

COMPRESSIVE SENSING IN DATA SEPARATION

BY

REEM SAEED ABDULLAH ALSHAMMARI

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

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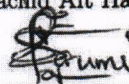
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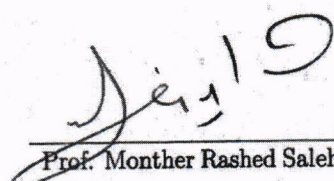
This thesis, written by **REEM SAEED ABDULLAH ALSHAMMARI** under the direction of her thesis adviser and approved by her thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

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*This thesis is dedicated to my late parents,
Saeed Abdullah Alshammari & Norah Nayef Alshammari,*

*Their reflection in me empowers me through the years to pursue
many endeavors leading me to such accomplishments.*

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THESIS ABSTRACT

NAME: Reem Saeed Abdullah Alshammari
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Compressed sensing is a fairly new mathematical paradigm designed to efficiently reconstruct large sparse dimensional signals using a small number of linear measurements. Sparse coding of signals combined with compressed sensing emerged as a powerful tool for separating data that are composed of several morphologically distinct constituents. In this thesis, combining adaptive incoherent sparse coding with compressive sensing, we propose an efficient method for data separation. With a prior knowledge of the characteristics of the distinct constituents in the given data, we carefully select adequate frames to sparsify each component, i.e., the vector of the coefficients that represent the component on that frame is sparse. We then use compressive sensing to extract each component using adequate ℓ_1 -optimization criteria. Applications to audio signals and images separation will be provided.

ملخص الرسالة

الاسم: ريم سعيد عبدالله النماصي الشمري

عنوان الدراسة: الاستشعار المضغوط في فصل البيانات

التخصص: الرياضيات

تاريخ الدرجة العلمية: ٢٠٢٣ مايو

“الاستشعار المضغوط” هو نموذج رياضي جديد تم تصميمه لإعادة بناء إشارات ذات أبعاد كبيرة باستخدام عدد صغير من القياسات الخطية. تعتبر نظرية الاستشعار المضغوط من أهم الطرق المستخدمة في تحليل البيانات وفصلها خطياً وذلك بالاعتماد على تمثيل البيانات في نظام خطي يمثلها كليا باستخدام القليل من الحسابات وذلك لتقليل الكلفة والوقت لتحليلها ومن ثم استعادة البيانات وفصلها بالحصول على معامل المنتج الأقل من خوارزمية التقليل باستخدام معيار ال ℓ_1 في البرمجة الخطية. نتناول في هذه الرسالة دراسة استقصائية مع التفصيل والبرهنة لما سيتم نصة من النظريات المستخدمة لتمكين الفصل ومفهوم الإطارات وما يترتب عليها من نتائج، ختاماً نعرض موجز ماتم دراسته وعرض التطبيقات المعاصرة باستخدام امثله لتحليل البيانات الصوتية الممثلة في متجه ذو بعد واحد وفصل الصور عن التشويش الناتج عن تداخل الموجات وفصلها وفق هذا المنهج.

CHAPTER 1

INTRODUCTION

Compressed sensing is a signal processing technique that allows for the efficient acquisition and reconstruction of signals by finding solutions to underdetermined linear systems [1]. This technique is based on the principle that the sparsity of a signal can be exploited to recover it from far fewer samples than required by traditional sampling methods such as the Nyquist–Shannon sampling theorem ([2][3]). Compressed sensing has a wide range of applications in various fields. For instance, it was used in photography with the advent of single pixel camera [4], in holography as a mean to improve image reconstruction by increasing the number of voxels one can infer from a single hologram [5], in face recognition, in Magnetic Resonance Imaging (MRI) to reduce the number of measurements([6],[2]).

Another application of compressed sensing is the Morphological component analysis (MCA) for signal separation ([7],[8]). In this approach a signal is assumed to be a superposition of morphological sub-components, each of which is approximately sparse in a certain dictionary [9]. The separation method aims to retrieve the components from the signal using two basic principles: Sparsity, which requires the signal

to be a superposition of sparse signals with respect to different dictionaries [10]. The second one is the incoherence of the dictionaries ([11][12]). The power of MCA is that under certain conditions on the coherence of the dictionaries, signal separation can be achieved efficiently using convex optimisation ([13],[14]).

1.1 Problem Statement and Description

Consider a signal x which belong to a Hilbert space $\mathcal{H}$ and assume that the signal can be split into L components

$$x = x_1 + x_2 + \dots x_L, \quad x_k \in \mathcal{H}. \quad (1.1)$$

Moreover, we assume that its constituent are *morphologically* different in the sense that for each x_k there exists a frame Φ_k of $\mathcal{H}$ such $x_k = \Phi_k \gamma_k$ where γ_k is sparse. An uniqueness assumption of the sparse solution $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_L)$ ensures the recovery of the signal components using the ℓ_0 -minimisation

$$\begin{aligned} \min \quad & \|\gamma\|_0 \\ \text{s.t.} \quad & [\Phi_1 | \Phi_2 | \dots | \Phi_L] \gamma^T = x. \end{aligned} \quad (1.2)$$

Thus the first step in this strategy is to purposefully select over-complete representation made up of a number of frames corresponding to the number of the unknown components to be extracted, each of which provides a sparse expansion of one of the components to be recovered [14][2]. The incoherence conditions of those frames are

subsequently represented as the morphological differences between the components. Nevertheless, the minimisation problem (3.2) is NP-hard and therefore, one should choose the frames in such a way that the recovery of the components could be obtained using the ℓ_1 -minimisation

$$\begin{aligned} \min \quad & \|\gamma\|_1 \\ \text{s.t.} \quad & [\Phi_1|\Phi_2|\dots|\Phi_L]\gamma^T = x. \end{aligned} \tag{1.3}$$

1.2 Thesis Outlines

This thesis is organized as follows, In chapter 2, we state basic properties on frame theory in Hilbert space, which provides a redundant series representations of vectors in Hilbert space. In chapter 3, we formulate the problem of signal separation using sparsity and coherence. Several examples of signal separation will be illustrated, In particular separation of signals composed of periodic and spikes signals and separation of signal composed of sparse orthogonal polynomials will be given. In chapter 4, we use Gábor frames as a time-frequency analysis tool for audio signals separation. Finally, a summary of the work resented in this thesis.

CHAPTER 2

FRAME THEORY IN HILBERT SPACES

A **frame** is a generalization of a basis that provides redundant representation of vectors in Hilbert spaces and is a fundamental notion in wavelet theory, sampling theory and many other applied areas [15]. In this chapter, we recall basic notions on Hilbert spaces and frame theory.

2.1 Preliminaries

2.1.1 Hilbert Spaces

In this section we review the main facts concerning Hilbert spaces. We begin with basic definitions, identities and inequalities that are frequently used in Hilbert spaces.

Let $\mathcal{H}$ be a real or complex K -linear space ($K = \mathbb{R}$ or $\mathbb{C}$) [16]. An inner-product in $\mathcal{H}$ is a function $\langle \cdot, \cdot \rangle: \mathcal{H} \times \mathcal{H} \longrightarrow K$ such that the following holds:

1. Linearity: for any $a, b \in K$ and for any $x, y, z \in \mathcal{H}$, $\langle ax+by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.
2. Complex conjugate symmetry: for any $x, y \in \mathcal{H}$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
3. Positivity: for any non-zero vector $x \in \mathcal{H}$, $\langle x, x \rangle > 0$.

A *normed* space $\mathcal{H}$ is a pair $(\mathcal{H}, \|\cdot\|)$, where $\mathcal{H}$ is a K -vector space and $\|\cdot\|$ is a function from $\mathcal{H}$ to $\mathbb{R}$, called a *norm* [16], that satisfies the following properties for all $x, y \in \mathcal{H}$ and $\alpha \in K$

1. Positive definiteness: $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.
2. Absolute homogeneity: $\|\alpha x\| = |\alpha| \|x\|$.
3. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$.

Definition 2.1 A **Hilbert space** $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a real or complex inner-product vector space that is complete with respect to the induced norm,

$$\|x\| = \sqrt{\langle x, x \rangle}, \quad x \in \mathcal{H}.$$

Example 1: The linear spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ endowed the inner-product of two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ as

$$\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i.$$

are Hilbert spaces. The induced norm is given by

$$||x|| = \sqrt{\sum_{i=1}^n |x_i|^2}.$$

Example 2: The space of measurable functions $f : [a, b] \longrightarrow \mathbb{R}$ with Lebesgue integrable squares and endowed with the inner-product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

is a real Hilbert space with induced norm

$$||f|| = \sqrt{\int_a^b f^2(x)dx}.$$

The defined Hilbert space is denote by $L^2([a, b])$.

Example 3: The space of complex sequence $x = (x_1, x_2, \dots, x_n, \dots)$ such that

$$\sum_{i=1}^{\infty} |x_i|^2 < \infty,$$

endowed with the inner-product $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i$ is a complex Hilbert space with induced norm

$$||x||_2 = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}.$$

The defined Hilbert space is denote by $\ell^2(\mathbb{C})$.

Definition 2.2 Let I be a subset of $\mathbb{N}$. A set of non-zero vectors $\{x_i\}_{i \in I}$ in a Hilbert

space $\mathcal{H}$ is called:

- **Linearly independent set** if for any scalars $\{a_i\}_{i \in I}$ such that $\sum_{i \in I} a_i x_i = 0$, we have $a_i = 0$ for all $i \in I$.
- **Spanning set** if $\text{span}\{x_i\}_{i \in I} = \mathcal{H}$.
- **Orthogonal set** if, for any $i \neq j$ in I , we have $\langle x_i, x_j \rangle = 0$.
- **Orthonormal set** if it is an orthogonal set and $\langle x_i, x_i \rangle = 1$ for any $i \in I$.

The space K^n ($K = \mathbb{R}$ or $\mathbb{C}$) can be endowed with many norms. For instance, to a vector vectors $x = (x_1, x_2, \dots, x_n)$ in K^n , we define the ℓ_p norms for $p \in [1, \infty]$ as

$$\|x\|_p = \begin{cases} (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}, & \text{if } p \in [1, \infty); \\ \max_{i=1, \dots, n} |x_i|, & \text{if } p = \infty. \end{cases}$$

For $p < 1$, $\|\cdot\|_p$ is not a norm. However, it is customary to define the ℓ_0 -norm of a vector $x \in K^n$ as

$$\|\mathbf{x}\|_0 := \text{card}(\text{supp}(\mathbf{x})) \tag{2.1}$$

where the support of a vector x is the index set of its nonzero entries, i.e.,

$$\text{supp}(\mathbf{x}) := \{i \in [N] \mid x_i \neq 0\} \quad \text{with} \quad [N] = \{1, 2, \dots, n\}.$$

The $\|\cdot\|_0$ is not a norm as it does not satisfy the homogeneity condition, i.e.; in general

$$\|\lambda x\|_0 \neq |\lambda| \|x\|_0.$$

The *unit ball* of a normed space $\mathcal{H}$ is defined by

$$\mathbb{B} = \{x \in \mathcal{H} \mid \|x\| \leq 1\}$$

Lemma 2.1 *Unit balls in normed spaces are convex [16].*

Proof. Let $x, y \in \mathbb{B}$ and $\theta \in (0, 1)$. By applying the triangle inequality and the homogeneity of the norm, we obtain

$$\|\theta x + (1 - \theta)y\| \leq \|\theta x\| + \|(1 - \theta)y\| = \theta\|x\| + (1 - \theta)\|y\| \leq 1.$$

This concludes the proof. ■

Unit balls of the ℓ_0 -”norm” are not convex. For instance, in $\mathbb{R}^2$, take the vectors $x = (1, 0)$ and $y = (0, 1)$ which are in the unit ball since $\|x\|_0 = \|y\|_0 = 1$. However, for any $\theta \in (0, 1)$, we have $\|\theta x + (1 - \theta)y\|_0 = 2$. Also the unit balls for the ℓ_p -”norms” with $p < 1$ are not convex. The convexity of the ℓ_p norms for $p > 1$ can be seen from Figure (2.1). However, for $p < 1$ the unit balls are not convex.

2.2 Frames in Hilbert Spaces

A **frame** is a generalization of a basis that provides redundant representation of vectors in Hilbert spaces and is a fundamental notion in wavelet theory, sampling theory and many other areas of engineering [15].

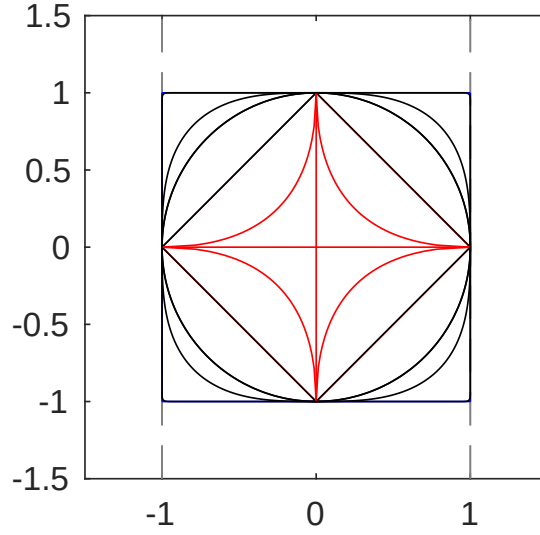


Figure 2.1: The Unit ball in $\mathbb{R}^2$ for the ℓ_p norms with $p = \infty, 100, 3, 2, 1$ respectively from the outer convex borders (black) to the inner non-convex unit balls $\ell_{\frac{1}{2}}$ and ℓ_0 (red).

Definition 2.3 A frame is a sequence $(\varphi_j)_{j \in J}$ ($J \subset \mathbb{N}$) of elements of a Hilbert space $\mathcal{H}$ such that there exist real numbers $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j \in J} |\langle \varphi_j, f \rangle|^2 \leq B\|f\|^2$$

for all $f \in \mathcal{H}$. The supremum of all A and the infimum of all B that satisfy the above inequality are called the **frame bounds** [17].

Definition 2.4 Given a frame $(\varphi_j)_{j \in J}$ in a Hilbert space $\mathcal{H}$ [18].

