i^* \{s^{*H} [\overline{\nabla_{zz} f(\xi^*, \eta_{1_i}^*)} + \nabla_{\bar{z}\bar{z}} f(\xi^*, \eta_{1_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}z} f(\xi^*, \eta_{1_i}^*)} + \nabla_{z\bar{z}} f(\xi^*, \eta_{1_i}^*)]\} \\
& - \lambda \sum_{i=1}^{k^*} \alpha_i^* \{s^{*H} [\overline{\nabla_{zz} g(\xi^*, \eta_{2_i}^*)} + \nabla_{\bar{z}\bar{z}} g(\xi^*, \eta_{2_i}^*)] + s^{*T} [\overline{\nabla_{\bar{z}z} g(\xi^*, \eta_{2_i}^*)} + \nabla_{z\bar{z}} g(\xi^*, \eta_{2_i}^*)]\} \\
& + s^{*H} [\mu^T \overline{\nabla_{zz} h(\xi^*, v^*)} + \mu^{*H} \nabla_{\bar{z}\bar{z}} h(\xi^*, v^*)] + s^{*T} [\mu^{*T} \overline{\nabla_{\bar{z}z} h(\xi^*, v^*)} + \mu^{*H} \nabla_{z\bar{z}} h(\xi^*, v^*)] < 0.
\end{aligned}$$

This contradicts the equality of (5.7), and the proof of the theorem is completed. ■

CHAPTER 6

CONCLUSION AND FUTURE WORK

This dissertation is devoted to studying various mathematical programming under uncertain data. In particular, the semi-infinite multi-objective fractional programming problem involving uncertainty. Employing the partial order relation between vectors, robust weakly efficient (efficient, properly efficient) solutions have been introduced and generalized over cones. A generalization of robust Slater constraint qualification has been stated. Hence, the robust Karush–Kuhn– Tucker necessary optimality conditions have been established for the considered problem constructed in the Dinkelbach parametrization approach. Then, the robust sufficient optimality conditions have been studied through some generalized convexity assumptions, The Mond-Weir type robust duality model has been formulated and the robust duality theorems for the robust semi-infinite multi-objective programming problem have been investigated. A numerical example has been provided to demonstrate these obtained results.

In addition, we have considered a non-smooth robust semi-infinite multi-objective fractional problem and employed an extended form of a robust type constraint qualification. Then, we investigated non-smooth robust necessary optimality conditions for two types of optimal solutions in the vector situation, namely, local isolated and local properly efficient solutions. After that, we reconstructed our non-smooth robust semi-infinite multi-objective fractional problem into a non-smooth robust semi-infinite parametric problem in terms of the generalized Dinkelbach parametrization approach by associating the problem with a non-negative parameter. Hence, we introduced the non-smooth robust necessary optimality conditions for this convex formulated problem under some conditions. Under the assumption of some generalized convexity of the functions under consideration, For the non-smooth robust semi-infinite multi-objective fractional problems, we propose the non-smooth robust sufficient optimality criteria. Also for this kind of mathematical programming problem, the robust dual problem has been formulated in Wolfe-type robust dual problem as well as Mond-Weir-type robust dual problem for local positively properly efficient solutions. For both robust dual formulations of the problem under consideration, the robust strong, robust weak, and robust converse duality theorems have been demonstrated.

Moreover, the vector optimization problem with vanishing constraints has been studied under uncertain data. Utilizing the robust strong stationary points concept, we have investigated both the robust first-order stationarity type necessary and sufficient optimality conditions for this problem. The linearization approach has been employed to construct a robust linearized vector programming problem and investigate the relations between the robust problem under consideration and its linearized form. The crucial results have been clarified with examples.

Furthermore, using robust optimization techniques, the uncertain non-differentiable fractional programming problem with complex variables has been investigated and hence its robust counterpart has been introduced as a robust minimax fractional programming problem. The Karush-Kuhn-Tucker (KKT)- robust necessary optimality conditions have been established. Then a non-differentiable fractional programming problem with uncertain data in complex space has been formulated along with its robust second-order duality. The robust duality theorems have been presented and proved.

In a future study, the robust semi-infinite multi-objective programming problems can be extended to involve uncertain data within the objective functions as well as the constraints functions.

