

TWO-DIMENSIONAL DIRECTION OF ARRIVAL ESTIMATION USING SPARSE ARRAYS

BY

IBRAHIM BAYOUMI ABOUMAHMOUD

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

ELECTRICAL ENGINEERING

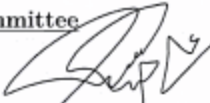
APRIL 2021

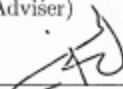
KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **IBRAHIM BAYOUMI ABOUMAHMOUD** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN ELECTRICAL ENGINEERING**.

Thesis Committee


Dr. Ali H. Mugaibel (Adviser)


Dr. Mohammad S. Albassoun (Member)


Dr. Suhail I. Al-Dharrab (Member)


Dr. Abdallah S. Al-Ahmari
Department Chairman


Dr. Suliman S. Al-Homidan
Dean of Graduate Studies

8/4/2021
Date



© Copyright by Ibrahim Bayoumi Aboumahmoud 2021
All Rights Reserved

To my beloved parents

ACKNOWLEDGMENTS

All the praises and thanks be to Allâh, the Most Gracious, the Most Merciful, the Most Generous, Who has taught (the writing) by the pen. He has taught man that which he knew not.

I would like to highly appreciate the encouragement and continuous support of my beloved parents which greatly helped me pursue this degree.

I am extremely delighted that I had the opportunity to work under the supervision of Prof. Ali H. Muqaibel. Not only for this thesis, but also for my senior design project when I was an undergraduate student. Working with him is a chance of critical thinking, with continuous supportive feedback that greatly helped me in finding the direction of improvement; in technical work and personal skills.

I also appreciate the support of Dr. Saleh A. Alawash, the expert in direction of arrival estimation who shared useful comments and ideas, and helped resolve many issues throughout my work on this thesis.

Special thanks go to Prof. Mohammad S. Alhassoun, who has provided critical reviews of many parts of this work, and also shared extremely useful experimental tips that helped me form a better understanding. He also continuously encouraged me and was a primary supporter of my use of L^AT_EX which I hope will delight the readers.

Many thanks also go to Dr. Suhail I. Al-Dharrab, who kindly accepted to be in the committee and provided valuable constructive comments.

I don't forget to also thank Mr. Mohamed H. Mohamed who helped in some parts of this research.

Last but certainly not least, I thank King Fahd University of Petroleum & Minerals and the Electrical Engineering Department for providing me the opportunity of conducting this research.

TABLE OF CONTENTS

ACKNOWLEDGMENTS	v
LIST OF TABLES	x
LIST OF FIGURES	xi
NOMENCLATURE	xiii
LIST OF ABBREVIATIONS	xv
ABSTRACT (ENGLISH)	xviii
ABSTRACT (ARABIC) ملخص	xx
CHAPTER 1 INTRODUCTION	1
1.1 Overview of Localization	2
1.2 Objectives	5
1.3 Organization of this Thesis	6
CHAPTER 2 TECHNICAL BACKGROUND	7
2.1 System Model	8
2.2 Statistical Correlation	9
2.2.1 Preliminaries	10
2.2.2 Second-order Statistics	11
2.2.3 Higher-order Statistics	11
2.3 Parameter Estimation	14

2.3.1	MUSIC	15
CHAPTER 3 DIRECTION FINDING ALGORITHMS IN 2D		20
3.1	Introduction	20
3.2	Two-dimensional DoA Estimation Essentials	24
3.2.1	2D System Model	25
3.2.2	Terminology of 2D Arrays	30
3.2.3	Coarray	31
3.2.4	Performance Metrics	33
3.2.5	2D-DoA Estimation Algorithms	37
3.3	Chapter Summary	39
CHAPTER 4 SPARSE PLANAR ARRAYS FOR DIRECTION FINDING: A COMPREHENSIVE REVIEW		40
4.1	Linear Arrays	42
4.1.1	Uniform Linear Array	42
4.1.2	Zero-redundancy Array	43
4.1.3	Nested Linear Array	46
4.1.4	Coprime Linear Array	47
4.2	Planar Arrays	47
4.2.1	Rectangular Array	48
4.2.2	L-shaped Array	49
4.2.3	V-shaped Array	49
4.2.4	Coprime Planar Array	50
4.2.5	Circular Array	50
4.3	Sparse Planar Arrays	52
4.3.1	Parallel Arrays	54
4.3.2	Non-Parallel Arrays	58
4.3.3	Planar Arrays	64
4.3.4	Other 2D Arrays	69
4.4	Comparative Evaluation of Sparse Planar Arrays	76

