

**STACKING SEQUENCE OPTIMIZATION OF
RECTANGULAR LAMINATED COMPOSITE PLATES**

BY

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A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

CIVIL ENGINEERING

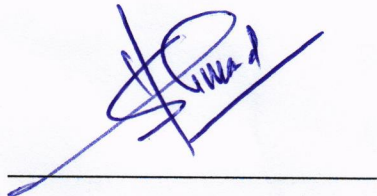
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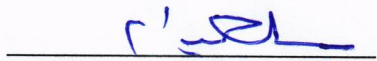
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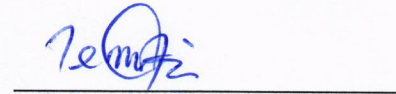


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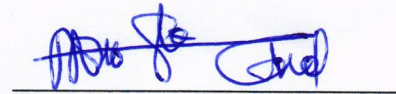
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2022

Dedicated
to
My parents, brothers & sisters
for their help and support

ACKNOWLEDGMENTS

First and foremost, all praise and glory be to ALLAH who gave me health, patience and determination to complete this work. All appreciation and gratitude to KFUPM for giving me the opportunity to pursue my master's degree with full financial support. I am very thankful to the CEE Department Chairman, Prof. Shamsad Ahmad, and all the teaching staff at the Civil Engineering Department for the time and great help and knowledge they offered.

I acknowledge, with deep gratitude and appreciation, the valuable time and guidance given to me by my advisor, Prof. Husain Jubran Al-Gahtani. for his continuous magnificent helpful guidance and support throughout the research that could not be shortened in few words. I am also thankful to my committee members, Dr. Faisal M. Mukhtar and Dr. Saheed K. Adekunle, for their constructive and valuable comments on this research.

I also acknowledge the great efforts of Dr. Madyan Shugaa and Dr. Abubakr Musa, where we combine a pleasurable and advantageous research team.

I would like to express my heartfelt love and gratitude to my parents and family in Palestine for always standing by my side and for their continuous support and help. Also, I could not forget my colleagues at KFUPM for their friendship during my university life.

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LIST OF SYMBOLS

The following are the list of symbols used in this dissertation. Other symbols are defined within the text.

t	:	lamina thickness.
h	:	The plate's thickness.
E	:	The elastic modulus.
ν	:	Poisson's ratio.
G	:	The shear modulus.
$[T]$	:	The transformation stiffness matrix.
$[Q]$	:	The reduced stiffness matrix.
$[A]$	:	The extensional stiffness matrix.
$[B]$	:	The coupling stiffness matrix.
$[D]$	:	The bending stiffness matrix.
X_t, Y_t	:	The tensile strengths along and normal to the fiber direction, respectively.
X_c, Y_c	:	The compressive strengths along and normal to the fiber direction, respectively.
S	:	The ultimate in-plane shear strength.
SR	:	Strength Ratio= <i>Maximum Load/Load Applied</i> .

- β : Load ratio = N_y/N_x .
- γ : Load ratio = N_{xy}/N_x .
- SR^* : Strength Ratio for a laminate with $h = 1 \text{ mm}$ and $N_x = 1 \text{ N/mm}$.
- h_{min} : The minimum laminate thickness without causing failure.
- $\theta_{optimum}$: The optimum fiber orientation.
- α : Load ratio after the transformation to the principal stresses.
- θ_t : The principal stress transformation angle.

ABSTRACT

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Thesis Title : [Stacking Sequence Optimization of Rectangular Laminated Composite Plates]
Major Field : [Civil Engineering]
Date of Degree : [May 2022]

Fiber-reinforced polymeric composites are experiencing more popularity with applications in almost every type of advanced engineering structure such as aircraft and automobiles. However, their high cost and complexity in analysis require special care to find the optimum design that takes into consideration all the design variables. Motivated by this, the present work is a stacking sequence optimization of symmetrically rectangular laminated composite plates subjected to in-plane loading. The classical lamination theory will be used for formulation. The main objective function is to maximize the strength capacity of the laminated plate. For this objective the strength ratio based on Tsai-Hill and Tsai-Wu failure criteria will be used. The fiber orientations for a given plate thickness is considered as the design variables. The aim is to construct a closed-form formula for the optimum strength design first to in-plane tension loading and then by adding the shear. An open code using Mathematica software will be used to solve the optimization problem effectively for the two failure criteria (Tsai-Wu and Tsai-Hill) and three different materials (Glass/Epoxy, Carbon/Epoxy and Kevlar/Epoxy). The outcomes are presented in graphical and closed-form formula suitable for design purposes with a numerical example to show the effectiveness and accuracy of the closed-form formulas for a wide range of winding angles, materials and load level.

ملخص الرسالة

الاسم الكامل: إبراهيم عوني إبراهيم الفقيه

عنوان الرسالة: تحسين تسلسل التراص للصفائح المركبة الطبقة ذات الشكل المستطيل

التخصص: الهندسة المدنية - إنشاءات

تاريخ الدرجة العلمية: ماي، 2022

أصبحت المركبات البوليمرية المقواة بالألياف تحظى بشعبية أكبر مع تطبيقات تقريباً في كل نوع من أنواع الهياكل الهندسية المتقدمة مثل الطائرات والسيارات. وبالرغم من ذلك فإن التكلفة العالية والصعوبة في التحليل لهذه المواد يتطلبان عناية خاصة للعثور على التصميم الأمثل الذي يأخذ في الاعتبار جميع متغيرات التصميم. بأخذ هذا في عين الاعتبار فإن العمل الحالي عبارة عن دراسة لتحسين تسلسل التراص للصفائح المركبة الطبقة المستطيلة الشكل المتناظرة المعرضة للتحميل داخل المستوى. للقيام بذلك سيتم استخدام نظرية الصفائح الكلاسيكية لصياغة المعادلات (CLT). ويتمثل الهدف الرئيسي لهذا البحث هو زيادة قوة الصفائح الطبقة الرقيقة إلى الحد الأقصى. لتحقيق هذا الهدف سيتم الاعتماد على معيارين للفشل وهما Tsai-Wu و Tsai-Hill. في حين ستكون اتجاهات الألياف داخل كل طبقة بمثابة متغيرات التصميم. الهدف هو بناء صيغة رياضية لتصميم الصفائح الطبقة لتحقيق أقصى قوة تحت تأثير قوى شد داخل المستوى ثم بإضافة قوى القص. سيتم استخدام كود تصميم باستخدام برنامج Mathematica لإيجاد أفضل ترارص للتحسين تحت تأثير كلا معياري الفشل (Tsai-Hill و Tsai-Wu) وثلاث مواد مختلفة (glass / epoxy و carbon / epoxy و Kevlar / epoxy). سيتم تقديم النتائج في صيغة رسومية ومعادلات رياضية مناسبة لأغراض التصميم مع أمثلة حسابية لإظهار فعالية ودقة النتائج لمجموعة واسعة من زوايا الدوران والمواد ومستوى التحميل.

INTRODUCTION

1.1 General

The demand for composite materials is surging every year with the increase in their applications in various areas. They are used extensively in civil engineering applications as in modular structures, towers, bridge structures, pipes and tanks. They are also used in automotive, aerospace, and other engineering applications [1]. These materials are particularly attractive due to their high-performance physical properties especially the high strength to weight ratios and their ability to withstand the hostile environment such as high radiation and high temperature. However, the anisotropy (presence of highly directional properties) of the composite materials makes their analysis and design relatively much more complicated than that of other structures [2]. One of the most popular composite materials is the laminated composite where layers that contained fibers oriented at different angles are placed over each other to form stacking sequence layers as shown in Figure 1.1.

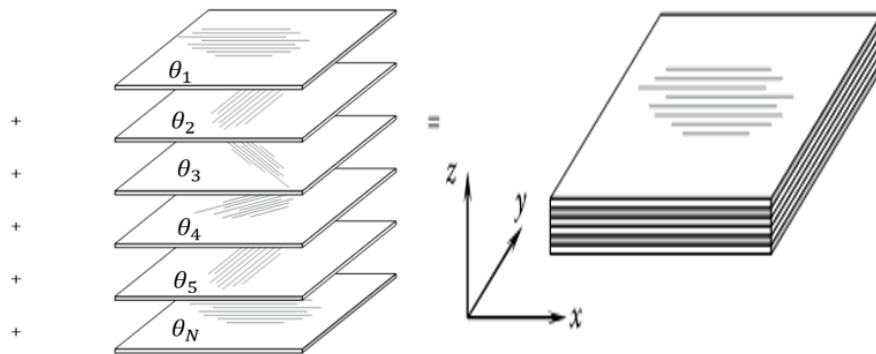


Figure 1.1: Schematic of a laminate.

Laminated composite materials have been becoming more common due to their excellent mechanical characteristics such as high strength and stiffness to weight ratios, as well as the ability to customize these properties through the selection of fiber orientations. For this type, However, the optimizing of the material system must be taking into consideration because of their relatively high cost compared with other materials. Many researchers, through the implementation of optimization methods, conducted studies to achieve the optimal designs of the composite laminated structures with different loads and boundary conditions. The studies attempt to minimize the material use either by reducing the weight, minimizing the laminate thickness, deflection, and cost, or by maximizing static strength and stiffness. The common designing variables are ply thickness, fiber orientations and stacking sequence [3].

The anisotropy of the composite laminates under several types of loading will result in a non-uniform distribution of the stresses because of the existence of bending-extension coupling. This as a result leads to a very complex failure process with no possibility, except in special cases, to find an analytical solution. Taking this complexity in the calculations into account makes the design process requires dedicated methods that takes into consideration the specificities of composite materials. All the design configurations, the large number of variables, failure mechanisms and the complex behavior have to be considered while designing [4].

In engineering applications, Composite laminated structures are exposed to a variety of loading conditions (in-plane and out-of-plane loads). Designing a general laminated plate is quite difficult as bending–stretching effects have to be considered. Due to this effect, it is not possible to obtain closed-form solutions, and thus many studies neglected the effect

through using symmetrically laminated angle-ply configurations [5]. In most of the studies, because of difficulty to achieve efficient design procedures, mathematical optimization strategies are required to select the optimal values within large set of constrained design variables [6].

None of the later mentioned cited works attempted to obtain a closed formula for the optimum fiber orientation angles. In the present work, optimum strength design of symmetrically laminated composite plate under in-plane tension (N_x and N_y) and shear (N_{xy}) loading is proposed utilizing the classical lamination theory (CLT) along with Tsai-Wu and Tsai-Hill failure criteria. The optimum design, for three different materials (glass/epoxy, carbon/epoxy and Kevlar/epoxy), has been found by taking failure criteria as the objective functions and the orientation angles as the design variable. To approach this, first a code based on analytical procedure has been introduced. This code is able to find the global optimum strength design through testing different laminate combinations. Once the critical strength ratio for each combination is found by taking the minimum of the strength ratio for each ply within the laminate. Then, out of all attempted combinations, the one yields the maximum strength ratio is selected to be the optimum design stacking sequence. This has been implemented for a wide range of winding angles, materials and load level. The outcomes are presented in graphical and closed-form formula suitable for design purposes. Finally, a numerical example will be performed to do verification for the results under different in-plane loading conditions.

1.2 Problem Statement

The structure to be optimized is a symmetric laminated composite plate subjected to in-plane loading (N_x , N_y and N_{xy}) where N_x and N_y are both in tension (Figure 1.2). The laminate has a total thickness h and contains N number of plies, each has a thickness of $t = h/N$ and has its own fiber orientation angle. The plies are assumed to be homogeneous and perfectly bonded. Moreover, the plate is assumed to behave according to the classical lamination theory (CLT).

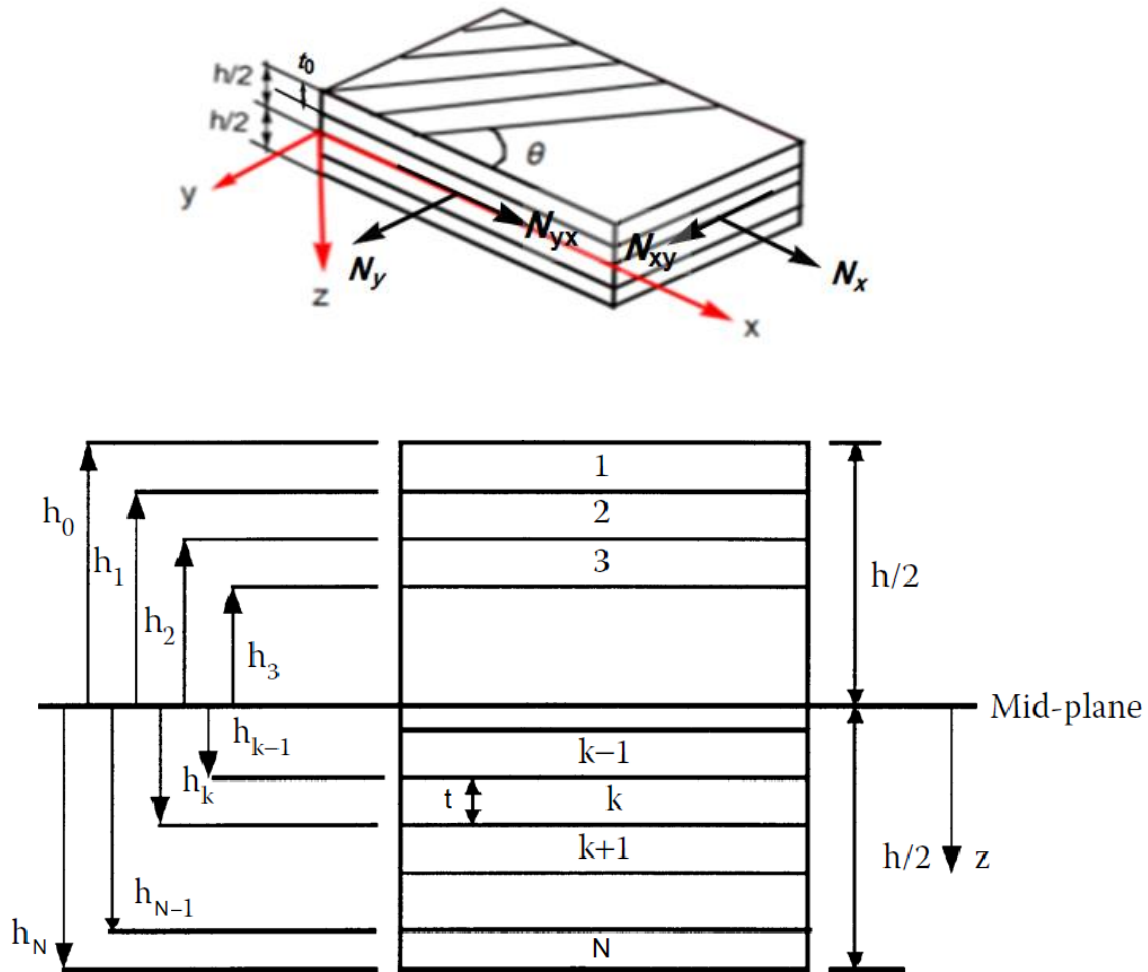


Figure 1.2: Laminated composite plate subjected to in-plane loading and Coordinate locations of plies.

The ultimate goal is to determine the optimum fiber orientation angle of each ply leading to the maximum strength of the whole laminate based on the first-ply failure theory. No constraint is imposed on the fiber orientations angles of the plies which means that all possible angles in the continuous interval $\{-90^\circ, 90^\circ\}$ are allowed. The two failure criteria of Tsai-Hill and Tsai-Wu are considered as design constraints and the optimization is performed for three composite systems: glass/epoxy, carbon/epoxy and Kevlar/epoxy.

1.3 Significance of the Research

Fiber reinforced polymeric composites are used in almost every type of advanced engineering structure, with applications ranging from space and aircraft through boats and automobiles to oil/gas pipelines and highway bridges. It has been noted that there is a growing interest by the local industry in these materials. A good example is the recent research and development activities conducted on the subject by Saudi Aramco who has signed with Baker Hughes a memorandum of understanding to create a new joint venture facility to develop non-metallic materials based on carbon and glass fibers reinforced plastics. Finding the optimal design of these material is considered a big advantage due to their complex analysis and high material/manufacturing cost. Therefore, the outcome of this study is expected to provide a valuable design tool of benefit to the relevant design industry and to the design engineers in this field.

1.4 Objectives

The main objective of this study is to come up with an optimum design of the lay-up of a fiber-reinforced laminated plate subjected to in-plane loading. The specific objectives are:

- 1- To formulate the optimization problem in terms of the objective function to be minimized/maximized, the design variables and parameters and the design constraints.
- 2- To develop a Mathematica code capable of performing the analysis of a symmetrically stacked laminate made of different fiber/epoxy materials and subjected to an in-plane loading.
- 3- To utilize the code in performing the global optimization leading for the optimum fiber orientation angle in each lamina, taking into account both failure criteria of Tsai-Hill and Tsai Wu.
- 4- To run a parametric study and generate the data required for driving closed-form solutions for the optimum fiber orientation angles for different materials and loading ratios.
- 5- To verify the accuracy of the developed closed-form solutions by comparison with the available results from the literature.

1.5 Thesis Outline

The introductory chapter of the report begins with a general background about the laminated composite materials and their applications. Then, it provides a description for the problem to be solved with its parameters, followed by a part that shows the significance and the advantages of this study.

The second chapter provides a detailed literature review for the topic. These studies are categorized based on the optimization objectives, the failure criteria used and the optimization techniques.

Chapter three contains the detailed formulation of the problem and materials of the laminated plates that has been used through the research. This consists first of the governing equations for a unidirectional lamina, followed by the formulation of laminates. Then, a section that introduce the failure criteria and their equations. The last part of this chapter includes all the mechanical and strength properties of the materials.

The followed four chapters are the main sections of this research. Chapter four provides the general guidelines for finding the optimum lay-up of the laminated plate under in-plane loading. It contains the specific formulation for the in-plane loading and the methodology of optimization that include the optimization statement and computer implementation. chapter five then provides a closed-form expressions and figures for the optimum lay-up of a plate subjected to in-plane normal forces with a detailed discussions on the results. Chapter six expand the results found in the previous chapter so it can consider shear loading too. Chapter seven provide a numerical verification for the founded results in the previous chapters through comparing them with the literature review.

The final chapter provides concluding remarks and recommended topics that open up a horizon for research in the future.

LITERATURE REVIEW

2.1 General

Over the recent years, many researchers have studied maximizing the strength of laminated plates by adopting the first ply-failure approach to find the optimum design. Fibers are well known to play the main role in the load-bearing capacity of laminated plates. As a result, choosing a proper stacking sequence by adjusting the fiber orientation angles is essential to higher strength, structural stability, and improving the various mechanical properties. For a plate subjected to a combined stresses, Tsai-Hill and Tsai-Wu are considered the most common failure criteria used by researchers for predicting failure [7].

2.2 Optimization of Plates Under In-plane Loading

Optimization of laminated composite plates under different loading and boundary conditions has drawn the interest of research community over the last decades. The usual optimization object is to design layer thickness, fiber orientations, and stacking sequence which will provide minimum weight, maximum stiffness, or strength [7,8,9,10,11]. Among those works, optimization of laminated plate under state of in-plane loadings by taking strength as an objective function or constraints has been studied intensively. For instance, Schmit and Farshi [12,13] presented an optimization for the minimum weight of a symmetric laminates subjected to in-plane loading with a stiffness and strength constraints. They used method of inscribed hyperspheres by considering the layer thicknesses at specified orientation ($0, \pm 45, 90$) and their results showed improvement in the strength of the laminate. Park [14,15] utilize two failure criteria (strain envelope and von mises) as an

objective function to design simple symmetric laminates by taking ply orientation angle as the design variable. He concluded with the advantage of using particular class of laminates (e.g., $[\pm\theta]_s$, $[\theta/0/-\theta]_s$) to obtain the most strength laminate under in-plane loading. Fukunag and Vanderplaa [16] used a mathematical programming method to do an effective strength optimization for a symmetric laminate under in-plane loading. They consider the total thickness of laminate as the objective function, Tsai-Wu as constrained and layer thicknesses and orientation angles as the design variables. Kim et al. [17,18] have used Tsai-Wu failure criterion as an optimization objective to obtain the stacking sequence and fiber orientations of a symmetrically laminated plate and cylindrical composite shell under various in-plane loading conditions. The studies are based on classical laminate theory (CLT) and used state space as a method to do the optimization. Song et al. [19] implement a finite element method based on shear deformation theory to design of symmetric laminates for maximum strength. They took Tsai-Hill theory as the objective function and ply orientation angles as the design variables to find the optimum sequence for different in-plane and transverse loadings. Their results show that it is valuable to use Tsai-Hill failure criterion to design composite laminates. Groenwold and Haftka [20] compare between taking failure index or safety factor as the objective function for both Tsai-Wu and Tsai-Hill criteria. They concluded that to find the optimum laminate design it is recommended to directly start by selecting the safety factor the objective function. Callahan and Weeks [21] used genetic algorithms (GAs) to design laminated plates that provide the minimum weight with the maximum strength and/or stiffness. They took stacking sequence and lamina orientations for common angles as the design variables. Akbulut and Sonmez [3,22] consider fiber orientation angles and lamina thickness as design variables to

minimize thickness (or weight) of laminated composite plates. The formulation is based on CLT with strength constraints based on Tsai–Wu and the maximum stress failure criteria. In the first study, they used Direct search simulated annealing (DSA) as an optimization tool to design laminated composite plates under in-plane loading. Whereas in the second one they modify the tool so it can perform design to both in-plane and out-of-plane loading. Nicholas et al. [23] utilize GAs to maximize the safety factor of laminated composite plates by considering both Tsai-Wu and the maximum stress failure criteria. They employed CLT for formulation and took in-plane loading only. Satheesh et al. [24,25] performed a study to find the minimum weight design of composite laminates by implementing genetic algorithms. They studied the effect of different failure criteria by proposing conservative failure envelope for both biaxial and combined biaxial and shear loading conditions. Khandan et al. [26] minimized the thickness of laminated composite plates using simulated annealing (SA) as an optimization tool. They applied Tsai-Wu failure and maximum stress criteria as a constrained under various loading cases. Sebaey et al. [27] viewed a part on using an Ant Colony Optimization algorithm to minimize the failure index of laminated plates subjected to biaxial tension loading. They used maximum strain and Tsai-Wu failure criteria.

