

STABILITY AND SYNCHRONIZATION OF A MEMRISTOR
BASED INTEGRO-DIFFERENTIAL SYSTEM IN NEURAL
NETWORKS THEORY

BY

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*To my parents, wife, children, brothers, all family and all friends who waited patiently
for me to come out of this study.*

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Dissertation Abstract

NAME: Ebrahim Saleh Mohammed Bakalah

TITLE OF STUDY: Stability and Synchronization of a Memristor Based
Integro-Differential System in Neural Networks Theory

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There are many results in the literature on the stability and synchronization of memristors in neural network theory. Under certain reasonable conditions, solutions of some memristive neural network systems of ordinary differential equations are exponentially or polynomially stable (synchronized). In contrast, there are very few studies of stability and synchronization of solutions of memristors based integro-differential systems with more general rates. In addition to the well-posedness mentioned in the literature below, we would like to contribute, in this dissertation, in covering some gap by investigating the stability and synchronization of solutions for wider classes of memristors based integro-differential systems in neural network theory which contain discrete and distributed delays. Reasonable sufficient conditions under which the stability and synchronization are achieved under general rates are determined. In particular, it is of great importance to weaken the existing hypotheses on the activation functions. Furthermore, it is very useful in applications to consider wider classes of kernel functions than the ones treated so far. In addition to the traditional approaches and methods adopted to tackle similar problems in the literature such as the implementation of different controllers, the Lyapunov method and the linear matrix inequalities method (LMIM), we introduce

new functionals and combine them with arguments such as the Filippov method in the context of the theory of differential inclusion. More precisely, we consider and study a memristive Hopfield neural network system (MHNNS) and a memristive high-order Cohen–Grossberg bidirectional associative memory neural network system (MCGBAMNNS).

ملخص الرسالة

الإسم الكامل: إبراهيم صالح محمد باكله

عنوان الرسالة: الإستقرار والتزامن للممرستر المبني على نظام تفاضلي تكاملي
في نظرية الشبكات العصبية

التخصص: الرياضيات

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توجد العديد من الدراسات العلمية السابقة حول استقرار ومزامنة الميمرستورز memristors في نظرية الشبكة العصبية. في ظل شروط معينة ، تكون حلول بعض أنظمة الشبكات العصبية الميمريستية للمعادلات التفاضلية العادية مستقرة بشكل أسي أو متعدد الحدود (متزامنة). في المقابل ، هناك عدد قليل جدًا من الدراسات حول استقرار وتزامن حلول الأنظمة التكاملية-التفاضلية القائمة على الميمرستورز مع معدلات أكثر عمومية. في هذه الرسالة ، نود المساهمة في تغطية الفجوة المذكورة من خلال التحقق من استقرار وتزامن الحلول لفئات أوسع من أنظمة التكامل التفاضلية القائمة على الميمرستورز في نظرية الشبكة العصبية التي تحتوي على تأجيلات زمنية منفصلة وموزعة. سيتم تحديد الظروف الكافية المعقولة التي يتم بموجبها تحقيق الاستقرار والتزامن وفقًا للمعدلات العامة. على وجه الخصوص ، من الأهمية بمكان التقليل من الفرضيات الموجودة حول وظائف التنشيط. علاوة على ذلك ، من المفيد جدًا في التطبيقات النظر في فئات أوسع من دوال النواة kernel functions عن تلك التي تمت معالجتها حتى الآن. بالإضافة إلى الأساليب والطرق التقليدية المعتمدة لمعالجة مشاكل مماثلة مثل تنفيذ وحدات تحكم مختلفة وطريقة ليابونوف Aleksander Mikhailovich Lyapunov وطريقة عدم مساواة المصفوفة الخطية (LMIM) ، نقدم وظائف جديدة ودمجها مع طرق شرح مثل طريقة فيلييوف Aleksei A. Filippov في سياق نظرية الشمول التفاضلي. بتعبير أدق ، نحن ندرس نظام الشبكة العصبية الميمريستية هوبفيلد memristor John Joseph Hopfield (MHNNS) ونظام الشبكة العصبية للذاكرة النقاوية ثنائية الاتجاه (MCGBAMNNS) كوهين – كروسبيرغ Michael Cohen و Stephen Grossberg.

Chapter 1

INTRODUCTION

In this chapter, we will introduce the concept of a memristor in neural network theory, present some of its history, a literature review and some related problems. We will also give some terminology and notation that is used in this thesis.

1.1 Memristors and Generalities

In electrical engineering and artificial intelligence, the concept of “memristor” refers to the contraction for memory resistors [2] (Yang, X. *et al.*, 2014). A memristor is a non-linear two-terminal electrical component element that limits or regulates the flow of electrical current in electrical circuits and remembers the amount of charge that has previously flowed through it. It points out to the existence of three basic circuit elements that define the connections between basic parameters such as current (i), voltage (v), charge (q) and magnetic flux (φ). These elements are the resistor, the capacitor and the inductor. However, a circuit element that determines the relation between q and φ was not defined yet.

In 1971, Chua, L. formulated the concept of the relation as a connection between q and φ interconnections [18] and named it memristor (memory and resistor). In 2008, researchers at the H-P laboratory created another type of memristor [19] (Strukov, D. *et al.*, 2008).

Memristors play a significant role in neural networks theory. They are different from the other circuit elements because they have a non-volatile memory effect - they can return memory without energy. So, they are not elements that store energy (see Chen,

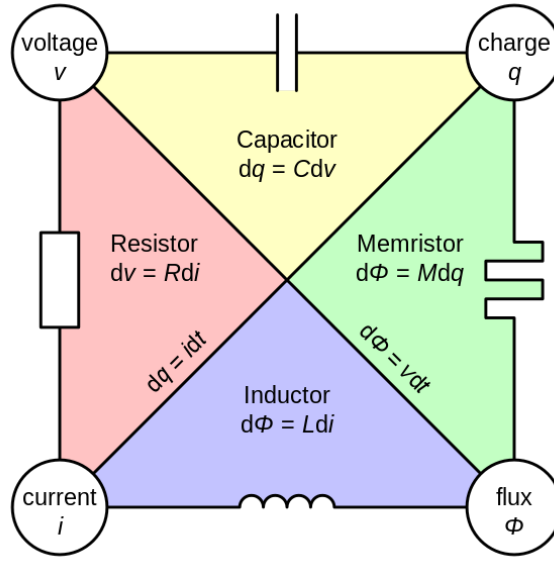


Figure 1.1: The Relations Between Memristor (M), Resistor (R), Capacitor (C), Inductor (L), Voltage (v), Current (i), Charge (q) and Magnetic Flux (φ)
(see Chen *et al.*, 2017 [111] and references therein).

C. *et al.*, 2017 [3] – Park, J. H., 2006 [12] and Hopfield, J. J., 1982 [55] – Hopfield, J. J. *et al.*, 1986 [58]). There are several applications of memristors in real-life scenario. For instance, IT companies use memristors to create smaller, faster and low-power computers that do not require transferring data between volatile and non-volatile memory. An essential general utilization of a memristor can be observed in the area of electrical engineering and it mainly deals with the synchronizations and synapses. In electrical circuits, usually resistors are utilized to imitate and simulate synapses and synchronization through the neurons Wang, W. *et al.*, 2014 [20]. As though, synapses and synchronization play paramount role in the figuration of memories. However, the widespread resistors can not fulfill the mission of memory. The tacky information is that the human brain has more than side split billion of neurons and, in fact, there are more than twenty thousands synapses and synchronization for any neuron. Consequently, by seeking to makeup a brain such as machine, practical electrical circuits fulfillment of synapses and synchronization is of specific significance. If we assume that the resistors are replaced by the memories, the customary synthetically neural network will turn into what is called a memristor based neural networks. Even after the memristor based neural networks is able to recollect its genetical situation, it is extremely convenient to mimic and imitate the assignment of the human brain Jo, S. H. *et al.*, 2010 [21]. Furthermore, an important aspect of the memristor is the dynamical behavior of a resistor with memory. One of the common features

in serious dynamical conductance of non-linear neural network systems are synapses and synchronization Hua, C. *et al.*, 2016 [22]. Due to considerable usage in diverse fields and scopes can be found in secure communication Milanovic, V. and Zaghloul, M. E., 1996 [23], bidirectional associative memory Tan, Z. and Ali, M. K., 2001 [24] and image encryption. Synapses and synchronization control challenge of neural networks have magnetized supreme attentiveness of researchers. In 2008, Itoh, M and Chua, L. [25] obtained various kinds of frequencies and oscillators by applying a memristor to subrogate the diode in Chua's electrical circuit. Subsequently, numerous memristive chaotic systems Wang, L. *et al.*, 2012 [26] and Muthuswamy, B. and Kokate, P. P., 2009 [27] were suggested by employing a similar approach. Therefore, massive performances have been acquired in the field of synapses and synchronization control of memristive neural networks systems (see also Guo, Z. *et al.*, 2015 [28] – Abdurahman, A. *et al.*, 2015 [31] and references therein). In 1983 [38], Michael Cohen and Stephen Grossberg introduced the memristive Cohen–Grossberg neural network system (MCGNNS) and showed that it is a generalization of the memristive Hopfield neural network system (MHNNS) and memristive bidirectional associative memory neural network system (MBAMNNS) (see Yang, X. *et al.*, 2014 [2] and Yao, W. *et al.*, 2022 [139] and references therein).

Over the years, memristors have been studied and expressed in differential equations by many researchers. For instance, a memristive Hopfield neural network system (MHNNS) described by

$$\begin{cases} z'_i(t) = -\alpha_i z_i(t) + \sum_{j=1}^m \beta_{ij}(z_i(t)) \eta_j(z_j(t)) + \sum_{j=1}^m \gamma_{ij}(z_i(t)) \zeta_j(z_j(t - \tau)) \\ \quad + \sum_{j=1}^m \mu_{ij}(z_i(t)) \int_0^\infty \kappa_{ij}(s) \xi_j(z_j(t - s)) ds + I_i(t), \forall t > 0, i = 1, \dots, m, \end{cases} \quad (1.1)$$

was discussed by Qiu, R. *et al.*, 2022 [158], Pham, V. *et al.*, 2016 [159], Li, R. and Ding, R., 2022 [160], Ma, T. *et al.*, 2022 [161] and He, S. *et al.*, 2023 [162] and references therein. A memristive bidirectional associative memory neural network system presented by, for all $t > 0$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$,

$$\left\{ \begin{array}{l} u'_i(t) = -\alpha_{1i}u_i(t) + \sum_{j=1}^m \beta_{1ij}(u_i(t))\eta_{1j}(v_j(t)) + \sum_{j=1}^m \gamma_{1ij}(u_i(t))\zeta_{1j}(v_j(t - \tau_1)) \\ \quad + \sum_{j=1}^m \mu_{1ij}(u_i(t)) \int_0^\infty \kappa_{1ij}(s)\xi_{1j}(v_j(t - s))ds + I_i(t) \\ v'_j(t) = -\alpha_{2j}v_j(t) + \sum_{i=1}^n \beta_{2ij}(v_j(t))\eta_{2i}(u_i(t)) + \sum_{i=1}^n \gamma_{2ij}(v_j(t))\zeta_{2i}(u_i(t - \tau_2)) \\ \quad + \sum_{i=1}^n \mu_{2ij}(v_j(t)) \int_0^\infty \kappa_{2ij}(s)\xi_{2i}(u_i(t - s))ds + J_j(t), \end{array} \right. \quad (1.2)$$

was studied by many researchers such as Sakthivel, R. *et al.*, 2016 [163], Zhao, Y. *et al.*, 2019 [164], Zhao, Y. *et al.*, 2021 [165], Yuan, M. *et al.*, 2018 [166] and Zheng, M. *et al.*, 2018 [167] and references therein. Hui, M. *et al.*, 2020 [168], Zhu, S. *et al.*, 2023 [169], Ren, F. *et al.*, 2021 [170], Yao, W. *et al.*, 2022 [171], Hui, M. *et al.*, 2019 [172], Zhou, Y. *et al.*, 2019 [173], Wei, R. *et al.*, 2018 [174] and Qin, S. *et al.*, 2019 [175] studied the memristive Cohen–Grossberg neural network system given by the following system, for all $t > 0$ and $i = 1, \dots, m$,

$$\left\{ \begin{array}{l} w'_i(t) = -\chi_i(w_i(t)) \left[\alpha_i w_i(t) - \sum_{j=1}^m \beta_{ij}(w_i(t))\eta_j(u_j(t)) - \sum_{j=1}^m \gamma_{ij}(w_i(t))\zeta_j(w_j(t - \tau)) \right. \\ \quad \left. - \sum_{j=1}^m \mu_{ij}(w_i(t)) \int_0^\infty \kappa_{ij}(s)\xi_j(w_j(t - s))ds - I_i(t) \right]. \end{array} \right. \quad (1.3)$$

In 1960, Filippov, A. F. [44] and Filippov, A. F., 2013 [45] presented a novel method to convert a differential equation with discontinuous right hand side into a differential inclusion and applied the theory of differential inclusion to come up with stability (see also Aubin, J. P. and Cellina, A., 2012 [46] and references therein). In addition, several mathematical methods and technical analyses in neural networks systems have been used to obtained stability and synchronization results such as linear matrix inequalities method (LMIM), differential inequities, LaSalle method, invariance principle Singh, V. 2004 [47] – Tao, Q. *et al.*, 2001 [54], Lyapunov functionals, Lyapunov–Karasovskii method, contracting mapping theory, Halanay inequalities, etc.

In 2014, Yang, X. *et al.* [2] formulated a relationship between the memristor in neural network theory and differential equations with discontinuous right hand side (see also Liu, X. and Cao, J., 2011 [43]). The value (memristor) depends on the magnitude and polarity of the voltage applied to it and the length of the time that the voltage has been applied.

When the voltage is turned off, the memristor remembers its most recent value until it is turned on next time.

There are different types of stability and synchronization results for memristive Cohen–Grossberg neural network systems, memristive bidirectional associative memory neural network system and memristive Hopfield neural network systems. In the literature, we can find several results on the asymptotic, polynomial and exponential stability and synchronization. These results are achieved for large values of time. On the other side, in some situations, it is possible to obtain these results in a finite time. Actually, the finite time synchronization is important and essential to be investigated and examined. It helps to enable critical functions (e.g. handovers between cell towers, time stamping of financial transactions, highly accurate monitoring of electrical grids,...etc) at distributed nodes that require a precise frequency and time reference. For more details on stability and synchronization results for a memristor, we refer to Dong, X. and Wang, X., 2017 [138], Yao, W. *et al.*, 2020 [139], Wang, L. *et al.*, 2014 [140], Shi, Y. and Cao, J., 2020 [141], Hu, J. and Wang, J., 2010 [142], Li, G. *et al.*, 2018 [143], Wen, S. *et al.*, 2015 [144], Wu, A. and Zeng, Z., 2015 [145], Othmani, S. and Tatar, N.E., 2021 [81], Othmani, S. *et al.*, 2021 [82], Wu, A. *et al.*, 2014 [146], Muhammadhaji, A. and Teng, Z., 2020 [147], Wang, L. *et al.*, 2016 [148], Maharajan, C. *et al.*, 2018 [149], Xiong, W. *et al.*, 2017 [150], Yang, J. *et al.*, 2017 [151], Lai, Q. *et al.*, 2022 [152], Hu, S. G. *et al.*, 2015 [153], Xu, Q. *et al.*, 2018 [154], Yu, F. *et al.*, 2022 [155], Yu, F. *et al.*, 2022 [156] and Kong, D. *et al.*, 2019 [157] and references therein.

It has been made clear in the applications that memristive neural network systems with discrete and distributed delays are of paramount importance and should, therefore, be considered further. In spite of that, we can find only few studies in the literature treating the dynamics of memristive neural networks with delay effects. In particular, there are very few studies on the stability and synchronization memristive high–order Cohen–Grossberg bidirectional associative memory neural network systems and memristive Hopfield neural network systems containing discrete and distributed delays. Furthermore, because of a high–approximation property, quicker convergence rate, higher storage capability and the ability of solving optimization problems, high–order neural networks (HONNs) became more efficient than low–order neural networks. This kind of networks have been applied in many science and engineering areas to cite Kamruzzaman, J. and Sarker, R. A., 2006 [134], Karayiannis, N. B. and Venetsanopoulos, A. N., 1995 [135] and Psaltis, D. *et al.*, 1988 [136] and references therein. There are numbers of noteworthy

results on high-order neural network in the literature. For instance, Dong, Z. *et al.*, 2020 [93], Gu, H. *et al.*, [86], Dong, X. and Wang, X., 2017 [138], Dong, Z. *et al.*, 2020 [93], Gu, H. *et al.*, 2011 [86], Ren, F. and Cao, J., 2006 [137], Sheng, Y. *et al.*, 2020 [88], Xia, Y., 2010 [95] and Zu, J. and Zu, J. *et al.*, [96]. The authors investigated the exponential stability of the stationary point for delayed high-order neural network systems through the Lyapunov functionals method, linear matrix inequalities method, theory of differential equations and differential inequalities. The exponential stability of the steady point for a high-order Cohen–Grossberg neural network system with delays was proved by Chen, Z. *et al.*, 2007 [92]. Lou, X. Y. and Cui, B. T., 2007 [94] studied the global asymptotic stability of a high-order Cohen–Grossberg neural network system with delays using linear matrices inequalities method and constructing Lyapunov functionals. The dissipativity of a high-order bidirectional associative memory Cohen–Grossberg neural network system involving mixed delays was shown by Aouiti, C. *et al.*, 2020 [91] using Lyapunov–Karasovskii functionals method and linear matrices inequalities method.

Motivated by the above applications and functions of memristors in neural network theory and in addition to existence, uniqueness and continuity of such solutions of memristive Hopfield neural network system and memristive high-order Cohen–Grossberg bidirectional associative memory neural network system in Wang, X., 2019 [176], Chen, Y. *et al.*, 2023 [177], Yang, D. *et al.*, 2016 [178], Sheng, Y. *et al.*, 2019 [179], Sun, B. *et al.*, 2020 [180], Li, H. *et al.*, 2022 [181], Li, X. *et al.*, 2020 [182], Wu, H. *et al.*, 2016 [183], Yang, W. *et al.*, 2018 [184], He, W. and Chu, L., 2017 [185], Stamov, G. *et al.*, 2020 [186], Manikandan, M. *et al.*, 2021 [187], Chen, S. *et al.*, 2021 [188], Zhou, F., 2012 [189], XYang, W., 2014 [190], Bao, H., 2016 [191], Zhang, Z. and Wang, L., 2011 [201], Li, X., 2009 [202], Balasubramaniam, P. and Ali, M. S., 2011 [203], Tian, A. *et al.*, 2010 [204], Li, Y. and Fan, X., 2009 [205], [206], Xiang, H. *et al.*, 2011 [207], Jian, J. and Zhao, Z., 2015 [208], Liu, Q. and Xu, R., 2011 [209], Li, X. and Fu, X., 2011 [210], Wang, D. *et al.*, 2013 [211] and Zhao, Y. and Ren, S., 2020 [212], this dissertation will focus on discrete and distributed delays for memristors. In particular, we will examine the stability and synchronization of some memristive neural network systems including memristive Hopfield neural network system, memristive Cohen–Grossberg neural networks and bidirectional associative memory neural networks (BAMNN) that consists of two state variables. Specifically, we wish to weaken the hypotheses on the activation (signal transmission) functions, kernel (relaxation) functions and memristive connection weigh functions. Our approach will allow us to extend previous results to a wider class of kernel functions and activation functions and, therefore, solve a larger set of problems.

We shall use the Filippov method, Lyapunov functionals method, differential equation and differential inclusion and some technical analysis to achieve our goals.

1.2 Research Objectives

We study the stability and synchronization of solutions for two memristors based integro-differential systems in neural network theory. More precisely, we consider the following systems:

- A memristive Hopfield neural network system involving discrete and distributed delays. Stability and synchronization results for this kind of system will be established. This system is expressed by

$$\begin{cases} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(x_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t - \tau)) \\ \quad + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds + I_i, \quad \forall t > 0, i = 1, \dots, m, \\ x_i(s) = \varphi_i(s), \quad -\tau \leq s \leq 0, i = 1, \dots, m, \end{cases} \quad (1.4)$$

where for all $t > 0$ and $i, j = 1, 2, \dots, m$, $x_i(t)$ denotes the solution (state) of the problem (P_1) , a_i is a non-negative neuron self-inhibition, b_{ij} , c_{ij} and d_{ij} are the memristive connection weights functions, g_j indicates the activation function (signal transmission functions), I_i represents the external input, τ indicates the discrete delay, k_{ij} is the distributed delay kernel function and φ_i is a real-valued bounded continuous function.

- A memristive Cohen–Grossberg bidirectional associative memory neural network system of high-order type and with discrete and distributed delays. More precisely, we treat the global stability, uniform energy estimation, local stability and adaptive synchronization of the system

$$\left\{ \begin{array}{l} x'_i(t) = p_i(x_i(t)) \left[-\Lambda_i(x_i(t)) + \sum_{j,l=1}^m a_{jil}(x_i(t)) f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^m b_{jil}(x_i(t)) \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds + I_i \right], \\ y'_j(t) = q_j(y_j(t)) \left[-\bar{\Lambda}_j(y_j(t)) + \sum_{i,r=1}^n \bar{a}_{ijr}(y_j(t)) \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^n \bar{b}_{ijr}(y_j(t)) \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds + J_j \right], \\ \text{for all } t > 0, \quad i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, m, \end{array} \right. \quad (1.5)$$

equipped with the following initial data (initial conditions):

$$\begin{cases} x_i(t) = \phi_i(t), & t \leq 0, \\ y_j(t) = \varphi_j(t), & t \leq 0, \end{cases} \quad (1.6)$$

where n and m are the number of neurons of first and second layer, respectively. For all $t > 0$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, x_i and y_j are the state (solutions) functions of the i^{th} neuron from the neural layer L_x and the j^{th} neuron from the neural layer L_y at time t , respectively, p_i and q_j are the amplification functions of the system, Λ_j and $\bar{\Lambda}_j$ are some appropriate non-negative neuron self-inhibition functions, f_j , g_j , $\bar{f}_i$ and $\bar{g}_i$ indicate to the activation functions (signal transmission functions), I_i and J_j represent the external inputs, τ and θ refer to the discrete delays, k_{jil} , h_{jil} , $\bar{k}_{ijr}$ and $\bar{h}_{ijr}$ are the distributed delay kernel functions, ϕ_i and ψ_j are real-valued bounded continuous functions and the memristive connection weights functions a_{jil} , b_{jil} , $\bar{a}_{jil}$ and $\bar{b}_{ijr}$ are continuous functions and assumed to satisfy the following conditions:

$$a_{jil}(x_i(t)) = \begin{cases} \hat{a}_{jil}, & |x_i(t)| > T_i, \\ \check{a}_{jil}, & |x_i(t)| \leq T_i, \end{cases} \quad (1.7)$$

$$b_{jil}(x_i(t)) = \begin{cases} \hat{b}_{jil}, & |x_i(t)| > T_i, \\ \check{b}_{jil}, & |x_i(t)| \leq T_i, \end{cases} \quad (1.8)$$

$$\bar{a}_{ijr}(y_j(t)) = \begin{cases} \hat{a}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ \check{a}_{ijr}, & |y_j(t)| \leq \bar{T}_j, \end{cases} \quad (1.9)$$

and

$$\bar{b}_{ijr}(y_j(t)) = \begin{cases} \hat{b}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ \check{b}_{ijr}, & |y_j(t)| \leq \bar{T}_j, \end{cases} \quad (1.10)$$

for some positive switching jumps T_i and T_j and $\hat{a}_{jil}, \check{a}_{jil}, \hat{b}_{jil}, \check{b}_{jil}, \hat{a}_{ijr}, \check{a}_{ijr}, \hat{b}_{ijr}$ and $\check{b}_{ijr}$ are real numbers. Numerical illustration will be provided.

1.3 Methodology

The main procedures and techniques we used to achieve our objectives include the following steps:

- Employing the Filippov method to convert differential equations with discontinuous right hand sides into the differential inclusions.
- Using theory of differential inclusion and some tools to treat terms with discontinuities at some points.
- Using different types of controls.
- Utilizing the energy methods via the Lyapunov functionals.
- Appealing to some inequalities such as Young inequality, Cauchy–Schwarz inequality, Hôlder inequality,...etc.

1.4 Literature Review

The history goes back to at least 1980s. In 1983, Gopalsamy, K. [69] studied the following linear integro–differential Hopfield neural networks system:

$$x'_i(t) = \sum_{j=1}^n a_{ij}x_j(t - \tau_{ij}) + \sum_{j=1}^n \int_{-\infty}^t k_{ij}(t-s)x_j(s)ds + f_i(t), \quad i = 1, 2, \dots, n, t > 0, \quad (1.11)$$

and showed that

$$\|x(t)\| = \sum_{j=1}^n |x_j(t)| \longrightarrow 0, \text{ as } t \longrightarrow \infty,$$

where

$$a_{ii} < 0, i = 1, 2, \dots, n,$$

$$\int_0^\infty s|k_{ij}(s)|ds < \infty$$

and

$$\int_0^\infty |k_{ij}(s)|ds < \infty, i, j = 1, 2, \dots, n,$$

$\|f(t)\| = \sum_{i=1}^n |f_i(t)|$ is bounded for all $t \in [0, \infty)$ and $\|f(t)\| \in L^1([0, \infty))$. $\mu(A) + \|K^*\| = -\beta < 0$ for some positive constant β , where A denotes a real $n \times n$ constant matrix with entries $a_{ij}(i, j = 1, 2, \dots, n)$, $\mu(A)$ represents the matrix measure of A and K^* denotes the matrix with entries

$$k_{ij} = \int_0^\infty |k_{ij}(s)|ds,$$

$\tau_{ij}(i \neq j, i, j = 1, 2, \dots, n)$ are non-negative known constants.

In 2002, Li, X. P. and Jiao, L. C. [70] imposed conditions ensuring the existence of solutions for the following delayed cellular neural networks system:

$$x'_i(t) = -x_i(t) + \sum_{i=1}^n a_{ij}f_j(x_j(t)) + \sum_{i=1}^n b_{ij}f_j(x_j(t - \tau_j)) + I_i, \quad i = 1, 2, \dots, n,$$

on a given interval having global exponential behavior

$$\|x(t) - x^*\| \leq M\|\psi - x^*\|e^{-\varepsilon t}, \quad \varepsilon > 0, M \geq 1, t > 0.$$

Hopfield neural network is a special kind of neural networks whose is different from other neural networks. It is calculated by converging iterative process. It has just one layer of neurons relating to the size of the input and output, which must be the same. When such a network recognizes, for example, digits, we present a list of correctly rendered digits to the network. Subsequently, the network can transform a noise input to the relating perfect output.

In 1980s, John Joseph Hopfield introduced an artificial neural network (called Hopfield neural network) to collect and retrieve memory like the human brain. Here, a neuron is either on or off situation. The state of a neuron(on +1 or off 0) will be restored, relying on the input it receives from the other neuron. A Hopfield neural network is at first prepared to store various patterns or memories. Afterward, it is ready to recognize any of the learned patterns by uncovering partial or even some corrupted data about that pattern, i.e., it eventually settles down and restores the closest pattern. Thus, similar to the human brain, the Hopfield model has stability in pattern recognition Hopfield, J. J., 1982 [55], Hopfield, J. J., 1284 [56], Hopfield, J. and Tank, D., 1985 [57] and Hopfield, J. and Tank, D., 1986 [58]. The Hopfield neural network is a useful model and has many applications in real life. It has been proved useful in content-addressable memories Verleysen, M. *et al.*, 1989 [125] and combinatorial optimization problems, such as the travelling salesman problem and the location allocation problem Hopfield, J. and Tank, D., 1985 [57] and Du, K. *et al.*, 2014 [126]. Previous Hopfield network was realized by constructing complementary metal-oxide-semiconductor circuits as the synapses at a cost of large chip area and power consumption Verleysen, M. *et al.*, 1989 [125]. So far, in the existing papers in the literature, for instance, Zhang, R. *et al.*, 2018 [127], Yang, X. and Ho, D., 2015 [128], Guo, X. *et al.*, 2014 [129], Rakkiyappan R. *et al.*, 2014 [130], Abdurahman, A. Jiang, H. 2016 [131] and Zhang, G. *et al.*, [132] some conditions are imposed on the kernel functions and the different parameters to obtain exponential stability. We are aiming here to establish stability and synchronization of the above memristive hopfield neural network system for a wider class of the kernel functions and with a general decay rate of stability.

The stability and synchronization of solutions for a memristor based integro-differential

system in neural networks theory have been studied by many authors and researchers e.g., Yang, X. *et al.*, 2014 [2] proved global exponential synchronization for a class of memristor-based Cohen–Grossberg neural network system:

$$\begin{aligned} u'_i(t) = & -a_i(u_i(t)) \left[b_i(u_i(t)) - \sum_{j=1}^m w_{ij}(u_j(t)) f_j(u_j(t)) + c_{ij}(u_i(t)) f_j(u_j(t - \tau_{ij}(t))) \right. \\ & \left. + d_{ij}(u_i(t)) \int_{-\infty}^0 K_{ij}(t-s) f_j(u_j(s)) ds \right] + I_i, \quad i = 1, 2, \dots, m, \end{aligned} \quad (1.12)$$

with time-varying discrete delays and unbounded distributed delays based on the characteristics of amplification function. By utilizing feedback controllers and constructing suitable Lyapunov functionals, Chen, C. *et al.*, 2017 [3]–[111] discussed the finite time synchronization problem of a memristor-based neural network system (MNNS) of (1.12) with mixed delays (see also Chua, L., 1971 [18]). Dissipativity analysis of neutral-type memristive neural network system with two additive time-varying and leakage delays

$$\begin{aligned} x'_i(t) = & -c_i x_i(t - \eta(t)) + \sum_{j=1}^m a_{ij}(x_i(t)) f_j(x_j(t)) + \sum_{j=1}^m b_{ij}(x_i(t)) f_j(x_j(t - \tau_{j1}(t)) - \tau_{j2}(t)) \\ & + \sum_{j=1}^m d_{ij}(x_i(t)) \int_{t-\delta_2(t)}^{t-\delta_1(t)} f_j(x_j(s)) ds + e_i x'_i(t - h(t)) + u_i(t), \quad i = 1, 2, \dots, n \end{aligned} \quad (1.13)$$

was studied by Yang, C. *et al.*, 2019 [6] by applying a suitable Lyapunov–Krasovskii functional (LKF), some integral inequality techniques, linear matrix inequalities (LMIs) and free-weighting matrix method. Song, Q., 2009 [32] investigated the global exponential synchronization of a coupled connected neural network system with both discrete and distributed delays

$$\begin{aligned} x'_i(t) = & -Dx_i(t) + Af(x_i(t)) + Bf(x_i(t - \sigma)) + C \int_{t-\tau}^t f(x_i(s)) ds + I(t) + \sum_{j=1}^m \alpha_{ij} P x_j(t) \\ & + \sum_{j=1}^m \beta_{ij} Q x_j(t - \sigma) + \sum_{j=1}^m \gamma_{ij} R \int_{t-\tau}^t x_j(s) ds, \quad i = 1, 2, \dots, m, \end{aligned} \quad (1.14)$$

under mild condition, assuming neither the differentiability and strict monotonicity for the activation functions nor the diagonal for the inner coupling matrices, a new Lyapunov–Krasovskii functional, the theory of Kronecker product of matrices and the linear matrix

inequality (LMI) technique were employed. The global asymptotic stability question of

$$u'_i(t) = -d_i u_i(t) + \sum_{j=1}^m a_{ij} f_j(u_j(t)) + \int_{t-\tau}^t c_{ij} h_j(u_j(s)) ds + I_i, \quad i = 1, 2, \dots, m, \quad (1.15)$$

was studied by Wang, Z. *et al.*, 2011 [33] for a class of recurrent neural networks with distributed delays satisfying Lebesgue–Stieljies measures on the basis of linear matrix inequality. Using Lyapunov stability theory and some technical analysis and inequalities, global exponential synchronization for a class of switched neural network system with the indicator function

$$\begin{aligned} x'(t) = & -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) \\ & + D \int_{-\infty}^t p(t-s)f(x(s))ds + I(t), \quad \text{for all } t > 0, \end{aligned} \quad (1.16)$$

where C , A , B and D are matrices, and with time-varying delays and unbounded distributed delays via impulsive control method was obtained by Yang, X. *et al.*, 2011 [34]. Based on a new Lyapunov–Krasovskii functional candidate, a new analysis technique and using a class of simple discontinuous state feedback controller Yang, X., 2014 [35] established the finite-time synchronization of the neural network system

$$\begin{aligned} u'_i(t) = & -c_i u_i(t) + \sum_{j=1}^m a_{ij}(x_i(t)f_j(x_j(t)) + \sum_{j=1}^m b_{ij}(x_i(t))f_j(x_j(t - \tau_{ij}(t)))) \\ & + \sum_{j=1}^m d_{ij}(x_i(t)) \int_{-\infty}^t K_{ij}(t-s)f_j(u_j(s))ds + J_i(t), \quad i = 1, 2, \dots, m. \end{aligned} \quad (1.17)$$

Under the framework of Filippov method, by designing simple controller, applying some new analytical techniques, and constructing some new Lyapunov–Krasovskii functionals, global existence of solutions to the neural network system

$$x'(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) + D \int_{-\infty}^t K(t-s)f(x(s))ds + J, \quad (1.18)$$

with discontinuous activation functions and mixed delays was proved by Yang, X. *et al.*, 2015 [36]. Song, Y. and Wen, S., 2015 [37] studied the synchronization control of stochastic memristor-based neural network system

$$\begin{aligned}
x'_i(t) = & -d_i x_i(t) + \sum_{j=1}^m a_{ij}(x_i(t)) f_j(x_j(t)) + \sum_{j=1}^m b_{ij}(x_i(t)) f_j(x_j(t - \tau(t))) \\
& + \sum_{j=1}^m c_{ij}(x_i(t)) \int_{t-\rho(t)}^t f_j(x_j(s)) ds \\
& + \beta_i(t, x_i(t), x_i(t - \tau(t))) dw_i(t), \quad i = 1, 2, \dots, m,
\end{aligned} \tag{1.19}$$

with mixed delays based on the drive–response concept, the stochastic differential inclusions theory and Lyapunov functional method. Absolute stability of global pattern formation and parallel memory for a class of n –dimensional storage competitive neural network system

$$x'_i(t) = \alpha_i(x_i(t)) \left[\beta_i(x_i(t)) - \sum_{j=1}^m \gamma_{ij} \xi_j(t) \right] \quad \forall t > 0, i = 1, 2, \dots, m, \tag{1.20}$$

was studied by Cohen, M. A. and Grossberg, S., 1983 [38]. Based on sufficient synchronization criteria, the eigenvalues of the weight matrix belonging to the neural network system and an idea of structured inverse eigenvalue problem. Mahdavi, N. and Kurths, J., 2014 [39] considered a delay system presented by

$$x_i(t) = w_{ii}x_i(t - 1) + \sum_{\substack{j=1 \\ j \neq i}}^N w_{ij}x_j(t - \tau), \quad i = 1, 2, \dots, N, \tau \geq 0. \tag{1.21}$$

The authors developed this system by a new learning rule for the coupling weights tuning of Hopfield like chaotic neural networks such a way that all neurons behave in a synchronous manner, while the desirable structure of the network is preserved during the learning process. This proposed learning rule is based on sufficient synchronization criteria, on the eigenvalues of the weight matrix belonging to the neural network and the idea of structured inverse eigenvalue problem. The developed learning rule not only synchronizes all neurons outputs with each other in a desirable topology, but also enables researchers to enhance the synchronizability of the network by choosing the appropriate set of weight matrix eigenvalues. Specifically, this method is evaluated by performing simulations on the scale–free topology. Kamel, M. S. and Xia, Y., 2009 [40] discussed the optimization problem of cooperative recurrent neural network system

$$x'(t) = - \begin{cases} \sum_{i \in \rho(x)} \nabla f_i(x), & \text{if } x \notin \hat{S}, \\ \nabla f(x), & \text{if } x \in \hat{S}, \end{cases} \quad (1.22)$$

where $x \in \mathbb{R}^n$, $\hat{S} = \{x \in \mathbb{R}^n : f(x) \leq 0\}$ and $\rho(x) = \{1 \leq i \leq m : f_i(x) > 0\}$. Existence and global exponential stability of periodic solutions for impulsive Cohen–Grossberg neural network system (CGNNS)

$$\begin{aligned} x'_i(t) = & -a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^m c_{ij}(t) f_j(x_j(t - \tau_{ij}(t))) \right. \\ & \left. + d_{ij}(t) \int_{-\infty}^0 K_{ij}(t-s) g_j(x_j(s)) ds \right] + I_i, \quad i = 1, 2, \dots, m, \end{aligned} \quad (1.23)$$

with mixed delays was obtained by Yang, X. *et al.*, 2008 [41], using Leray–Schauder’s fixed point theorem, differential inequality techniques, and Lyapunov functionals. Under two new Lyapunov functions and the invariance principle of stochastic differential equations, the problem of lag synchronization for a kind of chaotic neural network system

$$\begin{aligned} x'_i(t) = & -c_i u_i(t) + \sum_{j=1}^m a_{ij} f_j(x_j(t)) + \sum_{j=1}^m b_{ij} f_j(x_j(t - \tau)) \\ & + \sum_{j=1}^m d_{ij} \int_{t-\eta}^t f_j(x_j(s)) ds + I_i, \quad i = 1, 2, \dots, m, \end{aligned} \quad (1.24)$$

was studied by Yang, X. *et al.*, 2011 [42] with discrete and distributed delays (see also references therein).

Next, we mention some results on neural network systems in the form of propositions.

Proposition 1.4.1 (*Zhang, G. et al., 2017 [121]*)

Consider the memristive Hopfield neural network system (MHNNS)

$$\begin{aligned} x'_i(t) = & -b_i x_i(t) + \sum_{j=1}^n c_{ij}(x_i(t)) f_j(x_j(t)) + \sum_{j=1}^n d_{ij}(x_i(t)) f_j(x_j(t) - \tau(t)) + I_i, \\ & \text{for all } t > 0 \text{ and } i = 1, 2, \dots, n, \end{aligned} \quad (1.25)$$

where $x_i(t)$ are the state of the i th neuron, b_i are positive constants, $f_i(\cdot)$ denote the activation functions of the i th neuron at time t and assumed to be Lipschitz, $\tau(t)$ is the delay that satisfies $0 \leq \tau(t) \leq \tau$ and I_i denote the external inputs on the i th neuron. Let $x_i(s) = \varphi(s)$, $x'_i(s) = \psi(s)$, for $-\tau \leq s \leq 0$, where φ and ψ are bounded and continuous functions, $c_{ij}(x_i(t))$ and $d_{ij}(x_i(t))$ are the feedback connection weights and the delay feedback connection weight, respectively. Then, there exists a globally exponentially stable solution of the system.

The technique used in the proof of Proposition 1.4.1 is based on the theory of differential equations and Lyapunov method.

Proposition 1.4.2 (Nie, X. and Zheng, W. X., 2015 [72])

Consider the memristive Cohen-Grossberg neural network system (MCGNNS)

$$u'_i(t) = -a_i(u_i(t)) \left[b_i u_i(t) - \sum_{j=1}^n h_{ij}(u_i(t)) f_j(u_j(t)) - I_i \right], \quad (1.26)$$

where $i = 1, 2, \dots, n$, $u(t) = (u_1(t), u_2(t), \dots, u_n(t)) \in \mathbb{R}^n$ are the state vector, $b_i > 0$, $a_i(u_i(t))$ represent the amplification functions, $I = (I_1, I_2, \dots, I_n)$ is the input vector, $f_j(\cdot)$ are the activation functions and $h_{ij}(u_j(t))$ denote the connection weight functions of the neural networks system assumed bounded. Assume that $a_i(r)$ are continuous functions and there exist positive constants a_{i1} and a_{i2} such that $0 < a_{i1} \leq a_i(r) < a_{i2}$, for all $r \in \mathbb{R}$, $i = 1, 2, \dots, n$. Then, the (MCGNNS) (1.26) has 3^n exponentially stable equilibrium points out of 5^n .

In the proof of Proposition 1.4.2, the technique used relies on fixed point theorems and non-smooth analysis theory as well as the aid of set-valued maps theories and differential inclusions.

Proposition 1.4.3 (Liang, J. and Cao, J., 2004 [73])

Consider the bidirectional associative memory neural network system (BAMNNS)

$$\begin{cases} x'_i(t) = -c_i x_i(t) + \sum_{j=1}^p a_{ji} \int_{-\infty}^t k_{ji}(t-s) f_j(y_j(s)) ds + I_i, & i = 1, 2, \dots, n, \\ y'_j(t) = -d_j y_j(t) + \sum_{i=1}^n b_{ij} \int_{-\infty}^t v_{ij}(t-s) f_i(x_i(s)) ds + J_j, & j = 1, 2, \dots, p, \end{cases} \quad (1.27)$$

where $x = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$, $y = [y_1, y_2, \dots, y_p]^T \in \mathbb{R}^p$, c_i and d_j are non-negative constants, $f_i(\cdot)$ are the activation functions assumed to be bounded and Lipschitz condition, I_i and J_j are the external inputs and the connection weight functions a_{ji} and b_{ij} are real numbers and the kernel functions k_{ji} and v_{ij} for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, p$, are real valued non-negative continuous functions defined on $[0, +\infty)$ and satisfy

$$\int_0^\infty k_{ji}(s) ds = 1 \text{ and } \int_0^\infty s k_{ji}(s) ds < \infty,$$

$$\int_0^\infty v_{ij}(s) ds = 1 \text{ and } \int_0^\infty s v_{ij}(s) ds < \infty.$$

Suppose there exist positive constants $\lambda_i > 0$ and $\lambda_{n+j} > 0$ (for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, p$) such that

$$-r \lambda_i c_i + (r-1) \sum_{j=1}^p \lambda_i |a_{ji}| + \sum_{j=1}^p \lambda_{n+j} \mu_i^r |b_{ij}| < 0, \quad i = 1, 2, \dots, n,$$

$$-r \lambda_{n+j} d_j + (r-1) \sum_{i=1}^n \lambda_{n+j} |b_{ij}| + \sum_{i=1}^n \lambda_i \mu_j^r |a_{ji}| < 0, \quad j = 1, 2, \dots, p.$$

Then, the equilibrium is globally asymptotically stable.

Suitable Laypunov functionals are utilized in the proof of Proposition 1.4.3 and some technical analysis and inequalities to obtain the globally asymptotic stability for (BAMNNS) defined in (1.27).

1.5 Some Related Problems

In 2017, Chen, C. *et al.* [3] (see also [111] and [5]) considered a memristive Hopfield neural networks system (MHNNS) with mixed delays of the form

$$\begin{cases} x'_i(t) = -d_i x_i(t) + \sum_{j=1}^m a_{ij}(x_i(t)) f_j(x_j(t)) + \sum_{j=1}^m b_{ij}(x_i(t)) f_j(x_j(t - \tau(t))) \\ \quad + \sum_{j=1}^m c_{ij}(x_i(t)) \int_{t-\rho(t)}^t f_j(x_j(s)) ds + I_i, \quad \forall t \geq 0, i = 1, \dots, n, \end{cases} \quad (1.28)$$

where for all $i, j = 1, 2, \dots, m$, $x_i(t)$ represent for the voltage of the capacity C_i , $d_i > 0$ stand for neuron self-inhibitions, $f_i(\cdot)$ denote the activation functions, τ and ρ denote discrete delay and distributed delay, respectively, such that $0 \leq \tau(t) \leq \tau_1$, $0 \leq \rho(t) \leq \tau_2$ and $\dot{\tau}(t) \leq \sigma \leq 1$, I_i represent the external inputs. The initial value corresponding to the system (1.28) is $\varphi = (\varphi_1(s), \varphi_2(s), \dots, \varphi_n(s))^T \in C([- \tau, 0], \mathbb{R}^n)$, where $\tau = \max\{\tau_1, \tau_2\}$ and $C([- \tau, 0], \mathbb{R}^n)$ denotes the Banach space of all continuous functions that map $[- \tau, 0]$ into $\mathbb{R}^n$, the functions $a_{ij}(x_i(t))$, $b_{ij}(x_i(t))$ and $c_{ij}(x_i(t))$ are the memristive connection weights assumed to be bounded by some positive constants.

In 2020, Tatar, N-E. [59] (see also Tatar, N-E. [60]-[68]) considered the following high-order neural networks system (HONNS)

$$\begin{aligned} x'_i(t) &= \nu_i x'_i(t - \tau_i(t)) - a_i(t) x_i(t) + \sum_{j=1}^m b_{ij}(t) f_j(x_j(t)) + \sum_{j=1}^m c_{ij}(t) g_j(x_j(t - \tau_{ij}(t))) \\ &\quad + \sum_{j=1}^m \sum_{l=1}^m d_{ijl}(t) g_j(x_j(t - \tau_{ij}(t))) g_l(x_l(t - \tau_{il}(t))) \\ &\quad + \sum_{j=1}^m \sum_{l=1}^m m_{ijl}(t) \int_0^\infty h_{ijl}(s) g_j(x_j(t - s)) ds \int_0^\infty k_{ijl}(s) g_l(x_l(t - s)) ds + I_i, \\ &\text{for all } t \geq 0, i = 1, 2, \dots, m, \\ x_j(t) &= x_{0j}(t), \quad t \in (-\infty, 0], \end{aligned} \quad (1.29)$$

where for all $i, j, l = 1, 2, \dots, m$, $x_{0j}(t)$ are given continuous functions, the terms $\nu_i x'_i(t - \tau_i(t))$ are the neutral delays, b_{ij} are real constants, $a_i(t)$ are non-negative functions, $c_i(t)$ are continuous functions, f_j, g_j and g_l are the activation functions, $I_i(t)$ are the external inputs and h_{ijl} and k_{ijl} are the kernel functions which are assumed to be continuous and integrable. Problem (1.29) is more general than the simple problems without neutral delays that presented in the system

$$\left\{ \begin{array}{l} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^m \sum_{l=1}^m b_{ijl}(t) g_j(x_j(t-\tau)) g_l(x_l(t-\tau)) \\ \quad + \sum_{j=1}^m \sum_{l=1}^m c_{ijl}(t) \int_0^\infty v_{ijl}(s) h_j(x_j(t-s)) ds \int_0^\infty w_{ijl}(s) h_l(x_l(t-s)) ds + I_i, \\ \quad \forall t \geq 0, i = 1, 2, \dots, m. \\ x_j(t) = x_{0j}(t), \quad t \in (-\infty, 0]. \end{array} \right. \quad (1.30)$$

By assuming that for all $i, j, l = 1, 2, \dots, m$, the kernel functions v_{ijl} and w_{ijl} are non-negative and continuous summable functions such that

$$\int_0^\infty v_{ijl}(s) ds = \bar{v}_{ijl} \text{ and } \int_0^\infty v_{ijl}(s) \chi(s) ds < \infty$$

and

$$\int_0^\infty w_{ijl}(s) ds = \bar{w}_{ijl} \text{ and } \int_0^\infty w_{ijl}(s) \chi(s) ds < \infty,$$

for some positive continuous function $\chi(t)$ where $\varphi(t) = \frac{\chi'(t)}{\chi(t)}$ is a decreasing function for all $t \geq 0$, and by utilizing suitable Lyapunov functionals in addition to some technical inequalities, the author proved some exponential stability and synchronization results depending on the rate of the decay of the delay kernel functions for both bounded and unbounded activation functions.