4.4.1	Concluding Remarks	82
CHAPTER 5 DESIGN OF SPARSE PLANAR ARRAYS MAXIMIZ-		
	ING LAGS	84
5.1	Arrays with contiguous virtual elements in the 4-DC	84
5.1.1	Proposed Arrays	86
5.1.2	Evaluating the 4-DC	89
5.1.3	Evaluating the Estimation Performance	94
5.2	Arrays with constrained number of sensors	96
5.2.1	Four-element sensor arrays	97
5.2.2	DoA Performance Simulation Results	98
CHAPTER 6 CONCLUSIONS AND FUTURE WORK		101
6.1	Conclusions	101
6.2	Future Work	103
APPENDIX A SUPPLEMENTARY MATERIAL		105
A.1	Useful Matrix Operations	105
A.1.1	The Kronecker Product	105
A.1.2	The vec Operator	106
A.2	Properties of Cumulants	107
REFERENCES		109
VITAE		128

LIST OF TABLES

3.1	Mapping between some symbols in cited papers and their equivalent in this document	28
4.1	Comparison between physical sensors and achievable degrees of freedom for sparse array geometries	77
4.2	Comparison between array geometries in terms of computational complexity, mutual coupling, and aperture	78

LIST OF FIGURES

1.1	Local positioning system (LPS) classification	3
1.2	General classification of antenna arrays used for direction of arrival estimation.	4
2.1	Two main steps of any direction of arrival (DoA) estimation algorithm.	8
2.2	The system model showing a single source $s_k[t]$ and N receiving an- tennas in a uniform linear array (ULA).	8
2.3	Estimation of four DoAs using (a) $\mathbf{R}_{\mathbf{xx}}$ (2.11) and (b) $\mathbf{C}_1$ (2.18). . . .	17
3.1	Illustration of important geometrical variables.	28
4.1	Overview of antenna arrays described in chapter 4.	41
4.2	Normalized power gain of a ULA with $N_t = 6$ elements.	43
4.3	The four zero-redundancy linear arrays.	45
4.4	Examples of linear antenna arrays.	45
4.5	Examples of planar antenna arrays.	51
4.6	Classification of sparse two-dimensional (2D) geometries based on the constituent arrays and the associated literature.	52
4.7	Timeline of papers using 2D sparse array geometries	52
4.8	Examples of 2D sparse arrays constructing using parallel linear arrays.	55
4.9	Examples of 2D sparse arrays constructed using non-parallel linear arrays.	59
4.10	Examples of sparse arrays constructed using planar subarrays.	64
4.11	Additional examples of 2D sparse arrays.	69
5.1	Examples of proposed sparse arrays constructed using three linear arrays.	87

5.2	Example of the fourth order difference coarray (4-DC) of the array in Fig. 5.1f.	90
5.3	Unique and consecutive lags vs number of physical sensors.	92
5.4	Ratio of consecutive lags to unique lags for the arrays in Fig. 5.3. . . .	93
5.5	Aperture (on the y axis) of the arrays in Fig. 5.3. The aperture on the x -axis is double the shown value for T-shaped arrays.	93
5.6	Performance of the six arrays in Fig. 5.1 using ESPRIT on 4-DC without mutual coupling.	95
5.7	Performance of the six arrays in Fig. 5.1 using ESPRIT on 4-DC with mutual coupling.	96
5.8	Examples of 2D sparse arrays with four elements and nine consecutive lags.	98
5.9	Weight function of the difference coarray of the arrays in Fig. 5.8 . . .	98
5.10	Performance of the four arrays in Fig. 5.8 using ESPRIT on 2-D and 4-DC with 500 snapshots.	100
5.11	Performance of the four arrays in Fig. 5.8 using ESPRIT on 2-D and 4-DC at 0dB SNR.	100

NOMENCLATURE

Operations

$(\cdot)^*$	Complex Conjugate Operator
$\text{cum}[\cdot]$	Cumulant Operator
$\mathbb{E}[\cdot]$	Expectation Operator
$(\cdot)^H$	Hermitian Operator
$\otimes$	Kronecker Product
$(\cdot)^T$	Transpose Operator

