ASYMPTOTIC BEHAVIOR STABILITY FOR SOME GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS

BY

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To my parents, husband, children, sisters and brothers who waited patiently for me to come out of this study.

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THESIS ABSTRACT

NAME:	Fatimah Yousef Muhammed Alkhaldi
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Of concern is a generalized fractional derivative which covers many of the existing non-integer derivatives in the literature so far such as Riemann-Liouville, Caputo, Prabhakar, Erdelyi-Kober and Hadamard derivative. First, we prove an existence result in a appropriate space for an important class of fractional differential problems with nonlinear sources. This family of problems involves both discrete delays and neutral type delays. The main tool is Krasnoselskii fixed point theorem applied to the corresponding integral equation which turns out to be equivalent to the fractional differential problem in a suitable space.

For a class of fractional differential problems (involving the above mentioned kinds of derivatives) with a large class of right-hand sides which may contain polynomials in the state, it is shown that solutions are stable. Moreover, it is established that the rate of stability is of the order a negative power of the function with respect to which the fractional derivative is calculated. In fact,

we ascertain that the absolute value of the state is bounded by this latter expression. To this end, we generalize a well-known inequality to cope with the current general situation.

Finally, we demonstrate that a polynomial source, which usually may cause blow-up in finite time for solutions of differential equations, is responsible for non-existence of a nontrivial global solution. This is shown, for our generalized fractional derivative, despite the presence of lower-order terms which may play the role of dampers. We use the technique developed by Mitidieri and Pohozaev based on an appropriate choice of a test function which we will determine.

ملخص الرسالة

الاسم الكامل: **فاطمة يوسف محمد الخالدي**

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الاشتقاق الكسري المعمم هو مصدر اهتمام للبحاثين حيث يغطي العديد من المشتقات غير الصحيحة الموجودة حتى الآن مثل مشتقات ريمان-ليوفيل وبرابهاكار وكابوتو واردیلي-كوبر وهارامر.
في هذه الأطروحة تطرقنا لهذا النوع من الاشتقاق

أولا، أثبتنا وجود نتيجة وحل في فضاء مناسب لفئة مهمة من مشاكل التفاضل الجزئي مع المصادر غير الخطية. تتضمن هذه المجموعة من المشاكل كلاً من التأخيرات المنفصلة والتأخيرات من النوع المحايد. الأداة الرئيسية هي تطبيق نظرية النقطة الثابتة كرانسولسيسكي على المعادلة التكاملية المقابلة والتي تبين أنها معادلة لمسألة التفاضل الكسري التي تكون متحققة في فضاء مناسب.

بالنسبة الى الفئة من مسائل التفاضل الكسري (تشمل نوع الاشتقاق السابق ذكره) مع فئة كبيرة من الجهه اليمنى التي قد تحتوي على كثيرات حدود، أتضح أن الحلول مستقرة .

أخيرًا ، وضحنا أن مصدرًا متعدد الحدود ، والذي قد يتسبب عادةً لحدوث انفجاراً خلال وقت محدد لحلول المعادلات التفاضلية ، وهو المسؤول عن عدم وجود حل عالمي غير بديهي. يظهر هذا، بالنسبة لمشتقنا الكسري المُنشأ، على الرغم من وجود شروط قد تلعب دور المخمدات. ولقد أستخدمنا لذلك التقنية التي طورها ميديتيري وبوهوزاف لاختيار الاختبار المناسب والتي سوف نحددها فيما بعد.

INTRODUCTION

1 Introduction

It is by now well-admitted that fractional calculus is the best way to deal with materials and phenomena depending on the entire prehistory of the concerned state. The other way is using nonlinear models which is often a cumbersome and costly approach. Indeed, there are many phenomena which cannot be described in Newtonian terms. Fractional calculus provides a nonlocal term (usually referred to as a memory term as well) to account for this dependence. Therefore, fractional derivatives are now considered to be an appropriate manner to describe such hereditary phenomena which occur, for instance, in fractal media.

In this thesis, we will generalize three different results in such a way that they become special cases of ours. These three results have been proved for different types of fractional derivatives such as Riemann-Liouville, Caputo, Hadamard and/or Hilfer fractional derivative. Here we consider a 'generalized' fractional derivative which encloses all these fractional derivatives and others as special cases. This kind of fractional derivatives goes back at least to Edelyi (1964), Dzhrbashyan (1967), Osler (1970) and Krasnov (1977) and was reported in the book of Samko, Kilbas and Marichev [72]. In this book it was called a 'fractional derivative of a function with respect to another function'. The corresponding 'generalized' fractional integral is defined (roughly) as follows

$$I_{a+}^{\alpha;\phi}\, g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \phi'(s)\,(\phi(t) - \phi(s))^{\alpha-1} g(s)\, ds,\ t \geq a,$$

the Riemann-Liouville fractional derivative

$$^{RL}\, D_{a^+}^{\alpha,\phi} g(t) := \left(\frac{1}{\phi'(t)} \frac{d}{dt} \right)^n I_{a^+}^{n-\alpha;\phi}\, g(t),\ t \geq a$$

and the Caputo version

$$^{c}\, D_{a^+}^{\alpha,\phi} g(t) := I_{a^+}^{n-\alpha;\phi}\, g_{\phi}^{[n]}(t),\ t \geq a$$

where

$$g_{\phi}^{[n]}(t) := \left(\frac{1}{\phi'(t)} \frac{d}{dt} \right)^n g(t).$$

It can be easily seen that by putting $\phi(t) = t$ or $\phi(t) = \ln t$, we recover the well-known Riemann-Liouville fractional derivative (or Caputo fractional derivative) and the Hadamard fractional derivative. Apparently, researchers unaware of this 'generalized' derivative are referring to the case $\phi(t) = t^{\sigma}$ as the 'Katugampola' derivative.

1.1 The Problems

This subsection contains briefly the problems we intend to investigate.

Problem 1: We consider the following initial value problem

$$\begin{cases} D_{t_0}^{\alpha;\phi}\big(x\,(t) - k(t,x_t)\big) = g\,(t,x_t)\,,\ t \in (t_0,\infty), \\ x_{t_0} = \chi \end{cases}$$

where $D^{\alpha;\phi}x(t)$ is the generalized Caputo fractional derivative of order $0 < \alpha < 1$ of the function $x(t)$ with respect to the function $\phi(t)$ defined below (Definition 3.5.1) and x_t accounts for the delayed function of $x(t)$. Here $k(t,x_t)$ describes the neural delay which often is unavoidable and cannot be neglected without serious consequences in practical problems. Due to these two kinds of delays, this problem is often referred to as a 'neutral fractional functional differential' problem.

Problem 2: Our second problem is

$$\begin{cases} D_{t_0}^{\alpha,\beta;\phi}x\,(t) = g\,\big(t,x(t)\big)\,,\ t \in (t_0,\infty), \\ I_{t_0}^{(1-\alpha)(1-\beta)}x(t)\Big|_{t=t_0} = b \end{cases}$$

where $D_{t_0}^{\alpha;\phi}x(t)$ is the Phi-Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ of the function $x(t)$ with respect to the function $\phi(t)$ and $I_{t_0}^{(1-\alpha)(1-\beta)}x(t)$ is the fractional integral defined below. This kind of fractional derivative is obtained in the same way the Hilfer derivative is derived from the Riemann-Liouville derivative and the Caputo derivative. Therefore, this derivative interpolates the Riemann-Liouville generalized fractional derivative ($\beta = 0$) and the Caputo generalized fractional derivative ($\beta = 1$).

Problem 3: The third problem concerns the two-terms fractional differential inequality

$$\begin{cases} D_{t_0}^{\alpha;\phi} x\left(t\right) + D_{t_0}^{\beta;\phi} x\left(t\right) \geq \left(\phi(t) - \phi(t_0)\right)^{\gamma} \left|x(t)\right|^{m}, \ t > t_0, \ 0 < \beta \leq \alpha \\ x^{(i)}(t_0) = b \in R, \ i = 0, 1, 2, ...n - 1, \ n = -[-\alpha] \end{cases}$$

in presence of a polynomial source which is often called blow up term.

1.2 Objectives and methodology

Objective 1: For the first problem, we will establish an existence result in the presence of both delays (discrete and neutral). We will first prove the equivalence of the fractional differential problem with its corresponding (Volterra) integral equation in a suitable space. Then, we use the Krasnoselskii fixed point theorem to obtain the existence of a solution.

Objective 2: The question of stability is addressed for the second problem. A class of functions $g\left(t, x\right)$ is determined for which solutions are stable. More precisely, for functions $g\left(t, x\right)$ satisfying a condition of the form

$$\left|g\left(t, x\right)\right| \leq \left(\phi(t) - \phi(t_0)\right)^{\mu} \chi(t) \left|x(t)\right|^{m}, \ \mu \geq 0, \ m > 1, \ t > t_0,$$

it is proved that solutions decay with the rate

$$\left|x\left(t\right)\right| \leq C \left(\phi(t) - \phi(t_0)\right)^{-\rho}, \ \rho > 0, \ t > t_0.$$

This result is in line with what was obtained in the earlier works for special cases of fractional derivatives (see literature review below). To prove this result, we were forced to establish first an inequality of the form

$$[\phi(t)-\phi(a)]^{\lambda}\int_0^1 (1-\xi)^{\gamma-1}\,\xi^{\lambda-1}\left\{\phi^{-1}[\phi(a)+\xi\phi(t)-\phi(a))]\right\}^{\omega}d\xi \le U$$

which is in fact a generalization of an existing one with $\phi(t)=t$, also proven in the case $\phi(t)=\ln t$ (see literature review below). In addition to this inequality we utilize several adequate estimations on the corresponding integral equation.

Objective 3: Regarding the third problem, we prove that nontrivial solutions cannot exist globally. This is established for $\beta \le \alpha$. The case $\beta=\alpha$ reduces to only one fractional derivative. If $\beta<\alpha$, then the lower-order term $D_{t_0}^{\beta;\phi}x\,(t)$ might help stabilizing the system. Therefore, it will enter into competition with the blow-up term $\left|x(t)\right|^m$. Hence, it is interesting to shed some light about what will happen in this situation. The main tool here is the technique due to Mitidieri and Pohozaev. To be able to use this technique we had to come up with a satisfactory test function to work with.

1.3 Dissertation outline

After the above introduction where we explained briefly the need of fractional calculus, stated our problems and specified our objectives, we shall describe rest of the document. There remains six other chapters. Chapter 2 contains a literature review of previous works on existence, asymptotic behavior, and nonexistence of solutions for fractional differential problems. In Chapter 3 we gather all the material needed in this document. Chapter 4 is devoted to the existence of solutions for the fractional neutral functional differential problem (Problem 1). In Chapter 5 we state and prove our asymptotic behavior result for Problem 2. The nonexistence of nontrivial global solutions for Problem 3 is shown in Chapter 6. Finally, in Chapter 7 we give an overall conclusion on the possible generalizations, extensions and future works.

LITERATURE REVIEW

2 Literature Review

In this chapter, we review works concerning existence, nonexistence, blow-up, and the asymptotic behavior of solutions of some differential equations of integer and noninteger orders. Section 2.1 contains results on existence of solutions for some classes of fractional differential equations. In Section 2.2, we survey recent results on the asymptotic behavior of solutions for some FDEs. Section 2.3 is devoted to some nonexistence and blow-up results existing in the literature.

2.1 Existence of solutions

The most studied topic in differential equations in general is without doubt the well-posedness question. Fractional differential equations are not an exception. Most of the existing results so far are concerned with existence and uniqueness of solutions. Different methods such as fixed point theorems have been used. This is documented in many articles and even books such as Diethelm [24], Kilbas, Srivastava, and Trujillo [46], Zhou [78] and Ahmad, Alsaedi, Ntouyas and Tariboon [6] to cite few. In [16] Boutiara et al. discussed the existence of solutions for a new class of hybrid Hilfer fractional differential with boundary hybrid conditions. They used the hybrid fixed point theorem for the sum of three operators to achieve their goal (see also

Sitho et al. [73]). El Borai and Abbas proved in [25] existence theorems for some integro-differential equations of fractional orders under mixed generalized Lipschitz and Caratheodory conditions. Niazi et al. [61] studied the existence of solutions of initial value problems for a class of hybrid fractional neutral differential equations. They also used the hybrid fixed point theorem for the sum of three operators to prove their result. Abbas [1] established and existence and uniqueness result for a class of nonlinear sequential ψ-Hilfer fractional differential equations. The main tool was Schaefer's fixed point theorem and a modified version of the contraction principle.

More precisely, in [13], Baleanu at al. proved some existence results for the fractional neutral functional differential equations with bounded delay of the form

$$\begin{cases} {}^{C}D^{\alpha}\left(w\left(t\right)-h(t,w_t)\right)=g\left(t,w_t\right),\ t\in(t_0,\infty), \\ w_{t_0}=\chi \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative.

In [22], Benchohra et al. considered the IVP for a class of fractional neutral functional differential equations with infinite delay of the form

$$\begin{cases} {}^{RL}D^{\alpha}w\left(t\right)=g\left(t,w_t\right),\ t\in(0,\infty), \\ w(t)=\chi,\ t\in(-\infty,0) \end{cases}$$

where ${}^{RL}D^{\alpha}$ is the Riemann-Liouville fractional derivative.

While, in [77] Zhou et al. investigated the existence and uniqueness for fractional functional differential equations with unbounded and infinite delay. Kassim, Furati and Tatar in [34] studied the existence of solutions for

$$\begin{cases} \left({}_{H}\mathcal{D}_{a^{+}}^{\alpha,\beta}y\right)(w) = g\,(w,y)\,,\ w > a > 0 \\ \left(\mathcal{J}_{a^{+}}^{(1-\beta)(1-\alpha)}y\right)(a) = c \end{cases}$$

where ${}_{H}\mathcal{D}_{a^{+}}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative.

