

**DISTRIBUTIONAL MODELS FOR STOCK RETURNS OF
CRUDE OIL PRICES**

BY

HAADY JALLOW

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

APPLIED STATISTICS

NOVEMBER 2023

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **HAADY JALLOW** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN APPLIED STATISTICS**.

Thesis Committee

Dr. Emmanuel Osita Afuecheta (Adviser)

Dr. Nasir Abbas (Member)

Dr. Jimoh Olawale Ajadi (Member)

Prof. Monther Rashed Alfuraidan
Department Chairman

Prof. Suliman Al-Homidan
Dean of Graduate Studies

Date



ACKNOWLEDGMENTS

I express my gratitude to my mentor, Dr. Emmanuel Osita Affuecheta, for his knowledge, suggestions, criticism, encouragement, and time. I sincerely thank him for his assistance; without it, the project would not have been possible. My special thanks go to my committee members: Dr Jimo Olawale Adjadi and Dr Nasir Abbas for their support and encouragement. To all my lecturers (Dr. Cherif AT Aidra, Prof. Riaz,...etc), I am truly indebted for your contribution and support. Your enthusiasm, tenacity, and unwavering quest of perfection serve as a wonderful source of inspiration. My sincere gratitude to my parents, who have always provided me with unwavering support and encouragement through all of life's challenges. I would like to thank my family and friends, who helped and encouraged me throughout my life. Special thanks to KFUPM, chairman of mathematics department (Prof. Monther Rashed Alfuraidean) and his members for giving me the opportunity to complete a master's degree in Applied Statistics. Above all, I am saying "Alhamdulillah" to the Great Almighty Allah, the Author of knowledge and wisdom for His countless love and blessings.

TABLE OF CONTENTS

ACKNOWLEDGMENTS	iv
LIST OF TABLES	vii
LIST OF FIGURES	viii
ABSTRACT (ENGLISH)	x
ABSTRACT (ARABIC)	xii
CHAPTER 1 INTRODUCTION	1
1.1 STYLIZED FACTS OF FINANCIAL DATA	7
1.2 RISK MEASURES	10
1.2.1 VALUE AT RISK (VaR)	11
1.2.2 EXPECTED SHORTFALL (ES)	12
CHAPTER 2 LITERATURE REVIEW	14
CHAPTER 3 RESEARCH METHODOLOGY	21
3.1 Dagum Distribution	25
3.2 Rayleigh Distribution	28
3.3 Loglogistic Distribution	31
3.4 Parameter Estimation methods	34
3.5 Maximum Likelihood Estimation	39
3.6 Kurtosis and Skewness	41

3.7	Bootstrap	42
3.7.1	Bootstrap algorithm	45
3.7.2	Bias	46
CHAPTER 4 DATA ANALYSIS, RESULT AND INTERPRETA-		
	TION	47
4.1	Estimated Results and Discussion	52
4.2	Kurtosis and Skewness plots	54
4.3	VaR plots of BT and WTI's daily log-returns	55
CHAPTER 5 SUMMARY, CONCLUSION AND RECOMMENDA-		
	TION	66
REFERENCES		68
VITAE		79

LIST OF TABLES

4.1	Summary statistics of daily log-returns of BT and WTI.	49
4.2	Estimated parameters of the fitted models on log-returns of BT crude oil prices	53
4.3	Estimated parameters of the fitted models on log-returns of WTI crude oil prices	53
4.4	The values of the five selection criteria and the log-likelihood for the proposed models	54

LIST OF FIGURES

1.1	Illustration of VaR and ES.	6
3.1	Diagram of Least Square methods	37
4.1	Histograms of BT Crude Oil and WTI Crude Oil Prices	50
4.2	Box plots of daily stock prices for BT and WTI crude oil prices	51
4.3	Time series plots of daily log-returns of BT and WTI stock	52
4.4	Graphs of kurtosis value versus ν for the model based on $g(.) \sim \text{Dagum}$ distribution	55
4.5	Graphs of kurtosis value versus ν for the model based on $g(.) \sim$ Rayleigh distribution	56
4.6	Graphs of kurtosis value versus ν for the model based on $g(.) \sim \text{Loglo-}$ gistic distribution	57
4.7	VaR plots of BT's log-returns and WTI's log returns based on the models	58
4.8	VaR plots of BT's log-returns and WTI's log returns based on the models	58
4.9	Parameter estimates of model based on Dagum distribution on BT's log-returns	60
4.10	Parameter estimates of model based on Dagum distribution on WTI's log-returns	61
4.11	Parameter estimates of model based on Rayleigh distribution on BT's log-returns	62
4.12	Parameter estimates of model based on Rayleigh distribution on WTI's log-returns	63

4.13	Parameter estimates of model based on Loglogistic distribution on BT's log-returns	64
4.14	Parameter estimates of model based on Loglogistic distribution on WTI's log-returns	65

THESIS ABSTRACT

NAME: HAADY JALLOW
TITLE OF STUDY: DISTRIBUTIONAL MODELS FOR STOCK RETURNS
OF CRUDE OIL PRICES
MAJOR FIELD: APPLIED STATISTICS
DATE OF DEGREE: NOVEMBER 2023

An essential challenge in financial risk management is the characterization and quantification of financial market risks as well as the accompanying volatility. Value at Risk (VaR) and Expected Shortfall (ES) are two of the most widely used methods for assessing financial risk. However, the effectiveness of these risk assessment methodologies, is dependent on the probability distributions that are assume for the behavior of the financial returns. In this thesis, mixture models for the conditional distribution of daily stock returns are proposed. The models are dependent on the Gosset's distribution (Student's t) with its variance assumed to be a random variable. The mathematical properties of these models, including their moments, CDFs, PDFs, and characteristic functions are derived. To illustrate the flexibility and usefulness of the proposed distributions, these models are employed to analyze crude oil stock prices.

We employed the MLE approach to estimate our model parameters. To access the accuracy of our estimation method and stability of the parameters, we used the bootstrapping technique.

ملخص الرسالة

الاسم: هادي جالو

عنوان الدراسة: النماذج التوزيعية لعوائد الأسهم لأسعار النفط الخام

التخصص: الإحصاء التطبيقي

تاريخ الدرجة العلمية: نوفمبر 2023

يتمثل أحد التحديات الأساسية في إدارة المخاطر المالية في توصيف مخاطر الأسواق المالية وتحديد كميا بالإضافة إلى التقلبات المصاحبة لها. القيمة المعرضة للخطر والنقص المتوقع هما من أكثر الطرق استخداما لتقييم المخاطر المالية. ومع ذلك ، فإن فعالية منهجيات تقييم المخاطر هذه ، تعتمد على التوزيعات الاحتمالية التي تفترض لسلوك العوائد المالية. ومع ذلك ، فإن فعالية منهجيات تقييم المخاطر هذه ، تعتمد على التوزيعات الاحتمالية التي تفترض لسلوك العوائد المالية. في هذه الأطروحة ، يتم اقتراح نماذج خليط للتوزيع الشرطي لعوائد الأسهم اليومية. تعتمد النماذج على توزيع t للطالب مع افتراض تباينه كمتغير عشوائي يتم اشتقاق الخصائص الرياضية لهذه النماذج ، بما في ذلك لحظاتها ، ووظائف التوزيع التراكمي ، والوظائف المميزة. لتوضيح مرونة وفائدة التوزيعات المقترحة ، يتم استخدام هذه النماذج لتحليل أسعار مخزون النفط الخام. استخدمنا طريقة الاحتمال الأقصى لتقدير معلمات نموذجنا. للوصول إلى دقة طريقة التقدير الخاصة بنا واستقرار المعلمات ، استخدمنا تقنية التمهيد.

CHAPTER 1

INTRODUCTION

Crude oil is a mixture of hydrocarbon and organic materials that are formed from the remains of plants and animals. It is the primary fuel source in the modern world. It is typically liquid in nature and is collected through drilling and pumping before being transferred via pipelines and tankers. Its industry is the biggest and most important in the entire world of commerce, and its economic price is one of the most intricate and unpredictable systems on the commodities market [1].

The two important oil contracts monitored by oil participants in the oil market are West Texas Intermediate (WTI) crude and Brent (BT) Crude. In Northwest Europe, Africa and Middle East, Brent Crude oil is typically refined and consumed in large amounts. It is an ideal alternative for generating gasoline and middle distillates due to its features, such as being a light blend of crude oils from several locations in the North Sea. When compared to WTI crude oil, BT's crude oil API gravity of 38.8 indicates that it is less sweet [2].

Similar to BT, WTI is a sweet and light crude oil. Because of its high quality, it

is suited for the manufacturing of huge amounts of gasoline. It has an API gravity of 39.6 and a sulphur content of about 0.24%. USA, the nation that consumes the most gasoline, refines WTI [3].

Since crude oil is considered as the most significant commodity in the world market and given the need of an increasing amount of oil to support economic growth and subsequent rise in energy consumption, emerging industrial markets like China, India, and Latin America have a major impact on its price. However, due to the difficulty of discovering and developing new oil reserves in recent years, the supply is limited and cannot be increased further. As a result, the price of the commodity has a significant impact on the growth of the global economy, since it is needed in all industrialized countries. It also has the necessary raw material to produce plastics, cosmetics, pharmaceuticals products and other basic materials which makes it more essential as a commodity in the market.

Due to high demand for crude oil and living in a high data-rich environment, the economies and financial markets are more integrated than ever before, which has greatly enhanced the role of quantitative methods in business and finance for decision making. This market integration has made risk characterization and estimation more demanding. In business and financial management, risk quantification is generally focused on defining price/return as well as characterizing uncertainty caused by market fluctuations. Given the importance of risk and their numerous effects in current day economic scenario, several risk measures and their estimation methods have been developed in the last few decades. See for example, Lori and Gabbi [4] and the

references therein. The return, which is often seen as a random variable, is the primary attribute of every financial asset. The dispersion in the spread of the outcomes of the variable of interest is known as asset's volatility and this plays a vital role in many financial applications and related areas.

Volatility in financial term means the rate at which security prices changes over a given period. That is, the amount of uncertainty or risk related to the size of change in a security's value such as stocks, bonds, or commodities, over a specific period of time. It helps financial investors to estimate the fluctuations that may happen to stock or security prices in the future. A price movement that is reasonably consistent and predictable is implied by low volatility, whereas high volatility denotes substantial price swings. Events in the economy and politics, market sentiment, supply and demand dynamics, and investment behavior are just a few examples of the variables that might affect volatility. Sometimes in risk and market risk management; volatility plays a crucial part as Traders and investors often use it as an important factor in pricing financial derivatives[5],[6]. A volatility parameter is used as the price evaluation criterion in all contemporary option pricing systems, and both general portfolio management and risk management assessment.

Financial institutions/markets must be able to forecast the future values of the controlled assets, in addition to understanding their current volatility, and for businesses that manage portfolios and trade options, predicting volatility is particularly crucial. The intricate relationships between financial indicators, which are frequently confusing and counter intuitive, make it difficult to predict how their values will behave in

the future. This makes predicting the volatility's behaviour difficult, even for financial professionals. The relationship between the existing values of financial indicators and their anticipated future values can be constructed with the aid of mathematical models. Quantitative projections based on models can give financial organizations a useful indication of a potential market trend; even though some experts think future events are unforeseeable, there is some empirical evidence to the contrary. For example, it is well known fact that volatility clustering is an important characteristic of financial data, it refers to the observation, first noted by Mandelbrot (1963), that "large changes tend to be followed by large changes, of either sign, or small changes tend to be followed by small changes." Thus, it is a conditional in financial time series where bigger shocks tend to be followed by big shock in both direction, and small shocks tend to follow small shocks [6]. Shocks are arbitrary, unpredictable events that have a significant effect on the economy and are things outside the purview of economic models.

To appropriately describe and capture the evaluations of volatility, it is important to investigate models based on distributions[7]. Specifically, Assuming we take X to be a continuous random variable that represent the financial data series of interest; in our case, log-returns of WTI and BT Crude Oil Stock prices. Suppose the probability density function and cumulative distribution function of the random variable X are $f_x(x)$ and $F_x(x)$, respectively. Then, the VaR and ES of X for a given p are , respectively defined mathematically as:

$$VaR_p(x) = F^{-1}(p) \tag{1.1}$$

or

$$VaR_p(x) = \inf\{\mu : F(\mu) \geq p\} \quad (1.2)$$

and

$$ES_p(x) = \frac{1}{p} \int_0^p VaR_t(x) dt \quad (1.3)$$

where $p = \alpha \in [0, 1]$. Obviously, from equation(1.1) to equation(1.3) we can see that these two measures are tail quantiles. For instance, VaR is the quantile of F exceeded with probability $1 - p$. Figure 1.1 clearly show the definition given by equations (1.1) to equation (1.3).

Although the applications of these two measure have been extensive; however, their overall performance depends on the choice of the probability distribution assume for them. For instance, financial data are known to have larger tails than that of the normal distribution. Therefore, if one employs normal type of model as a candidate distribution for financial data, this will amount to poor representation of the data and by extension yield poor estimates for VaR and ES. Thus, to accurately estimate risk that are associated with financial data of any kind using VaR and ES, it is reasonable to assume a heavy tail distribution [8]. One of the most widely used heavy tail financial return model is the Student's t distribution. William Gosset made the discovery of the Student's t distribution in 1908. We can derive this distribution from the normal random variable. For instance, if we let $Z_1, Z_2, \dots, Z_n$ to represent independent and identically distributed normal random variables, $N(\mu, \sigma^2)$. Suppose that $\bar{Z}$ and S^2 are the associated sample mean and variance defined by

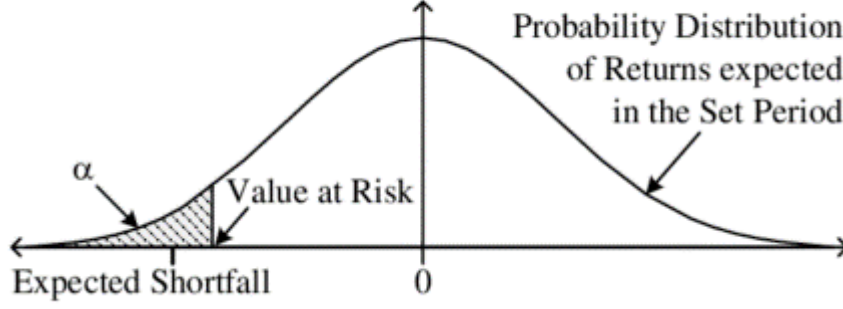


Figure 1.1: Illustration of VaR and ES.

$$\bar{Z} = \frac{1}{n} \sum_{i=0}^n Z_i \quad (1.4)$$

and

$$S^2 = \frac{1}{n-1} \sum_{i=0}^n (Z_i - \bar{Z})^2 \quad (1.5)$$

In that case $X = \frac{\bar{Z} - \mu}{S/\sqrt{n}}$ becomes a Student's t random variable with $n - 1$ degrees of freedom. Now, the resulting probability density and cumulative distribution functions of a Student's t random variable with $v > 0$ degrees of freedom are

$$f(x) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi\sigma^2}\Gamma(\frac{v}{2})} \left(1 + \frac{x^2}{v\sigma^2}\right)^{-\frac{v+1}{2}} \quad (1.6)$$

for $-\infty < x < \infty$ and where σ^2 denotes the variance.

$$F_X(x) = \frac{1}{2} + \frac{x\Gamma(\frac{v+1}{2})}{\sqrt{\pi v\sigma^2}\Gamma(\frac{v}{2})} {}_2F_1\left(\frac{1}{2}; \frac{v+1}{2}; \frac{3}{2}; \frac{-x^2}{\sigma^2 v}\right) \quad (1.7)$$

for $-\infty < x < \infty$, where ${}_pF_q(;;)$ denotes the generalized hypergeometric function

defined by:

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k z^k}{(b_1)_k (b_2)_k \dots (b_q)_k k!};$$

The fundamental drawback of equation (1.6) in financial econometric modeling, however, is that it cannot model asymmetric data [9]. Recently, more effort has been put towards generalizing this distributional model through econometric research. See for example this [10], and [11]. See also Li and Nadarajah [12] for a thorough examination of Student's t and its generalization. Although the formulations of these generalizations use various methodologies, they have expanded applications of the Student's t distribution to a number of additional domains. However, only few have employed the concept of scale mixing method. For example, see [6] and the references therein. With respect to distributional model development in finance, scale mixing method entails randomizing one of the parameters of the so called "parent" distribution. Distributions arising from this approach have shown to give answers to some of the stylized facts of asset returns such as Heavy-tailness and volatility clustering. See Figure 4.1 for pictorial/graphical representation of volatility clustering. The corresponding boxplots for the prices are given in Figure 4.1.

1.1 STYLIZED FACTS OF FINANCIAL DATA

Stylized facts are empirical observations that have been condensed into a theory-like format and are typically representative in a wide range of situations. Stylised facts are obtained by choosing a common denominator between the characteristics noted in the examinations of various markets and instruments. There is little doubt that

doing so increases generality but tends to impair the precision of statements about asset returns [13].

- Lack of autocorrelations: (Linear) autocorrelations of asset returns are usually negligible, with the exception of very short intraday time scales (less than 20 minutes), where microstructure effects are considerable.
- Tail dependence between different assets is high, marking copula fitting worthwhile.
- Fat-tails: The distribution of fluctuations indicated by the returns of market indices has slow-decaying tails, and the distributions appear to maintain the same functional structure for a variety of time scales. Although [14] discovered that the tail of the distribution is best explained by a power law, studies by Andersen and [15] suggest that the distribution of volatility is log-normal. Leptokurtic returns on assets are typical.
- Heterogeneity: In speculative returns, volatility clustering is pervasive, and its non-stationarity has been recognized since Houthakker (1961). Since the standard deviation of price movements, or volatility, is time-dependent in real markets, empirical investigations of financial data demonstrate that price changes cannot be explained by a strict-sense stationary stochastic process. As a result, the type of stationarity that exists in financial markets is, at most, asymptotic. The asymptotic PDF of price changes is discovered through the examination of sufficiently extended time series. The stochastic process' large-time statistical

characteristics are provided by the asymptotic PDF.