In 1983, Cohen and Grossberg [74] introduced the system of ordinary differential equations described by

$$u'_i(t) = -\alpha_i(u_i(t)) \left[\beta_i(u_i(t)) - \sum_{j=1}^m \gamma_{ij} f_j(x_j(t)) \right], \quad i = 1, 2, \dots, m, \quad (1.31)$$

where for all $i, j = 1, 2, \dots, m$, $m \geq 2$ is the number of the neurons in the network, $u_i(t)$ denotes the state variable of the i th neuron at time t , $f_j(u_j(t))$ denotes the activation function of the j th neuron at time t ; the feedback matrix $\Gamma := (\gamma_{ij})_{m \times m}$ indicates the strength of the neuron interconnections within the network, $\alpha_i(u_i(t))$ and $\beta_i(u_i(t))$ represent an amplification and an appropriately behaved function, respectively such that the solutions of the model remain bounded to investigate the absolute stability of global

pattern formation of such system.

In 2003, Hwang, C. *et al.* [75] generalized the previous Cohen–Grossberg neural network by adding variable delays as the follows

$$u' = -\alpha_i(u_i(t)) \left[\beta_i(u_i(t)) - \sum_{j=1}^m \gamma_{ij}(u_{ij}(t))g_j(u_j(t)) - \sum_{j=1}^m \mu_{ij}(u_{ij}(t))g_j(u_j(t - \tau_{i,j}(t))) \right], \quad (1.32)$$

where $g_j(u_j(t))$ describes the manner in which the neurons respond to each other. It is assumed to be a sigmoid function, which indicates that the activation function is monotone in nature and smooth. The authors established the exponential stability provided that $\alpha_i(u_i(t))$, where $i = 1, 2, \dots, m$, is bounded, positive, and locally Lipschitz continuous function, such that $0 < n_i \leq \alpha_i(u_i(t)) \leq N < \infty$, for all $u_i \in \mathbb{R}$, for some positive constants n_i and N_i , $\beta_i(u_i(t))$ and $\beta_i^{-1}(u_i(t))$, where $i = 1, 2, \dots, m$, is locally Lipschitz continuous function, such that $\beta'_i(u_i(t)) \geq a_i > 0$, for all $u_i(t) \in \mathbb{R}$, for some positive constant a_i and the signal transmission function $g_j : \mathbb{R} \rightarrow \mathbb{R}$, where $i = 1, 2, \dots, m$, is bounded and satisfies the Lipschitz condition with Lipschitz constant $L_i > 0$, i.e.,

$$|g_j(u(t)) - g_j(v(t))| \leq L_i |u(t) - v(t)|, \quad \forall u, v \in \mathbb{R}. \quad (1.33)$$

In 2014, Yang, X. *et al.* [2] considered a memristive based Cohen–Grossberg neural networks system (MCGNNS) with time varying discrete delays and unbounded distributed delays of the form

$$\begin{cases} x'_i(t) = -a_i(x_i(t)) \left\{ b_{ij}(x_i(t)) - \sum_{j=1}^m w_{ij}(x_i(t))f_j(x_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t))f_j(x_j(t - \tau_{i,j}(t))) \right. \\ \left. + \sum_{j=1}^m d_{ij}(x_i(t)) \int_{-\infty}^0 k_{ij}(t-s)f_j(x_j(s))ds - I_i \right\}, \quad \forall t > 0, i = 1, \dots, n, \end{cases} \quad (1.34)$$

where for all $i, j = 1, 2, \dots, n$, $x_i(t)$ denote the state variable of the i th neuron at time t , I_i are the external input to the i th neuron, $a_i(x_i(t))$ and $b_i(x_i(t))$ represent the amplification functions and appropriately behaved functions at time t , respectively, the time varying delay $\tau_{i,j}(t)$ correspond to the finite speed of signal transmission, $k_{ij}(t)$ are non-negative bounded kernel functions defined on $[0, \infty)$, the functions $w_{ij}(x_i(t))$, $c_{ij}(x_i(t))$

and $d_{ij}(x_i(t))$ are the memristive connection weights of the neural network which are assumed to be bounded by some positive constants.

The following assumptions are made

(A₁) There exist constants $\tau_{i,j} > 0$ such that $0 \leq \tau_{i,j}(t) \leq \tau_{i,j}$, $i, j = 1, 2, \dots, n$, $t \in \mathbb{R}$.

(A₂) The functions $a_i(x)$ are continuous and there exist positive constants a_1 and a_2 such that $0 < a_1 \leq a_i(x) \leq a_2$, $\forall x \in \mathbb{R}, i = 1, 2, \dots, n$.

(A₃) There exist constants l_i such that $b'_i(x) \geq l_i$, where $b'_i(x)$ denote the derivative of $b_i(x)$, $x \in \mathbb{R}$ and $b_i(0) = 0$, $i = 1, 2, \dots, n$.

(A₄) The activation functions are Lipschitz, i.e. $\exists L_i > 0$ such that

$$|f_i(x) - f_i(y)| \leq L_i|x - y|, \quad \forall x, y \in \mathbb{R}, x \neq y, i = 1, 2, \dots, n.$$

(A₅) There exist constants M_i such that $|f_i(x)| \leq M_i$, for bounded x , $x \in \mathbb{R}$, for all $i = 1, 2, \dots, n$.

(A₆) The kernel functions $k_{i,j} : [0, \infty) \rightarrow [0, \infty)$ are real-valued non-negative continuous functions and $\exists \beta_{i,j}$ such that

$$\int_0^\infty k_{i,j}(s)ds \leq \beta_{i,j}, \quad i, j = 1, 2, \dots, n.$$

The authors discussed the convergence rate and investigated the global exponential stability and synchronization using a novel state feedback controller (see [44]–[45]).

Remark 1.5.1

Under the previous assumptions (A₄) and (A₅) above, many authors in the literature investigated and examined the stability and finite time synchronization based on Filippov

method [44], the theory of differential inclusion [45], set valued-maps [46] and different finite time synchronization analysis techniques, utilizing feedback controllers and suitable Lyapunov functionals.

In 2017, Yucel, E. [76], considered a memristive based Cohen–Grossberg neural networks system (MCGNNS) with time varying discrete delays and unbounded distributed delays of the form

$$\begin{cases} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^m w_{ji} g_j(y_j(t)) + \sum_{j=1}^m w_{ji}^* g_j(y_j(t - \tau_{ji})) + I_i \\ y'_j(t) = -b_j y_j(t) + \sum_{i=1}^n v_{ij} g_i(x_i(t)) + \sum_{i=1}^n v_{ij}^* g_i(x_i(t - \tau_{ij})) + J_j, \end{cases} \quad (1.35)$$

for all $i, j = 1, 2, \dots, m$. The given (BAM) neural network model consists of two layers, n denotes number of the neurons in the first layer and m denotes the number of neurons in the second layer. $x_i(t)$ is the state of the i^{th} neuron in the first layer and $y_j(t)$ is the state of the j^{th} neuron in the second layer. a_i and b_j denote the neuron charging time constants and passive decay rates, respectively; w_{ji} , w_{ji}^* , v_{ij} and v_{ij}^* are synaptic connection strengths; g_i and g_j represent the activation functions of the neurons and the propagational signal functions, respectively; and I_i and J_j are the exogenous inputs. Several new sufficient conditions that imply the existence, uniqueness and global robust stability of the equilibrium point for the class of BAM neural networks are obtained by the use of proper Lyapunov functionals and assuming

$$0 \leq \frac{g_i(u_1) - g_i(v_1)}{u_1 - v_1} \leq \ell_i, \quad 0 \leq \frac{g_j(u_2) - g_j(v_2)}{u_2 - v_2} \leq k_j \quad (1.36)$$

and

$$|g_i(u)| \leq L_i, \quad |g_j(v)| \leq L_j, \quad (1.37)$$

for all $u, v, u_1, u_2, v_1, v_2 \in \mathbb{R}$, for some positive constants ℓ_i , k_j , L_i and L_j .

In 2008, Wang, B. *et al.* [77], discussed a more general Cohen–Grossberg type BAM neural network model with time-varying delays and continuously distributed delays described by the following equations

$$\left\{ \begin{array}{l} x'_i(t) = \alpha_i(x_i(t)) \left[-a_i(x_i(t)) + \sum_{j=1}^n c_{1ji}(t)f_j(y_j(t)) + \sum_{j=1}^n c_{2ji}(t)f_j(y_j(t - \tau_{ji}(t))) \right. \\ \quad \left. + \sum_{j=1}^n w_{ji} \int_0^\infty K_{ji}(s)f_j(\lambda_j y_j(t - s))ds + I_i(t) \right] \quad i = 1, 2, \dots, m, \\ y'_j(t) = \beta_j(y_j(t)) \left[-b_j(y_j(t)) + \sum_{i=1}^m d_{1ij}(t)g_i(x_i(t)) + \sum_{i=1}^m d_{2ij}(t)g_i(x_i(t - \sigma_{ij}(t))) \right. \\ \quad \left. + \sum_{i=1}^m v_{ij} \int_0^\infty L_{ij}(s)g_i(h_i x_i(t - s))ds + J_j(t) \right] \quad j = 1, 2, \dots, n, \end{array} \right. \quad (1.38)$$

provided that $m \geq 2$ and $n \geq 2$.

Under the following conditions

- The functions $\alpha_i(x_i(t))$ and $\beta_j(y_j(t))$ are bounded, positive and satisfy

$$0 \leq \alpha_{1i} \leq \alpha_i(x_i(t)) \leq \alpha_{2i} < \infty, \quad 0 \leq \beta_{1j} \leq \beta_j(y_j(t)) \leq \beta_{2j} < \infty. \quad (1.39)$$

- The functions $a_i(x_i(t))$ and $b_j(y_j(t))$ are bounded and continuous; furthermore these functions are reversible and satisfy

$$a'_i(x_i(t)) \geq \zeta_i > 0, \quad b'_j(y_j(t)) \geq \eta_j > 0, \quad (1.40)$$

for some positive constants ζ_i and η_j .

- The activation functions f_j and g_i are bounded or globally Lipschitz continuous i.e., there exist Lipschitz constants m_i and n_j such that

$$|f_j(u) - f_j(v)| \leq n_j |u - v|, \quad |g_i(u) - g_i(v)| \leq m_i |u - v|, \quad \forall u, v \in \mathbb{R}. \quad (1.41)$$

- The time-varying delays $\tau_{ji}(t)$ and $\sigma_{ij}(t)$ are continuous, bounded and differentiable functions.

- K_{ji} and L_{ij} are continuous kernel functions and satisfy

$$\int_0^\infty K_{ji}(s)ds = k_{ji} > 0, \quad \int_0^\infty L_{ij}(s)ds = \ell_{ij} > 0. \quad (1.42)$$

Furthermore, $c_{1ji}(t)$, $c_{2ji}(t)$, $d_{1ij}(t)$, $d_{2ij}(t)$, $dw_{ij}(t)$, $v_{ij}(t)$, $I_i(t)$ and $J_j(t)$ are assumed to be continuous functions on $\mathbb{R}$. Based on Lyapunov stability theory, the authors investigated the global exponential stability of the equilibrium point for the above Cohen–Grossberg type BAM neural networks with time-varying delays and continuously distributed delays.

1.6 Main Contribution

In this work, we establish some stability and synchronization results for some history dependent systems of memristors in neural network theory. The few works on the stability and synchronization of solutions, we are aware of, impose severe conditions on the involved kernel functions and memristive connection weight functions. For instance, it is assumed that kernels are bounded or subexponential functions to guarantee the stability and synchronization results. Moreover, the memristive connection weight functions are assumed to be bounded. Our objective in this dissertation is to relax such conditions and obtain general decay rates covering exponential as well as polynomial decay rates.

Our results could be used to analyze the stability and synchronization of solutions for memristor based integro–differential systems in neural network theory. More precisely, we shall consider

- Memristive Hopfield neural network system modeled by

$$\begin{cases} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(x_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t - \tau)) \\ \quad + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds + I_i, \quad \forall t > 0, i = 1, \dots, m, \\ x_i(s) = \varphi_i(s), \quad -\infty < s \leq 0, i = 1, \dots, m, \end{cases} \quad (1.43)$$

where for all $t > 0$ and $i, j = 1, 2, \dots, m$, $x_i(t)$ denotes the solution (state) of the problem (P_1) , a_i is a non-negative neuron self-inhibition, b_{ij} , c_{ij} and d_{ij} are the memristive connection weights functions, g_j indicates the activation function (signal transmission functions), I_i represents the external input, τ indicates the discrete

delay, k_{ij} is the distributed delay kernel function and φ_i is a real-valued bounded continuous function. The following assumptions will be made:

1. The functions $g_j, j = 1, 2, \dots, m$, are Lipschitz continuous functions on $\mathbb{R}$ with Lipschitz constants G_j i.e.

$$|g_j(x) - g_j(y)| \leq G_j |x - y|, \quad \forall x, y \in \mathbb{R}, j = 1, 2, \dots, m.$$

2. The kernel functions $k_{ij} : [0, \infty) \rightarrow \mathbb{R}^+, i, j = 1, 2, \dots, m$, for all $t > 0$, are continuous and integrable functions such that

$$\int_0^\infty k_{ij}(s)ds = \bar{k}_{ij} \in (0, 1),$$

and

$$\int_0^\infty k_{ij}(s)\xi(s)ds < \infty,$$

for some positive and continuous function ξ such that $\varphi(t) := \frac{\xi'(t)}{\xi(t)}$ is a decreasing function.

3. For all $t > 0$ and $i, j = 1, 2, \dots, m$, the memristive connection weight functions b_{ij} , c_{ij} and d_{ij} are bounded by B_{ij} , C_{ij} and D_{ij} , respectively and satisfy the Lipschitz condition.

Relying on the energy estimation method such as the Lyapunov method, the theory of differential equations and some technical analysis and inequalities such as Young inequality, Cauchy–Schwarz inequality, Hölder inequality, ... etc., stability and synchronization results of this system will be established. This work extends some existing results in the literature, for instance Wu, A. *et al.*, 2012 [120], Zhang, G. *et al.*, 2013 [121], Wang, H. *et al.*, 2016 [122], Wen, S. *et al.*, 2012 [123] and Li, Y. *et al.*, 2019 [124] (see also references therein).

- Memristive high-order Cohen–Grossberg bidirectional associative memory neural network system described by

$$\left\{ \begin{array}{l} x'_i(t) = p_i(x_i(t)) \left[-\Lambda_i(x_i(t)) + \sum_{j,l=1}^m a_{jil}(x_i(t)) f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^m b_{jil}(x_i(t)) \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds + I_i \right], \\ y'_j(t) = q_j(y_j(t)) \left[-\bar{\Lambda}_j(y_j(t)) + \sum_{i,r=1}^n \bar{a}_{ijr}(y_j(t)) \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^n \bar{b}_{ijr}(y_j(t)) \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds + J_j \right], \\ \text{for all } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, m, \end{array} \right. \quad (1.44)$$

equipped with the following initial data (initial history)

$$\begin{cases} x_i(t) = \phi_i(t), & t \leq 0, \\ y_j(t) = \varphi_j(t), & t \leq 0, \end{cases} \quad (1.45)$$

where n and m are the number of the neurons of the first and the second layer, respectively. For all $t > 0$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, x_i and y_j are the state (solutions) functions of the i^{th} neuron from the neural layer L_x and the j^{th} neuron from the neural layer L_y at time t , respectively, p_i and q_j are the amplification functions of the system, Λ_j and $\bar{\Lambda}_j$ are some appropriate non-negative neuron self-inhibition functions, f_j , g_j , $\bar{f}_i$ and $\bar{g}_i$ indicate to the activation functions (signal transmission functions), I_i and J_j represent the external inputs, τ and θ refer to the discrete delays, k_{jil} , h_{jil} , $\bar{k}_{ijr}$ and $\bar{h}_{ijr}$ are the distributed delay kernel functions, ϕ_i and ψ_j are real-valued bounded continuous functions and the memristive connection weights functions a_{jil} , b_{ijr} , $\bar{a}_{jil}$ and $\bar{b}_{ijr}$ are continuous functions and assumed to satisfy the following conditions

$$a_{jil}(x_i(t)) = \begin{cases} \hat{a}_{jil}, & |x_i(t)| > T_i, \\ \check{a}_{jil}, & |x_i(t)| \leq T_i, \end{cases} \quad (1.46)$$

$$b_{jil}(x_i(t)) = \begin{cases} \hat{b}_{jil}, & |x_i(t)| > T_i, \\ \check{b}_{jil}, & |x_i(t)| \leq T_i, \end{cases} \quad (1.47)$$

$$\bar{a}_{ijr}(y_j(t)) = \begin{cases} \hat{a}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ \check{a}_{ijr}, & |y_j(t)| \leq \bar{T}_j, \end{cases} \quad (1.48)$$

and

$$\bar{b}_{ijr}(y_j(t)) = \begin{cases} \hat{b}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ \check{b}_{ijr}, & |y_j(t)| \leq \bar{T}_j, \end{cases} \quad (1.49)$$

for some positive switching jumps constants T_i and T_j and $\hat{a}_{jil}, \check{a}_{jil}, \hat{b}_{jil}, \check{b}_{jil}, \hat{a}_{ijr}, \check{a}_{ijr}, \hat{b}_{ijr}$ and $\check{b}_{ijr}$ are real numbers. It is further assumed that

1. The amplification functions p_i and q_j are positive and continuous functions such that

$$0 < \underline{p}_i \leq p_i(x) \leq \bar{p}_i, \quad \forall x \in \mathbb{R} \quad (1.50)$$

and

$$0 < \underline{q}_j \leq q_j(x) \leq \bar{q}_j, \quad \forall x \in \mathbb{R}. \quad (1.51)$$

for some positive constants $\underline{p}_i, \bar{p}_i, \underline{q}_j$ and $\bar{q}_j$.

2. The appropriate behaved (self-inhibition) or (self-signal) functions Λ_i and $\bar{\Lambda}_j$ are monotone increasing functions such that

$$\frac{\Lambda_i(x) - \Lambda_i(y)}{x - y} \geq \nu_i, \quad \forall x, y \in \mathbb{R}, x \neq y, \quad (1.52)$$

and

$$\frac{\bar{\Lambda}_j(u) - \bar{\Lambda}_j(v)}{u - v} \geq \nu_j \quad \forall x, y \in \mathbb{R}, x \neq y \quad (1.53)$$

for some positive constants ν_i and ν_j .

3. The activation (signal transmission) functions f_j , g_j , $\bar{f}_i$ and $\bar{g}_i$ are uniformly bounded, i.e.,

$$|f_j(x)| \leq F, \quad \forall x \in \mathbb{R}, \quad (1.54)$$

$$|g_j(x)| \leq G, \quad \forall x \in \mathbb{R}, \quad (1.55)$$

$$|\bar{f}_i(x)| \leq \bar{F}, \quad \forall x \in \mathbb{R}, \quad (1.56)$$

and

$$|\bar{g}_i(x)| \leq \bar{G}, \quad \forall x \in \mathbb{R}, \quad (1.57)$$

for some positive constants F , G , $\bar{F}$, and $\bar{G}$.

4. The signal propagation functions f_j , g_j , $\bar{f}_i$ and $\bar{g}_i$ are Lipschitz continuous with Lipschitz constants L_{1j} , L_{2j} , M_{1i} , M_{2i} , i.e.,

$$|f_j(x) - f_j(y)| \leq L_{1j}|x - y|, \quad \forall x, y \in \mathbb{R}, \quad (1.58)$$

$$|g_j(x) - g_j(y)| \leq L_{2j}|x - y|, \quad \forall x, y \in \mathbb{R}, \quad (1.59)$$

$$|\bar{f}_i(x) - \bar{f}_i(y)| \leq M_{1i}|x - y|, \quad \forall x, y \in \mathbb{R} \quad (1.60)$$

and

$$|\bar{g}_i(x) - \bar{g}_i(y)| \leq M_{2i}|x - y|, \quad \forall x, y \in \mathbb{R}. \quad (1.61)$$

5. The kernel (relaxation) functions $k_{jil}, h_{jil}, \bar{k}_{ijr}, \bar{h}_{ijr} : [0, \infty) \rightarrow \mathbb{R}^+, i, j = 1, 2, \dots, m$, for all $t > 0$, are continuous and integrable functions such that

$$\int_0^\infty k_{jil}(s)ds = \hat{k}_{jil}, \quad (1.62)$$

$$\int_0^\infty h_{jil}(s)ds = \hat{h}_{jil}, \quad (1.63)$$

$$\int_0^\infty \bar{k}_{ijr}(s)ds = \hat{\bar{k}}_{ijr}, \quad (1.64)$$

and

$$\int_0^\infty \bar{h}_{ijr}(s)ds = \hat{\bar{h}}_{ijr}. \quad (1.65)$$

6. The initial conditions (initial data) $\phi_i(t)$ and $\psi_j(t)$ for all $t \leq 0$ are continuous and assumed to be in $L^2(-\infty, 0)$.
7. Let $\eta_{\varepsilon jil}, \bar{\eta}_{\varepsilon jil}, \xi_{\varepsilon jil}$ and $\bar{\xi}_{\varepsilon jil}$ be non-negative and continuous functions such that, for $\varepsilon = 1, 2$

$$\lim_{t \rightarrow \infty} \eta_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq i \leq n \\ 1 \leq j, \ell \leq m}} \{\eta_{\varepsilon jil}(t)\} = \eta_\varepsilon^*, \quad (1.66)$$

$$\lim_{t \rightarrow \infty} \xi_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq i \leq n \\ 1 \leq j, \ell \leq m}} \{\xi_{\varepsilon jil}(t)\} = \xi_\varepsilon^*, \quad (1.67)$$

$$\lim_{t \rightarrow \infty} \bar{\eta}_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq j \leq m \\ 1 \leq i, r \leq n}} \{\bar{\eta}_{\varepsilon ijr}(t)\} = \bar{\eta}_\varepsilon^*, \quad (1.68)$$

and

$$\lim_{t \rightarrow \infty} \bar{\xi}_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq j \leq m \\ 1 \leq i, r \leq n}} \{\bar{\xi}_{\varepsilon ijr}(t)\} = \bar{\xi}_\varepsilon^*, \quad (1.69)$$

such that

$$\eta_{\varepsilon jil}(t) \int_0^\infty k_{\varepsilon 1jil}(\sigma - s) d\sigma \leq k_{\varepsilon jil}(t - s), \quad 0 \leq s \leq t, \quad (1.70)$$

$$\xi_{\varepsilon jil}(t) \int_0^\infty h_{\varepsilon 1jil}(\sigma - s) d\sigma \leq h_{\varepsilon jil}(t - s), \quad 0 \leq s \leq t, \quad (1.71)$$

$$\bar{\eta}_{\varepsilon ijr}(t) \int_0^\infty \bar{k}_{\varepsilon ijr}(\sigma - s) d\sigma \leq \bar{k}_{\varepsilon 2ijr}(t - s), \quad 0 \leq s \leq t, \quad (1.72)$$

and

$$\bar{\xi}_{\varepsilon ijr}(t) \int_0^\infty \bar{h}_{\varepsilon ijr}(\sigma - s) d\sigma \leq \bar{h}_{\varepsilon ijr}(t - s), \quad 0 \leq s \leq t. \quad (1.73)$$

8. Assume that the amplification functions p_i and q_j are Lipschitz continuous functions, i.e.,

$$|p_i(u) - p_i(v)| \leq \rho_i |u - v|, \quad \forall u, v \in \mathbb{R} \quad (1.74)$$

and

$$|q_j(u) - q_j(v)| \leq \bar{\rho}_j |u - v|, \quad \forall u, v \in \mathbb{R}, \quad (1.75)$$

for some positive constants ρ_i and $\bar{\rho}_j$.

9. Assume there exist positive constants ζ_i and $\bar{\zeta}_j$, such that

$$\zeta_{1i} \leq \frac{p_i(u)\Lambda_i(u) - p_i(v)\Lambda_i(v)}{u - v}, \quad \forall u, v \in \mathbb{R}, u \neq v, \quad (1.76)$$

and

$$\bar{\zeta}_j \leq \frac{q_j(u)\bar{\Lambda}_j(u) - q_j(v)\bar{\Lambda}_j(v)}{u - v}, \quad \forall u, v \in \mathbb{R}, u \neq v. \quad (1.77)$$

With the help of the Filippov method, the theory of differential inclusion, the theory of differential equations and utilizing with some technical analysis and inequalities, we will investigate global stability, uniform energy estimation, local stability and adaptive synchronization of the system. Moreover, numerical examples will be provided at the end. This work is an extension of existing results such as Abdurahman, A. *et al.*, 2015 [109], Zhou, Y. *et al.*, 2018 [110], Chen, C. *et al.*, 2017 [111], Zhang, Z. *et al.*, 2012 [112], Bai, C., 2008 [113], He, Q. *et al.*, 2014 [114], Benbourenane, J. *et al.*, 2014 [115], Cai, S. and Zhang, Q., 2016 [116], Chinnathambi, R. *et al.*, 2018 [117], Subramanian, K. and Muthukumar, P., 2018 [118] and Wang, J. *et al.*, 2018 [119] (see also references therein).

Such problems of memoristors arise in many areas and several applications in real-life scenarios. For instance, they appear in electrical circuits, optimal control, dynamical systems, communications, robotics, fluid mechanics, viscoelasticity, material science, quantum mechanics, bioengineering, biological models, medicine, cardiac tissues, finance, facial recognition, social media, aerospace, defence, healthcare, signature verification and handwriting analysis, IT companies,...etc.

Remark 1.6.1

*It is clear that from the memristive high-order Cohen–Grossberg bidirectional associative memory neural network system (1.44), the memristive connection weight functions $a_{ji}(x_i(t))$, $b_{ji}(x_i(t))$, $\bar{a}_{ijr}(y_j(t))$ and $\bar{b}_{ijr}(y_j(t))$ have discontinuities at some points. Then, the classical definition of the solution for differential equations can not apply here. To handle this problem, Filippov, A. F., 1960 [44] and Filippov, A. F., 2013 [45] developed a solution concept for the differential equation with a discontinuous right-hand side. Based on this definition, a differential equation with a discontinuous right-hand side has the same solution set as a certain differential inclusion (see also Clarke, F. H. *et al.*, 1998 [83] and references therein).*

Chapter 2

PRELIMINARIES

In this chapter, we introduce some definitions, lemmas and some terminologies that are used in this dissertation.

2.1 Definitions

In this subsection, we present some definitions adopted throughout this document.

Definition 2.1.1 (*Othmani, S. and Tatar, NE., 2021 [81]*)

A memristive high-order Cohen–Grossberg bidirectional associative memory neural network system (MHOCGBAMNNS) that introduced in (P₂) below is said to be arbitrarily (generally) stable, if there exists a non-negative decreasing continuous functions $\theta(t)$, such that $\lim_{t \rightarrow \infty} \theta(t) = 0$ and

$$\|u(t, \varphi) - u^*\| + \|v(t, \psi) - v^*\| \leq \theta(t), \quad \forall t \geq 0. \quad (2.1)$$

Definition 2.1.2 (*Cai, Z. and Huang, L., 2014 [79]*)

Suppose that for every point x of a set $X \subset \mathbb{R}$, there corresponds a non-empty set $G(x) \subset \mathbb{R}^n$, then $x \mapsto G(x)$ is called a set-valued map from $X \hookrightarrow \mathbb{R}^n$. G is said to have a fixed point, if there is $x \in X$ such that $x \in G(x)$.

Definition 2.1.3 (*Cai, Z. and Huang, L., 2014 [79]*)

Assume that the set of values of G is nonempty and to any set L enclosing $G(x_0)$, there

corresponds a neighborhood N of x_0 , such that $G(N) \subset L$. Then, G is said to be upper semi-continuous at $x_0 \in X \subset \mathbb{R}^n$. The image of $G(x)$ is closed (convex, compact) in case for any $x \in X$, $G(x)$ is closed (convex, compact).

We consider the differential equation

$$u'(t) = f(t, u(t)), \quad u \in \mathbb{R}, \quad (2.2)$$

where $f(t, u(t))$ is assumed to be discontinuous with respect to the second argument.

Definition 2.1.4 (Cai, Z. and Huang, L., 2014 [79])

Let $F : \mathbb{R} \times \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ be a set-valued map defined by

$$F(t, u) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(L)=0} \bar{co} [f(t, \mathbb{B}(u, \varepsilon)) \setminus L], \quad (2.3)$$

where $\bar{co}[\cdot]$ denotes the closure of the convex hull, $\mathbb{B}(x, \varepsilon)$ is a ball of center u and radius ε and $\mu(L)$ is the Lebesgue measure of the set L . An absolutely continuous vector-valued function $u(t)$ defined on $[0, T)$ that satisfies $u(0) = u_0$ and the following differential equation

$$u'(t) \in F(t, u(t)), \quad \forall t \in [0, T), \quad (2.4)$$

is called a solution of (2.2) with the initial data $u(0) = u_0$.

Remark 2.1.1

- In mathematics, a differential equation with discontinuous multi-valued right-hand side is called a differential inclusion, written as $x'(t) \in F(t, x(t))$, where $x(t)$ is the absolutely continuous state. So, a differential inclusion is defined as a generalization of the concept of ordinary differential equation, where $F(t, x(t))$ is a set-valued map rather than a single point in $\mathbb{R}^n$.
- The convex hull of a set of points S in n dimension is the intersection of all convex sets containing S (the smallest convex set containing S).

Definition 2.1.5

Consider the memristive Hopfield neural network system (IMHNNS) described in (P_1) . A function $u(\cdot)$ defined on $I := (0, \infty)$ is said to be a solution of (P_1) with the initial conditions $\varphi_i \in \mathbb{R}$, if $u(\cdot) \in C^1(I)$ and the MHNNS is fulfilled together with the given initial conditions.

Definition 2.1.6

A function $f : X \rightarrow \mathbb{R}$ is called upper semiconscious at a point $x_0 \in X$ if for every real $y > f(x_0)$, there exists a neighborhood U of x_0 such that $f(x) < y$, for all $x \in U$. Equivalently, f is upper semiconscious at x_0 if and only if

$$\limsup_{x \rightarrow x_0} f(x) \leq f(x_0), \quad (2.5)$$

where $\limsup$ is the limit superior of the function f at the point x_0 .

2.2 Lemmas

Here, in this subsection, we mention some useful lemmas.

Lemma 2.2.1 (Brauer, F. and Nohel, J. A., 1989 [213])

Consider the Cauchy problem

$$y'(t) = f(t, y(t)), \quad \forall t > 0, \quad \text{with} \quad y(s) = y_0, \quad \forall s \leq 0. \quad (2.6)$$

Assume that f and ∂f are continuous on the “box” $\Omega := \{(t, y) : |t - t_0| \leq a, |y - y_0| \leq b\}$. Then, there exists at most one solution on the interval $|t - t_0| \leq \alpha$, where $\alpha := \{a, \frac{b}{M}\}$ and M is the bound of f .

Lemma 2.2.2 (Monti, R., 2010 [214])

Let $I := (a, b)$, with $-\infty \leq a \leq b \leq \infty$, $\Omega := I \times \mathbb{R}^n$. Consider (2.6) such that $f \in C(\Omega; \mathbb{R}^n)$ and satisfying the local Lipschitz continuous property. Suppose that for any compact set $K \in I$, there exists a positive constant C such that

$$|f(t, y(t))| \leq C(1 + |y(t)|), \quad \forall t \in K, y \in \mathbb{R}^n. \quad (2.7)$$

Then, the Cauchy problem introduced in (2.6), for all $t_0 \in K$ and $y_0 \in \mathbb{R}^n$, has a unique global solution defined on I .

Lemma 2.2.3 (Monti, R., 2010 [214])

Assume that $f \in C^k(\Omega; \mathbb{R}^n)$, for $k \geq 0$, then any solution of the differential equation presented in (2.6) is of class C^{k+1} .

Lemma 2.2.4 (Continuation Solution) (Monti, R., 2010 [214])

Let $I = (a, b) \subset \mathbb{R}$ be an open interval with $-\infty < a < b < \infty$, $\Omega := I \times \mathbb{R}$ and $f \in C(\Omega; \mathbb{R})$ be a function satisfying the local Lipschitz condition in y . If $y \in C^1((a_0, b_0); \mathbb{R}^n)$ is the maximal solution to the Cauchy problem (2.6), for some interval $(a_0, b_0) \subset (a, b)$, then we have either

- $b_0 = b$ or
- $\lim_{x \rightarrow b_0} |y(t)| = \infty$

Remark 2.2.1

The function y defined in (2.6) is called maximal solution to the Cauchy problem.

Lemma 2.2.5 (Gronwall's inequality) ([214])

Let $I \subset \mathbb{R}$ be an interval, $t_0 \in I$ and $\xi \in C(I)$ be a non negative $\xi \geq 0$ continuous function. If there exist $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geq 0$, such that

$$\xi(t) \leq \alpha + \beta \int_{t_0}^t \xi(\sigma) d\sigma, \quad \forall t \in I, t \geq t_0, \quad (2.8)$$

then

$$\xi(t) \leq \alpha e^{\beta(t-t_0)}, \quad \forall t \in I, t \geq t_0. \quad (2.9)$$

Lemma 2.2.6 (*Pata, V., 2000 [80]*)

Let u, α, β be non-negative summable functions which satisfy the following differential inequality:

$$[u^2(t)]' + \varepsilon u^2(t) \leq \alpha(t) + \beta(t)u(t), \quad \forall t \in [\tau, \infty), \quad (2.10)$$

for some positive constant ε . Furthermore, assume that

$$\gamma = \sup_{\sigma \geq \tau} \int_{\sigma}^{\sigma+1} \beta(s) ds < \infty. \quad (2.11)$$

Then

$$u^2(t) \leq 2u^2(\tau)e^{-\varepsilon(t-\tau)} + 2 \int_{\tau}^t e^{-\varepsilon(t-s)} \alpha(s) ds + \frac{e^{\varepsilon} \gamma^2}{(1 - e^{-\frac{\varepsilon}{2}})^2}, \quad (2.12)$$

for any $t \in [\tau, \infty)$.

Lemma 2.2.7 (*Hoang, N. S. and Ramm, A. G., 2009 [1]*)

Let u be a non-negative $C^1([0, \infty))$ function satisfying the inequality

$$u'(t) \leq -p(t)u(t) + q(t)u^2(t) + r(t), \quad \forall t \geq 0, \quad (2.13)$$

where $p(t), q(t)$ and $r(t)$ are non-negative continuous functions. Assume that there exists $\mu(t) \in C^1(0, \infty)$, $\mu(t) > 0$ and $\lim_{t \rightarrow \infty} \mu(t) = \infty$ such that

$$q(t) \leq \frac{\mu(t)}{2} \left[p(t) - \frac{\mu'(t)}{\mu(t)} \right], r(t) \leq \frac{1}{2\mu(t)} \left[p(t) - \frac{\mu'(t)}{\mu(t)} \right] \text{ and } \mu(0)u(0) < 1. \quad (2.14)$$

Then,

$$0 \leq u(t) < \frac{1}{\mu(t)}, \quad \forall t > 0. \quad (2.15)$$

Notation 2.2.1

Let $\mathbb{R}^n$, ($n \geq 1$) be an n -dimensional real Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and

induced norm $\|\cdot\|$. Suppose that $\Omega \subseteq \mathbb{R}^n$, let us denote by

$$P(\Omega) = \{S \subset \Omega : (\text{convex}) \text{ non-empty and compact}\}. \quad (2.16)$$

The Kakutanni–fixed point theorem was the first fixed point result about set-valued mappings. It is a generalization of the fixed point theorem by Brouwer. The statement is as the following versions:

Lemma 2.2.8 (*Kakutanni Fixed Point Theorem (Zhang, L. L. et al., 2016 [220])*)

Let Ω be a compact convex subset of a Banach space X , if the set-valued map defined as $\varphi : \Omega \longrightarrow P(\Omega)$ is an upper semi-continuous (USC) convex compact map, then ϕ has a fixed point in Ω , i.e., there exists $x \in \Omega$, such that $x \in \varphi$.

Lemma 2.2.9 (*Kakutanni Fixed Point Theorem (Widder, A., 2009 [217])*)

Let K be a nonempty compact convex subset of $\mathbb{R}^n$. Let $T : K \longrightarrow \mathfrak{K}$ such that

- For each $x \in K$, $T(x)$ is nonempty closed and convex.
- T is upper semi-continuous.

Then, T has a fixed point.

Lemma 2.2.10 (*Kakutanni Fixed Point Theorem (Saveliev, P., 2000 [218])*)

Let X be a nonempty convex compact subset of a Hausdorff topological vector space and let $G : X \longrightarrow X$ be a multifunction with nonempty convex images and preimages relatively open in X . Then G has a fixed point.

For more details, you may refer to Okon, T., 2002 [215], Hannah Keese, 2014 [216], ORegan, D., and Precup, R., 2000 [219] and references therein.

Chapter 3

MEMRISTIVE HOPFIELD NEURAL NETWORK SYSTEM (MHNNS)

In this chapter, we establish the stability and the synchronization of memristive Hopfield neural network system.

3.1 Model Description

We consider, in this section, a memristive Hopfield neural network (MHNN) described by the following system

$$\begin{cases} x_i'(t) = -a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t))g_j(x_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t))g_j(x_j(t-\tau)) \\ \quad + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s)g_j(x_j(t-s))ds + I_i, & \forall t > 0, i = 1, \dots, m, \\ x_i(s) = \varphi_i(s), & -\infty < s \leq 0, i = 1, \dots, m, \end{cases} \quad (P_1)$$

where for all $t > 0$ and $i, j = 1, 2, \dots, m$, $x_i(t)$ denotes the solution (state) of the problem (P_1) , a_i is a non-negative neuron self-inhibition, b_{ij} , c_{ij} and d_{ij} are the memristive connection weights functions, g_j indicates the activation function (signal transmission functions), I_i represents the external input, τ indicates the discrete delay, k_{ij} is the distributed delay kernel function and φ_i is a real-valued bounded continuous function.

3.1.1 Assumptions

(A₁) Assume that the activation functions (signal transmission functions) $g_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2, \dots, m$, are continuously differentiable functions on $\mathbb{R}$. Assume, further, their first derivatives are bounded, i.e. they satisfy Lipschitz conditions on $\mathbb{R}$ with Lipschitz constants G_j , i.e. for all $x, y \in \mathbb{R}$ and $j = 1, 2, \dots, m$, we obtain

$$|g_j(x) - g_j(y)| \leq G_j |x - y|.$$

(A₂) The kernel functions $k_{ij} : [0, \infty) \rightarrow \mathbb{R}^+$, $i, j = 1, 2, \dots, m$, for all $t > 0$, are continuous and integrable functions such that

$$\int_0^\infty k_{ij}(s) ds = \bar{k}_{ij} \in (0, 1),$$

and

$$\int_0^\infty k_{ij}(s) \xi(s) ds < \infty,$$

for some positive and continuous function ξ such that $\varphi(t) := \frac{\xi'(t)}{\xi(t)}$ is a decreasing function.

(A₃) Assume that for all $t > 0$ and $i, j = 1, 2, \dots, m$, the memristive connection weight functions b_{ij} , c_{ij} and d_{ij} are continuously differentiable functions on $\mathbb{R}$ and bounded by B_{ij} , C_{ij} and D_{ij} , respectively. Assume, further, their first derivatives are also bounded.

3.2 Well-Posedness

We prove, in this section, the existence, uniqueness and the continuity of such solutions of the memristive Hopfield neural network system (MHNNS) using the theory of ordinary differential equations.

Assume that, for all $t > 0$, $i = 1, 2, \dots, m$,

$$\begin{cases} y_i'(t) = f_i(y_i(t)), \\ y_i(t_0) = \varphi_i(t) \end{cases} \quad (3.1)$$

Define the rectangular region $\mathbb{B}_{\rho, \varepsilon}$ by:

$$\mathbb{B}_{\rho, \varepsilon} := \{(t, y) : 0 \leq t \leq \rho, \|y - y_0\| \leq \varepsilon\},$$

where t is the time, $y(t) = (y_1(t), y_2(t), \dots, y_m(t))$ is the solution (state) of (3.1) below, $y_0(s) = (\varphi_1(s), \varphi_2(s), \dots, \varphi_m(s))$ is the initial data, ρ, ε are positive constants. By the norm $\|\cdot\|$, we mean for all $u \in C^1(\mathbb{R}_+; \mathbb{R}^m)$

$$\|y\|_\rho = \sup_{\substack{1 \leq i \leq m \\ 0 \leq t \leq \rho}} \{|y_i(t)|\}. \quad (3.2)$$

Theorem 3.2.1

Let the assumptions (A_1) – (A_3) be hold. For any initial condition φ_i , then there exists a unique function $y \in C^1(\mathbb{R}_+; \mathbb{R}^m)$ solution to the initial value problem (3.1) (to the memristive Hopfield neural network system described in (P_1) in the space).

Proof

It is obvious that the previous system in (P_1) can be recast, for all $t > 0$ in the form

$$W'(t) = F(W), \quad (3.3)$$

with

$$W(s) = W_0(s), \quad -\infty < s \leq 0, \quad (3.4)$$

where

$W(t) = (x_1(t), x_2(t), \dots, x_m(t))$, $W_0(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_m(t))$ and $F = (F_1, F_2, \dots, F_m)$.

Clearly, from the system (P_1) , we have for all $i = 1, 2, \dots, m$

$$\begin{aligned} F_i = & -a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(x_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t - \tau)) \\ & + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds + I_i. \end{aligned} \quad (3.5)$$

It is obvious that under the assumptions (A_1) – (A_3) , we obtain that the non-linear term F , in the right-hand side of (3.3), and ∂f are continuous on the closed and bounded rectangular region “box” $\mathbb{B}_{\rho, \varepsilon}$.

It turns out that, in view of Lemma 2.2.1, the existence and uniqueness of a local solution $W(t) \in \mathbb{B}_{\rho, \varepsilon}$, for all $t \in I := [0, \omega]$, where $\omega := \min \left\{ \rho, \frac{\varepsilon}{M} \right\}$.

Now, to pass from a local solution to a global one, it suffices to show that the local solution W is time independent i.e. bounded by some positive constant, so that the time can be extended from the interval I to the interval $[0, T]$, for any arbitrary time $T > 0$.

First, assume that $I = \mathbb{R}$ and $K := [0, \rho]$ is a compact subset of I . It is clear that $F \in C(\mathbb{R}^m)$. For all $t \in K$ and $W \in \mathbb{R}^m$ and we have

$$\begin{aligned}
\|F\| = \sup_{\substack{1 \leq i \leq m \\ 0 \leq t \leq \rho}} \{|F_i|\} &\leq \sup_{\substack{1 \leq i \leq m \\ 0 \leq t \leq \rho}} \left\{ \left| -a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(x_j(t)) \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t-\tau)) + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t-s)) ds \right. \right. \\
&\quad \left. \left. + I_i \right| \right\} \\
&\leq \sup_{\substack{1 \leq i \leq m \\ 0 \leq t \leq \rho}} \left\{ a_i |x_i(t)| + \sum_{j=1}^m |b_{ij}(x_i(t))| |g_j(x_j(t))| \right. \\
&\quad \left. + \sum_{j=1}^m |c_{ij}(x_i(t))| |g_j(x_j(t-\tau))| + \left| \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t-s)) ds \right| \right. \\
&\quad \left. + |I_i| \right\} \\
&\leq \sup_{\substack{1 \leq i \leq m \\ 0 \leq t \leq \rho}} \left\{ a_i |x_i(t)| + \sum_{j=1}^m G_j [B_{ij} + C_{ij} + D_{ij} \bar{k}_{ij}] + |I_i| \right\} \\
&\leq \sup_{\substack{1 \leq i \leq m \\ 0 \leq t \leq \rho}} \{a_i |x_i(t)|\} + M_1 \\
&= a \|W\| + M_1 \leq M(1 + \|W\|)
\end{aligned} \tag{3.6}$$

where $a := \max_{1 \leq i \leq m} \{a_i\}$, $M_1 := \max_{1 \leq i \leq m} \left\{ \sum_{j=1}^m G_j [B_{ij} + C_{ij} + D_{ij} \bar{k}_{ij}] + |I_i| \right\}$ and $M := \max \{a, M_1\}$.

Now, we need to show that the solution is independent on time. The proof is by a contradiction.

Let $W \in C^1(J; \mathbb{R}^m)$ be the maximal solution to the Cauchy problem (2.6), with $J := (a_0, b_0) \subset I$. Assume by contraction that $b_0 < b$. By Lemma (2.2.4), we have:

$$\lim_{t \rightarrow b_0} |W(t)| = \infty. \tag{3.7}$$

Now, let $K := [t_0, b_0]$ and $M > 0$ such that the inequality in (3.6) holds. From

$$W(t) = W_0 + \int_{t_0}^t F(s, W(s)) ds, \quad t \in J, \tag{3.8}$$

we get for all $t \in J$ with $t \geq t_0$

$$\begin{aligned}
|W(t)| &\leq |W_0| + \int_{t_0}^t |F(s, W(s))| ds \\
&\leq |W_0| + M(b_0 - t_0) + M \int_{t_0}^t |W(s)| ds \\
&\leq |W_0| + \int_{t_0}^t |F(s, W(s))| ds.
\end{aligned} \tag{3.9}$$

By the Gronwall Lemma (2.2.5), it follows that

$$|W(t)| \leq [|W_0| + M(b_0 - t_0)] e^{M(t-t_0)}, \quad \forall t \in (t_0, b_0). \tag{3.10}$$

Hence,

$$\lim_{t \rightarrow t_0} |W(t)| \leq \lim_{t \rightarrow t_0} [|W_0| + M(b_0 - t_0)] e^{M(t-t_0)} < \infty \tag{3.11}$$

which is a contradiction.

Therefore, in view of Lemma 2.2.2, the property of the non-linear term F gives that the existence and uniqueness of such solution W is global, that is the time can be extended to the interval $[0, T]$, for any $T > 0$. Hence, $W \in C^1(\mathbb{R}_+; \mathbb{R}^m)$. Therefore, our assumption implies that the existence and uniqueness of $x_i \in C^1(\mathbb{R}_+; \mathbb{R}^m)$, $i = 1, 2, \dots, m$.

3.3 Stability Result

In this section, we establish some sufficient conditions that ensure the stability of the memristive Hopfield neural network system (MHNNS) together with the initial data given in (P_1) .