2.2 Asymptotic behavior

As opposite to the well-posedness issue, the asymptotic behavior and stability for fractional differential problems is less studied. We mention works here. The behavior of solutions of the nonlinear problem

$$\begin{cases} D_{0}^{\alpha}w\,(t) = g\left(t,w\,(t)\right)\,,\ t > 0, \\ t^{1-\alpha}w\,(t)\,|_{t=0} = w_{0} \end{cases}$$

has been considered by Furati and Tatar in 2005 in [5]. They proved that solutions decay as a power type function on their interval of existence provided that $g\,(t,w)$ satisfies the condition

$$\left|g\,(t,w)\right| \leq t^{\mu}e^{-\sigma t}\chi(t)\left|w(t)\right|^{m},\ \mu \geq 0,\ m > 1,\ \sigma > 0,\ t > 0 \tag{2.1}$$

where $\chi(t)$ is a continuous function on $[0,\infty)$. In 2012, Furati, Kassim and Tatar in [34] studied the nonlinear fractional differential problem with Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$

$$\begin{cases} D^{\alpha,\beta} w\left(t\right) = g\left(t, w\left(t\right)\right), \ t > 0, \\ t^{(1-\alpha)(1-\beta)} w\left(t\right)|_{t=0} = w_0 \end{cases} \tag{2.2}$$

and showed that the solution of (2.2) also decays as a power function under the same condition (2.1) on the function $g(t, y)$. In 2013, Kassim and Tatar in [44] showed the stability for a differential problem with Hilfer-Hadamard fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ of the form

$$\begin{cases} D_a^{\alpha,\beta} w\left(t\right) = g\left(t, w\left(t\right)\right), \ t > 0, \\ \left(\ln \frac{t}{a}\right)^{(1-\alpha)(1-\beta)} w\left(t\right)|_{t=0} = w_0 \end{cases}$$

under the condition

$$\left|g\left(t, w\right)\right| \leq \left(\ln \frac{t}{a}\right)^{\mu} \chi(t) \left|w(t)\right|^{m}, \ \mu \geq 0, \ m > 1, \ t > a.$$

We mention also the following problem

$$\begin{cases} D_0^{\alpha} w\left(t\right) = g\left(t, w\left(t\right), D_0^{\beta} w\left(t\right)\right), \ 0 < \beta < \alpha < 1, \ t > 0 \\ I_0^{1-\alpha} w\left(t\right)|_{t=0} = w_0, \end{cases}$$

studied under the condition

$$\left|g\left(t, u, v\right)\right| \leq t^{\gamma} e^{-\sigma t} l\left(t\right) \chi_1\left(t^{1-\alpha}\left|u\right|\right) \chi_2\left(t^{1-(\alpha-\beta)}\left|v\right|\right)$$

or

$$\left|g\left(t, u, v\right)\right| \leq t^{\gamma_1} e^{-\sigma_1 t} l_1\left(t\right) \chi_1\left(t^{1-\alpha}\left|u\right|\right) + t^{\gamma_2} e^{-\sigma_2 t} l_2\left(t\right) \chi_2\left(t^{1-(\alpha-\beta)}\left|v\right|\right).$$

2.3 Non-existence

The same observation can be made for the blow up in finite time and non-existence of solutions. We can find only relatively fewer papers in the literature on these issues. Some of these publications were motivated by the problems

$$\begin{cases} w'(t) = t^{\gamma} \left| w(t) \right|^{m}, \ t > 0, \ m > 1 \\ w(0) = w_0 \end{cases}$$

and

$$\begin{cases} w'(t) + w(t) = w(t)^{m}, \ t > 0, \\ w(0) = w_0. \end{cases}$$

Under certain condition , these problem have

$$w(t) = \left[\frac{1-m}{1+\gamma} t^{1+\gamma} + w_0^{1-m} \right]^{\frac{1}{1-m}}$$

and

$$w(t) = \left[1 + (b^{1-m} - 1) e^{(m-1)t} \right]^{\frac{1}{1-m}}, t \geq 0$$

as solutions which blow-up in finite time $\frac{1}{1-m} \ln \left(1 - b^{1-m}\right), m, b > 1$, resp.

The two-term fractional problem

$$\begin{cases} D_0^{\alpha} w(t) + D_0^{\beta} w(t) \geq t^{\gamma} \left| w(t) \right|^{m}, \ t > 0, \\ I_0^{1-\alpha} w(t) \mid_{t=0} = w_0 \end{cases}$$

was investigated by [44]. The authors established sufficient conditions ensuring non-existence of nontrivial global solutions.

In case $\alpha = \beta$, $0 < \alpha < 1$, and Riemann-Liouville fractional derivative, that is

$$\begin{cases} {}^{RL}D_0^{\alpha} w(t) \geq t^{\gamma} \left| w(t) \right|^m, \ t > 0, \\ I_0^{1-\alpha} w(t) \left.\right|_{t=0} = w_0, \end{cases}$$

the problem was studied by Laskri and Tatar in [51]. The same problem with Caputo derivative is treated in [44] by Kassim et al.

The Hadamard version of this type of problems

$$\begin{cases} \mathcal{D}_a^{\alpha} w(t) + \mathcal{D}_a^{\beta} w(t) \geq \left(\log \frac{t}{a}\right)^{\gamma} \left| w(t) \right|^m, \ t > a > 0, \ m > 1, \ \gamma \in \mathbb{R}, \\ \mathcal{I}_a^{1-\alpha} w(t) \left.\right|_{t=a} = w_0, \end{cases}$$

where $\mathcal{D}_a^{\sigma}$ is the Hadamard fractional derivative may be found in [16].

The next lemma is an adaptation of a corresponding one in Michaski [55] and Kassim and N. Tatar in [44]. We are citing it here to compare with later.

Lemma 2.3.1: [44] *If* $\lambda, \ \nu, \ \omega > 0$, *then for any* $t > a$, $a > 0$ *we have*

$$\left(\log \frac{t}{a}\right)^{1-\nu} \int_a^t \left(\log \frac{t}{s}\right)^{\nu-1} \left(\log \frac{s}{a}\right)^{\lambda-1} \left(\frac{s}{a}\right)^{-\omega} \frac{ds}{s} \leq C$$

for some positive constant C.

PRELIMINARIES

3 Preliminaries

This chapter contains definitions, lemmas, propositions and properties of different fractional derivatives. We also fix the notation used in this document. Moreover, some other results needed in our proofs are presented here. For more details, we refer the reader to [65,44,71]. We designate by $J = [a, b] \subset \mathbb{R} = (-\infty, \infty)$, $\delta = t\frac{d}{dt}$, $n \in \mathbb{N}$ and assume that $0 \leq \gamma < 1$.

3.1 Underlying spaces

In this subsection we present some of the different spaces used in the theory of fractional calculus [46].

Definition 3.1.1 : [46] *Let $-\infty \leq a < b \leq \infty$. The space $L_p(a, b)$, $1 \leq p \leq \infty$ is the usual space of Lebesgue (real-valued) measurable functions g on J having $\|g\|_p < \infty$, where*

$$\|g\|_p = \left(\int_J |g(s)|^p \, ds \right)^{1/p}, \; 1 \leq p < \infty,$$

and $\|g\|_\infty$ is the essential supremum of the function $g(t)$.

Definition 3.1.2 : [46] *Let $-\infty < a < b < \infty$. The space $C^m(\Omega)$ contains all m times continuously differentiable functions g on J endowed with the* *norm*

$$\|g\|_{\mathbf{C}^m} = \sum\nolimits_{k=0}^{m} \left\|g^{(k)}\right\|_{\mathbf{C}} = \sum\nolimits_{k=0}^{m} \max_{x\in\Omega} \left|g^{(k)}\left(t\right)\right|, \ m = 0, 1, 2, ...$$

Definition 3.1.3 : [46] *Let J be a finite interval, the space $C_\gamma\left[a,b\right]$ consists of all continuous functions g on $(a,b]$, such that $(t-a)^\gamma g\left(t\right) \in C(J)$, i.e.*

$$C_\gamma(J) = \left\{g : (a,b] \to \mathbb{R}\text{:}\ (t-a)^\gamma g\left(t\right) \in C\left[a,b\right]\right\}.$$

In this space we adopt the norm

$$\|g\|_{C_\gamma} = \left\|(t-a)^\gamma g\left(t\right)\right\|_C.$$

Definition 3.1.4 : [46] *The space $C_\gamma^n(J)$ accounts for all continuously differentiable functions on $[a,b]$ up to order $n-1$ such that $g^{(n)} \in C_\gamma(J)$ endowed with the norm*

$$\|g\|_{C_\gamma^n} = \sum_{k=0}^{n-1} \left\|g^{(k)}\right\|_C + \left\|g^{(n)}\right\|_{C_\gamma}.$$

Definition 3.1.5 : *Let J be a finite interval, we introduce the space*

$$C_{\gamma,\phi}\left[a,b\right] = \left\{g : (a,b] \to \mathbb{R}\text{:}\ \left(\phi(t)-\phi(a)\right)^\gamma g\left(t\right) \in C(J)\right\}$$

Definition 3.1.6 : *The space $C_{\gamma,\phi}^{\alpha}(J)$ is defined as follows*

$$C_{\gamma,\phi}^{\alpha}\left[a,b\right] = \left\{g : (a,b] \to \mathbb{R}\text{:}g\left(t\right) \in C_{\gamma,\phi}(J),\ D^\alpha\left(\phi(t)-\phi(a)\right)^\gamma g\left(t\right) \in C(J)\right\}$$

Definition 3.1.7 : [46] *Let J be a finite interval, we introduce the space*

$$C_{\gamma,\log}[a,b] = \left\{ g : (a,b] \to \mathbb{R}\colon \left(\log \frac{t}{a}\right)^{\gamma} g(t) \in C(J) \right\}$$

with the norm

$$\|g\|_{C_{\gamma,\log}} = \left\| \left(\log \frac{t}{a}\right)^{\gamma} g(t) \right\|_{C}.$$

Definition 3.1.8: [46] *The space $C^{n}_{\delta,\gamma}(J)$ is defined as follows*

$$C^{n}_{\delta,\gamma}(J) = \left\{ g : \delta^{k} g \in C(J),\ k = 0,...,n-1,\ \delta^{n} g \in C_{\gamma,\log}(J) \right\}.$$

It is endowed with the norm

$$\|g\|_{C^{n}_{\delta,\gamma}} = \sum_{k=0}^{n-1} \left\| \delta^{k} g \right\|_{C} + \|\delta^{n} g\|_{C_{\gamma,\log}}.$$

3.2 Riemann-Liouville fractional integral and fractional derivative

The definitions of the Riemann-Liouville fractional integral and fractional derivative given by

Definition 3.2.1: [46] *Let J be a finite interval. The Riemann-Liouville left-sided fractional integral $I^{\alpha}_{a+}g$ of order $\alpha > 0$ is defined by*

$$\left(I^{\alpha}_{a^{+}} g\right)(t) := \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \frac{g(s)}{(t-s)^{1-\alpha}} ds,\ a < t < b,\ \alpha > 0$$

whenever the integral makes sense. Here $\Gamma(\alpha)$ *is the standard Euler Gamma function* $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,\ z > 0$ *where* $t^{z-1} = e^{(z-1)\log t}$.

Definition 3.2.2 : [46] *Let* J *be a finite interval. The Riemann-Liouville right-sided fractional integral* $I_{b\text{-}}^{\alpha}g$ *of order* $\alpha > 0$ *is defined by*

$$\left(I_{b^-}^{\alpha}g\right)(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{g(s)}{(s-t)^{1-\alpha}}ds,\ a \le t < b,\ \alpha > 0$$

whenever the integral makes sense.

Definition 3.2.3 : [46] *Let* J *be a finite interval. The Riemann-Liouville left-sided fractional derivative* $D_{a+}^{\alpha}g$ *of order* α, $0 \le \alpha < 1$ *is defined by*

$$\left(D_{a+}^{\alpha}g\right)(t) = \frac{d}{dt}\left(I_{a+}^{1-\alpha}g\right)(t).$$

Definition 3.2.4 : [46] *Let* J *be a finite interval. The Riemann-Liouville right-sided fractional derivative* $D_{b^-}^{\alpha}g$ *of order* α, $0 \le \alpha < 1$ *is defined by*

$$\left(D_{b^-}^{\alpha}g\right)(t) = -\frac{d}{dt}\left(I_{b^-}^{1-\alpha}g\right)(t).$$

Proposition 3.2.1 : [46] *Let* $g \in C_{\gamma}^{1}$, *then the fractional derivatives* $D_{a^+}^{\alpha}$ *and* D_{b}^{α}, $0 < \alpha < 1$ *exist on* $(a, b]$ and $[a, b)$ respectively. *Moreover, the relations below hold*

$$\left(D_{a^+}^{\alpha}g\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{g(a)}{(t-a)^{\alpha}} + \int_a^t \frac{g'(s)\,ds}{(t-s)^{\alpha}}\right]$$

and

$$\left(D_{b^-}^{\alpha}g\right)(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{g(b)}{(b-t)^{\alpha}}-\int_t^b\frac{g'(s)\,ds}{(s-t)^{\alpha}}\right].$$

The next property is often called the semigroup property.

Proposition 3.2.2 : [46] *For all functions* $g\in C_{\gamma}(J)$, $0\leq\gamma<1$ *and* $D_{a^+}^{\alpha}$, *it holds that*

$$D_{a+}^{\alpha}D_{a+}^{\beta}g(t)=D_{a+}^{\alpha+\beta}g(t),\ t\in(a,b].$$

If $\gamma=0$ *the identity is true for all* $t\in J$.

The application of a fractional derivative on the fractional integral of the same order is equal to the identity.

Proposition 3.2.3: [46] *For* $g\in C_{\gamma}(J)$, *we have*

$$D_{a+}^{\alpha}I_{a+}^{\alpha}g(t)=g(t),\ t\in(a,b].$$

If $\gamma=0$ *the identity is true for all* $t\in J$.

The inverse operation does not yield the identity necessarily.

Proposition 3.2.4: [46] *If* $g\in C_{\gamma}(J)$ *and* $I_{a+}^{1-\alpha}g\in C_{\gamma}^{1}(J)$, *then the identity below is verified*

$$\left(I_{a+}^{\alpha}D_{a+}^{\alpha}g\right)(t)=g(t)-\frac{\left(I_{a+}^{1-\alpha}g\right)(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1},\ t\in(a,b].$$

For different orders, we have the following result.

Proposition 3.2.5 : [46] *If* $g \in C_\gamma(J)$ and $\alpha > \beta > 0$ *then*

$$D_{a+}^{\beta} I_{a+}^{\alpha} g(t) = I_{a+}^{\alpha-\beta} g(t),\ t \in (a, b].$$

For continuous functions on J, *the identity holds on all the interval* J.

3.3 Caputo type fractional derivative

Here we define the apparently most used derivative in applications as it is appropriate for initial value problems.

Definition 3.3.1: [46] *Let* J *be a finite interval. The Caputo (left-sided) fractional derivative of order* α, $0 < \alpha < 1$, *is defined by*

$$^{C}D_{a+}^{\alpha} g\left(t\right) =^{RL} D_{a+}^{\alpha} \left[g\left(s\right) - g\left(a\right)\right]\left(t\right).$$

Notice that this definition provides the link between the two fractional derivatives.

Definition 3.3.2: [46] *The Caputo left-sided fractional derivative of order* α, $0 < \alpha < 1$, *is equal to*

$$^{C}D_{a+}^{\alpha} g(t) = I_{a+}^{1-\alpha} Dg(t)$$

where $D = \frac{d}{dt}$, *in case* $g \in AC(J)$.

Proposition 3.3.1 : [46] *The identity*

$$I_{a^+}^{\alpha}\, {}^{C}D_{a^+}^{\alpha} g\left(t\right) = g\left(t\right) - g\left(a\right)$$

holds for $g \in AC(J)$ *or* $g \in C^1(J)$.

