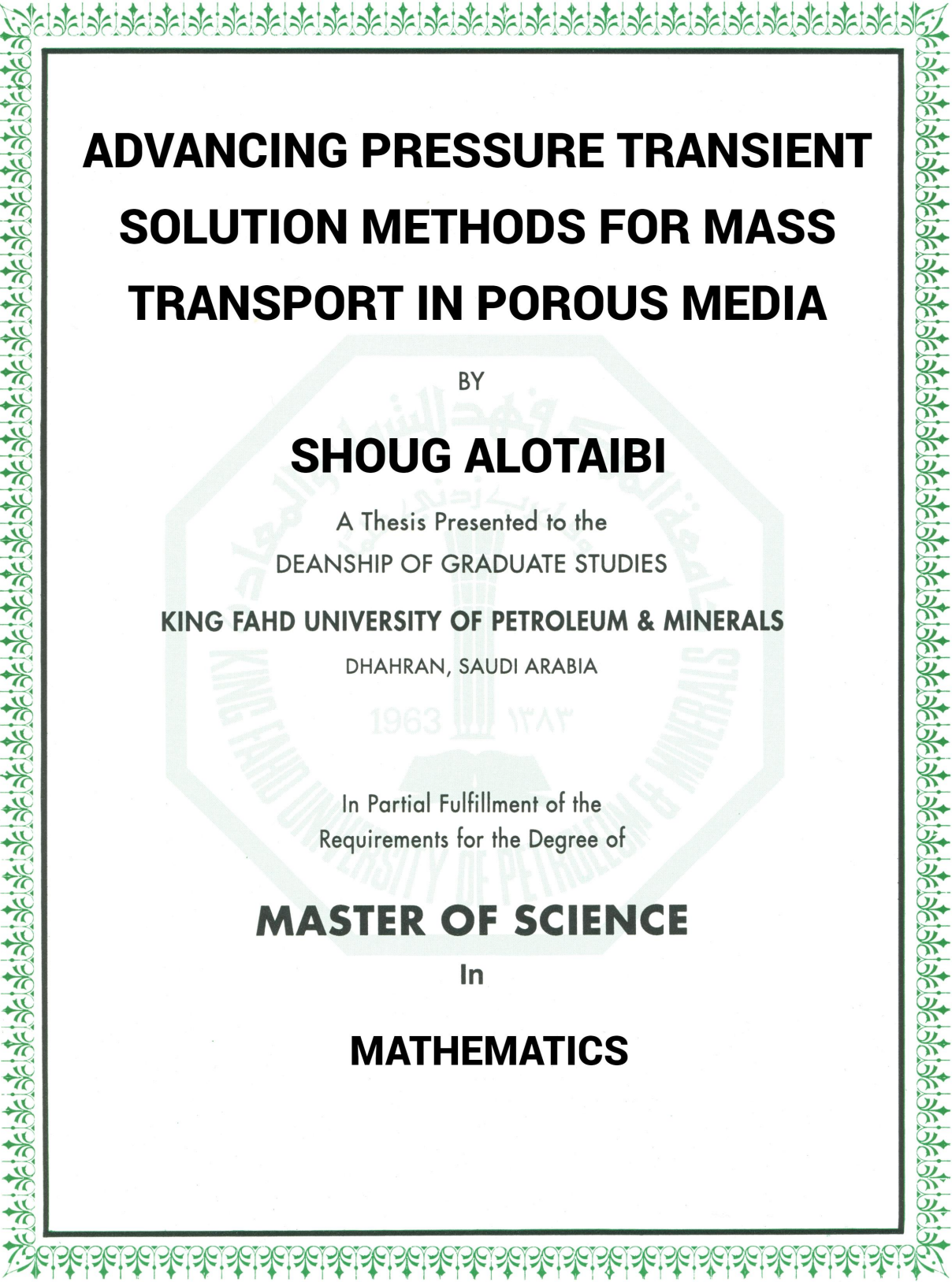




ADVANCING PRESSURE TRANSIENT SOLUTION METHODS FOR MASS TRANSPORT IN POROUS MEDIA

BY

SHOUG ALOTAIBI

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICS

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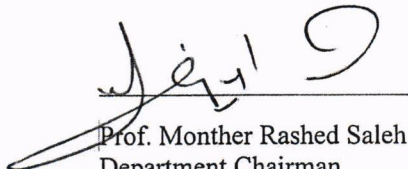
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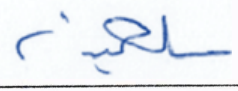
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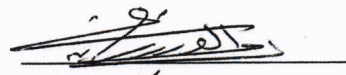
DEANSHIP OF GRADUATE STUDIES

This thesis, written by **SHOUG ALOTAIBI** under the direction of her thesis advisor and approved by her thesis committee, has been presented and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.


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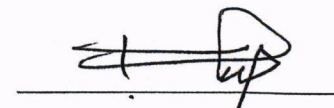
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2023

To my family

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LIST OF SYMBOLS

A	Cross-sectional area, [m^2]
β	Formation volume factor, [rb/stb]
D_h	Hydraulic diffusivity, [m^2/s]
D_x	Diffusivity in x-direction, [m^2/s]
D_y	Diffusivity in y-direction, [m^2/s]
h	Reservoir thickness, [m]
κ	Permeability, [Darcy]
m	Well strength, [m^2/s]
μ	Fluid viscosity, [Pa.s]
ϕ	Porosity, [-]
Φ	Potential function, [m^2/s^2]
ψ	Stream function, [m^2/s]
q_D	Fluid flux, [m/s]
q_w	Volumetric rate, [m^3/s]
q_x	Darcy velocity in x-direction, [m/s]
q_y	Darcy velocity in y-direction, [m/s]

ρ	Fluid density, $[kg/m^3]$
r_w	Wellbore radius, $[m]$
$\tilde{m}$	Fluid mass, $[kg]$
u_r	Fluid velocity in porous medium, $[m/s]$
u_{rad}	Radial velocity, $[m/s]$
u_θ	Tangential velocity, $[m/s]$
u_x	Velocity component in x-direction, $[m/s]$
u_y	Velocity component in y-direction, $[m/s]$
u	Fluid velocity, $[m/s]$
W_f	Half-length of a hydraulic fracture, $[m]$

LIST OF ABBREVIATIONS

CAM	complex Analysis Methods
GPT	Gaussian Pressure Transient

THESIS ABSTRACT

NAME: Shoug Alotaibi

TITLE OF STUDY: Advancing Pressure Transient Solution Methods for Mass
Transport in Porous Media

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In this thesis, we advance a recently developed Gaussian method for quantifying the propagation of pressure fronts in subsurface reservoirs. Gaussian solutions of the diffusion equation can be applied to visualize the flow paths due to the spatial distribution of the pressure gradient caused by engineering interventions (vertical wells, horizontal wells) in subsurface reservoirs for the extraction of resources (water, oil, gas, geothermal fluids). Having solved the temporal and spatial changes in the pressure field caused by the lowered pressure of a well's production system, the Gaussian method is extended and applied to compute and visualize velocity magnitude contours, streamlines and other relevant flow attributes in the vicinity of well systems that are depleting the pressure in a reservoir. The results produced with the GPT method are benchmarked against independent closed-form solutions based on complex Analysis

Methods (CAM) for the same initial conditions. We derive stream function and potential function solutions which allow instantaneous modeling of flow paths and pressure contour solutions for transient flows. Such analytical solutions for transient flows have not been derived before without time-stepping. The new closed-form solutions avoid the computational complexity of time-stepping, required when time-dependent flows are modeled by superposing steady-state solutions using CAM.

ملخص الرسالة

الاسم: شوق العتيبي

عنوان الدراسة: طريقة تحليلية لحساب الضغط العابر لتدفق السائل في الوسائط المسامية

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في هذا العمل , سوف نطور طريقة مستحدثة تقوم بحساب الضغط الموجود في الآبار الجوفية و تسمى بالطريقة الغاوسية (Gaussian Method). حيث يتم استعمال الحلول الناتجة من الطريقة الغاوسية لتمثيل مسار التدفق الناتج عن التدخلات الهندسية اثناء استخراج الموارد الطبيعية من الابار الجوفية مثل: الماء , النفط , الغاز ... إلخ. و للتحقق من صلاحية الطريقة المستخدمة في هذا البحث , تمت المقارنة بين نتائج الدراسة الحالية الناتجة من استعمال الطريقة الغاوسية التحليلية مع طريقة تحليلية مستقلة تسمى طريقة التحليل المركب (Complex Analysis Method (CAM. تعتمد طريقة CAM على تقسيم الفترة الزمنية الى عدة خطوات لذلك يصبح ايجاد الحل اكثر تعقيداً. اضافة الى ذلك , طريقة CAM تعطي تمثيل لمسارات التدفق في حالة الاستقرار و التي ينعدم عندها تغير الضغط بتغير الزمن. خلافا لذلك فإن الطريقة المستعملة في هذه الدراسة تعطي حلول آنية كما تمثل مسارات التدفق لأي فترة زمنية منذ بداية انتشار الضغط و حتى حالة الاستقرار باستعمال معادلات تحليلية قياسية.

CHAPTER 1

INTRODUCTION

1.1 Motivation

Streamline simulations are routinely used to study fluid migration in a variety of subsurface reservoirs containing water, geothermal, or hydrocarbon resources. Advanced tools exist to model the migration of fluids in subsurface reservoirs and range from analytical [1]–[3] to numerical methods such as finite difference and finite element methods [4]–[6]. In addition to the modeling of fluid migration in the reservoir, finite difference methods are used to numerically solve the transient flow in complex pipe networks [7]. Changjun et al. [8] built a model for both steady-state and transient flow in condensate gas pipelines and solved the problem using a fourth order Runge-Kutta method and Gaussian elimination method. Tong et al [9] studied the behavior of the wellhead pressure for the transient non-isothermal wellbore flow. Agaie et al. [10] considered the variation of pressure, velocity, density and temperature with respect to time in the simulation of transient gas flow. In [11], a mathematical model was developed for nonisothermal inflow and lifting of the recovered geothermal fluid was

solved numerically using a fourth order Runge-Kutta method.

Although, numerical methods are widely used and provide efficient solutions in simulating the transient flow in these studies (and many others), analytical solution for transient flow aim to avoid the complexity and resolution issues due to gridding associated with the numerical methods. Numerical tools require gridding and meshing, which makes them relatively complex to use (involving translation of partial differential equations into numerical solution scheme format). In addition, computation times for sensitivity studies and well-placement optimization using numerical solution methods are relatively time-consuming (and therefore costly) as compared to closed-form solution methods.

Numerous analytical solutions (both approximations and exact) for transient flow have been proposed in earlier studies. For example, Wu and Pan [12] presented a set of analytical solutions for transient flow in unsaturated porous media. Lu et al [13] presented approximate analytical solution for transient gas flow from a vertically-fractured well, which can be efficiently used to evaluate the fractured reservoir properties. Amadei et al. [14] considered the effects of non-Newtonian Bingham viscoplastic fluids, and provided analytical solutions for non-transient flow of Bingham viscoplastic materials in rock fractures. Analytical methods based on complex analysis methods (CAM) combined with a time-stepping provide fast alternatives to visualize the streamlines and time-of-flight contours [1]. The basic solution method assumes flow in unbounded domains, but conformal mapping allows for flow path construction of flood displacement fronts in the presence of reservoir boundaries [2].

The present study further explores the potential of another fast method to model

fluid flow, which computes the local pressure gradient analytically based on the so-called Gaussian Pressure Transients (GPT) [15] and [16]. The GPT-method is explained and illustrated in chapters 1, 2. After solving the scalar pressure field in the reservoir space, Darcy’s Law is applied in the present study to directly compute the fluid flux from the reservoir into the well system. The method also can be used to compute vector-field-based flow paths as the fluid moves from the porous reservoir rock into hydraulically fractured well-systems.

The ultimate aim of the flow path simulations commonly is two-fold (1) predict the fluid extraction rate (well productivity) based on the flux of fluid through stream tubes toward the producer well, and (2) study where in the reservoir that fluid originated from. While many studies focus on the fluid extraction from reservoirs (water, oil, gas, geothermal fluids), additional applications are fluid injection for either disposal or storage of certain fluids (waste fluids, carbon-dioxide, hydrogen, natural gas, etc.). For such cases, the solution methods presented here are equally valid; a sign conversion can be applied to model the well injectivity using a certain pressure required to maintain a certain flux in the injection wells.

The results for pressure transients propagating from the selected well constellations, generated with the Gaussian Pressure Transients (GPT), are in the present study validated against known solutions computed with a suitable separate solution method. For example, the grid-less and meshless CAM, used in prior studies of flow in subsurface reservoirs [3], [17]–[20], can model flow paths under steady-state conditions, augmented with time-stepping to account for wells of variable strength (the CAM pressure field is static at each time step, without a true time-dependent pressure

transient at is basis).

In contrast to CAM, GPT-solutions can show the flow paths for both the transient period and a subsequent steady-state flow period; the steady-state will only occur when injection and production volumes are balanced. The comparison between CAM and GPT solutions is evaluated by constructing the flow paths for vertical well arrangements with regular spacing between them, using 2-, 5-, and 7-spot well pattern arrangements (Figure 1.1), essentially benchmarking GPT-solutions for these well constellations. Next, GPT-solutions are used to derive new explicit, analytical solutions for the stream function and potential function describing the time-dependent flow paths, which will shift spatially during the transient flow period.

Analytical stream and potential function solutions in the domain of complex analysis and complex potentials are limited to steady state flow and can only account for time-dependency by superposing steady-state solutions with small incremental time-steps. The accuracy of such solutions is determined by the product of the local velocity and the duration of the finite time-step, which may lead to inaccuracy in spite of using the closed-form CAM solution method [17]–[19]. In the present work, new GPT-solutions allow us to visualize the solutions of the stream and potential functions for both the steady-state and transient stages of the flow development. Furthermore, deriving stream and potential functions from only the imposed bottom-hole pressures of the well system used for the extraction of natural resources like oil, gas, and water from subsurface reservoirs provides a very powerful new tool, because it can be used to optimize the well positioning, adjust pressures if well rates become sub-economic and find the optimum solution, all of which will contribute to more efficient and sus-

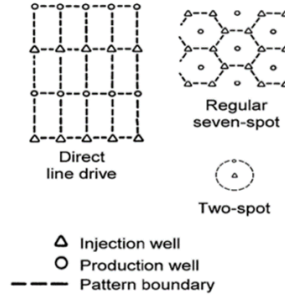


Figure 1.1: Sketch of well patterns commonly used in the development of oil and gas fields with injection wells (triangles) for waterflooding to support the pressure and sweep in the reservoir toward the producer wells (circles). Adapted from <http://www.petrowiki.org/waterflooding>.

tainable utilization of natural resources.

1.2 Preliminaries

The presented study depends on the analytical GPT solution of the pressure diffusion equation presented in [15]. In order to discuss the GPT-solution for pressure diffusion equation, we will review some basic differential equations that govern the flow of fluids in porous media. We will start with some terminologies and notations used in the derivation of the pressure diffusion equation and other flow equations. Then, we will review the derivation of the pressure equation as a diffusion-type equation governs the way that the fluid pressure changes due to both time-change and spatial-change throughout the region. Finally, we will introduce the derivation of GPT-solution as presented in [15]. In the consequent chapters (Chapters 2-4), we will advance this method and utilize it in some applications.

1.2.1 Terminologies

The basic equations presented in this study depend on fluid and media (rock) properties of porous media. In this section, we introduce these properties with some illustrated examples.

Fluid properties

Density (ρ) is a mass per unit volume with SI unit given by kg/m^3 [21].

The fluid density can be categorized as compressible or incompressible fluids. When the fluid density is affected by temperature and pressure we call the fluid compressible and when there is no affect of pressure and temperature or the affect is barely noticeable the fluid is considered as incompressible. For example, if we fixed the pressure and we apply a temperature change on water, the water density changes from $635 kg/m^3$ at $30C$ to $618 kg/m^3$ at $80C$.

On the other hand, if we fixed the the temperature at $20C$ and we apply pressure on water density, the density of water changes from $635 kg/m^3$ at 1 atm to $642 kg/m^3$ at 100 atm. In both examples the rate of change in water density are very small and it can be overlooked. Therefore, we can consider liquids as incompressible fluids. For gases, most of them are compressible since their density is proportional to pressure and inversely proportional to temperature.

Viscosity (μ) is a fluid resistance to deformation after applying some type of forces such as pressure or heat [22]. For example, the viscosity of honey being higher than the viscosity of water, water flows and fills a glass faster than honey under gravity

force.

Viscous fluids can be categorized as newtonian and non-newtonian fluids depending on the change of the rate of a fluid deformation to the applied force. For newtonian fluids, the change of the rate of deformation is proportional to the force such as water, oil and air. On the other hand, for non-newtonian fluids the change of the rate of deformation to the force is not linear such as blood and liquid plastic.

Compressibility (c_f) is the change of density, ρ , with respect to pressure, P , and is given by [21]:

$$c_f = \frac{1}{\rho} \frac{\partial \rho}{\partial P}. \quad (1.1)$$

For constant density the fluid is considered incompressible.

Rock properties

Permeability (κ) is the ability of a fluid to transmit through a rock. Figure 1.2 shows the difference between high and low permeability where the fluid in high permeable section (left) flows in larger quantities than the low permeable section (right) [23].

Porosity (ϕ) is the ratio of the pore volume, V_p , to the total rock volume, V_t , which is given by [23]:

$$\phi = \frac{V_p}{V_t} \quad (1.2)$$

The values of porosity of rocks can vary from 0 to 1. For example, if we want to compute the porosity of the cross sectional area in Figure 1.2 with knowing the total volume of this area ($30m^3$) and the volume of the rocks ($V_r = 10m^3$). To find the pore volume we shall subtract the volume of the rocks from the total volume. So the

porosity will be :

$$\phi = \frac{V_p}{V_t} = \frac{V_t - V_r}{V_t} = \frac{30 - 10}{30} \approx 0.6667. \quad (1.3)$$

Rock compressibility (c_ϕ) is the change of porosity, ϕ , with respect to pressure, P , and is given by [23]:

$$c_\phi = \frac{1}{\phi} \frac{\partial \phi}{\partial P}. \quad (1.4)$$

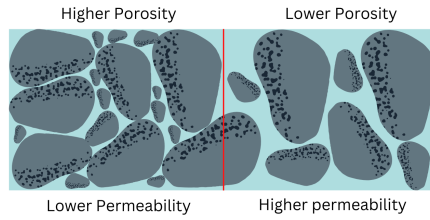


Figure 1.2: Schematic of a porous media at the micro scale

1.2.2 Pressure diffusion equation for fluid flow in porous media

The flow of fluids in porous media is governed by (1) the principle of conservation of mass and (2) conservation of momentum.