- **Scaling:** Comparisons and differences between price variations seen at various time scales (seconds, minutes, hours, days, months, and years) indicate important links between the various scales.
- **Correlation:** The significant correlation between price variability and "market activity" is a frequently observed phenomenon in the field of financial returns. Trading volume, return volatility, and bid-ask spreads peak around the opening and end of trade. During trading hours, there is a greater range in returns per unit of time compared to non-trading hours. Large return innovations are accompanied by particularly high trading volume and spreads. Public information releases also usually coincide with very high volume, which theoretically may create price increases without any trading.
- **Volatility clustering:** The rate of fluctuation in asset prices over a specific period is known as volatility. Basically, it refers to the degree of risk or uncertainty associated with the size of the movement in a security's value. Financial investors might forecast future stock price swings with the use of this information. And it is employed in market risk management and risk assessment. Volatility clustering is a crucial aspect of financial data, as is widely acknowledged. Observation made by Mandelbrot in 1963 that big changes tend to be followed by large changes, of either sign, and small changes tend to be followed by little changes is referred to as volatility clustering. Therefore, it is a precondition that bigger shocks tend to be followed by bigger shocks in both directions and smaller

shocks tend to be followed by smaller shocks in financial time series. Shocks are unforeseen, ad hoc occurrences that have a significant influence on the economy yet are outside the purview of economic models. Flexible models built on distributions should be taken into consideration to accurately represent and capture the volatility of the evaluations [6].

1.2 RISK MEASURES

Another important aspect of using financial data is risk measurement. Risk is characterized by return volatility in the Capital Asset Pricing Model (CAPM). One of the main issues with risk theory is the quantification of risk. Risk management techniques are frequently applied for this. There are many alternative risk measures available right now, such as simple probability of an adverse event, second order measures (variance or standard deviation, beta), quantile measures like value-at-risk and its derivatives, conditional value-at-risk [16],[17], expected shortfall [16], see [18]. Value at risk is the most famous measure of financial risk followed by expected shortfall. If the distribution of the data is continuous then expected shortfall is equivalent to tail conditional expectation. We are going to consider value at risk and expected shortfall to measure the risk associated with the available data set.

1.2.1 VALUE AT RISK (VaR)

It is a financial metric that estimates the risk of an investment. It is a statistical technique used to measure the amount of potential loss that could happen in an investment's portfolio over a specified period. Thus, it is a single, statistically summarized indicator of potential portfolio losses. VaR is a measurement of losses brought on by "normal" market fluctuations. Only with a certain low likelihood do losses higher than the value at risk occur. Subject to the simplifying assumptions used in its computation, It reports to regulators and combines all of the hazards in a portfolio into a single figure appropriate for use in the boardroom, or publication in an annual report. The idea of VaR is simple to comprehend once the use of a statistical metric has been overcome. It is merely a way to convey the size of the portfolio's anticipated losses. Even though the VaR concept is very straight forward, calculating it is challenging. To calculate a portfolio's VaR, three methodologies were initially developed; i) the parametric technique(also known as Variance-covariance technique), ii) the simulation approach based on historical values iii) the semi-parametric method which is based on the monte carlo simulation. All of these methodologies often referred to as standard models, have numerous drawbacks, which have prompted the creation of fresh ideas. According to [9], VaR has a number of drawbacks. For example, [19] demonstrated that value at risk ignores any loss above the VaR level and cannot meet one of the coherence axioms since it is not sub-additive. Furthermore, [20] have added two more drawbacks. One is that value at risk's information may deceive rational investors who want to maximize expected utility. Another is that it might be chal-

lenging to employ VaR when investors attempt to optimize their portfolios, [21]. The three approaches of VaR estimations are:

- Parametric Method: Risk is measured using parametric methods by fitting probability curves to the data and calculating the VaR from the fitted curve. Riskmetrics from Morgan 1996 was the first model to estimate VaR using a parametric approach. This model assumes that the return portfolio follows a normal distribution. Volatility models and density functions are also parametric methods used for VaR estimation.
- Non-parametric Method: The non-parametric approaches aim to calculate VaR of a portfolio without making any firm assumptions about the distribution of returns. The non-parametric approaches are historical simulation and non-parametric density estimation.
- Semi-parametric method: This method combines the non-parametric and the parametric approach. Volatility-weight historical simulation, filtered historical simulation (FHS), Conditional Autoregressive Value at Risk (CaViaR) method and the approach based on extreme value theory are the most important semi-parametric methodologies.

1.2.2 EXPECTED SHORTFALL (ES)

A financial matrix called expected shortfall assesses the risk of an investment when it exceeds VaR at a particular level. The projected return on the portfolio in the worst $k\%$ of scenarios is the expected shortfall "at $k\%$ level". Contrary to the more popular

VaR, it is a substitute that is more sensitive to the nature of the tail of the return distribution on a portfolio. The predicted shortfall, assuming VaR is measured with a 95% confidence level, would be the average loss in the 5% of instances where the fund exceeds its VaR. The ES is calculated by taking the average of the losses that exceed the VaR [22]. It provides information about the distribution of the losses beyond the VaR threshold and helps to quantify the potential losses that may exceed the VaR estimate. In simple terms expected shortfall is a measure of the amount of money that a portfolio or investment is expected to lose beyond a certain level of confidence. It is an essential tool for risk managers, traders, and investors who need to manage risk beyond the traditional VaR limits. It has the following properties:

- 1 The ES (ES_q) increases as q decreases.
- 2 The 100% quantile ES (ES_1) equals negative of the expected value of the portfolio.
- 3 For a given portfolio, the expected shortfall ES_q is greater than or equal to the value at Risk VaR_q at the same q level.

CHAPTER 2

LITERATURE REVIEW

The Student's t distribution due to Gosset is the most popular model used for characterizing economics and financial data. The popularity of this distribution in modeling financial data is due to the fact that, it manages small sample sizes and outliers better than other renowned distributions like the normal distribution. Student's t inability to model asymmetric data, however, is one of its biggest flaws. Given this shortcoming, the Student's t distribution has undergone a lot of generalizations recently so as to accommodate more intricate data structures. For example, [23] proposed a generalization of the student's t distribution and their generalization includes the student's t distribution and the generalized error distribution (GED) as special cases. The flexibility of their model was demonstrated by modeling volatility associated with daily observations on returns from `iShare silver Trust`. [24] developed a general form of student's t and employed it for static regression techniques. Their proposed model encompasses the power exponential or the so called Box-Xiao distribution, the Gaussian distribution, the Laplace distribution, and the student's t distribution as

special cases. A skewed version of the generalization of the student's t distribution due [24] was later proposed by [25]. [25]'s extension of the t distribution contains a number of other distributional models, including: the power exponential, Laplace, normal, and uniform distributions, as well as the symmetric and skewed student's t distributions. The flexibility of the proposed model was done using financial data. In specific terms, the study derives the mathematical moments and other characteristics of the distribution, evaluates their applicability to fit the empirical distribution of a number of financial series with skewness and excess kurtosis. These include: three financial stock indices, two financial exchange rates and one precious stone commodity. Financial stock indices are the S&P500 for US; the TSE300 for Canada and the Topix for Japan. Exchange rates are the Canadian dollar and Japanese yen against the U.S. dollar exchange. The commodity is the gold prices. In all cases, the skewed extension of the general form of the student's t consistently offered a strong fit to the empirical distribution of these financial series.

In a bid to account for both signs (i.e positive and negative) degrees of freedom, [26] proposed a natural extension of the student's t distribution. Their proposed model shows a variety of interesting mathematical characteristics. For instance, under the classic student's t distributional framework, this model may be used to describe the uncertainty in the distributional tail and to characterize both the exponential and heavy-tailed distributional properties. They utilized pharmaceutical data that has to do with drug-induced liver toxicity problems to demonstrate their model's application. [27] proposed a folded student's t distribution based on a practical problem

in a validation process. They also studied the relationship between their proposed model and other distributions. [28] proposed and discussed the discrete student's t distribution. He noted that the student's t yields a larger kurtosis when compared numerically to the symmetric binomial. [29] derived a doubly truncated Student's t distribution using scale mixture based on doubly truncated standard normal distribution. Nevertheless, by putting two scaled halves of the Student's t distribution together, a skewed variant of the Student's t distribution was proposed by Fernandez and Steel in 1998. Similarly, [30] proposed a skewed variant of t distribution and they illustrated the flexibility of their proposed model with two data sets. More also, [31] discussed and proposed the skewed version of the Student's t distribution. In their work, they thoroughly examined the probabilistic properties as well as inferential and other statistical issues of their proposed model. Real life example was performed to some multivariate statistics issues. Following the $\sin h - \arcsin h$ transformation due to [32] for generating class of distributions, [33] employed this transformation to develop a version of their skewed t distribution. The fundamental characteristics of the distributions of the $\sin h - \arcsin h$ family were studied. They equally illustrated the flexibility of the proposed distribution by employing it to characterize the heavy-tailed and skew data based on fibre glass strengths. By employing a new skewing procedure, [34] developed a variant of generalization of the skewed distribution due to [24]. As opposed to earlier skew versions of the generalization of the t distribution, their proposed model encompasses some of the most popular density functions such as the normal distribution, the classical Student's t , the power exponential distribution, the

Laplace distribution etc. Their proposed model is also a good candidate distribution for describing bimodal data sets. In the same fashion, [10] generated a distribution called generalized hyperbolic skew student's t distribution which has an important property in its tails. That is one of its tail behaves exponentially while the other has polynomial behaviour. The model is a special case of the GH family of distributions. They employed their proposed model to quantify different stock indices, including Norwegian stock, International bonds, as well as exchange rates associated with their countries of interest. Their proposed distribution performed better than the normal inverse Gaussian distribution. In the same vein, [35] proposed and examined the mathematical characteristics of a family of generalised beta distributions, their model includes other notable distributions such skewed- t . They employed this distribution to study the size distribution of income. In terms of measures of goodness of fit, they compared their proposed model with the commonly used generalised beta distributions of the first and second types. Also, based on the beta distribution, [36] proposed the beta skewed t distribution. Recently, [37] developed the asymmetric Student's t distribution. This generalization is arguably, the most general form of the Student's t distribution to date, as many notable extensions of the Student distribution are particular cases of the asymmetric Student's t distribution. For instance, the classical distributional model due to Gosset (1908), the generalised Student's t model due to [10], the generalised Student's t distribution owing to Hansen (1994), [38], and others. More recently, [39] put forward the Kumaraswamy version of skewed t distribution. They equally performed a thorough review of generalization of Gosset's distribution.

Their review comprised of over 30 extensions based on the distribution due to Gosset. They equally used 11 out of the thirty generalized forms of the Student's t distribution and performed a comparison among them using a set of data from Internal finance corporation. The distributions were AST, FSST, JFST, ACST, ST, KT, ACIST, BST, KST, STN, NCST. Using the maximum likelihood method of fitting the distributions, AST gave the best fit while the NCIST and ACIST performed gave the worst fit.

It is worth mentioning that these extensions are done by either twerking or reparameterization of the base line distribution. However, there are other methods like scale mixing techniques. And several authors have employed this approach with the base distribution being considered to be normal. For example, [40] proposed the smooth adaptive method based on Gaussian mixture (SAGM) model as a versatile approach for regression density estimation. They used their model to describe daily stock market return with the sole aim of capturing volatility via lagged returns. Similar work in Bayesian context was also done by [41]. The R package called bayesGARCH was one of the outcomes of the projects. An emperical application of that package was illustrated on foreign exchange rate log-returns. It is also important to mention that numerous scale combinations of normal distribution have been suggested in the literature. For instance [42] were among the first to introduce variance-mean mixture(VMM) distributions and later it was studied extensively by blaesild (1981) and [43] in order to build models that can combine skewness with heavy tails. [44] extended it and proposed the VMM of the multivariate skew normal. Similarly, by using the approach of compounding McDonald (1982) developed distributions for stock

returns and income, as well as models for unemployment duration. [45] propose a Robust mixture model using the t -distribution. The model was applied to a set of data that has atypical/unusual observations. They found that the model outperform the mixture of normal distribution. [46] propose exponentiated generalized exponential Dagum distribution (EGEDD). The properties was also studied with some applications. [47] proposed a new extended Rayleigh distribution with application to covid-19 data. They generated a new distribution by combining Rayleigh distribution and extended odd weibull family called odd weibull Rayleigh (EOWR). A comparison on the two estimations were perform, bayesian estimation method and classical method. [48] proposed a new class of generalized Dagum distribution called Gamma Dagum with application to income and lifetime data. The proposed model was compared with gamma exponentiated Weibull distribution. Nadarajah [49] generated sixteen normal variance models, these models where generated by taking the normal conditional stock return distribution with its variance being random variable and compounded with the distribution of the variance. Many authors have studied the properties of the normal variance model and many of its extension have being proposed too. [6] proposed six models based on classical model due Gosset (1908) and performed a comparison between the six distributions and the two well known distribution within the GARCH(1,1) model with generalized hyperbolic and asymmetric Student's t innovations. The compound generalized gamma innovations and Frechet innovations performed better than the well known models. [50] introduced an EGARCH model in which the dynamic equation for the logarithm of scale is driven by the conditional

score. In their set up, they employed the generalized t distribution as the distributional models for innovations. The proposed model is very flexible, especially for accounting for outliers in a set of data. As noted earlier, the Student t and Gaussian mixture model have been popular in modeling stock indices. However, financial data are known to have large tails. Due to their lack of heavy tails, normal type distributions might not be appropriate models for these class of financial data sets. Hence the motivation for this work.

CHAPTER 3

RESEARCH METHODOLOGY

Now, we begin by defining the fundamental framework of the proposed model. The functional form is based on scale mixing method, which give rise to a class of distribution known as compound model. More formally, let $\{X_i, i = 1, 2, \dots, n\}$ denote a stationary financial series that represent the asset return of BT crude oil and WTI crude oil prices. Then, if we assume that the conditional distribution for these asset returns of BT crude oil and WTI crude oil prices follow the Student' t with the PDF given by

$$f(x|\sigma^2) = \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi\sigma^2}\Gamma(\frac{v}{2})} \left(1 + \frac{x^2}{v\sigma^2}\right)^{-\frac{v+1}{2}} \quad (3.1)$$

where $-\infty < x < \infty$ and σ^2 denotes the variance or the scale parameter of the model.

If we randomize σ^2 such that we assume it to follow another distribution, say $g(\cdot)$.

Then, we can specify the unconditional density function for these asset returns as

$$f(x) = \int_0^\infty \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi\sigma^2}\Gamma(\frac{v}{2})} \left(1 + \frac{x^2}{v\sigma^2}\right)^{-\frac{v+1}{2}} g(\sigma^2) d\sigma^2 \quad (3.2)$$

Equation(3.2) appeared to be complicate, for simplicity let $\tau = \sigma^2$, we rewrite it

as

$$f(x) = \int_0^\infty \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi\tau}\Gamma(\frac{v}{2})} \left(1 + \frac{x^2}{v\tau}\right)^{-\frac{v+1}{2}} g(\tau) d\tau \quad (3.3)$$

simplifying the above equation (3.3);

$$\begin{aligned} f(x) &= \int_0^\infty \frac{\Gamma(\frac{v+1}{2})}{\sqrt{v\pi\tau}\Gamma(\frac{v}{2})} \left(\frac{x^2}{v\tau}\right)^{-\frac{v+1}{2}} \left(1 + \frac{v\tau}{x^2}\right)^{-\frac{v+1}{2}} g(\tau) d\tau \\ &= \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \int_0^\infty \tau^{\frac{v}{2}} \left(1 + \frac{v\tau}{x^2}\right)^{-\frac{v+1}{2}} g(\tau) d\tau \end{aligned} \quad (3.4)$$

Using the series expansion

$$(1 + p)^{-q} = \sum_{k=0}^{\infty} \binom{-q}{k} p^k, \quad (3.5)$$

therefore we can rewrite f(x) as follows

$$\begin{aligned} &= \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \int_0^\infty \tau^{\frac{v}{2}} \sum_{k=0}^{\infty} \binom{-\frac{v+1}{2}}{k} \left(\frac{v\tau}{x^2}\right)^k g(\tau) d\tau \\ &= \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \int_0^\infty \tau^{\frac{v}{2}+k} \sum_{k=0}^{\infty} \binom{-\frac{v+1}{2}}{k} \frac{v^k}{x^{2k}} g(\tau) d\tau \\ &= \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \sum_{k=0}^{\infty} \binom{-\frac{v+1}{2}}{k} \frac{v^k}{x^{2k}} \int_0^\infty \tau^{\frac{v}{2}+k} g(\tau) d\tau \end{aligned} \quad (3.6)$$

we can go further and simplify the term $\sum_{k=0}^{\infty} \binom{-\frac{v+1}{2}}{k} \frac{v^k}{x^{2k}}$

$$f(x) = \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \sum_{k=0}^{\infty} \binom{v+1}{k} \frac{(-v)^k}{k!x^{2k}} \int_0^\infty \tau^{\frac{v}{2}+k} g(\tau) d\tau \quad (3.7)$$

For simplicity let $\omega = \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \sum_{k=0}^{\infty} \left(\frac{v+1}{2}\right)_k \frac{(-v)^k}{k!x^{2k}}$ then

$$f_X(x) = \omega \int_0^{\infty} \tau^{\frac{v}{2}+k} g(\tau) d\tau \quad (3.8)$$

We note here that equation (3.7), which is in general form will be applied to generate models model for the log-returns of BT crude oil and WTI crude oil prices. Though not in a close form yet, however once we define specific model expressions for each possible mixing distribution we obtain the close form of the required models. But before then, we have to obtain the general form of some of the properties of (3.7).