Among the optimization methods, genetic algorithms (GAs) are considered the most commonly used method to design composite laminates for strength such as in [21,23,24,28,29]. Other methods have been also used such as simulated annealing (SA), for example, in [3,22,26]. Regarding the failure criteria, Tsai-Wu is frequently used and seems to be very popular such as in [20,17,18,30]. Tsai-Hill criterion is also commonly used as in [19,28]. However, some studies showed that Tsai-Wu could lead to a false

optimum design under certain loading cases. So, to avoid this condition many studies have used Tsai-Wu with other failure criteria such as the maximum stress criterion as in [3,22,23,24,26,27] or Tsai-Hill as in [20].

Laminated composite pressure vessels is considered a main application for laminated composite materials under in-plane loading. It is known that using composite materials for designing a cylindrical pressure vessel is favorable because the loads are mainly dominated by tension in the fibers (the axial loads (σ_l) and hoop loads (σ_h)) [31]. A lot of research has been done on the analysis and optimization of composite pressure vessels for different loading scenarios. The usual optimization object is to design layer thickness, fiber orientations, and stacking sequence which will provide minimum weight, maximum stiffness, or strength. For instance, Kim et al.'s [18] did a study based on classical laminate theory (CLT) to find the optimum design of cylindrical composite shell under different values of axial force and internal pressure. They consider the sequence to be symmetric and took Tsai-Wu failure criterion as an objective function to find the optimum stacking sequence. Fukunaga and Chou [32] used membrane theory to find the optimum design of symmetric-balanced laminated composite cylindrical pressure made of graphite/epoxy material. They consider both a stiffness and strength (Tsai-Wu failure criterion) constraints to perform the optimization. Adali et al. [33] conducted a study to optimize symmetrically laminated cylindrical pressure vessels under axial loading and torque focusing strength constraint. The study goal was to find the optimum fibers orientations and thicknesses for pressure vessels through implementing Tsai-Wu failure criterion. Parnas et al.'s [34] also investigated the optimum angle of a composite pressure vessel subjected to internal loading. They used an analytical procedure based CLT to develop the design which was

determined to be in a range of $[52.1^\circ, 54.2^\circ]$. Walker and Smith [35] also performed an approach to determine the optimal angle and the maximum load-carrying capacity of a symmetrically laminated cylindrical shell under a combination of in-plane forces and torque. They consider fiber orientations as they design variable and find the optimum design by minimizing failure index based on Tsai-Wu failure criterion. Using CLT theory, Mian et al. [31] presented an optimization for a pressure vessel made of three different composite materials (S-glass/epoxy, Kevlar/epoxy, and Carbon/epoxy) based on maximum stress and Tsai-Wu and failure criteria. They concluded that the angle-ply $[\pm\theta]_{ns}$ was the optimum lay-up and used a finite element code using ANSYS and MATLAB code to check the effectiveness of their results.

None of the presented works were able to construct a closed formula to find the optimum orientation angles under any certain of in-plane loading condition. In this thesis, the optimum strength design of symmetrically laminated composite plate under in-plane tension (N_x and N_y) and shear (N_{xy}) loading is proposed utilizing the classical lamination theory (CLT) along with Tsai-Wu and Tsai-Hill failure criteria. The optimum design, for three different materials (glass/epoxy, carbon/epoxy and Kevlar/epoxy), have been obtained by taking the strength ratio as the objective function to be maximized and the orientation angles as the design variables. The strength ratio is based on the two popular failure criteria namely, Tsai-Wu and Tsai-Hill failure criteria. To approach this, the optimization problem is first formulated and implemented a Mathematica code capable of finding the global optimum fiber orientation angles through testing all possible lay-up combinations. The results of the optimization analysis are presented in terms of graphical and closed-form formulae suitable for design purposes. Finally, numerical examples are

presented to confirm the accuracy of the design formulae for different materials and in-plane loadings.

PROBLEM FORMULATION

3.1 Macro-Mechanical Behavior of a Lamina

A lamina, or ply, is a thin plane layer of a composite material (unidirectional fibers within a matrix) that has a general thickness in the order of 0.125 mm. It is important to understand the behavior of such lamina under different loads as it is considered the building block of the laminate (a stacking of laminas in the direction of their thickness). Looking at its properties, A unidirectional lamina is considered to fall under orthotropic material (material with three mutually directional properties). Now, referring to [1,36] let us consider a single "lamina" of continuous fiber arranged in a rectangular array where the principal material axes 1 is coinciding with the fibers direction, 2 is normal to the fibers direction and 3 is completing the right-handed-rule system of coordinates (Figure 3.1).

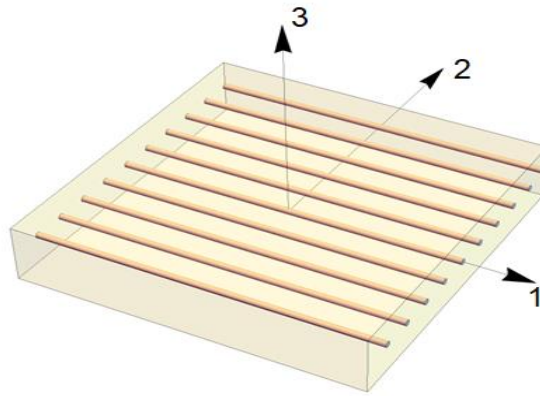


Figure 3.1: Typical lamina and the principal coordinate axes.

one can assume the lamina to be under plane stress conditions as it is thin and does not carry any out-of-plane loads ($\sigma_3 = \tau_{13} = \tau_{23} = 0$). Therefore, the 2-D strain-stress relations are obtained by omitting the out-of-plane from the 3-D relations of the orthotropic material. As a result, the strain-stress relations can be written as:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \quad (3.1)$$

Where S_{ij} represent the elements of the compliance matrix that involves 4 independent elastic constants. These constants can be given in term of engineering constants (E_1, E_2, ν_{12} and G_{12}) as:

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2}, \text{ and}$$

$$S_{66} = \frac{1}{G_{12}}.$$

Now, for the purpose of finding the stress-strain relations using the stiffness matrix, one can easily take the inverse of the compliance matrix, resulting in:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} \quad (3.2)$$

where Q_{ij} are the stiffness coefficients which can be given in term of engineering constants as:

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}},$$

$$Q_{12} = \frac{E_1 \nu_{21}}{1 - \nu_{12} \nu_{21}},$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12} \nu_{21}}, \text{ and}$$

$$Q_{66} = G_{12}.$$

Since a laminate consists of a number of laminae each with different fiber direction. Thus, it is necessary to develop the stress–strain relationship for an angle lamina. Figure 3.2 shows an angle lamina with the local axis (1–2 coordinate system), the global axes (x–y coordinate system) and the angle measured positive counterclockwise between the two axes (θ).

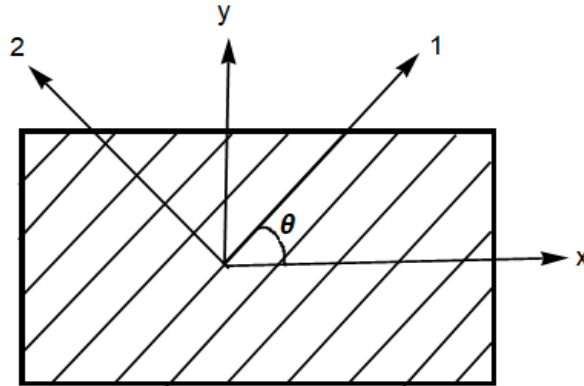


Figure 3.2: Local and global axes of an angle lamina.

Now, we would like to derive the stiffness matrices relating stress to strain with respect to the x-y coordinate system. First, using the transformation matrix [T] the stress components between the global and local axis can be given by the following relations:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (3.3)$$

Where:

$$[T] = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \quad (3.4)$$

$$c = \text{Cos}(\theta) \text{ and } s = \text{Sin}(\theta)$$

While the global and local strain components referred by the following transformation relations:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = [T] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy}/2 \end{bmatrix} \quad (3.5)$$

Which can be rewritten as:

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = [R]^{-1}[T][R] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (3.6)$$

Where:

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (3.7)$$

Now, by taken the inverse of the $[T]$ matrix and using the stress–strain Eq. (3.2) in the local axes, Eq. (3.3) can be written as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1}[Q] \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} \quad (3.8)$$

Then, Eq. (3.6) can be substituted in Eq. (3.8) which results in:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1}[Q][R]^{-1}[T][R] \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (3.9)$$

The multiplication of these 5 matrices gives:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (3.10)$$

where $\bar{Q}_{ij}$ are the elements of the transformed reduced stiffness matrix, which can be given by:

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})c^2s^2$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4)$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})cs^3$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})c^2s^2 + Q_{66}(c^4 + s^4)$$

From the previous relations, it is obvious that for the unidirectional lamina, the normal stresses applied in the 1-2 direction do not result in any shearing strains in the 1-2 plane because $Q_{16} = Q_{26} = 0$. Also, the shearing stresses applied in the 1-2 direction do not result in any normal strains in the 1-2 plane. However, for the angle lamina there is a coupling between strains and stresses normal and shearing components. Therefore,

applying a normal stress only to an angle lamina will result in shear strains and vice versa. This relation between stress-strain in the global axes is called a generally orthotropic lamina.

3.2 Macro-Mechanical Behavior of a Laminate

A laminate, as mentioned previously, is constructed by stacking a group of laminas (plies) over each other's. Each of these plies can have its own material, thickness, location in the laminate and orientation angle with a reference axis (Figure 3.3). In this thesis all plies in a laminate will be made from the same material and will have the same thickness. For the purpose of identifying laminate, each ply is represented by its orientation angle and separated from other plies by a slash sign. Also, a special notation is used for symmetric laminates (The plies above and below the midplane have the same material, orientation and thickness), For example $[0/-45/90]_s$ represent a laminate with the following sequence $[0/-45/90/90/-45/0]$ while $[\pm 45]_s$ is equivalent to $[45/-45/-45/45]$.

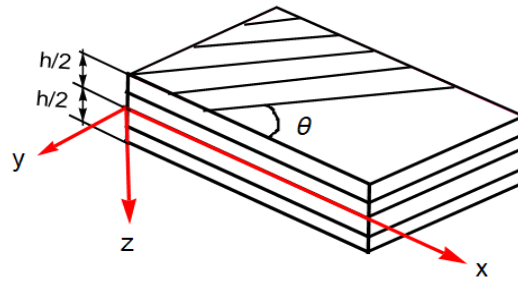


Figure 3.3: Schematic of a laminate

Based on the Classical Lamination Theory (CLT) by referring [1,36], the analysis of laminated composite plates with specific assumptions can be correctly found. These assumptions include that each lamina should be elastic, homogenous, orthotropic and no slip happens between lamina interfaces. Moreover, the laminate should be thin (small

thickness compared to the others lateral dimensions), loaded only in its plane loading (plane stress condition) and displacements are small ($u, v, w \ll h$) and continuous throughout the laminate. Figure 3.4 shows a section of the laminate in the Cartesian $x - y - z$ coordinate system normal to the y -axis before and after deformation with z -axis being downward and start from the midplane.

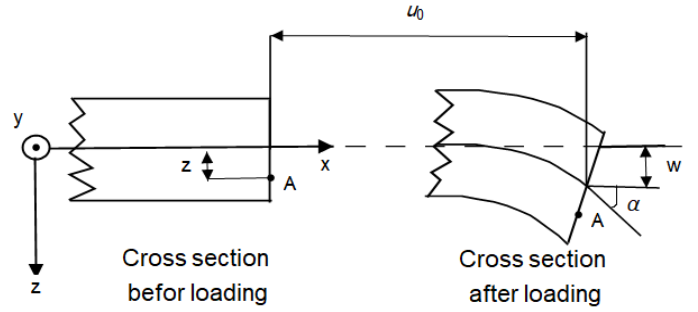


Figure 3.4: Laminate section before and after deformation.

Assume u_0 and v_0 to be the midplane displacements in the x and y directions, respectively, and w to be the out-of-plane displacement in the z direction (CLT assumed all point through thickness to have the same deflection). Thus, the displacements u and v at any point are

$$u = u_0 - z \frac{\partial w}{\partial x} \quad (3.11)$$

$$v = v_0 - z \frac{\partial w}{\partial y} \quad (3.12)$$

Where z is the coordinate of a general point through the thickness of the cross section. Now, using the definitions of the three strains for small displacements in the $x - y$ plane yield

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \quad (3.13)$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2} \quad (3.14)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \quad (3.15)$$

Thus, the strains varying linearly through the thickness of the plate and depend on the midspan strains $(\epsilon_x^0, \epsilon_y^0, \gamma_{xy}^0)$ and curvatures $(\kappa_x, \kappa_y, \kappa_{xy})$. They may be express as:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.16)$$

Where:

$$\begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{bmatrix} \quad (3.17)$$

And

$$\begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (3.18)$$

Now, if the strains are known at any point through thickness, the stress-strain relations Eq. (3.10) for an individual layer k (located between $z = h_{k-1}$ and $z = h_k$) referred to the global laminate results in:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}_k \quad (3.19)$$

Where $\bar{Q}_{ij}$ are the off axis reduced stiffnesses. After the combination of Eqs. (3.16) and (3.19),

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + z \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.20)$$

From Eq. (3.20), it is obvious that for each lamina the stresses will vary linearly through the thickness. However, as the reduced stiffness matrix $[\bar{Q}]$ (depends on the material and angle orientation of the lamina θ_k) changes, the stresses components (unlike the strains) may jump from lamina to lamina within the laminate (Figure 3.5).

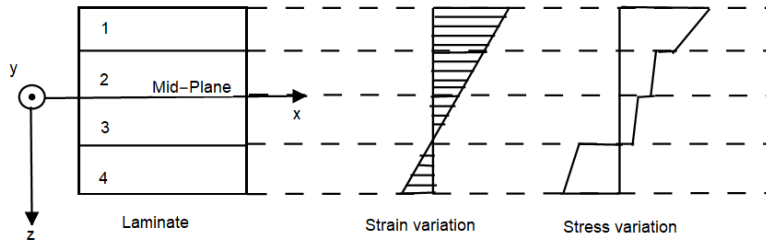


Figure 3.5: Strain and stress variation through the thickness of the laminate.

The only unknowns for finding the lamina strains and stresses are the midplane strains and plate curvatures. However, because of the stresses are not continuous through the laminate,

it is more convenient to deal with resultant forces and moments. This can be done by integrating the stresses in each lamina through the thickness of the laminate. Now, by considering a laminate with a thickness h and made of N plies each with a fixed thickness $t = \frac{h}{N}$ shown in Figure 3.6.

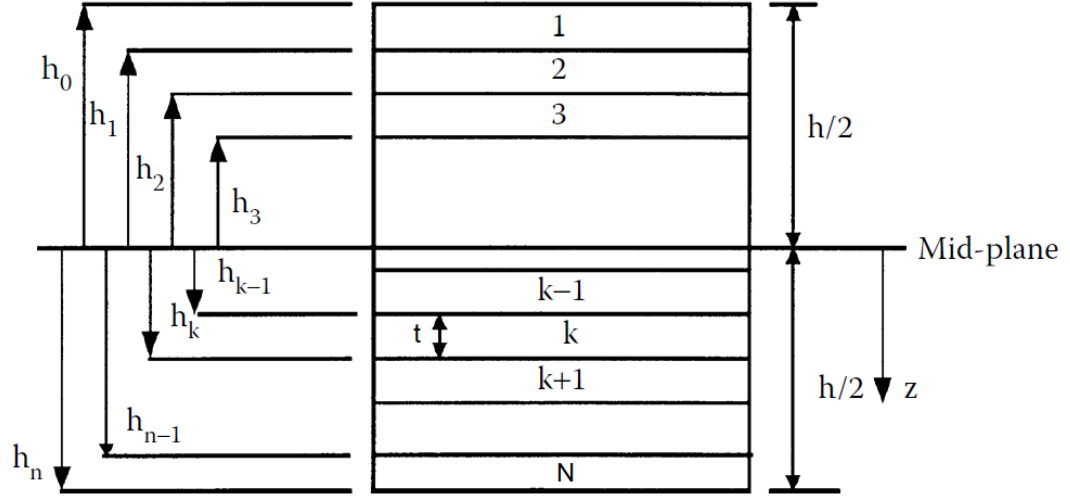


Figure 3.6: Coordinate locations of plies in a laminate [36].

Thus, the resultant forces and bending moments per unit width can be given as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^N \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz \quad (3.21)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^N \int_{h_{k-1}}^{h_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k z dz \quad (3.22)$$

Substituting Eq. (3.20) into Eqs. (3.21) and (3.22) with taking into account that the reduced stiffness matrix $[\bar{Q}]$ for each ply is constant as well as that midplane strains and curvatures are independent of the z -coordinate. The result equations can be written as

$$\begin{aligned}
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} &= \left\{ \sum_{k=1}^N \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{z_{k-1}}^{z_k} dz \right\} \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \\
&+ \left\{ \sum_{k=1}^N \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z dz \right\} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} &= \left\{ \sum_{k=1}^N \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z dz \right\} \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \\
&+ \left\{ \sum_{k=1}^N \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \int_{z_{k-1}}^{z_k} z^2 dz \right\} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}
\end{aligned} \tag{3.24}$$

After doing the integrations, the expressions above can be combined into one general expression that relates in plane force (N_x, N_y, N_{xy}) and moments (M_x, M_y, M_{xy}) per unit width to the midspan strains ($\epsilon_x^0, \epsilon_y^0, \gamma_{xy}^0$) and curvatures ($\kappa_x, \kappa_y, \kappa_{xy}$):

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{3.25}$$

Where this whole 6x6 matrix is called ABD matrix, A_{ij} are the extensional stiffnesses that relate in-plane loads to in-plane strains, D_{ij} are the bending stiffnesses that relate moments to curvatures and B_{ij} are the bending-extensional coupling stiffnesses that relate in - plane

loads to curvatures and moments to in - plane strains. These stiffnesses components are given in a mathematical way as:

$$A_{ij} = \sum_{k=1}^N \bar{Q}_{ij}(z_k - z_{k-1}) \quad (3.26)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^N \bar{Q}_{ij}(z_k^2 - z_{k-1}^2) \quad (3.27)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^N \bar{Q}_{ij}(z_k^3 - z_{k-1}^3) \quad (3.28)$$

Where i and j range over 1, 2, 6. In special cases some of these stiffnesses components are becoming zero. For example, if the laminates are symmetric (which is our concern in this thesis) all B_{ij} terms become zero (no bending-extensional coupling) and the whole expression divided into the following expressions:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \quad (3.29)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (3.30)$$

Thus, if a symmetric laminate is subjected to forces only, the midplane curvatures will be zero. Similarly, if it is subjected to moments only, the midplane strains will be zero. This uncoupling between extension and bending leads to a more stable plate and makes the process of analyzing and designing such laminates simpler. So, it is usually favorable to choose symmetric laminate to find the optimum design over other laminates.

3.3 Failure Criteria

3.3.1 General

Failure criteria is important to insure a successful design with efficient use of materials. Over the last years, various theories were proposed to study the failure behavior of composite material [1,36]. The concept of failure is based on the fact that the strength of a laminate is related to the strength of each lamina within it. This means that a shift from the global axes to the local axes of each lamina is required in order to study the failure of laminate under loading. In case a unidirectional lamina subjected to in-plane loading, there are total of five strength parameters. Four of them are for the tension and compression in both the longitudinal and transversal direction and the fifth is the shear strength parameter. These parameters can give in a symbolic way as X_t , X_c , Y_t , Y_c and S . Where the symbols “X” and “Y” represent the ultimate longitudinal strengths along the fiber direction (axis 1) and normal to it (axis 2), respectively; the subscripts “t” and “c” symbolize the tensile and compressive; S is the ultimate in-plane shear strength (Figure 3.7).

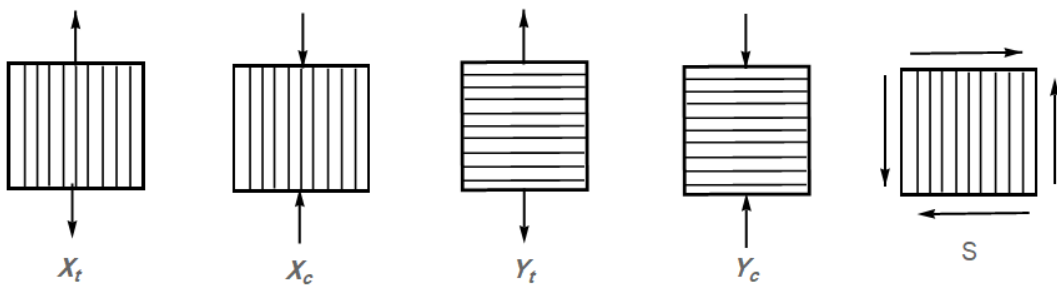


Figure 3.7: The lamina five basic strength parameters.