Conservation of mass

Conservation of mass law states that the difference between the mass flux entering the control volume region and the mass flux exiting the region will give us the amount of mass stored in the control volume [24]. Assume we have a cross sectional area of porous media in 1-Dimension, we can write the equation of conservation of mass

as [24]:

$$\underbrace{A(x)\rho(x)q_D(x)\Delta t}_{\text{Flux in}} - \underbrace{A(x+\Delta x)\rho(x+\Delta x)q_D(x+\Delta x)\Delta t}_{\text{Flux out}} = \underbrace{\tilde{m}(t+\Delta t) - \tilde{m}(t)}_{\text{Total mass stored}}, \quad (1.5)$$

where, q_D is the fluid flux, Δt change of time, Δx change of spatial variable, A represents the cross-sectional area and $\tilde{m}$ is the amount of fluid mass in the region of interest which is given by $\tilde{m} = \rho V_p = \rho \phi V_t = \rho \phi A \Delta x$.

Now back to equation (1.5), in 1-Dimension flow $A(x) = A = \text{constant}$. Therefore, A can be factored out from equation (1.5) and then we divide both sides by Δt and we let $\Delta t \rightarrow 0$

$$A[\rho q(x) - \rho q_D(x + \Delta x)] = \lim_{\Delta t \rightarrow 0} \frac{\tilde{m}(t + \Delta t) - \tilde{m}(t)}{\Delta t}. \quad (1.6)$$

we know that,

$$\lim_{\Delta t \rightarrow 0} \frac{\tilde{m}(t + \Delta t) - \tilde{m}(t)}{\Delta t} = \frac{\partial \tilde{m}}{\partial t} = \frac{\partial \rho \phi}{\partial t} A \Delta x. \quad (1.7)$$

We substitute equations (1.7) into equation (1.6) to get:

$$- A[\rho q_D(x + \Delta x) - \rho q_D(x)] = \frac{\partial \rho \phi}{\partial t} A \Delta x. \quad (1.8)$$

Now, divide equation (1.8) by $A \Delta x$ and let $\Delta x \rightarrow 0$ we will get the law of conservation of mass [24]:

$$\lim_{x \rightarrow 0} \frac{-A[\rho q_D(x + \Delta x) - \rho q_D(x)]}{\Delta x} = \frac{\partial \rho \phi}{\partial t} A \Delta x \quad (1.9)$$

$$-[\rho q_D(x + \Delta x) - \rho q_D(x)] = \frac{\partial \rho \phi}{\partial t} \Delta x \quad (1.10)$$

$$-\frac{\partial \rho q_D}{\partial x} = \frac{\partial \rho \phi}{\partial t}. \quad (1.11)$$

Darcy's law

We can express fluid flow through sectional areas of a porous medium by the volumetric rate given by [24]:

$$q_w = \frac{CA\Delta(P - \rho g z)}{L}, \quad (1.12)$$

where q_w is the volumetric flow rate, P is the pressure, ρ is the density, g is the gravitational acceleration, z is the vertical coordinate, L is the length of sample, C is the constant of proportionality, and A is the cross-sectional area of the sample. In equation (1.12), $P - \rho g z$ is the energy source. Therefore, what cause the flow to move from high energy to low energy is the gradient of $P - \rho g z$. The volumetric rate is proportional to permeability κ and it is inversely proportional to viscosity μ therefore, we can write the proportionality constant as $C = \frac{\kappa}{\mu}$. In general, the permeability depends on the direction of the spatial variable. However, in our study we will usually assume that κ remains constant in each direction. Next, we will divide equation (1.12) by A so we can work with the volumetric rate per unit area which is

called Darcy's velocity q_D [24]:

$$q_D = \frac{\kappa}{\mu} \frac{\Delta(P - \rho g z)}{L}. \quad (1.13)$$

Our analysis in this study is limited to flow in a horizontal plane (map view) at the reservoir level. Therefore, $\rho g z$ can be dropped from equation (1.13) and we rewrite it as [24]:

$$q_D = -\frac{\kappa}{\mu} \frac{\partial P}{\partial x}, \quad (1.14)$$

the minus sign is included to represent the fluid movement toward lower pressure.

Pressure diffusion equation

Pressure transient of a fluid flows through porous media is described by the pressure diffusion equation. R. W. Zimmerman firstly derived the pressure diffusion equation in [24] by combining conservation of mass and Darcy's law. In order to derive the pressure diffusion equation we start by applying Darcy's law on the left hand side of equation (1.11).

$$\begin{aligned} -\frac{\partial \rho q_D}{\partial x} &= -\frac{\partial}{\partial x} \left[-\frac{\rho \kappa}{\mu} \frac{\partial P}{\partial x} \right] \\ &= \frac{\kappa}{\mu} \left[\rho \frac{\partial^2 P}{\partial x^2} + \frac{\partial \rho}{\partial x} \frac{\partial P}{\partial x} \right] \\ &= \frac{\kappa}{\mu} \left[\rho \frac{\partial^2 P}{\partial x^2} + \frac{\partial \rho}{\partial P} \frac{\partial P}{\partial x} \frac{\partial P}{\partial x} \right] \\ &= \frac{\kappa \rho}{\mu} \left[\frac{\partial^2 P}{\partial x^2} + \frac{1}{\rho} \frac{\partial \rho}{\partial P} \left(\frac{\partial P}{\partial x} \right)^2 \right] \\ &= \frac{\kappa \rho}{\mu} \left[\frac{\partial^2 P}{\partial x^2} + c_f \left(\frac{\partial P}{\partial x} \right)^2 \right], \end{aligned} \quad (1.15)$$

where the part $c_f(\frac{\partial P}{\partial x})^2$ is always negligible since it is almost zero in liquids. Also, it is assumed that κ is constant in which we could factor it out in equation (1.15).

Now for the right hand side of equation (1.11) we apply the product and chain rule to get:

$$\begin{aligned}
\frac{\partial \rho \phi}{\partial t} &= \rho \frac{\partial \phi}{\partial t} + \phi \frac{\partial \rho}{\partial t} \\
&= \rho \frac{\partial \phi}{\partial P} \frac{\partial P}{\partial t} + \phi \frac{\partial \rho}{\partial P} \frac{\partial P}{\partial t} \\
&= \rho \phi \left[\frac{1}{\phi} \frac{\partial \phi}{\partial P} + \frac{1}{\rho} \frac{\partial \rho}{\partial P} \right] \frac{\partial P}{\partial t} \\
&= \rho \phi \underbrace{[c_\phi + c_f]}_{=c_t} \frac{\partial P}{\partial t}.
\end{aligned} \tag{1.16}$$

Now, we can substitute the resulting equations (1.15) and (1.16) into the conservation of mass equation given in (1.11) to get the final formula for pressure diffusion equation.

$$\begin{aligned}
\rho \phi c_t \frac{\partial P}{\partial t} &= \frac{\kappa \rho}{\mu} \frac{\partial^2 P}{\partial x^2} \\
\frac{\partial P}{\partial t} &= \frac{\kappa}{\mu \phi c_t} \frac{\partial^2 P}{\partial x^2},
\end{aligned} \tag{1.17}$$

where, the diffusion coefficient $\frac{\kappa}{\mu \phi c_t}$ in equation (1.17) is usually called the hydraulic diffusivity where it captures the reservoir properties [24].

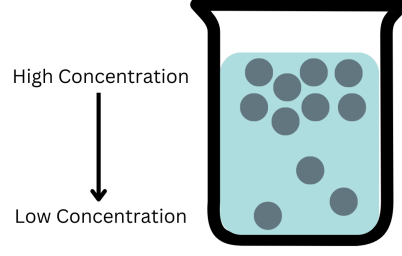


Figure 1.3: The diffusion from high concentration to low concentration

1.2.3 Gaussian pressure transient solution for pressure diffusion equation

Molecular diffusion

The molecular diffusion or simply diffusion is the movement of molecules randomly from high concentration area to low concentration area as shown in Figure 1.3.

The differential equation for mass concentration diffusion in 1-Dimension is given by [25]:

$$\frac{\partial C}{\partial t} = D_m \frac{\partial^2 C}{\partial x^2}, \quad (1.18)$$

where $C(x, t)$ is the mass concentration of the fluid in a certain location of a diffusing substance and D_m is the diffusion coefficient. The solution for equation (1.18) is given by [25]:

$$C(x, t) = \frac{A}{\sqrt{t}} e^{-\frac{x^2}{4D_m t}}, \quad (1.19)$$

where A is arbitrary constant with dimension $ML^{-1}T^{-1/2}$. We can find the value of A from the total amount of mass diffusing in an infinite number of 1D lateral directions

which is given by:

$$M(t) = \int_{-\infty}^{\infty} C(x, t) dx. \quad (1.20)$$

Substituting the general solution of concentration given by equation (1.19) and use substitution to solve the integral by taking $u = \frac{x}{2\sqrt{D_m t}}$ and $du = \frac{dx}{2\sqrt{D_m t}}$

$$M = 2A\sqrt{D_m} \int_{-\infty}^{\infty} e^{-u^2} du = 2A\sqrt{D_m} \sqrt{\pi} = 2A\sqrt{D_m \pi}. \quad (1.21)$$

Therefore, we substitute the value $A = \frac{M}{2\sqrt{D_m \pi}}$ in equation (1.19) to get the standard solution of fluid concentration subjected to a finite initial mass concentration C_0 at location $x = 0$ (see [25]).

$$C(x, t) = \frac{M}{\sqrt{4\pi D_m t}} e^{-\frac{x^2}{4D_m t}}. \quad (1.22)$$

This solution is called the Gaussian diffusion solution [15], based on using the well-known normal distribution function (also called Gaussian distribution function) with the inverse of the variance $\sigma_t^{-2} = \frac{1}{\sqrt{2D_m t}}$. Letting $x = 0$ in Equation (1.22) we obtain:

$$C_0 = \frac{M}{\sqrt{4\pi D_m t}}. \quad (1.23)$$

Therefore, Equation (1.22) can be written in term of C_0 as following:

$$C(x, t) = C_0 e^{-\frac{x^2}{4D_m t}}. \quad (1.24)$$

Now, considering the more general 3D diffusion equation for mass concentration with

isotropic diffusivity given by:

$$\frac{\partial C}{\partial t} = D_m \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right). \quad (1.25)$$

The solution will be a straightforward generalization of Equations (1.24) and (1.23) in 3D. Namely, the general solution of Equation (1.25) under the initial condition $C(0, t) = C_0$ ([25]) is:

$$C(x, y, z, t) = C_0 e^{-\frac{x^2+y^2+z^2}{4D_mt}}; \quad C_0 = \frac{M}{(4\pi D_mt)^{\frac{3}{2}}}. \quad (1.26)$$

Pressure diffusion

Let us consider the pressure diffusion equation given by Equation (1.17) with hydraulic diffusivity $D_h = \frac{\kappa}{\mu\phi c_t}$:

$$\frac{\partial P}{\partial t} = D_h \frac{\partial^2 P}{\partial x^2}, \quad (1.27)$$

which is identical to concentration diffusion Equation (1.18) except for the use of hydraulic diffusivity D_h in Equation (1.18) and molecular diffusivity D_m in Equation (1.27).

In reservoir studies, we are commonly dealing with a pre-existing pressure in the reservoir, P_0 , which is then suddenly changed by an engineering intervention (e.g., producer well and hydraulic fractures).

Thus, the pressure change, i.e, the difference between the well pressure and hydraulic

pressure (ΔP) is also imposed. We therefore define a local pressure change,

$$\Delta P(x_0, t) = \Delta P_0 = P_0 - P_{BH}, \quad (1.28)$$

where x_0 is the well location and P_{BH} is the pressure level of the well intervention.

The initial pressure change at time $t = 0$ is zero, $\Delta P(x, 0) = 0$. Now equation (1.27)

can be written as:

$$\frac{\partial \Delta P}{\partial t} = D_h \frac{\partial^2 \Delta P}{\partial x^2}, \quad (1.29)$$

subjected to the boundary condition $\Delta P(x_0, t) = \Delta P_0$ and initial condition

$\Delta P(x, 0) = 0$. The adaptive Gaussian diffusion solution is then given by:

$$\Delta P(x, t) = \Delta P_0 e^{-\left(\frac{x^2}{4D_h t}\right)} \quad (1.30)$$

Remark: Equation (1.30) is an approximate analytical solution for (1.29), the exact Gaussian solution for the pressure diffusion equation has a transient multiplier in front of the exponential term that was ignored in the visualizations presented in the following chapters, which means all results are still accurate but the time scales appearing as linear in the plots are in reality non-linear. The non-dimensional times specified in the plots can be quickly converted to the actual (non-linear) times by multiplying each time unit by their respective transient multipliers; the appropriate multipliers are $\frac{1}{\sqrt{4\pi D t}}$ for the 1D-diffusion cases, and $\frac{1}{4\pi D t}$ for the 2D-diffusion cases. The proof is given in Appendix B.

Equation (1.30) can now be used for tracking the spatial and temporal change of the pressure due to a unidirectional 1D-pressure diffusion in a reservoir layer, which shows how the pressure front advances from a planar source (such as a hydraulic fracture) into the reservoir along the x-direction. In Section 2.3 we will introduce Gaussian-solutions for 2D and 3D-diffusion cases. After computing the pressure change, $\Delta P(x, y, t)$, In the reservoir domain, the reservoir pressure can be computed by adding the pressure change to the pre-existing pressure in the reservoir before imposing the change source.

1.3 Thesis organization

In this thesis, we will apply a newly developed method that is used to compute the spatial pressure depletion in oil reservoirs for further development of advanced reservoir simulations.

This thesis is divided into 4 chapters. In the first chapter, we introduced the problem and essential terminologies and differential equations that plays a big role in deriving the GPT-solution which includes physical properties, conservation of mass equation, Darcy's law, and diffusion equation for concentration.

In the second chapter, we introduced the model development by assuming the basic assumption for the model, we explained the equation of motion governing the flow in porous media, and we gave examples for GPT- solution.

In the third chapter, we study the evolution of pressure transient with time to the

steady-state. We demonstrate and visualize the flow paths for the applications of the pressure transient solution by superpose the basic pressure transient solution and produce simple synthetic cases of multiple vertical wells (producers and injectors) in various flow cell configurations and benchmark it against solutions for the same initial conditions with independent methods for both homogeneous and heterogeneous diffusivity. We also derive the stream function and potential function solutions for transient flow to visualize the streamlines and equipotential lines at different time stages for both homogeneous and heterogeneous diffusivity. Lastly, we will end with a conclusion.

CHAPTER 2

METHODOLOGY

This chapter develops step-wise the key elements required to develop our models. First, we explain the basic assumptions (Section 2.1), equation of motion governing the flow in porous media (Section 2.2), GPT equations with examples of GPT solutions and superposed for multiple sources (Section 2.3).

2.1 Basic assumption

In the Gaussian solution of the pressure-diffusivity equation, the hydraulic diffusivity is the key controlling parameter for all fluid transfer [15], [16]. This approach turns out to greatly simplify the computation of pressure transient solutions for a wide range of practical applications. In this study, the fluid transfer in hypothetical reservoirs will be modelled based on the following initial assumptions:

- The porous medium hosting the reservoir, is considered to be homogeneous and possesses either isotropic or anisotropic diffusivity.
- The fluid flow in the reservoir is solely due to pressure gradients caused by the

engineering interventions (such as the drilling of injection and production wells).

- The fluid in the reservoir is assumed to be single phase incompressible (variations in density are negligible), isotropic (the properties are not dependent on the direction along which they are measured) and of Newtonian viscosity (which means the fluid does not change when we apply a stress and its magnitude is insensitive to pressure changes).

2.2 The fluid motion model in porous medium

The motion of the fluid is described by incompressible Navier-Stokes equations (see [26]).

$$\rho\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = -\nabla P + \rho g + \mu \nabla^2 u, \quad (\text{balance of momentum}) \quad (2.1)$$

$$\nabla \cdot u = 0. \quad (\text{incompressibility condition}) \quad (2.2)$$

In equations (2.1) and (2.2):

- $u = u(x, y, t)$ is the fluid velocity, where (x, y) , in two-dimension, is the spatial coordinate and t is the time variable.
- ρ is the fluid density and μ is the fluid viscosity, and they are assumed constants as stated in Section 2.1.
- $P = P(x, y, t)$ is a scalar function represents the pressure of the fluid and g is the gravity.

- The operator ∇ in $2D$ is $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ and Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
- The term $(u \cdot \nabla)u$ is called the convection term and the term $\mu \nabla^2 u$ is the diffusion term. Therefore, equation (2.1) belongs to the class of convection–diffusion equations.
- Equation (2.2) is the mass conservation equation, also called continuity equation in case of incompressible flow (i.e. with constant density ρ).

The right-hand side of Equation (2.1) represents the various forces acting on the fluid body: (1) the pressure force which drives the motion; (2) the body force which is due to gravity and (3) viscous force which resists the motion. The left-hand side of Equation (2.1) represents the time rate of change of velocity in the first term and convective acceleration in the second term. This convection term makes equation (2.1) nonlinear and hence a closed form solution is applicable only under special conditions. For example, if we assume the convection acceleration is small enough to be neglected (i.e. $(u \cdot \nabla)u \approx 0$) and ignoring the body force, the steady state case of Equation (2.1) can be written as follows:

$$-\nabla P + \mu \nabla^2 u = 0. \quad (2.3)$$

For a porous medium, Equation (2.3) is solved in macroscopic scale considering the properties of the porous medium such as the porosity, ϕ_{ij} , and the permeability tensor κ_{ij} which is a symmetric matrix, where i and j correspond to the spatial coordinates of the porous medium, each component represents the proportionality constant between the fluid velocity in the i –th direction and the pressure gradient in the j –th direction.

On the other hand, the porosity tensor ϕ_{ij} is also a symmetric where each component represents the ratio of the void space in a certain direction to the total volume of the porous medium.

Based on Darcy law, the fluid velocity (in ms^{-1}) due to a given pressure gradient in a porous medium is given by:

$$u_r(x, y, t) = -\frac{\kappa_{ij}}{\mu\phi_{ij}}\nabla P(x, y, t). \quad (2.4)$$

The derivation of Equation (2.4) from equation (2.3) is obtained by averaging over a cross section of the tube representing the pore in the medium, see [27]. If we remove the porosity, ϕ_{ij} , from Equation (2.4) we will obtain the fluid flux (also called Darcy velocity) as it will occur at the reservoir level into the well system:

$$q_D = -\frac{\kappa_{ij}}{\mu}\nabla P. \quad (2.5)$$

The relation between the fluid velocity, u_r , in the pore space of the reservoir and the Darcy velocity, q_D , into the well system is then given by $u\phi = q_D$.