The cumulative distribution function, the k^{th} moment and the characteristic function of X corresponding to (3.8) all in general form are defined by

$$F_X(x) = \frac{1}{2} + \frac{x\Gamma(\frac{v+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{v}{2})} \int_0^{\infty} \frac{1}{\sqrt{\tau}} {}_2F_1\left(\frac{1}{2}, \frac{v+1}{2}; \frac{3}{2}; \frac{-x^2}{\tau\nu}\right) g(\tau) d\tau \quad (3.9)$$

for $-\infty < x < \infty$. Using the series expansion

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{\Gamma(b-a)\Gamma(c)(-z)^{-a}}{\Gamma(b)\Gamma(c-a)} \sum_{k=0}^{\infty} \frac{(a)_k(a-c+1)_k z^{-k}}{k!(a-b+1)_k} \\ &+ \frac{\Gamma(a-b)\Gamma(c)(-z)^{-b}}{\Gamma(a)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{(b)_k(b-c+1)_k z^{-k}}{k!(b-a+1)_k}, \end{aligned} \quad (3.10)$$

using equation (3.10), 3.9 can be written as

$$F_X(x) = \frac{1}{2} + \frac{x\Gamma(\frac{v+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{v}{2})} \int_0^\infty \frac{1}{\sqrt{\tau}} \frac{\Gamma(\frac{-v}{2})\Gamma(\frac{3}{2})(\frac{x^2}{\tau\nu})^{-\frac{\nu+1}{2}}}{\Gamma(\frac{1}{2})\Gamma(1-\frac{v}{2})} \sum_{k=0}^\infty \frac{(\frac{1+v}{2})_k(\frac{\nu}{2})_k}{(1+\frac{\nu}{2})_k k!} \left(\frac{-x^2}{\tau\nu}\right)^{-k} g(\tau) d\tau$$

$$F_X(x) = \frac{1}{2} - \frac{v^{\frac{v}{2}-1}\Gamma(\frac{v+1}{2})}{x^v\sqrt{\pi}\Gamma(\frac{v}{2})} \sum_{k=0}^\infty \frac{(\frac{1+v}{2})_k(\frac{\nu}{2})_k}{(1+\frac{\nu}{2})_k k!} \left(\frac{x^2}{v}\right)^{-k} \int_0^\infty \tau^{(k+\frac{v}{2})} g(\tau) d\tau \quad (3.11)$$

The k^{th} moment of X is given as

$$E(X^k) = E[E(X^k|\tau)] = \left[\int_0^\infty \tau^{\frac{k}{2}} g(\tau) d\tau \right] \frac{1 + (-1)^k}{2\sqrt{\pi}\Gamma(\frac{v}{2})} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{v-k}{2}\right) v^{\frac{k}{2}} \quad (3.12)$$

On the condition that $0 < k < v$.

$$E[\exp(itX)] = \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{\Gamma(\frac{v}{2})} \sum_{k=0}^\infty \frac{(\sqrt{v}|t|)^k}{2^{2k} k! \Gamma(k - \frac{v}{2} + 1)} \left[\int_0^\infty \tau^k g(\tau) d\tau \right]$$

$$- \frac{(\sqrt{v}|t|)^{\frac{3v}{4}} \csc(\frac{\pi v}{2})}{2^v \Gamma(\frac{v}{2})} \sum_{k=0}^\infty \frac{(\sqrt{v}|t|)^k}{2^{2k} k! \Gamma(k + \frac{v}{2} + 1)} \left[\int_0^\infty \tau^{k+\frac{v}{2}} g(\tau) d\tau \right]$$

$$= \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{\Gamma(\frac{v}{2})\Gamma(1-\frac{v}{2})} \sum_{k=0}^\infty \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1-\frac{v}{2})_k} \left[\int_0^\infty \tau^k g(\tau) d\tau \right]$$

$$- \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{2^v \Gamma(\frac{v}{2})\Gamma(1+\frac{v}{2})} \sum_{k=0}^\infty \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1+\frac{v}{2})_k} \left[\int_0^\infty \tau^{k+\frac{v}{2}} g(\tau) d\tau \right] \quad (3.13)$$

respectively. Notably, these equations: (3.7), (3.11), (3.12) and (3.13) are not in a close form, but their close form expressions will be obtained with the specific model expressions for each possible mixing distribution. In our case, we consider the following distributions: Dagum distribution, Loglogistic distribution and Rayleigh distribution

as our choices for $g(\cdot)$.

3.1 Dagum Distribution

Taking $g(\tau)$ to be a Dagum distribution with

$$g(\tau) = \frac{ap}{\tau} \left[\frac{\left(\frac{\tau}{b}\right)^{ap}}{\left(\left(\frac{\tau}{b}\right)^a + 1\right)^{p+1}} \right]$$

for $x > 0$ where $a, b, p > 0$ This distribution is a continuous probability distribution defined over positive real numbers. This distribution is mostly applied in the study of income distribution. It is used in modeling inequality and poverty index, survival analysis and reliability studies [51] and more of its applications see [52], [53], [54] and [55]. The PDF, CDF and moments of X as $g(\tau)$ follows Dagum distribution is given below respectively:

$$f_X(x) = \omega \int_0^\infty \tau^{\frac{v}{2}+k} \frac{ap}{\tau} \left[\frac{\left(\frac{\tau}{b}\right)^{ap}}{\left(\left(\frac{\tau}{b}\right)^a + 1\right)^{p+1}} \right] d\tau$$

if we take $n = k + \frac{v}{2}$

$$f_X(x) = \omega \frac{ap}{b^{ap}} \int_0^\infty \tau^{n-1+ap} \left[\frac{1}{\left(\left(\frac{\tau}{b}\right)^a + 1\right)^{p+1}} \right] d\tau$$

$$\text{let } u = \frac{\tau}{b} \Rightarrow \tau = bu, du = \frac{d\tau}{b} \Rightarrow d\tau = bdu$$

Substituting τ , $d\tau$ we have

$$\begin{aligned}
f_X(x) &= \omega \frac{ap}{b^{ap}} \int_0^\infty (bu)^{n-1+ap} \left[\frac{1}{((u)^a + 1)^{p+1}} \right] b du \\
&= \omega ap b^n \int_0^\infty \frac{u^{n-1+ap}}{(u^a + 1)^{p+1}} du
\end{aligned} \tag{3.14}$$

using the given identity below in equation (3.14):

$$\int_0^\infty x^{u-1} (1 + bx^p)^{-v} dx = \frac{1}{p} b^{-\frac{u}{p}} \beta \left[\frac{u}{p}, v - \frac{u}{p} \right] \tag{3.15}$$

where

$$u=n+ap, \quad p=a, \quad v=p+1, \quad b=1$$

$$\begin{aligned}
f_X(x) &= \omega ap b^n \int_0^\infty \frac{u^{n-1+ap}}{(u^a + 1)^{p+1}} du = \omega \frac{1}{a} ap b^n \beta \left[\frac{n+ap}{a}, p+1 - \frac{n+ap}{a} \right] \\
&= \omega p b^n \frac{\Gamma(\frac{n+ap}{a}) \Gamma(p+1 - \frac{n+ap}{a})}{\Gamma(\frac{n+ap}{a} + p+1 - \frac{n+ap}{a})} \\
&= \omega p b^n \frac{\Gamma(\frac{n+ap}{a}) \Gamma(\frac{a-n}{a})}{\Gamma(p+1)} \\
&= \omega b^n \frac{\Gamma(\frac{n+ap}{a}) \Gamma(\frac{a-n}{a})}{\Gamma(p)}
\end{aligned} \tag{3.16}$$

Now, substituting ω in equation (3.16) we have our new model base on Dagum distri-

bution

$$f_X(x) = \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \sum_{k=0}^{\infty} \left(\frac{v+1}{2}\right)_k \frac{(-v)^k}{k!x^{2k}} b^{\frac{v}{2}+k} \frac{\Gamma(\frac{\frac{v}{2}+k+ap}{a})\Gamma(\frac{a-k-\frac{v}{2}}{a})}{\Gamma(p)} \quad (3.17)$$

CDF of the model

$$F_X(x) = \frac{1}{2} - \frac{v^{\frac{v}{2}-1}\Gamma(\frac{v+1}{2})}{x^v\sqrt{\pi}\Gamma(\frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\frac{1+v}{2})_k}{(\frac{v}{2})_k} k! (1 + \frac{v}{2})_k \left(\frac{x^2}{v}\right)^{-k} b^{\frac{v}{2}+k} \frac{\Gamma(\frac{v+2k}{2a} + p)\Gamma(1 - \frac{v+2k}{2a})}{\Gamma(p)} \quad (3.18)$$

the k^{th} moment of x given as follows;

$$E(X^k) = E[E(X^k|\tau)] = \left[\int_0^{\infty} \tau^{\frac{k}{2}} \frac{ap}{\tau} \left(\frac{(\frac{\tau}{b})^{ap}}{((\frac{\tau}{b})^a + 1)^{p+1}} \right) d\tau \right] \frac{1 + (-1)^k}{2\sqrt{\pi}\Gamma(\frac{v}{2})} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{v-k}{2}\right) v^{\frac{k}{2}} \quad (3.19)$$

solving the integral part we have:

let

$$u = \frac{\tau}{b}; ub = \tau; bdu = d\tau$$

$$b^{\frac{k}{2}}ap \int_0^{\infty} \frac{u^{\frac{k}{2}+ap}}{(u^a + 1)^{p+1}} du = b^{\frac{k}{2}}ap \int_0^{\infty} u^{\frac{k}{2}+ap+1-1} (u^a + 1)^{-(p+1)}$$

using the following identity

$$\int_0^{\infty} x^{u-1} (1 + bx^p)^{-v} dx = \frac{1}{p} b^{-\frac{u}{p}} B\left[\frac{u}{p}, v - \frac{u}{p}\right]$$

we have

$$pb^{\frac{k}{2}}B\left[\frac{\frac{k}{2} + ap + 1}{a}, p + 1 - \frac{\frac{k}{2} + ap + 1}{a}\right] \quad (3.20)$$

$$\therefore E(X^k) = \frac{1 + (-1)^k}{2\sqrt{\pi}\Gamma(\frac{v}{2})} \Gamma(\frac{k+1}{2}) \Gamma(\frac{v-k}{2}) v^{\frac{k}{2}} p b^{\frac{k}{2}} B\left[\frac{\frac{k}{2} + ap + 1}{a}, p + 1 - \frac{\frac{k}{2} + ap + 1}{a}\right] \quad (3.21)$$

the characteristic function is given as:

$$E[\exp(itX)] = \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{\Gamma(\frac{v}{2})\Gamma(1 - \frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1 - \frac{v}{2})_k} b^k \frac{\Gamma(\frac{k}{2} + p) \Gamma(1 - \frac{k}{2})}{\Gamma(p)} \\ - \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{2^v \Gamma(\frac{v}{2}) \Gamma(1 + \frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1 + \frac{v}{2})_k} b^{\frac{v}{2} + k} \frac{\Gamma(\frac{\frac{v}{2} + k + ap}{a}) \Gamma(\frac{a - k - \frac{v}{2}}{a})}{\Gamma(p)} \quad (3.22)$$

3.2 Rayleigh Distribution

Taking $g(\tau)$ to be a Rayleigh Distribution with

$$g(\tau) = \frac{\tau}{b^2} \exp\left[-\frac{1}{2}\left(\frac{\tau}{b}\right)^2\right]$$

$x > 0$ where b is the scale parameter of the distribution. Some of its applications were employed in the field of ballistics(measuring weapon's precision), physical oceanography [56], control charts [57], [58]. See [59], [60], [61], [62], [63] for more application of the distribution. The PDF, CDF and moments of X as $g(\tau)$ follows Rayleigh distribution is given below respectively. The unconditional distribution of $f_X(x)$ is given as

$$f_X(x) = \omega \int_0^{\infty} \tau^{\frac{v}{2} + k} \frac{\tau}{b^2} \exp\left[-\frac{1}{2}\left(\frac{\tau}{b}\right)^2\right] d\tau \quad (3.23)$$

letting $n = k + \frac{v}{2}$ as before and let $u = \frac{1}{2}(\frac{\tau}{b})^2; \frac{du}{d\tau} = \frac{\tau}{b^2} \Rightarrow du = \frac{\tau d\tau}{b^2}$

$$\tau = 2^{\frac{1}{2}} u^{\frac{1}{2}} b$$

$$d\tau = \frac{b^2 du}{\tau} \Rightarrow d\tau = \frac{b^2 du}{2^{\frac{1}{2}} u^{\frac{1}{2}} b}$$

$$f_X(x) = \omega \frac{1}{b^2} \int_0^\infty \tau^{n+1} \exp[-\frac{1}{2}(\frac{\tau}{b})^2] d\tau \quad (3.24)$$

we replace τ and $d\tau$

$$\begin{aligned} &= \omega \frac{1}{b^2} \int_0^\infty \frac{(2^{\frac{1}{2}} u^{\frac{1}{2}} b)^{n+1} \exp(-u)}{2^{\frac{1}{2}} u^{\frac{1}{2}} b} b^2 du \\ &= \omega 2^n b^n \int_0^\infty u^{\frac{n}{2}} \exp(-u) du \end{aligned}$$

Rewriting the function as

$$= \omega 2^n b^n \int_0^\infty u^{\frac{n}{2}+1-1} \exp(-u) du \quad (3.25)$$

The integral part in equation (3.25) that is

$$\int_0^\infty u^{\frac{n}{2}+1-1} \exp(-u) du$$

is a Gamma function

$\therefore$

$$f_X(x) = \omega 2^n b^n \int_0^\infty u^{\frac{n}{2}+1-1} \exp(-u) du = \omega 2^n b^n \Gamma(\frac{n}{2} + 1) = \omega 2^n b^n (\frac{n}{2}) \Gamma(\frac{n}{2}) \quad (3.26)$$

Now, substituting ω in equation (3.26) we have our new model base on Rayleigh distribution

$$f_X(x) = \frac{\Gamma(\frac{v+1}{2})v^{\frac{v}{2}}}{\sqrt{\pi}\Gamma(\frac{v}{2})x^{v+1}} \sum_{k=0}^{\infty} \left(\frac{v+1}{2}\right)_k \frac{(-v)^k}{k!x^{2k}} 2^{(\frac{k}{2}+\frac{v}{4})} b^{(k+\frac{v}{2})} \left(\frac{k}{2} + \frac{v}{4}\right) \Gamma\left(\frac{k}{2} + \frac{v}{4}\right) \quad (3.27)$$

CDF of the model:

$$F_X(x) = \frac{1}{2} - \frac{v^{\frac{v}{2}-1}\Gamma(\frac{v+1}{2})}{x^v\sqrt{\pi}\Gamma(\frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\frac{1+v}{2})_k}{(\frac{v}{2})_k} k! \left(1 + \frac{v}{2}\right)_k \left(\frac{x^2}{v}\right)^{-k} 2^{(\frac{k}{2}+\frac{v}{4})} b^{(k+\frac{v}{2})} \left(\frac{k}{2} + \frac{v}{4}\right) \Gamma\left(\frac{k}{2} + \frac{v}{4}\right) \quad (3.28)$$

Using equation (3.12) we have the k^{th} moment is given as:

$$E(X^k) = E[E(X^k|\tau)] = \left[\int_0^{\infty} \tau^{\frac{k}{2}} \frac{\tau}{b^2} \exp\left(-\frac{\tau^2}{2b^2}\right) d\tau \right] \frac{1 + (-1)^k}{2\sqrt{\pi}\Gamma(\frac{v}{2})} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{v-k}{2}\right) v^{\frac{k}{2}} \quad (3.29)$$

solving the integral part;

$$\text{let } u = \frac{\tau^2}{2b^2};$$

$$\frac{du}{d\tau} = \frac{2\tau}{2b^2} \Rightarrow \frac{\tau}{b^2}$$

$$\frac{b^2 du}{\tau} = d\tau \text{ and } \tau = \sqrt{2b^2 u}$$

After substituting u and du we have

$$(2b^2)^{\frac{k}{4}} \int_0^{\infty} u^{\frac{k}{4}} \exp(-u) du = (2b^2)^{\frac{k}{4}} \int_0^{\infty} u^{\frac{k}{4}+1-1} \exp(-u) du = (2b^2)^{\frac{k}{4}} \Gamma\left(\frac{k}{4} + 1\right)$$

$$\therefore E(X^k) = \frac{1 + (-1)^k}{2\sqrt{\pi}\Gamma(\frac{v}{2})} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{v-k}{2}\right) v^{\frac{k}{2}} (2b^2)^{\frac{k}{4}} \Gamma\left(\frac{k}{4} + 1\right) \quad (3.30)$$

the characteristic function is given as:

$$\begin{aligned} E[\exp(itX)] &= \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{\Gamma(\frac{v}{2})\Gamma(1 - \frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1 - \frac{v}{2})_k} 2^k b^k \left(\frac{k}{2}\right) \Gamma\left(\frac{k}{2}\right) \\ &- \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{2^v \Gamma(\frac{v}{2})\Gamma(1 + \frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1 + \frac{v}{2})_k} 2^{k+\frac{v}{2}} b^{k+\frac{v}{2}} \left(\frac{v+2k}{4}\right) \Gamma\left(\frac{v+2k}{4}\right) \end{aligned} \quad (3.31)$$

3.3 Loglogistic Distribution

Taking $g(\tau)$ to be a Loglogistic Distribution the pdf is

$$g(\tau) = \frac{\alpha \tau^{\alpha-1}}{\beta^\alpha [1 + \frac{\tau}{\beta}]^2}$$

where $x > 0$, $\alpha > 0$, $\beta > 0$. In survival analysis, this distribution is used as a parametric model for events whose rate rises initially and falls over time, such as the mortality rate from cancer after diagnosis or treatment. It has been applied to hydrology models of precipitation and stream flow [64], modeling of income or wealth distribution, as well as modeling data transmission times in networking that takes into account the network and the software [65]. The PDF, CDF and moments of X

as $g(\tau)$ follows Log-logistic distribution is given below respectively.

$$\begin{aligned}
f_X(x) &= \omega \int_0^\infty \tau^n g(\tau) d\tau \\
&= \omega \int_0^\infty \tau^n \frac{\alpha \tau^{\alpha-1}}{\beta^\alpha [1 + \frac{\tau}{\beta}]^2} d\tau
\end{aligned} \tag{3.32}$$