Scalars

$d, (d_x, d_y)$	Inter-element Spacing (along x , along y)
D	Effective Array Aperture
$M, (M_x, M_y)$	Number of Elements of a Sensor array (along x , along y)
$N, (N_x, N_y)$	Number of Elements of a Sensor Array (along x , along y)
N_i	Number of Elements in Subarray i
N_t	Total Number of Elements in a Sensor Array
k	The k th Source
K	Number of Sources
t	Time Sample
T	Total Number of Samples (Snapshots)
λ	Wavelength of Operation
$\phi_k \in [0, 2\pi]$	Azimuth Angle for the k th source (measured from the positive x -axis)

$\theta_k \in [0, \pi]$	Elevation Angle for the k th source (measured from the positive z -axis)
$\alpha_k \in [0, \pi]$	Electrical Angle for the k th source (related to ϕ_k and θ_k)
$\beta_k \in [0, \pi]$	Electrical Angle for the k th source (related to ϕ_k and θ_k)
$u_k = \cos(\beta_k)$	Direction Cosine on the x -axis
$v_k = \cos(\alpha_k)$	Direction Cosine on the y -axis

Vectors and Matrices

n	Noise Vector
s	Source Signals Vector
x	Array Output Vector
A	Array Manifold Matrix
C	Mutual Coupling Matrix
C_i	Cumulant Matrix (i is any subscript)
R	Covariance Matrix

Others

P	MUSIC Pseudo-spectrum
$\mathbb{D}$	Difference Coarray Set
$\mathbb{D}_4$	Fourth-order Difference Coarray Set
$\mathbb{S}$	Set of Array Elements

LIST OF ABBREVIATIONS

1D	One-Dimensional
2D	Two-Dimensional
2-DC	Second Order Difference Coarray
3D	Three-Dimensional
4-DC	Fourth Order Difference Coarray
AMM	Array Manifold Matrix
ANM	Atomic Norm Minimization
CcRA	Concentric Rectangular Array
CGF	Cumulant Generating Function
CLA	Coprime Linear Array
CPA	Coprime Planar Array
DA	Direct Augmentation
DF	Direction Finding
DMM	DoA Matrix Method
DoA	Direction of Arrival
DOF	Degrees of Freedom
EM	Electromagnetic
ESPRIT	Estimation of Signal Parameters Via Rotational Invariance Techniques

EVD	Eigen-Value Decomposition
FOC	Fourth Order Cumulants
GCPA	Generalized Coprime Planar Array
GPS	Global Positioning System
HOBA	Half Open Box Array
HOBA-2	Half Open Box Array with Two Layers
LA	Linear Array
LES	Linear Equispaced Array
LMR	Linear Minimum Redundancy
LPS	Local Positioning System
MCM	Mutual Coupling Matrix
MGF	Moment Generating Function
MIMO	Multiple Input Multiple Output
MRA	Minimum Redundancy Array
MRMSE	Maximum Root Mean Square Error
MSAE	Mean Square Angular Error
MSE	Mean Square Error
MUSIC	MUltiple SIgnal Classification
NCPA	Nested Coprime Planar Array
NLA	Nested Linear Array
NPA	Nested Planar Array
NULA	Nonuniform Linear Array

OBA	Open Box Array
OP	Oblique Projection
PCA	Parallel Coprime Array
PM	Propagator Method
PNA	Parallel Nested Array
POBA	Partial Open Box Array
POBA-L	Partial Open Box Array with L Levels
PS	Peak Search
PSS	Partial Spectral Search
RF	Radio Frequency
RFID	Radio-Frequency IDentification
RMSE	Root Mean Square Error
SDR	Software Defined Radio
SNR	Signal-To-Noise Ratio
SOS	Second Order Statistics
SS	Spatial Smoothing
SVD	Singular-Value Decomposition
UCPA	Unfolded Coprime Planar Array
ULA	Uniform Linear Array
URA	Uniform Rectangular Array

THESIS ABSTRACT

NAME: Ibrahim Bayoumi Aboumahmoud

TITLE OF STUDY: Two-Dimensional Direction of Arrival Estimation using
Sparse Arrays

MAJOR FIELD: Electrical Engineering

DATE OF DEGREE: April 2021

Direction of arrival (DoA) estimation is essential in many applications such as radars and mobile telecommunications systems. Among the best ways of performing DoA estimation is using an array of sensors (antennas) arranged in a linear, planar or three-dimensional geometry. The first is the most common in research and applications. However, planar arrays are gaining more attention recently. Linear arrays can only estimate a single angle, azimuth for example; whereas planar arrays can estimate elevation and azimuth, i.e. two-dimensional DoA estimation is possible. If an array is placed uniformly on a grid, it is called uniform; and if some elements of such a grid are missing, the array is sparse. For instance, the uniform linear array (ULA) is the simplest and most ubiquitous antenna array in use.