As there are various type of failure theories, in this thesis a type called interactive failure theories will be used. This type of failure criteria allow interaction among the stresses and

give a prediction of strength in a single expression. They also consider giving good results compare to the experimental tests.

3.3.2 Tsai-Hill Failure criterion

The first interactive failure theory that will be considered is called Tsai-Hill [1,36]. This theory is an extension of the distortion energy failure theory (von Mises yield criterion) for isotropic materials as applied to anisotropic material theory. For an isotropic material with a two-dimensional stress condition referred to the principal stress direction (axis 1 and 2) This can be presented as:

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_{yp}^2 \quad (3.31)$$

Where σ_{yp} is the yield stress. Hill did a modification on the formula to take the anisotropy of laminated composite material and proposed the following expression:

$$A\sigma_1^2 + B\sigma_2^2 + C\sigma_1\sigma_2 + D\tau_{12}^2 = 1 \quad (3.32)$$

Where these constants A, B, C and D are the material parameters. Azzi and Tsai used this criterion for orthotropic material through giving each previously mentioned constants values based on different scenarios such as $\sigma_1 = X_t, \sigma_2 = \tau_{12} = 0$ and $\sigma_2 = Y_t, \sigma_1 = \tau_{12} = 0$ concluded with the following values:

$$A = \frac{1}{X_t^2}, B = \frac{1}{Y_t^2}, C = -\frac{1}{X_t X_t}, D = \frac{1}{S^2} \quad (3.33)$$

Substituting these values in Eq. (3.32) result in Tsai Hill criterion:

$$\left(\frac{\sigma_1}{X_t}\right)^2 - \left(\frac{\sigma_1\sigma_2}{X_tX_t}\right) + \left(\frac{\sigma_2}{Y_t}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 = 1 \quad (3.34)$$

The disadvantage of this failure criteria is that it does not distinguish tensile strengths from compressive strengths. To withstand this problem a second failure criteria will be used which is called Tsai-Wu.

3.3.3 Tsai-Wu Failure criterion

Tsai-Wu theory is more general and capable of distinguish between tensile and compressive strengths [1,36]. It is based on the theory of total strain energy. Tsai-Wu considered a lamina in plane stress condition to be failed if the following expression is violated.

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 < 1 \quad (3.35)$$

The strength coefficients $F_{11}, F_{22}, F_{66}, F_{12}, F_1$ and F_2 has been found through taking different scenarios of applied stresses in each direction. They expressed in term of the strength properties of the material as follow:

$$\begin{aligned} F_1 &= \frac{1}{X_t} - \frac{1}{X_c}, & F_2 &= \frac{1}{Y_t} - \frac{1}{Y_c}, & F_{11} &= \frac{1}{X_tX_c} \\ F_{22} &= \frac{1}{Y_tY_c}, & F_{12} &= -\frac{1}{2}\sqrt{F_{11}F_{22}}, & F_{66} &= \frac{1}{S^2} \end{aligned} \quad (3.36)$$

Where for both failure criteria, the principal stresses ($\sigma_1^k, \sigma_2^k, \tau_{12}^k$) can be found using a tensor transformation of the global stresses ($\sigma_x^k, \sigma_y^k, \tau_{xy}^k$).

3.4 Materials

A total of three materials will be considered in this thesis which are glass/epoxy, carbon/epoxy and Kevlar/epoxy. These materials are chosen specially because they are

heavily used in industry. Besides that, it is also possible to study the changes in the optimum design by changing the mechanical properties and strengths. Table 3.1 below shows the Material and Strength Properties for each of the materials [36].

Table 3.1: Typical Mechanical Properties of a Unidirectional Lamina

Material Properties	Symbol	Unit	Graphite/ epoxy (T300/5208)	Glass/epoxy (Gl/Ep)	Aramid/epoxy (Ar/Ep)
Longitudinal elastic modulus	E_1	GPa	181	38.6	76
Transverse elastic modulus	E_2	GPa	10.3	8.27	5.5
Shear modulus	G_{12}	GPa	7.17	4.14	2.3
Major Poisson's ratio	ν_{12}	—	0.28	0.26	0.34
Ultimate longitudinal tensile strength	X_t	MPa	1500	1062	1400
Ultimate longitudinal compressive strength	X_c	MPa	1500	610	235
Ultimate transverse tensile strength	Y_t	MPa	40	31	12
Ultimate transverse compressive strength	Y_c	MPa	246	118	53
Ultimate in-plane shear strength	S	MPa	68	72	34

LAY-UP OPTIMIZATION OF PLATE UNDER IN-PLANE LOADING

4.1 General

Based on the Classical Lamination Theory (CLT), the analysis of laminated composite plates with a small deformations and small thickness compared to the others lateral dimensions can be correctly found. The theory assumed that the laminate is thin and the bonds between laminae are perfect and infinitesimally thin. So, in this case it is possible to assume each ply to be under plane stress condition and consider out of plane stresses components $(\sigma_z, \tau_{xz}, \tau_{yz})$ as zero. Because, In the present study the laminate is considered to be symmetric and subjected to in-plane loading only, curvature terms become zero. As a result, the strains in each ply $(\epsilon_x, \epsilon_y, \gamma_{xy})$ will equal the midspan strains $(\epsilon_x^0, \epsilon_y^0, \gamma_{xy}^0)$ and the laminate mechanical response become independent of the lay-up order.

4.2 Reduction of Formulation for a Symmetric Laminate

For a symmetric laminate subjected to in-plane loading, the order of lay-up becomes unimportant as the laminate curvatures are zeros. Therefore, the integration procedure to get the resultant forces per unit width can be replaced by summation over half of the laminate, i.e.:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^N \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz = 2t \sum_{k=1}^m \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k \quad (4.1)$$

where t is the thickness of an individual ply, and $m = N/2$ represents the number of plies in one in a half thickness of the symmetric laminate. Now, Eq. (3.29) which gives the

relations between the in-plane forces and the strains is reduced for a symmetrically laminated plate to:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (4.2)$$

Where A_{ij} are the extensional stiffnesses that relate in-plane loads to in-plane strains. These stiffnesses components are given in a mathematical way for symmetric laminate as under in-plane loading as:

$$A_{ij} = 2 t \sum_{k=1}^m \bar{Q}_{ij} \quad (4.3)$$

Now, if we have the loading (N_x, N_y, N_{xy}) , one may get the global strain components $(\epsilon_x, \epsilon_y, \gamma_{xy})$ using Eq. (4.2), and then finding the global stresses components $(\sigma_x^k, \sigma_y^k, \tau_{xy}^k)$ using Eq. (3.10). Finally, the local stress components in each ply are obtained from:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}_k = [T]_k \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k \quad (4.4)$$

4.3 Optimization Statement

If the problem is approached as a classical optimization problem with an objective function, design variables and parameters and constraints, it can be described as detailed below:

- 1- Objective function: the total thickness of the laminate, $h = 2 m t$, to be minimized where m the number of plies in one portion of symmetric ($N/2$) and t is the thickness of each ply.
- 2- Design variables: the fiber orientation angle in each ply, $\theta_p, p = 1, m$. θ_p can take any value in the continuous interval $\{-90^\circ, 90^\circ\}$.
- 3- Design parameters: The elastic and strength properties of a unidirectional lamina as given in Table 3.1.
- 4- Design constraint: In addition to the constraint on θ_p interval, the local stresses (normal and shear stresses with respect to the fiber direction) must not violate the failure criteria. This constraint is given below for the two most popular composite failure theories of Tsai-Hill and Tsai-Wu (Eq. (3.34 and 3.35)):

$$\left(\frac{\sigma_1}{X_t}\right)^2 - \left(\frac{\sigma_1\sigma_2}{X_tX_t}\right) + \left(\frac{\sigma_2}{Y_t}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 < 1. \quad (Tsai - Hill) \quad (4.5)$$

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + 2F_{12}\sigma_1\sigma_2 < 1. \quad (Tsai - Wu) \quad (4.6)$$

However, a more efficient approach to the optimization problem can be derived through implementing the concept of strength ratio (SR) which is a measure of how much the load can be increased if the lamina is safe or how much the load should be decreased if the lamina has failed. In order to obtain a safe structure, SR of each layer within the laminate should be greater than 1. The formula of SR depends on the adopted failure theory but can be generally represented by the following simple relation:

$$SR = \frac{\text{Maximum Load Which Can Cause Failure}}{\text{Load Applied}} \quad (4.7)$$

In the following, we will show how SR can be included in the design constraints of the two theories of Tsai-Hill and Tsai-Wu. Starting with Tsai-Hill criterion, one would still like to know by what strength ratio SR the stresses can be multiplied so as to cross the envelope, producing failure. Simply multiplying all the stresses from the Hill failure envelope by the value SR yields the following equation:

$$\left(\left(\frac{\sigma_1}{X_t} \right)^2 - \left(\frac{\sigma_1 \sigma_2}{X_t X_t} \right) + \left(\frac{\sigma_2}{Y_t} \right)^2 + \left(\frac{\tau_{12}}{S} \right)^2 \right) SR^2 < 1 \quad (4.8)$$

Solving for SR , we get:

$$SR_{TH} = \frac{1}{\sqrt{\left(\frac{\sigma_1}{X_t} \right)^2 - \left(\frac{\sigma_1 \sigma_2}{X_t X_t} \right) + \left(\frac{\sigma_2}{Y_t} \right)^2 + \left(\frac{\tau_{12}}{S} \right)^2}} \quad (4.9)$$

Doing the same thing for the constraint of Tsai-Wu criterion, we get:

$$(F_1 \sigma_1 + F_2 \sigma_2) SR + (F_{11} \sigma_1^2 + F_{22} \sigma_2^2 + F_{66} \tau_{12}^2 + 2F_{12} \sigma_1 \sigma_2) SR^2 < 1 \quad (4.10)$$

Solving for SR , we get:

$$SR_{TW} = \frac{-b + \sqrt{b^2 + 4a}}{2a} \quad (4.11)$$

Where:

$$a = F_{11} \sigma_1^2 + F_{22} \sigma_2^2 + F_{66} \tau_{12}^2 + 2F_{12} \sigma_1 \sigma_2 \quad (4.12)$$

$$b = F_1 \sigma_1 + F_2 \sigma_2 \quad (4.13)$$

Now, the objective function becomes SR to be maximized and the only constraint is the angle interval. The question is: does this lead to a minimization of the laminate thickness? The answer is yes, as explained below.

Let us perform the optimization of the laminate lay-up based on a laminate thickness of unity and subjected to in-plane loading $NL = N_x \{1, \beta, \gamma\}$, where $\beta = N_y/N_x$ and $\gamma = N_{xy}/N_x$. Also Setting $N_x = 1$ and denoting all variables by star, the in-plane forces to the mid-plane strains become:

$$\{NL^*\} = [A^*][\epsilon^*] \quad (4.14)$$

Where A^* contains $\{A_{11}^* \dots, A_{33}^*\}$, and ϵ^* contains $\{\epsilon_x^*, \epsilon_y^*, \gamma_{xy}^*\}$ and $NL^* = \{1, \beta, \gamma\}$.

Then Eq. (4.14) becomes:

$$[\epsilon^*] = [A^*]^{-1}[NL^*] \quad (4.15)$$

The global stresses in the k^{th} ply are given by

$$[\sigma_g^*]_k = [\bar{Q}]_k [A^*]^{-1} [NL^*] \quad (4.16)$$

The local stresses in the k^{th} ply are given by

$$[\sigma_l^*]_k = [T]_k [\bar{Q}]_k [A^*]^{-1} [NL^*] \quad (4.17)$$

Using (4.17) in (4.8), we get for the Tsai-Hill criterion:

$$\left(\left(\frac{\sigma_1^*}{X_t} \right)^2 - \left(\frac{\sigma_1^* \sigma_2^*}{X_t X_t} \right) + \left(\frac{\sigma_2^*}{Y_t} \right)^2 + \left(\frac{\tau_{12}^*}{S} \right)^2 \right) SR^{*2} = 1 \quad (4.18)$$

If the thickness h and load N_x is retained the equation becomes:

$$\left(\left(\frac{\sigma_1^*}{X_t} \right)^2 - \left(\frac{\sigma_1^* \sigma_2^*}{X_t X_t} \right) + \left(\frac{\sigma_2^*}{Y_t} \right)^2 + \left(\frac{\tau_{12}^*}{S} \right)^2 \right) \left(\frac{N_x SR}{h} \right)^2 = 1 \quad (4.19)$$

Therefore, we get:

$$SR^* = \frac{N_x SR}{h} \quad (4.20)$$

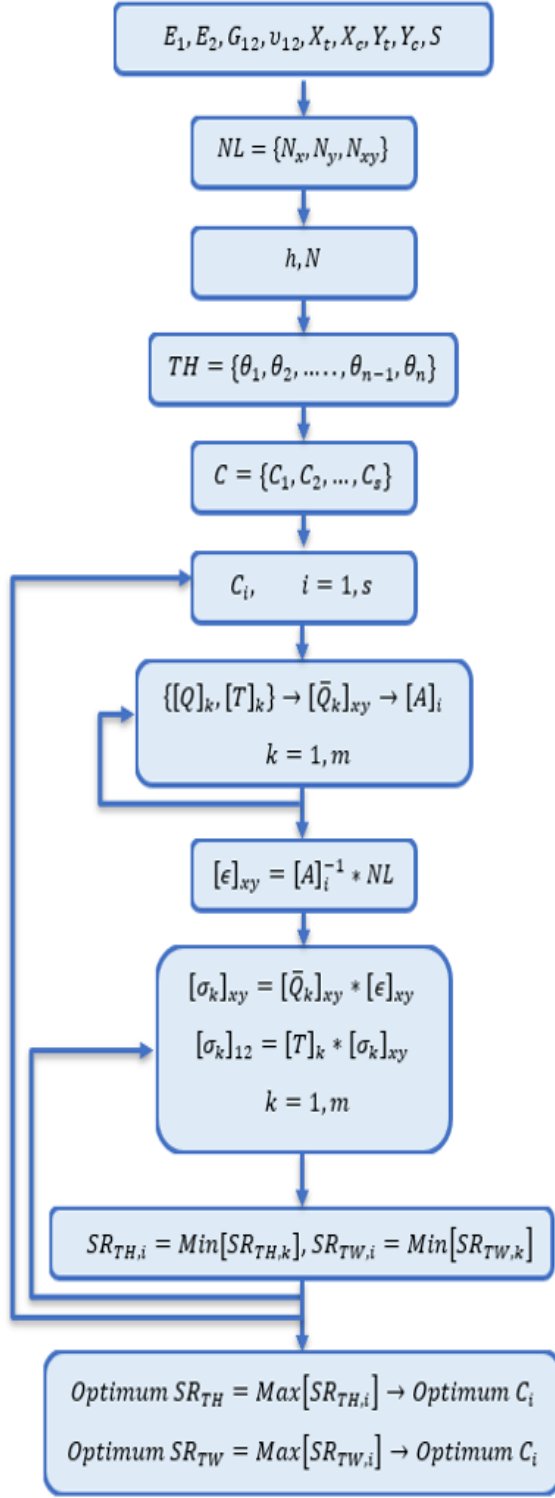
The minimum laminate thickness should be the one that corresponds to an SR value of 1, i.e.:

$$h_{min} = \frac{N_x}{SR^*} \quad (4.21)$$

For a laminate having a thickness = 1 and subjected to in-plane loading $N_x \{1, \beta, \gamma\}$, find the optimum fiber orientation angle in each ply, $-90^\circ < \theta_p < 90^\circ$ that maximizes the strength ratio SR^* . Once $\theta_p, p = 1, N$ are obtained, the number of plies N is the minimum integer greater $\geq h_{min}/t$, where t is the thickness of each ply.

4.4 Computer Implementation

In order to find the optimum fiber orientation angle in each ply as per the optimization statement stated above, one needs to run a huge number of loops depending on the number of angles selected in the interval $\{-90, 90\}$ and the number of laminas. As an example, if the angles are selected in an increment of 1° , i.e., 180 angles, then the number of possible stacking sequences = $180^{N/2}$, where N is the number of laminas. The algorithm of Mathematica code performing the search procedure is given by flow chart in Figure 4.1. This code has the ability to find the optimum fiber orientation angles by testing all the possible combinations for a specific number of piles (N).



Engineering constants and strengths of lamina

The internal in-plane loading NL

Laminate thickness and total number of plies

Choose all the possible orientation angles (n) ,

For example: $TH = \{-89, -88, \dots, 0, \dots, 89, 90\}$.

Then, use TH to Construct all the possible symmetric laminate combinations $s = (n)^{(N/2)}$.

Then, take one combination at a time.

Find the off axis reduced stiffnesses $[\bar{Q}]_{xy}$ for each lamina using $[Q]_k$ and $[T]_k$. Then, find the extensional stiffnesses matrix $[A]$.

Find the global strains $[\epsilon]_{xy}$ using $[A]_i$ and NL .

Find the global stresses $[\sigma_k]_{xy}$ for each lamina and use $[T]_k$ to find the local stresses

Find the failure SR for each combination by taking the minimum in each lamina $SR_{i,k}$ for both Tsai-Hill and Tsai-Wu criteria.

Find the optimum SR by taking the maximum SR among the combinations for both criteria and obtain the sequence that provide this SR.

Figure 4.1: Flow chart for finding the optimum sequence and strength ratio (SR).

The code implementing the above algorithm is given in Appendix A.

4.5 Optimization Results

The code developed in the previous section is run for many cases involving laminates with different number of plies and made of different materials by employing the two failure criteria. A representative sample of the results is given in Tables 4.1 and 4.2 which are enough to give a clue for the next task of deriving the closed-form expressions for the optimum lay-up.

Table 4.1: The optimum SR and Sequence of $T300/5208$ with variable load and $N = 4$ piles.

Material	$T300/5208$					
Laminate Thickness (h)	1 mm					
The internal loading (NL)	$\{1, 0, 0\} N/mm$		$\{1, 2, 0\} N/mm$		$\{1, 5, 2\} N/mm$	
Failure Criteria	Tsai-Hill	Tsai-Wu	Tsai-Hill	Tsai-Wu	Tsai-Hill	Tsai-Wu
The optimum SR	1500	1522.38	217.04	223.07	232.182	228.237
The optimum sequence	$[0,0]_s$	$[\pm 5.6]_s$	$[\pm 53.4]_s$	$[\pm 54.2]_s$	$[57.6, 77.4]_s$	$[56.1, 78.9]_s$

Table 4.2: The optimum SR and Sequence of $T300/5208$ with variable load and $N = 8$ piles.

Material	$T300/5208$					
Laminate Thickness (h)	1 mm					
The internal loading (NL)	$\{1, 0, 0\} N/mm$		$\{1, 2, 0\} N/mm$		$\{1, 5, 2\} N/mm$	
Failure Criteria	Tsai-Hill	Tsai-Wu	Tsai-Hill	Tsai-Wu	Tsai-Hill	Tsai-Wu
The optimum SR	1500	1522.38	217.04	223.07	232.182	228.237
The optimum sequence	$[0,0,0,0]_s$	$[\pm 5.6, \pm 5.6]_s$	$[\pm 53.4, \pm 53.4]_s$	$[\pm 54.2, \pm 54.2]_s$	$[57.6, 77.4, 57.6, 77.4]_s$	$[56.1, 78.9, 56.1, 78.9]_s$

following can be concluded from the results in the above two tables:

- For $\gamma = 0$, the optimum stacking sequence corresponds to a symmetric angle-ply laminate. The stacking sequence takes the form: $[(\pm\theta)_{m/2}]_s$, regardless of the loading ratio, the number of plies and the failure criterion.
- The angle-ply pattern is consistent but the value of θ changes depending on the value of β and the type of material and the failure criterion with the effect of β being more significant than the other two factors.
- For $\gamma \neq 0$, the optimum stacking sequence takes the form: $[(\theta_1/\theta_2)_{m/2}]_s$.
- The above observations can be utilized in reducing the huge number of angle combinations as required by the global optimization code and make it possible to reach closed-form expressions for the optimum fiber orientation angles.

CLOSED-FORM EXPRESSIONS OF THE OPTIMUM LAY-UP ANGLES FOR PLATE UNDER PURE IN- PLANE NORMAL FORCES

5.1 Methodology of Optimization

The results in the previous chapter suggest that obtaining a closed form solution for the optimum angle is possible at least for the case of pure normal forces, i.e. $\gamma = 0$. The procedure and results are discussed below.