- The frame $(\varphi_j)_{j \in J}$ is said to be a **tight frame** if its frame bounds A and B are equal.
- The frame $(\varphi_j)_{j \in J}$ is said to be a **Parseval frame** if its frame bounds satisfies $A = B = 1$

- The frame $(\varphi_j)_{j \in J}$ is said to be a **Riesz basis** if it is also a basis for $\mathcal{H}$
- The frame $(\varphi_j)_{j \in J}$ is said to be a **normalized frame** if each elements φ_j is a unit vector, i.e.; $\|\varphi_j\| = 1$ for any $j \in J$.

Example 1: Consider the sequence x_1, x_2 and x_3 in the Hilbert space $\mathbb{R}^2$

$$x_1 = (1, 0), \quad x_2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad x_3 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

We have

$$\sum_{i=1}^3 |\langle x, x_i \rangle|^2 = \frac{3}{2} \|x\|^2 \quad \text{for all } x \in \mathbb{R}^2.$$

Thus (cx_1, cx_2, cx_3) , where $c = \sqrt{3/2}$, is a Parseval frame called **Mercedes-Benz frame** (see Figure 2.2).

Example 2: Let $\Phi = (e_n)_{n=1}^\infty$ be an orthonormal basis of a Hilbert space. Clearly Φ is a frame with frame bound $A = B = 1$. Consider the sequence

$$E_2 = \left(e_1, \frac{e_2}{2}, \frac{e_3}{3}, \dots\right).$$

E_2 is a basis for the Hilbert space but not a frame since for $f = e_k$, we have

$$\sum_{j=1}^{\infty} \left\langle \frac{e_j}{j}, e_k \right\rangle = \frac{1}{k}$$

and $\|f\| = 1$ and there does not exist a positive constant A such that

$$A \leq \frac{1}{k} \quad \text{for any } k \in \mathbb{N}^*.$$

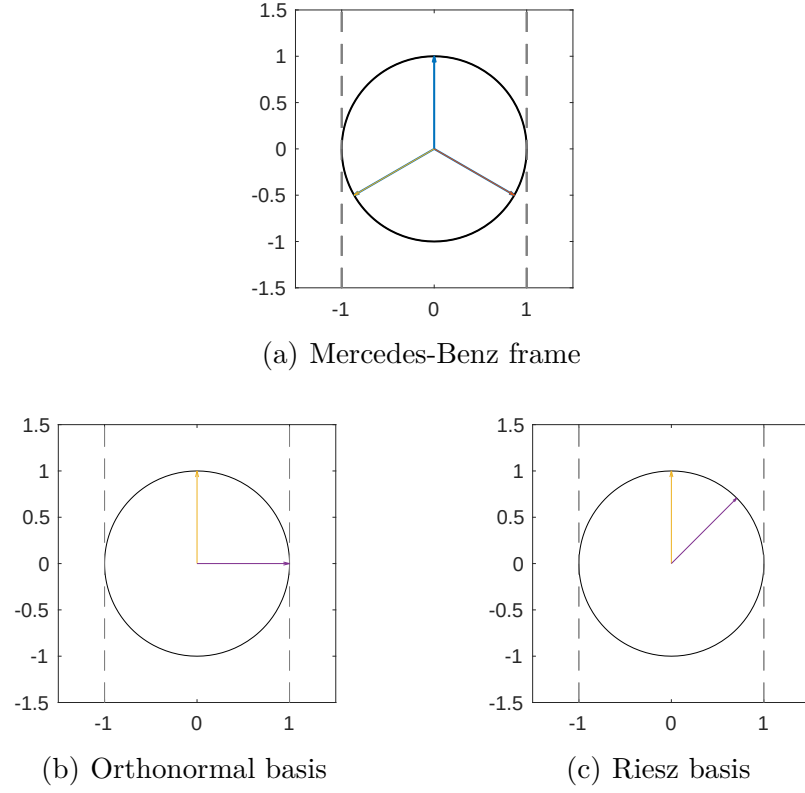


Figure 2.2: Tight frames in $\mathbb{R}^2$.

The previous example shows that a basis in a Hilbert space is not necessarily a frame. However, it is the case for finite dimensional Hilbert spaces[16]. We have,

Proposition 2.1 *A basis $\Phi = (\varphi_i)_{i=1}^n$ of a finite dimensional Hilbert space $\mathcal{H}$ is a frame.*

Proof. Let $x = \sum_{i=1}^n x_i \varphi_i$ be an element of $\mathcal{H}$. We have

$$\sum_{i=1}^n |\langle \varphi_i, x \rangle|^2 = \sum_{i=1}^n |x_i|^2 |\langle \varphi_i, \varphi_i \rangle|^2.$$

Denoting by

$$A = \max_{i,j=1,\dots,n} |\langle \varphi_i, \varphi_j \rangle|^2, \quad B = \min_{i,j=1,\dots,n} |\langle \varphi_i, \varphi_j \rangle|^2,$$

we have

$$B||x||^2 \leq \sum_{i=1}^n |\langle \varphi_i, x \rangle|^2 \leq A||x||^2.$$

Therefore, Φ is a frame in $\mathcal{H}$. ■

Moreover, we have the following.

Theorem 2.1 *A normalized frame of a Hilbert space $\mathcal{H}$ is an orthonormal basis if and only if it is a tight frame [19].*

Proof. Suppose $\{v_j\}_{j \in J}$ is an orthonormal basis of the Hilbert space $\mathcal{H}$. Then for any $u \in \mathcal{H}$, we have the basis expansion

$$u = \sum_{j \in J} \langle v_j, u \rangle v_j.$$

Using orthonormality, we have

$$||u||^2 = \sum_{j \in J} |\langle v_j, u \rangle|^2.$$

Thus $(v_j)_{j \in J}$ is a tight frame. For the other direction assume that $\{v_j\}_{j \in J}$ is a normalized tight frame. Then for any $f \in \mathcal{H}$, we have

$$||f||^2 = \sum_{j \in J} |\langle \phi_j, f \rangle|^2. \tag{2.2}$$

Thus in particular, for any $i \in J$ we have

$$||\phi_i||^2 = 1 = 1 + \sum_{i \neq j} |\langle \phi_j, \phi_i \rangle|^2.$$

This implies that $|\langle \phi_j, \phi_i \rangle| = \delta_{ij}$ and the frame vectors are orthonormal. Consider the difference

$$d = f - \sum_{j \in J} \langle \phi_j, f \rangle \phi_j.$$

From (2.2), we have

$$\begin{aligned} \|d\|^2 &= \|f\|^2 + \sum_{j \in J} |\langle \phi_j, f \rangle|^2 - 2 \left\langle f, \sum_{j \in J} \langle \phi_j, f \rangle \phi_j \right\rangle \\ &= 2 \sum_{j \in J} |\langle \phi_j, f \rangle|^2 - 2 \sum_{j \in J} |\langle \phi_j, f \rangle|^2 = 0. \end{aligned}$$

Thus, $d = 0$ and $f = \sum_{j \in J} \langle \phi_j, f \rangle \phi_j$. Therefore $\{\phi_j\}$ is an orthonormal basis of $\mathcal{H}$. **■**

2.2.1 Frame Operators

Given a frame $(\varphi_j)_{j \in J}$ in a Hilbert space $\mathcal{H}$, the **analysis operator** of $(\varphi_j)_{j \in J}$ is the linear operator $F : \mathcal{H} \mapsto \ell^2(\mathbb{C})$ defined by,

$$F(f) = (\langle \varphi_j, f \rangle)_{j \in J}.$$

The frame operator is well defined since we have,

$$A\|f\|^2 \leq \|F(f)\|_2^2 \leq B\|f\|^2. \quad (2.3)$$

Proposition 2.2 *The frame operator F is injective [15].*

Proof. let f and g in $\mathcal{H}$ such that $F(f) = F(g)$. Thus, by linearity we have

$F(f - g) = 0$ and using (2.3), we obtain

$$A\|f - g\| \leq 0 \leq B\|f - g\|^2.$$

Therefore, $\|f - g\| = 0$ and hence $f = g$. ■

Proposition 2.3 *Let $(\varphi_j)_{j \in J}$ be a frame in a Hilbert space $\mathcal{H}$. Then $\text{span}((\varphi_j)_{j \in J}) = \mathcal{H}$ [20].*

Proof. Let us assume that the linear space $V := \text{span}((\varphi_j)_{j \in J})$ is strictly included in $\mathcal{H}$. Then there exist a non-zero element $f \in \mathcal{H}$ that is orthogonal to every element of V , that is for all $j \in J$

$$\langle \varphi_j, f \rangle = 0.$$

Thus $F(f) = 0$. From the frame inequality (2.3), we obtain

$$A\|f\|^2 \leq \|F(f)\|_2^2 = 0 \leq B\|f\|^2.$$

Thus $\|f\|^2 = 0$ and $f = 0$. This contradiction concludes the proof. ■

Proposition 2.4 *Let $\Phi = (\varphi_j)_{j \in J}$ be a frame in a Hilbert space $\mathcal{H}$ and F its frame operator. The adjoint $D : \ell_2(\mathbb{C}) \longrightarrow \mathcal{H}$ of F is given by*

$$D(c) = \sum_{j \in J} c_j \varphi_j.$$

Proof. We have

$$\langle F(f), c \rangle = \sum_{j \in J} \langle f, \varphi_j \rangle c_j = \left\langle f, \sum_{j \in J} \varphi_j c_j \right\rangle = \langle f, D(c) \rangle.$$

This conclude the proof. |

The operator D is called the *synthesis* operator and it is usually denoted by the same letter as the frame itself, i.e.; Φ .

2.2.2 Frames in Finite Dimensions

In this section, we show that any finite spanning set in K^n ($K = \mathbb{R}$ or $\mathbb{C}$) is a frame.

Let A be an $(m \times n)$ complex matrix. The eigenvalues of the Hermitian matrix A^*A are real. Moreover, if λ is an eigenvalue of A^*A , then we have

$$||Ax||^2 = \langle Ax, Ax \rangle = \langle x, A^*Ax \rangle = \lambda ||x||^2.$$

Thus, all the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of A^*A are non-negative. The numbers $\sigma_i = \sqrt{\lambda_i}, i = 1, 2, \dots, n$ are called **the singular values** of the matrix A . The number of non-zero singular values of A equals the rank of A . Let $(v_1, v_2, \dots, v_n)$ an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvector of A^*A with eigenvalue σ_i^2 . Let $x = \sum_{i=1}^n x_i v_i$. We have

$$||Ax||^2 = \langle x, A^*Ax \rangle = \sum_{i=1}^n \sigma_i^2 x_i^2.$$

Thus we have

$$\sigma_n^2 ||x||^2 \leq ||Ax||^2 \leq \sigma_1^2 ||x||^2. \quad (2.4)$$

[21]. Based on the (2.4), we prove the following.

Proposition 2.5 *Any finite spanning set in K^n ($K = \mathbb{R}$ or $\mathbb{C}$) constitutes a frame [16].*

Proof. Suppose $\{\phi_j\}_{j=1}^m \subset K^n$ such that $\text{span}(\{\phi_j\}) = K^n$. Define the operator $G : K^n \mapsto K^m$ by $(Gx)_j = \langle \phi_j, x \rangle$, $j = 1, 2, \dots, m$. G can be written as a matrix $G \in K^{m \times n}$. This matrix has rank n , since the span of its rows is all of K^n by assumption; therefore all the singular values of G are nonzero. We denote the largest singular value σ_1 , and the smallest by σ_n . Then, from (2.4), for any $x \in K^n$,

$$\sigma_n^2 ||x||^2 \leq ||G(x)||^2 \leq \sigma_1^2 ||x||^2$$

Thus, F is a frame operator and $\{\phi_j\}_{j=1}^m$ is a frame in K^n . The frame bounds are given by the squares of the smallest and largest singular values of G . ■

We can associate a matrix to every finite frame in $\mathbb{R}^n$ or $\mathbb{C}^n$. The analysis and the synthesis operators can be easily defined in term of the associated matrix [22].

Let $\Phi = (x_i)_{i=1}^k$ be a frame of K^n , the matrix Θ associated with the frame is obtained by viewing the vectors $x_i, i = 1, \dots, k$, as the columns of the matrix Φ . The synthesis operator is given simply by Φx .

CHAPTER 3

DATA SEPARATING BY SPARSE REPRESENTATION

This chapter serves as the main goal of this thesis, where separation of mixed signals will be achieved by taking advantage of the sparsity of the components of the signal in adapted frames and using ℓ_1 -minimization techniques. The general problem is as follow: Consider a signal x which belong to a Hilbert space $\mathcal{H}$ and assume that the signal can be split into L components

$$x = x_1 + x_2 + \dots x_L, \quad x_k \in \mathcal{H}. \quad (3.1)$$

Moreover, we assume that its constituent are *morphologically* different in the sense that for each x_k there exists a frame Φ_k of $\mathcal{H}$ such that, $x_k = \Phi_k \gamma_k$ where γ_k is sparse. An assumption uniqueness of the sparse solution $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_L)$ ensures

the recovery of the signal components using the ℓ_0 -minimisation,

$$\begin{aligned} \min \quad & \|\gamma\|_0 \\ \text{s.t.} \quad & [\Phi_1|\Phi_2|\dots|\Phi_L]\gamma^T = x. \end{aligned} \tag{3.2}$$

Thus the first step in this strategy is to purposefully select over-complete representation made up of a number of frames corresponding to the number of the unknown components to be extracted, each of which provides a sparse expansion of one of the components to be recovered [14][2]. The incoherence conditions of those frames are subsequently represented as the morphological differences between the components. Nevertheless, the minimisation problem (3.2) is NP-hard and therefore, one should choose the frames in such a way that the recovery of the components could be obtained using the ℓ_1 -minimisation,

$$\begin{aligned} \min \quad & \|\gamma\|_1 \\ \text{s.t.} \quad & [\Phi_1|\Phi_2|\dots|\Phi_L]\gamma^T = x. \end{aligned} \tag{3.3}$$

In the first part of this chapter we give conditions on the frame for the uniqueness of the sparse solution. in the second part we study conditions on the frames ensuring recovery using the ℓ_1 -minimisation (4.7). Applications to separation of periodic signal and spikes, image inpainting and smoothing will be given. Finally separation of signals with components sparse orthogonal polynomials, spikes and periodic components will be illustrated.