In reservoir models, Equation (2.5) quantifies the instream of fluid into the well system at the reservoir level. The volumetric rate of inflow from the reservoir requires multiplication of the flux rate by the vertical exposure of the wellbore to the reservoir space of thickness h . In addition, hydrocarbons in the reservoir have a different volume than at surface conditions, which is accounted for by the so-called formation volume

factor β . Hence, with isotropic porosity and permeability, the volumetric rate of the inflow from a well with radius r_w is given by (see [16])

$$q_w(x, y, t) = -\frac{2\pi r_w}{\beta} \frac{\kappa h}{\phi \mu} \nabla P(x, y, t). \quad (2.6)$$

In Equation (2.6) the parameter κh represents the reservoir conductivity which governs how fast fluid can flow to/from the well. If we only study the pressures in the reservoir, we do not need to convert to fluid volumes produced at surface conditions (β can be conveniently dropped). Neither do we need to multiply with reservoir thickness (h can be conveniently dropped), if we confine our analysis to flow in a horizontal plane (map view) at the reservoir level. The radius of the well is only needed in the expression when we aim to compute the volumes collected at the well (so $2\pi r_w$ can be dropped as well). We retain for our initial analysis the fluid flux formula of Equation (2.4), which describes fluid mass transfer in the reservoir space. Another crucial simplifying step made here is to rather than model with the dimensional hydraulic diffusivity, the present study uses normalized quantities (see Appendix), which speeds up the subsequent computations. However, for translation to any real-well cases, properly scaled dimensional inputs should be used for the hydraulic diffusivity and the formation volume factor.

In our presented work, Equation (2.4) is used to compute the fluid flux from the reservoir into the well-system and any hydraulic fractures, and then we plot the flow paths. Although, well known numerical grid-based methods can be used to solve

Equation (2.4), they have limit resolution comparing to analytic solutions. To solve Equation (2.4) analytically, we need to know what is the pressure gradient everywhere in the reservoir due to the pressure disturbance and its transient advance, caused by the well system. The spatial and temporal pressure gradients can be solved for by the GPT method introduced in Weijermars [15]

2.3 Gaussian pressure transient solution

In this section, we will consider a reservoir with a single vertical well or a single hydraulic horizontal fracture and multiple hydraulic fractures. We distinguish two special cases for the pressure diffusivity: isotropic and anisotropic.

2.3.1 Gaussian pressure transient solution with isotropic diffusivity

In this case the diffusivity is the same in every direction (for 2D and 3D diffusion sources) and is represented by a scalar D_h . Therefore, the general solution of Equation (1.17) for the spatial and temporal change of the pressure for the isotropic pressure diffusivity case is given by the following forms [15]:

$$\Delta P = \Delta P_0 e^{-\left(\frac{x^2+y^2}{4D_h t}\right)}, \quad (2D\text{- Diffusion}) \quad (2.7)$$

$$\Delta P = \Delta P_0 e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)}. \quad (3D\text{- Diffusion}) \quad (2.8)$$

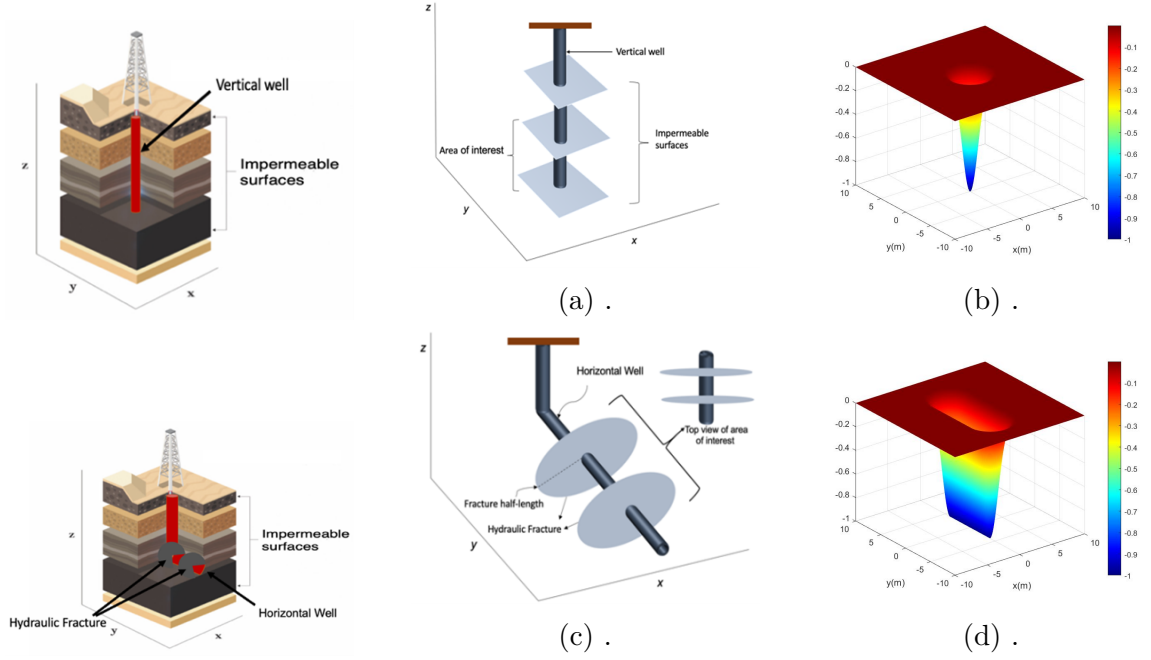


Figure 2.1: The GPT solution for isotropic pressure diffusivity D_h (Case A) for a fixed time variable t . (a) Schematic of the vertical producer well. (b) The pressure-change profile in a vertical producer well ($\Delta P_0 = -1$), centered at the origin with radius $r = \sqrt{(x^2 + y^2)}$. (c) Schematic of the horizontal hydraulically fractured well. (d) The pressure change profile induced by a single hydraulic fracture of half-length $W_f = 4$, i.e., the fracture is represented by the interval $[-4, 4]$.

In Figure 2.1, we represent the GPT solution for the 2D-diffusion in a horizontal reservoir produced with a single vertical well, using the solution form given by Equation (2.7). The vertical scale in Figure 2.1b shows the pressure transient probability profile starting from the pressure drop source. For simplicity, the isotropic pressure diffusivity D_h was non-dimensionalized (see Appendix A) in such a way that $D_h = 1$. For the single vertical well (Top row of figure 2.1), the pressure transient contours will advance in the reservoir expanding as concentric circles of radius $r = \sqrt{x^2 + y^2}$ in a 2D map view (Figure 2.1a) and as a cone in a 3D perspective (Figure 2.1b). The reservoir pressure drops $\Delta P_0 = 0$ in Equation (2.7) is entered as a negative z-value. Further away from the pressure drop source, $\Delta P(x, y, t) = 0$, if the time used is relatively short as compared to the diffusion rate which means that there is no pressure change in the regions further away from the pressure drop source (red top surface in Figure 2.1b).

The solution for a finite length, single hydraulic fracture (Figure 2.1) is constructed by superposing 1D-diffusion with the 2D solution of radial flow for the fracture tips (bottom row of Figure 2.1).

The pressure change induced by a single hydraulic fracture (Bottom row of figures 2.1) of half-length W_f (Figure 2.2) is represented in a 2D map view by elliptical pressure contours. to describe the physical process, when the transient propagates in all 2D-directions away from the initial pressure drop source, we will apply the following conditional modifications in all presented examples of pressure diffusion initiated from the hydraulic fractures:

- For $y \geq W_f$, use $y \rightarrow y - W_f$;
- For $-W_f < y < W_f$, use $y \rightarrow 0$;
- For $y \leq -W_f$, use $y \rightarrow y + W_f$.

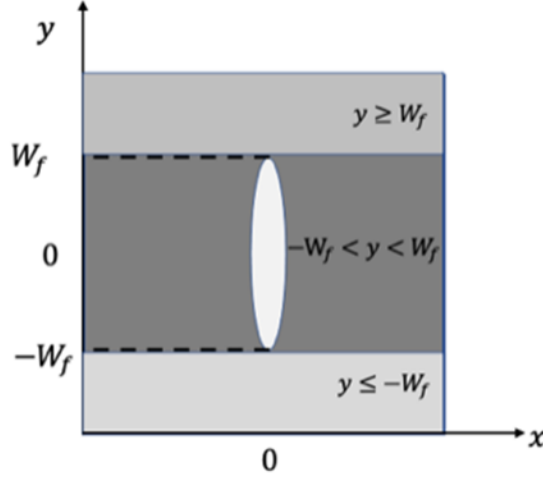


Figure 2.2: A hydraulic fracture, parallel to y-direction with half-length W_f , divides the reservoir region into three parts: the upper part $y \geq W_f$, the intermediate part $-W_f < y < W_f$ and the lower part $y \leq -W_f$. The actual flow at the fracture tips is computed by superposing a 2D-diffusion solution.

The GPT-solution used in the present study is the approximate solution of Equation (19) in Weijermars [16], while an exact solution is also given in that same study [16] in Equation (B18). The approximate solution remains a very simple expression but requires the above conditional modification, to make sure that pressure transient propagates with a uniform rate in all directions away from the initial pressure drop source.

2.3.2 Gaussian pressure transient solution with an-isotropic diffusivity

This case considers a layered reservoirs in which the hydraulic diffusivity is not a scalar but a tensor quantity sensitive to direction in the (x, y) -plane represented by the 2D-diffusivity tensor, D_{ij} . If we consider the case when $D_{ij} = 0$ for $i \neq j$, then the diffusivity in each direction will be denoted by D_x and D_y .

Considering the 2D-case, the spatial and temporal change of the pressure for the an-isotropic pressure diffusivity case is given by [15]:

$$\Delta P = \Delta P_0 e^{-\frac{1}{4t}(\frac{x^2}{D_x} + \frac{y^2}{D_y})}. \quad (2.9)$$

Figure 2.3 shows map views of the pressure in the target zone for the vertical well (top row) and horizontal hydraulic fracture (bottom row) and it represents the an-isotropic pressure diffusivity where the pressure change is computed using Equation (2.9). We can see the effect of the larger diffusivity component on the pressure profile around the wellbore. For both type of well systems, the pressure magnitude-contours representing the advance of the pressure transient at a certain time will show stretching in the y-direction if $(D_x < D_y)$ (left column in Figure 2.3), and stretching occurs in the x-direction if $(D_x > D_y)$ (right column in Figure 2.3).

2.3.3 Multiple pressure sources

In case the reservoir is developed with well engineering interventions with multiple sources of pressure changes, say n vertical wells and/or n hydraulic fractures, with pre-existing reservoir pressure P_0 , then the transient changes in the reservoir pressure (at any time t) can be determined using the sum of Gaussian solutions for the pressure change due to each individual source in the system and is given by [15]:

$$P(x, y, t) = P_0 + \sum_{k=1}^n \Delta P(x_k, y_k, t). \quad (2.10)$$

Figure 2.4 shows how the pressure depletion fronts will march forward as the time is advancing in a reservoir containing four parallel hydraulic fracture with the same half-lengths. For simplicity, the four fractures are aligned with one of the Cartesian axes. The pressure changes $\Delta P(x_k, y_k, t)$ (Which is given by (2.7)), $k = 1, 2, 3, 4$ for individual hydraulic fracture after the first 100 hours of production is shown in Figure 2.4 (top left). After 1000 hours, the pressure depletion fronts start to meet and merge (Figure 2.4 (top right)). Then, after 5000 hours the pressure in the center largely depleted and the pressure transient advances from the center between the hydraulic fracture (Figure 2.4 (bottom left)). As time goes on, the pressure depletion area will almost cover the whole reservoir domain (Figure 2.4 (bottom right)).

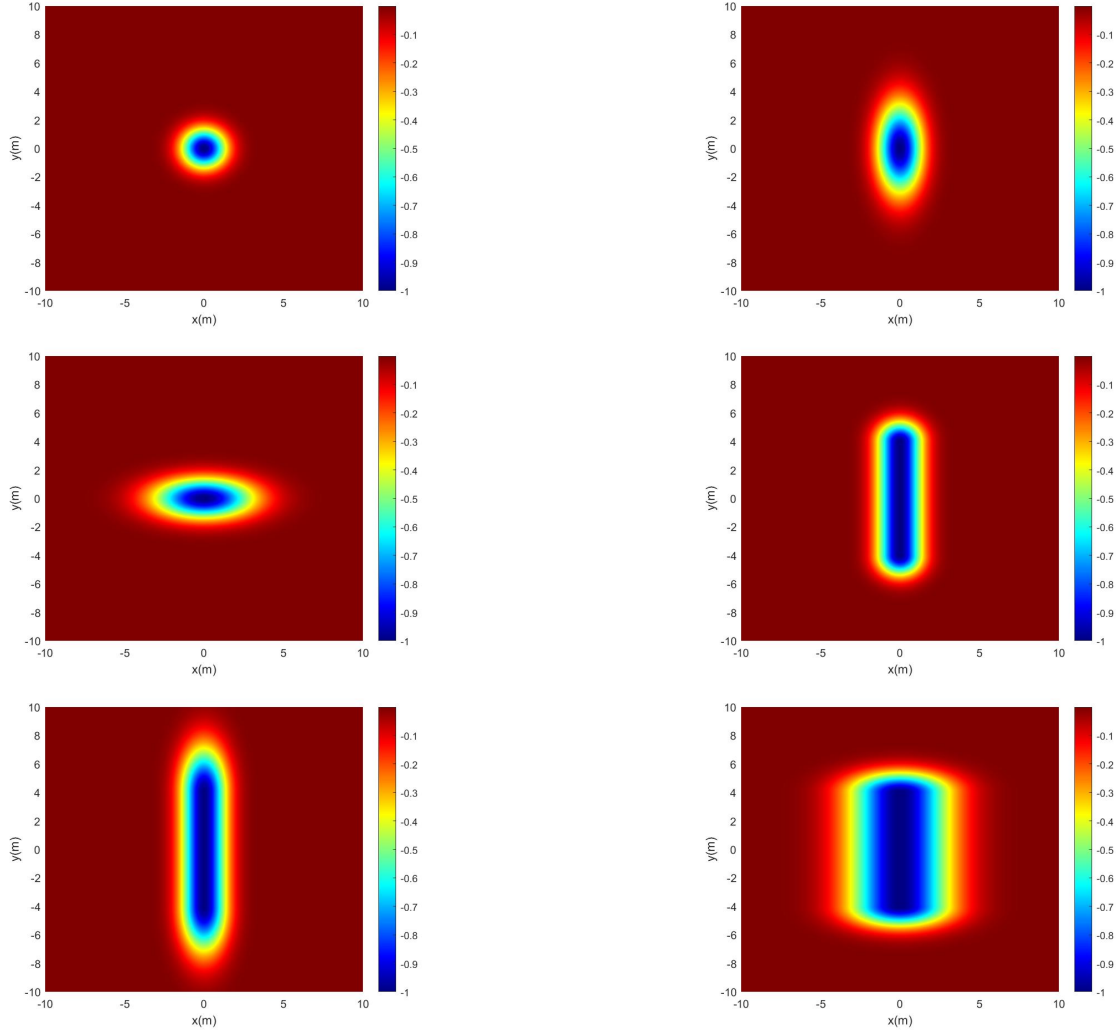


Figure 2.3: The map views of the GPT solution for isotropic pressure diffusivity (Case A) and an- isotropic pressure diffusivity (Case B). First row: vertical well centered at the origin with radius $r = \sqrt{(x^2 + y^2)}$ and (a) ($D_x = 1 < D_y = 5$) and (b) ($D_x = 5 > D_y = 1$). Second row: hydraulic fracture with half-length $W_f = 4$ and (c) ($D_x = 1 < D_y = 5$) and (d) ($D_x = 5 > D_y = 1$).

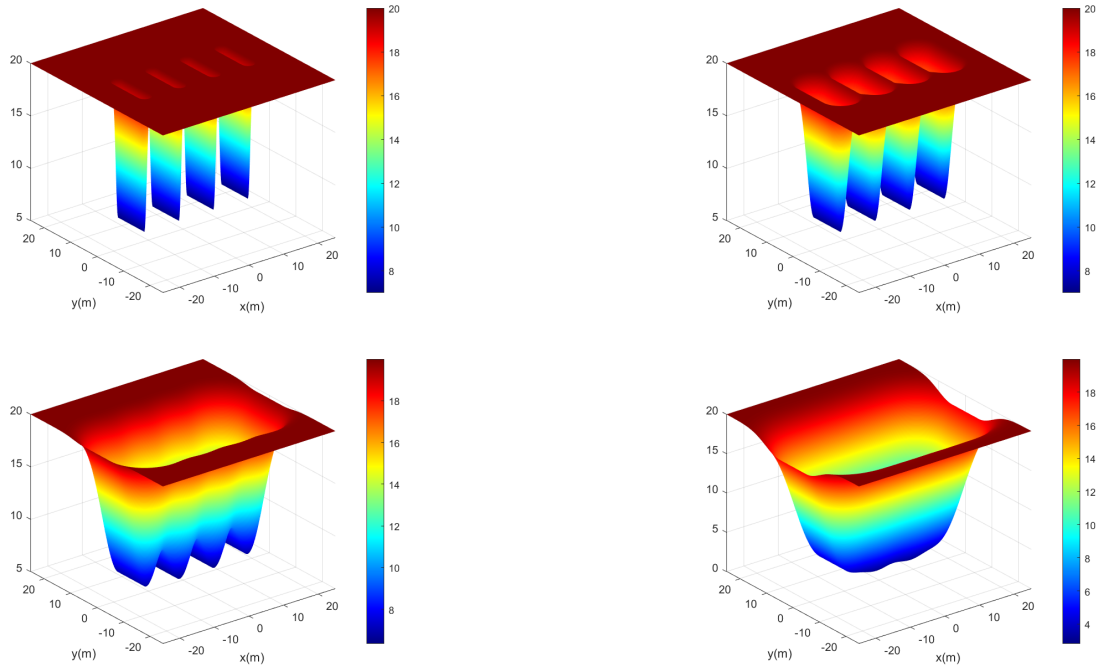


Figure 2.4: Four hydraulic fractures (with half-length $W_f = 4$) for one treatment that depends on time. From left to right $t=100\text{hrs}$ (4.2 days), 1000hrs (1.4 months), 5000hrs (6.8 months), 10000hrs (1.14 years). $D_h = 1$.