3.4 Hadamard type fractional integral and fractional derivatives

The definitions of the Hadamard type fractional integrals and fractional derivatives are given here just because of the literature review.

Definition 3.4.1: [5] *Let* $0 \leq a < b \leq \infty$. *The expressions*

$$\left(\mathcal{I}_{a+}^{\alpha} g\right)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)\, ds}{s}, \ a < t < b$$

and

$$\left(\mathcal{I}_{b-}^{\alpha} g\right)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{g(s)\, ds}{s}, \ a < t < b$$

are called the left-sided (right-sided, resp.) Hadamard fractional integral of order $\alpha > 0$.

Definition 3.4.2 : [6] *The expressions*

$$\left(\mathcal{D}_{a+}^{\alpha} g\right)(t) := \delta \left(\mathcal{I}_{a^+}^{1-\alpha} g\right)(t),$$

and

$$\left(\mathcal{D}_{b-}^{\alpha} g\right)(t) := -\delta \left(\mathcal{I}_{b^-}^{1-\alpha} g\right)(t),$$

are called the left-sided (right-sided, resp.) Hadamard fractional derivative of order $0 \leq \alpha < 1$.

The following fractional derivative has been introduced first in [34].

Definition 3.4.3: [5] *We define the left-sided fractional derivative of order* α, $(0 < \alpha < 1)$ *and type* $0 \leq \beta \leq 1$ *by the expression*

$$\left({}_H\mathcal{D}_{a^+}^{\alpha,\beta} g\right)(t) = \left(\mathcal{J}_{a^+}^{\beta(1-\alpha)} \cdot {}_H\mathcal{D}_{a^+}^{\alpha+\beta-\alpha\beta} g\right)(t)$$

where ${}_H\mathcal{D}_{a^+}^{\alpha+\beta-\alpha\beta}$ is the Hadamard fractional derivative defined above.

3.5 Riemann-Liouville type generalized fractional derivative

We are now ready to introduce the "generalized" fractional derivative used in this thesis. In addition to the definitions, we present some of its properties. In fact, all the properties of Riemann-Liouville and Caputo derivatives have been extended to this type of derivatives including the one mentioned above. These derivatives are also known as fractional derivatives of a function with respect to another function, say ϕ. In such a case, it is referred to as a ϕ-fractional derivative as well. Let $\alpha > 0$, $g : J \to R$ be an integrable function, and $\phi : J \to R$ be an increasing differentiable function such that $\phi'(t) \neq 0$, for all $t \in J$. The ϕ-Riemann–Liouville left-sided fractional integral of g is defined by

$$I_{a+}^{\alpha;\phi} g\ (t) = \frac{1}{\Gamma(\alpha)} \int_a^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1} g(s) ds \tag{3.1}$$

and the ϕ-Riemann–Liouville left-sided fractional derivative of g by

$$\begin{aligned} D_{a+}^{\alpha;\phi} g\ (t) &= \left(\tfrac{1}{\phi'(x)} \tfrac{d}{dt}\right)^n I_{a+}^{n-\alpha} g(t) \\ &= \tfrac{1}{\Gamma(n-\alpha)} \left(\tfrac{1}{\phi'(t)} \tfrac{d}{dt}\right)^n \int_a^t \phi'(s)(\phi(t)-\phi(s))^{n-\alpha-1} g\ (s) ds, \end{aligned} \tag{3.2}$$

respectively, where $n = [\alpha] + 1$. Similar formulas can be presented for the right-sided fractional integral and right-sided fractional derivative

$$I_{b-}^{\alpha;\phi} g\ (t) = \frac{1}{\Gamma(\alpha)} \int_t^b \phi'(s)(\phi(s)-\phi(t))^{\alpha-1} ds$$

and

$$\begin{aligned} D_{b-}^{\alpha;\phi} g(t) &= \left(-\tfrac{1}{\phi'(x)} \tfrac{d}{dt}\right)^n I_{b-}^{n-\alpha} g(t) \\ &= \tfrac{1}{\Gamma(n-\alpha)} \left(-\tfrac{1}{\phi'(t)} \tfrac{d}{dt}\right)^n \int_t^b \phi'(s)(\phi(s)-\phi(t))^{n-\alpha-1} g(s) ds, \end{aligned}$$

resp.

The semigroup property is valid for this type of fractional integrals: if $\alpha, \beta > 0$, then

$$I_{a+}^{\alpha;\phi} I_{a+}^{\beta;\phi} g\ (t) = I_{a+}^{\alpha+\beta;\phi} g(t) \ and \ I_{b-}^{\alpha;\phi} I_{b-}^{\beta;\phi} g(t) = I_{b-}^{\alpha+\beta;\phi} g(t).$$

3.6 Caputo-type generalized fractional derivative

The Caputo version of this ϕ-fractional derivative is given next.

Definition 3.6.1: [7] *Let $-\infty \leq a < b \leq \infty$, $g, \phi \in C^n(J)$ be two functions such that ϕ is increasing and $\phi'(t) \neq 0$, for all $t \in J$. The left ϕ-Caputo fractional derivative of g of order α is given by*

$$^C D_{a+}^{\alpha;\phi} g(t) = I_{a+}^{n-\alpha} \left(\frac{1}{\phi'(t)} \frac{d}{dt}\right)^n g(t)$$

and the right ϕ-Caputo fractional derivative of g by

$$^{C}D_{b-}^{\alpha;\phi}g\ (t) = I_{b-}^{n-\alpha}\left(-\frac{1}{\phi'(t)}\frac{d}{dt}\right)^{n} g\ (t)$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, $n = \alpha$ for $\alpha \in \mathbb{N}$.

To simplify notation, we will use

$$g_{\phi}^{[n]}(t) := \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^{n} g(t). \tag{3.3}$$

From the definition it is clear that, given $\alpha = m \in \mathbb{N}$,

$$^{C}D_{a+}^{\alpha;\phi}g\ (t) = g_{\phi}^{[m]}(t) \text{ and } {}^{C}D_{b-}^{\alpha;\phi}g(t) = (-1)^{m}\ g_{\phi}^{[m]}(t)$$

and if $\alpha \notin \mathbb{N}$, then

$$^{C}D_{a+}^{\alpha;\phi}g\ (t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t} \phi'(t)(\phi(t)-\phi(t))^{n-\alpha-1}g_{\phi}^{[n]}(t)dt \tag{3.4}$$

and

$$^{C}D_{b-}^{\alpha;\phi}g\ (t) = \frac{1}{\Gamma(n-\alpha)}\int_{t}^{b} \phi'(t)(\phi(t)-\phi(t))^{n-\alpha-1}(-1)^{m}\ g_{\phi}^{[n]}(t)dt.$$

In particular, when $\alpha \in (0, 1)$ we have

$$^{C}D_{a+}^{\alpha;\phi}g\ (t) = \frac{1}{\Gamma(1-\alpha)}\int_{a}^{t} (\phi(t)-\phi(t))^{-\alpha}g'(t)dt \tag{3.5}$$

and

$$^{C}D_{b-}^{\alpha;\phi}g\ (t) = \frac{-1}{\Gamma(1-\alpha)}\int_{t}^{b} (\phi(t)-\phi(t))^{-\alpha}g'(t)dt.$$

Theorem 3.6.1: [7] *For $\beta > 0$ and $\lambda \in \mathbb{R}\backslash\{0\}$, we have the following two formulas:*

$$I_{a+}^{\alpha;\phi}\left(\phi\left(t\right)-\phi\left(a\right)\right)^{\beta-1}=\frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta+\alpha\right)}\left(\phi\left(t\right)-\phi\left(a\right)\right)^{\beta+\alpha-1}$$

and

$$I_{a+}^{\alpha;\phi}E_{\alpha}\left(\phi\left(t\right)-\phi\left(a\right)\right)^{\alpha}=\frac{1}{\lambda}\left(E_{\alpha}(\lambda\left(\phi\left(t\right)-\phi\left(a\right)\right))^{\alpha}-1\right).$$

Lemma 3.6.1 : [7] *Given $\beta \in [0,\infty)$, consider the functions $g\ (t) = (\phi(t)-\phi(a))^{\beta-1}$ and $g(t) = (\phi(b)-\phi(t))^{\beta-1}$, where $\beta > n$. Then, for $\alpha > 0$,*

$$\begin{aligned}{}^{C}D_{a+}^{\alpha;\phi}g\ (t) &= \tfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\phi(t)-\phi(a)\right)^{\beta-\alpha-1},\\ {}^{C}D_{b-}^{\alpha;\phi}g(t) &= \tfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\phi(b)-\phi(t)\right)^{\beta-\alpha-1}.\end{aligned}$$

Lemma 3.6.2: [7] *Given $\lambda \in R$ and $\alpha > 0$, consider the functions $g\ (t) = E_{\alpha}\left(\lambda\left(\phi\left(t\right)-\phi\left(a\right)\right)^{\alpha}\right)$ and $g(t) = E_{\alpha}\left(\lambda\left(\phi\left(b\right)-\phi\left(t\right)\right)^{\alpha}\right)$, where E_{α} is the Mittag–Leffler function. Then, ${}^{C}D_{a+}^{\alpha;\phi}g\ (t) = \lambda g\ (t)$ and ${}^{C}D_{b-}^{\alpha;\phi}g\ (t) = \lambda g(t)$.*

Theorem 3.6.2: [7] *Given a function $g \in C^{n}(J)$ and $\alpha > 0$, it holds that*

$$I_{a+}^{\alpha;\phi}\,{}^{C}D_{a+}^{\alpha;\phi}g(t) = g(t)-\sum_{k=0}^{n-1}\frac{g_{\ \phi}^{[k]}\left(a\right)}{k!}\left(\phi\left(t\right)-\phi\left(a\right)\right)^{k}$$

and

$$I_{b-}^{\alpha;\phi}\,{}^{C}D_{b-}^{\alpha;\phi}g(t) = g(t)-\sum_{k=0}^{n-1}\left(-1\right)^{k}\frac{g_{\ \phi}^{[k]}\left(b\right)}{k!}\left(\phi\left(b\right)-\phi\left(t\right)\right)^{k}.$$

In particular, given $\alpha \in (0,1)$ we have

$$I_{a+}^{\alpha;\phi}\, {}^{C}D_{a+}^{\alpha;\phi} g\left(t\right) = g(t) - g(a) \text{ and } I_{b-}^{\alpha;\phi}\, {}^{C}D_{b-}^{\alpha;\phi} g(t) = g(t) - g\left(b\right).$$

Theorem 3.6.3: [7] *Given a function $g \in C^1(J)$ and $\alpha > 0$, the following identities hold*

$${}^{C}D_{a+}^{\alpha;\phi} I_{a+}^{\alpha;\phi} g\left(t\right) = g\left(t\right) \text{ and } {}^{C}D_{b-}^{\alpha;\phi} I_{b-}^{\alpha;\phi} g\left(t\right) = g\left(t\right).$$

Theorem 3.6.4: [6] *Let $g \in C^n(J)$ and $\alpha > 0$. Then,*

$${}^{C}D_{a+}^{\alpha;\phi} g\left(t\right) = {}^{C}\, D_{a+}^{\alpha;\phi} g\left(t\right) \Leftrightarrow g\left(t\right) = g\left(t\right) + \sum_{k=0}^{n-1} c_k \left(\phi\left(t\right) - \phi\left(a\right)\right)^k$$

and

$${}^{C}D_{b-}^{\alpha;\phi} g\left(t\right) = {}^{C}\, D_{b-}^{\alpha;\phi} g\left(t\right) \Leftrightarrow g\left(t\right) = g\left(t\right) + \sum_{k=0}^{n-1} d_k \left(\phi\left(b\right) - \phi\left(t\right)\right)^k$$

where c_k and d_k are arbitrary constant.

Theorem 3.6.5: [7] *Let $\alpha, \beta > 0$ be such that there exists some $k \in \mathbb{N}$ with $\beta, \alpha + \beta \in [k-1, k]$. Then, for $g \in C^k(J)$ the following holds:*

$${}^{C}D_{a+}^{\alpha;\phi}\, {}^{C}D_{a+}^{\beta;\phi} g\left(t\right) = {}^{C}\, D_{a+}^{\alpha+\beta;\phi} g\left(t\right) \text{ and } {}^{C}D_{b-}^{\alpha;\phi}\, {}^{C}D_{b-}^{\beta;\phi} g\left(t\right) = (-1)^{[\alpha+\beta]}\, {}^{C}D_{b-}^{\alpha+\beta;\phi}\, g\left(t\right).$$

Theorem 3.6.6: [7] *Given $f \in C^n(J)$, $g \in C^n(J)$, and $\phi \in C^{n+1}(J)$ we have that for all $\alpha > 0$,*

$$\begin{aligned}\int_a^b f\left(t\right)^{C} D_{a+}^{\alpha;\phi} g(t)dt &= \int_a^b {}^{C}D_{b-}^{\alpha;\phi} \left(\frac{f(t)}{\phi'(t)}\right) g\left(t\right) \phi'\left(t\right) dt \\ &+ \left[\sum_{k=0}^{n-1} \left(-\frac{1}{\phi'(t)}\frac{d}{dt}\right)^k I_{b-}^{n-\alpha;\phi} \left(\frac{f(t)}{\phi'(t)}\right) g_{\phi}^{[n-k-1]}\left(t\right)\right]_{t=a}^{t=b}.\end{aligned}$$

3.7 Phi-Hilfer generalized fractional derivative

Definition 3.7.1: [1] *Let* $0 < \alpha < 1$, $0 \leq \beta \leq 1$ *and* $g, \phi \in C^1(J)$ *be two functions such that* $\phi'(t) \neq 0$ *for all* $t \in J$. *The left-sided* ϕ*-Hilfer derivative* $D_{a^+}^{\alpha,\beta;\phi}(\cdot)$ *of a function* g *of order* α *and type* β *is defined by*

$$D_{a^+}^{\alpha,\beta;\phi}\, g(t) = I^{\beta(1-\alpha);\phi} \left(\frac{1}{\phi'(t)} \frac{d}{dt} \right) I^{(1-\beta)(1-\alpha);\phi}\, g(t). \tag{3.6}$$

Clearly, it appears that

$$D_{a^+}^{\alpha,\beta;\phi}\, g(t) = I_{a^+}^{\beta(1-\alpha);\phi}\, D_{a^+}^{\gamma;\phi}\, g(t), \quad t > a,$$

where

$$D^{\phi;x}\, g(t) = \left(\frac{1}{\phi'(t)} \frac{d}{dt} \right) I^{(1-\beta)(1-\alpha);\phi}\, g(t)$$

and $\gamma = \alpha + \beta - \alpha\,\beta$.