3.1 Compressed Sensing

Let $\Phi = (\varphi_i)_{i \in I}$ be a frame in a $\mathbb{K}$ -Hilbert space $\mathcal{H}$ and denote by $\Phi : \ell_2(\mathbb{K}) \longrightarrow \mathcal{H}$ its synthesis operator. Given a sparsity s , the following are equivalent

- a) $x \in \ell_2(\mathbb{K})$ is the unique s -sparse solution of $\Phi z = y$ with $y = \Phi x$
- b) the sequence x is the unique solution of the ℓ_0 -minimization

$$\min_{z \in \ell_2(\mathbb{K})} \|z\|_0 \quad \text{subject to} \quad Dz = y \quad (3.4)$$

Indeed, let us assume that a) holds and that the minimization problem has a solution $x^* \in \ell_2(\mathbb{K})$ different from x with $\|x^*\| < s$. Thus, x^* is also an s -sparse sequence. This contradicts the uniqueness hypothesis. That b) implies a) is clear. Therefore, the existence and uniqueness of a sparse solution guarantees its recovery by the ℓ_0 -minimization problem (3.4). Denote by $N(\Phi)$ the space

$$N(\Phi) = \{x \in \ell_2(\mathbb{K}), \text{ and } \Phi x = 0\}$$

Theorem 3.1 *Let Φ be a frame in a $\mathbb{K}$ -Hilbert space $\mathcal{H}$ and Φ its synthesis operator.*

The following properties are equivalent

- (a) *Every s -sparse sequence $x \in \ell_2(\mathbb{K})$ is the unique s -sparse solution of $\Phi z = \Phi x$.*
- (b) $N(\Phi) \cap \{z \in \ell_2(\mathbb{K}) : \|z\|_0 \leq 2s\} = \{0\}$.

Proof. Let us assume that $a)$ holds. Let $v \in \ell_2(\mathbb{K})$ such that $\|v\| \leq 2s$. Then we can write the sequence v as $v = x - z$ with $\|x\| \leq s$ and $\|z\| \leq s$. Since $\Phi v = \Phi x - \Phi z = 0$, we obtain $\Phi x = \Phi z$ and thus (a) guarantee that $x = z$ and thus $v = 0$. Now, let us assume that (b) holds. Let x and z element of $\ell_2(\mathbb{K})$ that are both s -sparse and such that $\Phi x = \Phi z$. Thus, $x - z$ is $2s$ -sparse with $\Phi(x - z) = 0$. Thus, according to (b) , $x = z$. This concludes the proof. ■

One can use *uncertainty principles*, which limit the simultaneous sparsity of a vector in a Hilbert space formed by concatenation of orthonormal bases to give conditions of the uniqueness of the ℓ_0 -minimisation problem. We shall first define the notion of *mutual coherence* of a frame.

Definition 3.1 For a frame $\Phi = (\varphi_i)_{i \in I}$ in a Hilbert space $\mathcal{H}$, the mutual coherence of Φ is defined as

$$\mu(\Phi) = \max_{i,j \in I, i \neq j} \frac{|\langle \varphi_i, \varphi_j \rangle|}{\|\varphi_i\| \|\varphi_j\|}.$$

We have the following bounds on the mutual coherence of a frame.

Theorem 3.2 Let $\mathcal{H}$ be a finite dimensional Hilbert space of dimension N , and let $\Theta := [\Phi, \Psi]$ be a frame of $\mathcal{H}$ where Φ and Ψ are two orthonormal bases of $\mathcal{H}$. Then

$$\frac{1}{\sqrt{N}} \leq \mu(\Theta) \leq 1.$$

Proof. As the bases are orthonormal, by Cauchy-Schwartz inequality we have

$$|\langle \phi_i, \psi_j \rangle|^2 \leq \|\psi_i\| \|\phi_j\| = 1.$$

Thus, clearly $\mu(\Theta) \leq 1$. Each element ϕ_j , $j = 1, 2, \dots, N$, of the orthonormal basis Ψ can be written as $\psi_j = \sum_{i=1}^N \alpha_i \phi_i$, with $\alpha_i = \langle \phi_j, \phi_i \rangle$. Thus

$$1 = \|\psi_j\|^2 = \langle \psi_j, \psi_j \rangle = \sum_{i=1}^N \langle \phi_i, \psi_j \rangle \langle \phi_i, \psi_j \rangle = \sum_{i=1}^N \langle \phi_i, \psi_j \rangle^2.$$

If $\mu(\Phi, \Psi) < \frac{1}{\sqrt{N}}$, then,

$$\langle \phi_i, \psi_j \rangle^2 < \frac{1}{N} \quad \text{for any } i = 1, 2, \dots, N$$

Thus, we would have $\sum_{i=1}^N \langle \phi_i, \psi_j \rangle^2 < 1$. This contradicts the fact that $\sum_{i=1}^N \langle \phi_i, \psi_j \rangle^2 = 1$. This concludes the proof. ■

When the Hilbert space $\mathcal{H} = \mathbb{C}^n$ and the frame Φ is finite with $m \geq 2$ element such that $m \geq n$, then we have the *Welsh bound* [23] [24]

$$\sqrt{\frac{m-n}{n(m-1)}} \leq \mu(\Phi) \leq 1.$$

We are now in a position to give criteria for the uniqueness of a sparse solution.

Theorem 3.3 [25] *Let $\mathcal{H}$ be a finite dimensional Hilbert space, and let $\Theta := [\Phi, \Psi]$ be a frame of $\mathcal{H}$ where Φ and Ψ are two orthonormal bases of $\mathcal{H}$. Then for any $d \in N(\Theta)/\{0\}$, we have*

$$\|d\|_0 \geq \frac{2}{\mu(\Theta)}.$$

Proof. Let $d = (d_1, d_2, \dots, d_n, \tilde{d}_1, \dots, \tilde{d}_n) \in N(\Theta)$. Thus we have

$$\sum_{i=1}^n d_i \phi_i = \sum_{i=1}^n -\tilde{d}_i \psi_i.$$

Without loss of generality, we may assume that the element $S = \sum_{i=1}^n d_i \phi_i$ satisfies

$\|S\| = 1$. Thus we have

$$1 = \langle S, S \rangle = \sum_{i,j=1}^n d_i \tilde{d}_j \langle \phi_i, \psi_j \rangle \leq \mu(\Theta) \left(\sum_{i \in \text{supp}(d)} |d_i| \right) \left(\sum_{j \in \text{supp}(\tilde{d})} |\tilde{d}_j| \right).$$

By Cauchy-Schwartz inequality, we have

$$\sum_{i \in \text{supp}(d)} |d_i| \leq \sqrt{\|d\|_0}, \quad \text{and} \quad \sum_{i \in \text{supp}(\tilde{d})} |\tilde{d}_i| \leq \sqrt{\|\tilde{d}\|_0}.$$

Thus,

$$1 \leq \mu(\Theta) \sqrt{\|d\|_0 \|\tilde{d}\|_0}$$

Using the geometric-mean inequality, we obtain

$$1 \leq \mu(\Theta) \frac{\|d\|_0}{2}.$$

This concludes the proof. I

A consequence of the previous theorem is the following,

Proposition 3.1 *Let $\Theta := [\Phi, \Psi]$ be a frame of a finite dimensional Hilbert space $\mathcal{H}$*

where Φ and Ψ are two orthonormal bases of $\mathcal{H}$. Let $x \in \mathcal{H}$ and c be such that

$$\Theta c = x \quad \text{and} \quad \|c\|_0 < \frac{1}{\mu(\Theta)}.$$

Thus, the solution to (3.2) is unique.

Proof. Let $c_1 \neq c$ be such that

$$\Theta c_1 = x \quad \text{and} \quad \|c_1\|_0 < \frac{1}{\mu(\Theta)}.$$

Thus, we have $\Theta(c - c_1) = 0$ and

$$\|c_1 - c\|_0 \leq \|c_1\|_0 + \|c\|_0 \leq \frac{2}{\mu(\Theta)}$$

This contradicts Theorem 3.3. ■

The previous theorem can be generalized to frames that are the union of several orthonormal bases. More precisely, we have the following.

Theorem 3.4 *Let $\mathcal{H}$ be a finite dimensional Hilbert space, and let $\Theta := [\Phi_1, \Phi_2, \dots, \Phi_L]$ be a frame of $\mathcal{H}$ where $\Phi_k, k = 1, 2, \dots, L$, are orthonormal bases of $\mathcal{H}$. Then for any $d \in N(\Theta)/\{0\}$, we have*

$$\|d\|_0 \geq \left(1 + \frac{1}{L-1}\right) \frac{1}{\mu(\Theta)}.$$

Proof. Denote by $\Phi_k = (\phi_k^1, \phi_k^2, \dots, \phi_k^n)$ and let $d = (d_1, d_2, \dots, d_L) \in N(\Theta)/\{0\}$

where $d_k = (d_k^1, d_k^2, \dots, d_k^n)$. Thus we have

$$d_k^j = - \sum_{\ell=1, \ell \neq k; i=1}^{L,n} d_\ell^j < \phi_j^k, \phi_i^\ell > \quad \text{for any } k, j.$$

Thus, we have

$$|d_k^j| \leq \mu(\Theta) \sum_{\ell=1, \ell \neq k; i=1}^{L,n} |d_\ell^j|.$$

Summing over the non-zero coordinates of d_k , we obtain, for any k

$$\|d_k\|_1 \leq \mu(\Theta) \|d_k\|_0 \sum_{\ell \neq k} \|d_\ell\|_1.$$

Therefore, $(1 + \mu(\Theta) \|d_k\|_0) \|d_k\|_1 \leq \mu(\Theta) \|d_k\|_0 \|d\|_1$, or equivalently,

$$\|d_k\|_1 \leq \frac{\mu(\Theta) \|d_k\|_0}{(1 + \mu(\Theta) \|d_k\|_0)} \|d\|_1.$$

Summing over k we obtain

$$\|d\|_1 \leq \left(\sum_{k=1}^L \frac{\mu(\Theta) \|d_k\|_0}{1 + \mu(\Theta) \|d_k\|_0} \right) \|d\|_1.$$

Therefore, we have

$$\sum_{k=1}^L \frac{\mu(\Theta) \|d_k\|_0}{1 + \mu(\Theta) \|d_k\|_0} \geq 1. \tag{3.5}$$

Consider the function $g(y) = 1/(1 + y)$. From (3.18), we have

$$\sum_{k=1}^L g(\mu(\Theta) \|d_k\|_0) \leq L - 1.$$

Since the function g is concave in $[0, \infty)$, we obtain

$$g\left(\frac{\mu(\Theta)\|d\|_0}{L}\right) \leq \frac{L-1}{L}$$

This is translated into the claim of the theorem. |

Proposition 3.2 *Let $\Theta := [\Phi_1, \Phi_2, \dots, \Phi_L]$ be a frame of a finite dimensional Hilbert space $\mathcal{H}$ where $\Phi_k, k = 1, 2, \dots, L$, are orthonormal bases of $\mathcal{H}$. Let $x \in \mathcal{H}$ and c be such that*

$$\Theta c = x \quad \text{and} \quad \|c\|_0 < \left(1 + \frac{1}{L-1}\right) \frac{1}{2\mu(\Theta)}.$$

Thus, the solution to (3.2) is unique.

Proof. Let $c_1 \neq c$ be such that

$$\Theta c_1 = x \quad \text{and} \quad \|c_1\|_0 < \left(1 + \frac{1}{L-1}\right) \frac{1}{2\mu(\Theta)}.$$

Thus, we have $\Theta(c - c_1) = 0$ and

$$\|c_1 - c\|_0 < \|c_1\|_0 + \|c\|_0 \leq \left(1 + \frac{1}{L-1}\right) \frac{1}{\mu(\Theta)}$$

This contradicts Theorem 3.4. |

When the Hilbert space is given by $\mathbb{K}^m$ and the frame Φ is finite with cardinality N with $N > m$ then, denoting by A the matrix of the frame, the claims of Theorem 3.1 are also equivalent to the fact that *every set of $2s$ columns of the matrix A is linearly independent*. To show that, let us assume that there exists a set of $2s$ columns of the

matrix A that is linearly dependent. Denote by S the indices of the columns and by A_S the sub-matrix with the column indexed by S . Thus there exists a non-zero vector $\omega \in \mathbb{K}^{2s}$ such that $A_S \omega = 0$. Therefore, the vector $v \in \mathbb{K}^N$ is a non-zero vector such that $v_S = \omega$ is a $2s$ -sparse vector such that $Av = 0$. This contradicts the claim *b)* of Theorem 3.1. Accordingly, Theorem 3.1 implies that in order to recover the sparse vector x we should have $\text{Rank}(A) \geq 2s$; and since $\text{Rank}(A) \leq m$, we conclude that exact recovery of x is possible when $m \geq 2s$.

There are many examples of matrices that satisfy the conditions of Theorem 3.1.

Example 1: Vandermonde matrix

Let s be a positive integer and $t_N > t_{N-1} > \dots > t_1 > 0$. Denote by A the matrix of order $m \times N$ with $N > m \geq 2s$, defined by

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_N \\ \vdots & \vdots & \dots & \vdots \\ t_1^{m-1} & t_2^{m-1} & \dots & t_N^{m-1} \end{bmatrix}.$$

For any subset $S = \{j_1 < j_2 < \dots < j_{2s}\}$ of $\{1, 2, \dots, N\}$ of cardinality $2s$ we have

$$\det(A_S) = \prod_{\ell < k} (t_{j_\ell} - t_{j_k}) > 0.$$

Thus, the matrix A satisfies the requirement of Theorem 3.1 and ensure the recovery of any s -sparse vector.

The problem of recovering an s -space vector x with a measurement matrix A by

solving the ℓ_0 - minimization problem

$$\min_{x \in \mathbb{K}^N} \|x\|_0 \quad \text{subject to} \quad Ax = y \quad (3.6)$$

is an *NP-hard*. Indeed, solving (3.6) consist of solving

$$A_S^* A_S u = A_S^* y,$$

where S runs over all the subset of $\{1, 2, \dots, N\}$ of cardinality s and then check for each solution u if $A_S u = y$. The number of systems to be solved is given by $\binom{N}{s}$ which is a very large number for the method to be of any practical relevance.

In the next section we shall give condition on the measurement matrix A or the frame Φ for which the ℓ_0 -minimization problem give the same solution as the ℓ_1 -minimization problem

3.2 Recovery via ℓ_1 -Minimization

For a sequence $d = (d_1, d_2, \dots, d_n, \dots)$ of real numbers and a finite set $\Lambda \subset \mathbb{N}$, we denote by $1_\Lambda d$ the characteristic sequence of Λ , i.e.; the sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_n, \dots)$ where $\epsilon_i = d_i$ if $i \in \Lambda$ and $\epsilon_i = 0$ otherwise.