A new code can be established to find the optimum sequence for any ratio of loading. This will be done by applying a loading $NL = \{1, \beta, 0\}$. The sequence will be taken as $[\pm\theta]_{ns}$ to find the optimum angle $\theta_{optimum}$ for any value of β (Figure 5.1). The code implementing this algorithm is given in Appendix B. Finally, the SR resulted from taking $h = 1 \text{ mm}$ and $N_x = 1 \text{ N/mm}$ for the optimum sequence will be called SR^* and will be used later on to find the minimum thickness so that the laminate is safe. This thickness will be called h_{min} and its value with a chosen safety factor (SF) can be given from the liner relationship between SR^* and N_x as

$$h_{min} = SF * \frac{N_x}{SR^*} \quad (5.1)$$

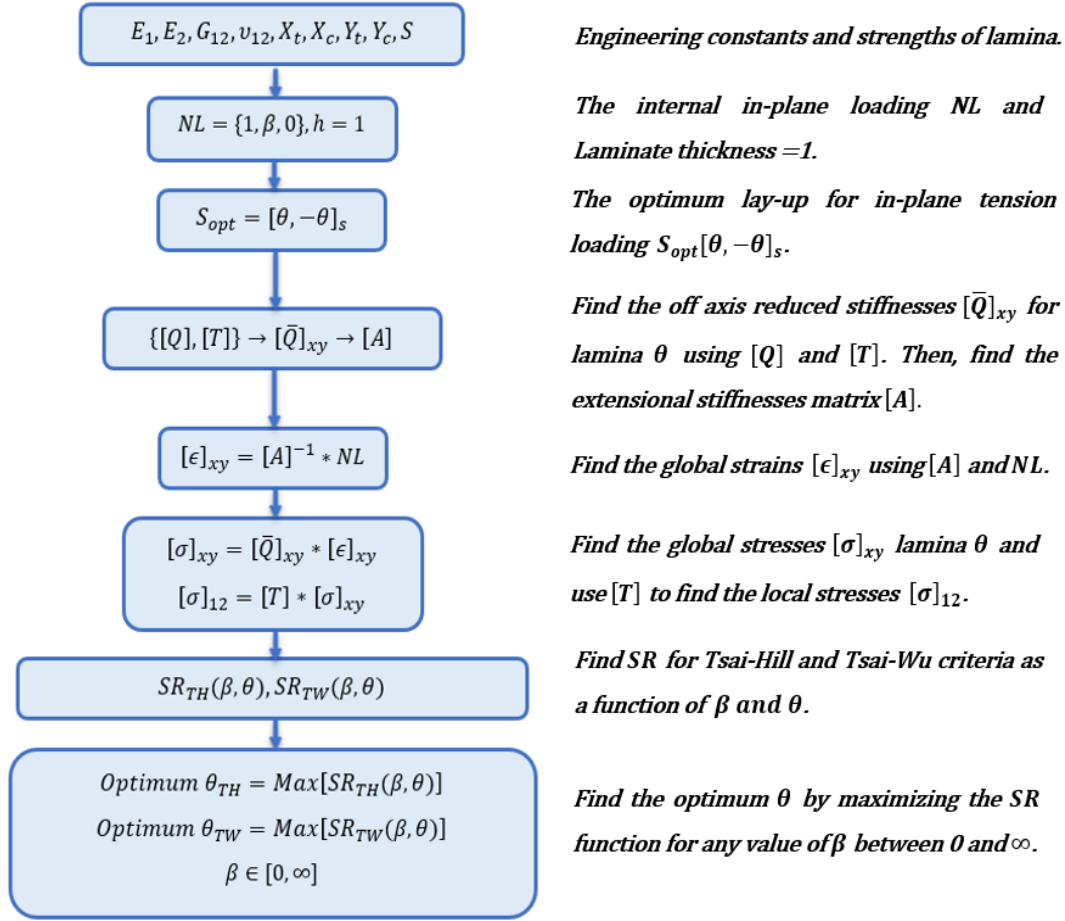


Figure 5.1: Flow chart for finding the optimum θ for any load ratio β .

5.2 Optimization Results and Discussion

The new code based on a symmetric 4 angle-ply laminate $[\theta/-\theta]_s$ can be used to generate closed-form expressions for SR for the three materials. The expressions are lengthy and it is not practical to show them within the limited space here. However, if the expressions are evaluated for a given load ratio β , the size of expressions are tremendously reduced. To show a typical expression, let $\beta = 1$, then SR as function of θ for the "Ar/Ep" laminate based on Tsai-Hill criterion becomes:

$$SR(\theta) = \frac{5.37511 + 1.94282 \cos[4\theta]}{\sqrt{1.2137 \times 10^8 + 1.50665 \times 10^8 \cos[4\theta] + 6.78565 \times 10^6 \cos[4\theta]}}$$

Similar expressions can be obtained for different data of materials, failure criteria and loading ratios.

The next step is getting the optimum values of θ that maximizes SR . Performing this step for three composite materials based on Tsai-Hill criterion, we get the three curves of Figure 5.2. The graph shows that the optimum angle is material dependent for β ranging from zero to about 30. For higher values of β , the Gl/Ep laminate yields a higher optimum value of θ compared to T300/5208 and Ar/Ep which almost coincides over the whole range of β . Another observation is that that $\theta_{optimum}$ for Gl/Ep laminate reaches the value of 90° at finite low β value of about 72 while for T300/5208 and Ar/Ep , it requires a much higher value of β to reach the value of 90° .

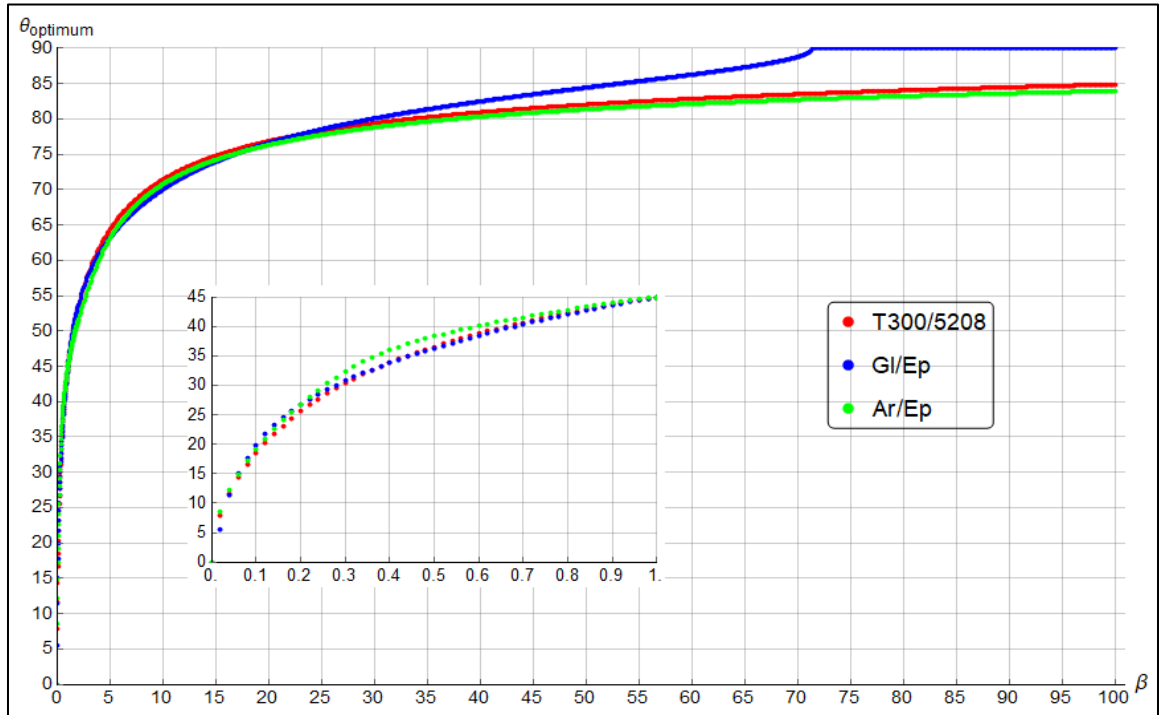


Figure 5.2: The change in the optimum angle versus β for the three materials based on Tsai-Hill.

Figure 5.3 represents the optimum angle based on Tsai-Wu failure criterion. The trend of the curves is similar to that obtained for Tsai-Hill failure criterion with the following differences: 1) the agreement among the three materials curves ends at a lower value of $\beta = 10$; 2) $\theta_{optimum}$ for Gl/Ep laminate reaches the value of 90° at lower value of $\beta = 32$ followed by Ar/Ep which reaches the value of 90° at around $\beta = 97$; 3) Unlike the curves for Tsai-Hill criterion where $\theta_{optimum} = 0$ for $\beta = 0$, the case here is not true for T300/5208 curve which gives $\theta_{optimum} = 5.6^\circ$ for $\beta = 0$. The above is attributed to the different elastic modulus ratio E_1/E_2 among the three materials which are 17.6, 13.8, and 4.7 for T300/5208, Ar/Ep and Gl/Ep , respectively.

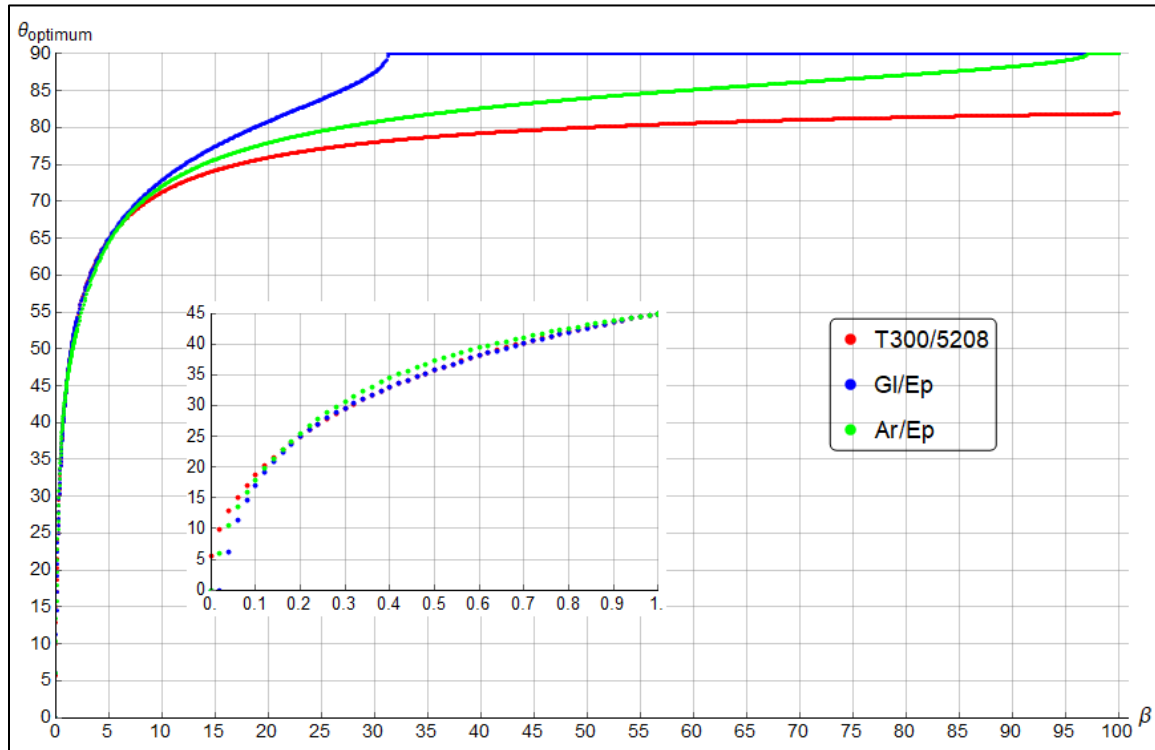


Figure 5.3: The change in the optimum angle versus β for the three materials based on Tsai-Wu.

The ultimate objective of this study is to come up with closed-form formulation for the optimum angles for each material and failure criteria. The shapes of the curves in Figures

5.2 and 5.3 suggest representing most of them by power-law formulas. Utilizing the symbolic computational capability of Mathematica software, we can try different power law and logarithmic relations until arriving at the ones with minimum error. The final equations after a series of trials are shown in Table 5.1 which yields the optimum angle with a maximum error of 1° compared to the analytical solution found by the code. The equations are chosen so that they fit the optimum orientation angles with the least number of coefficients.

The optimum angles can be also provided in a graphical way. The following figures (5.4-5.6) give $\theta_{optimum}$ and their corresponding SR^* which calculated by taken the optimum sequence for each β with $h = 1 \text{ mm}$ and $N_x = 1 \text{ N/mm}$ as mentioned previously. Each material has its own figure with the ability to find $\theta_{optimum}$ and $SR_{optimum}$ based on Tsai-Hill and Wu failure criteria. Each figure has been divided into two intervals to be able to read each value clearly.

Table 5.1: The optimum angle closed formula for each material based on both failure criteria.

Material	Failure Criteria	$\theta_{optimum}(\beta)$
T300/5208	Tsai-Hill	$\begin{cases} 0, & 0 \leq \beta \leq 0.003 \\ -5.785 + 65.5 \beta^{0.4} - 14.714 \beta, & 0.003 < \beta \leq 1 \\ 95.785 - 65.5 \left(\frac{1}{\beta}\right)^{0.4} + \frac{14.714}{\beta}, & 1 < \beta \leq 334 \\ 90, & \beta > 334 \end{cases}$
	Tsai-Wu	$\begin{cases} 5.325 + 94.246 \beta^{0.7} - 54.866 \beta, & 0 \leq \beta \leq 1 \\ 84.675 - 94.246 \left(\frac{1}{\beta}\right)^{0.7} + \frac{54.866}{\beta}, & \beta > 1 \end{cases}$

Gl/Ep	Tsai-Hill	$\begin{cases} 0, & 0 \leq \beta \leq 0.014 \\ 16.6 + 28.56 \beta^{0.5} - 1.0336 \ln(\beta)^2, & 0.014 < \beta \leq 1 \\ 73.4 - 28.56 \left(\frac{1}{\beta}\right)^{0.5} + 1.0336 \ln\left(\frac{1}{\beta}\right)^2, & 1 < \beta \leq 72 \\ 90, & \beta > 72 \end{cases}$
	Tsai-Wu	$\begin{cases} 0, & 0 \leq \beta \leq 0.032 \\ 112.23 - 69.33 \left(\frac{1}{\beta}\right)^{0.2} + \frac{1.7688}{\beta} - \frac{0.02955}{\beta^2}, & 0.032 < \beta \leq 1 \\ -22.23 + 69.33 \beta^{0.2} - 1.7688 \beta + 0.02955 \beta^2, & 1 < \beta \leq 32 \\ 90, & \beta > 32 \end{cases}$
Ar/Ep	Tsai-Hill	$\begin{cases} 0, & 0 \leq \beta \leq 0.0007 \\ -0.56 + 104.505 \beta^{0.6} - 61.242 \beta + 1.8354 \beta^2, & 0.0007 < \beta \leq 1 \\ 90.56 - 104.505 \left(\frac{1}{\beta}\right)^{0.6} + \frac{61.242}{\beta} - \frac{1.8354}{\beta^2}, & 1 < \beta \leq 1428 \\ 90, & \beta > 1428 \end{cases}$
	Tsai-Wu	$\begin{cases} 0, & 0 \leq \beta \leq 0.0103 \\ -59.74 + 95.64 \beta^{0.1} + 22.055 \beta - 13.475 \beta^2, & 0.0103 < \beta \leq 1 \\ 149.74 - 95.64 \left(\frac{1}{\beta}\right)^{0.1} - \frac{22.055}{\beta} + \frac{13.475}{\beta^2}, & 1 < \beta \leq 97 \\ 90, & \beta > 97 \end{cases}$

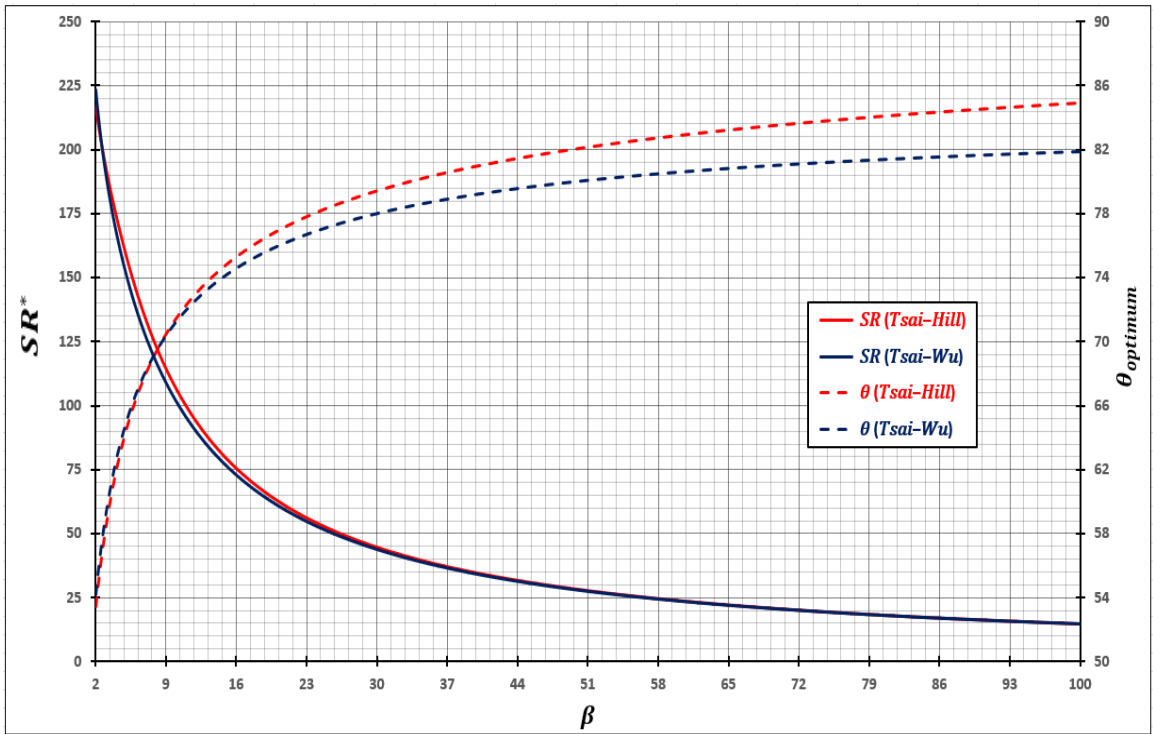
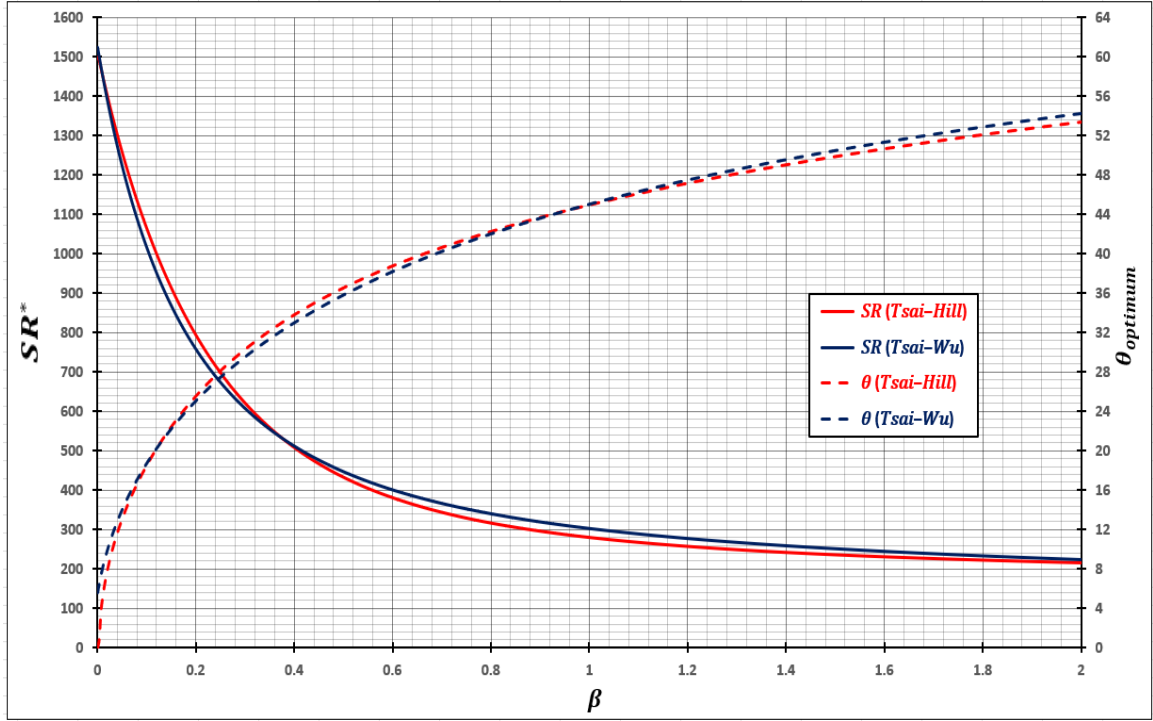


Figure 5.4: The optimum angle and its corresponding SR^* versus β for T300/5208 material.