Let:

$$u = \frac{\tau}{\beta}$$

$$\beta u = \tau$$

$$d\tau = \beta du$$

$$\begin{aligned}
&= \omega \frac{\alpha}{\beta^\alpha} \int_0^\infty \frac{(\beta u)^{n+\alpha-1}}{(1+u)^2} \beta du \\
&= \omega \frac{\alpha}{\beta^\alpha} \int_0^\infty \frac{\beta^{n+\alpha-1} u^{n+\alpha-1}}{(1+u)^2} \beta du \\
&= \omega \alpha \beta^n \int_0^\infty \frac{u^{n+\alpha-1}}{(1+u)^2} du
\end{aligned}$$

using the following identity:

$$\int_0^\infty \frac{x^{a-1}}{(1+bx)^v} dx = b^{-a} \beta(a, v-a) \tag{3.33}$$

where

$$a = \alpha + n$$

$$b = 1$$

$$v = 2$$

$$\begin{aligned}
f_X(x) &= \omega \alpha \beta^n (1)^{-a-\alpha} \beta [n + \alpha, 2 - n - \alpha] \\
&= \omega \alpha \beta^n \beta [n + \alpha, 2 - n - \alpha] \\
f_X(x) &= \omega \alpha \beta^n \frac{\Gamma(n + \alpha) \Gamma(2 - n - \alpha)}{\Gamma(n + \alpha + 2 - n - \alpha)} \\
&= \omega \alpha \beta^n \frac{\Gamma(n + \alpha) \Gamma(2 - n - \alpha)}{\Gamma(2)}
\end{aligned} \tag{3.34}$$

replacing ω and n we have

$$f_X(x) = \frac{\Gamma(\frac{v+1}{2}) v^{\frac{v}{2}}}{\sqrt{\pi} \Gamma(\frac{v}{2}) x^{v+1}} \sum_{k=0}^{\infty} \left(\frac{v+1}{2} \right)_k \frac{(-v)^k}{k! x^{2k}} \alpha \beta^{k+\frac{v}{2}} \frac{\Gamma(k + \frac{v}{2} + \alpha) \Gamma(2 - (k + \frac{v}{2}) - \alpha)}{\Gamma(2)} \tag{3.35}$$

the CDF is as follows

$$F_X(x) = \frac{1}{2} - \frac{v^{\frac{v}{2}-1} \Gamma(\frac{v+1}{2})}{x^v \sqrt{\pi} \Gamma(\frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\frac{1+v}{2})_k}{(\frac{v}{2})_k} k! \left(1 + \frac{v}{2} \right)_k \left(\frac{x^2}{v} \right)^{-k} \alpha \beta^{k+\frac{v}{2}} \frac{\Gamma(k + \frac{v}{2} + \alpha) \Gamma(2 - (k + \frac{v}{2}) - \alpha)}{\Gamma(2)} \tag{3.36}$$

the k^{th} moment is given as:

$$E(X^k) = E[E(X^k|\tau)] = \left[\int_0^{\infty} \tau^{\frac{k}{2}} \frac{\alpha \tau^{\alpha-1}}{\beta^{\alpha} (1 + \frac{\tau}{\beta})^2} d\tau \right] \frac{1 + (-1)^k}{2\sqrt{\pi} \Gamma(\frac{v}{2})} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{v-k}{2}\right) v^{\frac{k}{2}} \tag{3.37}$$

solving the integral part; let $u = \frac{\tau}{\beta}$; $\tau = \beta u$ $du = \frac{1}{\beta} d\tau$; $\beta du = d\tau$ we substitute the variable in the integral we have

$$\int_0^{\infty} \tau^{\frac{k}{2}} \frac{\alpha (\beta u)^{\alpha-1}}{\beta^{\alpha} (1 + u)^2} \beta du$$

$$\begin{aligned}
&= \int_0^\infty (\beta u)^{\frac{k}{2}} \frac{\alpha(\beta u)^{\alpha-1}}{\beta^\alpha(1+u)^2} \beta du \\
&= \alpha \beta^{\frac{k}{2}} \int_0^\infty \frac{u^{\frac{k}{2}+\alpha-1}}{(1+u)^2} = \alpha \beta^{\frac{k}{2}} \beta \left[\frac{k}{2} + \alpha, 2 - \frac{k+2\alpha}{2} \right]
\end{aligned}$$

$$\therefore E(X^k) = \frac{1+(-1)^k}{2\sqrt{\pi}\Gamma(\frac{v}{2})} \Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{v-k}{2}\right) v^{\frac{k}{2}} \alpha \beta^{\frac{k}{2}} \beta \left[\frac{k}{2} + \alpha, 2 - \frac{k+2\alpha}{2} \right] \quad (3.38)$$

the characteristic function is given as:

$$\begin{aligned}
E[\exp(itX)] &= \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{\Gamma(\frac{v}{2})\Gamma(1-\frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1-\frac{v}{2})_k} \beta^k \frac{\Gamma(k+\alpha)\Gamma(2-k-\alpha)}{\Gamma(2)} \\
&- \frac{(\sqrt{v}|t|)^{-\frac{v}{4}} \csc(\frac{\pi v}{2})}{2^v \Gamma(\frac{v}{2})\Gamma(1+\frac{v}{2})} \sum_{k=0}^{\infty} \frac{(\sqrt{v}|t|)^k}{2^{2k} k! (1+\frac{v}{2})_k} \beta^{k+\frac{v}{2}} \frac{\Gamma(k+\alpha+\frac{v}{2})\Gamma(2-k-\alpha-\frac{v}{2})}{\Gamma(2)} \quad (3.39)
\end{aligned}$$

3.4 Parameter Estimation methods

There are several methods that can be used in estimating the unknown parameters of a given model. And some of the most popular methods in the literature are: methods of moments, L-moments approach, maximum likelihood estimation approach, least squares approach, probability weighted moments approach, mixed moments approach, and so forth. For more information on the given methodologies see [66], [67], [68] and [69].

- Method of moments: This is a technique used in statistics to estimate population parameters. The population moments are first expressed as functions

of the relevant parameters and the sample moments are then equated to the expressions. There are exactly as many of these equations as the number of parameters needed to be estimated. Then, the equations are solved for the relevant parameters. The solutions are estimates of those parameters. Suppose X is a distribution of sample size n , that is $X = (x_1, x_2, x_3, \dots, x_n)$. Thus X is a sequence of independent random variables, each with the distribution of X . This method construct estimators of the parameters that is based on matching the sample moments with corresponding distribution moments.

$$\mu^j(\theta) = E(x^j); j \in N_+$$

so that $\mu^j(\theta)$ is the j th moment of X about 0.

- Probability weighted moments (PWM) Approach: This method was first developed by [70] who also shown how effective it is for obtaining explicit expressions for distributional parameters whose inverse forms, $X = X(F)$, can be explicitly defined. Let a probability distribution function be denoted as $F = F(X) = P(X \leq x)$. The PWMs of this function can be defined as

$$M_{i,j,k} = E [x^j F^j (1 - F)^k] = \int_0^1 [x(F)]^i F^j (1 - F)^k dF \quad (3.40)$$

where $M_{i,j,k}$ is PWM of order (i, j, k) , E is the expectation operator and i, j and k are real numbers. If $j = k = 0$ and i is a non-negative integer then $M_{i,0,0}$ represents the conventional moment about origin of order i . If $M_{i,0,0}$ exists and X

is a continuous function of F , then $M_{i,j,k}$ exists for all non-negative real numbers j and k . The two useful sets of $M_{i,j,k}$ are defined as:

$$a_k = M_{i,0,k} = \int_0^1 x(F)(1-F)^k dF; k = 0, 1, 2... \quad (3.41)$$

$$b_r = M_{i,r,0} = \int_0^1 x(F)(1-F)^r dF; r = 0, 1, 2... \quad (3.42)$$

The two sets a_k and b_r are linear combinations of each other. See [71] and the references therein.

- **Technique of Mixed Moment:** This was put out by Rao (1980, 1983) [72] for fitting a log-Pearson type 3 distribution. Any distribution with a log-probability can be used with the MIXM technique. The MIXM approach, as its name implies, combines the moments of real and logarithmically converted data.
- **Method of least squares:** Finding the line that best fits a set of data written as an equation such as $y = mx + b$ can be done statistically using the method of least squares. The curve of the equation is represented by the regression line. The goal of this strategy is to minimize the sum of squared errors as much as possible. This method is frequently used in data fitting, where the best fit result is supposed to reduce the sum of squared errors, which is defined as the difference between observed and fitted values. The sum of squared errors helps identify variation in observable data. The method of least squares analysis begins with a set of data points to be plotted on the graph of the XY plane. The method of least squares problems is divided into two categories: Linear or ordinary least

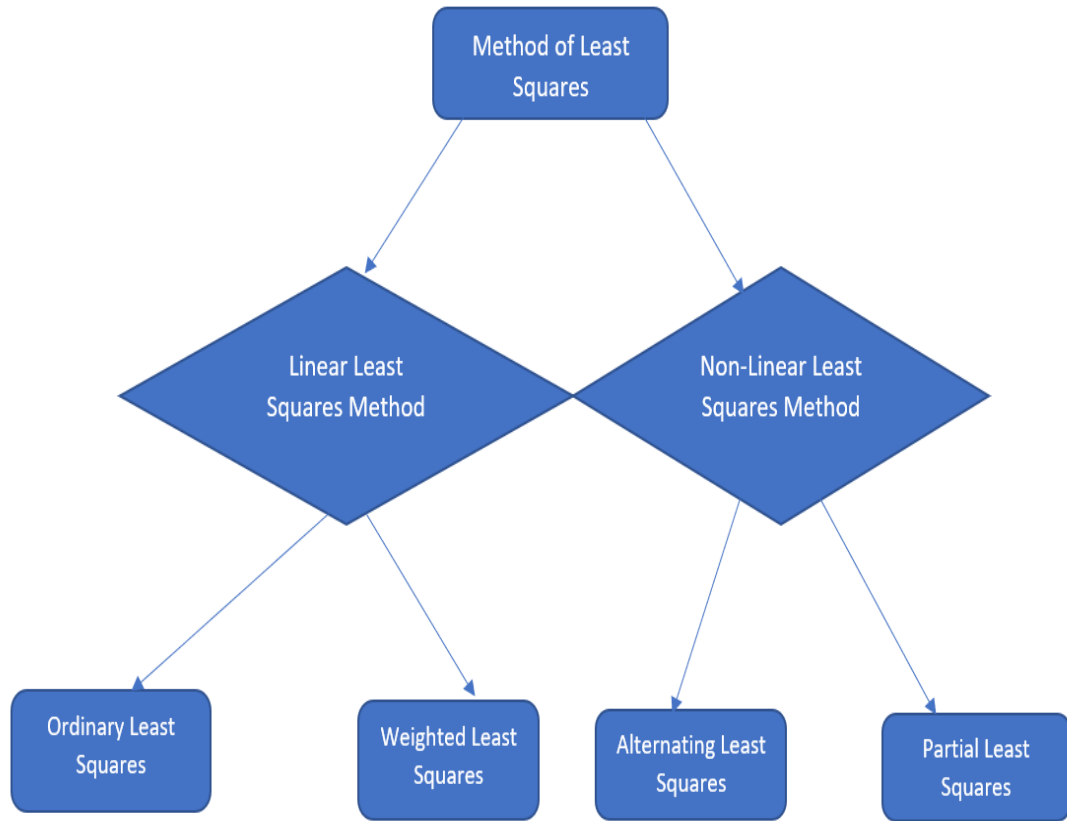


Figure 3.1: Diagram of Least Square methods

square method and non-linear least square method. These are further classified as ordinary least squares, weighted least squares, alternating least squares and partial least squares.

- L-moment: [67] created the L-moments method, which has since gained a lot of popularity for use in characterizing probability distributions, summarizing observed data samples, estimating parameters or fitting probability distributions to data, estimating intervals, and testing of hypotheses about distributional form. The theory gives rise to several potential advances like the measurements

of skewness and kurtosis and involves well-known techniques like the use of order statistics and Gini's mean difference statistic. [67] defined the r -th L-moment of X as a linear function of expected order statistics as:

$$L_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E\{X_{r-k:r}\}, r = 1, 2, 3, \dots \quad (3.43)$$

where $E.$ is the expectation of an order statistic and is equal to

$$E(X_{j:r}) = \binom{r}{j} \int x \{F(x)\}^{j-1} \{1 - F(x)\}^{r-j} dF(x)$$

Given that X is a real value ordered random variate of sample size n , such that $x_{1:n} \leq x_{2:n} \leq x_{3:n} \leq x_{4:n} \dots \leq x_{n:n}$ with CDF $F(x)$ and quantile function $X(F)$.

- Maximum Likelihood Estimation (MLE): This method is considered among the most effective techniques for unknown parameter estimation because of the following reasons: (i) it is easy to compute mathematically; (ii) it is asymptotically unbiased; (iii) it has minimal variance and (iv) it has the invariance property.

That is, the MLE of a transformed function or parameter is equal to MLE of the original function or parameter transformed. For more details see [73], [74], [75].

3.5 Maximum Likelihood Estimation

In this work, MLE method was used for parameter estimation of the proposed models.. Formally, suppose $x_1, x_2, \dots, x_n$ are independent and identically distributed observations, then the likelihood is given as:

$$L(\Theta) = \prod_{i=0}^n f(x_i; \Theta) \quad (3.44)$$

In practice, it is easier to work with the log of the likelihood function, therefore taking the log likelihood we have:

$$\log L(\Theta) = \sum_{i=0}^n \log f(x_i, \Theta) \quad (3.45)$$

where $\Theta = (\theta_1 \dots \theta_k)'$ is the parameter vector. Consequently, the optimal estimates for Θ are $\hat{\Theta} = (\hat{\theta}_1, \hat{\theta}_2 \dots \hat{\theta}_k)'$. The standard errors of the estimates are obtained through the help of the hessian matrix. The hessian matrix is a matrix of second-order partial derivatives of a scalar-value function of multiple variables. It indicates the local shape of the log-likelihood surface near the optimal value. We can use the matrix to estimate the covariance matrix of the parameters, which in turn is used to get estimates of the standard errors of the unknown parameters estimated. The hessian at the optimal MLE value is related to the covariance of the parameters. The inverse matrix of the hessian is related to the covariance matrix of the parameters. Statistical software often minimizes the negative log-likelihood function; which gives a positive definite for the hessian at the minimum and its inverse, which is the estimate of the covariance matrix

of the parameters. The standard errors of the estimated parameters are obtained by taking the square roots of the diagonal elements of the covariance. Mathematically the hessian matrix can be express as:

$$H(\hat{\theta}) = \begin{bmatrix} \frac{d^2 f}{d\hat{\theta}_1^2} & \frac{d^2 f}{d\hat{\theta}_1 d\hat{\theta}_2} & \cdots & \frac{d^2 f}{d\hat{\theta}_1 d\hat{\theta}_n} \\ \frac{d^2 f}{d\hat{\theta}_2 d\hat{\theta}_1} & \frac{d^2 f}{d\hat{\theta}_2^2} & \cdots & \frac{d^2 f}{d\hat{\theta}_2 d\hat{\theta}_n} \\ \cdots & & & \\ \cdots & & & \\ \frac{d^2 f}{d\hat{\theta}_n d\hat{\theta}_1} & \frac{d^2 f}{d\hat{\theta}_n d\hat{\theta}_2} & \cdots & \frac{d^2 f}{d\hat{\theta}_n^2} \end{bmatrix}. \quad (3.46)$$

So, the diagonal matrix represents the variances of the estimated parameters of the model. The models given by equations (3.17), (3.27), and (3.35) were fitted by taking the negative log-likelihood; by the help of the Optim function in R software. The results of the fitted models are given in Table (4.2) and (4.3). Since the distributions under consideration are not nested, the following accepted standards are used to differentiate between them:

- the Akaike information criterion (AIC) due to [76] given as,

$$AIC = 2k - 2 \log L(\hat{\Theta});$$

- Schwarz et al.'s (1978) Bayesian information criterion (BIC) given as,

$$BIC = k \log n - 2 \log L(\hat{\Theta});$$

- Hurvich and Tsai's (1989) corrected Akaike information criterion (AICc) given as,

$$AICc = AIC + \frac{2k(k+1)}{n-k-1};$$

- the Hannan-Quinn criterion (HQC) due to [77] given as,

$$HQC = -2 \log L(\hat{\Theta}) + 2k \log \log n;$$

- the consistent Akaike information criterion (CAIC) due to Bozdogan (1987), defined as,

$$CAIC = -2 \log L(\hat{\Theta}) + k(\log n + 1).$$

A smaller value of any of these criteria will suggest that the model will be preferred. [73], [74], [75].

3.6 Kurtosis and Skewness

Skewness is the degree of observed in a probability distribution that deviates from the symmetrical normal distribution (bell curve) in a given set of data. The values in the set of data normally fall at the first quartile (Q1) and the third quartile (Q3). Normal distribution has zero skewness indicating that the data set is symmetrically distributed; its zero skewness helps to know the if a set of data is skew or not. A skew data set is called an asymmetrical data. An asymmetrical data can either be right skew or left skew. A data set is positively skewed if the *mean* > *median* > *mode*.

while a negative skewed data set has $mode > median > mean$. A left-skewed distribution has a long left tail. It is one in which the majority of data are plotted on the right side of the graph while the tail of the distribution is spreading to the left. Kurtosis measures the outliers in a given data set or it measures the tailedness of a distribution. Tailedness refers to how often the outliers occur. Peakedness in a data distribution is the degree to which data values are concentrated around the mean. High kurtosis data sets exhibit a prominent peak near the mean, a quick drop, and hefty tails. Low kurtosis data sets feature a flat top near the mean rather than a high peak. Each of the three compound distributions has zero skewness by definition. The kurtosis values of the distributions are generated by using equation (3.21), (3.30) and (3.38). It is the fourth standardized moment, defined as:

$$E \left[\left(\frac{X_i - \mu}{\sigma} \right)^4 \right] \quad (3.47)$$

$$= \frac{E[(X_i - \mu)^4]}{[E(X_i - \mu)^2]^2} \quad (3.48)$$

Where the numerator of equation (4.7) is the fourth moment, while the denominator is the squared of the second moment of the distributions. The values of the kurtosis for each model is plotted against the degree of freedom parameter ν .

3.7 Bootstrap

The bootstrap is a technique for evaluating some of the statistical estimator's characteristics in a non-parametric setting. This means that we do not presume that the data

came from any particular parametric distribution, such as the normal or exponential distribution. The bootstrap is typically employed to evaluate the variance of a statistical estimate, though this is not its only application. The bootstrap has parametric variations, but for some reasons, we shall stick to the non parametric version.

Assume we have an estimator $\hat{\theta}(x)$ of some population parameter θ based on a sample of size n ; $x_1, x_2, \dots, x_n$ from an unknown distribution F . Suppose X represent a random variable with CDF, F and PDF, f . So here are some instances of θ options and their estimators:

•

$$\theta = E_f[X] \quad (3.49)$$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i \quad (3.50)$$

•

$$\theta = \text{var}_f(X) \quad (3.51)$$

$$\hat{\theta} = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \quad (3.52)$$

•

$$\theta = P_f(X > c) \quad (3.53)$$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n 1_{x_i > c} \quad (3.54)$$

It would be ideal if we could collect more samples from F . The estimator for these samples might then be evaluated as well. After that, we may acquire the estimator's sample variance and use it to calculate its actual variance. Sadly, we frequently have to settle for working with a single sample due to practical limitations (money, time). To calculate the estimator's variance in this situation, we can utilise bootstrapping based on one sample. The fundamental principle is that we sample with replacement from the sample we acquired rather than from the population of interest. Specifically, we assume that the sample accurately represents the population in question.