To get angle estimates from any array, an algorithm for DoA estimation is re-

quired. Many algorithms have been proposed for DoA estimation in the past decades. However, some of them are usable with a variety of antenna arrays with appropriate modifications. The most famous algorithms for DoA estimation are Multiple Signal Classification (MUSIC) and estimation of signal parameters via rotational invariance techniques (ESPRIT). These algorithms, among others, utilize estimated statistics from the finite samples available from the array output. Second order statistics (auto-correlation) estimates are very useful in many applications, but can be surpassed by higher order statistics in scenarios like colored Gaussian noise. Recently, fourth order cumulants, a higher order statistics estimate is gaining attention in DoA estimation.

This thesis provides a comprehensive review of 2D sparse sensor arrays used for 2D-DoA estimation. The meaning of sparse is clarified and the properties of the sparse arrays are explored in addition to performance metrics to compare the arrays. The presented comprehensive review of more than 30 papers using sparse planar arrays is hopefully useful for researchers aiming at finding new array designs. Key selected properties of the reviewed arrays are summarized in two tables at the end of this review. Furthermore, this thesis proposes new sparse sensor arrays and discusses some of their desired properties. In particular, some arrays with large number of consecutive lags in the fourth order difference coarray (4-DC) are investigated. In addition, design of arrays with limited number of sensors in presence of mutual coupling is examined.

ملخص الرسالة

الاسم: إبراهيم بن بيومي أبو محمود

عنوان الدراسة: التقدير ثنائي البعد لجهة الوصول باستخدام مصفوفات منشورة

التخصص: الهندسة الكهربائية

تاريخ الدرجة العلمية: رمضان ١٤٤٢ هـ

تقدير اتجاه الوصول أساس لعدد من التطبيقات مثل الرادارات وأنظمة الاتصالات النقالة. أحد أفضل أساليب تقدير اتجاه الوصول هو استخدام عدة مستشعرات (هوائيات) مصفوفة خطية، على مستوى، أو في الأبعاد الثلاثة. الأولى الأشهر في البحث والتطبيق. مع ذلك، الاهتمام بالمصفوفات الخطية تزايد مؤخرًا. المصفوفات الخطية يمكنها تقدير زاوية واحدة، السميت مثلاً. أما المصفوفات المستوية يمكنها تقدير زاويتي السميت والارتفاع، وهذا هو التقدير ثنائي البعد. إذا كانت الهوائيات مصفوفة بمسافات متساوية فهي منتظمة، وإذا أزيلت بعض الهوائيات من هذه المصفوفة، فهي منشورة. أيسر وأشهر مصفوفة استعمالاً هي المصفوفة الخطية المنتظمة.

لاستخراج تقدير الزوايا من مصفوفة لا بد من خوارزمية. اقترح عدد من الخوارزميات لتقدير اتجاه الوصول في العقود الماضية، منها ما يستعمل مع أنواع شتى من المصفوفات بعد تعديلات مناسبة. أشهر خوارزميتان لتقدير اتجاه الوصول هما تصنيف الإشارة المتعدد (MUSIC) و تقدير معاملات الإشارة عبر تقنيات الثبات الدوراني (ESPRIT). هاتان الخوارزميتان وغيرهما يستخدمان الإحصائيات المقدرة من العينات المحدودة المتوفرة من خرج المصفوفة. إحصائيات الدرجة الثانية (autocorrelation) مفيدة في كثير من التطبيقات، بيد أن إحصائيات الدرجات الأعلى قد تغلبها في حالات مثل ضوضاء غاوس الملونة. مؤخراً، لاقت تراكميات الدرجة الرابعة، إحدى إحصائيات الدرجات الأعلى، اهتماماً في تقدير اتجاه

الوصول.