Another programming work that could be interesting to investigate is the complex non-differentiable multi-objective fractional optimization problems confined with uncertain data using robust optimization methods and constructing the robust second-order duality for this problem. The question is to what extent it is reasonable to extend the results obtained in this research to the preceding problem that could be considered as follows:

$$\min_{\zeta} \left(\frac{Re[f_1(\zeta, \eta_1) + (z^H A_1 z)^{\frac{1}{2}}]}{Re[g_1(\zeta, \gamma_1) - (z^H B_1 z)^{\frac{1}{2}}]}, \dots, \frac{Re[f_l(\zeta, \eta_l) + (z^H A_l z)^{\frac{1}{2}}]}{Re[g_l(\zeta, \gamma_l) - (z^H B_l z)^{\frac{1}{2}}]} \right) \quad (\text{RCMFP})$$

$$\text{Subject to} \quad -h(\zeta, v) \in T, \quad \zeta = (z, \bar{z}) \in \mathbb{C}^{2n},$$

where $f_i(\zeta, \eta_i), g_i(\zeta, \gamma_i) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}$, $i = 1, \dots, l$, and $h(\zeta, v) : \mathbb{C}^{2n} \times \mathbb{C}^{2m} \rightarrow \mathbb{C}^p$ are analytic functions at each $\zeta = (z, \bar{z}) \in Q \subset \mathbb{C}^{2n}$ such that Q is a linear manifold over $\mathbb{R}$. Assume the functions f_i, g_i for each $i = 1, \dots, l$, and h are continuous functions on $\eta_i \in Y_i \subset \mathbb{C}^{2m}, \gamma_i \in U_i \subset \mathbb{C}^{2m}$ and $v \in V \subset \mathbb{C}^{2m}$ are imprecise parameters that fit into the convex and non-empty compact uncertainty sets Y_i, U_i and V , respectively. The matrices A_i and B_i are $n \times n$ complex positive semidefinite hermitian matrices for $i = 1, \dots, l$. The constraints are subject to a polyhedral cone T in $\mathbb{C}^p$.

REFERENCES

- [1] Abbassi, A., A robust optimization approach for multi-objective linear programming under uncertainty. *Journal of Mathematical Modeling*, 2023.
- [2] Achtziger, W. and Kanzow, C., Mathematical programs with vanishing constraints: optimality conditions and constraint qualifications. *Mathematical Programming*, 114: 69-99, 2008.
- [3] Ahmad, I., Kaur, A. and Sharma, M.K., Robust duality for generalized convex non-smooth vector programs with uncertain data in constraints. *RAIRO-Operations Research*, 55(4): 2181-2188, 2021.
- [4] Ahmad, I., Kaur, A. and Sharma, M.K., Robust optimality conditions and duality in semi-infinite multiobjective programming. *Acta Mathematica Universitatis Comenianae*, 91(1): 87-99, 2022.
- [5] Ahmad, I., Kummari, K. and Al-Homidan, S., Unified duality for mathematical programming problems with vanishing constraints. *International Journal of Nonlinear Analysis and Applications*, 13(2): 3191-3201, 2022.
- [6] Amahroq, T., Penot, J.P. and Syam, A., On the subdifferentiability of the difference of two functions and local minimization. *Set-Valued Analysis*, 16(4): 413-427, 2008.

- [7] Antczak, T., On nonsmooth (ϕ, ρ) -invex multiobjective programming in finite-dimensional Euclidean spaces. *Journal of Advanced Mathematical Studies*, 7(1): 127-129, 2014.
- [8] Antczak, T., Parametric approach for approximate efficiency of robust multiobjective fractional programming problems. *Mathematical Methods in the Applied Sciences*, 44(14): 11211-11230, 2021.
- [9] Antczak, T., Optimality conditions and Mond–Weir duality for a class of differentiable semi-infinite multiobjective programming problems with vanishing constraints. *4OR*, 20(3): 417-442, 2022.
- [10] Antczak, T., Pandey, Y., Singh, V. and Mishra, S. K., On approximate efficiency for nonsmooth robust vector optimization problems. *Acta Mathematica Scientia*, 40: 887-902, 2020.
- [11] Bazaraa, M.S., Sherali, H.D. and Shetty, C.M., *Nonlinear Programming: Theory And Algorithms*. John Wiley & Sons, 2013.
- [12] Beck, A. and Ben-Tal, A., Duality in robust optimization: primal worst equals dual best. *Operations Research Letters*, 37: 1-6, 2009.
- [13] Bector, C.R. and Chandra, S., Generalized bonvexity and higher order duality for fractional programming. *Opsearch*, 24(3): 143-154, 1987.
- [14] Ben-Tal, A., Boyd, S. and Nemirovski, A., Extending scope of robust optimization: Comprehensive robust counterparts of uncertain problems. *Mathematical Programming*, 107: 63-89, 2006.