Formally speaking, the main principle behind bootstrapping is to analyse the population by substituting the empirical distribution function $\hat{F}$ from the sample for the unknown distribution function F . As always, getting this is as easy as

1. Ordering the data; $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$
2. Setting $\hat{F}(x) = \frac{i}{n}; x_{(i)} \leq x < x_{(i+1)}$

Hence, $x_1^*, x_2^*, \cdots, x_n^*$ which is a bootstrap sample constructed from $x = x_1, x_2, \cdots, x_n$ is done as follows.

1. Simulate $u_1, \mu_2, \cdots, \mu_n$ from $U(0, 1)$
2. For $k = 1, 2, \cdots, n$, set $x_k^* = \hat{F}^{-1}(u_k)$

This means that the bootstrap sample (x^*) can be obtained from x using inversion of the empirical CDF $\hat{F}$. So, by just sampling x^* from x with replacement, this technique can be done in $\mathbf{R}$ without computing $\hat{F}$.

To get sample variances for the relevant estimator in our work, we therefore use the bootstrap procedure below.

3.7.1 Bootstrap algorithm

1. Estimate the empirical distribution $\hat{F}$
2. Then, simulate $\mathbf{B}$ samples $x^{*1}, x^{*2}, \dots, x^{*\mathbf{B}}$ each of size n from $\hat{F}$ following the earlier procedure outlined before now.
3. Calculate the population bootstrap variance as the sample variance.

$$var_{\mathbf{B}}(\hat{\theta}) = \frac{1}{\mathbf{B} - 1} \sum_{b=1}^{\mathbf{B}} \left(\hat{\theta}(\mathbf{x}^{*b}) - \theta^* \right)^2 \quad (3.55)$$

where

$$\theta^* = \frac{1}{\mathbf{B}} \sum_{b=1}^{\mathbf{B}} \hat{\theta}(\mathbf{x}^{*b}) \quad (3.56)$$

Basically, this procedure has three aspects, which include:

- (a) The empirical distribution $\hat{F}$ is used to replace F ;
- (b) Using a simulation based on the empirical distribution;
- (c) Estimating the estimator θ and its variance.

3.7.2 Bias

The bias of an estimator is defined as

$$bias = E[\hat{\theta}] - \theta, \quad (3.57)$$

that is the difference between the mean of the estimate of θ and its actual value θ .

Given that we are working with a sample, we determine the bias's bootstrap estimate,

which is given by

$$bias = E_{\hat{F}}[\hat{\theta}] - \hat{\theta} \quad (3.58)$$

Specifically, the discrepancy between the estimated value and the bootstrap distribution's predicted value. For large B , it is obvious that we may estimate $E_{\hat{F}}[\hat{\theta}]$ using the bootstrap sample as

$$E_{\hat{F}}[\hat{\theta}] \approx \frac{1}{B} \sum_{b=1}^B \hat{\theta}(\mathbf{x}^{*b}) \quad (3.59)$$

This is appropriate When a direct calculation of $E_{\hat{F}}[\hat{\theta}]$ cannot be done.

CHAPTER 4

DATA ANALYSIS

To illustrate the flexibility of our models, we employ two data sets: BT crude oil and WTI crude oil prices. These two data sets were obtained from Database Data-stream of KFUPM Business School. The Data period spans from 1st of January 2000 to 31st August 2022. Following the general tradition, we transformed our data to obtain the log-returns before fitting our proposed models. That is, the return, say r_t of an asset can be defined by

$$r_t = \frac{X_t - X_{t-1}}{X_{t-1}} \quad (4.1)$$

where X_t is the price of the asset at time t. Then, we can rewrite equation(4.1) as

$$X_t = (1 + r_t)X_{t-1} \quad (4.2)$$

Now, after taking log of equation (4.1) we have

$$\log(X_t) = \log(1 + r_t) + \log(X_{t-1}) \quad (4.3)$$

Therefore,

$$\begin{aligned}\log\left(\frac{X_t}{X_{t-1}}\right) &= \log(X_t) - \log(X_{t-1}) \\ &= \log(1 + r_t) \\ &\approx r_t\end{aligned}$$

if r_t is small.

Given the daily log-returns of BT crude oil prices and WTI crude oil prices, we obtained the basic summary statistics in Table 4.1. Here, we summarize the following statistics: the minimum, first quartile(Q1), median, mean, third quartile(Q3), maximum, standard deviation(SD), coefficient of variation, skewness, kurtosis, interquartile range(IQR), and range. The minimum and the first quartile values for each commodity's log-return are negative. WTI crude oil has the smallest minimum value compared to BT's crude oil log-return. The median is smallest for WTI price return. The coefficient of variation is higher for WTI crude oil than BT oil price return. This indicates that WTI's price returns have been significantly more volatile than BT's price return.

The price return of both commodities are negatively skewed, but BT price return has smaller negative skewness. This fits with the observation that the majority of financial data are negatively skewed. The authors [78] considers negative skewness of financial data as "large negative stock returns are more common than large positive ones, so

Statistics	Brent	West Texas
Min	-0.6436989	-1.4232030
Q1	-0.0110226	-0.118141
Median	0.0001814	0.0001347
Mean	0.0002239	0.0002034
Q3	0.0124232	0.0130852
Maximum	0.4120225	0.7029299
Skewness	-2.024694	-10.2755
Kurtosis	82.01139	541.6941
SD	0.02645719	0.03427319
Variance	0.0069993	0.001174652
CV	118.184	168.5399
Range	1.055721	2.126133
IQR	0.02344574	0.02489925

Table 4.1: Summary statistics of daily log-returns of BT and WTI.

stock returns are negatively skewed ... this shows up clearly in the pattern of extreme moves in stock prices in the postwar period”. Also, see [2] for more details on the negative skewness of financial returns

The kurtosis value for both commodities are above three, which corresponds to the normal distribution. The largest kurtosis is for WTI crude oil. This is consistent with the excess kurtosis found in the majority of financial data. The authors [78] talked about excess kurtosis of financial data as “extreme stock market movements are more common than would be expected if stock returns were drawn from a normal distribution, so stock returns have excess kurtosis. This is not just the result of changing volatility, because excess kurtosis remains after one normalizes returns by their estimated conditional standard deviations”.

In addition to the computed statistics, the box plots of the data sets are given in Figure 4.2, from these plots, we can observe that prices were volatile in certain period but stable in other periods.

Figure 4.1 shows the histograms of the BT crude oil prices and WTI crude oil prices at the top respectively, and the histograms of the log-returns at the bottom respectively. The histograms of crude oil prices have multiples peaks and the shapes are slightly skewed to the right, we can assume that the data have a multi-modal distribution. While the histograms of the price returns of the commodities appears symmetrical about zero.

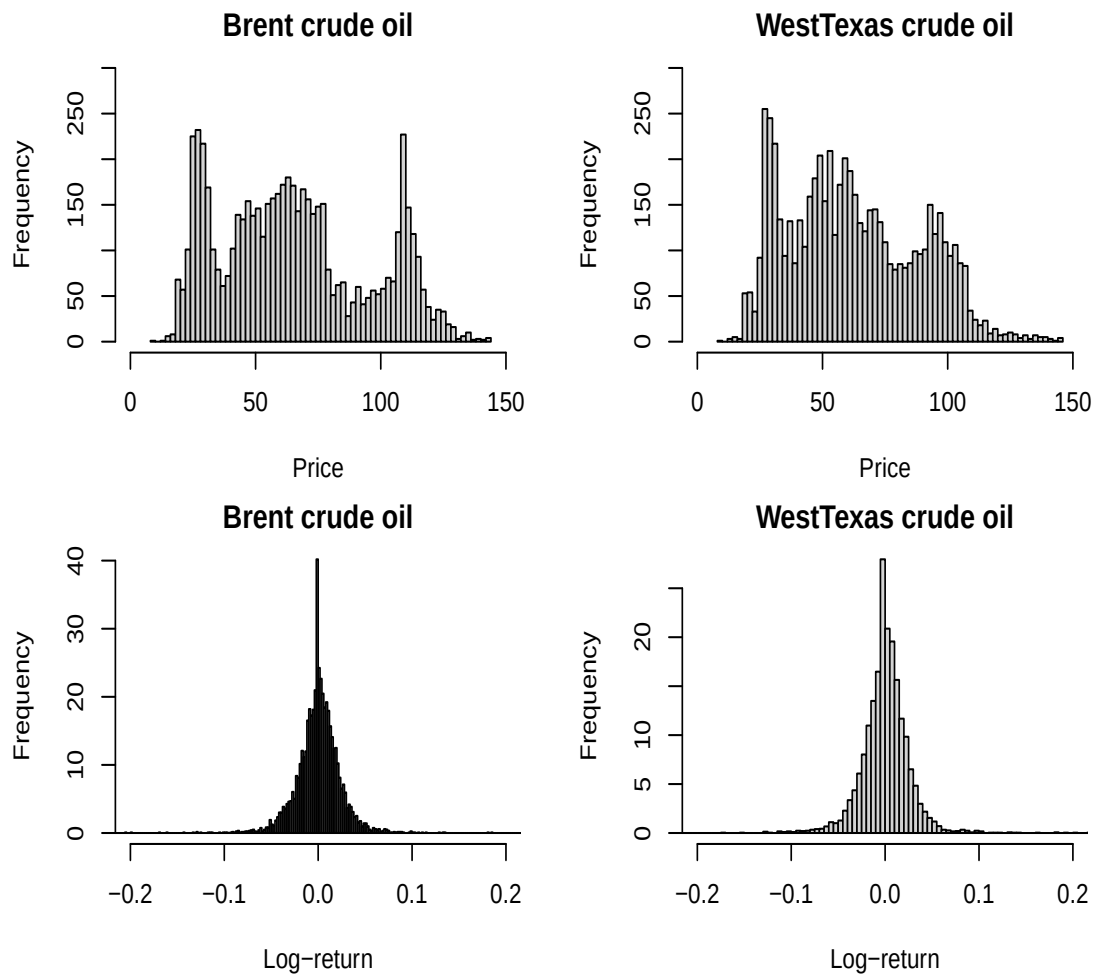


Figure 4.1: Histograms of BT Crude Oil and WTI Crude Oil Prices

Figure 4.2 shows the box plots of the log-returns of BT crude oil prices and WTI crude oil prices for period from 2000 to 2022 respectively. In the box plots we can

see multiples outliers in the values of log-returns on crude oil prices for every year. More dispersion on BT's price returns occurs during the years: 2000, 2001, 2004, 2008, 2009, 2015, 2016, and 2020. We visualised more outliers occurring in year 2020, this maybe due to the pandemic (Covid-19).

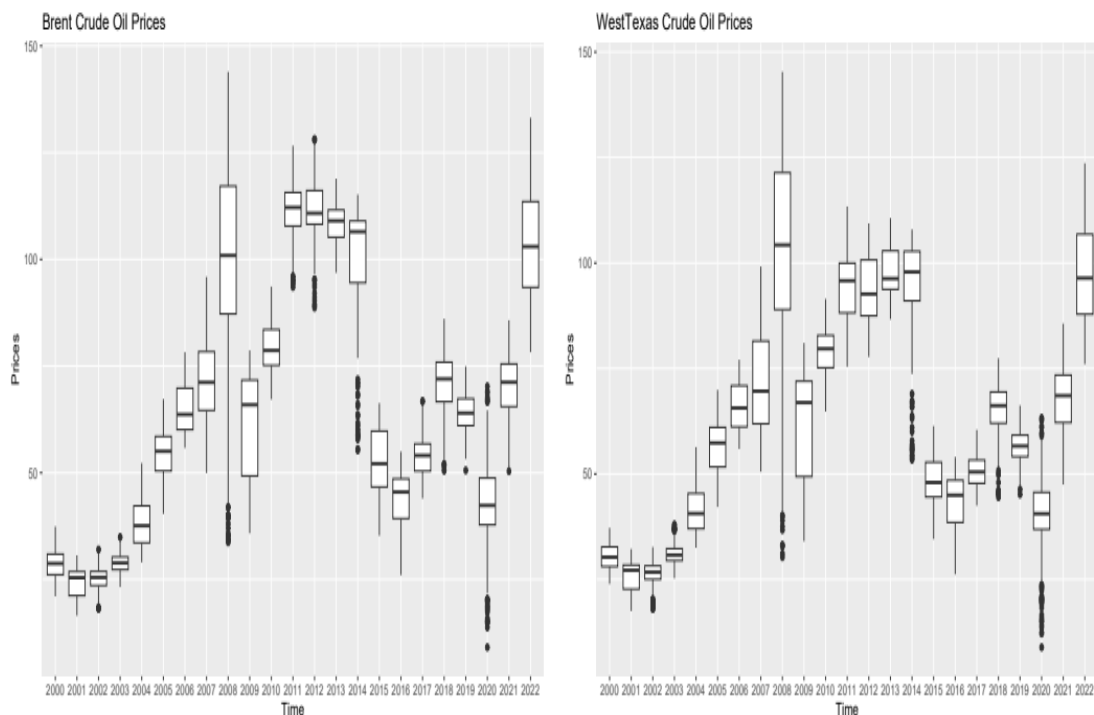


Figure 4.2: Box plots of daily stock prices for BT and WTI crude oil prices

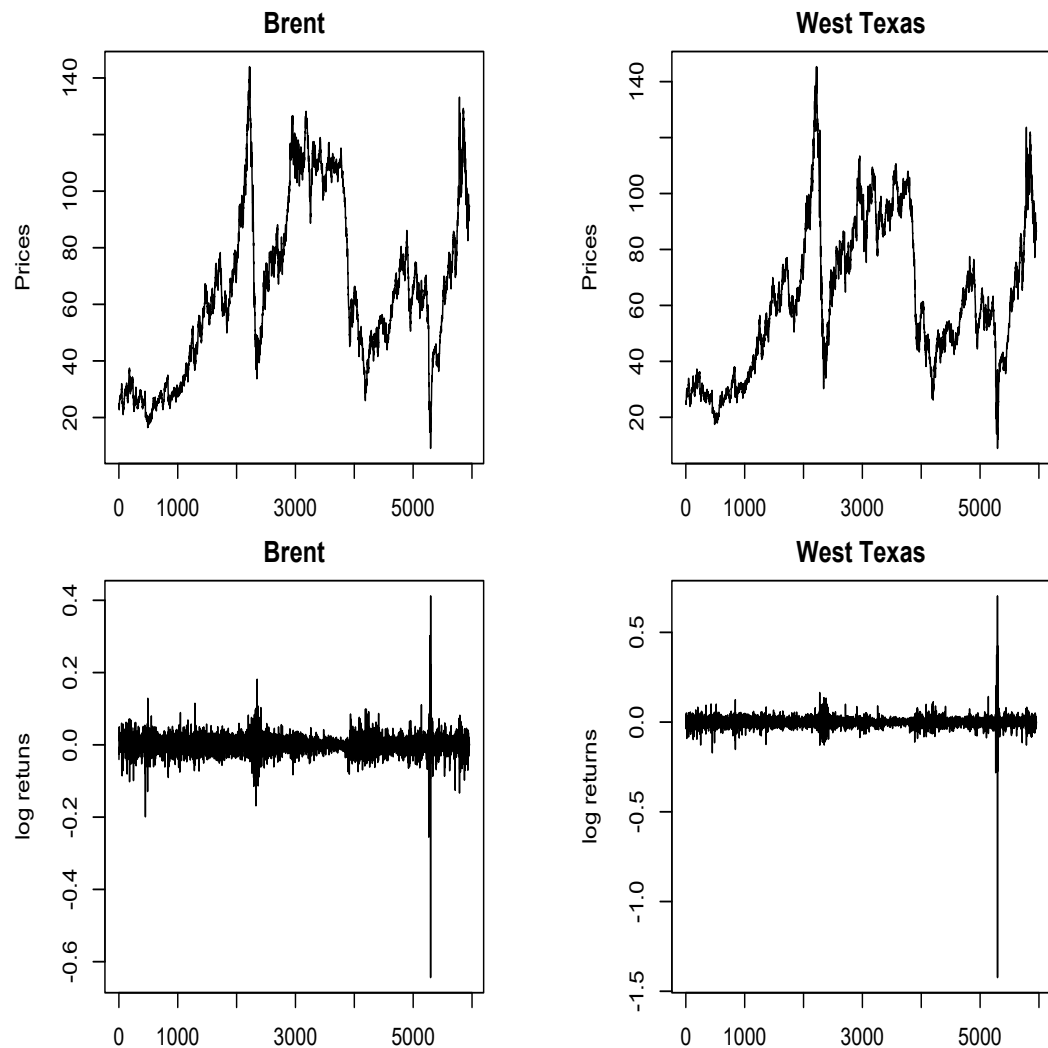


Figure 4.3: Time series plots of daily log-returns of BT and WTI stock

4.1 Estimated Results and Discussion

Based on the log-returns of BT crude oil prices and WTI crude oil prices, our models were fitted . The method of maximum likelihood was used for fitting each model. The models given by (3.17),(3.27) and (3.35) were fitted by directly minimizing the negative their log-likelihood functions. Tables 4.2 and 4.3 gives the estimated parameters of the models. While Table 4.4 provides the log-likelihood values as well as the five fitted models selection criteria. We visualize that model given by (3.27) gives the

best fit. It gives the least for AIC, BIC, consistent AIC, corrected AIC and HQC for each commodity price return. The model given by (3.35) fits better to log-return of BT crude oil price than the model given by (3.17); while the aforementioned model fits better than the former to WTI crude oil log returns. The three models were compared to the two well-known generalized models which are Generalized hyperbolic distribution(GHD) and Assymetric student's t distribution(AST). The model given by (3.27) out performed AST model and have almost equal performance with GHD model for fitting the two commodities' log-returns. So the three developed models can serve as an alternative to generalized models which are: GHD and AST. The selection criteria values of the models are given in table 4.4.