Let $\Phi = (\varphi_i)_{i \in I}$ be a frame of a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$. and consider the

following minimization problems

$$\begin{aligned} \min \quad & \|c\|_0 \\ \text{s.t.} \quad & \Phi c = x \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \min \quad & \|c\|_1 \\ \text{s.t.} \quad & \Phi c = x \end{aligned} \tag{3.8}$$

The main goal of this chapter is to characterize conditions on the frame Φ such that the solutions of the two minimization problems (3.7) and (3.8) coincide.

Definition 3.2 *Let $\Phi = (\varphi_i)_{i \in I}$ be a frame of a Hilbert space $\mathcal{H}$ and let $N(\Phi)$ denote the null space of the associated synthesis operator. Then Φ is said to satisfy the **Null space property of order k** if*

$$\|1_\Lambda d\|_1 < \frac{1}{2} \|d\|_1.$$

for all $d \in N(\Phi) \setminus \{0\}$ and for all sets $\Lambda \subseteq I$ with $|\Lambda| \leq k$.

Proposition 3.3 *Let $\Phi = (\varphi_i)_{i \in I}$ be a frame of a Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$. Then the following conditions are equivalent:*

- (i) *All vectors c with $\|c\|_0 \leq k$ are unique solutions of the ℓ_1 -minimization.*

$$\begin{aligned} \min \quad & \|c\|_1 \\ \text{s.t.} \quad & \Phi c = x \end{aligned}$$

- (ii) The frame Φ satisfies the null space property of order k .

Proof. Let us assume that (i) holds and let $d \in N(\Phi)/\{0\}$ and $\Lambda \subseteq I$, $|\Lambda| \leq k$ be an arbitrary subset of I . Since $d \in N(\Phi)/\{0\}$ we have $\Phi d = 0$. Moreover, since $1_\Lambda + 1_{\Lambda^c} = d$, we have

$$\Phi(1_\Lambda d) = \Phi(-1_{\Lambda^c}).$$

As $\|1_\Lambda d\|_0 \leq k$, by (i), $1_\Lambda d$ is thus the unique ℓ_1 -solution to the optimization problem

$$\begin{aligned} \min \quad & \|c\|_1 \\ \text{s.t.} \quad & x = \Phi(1_\Lambda d). \end{aligned}$$

Therefore, $\|1_\Lambda d\|_1 < \|1_{\Lambda^c}\|_1$. In other words,

$$\|1_\Lambda d\|_1 < \frac{1}{2}\|d\|_1.$$

Since d and Λ were chosen arbitrary, we conclude that Φ satisfies the null space property of order k . Assume now that (ii) holds, and let c_1 be a vector such that $\|c_1\|_0 \leq k$ with $\text{supp}(c_1) = \Lambda$. Furthermore, Let c_2 be an arbitrary solution of $\Phi c = x$ with $x = \Phi c_1$. Set $d = c_2 - c_1$. Then $d \in N(\Phi)/\{0\}$ and

$$\begin{aligned} \|c_2\|_1 - \|c_1\|_1 &= \|1_{\Lambda^c} c_2\|_1 + \|1_\Lambda c_2\|_1 - \|1_\Lambda c_1\|_1 \\ &\geq \|1_{\Lambda^c} d\|_1 - \|1_\Lambda d\|_1 > 0. \end{aligned}$$

Thus $\|c_2\|_1 > \|c_1\|_1$. This concludes the proof █

Using Proposition 3.3, we show the following

Theorem 3.5 *Let $\Phi = (\varphi_i)_{i \in I}$ be a frame of a Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$ with $x \neq 0$. If $x = \Phi c$ and*

$$\|c\|_0 < \frac{1}{2} \left(1 + \frac{1}{\mu(\Phi)} \right), \quad (3.9)$$

then the solution of the ℓ_1 minimisation (3.8) coincides with the solution of the ℓ_0 minimisation (3.7).

Proof. Without loss of generality, we shall assume that the frame Φ is normalized.

We first prove that if (3.9) is satisfied then the solution of the ℓ_1 minimization problem (3.8) is unique. Let d be in the null space of the synthesis operator of Φ , i.e.; $d \in N(\Phi)$.

Thus, for any $i \in I$, we have

$$\langle \phi_i, \sum_{j \in I} d_j \phi_j \rangle = 0.$$

Thus,

$$d_i = - \sum_{j \in I, j \neq i} d_j \langle \phi_i, \phi_j \rangle.$$

Therefore,

$$|d_i| \leq \sum_{j \in I, j \neq i} |d_j| |\langle \phi_i, \phi_j \rangle| \leq \mu(\Phi)(\|d\|_1 - |d_i|),$$

and hence,

$$|d_i| \leq \left(1 + \frac{1}{\mu(\Phi)} \right)^{-1} \|d\|_1.$$

For any $\Lambda \subset I$ with $|\Lambda| = \|c\|_0$, we have

$$\|1_\Lambda d\|_1 \leq |\Lambda| \left(1 + \frac{1}{\mu(\Phi)} \right)^{-1} \|d\|_1 = \|c\|_0 \left(1 + \frac{1}{\mu(\Phi)} \right)^{-1} \|d\|_1 < \frac{1}{2} \|d\|_1.$$

Thus, the frame Φ satisfies the null space property of order $\|c\|_0$ and by Proposition 3.3 and the solution of the ℓ_1 minimization problem (3.8) is unique. Now, we shall prove that this solution coincides with the solution of the ℓ_0 minimisation problem (3.7) by showing it is the unique solution of (3.7). Let us assume that there is a solution $\tilde{c} \neq c$ with $\Phi\tilde{c} = x$ and $\|\tilde{c}\|_0 \leq \|c\|_0$. Thus we have

$$\|\tilde{c}\|_0 < \frac{1}{2} \left(1 + \frac{1}{\mu(\Phi)} \right),$$

and thus from the first part of the proof, $\tilde{c}$ is the unique solution to the ℓ_1 minimisation (3.8). This contradiction concludes the proof. ■

The bound given in (3.9) can be slightly improved if the frame Φ is a concatenation of orthonormal bases of a Hilbert space. We shall provide the result and the proof only for the concatenation of two orthonormal bases and state the general result for the concatenation of many orthonormal bases.

Theorem 3.6 *Let $[\Phi|\Psi]$ be a frame of a Hilbert space $\mathcal{H}$ where Φ and Ψ two orthonormal bases of $\mathcal{H}$. Let $x \in \mathcal{H}$ with $x \neq 0$. If $x = \Phi c$ and*

$$\|c\|_0 < \frac{\sqrt{2} - 0.5}{\mu(\Phi)}, \tag{3.10}$$

then the solution of the ℓ_1 minimisation (3.8) coincides with the solution of the ℓ_0 minimisation (3.7).

Proof. It is sufficient to proof that the frame Φ satisfies the null space property of

order the right hand side of (3.15). Denote by

$$\Phi = (\phi_1, \phi_2, \dots, \phi_n), \quad \Psi = (\psi_1, \psi_2, \dots, \psi_n).$$

Let $D = (d_1, d_2, \dots, d_n, \tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n) \in N(\Phi)/\{0\}$. Thus we have

$$\sum_{i=1}^n d_i \phi_i + \sum_{j=1}^n \tilde{d}_j \psi_j = 0.$$

Thus,

$$\sum_{i=1}^n d_i \phi_i = \sum_{j=1}^n (-\tilde{d}_j) \psi_j$$

Let Λ be a subset of $\{1, 2, \dots, 2n\}$ such that

$$|\Lambda| \leq \frac{\sqrt{2} - 0.5}{\mu(\Phi)},$$

and consider the following optimization problem

$$\text{Min} \quad \frac{1}{2} \|D\|_1 - \|1_\Lambda D\|_1 \tag{3.11}$$

$$\sum_{i=1}^n d_i \phi_i = \sum_{j=1}^n (-\tilde{d}_j) \psi_j \tag{3.12}$$

We should show that the optimal value of the previous optimization is positive. Note that as $\|1_\Lambda D\|_1$ increases the minimum of the above expression decreases. To avoid the trivial solution, we restrict the solution of the minimization problem to vector D

such that $\|D\|_1 = 1$. Thus, we modify the minimization problem to

$$\begin{aligned} \text{Min} \quad & \frac{1}{2} - \|1_\Lambda D\|_1 \\ \sum_{i=1}^n d_i \phi_i &= \sum_{j=1}^n (-\tilde{d}_j) \psi_j \\ \|D\|_1 &= 1 \end{aligned}$$

The constraint

$$\sum_{i=1}^n d_i \phi_i = \sum_{j=1}^n (-\tilde{d}_j) \psi_j$$

implies that for $k = 1, 2, \dots, n$, we have

$$|d_k| \leq \mu(\Phi) \|\tilde{d}\|_1 \quad \text{and} \quad |\tilde{d}_k| \leq \mu(\Phi) \|d\|_1.$$

It is then sufficient to prove the positivity of the optimal value of the following minimization with weaker set of constraints:

$$\begin{aligned} \text{Min} \quad & \frac{1}{2} - \|1_\Lambda D\|_1 \\ \text{s.t.} \quad & |d_k| \leq \mu \|\tilde{d}\|_1, \\ & |\tilde{d}_k| \leq \mu \|d\|_1, \\ & \|D\|_1 = 1. \end{aligned}$$

Denote by $\Lambda_1 = \Lambda \cap \{1, 2, \dots, n\}$ and $\Lambda_2 = \Lambda \cap \{n+1, \dots, 2n\}$. Assume that $|\Lambda_1| = k_1$ and $|\Lambda_2| = k_2$. Note that $\Lambda = (\Lambda_1, \Lambda_2)$. To show that the optimal value of the previous

minimisation problem is positive, it is sufficient to show that the optimal value of the following minimisation problem

$$\begin{aligned}
\min \quad & -\sum_{i \in \Lambda_1} d_i - \sum_{j \in \Lambda_2} \tilde{d}_j \\
\text{s.t.} \quad & -d_k + \mu(\tilde{d}_1 + \dots + \tilde{d}_n) \geq 0, \quad k = 1, 2, \dots, n \\
& -\tilde{d}_k + \mu(d_1 + \dots + d_n) \geq 0, \quad k = 1, 2, \dots, n \\
& d_1 + \dots + d_n + \tilde{d}_1 + \dots + \tilde{d}_n \geq 1 \\
& -(d_1 + \dots + d_n + \tilde{d}_1 + \dots + \tilde{d}_n) \geq -1 \\
& d_i \geq 0, \tilde{d}_i \geq 0
\end{aligned}$$

is larger than $-1/2$. This linear programming problem can be re-written in its canonical form as

$$\begin{aligned}
\min \quad & C^T Z \\
\text{s.t.} \quad & AZ \geq B, \\
& Z \geq 0,
\end{aligned}$$

where

$$Z = [d_1, d_2, \dots, d_n, \tilde{d}_1, \dots, \tilde{d}_n], C = [-1_{\Lambda_1}, -1_{\Lambda_2}], B = [0, \dots, 0, 0, \dots, 0, 1, -1]^T,$$

and A is the $2n + 2 \times 2n$ matrix

$$A = \begin{bmatrix} -I_n & \mu J_n \\ \mu J_n & -I_n \\ e & e \\ -e & -e \end{bmatrix}.$$

where I_n is the identity matrix of order n , J_n the square matrix of order n with all entries equal to 1 and $e = (1, 1, \dots, 1)$. Since the primal and the dual problems yield the same optimal value, i.e.

$$\text{Minimum}\{C^T Z | AZ \geq B, Z \geq 0\} = \text{Maximum}\{\langle B^T U | A^T U \leq C, U \geq 0\},$$

we fulfill the original requirement of the primal problem by finding a feasible solution U such that $B^T U \geq -\frac{1}{2}$. Consequently the maximal value is guaranteed to be also above the value $-1/2$. Consider the Parameterized form of the feasible solution U as follows:

$$U^T = [1_{\Lambda_1}, \alpha 1_{\Lambda_2}, \beta, \delta],$$

where α, β and δ are real scalar parameters. We have $B^T U = \beta - \delta$ and the constraints of the dual problem $A^T U \leq C$ and $U \geq 0$ are given by the positivity of the parameters α, β and δ and the inequalities

$$\mu \alpha k_2 + \beta - \delta \leq 0, \quad (1 - \alpha) \mu k_1 + \beta - \delta \leq 0, \quad (3.13)$$

where we assume, without loss of generality, that $k_1 := |\Lambda_1| \leq k_2 := |\Lambda_2|$. We shall choose the parameters α, β, δ such that the inequalities in (3.18) are equalities and $\beta = 0$. Solving the resulting system for α , we find

$$\alpha = \frac{1 + \mu k_1}{1 + \mu k_2}.$$

Thus $0 \leq \alpha \leq 1$. Moreover, we have

$$-\delta = -\mu k_2 \left(\frac{1 + \mu k_1}{1 + \mu k_2} \right) \geq 0.$$

Thus, the requirement that $B^T U \geq -1/2$ can be satisfied if

$$2\mu^2 k_1 k_2 + \mu k_2 - 1 \leq 0, \tag{3.14}$$

The condition (3.14) is satisfied if

$$k_1 \leq \frac{1}{\mu} \left(\frac{1 - \mu k_2}{2\mu k_2} \right).$$

Adding k_2 to both sides, we obtain

$$k_1 + k_2 \leq \frac{1}{\mu} \left(\frac{1 - \mu k_2}{2\mu k_2} + \mu k_2 \right) = \frac{1}{\mu} \left(\frac{1 - \mu k_2 + 2k_2^2 \mu^2}{2\mu k_2} \right)$$

Since, the minimum of the function

$$f(x) = \frac{1 - x + 2x^2}{2x}$$

over the interval $(0, \infty)$ is given by $f(\sqrt{0.5}) = \sqrt{2} - 0.5$, we conclude that if

$$k_1 + k_2 \leq \frac{\sqrt{2} - 0.5}{\mu},$$

then the condition (3.14) is satisfied. This concludes the proof. ■

Generalisation of the previous theorem to frames composed of concatenation of many orthonormal bases is given as follows.