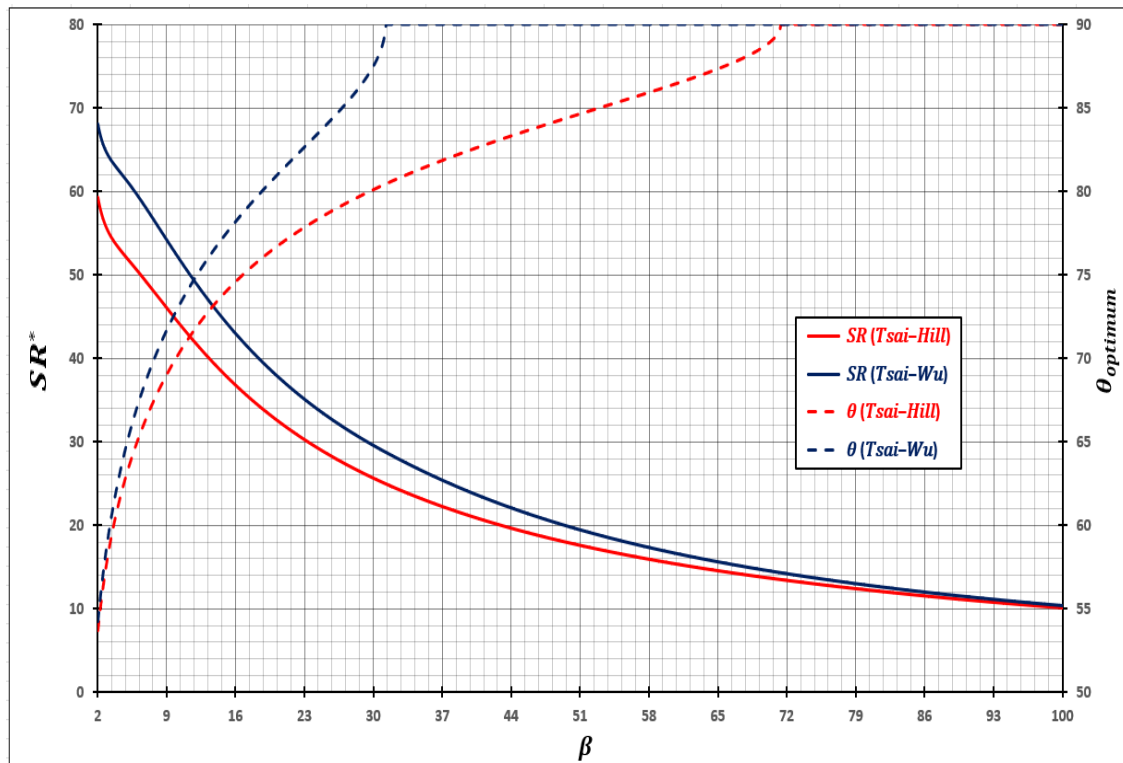
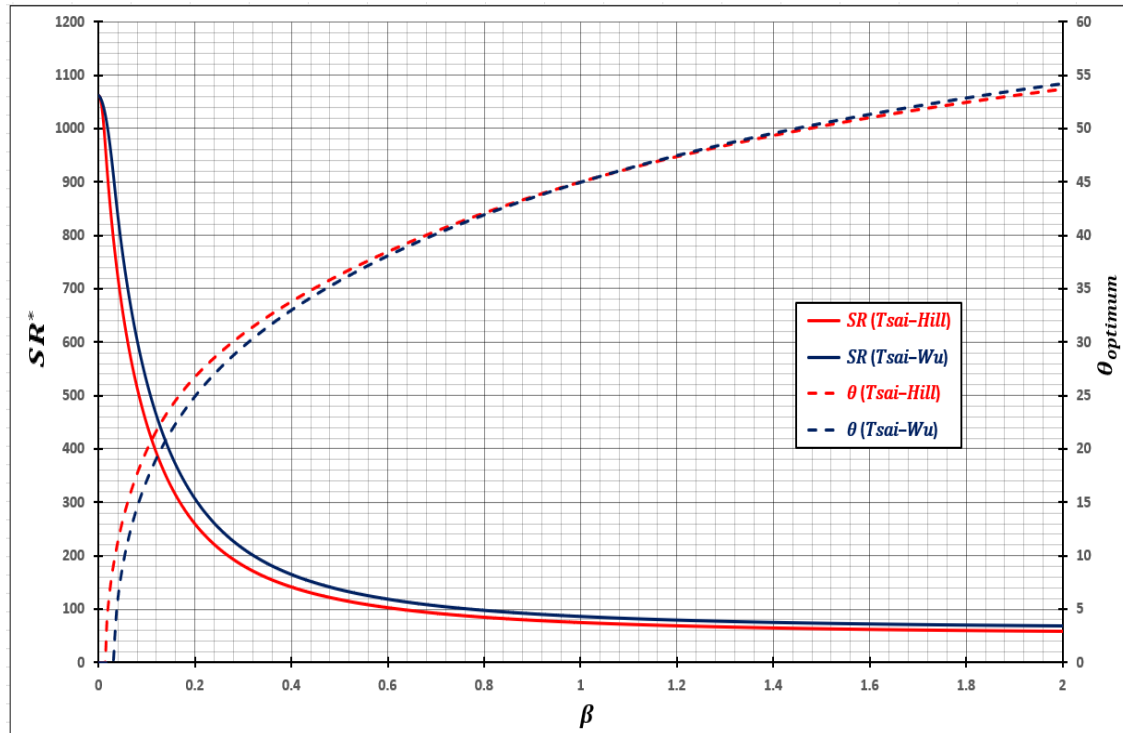


Figure 5.5: The optimum angle and its corresponding SR^* versus β for GL/Ep material.

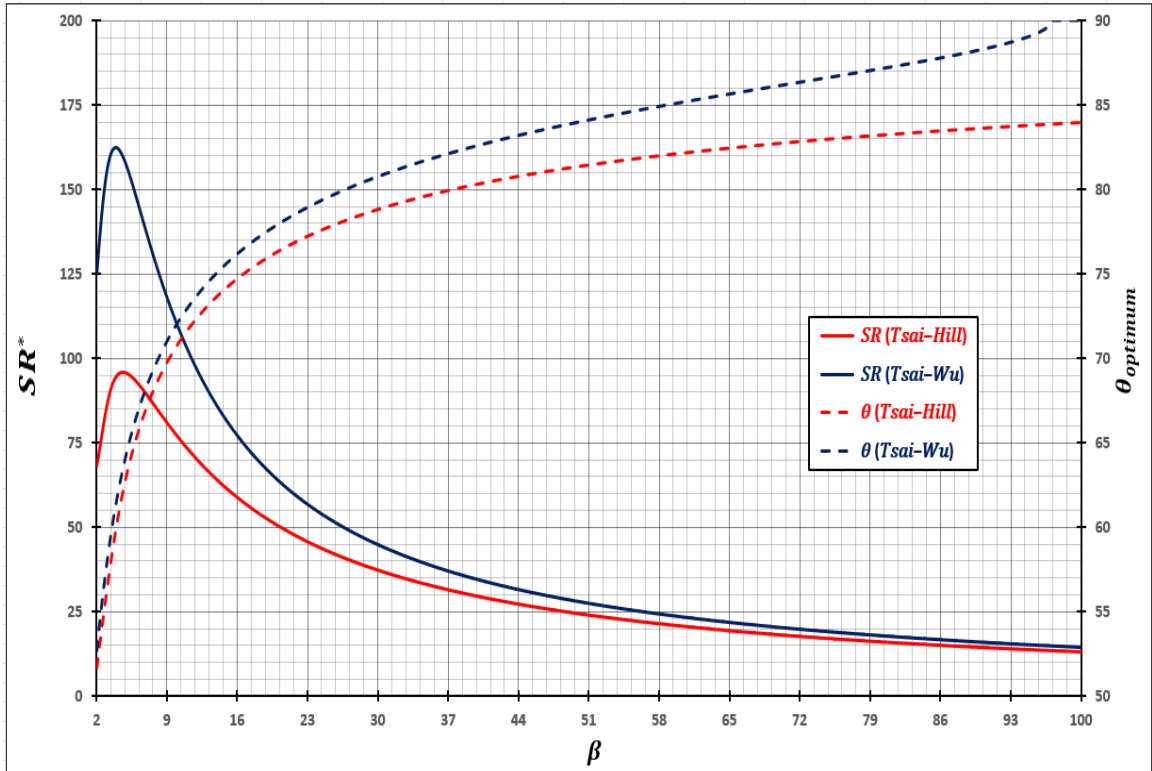
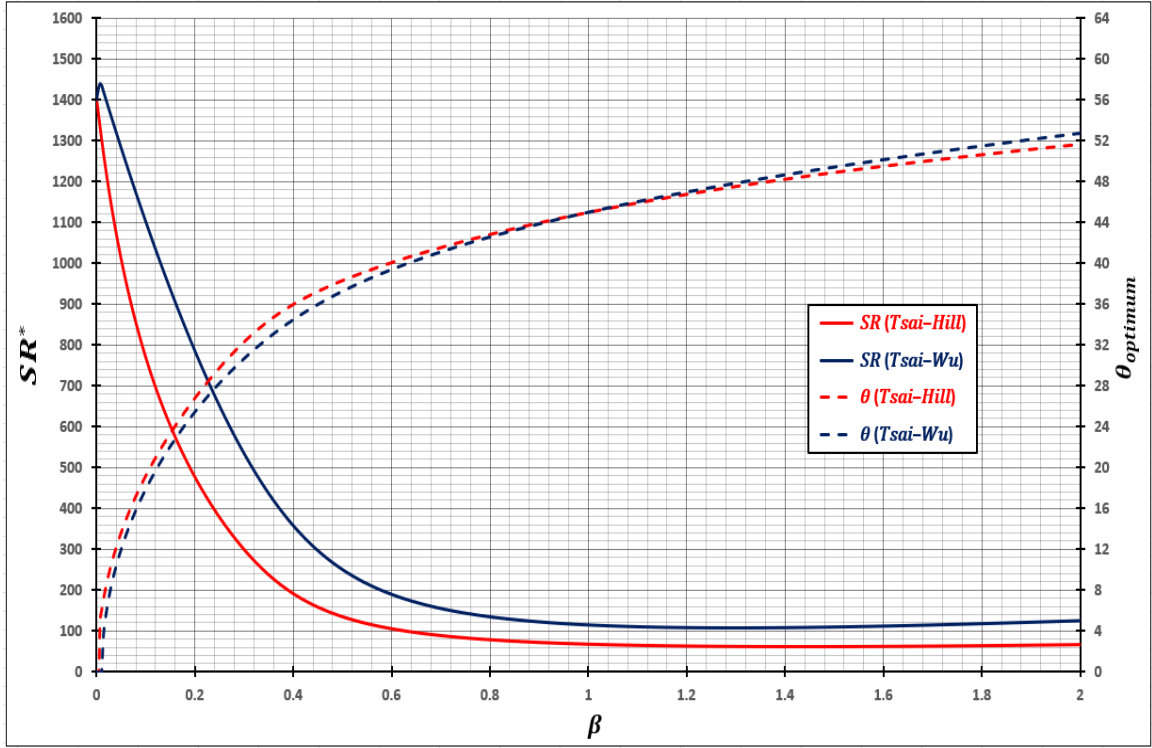


Figure 5.6: The optimum angle and its corresponding SR^* versus β for Ar/Ep material.

CLOSED-FORM EXPRESSIONS OF THE OPTIMUM LAY-UP ANGLES FOR PLATE UNDER IN-PLANE NORMAL AND SHEAR FORCES

In the previous chapter, the optimization of laminated plates under in-plane tension loading has been discussed. For this chapter, the purpose is to do optimization by adding the shear loading too. The formulation will not change as they are applicable for any in-plane conditions. The code used in the previous section will work the same way except now the shear loading will be added $NL = \{N_x, N_y, N_{xy}\}$. The final aim is the same which is constructing a closed-formula able to find the optimum ordination angles for different material, failure criteria and load level.

6.1 Methodology of Optimization

The previously derived closed formulas for the pure tensile in-plane loading can be utilized to derive similar formulas for the general in-plane loading including the shear by using the principle of stress transformation. First, we modify the loading to be $N_x\{1, \beta, \gamma\}$, where $\gamma = \frac{N_{xy}}{N_x}$. Figure 6.1 Shows the conditions of each loading case.

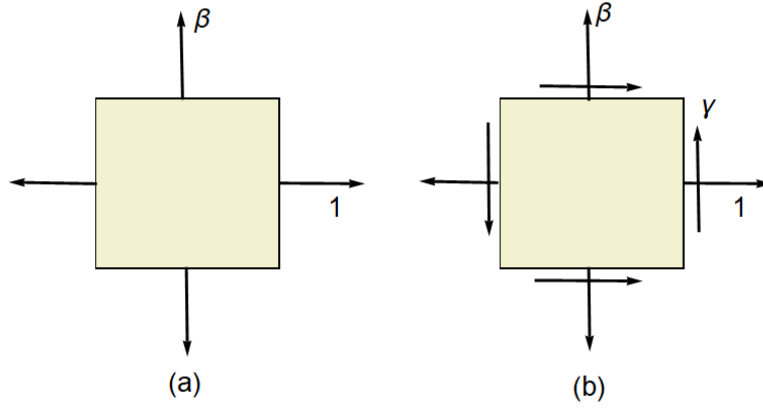


Figure 6.1: Laminated plate under (a) in-plane tension loading (b) in-plane tension and shear loading.

The procedure now is to transform the stress state of the element in Fig. 6.1-b to the one in Fig. 6.1-a for which we already have established the closed-form solution. This can be easily found by finding the principal stresses for the load vector $N_x\{1, \beta, \gamma\}$ resulting in:

$$N_x \left\{ \frac{1}{2} \left(1 + \beta - \sqrt{1 - 2\beta + \beta^2 + 4\gamma^2} \right), \frac{1}{2} \left(1 + \beta + \sqrt{1 - 2\beta + \beta^2 + 4\gamma^2} \right), 0 \right\}$$

The result of the transformation is shown in Figure 6.2. Now, the resulted case is free of shear loading and similar to that from case (a) loading except that the old β now will called α and it will have a value equal to:

$$\alpha = \begin{cases} \frac{1 + \beta - \sqrt{1 - 2\beta + \beta^2 + 4\gamma^2}}{1 + \beta + \sqrt{1 - 2\beta + \beta^2 + 4\gamma^2}}, & 0 \leq \beta \leq 1 \\ \frac{1 + \beta + \sqrt{1 - 2\beta + \beta^2 + 4\gamma^2}}{1 + \beta - \sqrt{1 - 2\beta + \beta^2 + 4\gamma^2}}, & 1 < \beta \end{cases} \quad (6.1)$$

The only constrained here is that both of these resulted principles must be in tension. This condition requires that $\frac{1}{2}(1 + \beta - \sqrt{1 - 2\beta + \beta^2 + 4\gamma^2}) \geq 0$ which means $-\sqrt{\beta} < \gamma < \sqrt{\beta}$.

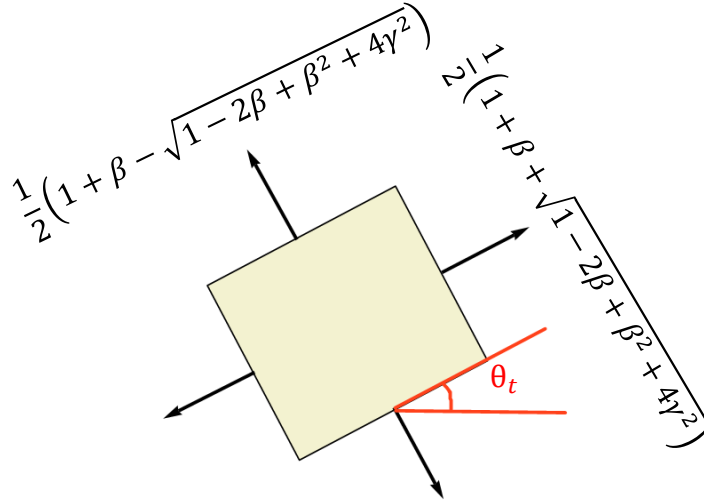


Figure 6.2: Laminated plate under the loading resulted from stress transformation of (b) case loading.

From Figure 6.2, it is obvious that the new optimum angles can be found by adding the angle obtained from principal stress transformation (θ_t) to that obtained from the in-plane tension loading with the new load ratio α .

6.2 Optimization Results and Discussion

Referring to the previous chapter where the optimum sequence for any load ratio β was taken as $[\pm\theta]_{ns}$, the new one with shear loading can be represented as $[\theta_1/\theta_2]_{ns}$. As the principal stress transformation angle (θ_t) can be given as:

$$\theta_t = \frac{1}{2} \tan^{-1} \left(\frac{2\gamma}{1-\beta} \right) \quad (6.2)$$

Now, θ_1 and θ_2 can be found for each material and failure criteria by adding θ_t to the optimum angles $[\pm\theta]_{ns}$ resulted from any load ratio α . The new updated closed formulas for each case of in-plane loading under the constrained $-\sqrt{\beta} < \gamma < \sqrt{\beta}$ is given in Table 6.1. Finally, the SR^* resulted from taking $h = 1 \text{ mm}$ and $N_x = 1 \text{ N/mm}$ for the optimum sequence will as previous except its value now will change for any value of β and γ .

Table 6.1: The optimum angle closed formula for each material based on both failure criteria.

Material	Failure Criteria	$\theta_{1optimum}, \theta_{2optimum}$
T300/5208	Tsai-Hill	$\theta_1 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.003 \\ -5.785 + 65.5 \alpha^{0.4} - 14.714 \alpha, & 0.003 < \alpha \leq 1 \\ 95.785 - 65.5 \left(\frac{1}{\alpha}\right)^{0.4} + \frac{14.714}{\alpha}, & 1 < \alpha \leq 334 \\ 90, & \alpha > 334 \end{cases}$ $\theta_2 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.003 \\ 5.785 - 65.5 \alpha^{0.4} + 14.714 \alpha, & 0.003 < \alpha \leq 1 \\ -95.785 + 65.5 \left(\frac{1}{\alpha}\right)^{0.4} - \frac{14.714}{\alpha}, & 1 < \alpha \leq 334 \\ -90, & \alpha > 334 \end{cases}$
	Tsai-Wu	$\theta_1 = \theta_p + \begin{cases} 5.325 + 94.246 \alpha^{0.7} - 54.866 \alpha, & 0 \leq \alpha \leq 1 \\ 84.675 - 94.246 \left(\frac{1}{\alpha}\right)^{0.7} + \frac{54.866}{\alpha}, & \alpha > 1 \end{cases}$ $\theta_2 = \theta_p + \begin{cases} -5.325 - 94.246 \alpha^{0.7} + 54.866 \alpha, & 0 \leq \alpha \leq 1 \\ -84.675 + 94.246 \left(\frac{1}{\alpha}\right)^{0.7} - \frac{54.866}{\alpha}, & \alpha > 1 \end{cases}$
Gl/Ep	Tsai-Hill	$\theta_1 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.014 \\ 16.6 + 28.56 \alpha^{0.5} - 1.0336 \ln(\alpha)^2, & 0.014 < \alpha \leq 1 \\ 73.4 - 28.56 \left(\frac{1}{\alpha}\right)^{0.5} + 1.0336 \ln\left(\frac{1}{\alpha}\right)^2, & 1 < \alpha \leq 72 \\ 90, & \alpha > 72 \end{cases}$ $\theta_2 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.014 \\ -16.6 - 28.56 \alpha^{0.5} + 1.0336 \ln(\alpha)^2, & 0.014 < \alpha \leq 1 \\ -73.4 + 28.56 \left(\frac{1}{\alpha}\right)^{0.5} - 1.0336 \ln\left(\frac{1}{\alpha}\right)^2, & 1 < \alpha \leq 72 \\ -90, & \alpha > 72 \end{cases}$
	Tsai-Wu	$\theta_1 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.032 \\ 112.23 - 69.33 \left(\frac{1}{\alpha}\right)^{0.2} + \frac{1.7688}{\alpha} - \frac{0.02955}{\alpha^2}, & 0.032 < \alpha \leq 1 \\ -22.23 + 69.33 \alpha^{0.2} - 1.7688 \alpha + 0.02955 \alpha^2, & 1 < \alpha \leq 32 \\ 90, & \alpha > 32 \end{cases}$ $\theta_2 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.032 \\ -112.23 + 69.33 \left(\frac{1}{\alpha}\right)^{0.2} - \frac{1.7688}{\alpha} + \frac{0.02955}{\alpha^2}, & 0.032 < \alpha \leq 1 \\ 22.23 - 69.33 \alpha^{0.2} + 1.7688 \alpha - 0.02955 \alpha^2, & 1 < \alpha \leq 32 \\ -90, & \alpha > 32 \end{cases}$

Material	Failure Criteria	$\theta_{1optimum}, \theta_{2optimum}$	
Ar/Ep	Tsai-Hill	$\theta_1 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.0007 \\ -0.56 + 104.505 \alpha^{0.6} - 61.242 \alpha + 1.8354 \alpha^2, & 0.0007 < \alpha \leq 1 \\ 90.56 - 104.505 \left(\frac{1}{\alpha}\right)^{0.6} + \frac{61.242}{\alpha} - \frac{1.8354}{\alpha^2}, & 1 < \alpha \leq 1428 \\ 90, & \alpha > 1428 \end{cases}$	
		$\theta_1 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.0007 \\ 0.56 - 104.505 \alpha^{0.6} + 61.242 \alpha - 1.8354 \alpha^2, & 0.0007 < \alpha \leq 1 \\ -90.56 + 104.505 \left(\frac{1}{\alpha}\right)^{0.6} - \frac{61.242}{\alpha} + \frac{1.8354}{\alpha^2}, & 1 < \alpha \leq 1428 \\ -90, & \alpha > 1428 \end{cases}$	
	Tsai-Wu	$\theta_1 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.0103 \\ -59.74 + 95.64 \alpha^{0.1} + 22.055 \alpha - 13.475 \alpha^2, & 0.0103 < \alpha \leq 1 \\ 149.74 - 95.64 \left(\frac{1}{\alpha}\right)^{0.1} - \frac{22.055}{\alpha} + \frac{13.475}{\alpha^2}, & 1 < \alpha \leq 97 \\ 90, & \alpha > 97 \end{cases}$	
		$\theta_2 = \theta_p + \begin{cases} 0, & 0 \leq \alpha \leq 0.0103 \\ 59.74 - 95.64 \alpha^{0.1} - 22.055 \alpha + 13.475 \alpha^2, & 0.0103 < \alpha \leq 1 \\ -149.74 + 95.64 \left(\frac{1}{\alpha}\right)^{0.1} + \frac{22.055}{\alpha} - \frac{13.475}{\alpha^2}, & 1 < \alpha \leq 97 \\ -90, & \alpha > 97 \end{cases}$	

To illustrate the change in the sequence $[\theta_1/\theta_2]_{ns}$ and its SR^* with the change of shear loading, graphs with a fixed $\beta = 2$ and a variable γ will be drawn for each material under both failure criteria (Figure 6.3- 6.11). The value of optimum angles calculated from the above equations could be less than -90 or greater than 90, but this is not a problem as for example 120 is equivalent to -60 and -120 is equivalent to 60 and so on. The figures are plotted based on this rule which gives the reason for the sudden jump in the values of angles. From figures the values of optimum angles change for every change in γ as they are both negatives or positives or with different sign. The value of SR^* also changes with being minimum at $\gamma = 0$ to be maximum at the limits $\gamma = -\sqrt{\beta}$ and $\sqrt{\beta}$. All the figures of the optimum angles are symmetric about the origin which means that if for a positive value of γ the optimum sequence is $[\theta_1/\theta_2]_{ns}$ it is $[-\theta_2/\theta_1]_{ns}$ for the same value of γ but

with a negative sign and vice versa. The SR^* is symmetric about the y-axis which mean that the above-mentioned change in sequence resulted in the same value of SR^* .