هذه الرسالة تقدم مراجعة شاملة لمصفوفات المستشعرات المنثورة على مستوى في تقدير بعدي اتجاه الوصول. وفيها توضيح معنى النثر ودراسة خصائص المصفوفات المنثورة بالإضافة إلى مؤشرات أداء لمقارنة المصفوفات. المقارنة المعروضة المحتوية مراجعة شاملة لأكثر من ٣٠ ورقة استعملت مصفوفات مستشعرات منثورة يرجى أن تكون مفيدة للباحثين الساعين إلى إيجاد تصميمات مصفوفات جديدة. بعض الخصائص المفتاحية المختارة للمصفوفات المراجعة لخصت في جدولين عند نهاية المراجعة. علاوة على ذلك، فالرسالة تقترح مصفوفات مستشعرات منثورة جديدة وتناقش بعض خصائصها المرغوبة. تحديداً، مسحت بعض المصفوفات ذات العدد الكبير من المسافات المختلفة في المصفوفة المكافئة من الدرجة الرابعة. وأيضاً، اختبر تصميم المصفوفات محدودة المستشعرات في وجود الاقتران المتبادل.

CHAPTER 1

INTRODUCTION

The term ‘direction of arrival (DoA) estimation’ refers to the processing done by a receiver to guess the direction of incoming signals. Humans, for example, estimate the source of sound after processing the two sound copies received by their two ears. Inspired by this, some researchers utilize multiple microphones to allow computers to estimate the direction of arrival of sounds [1–3]; this can be used, for example, to automatically move a camera to follow a speaker on the stage [4]. Similarly, other researchers utilize multiple antennas to estimate the direction of arrival of electromagnetic (EM) waves.

EM DoA was in use about a century ago. In 1922, a book entitled ‘Wireless Direction Finding’ was published [5], which was a continuation of earlier activity in this field. During the last century, technological advances in signal generation and amplification at high frequencies (nowadays millimeter wave and terahertz), along with advances in digital signal processing and the capabilities of processors has allowed more application of direction finding (DF) in a plethora of situations.

DoA estimation can be one-dimensional (1D), which finds an estimate of a single

arrival angle (the azimuth angle). Or, it can be two-dimensional (2D), whereby the azimuth and elevation angles are estimated. The latter case received less attention compared to the former; because more steps are required to get correct estimation. For example, sometimes the azimuth and elevation angles are estimated separately, then, an extra pairing step is required to match related azimuth and elevation angles to yield the correct directions in the space. This work aims to focus on 2D DoA using sparse arrays, defined in Chapter 4.

Two-dimensional (2D) DoA estimation is important in many applications including radio astronomy, mobile telecommunication systems, sonar, and radar [6]. Some specific examples are: 1) Interference mitigation by determining the direction of interference or jamming sources and not amplifying those directions [7]. 2) Determining locations of ground stations using airborne antenna array. 3) Estimating the channel of massive multiple input multiple output (MIMO) system. 4) Gesture sensing.

Although a single antenna can be utilized under specific conditions to estimate the DoA of some signals, most of the research activity in the past decades utilized an array of antennas for this task. The following sections briefly describe general categories of localization, and how DoA estimation and this work fit into this world.

1.1 Overview of Localization

Positioning and localization are very broad terms. For instance, global positioning system (GPS) and cellular towers can be used to *locate* devices. Another example is a moving robot locating itself indoors. More precisely, this moving robot is doing

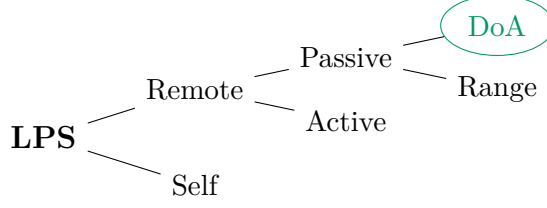


Figure 1.1: Local positioning system (LPS) classification

navigation, since it moves and relocates itself continuously. In the following discussion, two terms are used: anchor and target. The former refers to a fixed point at which the estimation is carried out. The latter denotes the object to be localized.

Apart from the GPSs, local positioning systems (LPSs) are of interest here. Fig 1.1 shows the classification of LPSs and highlights the focus of this work, which is DoA estimation. LPS can be classified to self or remote positioning. The former means the node locating itself relative to known references, while the latter refers to locating nodes (targets) at a remote anchor (e.g. antenna array). Remote positioning can be Active, if the targets cooperate, like Radio-Frequency IDentification (RFID); or passive like tracking radars [8].

Consider an anchor trying to find the location of a target. The anchor is mainly required to estimate the range (distance), and the direction. Here comes DoA Estimation, which is the focus of this work.