Theorem 3.7 *Let $\Theta = [\Phi_1 | \Phi_2 | \dots | \Phi_L]$ be a frame of a Hilbert space $\mathcal{H}$ where Φ_j , $j = 1, 2, \dots, L$, are orthonormal bases of $\mathcal{H}$. Let $x \in \mathcal{H}$ with $x \neq 0$. If $x = \Phi c$ and*

$$\|c\|_0 < \frac{1}{\mu(\Phi)} \left(\sqrt{2} - 1 - \frac{1}{2(L-1)} \right), \quad (3.15)$$

then the solution of the ℓ_1 minimisation (3.8) coincides with the solution of the ℓ_0 minimisation (3.7).

Another criteria when the sparse solution can be recovered with the ℓ_1 -optimisation is when the measurement matrix satisfies *the restricted isometry condition* introduced by Candés and Tao [26][2].

Definition 3.3 *A matrix $A \in \mathbb{C}^{m \times N}$ satisfies the restricted isometry property (RIP)*

of order s if there exists a $\delta_s \in (0, 1)$ such that

$$(1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2 \quad (3.16)$$

for all s -sparse $x \in \mathbb{C}^N$.

The restricted isometry constant δ_s of a matrix $A \in \mathbb{C}^{m \times N}$ is defined as the smallest $\delta_s > 0$ that satisfies 3.16 for any s -sparse signal $\|x\|_0 \leq s$.

Theorem 3.8 (Candès et al. 2006 [1]) *If the RIP δ_{2s} of a matrix $A \in \mathbb{C}^{m \times N}$ satisfies $\delta_{2s} < \frac{1}{3}$, then A satisfies the null space property of order s is satisfied. In particular, every s -sparse vector $x \in \mathbb{C}^N$ can be recovered by ℓ_1 -minimization.*

Proof. Let $v \in \text{Ker}(A) \setminus \{0\}$. Consider an index set S_0 of s largest modulus entries of the vector v . We partition the complement of S_0 as the following,

$$S_0^C = S_1 \cup S_2 \cup \dots, \quad (3.17)$$

where S_1 is an index set of s largest absolute entries of v in $[N] \setminus S_0$, S_2 is an index set of s largest absolute entries of v in $[N] \setminus (S_0 \cup S_1)$ etc. Since $v \in \text{Ker}(A) \setminus \{0\}$, we have $A(v) = 0$, implies that $A(v_{S_0}) = -A(v_1 + v_2 + \dots)$. so that

$$\begin{aligned} \|v_{S_0}\|_2^2 &\leq \frac{1}{1 - \delta_{2s}} \|A(v_{S_0})\|_2^2 \\ &= \frac{1}{1 - \delta_{2s}} \langle A(v_{S_0}), A(-v_{S_1}) + A(-v_{S_2}) + \dots \rangle \\ &= \frac{1}{1 - \delta_{2s}} \sum_{k \geq 1}^{h-1} \langle A(v_{S_0}), A(-v_{S_k}) \rangle. \end{aligned} \quad (3.18)$$

We have for all $u, v \subset \mathbb{C}^N$ with disjoint supports ($\text{supp}(u) \cap \text{supp}(v) = \emptyset$), and $|\text{supp}(u)| + |\text{supp}(v)| = s$, then

$$|\langle Au, Av \rangle| \leq \delta_s \|u\|_2 \|v\|_2. \quad (3.19)$$

Therefore,

$$\langle A(v_{s_0}), A(-v_{s_k}) \rangle \leq \delta_{2s} \|v_{s_0}\|_2 \|v_{s_k}\|_2. \quad (3.20)$$

Substituting (3.20) into (3.18) and dividing by $\|v_{s_0}\|_2$,

$$\|v_{s_0}\|_2 \leq \frac{\delta_{2s}}{1 - \delta_{2s}} \sum_{k \geq 1}^{h-1} \|v_{s_k}\|_2.$$

Since the s entries of v_{s_k} do not exceed the s entries of $v_{s_{k-1}}$ for $k \geq 1$, we have

$$|v_j| \leq \frac{1}{s} \sum_{\ell \in S_{k-1}} |v_\ell|, \quad \text{for all } j \in S_k$$

and,

$$\|v_{s_k}\|_2 = \left(\sum_{j \in S_k} |v_j|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{s}} \|v_{s_{k-1}}\|_1.$$

By Cauchy-Schwartz inequality, we obtain

$$\|v_{s_0}\|_1 \leq \sqrt{s} \|v_{s_0}\|_2 \leq \frac{\delta_{2s}}{1 - \delta_{2s}} \sum_{k \geq 1} \|v_{s_{k-1}}\|_1 \leq \frac{\delta_{2s}}{1 - \delta_{2s}} \left(\|v_{s_0^c}\|_1 + \|v_{s_0}\|_1 \right). \quad (3.21)$$

Since $\frac{\delta_{2s}}{1 - \delta_{2s}} < \frac{1}{2}$ by assumption, the null space property follows. ■

3.3 The Separation of Sinusoids and Spikes.

In this section we shall apply Theorem (3.6) to separate sparse sinusoidal signals and spikes using their distinct morphological features of one being periodic and the other transient.

The Fourier basis is orthonormal with respect to the Euclidean inner-product.

Indeed, we have

$$\langle \mathcal{F}_j, \mathcal{F}_k \rangle = \frac{1}{n} \sum_{t=0}^{n-1} e^{\frac{(2\pi i j t - 2\pi i k t)}{n}} = \frac{1}{n} \sum_{t=0}^{n-1} e^{\frac{((j-k)2\pi i t)}{n}} = \frac{1}{n} \sum_{t=0}^{n-1} A^t$$

where $A = e^{\frac{2\pi i}{n}(j-k)}$. Thus, if $k \neq j$, we have

$$\langle \mathcal{F}_j, \mathcal{F}_k \rangle = \frac{1}{n} \left(\frac{A^n - 1}{A - 1} \right) = 0,$$

while if $k = j$, we have $A = 1$ and thus $\langle \mathcal{F}_j, \mathcal{F}_k \rangle = 1$.

The *Dirac or the canonical basis* of $\mathbb{C}^n$ is defined by

$$\Psi = (\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n) \quad \text{with} \quad \mathcal{E}_j = (0, 0, \dots, 1, 0, \dots, 0).$$

Clearly the Dirac basis Ψ is an orthonormal basis of $\mathbb{C}^n$.

Theorem 3.9 *Let Φ be the Fourier basis for $\mathbb{C}^n$ and let Ψ be the Dirac basis for $\mathbb{C}^n$.*

Further, let $x \in \mathbb{C}^n$ be the signal

$$x = x_1 + x_2, \quad \text{where} \quad x_1 = \Phi \gamma_1 \quad \text{and} \quad x_2 = \Psi \gamma_2.$$

with the coefficient vectors $\gamma_1, \gamma_2 \in \mathbb{C}^n$. If

$$\|\gamma_1\|_0 + \|\gamma_2\|_0 < (\sqrt{2} - 0.5)\sqrt{n},$$

then the ℓ_1 minimization problem (3.8) recovers γ_1 and γ_2 uniquely, and hence extracts x_1 and x_2 from $\mathbf{x}$ precisely.

Proof. Consider the frame,

$$\Theta = [\Phi | \Psi] \tag{3.22}$$

. The mutual coherence of Θ is given by

$$\mu(\Theta) = \max_{1 \leq k, j \leq n} |\langle \mathcal{F}_k, \mathcal{E}_j \rangle| = \max_{1 \leq k, j \leq n} \left| \frac{1}{\sqrt{n}} e^{\frac{2\pi i k(j-1)}{n}} \right| = \frac{1}{\sqrt{n}}.$$

Then by theorem (3.8) the requirement

$$\|\gamma_1\|_0 + \|\gamma_2\|_0 < \frac{\sqrt{2} - 0.5}{\mu(\Theta)} = (\sqrt{2} - 0.5)\sqrt{n}. \tag{3.23}$$

ensures that the ℓ_1 minimization problem (3.8) recovers γ_1 and γ_2 uniquely. █

3.3.1 Separation of a Signals Composed of Sinusoidals and Spikes

An example of the Separation of a signal composed of sinusoidals and spikes is illustrated in Figure 3.1. In the example, the sinusoidal curve is given by

$$f(t) = \sin 2t - 5 \cos \left(\frac{t}{2} \right).$$

We sample uniformly the curve over the interval $[0, 4\pi]$ with a vector $x = (x_1, x_2, \dots, x_N)$ with $n = 3200$. The discrete Fourier transform $\hat{x}$ of the vector x satisfies $\|\hat{x}\|_0 = 4$. Thus according to Theorem 3.9, we can randomly and uniformly add as much as $(\sqrt{2} - 0.5)\sqrt{n} - 4 = 47.7157$ spikes and the ℓ_1 - minimization could separate the curve from the spikes.

Figure 3.2 show the result of increasing the number of spikes beyond the allowed bound given in Theorem 3.9 and use the ℓ_1 -minimisation to recover the signal.

This phenomena can be explained by viewing the spike as noise with small strength and the recovered signal as sparse recovery of noisy signals. In this respect, one should consider the ℓ_1 - minimisation,

$$\begin{aligned} \min \quad & \|c\|_1 \\ \text{s.t.} \quad & \|\Phi c - x\|_2 \leq \epsilon. \end{aligned} \tag{3.24}$$

where ϵ is the strength of the noise.

Within this context we have the following Theorem,

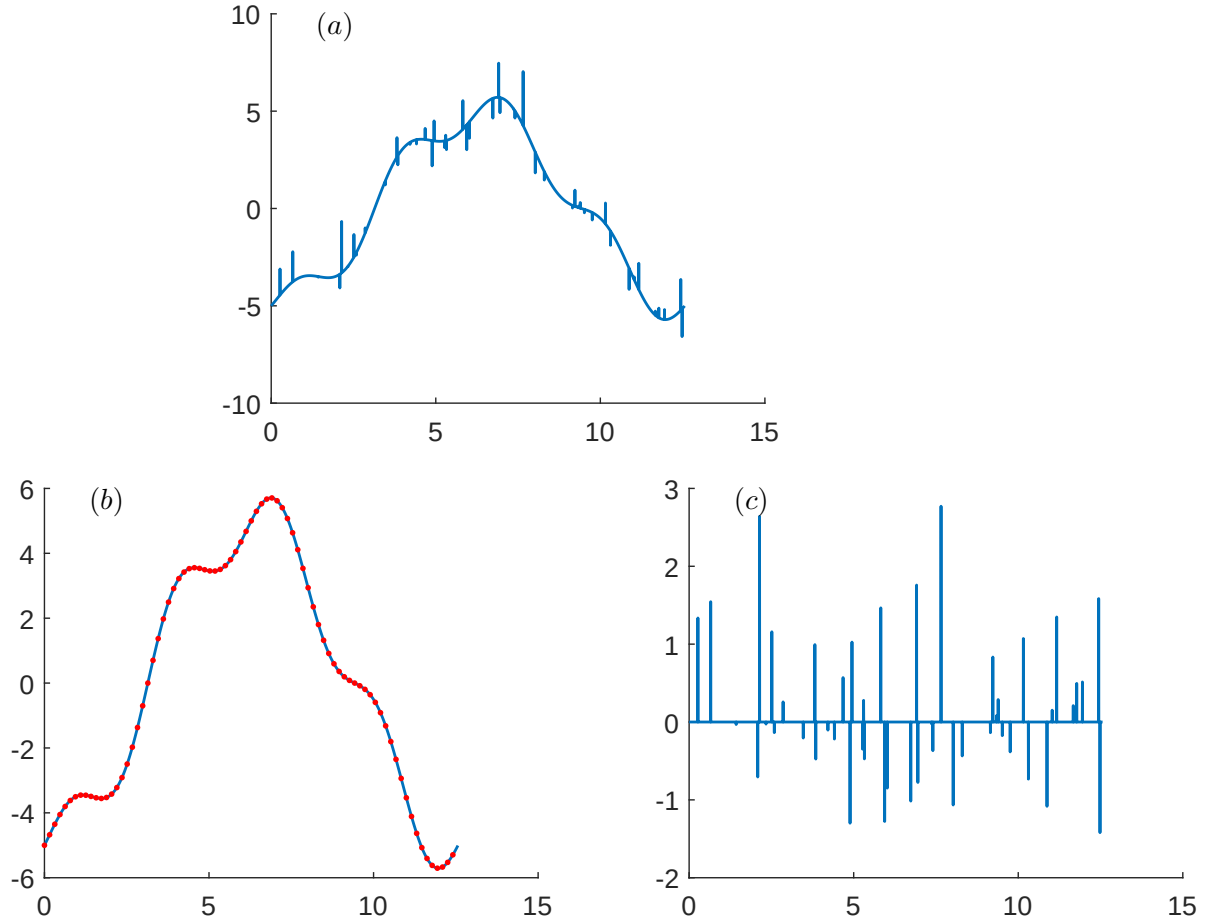


Figure 3.1: Sparse recovery of a signal composed of a sinusoidal curve and spikes. (a) the mixed signal, (b) the exact recovery of the signal (blue) and the original signal (red), (c) the exact recovery of the spikes.