CHAPTER 3

FLUID FLUX AND FLOW PATHS

In this chapter, we will use the fluid flux (see equation (2.5)) obtained from Darcy's Law (in 2D) for isotropic and single-phase flow to plot the flow paths as the fluid moves from the porous reservoir rocks into the well system at any time before the flow reaches the steady-state. We will use the Gaussian Pressure Transient GPT method to compute the pressure gradient ∇P of Equation (2.5),

The fluid flux (Darcy velocity) q_D [LT^{-1}] in Equation (2.5) is the flux from the porous formation into the open space of the wellbore across a unit area of wellbore exposed. Recall from Section 2.2, the average fluid velocity (u_r) in the porous reservoir is the Darcy velocity (q_D) divided by the effective porosity (ϕ) of the medium, i.e.

$$u_r = \frac{q_D}{\phi} = -\frac{k}{\phi\mu}\nabla P \quad (3.1)$$

which is similar to Equation (2.4) but for isotropic permeability (the off-diagonal

elements in the permeability tensor are zero and the diagonal elements are identical). Based on the fluid flux components $q_D=(q_x, q_y)$ and pressure gradient components $\nabla P = (\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y})$, we can split Equation (2.5) into two equations representing the Darcy fluid-flux (m/s) in x-direction and y-direction, respectively:

$$q_x = -\frac{k}{\mu} \frac{\partial P}{\partial x}, \quad (3.2)$$

$$q_y = -\frac{k}{\mu} \frac{\partial P}{\partial y}. \quad (3.3)$$

To plot the flow paths, we will consider some common well-configuration patterns which are either well-known or can be compared with Published CAM solutions for steady state. (CAM-solutions are only capable of modeling steady-state flow patterns, while GPT-based solutions can model both the initial transient flow and the final steady-state. The comparison between GPT and CAM results must be limited to the steady-state flow stages. Therefore, we need to study the evolution of pressure change to the steady-state for each well pattern in section 3.1.

3.1 Evolution of pressure transients to steady-state flow

In the procedure of extracting natural resources (water, oil, gas, geothermal fluids), the engineering interventions (vertical wells, horizontal wells) cause pressure changes in the wells of these resources. Studying the induced time-dependent pressure-changes using Gaussian pressure-transients and the relevant flow attributes at different time

stages is the scope of the current work.

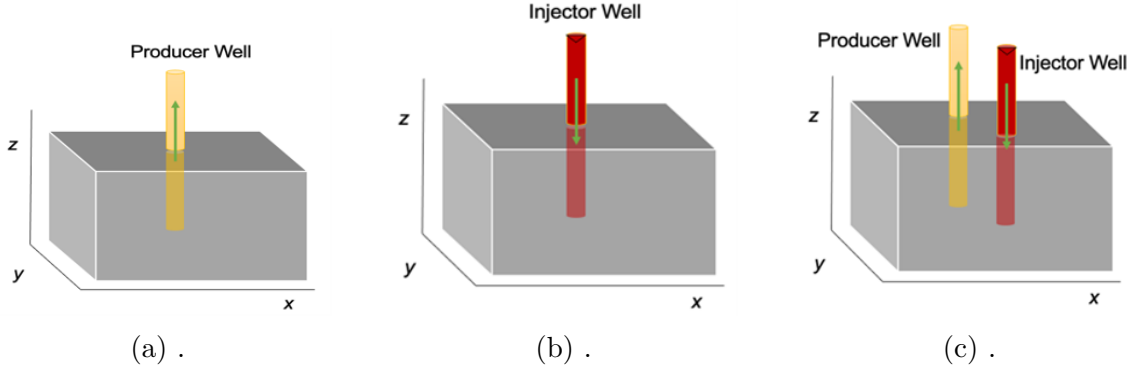


Figure 3.1: Three cases considered for studying the evolution of pressure transients to steady-state flow. (a) Case 1: a reservoir containing single producer. (b) Case 2: a reservoir containing single injector (c) Case 3: Two spot case, with one producer and one injector (a so-called doublet flow)

The pressure-change, ΔP , in the well passes through three main stages. The first stage is the initial stage before starting the production/injection procedures. At the initial stage the initial pressure is the pre-existing pressure in the reservoir denoted by P_0 and the pressure-change at this stage, ΔP_0 , is the difference between P_0 and the pressure level of the well intervention P_{BH} . Next stage is the transient stage when the pressure-change is still a function of the well geometry and time variable, $\Delta P = \Delta P(x, y, t)$. The pressure-change at this stage can be obtained analytically using the Gauss pressure transient solution (GPT), first named in [15]. Once the pressure gradient no longer changes with time ($\frac{\partial \Delta P}{\partial t} = 0$), the pressure reaches the third stage which is the steady-state stage. In this section, we track the GPT evolution over time until it either ceases due to pressure depletion or reaches the steady-state for the three cases of well systems considered (3.1). However, we will show in this section that, in some cases, reaching steady-state is not accurately achieved, especially for

the superposed well system. In such cases, the third stage for the pressure-change life would be called the pseudo-steady-state. The pressure gradient changes with respect to time, $\frac{\partial P}{\partial t}$, and will advance in space until a steady pattern is established sometime after the onset of the pressure disturbance due to the well intervention. As long as flow is still in the transient state, then the pressure rate is a function of the well geometry and time variable, $\frac{\partial P}{\partial t} = F(x, y, t)$. Regions where the pressure gradient no longer changes with time, $\frac{\partial P}{\partial t} = 0$, the flow is classified as a steady-state flow.

In this section, we track the GPT evolution over time until it either ceases altogether due to pressure depletion, or reaches a steady-state for the three cases of well systems considered (3.1)

For the case with a single producer (Figure 3.1a) or a single injector (Figure 3.1b), no steady-state flow will ever be reached, because the pressure transient will advance in the reservoir until the entire reservoir attains the bottomhole pressure (P_{BH}). The reservoir pressure is then said to be depleted, at which moment in time the pressure gradient ΔP will no longer exist and the flow in the reservoir will cease and the well will stop producing fluid from the subsurface. For cases with one producer and one injector (Figure 3.1c), a steady-state flow can be maintained in the region between the wells after the pressure gradient profile between the well pair has been established.

3.1.1 Case 1: Reservoir with single producer vertical well

Gaussian pressure transient solutions are plotted in Figure 3.2 for a variety of times since the introduction of the pressure perturbation in the bottom of the well. Complete depletion of the reservoir pressure is an asymptotic process, and will not occur until time becomes infinitely long.

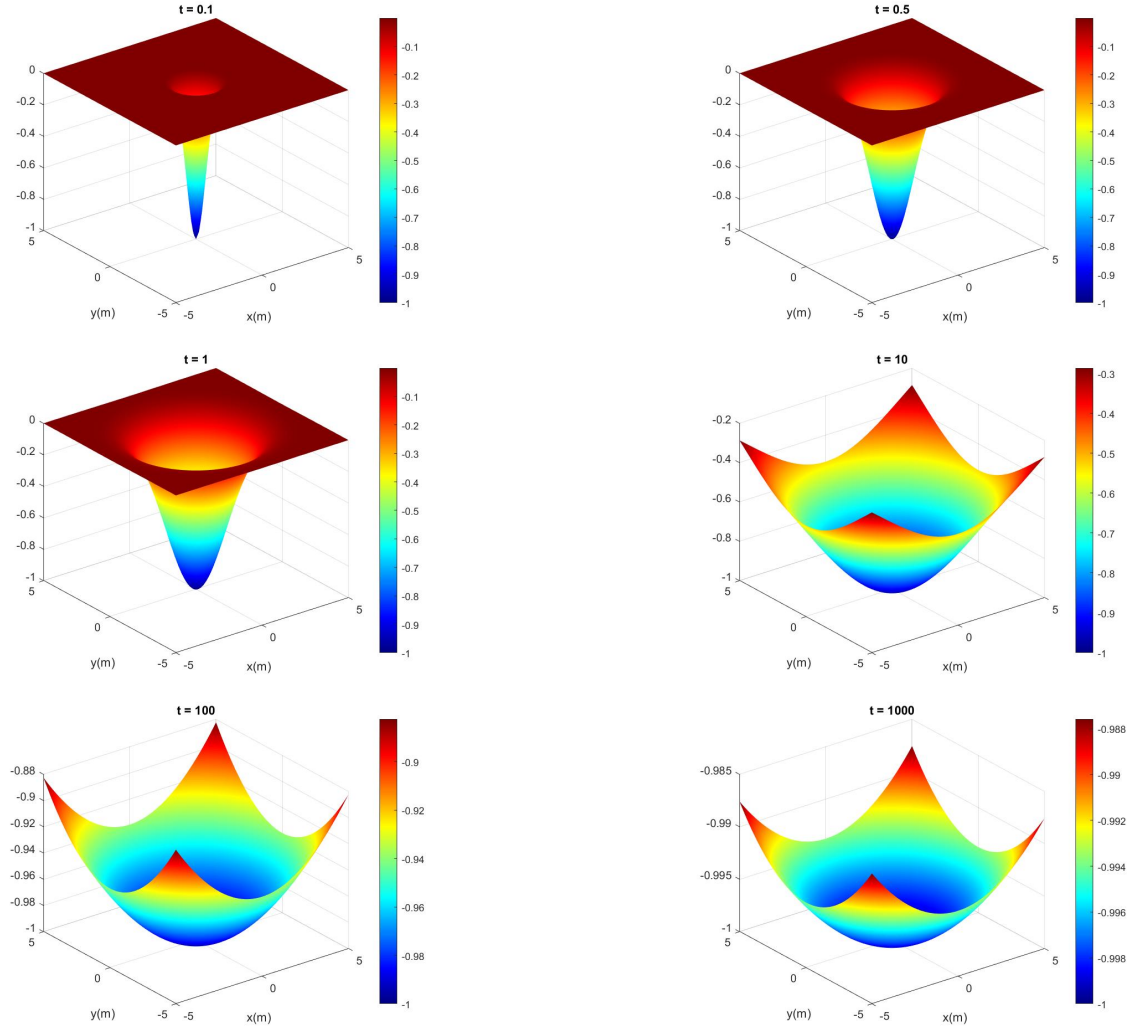


Figure 3.2: Case 1 – Producer. The GPT solution using Equation (2.10) for a single producer (Case 1) located at the origin with isotropic diffusivity $D_x = D_y = 1$, pre-existing pressure $P_0 = 0$ and $P_{BH} = -1$, at different times: top row (from left to right): $t = 0.1, 0.5, 1$. Bottom row (from left to right): $t = 10, 100, 1000$.

Figure 3.3 shows the evolution of the pressure depletion profiles at different times

as seen in a section containing the producer well located at the origin; as time passes the pressure in the reservoir will be progressively reduced to attain the bottomhole pressure until $\frac{\partial P}{\partial t} = 0$, which is when the flow ceases to exist. The pressure depletion curves converge to the horizontal y-axis of value $y = P_{BH}$ (the light blue line in Figure 3.3) for $t=1000$.

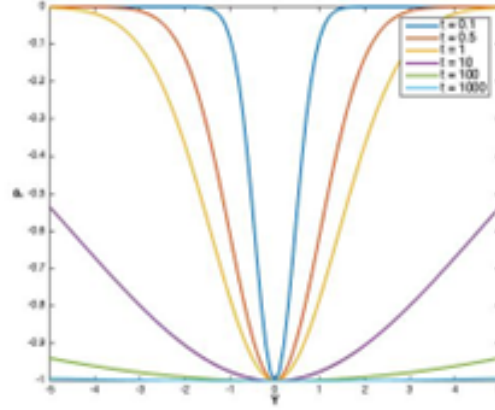


Figure 3.3: Pressure profile at $x = 0$ over different time ($t = 0.1, 0.5, 1, 10, 100, 1000$) for one producer (Case 1) located at the origin. The vertical scale is the pressure and horizontal axis is the distance of y values.

3.1.2 Case 2: Reservoir with single injector vertical well.

The GPT solution is plotted in Figure 3.4 for a variety of times since the introduction of the pressure perturbation in the bottom of an injection well. The maximum pressure may not exceed the bottomhole pressure, which will raise the reservoir pressure in an asymptotic process, and will in an infinitely large reservoir not occur until time becomes infinitely long.

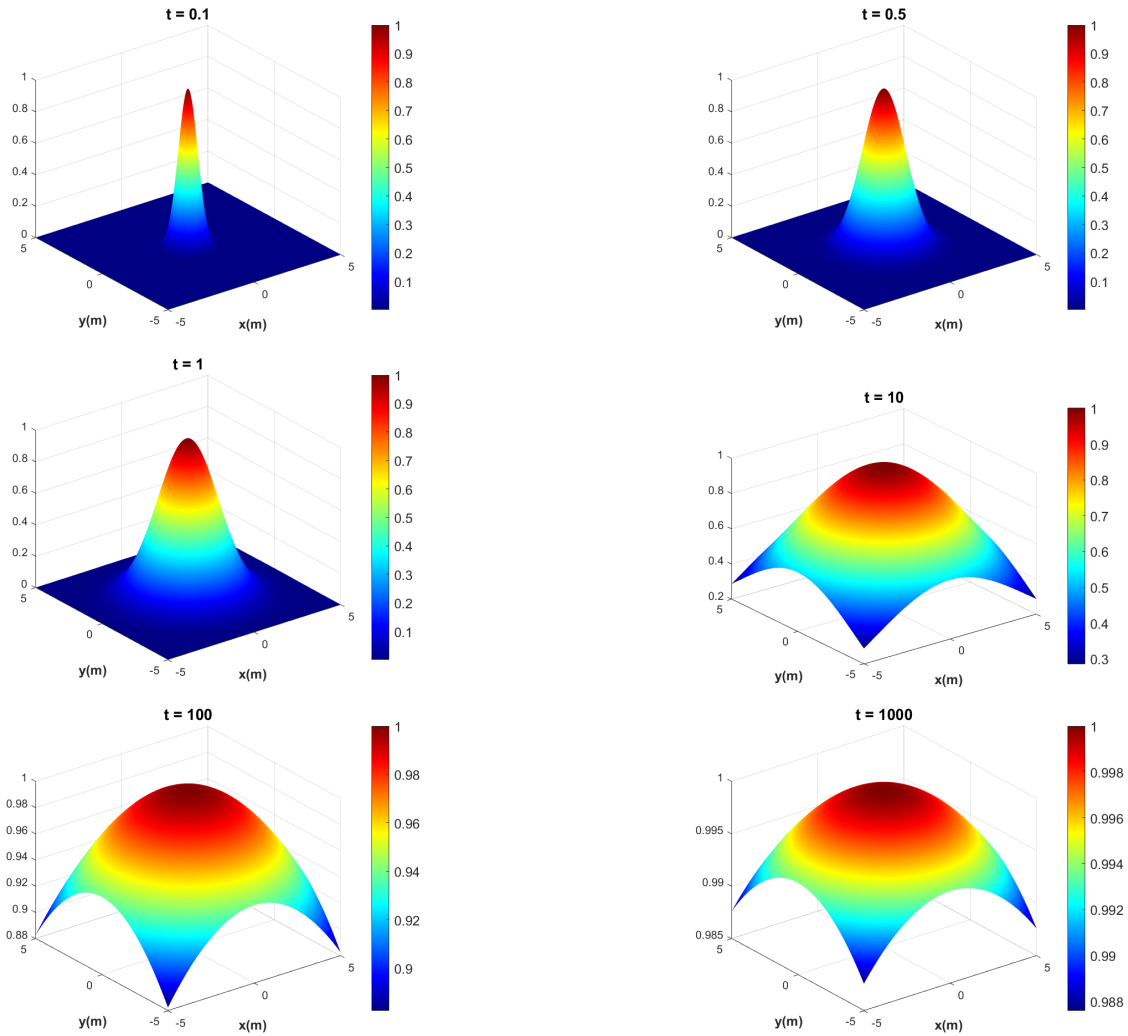


Figure 3.4: Case 2 – Injector. The GPT solution using Equation (2.10) for a single injector (Case 2) located at the origin with isotropic diffusivity $D_x = D_y = 1$, pre-existing pressure $P_0 = 0$ and $P_{BH} = -1$, at different times: top row (from left to right): $t = 0.1, 0.5, 1$, bottom row (from left to right) $t = 10, 100, 1000$

Figure 3.5 shows the pressure saturation profiles at different times, initiated by a step change in pressure by one injector well located at the origin at $x = 0$. Similar to the single producer case (Case1), the pressure depletion curve converges to the horizontal y-axis of value $y = P_{BH}$ (the light blue line in Figure 3.5) for $t=1000$.

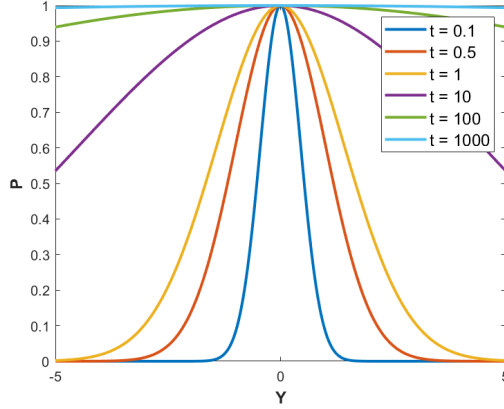


Figure 3.5: Pressure profile at $x = 0$ over different time ($t = 0.1, 0.5, 1, 10, 100, 1000$) for one injector (Case 2)

The computation of the $t_{limit-stop}$ for a single well (Case 1, Case2) can be done mathematically as following:

Recall that for a single well located at the origin, the GPT- solution is given by

$$P(x, y, t) = P_0 + \Delta P_0 e^{-\frac{x^2+y^2}{4D_h t}}. \quad (3.4)$$

At the $t_{limit-stop}$ there is no pressure gradient around the well location and the pressure depletion curve, when x is fixed as in Figures 3.5 and 3.3, is represented by a horizontal line. Thus, to compute the $t_{limit-stop}$ we compute $\frac{\partial P}{\partial y}$ and equate it with zero.

$$\frac{\partial P}{\partial y} = 0 + \Delta P_0 \frac{2y}{4D_h t} e^{-\frac{x^2+y^2}{4D_h t}}. \quad (3.5)$$

The LHS of (3.5) will be zero when we take t infinitely large ($t \rightarrow \infty$).