Model	Estimated Parameters				
DD	$\hat{\mu} = 2.080032e^{-6}$	$\hat{\nu} = 5.769778$	$\hat{b} = 0.9869273$	$\hat{\alpha} = 0.5005181$	$\hat{p} = 0.4646905$
RD	$\hat{\mu} = 2.8977e^{-5}$	$\hat{\nu} = 6.8601567409$	$\hat{b} = 0.0147153112$		
LD	$\hat{\mu} = -6.035347e^{-9}$	$\hat{\nu} = 5.75566$	$\hat{\alpha} = 0.01543194$	$\hat{\beta} = 0.5222887$	

Table 4.2: Estimated parameters of the fitted models on log-returns of BT crude oil prices

model	Estimated Parameters				
DD	$\hat{\mu} = -2.25293e^{-6}$	$\hat{\nu} = 5.620864$	$\hat{b} = 1.142578$	$\hat{\alpha} = 0.4602504$	$\hat{p} = 0.7386149$
RD	$\hat{\mu} = 1.148668e^{-4}$	$\hat{\nu} = 6.8546480201$	$\hat{b} = 1.55650573e^{-3}$		
LD	$\hat{\mu} = -3.189770e^{-6}$	$\hat{\nu} = 5.740948$	$\hat{\alpha} = 1.196077$	$\hat{\beta} = 0.3472037$	

Table 4.3: Estimated parameters of the fitted models on log-returns of WTI crude oil prices

Returns	Models	$-\log L(\hat{\theta})$	AIC	BIC	CAIC	AICc	HQC
BT	DD	-9074.505	-18139.01	-18105.55	-18100.55	-18139	-18127.39
	RD	-14358.94	-28711.88	-28691.8	-28688.8	-28711.88	-28704.91
	LD	-12142.37	-24276.74	-24249.97	-24245.97	-24276.73	-24267.44
	GHD	-14368.6	-28727.2	-28693.74	-28727.19	-28715.58	-28688.74
	AST	-5747.832	-11489.66	-11469.59	-11489.66	-11482.69	-11466.59
WTI	DD	-7917.492	-15824.98	-15791.52	-15786.52	-15824.97	-15813.36
	RD	-13982.36	-27958.72	-27938.64	-27935.64	-27958.72	-27951.75
	LD	-6701.455	-13394.91	-13368.14	-13364.14	-13394.9	-13385.61
	GHD	-14000.21	-27990.42	-27956.96	-27990.41	-27978.8	-27951.96
	AST	-5749.32	-11492.69	-11472.56	-11492.64	-11485.67	-11469.56

Table 4.4: The values of the five selection criteria and the log-likelihood for the proposed models

4.2 Kurtosis and Skewness plots

We observe that the models obtain have greater kurtosis than that of student's t distribution. The graphs in figure 4.4 depicts the kurtosis of the model based on the Dagum distribution, the model has greater kurtosis than that of student's t distribution. This indicates that the model can contain more information than that of the student's t distribution. We also observed that the kurtosis of the model based on Dagum distribution i.e equation (3.21) is a decreasing function as the parameter $\hat{a}$ increases given others constant. While the kurtosis is an increasing function as the $\hat{p}$ increases. In general, the kurtosis function increases as the parameters' values increase, though parameter $\hat{b}$ has no effect on the shape of the kurtosis. Figure (4.3) and figure (4.4) we can see that the kurtosis of the models based on Rayleigh (equation 3.30) and log-logistics (equation 3.38) respectively have improved the kurtosis of student's t distribution significantly, that is they can contain more information about a data set.

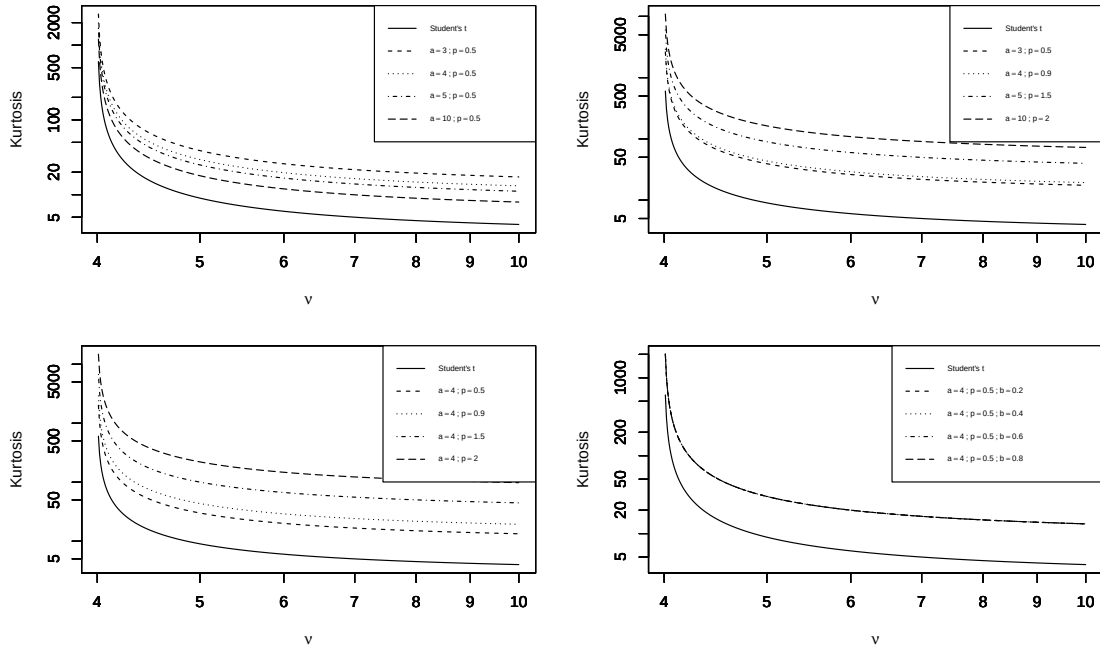


Figure 4.4: Graphs of kurtosis value versus ν for the model based on $g(.) \sim \text{Dagum}$ distribution

4.3 VaR plots of BT and WTI's daily log-returns

A VaR plot illustrates the range of possible losses that an investment or portfolio could encounter. It is a graphical representation of the VaR statistic. The VaR value is normally plotted on the y-axis and the likelihood of exceeding that value is typically plotted on the x-axis. Risk managers and investors frequently use VaR plots to comprehend the possible downside risk of their investments and to decide wisely on risk management and asset allocation. Figure 4.6 displays the VaR plots of the distributional models on the daily log-returns of BT and WTI crude oil prices. At the top left side of the graph we see that WTI crude oil is riskier than BT crude oil. The probability of the VaR to exceed a certain value of daily log-return of WTI crude oil price is higher than that of daily log-return of BT crude oil price. The graphs

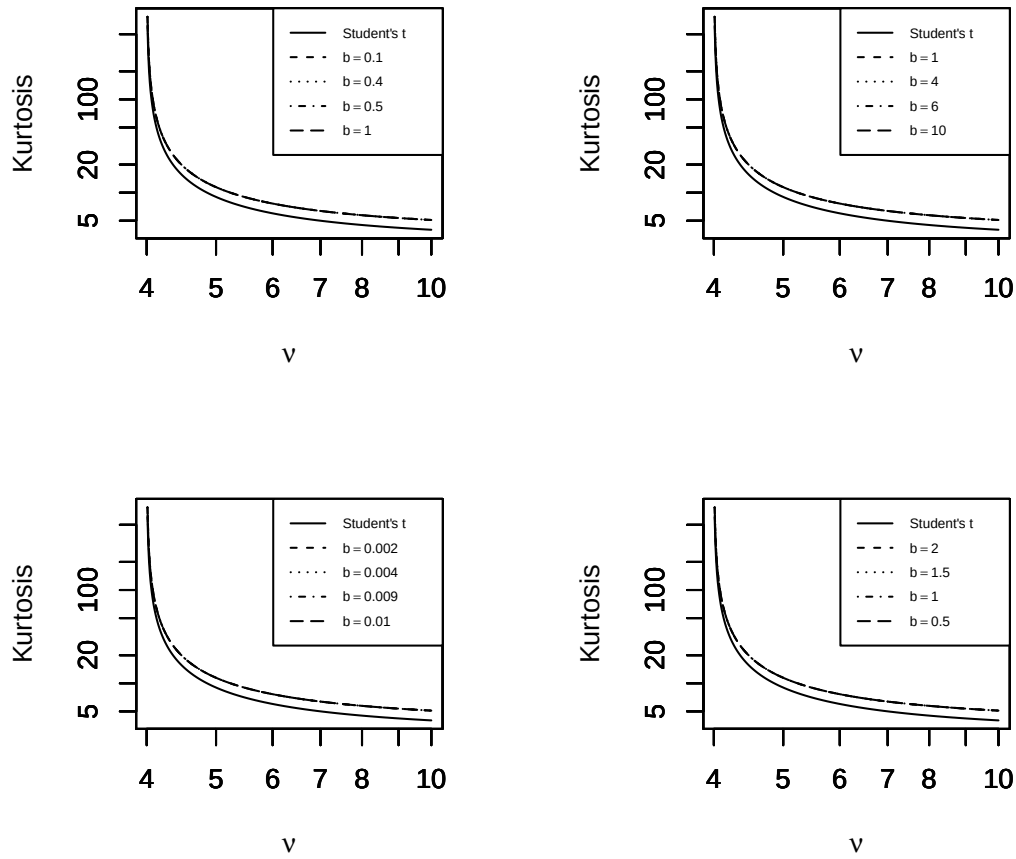


Figure 4.5: Graphs of kurtosis value versus ν for the model based on $g(.) \sim \text{Rayleigh}$ distribution

in Figure 4.6 shows that daily log-returns of WTI crude oil is riskier of that of daily log-returns of BT crude oil. We observed that the probability given by the model based on Dagum distribution(DD) for the VaR to exceed a certain value is higher than the other models in both log-returns, followed by the model based on Rayleigh distribution(RD) and the model based on Loglogistic distribution gives the smallest probability for a VAR to exceed a certain level.

Figure 4.7 shows the plots of the parameter estimates of the model based on Dagum distribution on BT's log-returns against the sample size(n), their bias, standard errors

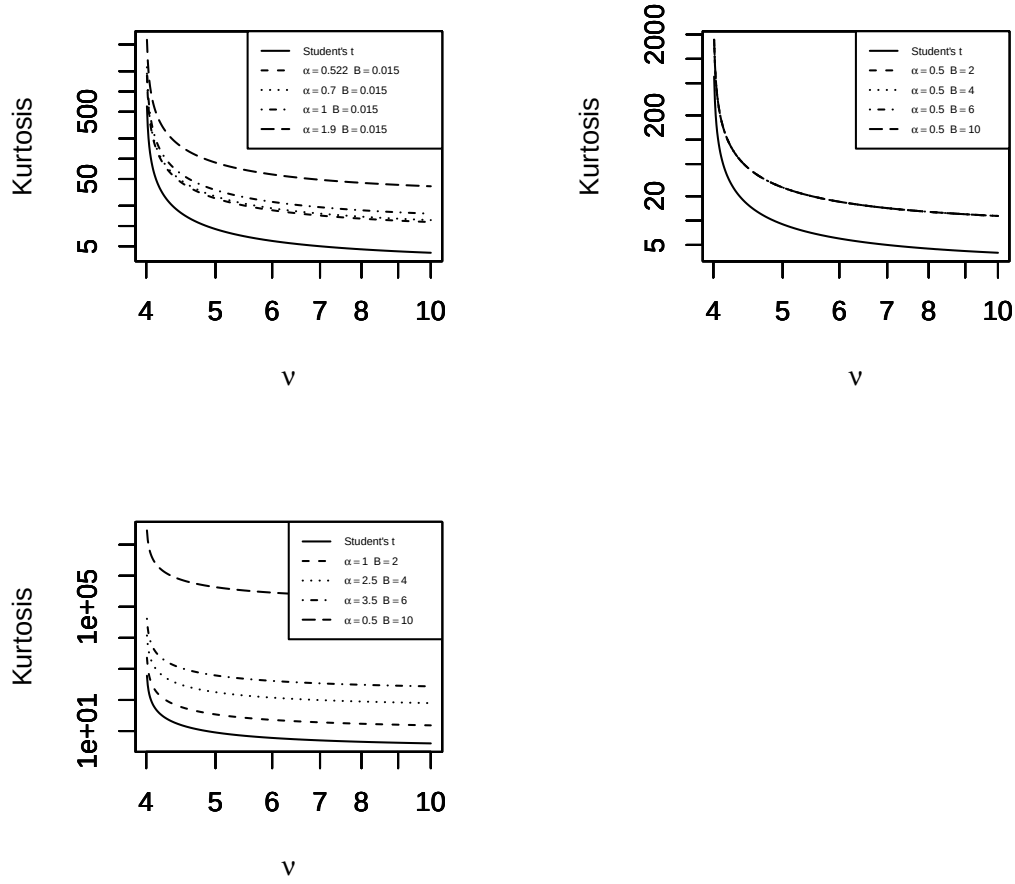


Figure 4.6: Graphs of kurtosis value versus ν for the model based on $g(.) \sim \text{Loglogistic}$ distribution

and mean squared errors are plotted against the sample sizes. We observed that the parameter $\hat{\nu}$ varies instantaneously with different sample sizes, while the remaining parameters are some how stable with different samples. The plot at the top right-hand side of Figure 4.7 shows the bias of the parameters with different sample sizes; we see that parameter $\hat{\nu}$ has a big variation while the biases of the remaining estimated parameters approaches zero with different sample sizes. At the bottom right-hand side we observe the mean squared error of $\hat{\nu}$ increases and decreases drastically between sample size of 5500 to 7000; which indicates the model is more accurate for samples

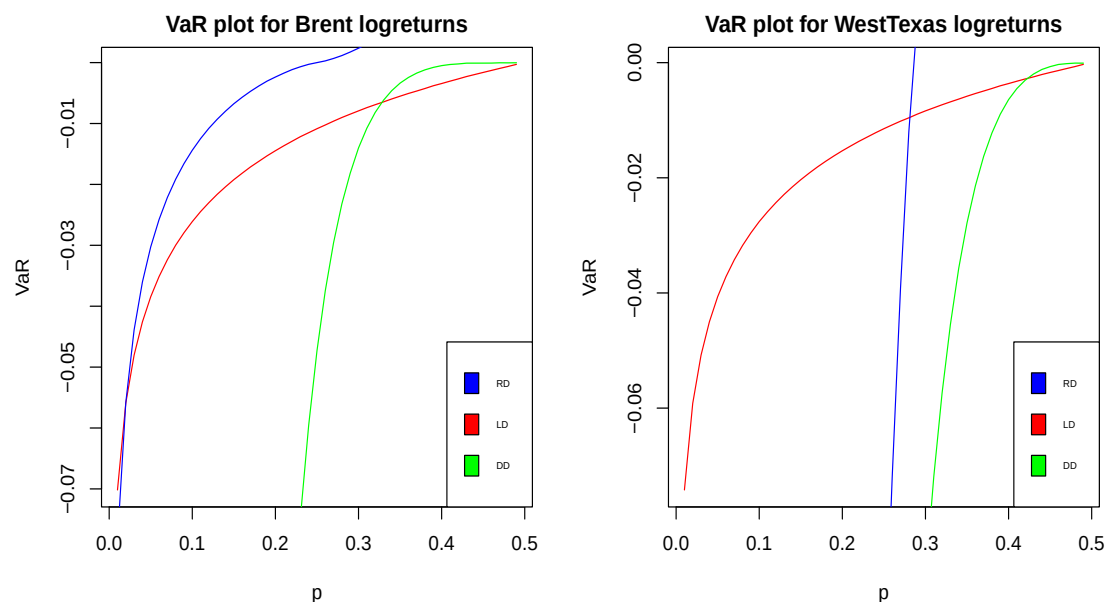


Figure 4.7: VaR plots of BT's log-returns and WTI's log returns based on the models

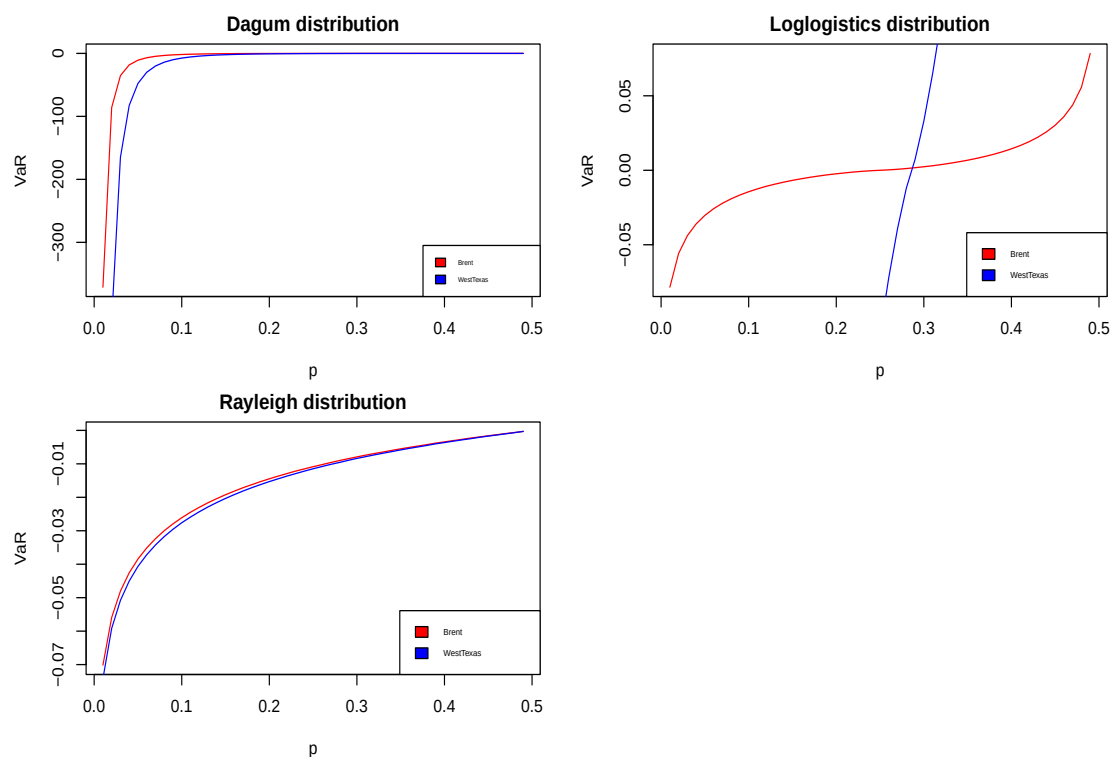


Figure 4.8: VaR plots of BT's log-returns and WTI's log returns based on the models

sizes ranging from 6000 downwards and 7000 upwards on Brent's log-returns. Figure 4.8 shows the plots of parameter estimates of the model based on Dagum distribution

on WTI's log-returns with different sample sizes. We observe that the mean squared errors of the parameters are almost zero with different samples sizes less than 7000.