Lemma 3.7.1: [1] *Let* $\alpha > 0$, $0 \leq \beta \leq 1$ *and* $0 \leq \gamma = \alpha + \beta - \alpha\,\beta < 1$. *If* $g \in L^1(a,b)$ *and* $D_{a^+}^{\beta(1-\alpha);\phi}\, g(.)$ *exists on* $L^1(J)$ *then*

$$D_{a^+}^{\alpha,\beta,\phi}\, I^{\alpha;\phi}\, g(t) = I_{a^+}^{\beta(1-\alpha);\phi}\, D_{a^+}^{\beta(1-\alpha);\phi}\, g(t), \quad t \in (a,b].$$

If $g \in C_{1-\gamma;\phi}(J)$ *and* $I_{a^+}^{\beta(1-\alpha);\phi}\, g \in C^1_{1-\gamma}(J)$, *then*

$$D_{a^+}^{\alpha,\beta;\phi}\, I_{a^+}^{\alpha;\phi}\, g(t) = g(t), \quad t \in (a,b]$$

where

$$C_{\gamma;\phi}(J) := \{g : (a,b] \to \mathbb{R} : (\phi(t) - \phi(a))^{\gamma}\, g(t) \in C(J)\}.$$

Lemma 3.7.2: [1] *Let* $g \in C^1(J, R), \alpha > 0$ *and* $0 \leq \beta < 1$. *Then*

$$D_{a^+}^{\alpha,\beta;\phi}\, I_{a^+}^{\alpha;\phi}\, g(t) = g(t),\ t \in [a, b].$$

Lemma 3.7.3: [1] *If* $g \in C^n[a, b]$, $n - 1 < \alpha < n$, $0 \leq \beta \leq 1$ *and* $\gamma = \alpha + n - \alpha\,\beta$, *then*

$$I_{a^+}^{\alpha;\phi}\, D_{a^+}^{\alpha,\beta;\phi}\, g(t) = g(t) - \sum_{k=1}^{n} \frac{(\phi(t) - \phi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^{n-k} I^{(1-\beta)(n-\alpha)}\, g(a)$$

for all $t \in (a, b]$.

Note: if $t = a$ then the numerator in the summation will be $\phi(a) - \phi(a) = 0$. Thus $I_{a^+}^{\alpha;\phi}\, D_{a^+}^{\alpha,\beta;\phi}\, g(t) = g(t)$, $t = a$.

Remark 3.7.1: *From the definition of* $\phi-$*Hilfer fractional derivative we can obtain many other fractional derivative such as,*

1-taking the limit $\beta \to 1$, we get

$$D_{a^+}^{\alpha,1;\phi}\, g(t) = I^{(1-\alpha);\phi} \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right) g(t) = {}^{C}D^{\alpha;\phi} g\,(t)$$

the $\phi-$Caputo fractional derivative with respect to another function.

2- Taking the limit $\beta \to 0$, we get

$$D_{a^+}^{\alpha,1;\phi}\, g(t) = \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right) I^{(1-\alpha);\phi} g(t) = {}^{RL}D^{\alpha;\phi} g\,(t)$$

the $\phi-$Riemann Liouville fractional derivative with respect to another function.

3- Consider $\phi(t) = t$ and taking the limit $\beta \to 1$, we get

$$D_{a^+}^{\alpha,1;t}\, g(t) = I^{(1-\alpha)}\frac{d}{dt}\, g(t) = {}^{C}D^{\alpha}g(t)$$

the usual Caputo fractional derivative.

4-Consider $\phi(t) = t$ and taking the limit $\beta \to 0$, we get

$$D_{a^+}^{\alpha,0;t}\, g(t) = \frac{d}{dt}I^{(1-\alpha)}\, g(t) = {}^{RL}D^{\alpha}g(t)$$

the usual Riemann Liouville fractional derivative.

5- For $\phi(t) = \ln t$ and taking the limit $\beta \to 0$, we have

$$D_{a^+}^{\alpha,1;\ln t}\, g(t) = \left(t\frac{d}{dt}\right)\frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{-\alpha} g(s)\,\frac{ds}{s} = {}^{H}D^{\alpha}g(t)$$

the Hadamard fractional derivative.

6- For $\phi(t) = \ln t$ and taking the limit $\beta \to 1$, we have

$$D_{a^+}^{\alpha,1;\ln t}\, g(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{-\alpha}\left(t\frac{d}{dt}\right) g(s)\,\frac{ds}{s} = {}^{CH}D^{\alpha}g(t)$$

the Caputo-Hadamard fractional derivative.

3.8 Bihari inequality

The inequality we present here, due to Bihari, plays a fundamental role in many studies of differential and integral equations.

Theorem 3.8.1: (Bihari inequality) [15] *Let φ, z and $h : [0, \infty) \to [0, \infty)$ be three continuous functions such that z* is *nondecreasing and $z(\varphi) > 0$ on $(0, \infty)$. If*

$$\varphi(t) \leq a + \int_0^t h(s)\, z\left(\varphi(s)\right) ds,\ t \geq 0,\ a \geq 0$$

then

$$\varphi(t) \leq P^{-1}\left(P(a) + \int_0^t h(s)\, ds\right),$$

for $0 \leq t \leq t_*$, where

$$P(r) = \int_{\rho_0}^{\rho} \frac{ds}{z(s)},\ \rho > 0,\ \rho_0 > 0,$$

and P^{-1} is the inverse of P whilst $t_* \geq 0$ is selected in such a manner that

$$P(a) + \int_0^t h(s)\, ds \in Dom\left(P^{-1}\right),\ t \in [0, t_*].$$

3.9 Some usefull concepts

We will present here some of the important definitions that we will use later.

Definition 3.9.1: [68] *let X, Y be Banach space and $T : D \subset X \to Y$. The operator T is said to be completly continuous if it is continuous and maps any bounded subset of D into a relatively compact subset of Y.*

Definition 3.9.2: [71] *A collection F of real-valued functions on a metric space X is said to be equicontinuous at the point $x \in X$ provided for each $\varepsilon > 0$, there is a $\delta > 0$ such that for every $f \in F$ and $x' \in X$, if $\rho(x', x) < \delta$, then If $\left|f(x') - f(x)\right| < \varepsilon$.*

The collection F is said to be equicontinuous on X provided it is equicontinuous at every point in X.

EXISTENCE OF SOLUTIONS

4 Existence of solutions

In this subsection we state and prove our first result on the existence of solutions for Problem 1. To this we need the following theorem.

Theorem 4.1.1: [**78**] *Krasnoselskii's fixed point theorem*

Let X be a Banach space, Ω a bounded closed convex subset of X and let S, V be mappings of Ω into X such that $Sz + Vw \in \Omega$ for every pair $z, w \in \Omega$. If S is a contraction and V is completely continuous, then the equation $Sz + Vz = z$ has a solution in Ω.

We recall below our Problem 1 mentioned in the introduction

$$\begin{cases} {}^{C}D_{t_0}^{\alpha;\phi}(x(t) - k(t, x_t)) = g(t, x_t),\ t \in (t_0, \infty),\ t_0 \geq 0,\ t \in (t_0, t_0 + a] \\ x_{t_0} = \chi, \end{cases} \tag{4.1}$$

where ${}^{C}D_{t_0}^{\alpha;\phi}$ is Caputo's fractional derivative with respect to the function ϕ of order $0 < \alpha < 1$ (see (3.5)) and $a > 0$. The functions $g, k : ([t_0, \infty) \times ([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n, \tau > 0$ are given functions satisfying some assumptions that will be determined later and

$$\chi \in C([-\tau, 0], \mathbb{R}^n).$$

If $x \in C([t_0 - \tau, t_0 + \delta], \mathbb{R}^n)$ for $\delta > 0$, then for any $t \in [t_0, t_0 + a]$ define x_t by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau, 0)$.

In the next lemma and theorem we will establish the existence of solutions for Problem 1. To this end, we start by defining

$$A(\delta,\gamma)$$

$$:= \left\{ x \in C\left([t_0-\tau, t_0+\delta], \mathbb{R}^n\right),\ x_{t_0} = \chi,\ \sup_{t_0\le t\le t_0+\delta} \left|x(t) - \chi(0)\right| \le \gamma \right\}$$

for $\delta, \gamma > 0$ and setting our assumptions

(H1) The function $g(t, x_t)$ is in $C_{\rho,\phi}[t_0, t_0+\delta]$, $0 \le \rho < 1$, $\rho < \alpha$

(H2) For any $x \in A(\delta,\gamma)$, $k(t, x_t) = k_1(t, x_t) + k_2(t, x_t)$ for some functions k_1 and k_2, where $k \in C^{\alpha}_{\rho,\phi}[t_0, t_0+\delta]$.

(H3) The function k_1 is continuous and for all $x, y \in A(\delta,\gamma)$, $t \in [t_0, t_0+\delta]$

$$\left|k_1(t,x) - k_1(t,y)\right| \le L|x-y|$$

where $0 < L < 1$.

(H4) The function k_2 is completely continuous (see Definition 3.9.1) and for any bounded set A in $A(\delta,\gamma)$ the set $\{t \to k_2(t, x_t) : x \in A\}$ is equicontinuous (see Definition 3.9.2) in $C([t_0, t_0+\delta], \mathbb{R}^n)$.

Lemma 4.1.1: *If there is $\delta \in (0,a)$ and $\gamma \in (0,\infty)$ such that (H1)-(H2) are satisfied, then, for $t \in (t_0, t_0+\delta)$, the fractional differential problem (4.1) is equivalent to the following integral equation*

$$
\begin{cases}
x(t) = \chi(0) - k(t_0, \chi) + k(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \phi'(s)(\phi(t) - \phi(s))^{\alpha-1} g(s, x_s)\, ds, \\
x_{t_0} = \chi
\end{cases}
\tag{4.2}
$$

in $C_{\rho,\phi}[t_0, t_0 + \delta]$, $0 \leq \rho < 1$, $\rho < \alpha$.

Proof: We need to show that, for fixed $t \in [t_0, t_0 + \delta]$,

$$\phi'(s)(\phi(t) - \phi(s))^{\alpha-1}(\phi(s) - \phi(t_0))^{-\rho} \in L^1\,[t_0, t_0 + \delta].$$

This follows from

$$
\begin{aligned}
\int_{t_0}^{t} \phi'(s)(\phi(t) - \phi(s))^{\alpha-1}\,(\phi(s) - \phi(t_0))^{-\rho} ds &= \Gamma(\alpha) I^{\alpha,\phi}(\phi(t) - \phi(t_0))^{-\rho} \\
&= \Gamma(\alpha)\tfrac{\Gamma(1-\rho)}{\Gamma(1-\rho-\alpha)}(\phi(t) - \phi(t_0))^{\alpha-\rho} \\
&\leq \Gamma(\alpha)\tfrac{\Gamma(1-\rho)}{\Gamma(1-\rho-\alpha)}(\phi(t_0 + \delta) - \phi(t_0))^{\alpha-\rho} < \infty.
\end{aligned}
$$

Next, it is easy to see that

$$
\begin{aligned}
&\left|\int_{t_0}^{t} (\phi'(s)(\phi(t) - \phi(s))^{\alpha-1} g(s, x_s) ds\right| \\
&\leq \int_{t_0}^{t} |\phi'(s)(\phi(t) - \phi(s))^{\alpha-1}(\phi(s) - \phi(t_0))^{-\rho}\ (\phi(s) - \phi(t_0))^{\rho}\ g(s, x_s)| ds \\
&\leq \int_{t_0}^{t}\ |\phi'(s)(\phi(t) - \phi(s)|^{\alpha-1}\,(\phi(s) - \phi(t_0))^{-\rho}|(\phi(s) - \phi(t_0))^{\rho}\ g(s, x_s)| ds \\
&\leq M \left\|\phi'(s)(\phi(t) - \phi(s))^{\alpha-1}(\phi(s) - \phi(t_0))^{-\rho}\right\|_{L^1(t_0,t)}.
\end{aligned}
$$

where M is a bound for $(\phi(s) - \phi(t_0))^{\rho}\ g(s, x_s)$.

Therefore, $\phi'(s)\,(\phi(t) - \phi(s))\,g(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in [t_0, t_0+\delta]$ and $x \in A(\delta, \gamma)$. Consequently, if x is a solution of (4.1) then x is a solution of equation (4.2) because if we apply $I^{\alpha;\phi}_{t_0^+}$ to both sides of (4.1) and utilize the identity in Theorem 3.5.1, we find

$$x(t) - k(t, x_t) - x(t_0) + k(t_0, x_{t_0}) = I^{\alpha,\phi}_{t_0^+} g(t, x_t).$$

Notice that

$$\lim_{t \to t_0} \int_{t_0}^{t} \phi'(s)[\phi(t) - \phi(s)]^{\alpha - 1} g(s, x_s)\, ds = 0.$$

This follows from the summability of the integrand shown above. Moreover,

$$\chi(0) = x_{t_0}(0) = x(t_0 + 0) = x(t_0).$$

Hence, $x(t)$ is solution of the integral problem

$$\begin{cases} x(t) = \chi(0) + k(t, x_t) - k(t_0, \chi) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \phi'(s)(\phi(t) - \phi(s))^{\alpha - 1} g(s, x_s)\, ds, \\ x_{t_0} = \chi. \end{cases}$$

Conversely, if (4.2) is satisfied, then for every $t \in (t_0, t_0 + \delta)$ we have, by virtue of Theorem 3.6.3

$$\begin{aligned} {}^C D^{\alpha;\phi}(x(t) - k(t, x_t)) &= {}^C D^{\alpha;\phi}\ [\chi(0) + k(t, x_t) - k(t_0, \chi) \\ &+ \tfrac{1}{\Gamma(\alpha)} \textstyle\int_{t_0}^{t} \phi'(s)(\phi(t) - \phi(s))^{\alpha - 1} g(s, x_s)\, ds - k(t, x_t)] \\ &= {}^C D^{\alpha;\phi}\ (\chi(0) - k(t_0, \chi) + I^{\alpha;\phi} g(t, x_t) \\ &= {}^C D^{\alpha;\phi}\ I^{\alpha;\phi} g(t, x_t) = g(t, x_t). \end{aligned}$$

This completes the proof.

Theorem 4.1.2: *Assume that there is $\delta \in (0, a)$ and $\gamma \in (0, \infty)$ such that **(H1)-(H5)** and $[\phi(t_0)]^{\alpha - \rho} < \frac{\gamma \Gamma(\alpha + 1 - \rho)}{3\Gamma(1 - \rho)}$ are satisfied. Then, (4.1) has at* *least one solution in* $C^{\alpha}_{\phi,\rho}\left([t_0, t_0+\eta], R^n\right)$ *for some positive number* $\eta$to be determined.