- [15] Ben-Tal, A., El Ghaoui, L. and Nemirovsk A., Robust Optimization, volume 28. Princeton University Press, 2009.
- [16] Ben-Tal, A. and Nemirovski, A., Robust optimization–methodology and applications. Mathematical Programming, 92: 453-480, 2002.
- [17] Bertsimas, D., Brown, D. B. and Caramanis, C., Theory and applications of robust optimization. SIAM review, 53(3): 464-501, 2011.
- [18] Branke, J., Creating robust solutions by means of evolutionary algorithms. In International Conference on Parallel Problem Solving from Nature, Springer, Berlin, Heidelberg, 119-128, 1998.
- [19] Chen, J., Al-Homidan, S., Ansari, Q.H., Li, J. and Lv, Y., Robust necessary optimality conditions for nondifferentiable complex fractional programming with uncertain data. Journal of Optimization Theory and Applications, 189(1): 221-243, 2021.
- [20] Chen, J., Kobis, E. and Yao, J.-C., Optimality conditions and duality for robust nonsmooth multiobjective optimization problems with constraints. Journal of Optimization Theory and Applications, 181: 411-436, 2019.
- [21] Chen, J.C., Lai, H.C. and Schaible, S., Complex fractional programming and the Charnes-Cooper transformation. Journal of Optimization Theory and Applications, 126(1): 203-213, 2005.
- [22] Chuong, T.D., Optimality and duality for proper and isolated efficiencies in multiobjective optimization. Nonlinear Analysis: Theory, Methods & Applications, 76: 93-104, 2013.

- [23] Chuong, T.D., Nondifferentiable fractional semi-infinite multiobjective optimization problems. *Operations Research Letters*, 44(2): 260-266, 2016.
- [24] Chuong, T. D., Optimality and duality for robust multiobjective optimization problems. *Nonlinear Analysis: Theory, Methods and Applications*, 134: 127-143, 2016.
- [25] Chuong, T.D., Robust optimality and duality in multiobjective optimization problems under data uncertainty. *SIAM Journal of Optimization*, 30: 1501-1526, 2020.
- [26] Chuong, T.D., Approximate solutions in nonsmooth and nonconvex cone constrained vector optimization. *Annals of Operations Research*, 311(2): 997-1015, 2022.
- [27] Chuong, T.D. and Yao, J.C., Isolated and proper efficiencies in semi-infinite vector optimization problems. *Journal of Optimization Theory and Applications*, 162: 447-462, 2014.
- [28] Clarke, F.H., *Optimization and Nonsmooth Analysis*. Wiley-Interscience, New York, 1983.
- [29] Crouzeix, J.P., Ferland, J.A. and Schaible, S., An algorithm for generalized fractional programs. *Journal of Optimization Theory and Applications*, 47(1): 35-49, 1985.
- [30] Datta, N. and Bhatia, D., Duality for a class of nondifferentiable mathematical programming problems in complex space. *Journal of Mathematical Analysis and Applications*, 101(1): 1-11, 1984.
- [31] Deb, K. and Gupta, H., Introducing robustness in multi-objective optimization. *Evolutionary Computation*, 14: 463-494, 2006.
- [32] Dinh, N., Goberna, M. A., Lopez, M. A. and Volle, M., Convexity and closedness in stable robust duality. *Optimization Letters*, 13: 325-339, 2019.

- [33] Dinkelbach, W., On nonlinear fractional programming. *Management Science*, 13(7): 492-498, 1967.
- [34] Doolittle, E. K., Kerivin, H. L. M. and Wiecek, M. M., Robust multiobjective optimization with application to internet routing. *Annals of Operations Research*, 271: 487-525, 2018.
- [35] Dorsch, D., Shikhman, V. and Stein, O., Mathematical programs with vanishing constraints: critical point theory. *Journal of Global Optimization*, 52: 591-605, 2012.
- [36] Ehrgott, M., *Multicriteria Optimization* (Vol. 491). Springer Science and Business Media, 2005.
- [37] Ehrgott, M., Ide, J. and Schöobel, A., Minmax robustness for multi-objective optimization problems. *European Journal of Operational Research*, 239: 17-31, 2014.
- [38] Fakhar, M., Mahyarinia, M. R. and Zafarani, J., On nonsmooth robust multiobjective optimization under generalized convexity with applications to portfolio optimization. *European Journal of Operational Research*, 265: 39-48, 2018.
- [39] Falk, J. E. Exact solutions of inexact linear programs. *Operations Research*, 24: 783-787, 1976.
- [40] Ferrero, O., On nonlinear programming in complex spaces. *Journal of Mathematical Analysis and Applications*, 164(2): 399-416, 1992.
- [41] Geoffrion, A.M., Proper efficiency and the theory of vector maximization. *Journal of Mathematical Analysis and Applications*, 22: 618- 630, 1968.