Definition 3.3.1 (*Chen, C. et al., 2017 [3]*)

A corresponding slave (response) system to the given master (drive) memristive Hopfield neural network system in (P_1) is said to be synchronized with the MHNNS in finite time,

if there exists a constant $t^* > 0$ such that

$$\lim_{t \rightarrow t^*} e_i(t) = 0 \text{ and } e_i(t) \equiv 0, \quad \forall t \geq t^*, i = 1, 2, \dots, m, \quad (3.12)$$

where $e_i(\cdot)$ is the synchronization error that will be explained in Section 3.4 below, and t^* is called the settling time.

Remark 3.3.1

System (P_1) and system (3.66) (in Section 3.4 below) are called master (drive) system and slave (response) system, respectively.

Definition 3.3.2 (Wan, L. and Zhou, Q., 2010 [107])

We denote an equilibrium solution (state) of the memristive Hopfield network network system (MHNNS) given in (P_1) by the constant vector $x^* = (x_1^*, x_2^*, \dots, x_m^*) \in \mathbb{R}^m$, $i = 1, 2, \dots, m$, where the components x_i^* are governed by the algebraic system

$$\begin{aligned} 0 &= -a_i x_i^* + \sum_{j=1}^m b_{ij}(x_i^*) g_j(x_j^*) + \sum_{j=1}^m c_{ij}(x_i^*) g_j(x_j^*) + \sum_{j=1}^m d_{ij}(x_i^*) \int_0^\infty k_{ij}(s) g_j(x_j^*) ds + I_i \\ &= -a_i x_i^* + \sum_{j=1}^m [b_{ij}(x_i^*) + c_{ij}(x_i^*) + \bar{k}_{ij} d_{ij}(x_i^*)] g_j(x_j^*) + I_i, \end{aligned} \quad (3.13)$$

for all $i = 1, 2, \dots, m$.

Define the Lyapanouv functional $E(t)$ by

$$E(t) := \sum_{i=1}^5 E_i(t) \quad \forall t > 0, \quad (3.14)$$

where $E_1(t)$, $E_2(t)$, $E_3(t)$, $E_4(t)$ and $E_5(t)$ are defined in (3.33), (3.45), (3.47), (3.49) and (3.50) below, respectively.

Theorem 3.3.1

Consider the memristive Hopfield neural network system together with the initial data described in (P_1) . Let A_1 – A_3 be satisfied. If for all $i = 1, 2, \dots, m$,

$$e^{\alpha\tau}\lambda_{1i} + \sum_{j=1}^m \left[\lambda_{2ji}V_{ji}(0) + \frac{20m}{a} \left(L_{1ji}^2 + L_{2ji}^2 + \frac{k_{ji}^2}{5} L_{3ji}^2 \right) g_i^2(x_i^*) + G_j^2 b_{ij}^2(x_j^*) \right] \leq a_i - b_2, \quad (3.15)$$

for some positive constant b_2 and $a := \min_{1 \leq i \leq m} \{a_i\}$. Then, the Lyapunov functional $E(t)$ defined above in (3.14) satisfies

$$E(t) < \frac{1}{\sigma(t)}, \quad t \geq t^* > 0, \quad (3.16)$$

provided that

$$\alpha(t) \leq \frac{\sigma(t)}{2} \left(\beta(t) - \frac{\sigma'(t)}{\sigma(t)} \right) \quad \text{and} \quad E(0)\sigma(0) < 1, \quad (3.17)$$

for some positive continuous function σ and non-negative continuous functions α and β .

Proof

Let us denote by

$$y_i(t) = x_i(t) - x_i^*. \quad (3.18)$$

for all $t > 0$ and $i = 1, 2, \dots, m$.

From the systems defined in (P_1) and (3.13), we obtain

$$\begin{aligned} y_i'(t) = & -a_i y_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(x_j(t)) - \sum_{j=1}^m b_{ij}(x_i^*) g_j(x_j^*) \\ & + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t - \tau)) - \sum_{j=1}^m c_{ij}(x_i^*) g_j(x_j^*) \\ & + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds - \sum_{j=1}^m \bar{k}_{ij} d_{ij}(x_i^*) g_j(x_j^*) ds. \end{aligned} \quad (3.19)$$

Now, let us write the terms in the right hand side of (3.19) as follows:

$$\begin{aligned}
& b_{ij}(x_j(t))g_j(x_j(t)) - b_{ij}(x_j^*)g_j(x_j^*) \\
&= [g_j(x_j(t)) - g_j(x_j^*)] b_{ij}(x_j(t)) + g_j(x_j^*) [b_{ij}(x_j(t)) - b_{ij}(x_j^*)] \\
&= [g_j(x_j(t)) - g_j(x_j^*)] [b_{ij}(x_j(t)) - b_{ij}(x_j^*)] + b_{ij}(x_j^*) [g_j(x_j(t)) - g_j(x_j^*)] \\
&\quad + g_j(x_j^*) [b_{ij}(x_j(t)) - b_{ij}(x_j^*)].
\end{aligned} \tag{3.20}$$

Since the activation functions are Lipschitz continuous, we get:

$$\begin{aligned}
& b_{ij}(x_j(t))g_j(x_j(t)) - b_{ij}(x_j^*)g_j(x_j^*) \\
&= \tilde{g}_j(y_j(t))\tilde{b}_{ij}(y_j(t)) + b_{ij}(x_j^*)\tilde{g}_j(y_j(t)) + g_j(x_j^*)\tilde{b}_{ij}(y_j(t)) \\
&\leq |\tilde{b}_{ij}(y_j(t))|G_j|y_j(t)| + |b_{ij}(x_j^*)|G_j|y_j(t)| + |g_j(x_j^*)||\tilde{b}_{ij}(y_j(t))|.
\end{aligned} \tag{3.21}$$

So,

$$\begin{aligned}
& 2|y_i(t)||\tilde{b}_{ij}(y_j(t))|G_j|y_j(t)| \\
&\leq |y_i(t)| \left(G_j^2|y_j(t)|^2 + \tilde{b}_{ij}^2(y_j(t)) \right) \\
&= |y_i(t)|G_j^2|y_j(t)|^2 + |y_i(t)||\tilde{b}_{ij}^2(y_j(t))| \\
&\leq \delta_1|y_i(t)|^2 + \frac{G_j^4}{4\delta_1}|y_j(t)|^4 + \delta_1|y_i(t)|^2 + \frac{1}{4\delta_1}\tilde{b}_{ij}^4(y_j(t)) \\
&\leq 2\delta_1|y_i(t)|^2 + \frac{G_j^4}{4\delta_1}|y_j(t)|^4 + \frac{1}{4\delta_1}\tilde{b}_{ij}^4(y_j(t)),
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
2|y_i(t)| \sum_{j=1}^m |\tilde{b}_{ij}(y_j(t))|G_j|y_j(t)| &\leq |y_i(t)| \sum_{j=1}^m \left(G_j^2|y_j(t)|^2 + \tilde{b}_{ij}^2(y_j(t)) \right) \\
&\leq 2\delta_1 m |y_i(t)|^2 + \sum_{j=1}^m \frac{G_j^4}{4\delta_1} |y_j(t)|^4 + \sum_{j=1}^m \frac{1}{4\delta_1} \tilde{b}_{ij}^4(y_j(t)).
\end{aligned} \tag{3.23}$$

Similarly,

$$2|y_i(t)||b_{ij}(x_j^*)|G_j|y_j(t)| \leq \delta_2|y_i(t)|^2 + \frac{G_j^2 b_{ij}^2(x_j^*)}{\delta_2} |y_j(t)|^2, \tag{3.24}$$

and likewise

$$2|y_i(t)||g_j(x_j^*)||\tilde{b}_{ij}(y_j(t))| \leq \delta_2|y_i(t)|^2 + \frac{g_j^2(x_j^*)}{\delta_2}\tilde{b}_{ij}^2(y_j(t)). \quad (3.25)$$

Also,

$$\begin{aligned} & c_{ij}(x_j(t))g_j(x_j(t-\tau)) - c_{ij}(x_j^*)g_j(x_j^*) \\ = & [g_j(x_j(t-\tau)) - g_j(x_j^*)] c_{ij}(x_j(t)) + g_j(x_j^*) [c_{ij}(x_j(t)) - c_{ij}(x_j^*)] \\ = & [g_j(x_j(t-\tau)) - g_j(x_j^*)] [c_{ij}(x_j(t)) - c_{ij}(x_j^*)] + c_{ij}(x_j^*) [g_j(x_j(t-\tau)) - g_j(x_j^*)] \\ & + g_j(x_j^*) [c_{ij}(x_j(t)) - c_{ij}(x_j^*)]. \end{aligned} \quad (3.26)$$

As the activation functions are Lipschitz continuous,

$$\begin{aligned} & c_{ij}(x_j(t))g_j(x_j(t-\tau)) - c_{ij}(x_j^*)g_j(x_j^*) \\ = & [g_j(x_j(t-\tau)) - g_j(x_j^*)] [c_{ij}(x_j(t)) - c_{ij}(x_j^*)] + c_{ij}(x_j^*) [g_j(x_j(t-\tau)) - g_j(x_j^*)] \\ & + g_j(x_j^*) [c_{ij}(x_j(t)) - c_{ij}(x_j^*)] \\ = & \tilde{g}_j(y_j(t-\tau))\tilde{c}_{ij}(y_j(t)) + c_{ij}(x_j^*)\tilde{g}_j(y_j(t-\tau)) + g_j(x_j^*)\tilde{c}_{ij}(y_j(t)) \\ \leq & |\tilde{c}_{ij}(y_j(t))|G_j|y_j(t-\tau)| + |c_{ij}(x_j^*)|G_j|y_j(t-\tau)| + |g_j(x_j^*)||\tilde{c}_{ij}(y_j(t))|. \end{aligned} \quad (3.27)$$

The terms in the previous inequality (3.27) can be estimated as the following expressions:

$$\begin{aligned} & 2|y_i(t)||\tilde{c}_{ij}(y_j(t))|G_j|y_j(t-\tau)| \\ \leq & |y_i(t)|(\tilde{c}_{ij}^2(y_j(t)) + G_j^2|y_j(t-\tau)|^2) \\ = & |y_i(t)|\tilde{c}_{ij}^2(y_j(t)) + |y_i(t)|G_j^2|y_j(t-\tau)|^2 \\ \leq & \delta_3|y_i(t)|^2 + \frac{1}{4\delta_3}|\tilde{c}_{ij}^4(y_j(t))| + \delta_3|y_i(t)|^2 + \frac{G_j^4}{4\delta_3}|y_j(t-\tau)|^4 \\ \leq & 2\delta_3|y_i(t)|^2 + \frac{G_j^4}{4\delta_3}|y_j(t-\tau)|^4 + \frac{1}{4\delta_3}\tilde{c}_{ij}^4(y_j(t)), \end{aligned} \quad (3.28)$$

thus,

$$\begin{aligned}
& 2|y_i(t)| \sum_{j=1}^m |\tilde{c}_{ij}(y_j(t))| G_j |y_j(t - \tau)| \\
& \leq |y_i(t)| \sum_{j=1}^m (\tilde{c}_{ij}^2(y_j(t)) + G_j^2 |y_j(t - \tau)|^2) \\
& \leq 2\delta_3 m |y_i(t)|^2 + \sum_{j=1}^m \frac{G_j^4}{4\delta_3} |y_j(t - \tau)|^4 + \sum_{j=1}^m \frac{1}{4\delta_3} \tilde{c}_{ij}^4(y_j(t)), \tag{3.29}
\end{aligned}$$

Similarly,

$$2|y_i(t)| |c_{ij}(x_j^*)| G_j |y_j(t - \tau)| \leq \delta_4 |y_i(t)|^2 + \frac{c_{ij}^2(x_j^*)}{\delta_4} G_j^2 |y_j(t - \tau)|^2, \tag{3.30}$$

and likewise

$$2|y_i(t)| |g_j(x_j^*)| |\tilde{c}_{ij}(y_j(t))| \leq \delta_4 |y_i(t)|^2 + \frac{g_j^2(x_j^*)}{\delta_4} \tilde{c}_{ij}^2(y_j(t)). \tag{3.31}$$

Regarding the remaining two terms in (3.19), we modify the estimation as follows:

$$\begin{aligned}
& \sum_{j=1}^m \left[d_{ij}(x_j(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds - \bar{k}_{ij} d_{ij}(x_j^*) g_j(x_j^*) \right] \\
& = \sum_{j=1}^m \left\{ \int_0^\infty k_{ij}(s) [g_j(x_j(t - s)) - g_j(x_j^*)] ds [d_{ij}(x_j(t)) - d_{ij}(x_j^*)] \right. \\
& \quad \left. + d_{ij}(x_j^*) \int_0^\infty k_{ij}(s) [g_j(x_j(t - s)) - g_j(x_j^*)] ds \right\} \\
& \quad + \sum_{j=1}^m \left\{ \left[\int_0^\infty k_{ij}(s) g_j(x_j^*) ds \right] [d_{ij}(x_j(t)) - d_{ij}(x_j^*)] \right\},
\end{aligned}$$

or

$$\begin{aligned}
& \sum_{j=1}^m \left[d_{ij}(x_j(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds - \bar{k}_{ij} d_{ij}(x_j^*) g_j(x_j^*) \right] \\
& = \sum_{j=1}^m \left\{ \tilde{d}_{ij}(y_j(t)) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t - s)) ds \right. \\
& \quad \left. + d_{ij}(x_j^*) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t - s)) ds + \bar{k}_{ij} g_j(x_j^*) \tilde{d}_{ij}(y_j(t)) \right\}. \tag{3.32}
\end{aligned}$$

We define the energy functional $E_1(t)$ by:

$$E_1(t) := \sum_{i=1}^m |y_i(t)|^2, \quad \forall t \geq 0. \quad (3.33)$$

Hence, along solutions of the system, we entail:

$$\begin{aligned} E_1'(t) = & 2 \sum_{i=1}^m y_i(t) \left\{ -a_i y_i(t) + \sum_{j=1}^m [b_{ij}(x_j(t))g_j(x_j(t)) - b_{ij}(x_j^*)g_j(x_j^*)] \right. \\ & + \sum_{j=1}^m [c_{ij}(x_j(t))g_j(x_j(t-\tau)) - c_{ij}(x_j^*)g_j(x_j^*)] \\ & \left. + \sum_{j=1}^m \left[d_{ij}(x_j(t)) \int_0^\infty k_{ij}(s)g_j(x_j(t-s))ds - \bar{k}_{ij}d_{ij}(x_j^*)g_j(x_j^*) \right] \right\}, \quad (3.34) \end{aligned}$$

or

$$\begin{aligned} E_1'(t) \leq & -2 \sum_{i=1}^m a_i y_i^2(t) + \sum_{i=1}^m \sum_{j=1}^m 2|y_i(t)| |b_{ij}(x_j(t))g_j(x_j(t)) - b_{ij}(x_j^*)g_j(x_j^*)| \\ & + \sum_{i=1}^m \sum_{j=1}^m 2|y_i(t)| |c_{ij}(x_j(t))g_j(x_j(t-\tau)) - c_{ij}(x_j^*)g_j(x_j^*)| \\ & + \sum_{i=1}^m \sum_{j=1}^m 2|y_i(t)| \left| d_{ij}(x_j(t)) \int_0^\infty k_{ij}(s)g_j(x_j(t-s))ds - \bar{k}_{ij}d_{ij}(x_j^*)g_j(x_j^*) \right|. \quad (3.35) \end{aligned}$$

By substituting the all previous estimations in the inequality (3.35), we obtain:

$$\begin{aligned}
E'_1(t) \leq & -2 \sum_{i=1}^m a_i y_i^2(t) + \sum_{i=1}^m 2\delta_1 m |y_i(t)|^2 + \sum_{j=1}^m \frac{mG_j^4}{4\delta_1} |y_j(t)|^4 + \sum_{i,j=1}^m \frac{1}{4\delta_1} \tilde{b}_{ij}^4(y_j(t)) \\
& + \sum_{i=1}^m \delta_2 m |y_i(t)|^2 + \sum_{i,j=1}^m \frac{G_j^2 b_{ij}^2(x_j^*)}{\delta_2} |y_j(t)|^2 + \sum_{i=1}^m \delta_2 m |y_i(t)|^2 \\
& + \sum_{i,j=1}^m \frac{g_j^2(x_j^*)}{\delta_2} \tilde{b}_{ij}^2(y_j(t)) + \sum_{i=1}^m 2\delta_3 m |y_i(t)|^2 + \sum_{j=1}^m \frac{mG_j^4}{4\delta_3} |y_j(t-\tau)|^4 \\
& + \sum_{i,j=1}^m \frac{1}{4\delta_3} \tilde{c}_{ij}^4(y_j(t)) + \sum_{i=1}^m \delta_4 m |y_i(t)|^2 + \sum_{i,j=1}^m \frac{G_j^2 c_{ij}^2(x_j^*)}{\delta_4} |y_j(t-\tau)|^2 \\
& + \sum_{i=1}^m \delta_4 m |y_i(t)|^2 + \sum_{i,j=1}^m \frac{g_j^2(x_j^*)}{\delta_4} \tilde{c}_{ij}^2(y_j(t)) \\
& + \sum_{i=1}^m 2|y_i(t)| \sum_{j=1}^m \tilde{d}_{ij}(y_j(t)) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \\
& + \sum_{i=1}^m 2|y_i(t)| \sum_{j=1}^m d_{ij}(x_j^*) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \\
& + \sum_{i=1}^m 2|y_i(t)| \sum_{j=1}^m \bar{k}_{ij} g_j(x_j^*) \tilde{d}_{ij}(y_j(t)), \tag{3.36}
\end{aligned}$$

or

$$\begin{aligned}
E'_1(t) \leq & \sum_{i=1}^m \left[-2a_i + 2\delta_1 m + \delta_2 m + \sum_{j=1}^m \frac{b_{ij}^2(x_j^*)}{\delta_2} G_j^2 + \delta_2 m + 2\delta_3 m + \delta_4 m + \delta_4 m \right] |y_i(t)|^2 \\
& + \sum_{i,j=1}^m \frac{mG_j^4}{4\delta_1} |y_j(t)|^4 + \sum_{i,j=1}^m \frac{G_j^2 c_{ij}^2(x_j^*)}{\delta_4} |y_j(t-\tau)|^2 + \sum_{i,j=1}^m \frac{mG_j^4}{4\delta_3} |y_j(t-\tau)|^4 \\
& + \sum_{i,j=1}^m \left[\frac{1}{4\delta_1} \tilde{b}_{ij}^4(y_j(t)) + \frac{g_j^2(x_j^*)}{\delta_2} \tilde{b}_{ij}^2(y_j(t)) + \frac{1}{4\delta_3} \tilde{c}_{ij}^4(y_j(t)) + \frac{g_j^2(x_j^*)}{\delta_4} \tilde{c}_{ij}^2(y_j(t)) \right] \\
& + \sum_{i=1}^m 2|y_i(t)| \sum_{j=1}^m \tilde{d}_{ij}(y_j(t)) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \\
& + \sum_{i=1}^m 2|y_i(t)| \sum_{j=1}^m d_{ij}(x_j^*) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \\
& + \sum_{i=1}^m 2|y_i(t)| \sum_{j=1}^m \bar{k}_{ij} g_j(x_j^*) \tilde{d}_{ij}(y_j(t)). \tag{3.37}
\end{aligned}$$

Let us modify the estimations of the last 3 terms as follows:

$$\begin{aligned}
& 2|y_i(t)| \left[\tilde{d}_{ij}(y_j(t)) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \right] \\
& \leq |y_i(t)| \left[\tilde{d}_{ij}^2(y_j(t)) + \left(\int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \right)^2 \right] \\
& \leq |y_i(t)| \tilde{d}_{ij}^2(y_j(t)) + G_j^2 \bar{k}_{ij} |y_i(t)| \int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds \\
& \leq \delta_5 |y_i(t)|^2 + \frac{1}{4\delta_5} \tilde{d}_{ij}^4(y_j(t)) + \delta_5 |y_i(t)|^2 + \frac{G_j^4 \bar{k}_{ij}^2}{4\delta_5} \left(\int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds \right)^2 \\
& \leq 2\delta_5 |y_i(t)|^2 + \frac{1}{4\delta_5} \tilde{d}_{ij}^4(y_j(t)) + \frac{G_j^4 \bar{k}_{ij}^3}{4\delta_5} \int_0^\infty k_{ij}(s) |y_j(t-s)|^4 ds, \tag{3.38}
\end{aligned}$$

so, we have:

$$\begin{aligned}
& 2|y_i(t)| \sum_{j=1}^m \tilde{d}_{ij}(y_j(t)) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \\
& \leq 2\delta_5 m |y_i(t)|^2 + \sum_{j=1}^m \frac{1}{4\delta_5} \tilde{d}_{ij}^4(y_j(t)) + \sum_{j=1}^m \frac{G_j^4 \bar{k}_{ij}^3}{4\delta_5} \int_0^\infty k_{ij}(s) |y_j(t-s)|^4 ds. \tag{3.39}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& 2|y_i(t)| d_{ij}(x_j^*) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \\
& \leq |y_i(t)|^2 + \left(d_{ij}(x_j^*) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \right)^2 \\
& \leq |y_i(t)|^2 + G_j^2 \bar{k}_{ij} d_{ij}^2(x_j^*) \int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds, \tag{3.40}
\end{aligned}$$

and thus,

$$\begin{aligned}
& 2|y_i(t)| \sum_{j=1}^m d_{ij}(x_j^*) \int_0^\infty k_{ij}(s) \tilde{g}_j(y_j(t-s)) ds \\
& \leq \delta_6 m |y_i(t)|^2 + \frac{1}{\delta_6} \sum_{j=1}^m G_j^2 \bar{k}_{ij} d_{ij}^2(x_j^*) \int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds. \tag{3.41}
\end{aligned}$$

In the same way,

$$2|y_i(t)|\bar{k}_{ij}g_j(x_j^*)\tilde{d}_{ij}(y_j(t)) \leq |y_i(t)|^2 + \bar{k}_{ij}^2g_j^2(x_j^*)\tilde{d}_{ij}^2(y_j(t)), \quad (3.42)$$

then,

$$2|y_i(t)|\sum_{j=1}^m \bar{k}_{ij}g_j(x_j^*)\tilde{d}_{ij}(y_j(t)) \leq \delta_7 m|y_i(t)|^2 + \frac{1}{\delta_7} \sum_{j=1}^m \bar{k}_{ij}^2g_j^2(x_j^*)\tilde{d}_{ij}^2(y_j(t)). \quad (3.43)$$

Hence, $E'_1(t)$ can be evaluated by:

$$\begin{aligned} E'_1(t) \leq & \sum_{i=1}^m \left[-2a_i + 2\delta_1 m + 2\delta_2 m + \sum_{j=1}^m \frac{G_j^2}{\delta_2} b_{ij}^2(x_j^*) + 2\delta_3 m + 2\delta_4 m + 2\delta_5 m \right. \\ & \left. + \delta_6 m + \delta_7 m \right] |y_i(t)|^2 \\ & + \sum_{j=1}^m \frac{mG_j^4}{4\delta_1} |y_j(t)|^4 + \sum_{i,j=1}^m \frac{G_j^2 c_{ij}^2(x_j^*)}{\delta_4} |y_j(t-\tau)|^2 + \sum_{j=1}^m \frac{mG_j^4}{4\delta_3} |y_j(t-\tau)|^4 \\ & + \sum_{i,j=1}^m \left[\frac{1}{4\delta_1} \tilde{b}_{ij}^4(y_j(t)) + \frac{g_j^2(x_j^*)}{\delta_2} \tilde{b}_{ij}^2(y_j(t)) + \frac{1}{4\delta_3} \tilde{c}_{ij}^4(y_j(t)) + \frac{g_j^2(x_j^*)}{\delta_4} \tilde{c}_{ij}^2(y_j(t)) \right. \\ & \left. + \frac{1}{4\delta_5} \tilde{d}_{ij}^4(y_j(t)) + \frac{1}{\delta_7} \bar{k}_{ij}^2 g_j^2(x_j^*) \tilde{d}_{ij}^2(y_j(t)) \right] \\ & + \frac{1}{\delta_6} \sum_{i,j=1}^m G_j^2 \bar{k}_{ij} d_{ij}^2(x_j^*) \int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds \\ & + \sum_{i,j=1}^m \frac{G_j^4 \bar{k}_{ij}^3}{4\delta_5} \int_0^\infty k_{ij}(s) |y_j(t-s)|^4 ds. \end{aligned} \quad (3.44)$$

Next, we define the functional $E_2(t)$ by

$$E_2(t) := e^{-\alpha t} \sum_{j=1}^m \lambda_{1j} \int_{t-\tau}^t e^{\alpha(s+\tau)} y_j^2(s) ds, \quad \forall t > 0, \quad (3.45)$$

for some constants $\alpha > 0$ and $\lambda_{1j} > 0$, $j = 1, 2, \dots, m$, to be determined later.

The derivative of the functional $E_2(t)$, is equal to

$$\begin{aligned}
E_2'(t) &= -\alpha e^{-\alpha t} \sum_{j=1}^m \lambda_{1j} \int_{t-\tau}^t e^{\alpha(s+\tau)} y_j^2(s) ds \\
&\quad + e^{-\alpha t} \sum_{j=1}^m \lambda_{1j} (e^{\alpha(t+\tau)} y_j^2(t) - e^{\alpha(t-\tau+\tau)} y_j^2(t-\tau)) \\
&= -\alpha E_2(t) + e^{\alpha\tau} \sum_{j=1}^m \lambda_{1j} y_j^2(t) - \sum_{j=1}^m \lambda_{1j} y_j^2(t-\tau), \quad \forall t \geq 0. \tag{3.46}
\end{aligned}$$

The functional $E_3(t)$ we introduce is

$$E_3(t) := \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t V_{ij}(t-s) y_j^2(s) ds, \tag{3.47}$$

where

$$V_{ij}(t) := \frac{1}{\xi(t)} \int_t^\infty k_{ij}(s) \xi(s) ds, \quad \forall t > 0,$$

for some constants $\lambda_{2ij} > 0$, $i, j = 1, 2, \dots, m$ to be determined later.

Remark 3.3.2

We notice that the functionals $V_{ij}(t)$, $i, j = 1, 2, \dots, m$ are well defined by our assumption (A_2) .

Differentiating the functional $E_3(t)$, we obtain

$$\begin{aligned}
E_3'(t) &= \sum_{i,j=1}^m \lambda_{2ij} V_{ij}(0) y_j^2(t) + \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t V_{ij}'(t-s) y_j^2(s) ds \\
&= \sum_{i,j=1}^m \lambda_{2ij} V_{ij}(0) y_j^2(t) + \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t \left[\frac{-\xi'(t-s)}{\xi^2(t-s)} \int_{t-s}^{\infty} k_{ij}(\sigma) \xi(\sigma) d\sigma \right. \\
&\quad \left. - \frac{1}{\xi(t-s)} k_{ij}(t-s) \xi(t-s) \right] y_j^2(s) ds \\
&= \sum_{i,j=1}^m \lambda_{2ij} V_{ij}(0) y_j^2(t) \\
&\quad + \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t \left[\frac{-\xi'(t-s)}{\xi^2(t-s)} \int_{t-s}^{\infty} k_{ij}(\sigma) h(\sigma) d\sigma - k_{ij}(t-s) \right] y_j^2(s) ds \\
&= \sum_{i,j=1}^m \lambda_{2ij} V_{ij}(0) y_j^2(t) - \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t [\varphi(t-s) V_{ij}(t-s) + k_{ij}(t-s)] y_j^2(s) ds. \\
&\leq \sum_{i,j=1}^m \lambda_{2ij} V_{ij}(0) y_j^2(t) - \varphi(t) \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t V_{ij}(t-s) y_j^2(s) ds \\
&\quad - \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t k_{ij}(t-s) y_j^2(s) ds \\
&\leq \sum_{i,j=1}^m \lambda_{2ij} V_{ij}(0) y_j^2(t) - \varphi(t) E_3(t) - \sum_{i,j=1}^m \lambda_{2ij} \int_{-\infty}^t k_{ij}(t-s) y_j^2(s) ds. \tag{3.48}
\end{aligned}$$

Let us define the functionals $E_4(t)$ and $E_5(t)$ by

$$E_4(t) := e^{-\theta t} \sum_{j=1}^m \lambda_{3j} \int_{t-\tau}^t e^{\theta(s+\tau)} y_j^4(s) ds, \quad \forall t > 0, \tag{3.49}$$

and

$$E_5(t) := \sum_{i,j=1}^m \lambda_{4ij} \int_{-\infty}^t V_{ij}(t-s) y_j^4(s) ds, \quad \forall t > 0, \tag{3.50}$$

for some constants $\theta > 0$, $\lambda_{3j} > 0$ and $\lambda_{4ij} > 0$ to be determined later and $V_{ij}(t)$ is defined as before for all $i, j = 1, 2, \dots, m$.

A differentiation of $E_4(t)$ implies

$$\begin{aligned}
E'_4(t) &= -\theta e^{-\theta t} \sum_{j=1}^m \lambda_{3j} \int_{t-\tau}^t e^{\theta(s+\tau)} y_j^4(s) ds \\
&\quad + e^{-\theta t} \sum_{j=1}^m \lambda_{3j} [e^{\theta(t+\tau)} y_j^4(t) - e^{\theta t} y_j^4(t-\tau)] \\
&= -\theta E_4(t) + e^{\theta \tau} \sum_{j=1}^m \lambda_{3j} y_j^4(t) - \sum_{j=1}^m \lambda_{3j} y_j^4(t-\tau)
\end{aligned} \tag{3.51}$$

and the derivative of $E_5(t)$ is estimated by

$$E'_5(t) \leq \sum_{i,j=1}^m \lambda_{4ij} V_{ij}(0) y_j^4(t) - \varphi(t) E_5(t) - \sum_{i,j=1}^m \lambda_{4ij} \int_{-\infty}^t k_{ij}(t-s) y_j^4(s) ds, \tag{3.52}$$

where φ is as defined in the assumption (A₂).

Now, let us define the Lyapunov functional $E(t)$ by

$$E(t) := \sum_{i=1}^5 E_i(t), \quad \forall t > 0. \tag{3.53}$$

Thus, collecting the previous estimations (3.44), (3.46), (3.48), (3.51) and (3.52), we obtain

$$\begin{aligned}
E'(t) \leq & \sum_{i=1}^m \left[-2a_i + 2m \sum_{\ell=1}^5 \delta_\ell + \sum_{j=1}^m \frac{G_j^2}{\delta_2} b_{ij}^2(x_j^*) + \delta_6 m + \delta_7 m \right] |y_i(t)|^2 \\
& + \sum_{j=1}^m \frac{mG_j^4}{4\delta_1} |y_j(t)|^4 + \sum_{i,j=1}^m \frac{G_j^2}{\delta_4} c_{ij}^2(x_j^*) |y_j(t-\tau)|^2 + \sum_{j=1}^m \frac{mG_j^4}{4\delta_3} |y_j(t-\tau)|^4 \\
& + \sum_{i,j=1}^m \left[\frac{1}{4\delta_1} \tilde{b}_{ij}^4(y_j(t)) + \frac{g_j^2(x_j^*)}{\delta_2} \tilde{b}_{ij}^2(y_j(t)) + \frac{1}{4\delta_3} \tilde{c}_{ij}^4(y_j(t)) + \frac{g_j^2(x_j^*)}{\delta_4} \tilde{c}_{ij}^2(y_j(t)) \right. \\
& \quad \left. + \frac{1}{4\delta_5} \tilde{d}_{ij}^4(y_j(t)) + \frac{1}{\delta_7} \bar{k}_{ij}^2 g_j^2(x_j^*) \tilde{d}_{ij}^2(y_j(t)) \right] \\
& + \frac{1}{\delta_6} \sum_{i,j=1}^m G_j^2 \bar{k}_{ij} d_{ij}^2(x_j^*) \int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds \\
& + \sum_{i,j=1}^m \frac{G_j^4 \bar{k}_{ij}^3}{4\delta_5} \int_0^\infty k_{ij}(s) |y_j(t-s)|^4 ds \\
& - \alpha E_2(t) + e^{\alpha\tau} \sum_{j=1}^m \lambda_{1j} y_j^2(t) - \sum_{j=1}^m \lambda_{1j} y_j^2(t-\tau) + \sum_{i,j=1}^m \lambda_{2ij} V_{ij}(0) y_j^2(t) \\
& - \varphi(t) E_3(t) - \sum_{i,j=1}^m \lambda_{2ij} \int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds - \theta E_4(t) + e^{\theta\tau} \sum_{j=1}^m \lambda_{3j} y_j^4(t) \\
& - \sum_{j=1}^m \lambda_{3j} y_j^4(t-\tau) + \sum_{i,j=1}^m \lambda_{4ij} V_{ij}(0) y_j^4(t) - \varphi(t) E_5(t) \\
& - \sum_{i,j=1}^m \lambda_{4ij} \int_0^\infty k_{ij}(s) |y_j(t-s)|^4 ds, \tag{3.54}
\end{aligned}$$

or

$$\begin{aligned}
E'(t) \leq & \sum_{i=1}^m \left[-2a_i + 2m \sum_{\ell=1}^5 \delta_\ell + \delta_6 m + \delta_7 m + e^{\alpha\tau} \lambda_{1i} + \sum_{j=1}^m \left(\lambda_{2ji} V_{ji}(0) + \frac{g_i^2(x_i^*)}{\delta_2} L_{1ji}^2 \right. \right. \\
& \left. \left. + \frac{g_i^2(x_i^*)}{\delta_4} L_{2ji}^2 + \frac{1}{\delta_7} \bar{k}_{ji}^2 g_i^2(x_i^*) L_{3ji}^2 + \frac{G_j^2}{\delta_2} b_{ij}^2(x_j^*) \right) \right] |y_i(t)|^2 \\
& + \sum_{j=1}^m \left[\frac{mG_j^4}{4\delta_1} + e^{\theta\tau} \lambda_{3j} + \sum_{i=1}^m \left(\lambda_{4ij} V_{ij}(0) + \left(\frac{1}{4\delta_1} + \frac{1}{4\delta_5} \right) L_{1ij}^4 + \frac{1}{4\delta_3} L_{3ij}^4 \right) \right] |y_j(t)|^4 \\
& + \sum_{j=1}^m \left[\sum_{i=1}^m \frac{G_j^2}{\delta_4} c_{ij}^2(x_j^*) - \lambda_{1j} \right] |y_j(t-\tau)|^2 + \sum_{j=1}^m \left[\frac{mG_j^4}{4\delta_3} - \lambda_{3j} \right] |y_j(t-\tau)|^4 \\
& + \sum_{i,j=1}^m \left[\frac{1}{\delta_6} G_j^2 \bar{k}_{ij} d_{ij}^2(x_j^*) - \lambda_{2ij} \right] \int_0^\infty k_{ij}(s) |y_j(t-s)|^2 ds \\
& + \sum_{i,j=1}^m \left[\frac{G_j^4 \bar{k}_{ij}^3}{4\delta_5} - \lambda_{4ij} \right] \int_0^\infty k_{ij}(s) |y_j(t-s)|^4 ds \\
& - \alpha E_2(t) - \varphi(t) E_3(t) - \theta E_4(t) - \varphi(t) E_5(t), \tag{3.55}
\end{aligned}$$

We need, for some positive constant b_2 ,

$$\left\{ \begin{aligned} & -2a_i + 2m \sum_{\ell=1}^5 \delta_\ell + \delta_6 m + \delta_7 m + e^{\alpha\tau} \lambda_{1i} + \sum_{j=1}^m \left[\lambda_{2ji} V_{ji}(0) + \frac{g_i^2(x_i^*)}{\delta_2} L_{1ji}^2 + \frac{g_i^2(x_i^*)}{\delta_4} L_{2ji}^2 \right. \\ & \left. + \frac{1}{\delta_7} \bar{k}_{ji}^2 g_i^2(x_i^*) L_{3ji}^2 + \frac{G_j^2}{\delta_2} b_{ij}^2(x_j^*) \right] \leq -b_2, \\ & \sum_{i=1}^m \frac{G_j^2}{\delta_4} c_{ij}^2(x_j^*) - \lambda_{1j} \leq 0, \\ & \frac{mG_j^4}{4\delta_3} - \lambda_{3j} \leq 0, \\ & \frac{1}{\delta_6} G_j^2 \bar{k}_{ij} d_{ij}^2(x_j^*) - \lambda_{2ij} \leq 0 \text{ and} \\ & \frac{G_j^4 \bar{k}_{ij}^3}{4\delta_5} - \lambda_{4ij} \leq 0. \end{aligned} \right. \tag{3.56}$$

From the previous inequalities (3.56), it is obvious that the first inequality is guaranteed by our assumption in (3.15).

Let us choose $\delta_r = \frac{1}{20m} a_i$, $r = 1, 2, \dots, 5$ and $\delta_s = \frac{1}{4m} a_i$, $s = 6, 7$. Then, we pick λ_{1ij} , λ_{2ij} , λ_{3j} and λ_{4ij} , such that

$$\begin{cases} \lambda_{1j} = \sum_{j=1}^m \frac{G_j^2}{\delta_4} c_{ij}^2(x_j^*), \\ \lambda_{3j} = \frac{mG_j^4}{4\delta_3}, \\ \lambda_{2ij} = \frac{1}{\delta_6} G_j^2 \bar{k}_{ij} d_{ij}^2(x_j^*) \text{ and} \\ \lambda_{4ij} = \frac{G_j^4 \bar{k}_{ij}^3}{4\delta_5}. \end{cases} \quad (3.57)$$

We end up with

$$\begin{aligned} E'(t) &\leq -b_2 E_1(t) - \alpha E_2(t) - \varphi(t) E_3(t) - \theta E_4(t) - \varphi(t) E_5(t) \\ &\quad + \sum_{j=1}^m \left[\frac{mG_j^4}{4\delta_1} + e^{\theta\tau} \lambda_{3j} + \sum_{i,j=1}^m \left(\lambda_{4ij} V_{ij}(0) + \left(\frac{1}{4\delta_1} + \frac{1}{4\delta_5} \right) L_{1ij}^4 + \frac{1}{4\delta_3} L_{3ij}^4 \right) \right] |y_j(t)|^4 \\ &\leq -b_2 E_1(t) - \alpha E_2(t) - \varphi(t) E_3(t) - \theta E_4(t) - \varphi(t) E_5(t) \\ &\quad + \sum_{j=1}^m \left[\frac{mG_j^4}{4\delta_1} + e^{\theta\tau} \lambda_{3j} + \sum_{i=1}^m \left(\lambda_{4ij} V_{ij}(0) + \left(\frac{1}{4\delta_1} + \frac{1}{4\delta_5} \right) L_{1ij}^4 + \frac{1}{4\delta_3} L_{3ij}^4 \right) \right] E^2(t) \\ &= -b_2 E_1(t) - \alpha E_2(t) - \varphi(t) E_3(t) - \theta E_4(t) - \varphi(t) E_5(t) \\ &\quad + \eta E^2(t), \quad \forall t > 0, \end{aligned} \quad (3.58)$$

where

$$\eta := \sum_{j=1}^m \left[\frac{mG_j^4}{4\delta_1} + e^{\theta\tau} \lambda_{3j} + \sum_{i=1}^m \left(\lambda_{4ij} V_{ij}(0) + \left(\frac{1}{4\delta_1} + \frac{1}{4\delta_5} \right) L_{1ij}^4 + \frac{1}{4\delta_3} L_{3ij}^4 \right) \right]. \quad (3.59)$$

Case 1

If $\lim_{t \rightarrow \infty} \varphi(t) = 0$, there exists $t^* > 0$, such that $\varphi(t) \leq \gamma_1 := \min \{b_2, \alpha, \theta\}$, $\forall t \geq t^*$. Thus, the inequality (3.58) gives

$$E'(t) \leq -\varphi(t) E(t) + \eta E^2(t), \quad \forall t \geq t^* > 0. \quad (3.60)$$

In view of Lemma 2.2.7, we infer that

$$E(t) < \frac{1}{\sigma(t)}, \quad \forall t \geq t^* > 0, \quad (3.61)$$

provided that

$$\eta \leq \frac{\sigma(t)}{2} \left[\varphi(t) - \frac{\sigma'(t)}{\sigma(t)} \right] \quad \text{and} \quad E(0)\sigma(0) < 1. \quad (3.62)$$

Case 2

If $\lim_{t \rightarrow \infty} \varphi(t) = \bar{\varphi} > 0$, then we get

$$\begin{aligned} E'(t) &\leq -b_2 E_1(t) - \alpha E_2(t) - \bar{\varphi} E_3(t) - \theta E_4(t) - \bar{\varphi} E_5(t) + \eta E^2(t) \\ &\leq -\gamma_2 E(t) + \eta E^2(t), \quad \forall t > 0, \end{aligned} \quad (3.63)$$

where $\gamma_2 := \min \{b_2, \alpha, \theta, \bar{\varphi}\}$. Again, in view of Lemma 2.2.7, we obtain

$$E(t) < \frac{1}{\sigma(t)}, \quad \forall t \geq t^* > 0, \quad (3.64)$$

provided that

$$\eta \leq \frac{\sigma(t)}{2} \left[\gamma_2 - \frac{\sigma'(t)}{\sigma(t)} \right] \quad \text{and} \quad E(0)\sigma(0) < 1. \quad (3.65)$$

3.4 Synchronization Result

In this section, we show that the synchronization of the memristive Hopfield neural network system equipped with the initial data given in (P_1) can be inferred from the above stability result. Assume that the system defined in (P_1) is the master (drive) system, then the corresponding slave (response) system is defined as

$$\begin{cases} y_i'(t) = -a_i y_i(t) + \sum_{j=1}^m b_{ij}(y_i(t)) g_j(y_j(t)) + \sum_{j=1}^m c_{ij}(y_i(t)) g_j(y_j(t - \tau)) \\ \quad + \sum_{j=1}^m d_{ij}(y_i(t)) \int_0^\infty k_{ij}(s) g_j(y_j(t - s)) ds + I_i + \kappa_i(t), \quad \forall t > 0, i = 1, \dots, m, \\ y_i(s) = \varphi_i(s), \quad -\infty \leq s \leq 0, \end{cases} \quad (3.66)$$

where $\kappa_i(t)$, for all $t > 0$ and $i = 1, 2, \dots, m$, are the control inputs of the MHNNS.

Now, we introduce an error dynamical system e_i , $i = 1, 2, \dots, m$ by

$$\left\{ \begin{aligned} e_i'(t) = & -a_i e_i(t) + \sum_{j=1}^m b_{ij}(x_i(t))g_j(x_j(t)) - \sum_{j=1}^m b_{ij}(y_i(t))g_j(y_j(t)) \\ & + \sum_{j=1}^m c_{ij}(x_i(t))g_j(x_j(t-\tau)) - \sum_{j=1}^m c_{ij}(y_i(t))g_j(y_j(t-\tau)) \\ & + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s)g_j(x_j(t-s))ds \\ & - \sum_{j=1}^m d_{ij}(y_i(t)) \int_0^\infty k_{ij}(s)g_j(y_j(t-s))ds + \kappa_i(t), \\ & \forall t > 0, i = 1, \dots, m, \end{aligned} \right. \quad (3.67)$$

where the synchronization error is defined as $e_i(t) := y_i(t) - x_i(t)$.

If we select the control inputs as

$$\kappa_i(t) = -C(y_i(t) - x_i(t)), \quad (3.68)$$

for some positive constant C , then the synchronization of the MHNN system can be reduced to the stability matter mentioned above. Actually, this negative feedback adds up with the existing dissipation coefficient a_i , for all $i = 1, 2, \dots, m$, thus the convergence rate is improved.

Chapter 4

MEMRISTIVE HIGH-ORDER COHEN-GROSSBERG BIDIRECTIONAL ASSOCIATIVE MEMORY NEURAL NETWORK SYSTEM (MHOCGBAMNNS)

In this chapter, we will prove an exponential as well as another type of stability for memristive high-order Cohen-Grossberg bidirectional associative memory neural network system containing discrete and distributed delays and with bounded and unbounded activation functions.

4.1 Model Description and Preliminaries

We consider, in this section, a memristive high-order Cohen-Grossberg bidirectional associative memory neural network system (MHOCGBAMNNS) described by the following system, for all $t > 0$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$,

$$\left\{ \begin{array}{l} x'_i(t) = p_i(x_i(t)) \left[-\Lambda_i(x_i(t)) + \sum_{j,l=1}^m a_{jil}(x_i(t)) f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^m b_{jil}(x_i(t)) \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds + I_i \right], \\ y'_j(t) = q_j(y_j(t)) \left[-\bar{\Lambda}_j(y_j(t)) + \sum_{i,r=1}^n \bar{a}_{ijr}(y_j(t)) \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^n \bar{b}_{ijr}(y_j(t)) \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds + J_j \right], \end{array} \right. \quad (P_2)$$

equipped with the following initial data (initial history)

$$\begin{cases} x_i(t) = \phi_i(t), & t \leq 0, \\ y_j(t) = \varphi_j(t), & t \leq 0, \end{cases} \quad (4.1)$$

where n and m are the number of the neurons of the first and the second layer, respectively. For all $t > 0$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, x_i and y_j are the state (solutions) functions of the i^{th} neuron from the neural layer L_x and the j^{th} neuron from the neural layer L_y at time t , respectively, p_i and q_j are the amplification functions of the system, Λ_j and $\bar{\Lambda}_j$ are some appropriate behaved non-negative neuron self-inhibition functions, f_j , g_j , $\bar{f}_i$ and $\bar{g}_i$ indicate to the activation functions (signal transmission functions), I_i and J_j represent the external inputs, τ and θ refer to the discrete delays, k_{jil} , h_{jil} , $\bar{k}_{ijr}$ and $\bar{h}_{ijr}$ are the distributed delay kernel functions, ϕ_i and ψ_j are real-valued bounded continuous functions and the memristive connection weights functions a_{jil} , b_{ijr} , $\bar{a}_{jil}$ and $\bar{b}_{ijr}$ are continuous functions and assumed to satisfy the following conditions

$$a_{jil}(x_i(t)) = \begin{cases} \hat{a}_{jil}, & |x_i(t)| > T_i, \\ \check{a}_{jil}, & |x_i(t)| \leq T_i, \end{cases} \quad (4.2)$$

$$b_{jil}(x_i(t)) = \begin{cases} \hat{b}_{jil}, & |x_i(t)| > T_i, \\ \check{b}_{jil}, & |x_i(t)| \leq T_i, \end{cases} \quad (4.3)$$

$$\bar{a}_{ijr}(y_j(t)) = \begin{cases} \hat{\bar{a}}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ \check{\bar{a}}_{ijr}, & |y_j(t)| \leq \bar{T}_j, \end{cases} \quad (4.4)$$

and

$$\bar{b}_{ijr}(y_j(t)) = \begin{cases} \hat{\bar{b}}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ \check{\bar{b}}_{ijr}, & |y_j(t)| \leq \bar{T}_j, \end{cases} \quad (4.5)$$

for some positive switching jumps constants T_i and T_j and $\hat{a}_{jil}, \check{a}_{jil}, \hat{b}_{jil}, \check{b}_{jil}, \hat{a}_{ijr}, \check{a}_{ijr}, \hat{\bar{b}}_{ijr}$ and $\check{\bar{b}}_{ijr}$ are real numbers.