Proof: From **(H3),** for any $x \in A(\delta,\gamma)$, the fact

$$k(t,x_t) = k_1(t,x_t) + k_2(t,x_t)$$

implies that

$$\begin{aligned} x(t) &= \chi(0) - k_1(t_0,\chi) - k_2(t_0,\chi) + k_1(t,x_t) + k_2(t,x_t) \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \phi'(s)\left(\phi(t)-\phi(s)\right)^{\alpha-1} g(s,x_s)\, ds. \end{aligned}$$

Let $\tilde{\chi} \in A(\delta,\gamma)$ such that $\widetilde{\chi}_{t_0} = \chi$ and $\widetilde{\chi}(t_0+t) = \chi(0)$, $\forall t \in [0,\delta]$. If x is a solution of (4.1) and

$$x(t_0+t) = \widetilde{\chi}(t_0+t) + y(t),\ t \in [-\tau,\delta]$$

then

$$x_{t_0+t} = \widetilde{\chi}_{t_0+t} + y_t,\ t \in [0,\delta]$$

and

$$y(t) = x(t_0+t) - \widetilde{\chi}\,(t_0+t) = x(t_0+t) - \chi(0).$$

Therefore, we can write

$$\begin{aligned} y(t) &= -k_1(t_0,\chi) - k_2(t_0,\chi) + k_1(t+t_0, x_{t+t_0}) + k_2(t+t_0, x_{t+t_0}) \\ &+ \tfrac{1}{\Gamma(\alpha)} \int_{t_0}^{t_0+t} \phi'(s)(\phi(t_0+t)-\phi(s))^{\alpha-1} g(s,x_s)\, ds \qquad . \end{aligned}$$

Let, $s = t_0 + \theta \leftrightarrow \theta = s - t_0$, then $d\theta = ds$ and

$$\begin{aligned} &\int_{t_0}^{t_0+t} \phi'(s)(\phi(t_0+t) - \phi(s))^{\alpha-1}\, g(s, x_s)\, ds \\ &= \int_0^t \phi'(t_0+\theta)[\phi(t_0+t) - \phi(t_0+\theta)]^{\alpha-1} g(t_0+\theta, x_{t_0+\theta})\, d\theta \\ &= \int_0^t \phi'(t_0+s)[\phi(t_0+t) - \phi(t_0+s)]^{\alpha-1} g(t_0+s, x_{t_0+s})\, ds. \end{aligned}$$

We infer that

$$\begin{aligned} y(t) &= -k_1(t_0, \chi) - k_2(t_0, \chi) + k_1(t+t_0, y_t + \widetilde{\chi}_{t_0+t}) + k_2(t+t_0, y_t + \widetilde{\chi}_{t_0+t}) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \phi'(t_0+s)[\phi(t_0+t) - \phi(t_0+s)]^{\alpha-1} g(t_0+s, y_s + \widetilde{\chi}_{t_0+s})\, ds. \end{aligned} \tag{4.3}$$

By the continuity of k_1, k_2 and x_t, there exists $\delta' > 0$ such that when $0 < t < \delta'$, it holds that

$$\left|k_1(t_0+t,\, y_t + \widetilde{\chi}_{t+t_0}) - k_1(t_0, \chi)\right| < \frac{\gamma}{3}$$

and

$$|k_2(t_0+t, y_t + \widetilde{\chi}_{t+t_0}) - k_2(t_0, \chi)| < \frac{\gamma}{3}.$$

Choose

$$\eta = \min\left\{\delta, \delta', \phi^{-1}\left[\frac{\gamma\Gamma(\alpha+1-\rho)}{3\Gamma(1-\rho)}\right]^{\frac{1}{\alpha-\rho}}\right\}.$$

We introduce the space

$$E(\eta, \gamma) := \left\{y \in C\left([-\tau, \eta], R^n\right) : y(s) = 0 \; for \; s \in [-\tau, 0] \; and \; \|y\| \le \gamma\right\}.$$

It is clear that $E(\eta,\gamma)$ is a closed bounded and convex subset of $C\left([-\tau,\eta], R^n\right)$.

Next, we define the operators S and U as follows

$$Sy(t) = \begin{cases} 0, \ t \in [-\tau, 0], \\ -k_1(t_0,\chi) + k_1(t_0+t, y_t + \widetilde{\chi}_{t+t_0}), \ t \in [0,\eta], \end{cases}$$

and

$$Uy(t) = \begin{cases} 0, \ t \in [-\tau, 0], \\ -k_2(t_0,\chi) + k_2(t_0+t, y_t + \widetilde{\chi}_{t+t_0}) \\ +\frac{1}{\Gamma(\alpha)} \int_0^t \phi'(t_0+s)[\phi(t_0+t) - \phi(t_0+s)]^{\alpha-1} \\ \times g(t_0+s, y_s + \widetilde{\chi}_{t_0+s})\, ds, \ t \in [0,\eta]. \end{cases}$$

Obviously

$$Sy(t) + Uy(t) = y$$

has a solution $y \in E(\eta,\gamma)$ is equivalent to y is a solution of (4.3). Therefore,

$$x(t_0+t) = y(t) + \widetilde{\chi}(t_0+t)$$

is a solution of (4.1) on $[0,\eta]$ and so, the existence of the solution of (4.1) is equivalent to the existence of a fixed point in $E(\eta,\gamma)$ for $Sy(t)+Uy(t) = y$.

Let us now show that $S+U$ has a fixed point by using Krasnoselskii's fixed point theorem. We need to show three things:

(a) $Sz + Uy \in E(\eta,\gamma)$ for all $z, y \in E(\eta,\gamma)$.

(b) S is a contraction operator.

(c) U is completely continuous.

Proof of (a):

We know from the definition of $E(\eta,\gamma)$ and the operators S and U that for every pair $z, y \in E(\eta,\gamma)$,

$$Sz + Uy \in C\left([-\tau, 0], R^n\right)$$

and

$$(Sz + Uy)(t) = 0, \ \forall t \in [-\tau, 0].$$

Now, for $t \in [0, \eta]$ it appears that

$$\left|Sz(t) + Uy(t)\right| = \left|-k_1(t_0, \chi) + k_1(t_0 + t, y_t + \widetilde{\chi}_{t+t_0}) - k_2(t_0, \chi)\right.$$

$$+k_2(t_0 + t, y_t + \widetilde{\chi}_{t+t_0}) + \tfrac{1}{\Gamma(\alpha)} \ + \textstyle\int_0^t \phi^{'}(t_0 + s)[\phi(t_0 + t) - \phi(t_0 + s)]^{\alpha-1}$$

$$\left.\times g(t_0 + s, y_s + \widetilde{\chi}_{t_0+s})\, ds\right|$$

$$\leq \left|k_1(t_0 + t, y_t + \widetilde{\chi}_{t+t_0}) - k_1(t_0, \chi)\right| + \left|k_2(t_0 + t, y_t + \widetilde{\chi}_{t+t_0}) - k_2(t_0, \chi)\right|$$

$$+\tfrac{M}{\Gamma(\alpha)} \left\| \textstyle\int_0^t \phi^{'}(t_0 + s)[\phi(t_0 + t) - \phi(t_0 + s)]^{\alpha-1}(\phi(t_0 + s) - \phi(t_0))^{-\rho}\right\|_{L^1(0,t)}$$

and this is true since for $t_0 + s = \sigma$ we obtain,

$$\int_{t_0}^{t_0+t} \phi^{'}(\sigma)[\phi(t_0 + t) - \phi(\sigma)]^{\alpha-1}(\phi(\sigma) - \phi(t_0))^{-\rho} d\sigma$$

and by theorem 3.6.1 we get,

$$\leq \Gamma(\alpha)\frac{\Gamma(1-\rho)}{\Gamma(\alpha+1-\rho)}(\phi(t_0 + t) - \phi(t_0))^{\alpha-\rho}.$$

Therefore,

$$\phi'(t_0+s)[\phi(t_0+t)-\phi(t_0+s)]^{\alpha-1}(\phi(t_0+s)-\phi(t_0))^{-\rho} \in L^1(0,t).$$

As consequences,

$$\begin{aligned} &\left|Sz\left(t\right)+Uy\left(t\right)\right| \\ &\leq \tfrac{\gamma}{3}+\tfrac{\gamma}{3}+\tfrac{M}{\Gamma(\alpha)}\left\{\Gamma(\alpha)\tfrac{\Gamma(1-\rho)}{\Gamma(\alpha+1-\rho)}[\phi(t_0+t)-\phi(t_0)]^{\alpha-\rho}\right\}. \end{aligned}$$

As $\phi\left(t\right)$ is increasing, it is clear that $\phi\left(t_0+t\right)-\phi\left(t\right)\leq\phi\left(t_0+t\right)$ and

$$0\leq\left[\phi\left(t_0+t\right)-\phi\left(t\right)\right]^{\alpha-\rho}\leq\left[\phi\left(t_0+t\right)\right]^{\alpha-\rho}\leq\left[\phi\left(t_0+\eta\right)\right]^{\alpha-\rho}.$$

By virtue of the assumptions in the statement of the theorem, we select η small enough so that

$$\frac{\Gamma(1-\rho)}{\Gamma(\alpha+1-\rho)}[\phi(t_0+\eta)]^{\alpha-\rho}\leq\frac{\gamma}{3}$$

and

$$|Sz+Uy|\leq\frac{\gamma}{3}+\frac{\gamma}{3}+\frac{\gamma}{3}=\gamma.$$

We deduce that

$$\|Sz+Uy\|=\sup_{t\in[0,\eta]}\left|\left(Sz\right)\left(t\right)+\left(Uy\right)\left(t\right)\right|\leq\gamma.$$

That is, $Sz+Uy\in E\left(\eta,\gamma\right)$ for all $z,y\in E\left(\eta,\gamma\right)$.

Proof of (b)

For $y', y'' \in E(\eta,\gamma)$; $y'_t + \widetilde{\chi}_{t+t_0}, y''_t + \widetilde{\chi}_{t+t_0} \in A(\delta,\gamma)$, we have in view of

(H4)

$$\begin{aligned}
&\left|Sy'(t) - Sy''(t)\right| \\
&= \left|-k_1(t_0,\chi) + k_1(t_0+t, y'_t + \widetilde{\chi}_{t+t_0}) + k_1(t_0,\chi) - k_1(t_0+t, y''_t + \widetilde{\chi}_{t+t_0})\right| \\
&\le L\left|y' - y''\right|
\end{aligned}$$

where $L \in (0,1)$.

Proof of (c)

We designate by

$$U_1 y(t) := \begin{cases} 0,\ t \in [-\tau, 0], \\ -k_2(t_0,\chi) + k_2(t_0+t, y_t + \widetilde{\chi}_{t+t_0}),\ t \in [0,\eta] \end{cases}$$

and

$$U_2 y(t) := \begin{cases} 0,\ t \in [-\tau, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t \phi'(t_0+s)[\phi(t_0+t) - \phi(t_0+s)]^{\alpha-1} \\ \times g(t_0+s, y_s + \widetilde{\chi}_{t_0+s})\, ds,\ t \in [0,\eta]. \end{cases}$$

Thus, $U = U_1 + U_2$. First, we claim that, since k_2 is completely continuous, U_1 is continuous because

$$\begin{aligned}
&\lim_{t\to 0}\left[-k_2(t_0,\chi) + k_2(t_0+t, y_t + \widetilde{\chi}_{t+t_0})\right] \\
&= -k_2(t_0,\chi) + k_2(t_0,\chi) = 0.
\end{aligned}$$

Notice that

$$\widetilde{\chi}_{t_0} = \chi \; and \; y_0 = x_{t_0} - \widetilde{\chi}_{t_0} = \chi - \chi = 0.$$

We infer that $\{U_1 y : y \in E(\eta, \gamma)\}$ is uniformly bounded by the definition of $E(\eta, \gamma)$. Moreover, from **(H3)** we can conclude that

$$\left\{t \to k_2(t, x_t) : x \in \Lambda\right\}$$

is equicontinuous for any bounded set Λ in $A(\delta, \gamma)$. Hence, U_1 is a completely continuous operator. In addition, for any $t \in (0, \eta)$ we have

$$\begin{aligned}
\left|U_2 y(t)\right| &\leq \tfrac{1}{\Gamma(\alpha)} \int_0^t \left|\phi'(s)[\phi(t_0+t) - \phi(s)]^{\alpha-1}[\phi(s+t_0) - \phi(t_0)]^{-\rho}\right| \\
&\quad \left|[\phi(s+t_0) - \phi(t_0)]^{\rho} \, g(t_0+s, y_s + \widetilde{\chi}_{s+t_0})\right| ds \\
&\leq M I^{\alpha;\phi}[\phi(t+t_0) - \phi(t_0)]^{-\rho} \\
&= M \tfrac{\Gamma(1-\rho)}{\Gamma(\alpha+1-\rho)}[\phi(t+t_0) - \phi(t_0)]^{\alpha-\rho}.
\end{aligned}$$

We entail that $\{U_2 y : y \in E(\eta, \gamma)\}$ is uniformly bounded.

It remains to demonstrate that $\{U_2 y : y \in E(\eta, \gamma)\}$ is equicontinuous. For any $0 \leq t_1 \leq t_2 < \eta$ and $y \in E(\eta, \gamma)$, we see that

$$\begin{aligned}
&\left|U_2 y(t_2) - U_2 y(t_1)\right| = \Big| \tfrac{1}{\Gamma(\alpha)} \int_0^{t_2} \phi'(t_0+s)[\phi(t_0+t_2) - \phi(t_0+s)]^{\alpha-1} \\
&\times g(t_0+s, y_s + \widetilde{\chi}_{s+t_0}) ds - \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \phi'(t_0+s)[\phi(t_0+t_1) - \phi(t_0+s)]^{\alpha-1} \\
&\times g(t_0+s, y_s + \widetilde{\chi}_{s+t_0}) ds \Big| \\
&\leq \tfrac{1}{\Gamma(\alpha)} \int_0^{t_1} \Big|\phi'(t_0+s)\left[\phi(t_0+t_2) - \phi(t_0+s))^{\alpha-1}\right. \\
&\left. -(\phi(t_0+t_1) - \phi(t_0+s))^{\alpha-1}\right] g(t_0+s, y_s + \widetilde{\chi}_{s+t_0}) \Big| ds \\
&+ \int_{t_1}^{t_2} \left|\phi'(t_0+s)(\phi(t_0+t_2) - \phi(t_0+s))^{\alpha-1} g(t_0+s, y_s + \widetilde{\chi}_{s+t_0})\right| ds
\end{aligned}$$

and so,

$$\left|U_2y\left(t_2\right)-U_2y\left(t_1\right)\right| \leq \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\phi^{'}(t_0+s)\left[(\phi(t_0+t_2)-\phi(t_0+s))^{\alpha-1}\right.\right.$$

$$\left.-(\phi(t_0+t_1)-\phi(t_0+s))^{\alpha-1}\right](\phi(t_0+s)-\phi(t_0))^{-\rho}$$

$$\times\left|(\phi(t_0+s)-\phi(t_0))^{\rho}g(t_0+s,y_s+\widetilde{\chi}_{s+t_0})\right|ds$$

$$+\tfrac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left|\phi^{'}(t_0+s)(\phi(t_0+t_2)-\phi(t_0+s))^{\alpha-1}(\phi(t_0+s)-\phi(t_0))^{-\rho}\right|$$

$$\left|(\phi(t_0+s)-\phi(t_0))^{\rho}\ g(t_0+s,y_s+\widetilde{\chi}_{s+t_0})\right|ds.$$