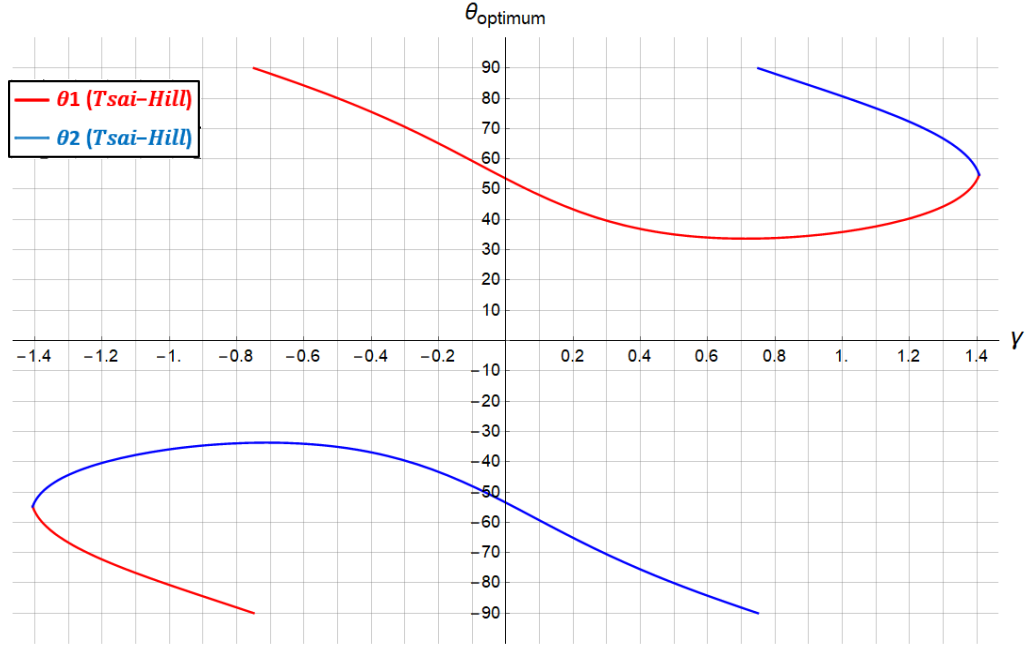


Figure 6.3: The optimum angles (θ_1/θ_2) at $\beta = 2$ for T300/5208 based on Tsai-Hill criteria.

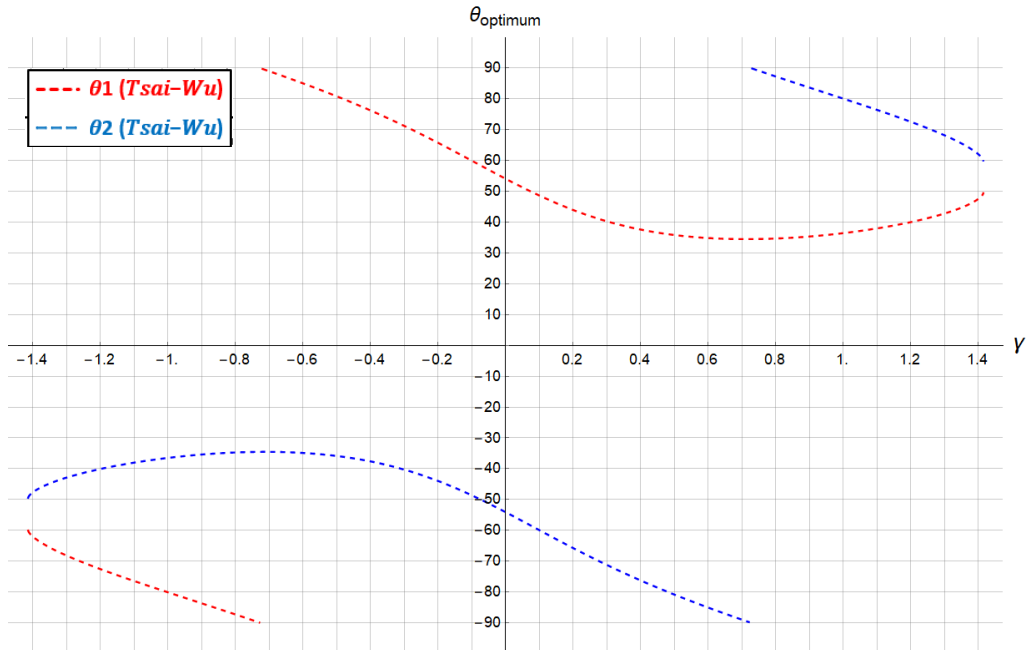


Figure 6.4: The optimum angles (θ_1/θ_2) at $\beta = 2$ for T300/5208 based on Tsai-Wu criteria.

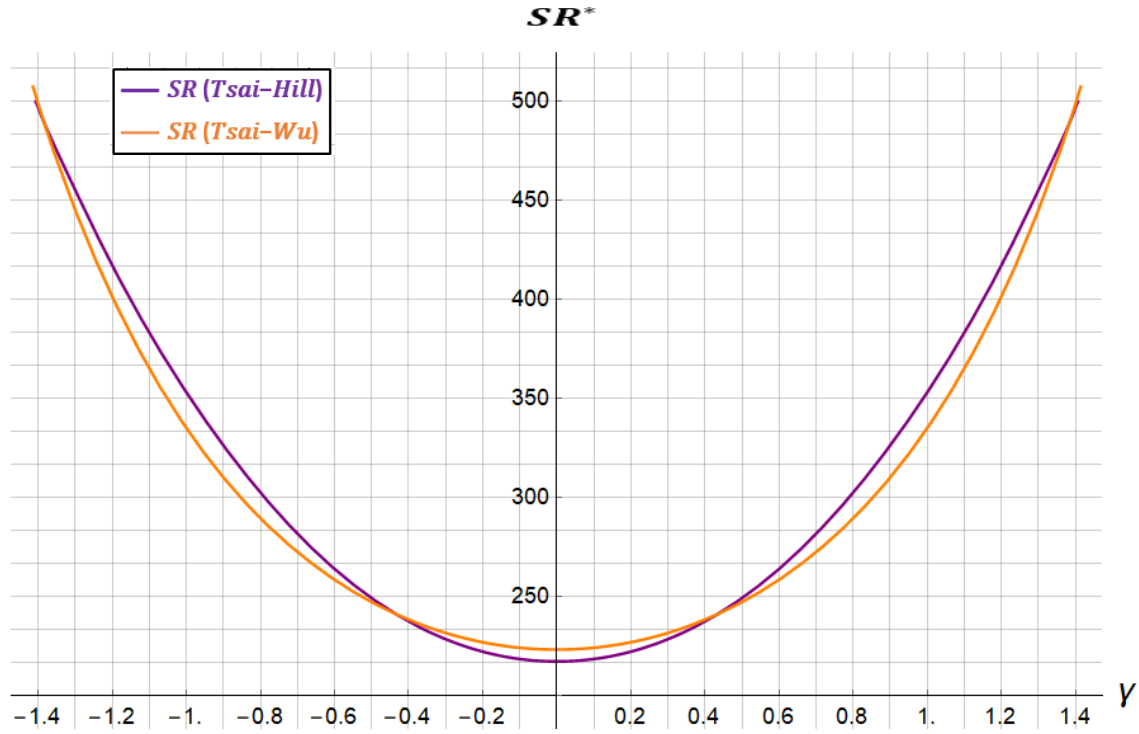


Figure 6.5: The optimum SR for the optimum $[\theta_1/\theta_2]_{ns}$ at $\beta = 2$ for T300/5208 based on both criteria.

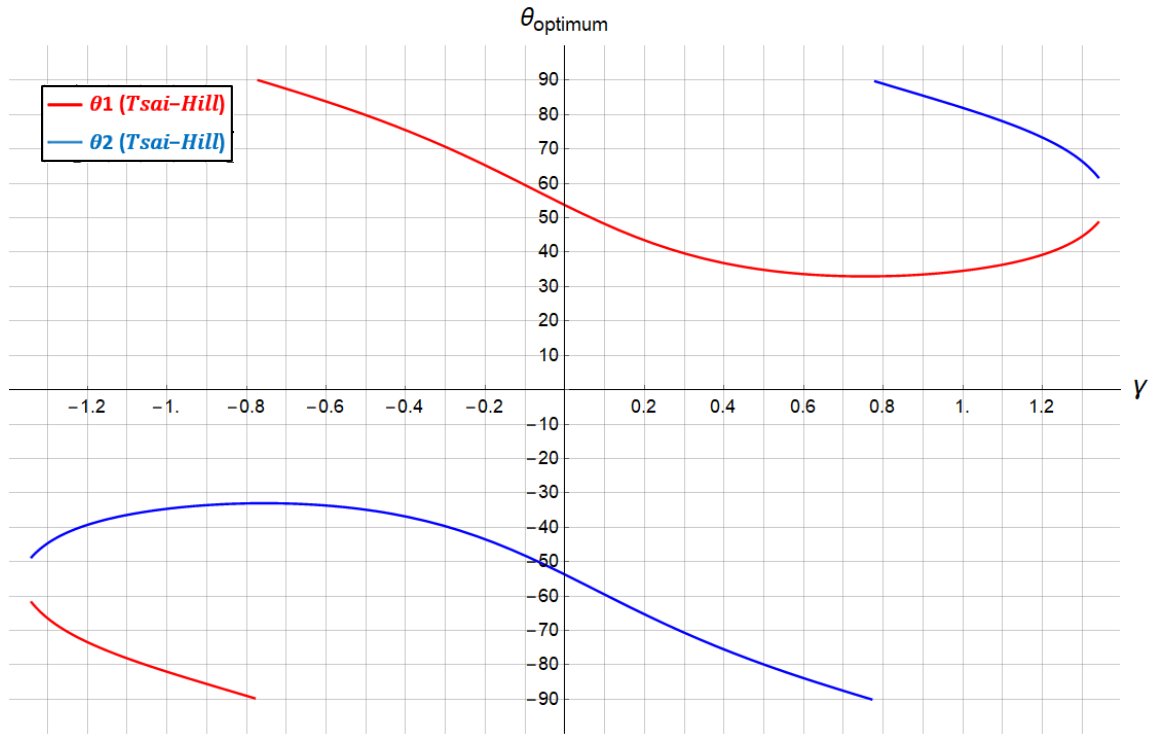


Figure 6.6: The optimum angles (θ_1/θ_2) at $\beta = 2$ for GI/E_p based on Tsai-Hill criteria.

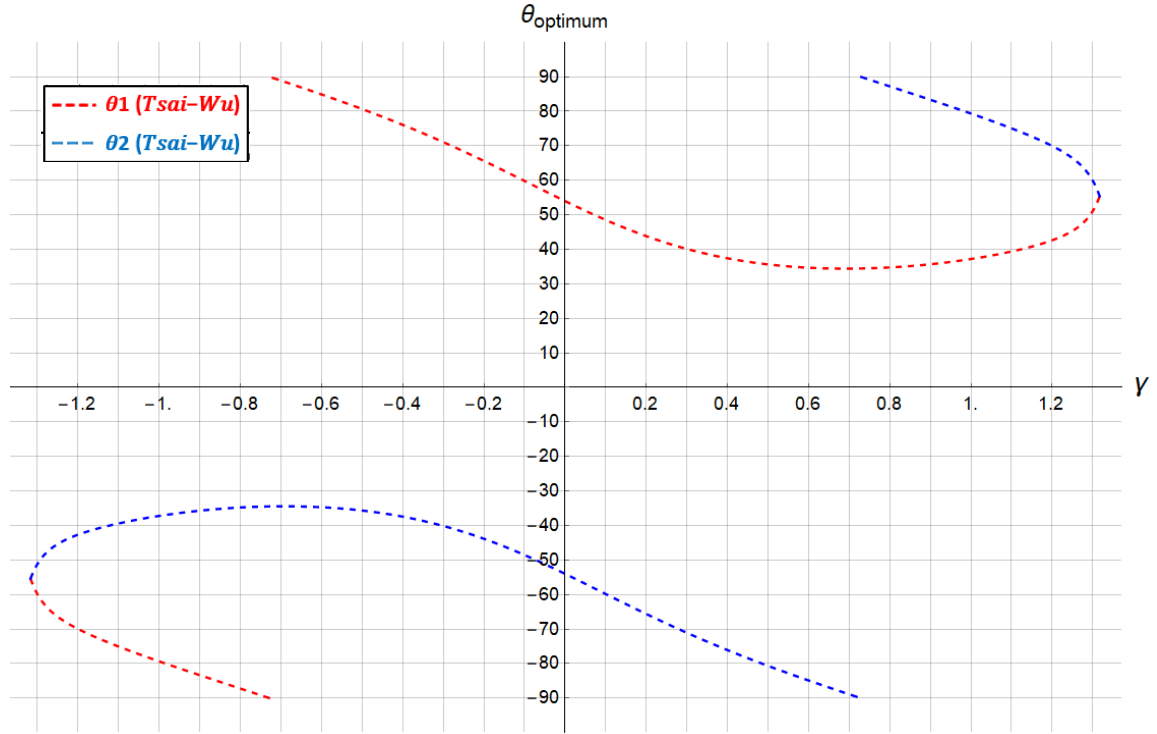


Figure 6.7: The optimum angles (θ_1/θ_2) at $\beta = 2$ for GI/Ep based on Tsai-Wu criteria.

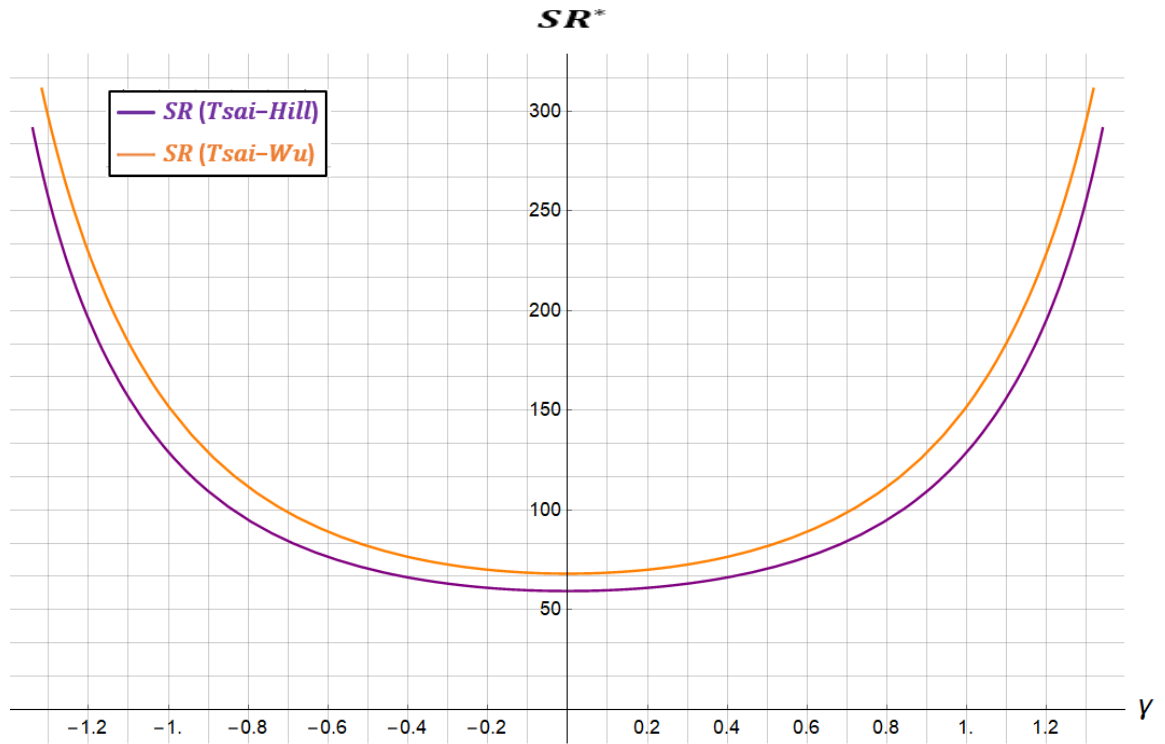


Figure 6.8: The optimum SR for the optimum $[\theta_1/\theta_2]_{ns}$ at $\beta = 2$ for GI/Ep based on both criteria.

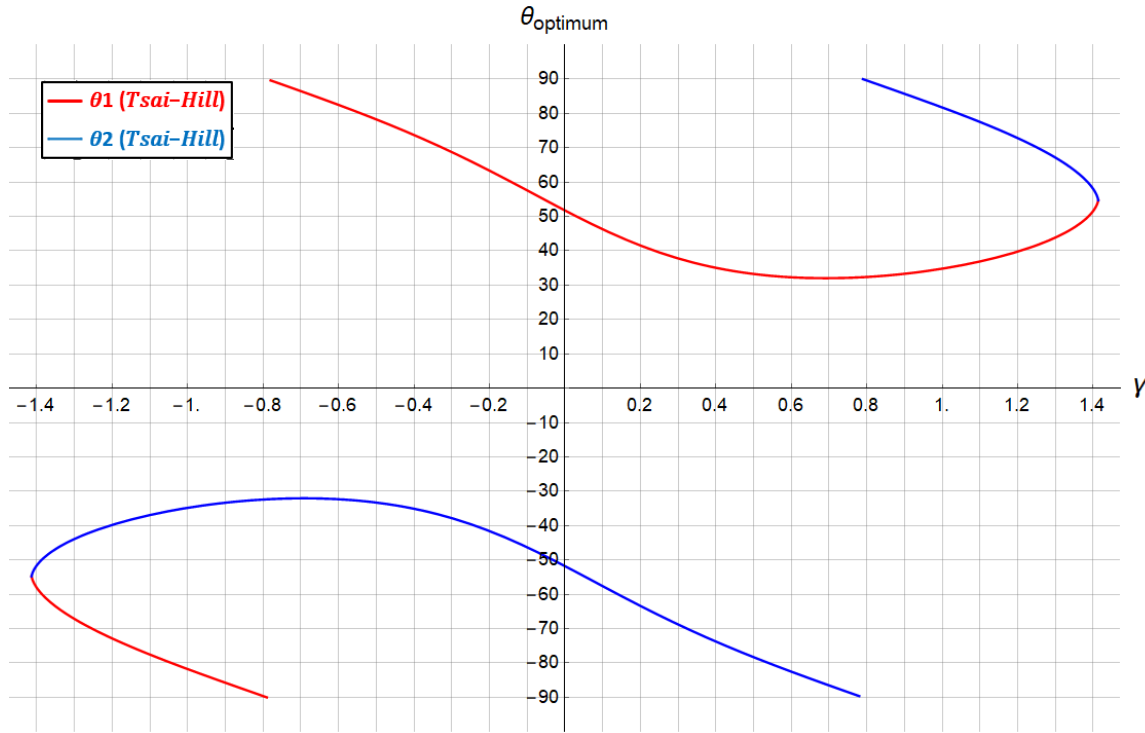


Figure 6.9: The optimum angles (θ_1/θ_2) at $\beta = 2$ for Ar/Ep based on Tsai-Hill criteria.