A single receiving antenna is not useful in determining DoA (unless it is directional and continuously rotated). The solution is to exploit diversity by using multiple antenna elements at the anchor. The vast majority of research and applications utilized a uniform linear array (ULA) (where antenna elements are equi-distant along a straight line) which work well with many algorithms including MULTiple SIGNAL Classification

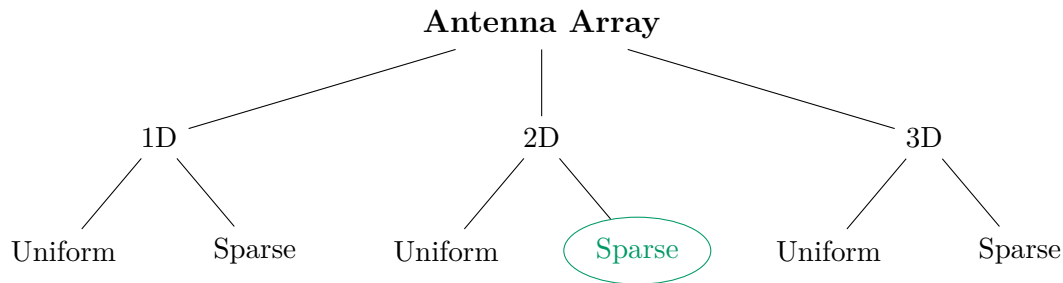


Figure 1.2: General classification of antenna arrays used for direction of arrival estimation.

(MUSIC) and estimation of signal parameters via rotational invariance techniques (ESPRIT). Newer designs skipped the equi-distance requirement, which allowed better performance for the same number of antenna elements, or the same performance for less elements [9].

Nevertheless, two-dimensional (2D) arrays received little attention compared to the linear array. Further, sparse arrays are promising, since the redundancy (repetitions) of antenna elements is minimized. Utilizing two-dimensional sparse arrays is expected to allow accurate two-dimensional DoA estimation. Fig 1.2 shows the three possible spacial arrangements of antenna elements, namely linear (1D), planar (2D), or in the 3D space. The 2D arrays arrangement is enough for two-dimensional DoA estimation, i.e. azimuth and elevation angles. However, some attempts of using 3D arrays are reported [10]. This work will focus on two-dimensional sparse arrays. To be specific, this thesis provides a comprehensive review of 2D sparse sensor arrays used for 2D DoA estimation. The meaning of sparse is clarified and the properties of the sparse arrays are explored in addition to performance metrics to compare the arrays. Furthermore, this thesis proposes new sparse sensor arrays and discusses some of their desired properties.

1.2 Objectives

The objectives of this thesis can be summarized as:

1. Prepare a comprehensive literature review on the use of 2D sparse arrays for DoA estimation, and identify relevant performance metrics.
2. Design a 2D sparse array for DoA estimation.
3. Test the designed array using computer simulations.
4. Verify the performance of a 2D array experimentally using software defined radio (SDR) (subject to availability of resources).

The first objective is satisfied through Chapter 3 and Chapter 4, where a detailed review is provided of sparse planar arrays geometries in the literature, and several performance metrics are defined and employed to compare the reviewed geometries. The second and third objective are the topic of Chapter 5 where proposed geometries that satisfy given design constraints are examined and compared. Finally, the fourth objective was quite difficult since all physical antenna channels must be synchronized for successful correlation of time samples. In addition, with about $150ms \pm 100ms$ on the laptop used for testing, the software was introducing extra mismatch on the coherency of the channels. Further, due to limited access to hardware, only 4 coherent channels was available, which is hardly useful to utilize in acceptable 2D DoA estimation.

1.3 Organization of this Thesis

Chapter 2 discusses the processing algorithms applied on the output of antenna arrays to estimate the DoA. Although the chapter focuses on 1D arrays, many concepts are extensible to 2D arrays. Then, Chapter 3 extends the model to the 2D array model, and introduces other concepts like the coarray and performance metrics. After that, Chapter 4 provides a comprehensive review of sparse 2D arrays. Next, Chapter 5 describes the design of proposed geometries and evaluates them. After that, Chapter 6 concludes the thesis and lists possible future extensions of this work. In addition, Appendix A demonstrates some useful matrix operations and lists properties of cumulants.