Figure 4.11 and Figure 4.12 show the plots of the parameter estimates of the model based on Rayleigh distribution on BT and WTI's log-returns respectively. At the top-right corner of Figure 4.11 we observed that the bias of $\hat{\nu}$ increases and decrease with different sample sizes, while $\hat{\mu}$ and $\hat{b}$ are nearly zero for different sample sizes(n). While in Figure 4.12 at the top-left side bias of $\hat{\nu}$ deviates from zero between sample sizes of 1000 to 2000, and henceforth it is nearly zero with the other parameters.

Furthermore, we applied the model based on log-logistics distribution for different sample sizes on BT and WTI's log-returns. Figure 4.13 and Figure 4.14 shows the plots of the estimated parameters respectively. The plot at the top-left in Figure 4.13 we observe that $\hat{\nu}$ increases and decreases with different sample sizes. The plot at the top-right in the Figure 4.13 shows that the bias of the parameter($\hat{\nu}$) is not constant for different sample sizes(n). The estimates of the model varies more on WTI's log-returns, Figure 4.14 depicts it vividly.

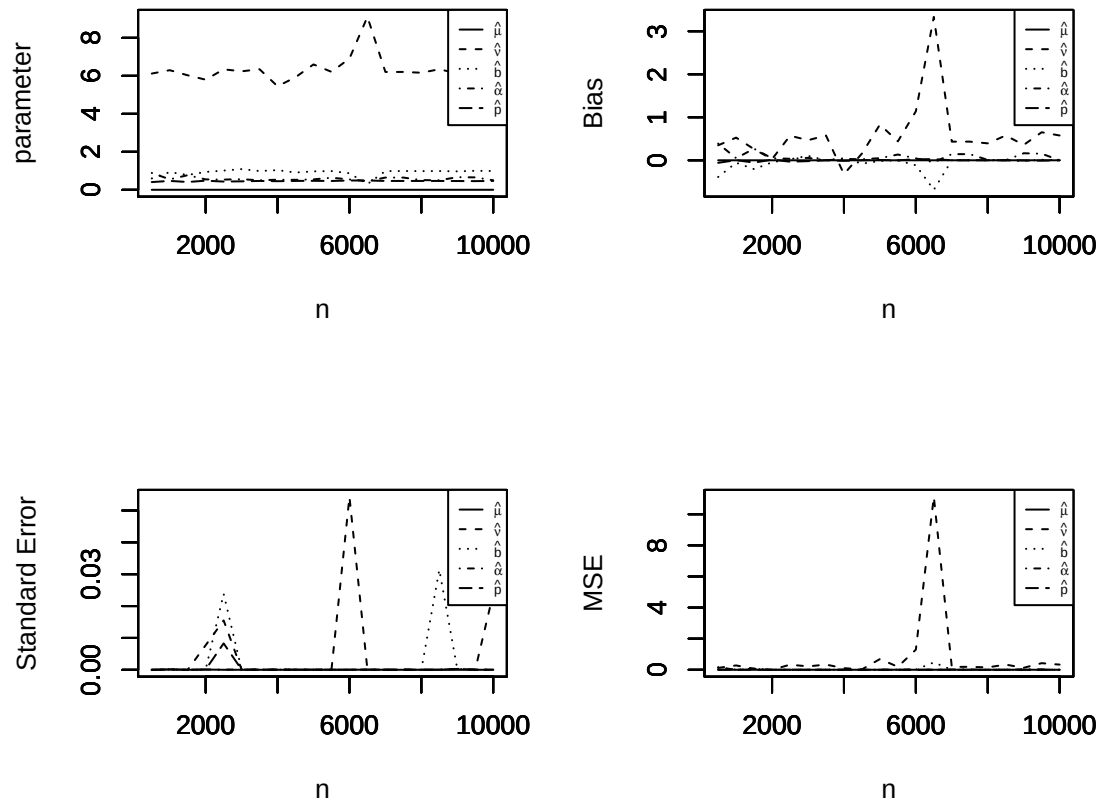


Figure 4.9: Parameter estimates of model based on Dagum distribution on BT's log-returns

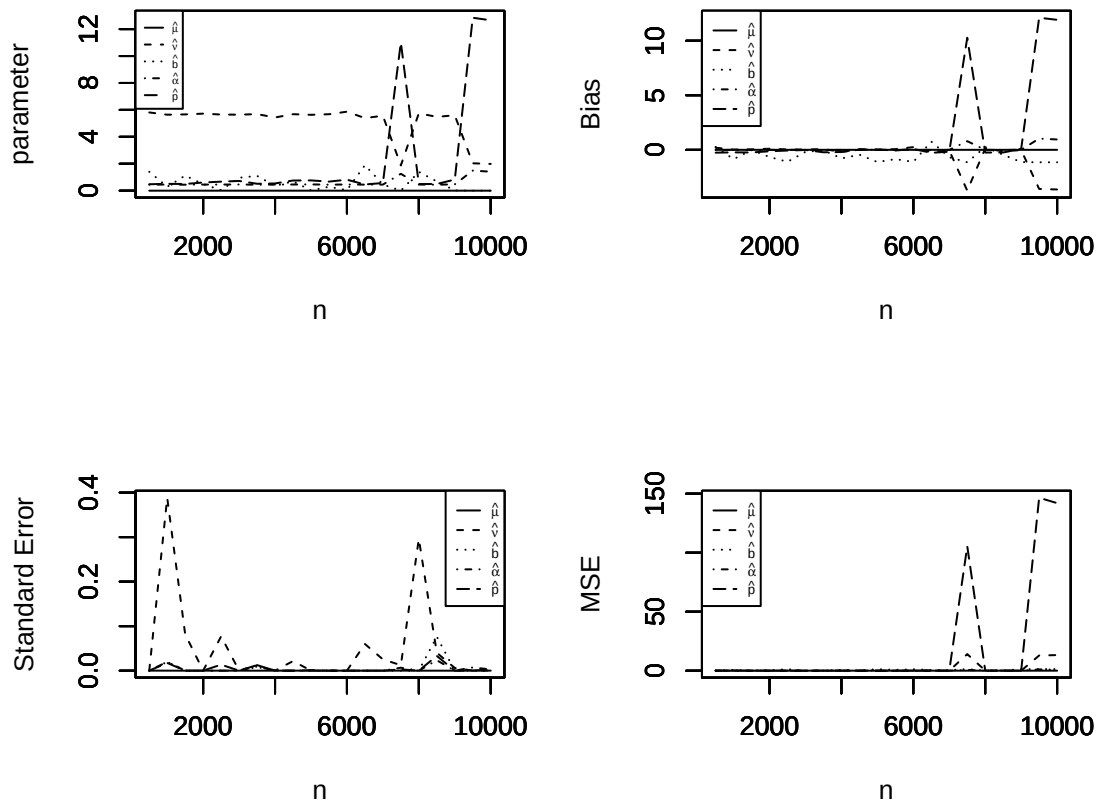


Figure 4.10: Parameter estimates of model based on Dagum distribution on WTI's log-returns

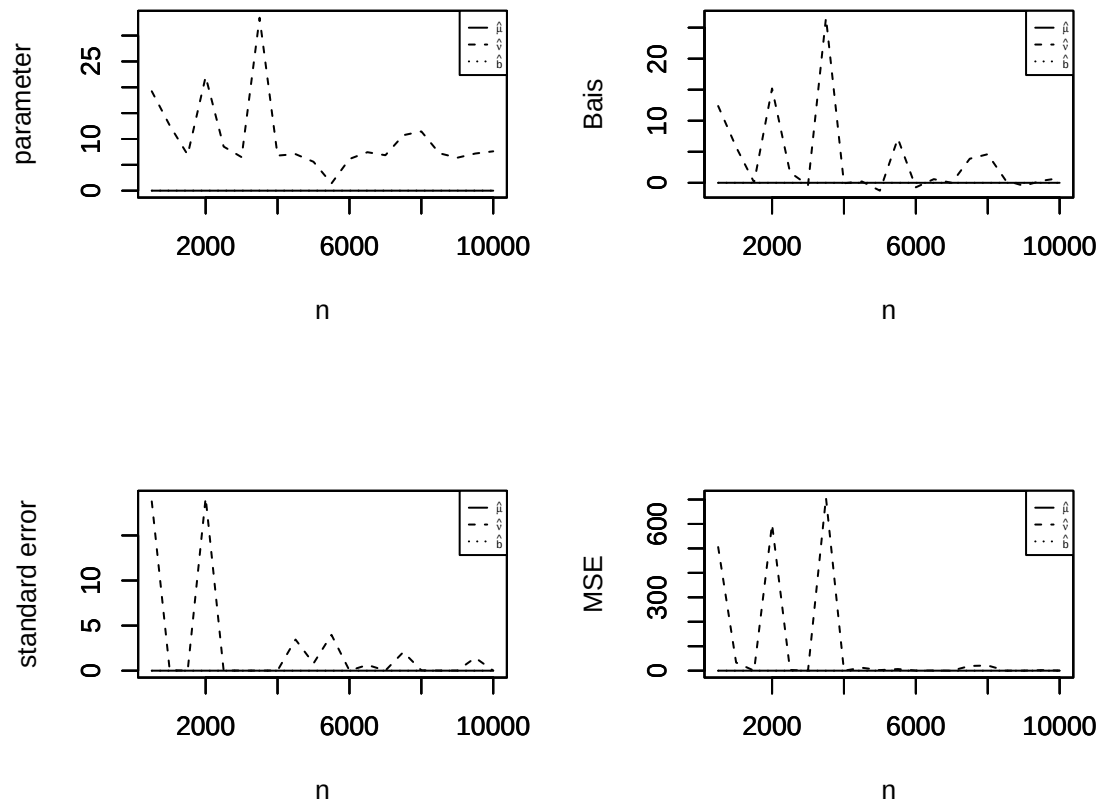


Figure 4.11: Parameter estimates of model based on Rayleigh distribution on BT's log-returns

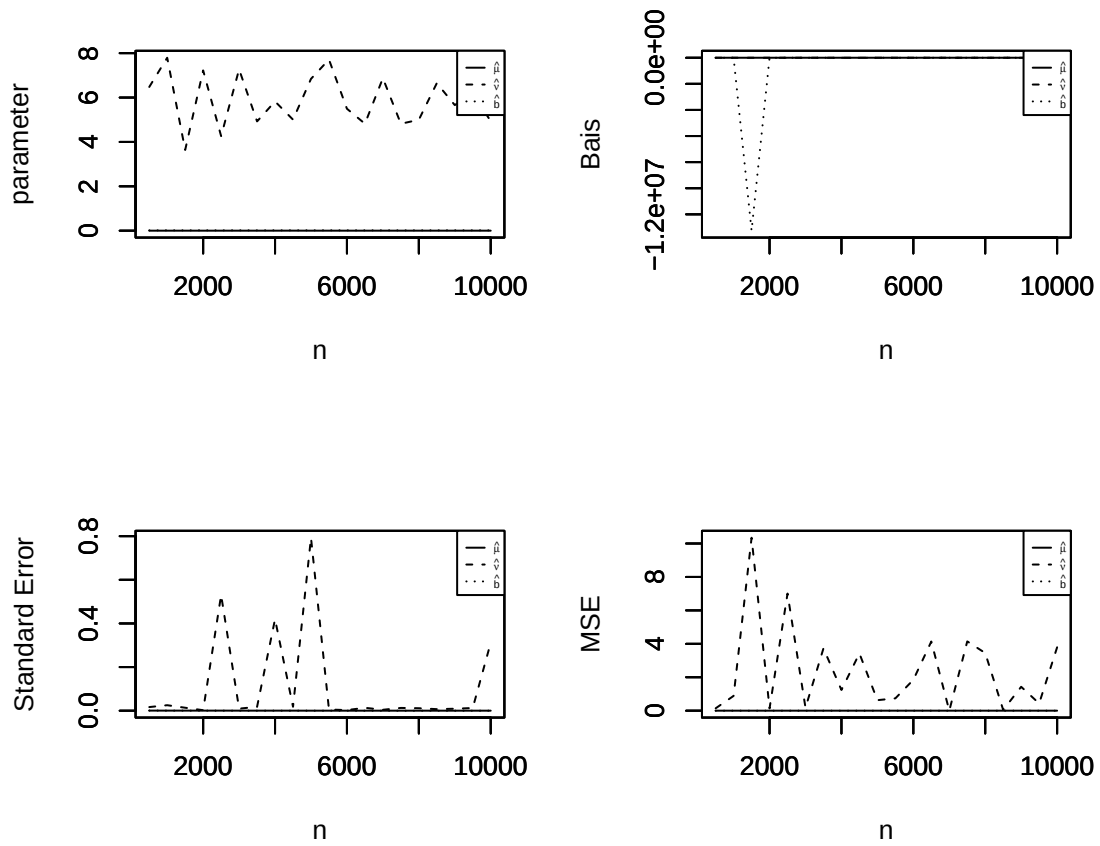


Figure 4.12: Parameter estimates of model based on Rayleigh distribution on WTI's log-returns

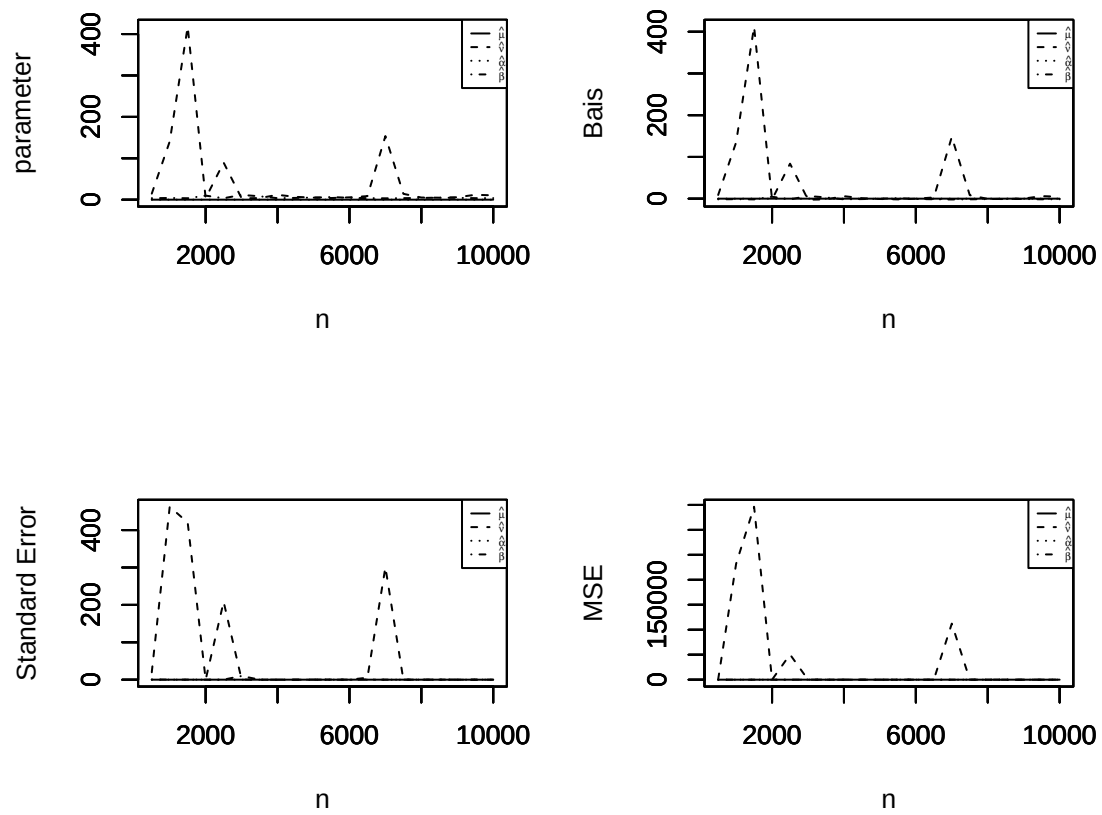


Figure 4.13: Parameter estimates of model based on Loglogistic distribution on BT's log-returns

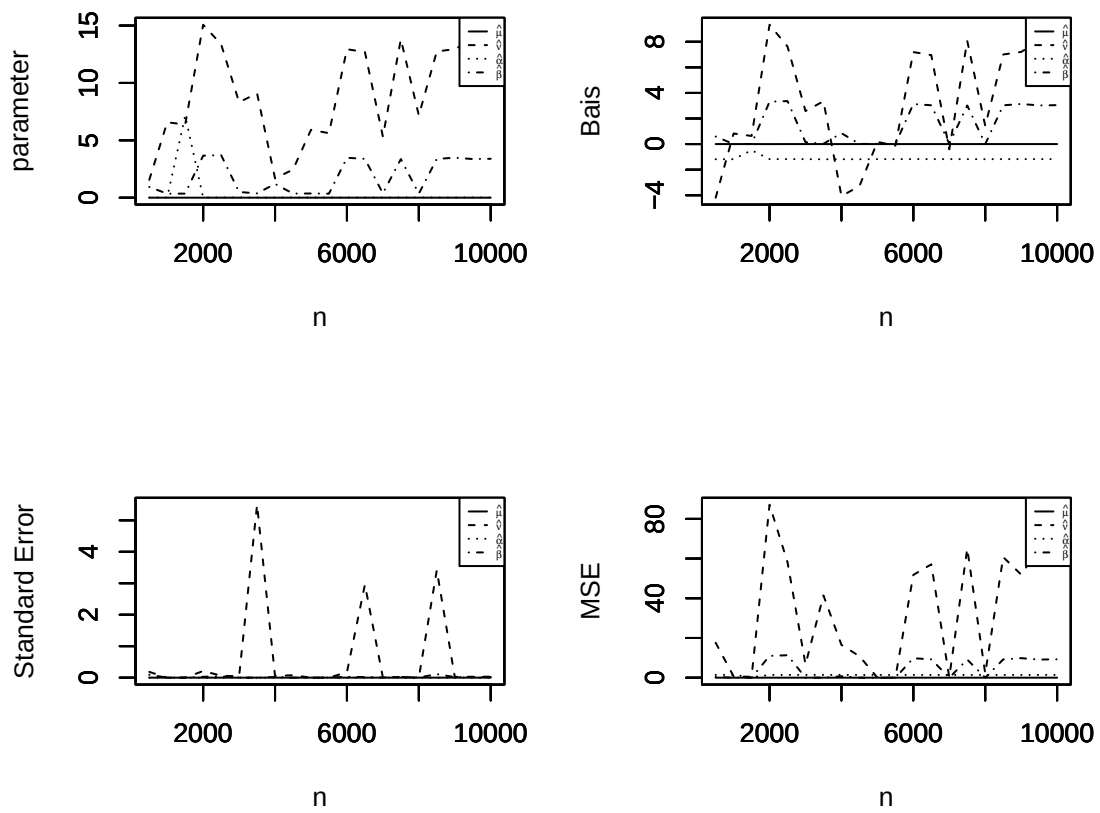


Figure 4.14: Parameter estimates of model based on Loglogistic distribution on WTI's log-returns

CHAPTER 5

SUMMARY AND CONCLUSION

Characterization of volatility associated with financial data is vital as this plays a crucial role in decision making for individuals, government, banks, businesses, etc. Having a good model that will best describe financial data worth consideration as this will guide financial managers in risk analysis and quantification. Financial data are generally known to exhibit some interesting characteristics such as heavy-taildness, and this makes the normal distribution unsuitable for modeling these class of data sets. This means that financial data such as daily stock returns display significant departure from normality. Thus, the need for a good candidate model[79]. We know that many of the main model such as Black-Scholes', option pricing model, mean-variance models depend on the normality assumption; but it is a well known fact that the normal distribution frequently fails to represent the distribution of stock return properly[79]. For instance, due to heavy-taildness of financial data, Students' t distribution serves as a good candidate distributional model in the literature for modeling stock returns. However, Student's t has its own limitation. For example,

it does not account for skewness in the data. As such, there are several extension or modifications to this model.