3.1.3 Case 3: Reservoir with two-spots production system

In this case, we consider a two-spot well constellation (Figure 3.1c), consisting of one producer located at (x_p, y_p) and one injector located at (x_i, y_i) , such that $x_p = x_i$ and $y_p = -y_i$. The classical 2D superposition Gaussian solution for the pressure-change induced by the two wells (for isotropic diffusivity case ($D_x = D_y = D_h$)) is given by:

$$\Delta P(x, y, t) = \Delta P_{0,i} e^{-\left(\frac{x_i^2 + y_i^2}{4D_h t}\right)} + \Delta P_{0,p} e^{-\left(\frac{x_p^2 + y_p^2}{4D_h t}\right)}, \quad (3.6)$$

where $\Delta P_{0,i}$ and $\Delta P_{0,p}$ are the bottomhole pressure for the injector well and producer well, respectively, assuming the pre-existing pressure in the reservoir $P_0 = 0$. If $|\Delta P_{0,i}| = |\Delta P_{0,p}|$, the system is called the balanced doublet flow, otherwise the system is called unbalanced doublet. In [28], the authors used the somewhat un-English term unsymmetrical rather than asymmetrical; we are fine with both and prefer to use the dynamic term ‘unbalanced doublet’.

The pressure transient advance is graphed in Figure 3.6 for a balanced doublet. In this case, $\Delta P(x, y, t)$ may not drop below the bottomhole pressure of the producer well, i.e. $\Delta P(x, y, t) < P_{BH}$ and a time limit stop, $t_{limit-stop}$, is then triggered for the producer well, otherwise the flow becomes non-physical. For the injector well, time $t_{limit-stop}$, occurs when $\Delta P(x, y, t) > P_{BH}$, otherwise the flow would become non-physical. For a doublet well pair, beyond time $t_{limit-stop}$, $\Delta P(x, y, t)$ would start to shift the pressure peaks beyond the injector location and pressure minima beyond the

producer location, which would be non-physical. The production time will continue, but the pressure gradient in the reservoir region between the two wells will have reached a steady-state and will remain so as long as the producers and injectors have balanced injection and production rates. In the case of uneven pressure quota, the system is either a net producer (Figure 3.7a) or a net injector (Figure 3.7b).

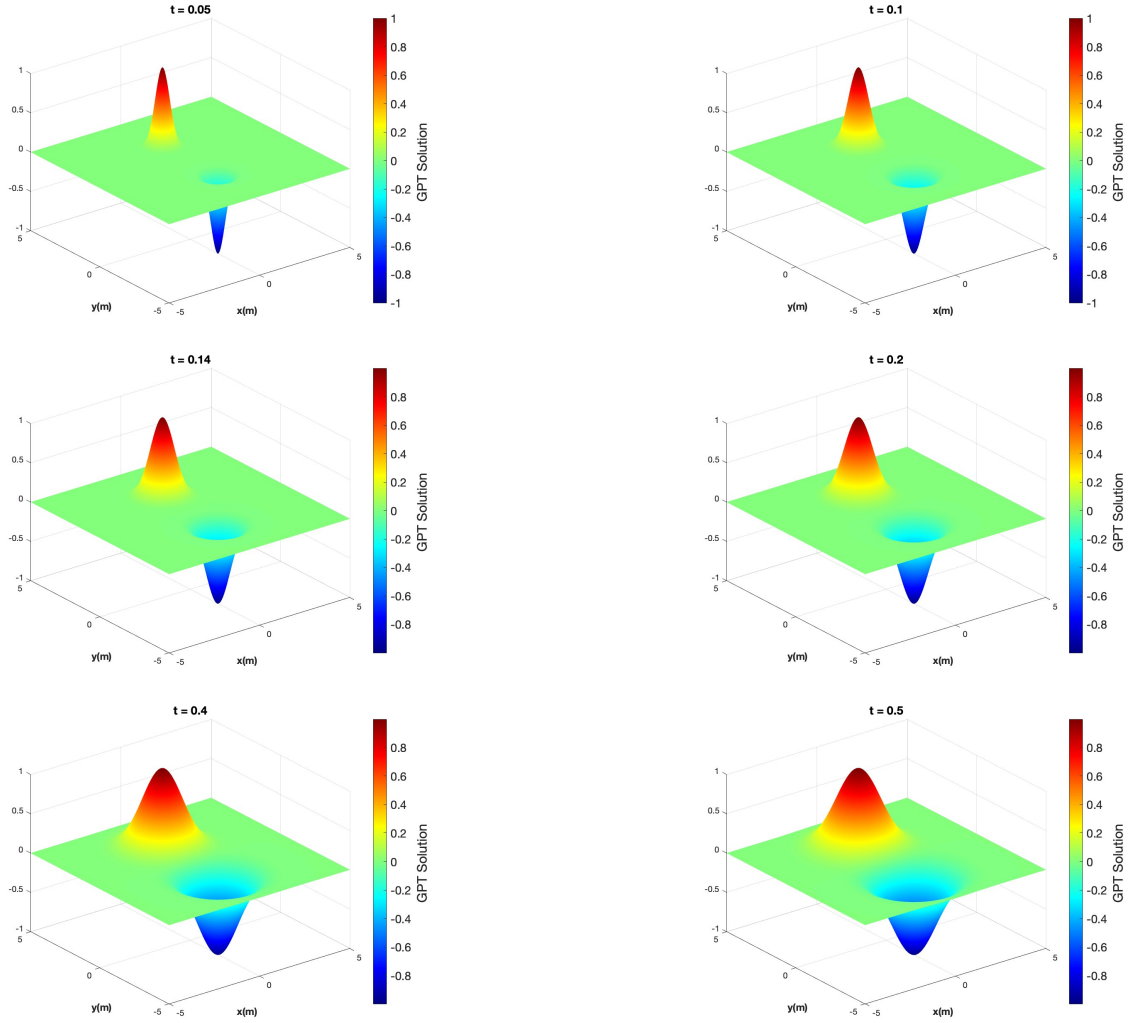
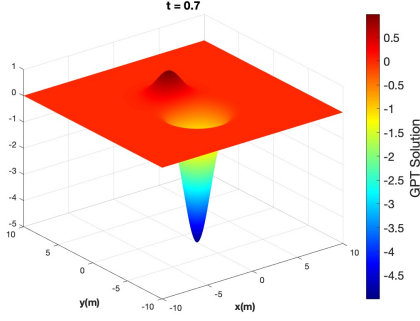
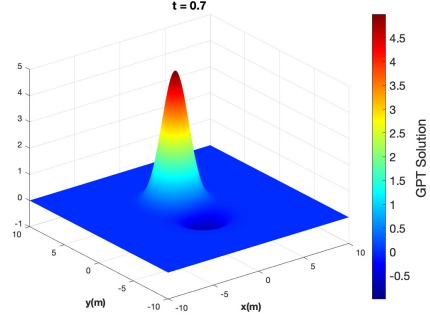


Figure 3.6: Case 3 – Balanced Doublet. The GPT solution using Equation (2.10), for one injector located at $(0,2)$ and one producer located at $(0,-2)$ with isotropic diffusivity $D_x = D_y = 1$, pre-existing pressure $P_0 = 0$ and $P_{BH} = 1$ for the injector and $P_{BH} = -1$ for the producer, at different times. Top row (from left to right): $t = 0.05, 0.1, 0.14$. Bottom row (from left to right): $t = 0.2, 0.4, 0.5$

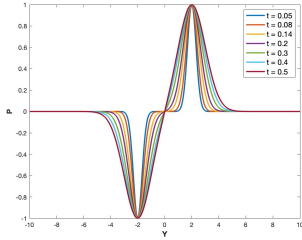


(a) .

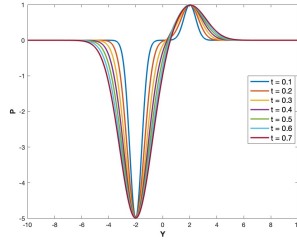


(b) .

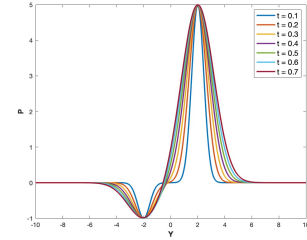
Figure 3.7: Case 3 – Unbalanced Doublet. The GPT solution for one injector located at $(0,2)$ and one producer located at $(0,-2)$ when (a) $P_{BH} = 1$ for the injector and -5 for the producer (net producer case), (b) $P_{BH} = 5$ for the injector and -1 for the producer (net injector case).



(a) .



(b) .



(c) .

Figure 3.8: Case 3: Pressure profile at $x = 0$ over different time for a producer located at $(0, -2)$ and injector located at $(0,2)$. The vertical scale is the pressure and horizontal axis is the distance (y -values). (a) Balanced Doublet. (b) Unbalanced Doublet, $P_{BH} = 1$ for the injector and -5 for the producer (net producer case). (c) Unbalanced Doublet, $P_{BH} = 5$ for the injector and -1 for the producer (net injector case).

Figure 3.8b show the pressure transient profiles for different times near a producer located at $(0, -2)$ and injector located at $(0,2)$, here x is fixed at $x = 0$ and $y \in [-10, 10]$. In Figure 3.8c the P_{BH} is -5 for the producer well and 1 for the injector well. Whilst in Figure 3.8c, the P_{BH} is -1 for the producer well and 5 for the injector well.

For the balanced doublet (Figure 3.8a) the flow between the two wells will

eventually reach a steady-state, when the pressure profile is represented by a straight line joining the points $(2, 1)$ and $(-2, -1)$, i.e., the pressure profile has become stable and no longer transient – no change occurs with respect to time. The flow in the region between the two wells (i.e., at $y \in [-2, 2]$), reaches that steady-state at $t_{limit-stop} \approx 0.5$. For $t > t_{limit-stop}$ the max and/or min value(s) of $P(x, y, t)$ would start(s) shifting beyond the original well locations and the flow becomes non-physical. For the unbalanced doublets, the flow (in the region between the two wells) needs longer time to reach the steady-state than for the balanced doublet case. Figures 3.8b and 3.8c show that the pressure transient period stops at $t_{limit-stop} \approx 0.7$ when the flow reaches the steady-state.

Remark: Several non-physical aspects came to light when superposing pressure-transients using Equation (2.10):

- In Figure 3.9 we arrest the pressure transient advance between a pair of producer wells to avoid the pressure to drop below the bottomhole pressure as in Figure 3.9b, the far-field pressure advance is also frozen, Figure 3.9a. This can be overridden by continuing the advance of the pressure front in the far-field region, but leads to an outer region with the time counter increasing, while in the inner region the time is already stopped (at $t_{limit-stop}$) when the bottomhole pressure is reached.
- For an injector and producer well pair, $t_{limit-stop}$ is introduced when the

pressure profile between the wells has been established (Figure 3.8a). However, the far-field pressure advance is frozen, and the far-field pressure front would no longer advance in the far-field when the pressure profile in the inner region between the doublet has been established (a pseudo-steady-state).

This paradox conclusion of the $t_{limit-stop}$, is a result of using a classical superposition method according to Equation (2.10). To avoid the non-physical effects with conflicting time counters in the inner regions and the far-field flow regions, a modified superposition method has been developed, which mitigates this issue (work under review).

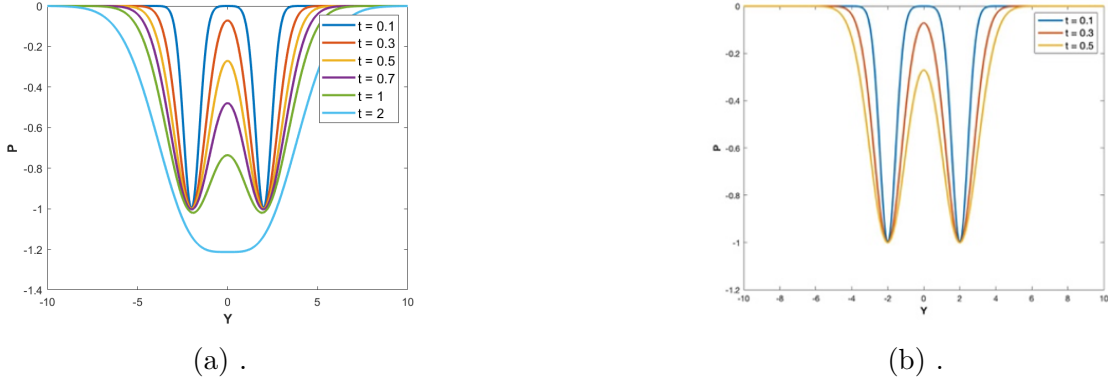


Figure 3.9: Pressure profiles for two producers, one located at $(0, -2)$ and the other located at $(0, 2)$ with $P_{BH} = -1$ for both producers. (a) The pressure profiles at various times; without any time-limit-stop the pressure profile would drop below the P_{BH} , which is non-physical. (b) The solution for the pressure advance remains above the P_{BH} everywhere when frozen at $t_{limit-stop} = 0.5$.

As for single well cases, we can compute the $t_{limit-stop}$ for superposed case (two-well system) mathematically by computing the partial derivative with respect y (x is

fixed). From (3.6) we deduce

$$\frac{\partial P}{\partial y} = 0 - \Delta P_{0,i} \frac{2(y - y_i)}{4D_h t} e^{-\left(\frac{(x-x_i)^2 + (y-y_i)^2}{4D_h t}\right)} - \Delta P_{0,p} \frac{2(y - y_p)}{4D_h t} e^{-\left(\frac{(x-x_p)^2 + (y-y_p)^2}{4D_h t}\right)}, \quad (3.7)$$

we have, $\Delta P_{0,pi} = -\Delta P_{0,p} = 1$ and $D_h = 1$ Therefore,

$$\frac{\partial P}{\partial y} = -\frac{(y - y_i)}{2t} e^{-\left(\frac{(x-x_i)^2 + (y-y_i)^2}{4t}\right)} + \frac{(y - y_p)}{2t} e^{-\left(\frac{(x-x_p)^2 + (y-y_p)^2}{4t}\right)}. \quad (3.8)$$

The $t_{limit-stop}$ is the time when the pressure depletion curve is represented by a straight line passes the midpoint between the wells location (see Figure 3.8a). Therefore, we compute $\frac{\partial P}{\partial y}$ at the midpoint $\alpha = \left(\frac{x_i+x_p}{2}, \frac{y_i+y_p}{2}\right)$:

$$\frac{\partial P}{\partial y} = 0 - \Delta P_{0,i} \frac{2(y - y_i)}{4D_h t} e^{-\left(\frac{(x-x_i)^2 + (y-y_i)^2}{4D_h t}\right)} - \Delta P_{0,p} \frac{2(y - y_p)}{4D_h t} e^{-\left(\frac{(x-x_p)^2 + (y-y_p)^2}{4D_h t}\right)}, \quad (3.9)$$

we have, $\Delta P_{0,pi} = -\Delta P_{0,p} = 1$ and $D_h = 1$ Therefore,

$$\frac{\partial P}{\partial y} = -\frac{(y - y_i)}{2t} e^{-\left(\frac{(x-x_i)^2 + (y-y_i)^2}{4t}\right)} + \frac{(y - y_p)}{2t} e^{-\left(\frac{(x-x_p)^2 + (y-y_p)^2}{4t}\right)}. \quad (3.10)$$

Then,

$$\frac{\partial P}{\partial y}|_{\alpha} = 2 \frac{(y_i - y_p)}{4t} e^{-\left(\frac{(x_i-x_p)^2 + (y_i-y_p)^2}{16t}\right)}. \quad (3.11)$$

Based on our assumption that $y_i = -y_p$ and $x_i = x_p$. Equation (3.11) can be written

as:

$$\frac{\partial P}{\partial y}|_{\alpha} = \frac{y_i}{t} e^{-\frac{y_i^2}{4t}}. \quad (3.12)$$

For the specific example given in Figure 3.8a, the slope of the straight line passes the midpoint is $1/2$. Thus, from (3.12) we have

$$\frac{2}{t} e^{-\frac{1}{t}} = 1/2. \quad (3.13)$$

Equation (3.13) is non-linear equation that can be solved for t using non-linear solver, such as Newton method that is used to numerically compute the root of a nonlinear function $f(x)$ using the following formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (3.14)$$

Applying the above formula for $f(t) = \frac{e^{-\frac{1}{t}}}{t} - \frac{1}{4} = 0$, with initial guess $t_0 = 0.2$, we get after 5 iterations that $t = 0.4644 \approx 0.5$ which meets the simulation finding of the time $t_{limit-stop} = 0.5$ for two-spots system.

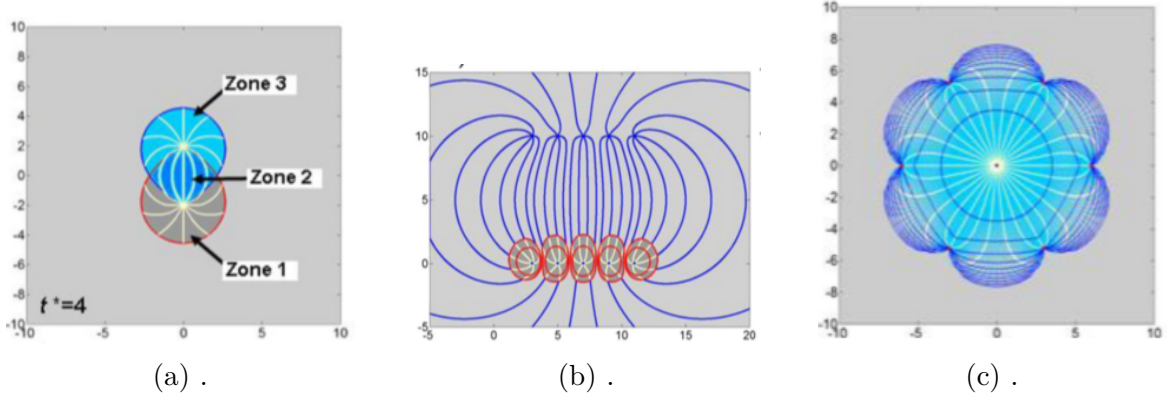


Figure 3.10: Steady-state CAM solutions. a) Doublet flow paths (white lines) between one injector located at (0,2) and one producer located at (0,-2). The colors show the advance of a waterfront from the injector well (blue) and oil-withdrawal region (red contour, dark gray zone) (after [9]). b) Direct line drive with 5 producers located at $y=10$ and 5 injectors at $y=0$. (after [3]). c) Flow paths for 7-spot pattern, one injector located at (0,0), surrounded by six producers. Water front advance contours in dark blue; after [9].

3.2 Fluid flux and flow paths

Figure 3.10 shows typical flow path patterns in subsurface reservoirs (penetrated by two or more wells) as was visualized in previous studies using CAM [3], [18]

3.2.1 Case A: Doublet (Two spot array)

The doublet flow pattern of Figure 3.10a corresponds to a balanced well pair as used in Case 3 of Section 3.1. In Figure 3.10a, the particle paths along the steady-state flow paths were traced with the CAM code using a Eulerian time-stepping schedule [18]. The front of the flow paths is outlined by time-of-flight (TOF) contours. The blue outline in Figure 3.10a is for fluid advance of injected fluid, the red TOF outline is for fluid withdrawal.