CHAPTER 2

TECHNICAL BACKGROUND

This chapter describes essential technical background for DoA estimation. In particular, this chapter describes the signal model assuming a linear array for simplicity, and also discusses two essential steps for any DoA estimation algorithm that is applied on a sensor array, namely: 1) Statistical Correlation and 2) Parameter Estimation. Fig. 2.1 shows the block diagram of the two steps, where $\mathbf{x}$ is the sampled output of the sensor array, $\mathbf{R}$ and $\mathbf{C}$ represent correlation matrices that are built based on estimated temporal statistics of the array output $\mathbf{x}$, and finally a set of estimated azimuth and elevation angles is produced. The estimation is successful if the estimated angles match the true angles. Note that in this chapter, only azimuth (θ_k) is considered since linear sensor arrays are used to facilitate easy introductory presentation. Note also that some assumptions about the sources like their number or statistical distribution are usually known at the parameter estimation step, whether assumed or estimated independently. In addition, the geometry of the sensor array (e.g. 1D or 2D, sparse or uniform) is also known at the parameter estimation step.

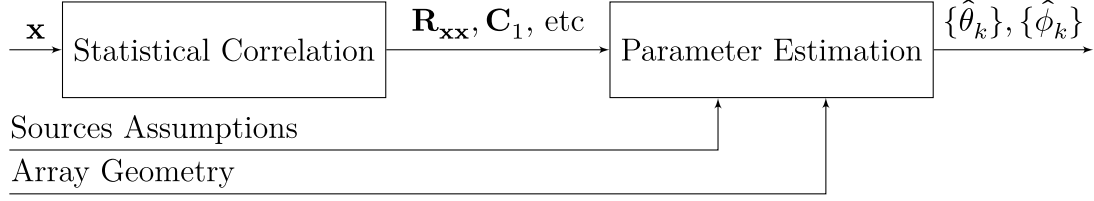


Figure 2.1: Two main steps of any DoA estimation algorithm.

2.1 System Model

Consider the model in Fig 2.2 where a narrow-band, zero-mean, stationary source exist in the far-field, and its angle of arrival θ_k at the ULA is to be estimated. In general, many similar sources exist in the far-field at different angles θ_k , and the total number of sources is K . Assume the sources signals are given by $\mathbf{s}[t] = [s_1[t], s_2[t], \dots, s_K[t]]^T$, and the output of array elements is given by $\mathbf{x}[t] = [x_1[t], x_2[t], \dots, x_N[t]]^T$.

Each array element receives a superposition of all sources. If the separation between array elements is d , then the phase delay of source k at the sensor array is represented by the array steering vector as

$$\mathbf{a}(\theta_k) = [1 \ e^{j2\pi d \sin(\theta_k)/\lambda} \ \dots \ e^{j2\pi n d \sin(\theta_k)/\lambda} \ \dots \ e^{j2\pi (N-1) d \sin(\theta_k)/\lambda}]^T \quad (2.1)$$

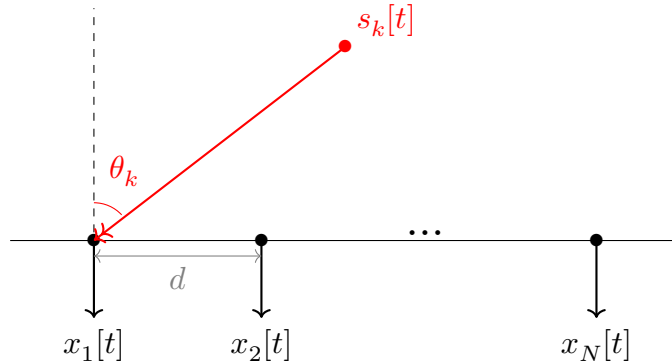


Figure 2.2: The system model showing a single source $s_k[t]$ and N receiving antennas in a ULA.

where $\frac{2\pi}{\lambda}$ is the wave number, $n = 0, 1, \dots, N-1$, and $k = 1, 2, \dots, K$.

Therefore, the output of the array can be represented by

$$\mathbf{x}[t] = [\mathbf{a}(\theta_1) \dots \mathbf{a}(\theta_k) \dots \mathbf{a}(\theta_K)] \mathbf{s}[t] + \mathbf{n}[t] \quad (2.2)$$

or more compactly

$$\mathbf{x}[t] = \mathbf{A}\mathbf{s}[t] + \mathbf{n}[t] \quad (2.3)$$

where $\mathbf{n}[t] \in \mathbb{C}^{N \times 1}$ is the additive white Gaussian noise, and $\mathbf{A} = [\mathbf{a}(\theta_1) \dots \mathbf{a}(\theta_k) \dots \mathbf{a}(\theta_K)] \in \mathbb{C}^{N \times K}$. This model will be used throughout this chapter.

The array output $\mathbf{x}[t]$ is sampled to yield T snapshots. The objective is to estimate the angles of arrival θ_k from the available snapshots of the array output. To do that, statistical estimation is employed with second-order and higher-order statistics.