4.1.1 Assumptions

In this subsection, we introduce some assumptions for our model.

(A₁) The amplification functions p_{1i} and q_{2j} are positive and continuous functions such that

$$0 < \underline{p}_i \leq p_i(x) \leq \bar{p}_i, \quad \forall x \in \mathbb{R} \quad (4.6)$$

and

$$0 < \underline{q}_j \leq q_j(x) \leq \bar{q}_j, \quad \forall x \in \mathbb{R}. \quad (4.7)$$

for some positive constants $\underline{p}_i, \bar{p}_i, \underline{q}_j$ and $\bar{q}_j$.

(A₂) The appropriately behaved self-inhibition (self-signal) functions Λ_i and $\bar{\Lambda}_j$ are monotone and increasing functions such that

$$\frac{\Lambda_i(x) - \Lambda_i(y)}{x - y} \geq \nu_i, \quad \forall x, y \in \mathbb{R}, x \neq y, \quad (4.8)$$

and

$$\frac{\bar{\Lambda}_j(x) - \bar{\Lambda}_j(y)}{x - y} \geq \nu_j \quad \forall x, y \in \mathbb{R}, x \neq y \quad (4.9)$$

for some positive constants ν_i and ν_j .

(A₃) The activation (signal–transmission) functions f_j , g_j , $\bar{f}_i$ and $\bar{g}_i$ are uniformly bounded, i.e.,

$$|f_j(x)| \leq F, \quad \forall x \in \mathbb{R}, \quad (4.10)$$

$$|g_j(x)| \leq G, \quad \forall x \in \mathbb{R}, \quad (4.11)$$

$$|\bar{f}_i(x)| \leq \bar{F}, \quad \forall x \in \mathbb{R}, \quad (4.12)$$

and

$$|\bar{g}_i(x)| \leq \bar{G}, \quad \forall x \in \mathbb{R}, \quad (4.13)$$

for some positive constants F , G , $\bar{F}$, and $\bar{G}$.

(A₄) The signal–propagation functions f_j , g_j , $\bar{f}_i$ and $\bar{g}_i$ are Lipschitz continuous with Lipschitz constants L_{1j} , L_{2j} , M_{1i} , M_{2i} , i.e.,

$$|f_j(x) - f_j(y)| \leq L_{1j}|x - y|, \quad \forall x, y \in \mathbb{R}, \quad (4.14)$$

$$|g_j(x) - g_j(y)| \leq L_{2j}|x - y|, \quad \forall x, y \in \mathbb{R}, \quad (4.15)$$

$$|\bar{f}_i(x) - \bar{f}_i(y)| \leq M_{1i}|x - y|, \quad \forall x, y \in \mathbb{R} \quad (4.16)$$

and

$$|\bar{g}_i(x) - \bar{g}_i(y)| \leq M_{2i}|x - y|, \quad \forall x, y \in \mathbb{R}. \quad (4.17)$$

(A₅) The kernel (relaxation) functions $k_{jil}, h_{jil}, \bar{k}_{ijr}, \bar{h}_{ijr} : [0, \infty) \longrightarrow \mathbb{R}^+, i, j = 1, 2, \dots, m$, for all $t > 0$, are continuous and integrable functions such that

$$\int_0^\infty k_{jil}(s)ds = \hat{k}_{jil}, \quad (4.18)$$

$$\int_0^\infty h_{jil}(s)ds = \hat{h}_{jil}, \quad (4.19)$$

$$\int_0^\infty \bar{k}_{ijr}(s)ds = \hat{\bar{k}}_{ijr}, \quad (4.20)$$

and

$$\int_0^\infty \bar{h}_{ijr}(s)ds = \hat{\bar{h}}_{ijr}. \quad (4.21)$$

(A₆) The initial conditions (initial data) $\phi_i(t)$ and $\psi_j(t)$ for all $t \leq 0$ are continuous and assumed to be in $L^2(-\infty, 0)$.

(A₇) Let $\eta_{\varepsilon jil}, \bar{\eta}_{\varepsilon jil}, \xi_{\varepsilon jil}$ and $\bar{\xi}_{\varepsilon jil}$ be non-negative and continuous functions such that, for $\varepsilon = 1, 2$

$$\lim_{t \rightarrow \infty} \eta_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq i \leq n \\ 1 \leq j, \ell \leq m}} \{\eta_{\varepsilon jil}(t)\} = \eta_\varepsilon^*, \quad (4.22)$$

$$\lim_{t \rightarrow \infty} \xi_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq i \leq n \\ 1 \leq j, \ell \leq m}} \{\xi_{\varepsilon jil}(t)\} = \xi_\varepsilon^*, \quad (4.23)$$

$$\lim_{t \rightarrow \infty} \bar{\eta}_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq j \leq m \\ 1 \leq i, r \leq n}} \{\bar{\eta}_{\varepsilon ijr}(t)\} = \bar{\eta}_\varepsilon^*, \quad (4.24)$$

and

$$\lim_{t \rightarrow \infty} \bar{\xi}_\varepsilon(t) = \lim_{t \rightarrow \infty} \min_{\substack{1 \leq j \leq m \\ 1 \leq i, r \leq n}} \{\bar{\xi}_{\varepsilon ijr}(t)\} = \bar{\xi}_\varepsilon^*, \quad (4.25)$$

such that

$$\eta_{\varepsilon jil}(t) \int_0^\infty k_{\varepsilon jil}(\sigma - s) d\sigma \leq k_{\varepsilon jil}(t - s), \quad 0 \leq s \leq t, \quad (4.26)$$

$$\xi_{\varepsilon jil}(t) \int_0^\infty h_{\varepsilon jil}(\sigma - s) d\sigma \leq h_{\varepsilon jil}(t - s), \quad 0 \leq s \leq t, \quad (4.27)$$

$$\bar{\eta}_{\varepsilon ijr}(t) \int_0^\infty \bar{k}_{\varepsilon ijr}(\sigma - s) d\sigma \leq \bar{k}_{\varepsilon ijr}(t - s), \quad 0 \leq s \leq t, \quad (4.28)$$

and

$$\bar{\xi}_{\varepsilon ijr}(t) \int_0^\infty \bar{h}_{\varepsilon ijr}(\sigma - s) d\sigma \leq \bar{h}_{\varepsilon ijr}(t - s), \quad 0 \leq s \leq t. \quad (4.29)$$

(A₈) Assume that the amplification functions p_i and q_j are Lipschitz continuous functions, i.e.,

$$|p_i(u) - p_i(v)| \leq \rho_i |u - v|, \quad \forall u, v \in \mathbb{R} \quad (4.30)$$

and

$$|q_j(u) - q_j(v)| \leq \bar{\rho}_j |u - v|, \quad \forall u, v \in \mathbb{R}, \quad (4.31)$$

for some positive constants ρ_i and $\bar{\rho}_j$.

(A₉) Assume there exist positive constants ζ_i and $\bar{\zeta}_j$, such that

$$\zeta_{1i} \leq \frac{p_i(u)\Lambda_i(u) - p_i(v)\Lambda_i(v)}{u - v}, \quad \forall u, v \in \mathbb{R}, u \neq v, \quad (4.32)$$

and

$$\bar{\zeta}_j \leq \frac{q_j(u)\bar{\Lambda}_j(u) - q_j(v)\bar{\Lambda}_j(v)}{u - v}, \quad \forall u, v \in \mathbb{R}, u \neq v. \quad (4.33)$$

In view of set-valued maps and differential inclusions theories Clarke, F. H. *et al.*, 1998 [83], Filippov, A. F., 1960 [44] and Filippov, A. F., 2013 [45], we rewrite the system (P_2) as

$$\begin{cases} x'_i(t) \in p_i(x_i(t)) \left[-\Lambda_i(x_i(t)) + \sum_{j,l=1}^m \bar{c}o[a_{jil}(x_i(t))] f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^m \bar{c}o[b_{jil}(x_i(t))] \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds + I_i \right], \\ y'_j(t) \in q_j(y_j(t)) \left[-\bar{\Lambda}_j(y_j(t)) + \sum_{i,r=1}^n \bar{c}o[\bar{a}_{ijr}(y_j(t))] \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^n \bar{c}o[\bar{b}_{ijr}(y_j(t))] \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds + J_j \right], \end{cases} \quad (4.34)$$

where

$$\bar{c}o[a_{jil}(x_i(t))] = \begin{cases} \hat{a}_{jil}, & |x_i(t)| > T_i, \\ [a_{*jil}, a_{*jil}^*], & |x_i(t)| = T_i, \\ \check{a}_{jil}, & |x_i(t)| < T_i, \end{cases} \quad \bar{c}o[b_{jil}(x_i(t))] = \begin{cases} \hat{b}_{jil}, & |x_i(t)| > T_i, \\ [b_{*jil}, b_{*jil}^*], & |x_i(t)| = T_i, \\ \check{b}_{jil}, & |x_i(t)| < T_i, \end{cases}$$

$$\bar{c}o[\bar{a}_{ijr}(y_j(t))] = \begin{cases} \hat{\bar{a}}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ [\bar{a}_{*ijr}, \bar{a}_{*ijr}^*], & |y_j(t)| = \bar{T}_j, \\ \check{\bar{a}}_{ijr}, & |y_j(t)| < \bar{T}_j, \end{cases} \quad \bar{c}o[\bar{b}_{ijr}(y_j(t))] = \begin{cases} \hat{\bar{b}}_{ijr}, & |y_j(t)| > \bar{T}_j, \\ [\bar{b}_{*ijr}, \bar{b}_{*ijr}^*], & |y_j(t)| = \bar{T}_j, \\ \check{\bar{b}}_{ijr}, & |y_j(t)| < \bar{T}_j, \end{cases}$$

with

$$a_{*jil} = \min\{\check{a}_{jil}, \hat{a}_{jil}\}, \quad a_{*jil}^* = \max\{\check{a}_{jil}, \hat{a}_{jil}\}, \quad b_{*jil} = \min\{\check{b}_{jil}, \hat{b}_{jil}\}, \quad b_{*jil}^* = \max\{\check{b}_{jil}, \hat{b}_{jil}\},$$

$$\bar{a}_{*ijr} = \min\{\check{\bar{a}}_{ijr}, \hat{\bar{a}}_{ijr}\}, \quad \bar{a}_{*ijr}^* = \max\{\check{\bar{a}}_{ijr}, \hat{\bar{a}}_{ijr}\}, \quad \bar{b}_{*ijr} = \min\{\check{\bar{b}}_{ijr}, \hat{\bar{b}}_{ijr}\}, \quad \bar{b}_{*ijr}^* = \max\{\check{\bar{b}}_{ijr}, \hat{\bar{b}}_{ijr}\}.$$

Moreover, there exist $\alpha_{jil} \in \bar{co}[a_{jil}(x_i(t))]$, $\beta_{jil} \in \bar{co}[b_{jil}(x_i(t))]$, $\bar{\alpha}_{ijr} \in \bar{co}[\bar{a}_{ijr}(y_j(t))]$, $\bar{\beta}_{ijr} \in \bar{co}[\bar{b}_{ijr}(y_j(t))]$, then, the system (4.34) is equivalent to the following system

$$\begin{cases} x'_i(t) = p_i(x_i(t)) \left[-\Lambda_i(x_i(t)) + \sum_{j,l=1}^m \alpha_{jil} f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^m \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds + I_i \right], \\ y'_j(t) = q_j(y_j(t)) \left[-\bar{\Lambda}_j(y_j(t)) + \sum_{i,r=1}^n \bar{\alpha}_{ijr} \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^n \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds + J_j \right]. \end{cases} \quad (4.35)$$

By setting $\alpha_{jil}^* = \max\{|a_{*jil}|, |a_{jil}^*|\}$, $\beta_{jil}^* = \max\{|b_{*jil}|, |b_{jil}^*|\}$, $\bar{\alpha}_{ijr}^* = \max\{|\bar{a}_{*ijr}|, |\bar{a}_{ijr}^*|\}$, $\bar{\beta}_{ijr}^* = \max\{|\bar{b}_{*ijr}|, |\bar{b}_{ijr}^*|\}$. In what remains of this work, we adopt the notation $\sum_{i,r,j,l=1}^{n,m} = \sum_{i,r=1}^n \sum_{j,l=1}^m$.

4.2 Well-Posedness

In this section, we give the proof of the existence, uniqueness and the continuity of such solutions of the memristive high-order Cohen-Grossberg bidirectional associative memory neural network system (MHOCGBAMNNS) using Kakutanni fixed point theorem for the set-valued maps.

Theorem 4.2.1

Under the assumptions (A_1) – (A_9) , the existence and uniqueness of the equilibrium point of memristive high-order Cohen-Grossberg bidirectional associative memory neural network system given in (P_2) is guaranteed.

Proof

Let $\Omega = \mathbb{R}^{n+m}$ and $w = (w_1, w_2, \dots, w_{n+m})^T = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)^T$ and let $\|w\|_1 = \sum_{i=1}^{n+m} |w_i| = \sum_{i=1}^n |x_i| + \sum_{j=1}^m |y_j|$ be a norm. Thus, Ω defines a Banach space equipped with the norm $\|\cdot\|_1$.

It is clear that the existence and uniqueness of a steady state (equilibrium point) for memristive high-order Cohen–Grossberg bidirectional associative memory neural network system (MHOCGBAM) given in (P_2) is equivalent to the existence and uniqueness of the solution for the following algebraic inclusion system

$$\left\{ \begin{array}{l} x_i \in \Lambda_i^{-1} \left(\sum_{j,l=1}^m \bar{c}o[a_{jil}(x_i)] f_j(y_j) f_l(y_l) + \sum_{j,l=1}^m \bar{c}o[b_{jil}(x_i)] \right. \\ \quad \times \int_0^\infty k_{jil}(s) g_j(y_j) ds \int_0^\infty h_{jil}(s) g_l(y_l) ds + I_i \Big), \quad i = 1, 2, \dots, n, \\ y_j \in \bar{\Lambda}_j^{-1} \left(\sum_{i,r=1}^n \bar{c}o[\bar{a}_{ijr}(y_j)] \bar{f}_i(x_i) \bar{f}_r(x_r) + \sum_{i,r=1}^n \bar{c}o[\bar{b}_{ijr}(y_j)] \right. \\ \quad \times \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r) ds + J_j \Big), \quad j = 1, 2, \dots, m, \end{array} \right. \quad (4.36)$$

or,

$$\left\{ \begin{array}{l} x_i \in \Lambda_i^{-1} \left(\sum_{j,l=1}^m \bar{c}o[a_{jil}(x_i)] f_j(y_j) f_l(y_l) + \sum_{j,l=1}^m \bar{c}o[b_{jil}(x_i)] \right. \\ \quad \times \hat{k}_{jil} \hat{h}_{jil} g_j(y_j) g_l(y_l) + I_i \Big), \quad i = 1, 2, \dots, n, \\ y_j \in \bar{\Lambda}_j^{-1} \left(\sum_{i,r=1}^n \bar{c}o[\bar{a}_{ijr}(y_j)] \bar{f}_i(x_i) \bar{f}_r(x_r) + \sum_{i,r=1}^n \bar{c}o[\bar{b}_{ijr}(y_j)] \right. \\ \quad \times \hat{k}_{ijr} \hat{h}_{ijr} \bar{g}_i(x_i) \bar{g}_r(x_r) + J_j \Big), \quad j = 1, 2, \dots, m. \end{array} \right. \quad (4.37)$$

Let Γ be a compact convex subset $\Gamma \subseteq \Omega$ defined by

$$\Gamma := \left\{ w = (w_1, w_2, \dots, w_{n+m})^T \in \Omega, \quad \|w\|_1 \leq \delta \right\},$$

where

$$\delta := \sum_{i=1}^n \left\{ \frac{F^2}{\nu_i} \sum_{j,l=1}^m \alpha_{jil}^* + \frac{G^2}{\nu_i} \sum_{j,l=1}^m \beta_{jil}^* \hat{k}_{jil} \hat{h}_{jil} + \frac{I_i}{\nu_i} \right\} \\ + \sum_{j=1}^m \left\{ \frac{\bar{F}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\alpha}_{ijr}^* + \frac{\bar{G}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\beta}_{ijr}^* \hat{k}_{ijr} \hat{h}_{ijr} + \frac{J_j}{\bar{\nu}_j} \right\}.$$

It is obvious that Γ is a compact convex subset of the Banach space Ω .

If for all $w \in \Omega$, we define a set-valued map $\psi : \Omega \longrightarrow P(\Omega)$ as

$$\psi(w) = (\psi_1(w), \psi_1(w), \dots, \psi_{n+m}(w))^T,$$

where

$$\left\{ \begin{array}{l} \psi_i(w) = \Lambda_i^{-1} \left(\sum_{j,l=1}^m \bar{c}o[a_{jil}(x_i)] f_j(y_j) f_l(y_l) + \sum_{j,l=1}^m \bar{c}o[b_{jil}(x_i)] \right. \\ \quad \left. \times \hat{k}_{jil} \hat{h}_{jil} g_j(y_j) g_l(y_l) + I_i \right), \quad i = 1, 2, \dots, n, \\ \psi_{n+j}(w) = \bar{\Lambda}_j^{-1} \left(\sum_{i,r=1}^n \bar{c}o[\bar{a}_{ijr}(y_j)] \bar{f}_i(x_i) \bar{f}_r(x_r) + \sum_{i,r=1}^n \bar{c}o[\bar{b}_{ijr}(y_j)] \right. \\ \quad \left. \times \hat{k}_{ijr} \hat{h}_{ijr} \bar{g}_i(x_i) \bar{g}_r(x_r) + J_j \right), \quad j = 1, 2, \dots, m. \end{array} \right. \quad (4.38)$$

It is clear that $\psi(w)$ is an upper semi-continuous map with compact, nonempty convex values.

For all $w \in \Gamma$ and $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_{n+m})^T \in \psi(w)$, then there exist $\alpha_{jil} \in \bar{c}o[a_{jil}(x_i)]$, $\beta_{jil} \in \bar{c}o[b_{jil}(x_i)]$, $\bar{\alpha}_{ijr} \in \bar{c}o[\bar{a}_{ijr}(y_j)]$, $\bar{\beta}_{ijr} \in \bar{c}o[\bar{b}_{ijr}(y_j)]$ such that

$$\left\{ \begin{array}{l} \varpi_i = \Lambda_i^{-1} \left(\sum_{j,l=1}^m \alpha_{jil} f_j(y_j) f_l(y_l) + \sum_{j,l=1}^m \beta_{jil} \right. \\ \quad \left. \times \hat{k}_{jil} \hat{h}_{jil} g_j(y_j) g_l(y_l) + I_i \right) \in \psi_i(w), \quad i = 1, 2, \dots, n, \\ \varpi_{n+j} = \bar{\Lambda}_j^{-1} \left(\sum_{i,r=1}^n \bar{\alpha}_{ijr} \bar{f}_i(x_i) \bar{f}_r(x_r) + \sum_{i,r=1}^n \bar{\beta}_{ijr} \right. \\ \quad \left. \times \hat{k}_{ijr} \hat{h}_{ijr} \bar{g}_i(x_i) \bar{g}_r(x_r) + J_j \right) \in \psi_{n+j}(w), \quad j = 1, 2, \dots, m. \end{array} \right. \quad (4.39)$$

According to the assumptions (A₂) and (A₃) and the relation (4.39), we end up with

$$\left\{ \begin{array}{l} \nu_i |\varpi_i| \leq F^2 \sum_{j,l=1}^m |\alpha_{jil}| + G^2 \sum_{j,l=1}^m |\beta_{jil}| \hat{k}_{jil} \hat{h}_{jil} + I_i, \quad i = 1, 2, \dots, n, \\ \bar{\nu}_j |\varpi_{n+j}| \leq \bar{F}^2 \sum_{i,r=1}^n |\bar{\alpha}_{ijr}| + \bar{G}^2 \sum_{i,r=1}^n |\bar{\beta}_{ijr}| \hat{k}_{ijr} \hat{h}_{ijr} + J_j, \quad j = 1, 2, \dots, m, \end{array} \right. \quad (4.40)$$

or,

$$\begin{cases} |\varpi_i| \leq \frac{F^2}{\nu_i} \sum_{j,l=1}^m \alpha_{jil}^* + \frac{G^2}{\nu_i} \sum_{j,l=1}^m \beta_{jil}^* \hat{k}_{jil} \hat{h}_{jil} + \frac{I_i}{\nu_i}, & i = 1, 2, \dots, n, \\ |\varpi_{n+j}| \leq \frac{\bar{F}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\alpha}_{ijr}^* + \frac{\bar{G}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\beta}_{ijr}^* \hat{k}_{ijr} \hat{h}_{ijr} + \frac{J_j}{\bar{\nu}_j}, & j = 1, 2, \dots, m. \end{cases} \quad (4.41)$$

By adding the sums $\sum_{i=1}^n$ on both sides of the first inequality of (4.41), and $\sum_{j=1}^m$ on both sides of the second inequality of (4.41), we obtain

$$\begin{cases} \sum_{i=1}^n |\varpi_i| \leq \sum_{i=1}^n \left\{ \frac{F^2}{\nu_i} \sum_{j,l=1}^m \alpha_{jil}^* + \frac{G^2}{\nu_i} \sum_{j,l=1}^m \beta_{jil}^* \hat{k}_{jil} \hat{h}_{jil} + \frac{I_i}{\nu_i} \right\}, \\ \sum_{j=1}^m |\varpi_{n+j}| \leq \sum_{j=1}^m \left\{ \frac{\bar{F}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\alpha}_{ijr}^* + \frac{\bar{G}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\beta}_{ijr}^* \hat{k}_{ijr} \hat{h}_{ijr} + \frac{J_j}{\bar{\nu}_j} \right\}. \end{cases} \quad (4.42)$$

By summing the inequalities of (4.42), we conclude

$$\begin{cases} \sum_{i=1}^{n+m} |\varpi_i| = \sum_{i=1}^n |\varpi_i| + \sum_{j=1}^m |\varpi_{n+j}| \leq \sum_{i=1}^n \left\{ \frac{F^2}{\nu_i} \sum_{j,l=1}^m \alpha_{jil}^* + \frac{G^2}{\nu_i} \sum_{j,l=1}^m \beta_{jil}^* \hat{k}_{jil} \hat{h}_{jil} + \frac{I_i}{\nu_i} \right\} \\ + \sum_{j=1}^m \left\{ \frac{\bar{F}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\alpha}_{ijr}^* + \frac{\bar{G}^2}{\bar{\nu}_j} \sum_{i,r=1}^n \bar{\beta}_{ijr}^* \hat{k}_{ijr} \hat{h}_{ijr} + \frac{J_j}{\bar{\nu}_j} \right\} = \delta, \end{cases} \quad (4.43)$$

which implies

$$\|\varpi\|_1 \leq \delta, \quad \forall w \in \Omega. \quad (4.44)$$

Therefore, for all $w \in \Gamma$, $\varpi \in \psi(w)$, we deduce that $\varpi \in \Gamma$. Thus $\psi(w) \in P(\Gamma)$, $\forall w \in \Gamma$ such that $\psi : \Gamma \rightarrow P(\Gamma)$.

This allows us to apply the theorem of Kakutanni–fixed point to ensure the existence and uniqueness of an equilibrium point for the memristive high–order Cohen–Grossberg bidirectional associative memory neural network system (P_2) .

Remark 4.2.1

We note here that for the case of unbounded activation functions, we need to add either $f(0) = 0$ or $f(x) = a|x| + b$, for some constants a and b on the activation functions. For more details, see Wu, A. and Zeng, Z., 2015 [145] and Zhang, L. et al., 2016 [220] and references therein.

4.3 Global Stability

This section includes a global stability of system (P_2) for bounded activation functions. Higher order features are a source of non-linear terms that will be transformed into linear ones using some analytical techniques.

Definition 4.3.1

The point (x^*, y^*) is an equilibrium of the system (P_2) , if

$$\left\{ \begin{array}{l} 0 \in p_i(x_i^*) \left[-\Lambda_i(x_i^*) + \sum_{j,l=1}^m \bar{c}o[a_{jil}(x_i^*)] f_j(y_j^*) f_l(y_l^*) + \sum_{j,l=1}^m \bar{c}o[b_{jil}(x_i^*)] \right. \\ \quad \times \int_0^\infty k_{jil}(s) g_j(y_j^*) ds \int_0^\infty h_{jil}(s) g_l(y_l^*) ds + I_i \Big], \\ 0 \in q_j(y_j^*) \left[-\bar{\Lambda}_j(y_j^*) + \sum_{i,r=1}^n \bar{c}o[\bar{a}_{ijr}(y_j^*)] \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) + \sum_{i,r=1}^n \bar{c}o[\bar{b}_{ijr}(y_j^*)] \right. \\ \quad \times \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i^*) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r^*) ds + J_j \Big], \end{array} \right. \quad (4.45)$$

which is equivalent to the following system

$$\begin{aligned} 0 &= -\Lambda_i(x_i^*) + \sum_{j,l=1}^m \alpha_{jil} f_j(y_j^*) f_l(y_l^*) + \sum_{j,l=1}^m \beta_{jil} \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) + I_i, \\ 0 &= -\bar{\Lambda}_j(y_j^*) + \sum_{i,r=1}^n \bar{\alpha}_{ijr} \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) + \sum_{i,r=1}^n \bar{\beta}_{ijr} \hat{\bar{k}}_{ijr} \bar{g}_i(x_i^*) \hat{\bar{h}}_{ijr} \bar{g}_r(x_r^*) + J_j, \end{aligned} \quad (4.46)$$

where $\alpha_{jil} \in \bar{c}o[a_{jil}(x_i^*)]$, $\beta_{jil} \in \bar{c}o[b_{jil}(x_i^*)]$, $\bar{\alpha}_{ijr} \in \bar{c}o[\bar{a}_{ijr}(y_j^*)]$, $\bar{\beta}_{ijr} \in \bar{c}o[\bar{b}_{ijr}(y_j^*)]$.

The stability of the stationary state of system (4.38) is equivalent to the stability of the

origin state of the system

$$\left\{ \begin{aligned} u'_i(t) &= \tilde{p}_i(u_i(t)) \left[-\tilde{\Lambda}_i(u_i(t)) + \sum_{j,l=1}^m \alpha_{jil} f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ &\quad - \sum_{j,l=1}^m \alpha_{jil} f_j(y_j^*) f_l(y_l^*) \\ &\quad + \sum_{j,l=1}^m \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds \\ &\quad \left. - \sum_{j,l=1}^m \beta_{jil} \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) \right], \\ v'_j(t) &= \tilde{q}_j(v_j(t)) \left[-\tilde{\Lambda}_j(v_j(t)) + \sum_{i,r=1}^n \bar{\alpha}_{ijr} \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\ &\quad - \sum_{i,r=1}^n \bar{\alpha}_{ijr} \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) \\ &\quad + \sum_{i,r=1}^n \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds \\ &\quad \left. - \sum_{i,r=1}^n \bar{\beta}_{ijr} \hat{\bar{k}}_{ijr} \bar{g}_i(x_i^*) \hat{\bar{h}}_{ijr} \bar{g}_r(x_r^*) \right], \end{aligned} \right. \quad (4.47)$$

the initial data are given by

$$\begin{cases} u_i(t) = \tilde{\phi}_i(t) = \phi_i(t) - x_i^*, & t \leq 0, \\ v_j(t) = \tilde{\varphi}_j(t) = \varphi_j(t) - y_j^*, & t \leq 0, \end{cases} \quad (4.48)$$

where

$$\begin{aligned} u_i(t) &= x_i(t) - x_i^*, \quad v_j(t) = y_j(t) - y_j^*, \\ \tilde{p}_i(u_i(t)) &= p_i(u_i(t) + x_i^*), \quad \tilde{q}_j(v_j(t)) = q_j(v_j(t) + y_j^*), \\ \tilde{\Lambda}_i(u_i(t)) &= \Lambda_i(x_i(t)) - \Lambda_i(x_i^*), \quad \tilde{\Lambda}_j(v_j(t)) = \bar{\Lambda}_j(y_j(t)) - \bar{\Lambda}_j(y_j^*). \end{aligned} \quad (4.49)$$

Let A and $\bar{A}$ be positive constants so that

$$\begin{aligned}
A &= \min_{1 \leq i \leq n} \left\{ \underline{p}_i \nu_i - \sum_{r;j=1}^{n;m} \left[(1 + \epsilon_2) \bar{\sigma}_{1ijr} + \frac{\hat{\sigma}_{2ijr}}{1-\delta} + \frac{\hat{\sigma}_{3rji}}{1-\delta} \right] \right\}, \\
\bar{A} &= \min_{1 \leq j \leq m} \left\{ \underline{q}_j \bar{\nu}_j - \sum_{i;l=1}^{n;m} \left[(1 + \epsilon_1) \sigma_{1jil} + \frac{\hat{\sigma}_{2jil}}{1-\delta} + \frac{\hat{\sigma}_{3lij}}{1-\delta} \right] \right\}, \\
\hat{\sigma}_{2jil} &= \sigma_{2jil} \hat{k}_{jil}, \quad \hat{\sigma}_{2ijr} = \bar{\sigma}_{2ijr} \hat{k}_{ijr}, \quad \hat{\sigma}_{3lij} = \sigma_{3lij} \hat{h}_{lij}, \quad \hat{\sigma}_{3rji} = \bar{\sigma}_{3rji} \hat{h}_{rji}, \\
\sigma_{1jil} &= \frac{F^2 m^2 \bar{p}_i^2 L_{1j}^2}{\gamma_1} \left(\alpha_{jil}^* + \alpha_{lij}^* \right)^2, \quad \bar{\sigma}_{1ijr} = \frac{\bar{F}^2 n^2 \bar{q}_j^2 M_{1i}^2}{\bar{\gamma}_1} \left(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^* \right)^2, \\
\sigma_{2jil} &= \frac{2G^2 m^2 \bar{p}_i^2}{\gamma_2} \left(L_{2j} \beta_{jil}^* \hat{h}_{jil} \right)^2 \hat{k}_{jil}, \quad \bar{\sigma}_{2ijr} = \frac{2\bar{G}^2 n^2 \bar{q}_j^2}{\bar{\gamma}_2} \left(M_{2i} \bar{\beta}_{ijr}^* \hat{h}_{ijr} \right)^2 \hat{k}_{ijr}, \\
\sigma_{3lij} &= \frac{2G^2 m^2 \bar{p}_i^2}{\gamma_2} \left(L_{2j} \beta_{lij}^* \hat{k}_{lij} \right)^2 \hat{h}_{lij}, \quad \bar{\sigma}_{3rji} = \frac{2\bar{G}^2 n^2 \bar{q}_j^2}{\bar{\gamma}_2} \left(M_{2i} \bar{\beta}_{rji}^* \hat{k}_{rji} \right)^2 \hat{h}_{rji}, \\
\gamma_1 &= \gamma_2 = \min_{1 \leq i \leq n} \frac{\underline{p}_i \nu_i}{2}, \quad \bar{\gamma}_1 = \bar{\gamma}_2 = \min_{1 \leq j \leq m} \frac{\underline{q}_j \bar{\nu}_j}{2}, \\
\rho(t) &= \inf \{ \eta_1(t), \bar{\eta}_1(t), \xi_1(t), \bar{\xi}_1(t) \}, \quad t \geq 0, \quad \lim_{t \rightarrow \infty} \rho(t) = \rho^*.
\end{aligned} \tag{4.50}$$

The principal theorem determining sufficient conditions for global stability of the neurodynamic system (P_2) is presented below.

Theorem 4.3.1

Assuming that assumptions (A_1) – (A_7) and

$$\begin{aligned}
&\sum_{r;j=1}^{n;m} \left[(1 + \epsilon_2) \bar{\sigma}_{1ijr} + \frac{\hat{\sigma}_{2ijr}}{1-\delta} + \frac{\hat{\sigma}_{3rji}}{1-\delta} \right] \leq \underline{p}_i \nu_i - A, \\
&\sum_{i;l=1}^{n;m} \left[(1 + \epsilon_1) \sigma_{1jil} + \frac{\hat{\sigma}_{2jil}}{1-\delta} + \frac{\hat{\sigma}_{3lij}}{1-\delta} \right] \leq \underline{q}_j \bar{\nu}_j - \bar{A}
\end{aligned}$$

hold, we entail that

- If $0 < \rho^* \leq \infty$, then for some constants $C_1, C_2 > 0$,

$$V(t) \leq C_1 e^{-C_2 t}, \quad t \geq 0.$$

- If $\rho^* = 0$, then for some constant $C_3 > 0$,

$$V(t) \leq C_3 e^{-\delta \int_0^t \rho(s) ds}, \quad t \geq 0.$$

Proof

It is obvious that

$$\begin{aligned}
& \sum_{j,l=1}^m \alpha_{jil} \left[f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) - f_j(y_j^*) f_l(y_l^*) \right] \\
&= \sum_{j,l=1}^m \tilde{f}_j(v_j(t-\theta)) \left[\alpha_{jil} f_l(y_l(t-\theta)) + \alpha_{lij} f_l(y_l^*) \right],
\end{aligned} \tag{4.51}$$

$$\begin{aligned}
& \sum_{i,r=1}^n \bar{\alpha}_{ijr} \left[\bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) - \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) \right] \\
&= \sum_{i,r=1}^n \tilde{\bar{f}}_i(u_i(t-\tau)) \left[\bar{\alpha}_{ijr} \bar{f}_r(x_r(t-\tau)) + \bar{\alpha}_{rji} \bar{f}_r(x_r^*) \right],
\end{aligned} \tag{4.52}$$

$$\begin{aligned}
& \sum_{j,l=1}^m \beta_{jil} \left\{ \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds - \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) \right\} \\
&= \sum_{j,l=1}^m \int_0^\infty \left[\beta_{jil} \left(\int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds \right) k_{jil}(s) + \beta_{lij} \hat{k}_{lij} g_l(y_l^*) h_{lij}(s) \right] \tilde{g}_j(v_j(t-s)) ds,
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
& \sum_{i,r=1}^n \bar{\beta}_{ijr} \left\{ \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds - \hat{\bar{k}}_{ijr} \bar{g}_i(x_i^*) \hat{\bar{h}}_{ijr} \bar{g}_r(x_r^*) \right\} \\
&= \sum_{i,r=1}^n \int_0^\infty \left[\bar{\beta}_{ijr} \left(\int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds \right) \bar{k}_{ijr}(s) + \bar{\beta}_{rji} \hat{\bar{k}}_{rji} \bar{g}_r(x_r^*) \bar{h}_{rji}(s) \right] \tilde{\bar{g}}_i(u_i(t-s)) ds,
\end{aligned} \tag{4.54}$$

where

$$\begin{aligned}
\tilde{f}_j(v_j(t)) &= f_j(y_j(t)) - f_j(y_j^*), & \tilde{\bar{f}}_i(u_i(t)) &= \bar{f}_i(x_i(t)) - \bar{f}_i(x_i^*), \\
\tilde{g}_j(v_j(t)) &= g_j(y_j(t)) - g_j(y_j^*), & \tilde{\bar{g}}_i(u_i(t)) &= \bar{g}_i(x_i(t)) - \bar{g}_i(x_i^*).
\end{aligned}$$

Firstly, we introduce the Lyapunov functional

$$V_1(t) = \sum_{i=1}^n u_i^2(t) + \sum_{j=1}^m v_j^2(t).$$

In view of the relations (4.51)–(4.54) and assumptions (A₁)–(A₅), the derivative of $V_1(t)$ yields

$$\begin{aligned}
V_1'(t) &\leq -2 \sum_{i=1}^n \underline{p}_i \nu_i u_i^2(t) + 2 \sum_{i=1}^n \bar{p}_i |u_i(t)| \sum_{j,l=1}^m FL_{1j}(\alpha_{jil}^* + \alpha_{lij}^*) |v_j(t - \theta)| \\
&+ 2 \sum_{i=1}^n \bar{p}_i |u_i(t)| \sum_{j,l=1}^m \int_0^\infty GL_{2j} \left[\beta_{jil}^* \hat{h}_{jil} k_{jil}(s) + \beta_{lij}^* \hat{k}_{lij} h_{lij}(s) \right] |v_j(t - s)| ds \\
&- 2 \sum_{j=1}^m \underline{q}_j \bar{\nu}_j v_j^2(t) + 2 \sum_{j=1}^m \bar{q}_j |v_j(t)| \sum_{i,r=1}^n \bar{F} M_{1i}(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*) |u_i(t - \tau)| \\
&+ 2 \sum_{j=1}^m \bar{q}_j |v_j(t)| \sum_{i,r=1}^n \int_0^\infty \bar{G} M_{2i} \left[\bar{\beta}_{ijr}^* \hat{h}_{ijr} \bar{k}_{ijr}(s) + \bar{\beta}_{rji}^* \hat{k}_{rji} \bar{h}_{rji}(s) \right] |u_i(t - s)| ds.
\end{aligned} \tag{4.55}$$

Moreover, we have

$$\begin{aligned}
2F\bar{p}_i |u_i(t)| \sum_{j,l=1}^m L_{1j}(\alpha_{jil}^* + \alpha_{lij}^*) |v_j(t - \theta)| &\leq \gamma_1 u_i^2(t) + \frac{F^2 \bar{p}_i^2}{\gamma_1} \left(\sum_{j,l=1}^m L_{1j}(\alpha_{jil}^* + \alpha_{lij}^*) |v_j(t - \theta)| \right)^2 \\
&\leq \gamma_1 u_i^2(t) + \frac{F^2 \bar{p}_i^2 m^2}{\gamma_1} \sum_{j,l=1}^m L_{1j}^2(\alpha_{jil}^* + \alpha_{lij}^*)^2 v_j^2(t - \theta).
\end{aligned} \tag{4.56}$$

In the same way, we get

$$\begin{aligned}
&2\bar{F}\bar{q}_j |v_j(t)| \sum_{i,r=1}^n M_{1i}(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*) |u_i(t - \tau)| \\
&\leq \bar{\gamma}_1 v_j^2(t) + \frac{\bar{F}^2 \bar{q}_j^2 n^2}{\bar{\gamma}_1} \sum_{i,r=1}^n M_{1i}^2(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*)^2 u_i^2(t - \tau),
\end{aligned} \tag{4.57}$$

$$\begin{aligned}
&2\bar{p}_i G |u_i(t)| \sum_{j,l=1}^m \int_0^\infty \left(\beta_{jil}^* \hat{h}_{jil} k_{jil}(s) + \beta_{lij}^* \hat{k}_{lij} h_{lij}(s) \right) L_{2j} |v_j(t - s)| ds \\
&\leq \gamma_2 u_i^2(t) + \frac{2\bar{p}_i^2 G^2 m^2}{\gamma_2} \sum_{j,l=1}^m \left(L_{2j} \beta_{jil}^* \hat{h}_{jil} \right)^2 \hat{k}_{jil} \int_0^\infty k_{jil}(s) v_j^2(t - s) ds \\
&\quad + \frac{2\bar{p}_i^2 G^2 m^2}{\gamma_2} \sum_{j,l=1}^m \left(L_{2j} \beta_{lij}^* \hat{k}_{lij} \right)^2 \hat{h}_{lij} \int_0^\infty h_{lij}(s) v_j^2(t - s) ds
\end{aligned} \tag{4.58}$$

and

$$\begin{aligned}
&2\bar{q}_j \bar{G} |v_j(t)| \sum_{i,r=1}^n \int_0^\infty \left(\bar{\beta}_{ijr}^* \hat{h}_{ijr} \bar{k}_{ijr}(s) + \bar{\beta}_{rji}^* \hat{k}_{rji} \bar{h}_{rji}(s) \right) M_{2i} |u_i(t - s)| ds \\
&\leq \bar{\gamma}_2 v_j^2(t) + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\bar{\gamma}_2} \sum_{i,r=1}^n \left(M_{2i} \bar{\beta}_{ijr}^* \hat{h}_{ijr} \right)^2 \hat{k}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t - s) ds \\
&\quad + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\bar{\gamma}_2} \sum_{i,r=1}^n \left(M_{2i} \bar{\beta}_{rji}^* \hat{k}_{rji} \right)^2 \hat{h}_{rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t - s) ds.
\end{aligned} \tag{4.59}$$

Therefore, from the relations (4.55), (4.56)–(4.59), we find

$$\begin{aligned}
V_1'(t) \leq & \sum_{i=1}^n \left[-2\underline{p}_i \nu_i + \gamma_1 + \gamma_2 \right] u_i^2(t) + \sum_{i,j,l=1}^{n;m} \left[\frac{F^2 \bar{p}_i^2 m^2}{\gamma_1} L_{1j}^2 (\alpha_{jil}^* + \alpha_{lij}^*)^2 v_j^2(t - \theta) \right. \\
& + \frac{2\bar{p}_i^2 G^2 m^2}{\gamma_2} \left(L_{2j} \beta_{jil}^* \hat{h}_{jil} \right)^2 \hat{k}_{jil} \int_0^\infty k_{jil}(s) v_j^2(t - s) ds + \frac{2\bar{p}_i^2 G^2 m^2}{\gamma_2} \left(L_{2j} \beta_{lij}^* \hat{k}_{lij} \right)^2 \hat{h}_{lij} \\
& \times \int_0^\infty h_{lij}(s) v_j^2(t - s) ds \left. \right] + \sum_{j=1}^m \left[-2\underline{q}_j \bar{\nu}_j + \bar{\gamma}_1 + \bar{\gamma}_2 \right] v_j^2(t) + \sum_{i,r;j=1}^{n;m} \left[\frac{\bar{F}^2 \bar{q}_j^2 n^2}{\bar{\gamma}_1} M_{1i}^2 \right. \\
& \times (\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*)^2 u_i^2(t - \tau) + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\bar{\gamma}_2} \left(M_{2i} \bar{\beta}_{ijr}^* \hat{\bar{h}}_{ijr} \right)^2 \hat{\bar{k}}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t - s) ds \\
& \left. + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\bar{\gamma}_2} \left(M_{2i} \bar{\beta}_{rji}^* \hat{\bar{k}}_{rji} \right)^2 \hat{\bar{h}}_{rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t - s) ds \right]. \quad (4.60)
\end{aligned}$$

Next, we construct the following Lyapunov functionals

$$\begin{aligned}
V_2(t) &= e^{-\varepsilon_1 t} \sum_{i,j,l=1}^{n;m} \sigma_{1jil} \int_{t-\theta}^t e^{\varepsilon_1(s+\theta)} v_j^2(s) ds, \quad V_3(t) = e^{-\varepsilon_2 t} \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{1ijr} \int_{t-\tau}^t e^{\varepsilon_2(s+\tau)} u_i^2(s) ds, \\
V_p(t) &= \sum_{j=1}^m \int_{-\infty}^t \left(\int_t^\infty \mathfrak{L}_{pj}(\lambda - s) d\lambda \right) v_j^2(s) ds = \sum_{j=1}^m \int_0^\infty \mathfrak{L}_{pj}(s) \int_{t-s}^t v_j^2(\lambda) d\lambda ds, \quad p = 4, 6, \\
V_q(t) &= \sum_{i=1}^n \int_{-\infty}^t \left(\int_t^\infty \bar{\mathfrak{L}}_{qi}(\lambda - s) d\lambda \right) u_i^2(s) ds = \sum_{i=1}^n \int_0^\infty \bar{\mathfrak{L}}_{qi}(s) \int_{t-s}^t u_i^2(\lambda) d\lambda ds, \quad q = 5, 7, \quad (4.61)
\end{aligned}$$

with ε_1 and ε_2 are picked too small such that $e^{\varepsilon_1 \theta} < 1 + \varepsilon_1$, $e^{\varepsilon_2 \tau} < 1 + \varepsilon_2$, $\mathfrak{L}_{4j}(t) = \sum_{i;l=1}^{n;m} \sigma_{2jil} k_{jil}(t)$, $\mathfrak{L}_{6j}(t) = \sum_{i;l=1}^{n;m} \sigma_{3lij} h_{lij}(t)$, $\bar{\mathfrak{L}}_{5i}(t) = \sum_{r;j=1}^{n;m} \bar{\sigma}_{2ijr} \bar{k}_{ijr}(t)$, $\bar{\mathfrak{L}}_{7i}(t) = \sum_{r;j=1}^{n;m} \bar{\sigma}_{3rji} \bar{h}_{rji}(t)$, for some positive constants ε_1 and ε_2 .