In the GPT-solution of Figure 3.11, the flow paths are mapped, together with the normalized pressure magnitude-contours. The yellow dot marks the producer location and the red triangle is for the injector. The injector and producer are spaced by 4 length units. The white flow paths (streamlines) are traced with MATLAB using $-\nabla P$ from Darcy's Law. Note that the pressure contours everywhere are perpendicular to the flow paths (as expected in the absence of inertia).

An important feature of the GPT-method is that it can visualize the flow during the earlier period of pressure transient advance in a reservoir, before the steady-state flow is established. The advance of the pressure gradient into the reservoir space is shown in Figure 3.11, together with the instantaneous flow paths. Note that before the pressure transient fronts of the respective wells meet, the flow paths of the injector and producer wells are perfectly radial (top row of Figure 3.11). After pressure interference starts occurring, the shape of the pressure contours is no longer concentric (bottom row of Figure 3.11) and flow paths begins to attain the well-known steady-state pattern of a doublet flow.

3.2.2 Case B: Direct line drive (five-spot well array)

The direct line drive of Figure 3.10c used a balanced well pairs, with 5 injectors placed on $y=0$ and 5 producers placed on $y= 10$. The GPT- solution was used to generate the transient Gaussian pressure profiles (colored contours) and flow paths (white lines) (Figure 3.12). Initially, the flow pattern of the lower injectors (red triangles) and upper producer row (yellow circles) are not connected and will lead to pressure profile development similar to that portrayed in Figure 3.11a for the balanced doublet. Once

steady-state pressure profile is established (Figure 3.12d), the flow of the direct line drive well-array will resemble that of the steady-state doublet (Figure 3.10c).

3.2.3 Case C: 7-spot well pattern

The 7-spot well pattern of Figure 3.10c had one strong injector in the center and six producers spaced equally around the central injector well. The injection rate of the single injector was equal to that of the six producers combined. The same well strengths were used in the realization of the GPT flow-paths in Figure 3.13, with one strong injector (red triangle) in the center, surrounded by six producers (6 yellow circles). The advance of the pressure change contours is shown, and the flow is evenly fanning radially outward the central injector and toward the surrounded six producers at early time before the flows approach each other (Figure 3.13a). Once the pressure gradient is extended in the circular region centered by the injector well, the flow paths are no longer radial (Figures 3.13b, 3.13c and 3.13d). The final stage (Figure 3.13d) is when the flow pattern becomes steady-state and similar to that seen in Figure 3.10c. However, it is worth noting again that the solution of Figure 3.10c was produced with CAM using time-stepping superposition, while tracking the advance of certain fluid tracers particles starting from the injector wells, as they move toward the producer wells. But this procedure is constrained by the superposition of steady-state solutions. In contrast, Figure 3.13 show the evolution starting from transient flow while the pressure transient advances to ultimately establish a steady-state flow.

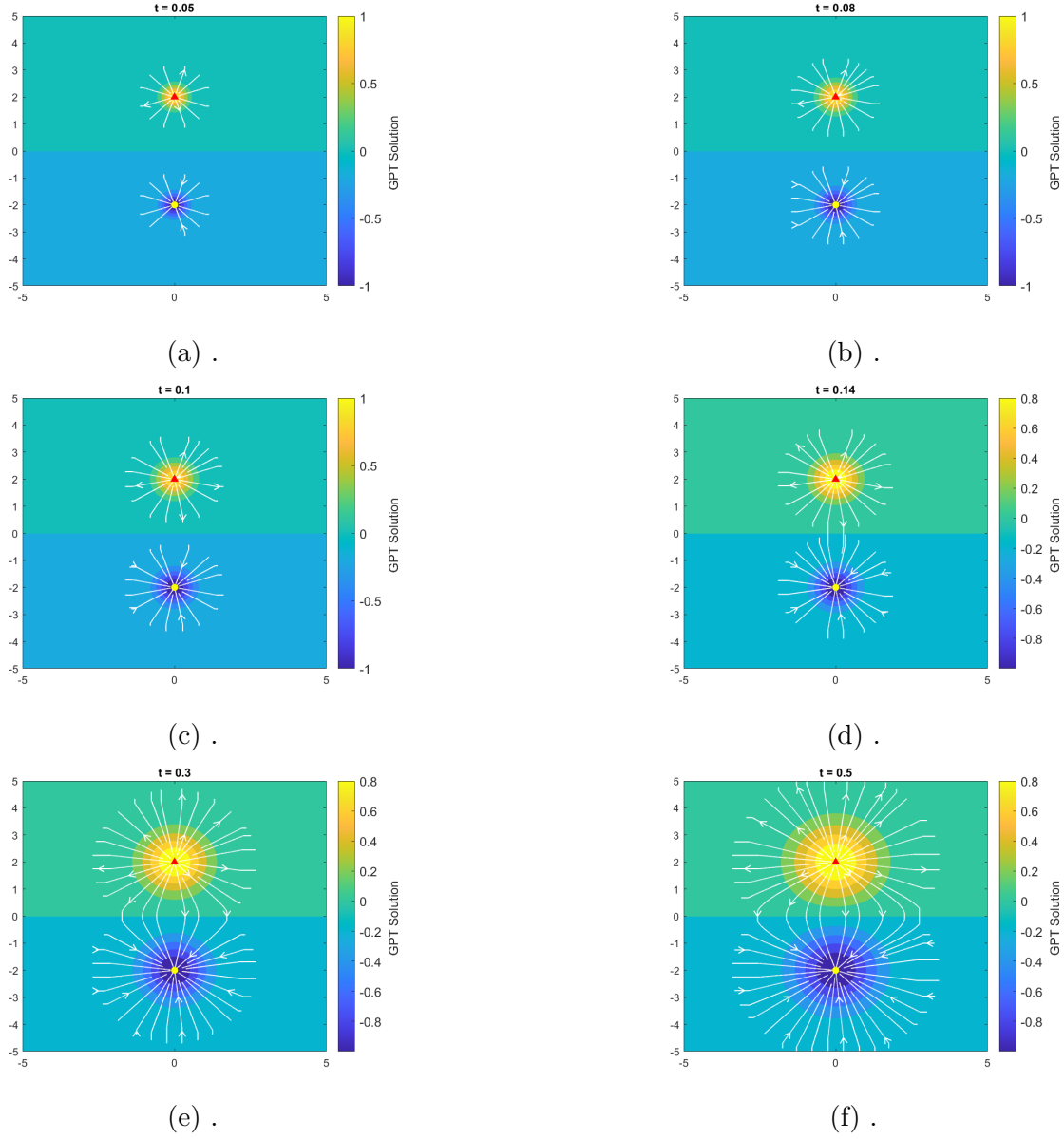
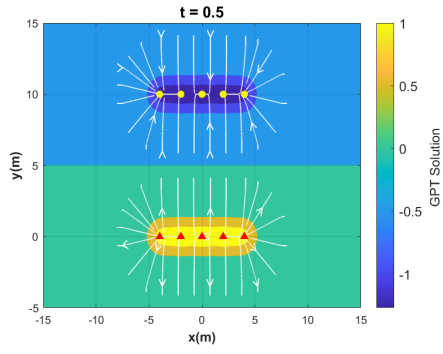
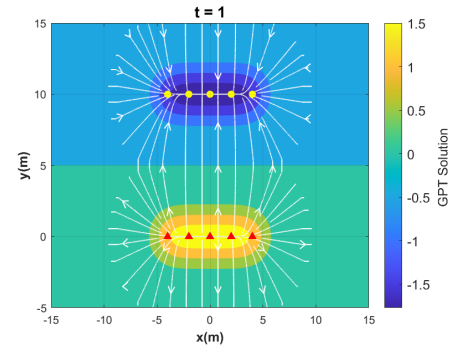


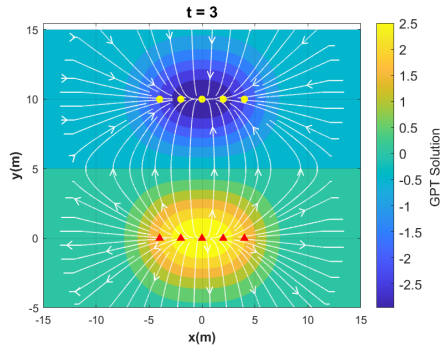
Figure 3.11: Transient streamlines (white lines) for balanced doublet with injector located at (0,2) and producer at (0,-2) with initial pressure change $\Delta P_0 = -1$ for the producer and $\Delta P_0 = 1$ for the injector. Shown are flow patterns and pressure fronts at different times indicated on the top of each plot, $t_{limit-stop} \approx 0.5$. Pressure contour-scale was normalized.



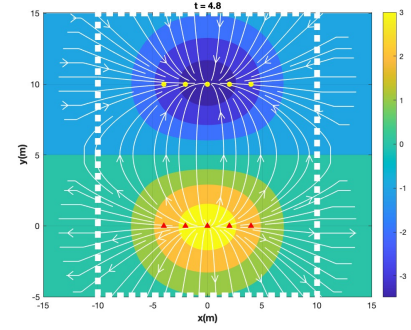
(a) .



(b) .



(c) .



(d) .

Figure 3.12: GPT-based solutions for the advance of the pressure transient and corresponding flow paths for five producers parallel to five injectors at different times; (a) $t=0.5$, (b) $t=1$, and (c) $t=3$.

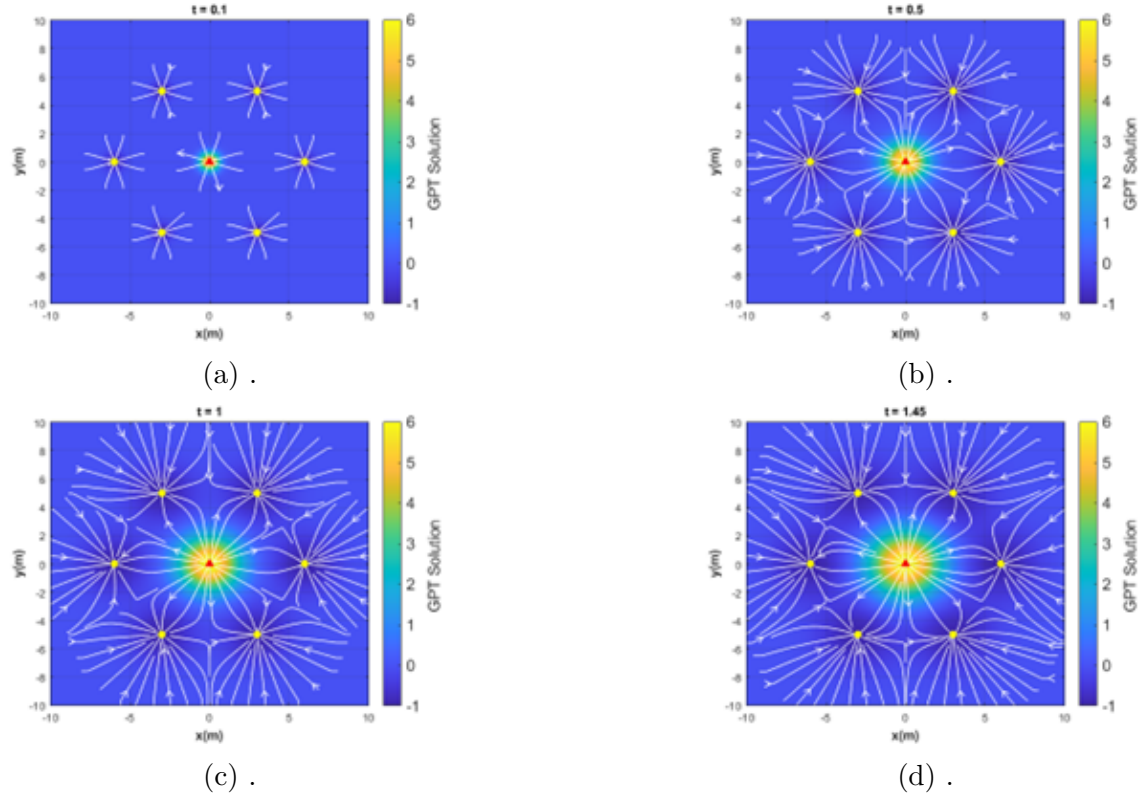


Figure 3.13: Flow paths visualization for 7-spot pattern, one injector located at (0,0) and six producers. $\Delta P_0 = -1$ for the producers, and $\Delta P_0 = 6$ for the central injector. Solutions are for different times since onset of the pressure perturbation: (a) $t=0.1$, (b) $t=0.5$, (c) $t=1$, and d) $t_{limit-stop} = 1.45$.

CHAPTER 4

DERIVATION OF TRANSIENT STREAM FUNCTION AND POTENTIAL FUNCTION

In Chapter 3, transient solutions for flow paths and pressure contours were computed based on gridded solution points solved after computing $-\nabla P$ using MATLAB. However, true analytical GPT-solutions can be obtained by deriving the stream function and potential function for transient flows. The visualization of fluid flow in porous media commonly based on numerical methods and semi-analytical methods that require space and time discretization. For time-dependent or so-called transient flow problems, tracing the rapidly changing flow paths is an additional challenge. In the present study, we present analytical solutions for the stream function, which can compute both the steady-state and the transient flow paths in porous media. Likewise, new potential function solutions are derived for computing the pressure gradients for both the

steady-state and the transient flow stage. Our new analytical solutions for stream and potential functions are based on analytical computations of the local pressure transient using the GPT solution [15], [16]. The GPT-solution is a space-time-dependent quantity that allows us to compute the instantaneous pressure gradient around the wells. Using GPT-solutions, one can visualize the flow paths at any time during the transient flow period. Here, the transient flow period refers to the period when there is an advancing pressure-gradient, causing the transient flow paths to shift over time. The essential fundamentals are given (Section 4.1), before applying those to the GPT solution to derive the new analytical stream function and potential function solutions for transient flow (Sections 4.2 and onward).

4.1 Fundamentals

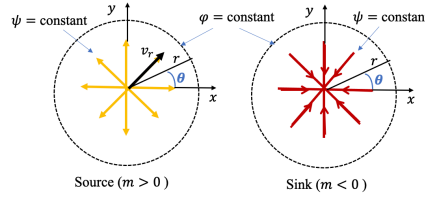


Figure 4.1: The streamlines (arrows) and equipotential lines (dashed circles) for source/sink flow. (a) source (the flow rate $m > 0$) (b) sink (The flow rate $m < 0$).

The transient GPT solutions of the temporal and spatial pressure changes [15] and [16] can be used to compute the potential function of the time-dependent fluid flow. The fluid flow in our study is subject to an incompressibility assumption. However, the flow does not need to be inviscous (see arguments in Weijermars [17]). The flow can be described by combining the stream function $\psi(x, y, t)$ and velocity potential function

$\Phi(x, y, t)$ to satisfy Cauchy-Riemann equations [17]:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \quad \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}. \quad (4.1)$$

The stream function is a scalar function which can be solved for constant values to plot the streamlines; its spatial derivatives are related to the fluid velocity components, u_x and u_y as follows [17]:

$$u_x = \frac{\partial \Psi}{\partial y}, \quad (4.2)$$

$$u_y = -\frac{\partial \Psi}{\partial x}. \quad (4.3)$$

Consider a reservoir with a single source/sink located at the origin (see Figure 4.1). The GPT solution of a single source/sink is given by Equation (2.10) for $n = 1$ with $\Delta P < 0$ for sink (producer well case) and $\Delta P > 0$ for the source flow (injector well case). Typically, to obtain the stream function, one can integrate Equation (4.2) with respect to y and Equation (4.3) with respect to x (see for example Weijermars [29]). However, to avoid the complexity of integrating the exponential function, which appears in the velocity components by differentiating the transient pressure (see Equation (2.10)) one can, alternatively, consider the radial and tangential velocity components (u_{rad} , u_θ) of the source/sink flow given by (see [21]):

$$u_{rad} = \frac{m}{2\pi r}, \quad u_\theta = 0, \quad (4.4)$$

where m is a scaling space-constant represents the strength (volumetric flow rate

per unit length- m^2/s). If m is positive the fluid is flowing radially outward the origin (source flow) and if m is negative the fluid is flowing radially toward the origin (sink flow) (see Figure 4.1). To solve and find the well strength, m , we use the stream function and velocity potential for the source/sink flow obtained from integrating u_{rad} in Equation (??) with respect to r and integrating u_{rad} with respect to θ to get (see [21] for derivation):

$$\Psi(r, \theta) = \frac{m}{2\pi} \theta, \quad (4.5)$$

$$\phi(r, \theta) = \frac{m}{2\pi} \ln r. \quad (4.6)$$

Having obtained the stream and potential functions of Equations (4.5), (4.6), one may revert to Cartesian coordinates if needed, using transformation from polar coordinates:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}. \quad (4.7)$$

Substituting Equations (4.7) in (4.5), (4.6) gives:

$$\Psi(x, y) = m \tan^{-1} \frac{y}{x}, \quad (4.8)$$

$$\phi(x, y) = \frac{m}{2\pi} \ln \sqrt{x^2 + y^2}. \quad (4.9)$$

The fundamental set of fluid dynamic principles and equations of Section ?? will be used in Section ?? to derive the stream function and potential function solutions. The goal now is to compute the strength m based on the pre-computed transient

pressure using GPT solutions.

4.2 Transient GPT stream function and potential function

The well-strength m in Equations (4.8), (4.9) for a vertical well, draining over its full length a reservoir of finite thickness, h is given by $m = q_w/h$, where q_w is the volumetric flow rate as given in Equation (2.6). Substituting the gradient of the GPT solution in Equation (2.6), the volumetric production rate of a well with radius r is given by:

$$q_w(t) = \frac{h\kappa\pi r^2}{\mu\beta t D_h} \Delta P_0 e^{-(\frac{r^2}{4D_h t})}. \quad (4.10)$$

Figure 4.2 shows the reduction of the production rate over time for a vertical injector well, located at the origin with radius $r = 1$.

As shown by Figure 4.2, the flow rate is increasing, initially, when $0 < t < 1$, but after $t = 1$ it decreases rapidly and continue decreasing until it becomes a constant close to 0 for a very large time. The transient flow behavior of the flow rate, for both the sink and source flows, is consist with our previous discussion in 3.1 where we concluded that for a reservoir with a single well (Case 1 for producer and Case 2 for injector) no steady-state flow will ever be reached and that $t_{limit-stop} \rightarrow \infty$.