Theorem 3.10 *Let Φ be a frame in a Hilbert space $\mathcal{H}$. Assume the sparsity s of a signal x in $\mathcal{H}$ satisfies*

$$s < \frac{1 + \mu(\Phi)}{4\mu(\Phi)}.$$

Given $y = \Phi x + n$ with $\|n\|_2 \leq \epsilon$. Then the solution $\hat{x}$ of the minimisation problem

(3.24) satisfies

$$\|x - \hat{x}\|_2 \leq \frac{2\epsilon}{\sqrt{1 - \mu(\Phi)(4s - 1)}}.$$

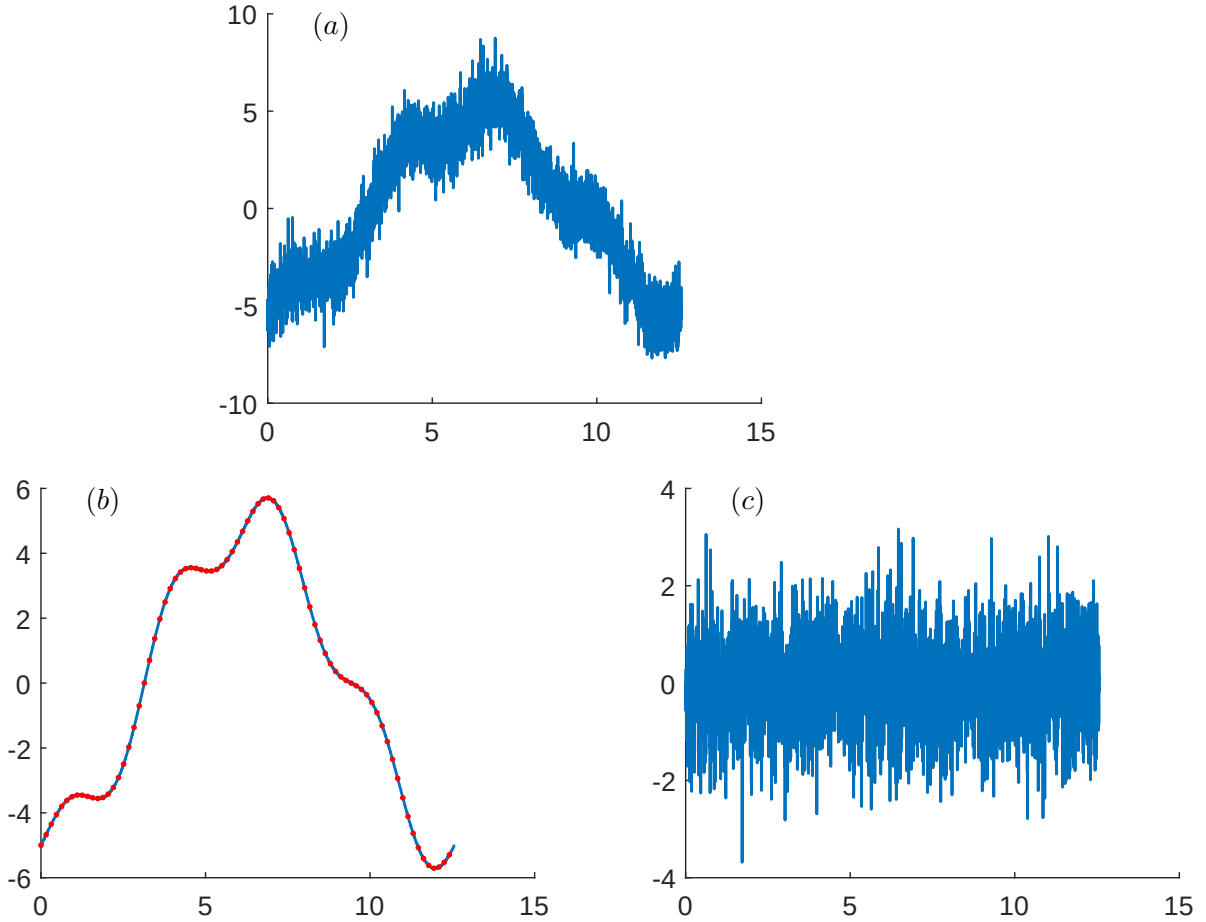


Figure 3.2: Sparse recovery of a signal composed of a sinusoidal curve and spikes when the number of spikes violates the condition of Theorem 3.9. (a) the mixed signal, (b) the exact recovery of the signal (blue) and the original signal (red), (c) the exact recovery of the spikes.

Application 2: Image In-painting and Smoothing

In the following, we apply Theorem 3.9 for image in-painting and smoothing. Consider the chessboard shown in Figure 3.3. Due to the periodicity of the image, it is expected to have sparse representation in the Fourier domain. Adding writing to the image and viewing the writing as spikes in the Dirac frame, we can recover the chessboard using the ℓ_1 -optimisation with respect to the frame 3.22. (See Figure 3.4)

An example of the recovery of an image of periodic ripples corrupted with noise



Figure 3.3: The original chessboard (left) and the same chessboard with some writing in it (right). The goal is to recover the original chessboard using the ℓ_1 -optimisation with the frame 3.22.



Figure 3.4: The result of inpainting of the chessboard in the left.

using the same frame 3.22 is illustrated in the Figure 3.5.



Figure 3.5: Smoothing of periodic ripples using the ℓ_1 -optimisation with respect to the frame 3.22.

3.4 The Separation of Orthogonal Polynomials and Spikes

Let ν be a positive Borel measure with finite moments of all orders, i.e.;

$$\int x^k d\nu < \infty \quad \text{for any } k \geq 0.$$

Denote by $\pi_0, \pi_1, \dots, \pi_n, \dots$ the orthonormal polynomials associated with the measure ν . Denote by $(\tau_1, \tau_2, \dots, \tau_n)$ the zeros of the orthogonal polynomial π_n and by $(\omega_1, \omega_2, \dots, \omega_n)$ the weights of the Gaussian quadrature with respect to ν . Thus, for any polynomial f of degree at most $2n - 1$, we have

$$\int f d\nu = \sum_{k=1}^n \omega_k f(\tau_k).$$

Define on $\mathbb{R}^n$ the inner-product

$$\langle x, y \rangle_\nu = \sum_{k=1}^n \omega_k x_k y_k.$$

Denote by $\Phi = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ with

$$\mathcal{F}_k = (\pi_k(\tau_1), \pi_k(\tau_2), \dots, \pi_k(\tau_n)), \quad k = 0, 1, \dots, n-1.$$

Proposition 3.4 *The sequence Φ is an orthonormal basis of $\mathbb{R}^n$ with respect to the*

inner-product $\langle \cdot, \cdot \rangle_\nu$.

Proof. Using Gaussian quadrature and orthogonality, we have

$$\langle \mathcal{F}_k, \mathcal{F}_j \rangle_\nu = \sum_{i=1}^n \omega_i \pi_k(\tau_i) \pi_j(\tau_i) = \int \pi_k(x) \pi_j(x) d\nu = \delta_{kj}.$$

This concludes the proof. ■

Consider now the weighted Dirac basis

$$\Psi_\omega = (\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n) \quad \text{with} \quad \mathcal{E}_j = \left(0, 0, \dots, \frac{1}{\sqrt{\omega_j}}, 0, \dots, 0 \right).$$

Clearly Ψ_ω is an orthonormal basis of $\mathbb{R}^n$ with respect to the inner-product $\langle \cdot, \cdot \rangle_\nu$.

With these two bases we have the following theorem.

Theorem 3.11 *Let Φ be the orthogonal polynomial basis for $\mathbb{R}^n$ and let Ψ_ν be the weighted Dirac basis for $\mathbb{R}^n$. Further, let $x \in \mathbb{R}^n$ be the signal*

$$x = x_1 + x_2, \quad \text{where} \quad x_1 = \Phi \gamma_1 \quad \text{and} \quad x_2 = \Psi_\nu \gamma_2.$$

with the coefficient vectors $\gamma_1, \gamma_2 \in \mathbb{R}^n$. If

$$\|\gamma_1\|_0 + \|\gamma_2\|_0 < \frac{\sqrt{2} - 0.5}{\max_{0 \leq k \leq n-1, 1 \leq j \leq n} \sqrt{\omega_j} |\pi_k(\tau_j)|},$$

then the ℓ_1 minimization problem (3.3) recovers γ_1 and γ_2 uniquely, and hence extracts x_1 and x_2 from $\mathbf{x}$ precisely.

Proof. Consider the frame $\Theta = [\Phi|\Psi_\nu]$. The mutual coherence of Θ is given by

$$\mu(\Theta) = \max_{0 \leq k \leq n-1, 1 \leq j \leq n} |\langle \mathcal{F}_k, \mathcal{E}_j \rangle| = \max_{0 \leq k \leq n-1, 1 \leq j \leq n} \sqrt{\omega_j} \pi_k(\tau_j).$$

Then by theorem (3.8) the requirement

$$\|\gamma_1\|_0 + \|\gamma_2\|_0 < \frac{\sqrt{2} - 0.5}{\mu(\Theta)} = \frac{\sqrt{2} - 0.5}{\max_{0 \leq k \leq n-1, 1 \leq j \leq n} \sqrt{\omega_j} |\pi_k(\tau_j)|}. \quad (3.25)$$

ensures that the ℓ_1 minimization problem (3.3) recovers γ_1 and γ_2 uniquely. ■

We shall apply the previous theorem to the separation of Jacobi orthogonal polynomials and spikes. Jacobi orthogonal polynomials $P_k^{(\alpha, \beta)}(\cdot)$, $k = 0, 1, \dots$, are orthogonal polynomials over the interval $[-1, 1]$ with respect to the weight function

$$\omega(x) = (1-x)^\alpha (1+x)^\beta, \quad \alpha > -1, \beta > -1.$$

Thus, for $n \neq m$, we have

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) \omega(x) dx = 0.$$

Special cases of Jacobi polynomials are: *Legendre polynomials* $P_n^{(1,0)}$, *Chebyshev polynomials* $P_n^{(-1/2, -1/2)}$ and *Gegenbauer polynomials* $P_n^{(\alpha-1/2, \alpha-1/2)}$. Figure 3.6 shows the mutual incoherence of the frame $\Theta = [\Phi|\Psi_\nu]$ for Legendre, Chebyshev and Gegenbauer ($\alpha = 1$) polynomials for various values of the degree n ($n = 3, 4, \dots, 30$).

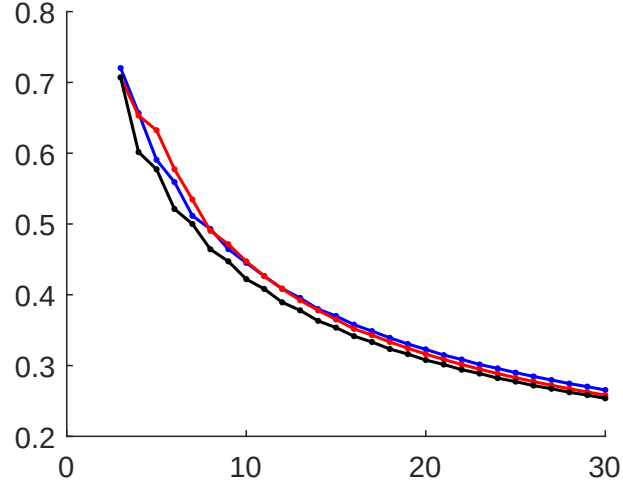


Figure 3.6: Mutual incoherence of the frame Θ when the frame Φ correspond to Legendre polynomial (blue), Chebyshev polynomials (red) and Gegenbauer polynomial with parameter $\alpha = 1$ (black).

Consider now the polynomial (see Figure 3.7)

$$P(t) = L_5(t) - L_6(t), \quad t \in [-1, 1],$$

where L_k are the orthonormal Legendre polynomials.

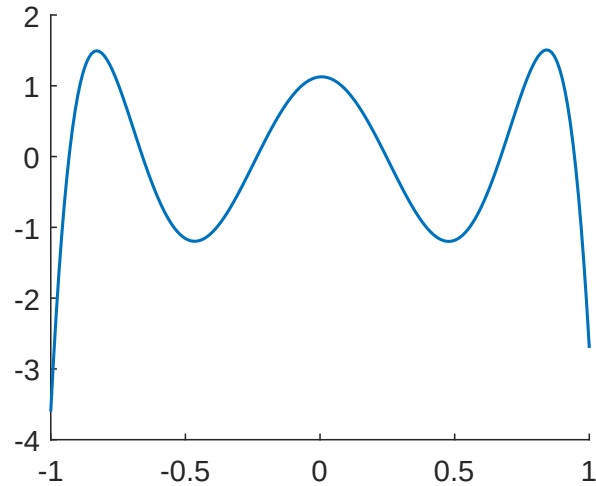


Figure 3.7: The graph of the polynomial $P(t) = L_5(t) - L_6(t)$, where L_k are the orthonormal Legendre polynomials.

The vector $x = (P(\tau_1), P(\tau_2), \dots, P(\tau_n))$, with $n = 30$, is sparse with respect to the frame Φ , i.e.,

$$x = \Phi\gamma_1 \quad \text{with} \quad \|\gamma_1\|_0 = 2.$$

Moreover, for $n = 30$ the mutual incoherence $\mu(\Theta) = 0.26$. Thus according to Theorem 3.11, we can add as much as $(\sqrt{2}-0.5)/0.26-2 = 1.447$ spikes and the ℓ_1 -minimization problem can recover the original signal exactly. Figure 3.8 shows the results of adding 7 spikes randomly at the zeros of the Legendre polynomials and the recovered signal.

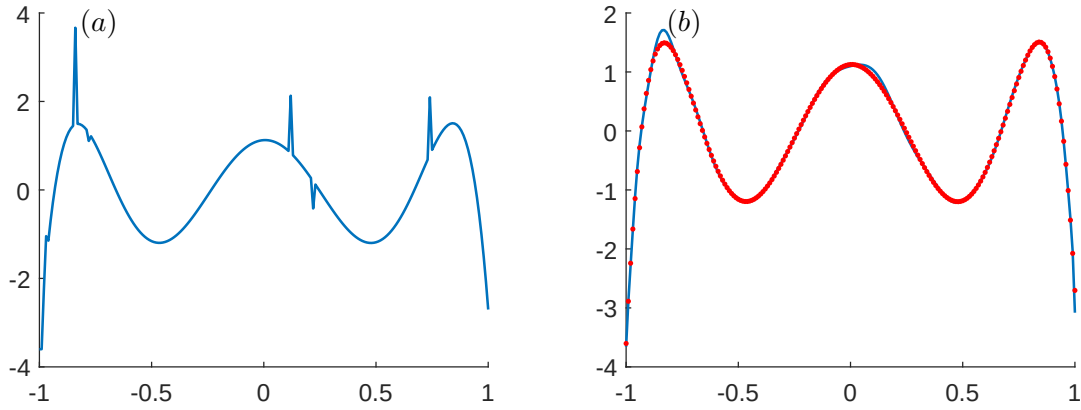


Figure 3.8: Sparse recovery of a signal composed of a sparse Legendre polynomial and spikes randomly located at the zeros of the degree $n = 30$ Legendre polynomial. (a) the mixed signal, (b) recovery of the signal (blue) and the original signal (red).