- [42] Georgiev, P.G., Luc, D.T. and Pardalos, P.M., Robust aspects of solutions in deterministic multiple objective linear programming. *European Journal of Operational Research*, 229(1): 29-36, 2013.
- [43] Ginchev, I., Guerraggio, A. and Rocca, M., From scalar to vector optimization. *Applications of Mathematics*, 51(1): 5-36, 2006.
- [44] Goberna, M. A., Guerra-Vazquez, F. and Todorov, M. I., Constraint qualifications in linear vector semi-infinite optimization. *European Journal of Operational Research*, 227:12-21, 2013.
- [45] Goberna, M.A., Jeyakumar, V., Li, G. and Vicente-Perez, J., Robust solutions of multi-objective linear semi-infinite programs under constraint data uncertainty. *SIAM Journal on Optimization*, 24: 1402-1419, 2014.
- [46] Goberna, M.A., Jeyakumar, V., Li, G. and Vicente-Pérez, J., Robust solutions to multi-objective linear programs with uncertain data. *European Journal of Operational Research*, 242(3): 730-743, 2015.
- [47] Goberna, M.A., Jeyakumar, V., Li, G. and Vicente-Perez, J., Guaranteeing highly robust weakly efficient solutions for uncertain multi-objective convex programs. *European Journal of Operational Research*, 270: 40-50, 2018.
- [48] Goerigk, M., Schmidt, M., Schobel, A., Knoth, M. and Muller-Hannemann, M., The price of strict and light robustness in timetable information. *Transportation Science*, 48:225-242, 2014.
- [49] Güzel, N., A proposal to the solution of multiobjective linear fractional programming problem. In *Abstract and Applied Analysis (Vol. 2013)*, Hindawi, 2013.

- [50] Guu, S.M., Singh, Y. and Mishra, S.K., On strong KKT type sufficient optimality conditions for multiobjective semi-infinite programming problems with vanishing constraints. *Journal of Inequalities and Applications*, 2017(1): 1-9, 2017.
- [51] Hoheisel, T. and Kanzow, C., First-and second-order optimality conditions for mathematical programs with vanishing constraints. *Applications of Mathematics*, 52(6): 495-514, 2007.
- [52] Hoheisel, T. and Kanzow, C., Stationary conditions for mathematical programs with vanishing constraints using weak constraint qualifications. *Journal of Mathematical Analysis and Applications*, 337(1): 292-310, 2008.
- [53] Hoheisel, T. and Kanzow, C., On the Abadie and Guignard constraint qualifications for mathematical programs with vanishing constraints. *Optimization*, 58(4): 431-448, 2009.
- [54] Hu, Q., Chen, Y. and Jian, J., Second-order duality for non-differentiable minimax fractional programming. *International Journal of Computer Mathematics*, 89(1): 11-16, 2012.
- [55] Huang, L. and Chen, J., Weighted robust optimality of convex optimization problems with data uncertainty. *Optimization Letters*, 1-17, 2019.
- [56] Huang, T.Y., Second-order duality for a non-differentiable minimax programming in complex spaces. *International Journal of Computer Mathematics*, 94(12): 2508-2519, 2017.
- [57] Huang, T.Y., Second-Order Parametric Free Dualities for Complex Minimax Fractional Programming. *Mathematics*, 8(1), 67, 2020.