From the assumptions (A₅) and (A₇) on the kernels, the derivatives of $V_i(t)$, $i = 2, 3, \dots, 7$ are given by

$$V_2'(t) = -\varepsilon_1 V_2(t) + e^{\varepsilon_1 \theta} \sum_{i,j,l=1}^{n;m} \sigma_{1jil} v_j^2(t) - \sum_{i,j,l=1}^{n;m} \sigma_{1jil} v_j^2(t - \theta), \quad (4.62)$$

$$V_3'(t) = -\varepsilon_2 V_3(t) + e^{\varepsilon_2 \tau} \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{1ijr} u_i^2(t) - \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{1ijr} u_i^2(t - \tau), \quad (4.63)$$

$$\begin{aligned}
V_4'(t) &= \sum_{i,j,l=1}^{n;m} \sigma_{2jil} \int_0^\infty k_{jil}(s) \left[v_j^2(t) - v_j^2(t - s) \right] ds = \sum_{i,j,l=1}^{n;m} \sigma_{2jil} \hat{k}_{jil} v_j^2(t) \\
&- \sum_{i,j,l=1}^{n;m} \sigma_{2jil} \int_0^\infty k_{jil}(s) v_j^2(t - s) ds = \sum_{i,j,l=1}^{n;m} \hat{\sigma}_{2jil} v_j^2(t) - \sum_{i,j,l=1}^{n;m} \sigma_{2jil} \int_0^\infty k_{jil}(s) v_j^2(t - s) ds,
\end{aligned}$$

or

$$\begin{aligned}
V_4'(t) &= \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{2lij} \int_0^\infty k_{lij}(s) \left[v_j^2(t) - v_j^2(t-s) \right] ds \\
&= \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{2lij} \hat{k}_{lij} v_j^2(t) - \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{2lij} \int_0^\infty k_{lij}(s) v_j^2(t-s) ds \\
&= \sum_{i=1}^n \sum_{j,l=1}^m \hat{\sigma}_{2lij} v_j^2(t) - \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{2lij} \int_0^\infty k_{lij}(s) v_j^2(t-s) ds,
\end{aligned}$$

In similar way, we obtain

$$\begin{aligned}
V_5'(t) &= \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) \left[u_i^2(t) - u_i^2(t-s) \right] ds \\
&= \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \hat{\bar{k}}_{ijr} u_i^2(t) - \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds \\
&= \sum_{j=1}^m \sum_{i,r=1}^n \hat{\bar{\sigma}}_{2ijr} u_i^2(t) - \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds,
\end{aligned}$$

$$\begin{aligned}
V_6'(t) &= \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \int_0^\infty h_{lij}(s) \left[v_j^2(t) - v_j^2(t-s) \right] ds \\
&= \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \hat{h}_{lij} v_j^2(t) - \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \int_0^\infty h_{lij}(s) v_j^2(t-s) ds \\
&= \sum_{i=1}^n \sum_{j,l=1}^m \hat{\sigma}_{3lij} v_j^2(t) - \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \int_0^\infty h_{lij}(s) v_j^2(t-s) ds,
\end{aligned}$$

$$\begin{aligned}
V_7'(t) &= \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \int_0^\infty \bar{h}_{rji}(s) \left[u_i^2(t) - u_i^2(t-s) \right] ds \\
&= \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \hat{\bar{h}}_{rji} u_i^2(t) - \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t-s) ds \\
&= \sum_{j=1}^m \sum_{i,r=1}^n \hat{\bar{\sigma}}_{3rji} u_i^2(t) - \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t-s) ds,
\end{aligned}$$

where

$$\hat{\sigma}_{2jil} = \sigma_{2jil} \hat{k}_{jil}, \quad \hat{\sigma}_{2ijr} = \bar{\sigma}_{2ijr} \hat{\bar{k}}_{ijr}, \quad \hat{\sigma}_{3lij} = \sigma_{3lij} \hat{h}_{lij}, \quad \hat{\sigma}_{3rji} = \bar{\sigma}_{3rji} \hat{\bar{h}}_{rji}.$$

In the same manner, we obtain

$$\begin{aligned}
V'_5(t) &= \sum_{j=1}^m \sum_{i,r=1}^n \hat{\sigma}_{2ijr} u_i^2(t) - \delta \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds \\
&\quad - (1-\delta) \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds \\
&\leq \sum_{j=1}^m \sum_{i,r=1}^n \hat{\sigma}_{2ijr} u_i^2(t) - \delta \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \bar{\eta}_{1ijr}(t) \int_{-\infty}^t \left(\int_t^\infty \bar{k}_{ijr}(\lambda-s) d\lambda \right) u_i^2(s) ds \\
&\quad - (1-\delta) \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds \\
&\leq \sum_{i,r;j=1}^{n;m} \hat{\sigma}_{2ijr} u_i^2(t) - \delta \bar{\eta}_1(t) V_5(t) - (1-\delta) \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds,
\end{aligned}$$

$$\begin{aligned}
V'_6(t) &= \sum_{i=1}^n \sum_{j,l=1}^m \hat{\sigma}_{3lij} v_j^2(t) - \delta \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \int_0^\infty h_{lij}(s) v_j^2(t-s) ds \\
&\quad - (1-\delta) \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \int_0^\infty h_{lij}(s) v_j^2(t-s) ds \\
&\leq \sum_{i=1}^n \sum_{j,l=1}^m \hat{\sigma}_{3lij} v_j^2(t) - \delta \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \xi_{1lij}(t) \int_{-\infty}^t \left(\int_t^\infty h_{lij}(\lambda-s) d\lambda \right) v_j^2(s) ds \\
&\quad - (1-\delta) \sum_{i=1}^n \sum_{j,l=1}^m \sigma_{3lij} \int_0^\infty h_{lij}(s) v_j^2(t-s) ds \\
&\leq \sum_{i,j,l=1}^{n;m} \hat{\sigma}_{3lij} v_j^2(t) - \delta \xi_1(t) V_6(t) - (1-\delta) \sum_{i,j,l=1}^{n;m} \sigma_{3lij} \int_0^\infty h_{lij}(s) v_j^2(t-s) ds,
\end{aligned}$$

$$\begin{aligned}
V'_7(t) &= \sum_{j=1}^m \sum_{i,r=1}^n \hat{\sigma}_{3rji} u_i^2(t) - \delta \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t-s) ds \\
&\quad - (1-\delta) \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t-s) ds \\
&\leq \sum_{j=1}^m \sum_{i,r=1}^n \hat{\sigma}_{3rji} u_i^2(t) - \delta \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \bar{\xi}_{1rji}(t) \int_{-\infty}^t \left(\int_t^\infty \bar{h}_{rji}(\lambda-s) d\lambda \right) u_i^2(s) ds \\
&\quad - (1-\delta) \sum_{j=1}^m \sum_{i,r=1}^n \bar{\sigma}_{3rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t-s) ds \\
&\leq \sum_{i,r;j=1}^{n;m} \hat{\sigma}_{3rji} u_i^2(t) - \delta \bar{\xi}_1(t) V_7(t) - (1-\delta) \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{3rji} \int_0^\infty \bar{h}_{rji}(s) u_i^2(t-s) ds.
\end{aligned}$$

$$\begin{aligned}
V'_p(t) &\leq \sum_{j=1}^m \kappa_{pj} v_j^2(t) - \delta \mathfrak{R}_p(t) V_p(t) - (1-\delta) \sum_{j=1}^m \int_0^\infty \mathfrak{L}_{pj}(s) v_j^2(t-s) ds, \quad p = 4, 6, \\
V'_q(t) &\leq \sum_{i=1}^n \bar{\kappa}_{qi} u_i^2(t) - \delta \mathfrak{R}_q(t) V_q(t) - (1-\delta) \sum_{i=1}^n \int_0^\infty \bar{\mathfrak{L}}_{qi}(s) u_i^2(t-s) ds, \quad q = 5, 7,
\end{aligned} \tag{4.64}$$

with $0 < \delta < 1$, $\kappa_{4j} = \sum_{i;l=1}^{n;m} \hat{\sigma}_{2ijl}$, $\kappa_{6j} = \sum_{i;l=1}^{n;m} \hat{\sigma}_{3lij}$, $\bar{\kappa}_{5i} = \sum_{r;j=1}^{n;m} \hat{\sigma}_{2ijr}$, $\bar{\kappa}_{7i} = \sum_{r;j=1}^{n;m} \hat{\sigma}_{3rji}$, $\mathfrak{R}_4(t) = \eta_1(t)$, $\mathfrak{R}_6(t) = \xi_1(t)$, $\bar{\mathfrak{R}}_5(t) = \bar{\eta}_1(t)$, $\bar{\mathfrak{R}}_7(t) = \bar{\xi}_1(t)$.

Then, we present the Lyapunov functional

$$V(t) = \sum_{l=1}^3 V_l(t) + \frac{1}{1-\delta} \sum_{k=4}^7 V_k(t). \quad (4.65)$$

Summing up the estimations (4.60)–(4.64) and taking into account the conditions set out in Theorem 4.3.1, we end up with

$$\begin{aligned} V'(t) \leq & \sum_{i=1}^n \left\{ -\underline{p}_i \nu_i + \sum_{r;j=1}^{n;m} \left[(1 + \epsilon_2) \bar{\sigma}_{1ijr} + \frac{\hat{\sigma}_{2ijr}}{1-\delta} + \frac{\hat{\sigma}_{3rji}}{1-\delta} \right] \right\} u_i^2(t) \\ & + \sum_{j=1}^m \left\{ -\underline{q}_j \bar{\nu}_j + \sum_{i;l=1}^{n;m} \left[(1 + \epsilon_1) \sigma_{1jil} + \frac{\hat{\sigma}_{2jil}}{1-\delta} + \frac{\hat{\sigma}_{3lij}}{1-\delta} \right] \right\} v_j^2(t) \\ & - \varepsilon_1 V_2(t) - \varepsilon_2 V_3(t) - \frac{\delta \rho(t)}{1-\delta} \sum_{k=4}^7 V_k(t), \end{aligned} \quad (4.66)$$

or

$$V'(t) \leq -BV_1(t) - \varepsilon_1 V_2(t) - \varepsilon_2 V_3(t) - \frac{\delta \rho(t)}{1-\delta} \sum_{k=4}^7 V_k(t), \quad (4.67)$$

where $B = \min\{A, \bar{A}\}$.

In the last part, via two cases based on the limit of the function $\rho(t)$, the derivative of the functional $V(t)$ is examined.

Case 1

If $0 < \rho^* < \infty$, then, there exists $t^* > 0$ so that

$$\rho(t) \geq \rho^*, \quad \forall t \geq t^*.$$

Thus, we have the estimation

$$V'(t) \leq -BV_1(t) - \varepsilon_1 V_2(t) - \varepsilon_2 V_3(t) - \frac{\delta \rho^*}{(1-\delta)} \sum_{k=4}^7 V_k(t), \quad t \geq t^*. \quad (4.68)$$

Let $C = \min \left\{ B, \varepsilon_1, \varepsilon_2, \frac{\delta \rho^*}{(1-\delta)} \right\}$, then

$$V'(t) \leq -CV(t), \quad t \geq t^*, \quad (4.69)$$

which implies that

$$V(t) \leq V(t^*)e^{-C(t-t^*)} \leq V(0)e^{-C(t-t^*)}, \quad t \geq t^*. \quad (4.70)$$

For $\rho^* = \infty$, there exist $t^* > 0$ and $\hat{\rho} > 0$ such that $\rho(t) \geq \hat{\rho}$, $t \geq t^*$, and we proceed with a similar argument. Furthermore, for $t \in [0, t^*]$,

$$V(t) \leq V(0) \leq V(0)e^{C(t^*-t)} \leq V(0)e^{-C(t-t^*)} \leq \left(V(0)e^{Ct^*}\right)e^{-Ct}.$$

Case 2

If $\rho^* = 0$, then, we can find $\bar{t} > 0$ such that

$$\frac{\delta}{(1-\delta)}\rho(t) \leq \min\{B, \varepsilon_1, \varepsilon_2\}, \quad t \geq \bar{t}. \quad (4.71)$$

Hence, the estimation (4.68) may be written as

$$V'(t) \leq -\delta^*\rho(t)V(t), \quad t \geq \bar{t}, \quad (4.72)$$

for some $\delta^* := \frac{\delta}{(1-\delta)} > 0$. This implies that

$$V(t) \leq V(\bar{t})e^{-\delta^* \int_{\bar{t}}^{\infty} \rho(s)ds} \leq V(0)e^{-\delta^* \int_{\bar{t}}^{\infty} \rho(s)ds}, \quad t \geq \bar{t}. \quad (4.73)$$

For $0 \leq t \leq \bar{t}$, we obtain

$$V(t) \leq V(0) \leq V(0)e^{\delta^* \int_t^{\bar{t}} \rho(s)ds} \leq V(0)e^{-\delta^* \int_t^{\bar{t}} \rho(s)ds}.$$

In the rest of this work, the system (4.45) satisfied by the equilibrium point (x^*, y^*) is equivalent to

$$\begin{aligned} 0 &= -\Lambda_i(x_i^*) + \sum_{j,l=1}^m \alpha_{jil}^\diamond f_j(y_j^*) f_l(y_l^*) + \sum_{j,l=1}^m \beta_{jil}^\diamond \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) + I_i, \\ 0 &= -\bar{\Lambda}_j(y_j^*) + \sum_{i,r=1}^n \bar{\alpha}_{ijr}^\diamond \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) + \sum_{i,r=1}^n \bar{\beta}_{ijr}^\diamond \hat{k}_{ijr} \bar{g}_i(x_i^*) \hat{h}_{ijr} \bar{g}_r(x_r^*) + J_j, \end{aligned} \quad (4.74)$$

where $\alpha_{jil}^\diamond \in \bar{co}[a_{jil}(x_i^*)]$, $\beta_{jil}^\diamond \in \bar{co}[b_{jil}(x_i^*)]$, $\bar{\alpha}_{ijr}^\diamond \in \bar{co}[\bar{a}_{ijr}(y_j^*)]$, $\bar{\beta}_{ijr}^\diamond \in \bar{co}[\bar{b}_{ijr}(y_j^*)]$.

4.4 Uniform Energy Estimation

In this section, we will discuss the uniform energy estimation for the neurodynamic system (P_2) through a result in Pata, V., 2000 [80].

To shift the equilibrium point to the origin, we use the transformation

$$u_i(t) = x_i(t) - x_i^*, \quad v_j(t) = y_j(t) - y_j^*,$$

then, the system (P_2) may be rewritten as

$$\left\{ \begin{array}{l} u'_i(t) = \tilde{p}_i(u_i(t)) \left[-\tilde{\Lambda}_i(u_i(t)) + \sum_{j,l=1}^m \alpha_{jil} f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ \quad - \sum_{j,l=1}^m \alpha_{jil}^\diamond f_j(y_j^*) f_l(y_l^*) \\ \quad + \sum_{j,l=1}^m \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds \\ \quad \left. - \sum_{j,l=1}^m \beta_{jil}^\diamond \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) \right], \\ v'_j(t) = \tilde{q}_j(v_j(t)) \left[-\tilde{\Lambda}_j(v_j(t)) + \sum_{i,r=1}^n \bar{\alpha}_{ijr} \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) - \right. \\ \quad \sum_{i,r=1}^n \bar{\alpha}_{ijr}^\diamond \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) \\ \quad + \sum_{i,r=1}^n \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds \\ \quad \left. - \sum_{i,r=1}^n \bar{\beta}_{ijr}^\diamond \hat{\bar{k}}_{ijr} \bar{g}_i(x_i^*) \hat{\bar{h}}_{ijr} \bar{g}_r(x_r^*) \right], \end{array} \right. \quad (4.75)$$

with the initial data

$$\left\{ \begin{array}{l} u_i(t) = \tilde{\phi}_i(t) = \phi_i(t) - x_i^*, \quad t \leq 0, \\ v_j(t) = \tilde{\varphi}_j(t) = \varphi_j(t) - y_j^*, \quad t \leq 0, \end{array} \right. \quad (4.76)$$

where

$$\begin{aligned} \tilde{p}_i(u_i(t)) &= p_i(u_i(t) + x_i^*), \quad \tilde{q}_j(v_j(t)) = q_j(v_j(t) + y_j^*), \\ \tilde{\Lambda}_i(u_i(t)) &= \Lambda_i(x_i(t)) - \Lambda_i(x_i^*), \quad \tilde{\Lambda}_j(v_j(t)) = \bar{\Lambda}_j(y_j(t)) - \bar{\Lambda}_j(y_j^*). \end{aligned} \quad (4.77)$$

Setting ϑ and $\bar{\vartheta}$ be positive constants such that

$$\begin{aligned}
\vartheta &= \min_{1 \leq i \leq n} \left\{ \underline{p}_i \nu_i - \sum_{r;j=1}^{n;m} \left[(1 + \varepsilon_2) \bar{\lambda}_{1ijr} + \frac{\hat{\lambda}_{2ijr}}{1-\sigma} + \frac{\hat{\lambda}_{3rji}}{1-\sigma} \right] \right\}, \\
\bar{\vartheta} &= \min_{1 \leq j \leq m} \left\{ \underline{q}_j \bar{\nu}_j - \sum_{i;l=1}^{n;m} \left[(1 + \varepsilon_1) \lambda_{1jil} + \frac{\hat{\lambda}_{2jil}}{1-\sigma} + \frac{\hat{\lambda}_{3lij}}{1-\sigma} \right] \right\}, \\
\hat{\lambda}_{2jil} &= \lambda_{2jil} \hat{k}_{jil}, \quad \hat{\lambda}_{2ijr} = \bar{\lambda}_{2ijr} \hat{k}_{ijr}, \quad \hat{\lambda}_{3lij} = \lambda_{3lij} \hat{h}_{lij}, \quad \hat{\lambda}_{3rji} = \bar{\lambda}_{3rji} \hat{h}_{rji}, \\
\lambda_{1jil} &= \frac{F^2 m^2 \bar{p}_i^2 L_{1j}^2}{\gamma_1} (\alpha_{jil}^* + \alpha_{lij}^*)^2, \quad \bar{\lambda}_{1ijr} = \frac{\bar{F}^2 n^2 \bar{q}_j^2 M_{1i}^2}{\bar{\gamma}_1} (\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*)^2, \\
\lambda_{2jil} &= \frac{2G^2 m^2 \bar{p}_i^2}{\gamma_2} (L_{2j} \beta_{jil}^* \hat{h}_{jil})^2 \hat{k}_{jil}, \quad \bar{\lambda}_{2ijr} = \frac{2\bar{G}^2 n^2 \bar{q}_j^2}{\bar{\gamma}_2} (M_{2i} \bar{\beta}_{ijr}^* \hat{h}_{ijr})^2 \hat{k}_{ijr}, \\
\lambda_{3lij} &= \frac{2G^2 m^2 \bar{p}_i^2}{\gamma_2} (L_{2j} \beta_{lij}^* \hat{k}_{lij})^2 \hat{h}_{lij}, \quad \bar{\lambda}_{3rji} = \frac{2\bar{G}^2 n^2 \bar{q}_j^2}{\bar{\gamma}_2} (M_{2i} \bar{\beta}_{rji}^* \hat{k}_{rji})^2 \hat{h}_{rji}, \\
\gamma_1 &= \gamma_2 = \gamma_3 = \gamma_4 = \min_{1 \leq i \leq n} \frac{\underline{p}_i \nu_i}{4}, \quad \bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = \bar{\gamma}_4 = \min_{1 \leq j \leq m} \frac{\underline{q}_j \bar{\nu}_j}{4}, \\
\omega(t) &= \inf\{\eta_1(t), \bar{\eta}_1(t), \xi_1(t), \bar{\xi}_1(t)\}, \quad t \geq 0, \quad \lim_{t \rightarrow \infty} \omega(t) = \omega^*, \quad \omega^* > 0.
\end{aligned} \tag{4.78}$$

The next outcome presents sufficient conditions depending on the system parameter that ensure the uniform estimation of solutions of system (P_2) .

Theorem 4.4.1

Under the assumptions $(A_1)-(A_7)$ and

$$\begin{aligned}
\sum_{r;j=1}^{n;m} \left[(1 + \varepsilon_2) \bar{\lambda}_{1ijr} + \frac{\hat{\lambda}_{2ijr}}{1-\sigma} + \frac{\hat{\lambda}_{3rji}}{1-\sigma} \right] &\leq \underline{p}_i \nu_i - \vartheta, \\
\sum_{i;l=1}^{n;m} \left[(1 + \varepsilon_1) \lambda_{1jil} + \frac{\hat{\lambda}_{2jil}}{1-\sigma} + \frac{\hat{\lambda}_{3lij}}{1-\sigma} \right] &\leq \underline{q}_j \bar{\nu}_j - \bar{\vartheta},
\end{aligned} \tag{4.79}$$

we entail that there exists $T > 0$ and for some constants $\mathfrak{C}_i > 0$, $i = 1, 2, 3$

$$u(t) \leq (\mathfrak{C}_1 - \mathfrak{C}_3) e^{-\mathfrak{C}_2(t-T)} + \mathfrak{C}_3, \quad v(t) \leq (\mathfrak{C}_1 - \mathfrak{C}_3) e^{-\mathfrak{C}_2(t-T)} + \mathfrak{C}_3, \quad t \geq T, \tag{4.80}$$

where $u(t) = \sum_{i=1}^n |u_i(t)|$, $v(t) = \sum_{j=1}^m |v_j(t)|$.

Proof

Obviously, we have

$$\begin{aligned}
&\sum_{j,l=1}^m \left[\alpha_{jil} f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) - \alpha_{jil}^\diamond f_j(y_j^*) f_l(y_l^*) \right] \\
&= \sum_{j,l=1}^m \left\{ \tilde{f}_j(v_j(t-\theta)) \left[\alpha_{jil} f_l(y_l(t-\theta)) + \alpha_{lij}^\diamond f_l(y_l^*) \right] + (\alpha_{jil} - \alpha_{jil}^\diamond) f_j(y_j^*) f_l(y_l(t-\theta)) \right\},
\end{aligned} \tag{4.81}$$

$$\begin{aligned}
& \sum_{i,r=1}^n \left[\bar{\alpha}_{ijr} \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) - \bar{\alpha}_{ijr}^\diamond \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) \right] \\
= & \sum_{i,r=1}^n \left\{ \tilde{f}_i(u_i(t-\tau)) \left[\bar{\alpha}_{ijr} \bar{f}_r(x_r(t-\tau)) + \bar{\alpha}_{rji}^\diamond \bar{f}_r(x_r^*) \right] + (\bar{\alpha}_{ijr} - \bar{\alpha}_{ijr}^\diamond) \bar{f}_i(x_i^*) \bar{f}_r(x_r(t-\tau)) \right\},
\end{aligned} \tag{4.82}$$

$$\begin{aligned}
& \sum_{j,l=1}^m \left\{ \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds - \beta_{jil}^\diamond \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) \right\} \\
= & \sum_{j,l=1}^m \left\{ \int_0^\infty \left[\beta_{jil} \left(\int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds \right) k_{jil}(s) + \beta_{lij}^\diamond \hat{k}_{lij} g_l(y_l^*) h_{lij}(s) \right] \tilde{g}_j(v_j(t-s)) ds \right. \\
& \left. + (\beta_{jil} - \beta_{jil}^\diamond) \hat{k}_{jil} g_j(y_j^*) \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds \right\},
\end{aligned} \tag{4.83}$$

$$\begin{aligned}
& \sum_{i,r=1}^n \left\{ \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds - \bar{\beta}_{ijr}^\diamond \hat{\bar{k}}_{ijr} \bar{g}_i(x_i^*) \hat{\bar{h}}_{ijr} \bar{g}_r(x_r^*) \right\} \\
= & \sum_{i,r=1}^n \left\{ \int_0^\infty \left[\bar{\beta}_{ijr} \left(\int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds \right) \bar{k}_{ijr}(s) + \bar{\beta}_{rji}^\diamond \hat{\bar{k}}_{rji} \bar{g}_r(x_r^*) \bar{h}_{rji}(s) \right] \tilde{g}_i(u_i(t-s)) ds \right. \\
& \left. + (\bar{\beta}_{ijr} - \bar{\beta}_{ijr}^\diamond) \hat{\bar{k}}_{ijr} \bar{g}_i(x_i^*) \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds \right\},
\end{aligned} \tag{4.84}$$

with

$$\begin{aligned}
\tilde{f}_j(v_j(t)) &= f_j(y_j(t)) - f_j(y_j^*), & \tilde{\bar{f}}_i(u_i(t)) &= \bar{f}_i(x_i(t)) - \bar{f}_i(x_i^*), \\
\tilde{g}_j(v_j(t)) &= g_j(y_j(t)) - g_j(y_j^*), & \tilde{\bar{g}}_i(u_i(t)) &= \bar{g}_i(x_i(t)) - \bar{g}_i(x_i^*).
\end{aligned} \tag{4.85}$$

Firstly, we present the Lyapunov functional

$$E_1(t) = \sum_{i=1}^n u_i^2(t) + \sum_{j=1}^m v_j^2(t). \tag{4.86}$$

By virtue of the estimations (4.81)–(4.84) and assumptions (A1)–(A5), we obtain

$$\begin{aligned}
E'_1(t) \leq & -2 \sum_{i=1}^n \underline{p}_i \nu_i u_i^2(t) + 2 \sum_{i=1}^n \bar{p}_i |u_i(t)| \sum_{j,l=1}^m FL_{1j}(\alpha_{jil}^* + \alpha_{lij}^*) |v_j(t - \theta)| \\
& + 2F^2 \sum_{i=1}^n \bar{p}_i |u_i(t)| \sum_{j,l=1}^m |\check{a}_{jil} - \hat{a}_{jil}| + 2 \sum_{i=1}^n \bar{p}_i |u_i(t)| \sum_{j,l=1}^m \int_0^\infty GL_{2j} \left[\beta_{jil}^* \hat{h}_{jil} k_{jil}(s) \right. \\
& \left. + \beta_{lij}^* \hat{k}_{lij} h_{lij}(s) \right] |v_j(t - s)| ds + 2G^2 \sum_{i=1}^n \bar{p}_i |u_i(t)| \sum_{j,l=1}^m \hat{k}_{jil} \hat{h}_{jil} |\check{b}_{jil} - \hat{b}_{jil}| \\
& - 2 \sum_{j=1}^m \underline{q}_j \bar{\nu}_j v_j^2(t) + 2 \sum_{j=1}^m \bar{q}_j |v_j(t)| \sum_{i,r=1}^n \bar{F} M_{1i}(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*) |u_i(t - \tau)| \\
& + 2\bar{F}^2 \sum_{j=1}^m \bar{q}_j |v_j(t)| \sum_{i,r=1}^n |\check{a}_{ijr} - \hat{a}_{ijr}| + 2 \sum_{j=1}^m \bar{q}_j |v_j(t)| \sum_{i,r=1}^n \int_0^\infty \bar{G} M_{2i} \left[\bar{\beta}_{ijr}^* \hat{h}_{ijr} \bar{k}_{ijr}(s) \right. \\
& \left. + \bar{\beta}_{rji}^* \hat{k}_{rji} \bar{h}_{rji}(s) \right] |u_i(t - s)| ds + 2\bar{G}^2 \sum_{j=1}^m \bar{q}_j |v_j(t)| \sum_{i,r=1}^n \hat{k}_{ijr} \hat{h}_{ijr} |\check{b}_{ijr} - \hat{b}_{ijr}|.
\end{aligned} \tag{4.87}$$

Besides, we find

$$2F^2 \bar{p}_i |u_i(t)| \sum_{j,l=1}^m |\check{a}_{jil} - \hat{a}_{jil}| \leq \gamma_3 u_i^2(t) + \frac{m^2 F^4 \bar{p}_i^2}{\gamma_3} \sum_{j,l=1}^m (\check{a}_{jil} - \hat{a}_{jil})^2, \tag{4.88}$$

$$2\bar{F}^2 \bar{q}_j |v_j(t)| \sum_{i,r=1}^n |\check{a}_{ijr} - \hat{a}_{ijr}| \leq \bar{\gamma}_3 v_j^2(t) + \frac{n^2 \bar{F}^4 \bar{q}_j^2}{\bar{\gamma}_3} \sum_{i,r=1}^n (\check{a}_{ijr} - \hat{a}_{ijr})^2, \tag{4.89}$$

$$2G^2 \bar{p}_i |u_i(t)| \sum_{j,l=1}^m \hat{k}_{jil} \hat{h}_{jil} |\check{b}_{jil} - \hat{b}_{jil}| \leq \gamma_4 u_i^2(t) + \frac{m^2 G^4 \bar{p}_i^2}{\gamma_4} \sum_{j,l=1}^m \hat{k}_{jil}^2 \hat{h}_{jil}^2 (\check{b}_{jil} - \hat{b}_{jil})^2, \tag{4.90}$$

$$2\bar{G}^2 \bar{q}_j |v_j(t)| \sum_{i,r=1}^n \hat{k}_{ijr} \hat{h}_{ijr} |\check{b}_{ijr} - \hat{b}_{ijr}| \leq \bar{\gamma}_4 v_j^2(t) + \frac{n^2 \bar{G}^4 \bar{q}_j^2}{\bar{\gamma}_4} \sum_{i,r=1}^n \hat{k}_{ijr}^2 \hat{h}_{ijr}^2 (\check{b}_{ijr} - \hat{b}_{ijr})^2. \tag{4.91}$$

According to the relations (4.56)–(4.59) and (4.88)–(4.91), (4.87) yields

$$\begin{aligned}
E'_1(t) \leq & \sum_{i=1}^n \left[-2\underline{p}_i \nu_i + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \right] u_i^2(t) + \sum_{i,j,l=1}^{n;m} \left[\frac{F^2 \bar{p}_i^2 m^2}{\gamma_1} L_{1j}^2 (\alpha_{jil}^* + \alpha_{lij}^*)^2 v_j^2(t - \theta) \right. \\
& + \frac{2\bar{p}_i^2 G^2 m^2}{\gamma_2} \left(L_{2j} \beta_{jil}^* \hat{h}_{jil} \right)^2 \hat{k}_{jil} \int_0^\infty k_{jil}(s) v_j^2(t - s) ds + \frac{2\bar{p}_i^2 G^2 m^2}{\gamma_2} \left(L_{2j} \beta_{lij}^* \hat{k}_{lij} \right)^2 \hat{h}_{lij} \int_0^\infty h_{lij}(s) \\
& \times v_j^2(t - s) ds \left. + \sum_{j=1}^m \left[-2\underline{q}_j \bar{\nu}_j + \bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_4 \right] v_j^2(t) + \sum_{i,r;j=1}^{n;m} \left[\frac{\bar{F}^2 \bar{q}_j^2 n^2}{\bar{\gamma}_1} M_{1i}^2 (\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*)^2 \right. \right. \\
& \times u_i^2(t - \tau) + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\bar{\gamma}_2} \left(M_{2i} \bar{\beta}_{ijr}^* \hat{h}_{ijr} \right)^2 \hat{k}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t - s) ds + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\bar{\gamma}_2} \left(M_{2i} \bar{\beta}_{rji}^* \hat{k}_{rji} \right)^2 \hat{h}_{rji} \\
& \left. \left. \times \int_0^\infty \bar{h}_{rji}(s) u_i^2(t - s) ds \right] \right] + C,
\end{aligned} \tag{4.92}$$

with

$$\begin{aligned}
C = & \frac{m^2 F^4 \bar{p}_i^2}{\gamma_3} \sum_{j,l=1}^m (\check{a}_{jil} - \hat{a}_{jil})^2 + \frac{n^2 \bar{F}^4 \bar{q}_j^2}{\bar{\gamma}_3} \sum_{i,r=1}^n (\check{\bar{a}}_{ijr} - \hat{\bar{a}}_{ijr})^2 \\
& + \frac{m^2 G^4 \bar{p}_i^2}{\gamma_4} \sum_{j,l=1}^m \hat{k}_{jil}^2 \hat{h}_{jil}^2 (\check{b}_{jil} - \hat{b}_{jil})^2 + \frac{n^2 \bar{G}^4 \bar{q}_j^2}{\bar{\gamma}_4} \sum_{i,r=1}^n \hat{k}_{ijr}^2 \hat{h}_{ijr}^2 (\check{\bar{b}}_{ijr} - \hat{\bar{b}}_{ijr})^2.
\end{aligned} \tag{4.93}$$

Next, we define the Lyapunov functionals

$$\begin{aligned}
E_2(t) &= e^{-\kappa_1 t} \sum_{i,j,l=1}^{n;m} \lambda_{1jil} \int_{t-\theta}^t e^{\kappa_1(s+\theta)} v_j^2(t) ds, \quad E_3(t) = e^{-\kappa_2 t} \sum_{i,r;j=1}^{n;m} \bar{\lambda}_{1ijr} \int_{t-\tau}^t e^{\kappa_2(s+\tau)} u_i^2(t) ds, \\
E_p(t) &= \sum_{j=1}^m \int_{-\infty}^t \left(\int_0^\infty \mathfrak{K}_{pj}(\mu - s) d\mu \right) v_j^2(s) ds = \sum_{j=1}^m \int_0^\infty \mathfrak{K}_{pj}(s) \int_{t-s}^t v_j^2(\mu) d\mu ds, \quad p = 4, 6, \\
E_q(t) &= \sum_{i=1}^n \int_{-\infty}^t \left(\int_0^\infty \bar{\mathfrak{K}}_{qi}(\mu - s) d\mu \right) u_i^2(s) ds = \sum_{i=1}^n \int_0^\infty \bar{\mathfrak{K}}_{qi}(s) \int_{t-s}^t u_i^2(\mu) d\mu ds, \quad q = 5, 7,
\end{aligned} \tag{4.94}$$

with κ_1 and κ_2 are chosen so small that $e^{\kappa_1 \theta} < 1 + \varepsilon_1$, $e^{\kappa_2 \tau} < 1 + \varepsilon_2$, $\mathfrak{K}_{4j}(t) = \sum_{i;l=1}^{n;m} \lambda_{2jil} k_{jil}(t)$, $\mathfrak{K}_{6j}(t) = \sum_{i;l=1}^{n;m} \lambda_{3lij} h_{lij}(t)$, $\bar{\mathfrak{K}}_{5i}(t) = \sum_{r;j=1}^{n;m} \bar{\lambda}_{2ijr} \bar{k}_{ijr}(t)$, $\bar{\mathfrak{K}}_{7i}(t) = \sum_{r;j=1}^{n;m} \bar{\lambda}_{3rji} \bar{h}_{rji}(t)$.

In view of the assumptions (A₅) and (A₇), the derivatives of $E_s(t)$, $s = 2, 3, \dots, 7$ are given by

$$E'_2(t) = -\kappa_1 E_2(t) + e^{\kappa_1 \theta} \sum_{i;j,l=1}^{n;m} \lambda_{1jil} v_j^2(t) - \sum_{i;j,l=1}^{n;m} \lambda_{1jil} v_j^2(t - \theta), \tag{4.95}$$

$$E'_3(t) = -\kappa_2 E_3(t) + e^{\kappa_2 \tau} \sum_{i,r;j=1}^{n;m} \bar{\lambda}_{1ijr} u_i^2(t) - \sum_{i,r;j=1}^{n;m} \bar{\lambda}_{1ijr} u_i^2(t - \tau), \tag{4.96}$$

$$\begin{aligned}
E'_p(t) &\leq \sum_{j=1}^m \mathfrak{M}_{pj} v_j^2(t) - \sigma \mathfrak{N}_p(t) E_p(t) - (1 - \sigma) \sum_{j=1}^m \int_0^\infty \mathfrak{K}_{pj}(s) v_j^2(t - s) ds, \quad p = 4, 6, \\
E'_q(t) &\leq \sum_{i=1}^n \bar{\mathfrak{M}}_{qi} u_i^2(t) - \sigma \bar{\mathfrak{N}}_q(t) E_q(t) - (1 - \sigma) \sum_{i=1}^n \int_0^\infty \bar{\mathfrak{K}}_{qi}(s) u_i^2(t - s) ds, \quad q = 5, 7,
\end{aligned} \tag{4.97}$$

with $0 < \sigma < 1$, $\mathfrak{M}_{4j} = \sum_{i;l=1}^{n;m} \hat{\lambda}_{2jil}$, $\mathfrak{M}_{6j} = \sum_{i;l=1}^{n;m} \hat{\lambda}_{3lij}$, $\bar{\mathfrak{M}}_{5i} = \sum_{r;j=1}^{n;m} \hat{\lambda}_{2ijr}$, $\bar{\mathfrak{M}}_{7i} = \sum_{r;j=1}^{n;m} \hat{\lambda}_{3rji}$, $\mathfrak{N}_4(t) = \eta_1(t)$, $\mathfrak{N}_6(t) = \xi_1(t)$, $\bar{\mathfrak{N}}_5(t) = \bar{\eta}_1(t)$, $\bar{\mathfrak{N}}_7(t) = \bar{\xi}_1(t)$.

Finally, we introduce the Lyapunov functional

$$E(t) = \sum_{l=1}^3 E_l(t) + \frac{1}{1-\sigma} \sum_{k=4}^7 E_k(t). \quad (4.98)$$

According to the estimates (4.92)–(4.97) and conditions outlined in Theorem 4.4.1, we have

$$\begin{aligned} E'(t) &\leq \sum_{i=1}^n \left\{ -\underline{p}_i \nu_i + \sum_{r;j=1}^{n;m} \left[(1+\varepsilon_2) \bar{\lambda}_{1ijr} + \frac{\hat{\lambda}_{2ijr}}{1-\sigma} + \frac{\hat{\lambda}_{3rji}}{1-\sigma} \right] \right\} u_i^2(t) \\ &\quad + \sum_{j=1}^m \left\{ -\underline{q}_j \bar{\nu}_j + \sum_{i;l=1}^{n;m} \left[(1+\varepsilon_1) \lambda_{1jil} + \frac{\hat{\lambda}_{2jil}}{1-\sigma} + \frac{\hat{\lambda}_{3lij}}{1-\sigma} \right] \right\} v_j^2(t) \\ &\quad - \kappa_1 E_2(t) - \kappa_2 E_3(t) - \frac{\sigma \omega(t)}{1-\sigma} \sum_{k=4}^7 E_k(t) + C, \end{aligned} \quad (4.99)$$

or

$$E'(t) \leq -\Lambda E_1(t) - \kappa_1 E_2(t) - \kappa_2 E_3(t) - \frac{\sigma \omega(t)}{1-\sigma} \sum_{k=4}^7 E_k(t) + C, \quad (4.100)$$

with $\Lambda = \min \{ \vartheta, \bar{\vartheta} \}$.

As $0 < \omega^* < \infty$, then there exists $T > 0$ so that

$$\omega(t) \geq \omega^*, \quad \forall t \geq T.$$

Hence, the estimation (4.100) yields

$$E'(t) \leq -\Lambda E_1(t) - \kappa_1 E_2(t) - \kappa_2 E_3(t) - \frac{\sigma \omega^*}{(1-\sigma)} \sum_{k=4}^7 E_k(t) + C, \quad t \geq T. \quad (4.101)$$

Let $\hat{C} = \min \left\{ \Lambda, \kappa_1, \kappa_2, \frac{\sigma \omega^*}{(1-\sigma)} \right\}$, then

$$E'(t) \leq -\hat{C} E(t) + C, \quad t \geq T, \quad (4.102)$$

From Lemma 2.2.6, we obtain

$$E(t) \leq 2E(T)e^{-\hat{C}(t-T)} + 2C \int_T^t e^{-\hat{C}(t-s)} ds, \quad t \in [T, \infty).$$

For $\omega^* = \infty$, there exist $T > 0$ and $\hat{\omega} > 0$ such that $\omega(t) \geq \hat{\omega}$, $t \geq T$, and we use the same argument.

Remark 4.4.1

- *The assumptions used here are not as constraining as those of (H_2) in Wen, S., et al., 2012 [123] and (A_8) in Zhou, Y., et al., 2018 [110], which was utilized to study the stability of the stationary point for memristive neurodynamic systems.*
- *High-order characteristics led to non-linear terms that are harder to process than linear ones deriving from first order features. To overcome this problem, analytical techniques are utilized to convert them into linear terms excluding the condition $f(0) = 0$ proposed in Gen, H., et al., 2011 [86] and in Ren, F. and Cao, J., 2006 [137].*
- *The stability result obtained here is more general than those in Wen, S., et al., 2012 [123], Sheng, Y., et al. 2020 [88] and Wu, A. and Zeng, Z., 2014 et al. [89]. Moreover, the global stability derived in and Wu, A. and Zeng, Z., 2014 [90] is enhanced here by processing a model that encompasses the continuous history and by using Lyapunov's direct method only.*

Remark 4.4.2

- *If $\rho^* = \infty$. Then, there exist $t^* > 0$ and a positive constant $\hat{\rho}$ such that*

$$\rho(t) \geq \hat{\rho}, t \geq t^*.$$

- *The estimations on $[0, t^*]$ and $[0, \bar{t}]$ are obtained by using the continuity and decreasingness of the functional $V(t)$.*

4.5 Unbounded Activation Functions

In the case of unbounded activation functions, we will discuss a stability result of system (P_2) which is generally more challenging to handle, due to non-linearities stemming from high-order terms. As a remedy, we shall use a first order inequality given in Hoang, N. S. and Ramm, A. G., 2009 [1].

Set A and $\bar{A}$ be positive constants so that

$$\begin{aligned}
A &= \min_{1 \leq i \leq n} \left\{ \underline{p}_i \nu_i - \sum_{r;j=1}^{n;m} \left[(1 + \epsilon_2) \bar{\sigma}_{1ijr} + \frac{\hat{\sigma}_{2ijr}}{1-\delta} + \frac{\hat{\sigma}_{3rji}}{1-\delta} \right] \right\}, \\
\bar{A} &= \min_{1 \leq j \leq m} \left\{ \underline{q}_j \bar{\nu}_j - \sum_{i;l=1}^{n;m} \left[(1 + \epsilon_1) \sigma_{1jil} + \frac{\hat{\sigma}_{2jil}}{1-\delta} + \frac{\hat{\sigma}_{3lij}}{1-\delta} \right] \right\}, \\
\hat{\sigma}_{2jil} &= \sigma_{2jil} \hat{k}_{jil}, \quad \hat{\sigma}_{3lij} = \sigma_{3lij} \hat{h}_{lij}, \quad \hat{\sigma}_{2ijr} = \bar{\sigma}_{2ijr} \hat{k}_{ijr}, \quad \hat{\sigma}_{3rji} = \bar{\sigma}_{3rji} \hat{h}_{rji}, \\
\hat{\sigma}_{5jil} &= \sigma_{5jil} \hat{k}_{jil}, \quad \hat{\sigma}_{6lij} = \sigma_{6lij} \hat{h}_{lij}, \quad \hat{\sigma}_{5ijr} = \bar{\sigma}_{5ijr} \hat{k}_{ijr}, \quad \hat{\sigma}_{6rji} = \bar{\sigma}_{6rji} \hat{h}_{rji}, \\
\bar{\sigma}_{1ijr} &= \frac{\bar{q}_j^2 M_{1i}^2 \bar{f}_r^2(x_r^*)}{\bar{\gamma}_2} \left(\bar{\alpha}_{ijr}^{*2} + \bar{\alpha}_{rji}^{*2} \right), \quad \sigma_{1jil} = \frac{\bar{p}_i^2 L_{1j}^2 f_l^2(y_l^*)}{\gamma_2} \left(\alpha_{jil}^{*2} + \alpha_{lij}^{*2} \right), \\
\bar{\sigma}_{2ijr} &= \frac{\left(\bar{q}_j \bar{\beta}_{ijr}^* M_{2i} \hat{h}_{ijr} \bar{g}_r(x_r^*) \right)^2}{\bar{\gamma}_4} \hat{k}_{ijr}, \quad \sigma_{2jil} = \frac{\left(\bar{p}_i \beta_{jil}^* L_{2j} \hat{h}_{jil} g_l(y_l^*) \right)^2}{\gamma_4} \hat{k}_{jil}, \\
\bar{\sigma}_{3rji} &= \frac{\left(\bar{q}_j \bar{\beta}_{rji}^* \hat{k}_{rji} \bar{g}_r(x_r^*) M_{2i} \right)^2}{\bar{\gamma}_4} \hat{h}_{rji}, \quad \sigma_{3lij} = \frac{\left(\bar{p}_i \beta_{lij}^* \hat{k}_{lij} g_l(y_l^*) L_{2j} \right)^2}{\gamma_4} \hat{h}_{lij}, \\
\bar{\sigma}_{4ijr} &= \frac{\bar{q}_j^2 M_{1i}^4}{4\bar{\gamma}_1} \left(\bar{\alpha}_{ijr}^{*2} + \bar{\alpha}_{rji}^{*2} \right), \quad \sigma_{4jil} = \frac{\bar{p}_i^2 L_{1j}^4}{4\gamma_1} \left(\alpha_{jil}^{*2} + \alpha_{lij}^{*2} \right), \quad \bar{\sigma}_{5ijr} = \frac{\bar{q}_j^2 \bar{\beta}_{ijr}^{*2} M_{2i}^4 \hat{k}_{ijr}^3}{\bar{\gamma}_3}, \\
\sigma_{5jil} &= \frac{\bar{p}_i^2 \beta_{jil}^{*2} L_{2j}^4 \hat{k}_{jil}^3}{\gamma_3}, \quad \bar{\sigma}_{6rji} = \frac{\bar{q}_j^2 \bar{\beta}_{rji}^{*2} M_{2i}^4 \hat{h}_{rji}^3}{\bar{\gamma}_3}, \quad \sigma_{6lij} = \frac{\bar{p}_i^2 \beta_{lij}^{*2} L_{2j}^4 \hat{h}_{lij}^3}{\gamma_3}, \\
\chi(t) &= \inf \left\{ \eta_1(t), \bar{\eta}_1(t), \xi_1(t), \bar{\xi}_1(t), \eta_2(t), \bar{\eta}_2(t), \xi_2(t), \bar{\xi}_2(t) \right\}, \quad \forall t \geq 0, \quad \lim_{t \rightarrow \infty} \chi(t) = \chi^*, \\
\gamma_1 &= \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \min_{1 \leq i \leq n} \frac{\underline{p}_i \nu_i}{10m^2}, \\
\bar{\gamma}_1 &= \bar{\gamma}_2 = \bar{\gamma}_3 = \bar{\gamma}_4 = \bar{\gamma}_5 = \min_{1 \leq j \leq m} \frac{\underline{q}_j \bar{\nu}_j}{10n^2}.
\end{aligned}$$

Now, a sufficient criteria for the local stability of the equilibrium point of system (P_2) is inferred.