3.5 The Separation of Discrete Orthogonal Polynomials and Sinusoidal Signals

Let $x_0, x_1, \dots, x_n$ be a uniform subdivision of an interval $[a, b]$, i.e.; $x_0 = a$ and

$$x_k = x_0 + kh, \quad \text{with} \quad h = \frac{b-a}{n}$$

Let $(\varphi_0, \varphi_1, \dots, \varphi_n, \dots)$ be the orthonormal polynomials with respect to the inner-product

$$\langle f, g \rangle_\rho = \sum_{k=0}^n \rho_k f(x_k) g(x_k).$$

Denote by $\Phi = (\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{n-1})$ with

$$\mathcal{F}_k = (\varphi_k(x_0), \varphi_k(x_1), \dots, \varphi_k(x_n)), \quad k = 0, 1, \dots, n.$$

We have

$$\langle \mathcal{G}_k, \mathcal{G}_j \rangle_\rho = \sum_{i=0}^n \rho_i \varphi_k(x_i) \varphi_j(x_i) = \langle \varphi_k, \varphi_j \rangle_\rho = \delta_{ij}$$

Thus Φ is an orthonormal basis of $\mathbb{R}^{n+1}$ with respect to the inner-product $\langle \cdot, \cdot \rangle_\rho$.

Consider now the weighted Dirac basis,

$$\Psi_\omega = (\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_n) \quad \text{with} \quad \mathcal{E}_j = \left(0, 0, \dots, \frac{1}{\sqrt{\rho_j}}, 0, \dots, 0 \right).$$

Clearly Ψ_ω is an orthonormal basis of $\mathbb{R}^n$ with respect to the inner-product $\langle \cdot, \cdot \rangle_\rho$.

With these two bases and using the same argument as in the proof of Theorem 3.11,

we obtain

Theorem 3.12 *Let Φ be the discrete orthogonal polynomial basis for $\mathbb{R}^n$ and let Ψ_ρ be the weighted Dirac basis for $\mathbb{R}^n$. Further, let $x \in \mathbb{R}^n$ be the signal*

$$x = x_1 + x_2, \quad \text{where} \quad x_1 = \Phi\gamma_1 \quad \text{and} \quad x_2 = \Psi_\rho\gamma_2.$$

with the coefficient vectors $\gamma_1, \gamma_2 \in \mathbb{R}^n$. If

$$\|\gamma_1\|_0 + \|\gamma_2\|_0 < \frac{\sqrt{2} - 0.5}{\max_{0 \leq k \leq n-1, 1 \leq j \leq n} \sqrt{\rho_j} |\varphi_k(x_j)|},$$

then the ℓ_1 minimization problem (4.7) recovers γ_1 and γ_2 uniquely, and hence extracts x_1 and x_2 from $\mathbf{x}$ precisely.

Consider the case where $\rho_i = 1$ for $i = 0, 1, \dots, n$. The orthogonal polynomials φ_k are the discrete Legendre polynomials. The mutual coherence between Φ and the Fourier basis is given by

$$| \langle \mathcal{F}_k, \mathcal{G}_\ell \rangle | = \left| \sum_{j=0}^n \frac{1}{\sqrt{n}} \varphi_\ell(x_j) e^{2\pi i h j / n} \right| = |(\hat{\varphi}_\ell(x_j))|.$$

Therefore, we obtain the following

Theorem 3.13 *Let Φ be the discrete Legendre polynomial basis for $\mathbb{C}^n$ and let $\mathcal{G}$ be the Fourier basis for $\mathbb{C}^n$. Further, let $x \in \mathbb{R}^n$ be the signal*

$$x = x_1 + x_2, \quad \text{where} \quad x_1 = \Phi\gamma_1 \quad \text{and} \quad x_2 = \mathcal{G}\gamma_2.$$

with the coefficient vectors $\gamma_1, \gamma_2 \in \mathbb{C}^n$. If

$$\|\gamma_1\|_0 + \|\gamma_2\|_0 < \frac{\sqrt{2} - 0.5}{\max_{0 \leq k \leq n-1, 1 \leq j \leq n} |\hat{\phi}_k(x_j)|},$$

where $(\hat{\phi}_k(x_j))_{j=1}^n$ is the discrete Fourier transform of $(\phi_k(x_j))_{j=1}^n$, then the ℓ_1 minimization problem (4.7) recovers γ_1 and γ_2 uniquely, and hence extracts x_1 and x_2 from $\mathbf{x}$ precisely.

As application we consider a signal which is a mixture of a sparse signal in discrete Legendre polynomials and a periodic function that is sparse in its Fourier domain, i.e.; (See Figure 3.9).

$$f(t) = 3L_2(t) - 5L_4(t) + \sin(4\pi t), \quad t \in [0, 1].$$

Using the ℓ_1 -minimization, we recover exactly the components of the mixture (See

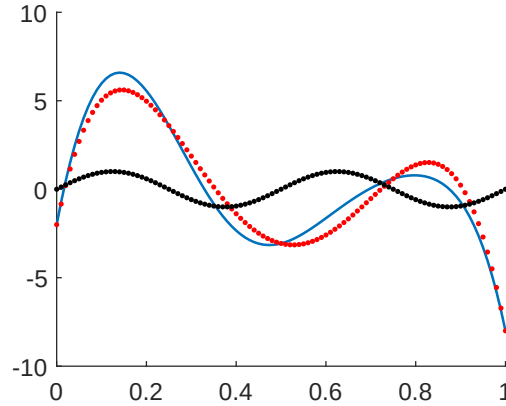


Figure 3.9: Mixture of a periodic and a sparse signal in discrete Legendre polynomials

Figure 3.10).

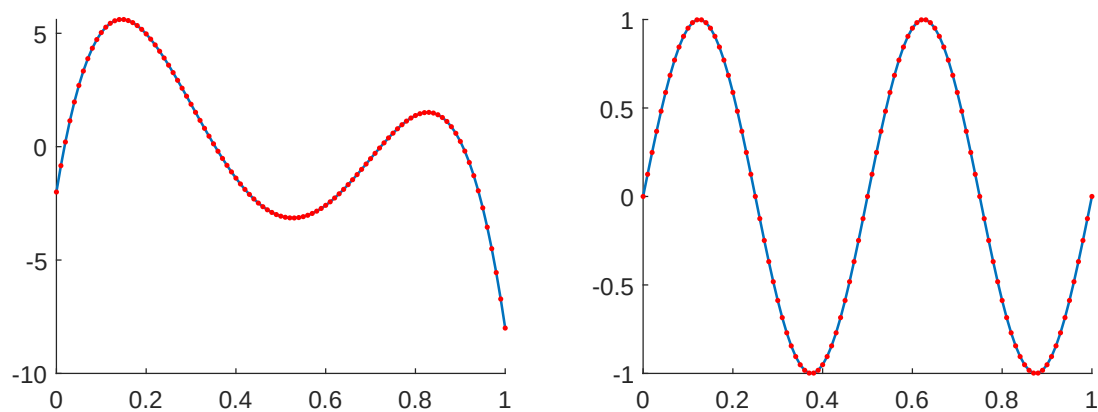


Figure 3.10: Sparse recovery of a signal composed of a sparse discrete Legendre polynomial and a periodic signal.

CHAPTER 4

GABOR FRAMES

Gábor frames are a set of functions that are used in signal processing to synthesize signals, decomposing signals into building blocks that are localized in time and frequency [16]. They have been extensively studied in time-frequency analysis over the last 30 years and are commonly used in science and engineering [27]. The Gábor transform is a special case of the short-time Fourier transform that is used to determine the sinusoidal frequency and phase content of local sections of a signal as it changes over time using a Gaussian window function that is modulated by a sinusoidal wave. It was named after Dennis Gábor who first introduced the windowed Fourier transform in 1949 [28] in order to measure localized frequency components of sounds and suggested the representation of a signal in a joint time-frequency domain [29].

4.1 The Time-Frequency Representation of Signals

Signals are represented mathematically as functions of one or more independent variables. We are concerned with the one dimensional signals involving a single indepen-

dent variable, time. There are two types of signals:

- Continuous time signal where the variable of time is continuous.
- Discrete time signal where the variable of time is discrete.

Analyzing a speech or music on a digital computer requires the use of a discrete time sequence, representing successive values of the continuous speech signal at a discrete points of time [30]. For the discrete time analysis, consider the sampling of f onto N data samples, for a given signal $f(t), t \geq 0$,

$$(f_0, f_1, f_2, \dots, f_k, \dots, f_{N-1}). \quad (4.1)$$

Since the integrand exists only at the sample points for the discrete time, the Fourier transform is evaluated at each time sample modulated by each frequency,

$$F(j\omega) = f[0]e^{-j0} + f[1]e^{-j\omega T} + f[2]e^{-j\omega 2T} + \dots + f[k]e^{-j\omega kT} + \dots + f[N-1]e^{-j\omega(N-1)T}.$$

To define the continuous Gábor transform, consider the following class of linear bounded operators on $L^2(\mathbb{R})$:

- For $a \in \mathbb{R}$, the operator T_a , called translation by a , is defined by

$$(T_a f)(x) := f(x - a), x \in \mathbb{R}.$$

- For $b \in \mathbb{R}$, the operator E_b , called modulation by b , is defined by

$$(E_b f)(x) := e^{2\pi i b x} f(x), x \in \mathbb{R}.$$

- For $c > 0$, the operator D_c , called dilation by c , is defined by

$$(D_c f)(x) := \frac{1}{\sqrt{c}} f\left(\frac{x}{c}\right), x \in \mathbb{R}.$$

Definition 4.1 ([31]) *The Gábor transform (short time Fourier transform) of f with respect to the window function g is defined by*

$$\mathcal{G}[f](t, \omega) = \tilde{f}_g(t, \omega) = \int_{-\infty}^{\infty} f(\tau) g(\tau - t) e^{-i\omega\tau} d\tau. \quad (4.2)$$

where $f, g \in L^2(\mathbb{R})$.

4.2 Finite Discrete Gábor Analysis

In practice, one should work with discrete version of a signal in finite time and the analogue of the Gábor transform for discrete signals should be defined. Thus, considering the interpretation of $f \in \mathbb{C}^N$, $N \in \mathbb{N}$ sampled as 4.1, We define the following operators for the finite discrete signals

- The **Shift** operator $T : \mathbb{C}^N \mapsto \mathbb{C}^N$ is defined by

$$Tf = T(f_0, f_1, f_2, \dots, f_{N-1})^T = (f_{N-1}, f_0, \dots, f_{N-2})^T.$$

- The **Translation** operator T_k by $k \in \{0, 1, \dots, N-1\}$ is defined by

$$T_k f(n) = T^k f(n) = f(n-k), \quad n \in \{0, 1, \dots, N-1\}.$$

- The **Modulation** operators $M_l : \mathbb{C}^N \mapsto \mathbb{C}^N$, $l = 0, 1, \dots, N-1$. are defined by

$$M_l f = (e^{2\pi i l 0/N} f_0, e^{2\pi i l 1/N} f_1, \dots, e^{2\pi i l (N-1)/N} f_{N-1}), \quad f \in \mathbb{C}^N.$$

Definition 4.2 *The discrete Gabor transform or Short Time Fourier Transform $V_g :$*

$\mathbb{C}^N \mapsto \mathbb{C}^{N \times N}$ of $f \in \mathbb{C}^N$ with respect to a discrete window $g \in \mathbb{C}^N \setminus \{0\}$ is defined as,

$$(V_g f)(k, l) = \sum_{n=0}^{N-1} f(n) g(n-k) e^{-2\pi i l n/N}.$$

Definition 4.3 *A Gábor frame $\Phi_g = (\phi_{k,\ell}), k, \ell = 0, 1, \dots, N-1$ in $\mathbb{C}^N$ with respect to a discrete window $g \in \mathbb{C}^N \setminus \{0\}$, is given by*

$$\phi_{k,\ell} = e^{-2\pi i \ell j/N} g(j-k), \quad j = 0, 1, \dots, N-1.$$

To demonstrate the windowed Fourier transform in extracting frequency information of a localized time window, we consider the following time signal

$$s(t) = 3\sin(2t) + 0.5\tanh(0.5(t-3)) + 0.2e^{-(t-4)^2} + 1.5\sin(5t) + 4\cos\left(\frac{3(t-6)^2}{10}\right) + \left(\frac{t}{20}\right)^3,$$

with the Gaussian window function

$$g(t) = e^{-a(t-b)^2}. \quad (4.3)$$

The Fourier transform of the windowed section of $s(t)$ then gives the local frequency

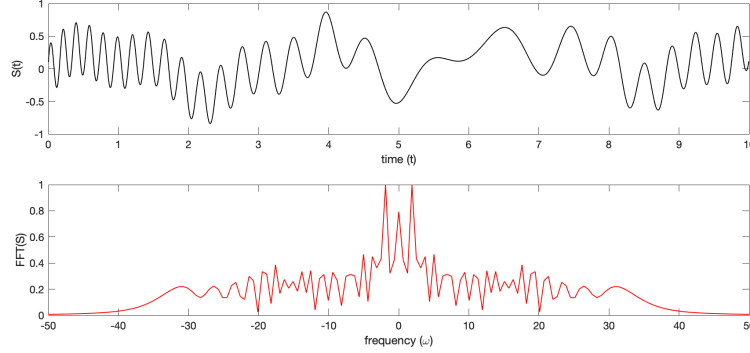


Figure 4.1: Time signal $s(t)$ and its Fourier transform.

content. By shifting the window to another position continuously at discrete time samples, we can analyse the whole signal's time domain to extract local frequency information from the Fourier transform of the signal, where the parameter " a " indicates the width of the window, and the parameter " b " is used to translate the window along the whole time axis. Figure (4.2) and Figure (4.3) shows the trade-off of time versus frequency resolution as determined by the choice of the analysis window function. Wide window in time produces better frequency resolution at the expense of reducing the time resolution which affects the tractability of the frequency peaks [32].

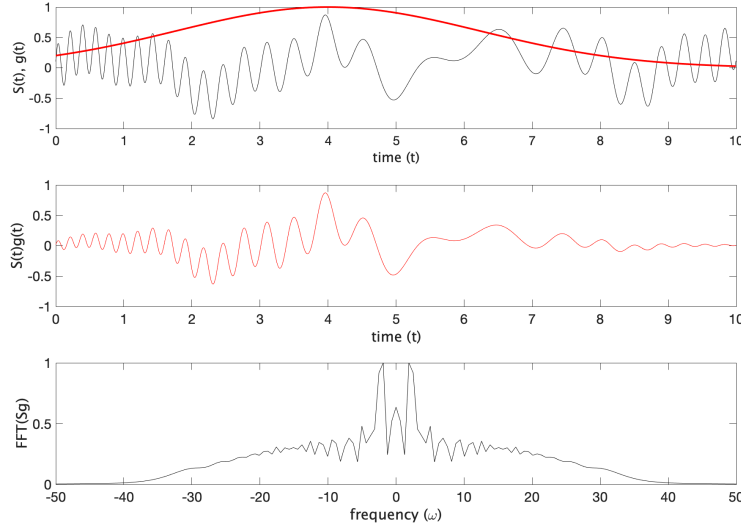


Figure 4.2: Gábor transform of $s(t)$ with respect to the window function 4.3 for $a = 0.1$.