- [58] Ide, J. and Kobis, E., Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations. *Mathematical Methods of Operations Research*, 80: 99-127, 2014.
- [59] Jeyakumar, V., Li, G. and Lee, G. M., Robust duality for generalized convex programming problems under data uncertainty. *Nonlinear Analysis: Theory, Methods and Applications*, 75:1362-1373, 2012.
- [60] Jeyakumar, V. and Li, G. Y., Strong duality in robust convex programming: complete characterizations. *SIAM Journal on Optimization*, 20: 3384-3407, 2010.
- [61] Jeyakumar, V., Lee, G.H. and Guoyin Li, G., Characterizing robust solution sets of convex programs under data uncertainty. *Journal of Optimization Theory and Applications*, 164: 407-435, 2015.
- [62] Joshi, B.C. and Minocha, A., On semi-infinite programming problems and strong KKT type sufficient optimality conditions. *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics*, 85(1): 87-94, 2023.
- [63] Kim, M. H., Duality theorem and vector saddle point theorem for robust multiobjective optimization problems. *Communications of the Korean Mathematical Society*, 28: 597-602, 2013.
- [64] Koopmans, T. C., An analysis of production as an efficient combination of activities. In T. C. Koopmans (Ed.), *Activity Analysis of Production and Allocation*, Cowles Commission for Research in Economics, Wiley, Monograph No. 13: 33-97, 1951.
- [65] Kuroiwa, D. and Lee, G. M., On robust multiobjective optimization. *Vietnam Journal of Mathematics*, 40: 305-317, 2012.

- [66] Lai, H.C. and Huang, T.Y., Optimality conditions for a nondifferentiable minimax programming in complex spaces. *Nonlinear Analysis: Theory, Methods & Applications*, 71(3-4): 1205-1212, 2009.
- [67] Lai, H.C. and Huang, T.Y., Optimality conditions for nondifferentiable minimax fractional programming with complex variables. *Journal of Mathematical Analysis and Applications*, 359(1): 229-239, 2009.
- [68] Lai, H.C. and Huang, T.Y., Nondifferentiable minimax fractional programming in complex spaces with parametric duality. *Journal of Global Optimization*, 53(2): 243-254, 2012.
- [69] Lai, H.C., Liu, J.C. and Schaible, S., Complex minimax fractional programming of analytic functions. *Journal of Optimization Theory and Applications*, 137(1): 171-184, 2008.
- [70] Lee, J. H. and Lee, G. M., On optimality conditions and duality theorems for robust semi-infinite multiobjective optimization problems. *Annals of Operations Research*, 269: 419-438, 2018.
- [71] Lee, J. H. and Lee, G.M., On ϵ -solutions for robust semi-infinite optimization problems. *Positivity*, 23: 651-669, 2019.
- [72] Levinson, N., Linear programming in complex space. *Journal of Mathematical Analysis and Applications*, 14(1): 44-62, 1966.
- [73] Li, X.B., Wang, Q.L and Lin, Z., Optimality conditions and duality conditions and duality for minimax fractional programming problems with data uncertainty. *Journal of Industrial and Management Optimization*, 15: 1133-1151, 2019.

- [74] Li, W. and Yu, G., Optimality and duality of approximate quasi weakly efficient solution for nonsmooth vector optimization problems. *Mathematical Problems in Engineering*, 2022.
- [75] Luc, D. T., *Multiobjective Linear Programming: An Introduction*, 49-82, 2016.
- [76] Mangasarian, O.L., Second-and higher-order duality in nonlinear programming. *Journal of Mathematical Analysis and Applications*, 51(3): 607-620, 1975.
- [77] Mangasarian, O.L., *Nonlinear Programming*. Society for Industrial and Applied Mathematics, 1994.
- [78] Mishra, S.K., Singh, V., Laha, V. and Mohapatra, R.N., On constraint qualifications for multiobjective optimization problems with vanishing constraints. *Optimization Methods, Theory and Applications*, 95-135, 2015.
- [79] Mishra, S.K., Singh, V. and Laha, V., On duality for mathematical programs with vanishing constraints. *Annals of Operations Research*, 243: 249-272, 2016.
- [80] Mond, B., An extension of the transposition theorems of Farkas and Eisenberg. *Journal of Mathematical Analysis and Applications*, 32(3): 559-566, 1970.
- [81] Mond, B. and Craven, B.D., A class of nondifferentiable complex programming problems. *Mathematische Operationsforschung und Statistik*, 6(4): 581-591, 1975.
- [82] Mordukhovich, B.S., *Variational analysis and generalized differentiation I: Basic Theory*. Berlin: Springer, 2006.
- [83] Mordukhovich, B.S., *Variational analysis and generalized differentiation II: Applications (Vol. 331)*. Berlin: Springer, 2006.