Theorem 4.5.1

Let assumptions that $(A_1), (A_2), (A_4)-(A_7)$ and

$$\begin{aligned}
\sum_{r;j=1}^{n;m} \left[(1 + \epsilon_2) \bar{\sigma}_{1ijr} + \frac{\hat{\sigma}_{2ijr}}{1-\delta} + \frac{\hat{\sigma}_{3rji}}{1-\delta} \right] &\leq \underline{p}_i \nu_i - A, \\
\sum_{i;l=1}^{n;m} \left[(1 + \epsilon_1) \sigma_{1jil} + \frac{\hat{\sigma}_{2jil}}{1-\delta} + \frac{\hat{\sigma}_{3lij}}{1-\delta} \right] &\leq \underline{q}_j \bar{\nu}_j - \bar{A}
\end{aligned} \tag{4.103}$$

hold. The assertions below are true

- If $0 < \chi^* \leq \infty$ and there exist $t^* > 0$ and a positive function $\omega(t)$ with

$$\begin{aligned}
\lim_{t \rightarrow \infty} \omega(t) &= \infty \text{ such that for some positive constants } C, \hat{C}, \quad C \leq \frac{\omega(t)}{2} \left(\chi(t) - \frac{\omega'(t)}{\omega(t)} \right), \quad C \leq \\
&\frac{1}{2\omega(t)} \left(\hat{C} - \frac{\omega'(t)}{\omega(t)} \right), \text{ then}
\end{aligned}$$

$$u(t) < \frac{1}{\omega(t)}, \quad v(t) < \frac{1}{\omega(t)}, \quad \text{for all } t \geq t^*.$$

- If $\chi^* = 0$ and there exists $\bar{t} > 0$ and a positive function $\bar{\omega}(t)$ with $\lim_{t \rightarrow \infty} \bar{\omega}(t) = \infty$ such that $\tilde{C} \leq \frac{\bar{\omega}(t)}{2} \left(\bar{C} - \frac{\bar{\omega}'(t)}{\bar{\omega}(t)} \right)$, $\tilde{C} \leq \frac{1}{2\bar{\omega}(t)} \left(\check{C} - \frac{\bar{\omega}'(t)}{\bar{\omega}(t)} \right)$ holds for some positive constants $\bar{C}, \bar{C}, \check{C}$, then,

$$u(t) < \frac{1}{\bar{\omega}(t)}, \quad v(t) < \frac{1}{\bar{\omega}(t)}, \quad \text{for all } t \geq \bar{t}.$$

Proof

Firstly, using the assumption (A₄), we obtain the following estimations

$$\begin{aligned} & \left[\alpha_{jil} f_j(y_j(t - \theta)) f_l(y_l(t - \theta)) - \alpha_{jil}^\diamond f_j(y_j^*) f_l(y_l^*) \right] = \alpha_{jil} \left[f_j(y_j(t - \theta)) - f_j(y_j^*) \right] \\ & \quad \times \left[f_l(y_l(t - \theta)) - f_l(y_l^*) \right] + \alpha_{jil} f_l(y_l^*) \left[f_j(y_j(t - \theta)) - f_j(y_j^*) \right] + \alpha_{jil} f_j(y_j^*) \\ & \quad \times \left[f_l(y_l(t - \theta)) - f_l(y_l^*) \right] + (\alpha_{jil} - \alpha_{jil}^\diamond) f_j(y_j^*) f_l(y_l^*) \\ & \leq \alpha_{jil}^* L_{1j} L_{1l} |v_j(t - \theta)| |v_l(t - \theta)| + \alpha_{jil}^* L_{1j} |f_l(y_l^*)| |v_j(t - \theta)| \\ & \quad + \alpha_{jil}^* L_{1l} |f_j(y_j^*)| |v_l(t - \theta)| + |\check{\alpha}_{jil} - \hat{\alpha}_{jil}| |f_j(y_j^*)| |f_l(y_l^*)|, \end{aligned} \quad (4.104)$$

and

$$\begin{aligned} & \left[\bar{\alpha}_{ijr} \bar{f}_i(x_i(t - \tau)) \bar{f}_r(x_r(t - \tau)) - \bar{\alpha}_{ijr}^\diamond \bar{f}_i(x_i^*) \bar{f}_r(x_r^*) \right] \\ & \leq \bar{\alpha}_{ijr}^* M_{1i} M_{1r} |u_i(t - \tau)| |u_r(t - \tau)| + \bar{\alpha}_{ijr}^* M_{1i} |\bar{f}_r(x_r^*)| |u_i(t - \tau)| \\ & \quad + \bar{\alpha}_{ijr}^* M_{1r} |\bar{f}_i(x_i^*)| |u_r(t - \tau)| + |\check{\bar{\alpha}}_{ijr} - \hat{\bar{\alpha}}_{ijr}| |\bar{f}_i(x_i^*)| |\bar{f}_r(x_r^*)|. \end{aligned} \quad (4.105)$$

Moreover, we have

$$\begin{aligned} & 2\bar{p}_i |u_i(t)| \sum_{j,l=1}^m \alpha_{jil}^* L_{1j} L_{1l} |v_j(t - \theta)| |v_l(t - \theta)| \\ & \leq |u_i(t)| \sum_{j,l=1}^m \left(\alpha_{jil}^* \bar{p}_i L_{1j}^2 v_j^2(t - \theta) + \alpha_{jil}^* \bar{p}_i L_{1l}^2 v_l^2(t - \theta) \right) \\ & \leq 2\gamma_1 m^2 u_i^2(t) + \sum_{j,l=1}^m \frac{\alpha_{jil}^{*2} \bar{p}_i^2 L_{1j}^4}{4\gamma_1} v_j^4(t - \theta) + \sum_{j,l=1}^m \frac{\alpha_{jil}^{*2} \bar{p}_i^2 L_{1l}^4}{4\gamma_1} v_l^4(t - \theta), \end{aligned} \quad (4.106)$$

in the same way, we get

$$\begin{aligned} & 2\bar{q}_j |v_j(t)| \sum_{i,r=1}^n \bar{\alpha}_{ijr}^* M_{1i} M_{1r} |u_i(t - \tau)| |u_r(t - \tau)| \\ & \leq 2\gamma_1 n^2 v_j^2(t) + \sum_{i,r=1}^n \frac{\bar{\alpha}_{ijr}^{*2} \bar{q}_j^2 M_{1i}^4}{4\gamma_1} u_i^4(t - \tau) + \sum_{i,r=1}^n \frac{\bar{\alpha}_{ijr}^{*2} \bar{q}_j^2 M_{1r}^4}{4\gamma_1} u_r^4(t - \tau), \end{aligned} \quad (4.107)$$

$$2\bar{p}_i|u_i(t)| \sum_{j,l=1}^m \alpha_{jil}^* L_{1j} |f_l(y_l^*)| |v_j(t-\theta)| \leq \gamma_2 m^2 u_i^2(t) + \sum_{j,l=1}^m \frac{\alpha_{jil}^{*2} \bar{p}_i^2 L_{1j}^2 f_l^2(y_l^*)}{\gamma_2} v_j^2(t-\theta), \quad (4.108)$$

$$2\bar{p}_i|u_i(t)| \sum_{j,l=1}^m \alpha_{jil}^* L_{1l} |f_j(y_j^*)| |v_l(t-\theta)| \leq \gamma_2 m^2 u_i^2(t) + \sum_{j,l=1}^m \frac{\alpha_{jil}^{*2} \bar{p}_i^2 L_{1l}^2 f_j^2(y_j^*)}{\gamma_2} v_l^2(t-\theta), \quad (4.109)$$

$$2\bar{q}_j|v_j(t)| \sum_{i,r=1}^n \bar{\alpha}_{ijr}^* M_{1i} |\bar{f}_r(x_r^*)| |u_i(t-\tau)| \leq \bar{\gamma}_2 n^2 v_j^2(t) + \sum_{i,r=1}^n \frac{\bar{\alpha}_{ijr}^{*2} \bar{q}_j^2 M_{1i}^2 \bar{f}_r^2(x_r^*)}{\bar{\gamma}_2} u_i^2(t-\tau), \quad (4.110)$$

$$2\bar{q}_j|v_j(t)| \sum_{i,r=1}^n \bar{\alpha}_{ijr}^* M_{1r} |\bar{f}_i(x_i^*)| |u_r(t-\tau)| \leq \bar{\gamma}_2 n^2 v_j^2(t) + \sum_{i,r=1}^n \frac{\bar{\alpha}_{ijr}^{*2} \bar{q}_j^2 M_{1r}^2 \bar{f}_i^2(x_i^*)}{\bar{\gamma}_2} u_r^2(t-\tau), \quad (4.111)$$

$$2\bar{p}_i|u_i(t)| \sum_{j,l=1}^m |f_j(y_j^*)| |f_l(y_l^*)| |\check{a}_{jil} - \hat{a}_{jil}| \leq 2\gamma_3 m^2 u_i^2(t) + \sum_{j,l=1}^m \frac{\bar{p}_i^2 f_j^2(y_j^*) f_l^2(y_l^*)}{2\gamma_3} (\check{a}_{jil} - \hat{a}_{jil})^2 \quad (4.112)$$

and

$$2\bar{q}_j|v_j(t)| \sum_{i,r=1}^n |\bar{f}_i(x_i^*)| |\bar{f}_r(x_r^*)| |\check{a}_{ijr} - \hat{a}_{ijr}| \leq 2\bar{\gamma}_3 n^2 v_j^2(t) + \sum_{i,r=1}^n \frac{\bar{q}_j^2 \bar{f}_i^2(x_i^*) \bar{f}_r^2(x_r^*)}{2\bar{\gamma}_3} (\check{a}_{ijr} - \hat{a}_{ijr})^2. \quad (4.113)$$

Furthermore,

$$\begin{aligned} & \sum_{j,l=1}^m \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds - \sum_{j,l=1}^m \beta_{jil}^\diamond \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) \\ &= \sum_{j,l=1}^m \beta_{jil} \left\{ \int_0^\infty k_{jil}(s) [g_j(y_j(t-s)) - g_j(y_j^*)] ds \int_0^\infty h_{jil}(s) [g_l(y_l(t-s)) - g_l(y_l^*)] ds \right. \\ & \quad + \hat{h}_{jil} g_l(y_l^*) \int_0^\infty k_{jil}(s) [g_j(y_j(t-s)) - g_j(y_j^*)] ds + \hat{k}_{jil} g_j(y_j^*) \int_0^\infty h_{jil}(s) \\ & \quad \times [g_l(y_l(t-s)) - g_l(y_l^*)] ds \Big\} + \sum_{j,l=1}^m (\beta_{jil} - \beta_{jil}^\diamond) \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) \end{aligned} \quad (4.114)$$

and

$$\begin{aligned}
& \sum_{i,r=1}^n \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds - \sum_{i,r=1}^n \bar{\beta}_{ijr}^\diamond \hat{k}_{ijr} \bar{g}_i(x_i^*) \hat{h}_{ijr} \bar{g}_r(x_r^*) \\
&= \sum_{i,r=1}^n \bar{\beta}_{ijr} \left\{ \int_0^\infty \bar{k}_{ijr}(s) [\bar{g}_i(x_i(t-s)) - \bar{g}_i(x_i^*)] ds \int_0^\infty \bar{h}_{ijr}(s) [\bar{g}_r(x_r(t-s)) - \bar{g}_r(x_r^*)] ds \right. \\
&\quad + \hat{h}_{ijr} \bar{g}_r(x_r^*) \int_0^\infty \bar{k}_{ijr}(s) [\bar{g}_i(x_i(t-s)) - \bar{g}_i(x_i^*)] ds + \hat{k}_{ijr} \bar{g}_i(x_i^*) \int_0^\infty \bar{h}_{ijr}(s) \\
&\quad \times [\bar{g}_r(x_r(t-s)) - \bar{g}_r(x_r^*)] ds \left. \right\} + \sum_{i,r=1}^n (\bar{\beta}_{ijr} - \bar{\beta}_{ijr}^\diamond) \hat{k}_{ijr} \bar{g}_i(x_i^*) \hat{h}_{ijr} \bar{g}_r(x_r^*).
\end{aligned} \tag{4.115}$$

By virtue of the assumption (A₄), one has

$$\begin{aligned}
& \sum_{j,l=1}^m \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds - \sum_{j,l=1}^m \beta_{jil}^\diamond \hat{k}_{jil} g_j(y_j^*) \hat{h}_{jil} g_l(y_l^*) \\
&\leq \sum_{j,l=1}^m \beta_{jil}^* L_{2j} L_{2l} \int_0^\infty k_{jil}(s) |v_j(t-s)| ds \int_0^\infty h_{jil}(s) |v_l(t-s)| ds + \sum_{j,l=1}^m \beta_{jil}^* L_{2j} \hat{h}_{jil} g_l(y_l^*) \\
&\quad \times \int_0^\infty k_{jil}(s) |v_j(t-s)| ds + \sum_{j,l=1}^m \beta_{jil}^* L_{2l} \hat{k}_{jil} g_j(y_j^*) \int_0^\infty h_{jil}(s) |v_l(t-s)| ds \\
&\quad + \sum_{j,l=1}^m |\check{\beta}_{jil} - \hat{\beta}_{jil}| \hat{k}_{jil} |g_j(y_j^*)| \hat{h}_{jil} |g_l(y_l^*)|
\end{aligned} \tag{4.116}$$

and

$$\begin{aligned}
& \sum_{i,r=1}^n \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds - \sum_{i,r=1}^n \bar{\beta}_{ijr}^\diamond \hat{k}_{ijr} \bar{g}_i(x_i^*) \hat{h}_{ijr} \bar{g}_r(x_r^*) \\
&\leq \sum_{i,r=1}^n \bar{\beta}_{ijr}^* M_{2i} M_{2r} \int_0^\infty \bar{k}_{ijr}(s) |u_i(t-s)| ds \int_0^\infty \bar{h}_{ijr}(s) |u_r(t-s)| ds + \sum_{i,r=1}^n \bar{\beta}_{ijr}^* M_{2i} \hat{h}_{ijr} \bar{g}_r(x_r^*) \\
&\quad \times \int_0^\infty \bar{k}_{ijr}(s) |u_i(t-s)| ds + \sum_{i,r=1}^n \bar{\beta}_{ijr}^* M_{2r} \hat{k}_{ijr} \bar{g}_i(x_i^*) \int_0^\infty \bar{h}_{ijr}(s) |u_r(t-s)| ds \\
&\quad + \sum_{i,r=1}^n |\check{\beta}_{ijr} - \hat{\beta}_{ijr}| \hat{k}_{ijr} |\bar{g}_i(x_i^*)| \hat{h}_{ijr} |\bar{g}_r(x_r^*)|.
\end{aligned} \tag{4.117}$$

On the other hand, from (A₅), we find

$$\begin{aligned}
& 2\bar{p}_i|u_i(t)| \sum_{j,l=1}^m \beta_{jil}^* L_{2j} L_{2l} \int_0^\infty k_{jil}(s) |v_j(t-s)| ds \int_0^\infty h_{jil}(s) |v_l(t-s)| ds \\
& \leq |u_i(t)| \left(\sum_{j,l=1}^m \beta_{jil}^* \bar{p}_i L_{2j}^2 \hat{k}_{jil} \int_0^\infty k_{jil}(s) v_j^2(t-s) ds + \sum_{j,l=1}^m \beta_{jil}^* \bar{p}_i L_{2l}^2 \hat{h}_{jil} \int_0^\infty h_{jil}(s) v_l^2(t-s) ds \right) \\
& \leq 2\gamma_3 m^2 u_i^2(t) + \sum_{j,l=1}^m \frac{\beta_{jil}^* \bar{p}_i^2 L_{2j}^4 \hat{k}_{jil}^3}{4\gamma_3} \int_0^\infty k_{jil}(s) v_j^4(t-s) ds + \sum_{j,l=1}^m \frac{\beta_{jil}^* \bar{p}_i^2 L_{2l}^4 \hat{h}_{jil}^3}{4\gamma_3} \int_0^\infty h_{jil}(s) v_l^4(t-s) ds,
\end{aligned} \tag{4.118}$$

$$\begin{aligned}
& 2\bar{q}_j|v_j(t)| \sum_{i,r=1}^n \bar{\beta}_{ijr}^* M_{2i} M_{2r} \int_0^\infty \bar{k}_{ijr}(s) |u_i(t-s)| ds \int_0^\infty \bar{h}_{ijr}(s) |u_r(t-s)| ds \\
& \leq |v_j(t)| \left(\sum_{i,r=1}^n \bar{\beta}_{ijr}^* \bar{q}_j M_{2i}^2 \hat{k}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds + \sum_{i,r=1}^n \bar{\beta}_{ijr}^* \bar{q}_j M_{2r}^2 \hat{h}_{ijr} \int_0^\infty \bar{h}_{ijr}(s) u_r^2(t-s) ds \right) \\
& \leq 2\gamma_3 n^2 v_j^2(t) + \sum_{i,r=1}^n \frac{\bar{\beta}_{ijr}^* \bar{q}_j^2 M_{2i}^4 \hat{k}_{ijr}^3}{4\gamma_3} \int_0^\infty \bar{k}_{ijr}(s) u_i^4(t-s) ds + \sum_{i,r=1}^n \frac{\bar{\beta}_{ijr}^* \bar{q}_j^2 M_{2r}^4 \hat{h}_{ijr}^3}{4\gamma_3} \int_0^\infty \bar{h}_{ijr}(s) u_r^4(t-s) ds,
\end{aligned} \tag{4.119}$$

$$\begin{aligned}
& 2\bar{p}_i|u_i(t)| \sum_{j,l=1}^m \beta_{jil}^* L_{2j} \hat{h}_{jil} g_l(y_l^*) \int_0^\infty k_{jil}(s) |v_j(t-s)| ds \\
& \leq \gamma_4 m^2 u_i^2(t) + \sum_{j,l=1}^m \left(\beta_{jil}^* L_{2j} \bar{p}_i \hat{h}_{jil} g_l(y_l^*) \right)^2 \frac{\hat{k}_{jil}}{\gamma_4} \int_0^\infty k_{jil}(s) v_j^2(t-s) ds,
\end{aligned} \tag{4.120}$$

$$\begin{aligned}
& 2\bar{p}_i|u_i(t)| \sum_{j,l=1}^m \beta_{jil}^* L_{2l} \hat{k}_{jil} g_j(y_j^*) \int_0^\infty h_{jil}(s) |v_l(t-s)| ds \\
& \leq \gamma_4 m^2 u_i^2(t) + \sum_{j,l=1}^m \left(\beta_{jil}^* L_{2l} \bar{p}_i \hat{k}_{jil} g_j(y_j^*) \right)^2 \frac{\hat{h}_{jil}}{\gamma_4} \int_0^\infty h_{jil}(s) v_l^2(t-s) ds,
\end{aligned} \tag{4.121}$$

$$\begin{aligned}
& 2\bar{q}_j|v_j(t)| \sum_{i,r=1}^n \bar{\beta}_{ijr}^* M_{2i} \hat{h}_{ijr} \bar{g}_r(x_r^*) \int_0^\infty \bar{k}_{ijr}(s) |u_i(t-s)| ds \\
& \leq \bar{\gamma}_4 n^2 v_j^2(t) + \sum_{i,r=1}^n \left(\bar{\beta}_{ijr}^* M_{2i} \bar{q}_j \hat{h}_{ijr} \bar{g}_r(x_r^*) \right)^2 \frac{\hat{k}_{ijr}}{\bar{\gamma}_4} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t-s) ds,
\end{aligned} \tag{4.122}$$

$$\begin{aligned}
& 2\bar{q}_j|v_j(t)| \sum_{i,r=1}^n \bar{\beta}_{ijr}^* M_{2r} \hat{k}_{ijr} \bar{g}_i(x_i^*) \int_0^\infty \bar{h}_{ijr}(s) |u_r(t-s)| ds \\
& \leq \bar{\gamma}_4 n^2 v_j^2(t) + \sum_{i,r=1}^n \left(\bar{\beta}_{ijr}^* M_{2r} \bar{q}_j \hat{k}_{ijr} \bar{g}_i(x_i^*) \right)^2 \frac{\hat{h}_{ijr}}{\bar{\gamma}_4} \int_0^\infty \bar{h}_{ijr}(s) u_r^2(t-s) ds,
\end{aligned} \tag{4.123}$$

$$\begin{aligned}
& 2\bar{p}_i|u_i(t)| \sum_{j,l=1}^m \hat{k}_{jil} |g_j(y_j^*)| \hat{h}_{jil} |g_l(y_l^*)| |\check{b}_{jil} - \hat{b}_{jil}| \\
& \leq 2\gamma_5 m^2 u_i^2(t) + \frac{\bar{p}_i^2}{2\gamma_5} \sum_{j,l=1}^m \hat{k}_{jil}^2 g_j^2(y_j^*) \hat{h}_{jil}^2 g_l^2(y_l^*) (\check{b}_{jil} - \hat{b}_{jil})^2,
\end{aligned} \tag{4.124}$$

$$\begin{aligned}
& 2\bar{q}_j|v_j(t)| \sum_{i,r=1}^n \hat{k}_{ijr}|\bar{g}_i(x_i^*)|\hat{h}_{ijr}|\bar{g}_r(x_r^*)||\check{b}_{ijr} - \hat{b}_{ijr}| \\
& \leq 2\bar{\gamma}_5 n^2 v_j^2(t) + \frac{\bar{q}_j^2}{2\bar{\gamma}_5} \sum_{i,r=1}^n \hat{k}_{ijr}^2 \bar{g}_i^2(x_i^*) \hat{h}_{ijr}^2 \bar{g}_r^2(x_r^*) (\check{b}_{ijr} - \hat{b}_{ijr})^2.
\end{aligned} \tag{4.125}$$

Next, we introduce the functional

$$V_1(t) = \sum_{i=1}^n u_i^2(t) + \sum_{j=1}^m v_j^2(t).$$

According to (A₁), (A₂) and the relations (4.106)–(4.123), the derivative of $V_1(t)$ is given by

$$\begin{aligned}
V_1'(t) & \leq \sum_{i=1}^n \left\{ -2\bar{p}_i \nu_i + 2m^2(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5) \right\} u_i^2(t) + \sum_{j=1}^m \left\{ -2\bar{q}_j \bar{\nu}_j \right. \\
& + 2n^2(\bar{\gamma}_1 + \bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_4 + \bar{\gamma}_5) \left. \right\} v_j^2(t) + \sum_{i,j,l=1}^{n;m} \left\{ \frac{\bar{p}_i^2 L_{1j}^2 f_l^2(y_l^*)}{\gamma_2} (\alpha_{jil}^{*2} + \alpha_{lij}^{*2}) v_j^2(t - \theta) \right. \\
& + \frac{\bar{p}_i^2 L_{1j}^4}{4\gamma_1} (\alpha_{jil}^{*2} + \alpha_{lij}^{*2}) v_j^4(t - \theta) + \frac{\beta_{jil}^{*2} \bar{p}_i^2 L_{2j}^4 \hat{k}_{jil}^3}{4\gamma_3} \int_0^\infty k_{jil}(s) v_j^4(t - s) ds \\
& + \frac{\beta_{jil}^{*2} \bar{p}_i^2 L_{2l}^4 \hat{h}_{jil}^3}{4\gamma_3} \int_0^\infty h_{jil}(s) v_l^4(t - s) ds + \left(\beta_{jil}^* \bar{p}_i \hat{h}_{jil} g_l(y_l^*) L_{2j} \right)^2 \frac{\hat{k}_{jil}}{\gamma_4} \int_0^\infty k_{jil}(s) \\
& \times v_j^2(t - s) ds + \left(\beta_{jil}^* \bar{p}_i \hat{k}_{jil} g_j(y_j^*) L_{2l} \right)^2 \frac{\hat{h}_{jil}}{\gamma_4} \int_0^\infty h_{jil}(s) v_l^2(t - s) ds \left. \right\} \\
& + \sum_{i,r;j=1}^{n;m} \left\{ \frac{\bar{q}_j^2 M_{1i}^2 \bar{f}_r^2(x_r^*)}{\bar{\gamma}_2} (\bar{\alpha}_{ijr}^{*2} + \bar{\alpha}_{rji}^{*2}) u_i^2(t - \tau) + \frac{\bar{q}_j^2 M_{1i}^4}{4\bar{\gamma}_1} (\bar{\alpha}_{ijr}^{*2} + \bar{\alpha}_{rji}^{*2}) u_i^4(t - \tau) \right. \\
& + \frac{\bar{\beta}_{ijr}^{*2} \bar{q}_j^2 M_{2i}^4 \hat{k}_{ijr}^3}{4\bar{\gamma}_3} \int_0^\infty \bar{k}_{ijr}(s) u_i^4(t - s) ds + \frac{\bar{\beta}_{ijr}^{*2} \bar{q}_j^2 M_{2r}^4 \hat{h}_{ijr}^3}{4\bar{\gamma}_3} \int_0^\infty \bar{h}_{ijr}(s) u_r^4(t - s) ds \\
& + \left(\bar{\beta}_{ijr}^* \bar{q}_j \hat{h}_{ijr} \bar{g}_r(x_r^*) M_{2i} \right)^2 \frac{\hat{k}_{ijr}}{\bar{\gamma}_4} \int_0^\infty \bar{k}_{ijr}(s) u_i^2(t - s) ds \\
& + \left(\bar{\beta}_{ijr}^* \bar{q}_j \hat{k}_{ijr} \bar{g}_i(x_i^*) M_{2r} \right)^2 \frac{\hat{h}_{ijr}}{\bar{\gamma}_4} \int_0^\infty \bar{h}_{ijr}(s) u_r^2(t - s) ds \left. \right\} \\
& + \sum_{i,j,l=1}^{n;m} \bar{p}_i^2 \left[\frac{f_j^2(y_j^*) f_l^2(y_l^*)}{2\gamma_3} (\check{a}_{jil} - \hat{a}_{jil})^2 + \frac{\hat{k}_{jil}^2 g_j^2(y_j^*) \hat{h}_{jil}^2 g_l^2(y_l^*)}{2\gamma_5} (\check{b}_{jil} - \hat{b}_{jil})^2 \right] \\
& + \sum_{i,r;j=1}^{n;m} \bar{q}_j^2 \left[\frac{\bar{f}_i^2(x_i^*) \bar{f}_r^2(x_r^*)}{2\bar{\gamma}_3} (\check{a}_{ijr} - \hat{a}_{ijr})^2 + \frac{\hat{k}_{ijr}^2 \bar{g}_i^2(x_i^*) \hat{h}_{ijr}^2 \bar{g}_r^2(x_r^*)}{2\bar{\gamma}_5} (\check{b}_{ijr} - \hat{b}_{ijr})^2 \right].
\end{aligned} \tag{4.126}$$

Then, we present the Lyapunov functionals

$$\begin{aligned}
V_8(t) &= e^{-\varepsilon_3 t} \sum_{i,j,l=1}^{n;m} \sigma_{4jil} \int_{t-\theta}^t e^{\varepsilon_3(s+\theta)} v_j^4(s) ds, \quad V_9(t) = e^{-\varepsilon_4 t} \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{4ijr} \int_{t-\tau}^t e^{\varepsilon_4(s+\tau)} u_i^4(s) ds, \\
V_{10}(t) &= \sum_{i;j,l=1}^{n;m} \sigma_{5jil} \int_0^\infty k_{jil}(s) \int_{t-s}^t v_j^4(\lambda) d\lambda ds, \quad V_{11}(t) = \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{5ijr} \int_0^\infty \bar{k}_{ijr}(s) \int_{t-s}^t u_i^4(\lambda) d\lambda ds, \\
V_{12}(t) &= \sum_{i;j,l=1}^{n;m} \sigma_{6lij} \int_0^\infty h_{lij}(s) \int_{t-s}^t v_j^4(\lambda) d\lambda ds, \quad V_{13}(t) = \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{6rji} \int_0^\infty \bar{h}_{rji}(s) \int_{t-s}^t u_i^4(\lambda) d\lambda ds,
\end{aligned} \tag{4.127}$$

with ε_3 and ε_4 are chosen so small that $e^{\varepsilon_3\theta} < 1 + \varepsilon_3$ and $e^{\varepsilon_4\tau} < 1 + \varepsilon_4$, by virtue of the assumptions (A₅) and (A₇), whose derivatives are given by

$$\begin{aligned}
V'_8(t) &= -\varepsilon_3 V_8(t) + e^{\varepsilon_3\theta} \sum_{i;j,l=1}^{n;m} \sigma_{4jil} v_j^4(t) - \sum_{i;j,l=1}^{n;m} \sigma_{4jil} v_j^4(t-\theta), \\
V'_9(t) &= -\varepsilon_4 V_9(t) + e^{\varepsilon_4\tau} \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{4ijr} u_i^4(t) - \sum_{i,r;j=1}^{n;m} \bar{\sigma}_{4ijr} u_i^4(t-\tau),
\end{aligned} \tag{4.128}$$

$$\begin{aligned}
V'_p(t) &\leq \sum_{j=1}^m \kappa_{pj} v_j^4(t) - \delta \mathfrak{R}_p(t) V_p(t) - (1-\delta) \sum_{j=1}^m \int_0^\infty \mathfrak{L}_{pj}(s) v_j^4(t-s) ds, \quad p = 10, 12, \\
V'_q(t) &\leq \sum_{i=1}^n \bar{\kappa}_{qi} u_i^4(t) - \delta \bar{\mathfrak{R}}_q(t) V_q(t) - (1-\delta) \sum_{i=1}^n \int_0^\infty \bar{\mathfrak{L}}_{qi}(s) u_i^4(t-s) ds, \quad q = 11, 13,
\end{aligned} \tag{4.129}$$

$$\begin{aligned}
\text{where } 0 < \delta < 1, \quad \kappa_{10j} &= \sum_{i;l=1}^{n;m} \hat{\sigma}_{5jil}, \quad \kappa_{12j} = \sum_{i;l=1}^{n;m} \hat{\sigma}_{6lij}, \quad \bar{\kappa}_{11i} = \sum_{r;j=1}^{n;m} \hat{\sigma}_{5ijr}, \quad \bar{\kappa}_{13i} = \sum_{r;j=1}^{n;m} \hat{\sigma}_{6rji}, \quad \mathfrak{R}_{10}(t) = \\
\eta_2(t), \quad \mathfrak{R}_{12}(t) &= \xi_2(t), \quad \bar{\mathfrak{R}}_{11}(t) = \bar{\eta}_2(t), \quad \bar{\mathfrak{R}}_{13}(t) = \bar{\xi}_2(t), \quad \mathfrak{L}_{10j}(t) = \sum_{i;l=1}^{n;m} \sigma_{5jil} k_{jil}(t), \quad \mathfrak{L}_{12j}(t) = \\
\sum_{i;l=1}^{n;m} \sigma_{6lij} h_{lij}(t), \quad \bar{\mathfrak{L}}_{11i}(t) &= \sum_{r;j=1}^{n;m} \bar{\sigma}_{5ijr} \bar{k}_{ijr}(t), \quad \bar{\mathfrak{L}}_{13i}(t) = \sum_{r;j=1}^{n;m} \bar{\sigma}_{6rji} \bar{h}_{rji}(t).
\end{aligned}$$

After that, we construct the following Lyapunov functional

$$V(t) = \sum_{i=1}^3 V_i(t) + V_8(t) + V_9(t) + \frac{1}{1-\delta} \left[\sum_{j=4}^7 V_j(t) + \sum_{k=10}^{13} V_k(t) \right],$$

where $V_i(t)$, $i = 2, \dots, 7$ are defined in (4.61).

In view of the relations (4.62)–(4.64) and (4.126)–(4.129) and the conditions of Theorem 4.5.1, we find

$$\begin{aligned}
V'(t) &\leq -\tilde{A}V_1(t) - \varepsilon_1 V_2(t) - \varepsilon_2 V_3(t) - \varepsilon_3 V_8(t) - \varepsilon_4 V_9(t) \\
&\quad - \frac{\delta \chi(t)}{1-\delta} \left[\sum_{k=4}^7 V_k(t) + \sum_{l=10}^{13} V_l(t) \right] + \tilde{A}V^2(t) + \hat{A},
\end{aligned} \tag{4.130}$$

where $\tilde{A} = \min\{A, \bar{A}\}$,

$$\begin{aligned} \check{A} &= \max \left\{ \sum_{i,r;j=1}^{n;m} (1 + \epsilon_4) \bar{\sigma}_{4ijr} + \frac{\hat{\sigma}_{5ijr}}{1-\delta} + \frac{\hat{\sigma}_{6rji}}{1-\delta}, \sum_{i;j,l=1}^{n;m} (1 + \epsilon_3) \sigma_{4jil} + \frac{\hat{\sigma}_{5jil}}{1-\delta} + \frac{\hat{\sigma}_{6lij}}{1-\delta} \right\}, \\ \hat{A} &= \sum_{i;j,l=1}^{n;m} \bar{p}_i^2 \left[\frac{f_j^2(y_j^*) f_l^2(y_l^*)}{2\gamma_3} (\check{a}_{jil} - \hat{a}_{jil})^2 + \frac{\hat{k}_{jil}^2 g_j^2(y_j^*) \hat{h}_{jil}^2 g_l^2(y_l^*)}{2\gamma_5} (\check{b}_{jil} - \hat{b}_{jil})^2 \right] \\ &+ \sum_{i,r;j=1}^{n;m} \bar{q}_j^2 \left[\frac{\bar{f}_i^2(x_i^*) \bar{f}_r^2(x_r^*)}{2\bar{\gamma}_3} (\check{a}_{ijr} - \hat{a}_{ijr})^2 + \frac{\hat{k}_{ijr}^2 \bar{g}_i^2(x_i^*) \hat{h}_{ijr}^2 \bar{g}_r^2(x_r^*)}{2\bar{\gamma}_5} (\check{b}_{ijr} - \hat{b}_{ijr})^2 \right]. \end{aligned} \quad (4.131)$$

Finally, the estimation of the Lyapunov functional $V(t)$ is obtained by means of two cases depending on the limit of the function $\chi(t)$.

Case 1

If $0 < \chi^* < \infty$, then, there exists $t^* > 0$ such that

$$\chi(t) \geq \chi^*, \quad t \geq t^*.$$

Then, we obtain

$$\begin{aligned} V'(t) &\leq -\tilde{A}V_1(t) - \varepsilon_1 V_2(t) - \varepsilon_2 V_3(t) - \varepsilon_3 V_8(t) - \varepsilon_4 V_9(t) \\ &- \frac{\delta \chi^*}{(1-\delta)} \left[\sum_{k=4}^7 V_k(t) + \sum_{l=10}^{13} V_l(t) \right] + \check{A}V^2(t) + \hat{A}, \quad t \geq t^*. \end{aligned} \quad (4.132)$$

For $C = \min\{\tilde{A}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \frac{\delta \chi^*}{(1-\delta)}\}$, the relation (4.132) becomes

$$V'(t) \leq -CV(t) + \check{A}V^2(t) + \hat{A}, \quad t \geq t^*. \quad (4.133)$$

According to Theorem 2.2.7, we deduce

$$V(t) < \frac{1}{\omega(t)}, \quad t \geq t^*, \quad (4.134)$$

where $\omega(t)$ satisfies

$$\check{A} \leq \frac{\omega(t)}{2} \left(C - \frac{\omega'(t)}{\omega(t)} \right), \quad \check{A} \leq \frac{1}{2\omega(t)} \left(\hat{A} - \frac{\omega'(t)}{\omega(t)} \right), \quad (4.135)$$

or

$$\omega'(t) - C\omega(t) + 2\check{A} \leq 0, \quad \omega'(t) - \hat{A}\omega(t) + 2\check{A}\omega^2(t) \leq 0, \quad (4.136)$$

when $V(0)\omega(0) < 1$.

Case 2

If $\chi^* = 0$, then, there exists $\bar{t} > 0$ such that

$$\frac{\delta}{(1-\delta)}\chi(t) \leq \min \left\{ \tilde{A}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \right\}, \quad t \geq \bar{t}. \quad (4.137)$$

Then, we get

$$V'(t) \leq -\delta^*\chi(t)V(t) + \check{A}V^2(t) + \hat{A}, \quad t \geq \bar{t}, \quad (4.138)$$

for some positive constant $\delta^* := \frac{\delta}{(1-\delta)}$. In view of Lemma 2.2.7, we infer

$$V(t) < \frac{1}{\bar{\omega}(t)}, \quad t \geq \bar{t}, \quad (4.139)$$

where $\bar{\omega}(t)$ satisfies

$$\check{A} \leq \frac{\bar{\omega}(t)}{2} \left(\delta\chi(t) - \frac{\bar{\omega}'(t)}{\bar{\omega}(t)} \right), \quad \check{A} \leq \frac{1}{2\bar{\omega}(t)} \left(\hat{A} - \frac{\bar{\omega}'(t)}{\bar{\omega}(t)} \right), \quad (4.140)$$

or

$$(4.141)$$

$$\bar{\omega}'(t) - \delta\bar{\omega}(t)\chi(t) + 2\check{A} \leq 0, \quad \bar{\omega}'(t) - \hat{A}\bar{\omega}(t) + 2\check{A}\bar{\omega}^2(t) \leq 0, \quad (4.142)$$

when $V(0)\bar{\omega}(0) < 1$.

Remark 4.5.1

The unbounded case has not been developed owing to the complexity of the non-linear terms, which are addressed in the majority of the literature via the boundedness condition as in Aouiti, C., et al., 2020 [91], Chen, Z., et al., 2007 [92], Dong, Z. et al., 2020 [93], Lou, X. Y. and Cui, B. T., 2007 [94], Ren, F. and Cao, J., 2006 [137], Sheng, Y. et al., 2020 [88], Xia, Y. 2010 [95] and Zu, J. et al., 2020 [96] and references therein.

4.6 Adaptive Synchronization of the System

In this section, we will study the synchronization between the uncontrolled system (4.38) and the following controlled system.

$$\begin{cases} \bar{x}'_i(t) = p_i(\bar{x}_i(t)) \left[-\Lambda_i(\bar{x}_i(t)) + \sum_{j,l=1}^m a_{jil}(\bar{x}_i(t)) f_j(\bar{y}_j(t-\theta)) f_l(\bar{y}_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^m b_{jil}(\bar{x}_i(t)) \int_0^\infty k_{jil}(s) g_j(\bar{y}_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(\bar{y}_l(t-s)) ds + I_i \right] + \Phi_i(t), \\ \bar{y}'_j(t) = q_j(\bar{y}_j(t)) \left[-\bar{\Lambda}_j(\bar{y}_j(t)) + \sum_{i,r=1}^n \bar{a}_{ijr}(\bar{y}_j(t)) \bar{f}_i(\bar{x}_i(t-\tau)) \bar{f}_r(\bar{x}_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^n \bar{b}_{ijr}(\bar{y}_j(t)) \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(\bar{x}_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds + J_j \right] + \Psi_j(t), \end{cases} \quad (4.143)$$

the initial data are

$$\begin{cases} \bar{x}_i(t) = \bar{\phi}_i(t), \quad t \leq 0, \\ \bar{y}_j(t) = \bar{\varphi}_j(t), \quad t \leq 0, \end{cases} \quad (4.144)$$

$$a_{jil}(\bar{x}_i(t)) = \begin{cases} \hat{a}_{jil}, & |\bar{x}_i(t)| > T_i, \\ \check{a}_{jil}, & |\bar{x}_i(t)| \leq T_i, \end{cases} \quad b_{jil}(\bar{x}_i(t)) = \begin{cases} \hat{b}_{jil}, & |\bar{x}_i(t)| > T_i, \\ \check{b}_{jil}, & |\bar{x}_i(t)| \leq T_i, \end{cases} \quad (4.145)$$

$$\bar{a}_{ijr}(\bar{y}_j(t)) = \begin{cases} \hat{\bar{a}}_{ijr}, & |\bar{y}_j(t)| > \bar{T}_j, \\ \check{\bar{a}}_{ijr}, & |\bar{y}_j(t)| \leq \bar{T}_j, \end{cases} \quad \bar{b}_{ijr}(\bar{y}_j(t)) = \begin{cases} \hat{\bar{b}}_{ijr}, & |\bar{y}_j(t)| > \bar{T}_j, \\ \check{\bar{b}}_{ijr}, & |\bar{y}_j(t)| \leq \bar{T}_j. \end{cases} \quad (4.146)$$

The system (4.143) may be rewritten as

$$\left\{ \begin{array}{l} \bar{x}'_i(t) \in p_i(\bar{x}_i(t)) \left[-\Lambda_i(\bar{x}_i(t)) + \sum_{j,l=1}^m \bar{c}o[a_{jil}(\bar{x}_i(t))] f_j(\bar{y}_j(t-\theta)) f_l(\bar{y}_l(t-\theta)) \right. \\ \left. + \sum_{j,l=1}^m \bar{c}o[b_{jil}(\bar{x}_i(t))] \int_0^\infty k_{jil}(s) g_j(\bar{y}_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(\bar{y}_l(t-s)) ds + I_i \right] + \Phi_i(t), \\ \bar{y}'_j(t) \in q_j(\bar{y}_j(t)) \left[-\bar{\Lambda}_j(\bar{y}_j(t)) + \sum_{i,r=1}^n \bar{c}o[\bar{a}_{ijr}(\bar{y}_j(t))] \bar{f}_i(\bar{x}_i(t-\tau)) \bar{f}_r(\bar{x}_r(t-\tau)) \right. \\ \left. + \sum_{i,r=1}^n \bar{c}o[\bar{b}_{ijr}(\bar{y}_j(t))] \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(\bar{x}_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds + J_j \right] + \Psi_j(t), \end{array} \right. \quad (4.147)$$

where $\bar{c}o[a_{jil}(\bar{x}_i(t))]$, $\bar{c}o[b_{jil}(\bar{x}_i(t))]$, $\bar{c}o[\bar{a}_{ijr}(\bar{y}_j(t))]$ and $\bar{c}o[\bar{b}_{ijr}(\bar{y}_j(t))]$ are defined by

$$\bar{c}o[a_{jil}(\bar{x}_i(t))] = \begin{cases} \hat{a}_{jil}, & |\bar{x}_i(t)| > T_i, \\ [a_{*jil}, a_{jil}^*], & |\bar{x}_i(t)| = T_i, \\ \check{a}_{jil}, & |\bar{x}_i(t)| < T_i, \end{cases} \quad \bar{c}o[b_{jil}(\bar{x}_i(t))] = \begin{cases} \hat{b}_{jil}, & |\bar{x}_i(t)| > T_i, \\ [b_{*jil}, b_{jil}^*], & |\bar{x}_i(t)| = T_i, \\ \check{b}_{jil}, & |\bar{x}_i(t)| < T_i, \end{cases} \quad (4.148)$$

$$\bar{c}o[\bar{a}_{ijr}(\bar{y}_j(t))] = \begin{cases} \hat{\bar{a}}_{ijr}, & |\bar{y}_j(t)| > \bar{T}_j, \\ [\bar{a}_{*ijr}, \bar{a}_{ijr}^*], & |\bar{y}_j(t)| = \bar{T}_j, \\ \check{\bar{a}}_{ijr}, & |\bar{y}_j(t)| < \bar{T}_j, \end{cases} \quad \bar{c}o[\bar{b}_{ijr}(\bar{y}_j(t))] = \begin{cases} \hat{\bar{b}}_{ijr}, & |\bar{y}_j(t)| > \bar{T}_j, \\ [\bar{b}_{*ijr}, \bar{b}_{ijr}^*], & |\bar{y}_j(t)| = \bar{T}_j, \\ \check{\bar{b}}_{ijr}, & |\bar{y}_j(t)| < \bar{T}_j, \end{cases} \quad (4.149)$$

which is equivalent to the following system

$$\left\{ \begin{array}{l} \bar{x}'_i(t) = p_i(\bar{x}_i(t)) \left[-\Lambda_i(\bar{x}_i(t)) + \sum_{j,l=1}^m \alpha_{jil}^\triangleright f_j(\bar{y}_j(t-\theta)) f_l(\bar{y}_l(t-\theta)) \right. \\ \left. + \sum_{j,l=1}^m \beta_{jil}^\triangleright \int_0^\infty k_{jil}(s) g_j(\bar{y}_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(\bar{y}_l(t-s)) ds + I_i \right] + \Phi_i(t), \\ \bar{y}'_j(t) = q_j(\bar{y}_j(t)) \left[-\bar{\Lambda}_j(\bar{y}_j(t)) + \sum_{i,r=1}^n \bar{\alpha}_{ijr}^\triangleright \bar{f}_i(\bar{x}_i(t-\tau)) \bar{f}_r(\bar{x}_r(t-\tau)) \right. \\ \left. + \sum_{i,r=1}^n \bar{\beta}_{ijr}^\triangleright \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(\bar{x}_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds + J_j \right] + \Psi_j(t), \end{array} \right. \quad (4.150)$$

where $\alpha_{jil}^\triangleright \in \bar{c}o[a_{jil}(\bar{x}_i(t))]$, $\beta_{jil}^\triangleright \in \bar{c}o[b_{jil}(\bar{x}_i(t))]$, $\bar{\alpha}_{ijr}^\triangleright \in \bar{c}o[\bar{a}_{ijr}(\bar{y}_j(t))]$, $\bar{\beta}_{ijr}^\triangleright \in \bar{c}o[\bar{b}_{ijr}(\bar{y}_j(t))]$.

The synchronized error dynamical system is

$$\left\{ \begin{aligned}
e'_i(t) &= - \left[p_i(\bar{x}_i(t)) \Lambda_i(\bar{x}_i(t)) - p_i(x_i(t)) \Lambda_i(x_i(t)) \right] \\
&+ p_i(\bar{x}_i(t)) \sum_{j,l=1}^m \left[\alpha_{jil}^\triangleright f_j(\bar{y}_j(t-\theta)) f_l(\bar{y}_l(t-\theta)) - \alpha_{jil} f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right] \\
&+ p_i(\bar{x}_i(t)) \sum_{j,l=1}^m \left[\beta_{jil}^\triangleright \int_0^\infty k_{jil}(s) g_j(\bar{y}_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(\bar{y}_l(t-s)) ds \right. \\
&\quad \left. - \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds \right] \\
&+ \left(p_i(\bar{x}_i(t)) - p_i(x_i(t)) \right) \left[\sum_{j,l=1}^m \alpha_{jil} f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\
&\quad \left. + \sum_{j,l=1}^m \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds + I_i \right] + \Phi_i(t), \\
\bar{e}'_j(t) &= - \left[q_j(\bar{y}_j(t)) \bar{\Lambda}_j(\bar{y}_j(t)) - q_j(y_j(t)) \bar{\Lambda}_j(y_j(t)) \right] \\
&+ q_j(\bar{y}_j(t)) \sum_{i,r=1}^n \left[\bar{\alpha}_{ijr}^\triangleright \bar{f}_i(\bar{x}_i(t-\tau)) \bar{f}_r(\bar{x}_r(t-\tau)) - \bar{\alpha}_{ijr} \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right] \\
&+ q_j(\bar{y}_j(t)) \sum_{i,r=1}^n \left[\bar{\beta}_{ijr}^\triangleright \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(\bar{x}_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds \right. \\
&\quad \left. - \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds \right] \\
&+ \left[q_j(\bar{y}_j(t)) - q_j(y_j(t)) \right] \left\{ \sum_{i,r=1}^n \bar{\alpha}_{ijr} \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\
&\quad \left. + \sum_{i,r=1}^n \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds + J_j \right\} + \Psi_j(t),
\end{aligned} \right. \tag{4.151}$$

where

$e_i(t) = \bar{x}_i(t) - x_i(t)$, $\bar{e}_j(t) = \bar{y}_j(t) - y_j(t)$, the adaptive feedback controllers are given by

$$\Phi_i(t) = -\vartheta_i(t) e_i(t) - \pi_i(t) \text{sign}(e_i(t)), \quad \Psi_j(t) = -\bar{\vartheta}_j(t) \bar{e}_j(t) - \bar{\pi}_j(t) \text{sign}(\bar{e}_j(t)), \tag{4.152}$$

where the functions $\vartheta_i(t)$, $\pi_i(t)$, $\bar{\vartheta}_j(t)$ and $\bar{\pi}_j(t)$ are assumed to be non-negative continuous functions, for all $t > 0$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$ such that the updated laws of these functions satisfy

$$\left\{ \begin{aligned}
\vartheta'_i(t) &= \epsilon_i e^{\kappa_1 t} e_i^2(t), \quad \vartheta_i(0) = 0, \\
\pi'_i(t) &= o_i e^{\kappa_1 t} |e_i(t)|, \quad \pi_i(0) = 0,
\end{aligned} \right. \quad \left\{ \begin{aligned}
\bar{\vartheta}'_j(t) &= \bar{\epsilon}_j e^{\kappa_2 t} \bar{e}_j^2(t), \quad \bar{\vartheta}_j(0) = 0, \\
\bar{\pi}'_j(t) &= \bar{o}_j e^{\kappa_2 t} |\bar{e}_j(t)|, \quad \bar{\pi}_j(0) = 0,
\end{aligned} \right. \tag{4.153}$$

for some positive constants $\kappa_1, \kappa_2, \epsilon_i, \bar{\epsilon}_j, o_i, \bar{o}_j$.