In this thesis, we proposed three compound distributional models based on scale mixing of Student's t distribution. Their fundamental characteristics, including their PDF, CDF, characteristic functions, and moments have been obtained/derived. These developed distributions were used to analysis log-returns of WTI crude oil prices and BT Crude oil prices. The method of maximum likelihood was used in the parameter estimation. We assess the stability of the estimated parameter via bootstrapping approach. Comparison among the models was made based on the following selections criteria: Akaike information criterion (AIC), the Bayesian information criterion (BIC), the corrected Akaike information criterion (AICc), the Hannan-Quinn criterion (HQC), and the consistent Akaike information criterion (CAIC). The model based on Rayleigh distribution fits better than the others.

The models have greater kurtosis than that of Student's t distribution. This is an indication of its flexibility in characterization of financial data. This means that Value at Risk and Expected shortfall estimated based on this model will be more reliable than that calculated based on the Student's t distribution.

REFERENCES

- [1] J. G. Speight, *An introduction to petroleum technology, economics, and politics*. John Wiley & Sons, 2011.
- [2] S. Nadarajah, E. Afuecheta, and S. Chan, “Garch modeling of five popular commodities,” *Empirical Economics*, vol. 48, pp. 1691–1712, 2015.
- [3] R. J. Beran, “Introduction to efron (1979) bootstrap methods: Another look at the jackknife,” *Breakthroughs in Statistics: Methodology and Distribution*, pp. 565–568, 1992.
- [4] G. Gabbi and G. Iori, “New measures for a new normal in finance and risk management,” *The European Journal of Finance*, vol. 28, no. 13-15, pp. 1257–1262, 2022.
- [5] S.-H. Poon and C. W. J. Granger, “Forecasting volatility in financial markets: A review,” *Journal of economic literature*, vol. 41, no. 2, pp. 478–539, 2003.
- [6] E. Afuecheta, S. Chan, and S. Nadarajah, “Flexible models for stock returns based on student’s t distribution,” *The Manchester School*, vol. 87, no. 3, pp. 403–427, 2019.

- [7] E. Afuecheta, A. Semeyutin, S. Chan, S. Nadarajah, and D. Andrés Pérez Ruiz, “Compound distributions for financial returns,” *Plos one*, vol. 15, no. 10, p. e0239652, 2020.
- [8] E. Platen and R. Rendek, “Empirical evidence on student-t log-returns of diversified world stock indices,” *Journal of statistical theory and practice*, vol. 2, no. 2, pp. 233–251, 2008.
- [9] S. Nadarajah, B. Zhang, and S. Chan, “Estimation methods for expected shortfall,” *Quantitative Finance*, vol. 14, no. 2, pp. 271–291, 2014.
- [10] K. Aas and I. H. Haff, “The generalized hyperbolic skew student’s-t-distribution,” *Journal of financial econometrics*, vol. 4, no. 2, pp. 275–309, 2006.
- [11] L. Bauwens and S. Laurent, “A new class of multivariate skew densities, with application to generalized autoregressive conditional heteroscedasticity models,” *Journal of Business & Economic Statistics*, vol. 23, no. 3, pp. 346–354, 2005.
- [12] R. Li and S. Nadarajah, “A review of student’s-t distribution and its generalizations,” *Empirical economics*, vol. 58, pp. 1461–1490, 2020.
- [13] R. Cont, “Empirical properties of asset returns: stylized facts and statistical issues,” *Quantitative finance*, vol. 1, no. 2, p. 223, 2001.
- [14] Y. Liu and X. Yao, “Ensemble learning via negative correlation,” *Neural networks*, vol. 12, no. 10, pp. 1399–1404, 1999.

- [15] T. G. Andersen, T. Bollerslev, F. X. Diebold, and H. Ebens, “The distribution of realized stock return volatility,” *Journal of financial economics*, vol. 61, no. 1, pp. 43–76, 2001.
- [16] C. Acerbi and B. Szekely, “Back-testing expected shortfall,” *Risk*, vol. 27, no. 11, pp. 76–81, 2014.
- [17] A. E. Lim, J. G. Shanthikumar, and G.-Y. Vahn, “Conditional value-at-risk in portfolio optimization: Coherent but fragile,” *Operations Research Letters*, vol. 39, no. 3, pp. 163–171, 2011.
- [18] A. Novosyolov, “Measuring risk,” *Reliability: Theory & Applications*, vol. 3, no. 4 (11), pp. 115–119, 2008.
- [19] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, “Coherent measures of risk,” *Mathematical finance*, vol. 9, no. 3, pp. 203–228, 1999.
- [20] Y. Yamai, T. Yoshida *et al.*, “Comparative analyses of expected shortfall and value-at-risk: their estimation error, decomposition, and optimization,” *Monetary and economic studies*, vol. 20, no. 1, pp. 87–121, 2002.
- [21] S. Chan, S. Nadarajah, and E. Afuecheta, “An r package for value at risk and expected shortfall,” *Communications in Statistics-Simulation and Computation*, vol. 45, no. 9, pp. 3416–3434, 2016.
- [22] E. Afuecheta, C. Utazi, E. Ranganai, and C. Nnanatu, “An application of extreme value theory for measuring financial risk in brics economies,” *Annals of Data Science*, pp. 1–40, 2020.

- [23] A. Harvey and R.-J. Lange, “Modeling the interactions between volatility and returns,” 2015.
- [24] J. B. McDonald and W. K. Newey, “Partially adaptive estimation of regression models via the generalized t distribution,” *Econometric theory*, vol. 4, no. 3, pp. 428–457, 1988.
- [25] P. Theodossiou, “Financial data and the skewed generalized t distribution,” *Management Science*, vol. 44, no. 12-part-1, pp. 1650–1661, 1998.
- [26] I. Papastathopoulos and J. A. Tawn, “A generalised student’s t-distribution,” *Statistics & Probability Letters*, vol. 83, no. 1, pp. 70–77, 2013.
- [27] S. Psarakis and J. Panaretos, “The folded t distribution,” *Communications in Statistics-Theory and Methods*, vol. 19, no. 7, pp. 2717–2734, 1990.
- [28] J. Ord, “The discrete student’s t distribution,” *The Annals of Mathematical Statistics*, pp. 1513–1516, 1968.
- [29] H.-J. Kim, “Moments of truncated student-t distribution,” *Journal of the Korean Statistical Society*, vol. 37, no. 1, pp. 81–87, 2008.
- [30] M. C. Jones and M. Faddy, “A skew extension of the t-distribution, with applications,” *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 65, no. 1, pp. 159–174, 2003.
- [31] A. Azzalini and A. Capitanio, “Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution,” *Journal of the Royal*

- Statistical Society Series B: Statistical Methodology*, vol. 65, no. 2, pp. 367–389, 2003.
- [32] M. C. Jones and A. Pewsey, “Sinh-arcsinh distributions,” *Biometrika*, vol. 96, no. 4, pp. 761–780, 2009.
- [33] J. Rosco, M. Jones, and A. Pewsey, “Skew t distributions via the sinh-arcsinh transformation,” *Test*, vol. 20, pp. 630–652, 2011.
- [34] S. Acitas, B. Senoglu, and O. Arslan, “Alpha-skew generalized t distribution,” *Revista Colombiana de Estadística*, vol. 38, no. 2, pp. 353–370, 2015.
- [35] J. Sepanski and L. Kong, “A family of generalized beta distributions for income,” *arXiv preprint arXiv:0710.4614*, 2007.
- [36] S. OI, K. Adepoju, and O. Adeniji, “On beta skew-t distribution in modelling stock returns in nigeria,” *Int. J. Modern Math. Sci*, vol. 11, no. 2, pp. 94–102, 2014.
- [37] D. Zhu and J. W. Galbraith, “A generalized asymmetric student-t distribution with application to financial econometrics,” *Journal of Econometrics*, vol. 157, no. 2, pp. 297–305, 2010.
- [38] C. Fernández and M. F. Steel, “On bayesian modeling of fat tails and skewness,” *Journal of the american statistical association*, vol. 93, no. 441, pp. 359–371, 1998.

- [39] R. Li and S. Nadarajah, “The kumaraswamy skew g distributions,” *Submitted to the Bulletin of Brazilian Mathematical Society*, 2016.
- [40] M. Villani, R. Kohn, and P. Giordani, “Regression density estimation using smooth adaptive gaussian mixtures,” *Journal of Econometrics*, vol. 153, no. 2, pp. 155–173, 2009.
- [41] D. Ardia, *Financial Risk Management with Bayesian Estimation of GARCH Models Theory and Applications*. Springer, 2008.
- [42] O. Barndorff-Nielsen, “Hyperbolic distributions and distributions on hyperbolae,” *Scandinavian Journal of statistics*, pp. 151–157, 1978.
- [43] O. Barndorff-Nielsen, J. Kent, and M. Sørensen, “Normal variance-mean mixtures and z distributions,” *International Statistical Review/Revue Internationale de Statistique*, pp. 145–159, 1982.
- [44] O. Arslan, “Variance-mean mixture of the multivariate skew normal distribution,” *Statistical Papers*, vol. 56, pp. 353–378, 2015.
- [45] D. Peel and G. J. McLachlan, “Robust mixture modelling using the t distribution,” *Statistics and computing*, vol. 10, pp. 339–348, 2000.
- [46] S. Nasiru, P. N. Mwita, and O. Ngesa, “Exponentiated generalized exponential dagum distribution,” *Journal of King Saud University-Science*, vol. 31, no. 3, pp. 362–371, 2019.

- [47] H. M. Almongy, E. M. Almetwally, H. M. Aljohani, A. S. Alghamdi, and E. Hafez, "A new extended rayleigh distribution with applications of covid-19 data," *Results in Physics*, vol. 23, p. 104012, 2021.
- [48] B. O. Oluyede, S. Huang, and M. Pararai, "A new class of generalized dagum distribution with applications to income and lifetime data," *Journal of Statistical and Econometric Methods*, vol. 3, no. 2, p. 125, 2014.
- [49] S. Nadarajah, "Models for stock returns," *Quantitative Finance*, vol. 12, no. 3, pp. 411–424, 2012.
- [50] A. Harvey and R.-J. Lange, "Volatility modeling with a generalized t distribution," *Journal of Time Series Analysis*, vol. 38, no. 2, pp. 175–190, 2017.
- [51] M. I. Ekum, M. O. Adamu, and E. E. Akarawak, "T-dagum: A way of generalizing dagum distribution using lomax quantile function," *Journal of Probability and Statistics*, vol. 2020, pp. 1–17, 2020.
- [52] A. Ishaq and A. Abiodun, "A new generalization of dagum distribution with application to financial data sets," in *2020 International Conference on Data Analytics for Business and Industry: Way Towards a Sustainable Economy (ICDABI)*. IEEE, 2020, pp. 1–6.
- [53] I. Elbatal and G. Aryal, "Transmuted dagum distribution with applications." *Chilean Journal of Statistics (ChJS)*, vol. 6, no. 2, 2015.

- [54] H. J. K. Al-Bderi, “Estimate survival function by using dagum distribution,” in *Journal of Physics: Conference Series*, vol. 1591, no. 1. IOP Publishing, 2020, p. 012033.
- [55] F. Domma, F. Condino, and S. Giordano, “A new formulation of the dagum distribution in terms of income inequality and poverty measures,” *Physica A: Statistical Mechanics and its Applications*, vol. 511, pp. 104–126, 2018.
- [56] M. K. Ochi, “Wave statistics for the design of ships and ocean structures,” Tech. Rep., 1978.
- [57] M. P. Hossain, M. H. Omar, M. Riaz, and S. Y. Arafat, “On designing a new control chart for rayleigh distributed processes with an application to monitor glass fiber strength,” *Communications in Statistics-Simulation and Computation*, vol. 51, no. 6, pp. 3168–3184, 2022.
- [58] O. A. Adeoti, “Design of ewma control chart for monitoring transformed rayleigh distributed data,” *International Journal of Quality Engineering and Technology*, vol. 8, no. 4, pp. 335–350, 2022.
- [59] T. Lishamol and G. Jiju, “A generalized rayleigh distribution and its application,” *Biom Biostat Int J*, vol. 8, no. 4, pp. 139–143, 2019.
- [60] G. M. Cordeiro, C. T. Cristino, E. M. Hashimoto, and E. M. Ortega, “The beta generalized rayleigh distribution with applications to lifetime data,” *Statistical papers*, vol. 54, pp. 133–161, 2013.

- [61] A. E. Gomes, C. Q. da Silva, G. M. Cordeiro, and E. M. Ortega, “A new lifetime model: the kumaraswamy generalized rayleigh distribution,” *Journal of statistical computation and simulation*, vol. 84, no. 2, pp. 290–309, 2014.
- [62] S. Bashir, M. Rasul *et al.*, “A new weighted rayleigh distribution: properties and applications on lifetime time data,” *Open Journal of Statistics*, vol. 8, no. 03, p. 640, 2018.
- [63] N. Al-Noor and N. Assi, “Rayleigh-rayleigh distribution: properties and applications,” in *Journal of Physics: Conference Series*, vol. 1591, no. 1. IOP Publishing, 2020, p. 012038.
- [64] F. Ashkar and S. Mahdi, “Fitting the log-logistic distribution by generalized moments,” *Journal of Hydrology*, vol. 328, no. 3-4, pp. 694–703, 2006.
- [65] A. Gago-Benítez, J.-A. Fernández-Madrigal, and A. Cruz-Martín, “Log-logistic modeling of sensory flow delays in networked telerobots,” *IEEE Sensors Journal*, vol. 13, no. 8, pp. 2944–2953, 2013.
- [66] N. Kantas, A. Doucet, S. S. Singh, J. Maciejowski, and N. Chopin, “On particle methods for parameter estimation in state-space models,” 2015.
- [67] J. R. Hosking, “L-moments: Analysis and estimation of distributions using linear combinations of order statistics,” *Journal of the royal statistical society: series B (methodological)*, vol. 52, no. 1, pp. 105–124, 1990.

- [68] D. Luengo, L. Martino, M. Bugallo, V. Elvira, and S. Särkkä, “A survey of monte carlo methods for parameter estimation,” *EURASIP Journal on Advances in Signal Processing*, vol. 2020, no. 1, pp. 1–62, 2020.
- [69] E. Hooda, B. Hooda, and N. Tanwar, “Probability weighted moments (pwms) and partial probability weighted moments (ppwms) of type-ii extreme value distribution.”
- [70] J. A. Greenwood, J. M. Landwehr, N. C. Matalas, and J. R. Wallis, “Probability weighted moments: definition and relation to parameters of several distributions expressible in inverse form,” *Water resources research*, vol. 15, no. 5, pp. 1049–1054, 1979.
- [71] A. Helu, “The principle of maximum entropy and the probability-weighted moments for estimating the parameters of the kumaraswamy distribution,” *Plos one*, vol. 17, no. 5, p. e0268602, 2022.
- [72] B. Prakasa Rao, “Asymptotic theory for non-linear least squares estimator for diffusion processes,” *Statistics: A Journal of Theoretical and Applied Statistics*, vol. 14, no. 2, pp. 195–209, 1983.
- [73] I. J. Myung, “Tutorial on maximum likelihood estimation,” *Journal of mathematical Psychology*, vol. 47, no. 1, pp. 90–100, 2003.
- [74] F. S. Richards, “A method of maximum-likelihood estimation,” *Journal of the Royal Statistical Society: Series B (Methodological)*, vol. 23, no. 2, pp. 469–475, 1961.

- [75] A. Henningsen and O. Toomet, “maxlik: A package for maximum likelihood estimation in r,” *Computational Statistics*, vol. 26, pp. 443–458, 2011.
- [76] H. Akaike, “A new look at the statistical model identification,” *IEEE transactions on automatic control*, vol. 19, no. 6, pp. 716–723, 1974.
- [77] J. B. Quinn, “Technological innovation, entrepreneurship, and strategy,” *Sloan Management Review (pre-1986)*, vol. 20, no. 3, p. 19, 1979.
- [78] J. Y. Campbell and L. Hentschel, “No news is good news: An asymmetric model of changing volatility in stock returns,” *Journal of financial Economics*, vol. 31, no. 3, pp. 281–318, 1992.
- [79] A. Peiro, “The distribution of stock returns: international evidence,” *Applied financial economics*, vol. 4, no. 6, pp. 431–439, 1994.

VITAE

- Name: Haady Jallow
- Nationality: Gambian
- Date of Birth: 24th April, 1995
- Email: *haadyjallow95@gmail.com*
- Permanent Address: The Gambia
- Postal Address: Kuntair Village, Jokadou District, North Bank Region, The Gambia.