- [84] Mordukhovich, B.S., Nam, N.M. and Yen, N.D., Fréchet subdifferential calculus and optimality conditions in nondifferentiable programming. *Optimization*, 55(5-6): 685-708, 2006.
- [85] Pham, T.H., On isolated/properly efficient solutions in nonsmooth robust semi-infinite multiobjective optimization. *Bulletin of the Malaysian Mathematical Sciences Society*, 46(2): 73, 2023.
- [86] Preeti, Jayswal, A. and Arana-Jiménez, M., Robust saddle-point criteria for multi-dimensional control optimisation problems with data uncertainty. *International Journal of Control*, 92: 3288-3299, 2020.
- [87] Rahimi, M. and Soleimani-damaneh, M., Isolated efficiency in nonsmooth semi-infinite multi-objective programming. *Optimization*, 67(11): 1923-1947, 2018.
- [88] Rezaee, A., Characterization of isolated efficient solutions in nonsmooth multiobjective semi-infinite programming. *Iranian Journal of Science and Technology, Transactions A: Science*, 43: 1835-1839, 2019.
- [89] Rockafellar, R.T., *Convex analysis*. Princeton University Press, Princeton, 1970.
- [90] Shaker Ardakani, J., Farahmand Rad, S. and Kanzi, N., Characterization of Properly Efficient Solutions for Convex Multiobjective Programming with Nondifferentiable vanishing constraints. *Control and Optimization in Applied Mathematics*, 3(2): 49-58, 2018.
- [91] Singh, C., Convex programming with set-inclusive constraints and its applications to generalized linear and fractional programming. *Journal of Optimization Theory and Applications*, 38: 33-42, 1982.

- [92] Singh, V., Jayswal, A., Stancu-Minasian, I. and Rusu-Stancu, A.M., Isolated and proper efficiencies for semi-infinite multiobjective fractional problems. *Series A Applied Mathematics and Physics*, 83 : 111-124, 2021.
- [93] Soyster, A. L., Convex programming with set-inclusive constraints and applications to inexact linear programming. *Operations Research*, 21: 1154-1157, 1973.
- [94] Sun, X., Feng, X., Teo, K. L., Robust optimality, duality and saddle points for multiobjective fractional semi-infinite optimization with uncertain data. *Optimization Letters*, 1-20, 2021.
- [95] Sun, X., Fu, H. and Zeng, J., Robust approximate optimality conditions for uncertain nonsmooth optimization with infinite number of constraints. *Mathematics*, 7(1): 12, 2018.
- [96] Sun, X.K., Li, X.B., Long, X.J. and Peng, Z.Y., On robust approximate optimal solutions for uncertain convex optimization and applications to multi-objective optimization. *Pacific Journal of Optimization*, 13(4): 621-643, 2017.
- [97] Sun, X.K., Long, X.J., Fu, H.Y., Li, X.B., Some characterizations of robust optimal solutions for uncertain fractional optimization and applications. *Journal of Industrial and Management Optimization*, 13: 803, 2017.
- [98] Sun, X., Teo, K.L., Zeng, J. and Liu, L., Robust approximate optimal solutions for nonlinear semi-infinite programming with uncertainty. *Optimization*, 69(9): 2109-2129, 2020.

- [99] Suneja, S. K., Khurana, S. and Bhatia, M., Optimality and duality in vector optimization involving generalized type I functions over cones. *Journal of Global Optimization*, 49: 23-35, 2011.
- [100] Swarup, K. and Sharma, J.C., Programming with linear fractional functionals in complex spaces. *Journal of Applied Mathematics and Mechanics*, 12: 103-109, 1970.
- [101] Tung, L.T., Karush–Kuhn–Tucker optimality conditions and duality for multiobjective semi-infinite programming with vanishing constraints. *Annals of Operations Research*, 311(2): 1307-1334, 2022.
- [102] Wang, S., Chen, Z., and Liu, T., Distributionally robust hub location. *Transportation Science*, 54: 1189-1210, 2020.
- [103] Zeng, J., Xu, P., and Fu, H., On robust approximate optimal solutions for fractional semi-infinite optimization with uncertainty data. *Journal of Inequalities and Applications*, 45: 1-16, 2019.
- [104] Zhao, J and Bin, M, Karush-Kuhn-Tucker optimality conditions for a class of robust optimization problems with an interval-valued objective function. *Open Mathematics*, 18: 781–793, 2020.

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