Set $s+t_0=\sigma$, clearly we find

$$\left|U_2y\left(t_2\right)-U_2y\left(t_1\right)\right| \leq \tfrac{M}{\Gamma(\alpha)}\left[\int_{t_0}^{t_1+t_0}\phi^{'}(\sigma)\left[(\phi(t_0+t_2)-\phi(\sigma))^{\alpha-1}(\phi(\sigma)-\phi(t_0))^{-\rho}\right]d\sigma\right.$$

$$\left.-\int_0^{t_1}\phi^{'}(\sigma)\left[(\phi(t_0+t_1)-\phi(\sigma))^{\alpha-1}(\phi(\sigma)-\phi(t_0))^{-\rho}\right]d\sigma\right]$$

$$+\tfrac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}\phi^{'}(\sigma)(\phi(t_0+t_2)-\phi(\sigma))^{\alpha-1}(\phi(\sigma)-\phi(t_0))^{-\rho}d\sigma$$

Finally,

$$\left|U_2y\left(t_2\right)-U_2y\left(t_1\right)\right| \leq M\tfrac{\Gamma(1-\rho)}{\Gamma(\alpha+1-\rho)}(\phi(t_1+t_0)-\phi(t_0))^{\alpha-\rho}$$

$$-M\tfrac{\Gamma(1-\rho)}{\Gamma(\alpha+1-\rho)}(\phi(t_1)-\phi(t_0))^{\alpha-\rho}+M\tfrac{\Gamma(1-\rho)}{\Gamma(\alpha+1-\rho)}(\phi(t_2)-\phi(t_0))^{\alpha-\rho}.$$

which means that $\{U_2y : y\in E\left(\eta,\gamma\right)\}$ is equicontinuous and it is clear that U_2 is continuous. Therefore, U_2 is completely continuous. It follows that $U=U_1+U_2$ is a completely continuous operator. Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point in $E(\eta,\gamma)$, and hence the problem (4.1) has a solution $x(t)=\chi(0)+y(t-t_0)$ for all $t\in[t_0,t_0+\eta]$.

ASYMPTOTIC BEHAVIOR

5 Asymptotic behavior

To study the asymptotic behavior of the solutions of the problem

$$\begin{cases} D_a^{\alpha,\beta;\phi} y\,(t) = f\left(t, y(t)\right), \; t \in (a, \infty), \\ I_a^{(1-\alpha)(1-\beta)} y(t)\Big|_{t=a} = b \end{cases} \tag{5.1}$$

we assume $f(t, y(t))$ to be continuous on $(a, \infty) \times \mathbb{R}$ and

$$\left|f(t, y(t))\right| \leq (\phi(t) - \phi(a))^{\mu} Q(t)\, |y(t)|^{m}$$

for some numbers μ and m and a function $Q(t)$ to be determined later. The assumption below is needed in the next lemma.

(A) There exists a constant $U > 0$ such that, for $\lambda, \nu, \omega > 0$

$$[\phi(t) - \phi(a)]^{\lambda} \int_0^1 (1-\xi)^{\nu-1}\, \xi^{\lambda-1} \phi' \left\{\phi^{-1}[\phi(a) + \xi(\phi(t) - \phi(a))]\right\}^{\omega} d\xi \leq U$$

for $t > a$

Remark 5.1.1: *Notice that assumption **(A)** is satisfied, for instance, for $\phi(t) = \ln t$ (see [80]). It is also fulfilled by $\phi(t) = t$ in presence of an extra exponential term e^{-ws}, see Michalski [82].*

Lemma 5.1.1: *Assume that the assumption **(A)** holds and let $\lambda, \nu, \omega > 0$. Then, for any $t > a, a > 0$ we have*

$$(\phi(t) - \phi(a))^{1-\nu} \int_a^t (\phi(t) - \phi(s))^{\nu-1} (\phi(s) - \phi(a))^{\lambda-1} (\phi'(s))^{w+1} ds \leq U \tag{5.2}$$

where U is the positive constant in assumption **(A)** *above.*

Proof: Let $I(t)$ denote the left hand side of (5.2) and put

$$\xi = \frac{\phi(s)-\phi(a)}{\phi(t)-\phi(a)} \to \phi(s) = \phi(a) + \xi(\phi(t)-\phi(a)).$$

Then, $d\xi = \frac{\phi'(s)}{\phi(t)-\phi(a)} ds$ and

$$I\,(t) = (\phi(t)-\phi(a))^{1-\nu} \int_a^t [(\phi(t)-\phi(a)) - \xi(\phi(t)-\phi(a))]^{\nu-1}$$

$$\times[\xi(\phi(t)-\phi(a)]^{\lambda-1}\,[\phi'(s)]^{\omega}\phi'\,(s)\,ds$$

$$= (\phi(t)-\phi(a))^{1-\nu} \int_0^1 (\phi(t)(1-\xi) - \phi(a)(1-\xi))^{\nu-1}$$

$$\times\xi^{\lambda-1}\,(\phi(t)-\phi(a))^{\lambda-1}[\phi'(s)]^{\omega}\,(\phi(t)-\phi(a))\,d\xi$$

$$= (\phi(t)-\phi(a))^{1-\nu} \int_0^1 (1-\xi)^{\nu-1}\,(\phi(t)-\phi(a))^{\nu-1}\,\xi^{\lambda-1}$$

$$\times(\phi(t)-\phi(a))^{\lambda}\;[\phi'(s)]^{\omega}\,d\xi.$$

From the relation

$$\phi(s) = \phi(a) + \xi(\phi(t)-\phi(a)),$$

and as the function $\phi\,(t)$ is monotone increasing, we derive

$$s = \phi^{-1}[\phi(a) + \xi(\phi(t)-\phi(a))]$$

and

$$\phi'(s)^{\omega} = \{\phi'(\phi^{-1}[\phi(a) + \xi(\phi(t)-\phi(a))]\}^{\omega}.$$

Therefore, we obtain

$$I\,(t) = (\phi(t)-\phi(a))^{\lambda} \int_0^1 (1-\xi)^{\nu-1}\;\xi^{\lambda-1}$$

$$\times\{\phi'(\phi^{-1}[\phi(a) + \xi(\phi(t)-\phi(a))]\}^{\omega}\,d\xi \leq U.$$

The proof is complete.

Let p and q be conjugate exponents i.e. such that $\frac{1}{p} + \frac{1}{q} = 1$. Define $\lambda_1 := 1 + p[\mu - (1-\gamma)\, m]$ and $\lambda_2 = 1 + p(\alpha - 1)$ where $\gamma := \alpha + \beta - \alpha\beta$. If $\mu - (m-1)(1-\gamma) > 0$ and $q > \frac{1}{\alpha}$ then, $\lambda_1,\, \lambda_2 > 0$. Denote by L^* the positive real number

$$L^* := \frac{1}{(m-1)^{1/q}} \left(\frac{\Gamma(\gamma)}{|b|}\right)^{m(m-1)} 2^{\frac{1}{q}-m} \hat{C}_1^{-1}.$$

Theorem 5.1.1: *Suppose f satisfies the assumption above, $Q \in L^q$ and $\mu > (m-1)(1-\gamma)$. If*

$$\left\|Q(t)\right\|_q^{m-1} \left\|Q(t)\,(\phi(t) - \phi(a))^{-m(1-\alpha)}\right\|_q \leq L^*$$

for some $q > \frac{1}{\alpha}$, then, for any solution of Problem (5.1) there exists a positive constant C such that

$$\left|u(t)\right| \leq C(\phi(t) - \phi(a))^{\gamma - 1},\ t > a.$$

Proof: Let us consider the Volterra integral equation associated to (5.1) by applying $I_{a^+}^{\alpha,\phi}$ to both sides of (5.1). We get, using Lemma 3.7.3

$$y(t) = \frac{(\phi(t) - \phi(a))^{\gamma-1}}{\Gamma(\gamma)} \underbrace{I^{(1-\beta)(1-\alpha)}\, y(a)}_{=b} + I_{a^+}^{\alpha;\phi} f(t, y(t))$$

or,

$$y(t)=\frac{(\phi(t)-\phi(a))^{\gamma-1}}{\Gamma(\gamma)}\,b+\frac{1}{\Gamma(\alpha)}\int_a^t \phi'(s)(\phi(t)-\phi\,(s))^{\alpha-1}\,f\,(s,y(s))\,ds\,. \tag{5.3}$$

Multiplying both sides of (5.3) by $(\phi(t)-\phi(a))^{1-\gamma}$, we infer

$$\begin{gathered}(\phi(t)-\phi(a))^{1-\gamma}\,y(t)=\tfrac{b}{\Gamma(\gamma)}+\tfrac{(\phi(t)-\phi(a))^{1-\gamma}}{\Gamma(\alpha)}\\ \times\textstyle\int_a^t \phi'(s)\,(\phi(t)-\phi(s))^{\alpha-1}f(s,y(s)\,)ds.\end{gathered}$$

Thus, by virtue of our assumption **(A)**

$$\begin{gathered}(\phi(t)-\phi(a))^{1-\gamma}\,\big|y(t)\big|\le\tfrac{|b|}{\Gamma(\gamma)}+\tfrac{(\phi(t)-\phi(a))^{1-\gamma}}{\Gamma(\alpha)}\\ \times\textstyle\int_a^t \phi'(s)(\phi\,(t)-\phi(s))^{\alpha-1}(\phi(s)-\phi(a))^{\mu}\,Q(s)\,|y(s)|^m\,ds.\end{gathered} \tag{5.4}$$

Now let $V(t)$ denote the left hand side of (5.4). Multiplying by

$$(\phi(s)-\phi(a))^{(1-\gamma)\,m}\,(\phi(s)-\phi(a))^{-(1-\gamma)m}$$

inside the integral, yields

$$\begin{gathered}V(t)\le\tfrac{|b|}{\Gamma(\gamma)}+\tfrac{(\phi(t)-\phi(a))^{1-\gamma}}{\Gamma(\alpha)}\textstyle\int_a^t\phi'(s)\,(\phi(t)-\phi(s))^{\alpha-1}\\ \times(\phi(s)-\phi(a))^{\mu-(1-\gamma)\,m}\,Q(s)\underbrace{\Big(\phi(s)-\phi(a))^{(1-\gamma)\,m}|y(s)|^m\Big)}_{\le V^m(s)}ds.\end{gathered}$$

That is,

$$\begin{gathered}V(t)\le\tfrac{|b|}{\Gamma(\gamma)}+\tfrac{(\phi(t)-\phi(a))^{1-\gamma}}{\Gamma(\alpha)}\textstyle\int_a^t(\phi(t)-\phi(s))^{\alpha-1}\\ \times(\phi(s)-\phi(a))^{\mu-(1-\gamma)\,m}\phi'(s)Q(s)V^m(s)\;ds.\end{gathered} \tag{5.5}$$

By Hölder inequality with exponents p and q, we can write

$$\begin{aligned} &\textstyle\int_a^t (\phi(t)-\phi(a))^{\alpha-1}\,(\phi(s)-\phi(a))^{\mu-(1-\gamma)\,m}\,Q(s)\,V^m\,(s)\,\phi'(s)\,ds \\ &\le \left(\textstyle\int_a^t (\phi(t)-\phi(a))^{p(\alpha-1)}\,(\phi(s)-\phi(a))^{p(\mu-(1-\gamma)\,m)}\,[\phi'\,(s)]^{(p-1)+1}\right)^{1/p} \\ &\qquad \times\left(\textstyle\int_a^t Q^q(s)\;\; V^{qm}(s)\,ds\right)^{1/q}. \end{aligned} \tag{5.6}$$

Next, Lemma (5.1.1) implies

$$\begin{aligned} &\textstyle\int_a^t (\phi(t)-\phi(a))^{\alpha-1}\,(\phi(s)-\phi(a))^{\mu-(1-\gamma)\,m}\,Q(s)\,V^m\,(s)\,\phi'(s)\,ds \\ &\quad\le C_1\{[\phi(t)-\phi(a)]^{-p(1-\alpha)}\}^{1/p}\left(\textstyle\int_a^t Q^q(s)\,V^{qm}\,(s)\,ds)\right)^{1/q}. \end{aligned} \tag{5.7}$$

Combining (5.5) and (5.7) we get

$$V(t)\le \frac{|b|}{\Gamma(\alpha)}+\hat{C}_1(\phi(t)-\phi(a))^{1-\gamma}\left(\int_a^t Q^q\,(s)\,V^{qm}(s)\,ds\right)^{1/q} \tag{5.8}$$

where $\hat{C}_1=\frac{C_1}{\Gamma(\alpha)}$. At this stage we multiply both sides of (5.8) by $(\phi(t)-\phi(a))^{\beta(1-\alpha)}$ to obtain

$$(\phi(t)-\phi(a))^{\beta(1-\alpha)}\,V(t)\le\frac{|b|}{\Gamma(\gamma)}(\phi(t)-\phi(a))^{\beta(1-\alpha)}+\hat{C}_1\left(\int_a^t \phi^q(s)\,V^{qm}\,(s)\,ds\right)^{1/q}. \tag{5.9}$$

Let $Z(t)$ be the left hand side of (5.9). Insert the term $(\phi(s)-\phi(a))^{-qm\beta(1-\alpha)}\,(\phi(s)-\phi(a))^{qm\beta(1-\alpha)}$ inside the integral to get

$$\begin{aligned} Z(t)&\le \tfrac{|b|}{\Gamma(\gamma)}\,(\phi(t)-\phi(a))^{\beta(1-\alpha)} \\ &+\hat{C}_1(\textstyle\int_a^t Q^q(s)\,V^{qm}(s)\;(\phi(s)-\phi(a))^{qm\beta(1-\alpha)}\,(\phi(s)-\phi(a))^{-qm\beta(1-\alpha)})^{1/q} \\ &\le \tfrac{|b|}{\Gamma(\gamma)}(\phi(t)-\phi(a))^{\beta(1-\alpha)}+\hat{C}_1\left(\textstyle\int_a^t Q^q(s)\,Z^{qm}(s)\;(\phi(s))-\phi(a))^{-qm\beta(1-\alpha)}\,ds\right)^{1/q}. \end{aligned} \tag{5.10}$$

Now, raising both sides to power q, we arrive at

$$\begin{aligned}
&Z(t)^q \leq \Big[\tfrac{|b|}{\Gamma(\gamma)}(\phi(t)-\phi(a))^{\beta(1-\alpha)} \\
&+\hat{C}_1\left(\int_0^t Q^q(s)\, Z^{qm}(s)(\phi(s)-\phi(a))^{-qm\beta(1-\alpha)}ds\right)^{1/q}\Big]^q \\
&\leq 2^{q-1}\left(\left(\tfrac{|b|}{\Gamma(\gamma)}\right)^q(\phi(t)-\phi(a))^{q\beta(1-\alpha)}\right. \\
&\left.+\hat{C}_1^q\int_0^1 Q^q(s)\, Z^{qm}(s)(\phi(s)-\phi(a))^{-qm\beta(1-\alpha)}\right).
\end{aligned} \tag{5.11}$$