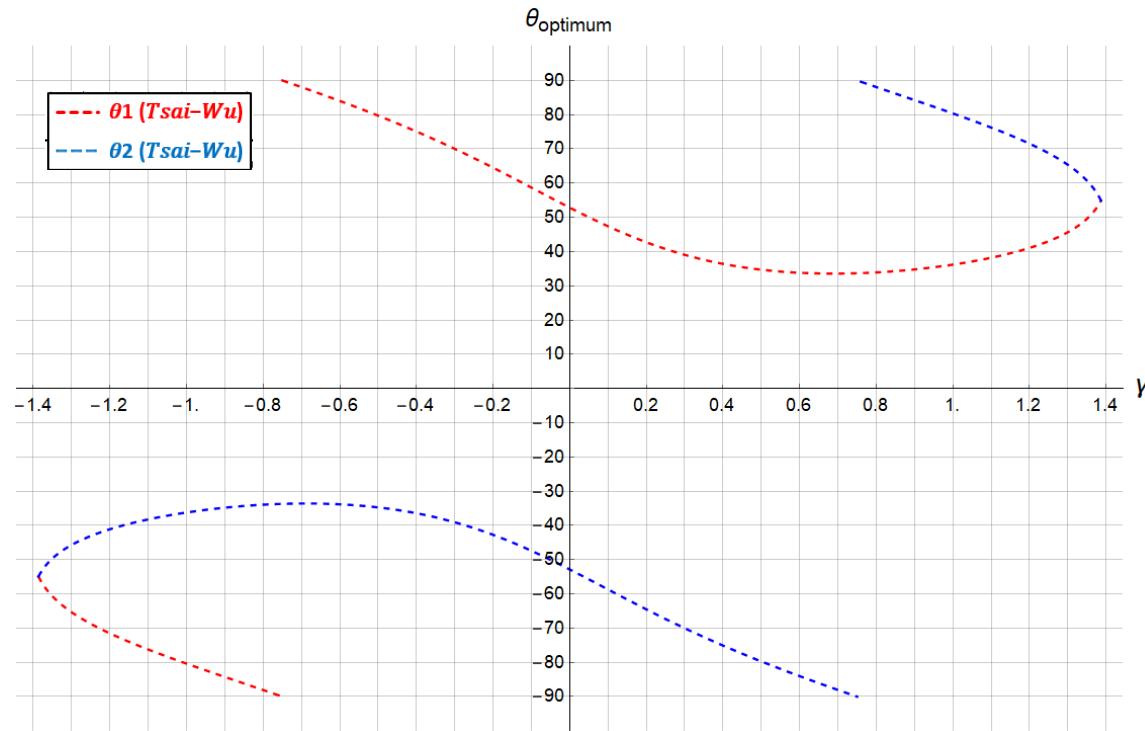


Figure 6.10: The optimum angles (θ_1/θ_2) at $\beta = 2$ for Ar/Ep based on Tsai-Wu criteria.

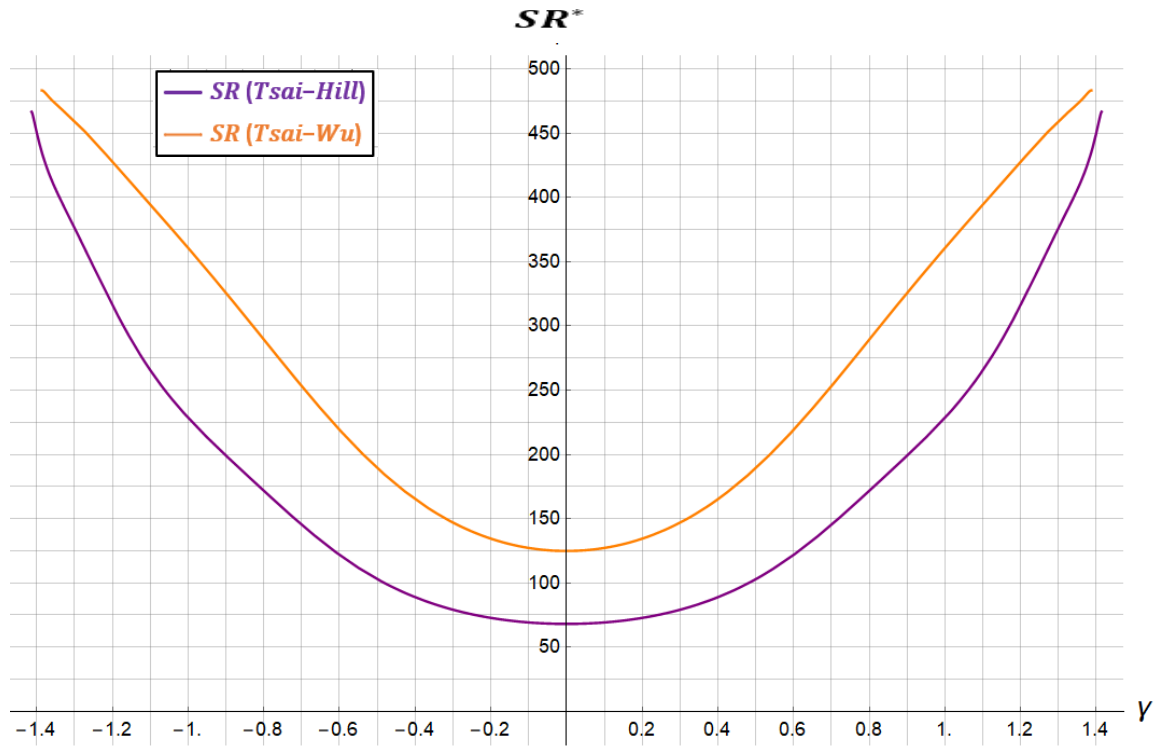


Figure 6.11: The optimum SR for the optimum $[\theta_1/\theta_2]_{ns}$ at $\beta = 2$ for Ar/Ep based on both criteria.

NUMERICAL VERIFICATION

7.1 Numerical Example 1

First, the adequacy of the code will be checked by comparing the optimum sequence obtained by the code with that obtained by Mian et. al, [31]. In [31], the structure to be optimized is a symmetric fiber-reinforced composite pressure vessel (Figure 7.1). The composite vessel has a diameter $D = 890 \text{ mm}$ and subjected to internal pressure $p = 2.07 \text{ MPa}$ and torsion $T = 283 \text{ kN.m}$ with no axial loading $F = 0$. The reference adapted 4 materials: aluminum with a density of 2.7 g/cm^3 and $\sigma_y = 242 \text{ MPa}$ as a reference and a three composite materials $S - \text{glass/Epoxy}$, Kevlar/Epoxy and Carbon/Epoxy with the properties shown in table 7.1. The thickness of the vessel was considered as the design variable and the objective functions were the failure criteria: von mises for aluminum and Tsai-Wu for the composite materials with a factor of safety = 2. Once the minimum thickness is found, the lightest weight can be found accordingly to choose the best material.

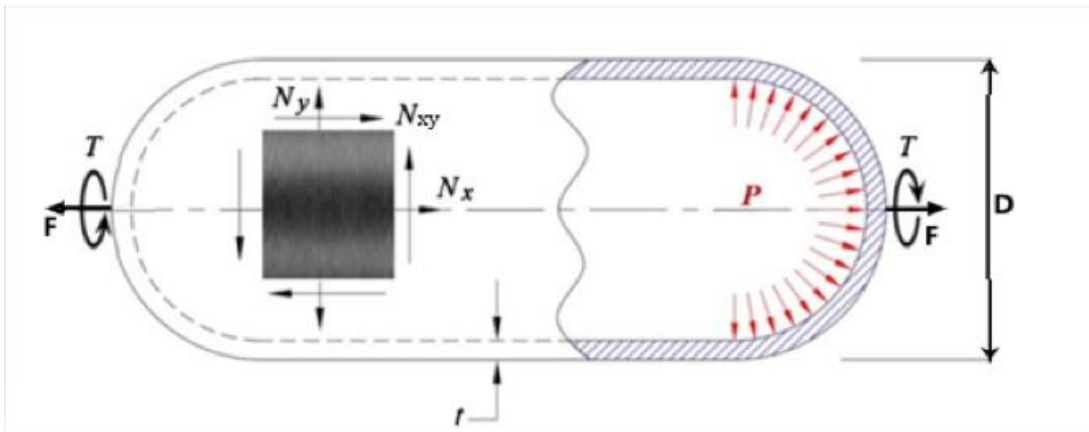


Figure 7.1: Thin-walled cylindrical pressure vessel under internal pressure, axial and torque loading [1].

Table 7.1: Materials mechanical and strength properties [31].

Material Properties	S-glass/Epoxy	Kevlar/Epoxy (Aramid 149)	Carbon/Epoxy (IM6/SC1081)
Density (ρ , g/cm ³)	2.0	1.38	1.6
Longitudinal modulus (E_1 , GPa)	43	87	177
Transverse modulus (E_2 , GPa)	8.9	5.5	10.8
In-plane shear modulus (G_{12} , GPa)	4.5	2.2	7.6
Major Poisson's ratio (ν_{12})	0.27	0.34	0.27
Minor Poisson's ratio (ν_{21})	0.06	0.02	0.02
Longitudinal tensile strength (F_{1t} , MPa)	1280	1280	2860
Transverse tensile strength (F_{2t} , MPa)	49	30	49
In-plane shear strength (F_6 , MPa)	69	49	83
Ultimate longitudinal tensile strain (ε_{1t}^u)	0.029	0.015	0.016
Ultimate transverse tensile strain (ε_{2t}^u)	0.006	0.005	0.005
Longitudinal compressive strength (F_{1c} , MPa)	690	335	1875
Transverse compressive strength (F_{2c} , MPa)	158	158	246

The analysis by Mian et. al. [31] is limited to three specific laminate lay-ups: cross-ply $[0_m/90_n]_s$, angle-ply $[\pm\theta]_{ns}$, $[90/\pm\theta]_{ns}$ and $[0/\pm\theta]_{ns}$. The formulation is based on CLT and the optimization is done using FEM method using ANSYS software and confirmed by MATLAB. Although, the reference used both Tsai-Wu and maximum stresses failure criteria. The results will be compared to Tsai-Wu failure criterion only from the present study. So, this may cause some differences in the results. Now, according to the theory of thin-walled cylindrical pressure and CLT the value of internal loadings (N_x, N_y, N_{xy}) in term of external loadings (P, F, T) can be given as follow:

$$N_x = \sigma_x t = \frac{F}{\pi D} + \frac{pD}{4} \quad (7.1)$$

$$N_y = \sigma_y t = \frac{pD}{2} \quad (7.2)$$

$$N_{xy} = \tau_{xy} t = \frac{2T}{\pi D^2} \quad (7.3)$$

Substituting the values of F, p, D and T yields:

$$N_x = 460 \text{ N/mm}, N_y = 920 \text{ N/mm}, N_{xy} = 228 \text{ N/mm}$$

The results of the optimization for the aluminum material were found as follow:

$$t_a = 7.36 \text{ mm}, \quad w_t = 7.36 * 2.7 = 19.872 \text{ g/cm}^2$$

Where t_a and w_t are the aluminum material minimum thickness and weight respectively.

For the composite materials, the results of the optimum design corresponding to the $[\pm\theta]_{ns}$ lay-up are given in Table 7.2.

Table 7.2: The optimum design of the three composite materials [31].

	<i>S – glass/Epoxy</i>	<i>Kevlar/Epoxy</i>	<i>Carbon/Epoxy</i>
Ply thickness (<i>mm</i>)	0.165	0.127	0.127
Optimized θ (<i>Deg</i>)	55	54	54
Optimized Layup	$[\pm 55]_{19s}$	$[\pm 54]_{9s}$	$[\pm 54]_{9s}$
Laminate thk. (<i>mm</i>)	12.54	4.58	4.58
Factor of Safety	2.09	2.10	2.06

Repeating the analysis using the code developed in this thesis, the following results (Table 7.3) are obtained for the lay-up $[\pm\theta]_{ns}$:

Table 7.3: The present study optimum design of the three composite materials with $[\pm\theta]_{ns}$ lay-up and Tsai-Wu.

	<i>S – glass/Epoxy</i>	<i>Kevlar/Epoxy</i>	<i>Carbon/Epoxy</i>
Ply thickness (<i>mm</i>)	0.165	0.127	0.127
Optimized θ (<i>Deg</i>)	55	54	54
<i>SR</i> ($h = 1 \text{ mm}$)	0.1745	0.48912	0.4964
Optimized Layup	$[\pm 55]_{18s}$	$[\pm 54]_{9s}$	$[\pm 54]_{9s}$
Laminate thk. (<i>mm</i>)	11.88	4.572	4.572
Factor of Safety	2.073	2.236	2.27

The code results show good agreements with those obtained in [31], despite the differences which are caused by taking Tsai-Wu only from the present study and the different

procedure in calculating the strength parameter F_{12} . Although, it has been proven in Ch. 6 that the optimum lay-up for combined normal-shear loading is $[\theta_1/\theta_2]_{ns}$ lay-up, the above comparative results were obtained for the $[\pm\theta]_{ns}$ lay-up for code accuracy verification. For the $[\theta_1/\theta_2]_{ns}$ lay-up which is unexploited in [31], the optimization results, as obtained by the present study, are given in Table 7.4.

Table 7.4: The present study optimum design of the three composite materials with $[\theta_1/\theta_2]_{ns}$ lay-up and Tsai-Wu.

	<i>S – glass/Epoxy</i>	<i>Kevlar/Epoxy</i>	<i>Carbon/Epoxy</i>
Ply thickness (<i>mm</i>)	0.165	0.127	0.127
Optimized θ (<i>Deg</i>)	-83, 38	-81, 37	-81, 36
<i>SR</i> ($h = 1\text{ mm}$)	0.28183	0.72121	0.741175
Optimized Layup	$[-83/38]_{11s}$	$[-81/38]_{6s}$	$[-81/36]_{6s}$
Laminate thk. (<i>mm</i>)	7.26	3.05	3.05
Factor of Safety	2.046	2.2	2.26

The results of Table 7.4 show a much better design for the pressure vessel with a noticeable smaller thickness. Figure 7.2 shows the differences in the minimum thicknesses by implementing $[\theta_1/\theta_2]_{ns}$ lay-up instead of the typical cross ply $[\pm\theta]_{ns}$ laminate. Another way to compare the three materials is through considering the weight as the optimization function. For this comparison the three materials relative weight per unit area saving compared with aluminum pressure vessel will be investigated. As the minimum weight for the aluminum was found from the density to be $w_t = 19.872\text{ g/cm}^2$, all others composite materials relative weight can be found by multiplying the minimum thickness found for both reference [31] and the code then dividing them by w_t . The results are shown in Figure 7.3.

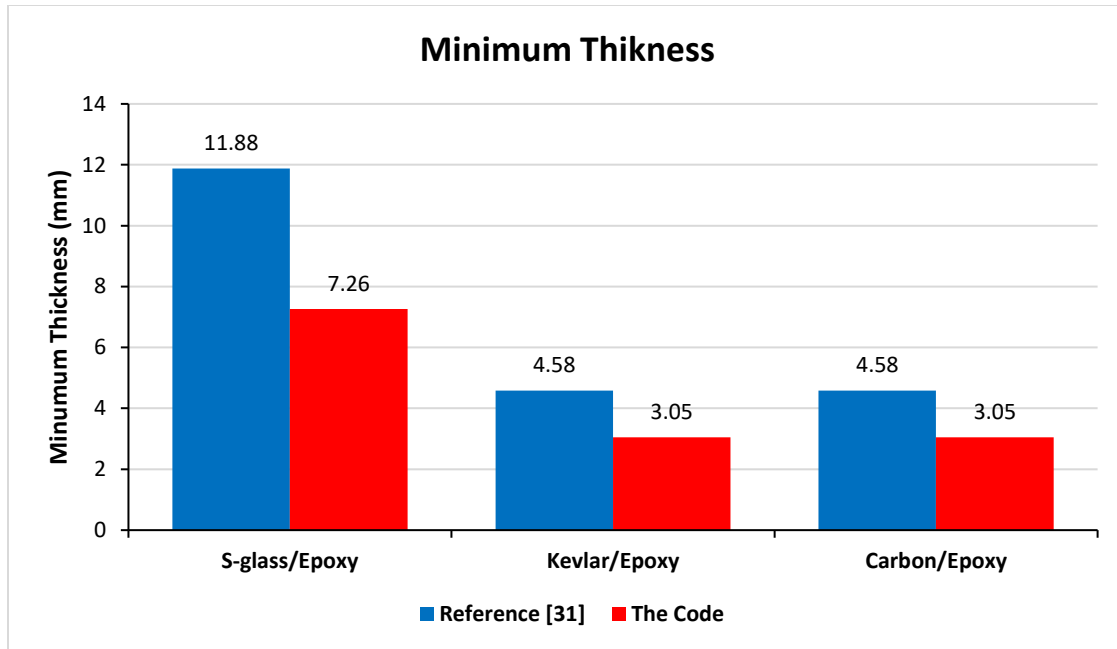


Figure 7.2: Comparison of the code with reference [31] for the minimum thickness.

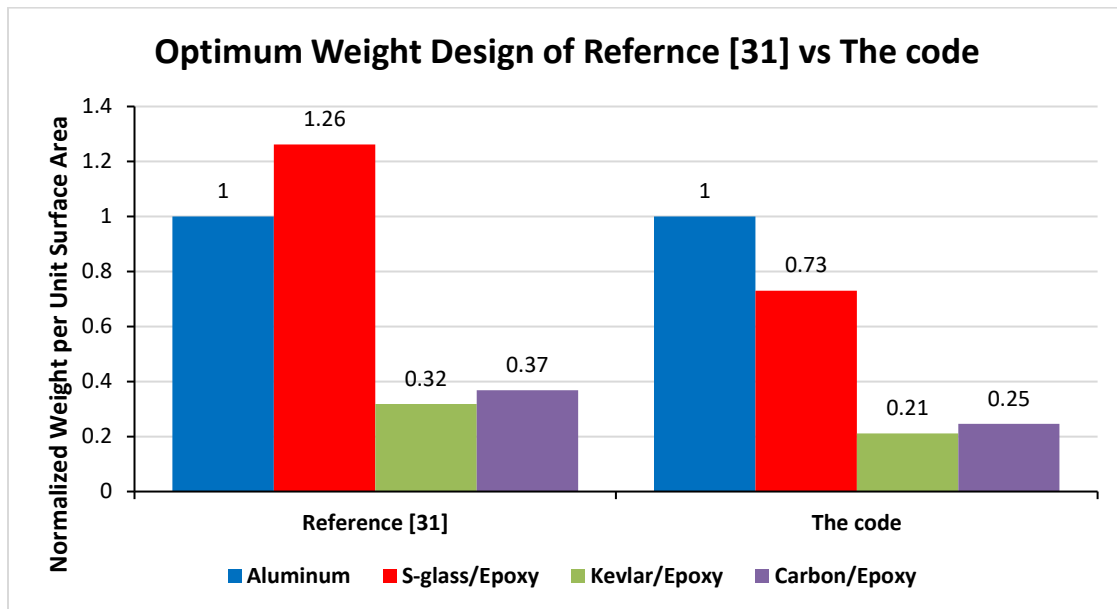


Figure 7.3: Comparison of the code with reference [31] for the minimum weight.

It can be seen from Figure 7.3 shows that as per the optimum design by reference [31], there are weight savings of 68% and 63% for Kevlar/epoxy and carbon/epoxy respectively, and a weight increment of 26% for S-glass/epoxy. On the other hand, the optimum design obtained by the code of the present study yields more economical designs

for the three materials with 79%, 75% and 27% weight saving for Kevlar/epoxy, carbon/epoxy and S-glass/epoxy, respectively.

The previous comparisons are based on Tsai-Wu failure criterion. In order to study the effect of the failure criterion type on the design, the above analysis is repeated but based on Tsai-Hill failure criterion. The results for the optimum lay-up and minimum thickness are given in Table 7.5.

Table 7.5: The present study optimum design based on Tsai-Hill of the three composite materials with $[\theta_1/\theta_2]_{ns}$ lay-up.

	<i>S – glass/Epoxy</i>	<i>Kevlar/Epoxy</i>	<i>Carbon/Epoxy</i>
Ply thickness (<i>mm</i>)	0.165	0.127	0.127
Optimized θ (<i>Deg</i>)	-81, 36	-79, 35	-80, 35
<i>SR</i> ($h = 1\text{ mm}$)	0.23202	0.46374	0.66473
Optimized Layup	$[-81/36]_{14s}$	$[-79/35]_{9s}$	$[-80/35]_{6s}$
Laminate thk. (<i>mm</i>)	9.24	4.58	3.05
Factor of Safety	2.14	2.2	2.03

From Table 7.5 it is clear that the employment of Tsai-Hill criterion results in a more conservative results for the minimum thickness compared to result based on Tsai-Wu criterion. However, all of them provide the same or better design than that provided by reference [31]. Figure 7.4 shows the comparison among the three composites in terms of the minimum thickness.

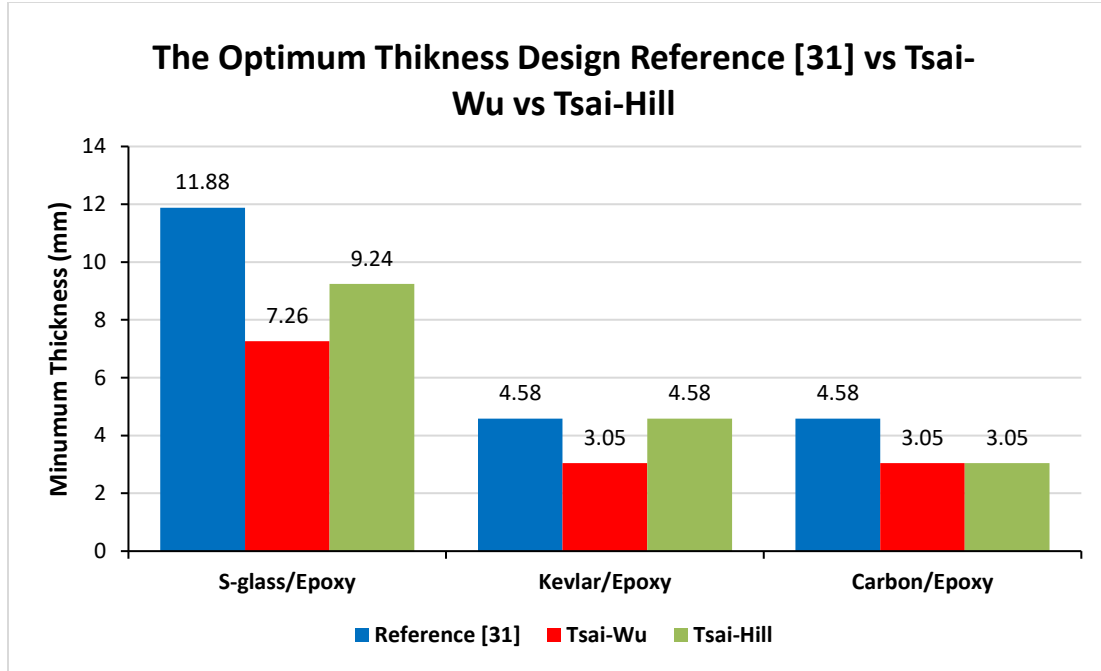


Figure 7.4: Comparison of the code with reference [31] for the minimum thickness based on Tsai-Hill criterion.