Assume that $\hat{A}$ and $\check{A}$ are positive constants such that

$$\begin{aligned}
\hat{A} &= \min_{1 \leq i \leq n} \left\{ \zeta_i + \omega_i - \rho_i \left[\sum_{j,l=1}^m \left(F^2(\alpha_{jil}^* + \alpha_{lij}^*) + G^2(\beta_{jil}^* \hat{k}_{jil}^2 + \beta_{lij}^* \hat{h}_{lij}^2) \right) + 2I_i \right] \right. \\
&\quad \left. - \sum_{r;j=1}^{n;m} \left[(1 + \bar{\varepsilon}) \bar{\chi}_{1ijr} + \frac{\hat{\chi}_{2ijr}}{1-\sigma} + \frac{\hat{\chi}_{3rji}}{1-\sigma} \right] \right\}, \\
\check{A} &= \min_{1 \leq j \leq m} \left\{ \bar{\zeta}_j + \bar{\omega}_j - \bar{\rho}_j \left[\sum_{i,r=1}^n \left(\bar{F}^2(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*) + \bar{G}^2(\bar{\beta}_{ijr}^* \hat{k}_{ijr}^2 + \bar{\beta}_{rji}^* \hat{h}_{rji}^2) \right) + 2J_j \right] \right. \\
&\quad \left. - \sum_{i;l=1}^{n;m} \left[(1 + \varepsilon) \chi_{1jil} + \frac{\hat{\chi}_{2jil}}{1-\sigma} + \frac{\hat{\chi}_{3lij}}{1-\sigma} \right] \right\}, \\
\hat{\chi}_{2jil} &= \chi_{2jil} \hat{k}_{jil}, \quad \hat{\chi}_{2ijr} = \bar{\chi}_{2ijr} \hat{k}_{ijr}, \quad \chi_{3lij} = \chi_{3lij} \hat{h}_{lij}, \quad \hat{\chi}_{3rji} = \bar{\chi}_{3rji} \hat{h}_{rji}, \\
\chi_{1jil} &= \frac{\bar{p}_i^2 F^2 m^2}{\lambda_1} L_{1j}^2 (\alpha_{jil}^* + \alpha_{lij}^*)^2, \quad \bar{\chi}_{1ijr} = \frac{\bar{q}_j^2 \bar{F}^2 n^2}{\lambda_1} M_{1i}^2 (\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*)^2, \\
\chi_{2jil} &= \frac{2\bar{p}_i^2 G^2 m^2}{\lambda_2} \left(L_{2j} \beta_{jil}^* \hat{h}_{jil} \right)^2 \hat{k}_{jil}, \quad \bar{\chi}_{2ijr} = \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\lambda_2} \left(M_{2i} \bar{\beta}_{ijr}^* \hat{h}_{ijr} \right)^2 \hat{k}_{ijr}, \\
\chi_{3lij} &= \frac{2\bar{p}_i^2 G^2 m^2}{\lambda_2} \left(L_{2j} \beta_{lij}^* \hat{k}_{lij} \right)^2 \hat{h}_{lij}, \quad \bar{\chi}_{3rji} = \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\lambda_2} \left(M_{2i} \bar{\beta}_{rji}^* \hat{k}_{rji} \right)^2 \hat{h}_{rji}, \\
\gamma_i &= \bar{p}_i \sum_{j,l=1}^m \left[F^2 |\check{a}_{jil} - \hat{a}_{jil}| + G^2 |\check{b}_{jil} - \hat{b}_{jil}| \hat{k}_{jil} \hat{h}_{jil} \right], \\
\bar{\gamma}_j &= \bar{q}_j \sum_{i,r=1}^n \left[\bar{F}^2 |\check{a}_{ijr} - \hat{a}_{ijr}| + \bar{G}^2 |\check{b}_{ijr} - \hat{b}_{ijr}| \hat{k}_{ijr} \hat{h}_{ijr} \right], \\
\lambda_1 &= \lambda_2 = \min_{1 \leq i \leq n} \frac{\zeta_i}{2}, \quad \bar{\lambda}_1 = \bar{\lambda}_2 = \min_{1 \leq j \leq m} \frac{\bar{\zeta}_j}{2}, \\
\delta(t) &= \inf \{ \eta_1(t), \bar{\eta}_1(t), \xi_1(t), \bar{\xi}_1(t) \}, \quad t \geq 0, \quad \lim_{t \rightarrow \infty} \delta(t) = \delta^*.
\end{aligned} \tag{4.154}$$

Under the adaptive feedback controllers given in (4.152), the synchronization of system (P_2) is reached in the following outcome.

Theorem 4.6.1

Assume that assumptions (A_1) , (A_3) – (A_9) and

$$\begin{aligned}
\rho_i &\left[\sum_{j,l=1}^m \left(F^2(\alpha_{jil}^* + \alpha_{lij}^*) + G^2(\beta_{jil}^* \hat{k}_{jil}^2 + \beta_{lij}^* \hat{h}_{lij}^2) \right) + 2I_i \right] \\
&\quad + \sum_{r;j=1}^{n;m} \left[(1 + \bar{\varepsilon}) \bar{\chi}_{1ijr} + \frac{\hat{\chi}_{2ijr}}{1-\sigma} + \frac{\hat{\chi}_{3rji}}{1-\sigma} \right] < \zeta_i + \omega_i, \\
\bar{\rho}_j &\left[\sum_{i,r=1}^n \left(\bar{F}^2(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*) + \bar{G}^2(\bar{\beta}_{ijr}^* \hat{k}_{ijr}^2 + \bar{\beta}_{rji}^* \hat{h}_{rji}^2) \right) + 2J_j \right] \\
&\quad + \sum_{i;l=1}^{n;m} \left[(1 + \varepsilon) \chi_{1jil} + \frac{\hat{\chi}_{2jil}}{1-\sigma} + \frac{\hat{\chi}_{3lij}}{1-\sigma} \right] < \bar{\zeta}_j + \bar{\omega}_j
\end{aligned} \tag{4.155}$$

hold, then

- If $0 < \delta^* \leq \infty$, then for some constants $\hat{C}, \check{C} > 0$

$$W(t) \leq \hat{C}e^{-\tilde{C}t}, \quad t \geq 0.$$

- If $\delta^* = 0$, we have for some constant $\check{C} > 0$

$$W(t) \leq \check{C}e^{-\sigma \int_0^t \delta(s)ds}, \quad t \geq 0.$$

Proof

Obviously, we have

$$\begin{aligned} & \sum_{j,l=1}^m \left[\alpha_{jil}^\triangleright f_j(\bar{y}_j(t-\theta)) f_l(\bar{y}_l(t-\theta)) - \alpha_{jil} f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right] \\ &= \sum_{j,l=1}^m \left[\alpha_{jil}^\triangleright f_l(\bar{y}_l(t-\theta)) + \alpha_{lij} f_l(y_l(t-\theta)) \right] \tilde{f}_j(\bar{e}_j(t-\theta)) \\ & \quad + \sum_{j,l=1}^m (\alpha_{jil}^\triangleright - \alpha_{jil}) f_j(y_j(t-\theta)) f_l(\bar{y}_l(t-\theta)), \end{aligned} \quad (4.156)$$

$$\begin{aligned} & \sum_{i,r=1}^n \left[\bar{\alpha}_{ijr}^\triangleright \bar{f}_i(\bar{x}_i(t-\tau)) \bar{f}_r(\bar{x}_r(t-\tau)) - \bar{\alpha}_{ijr} \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right] \\ &= \sum_{i,r=1}^n \left[\bar{\alpha}_{ijr}^\triangleright \bar{f}_r(\bar{x}_r(t-\tau)) + \bar{\alpha}_{rji} \bar{f}_r(x_r(t-\tau)) \right] \tilde{\bar{f}}_i(e_i(t-\tau)) \\ & \quad + \sum_{i,r=1}^n (\bar{\alpha}_{ijr}^\triangleright - \bar{\alpha}_{ijr}) \bar{f}_i(x_i(t-\tau)) \bar{f}_r(\bar{x}_r(t-\tau)), \end{aligned} \quad (4.157)$$

$$\begin{aligned} & \sum_{j,l=1}^m \left[\beta_{jil}^\triangleright \int_0^\infty k_{jil}(s) g_j(\bar{y}_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(\bar{y}_l(t-s)) ds \right. \\ & \quad \left. - \beta_{jil} \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(y_l(t-s)) ds \right] \\ &= \sum_{j,l=1}^m \int_0^\infty \left\{ \beta_{jil}^\triangleright \left(\int_0^\infty h_{jil}(s) g_l(\bar{y}_l(t-s)) ds \right) k_{jil}(s) \right. \\ & \quad \left. + \beta_{lij} \left(\int_0^\infty k_{lij}(s) g_l(y_l(t-s)) ds \right) h_{lij}(s) \right\} \tilde{g}_j(\bar{e}_j(t-s)) ds \\ &+ \sum_{j,l=1}^m (\beta_{jil}^\triangleright - \beta_{jil}) \int_0^\infty k_{jil}(s) g_j(y_j(t-s)) ds \int_0^\infty h_{jil}(s) g_l(\bar{y}_l(t-s)) ds, \end{aligned} \quad (4.158)$$

$$\begin{aligned}
& \sum_{i,r=1}^n \left[\bar{\beta}_{ijr}^{\triangleright} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(\bar{x}_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds \right. \\
& \quad \left. - \bar{\beta}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds \right] \\
& = \sum_{i,r=1}^n \int_0^\infty \left\{ \bar{\beta}_{ijr}^{\triangleright} \left(\int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds \right) \bar{k}_{ijr}(s) \right. \\
& \quad \left. + \bar{\beta}_{rji} \left(\int_0^\infty \bar{k}_{rji}(s) \bar{g}_r(x_r(t-s)) ds \right) \bar{h}_{rji}(s) \right\} \tilde{g}_i(e_i(t-s)) ds \\
& + \sum_{i,r=1}^n (\bar{\beta}_{ijr}^{\triangleright} - \bar{\beta}_{ijr}) \int_0^\infty \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^\infty \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds,
\end{aligned} \tag{4.159}$$

where

$$\begin{aligned}
\tilde{f}_j(\bar{e}_j(t-\theta)) &= f_j(\bar{y}_j(t-\theta)) - f_j(y_j(t-\theta)), \quad \tilde{f}_i(e_i(t-\tau)) = \bar{f}_i(\bar{x}_i(t-\tau)) - \bar{f}_i(x_i(t-\tau)), \\
\tilde{g}_j(\bar{e}_j(t-s)) &= g_j(\bar{y}_j(t-s)) - g_j(y_j(t-s)), \quad \tilde{g}_i(e_i(t-s)) = \bar{g}_i(\bar{x}_i(t-s)) - \bar{g}_i(x_i(t-s)).
\end{aligned} \tag{4.160}$$

Firstly, we introduce the functional

$$W_1(t) = \sum_{i=1}^n e_i^2(t) + \sum_{j=1}^m \bar{e}_j^2(t).$$

Using (A₃), (A₄), (A₅) and the same techniques as in previous sections, we find

$$\begin{aligned}
& 2\bar{p}_i |e_i(t)| \sum_{j,l=1}^m FL_{1j}(\alpha_{jil}^* + \alpha_{lij}^*) |\bar{e}_j(t-\theta)| \\
& \leq \lambda_1 e_i^2(t) + \frac{\bar{p}_i^2 F^2 m^2}{\lambda_1} \sum_{j,l=1}^m L_{1j}^2 (\alpha_{jil}^* + \alpha_{lij}^*)^2 \bar{e}_j^2(t-\theta),
\end{aligned} \tag{4.161}$$

$$\begin{aligned}
& 2\bar{q}_j |\bar{e}_j(t)| \sum_{i,r=1}^n \bar{F} M_{1i}(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*) |e_i(t-\tau)| \\
& \leq \bar{\lambda}_1 \bar{e}_j^2(t) + \frac{\bar{q}_j^2 \bar{F}^2 n^2}{\lambda_1} \sum_{i,r=1}^n M_{1i}^2 (\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*)^2 e_i^2(t-\tau),
\end{aligned} \tag{4.162}$$

$$\begin{aligned}
& 2\bar{p}_i |e_i(t)| \sum_{j,l=1}^m GL_{2j} \int_0^\infty \left(\beta_{jil}^* \hat{h}_{jil} k_{jil}(s) + \beta_{lij}^* \hat{k}_{lij} h_{lij}(s) \right) |\bar{e}_j(t-s)| ds \\
& \leq \lambda_2 e_i^2(t) + \frac{2\bar{p}_i^2 G^2 m^2}{\lambda_2} \sum_{j,l=1}^m \left(L_{2j} \beta_{jil}^* \hat{h}_{jil} \right)^2 \hat{k}_{jil} \int_0^\infty k_{jil}(s) \bar{e}_j^2(t-s) ds \\
& \quad + \frac{2\bar{p}_i^2 G^2 m^2}{\lambda_2} \sum_{j,l=1}^m \left(L_{2j} \beta_{lij}^* \hat{k}_{lij} \right)^2 \hat{h}_{lij} \int_0^\infty h_{lij}(s) \bar{e}_j^2(t-s) ds,
\end{aligned} \tag{4.163}$$

$$\begin{aligned}
& 2\bar{q}_j|\bar{e}_j(t)| \sum_{i,r=1}^n \bar{G}M_{2i} \int_0^\infty \left(\bar{\beta}_{ijr}^* \hat{h}_{ijr} \bar{k}_{ijr}(s) + \bar{\beta}_{rji}^* \hat{k}_{rji} \bar{h}_{rji}(s) \right) |e_i(t-s)| ds \\
& \leq \bar{\lambda}_2 \bar{e}_j^2(t) + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\lambda_2} \sum_{i,r=1}^n \left(M_{2i} \bar{\beta}_{ijr}^* \hat{h}_{ijr} \right)^2 \hat{k}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) e_i^2(t-s) ds \\
& \quad + \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\lambda_2} \sum_{i,r=1}^n \left(M_{2i} \bar{\beta}_{rji}^* \hat{k}_{rji} \right)^2 \hat{h}_{rji} \int_0^\infty \bar{h}_{rji}(s) e_i^2(t-s) ds.
\end{aligned} \tag{4.164}$$

By virtue of (A₁), (A₈), (A₉) and the relations (4.161)–(4.164), the derivative of $W_1(t)$ yields

$$\begin{aligned}
W_1'(t) & \leq \sum_{i=1}^n \left\{ -2\zeta_i + \lambda_1 + \lambda_2 + \rho_i \left[\sum_{j,l=1}^m \left(F^2(\alpha_{jil}^* + \alpha_{lij}^*) + G^2(\beta_{jil}^* \hat{k}_{jil}^2 + \beta_{lij}^* \hat{h}_{lij}^2) \right) + 2I_i \right] \right. \\
& \quad \left. - 2\vartheta_i(t) \right\} e_i^2(t) + \sum_{j=1}^m \left\{ -2\bar{\zeta}_j + \bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\rho}_j \left[\sum_{i,r=1}^n \left(\bar{F}^2(\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*) + \bar{G}^2(\bar{\beta}_{ijr}^* \hat{k}_{ijr}^2 \right. \right. \right. \\
& \quad \left. \left. \left. + \bar{\beta}_{rji}^* \hat{h}_{rji}^2) \right) + 2J_j \right] - 2\bar{\vartheta}_j(t) \right\} \bar{e}_j^2(t) + \sum_{i,j,l=1}^{n;m} \frac{\bar{p}_i^2 F^2 m^2}{\lambda_1} L_{1j}^2 (\alpha_{jil}^* + \alpha_{lij}^*)^2 \bar{e}_j^2(t-\theta) \\
& + \sum_{i,r;j=1}^{n;m} \frac{\bar{q}_j^2 \bar{F}^2 n^2}{\lambda_1} M_{1i}^2 (\bar{\alpha}_{ijr}^* + \bar{\alpha}_{rji}^*)^2 e_i^2(t-\tau) + \sum_{i,j,l=1}^{n;m} \frac{2\bar{p}_i^2 G^2 m^2}{\lambda_2} \left(L_{2j} \beta_{jil}^* \hat{h}_{jil} \right)^2 \hat{k}_{jil} \int_0^\infty k_{jil}(s) \bar{e}_j^2(t-s) ds \\
& + \sum_{i,r;j=1}^{n;m} \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\lambda_2} \left(M_{2i} \bar{\beta}_{ijr}^* \hat{h}_{ijr} \right)^2 \hat{k}_{ijr} \int_0^\infty \bar{k}_{ijr}(s) e_i^2(t-s) ds + \sum_{i,j,l=1}^{n;m} \frac{2\bar{p}_i^2 G^2 m^2}{\lambda_2} \left(L_{2j} \beta_{lij}^* \hat{k}_{lij} \right)^2 \hat{h}_{lij} \\
& \quad \times \int_0^\infty h_{lij}(s) \bar{e}_j^2(t-s) ds + \sum_{i,r;j=1}^{n;m} \frac{2\bar{q}_j^2 \bar{G}^2 n^2}{\lambda_2} \left(M_{2i} \bar{\beta}_{rji}^* \hat{k}_{rji} \right)^2 \hat{h}_{rji} \int_0^\infty \bar{h}_{rji}(s) e_i^2(t-s) ds \\
& + 2 \sum_{i=1}^n \bar{p}_i |e_i(t)| \sum_{j,l=1}^m \left[F^2 |\check{a}_{jil} - \hat{a}_{jil}| + G^2 |\check{b}_{jil} - \hat{b}_{jil}| \hat{k}_{jil} \hat{h}_{jil} \right] - 2 \sum_{i=1}^n \pi_i(t) |e_i(t)| \\
& + 2 \sum_{j=1}^m \bar{q}_j |\bar{e}_j(t)| \sum_{i,r=1}^n \left[\bar{F}^2 |\check{a}_{ijr} - \hat{a}_{ijr}| + \bar{G}^2 |\check{b}_{ijr} - \hat{b}_{ijr}| \hat{k}_{ijr} \hat{h}_{ijr} \right] - 2 \sum_{j=1}^m \bar{\pi}_j(t) |\bar{e}_j(t)|.
\end{aligned} \tag{4.165}$$

Next, we present the following Lyapunov functionals

$$\begin{aligned}
W_2(t) &= e^{-\kappa_1 t} \sum_{i=1}^n \left[\frac{1}{\epsilon_i} (\vartheta_i(t) - \frac{\omega_i}{2})^2 + \frac{1}{o_i} (\pi_i(t) - \gamma_i)^2 \right], \\
W_3(t) &= e^{-\kappa_2 t} \sum_{j=1}^m \left[\frac{1}{\bar{\epsilon}_j} (\bar{\vartheta}_j(t) - \frac{\bar{\omega}_j}{2})^2 + \frac{1}{\bar{o}_j} (\bar{\pi}_j(t) - \bar{\gamma}_j)^2 \right], \\
W_4(t) &= e^{-\kappa_3 t} \sum_{i,j,l=1}^{n;m} \chi_{1jil} \int_{t-\theta}^t e^{\kappa_3(s+\theta)} \bar{e}_j^2(s) ds, \\
W_5(t) &= e^{-\kappa_4 t} \sum_{i,r;j=1}^{n;m} \bar{\chi}_{1ijr} \int_{t-\tau}^t e^{\kappa_4(s+\tau)} e_i^2(s) ds, \\
W_6(t) &= \sum_{i,j,l=1}^{n;m} \chi_{2jil} \int_0^\infty k_{jil}(s) \int_{t-s}^t \bar{e}_j^2(\lambda) d\lambda ds, \\
W_7(t) &= \sum_{i,r;j=1}^{n;m} \bar{\chi}_{2ijr} \int_0^\infty \bar{k}_{ijr}(s) \int_{t-s}^t e_i^2(\lambda) d\lambda ds, \\
W_8(t) &= \sum_{i,j,l=1}^{n;m} \chi_{3lij} \int_0^\infty h_{lij}(s) \int_{t-s}^t \bar{e}_j^2(\lambda) d\lambda ds, \\
W_9(t) &= \sum_{i,r;j=1}^{n;m} \bar{\chi}_{3rji} \int_0^\infty \bar{h}_{rji}(s) \int_{t-s}^t e_i^2(\lambda) d\lambda ds,
\end{aligned} \tag{4.166}$$

with $\kappa_1 > 0, \kappa_2 > 0, \omega_i, \bar{\omega}_j, \gamma_i, \bar{\gamma}_j$ are constants, κ_3 and κ_4 are chosen so small that $e^{\kappa_3 \theta} < 1 + \varepsilon$ and $e^{\kappa_4 \tau} < 1 + \bar{\varepsilon}$.

According to (A₅) and (A₇), the derivatives of $V_k(t)$, $k = 2, 3, \dots, 9$ are given by

$$W_2'(t) = -\kappa_1 W_2(t) + \sum_{i=1}^n \left[(2\vartheta_i(t) - \omega_i) e_i^2(t) + 2(\pi_i(t) - \gamma_i) |e_i(t)| \right], \tag{4.167}$$

$$W_3'(t) = -\kappa_2 W_3(t) + \sum_{j=1}^m \left[(2\bar{\vartheta}_j(t) - \bar{\omega}_j) \bar{e}_j^2(t) + 2(\bar{\pi}_j(t) - \bar{o}_j) |\bar{e}_j(t)| \right], \tag{4.168}$$

$$W_4'(t) = -\kappa_3 W_4(t) + e^{\kappa_3 \theta} \sum_{i,j,l=1}^{n;m} \chi_{1jil} \bar{e}_j^2(t) - \sum_{i,j,l=1}^{n;m} \chi_{1jil} \bar{e}_j^2(t - \theta), \tag{4.169}$$

$$W_5'(t) = -\kappa_4 W_5(t) + e^{\kappa_4 \tau} \sum_{i,r;j=1}^{n;m} \bar{\chi}_{1ijr} e_i^2(t) - \sum_{i,r;j=1}^{n;m} \bar{\chi}_{1ijr} e_i^2(t - \tau), \tag{4.170}$$

$$\begin{aligned}
W_p'(t) &\leq \sum_{j=1}^m \mathfrak{B}_{pj} \bar{e}_j^2(t) - \sigma \mathfrak{E}_p(t) W_p(t) - (1 - \sigma) \sum_{j=1}^m \int_0^\infty \mathfrak{D}_{pj}(s) \bar{e}_j^2(t - s) ds, \quad p = 6, 8, \\
W_q'(t) &\leq \sum_{i=1}^n \bar{\mathfrak{B}}_{qi} e_i^2(t) - \sigma \bar{\mathfrak{E}}_q(t) W_q(t) - (1 - \sigma) \sum_{i=1}^n \int_0^\infty \bar{\mathfrak{D}}_{qi}(s) e_i^2(t - s) ds, \quad q = 7, 9,
\end{aligned} \tag{4.171}$$

$$\begin{aligned}
& \text{where } 0 < \sigma < 1, \quad \mathfrak{B}_{6j} = \sum_{i;l=1}^{n;m} \hat{\chi}_{2jil}, \quad \mathfrak{B}_{8j} = \sum_{i;l=1}^{n;m} \hat{\chi}_{3lij}, \quad \bar{\mathfrak{B}}_{7i} = \sum_{r;j=1}^{n;m} \hat{\chi}_{2ijr}, \quad \bar{\mathfrak{B}}_{9i} = \sum_{r;j=1}^{n;m} \hat{\chi}_{3rji}, \\
& \mathfrak{C}_6(t) = \eta_1(t), \quad \mathfrak{C}_8(t) = \xi_1(t), \quad \bar{\mathfrak{C}}_7(t) = \bar{\eta}_1(t), \quad \bar{\mathfrak{C}}_9(t) = \bar{\xi}_1(t), \quad \mathfrak{D}_{6j}(t) = \sum_{i;l=1}^{n;m} \chi_{2jil} k_{jil}(t), \\
& \mathfrak{D}_{8j}(t) = \sum_{i;l=1}^{n;m} \chi_{3lij} h_{lij}(t), \quad \bar{\mathfrak{D}}_{7i}(t) = \sum_{r;j=1}^{n;m} \bar{\chi}_{2ijr} \bar{k}_{ijr}(t), \quad \bar{\mathfrak{D}}_{9i}(t) = \sum_{r;j=1}^{n;m} \bar{\chi}_{3rji} \bar{h}_{rji}(t).
\end{aligned}$$

Then, we construct the Lyapunov functional

$$W(t) = \sum_{l=1}^5 W_l(t) + \frac{1}{1-\sigma} \sum_{k=6}^9 W_k(t).$$

From the relations (4.165)–(4.171) and hypotheses of Theorem 4.6.1, we infer that

$$W'(t) \leq -\check{A}W_1(t) - \sum_{l=1}^4 \kappa_l W_{l+1}(t) - \frac{\sigma\delta(t)}{1-\sigma} \sum_{k=6}^9 W_k(t), \quad (4.172)$$

where $\check{A} = \min\{\hat{A}, \check{A}\}$.

Finally, we discuss two cases which depend on the limit values of $\delta(t)$ to establish the estimation of the functional $W(t)$.

Case 1

If $0 < \delta^* < \infty$, then, a real number $T > 0$ can be found such that

$$\delta(t) \geq \delta^*, \quad t \geq T. \quad (4.173)$$

Then,

$$W'(t) \leq -\check{A}W_1(t) - \sum_{l=1}^4 \kappa_l W_{l+1}(t) - \frac{\sigma\delta^*}{(1-\sigma)} \sum_{k=6}^9 W_k(t), \quad t \geq T. \quad (4.174)$$

Furthermore,

$$W'(t) \leq -CW(t), \quad t \geq T, \quad C = \min \left\{ \check{A}, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \frac{\sigma\delta^*}{(1-\sigma)} \right\}, \quad (4.175)$$

which implies that

$$W(t) \leq W(T)e^{-C(t-T)} \leq W(0)e^{-C(t-T)}, \quad t \geq T. \quad (4.176)$$

For $t \in [0, T]$, we obtain

$$W(t) \leq W(0) \leq W(0)e^{C(T-t)} \leq W(0)e^{-C(t-T)} \leq \left(W(0)e^{CT}\right)e^{-Ct}.$$

A similar argument is used for $\delta^* = \infty$, using the fact that there exist $T > 0$ and $\bar{\delta} > 0$ such that $\delta(t) \geq \bar{\delta}$, $t \geq T$.

Case 2

If $\delta^* = 0$, then, we can find $\bar{T} > 0$ fulfilling

$$\frac{\sigma}{1-\sigma}\delta(t) \leq \min \left\{ \check{A}, \kappa_1, \kappa_2, \kappa_3, \kappa_4 \right\}, \quad t \geq \bar{T}. \quad (4.177)$$

Hence,

$$W'(t) \leq -\sigma^*\delta(t)W(t), \quad t \geq \bar{T}, \quad (4.178)$$

for some $\sigma^* := \frac{\sigma}{1-\sigma} > 0$. Consequently, we obtain

$$W(t) \leq W(\bar{T})e^{-\sigma^* \int_{\bar{T}}^t \delta(s)ds} \leq \frac{W(0)}{1-\sigma}e^{-\sigma^* \int_{\bar{T}}^t \delta(s)ds}, \quad t \geq \bar{T}. \quad (4.179)$$

When $0 \leq t \leq \bar{T}$, it is clear that

$$W(t) \leq W(0)e^{\sigma^* \int_t^{\bar{T}} \delta(s)ds} \leq W(0)e^{-\sigma^* \int_{\bar{T}}^t \delta(s)ds}.$$

Remark 4.6.1

- The asymptotic and exponential synchronization results obtained in Chen, C. et al., 2018 [97], Li, N. and Cao, J. 2015 [98], Liu, M., et al., 2019 [99], Wu, A. et al., 2012 [120], Yang, X. et al., 2014 [2], Zhang, G., et al., 2015 [132] and Zhang, G., et al., 2018 [127] are generalized here. In addition, the ψ -synchronization results have been established in Jia, S. et al., 2018 [103], Sader, S. [104] and Yang, M. and Jiang, M., 2020 [105] under constrained conditions on ψ function compared to the conditions on the function δ used here.
- The sufficient conditions obtained here are independent of the switching jumps compared to those in Cao, J. and Lu, J., 2006 [106], Chen, C., et al., 2018 [97] and Yan, M. and Jiang, M., 2020 [105]. Thus, the results can be applied to non-memristive neurodynamic systems.
- The adaptive controllers used here enhance the ones in Jia, S. et al., 2018 [103] and Wu, A. et al., 2012 [120] allowing faster synchronization and using small gains.

4.7 Numerical Illustration

In this section, we provide three examples validating the theoretical results in the bounded and unbounded cases.

Example 4.7.1

Consider a memristive higher-order CG BAM NN system as follows:

$$\left\{ \begin{array}{l} x'_i(t) = p_i(x_i(t)) \left[-\Lambda_i(x_i(t)) + \sum_{j,l=1}^2 a_{jil}(x_i(t)) f_j(y_j(t-\theta)) f_l(y_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^2 b_{jil}(x_i(t)) \int_0^t k_{jil}(s) g_j(y_j(t-s)) ds \int_0^t h_{jil}(s) g_l(y_l(t-s)) ds + I_i \right], \\ y'_j(t) = q_j(y_j(t)) \left[-\bar{\Lambda}_j(y_j(t)) + \sum_{i,r=1}^2 \bar{a}_{ijr}(y_j(t)) \bar{f}_i(x_i(t-\tau)) \bar{f}_r(x_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^2 \bar{b}_{ijr}(y_j(t)) \int_0^t \bar{k}_{ijr}(s) \bar{g}_i(x_i(t-s)) ds \int_0^t \bar{h}_{ijr}(s) \bar{g}_r(x_r(t-s)) ds + J_j \right], \end{array} \right. \quad (4.180)$$

$i, r, j, l = 1, 2, t \in [0, 10]$, where

$$\begin{aligned}
p_1(x) &= p_2(x) = 1 + \frac{1}{1+x^2}, \quad q_1(x) = q_2(x) = 2 - \frac{1}{1+x^2}, \\
\Lambda_1(x) &= 7.5x, \quad \Lambda_2(x) = 5.5x, \quad \bar{\Lambda}_1(x) = 6x, \quad \bar{\Lambda}_2(x) = 8x, \\
f_j(x) &= g_j(x) = \frac{1}{2}(|x+1| - |x-1|), \quad \bar{f}_i(x) = \bar{g}_i(x) = \tanh(x), \quad i, j = 1, 2, \\
k_{jil}(t) &= h_{jil}(t) = \bar{k}_{ijr}(t) = \bar{h}_{ijr}(t) = e^{-4t}, \quad i, r, j, l = 1, 2, \quad I_1 = 1.5, \quad I_2 = 0.5, \quad J_1 = 2, \\
J_2 &= 2.5, \quad \tau = 1, \quad \theta = 2, \quad x_1(s) = 1.25, \quad x_2(s) = 0.5, \quad y_1(s) = 0.75, \quad y_2(s) = 1, \quad s \in [-10, 0], \\
\end{aligned} \tag{4.181}$$

$$a_{111}(x_1(t)) = \begin{cases} -0.1, & |x_1(t)| > 1, \\ 0.09, & |x_1(t)| \leq 1, \end{cases} \quad a_{112}(x_1(t)) = \begin{cases} 0.16, & |x_1(t)| > 1, \\ -0.12, & |x_1(t)| \leq 1, \end{cases} \tag{4.182}$$

$$a_{211}(x_1(t)) = \begin{cases} -0.28, & |x_1(t)| > 1, \\ 0.17, & |x_1(t)| \leq 1, \end{cases} \quad a_{212}(x_1(t)) = \begin{cases} -0.19, & |x_1(t)| > 1, \\ 0.11, & |x_1(t)| \leq 1, \end{cases} \tag{4.183}$$

$$a_{121}(x_2(t)) = \begin{cases} 0.12, & |x_2(t)| > 1, \\ -0.1, & |x_2(t)| \leq 1, \end{cases} \quad a_{122}(x_2(t)) = \begin{cases} -0.11, & |x_2(t)| > 1, \\ 0.09, & |x_2(t)| \leq 1, \end{cases} \tag{4.184}$$

$$a_{221}(x_2(t)) = \begin{cases} -0.15, & |x_2(t)| > 1, \\ -0.12, & |x_2(t)| \leq 1, \end{cases} \quad a_{222}(x_2(t)) = \begin{cases} 0.36, & |x_2(t)| > 1, \\ -0.16, & |x_2(t)| \leq 1, \end{cases} \tag{4.185}$$

$$b_{111}(x_1(t)) = \begin{cases} 2.2, & |x_1(t)| > 1, \\ -1.4, & |x_1(t)| \leq 1, \end{cases} \quad b_{112}(x_1(t)) = \begin{cases} -1.5, & |x_1(t)| > 1, \\ -1.2, & |x_1(t)| \leq 1, \end{cases} \tag{4.186}$$

$$b_{211}(x_1(t)) = \begin{cases} -1.1, & |x_1(t)| > 1, \\ 0.7, & |x_1(t)| \leq 1, \end{cases} \quad b_{212}(x_1(t)) = \begin{cases} -1.7, & |x_1(t)| > 1, \\ -1, & |x_1(t)| \leq 1, \end{cases} \tag{4.187}$$

$$b_{121}(x_2(t)) = \begin{cases} -1.4, & |x_2(t)| > 1, \\ -1, & |x_2(t)| \leq 1, \end{cases} \quad b_{122}(x_2(t)) = \begin{cases} -2.1, & |x_2(t)| > 1, \\ 1.5, & |x_2(t)| \leq 1, \end{cases} \tag{4.188}$$

$$b_{221}(x_2(t)) = \begin{cases} -1.8, & |x_2(t)| > 1, \\ -1.4, & |x_2(t)| \leq 1, \end{cases} \quad b_{222}(x_2(t)) = \begin{cases} 2.9, & |x_2(t)| > 1, \\ 2.4, & |x_2(t)| \leq 1, \end{cases} \tag{4.189}$$

$$\bar{a}_{111}(y_1(t)) = \begin{cases} 0.125, & |y_1(t)| > 1, \\ 0.1, & |y_1(t)| \leq 1, \end{cases} \quad \bar{a}_{112}(y_1(t)) = \begin{cases} -0.11, & |y_1(t)| > 1, \\ 0.014, & |y_1(t)| \leq 1, \end{cases} \tag{4.190}$$

$$\bar{a}_{211}(y_1(t)) = \begin{cases} 0.31, & |y_1(t)| > 1, \\ -0.23, & |y_1(t)| \leq 1, \end{cases} \quad \bar{a}_{212}(y_1(t)) = \begin{cases} -0.16, & |y_1(t)| > 1, \\ -0.11, & |y_1(t)| \leq 1, \end{cases} \quad (4.191)$$

$$\bar{a}_{121}(y_2(t)) = \begin{cases} -0.31, & |y_2(t)| > 1, \\ 0.1, & |y_2(t)| \leq 1, \end{cases} \quad \bar{a}_{122}(y_2(t)) = \begin{cases} -0.14, & |y_2(t)| > 1, \\ 0.125, & |y_2(t)| \leq 1, \end{cases} \quad (4.192)$$

$$\bar{a}_{221}(y_2(t)) = \begin{cases} -0.23, & |y_2(t)| > 1, \\ -0.18, & |y_2(t)| \leq 1, \end{cases} \quad \bar{a}_{222}(y_2(t)) = \begin{cases} -0.22, & |y_2(t)| > 1, \\ -0.15, & |y_2(t)| \leq 1, \end{cases} \quad (4.193)$$

$$\bar{b}_{111}(y_1(t)) = \begin{cases} 2.4, & |y_1(t)| > 1, \\ -1.6, & |y_1(t)| \leq 1, \end{cases} \quad \bar{b}_{112}(y_1(t)) = \begin{cases} -1.2, & |y_1(t)| > 1, \\ 0.5, & |y_1(t)| \leq 1, \end{cases} \quad (4.194)$$

$$\bar{b}_{211}(y_1(t)) = \begin{cases} -0.94, & |y_1(t)| > 1, \\ -0.25, & |y_1(t)| \leq 1, \end{cases} \quad \bar{b}_{212}(y_1(t)) = \begin{cases} 1.3, & |y_1(t)| > 1, \\ 1, & |y_1(t)| \leq 1, \end{cases} \quad (4.195)$$

$$\bar{b}_{121}(y_2(t)) = \begin{cases} 1.8, & |y_2(t)| > 1, \\ 0.5, & |y_2(t)| \leq 1, \end{cases} \quad \bar{b}_{122}(y_2(t)) = \begin{cases} -1.6, & |y_2(t)| > 1, \\ -1, & |y_2(t)| \leq 1, \end{cases} \quad (4.196)$$

$$\bar{b}_{221}(y_2(t)) = \begin{cases} -1.4, & |y_2(t)| > 1, \\ 0.25, & |y_2(t)| \leq 1, \end{cases} \quad \bar{b}_{222}(y_2(t)) = \begin{cases} 2.6, & |y_2(t)| > 1, \\ 1.9, & |y_2(t)| \leq 1. \end{cases} \quad (4.197)$$

Obviously, $\underline{p}_i = \underline{q}_j = 1$, $\bar{p}_i = \bar{q}_j = 2$, $\nu_1 = 7.5$, $\nu_2 = 5.5$, $\bar{\nu}_1 = 6$, $\bar{\nu}_2 = 8$, $F = \bar{F} = G = \bar{G} = 1$, $L_{1j} = L_{2j} = M_{1i} = M_{2i} = 1$, $k_{jil} = h_{jil} = \hat{k}_{ijr} = \hat{h}_{ijr} = 0.25$, $\gamma_1 = \gamma_2 = 2.75$, $\bar{\gamma}_1 = \bar{\gamma}_2 = 3$, and choosing $\eta_{1jil}(t) = \xi_{1jil}(t) = \bar{\eta}_{1ijr}(t) = \bar{\xi}_{1ijr}(t) = 0.25$, $\varepsilon_1 = 0.005$, $\varepsilon_2 = 0.01$, $\sigma = 0.5$.

Thus, the assumptions (A_1) – (A_7) besides the inequalities stated in Theorem 4.3.1 are met. Via Theorem 4.3.1, the stationary point of neurodynamic system (4.180) can be globally stable. Fig. 4.1 and Fig. 4.2 illustrate the convergence of solutions of system (4.180) to the steady point, with the initial data $(x_1(s), x_2(s); y_1(s), y_2(s)) = (1.25, 0.5; 0.75, 1)$, $s \in [-10, 0]$ and with 40 initial data that selected at random, respectively.

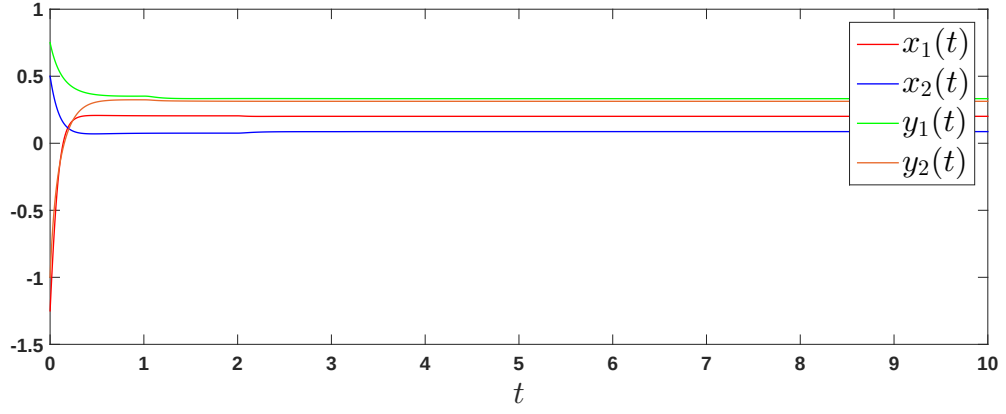


Figure 4.1: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$

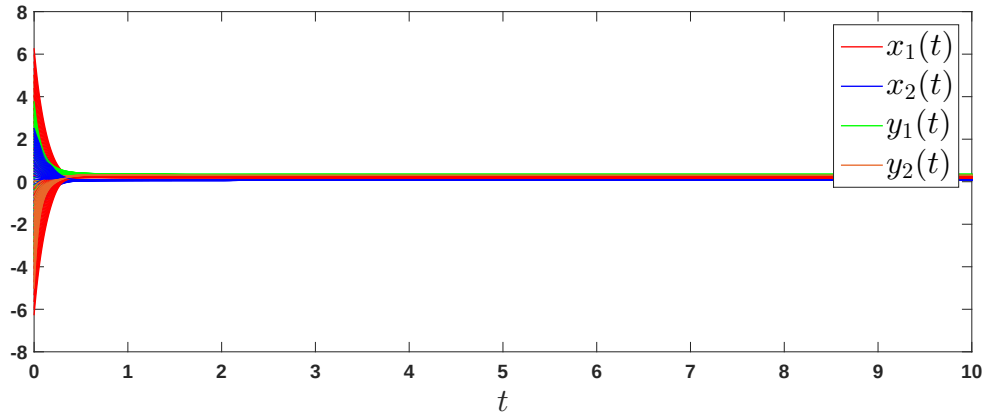


Figure 4.2: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ with Random Initial Data

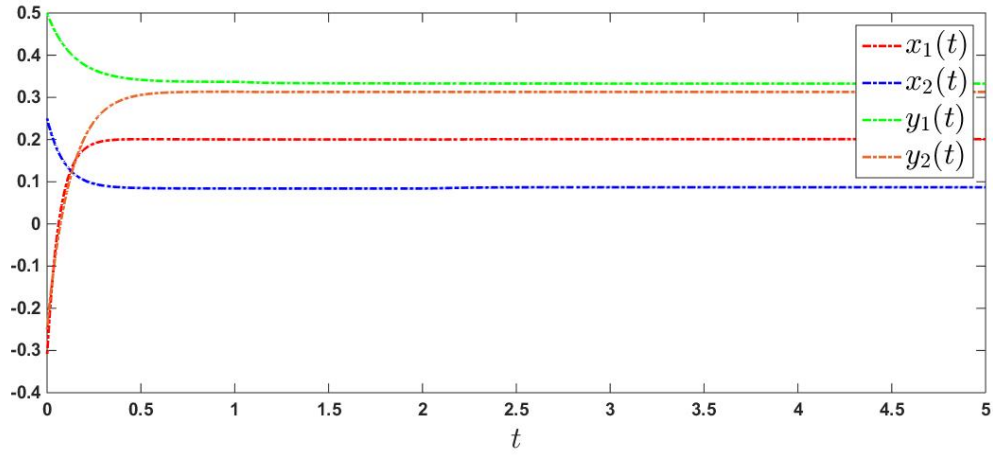


Figure 4.3: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$

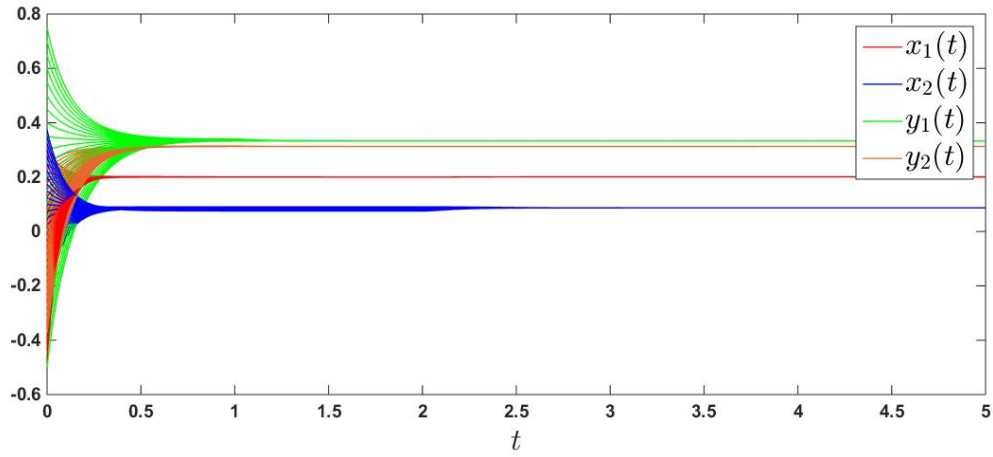


Figure 4.4: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ with Random Initial Data

Example 4.7.2

Consider the system (4.180) with the same all functions and parameters in Example (4.7.1) and with initial history $(0.3125, 0.25, 0.5, -0.25)$.

Thus, the assumptions (A_1) – (A_7) besides the inequalities stated in Theorem 4.3.1 are met. Via Theorem 4.3.1, the stationary point of neurodynamic system (4.180) can be globally stable.

Fig. 4.3 and Fig. 4.4 illustrate the convergence of solutions of system (4.180) to the steady point, with the initial data $(0.3125, 0.25, 0.5, -0.25)$ via interval $[-5, 0]$ and with 40 initial data that selected at random, respectively.

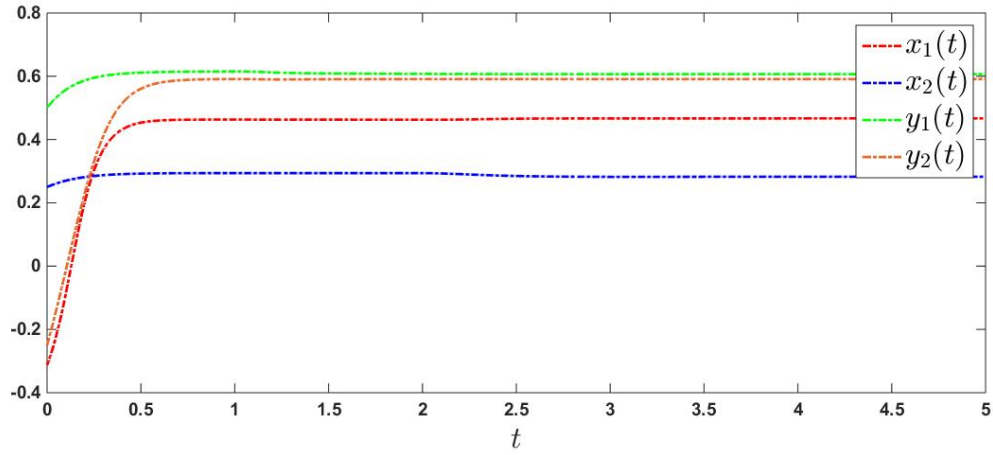


Figure 4.5: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$

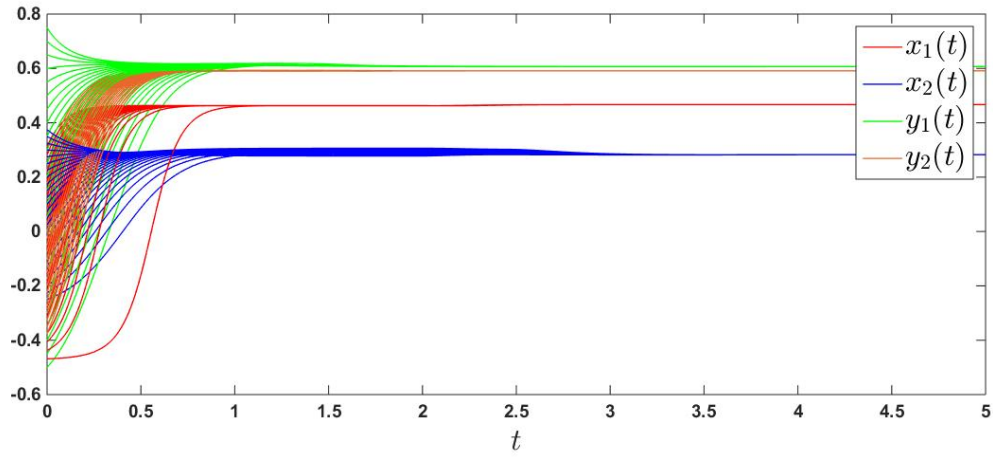


Figure 4.6: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ with Random Initial Data

Example 4.7.3

Consider the system (4.180) with the same all functions and parameters in Example (4.7.1), with initial history $[0.3125, 0.25, 0.5, -0.25]$ and with nonlinear appropriate behaved (self-inhibition) or (self-signal) functions

$$\Lambda_i(t) = \bar{\Lambda}_j(t) = \tanh(t). \quad (4.198)$$

Thus, the assumptions (A_1) – (A_7) besides the inequalities stated in Theorem 4.3.1 are met. Via Theorem 4.3.1, the stationary point of neurodynamic system (4.180) can be globally stable.