Let

$$w(t)=\hat{C}_1^q\int_a^t Q^q(s)\, Z^{qm}(s)\,(\phi(s)-\phi(a))^{-qm\beta(1-\alpha)}\,ds. \tag{5.12}$$

It is easy to see that $w(a)=0$ and

$$w'(t)=\hat{C}_1^q\; Q^q(t)\, Z^{qm}\,(t)\,(\phi(t)-\phi(a))^{-qm\beta(1-\alpha)}. \tag{5.13}$$

Since, ϕ and z, ψ continuous and nonnegative, w is is a continuous, nonnegative and non-decreasing function in $[a,\infty)$. Now, we will estimate the right hand side of (5.13) in terms of $w(t)$. From (5.11) and (5.12), we have

$$Z^q(t)\leq 2^{q-1}\left(\frac{|b|}{\Gamma(\gamma)}\right)^q(\phi(t)-\phi(a))^{q\beta(1-\alpha)}+w(t).$$

Raising to power m and using the inequality ([34])

$$\left(\sum_{i=1}^k a_i\right)^p \leq\; k^{p-1}\sum_{i=1}^k a_i^p,\;\; p\geq 1$$

we obtain

$$Z^{qm}\leq\Bigg[2^{qm-1}\left(\frac{|b|}{\Gamma(\gamma)}\right)^{qm}(\phi(t)-\phi(a))^{mq\beta(1-\alpha)}+w^m\,(t) \tag{5.14}$$

Finally, substituting (5.14) in (5.13), it appears that

$$\begin{aligned} w'(t) &\leq \hat{C}_1^q \, Q^q(t) \, (\phi(t) - \phi(a))^{-qm\beta(1-\alpha)} \\ &\times \left\{ 2^{mq-1} \left[\left(\tfrac{|b|}{\Gamma(\gamma)} \right)^{mq} (\phi(t) - \phi(a))^{mq\beta(1-\alpha)} + w^m(t) \right] \right\} \end{aligned}$$

or

$$w'(t) \leq 2^{mq-1} \, \hat{C}_1^q \, Q^q(t) \left(\frac{|b|}{\Gamma(\gamma)} \right)^{mq} + 2^{mq-1} \, \hat{C}_1^q \, Q^q(t) \, (\phi(t) - \phi(a))^{-qm\beta(1-\alpha)} w^m(t).$$

Next, we recall the lemma

Lemma 5.1.2: [34] *Let $a(t)$ and $b(t)$ be continuous positive functions defined on $[t_0, \infty)$, $t_0 \geq 0$, let $w : [0, \infty) \to [0, \infty)$ be a continuous monotonic nondecreasing function such that $w(0) = 0$ and $w(x) > 0$ for $x > 0$. If u is a positive differentiable function on $[t_0, \infty)$ that satisfies*

$$u'(t) \leq a(t) w(u(t)) + b(t), \quad t \in [t_0, \infty),$$

then we have,

$$u(t) \leq G^{-1} \left[G(u(t_0) + \int_{t_0}^{t} b(s) \, ds) + \int_{t_0}^{t} a(s) \, ds \right],$$

for the value of t for which the right-hand side is well-defined, where

$$G(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \quad r > r_0 > 0,$$

Applying Lemma 5.1.2, we get,

$$w(t) \leq G^{-1}\left[G\left(w(a) + \underbrace{2^{mq-1}\left(\frac{|b|}{\Gamma(\gamma)}\right)^{mq} \hat{C}_1^q \int_a^t Q^q(s)\, ds}_{=L(t)}\right)\right.$$

$$\left. + \underbrace{2^{mq-1}\, \hat{C}_1^q \int_a^t (\phi(s) - \phi(a))^{-qm(1-\alpha)} Q^q(s)\, ds}_{=K(t)}\right].$$

Thus,

$$w(t) \leq G^{-1}[G(L(t)) + K(t)]$$

and since

$$G(r) = \int_{r_0}^r \frac{ds}{w(s)} = \frac{r^{1-m}}{1-m} - \frac{r_0^{1-m}}{1-m}$$

it follows that

$$G^{-1}(y) = [r_0^{1-m} - (m-1)\, y]^{-\frac{1}{m-1}}.$$

That is,

$$\begin{aligned} w(t) &\leq G^{-1}\left[\tfrac{L(t)^{1-m}}{1-m} - \tfrac{L(a)^{1-m}}{1-m} + K(t)\right] \\ &\leq \left[L(a)^{1-m} - (m-1)\left[\tfrac{L(t)^{1-m}}{1-m} - \tfrac{L(a)^{1-m}}{1-m} + K(t)\right)\right]^{\frac{-1}{m-1}} \\ &\leq [L^{1-m}(t) - (m-1)\, K(t)]^{-\frac{1}{m-1}} \end{aligned}$$

as long as $L(t)^{m-1}\,K(t) < \frac{1}{m-1}$. In particular, if

$$\left\|Q(t)\right\|_q^{m-1}\left\|Q(t)\,(\phi(t)-\phi(a))^{-m\beta(1-\alpha)}\right\|_q \leq L^*$$

then $w(t) < k_1$ for some positive constant k_1 for all $t > a$ and from (5.10), we see that

$$Z(t) \leq \frac{|b|}{\Gamma(\gamma)}(\phi(t)-\phi(a))^{\beta(1-\alpha)} + k_1^{\frac{1}{q}}.$$

The definition of $Z(t)$gives

$$V(t) \leq \frac{|b|}{\Gamma(\gamma)} + k_1^{\frac{1}{q}}(\phi(t)-\phi(a))^{-\beta(1-\alpha)} \leq C.$$

This yields that

$$|y(t)| \leq C\;(\phi(t)-\phi(a))^{\gamma-1}, \quad t > a.$$

Observe that if $\phi(t) \to +\infty$ as $t \to +\infty$, then $|y(t)| \to 0$ as $t \to \infty$ with the rate $(\phi(t)-\phi(a))^{\gamma-1}$.

NON-EXISTENCE

6 Non-existence

Our aim here is to prove a nonexistence result for a generalized fractional differential problem. Namely, we will consider Problem 3 and establish sufficient condition insuring that no non-trivial solution exists globally. The proof of our result necessitates the preparation of the two lemmas below.

Lemma 6.1.1: *Let* $0 < \alpha < 1$, *and*

$$\psi(t) = \begin{cases} [\phi(T) - \phi(a)]^{-\lambda}[\phi(T) - \phi(t)]^{\lambda}\, \phi'(t), a \leq t \leq T,\ \lambda > 0, \\ 0, \qquad\qquad t > T \end{cases} \tag{6.1}$$

be a test function with $\lambda > \alpha - 1,\ 0 < \alpha < 1$. If $y \in C\,[a, T]$, *and* $\phi \in C^2[a, T],\ T > 0,$ *then*

$$\int_a^T \psi(t)\, {}^C D^{\alpha;\phi}\, y(t)\, dt = \int_a^T {}^C D_{T_-}^{\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \ y(t)\, \phi'(t)\, dt$$
$$-\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)} y(a)[\phi(T) - \phi(a)]^{1-\alpha}.$$

Proof: In view of the general fractional derivative property in Theorem 3.6.6, we have

$$\int_a^T \psi(t)\, {}^C D^{\alpha;\phi}\, y(t)\, dt \quad = \int_a^T {}^C D_{T_-}^{\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \ y(t)\, \phi'(t)\, dt$$
$$+ \left[\sum_{k=0}^{n-1} \left(-\frac{1}{\phi'(t)}\, \frac{d}{dt}\right)^k I_{T_-}^{n-\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \left(\frac{1}{\phi'(t)}\, \frac{d}{dt}\right)^{n-k-1} \ y(t)\right]_{t=a}^{t=T}.$$

Since $0 < \alpha < 1$, it follows that

$$\int_a^T \psi(t)\, {}^C D^{\alpha;\phi}\, y(t)\, dt = \int_a^T \ {}^C D_{T_-}^{\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \ y(t)\, \phi'(t)\, dt$$
$$+ \left[I_{T_-}^{1-\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) y(t) \right]_{t=a}^{t=T}$$

and by Theorem 3.6.1 and the definition of the function ψ in (6.1), it appears that

$$I_{T-}^{1-\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) = [\phi(T) \ - \phi\,(a)]^{-\lambda} I^{1-\alpha;\phi}\, [\phi(T) - \phi(t)]^{\lambda}$$
$$= (\phi(T) \ - \phi\,(a))^{-\lambda} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)} [\phi(T) - \phi(t)]^{\lambda+1-\alpha}.$$

Therefore,

$$\int_a^T \psi(t)\, {}^C D^{\alpha;\phi}\, y(t)\, dt \ = \int_a^T \ {}^C D_{T_-}^{\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \ \phi'(t)\, y(t) \ dt$$
$$+ \left[I_{T_-}^{1-\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) y(t) \right]_{t=a}^{t=T}$$
$$= \int_a^T \ {}^C D_{T_-}^{\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \ y(t)\, \phi'(t)\, dt$$
$$+(\phi(T) \ - \phi\,(a))^{-\lambda} \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)} [\phi(T) - \phi(t)]^{\lambda+1-\alpha} y(t) \right]_{t=a}^{t=T}.$$

Finally, as $\lambda + 1 - \alpha > 0$, we conclude that

$$\int_a^T \psi(t)\, {}^C D^{\alpha;\phi}\, y(t)\, dt = \int_a^T \ {}^C D_{T_-}^{\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \ y(t)\, \phi'(t)\, dt$$
$$- \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)} y(a) (\phi(T) - \phi(a))^{1-\alpha}.$$

Lemma 6.1.2: *Let ψ as in (6.1) with $\lambda > max\left\{\beta p, \beta - 1\right\}, \beta \geq 0$ and $p > 1$ then*

$$\int_a^T (\phi(t) - \phi(a))^{\gamma(1-p)} \ \left[\phi'(t)\right]^p \psi^{1-p}(t) \left[{}^C D_T^{\beta;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right) \right]^p \ dt$$
$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \ \frac{\Gamma(\gamma(1-p)+1)\Gamma(\lambda-\beta\, p+1)}{\Gamma(\gamma(1-p)+\lambda-\beta\, p+2)} [\phi(T) - \phi(a)]^{\gamma(1-p)-\beta\, p+1}.$$

Proof: By using the definition of ψ in (6.1) and Lemma 6.1.1, we find

$$\left[\phi'(t)\right]^p \left(\psi^{1-p}(t)\right) \left[D_T^{\beta;\phi}\left(\frac{\psi(t)}{\phi'(t)}\right)\right]^p$$

$$= [\phi(T) - \phi(a)]^{-\lambda(1-p)} [\phi(T) - \phi(t)]^{\lambda(1-p)} \phi'(t)$$

$$\times \left[(\phi(T) - \phi(a))^{-\lambda} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)} [\phi(T) - \phi(t)]^{\lambda-\beta}\right]^p.$$

That is

$$\left[\phi'(t)\right]^p \left(\psi^{1-p}(t)\right) \left[D_T^{\beta;\phi}\left(\frac{\psi(t)}{\phi'(t)}\right)\right]^p$$

$$= [\phi(T) - \phi(a)]^{-\lambda(1-p)-\lambda p} [\phi(T) - \phi(t)]^{\lambda(1-p)} \phi'(t)$$

$$\times \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-p)}\right]^p [\phi(T) - \phi(t)]^{\lambda p - \beta p}$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \left(\phi(T) - \phi(a)\right)^{-\lambda} \phi'(t) [\phi(T) - \phi(t)]^{\lambda-\beta p}.$$

Therefore,

$$\int_a^T [\phi(t) - \phi(a)]^{\gamma(1-p)} \left[\phi'(t)\right]^p \psi^{1-p}(t) \left[{}^C D^{\beta;\phi} \frac{\psi(t)}{\phi'(t)}\right]^p dt$$

$$= \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p [\phi(T) - \phi(a)]^{-\lambda}$$

$$\times \int_a^T \phi'(t) [\phi(t) - \phi(a)]^{\gamma(1-p)} [\phi(T) - \phi(t)]^{\lambda-\beta p} dt. \tag{6.3}$$

Let

$$s = \frac{\phi(t) - \phi(a)}{\phi(T) - \phi(a)}.$$

As

$$ds = \frac{\phi'(t)}{\phi(T) - \phi(a)}\, dt,$$

it is clear that

$$\int_a^T (\phi(t) - \phi(a))^{\gamma(1-p)} \left[\phi'(t)\right]^p \psi^{1-p}(t) \left[{}^C D^{\beta;\phi} \tfrac{\psi(t)}{\phi'(t)}\right]^p dt$$

$$= \left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \left[\phi(T) - \phi(a)\right]^{-\lambda} \int_0^1 s^{\gamma(1-p)} [\phi(T) - \phi(a)]^{\gamma(1-p)}$$

$$\times \left\{\phi(T) - s[\phi(T) - \phi(a)) + \phi(a)]\right\}^{\lambda-\beta p} [\phi(T) - \phi(a)] ds$$

$$= (\phi(T) - \phi(a))^{\gamma(1-p)-\lambda+1} \left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p$$

$$\times \int_0^1 s^{\gamma(1-p)} [\phi(T)(1-s) - \phi(a)(1-s)]^{\lambda-\beta p} ds$$

$$= [\phi(T) - \phi(a)]^{\gamma(1-p)-\lambda+1} \left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \int_0^1 s^{\gamma(1-p)} [\phi(T) - \phi(a)]^{\lambda-\beta p} s^{\lambda-\beta p} ds$$

$$= (\phi(T) - \phi(a))^{\gamma(1-p)-\beta p+1} \left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \int_0^1 s^{\gamma(1-p)} s^{\lambda-\beta p} ds$$

$$= (\phi(T) - \phi(a))^{\gamma(1-p)-\beta p+1} \left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \tfrac{\Gamma\left(\gamma(1-p)+1\right)\Gamma(\lambda-\beta p+1)}{\Gamma\left(\gamma(1-p)+\lambda-\beta p+2\right)} \quad .$$

Remark: We can rewire (6.3) to be

$$\left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \left[\phi(T) - \phi(a)\right]^{-\lambda} \Gamma(\lambda - \beta p + 1) I_{a^+}^{\lambda-\beta p+1;\phi} [\phi(T) - \phi(a)]^{\gamma(1-p)}$$

$$= \left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \left[\phi(T) - \phi(a)\right]^{-\lambda}$$

$$\times \Gamma(\lambda - \beta p + 1) \tfrac{\Gamma(\gamma(1-p)+1)}{\Gamma(\gamma(1-p)+\lambda-\beta p+2)} [\phi(T) - \phi(a)]^{\gamma(1-p)+\lambda-\beta p+1}$$

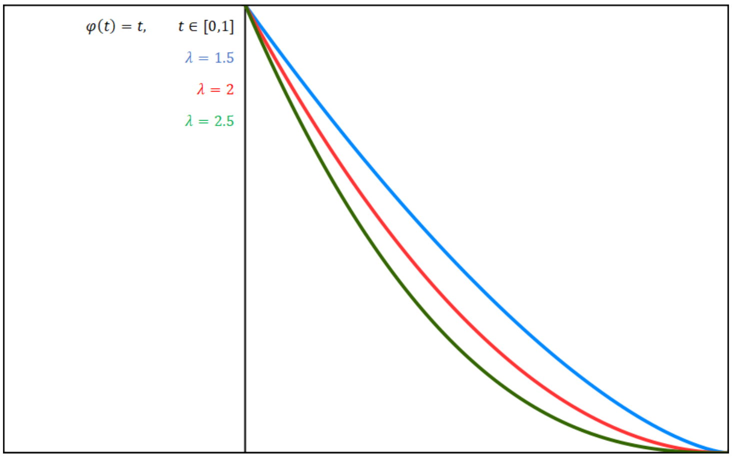
$$= (\phi(T) - \phi(a))^{\gamma(1-p)-\beta p+1} \left[\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^p \tfrac{\Gamma\left(\gamma(1-p)+1\right)\Gamma(\lambda-\beta p+1)}{\Gamma\left(\gamma(1-p)+\lambda-\beta p+2\right)},$$

which is the same result that we obtain above.