7.2 Numerical Example 2

In order to check present study results for the angles representing the optimum stacking sequence, let us consider the problem solved by Akbulut and Sonmez [22] who used the direct search simulated annealing method to minimize the thickness of laminated composite structures subject to in-plane loading. The material was assumed to be graphite/epoxy (T300/5308) with the mechanical and strength properties of $E_1 = 40.91 \text{ GPa}$, $E_2 = 9.88 \text{ GPa}$, $G_{12} = 2.84 \text{ GPa}$, $\nu_{12} = 0.292$, $X_t = 779 \text{ MPa}$, $X_c = 1134 \text{ MPa}$, $Y_t = 19 \text{ MPa}$, $Y_c = 131 \text{ MPa}$, $S = 75 \text{ MPa}$. Each ply has a thickness of 0.00127 mm and only Tsai-Wu failure criteria will be considered. As the loading is for tension loading only the optimum sequences is $[\pm\theta]_{ns}$. The detailed results obtained by the code of the present study are given in Table 7.6 While Table 7.7 gives the comparison between the two studies which shows good agreement between the two results.

Table 7.6: optimum angles according to the present study for material T300/5308.

Loading $\{N_x, N_y\}$	$\beta = \frac{N_y}{N_x}$	$\theta_{optimum}$	SR^*	$h_{min} = \frac{N_x}{SR^*}$	n
{10,5}	0.5	37°	73.403	0.13623	27
{20,5}	0.4	31°	174.728	0.1145	23
{40,5}	0.125	26°	401.168	0.0997	20
{80,5}	0.0625	20°	602.483	0.1328	27
{120,5}	0.04167	17°	676.94	0.1773	35

Table 7.7: optimum angles comparison between the present study and reference [22] for material T300/5308.

Loading $\{N_x, N_y\}$	Akbulut and Sonmez [22]		Present study	
	Optimized Layup	SR	Optimized Layup	SR
{10,5}	$[37_{27}/-37_{27}]_s$	1.0068	$[37_{27}/-37_{27}]_s$	1.0068
{20,5}	$[31_{23}/-31_{23}]_s$	1.0208	$[31_{23}/-31_{23}]_s$	1.0208
{40,5}	$[26_{20}/-26_{20}]_s$	1.0190	$[26_{20}/-26_{20}]_s$	1.0190
{80,5}	$[21_{25}/-19_{28}]_s$	1.0113	$[20_{27}/-20_{27}]_s$	1.0330
{120,5}	$[17_{35}/-17_{35}]_s$	1.0030	$[17_{35}/-17_{35}]_s$	1.0030

CONCLUSIONS AND RECOMMENDATION

8.1 Conclusions

The main objective of this thesis is to come up with closed-form expressions for the optimum stacking sequence of a laminated composite plate subjected to in-plane loading. The goal was achieved by implementing the classical lamination theory in Wolfram MATHEMATICA software which has the computational capabilities of handling the huge amount of symbolic mathematical manipulation of the problem and performing the global optimization that guarantees the achievement of the optimum design. In the optimization problem, the strength of the laminate represented by the strength ratio has been taken as the objective function to be maximized. The strength ratio was computed based on two failure criteria: Tsai-Wu and Tsai-Hill. Fiber angle orientation has been considered as the design variable with the constraint of being real numbers to be selected within the interval $\{-90^\circ, 90^\circ\}$ and forming a symmetric laminate. The three most widely used types of fiber-reinforced composites are considered, namely the glass/epoxy, the carbon/epoxy and the Aramid/epoxy are considered. The data generated by MATHEMATICA code was subjected to numerical experimentations that yielded simple design closed-form expressions and curves representing the optimum stacking sequence for different loadings, materials and failure criteria. Two types of verifications have been performed: The first one is for the code results against the results available in the literature. The second one is to check the accuracy of the closed-form expressions in predicting the code results through several numerical examples. Both verifications confirmed the accuracy of the code and

closed-form expressions. have been presented to verify the developed code and closed-form expressions.

The results of this study can be divided into two parts: one is related to laminates subjected to in-plane tensile loading ($\beta \neq 0, \gamma = 0$) and the other is related to laminates subjected to in-plane tensile and shear loadings ($\beta \neq 0, \gamma \neq 0$). Based on the optimization results and developed design formulae and design curves, the following conclusions are recognized for the first part:

- The optimum stacking sequence corresponds to a symmetric angle-ply laminate. The stacking sequence takes the form: $[(\pm\theta)_{m/2}]_s$, where m represents the number of plies in one portion of the symmetric laminate, regardless of the loading ratio (β), the material, the number of plies and the failure criterion. This observation was a key factor in reducing the huge number of angle combinations as required by the global optimization code and made it possible to reach closed-form expressions for the optimum fiber orientation angles.
- The angle-ply pattern $[(\pm\theta)_{m/2}]_s$ is consistent but the optimum value of θ (θ_{op}) changes depending on the value of β , the type of material and the failure criterion with the effect of β being more significant than the other two factors.
- The optimum angle is material independent for β less than 10 for Tsai-Wu criterion and 30 for Tsai-Hill criterion.
- Both Tsai-Hill and Tsai-Wu criteria yield almost the same value of strength ratio (SR^*) over the whole range of β for T300/5208 material. For Gl/Ep and Ar/Ep

materials, Tsai-Hill criterion yields a lower value of SR^* compared to Tsai-Wu especially for lower values of β and therefore Tsai-Hill criterion controls for conservative design.

For composite laminates subjected to normal and shear in-plane loading ($\beta \neq 0, \gamma \neq 0$), the following is concluded:

- The optimum stacking sequence takes the form: $[(\theta_1/\theta_2)_{m/2}]_s$, regardless of the loading ratios (β, γ) , the material, the number of plies and the failure criterion. The concept of stress transformation was used to modify the in-plane pure tension loading closed-form expressions to account for the shear loading.
- The angle-ply pattern $[(\theta_1/\theta_2)_{m/2}]_s$ is consistent but the optimum values of θ_1 and θ_2 changes depending on the value of α and θ_t .
- The values of the optimum angles change depending on the value of the load ratios (β, γ) , the type of material and the failure criterion with the effect of load ratios being more significant than the other factors.
- Tsai-Hill criterion yields lower value SR (more conservative), especially for Ar/Ep material.

A final observation that holds for both loading cases is that the strength capacity of T300/5208 and Ar/Ep are always higher than that of Gl/Ep regardless of the loading levels and ratios.

8.2 Recommendations

In light of the outcomes and the challenges experienced during this research, the following ideas are recommended for further investigation:

- 1- The optimal design has been found for symmetric laminated only. Non-symmetric laminates could be explored for possible further material saving.
- 2- The study can be extended to plates and shells subjected to more complicated loadings.
- 3- Confirming the obtained theoretical results with experimental work will enhance the usefulness of the proposed design formulas.

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Appendix A:

The Mathematica Code Implemented to Find the Optimum

Lay-Up Under Any In-Plane Loading, Laminate Thickness

and Number of Plies.

```
ClearAll["Global`*"];

MatProp[material_, units_] :=
If[units == "SI",
  If[material == "T300/5208", {181×10^9, 103×10^8, 717×10^7, 28/100},
  If[material == "G1/Ep", {38.6 10^9, 8.27×10^9, 4.14×10^9, 0.26},
  If[material == "Ar/Ep", {76.0 10^9, 5.50×10^9, 2.30×10^9, 0.34}]]];

MatStre[material_] :=
  If[material == "T300/5208", {1500, 1500, 40, 246, 68},
  If[material == "G1/Ep", {1062, 610, 31, 118, 72},
  If[material == "Ar/Ep", {1400, 235, 12, 53, 34}]]];

Tmat[theta_] := Module[{c, s, t}, If[NumberQ[theta], t = N[theta*Degree], t = theta];
  c = Cos[t];
  s = Sin[t];
  {{c^2, s^2, 2cs}, {s^2, c^2, -2cs}, {-cs, cs, c^2-s^2}}];

Qbmat[theta_, E1_, E2_, G12_, nu12_] :=
  Module[{t, nu21, Q, T, R, c, s}, If[NumberQ[theta], t = N[theta*Degree], t = theta];
  nu21 =  $\frac{nu12 E2}{E1}$ ;
  Q = {{ $\frac{E1}{1 - nu12 nu21}$ ,  $\frac{E1 nu21}{1 - nu12 nu21}$ , 0}, { $\frac{E2 nu12}{1 - nu12 nu21}$ ,  $\frac{E2}{1 - nu12 nu21}$ , 0}, {0, 0, G12}};
  c = Cos[t];
  s = Sin[t];
  T = {{c^2, s^2, 2cs}, {s^2, c^2, -2cs}, {-cs, cs, c^2-s^2}};
  R =  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$ ;
  Inverse[T].Q.Inverse[R].T.R];

Amat[theta_, thick_, E1_, E2_, G12_, nu12_] :=
  Module[{nt, z, h},
  nt = Length[theta]; h = thick;
  z = Range[-h/2, h/2,  $\frac{h}{2 nt}$ ];
  2 Sum[Qbmat[theta[[k]], E1, E2, G12, nu12] (z[[k+1]] - z[[k]]), {k, 1, nt}]]];
```

```

Do[Mat = Which[p == 1, "T300/5208", p == 2, "G1/Ep", p == 3, "Ar/Ep"];
Print[Style[Row[{"For Material ", Mat}], Purple, Bold, 24, Italic,
FontFamily -> "Courier"]];

{E1, E2, G12, v12} = MatProp[Mat, "SI"];
{E1, E2, G12} = {E1, E2, G12} * 10-6;

{Xt, Xc, Yt, Yc, S} = MatStre[Mat];

F11 = 1 / (Xc Xt);
F22 = 1 / (Yc Yt);
F66 = 1 / (S2);
F1 = 1 / Xt - 1 / Xc;
F2 = 1 / Yt - 1 / Yc;
F12 = -0.5  $\sqrt{F11 F22}$ ;

Nx = 1; Ny = 5; Nxy = 2;

h = 1; (* Thicknesses *)
nt = 4; (* Number of layers *)
t = h / nt; (* Single layer thickness *)
NL = {Nx, Ny, Nxy};

TH = Range[-89, 90, 1];
NN = Length[TH];
TH2 = Which[
nt == 2, Tuples[TH, 1],
nt == 4, Tuples[TH, 2],
nt == 8, Tuples[TH, 4],
nt == 16, Tuples[TH, 8]];

v = Table[Sort[TH2[[i]]], {i, 1, Length[TH2]}];
TH1 = DeleteDuplicates[v];

N1 = Length[TH1];
n = nt / 2;

Do[AA[i] = Amat[TH1[[i]], h, E1, E2, G12, v12];
e[i] = LinearSolve[AA[i], NL];

Do[Qb[s] = Qbmat[TH1[[i]][[s]], E1, E2, G12, v12], {s, 1, n}];
Do[σ[s] = Qb[s].e[i], {s, 1, n}];
Do[σx[s] = σ[s][[1]], {s, 1, n}];
Do[σy[s] = σ[s][[2]], {s, 1, n}];
Do[txy[s] = σ[s][[3]], {s, 1, n}];

Do[σg[s] = {σx[s], σy[s], txy[s]}, {s, 1, n}];
Do[σ1[s] = Tmat[TH1[[i]][[s]]].σg[s], {s, 1, n}];

```

```

Do[a1 = F11 * σ1[s] [[1]]2 + F22 * σ1[s] [[2]]2 + F66 * σ1[s] [[3]]2 + 2 F12 * σ1[s] [[1]] * σ1[s] [[2]] ;
b1 = F1 * σ1[s] [[1]] + F2 * σ1[s] [[2]] ;

SR1[i, s] = 
$$\frac{-b1 + \sqrt{b1^2 + 4 a1}}{2 a1} ;$$

SR2[i, s] =
1 / 
$$\sqrt{\left(\left(\frac{\sigma1[s] [[1]]}{x_t}\right)^2 - \left(\frac{\sigma1[s] [[1]] \sigma1[s] [[2]]}{x_t^2}\right) + \left(\frac{\sigma1[s] [[2]]}{y_t}\right)^2 + \left(\frac{\sigma1[s] [[3]]}{s}\right)^2}\right)} ;$$

(s, 1, n], {i, 1, N1}] ;
SRW = Flatten[Table[Min[Table[SR1[i, s], {s, 1, n}], {i, 1, N1}}];
SRH = Flatten[Table[Min[Table[SR2[i, s], {s, 1, n}], {i, 1, N1}}];

x1 = Flatten[Position[SRW, Max[SRW]]];
x2 = Flatten[Position[SRH, Max[SRH]]];

Print[Style[Row[{"Optimum Stacking Sequence For  $\beta =$ ",  $\beta$ }], Blue, Bold, 16,
  Italic, FontFamily → "Courier"]];
Print[
  Style[Row[{"Tsai-Wu:", Table[TH1[x1[[i]]], {i, 1, Length[x1]]}, "",
    " With Maximum SR=", Max[SRW]], Red, Bold, 16, Italic, FontFamily → "Courier"]];
Print[
  Style[Row[{"Tsai-Hill", Table[TH1[x2[[i]]], {i, 1, Length[x2]]}, "",
    " With Maximum SR=", Max[SRH]], Red, Bold, 16, Italic, FontFamily → "Courier"]];
Print[
  Style[
    Row[{"_____"}],
    Green, Bold, 20, Italic, FontFamily → "Courier"]];, {p, 1, 3}]

```

Appendix B:

The Mathematica Code Implemented to Find the Optimum

Lay-Up Under Any Load Ratio (β) Without Considering Any

Shear Loading.

```
ClearAll["Global`*"];

MatProp[material_, units_] :=
If[units == "SI",
  If[material == "T300/5208", {181*10^9, 103*10^8, 717*10^7, 28/100},
  If[material == "G1/Ep", {386*10^8, 827*10^7, 414*10^7, 26/100},
  If[material == "Ar/Ep", {76*10^9, 550*10^7, 230*10^7, 34/100}]]];

MatStre[material_] :=
  If[material == "T300/5208", {1500, 1500, 40, 246, 68},
  If[material == "G1/Ep", {1062, 610, 31, 118, 72},
  If[material == "Ar/Ep", {1400, 235, 12, 53, 34}]];

Tmat[theta_] := Module[{c, s, t}, If[NumberQ[theta], t = theta*Degree, t = theta];
  c = Cos[t];
  s = Sin[t];
  {{c^2, s^2, 2cs}, {s^2, c^2, -2cs}, {-cs, cs, c^2 - s^2}}];

Qbmat[theta_, E1_, E2_, G12_, nu12_] :=
  Module[{t, nu21, Q, T, R, c, s}, If[NumberQ[theta], t = theta*Degree, t = theta];
  nu21 =  $\frac{nu12 E2}{E1}$ ;
  Q = {{ $\frac{E1}{1 - nu12 nu21}$ ,  $\frac{E1 nu21}{1 - nu12 nu21}$ , 0}, { $\frac{E2 nu12}{1 - nu12 nu21}$ ,  $\frac{E2}{1 - nu12 nu21}$ , 0}, {0, 0, G12}};
  c = Cos[t];
  s = Sin[t];
  T = {{c^2, s^2, 2cs}, {s^2, c^2, -2cs}, {-cs, cs, c^2 - s^2}}];
  R =  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$ ;
  Inverse[T].Q.Inverse[R].T.R];

Amat[theta_, thick_, E1_, E2_, G12_, nu12_] :=
  Module[{nt, z, h},
  nt = Length[theta]; h = thick;
  z = Range[-h/2, h/2,  $\frac{h}{2 nt}$ ];
  2 Sum[Qbmat[theta[[k]], E1, E2, G12, nu12] (z[[k+1]] - z[[k]]), {k, 1, nt}];
```

```

Do[Mat = Which[p == 1, "T300/5208", p == 2, "G1/Ep", p == 3, "Ar/Ep"];

{E1, E2, G12, v12} = MatProp[Mat, "SI"];
{E1, E2, G12} = {E1, E2, G12} * 10-6;

{Xt, Xc, Yt, Yc, S} = MatStre[Mat];

F11 = 1 / (Xc Xt);
F22 = 1 / (Yc Yt);
F66 = 1 / (S2);
F1 = 1 / Xt - 1 / Xc;
F2 = 1 / Yt - 1 / Yc;
F12 = -1 / 2 * Sqrt[F11 F22];

h = 1; (* Thicknesses *)
nt = 4; (* Number of layers *)
t = h / nt; (* Single layer thickness *)
NL = {1, β, 0};

TH1 = {θ, -θ};
N1 = Length[TH1];
n = nt / 2;
AA = Amat[TH1, h, E1, E2, G12, v12];
ε0 = LinearSolve[AA, NL];
Qbθ = Qbmat[θ, E1, E2, G12, v12];
σgθ = Simplify[Qbθ.ε0];
Tθ = Tmat[θ];
σlθ = Simplify[Tθ.σgθ];

SRH[p] = 1 / Sqrt[ $\left(\frac{\sigma l \theta[[1]]}{Xt}\right)^2 - \left(\frac{\sigma l \theta[[1]] \sigma l \theta[[2]]}{Xt^2}\right) + \left(\frac{\sigma l \theta[[2]]}{Yt}\right)^2 + \left(\frac{\sigma l \theta[[3]]}{S}\right)^2$ ];

a1 = F11 * σlθ[[1]]2 + F22 * σlθ[[2]]2 + F66 * σlθ[[3]]2 + 2 F12 * σlθ[[1]] * σlθ[[2]];
b1 = F1 * σlθ[[1]] + F2 * σlθ[[2]];

SRW[p] =  $\frac{-b1 + \sqrt{b1^2 + 4 a1}}{2 a1}$ ; , {p, 1, 3}];

```

```

 $\beta\beta = \text{Table}[\mathbf{i}, \{\mathbf{i}, 0, 1, 1/10000\}];$ 
Do[If[p == 1, s = 0.14, If[p == 2, s = 12/100, s = 14/100]];
Do[SR[p] = SRH[p] /. { $\beta \rightarrow \beta\beta[[\mathbf{i}]]$ };
sol1 = FindMaximum[SR[p], { $\theta$ , s}];
SRmax[i] = sol1[[1]];
 $\theta_{\text{opt}}[\mathbf{i}] = \frac{180}{\pi} (\theta /. \text{sol1}[[2]]), \{\mathbf{i}, 1, \text{Length}[\beta\beta]\};$ 
DH1[p] = Table[{ $\beta\beta[[\mathbf{i}]]$ ,  $\theta_{\text{opt}}[\mathbf{i}]$ }, {i, 1, Length[ $\beta\beta$ ]}];, {p, 1, 3}];

Do[Do[SR1[p] = SRW[p] /. { $\beta \rightarrow \beta\beta[[\mathbf{i}]]$ };
sol1 = FindMaximum[SR1[p], { $\theta$ , 0.15}];
SRmax[i] = sol1[[1]];
 $\theta_{\text{opt}}[\mathbf{i}] = \frac{180}{\pi} (\theta /. \text{sol1}[[2]]), \{\mathbf{i}, 1, \text{Length}[\beta\beta]\};$ 
DW1[p] = Table[{ $\beta\beta[[\mathbf{i}]]$ ,  $\theta_{\text{opt}}[\mathbf{i}]$ }, {i, 1, Length[ $\beta\beta$ ]}];, {p, 1, 3}];

 $\beta\beta = \text{Join}[\text{Table}[\mathbf{i}, \{\mathbf{i}, 1, 2, 1/50\}], \text{Table}[\mathbf{i}, \{\mathbf{i}, 2, 600, 1/10\}]];$ 
Do[If[p == 3, s = 1.26, s = 1.4]; Do[SR[p] = SRH[p] /. { $\beta \rightarrow \beta\beta[[\mathbf{i}]]$ };
sol1 = FindMaximum[{SR[p]}, { $\theta$ , s}];
SRmax[i] = sol1[[1]];
 $\theta_{\text{opt}}[\mathbf{i}] = \frac{180}{\pi} (\theta /. \text{sol1}[[2]]), \{\mathbf{i}, 1, \text{Length}[\beta\beta]\};$ 
DH2[p] = Table[{ $\beta\beta[[\mathbf{i}]]$ ,  $\theta_{\text{opt}}[\mathbf{i}]$ }, {i, 1, Length[ $\beta\beta$ ]}];, {p, 1, 3}];

Do[Do[SR1[p] = SRW[p] /. { $\beta \rightarrow \beta\beta[[\mathbf{i}]]$ };
sol1 = FindMaximum[{SR1[p]}, { $\theta$ , 1.4}];
SRmax[i] = sol1[[1]];
 $\theta_{\text{opt}}[\mathbf{i}] = \frac{180}{\pi} (\theta /. \text{sol1}[[2]]), \{\mathbf{i}, 1, \text{Length}[\beta\beta]\};$ 
DW2[p] = Table[{ $\beta\beta[[\mathbf{i}]]$ ,  $\theta_{\text{opt}}[\mathbf{i}]$ }, {i, 1, Length[ $\beta\beta$ ]}];, {p, 1, 3}];

```

Vitae

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