Fig. 4.5 and Fig. 4.6 illustrate the convergence of solutions of system (4.180) to the steady point, with the initial data $(0.3125, 0.25, 0.5, -0.25)$ via interval $[-5, 0]$ and with 40

initial data that selected at random, respectively.

Example 4.7.4

Consider the memristive higher-order neurodynamic system (4.180) with $i, r, j, l = 1, 2$, $t \in [0, 20]$ where the functions $p_i(x_i(t))$, $q_j(y_j(t))$, $k_{jil}(t)$, $h_{jil}(t)$, $\bar{k}_{ijr}(t)$, $\bar{h}_{ijr}(t)$ and the parameters τ and θ are given the same as in system (4.180), the activation functions are taken as follows

$$f_j(x) = g_j(x) = \text{argsh}(x), \quad j = 1, 2, \quad \bar{f}_i(x) = \bar{g}_i(x) = \sqrt{1 + x^2}, \quad i = 1, 2, \quad (4.199)$$

and

$$\begin{aligned} \Lambda_1(x) &= x, \quad \Lambda_2(x) = 1.5x, \quad \bar{\Lambda}_1(x) = 2x, \quad \bar{\Lambda}_2(x) = 2.5x, \\ I_1 &= 0.08, \quad I_2 = 0.05, \quad J_1 = 0.06, \quad J_2 = 0.09, \\ x_1(s) &= 0.1, \quad x_2(s) = 0.2, \quad y_1(s) = 0.13, \quad y_2(s) = 0.24, \quad s \in [-20, 0], \end{aligned} \quad (4.200)$$

$$a_{111}(x_1(t)) = \begin{cases} -0.0023, & |x_1(t)| > 1, \\ -0.0019, & |x_1(t)| \leq 1, \end{cases} \quad a_{112}(x_1(t)) = \begin{cases} 0.002, & |x_1(t)| > 1, \\ 0.0031, & |x_1(t)| \leq 1, \end{cases} \quad (4.201)$$

$$a_{211}(x_1(t)) = \begin{cases} -0.0024, & |x_1(t)| > 1, \\ -0.0034, & |x_1(t)| \leq 1, \end{cases} \quad a_{212}(x_1(t)) = \begin{cases} -0.0017, & |x_1(t)| > 1, \\ -0.0021, & |x_1(t)| \leq 1, \end{cases} \quad (4.202)$$

$$a_{121}(x_2(t)) = \begin{cases} -0.0018, & |x_2(t)| > 1, \\ -0.0026, & |x_2(t)| \leq 1, \end{cases} \quad a_{122}(x_2(t)) = \begin{cases} 0.003, & |x_2(t)| > 1, \\ 0.002, & |x_2(t)| \leq 1, \end{cases} \quad (4.203)$$

$$a_{221}(x_2(t)) = \begin{cases} -0.0033, & |x_2(t)| > 1, \\ -0.0023, & |x_2(t)| \leq 1, \end{cases} \quad a_{222}(x_2(t)) = \begin{cases} -0.0024, & |x_2(t)| > 1, \\ -0.0016, & |x_2(t)| \leq 1, \end{cases} \quad (4.204)$$

$$b_{111}(x_1(t)) = \begin{cases} 0.012, & |x_1(t)| > 1, \\ 0.014, & |x_1(t)| \leq 1, \end{cases} \quad b_{112}(x_1(t)) = \begin{cases} -0.031, & |x_1(t)| > 1, \\ -0.022, & |x_1(t)| \leq 1, \end{cases} \quad (4.205)$$

$$b_{211}(x_1(t)) = \begin{cases} -0.018, & |x_1(t)| > 1, \\ -0.027, & |x_1(t)| \leq 1, \end{cases} \quad b_{212}(x_1(t)) = \begin{cases} -0.015, & |x_1(t)| > 1, \\ -0.013, & |x_1(t)| \leq 1, \end{cases} \quad (4.206)$$

$$b_{121}(x_2(t)) = \begin{cases} 0.019, & |x_2(t)| > 1, \\ 0.011, & |x_2(t)| \leq 1, \end{cases} \quad b_{122}(x_2(t)) = \begin{cases} -0.021, & |x_2(t)| > 1, \\ -0.032, & |x_2(t)| \leq 1, \end{cases} \quad (4.207)$$

$$b_{221}(x_2(t)) = \begin{cases} -0.024, & |x_2(t)| > 1, \\ -0.013, & |x_2(t)| \leq 1, \end{cases} \quad b_{222}(x_2(t)) = \begin{cases} -0.027, & |x_2(t)| > 1, \\ -0.024, & |x_2(t)| \leq 1, \end{cases} \quad (4.208)$$

$$\bar{a}_{111}(y_1(t)) = \begin{cases} -0.0015, & |y_1(t)| > 1, \\ -0.0012, & |y_1(t)| \leq 1, \end{cases} \quad \bar{a}_{112}(y_1(t)) = \begin{cases} 0.0032, & |y_1(t)| > 1, \\ 0.0028, & |y_1(t)| \leq 1, \end{cases} \quad (4.209)$$

$$\bar{a}_{211}(y_1(t)) = \begin{cases} -0.0036, & |y_1(t)| > 1, \\ -0.0022, & |y_1(t)| \leq 1, \end{cases} \quad \bar{a}_{212}(y_1(t)) = \begin{cases} -0.0024, & |y_1(t)| > 1, \\ -0.0026, & |y_1(t)| \leq 1, \end{cases} \quad (4.210)$$

$$\bar{a}_{121}(y_2(t)) = \begin{cases} -0.0011, & |y_2(t)| > 1, \\ -0.0014, & |y_2(t)| \leq 1, \end{cases} \quad \bar{a}_{122}(y_2(t)) = \begin{cases} 0.0026, & |y_2(t)| > 1, \\ 0.0015, & |y_2(t)| \leq 1, \end{cases} \quad (4.211)$$

$$\bar{a}_{221}(y_2(t)) = \begin{cases} -0.0041, & |y_2(t)| > 1, \\ -0.0026, & |y_2(t)| \leq 1, \end{cases} \quad \bar{a}_{222}(y_2(t)) = \begin{cases} -0.0023, & |y_2(t)| > 1, \\ -0.0025, & |y_2(t)| \leq 1, \end{cases} \quad (4.212)$$

$$\bar{b}_{111}(y_1(t)) = \begin{cases} 0.012, & |y_1(t)| > 1, \\ 0.014, & |y_1(t)| \leq 1, \end{cases} \quad \bar{b}_{112}(y_1(t)) = \begin{cases} -0.025, & |y_1(t)| > 1, \\ -0.032, & |y_1(t)| \leq 1, \end{cases} \quad (4.213)$$

$$\bar{b}_{211}(y_1(t)) = \begin{cases} -0.03, & |y_1(t)| > 1, \\ -0.023, & |y_1(t)| \leq 1, \end{cases} \quad \bar{b}_{212}(y_1(t)) = \begin{cases} -0.031, & |y_1(t)| > 1, \\ -0.029, & |y_1(t)| \leq 1, \end{cases} \quad (4.214)$$

$$\bar{b}_{121}(y_2(t)) = \begin{cases} 0.024, & |y_2(t)| > 1, \\ 0.021, & |y_2(t)| \leq 1, \end{cases} \quad \bar{b}_{122}(y_2(t)) = \begin{cases} -0.017, & |y_2(t)| > 1, \\ -0.022, & |y_2(t)| \leq 1, \end{cases} \quad (4.215)$$

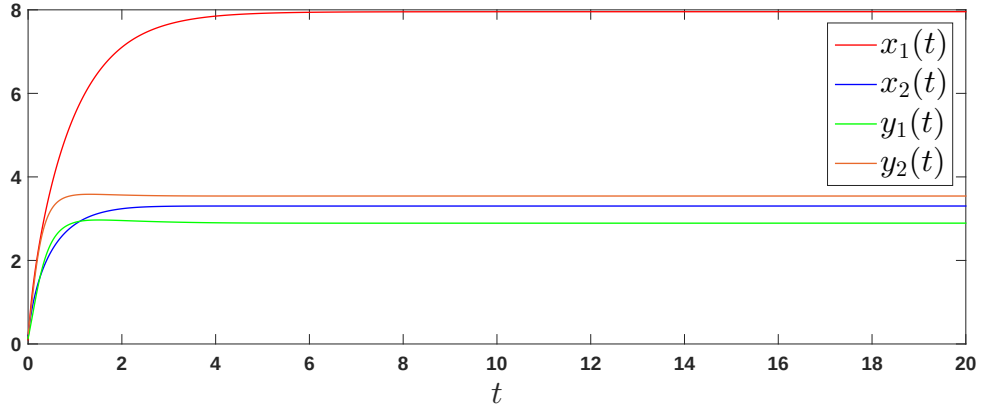


Figure 4.7: State Curves of $x_1(t), x_2(t), y_1(t)$ and $y_2(t)$

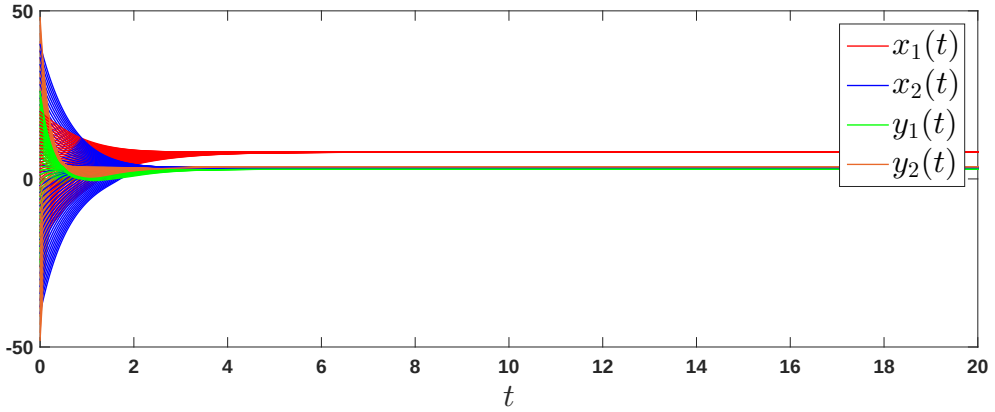


Figure 4.8: State Curves of $x_1(t), x_2(t), y_1(t)$ and $y_2(t)$ with Random Initial Data

$$\bar{b}_{221}(y_2(t)) = \begin{cases} -0.023, & |y_2(t)| > 1, \\ -0.019, & |y_2(t)| \leq 1, \end{cases} \quad \bar{b}_{222}(y_2(t)) = \begin{cases} -0.025, & |y_2(t)| > 1, \\ -0.027, & |y_2(t)| \leq 1. \end{cases} \quad (4.216)$$

Clearly, we can see that $L_{1j} = L_{2j} = M_{1i} = M_{2i} = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \frac{1}{40}$, $\bar{\gamma}_1 = \bar{\gamma}_2 = \bar{\gamma}_3 = \bar{\gamma}_4 = \bar{\gamma}_5 = \frac{1}{20}$, and by selecting $\eta_{\iota j \iota l}(t) = \xi_{\iota j \iota l}(t) = \bar{\eta}_{\iota i j r}(t) = \bar{\xi}_{\iota i j r}(t) = 0.5$, $\iota = 1, 2$, $\sigma = 0.5$. Then, the assumptions (A_1) , (A_2) , (A_4) – (A_7) and the conditions (4.103) are satisfied. Consequently, owing to Lemma 2.2.7, the steady state of system (4.180) is stable. The curves of solutions of system (4.180) are depicted in Fig. 4.7 and Fig. 4.8, with the initial data $(x_1(s), x_2(s); y_1(s), y_2(s)) = (0.1, 0.2; 0.13, 0.24)$, $s \in [-20, 0]$ and 40 initial data selected randomly, respectively.

Example 4.7.5

Consider the system (4.180) with the same all functions and parameters in Example

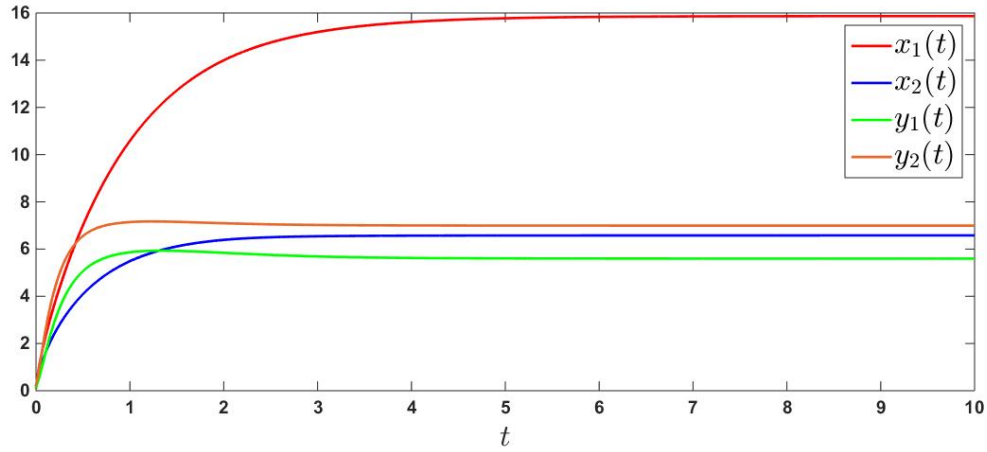


Figure 4.9: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$

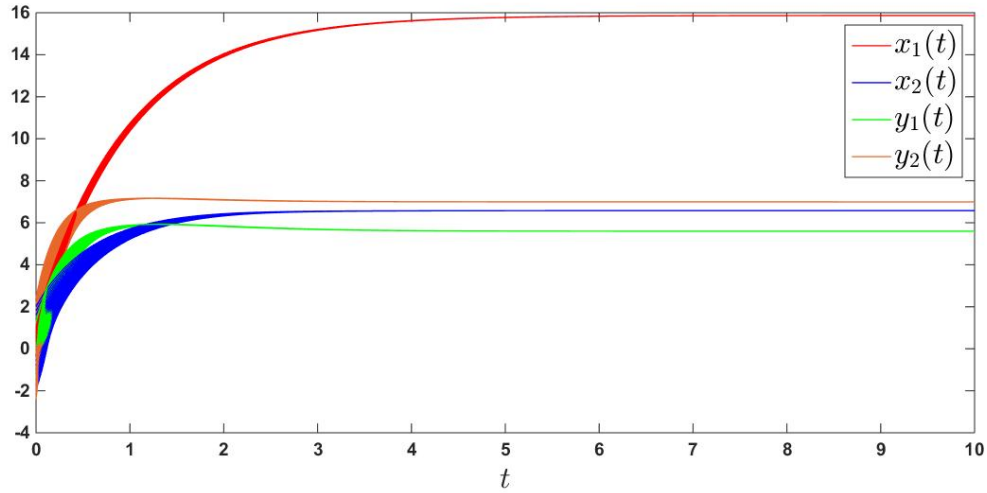


Figure 4.10: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ with Random Initial Data

(4.7.4), with initial history $(0.3125, 0.25, 0.5, -0.25)$.

The assumptions (A_1) , (A_2) , (A_4) – (A_7) and the conditions (4.103) are satisfied. Consequently, owing to Lemma 2.2.7, the steady state of system (4.180) is stable. The curves of solutions of system (4.180) are depicted in Fig. 4.9 and Fig. 4.10, with the initial data $(x_1(s), x_2(s); y_1(s), y_2(s)) = (0.3125, 0.25, 0.5, -0.25)$, $s \in [-10, 0]$ and 40 initial data selected randomly, respectively.

Example 4.7.6

Consider the system (4.180) with the same all functions and parameters in Example (4.7.4) and with initial history $(0.3125, 0.25, 0.5, -0.25)$ and with nonlinear appropriate

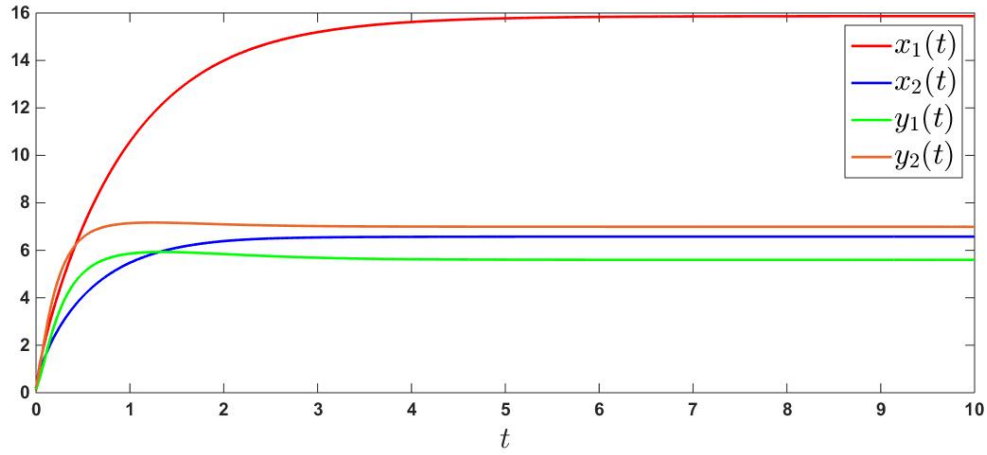


Figure 4.11: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$

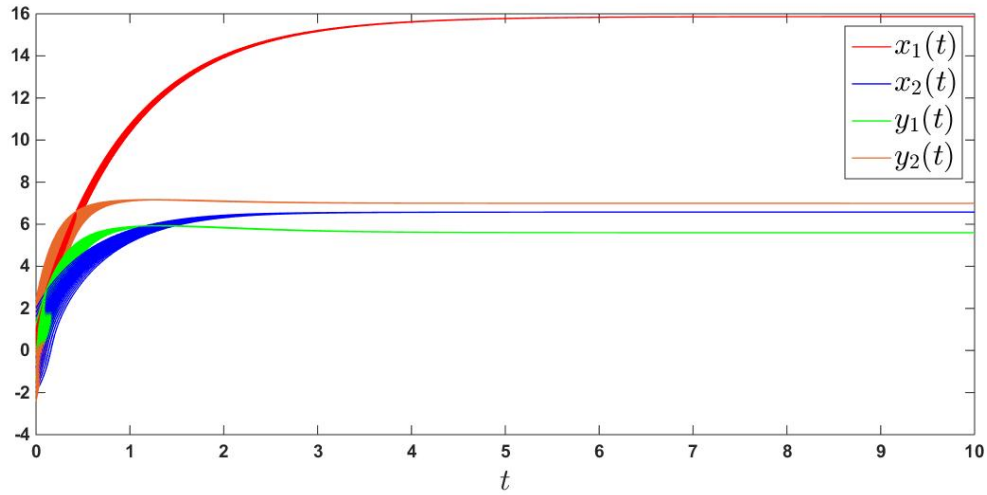


Figure 4.12: State Curves of $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ with Random Initial Data

behaved (self-inhibition) or (self-signal) functions

$$\Lambda_i(t) = \bar{\Lambda}_j(t) = \tanh(t). \quad (4.217)$$

The assumptions (A_1) , (A_2) , (A_4) – (A_7) and the conditions (4.103) are satisfied. Consequently, owing to Lemma 2.2.7, the steady state of system (4.180) is stable. The curves of solutions of system (4.180) are depicted in Fig. 4.11 and Fig. 4.12, with the initial data $(x_1(s), x_2(s); y_1(s), y_2(s)) = (0.3125, 0.25, 0.5, -0.25)$, $s \in [-10, 0]$ and 40 initial data selected randomly, respectively.

Example 4.7.7

Consider the memristive HOCGBAMNN system (4.180) in case of $i, r, j, l = 1, 2$, $t \in [0, 100]$ with

$$a_{111}(x_1(t)) = \begin{cases} 0.023, & |x_1(t)| > 0.1, \\ 0.019, & |x_1(t)| \leq 0.1, \end{cases} \quad a_{112}(x_1(t)) = \begin{cases} 2, & |x_1(t)| > 0.1, \\ 1.5, & |x_1(t)| \leq 0.1, \end{cases} \quad (4.218)$$

$$a_{211}(x_1(t)) = \begin{cases} 2.024, & |x_1(t)| > 0.1, \\ 3.034, & |x_1(t)| \leq 0.1, \end{cases} \quad a_{212}(x_1(t)) = \begin{cases} 0.017, & |x_1(t)| > 0.1, \\ 0.011, & |x_1(t)| \leq 0.1, \end{cases} \quad (4.219)$$

$$a_{121}(x_2(t)) = \begin{cases} 0.018, & |x_2(t)| > 0.1, \\ 0.016, & |x_2(t)| \leq 0.1, \end{cases} \quad a_{122}(x_2(t)) = \begin{cases} 2.3, & |x_2(t)| > 0.1, \\ 1.2, & |x_2(t)| \leq 0.1, \end{cases} \quad (4.220)$$

$$a_{221}(x_2(t)) = \begin{cases} 2.33, & |x_2(t)| > 0.1, \\ 4.23, & |x_2(t)| \leq 0.1, \end{cases} \quad a_{222}(x_2(t)) = \begin{cases} 0.024, & |x_2(t)| > 0.1, \\ 0.016, & |x_2(t)| \leq 0.1, \end{cases} \quad (4.221)$$

$$b_{111}(x_1(t)) = \begin{cases} -0.12, & |x_1(t)| > 0.1, \\ -0.1, & |x_1(t)| \leq 0.1, \end{cases} \quad b_{112}(x_1(t)) = \begin{cases} -0.31, & |x_1(t)| > 0.1, \\ -0.22, & |x_1(t)| \leq 0.1, \end{cases} \quad (4.222)$$

$$b_{211}(x_1(t)) = \begin{cases} -0.18, & |x_1(t)| > 0.1, \\ -0.14, & |x_1(t)| \leq 0.1, \end{cases} \quad b_{212}(x_1(t)) = \begin{cases} -0.15, & |x_1(t)| > 0.1, \\ -0.13, & |x_1(t)| \leq 0.1, \end{cases} \quad (4.223)$$

$$b_{121}(x_2(t)) = \begin{cases} -0.19, & |x_2(t)| > 0.1, \\ -0.11, & |x_2(t)| \leq 0.1, \end{cases} \quad b_{122}(x_2(t)) = \begin{cases} -0.21, & |x_2(t)| > 0.1, \\ -0.14, & |x_2(t)| \leq 0.1, \end{cases} \quad (4.224)$$

$$b_{221}(x_2(t)) = \begin{cases} -0.24, & |x_2(t)| > 0.1, \\ -0.13, & |x_2(t)| \leq 0.1, \end{cases} \quad b_{222}(x_2(t)) = \begin{cases} -0.27, & |x_2(t)| > 0.1, \\ -0.24, & |x_2(t)| \leq 0.1, \end{cases} \quad (4.225)$$

$$\bar{a}_{111}(y_1(t)) = \begin{cases} 0.015, & |y_1(t)| > 0.1, \\ 0.012, & |y_1(t)| \leq 0.1, \end{cases} \quad \bar{a}_{112}(y_1(t)) = \begin{cases} 1.72, & |y_1(t)| > 0.1, \\ 1.48, & |y_1(t)| \leq 0.1, \end{cases} \quad (4.226)$$

$$\bar{a}_{211}(y_1(t)) = \begin{cases} 1.76, & |y_1(t)| > 0.1, \\ 4.52, & |y_1(t)| \leq 0.1, \end{cases} \quad \bar{a}_{212}(y_1(t)) = \begin{cases} 0.024, & |y_1(t)| > 0.1, \\ 0.021, & |y_1(t)| \leq 0.1, \end{cases} \quad (4.227)$$

$$\bar{a}_{121}(y_2(t)) = \begin{cases} 0.011, & |y_2(t)| > 0.1, \\ 0.014, & |y_2(t)| \leq 0.1, \end{cases} \quad \bar{a}_{122}(y_2(t)) = \begin{cases} 1.86, & |y_2(t)| > 0.1, \\ 0.5, & |y_2(t)| \leq 0.1, \end{cases} \quad (4.228)$$

$$\bar{a}_{221}(y_2(t)) = \begin{cases} 1.9, & |y_2(t)| > 0.1, \\ 3.66, & |y_2(t)| \leq 0.1, \end{cases} \quad \bar{a}_{222}(y_2(t)) = \begin{cases} 0.023, & |y_2(t)| > 0.1, \\ 0.02, & |y_2(t)| \leq 0.1, \end{cases} \quad (4.229)$$

$$\bar{b}_{111}(y_1(t)) = \begin{cases} -0.12, & |y_1(t)| > 0.1, \\ -0.1, & |y_1(t)| \leq 0.1, \end{cases} \quad \bar{b}_{112}(y_1(t)) = \begin{cases} -0.25, & |y_1(t)| > 0.1, \\ -0.12, & |y_1(t)| \leq 0.1, \end{cases} \quad (4.230)$$

$$\bar{b}_{211}(y_1(t)) = \begin{cases} -0.3, & |y_1(t)| > 0.1, \\ -0.23, & |y_1(t)| \leq 0.1, \end{cases} \quad \bar{b}_{212}(y_1(t)) = \begin{cases} -0.31, & |y_1(t)| > 0.1, \\ -0.29, & |y_1(t)| \leq 0.1, \end{cases} \quad (4.231)$$

$$\bar{b}_{121}(y_2(t)) = \begin{cases} -0.24, & |y_2(t)| > 0.1, \\ -0.21, & |y_2(t)| \leq 0.1, \end{cases} \quad \bar{b}_{122}(y_2(t)) = \begin{cases} -0.17, & |y_2(t)| > 0.1, \\ -0.13, & |y_2(t)| \leq 0.1, \end{cases} \quad (4.232)$$

$$\bar{b}_{221}(y_2(t)) = \begin{cases} -0.23, & |y_2(t)| > 0.1, \\ -0.15, & |y_2(t)| \leq 0.1, \end{cases} \quad \bar{b}_{222}(y_2(t)) = \begin{cases} -0.25, & |y_2(t)| > 0.1, \\ -0.22, & |y_2(t)| \leq 0.1. \end{cases} \quad (4.233)$$

Let system (4.180) be the uncontrolled system and the associated controlled system is taken as

$$\begin{cases} \bar{x}'_i(t) = p_i(\bar{x}_i(t)) \left[-\Lambda_i(\bar{x}_i(t)) + \sum_{j,l=1}^2 a_{jil}(\bar{x}_i(t)) f_j(\bar{y}_j(t-\theta)) f_l(\bar{y}_l(t-\theta)) \right. \\ \quad \left. + \sum_{j,l=1}^2 b_{jil}(\bar{x}_i(t)) \int_0^t k_{jil}(s) g_j(\bar{y}_j(t-s)) ds \int_0^t h_{jil}(s) g_l(\bar{y}_l(t-s)) ds + I_i \right] + \Phi_i(t), \\ \bar{y}'_j(t) = q_j(\bar{y}_j(t)) \left[-\bar{\Lambda}_j(\bar{y}_j(t)) + \sum_{i,r=1}^2 \bar{a}_{ijr}(\bar{y}_j(t)) \bar{f}_i(\bar{x}_i(t-\tau)) \bar{f}_r(\bar{x}_r(t-\tau)) \right. \\ \quad \left. + \sum_{i,r=1}^2 \bar{b}_{ijr}(\bar{y}_j(t)) \int_0^t \bar{k}_{ijr}(s) \bar{g}_i(\bar{x}_i(t-s)) ds \int_0^t \bar{h}_{ijr}(s) \bar{g}_r(\bar{x}_r(t-s)) ds + J_j \right] + \Psi_j(t), \end{cases} \quad (4.234)$$

with

$$\Lambda_1(x) = x, \quad \Lambda_2(x) = 1.5x, \quad \bar{\Lambda}_1(x) = 2x, \quad \bar{\Lambda}_2(x) = 2.5x, \quad I_1 = I_2 = J_1 = J_2 = 0,$$

$$\varrho = \bar{\varrho} = 0.45.$$

The activation functions are

$$f_1(x) = \sin(|x| - 1),$$

$$g_1(x) = \sin(|x| - 1).$$

$$f_2(x) = \cos(|x| - 1),$$

$$g_2(x) = \cos(|x| - 1),$$

and

$$\bar{f}_i(x) = \bar{g}_i(x) = \tanh(|x| - 1),$$

for $i = 1, 2$.

The controllers $\Phi_i(t)$ and $\Psi_j(t)$ are defined in (4.152), and the other parameters are similar as in system (4.180) of Example 4.7.1. The initial data of systems (4.180) and (4.234) are $x_1(s) = -1$, $x_2(s) = 2$, $y_1(s) = 3$, $y_2(s) = -4$, $\bar{x}_1(s) = 0.25$, $\bar{x}_2(s) = -0.5$, $\bar{y}_1(s) = -1$, $\bar{y}_2(s) = 2.5$, $s \in [-100, 0]$, respectively. Seeing that, the trajectories of uncontrolled–controlled systems without controls are shown in Fig. 4.13.

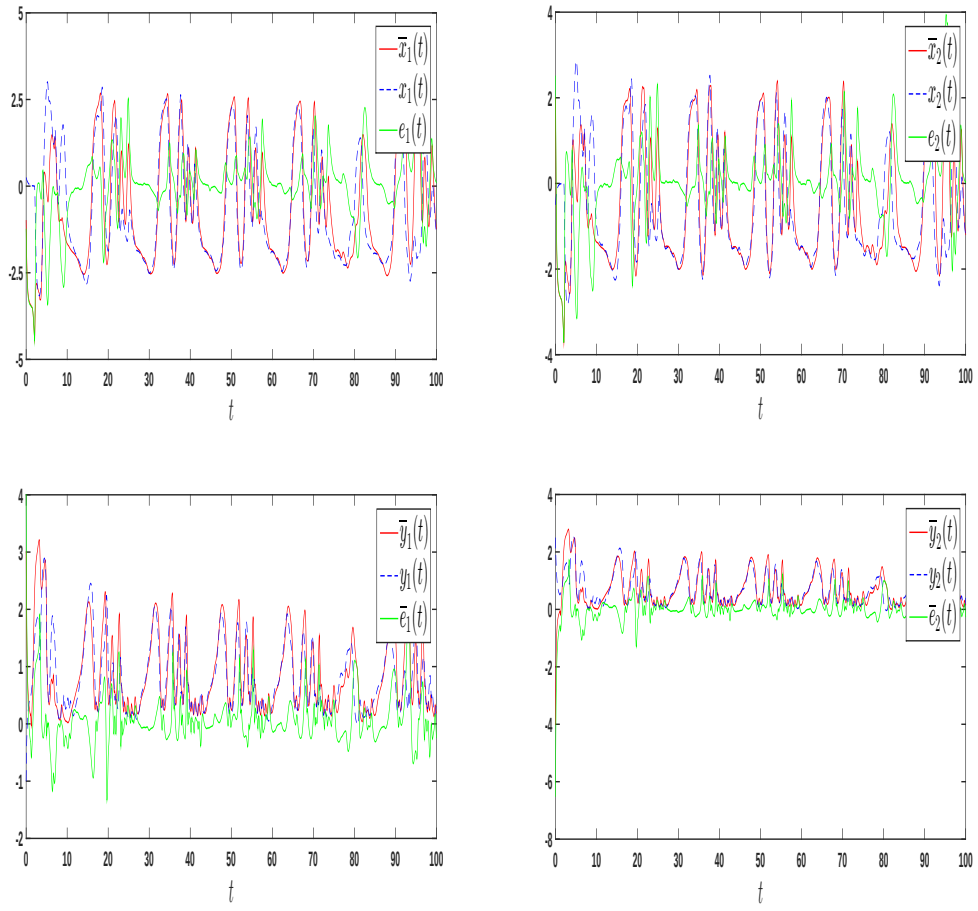


Figure 4.13: Trajectories of Systems (4.180), (4.234) and (4.151) without Control

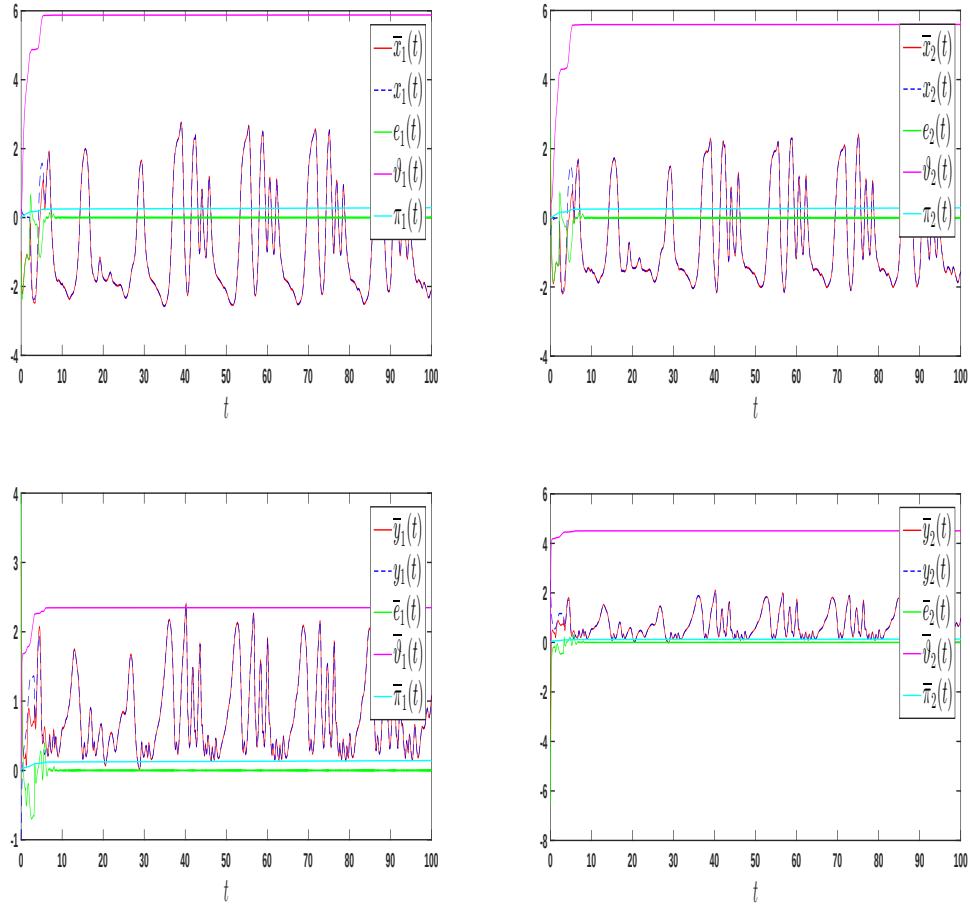


Figure 4.14: Trajectories of Systems (4.180), (4.234) and (4.151) and Control Gains $(\vartheta_1, \vartheta_2(t), \bar{\vartheta}_1(t), \bar{\vartheta}_2(t)), (\pi_1, \pi_2(t), \bar{\pi}_1(t), \bar{\pi}_2(t))$

Through simple calculation, $F = \bar{F} = G = \bar{G} = 1$, $L_{1j} = L_{2j} = M_{1i} = M_{2i} = 1$, $\rho_i = \bar{\rho}_j = 0.65$, $\lambda_1 = \lambda_2 = 0.5$, $\bar{\lambda}_1 = \bar{\lambda}_2 = 1$, and selecting $\eta_{1jil}(t) = \xi_{1jil}(t) = \bar{\eta}_{1ijr}(t) = \bar{\xi}_{1ijr}(t) = 0.125$, $\varepsilon_1 = 0.005$, $\varepsilon_2 = 0.01$, $\sigma = 0.5$, $\kappa_1 = \kappa_2 = 0.02$, $\epsilon_i = \bar{\epsilon}_j = 0.9$, $\alpha_i = \bar{\alpha}_j = 0.05$. Hence, we can see that assumptions (A_1) , (A_3) – (A_9) and the sufficient condition (4.155) are fulfilled. Then, by Theorem 4.6.1, the uncontrolled–controlled systems can be synchronized under the controller (4.152). Fig. 4.14 and Fig. 4.15 illustrate the state trajectories of the uncontrolled system, controlled system and error system besides to the control gains $\vartheta_i(t), \vartheta_j(t), \pi_i(t)$ and $\bar{\pi}_j(t)$. Fig. 4.16 shows the synchronization between systems (4.180) and (4.234).

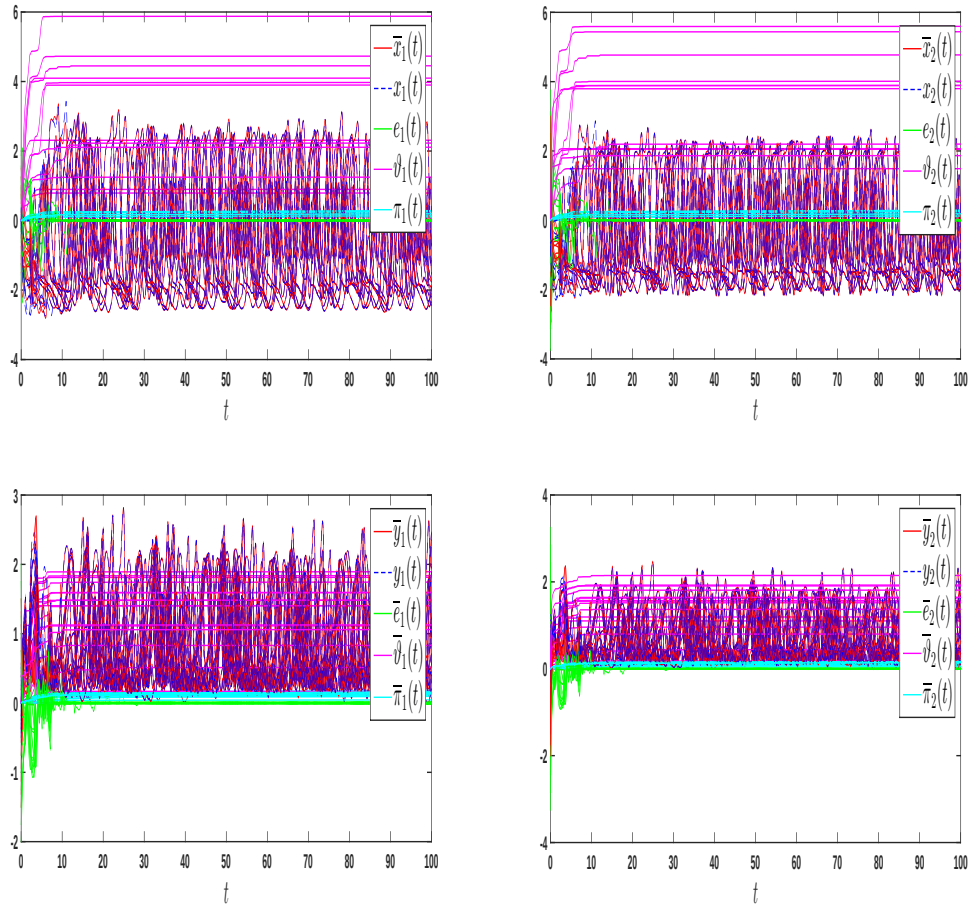


Figure 4.15: Trajectories of Systems (4.180), (4.234) and (4.151) and Control Gains $(\vartheta_1, \vartheta_2(t), \bar{\vartheta}_1(t), \bar{\vartheta}_2(t))$, $(\pi_1, \pi_2(t), \bar{\pi}_1(t), \bar{\pi}_2(t))$ for Various Data

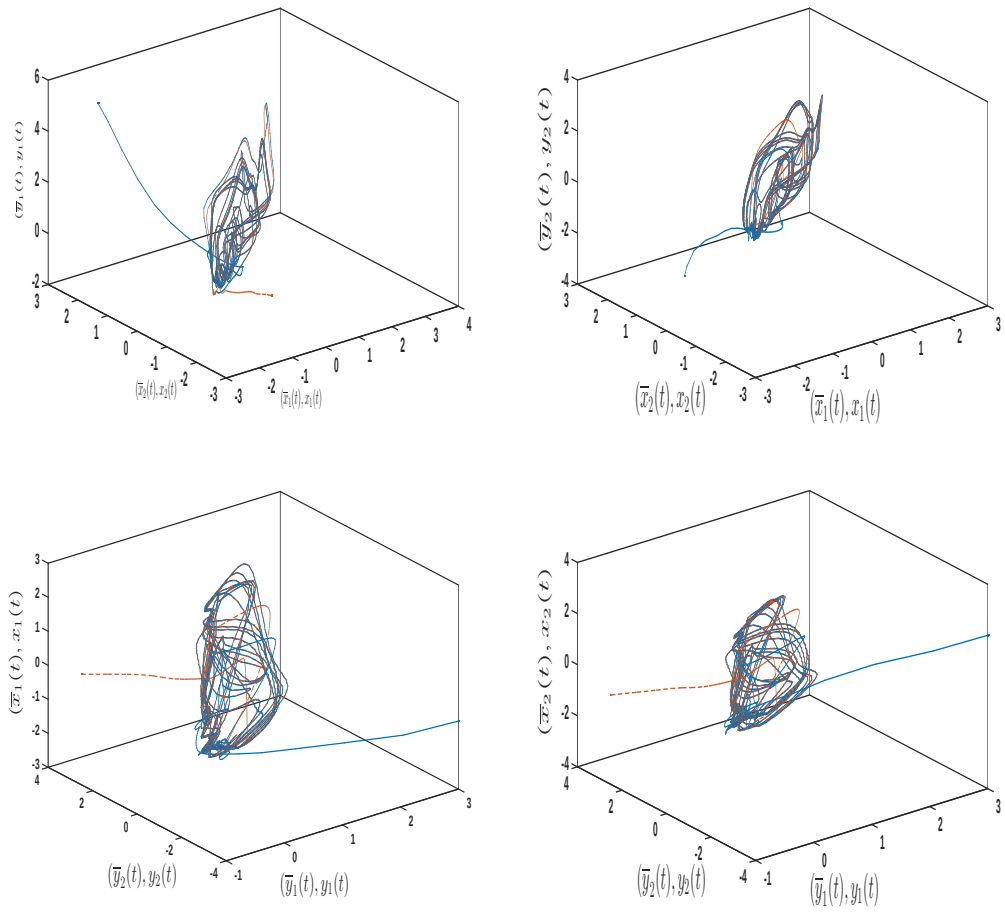


Figure 4.16: State Curves of Uncontrolled System (4.180) and Controlled System (4.234)

Chapter 5

CONCLUSION AND FUTURE WORK

In this chapter, we will conclude and provide some future work.

5.1 Conclusion

In this Ph.D. dissertation we established some general decay rates for some memristor based integro-differential systems in neural network theory. More precisely, we established some energy decay results for memristive high-order Cohen-Grossberg bidirectional associative memory neural network system (MHOCGBAMNNS) with a wider class of kernel functions and containing discrete and distributed delays. Furthermore, we considered a memristive Hopfield neural network system (MHNNS) which contains discrete and distributed delays. Stability and synchronization results were proved under reasonable conditions on the class of kernel functions and on the memristive connection weight functions. Our results are somehow extensions and generalizations of some results in the literature.

5.2 Future Work

As future investigations, we may address the following systems

- Memristive bidirectional associative memory neural network system (MBAMNNS)

introduced by

$$\left\{ \begin{array}{l} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(y_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(y_j(t - \tau)) \\ \quad + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(y_j(t - s)) ds + I_i, \\ y'_j(t) = -\tilde{a}_j y_j(t) + \sum_{i=1}^n \tilde{b}_{ij}(y_j(t)) \tilde{g}_i(x_i(t)) + \sum_{i=1}^n \tilde{c}_{ij}(y_j(t)) \tilde{g}_i(x_i(t - \tau)) \\ \quad + \sum_{i=1}^n \tilde{d}_{ij}(y_j(t)) \int_0^\infty \tilde{k}_{ij}(s) \tilde{g}_i(x_i(t - s)) ds + J_j, \\ \text{for all } t > 0, i = 1, \dots, m, j = 1, \dots, m. \end{array} \right. \quad (S_1)$$

- Memristive Cohen–Grossberg neural network system (MCGNNS) presented by

$$\left\{ \begin{array}{l} x'_i(t) = -a_i(x_i(t)) \left\{ b_i(x_i(t)) - \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t)) - \sum_{j=1}^m d_{ij}(x_i(t)) g_j(x_j(t - \tau)) \right. \\ \quad \left. - \sum_{j=1}^m h_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds - I_i \right\}, \\ \text{for all } t > 0, i = 1, \dots, m. \end{array} \right. \quad (S_2)$$

- Inertial memristive Hopfield neural network system (IMHNNS) described by

$$\left\{ \begin{array}{l} x''_i(t) = -p_i x'_i(t) - a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(x_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t - \tau)) \\ \quad + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds + I_i, \\ \text{for all } t > 0, i = 1, \dots, m, \end{array} \right. \quad (S_3)$$

- Inertial memristive bidirectional associative memory neural network system (IM-

BAMNNS) introduced by

$$\left\{ \begin{array}{l} x_i''(t) = -q_i x_i'(t) - a_i x_i(t) + \sum_{j=1}^m b_{ij}(x_i(t)) g_j(y_j(t)) + \sum_{j=1}^m c_{ij}(x_i(t)) g_j(y_j(t - \tau)) \\ \quad + \sum_{j=1}^m d_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(y_j(t - s)) ds + I_i, \\ y_j''(t) = -\tilde{q}_j y_j'(t) - \tilde{a}_j y_j(t) + \sum_{i=1}^n \tilde{b}_{ij}(y_j(t)) \tilde{g}_i(x_i(t)) + \sum_{i=1}^n \tilde{c}_{ij}(y_j(t)) \tilde{g}_i(x_i(t - \tau)) \\ \quad + \sum_{i=1}^n \tilde{d}_{ij}(y_j(t)) \int_0^\infty \tilde{k}_{ij}(s) \tilde{g}_i(x_i(t - s)) ds + J_j, \\ \text{for all } t > 0, i = 1, \dots, m, j = 1, \dots, m, \end{array} \right. \quad (S_4)$$

- Inertial memristive Cohen–Grossberg neural network system (IMCGNNS) given by

$$\left\{ \begin{array}{l} x_i''(t) = -r_i x_i'(t) - a_i(x_i(t)) \left\{ b_i(x_i(t)) - \sum_{j=1}^m c_{ij}(x_i(t)) g_j(x_j(t)) - \sum_{j=1}^m d_{ij}(x_i(t)) g_j(x_j(t - \tau)) \right. \\ \quad \left. - \sum_{j=1}^m h_{ij}(x_i(t)) \int_0^\infty k_{ij}(s) g_j(x_j(t - s)) ds - I_i \right\}, \\ \text{for all } t > 0, i = 1, \dots, m. \end{array} \right. \quad (S_5)$$

It is of paramount importance to address the above problems with wider classes of kernel functions, with unbounded memristive connection weight functions and with unbounded activation functions.

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