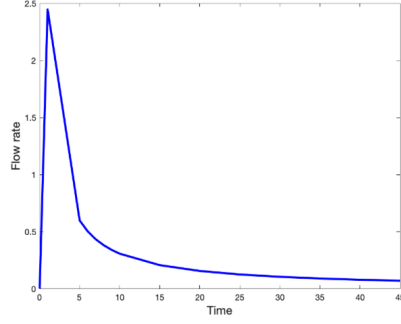


Figure 4.2: The flow rate (vertical axis) for source flow against different time level (horizontal axis) during the transient flow period.

Injector location	$(x_0, y_0) = (0, 0)$
P_{BHP}	$\Delta P_0 = 1$
Initial pressure	$P_0 = 0$
Diffusivity	$D_h = 1$
Time	$t \in 0.5, 1, 2$
Permeability	$\kappa = 1$
Fluid viscosity	$\mu = 1$

Table 4.1: Input parameters (non-dimensional) for transient flow shown in Figures 4.3

4.2.1 Transient stream function and potential function for single source/sink flows (Cases 1 and 2)

Next, we will use the flow rate $q_w(t)$ in equation (4.10) to visualize the streamlines and potential lines during the transient flow period. An important feature of the GPT-solution is that transient stream function and potential function can be solved for any time after the onset of flow. In Figure 4.3, we plot the streamlines, and potential lines for a source flow using the values as shown in Table 4.1.

The first column of Figure 4.3 shows streamline visualization (the straight radial lines in rainbow colors) with simple contouring for stream function given by Equation (4.8). The second column shows the equipotential lines as the concentric circles in

rainbow colors around the source position.

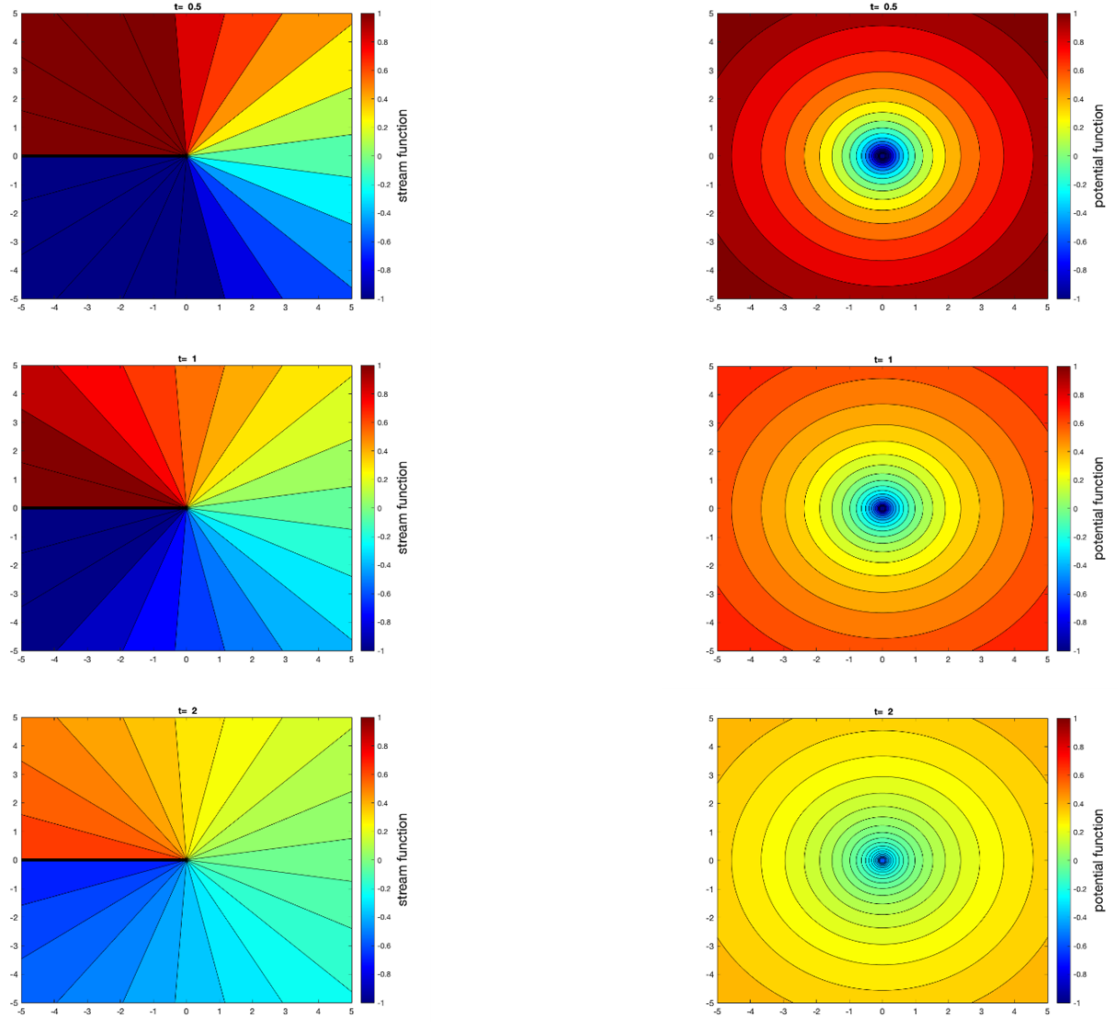


Figure 4.3: Streamlines (first column) and equipotential lines (second column) for a well at different time indicated on the top of each plot with time-dependent flow rate as defined in Equation (4.10) with $\Delta P_0 = 1$ and $r = 1$.

4.2.2 Transient stream and potential functions for balanced doublets flow

Consider a reservoir with a source (injector) located at (x_i, y_i) and a sink (producer) located at (x_p, y_p) , such that $x_p = x_i$ and $y_p = -y_i$ as illustrated in Figure 4.4.

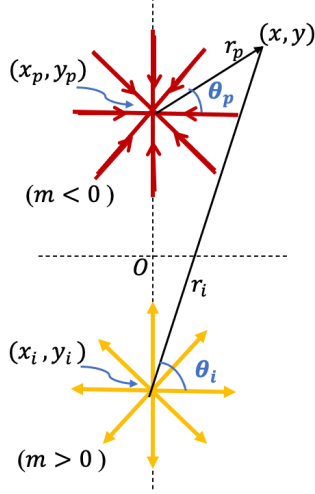


Figure 4.4: Superposed source located at (x_i, y_i) and sink located at (x_p, y_p) with $x_p = x_i$, $y_p = -y_i$. r_i and r_p are the distance between the point (x, y) to the centers (x_i, y_i) and (x_p, y_p) , respectively.

Assume that $|\Delta P_0|$ for the injector well equals the $|\Delta P_0|$ of the producer well (balanced doublet flow).

Before the pressure fields due to the source flow and sink flow meet each other (Figures 3.11a, 3.11b and 3.11c), the stream function and potential function for each individual flow are as given by Equations (4.5), (4.5) for the single source and sink flows. But as the pressure field meet (Figures 3.11d, 3.11e and ??), then the well pair obeys a combined stream function [(Equation (4.11))] and potential functions [Equation (4.12)] (see [21], [30]).

$$\Psi(r, \theta) = \frac{m}{2\pi}(\theta_i - \theta_p), \quad (4.11)$$

$$\phi(r, \theta) = \frac{m}{2\pi} \ln r_i - \ln r_p = \frac{m}{2\pi} \ln \frac{r_i}{r_p}. \quad (4.12)$$

where θ_i , θ_p , r_i and r_p are as shown in Figure 4.4.

Converting Equations (4.11), (4.12) into Cartesian coordinates gives the following equations:

$$\Psi(x, y) = \frac{m}{2\pi} \left(\tan^{-1} \frac{y - y_i}{x - x_i} - \tan^{-1} \frac{y - y_p}{x - x_p} \right), \quad (4.13)$$

$$\phi(x, y) = \frac{m}{4\pi} \ln \frac{(x - x_i)^2 + (y - y_i)^2}{(x - x_p)^2 + (y - y_p)^2}. \quad (4.14)$$

For the balanced doublet, the injection rate equals the production rate in magnitude and both are obtained using Equation (4.10). Therefore, the well strength in Equations (4.13), (4.14) is computed by dividing Equation (4.10) by the thickness h of the reservoir.

The analytical solutions of the stream and potential functions of the balanced doublet flow are shown in Figure 4.5 using the input parameters in Table 4.2. The streamlines (black contours) and potential lines (colored contours) are plotted after the doublet flow has been established at time $t=0.3, 0.4$ and 0.5 (the $t_{stop-limit}$). The streamlines for the sink flow and source flow are identical (as in Figure 4.3) before the pressure change fronts of the two wells reach each other. After the pressure transients meet, the streamlines start to curve up (for source flow) and down (for sink flow) establishing the typical steady-state doublet flow patterns (Figure 4.5). In our simulation, the moment when the two flows meet is considered to be at the time when there is a non-zero pressure gradient mid-way between the two wells on the line where their respective pressure transients meet (x-axis in our case), at which stage there comes into existence a flow across this line, for the first-time passing fluids to

Injector location	$(x_i, y_i) = (0, -2)$
Producer location	$(x_p, y_p) = (0, 2)$
Well pressure (injector)	$P_{BH} = 1$
Well pressure (producer)	$P_{BH} = -1$
Initial pressure	$P_0 = 0$
Diffusivity	$D_h = 1$
Time	$t \in 0.3, 0.4, 0.5$
Permeability	$\kappa = 1$
Fluid viscosity	$\mu = 1$

Table 4.2: Input parameters for transient balanced doublet flow shown in Figures 4.5

the other side of the half space initially occupied by the fluid.

4.2.3 Transient stream and potential functions for unbalanced doublets flow

For unbalanced doublet well rates, consider a reservoir with a source (injector) located at (x_i, y_i) with source strength $m > 0$ and a sink (producer) located at (x_p, y_p) with sink strength $-\alpha m$ for some positive integer α . The results of a well pair with unbalanced net rate will be the development of asymmetrical (or unsymmetrical [28]) flow patterns. The stream and potential functions for an unbalanced doublet flow in Cartesian coordinates are given by the following equations:

$$\Psi(x, y) = \frac{m}{2\pi} \left(\tan^{-1} \frac{y - y_i}{x - x_i} - \alpha \tan^{-1} \frac{y - y_p}{x - x_p} \right), \quad (4.15)$$

$$\phi(x, y) = \frac{m}{4\pi} \ln (x - x_i)^2 + (y - y_i)^2 - \alpha \ln (x - x_p)^2 + (y - y_p)^2. \quad (4.16)$$

Using the input parameters as in Table 4.3, we plot the analytical solutions for

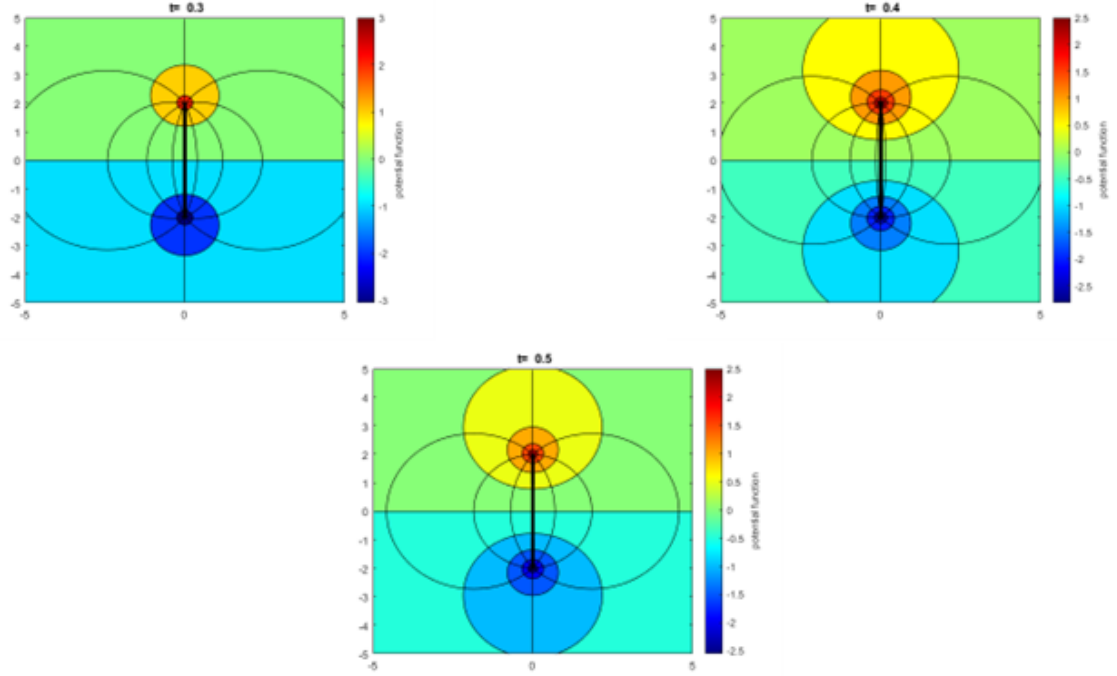


Figure 4.5: Streamlines (first column) and equipotential lines (second column) for balanced doublet flow at different time indicated on the top of each plot with time-dependent flow rate $q_w(t)$ computed using Equation (4.10). The producer (sink) is located at $(0, 2)$ and the injector (source) is located at $(0, -2)$.

stream and potential functions for an unbalanced doublet flow in Figure 4.6, when the sink strength is 5 times the source strength ($\alpha = 5$). Since the sink strength is larger (in magnitude) than the source strength the sink flow is dominant. Figure 4.7 shows the streamlines for the unbalanced doublet flow for different values of α ($\alpha = 2, 5, 9$) at time $t=0.5$ to illustrate the effect of the value of α on the behavior of the doublet flow. The doublet flow pattern is approaching sink flow pattern for larger values of α .

Injector location	$(x_i, y_i) = (0, -3)$
Producer location	$(x_p, y_p) = (0, 2)$
Well pressure (injector)	$P_{BH} = 1$
Well pressure (producer)	$P_{BH} = -1$
Initial pressure	$P_0 = 0$
Diffusivity	$D_h = 1$
Time	$t \in 0.01, 0.5$
Permeability	$\kappa = 1$
Fluid viscosity	$\mu = 1$

Table 4.3: Input parameters for the transient, unbalanced doublet flow shown in Figures 4.6

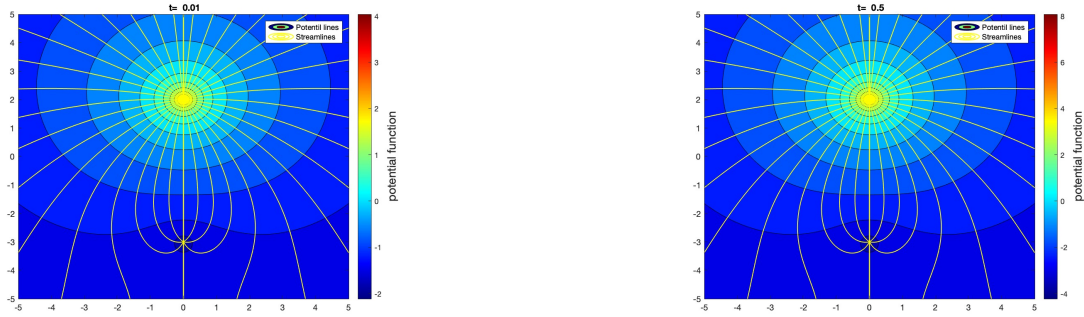


Figure 4.6: Streamlines (yellow contours) and equipotential lines (colored contours) for unbalanced doublet flow at $t = 0.5$ using equations (4.15), (4.16) with $\alpha = 5$. The source is located at $(0, -3)$ and the sink is located at $(0, 2)$.

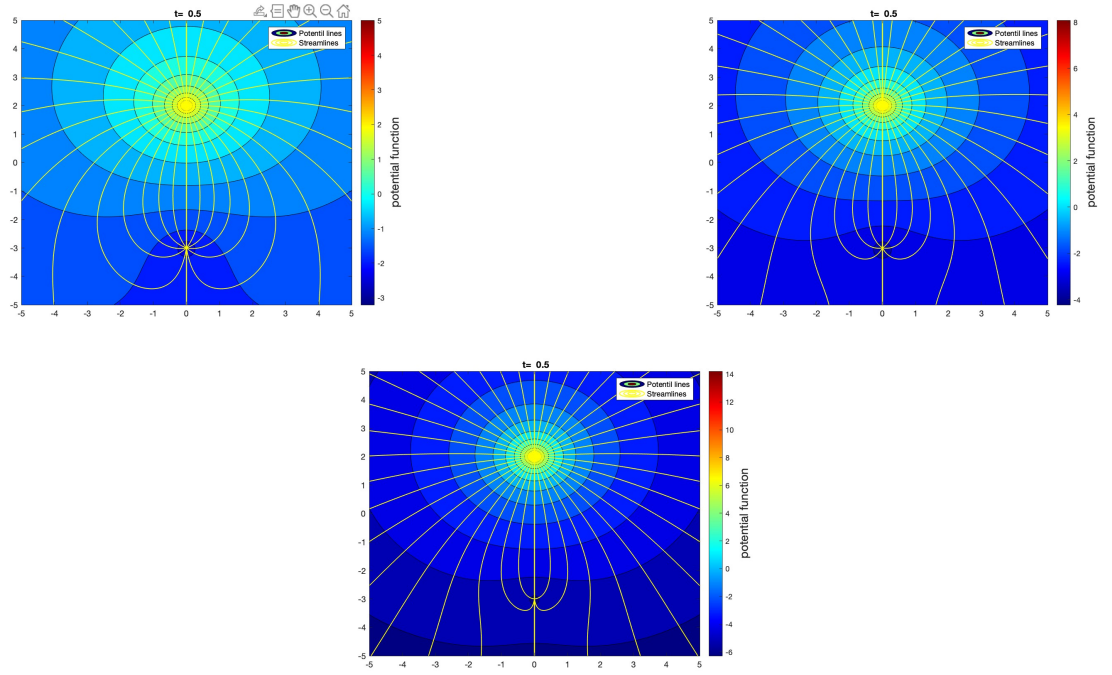


Figure 4.7: Streamlines (yellow contours) and equipotential lines (colored contours) for unbalanced doublet flow at $t = 0.5$ using equations (4.15), (4.16) at $t = 0.5$ with (a) $\alpha = 3$ (b) $\alpha = 5$ and (c) $\alpha = 9$. The source is located at $(0, -3)$ and the sink is located at $(0, 2)$.