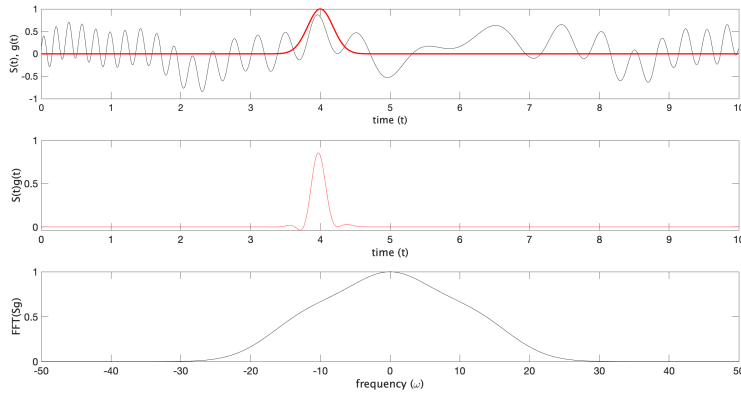


Figure 4.3: Gábor transform of $s(t)$ with respect to the window function 4.3 for $a = 15$.

4.3 The Spectrogram as a Time-Frequency Analysis

Tool

A *Spectrogram* is a visual representation of the spectrum of frequencies of a signal as it varies with time. It is a time-frequency analysis tool that is obtained by computing the short-time Fourier transform of a signal and plotting the magnitude of the resulting spectrum as a function of time and frequency. The spectrogram is useful for analyzing

signals that have time-varying frequency content, such as speech and music.

Definition 4.4 *The Spectrogram represents the value of the energy for each frequency l at each time k , that is*

$$SPEC_g(l, k) = |V_g f(l, k)|^2, \quad f \in \mathbb{C}^N. \quad (4.4)$$

In Figure 4.4, we illustrate the *Spectrogram* of the Gábor transform of the time signal $s(t)$ with respect to a Gaussian window function 4.3, Due to the Heisenberg uncer-

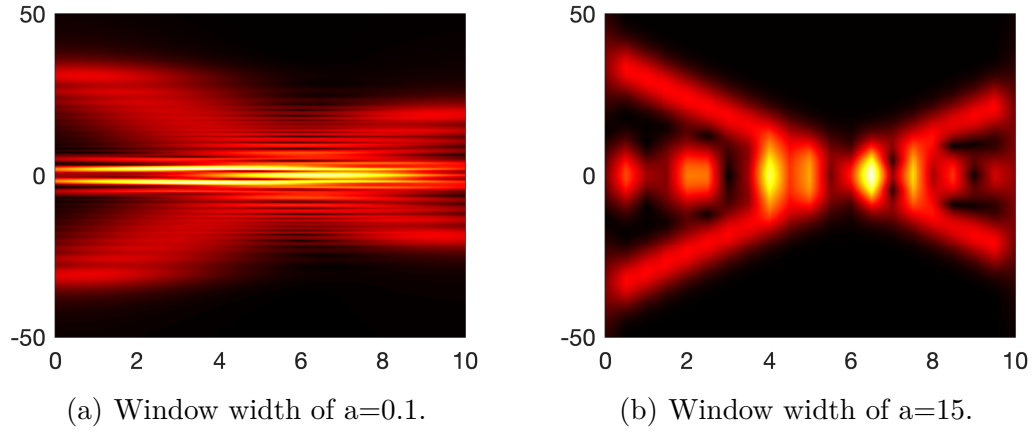


Figure 4.4: The *Spectrogram* of the time signal $s(t)$ analyzed with Gábor window $g(t)$, with respect to different width parameters.

tainty principle the wider (broader) the window is the more accurate the information frequency content at the expense of getting less accurate localization of the signal in time, [33][34].

4.3.1 Sparse Representation in Gábor Frames

A sparse representation in a Gábor frame is a way of representing a signal as a linear combination of a small number of Gábor functions. The Gabor functions are obtained by modulating a Gaussian window function with a sinusoidal wave. The sparse representation is obtained by selecting only a small number of the Gábor functions that are most important for representing the signal. This is useful for compressing signals and for denoising signals. In the following we provide an application of sparse signals in Gábor frames to audio separation.

Figure 4.5 shows a sampling of a signal representing a glockenspiel sound as $x = (x_1, x_2, \dots, x_N)$ with $n = 32768$ and its short time Fourier transform with respect to the discrete window function

$$g_k = \frac{1}{2} \left(1 - \frac{\cos(2\pi k)}{\omega - 1} \right), \quad (4.5)$$

computed with several windows of width

$$w = [192, 320, 576, 1088] \quad (4.6)$$

and overlapping factor $k = 2$. A sparse representation in Gábor frame of the signal x is achieved by finding a Gábor frame Φ_g that solve the LASSO problem

$$\min_{s \in \mathbb{C}^{2N}} \frac{1}{2} \|V_g^{-1}(s) - x\|_2^2 + \lambda \|s\|_1.$$

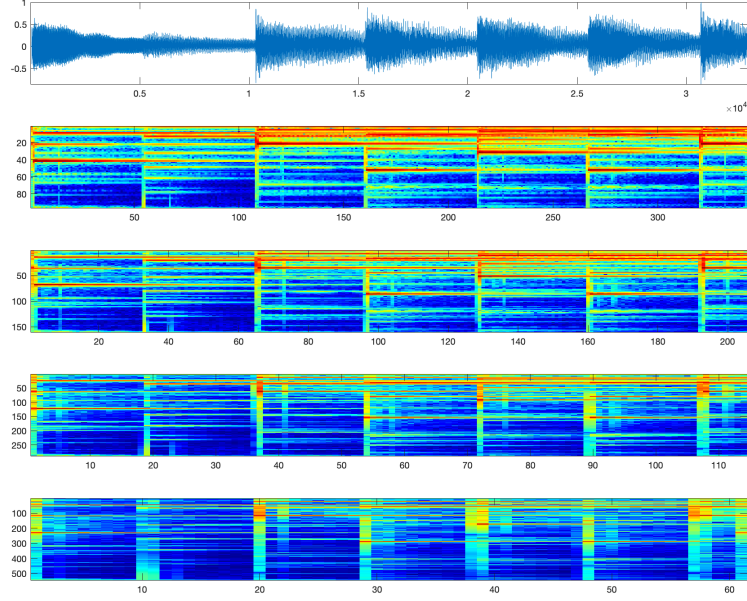


Figure 4.5: Short time Fourier transform of loaded input signal x , with several windows w .

4.3.2 Audio Separation Using Gábor Frames

consider a mixture of three audio signals of a bird, a glockenspiel and a male sounds, sampled with $n = 16384$ points. (See Figures 4.10 and 4.8).

The audio separation of the mixed signal is achieved by using sparsifying Gábor frames Φ_{g_k} for each sound and solve the ℓ_1 -minimisation problem

$$\begin{aligned} \min \quad & \|\gamma\|_1 \\ \text{s.t.} \quad & [\Phi_{g_1} | \Phi_{g_2} | \dots | \Phi_{g_L}] \gamma^T = x. \end{aligned} \tag{4.7}$$

This results in the audio separation as shown in Figure 4.10 (See [35],[36] for more details).

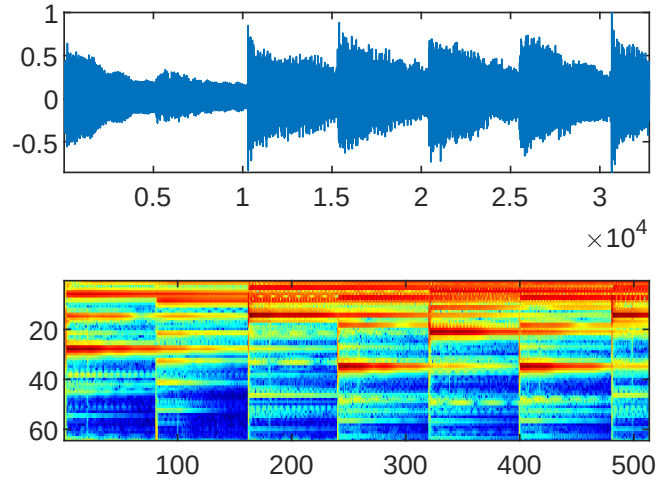


Figure 4.6: The (STFT) of the signal x with respect to the window function (4.5) with width $\omega = 128$.

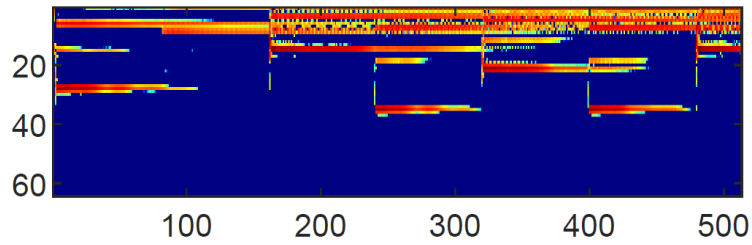


Figure 4.7: Sparsification of the spectrogram (4.6) using LASSO with $\lambda = 0.1$.

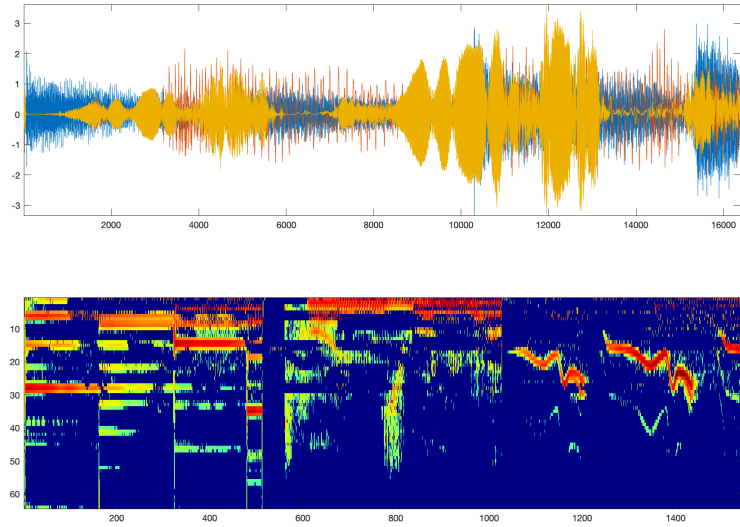


Figure 4.8: The three sources mixed together and their (STFT).

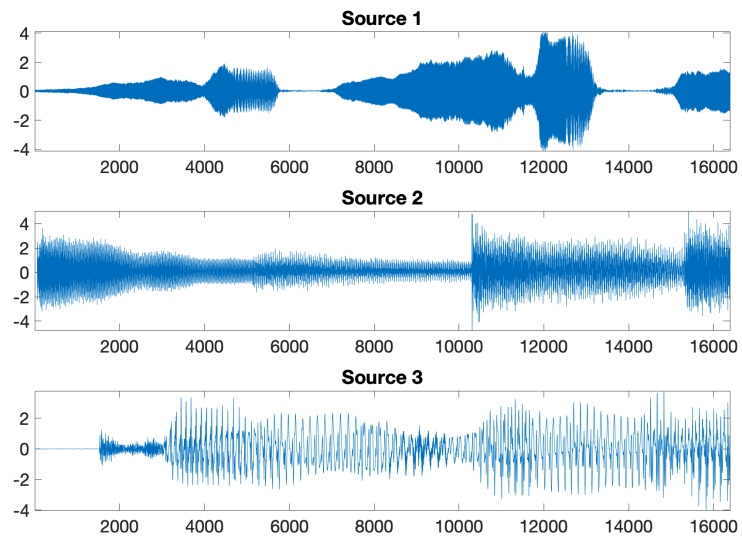


Figure 4.9: Original sources of a bird, glockenspiel and a male sounds signals.

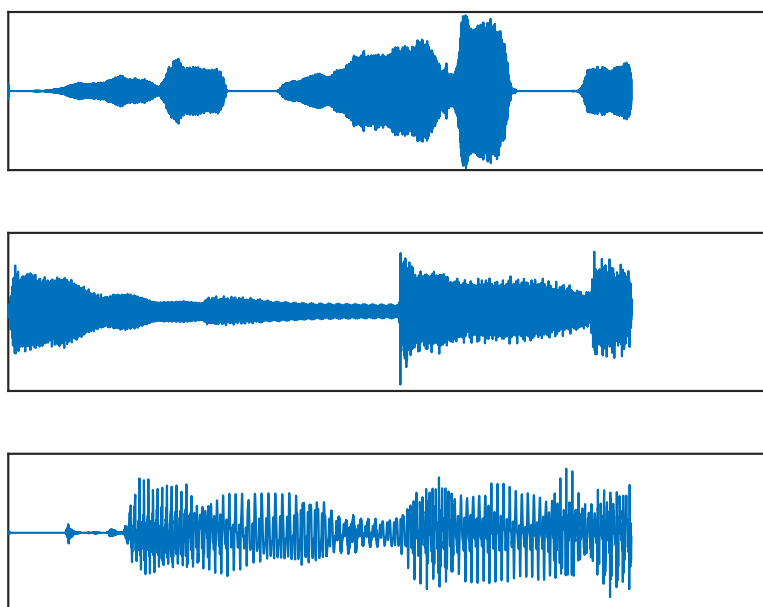


Figure 4.10: The recovered audio signals using (4.7).

CHAPTER 5

CONCLUSION

In this thesis we provided a detailed account on the use of morphological component analysis for signal separation. The main ingredient for successful separation is the sparsity of the components of the mixture in adequate frames and the incoherence of the frames. Under certain conditions on the coherence of the frames, we showed the possibility of extracting the components using convex optimisation criteria. We applied the method to the separation of mixture of periodic signals and spikes and also to the mixture of signals composed of periodic and sparse signal in specific bases of orthogonal polynomials. Separation of audio signals were achieved by choosing sparsifying Gábor frames to each audio component of the mixture.

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