Note: In the case where $\phi(t) = t, a = 0$. We have $\phi'(t) = 1$ and so the test function turn out to be

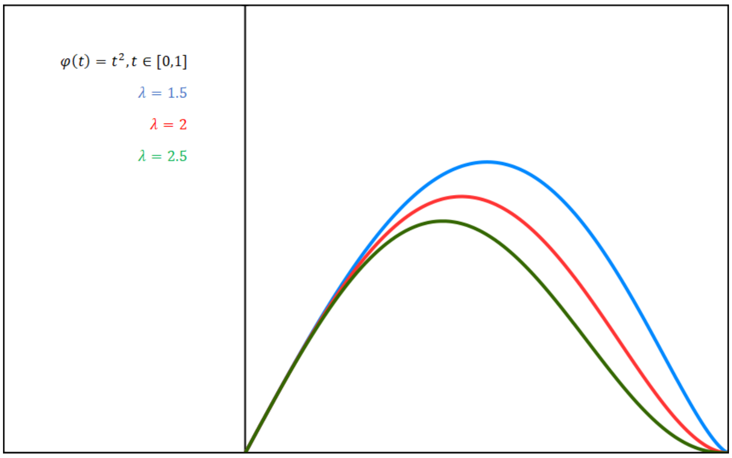
$$\psi(t) = \begin{cases} T^{-\lambda}(T-t)^{\lambda}\,, \; 0 \le t \le T, \, \lambda > 0, \\ 0, \qquad\qquad\qquad\qquad t > T \end{cases}$$

which is the classical test function (see the figure 1).



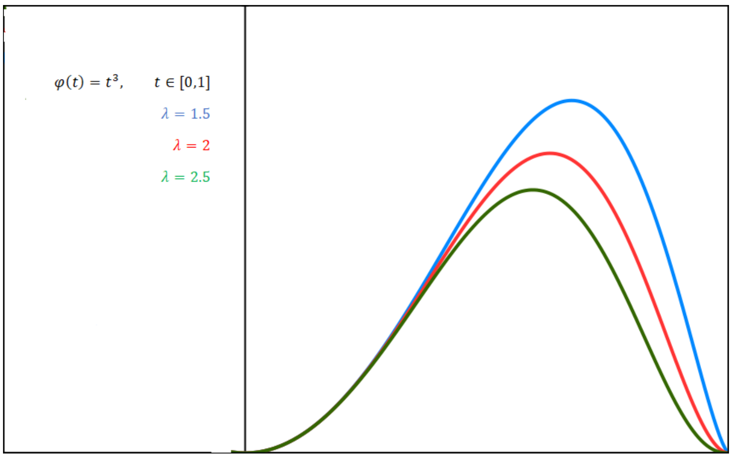
(figure 1)

We see that our test function is decreasing when $\phi(t) = t$. However, if $\phi(t) = t^2$, $t \in [0,1]$, then the test function ψ is increasing up to a certain point then it is starts to decrease till the endpoint 1 (see figure 2).



(figure 2)

Similarly, for $\phi(t) = t^3$ (see figure 3)



(figure 3)

In this section we study the nonexistence of a global solution of the initial value problem

$$\begin{cases} {}^C D_a^{\alpha;\phi}\, y(t) + {}^C D_a^{\beta;\phi} y(t) \geq (\phi(t) - \phi(a))^{\gamma}\, |y(t)|^m, \ 0 < \beta \leq \alpha < 1 \\ y(a) = b. \end{cases}$$

The main result of this section is stated in the following theorem.

Theorem 6.1.1: *Assume that* $m(1-\beta) < \gamma < m-1, m > 1$*. Then the problem (6.1) does not admit a global non trivial solution in* $C^n\,[0,\infty)$ *when* $b \geq 0$.

Proof:

Assume by contradiction that a solution exists for all time $t > 0$. Let ψ be as (6.1) with $\lambda > \frac{m\alpha}{m-1} - 1$. We multiply (6.2) by ψ and integrate over $[a, T]$, we obtain

$$\int_a^T \psi(t)\ {}^C D_a^{\alpha;\phi}\, y(t)\, dt + \int_a^T \psi(t)^C\, D_a^{\beta;\phi}\, y(t)\, dt \geq \int_0^T \psi(t) \left[(\phi(t) - \phi(a))^{\gamma}\, |y(t)|^m\right]\, dt. \tag{6.4}$$

By Lemma 6.1.1 we may write

$$\begin{aligned} \textstyle\int_a^T \psi(t)\, {}^C D_a^{\alpha;\phi}\, y(t) = \int_a^T \phi'(t)\, y(t)^C\, D_{T-}^{\alpha;\phi} \left(\frac{\psi(t)}{\phi'(t)}\right)\, dt \\ -\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+2)} [\phi(T) - \phi\,(a)]^{1-\alpha} y(a) \end{aligned} \tag{6.5}$$

and

$$\int_a^T \psi(t)\, {}^C D^{\beta;\phi}\, y(t) = \int_a^T \phi'(t)\, y(t)\, {}^C\, D^{\beta;\phi}_{T-}\, (\tfrac{\psi(t)}{\phi'(t)})\, dt \\ -\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda-\beta+2)}(\phi(T)-\phi\,(a))^{1-\beta} y(a). \tag{6.6}$$

In virtue of (6.4), (6.5) and (6.6) and $b \geq 0$ we deduce

$$I = \int_a^T \psi(t)\,(\phi(t)-\phi(a))^{\gamma}\,|y(t)|^m \leq \int_a^T \phi'(t) y(t)\, {}^C\, D^{\alpha;\phi}_{T-}\left(\tfrac{\psi(t)}{\phi'(t)}\right)\, dt \\ + \int_a^T \phi'\,(t)\, y(t)\, {}^C\, D^{\beta;\phi}_{T-}\left(\tfrac{\psi(t)}{\phi'(t)}\right)\, dt.$$

Now, multiply by

$$\psi^{1/m}\,[\phi(t)-\phi(a)]^{\gamma/m}\,\psi^{-\frac{1}{m}}\,[\phi(t)-\phi(a)]^{\frac{\gamma}{m}}$$

inside the integral on the right hand side of the last inquality to get

$$I \leq \int_a^T \phi'\,(t)\;\, y\,(t)\,[\phi(t)-\phi(a)]^{\frac{\gamma}{m}}\psi^{\frac{1}{m}}(t)\,\psi^{-\frac{1}{m}}(t)[\phi(t)-\phi(a)]^{-\frac{\gamma}{m}}\,{}^C D^{\alpha;\phi}_{T-}\,\left(\tfrac{\psi(t)}{\phi'(t)}\right)\, dt \\ + \int_a^T \phi'\,(t)\;\, y\,(t)\,[\phi(t)-\phi(a)]^{\frac{\gamma}{m}}\psi^{\frac{1}{m}}(t)\,\psi^{-\frac{1}{m}}(t)[\phi(t)-\phi(a)]^{-\frac{\gamma}{m}}\,{}^C D^{\beta;\phi}_{T-}\,\left(\tfrac{\psi(t)}{\phi'(t)}\right)\, dt\,.$$

Hölder inequality allows us to write

$$\begin{aligned} I &\leq \left(\int_a^T \psi(t)\,[\phi(t)-\phi(a)]^{\gamma}|y(t)|^m\right)^{\frac{1}{m}} \\ &\times \left(\int_a^T (\phi'(t))^{m'}\,\psi^{-\frac{m'}{m}}\,(t)\,\left[\phi\,(t)-\phi\,(a)\right]^{-\gamma\frac{m'}{m}}\,\left|{}^C D^{\alpha,\phi}_{T-}(\tfrac{\psi(t)}{\phi'(t)})\right|^{m'}\right)^{\frac{1}{m'}} \\ &+ \left(\int_a^T \psi(t)\,[\phi(t)-\phi(a)]^{\gamma}|y(t)|^m\right)^{\frac{1}{m}} \\ &\times \left(\int_a^T (\phi'(t))^{m'}\,\psi^{-\frac{m'}{m}}\,(t)\,\left[\phi\,(t)-\phi\,(a)\right]^{-\gamma\frac{m'}{m}}\,\left|{}^C D^{\beta,\phi}_{T-}(\tfrac{\psi(t)}{\phi'(t)})\right|^{m'}\right)^{\frac{1}{m'}} \end{aligned}$$

or

$$I^{\frac{1}{m'}} \leq \left(\int_a^T (\phi'(t))^{m'}(t)\ \psi^{\frac{-m'}{m}}(t)\,[\phi(t)-\phi(a)]^{\frac{-\gamma m'}{m}} \left| {}^C D_{T-}^{\alpha;\phi}\left(\frac{\psi(t)}{\phi'(t)}\right)\right|^{m'} \right)^{\frac{1}{m'}}$$
$$+ \left(\int_a^T (\phi'(t))^{m'}\,\psi^{\frac{-m'}{m}}(t)\,[\phi(t)-\phi(a)]^{\frac{-\gamma m'}{m}} \left| {}^C D_{T-}^{\beta;\phi}\left(\frac{\psi(t)}{\phi'(t)}\right)\right|^{m'} \right)^{\frac{1}{m'}}$$

where m' is the conjugate of m. Moreover, Lemma 6.1.2 implies

$$I^{\frac{1}{m'}} \leq \left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}\right]^{m'} (\phi(T)-\phi(a))^{\gamma(1-m')-\alpha m'+1} \frac{\Gamma(\gamma(1-m')+1)\Gamma(\lambda-\alpha m'+1)}{\Gamma(\gamma(1-m')+\lambda-\alpha m'+2)}^{\frac{1}{m'}}$$
$$+ \left(\left[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\beta)}\right]^{m'} (\phi(T)-\phi(a))^{\gamma(1-m')-\beta m'+1} \frac{\Gamma(\gamma(1-m')+1)\Gamma(\lambda-\beta m'+1)}{\Gamma(\gamma(1-m')+\lambda-\beta m'+2)}\right)^{\frac{1}{m'}}. \tag{6.7}$$

It suffices to raise both sides of (6.7) to the power m' to reach

$$I \leq c[(\phi(T)-\phi(a))^{\gamma(1-m')-\alpha m'+1} + (\phi(T)-\phi(a))^{\gamma(1-m')-\beta m'+1}]$$

for some constant $c > 0$. If $\gamma > m(1-\beta)-1$ we see that

$$\gamma(1-m')-\beta m'+1 < 0 \ \textit{ and } \ \gamma(1-m')-\alpha m'+1 < 0$$

and consequently

$$[\phi(T)-\phi(a)]^{\gamma(1-m')-\beta m'+1}, [\phi(T)-\phi(a)]^{\gamma(1-m')-\alpha m'+1} \to 0 \ \text{ as } T \to \infty.$$

Therefore,

$$\lim_{T\to\infty} \int_a^T \psi(t)[\phi(t)-\phi(a)]^{\gamma}\,|y(t)|^m = 0.$$

This is a contradiction.

7 Conclusion and future work

There is already a large number of different definitions of fractional derivatives in the market. Some of these fractional derivatives are equal on certain spaces and some others are identical under certain conditions. We are witnessing these last two decades a considerable increase in this number. A good number of these derivatives may be put under the umbrella of 'fractional derivative of a function with repect to another function'. Therefore, their study may be unified and generalized by investigating rather this latter kind of derivative. This will also avoid the duplication of publications and simplifies the dissemination. This is what we have done in this thesis for three types of problems and three types of questions. We have proved that existence of solutions, stability and non-existence of solutions shown already for special types of fractional derivatives such as the Riemann-Liouville, Caputo, Hadamard or Hilfer derivatives can be generalized to the 'generalized' fractional derivative considered in this thesis.

Obviously many things can be done in this respect such as considering systems of the form

$$\begin{cases} D_{0^+}^{\alpha_1,\phi} w(t) + \lambda_1 D_{a^+}^{\beta_1,\phi} w(t) = t^{\gamma_1} \left| z(t) \right|^q, \ t > 0 \\ D_{0^+}^{\alpha_2,\phi} z(t) + \lambda_2 D_{0^+}^{\beta_2,\phi} z(t) = t^{\gamma_2} \left| w(t) \right|^p, \ t > 0 \end{cases}$$

and

$$D_{0^+}^{\alpha_1,\phi}u\left(t\right)=h\left(t,D_{0^+}^{\beta_1,\phi}u\left(t\right),D_{0^+}^{\beta_2,\phi}v\left(t\right)\right),\ 0\leq\beta_1\leq\alpha_1<1,\ t>0$$

$$D_{0^+}^{\alpha_2,\phi}v\left(t\right)=g\left(t,D_{0^+}^{\beta_1,\phi}u\left(t\right),D_{0^+}^{\beta_2,\phi}v\left(t\right)\right),\ 0\leq\beta_2\leq\alpha_2<1,\ t>0.$$

Moreover, fractional integro-differential equations of the form

$$\left(D_{0^+}^{\alpha,\phi}w\right)(t)=g\left(t,\left(D_{0^+}^{\beta,\phi}w\right)(t),\int_0^t k\left(t,s,\left(D_{0+}^{\gamma}w\right)(s)\right)ds\right),\quad t>0$$

can be considered.

As future work, we may suggest tackling more general form of equations, inequalities and systems. In addition, it is important to weaken the imposed conditions so far. It is very much preferable to seek characterizations and optimality whenever possible.

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