CHAPTER 5

CONCLUSION

A method is presented for modeling fluid flow in porous media using , new solutions of the pressure diffusion equation based on Gaussian Pressure Transients (GPT). Fluid flow can be visualized for both the transient and steady-state stages by computing the local pressure gradient analytically. The ability to visualize the transient flow stage is important because it provides information about the flow paths that could help optimize well placement and design when extracting natural resources from subsurface reservoirs. The Gaussian solution method is without the computational complexity of time-stepping. GPT-solutions therefore provide a fast alternative to numerical solution methods, which are relatively complex to use and have long computation times, especially for sensitivity studies and well-placement optimization. The faster and less complex computations, make the GPT method an attractive option for optimizing natural resource extraction strategies.

Examples are given of how to use Gaussian solutions of the diffusion equation to visualize fluid flow paths in subsurface reservoirs as required for the extraction of natural resources like oil, gas, water, and geothermal fluids. The Gaussian method

is extended in our study by computing and visualizing velocity magnitude contours, streamlines, and other flow attributes in the vicinity of well systems that deplete pressure in a reservoir. The GPT-method uses the Gaussian probability function of Equation (2.10) which then gives the spatial and temporal of the pressure-change in 1D, 2D and 3D (Equations (1.30), (2.7) and (2.8)). The GPT-solution-form is a simple expression that provides a fast-analytical method to compute flow patterns with high resolution. To validate the performance of the GPT-solution for the transient pressure-change in the superposed well system, we compare the GPT-solution with the independent closed-form solutions based on an independent CAM, assuming the same initial conditions. The comparison shows a good match for the steady-state solution (see Chapter 3). CAM can only visualize steady-state flow paths. However, the GPT-solutions allow us to visualize the solutions of the stream and potential functions for both the steady-state and transient stages of the flow development, again without the computational complexity of time-stepping.

The transient pressure-changes in the superposed well-system can be rapidly computed by superimposing GPT-solutions for the pressure-transients due to individual well and/or fracture sources of pressure change in the system. Using the standard equations to compute the pressure derivatives, we visualized the flow paths in the reservoir during the pressure transient period. The GPT-method can predict well productivity and injectivity based on the applied bottomhole pressure and the applicable hydraulic diffusivity of the reservoir system and then helps tracing the fluid migration paths in the reservoir. The solutions presented are applicable to both fluid production cases and fluid injection such as used for the disposal or storage of waste

fluids, carbon dioxide, hydrogen, and natural gas.

The main points of the thesis are:

- Gaussian solutions of the diffusion equation can be used to visualize flow paths during transient pressure changes in subsurface reservoirs due to pressure gradients caused by engineering interventions.
- The Gaussian method was extended to compute and visualize velocity magnitude contours, streamlines, and other flow attributes in the vicinity of well systems that are depleting pressure in a reservoir.
- Stream function and potential function solutions were derived for instantaneous modeling of flow paths and pressure contour solutions of transient flows, without time-stepping.
- The GPT-method can, compute the local pressure gradient analytically based on Gaussian Pressure Transients, to model fluid flow and compute the fluid flux from the reservoir into the well system.
- The GPT-method can predict fluid extraction rates and study the fluid origin in reservoirs; the method can also model fluid injection in subsurface reservoirs when used for the disposal or storage of certain fluids.
- The computational efficiency of the analytical GPT-solution method compares favorably to numerical methods for sensitivity studies and well-placement optimization.

APPENDIX A

DIMENSIONAL SCALING OF PHYSICAL QUANTITIES

Dimensional analysis is a technique to transform an equation featuring physical quantities of certain units into non-dimensional form (see [22]). For example, consider the dimensional Gaussian Equation (1.30) for the advance of a 1D-diffusion along the x-axis front from a planar source in the origin:

$$\Delta P = \Delta P_0 e^{-\left(\frac{x^2}{4D_h t}\right)} \quad (\text{A.1})$$

This dimensional equation (with dimensional units) can be converted into a non-dimensional equation (with non-dimensional units), by dividing the variables by a typical unit with the corresponding dimension. For example, for the case of Equation (A.1), one should use dimensional unit measures for the fundamental physical quantities such as length ($x_0 = 1$ m) and time ($t_0 = 1$ s) for the divisions of the

Parameter	Description	Dimension
x, x_0	Length; characteristic length unit	$[L]$
t, t_0	Time; characteristic time unit	$[T]$
D_h	Hydraulic diffusivity	$[L^2T^{-1}]$
P	Pressure	$[ML^{-1}T^{-2}]$

Table A.1: Dimensional analysis

corresponding dimensions x and t to obtain the non-dimensional units x^* and t^* :

$$x^* = \frac{x}{x_0}, \quad t^* = \frac{t}{t_0}. \quad (\text{A.2})$$

The remaining parameters in Eq. (A.1), D_h and P , are so-called derived quantities with their dimensions being determined by certain fundamental quantities, as in Table A.1.

The hydraulic diffusivity, D_h , can now be non-dimensionalized as follows:

$$D_h^* = \frac{t_0 D_h}{x_0^2} \quad (\text{A.3})$$

Likewise, one may non-dimensionalize the pressure, P , by multiplying with the inverse of its fundamental units:

$$P^* = P \frac{x_0 t_0^2}{m_0} \quad (\text{A.4})$$

v However, because mass, m , does not feature as a separate fundamental quantity in Equation (A.1), using the typical unit mass, m_0 , is not very practical for non-dimensionalizing P . Instead, we may use an alternative approach and simply normalize the term $(P_0 - P_{BH})$ by the initial reservoir pressure, P_0 . Substituting all the proposed

normalizations, the non-dimensional form of Equation (A.1) becomes:

$$\begin{aligned}
P^*(x^*, t^*) &= \frac{P_0 - P_{BH}}{P_0} e^{-\frac{(x^* x_0)^2}{4 \frac{D_h^* x_0^2}{t_0} t^* t_0}} \\
&= \left(1 - \frac{P_{BH}}{P_0}\right) e^{-\frac{x^{*2}}{4 D_h^* t^*}}.
\end{aligned} \tag{A.5}$$

The principal purpose of the scaling procedure is to subsequently be able to work with non-dimensional units for x^* , t^* , and D_h^* , and corresponding dimensional quantities x , t , and D_h , can be computed anytime, using equations (A.2) and (A.3). For example, a non-dimensional diffusion travel distance $x^* = 10$ over a non-dimensional time of $t^* = 105$, when working in dimensional SI units with $x_0 = 1$ m and $t_0 = 1$ s, will correspond to dimensional quantities of $x = 10$ m and $t = 10^6$ s. When we do experimental runs, we can simply start from $t^* = 0$ and evaluate for a certain spatial domain bound by $x_{\max}^*$ how $P^*(x^*, t^*)$ changes over a distance $0 \leq x \leq x_{\max}^*$. Of course, for a 2-D diffusion process using the point sources in our study we also have $0 \leq y \leq y_{\max}^*$. The values of $x_{\max}^*$ and $y_{\max}^*$ depend on where the front of the pressure transient is studied for a given D_h^* .

For each set of model runs, D_h^* will normally not change, but is chosen as a typical value for the system studied. For example, in reservoir rocks the dimensional values of the hydraulic diffusivity in geological reservoirs may range from $10^{-2} \text{ m}^2 \text{ s}^{-1}$ for coarse sandstone formations (high permeability) and $10^{-10} \text{ m}^2 \text{ s}^{-1}$ for very tight shale formations (with ultra-low permeability). If we chose for our models an average dimensional value of $D_h = 10^{-6} \text{ m}^2 \text{ s}^{-1}$, the corresponding D_h^* will be 10^{-6} .

If instead of using $D_h^* = 10^{-6}$, we would use an arbitrary value of $D_h^* = 1$, that

means we are modeling a physical porous medium with diffusion rates much faster than occurs in oil and gas reservoirs. Such models $D_h^* = 1$ are still useful, but will have very short runtimes (t^* on the order of 10^n) to cover the diffusion distances that in geological reservoirs would require much longer times (t^* on the order of 10^n). Finally, if we work with non-dimensional velocities (as in Section 4), these can be normalized using x_0 and t_0 , because velocity has the dimensions of $[LT^{-1}]$.

APPENDIX B

DIFFUSION EQUATION

Recall the diffusion equation for pressure changes given by (1.29)

$$\frac{\partial \Delta P(x, t)}{\partial t} = D_h \frac{\partial^2 \Delta P(x, t)}{\partial x^2}, \quad (\text{B.1})$$

subjected to the boundary condition $\Delta P(0, t) = \Delta P_0$. In Section 1.2.3, we presented an approximate analytic solution given by:

$$\Delta P(x, t) = \Delta P_0 e^{-\left(\frac{x^2}{4D_h t}\right)} \quad (\text{B.2})$$

However, the exact Gaussian solution has a transient multiplier in front of the exponential term that was ignored in the visualizations presented in the previous chapters. In this Appendix we will show that the exact Gaussian solution with respective transient multiplier satisfy the diffusion equation for pressure change in 1D, 2D and 3D

B.1 1D Case

Diffusion equation for pressure changes in 1D is given by (B.1). The exact Gaussian pressure transient solution is given by

$$\Delta P(x, t) = \frac{\Delta P_0}{\sqrt{4\pi D_h t}} e^{-\left(\frac{x^2}{4D_h t}\right)}, \quad (\text{B.3})$$

where, $\Delta P_0 = P_0 - P_{BH}$ is a constant. To verify that (B.3) satisfies the diffusion equation (B.1), we will compute the time derivative for the solution (B.3) and the second derivative with respect to the spatial variable as following:

$$\begin{aligned} \frac{\partial \Delta P}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\Delta P_0}{\sqrt{4\pi D_h t}} e^{-\left(\frac{x^2}{4D_h t}\right)} \right) \\ &= \left[-\frac{1}{2} \frac{\Delta P_0}{\sqrt{4\pi D_h}} t^{-\frac{3}{2}} + \frac{\Delta P_0}{\sqrt{4\pi D_h t}} \left(\frac{x^2}{4D_h t^2} \right) \right] e^{-\left(\frac{x^2}{4D_h t}\right)} \\ &= \frac{\Delta P_0}{\sqrt{4\pi D_h t}} \left[-\frac{1}{2t} + \frac{x^2}{4D_h t^2} \right] e^{-\left(\frac{x^2}{4D_h t}\right)}. \end{aligned} \quad (\text{B.4})$$

The first derivative is given by:

$$\begin{aligned} \frac{\partial \Delta P}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{\Delta P_0}{\sqrt{4\pi D_h t}} e^{-\left(\frac{x^2}{4D_h t}\right)} \right) \\ &= \frac{\Delta P_0}{\sqrt{4\pi D_h t}} \frac{-2x}{4D_h t} e^{-\left(\frac{x^2}{4D_h t}\right)}. \end{aligned} \quad (\text{B.5})$$

Then,

$$\begin{aligned}\frac{\partial^2 \Delta P}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\Delta P_0}{\sqrt{4\pi D_h t}} \frac{-2x}{4D_h t} e^{-\left(\frac{x^2}{4D_h t}\right)} \right) \\ &= \frac{\Delta P_0}{\sqrt{4\pi D_h t}} \left(\frac{-1}{4D_h t} + \frac{x^2}{4D_h^2 t^2} \right) e^{-\left(\frac{x^2}{4D_h t}\right)}.\end{aligned}\tag{B.6}$$

Multiply by D_h

$$D_h \frac{\partial^2 \Delta P}{\partial x^2} = \frac{\Delta P_0}{\sqrt{4\pi D_h t}} \left(\frac{-1}{2t} + \frac{x^2}{4D_h t^2} \right) e^{-\left(\frac{x^2}{4D_h t}\right)}.\tag{B.7}$$

Thus (B.4)= (B.7) and (B.3) satisfies the pressure diffusion equation

B.2 2D Case

Diffusion Equation for pressure changes in 2D is given by

$$\frac{\partial \Delta P(x, y, t)}{\partial t} = D_h \left(\frac{\partial^2 \Delta P(x, y, t)}{\partial x^2} + \frac{\partial^2 \Delta P(x, y, t)}{\partial y^2} \right).\tag{B.8}$$

The exact Gaussian pressure transient solution is given by

$$\Delta P(x, y, t) = \frac{\Delta P_0}{4\pi D_h t} e^{-\left(\frac{x^2+y^2}{4D_h t}\right)},\tag{B.9}$$

where, $\Delta P_0 = P_0 - P_{BH}$ is constant. Next, we will compute the time derivative for the solution (B.9) and the second derivative with respect to the spatial variable to

prove that it satisfies the diffusion equation (B.8).

$$\begin{aligned}
\frac{\partial \Delta P}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\Delta P_0}{4\pi D_h t} e^{-\left(\frac{x^2+y^2}{4D_h t}\right)} \right) \\
&= \left[-\frac{\Delta P_0}{4\pi D_h t^2} + \frac{\Delta P_0}{4\pi D_h t} \left(\frac{x^2+y^2}{4D_h t^2} \right) \right] e^{-\left(\frac{x^2+y^2}{4D_h t}\right)} \\
&= \frac{\Delta P_0}{4\pi D_h t^2} \left[-1 + \frac{x^2+y^2}{4D_h t} \right] e^{-\left(\frac{x^2+y^2}{4D_h t}\right)}
\end{aligned} \tag{B.10}$$

The first derivative with respect to x is given by:

$$\begin{aligned}
\frac{\partial \Delta P}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{\Delta P_0}{4\pi D_h t} e^{-\left(\frac{x^2+y^2}{4D_h t}\right)} \right) \\
&= \frac{\Delta P_0}{4\pi D_h t} \frac{-2x}{4D_h t} e^{-\left(\frac{x^2+y^2}{4D_h t}\right)}.
\end{aligned} \tag{B.11}$$

Then,

$$\begin{aligned}
\frac{\partial^2 \Delta P}{\partial x^2} &= \frac{\Delta P_0}{8\pi D_h^2 t^2} \frac{\partial}{\partial x} \left(-x e^{-\left(\frac{x^2+y^2}{4D_h t}\right)} \right) \\
&= \frac{\Delta P_0}{8\pi D_h^2 t^2} \left(-1 + \frac{2x^2}{4D_h t} \right) e^{-\left(\frac{x^2+y^2}{4D_h t}\right)}
\end{aligned} \tag{B.12}$$

similarly,

$$\frac{\partial^2 \Delta P}{\partial y^2} = \frac{\Delta P_0}{8\pi D_h^2 t^2} \left(-1 + \frac{2y^2}{4D_h t} \right) e^{-\left(\frac{x^2+y^2}{4D_h t}\right)} \tag{B.13}$$

Then we sum the equations (B.12) and (B.13) and multiply it by D_h :

$$\begin{aligned}
D_h \left(\frac{\partial^2 \Delta P}{\partial x^2} + \frac{\partial^2 \Delta P}{\partial y^2} \right) &= \frac{\Delta P_0}{8\pi D_h^2 t^2} \left(-2 + 2 \frac{x^2+y^2}{4D_h t} \right) e^{-\left(\frac{x^2+y^2}{4D_h t}\right)} \\
&= \frac{\Delta P_0}{4\pi D_h t^2} \left(-1 + \frac{x^2+y^2}{4D_h t} \right) e^{-\left(\frac{x^2+y^2}{4D_h t}\right)}
\end{aligned} \tag{B.14}$$

Thus (B.10) equals to (B.14) and (B.9) satisfies the pressure diffusion equation

B.3 3D Case

Diffusion Equation for pressure changes in 3D is given by

$$\frac{\partial \Delta P(x, y, z, t)}{\partial t} = D_h \left(\frac{\partial^2 \Delta P(x, y, z, t)}{\partial x^2} + \frac{\partial^2 \Delta P(x, y, z, t)}{\partial y^2} + \frac{\partial^2 \Delta P(x, y, z, t)}{\partial z^2} \right). \quad (\text{B.15})$$

The exact Gaussian pressure transient solution is given by

$$\Delta P(x, y, z, t) = \frac{\Delta P_0}{(4\pi D_h t)^{\frac{3}{2}}} e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)}, \quad (\text{B.16})$$

where, $\Delta P_0 = P_0 - P_{BH}$ is constant. Next, we will compute the time derivative for the solution (B.16) and the second derivative with respect to the spatial variable to prove that it satisfies the diffusion equation (B.15).

$$\begin{aligned} \frac{\partial \Delta P}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{\Delta P_0}{(4\pi D_h t)^{\frac{3}{2}}} e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)} \right) \\ &= \left[-\frac{3}{2} \frac{\Delta P_0}{(4\pi D_h)^{3/2} t^{5/2}} + \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \left(\frac{x^2 + y^2 + z^2}{4D_h t^2} \right) \right] e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)} \\ &= \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \left[-\frac{3}{2t} + \frac{x^2 + y^2 + z^2}{4D_h t^2} \right] e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)} \end{aligned} \quad (\text{B.17})$$

The first derivative with respect to x is given by:

$$\begin{aligned}\frac{\partial \Delta P}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{\Delta P_0}{(4\pi D_h t)^{\frac{3}{2}}} e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)} \right) \\ &= \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \frac{-2x}{4D_h t} e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)}.\end{aligned}\quad (\text{B.18})$$

Then,

$$\begin{aligned}\frac{\partial^2 \Delta P}{\partial x^2} &= \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \frac{1}{2D_h t} \frac{\partial}{\partial x} (-x e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)}) \\ &= \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \frac{1}{2D_h t} \left(-1 + \frac{2x^2}{4D_h t}\right) e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)}\end{aligned}\quad (\text{B.19})$$

similarly,

$$\frac{\partial^2 \Delta P}{\partial y^2} = \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \frac{1}{2D_h t} \left(-1 + \frac{2y^2}{4D_h t}\right) e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)} \quad (\text{B.20})$$

And,

$$\frac{\partial^2 \Delta P}{\partial z^2} = \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \frac{1}{2D_h t} \left(-1 + \frac{2z^2}{4D_h t}\right) e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)} \quad (\text{B.21})$$

Then we sum the equations (B.12), (B.13) and (B.21) and multiply it by D_h :

$$\begin{aligned}D_h \left(\frac{\partial^2 \Delta P}{\partial x^2} + \frac{\partial^2 \Delta P}{\partial y^2} + \frac{\partial^2 \Delta P}{\partial z^2} \right) &= \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \frac{1}{2t} \left[-3 + 2 \frac{x^2 + y^2 + z^2}{4D_h t} \right] e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)} \\ &= \frac{\Delta P_0}{(4\pi D_h t)^{3/2}} \left[-\frac{3}{2t} + \frac{x^2 + y^2 + z^2}{4D_h t^2} \right] e^{-\left(\frac{x^2+y^2+z^2}{4D_h t}\right)}\end{aligned}\quad (\text{B.22})$$

Thus (B.17) equals to (B.22) and (B.16) satisfies the pressure diffusion equation

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