2.2 Statistical Correlation

This section reviews the definition of expectation and cumulants. Then, estimation of expectation and cumulants from measured data is explained. This section describes the first block of Fig. 2.1. Note that expectation is the most widely used statistical correlation step in DoA estimation. Nonetheless, in some occasions, cumulants offer some attractive features like immunity to Gaussian noise.

2.2.1 Preliminaries

Before introducing estimation of data statistics, a revisit of statistical notions is presented, mainly to serve as an introduction to cumulants, which will be useful in the analysis of fourth-order statistics in the context of DoA. The first statistical notion is the moment, which is defined as the expectation of a random variable X

$$\mu_n = \mathbb{E}[X^n] \quad (2.4)$$

where n is the order of the moment, and $\mathbb{E}$ stands for expectation. In particular, the first order moment (μ) is the mean. Next, the n th order central moment is defined as

$$\mu_{c,n} = \mathbb{E}[(X - \mu)^n] \quad (2.5)$$

note that $\mu_{c,2} = \sigma_X^2 = \mathbb{E}[(X - \mu)^2]$ is the variance of the random variable X .

However, another way of computing moments is through the moment generating function (MGF) given by

$$m_X(t) = \mathbb{E}[e^{tX}] \quad (2.6)$$

where the n th moment is computed by

$$\mu_n = \left. \frac{d^n}{dt^n} m_X(t) \right|_{t=0} \quad (2.7)$$

Assuming the random variable has an MGF, the cumulant generating function

(CGF) is defined as

$$c_X(t) = \ln(m_X(t)) \quad (2.8)$$

and the n th order cumulant is defined as [11, 12]

$$\kappa_n = \left. \frac{d^n}{dt^n} c_X(t) \right|_{t=0} \quad (2.9)$$

2.2.2 Second-order Statistics

A simple estimate is to calculate the autocorrelation of the array output $\mathbf{x}(t)$

$$\hat{\mathbf{R}}_{\mathbf{xx}} = \frac{1}{T} \sum_{t=1}^T \mathbf{x}(t) \mathbf{x}^H(t) \quad (2.10)$$

Recall that the autocorrelation is defined as

$$\mathbf{R}_{\mathbf{xx}} = \mathbb{E} [\mathbf{x} \cdot \mathbf{x}^H] \quad (2.11)$$

Autocorrelation is used in many DoA algorithms such as MUSIC and ESPRIT.

2.2.3 Higher-order Statistics

The term higher-order statistics refers to methods that utilize (3rd, 4th, 5th, 6th, or higher) order statistics. The fourth order statistics are very commonly used because they exhibit useful features, which will be examined in the next section. Nevertheless, other higher order statistics are used. In particular, $2q$ th order cumulants [13], where the resolution power and performance increase with increasing q . This subsection

introduces $2q$ th order cumulants, then focuses on the use of fourth-order cumulants for DoA estimation.

Consider a random vector $\mathbf{x} \in \mathbb{R}^{N \times 1}$ with random variables $\{x_i, 1 \leq i \leq N\}$, the Leonov-Shiryaev formula gives a link between the q th order cumulants of $\mathbf{x}$ and its moments of order $p \leq q$ [14]

$$\text{cum} [x_{i_1}, x_{i_2}, \dots, x_{i_q}] = \sum_{p=1}^q (-1)^{p-1} (p-1)! \mathbb{E} \left[\prod_{j \in S_1} x_{i_j} \right] \mathbb{E} \left[\prod_{j \in S_2} x_{i_j} \right] \dots \mathbb{E} \left[\prod_{j \in S_p} x_{i_j} \right] \quad (2.12)$$

where $1 \leq i_j \leq N$ for $1 \leq j \leq q$, and S_p is the partition in p set of $\{1, 2, \dots, q\}$. Note that if $\mathbf{x}$ is not zero-mean, the mean must be subtracted first.

If $\mathbf{x} \in \mathbb{C}^{N \times 1}$, then x_i is replaced by $x_i^{\varepsilon_i}$ where $\varepsilon_i = \pm 1$ and $x_i^1 = x, x_i^{-1} = x^*$ (the conjugate of x_i). Therefore, the simplified expressions for $q = 2, q = 3$, and $q = 4$ are [14, 15]

$$\begin{aligned} \text{cum} [x_{i_1}, x_{i_2}] &= \mathbb{E} [x_{i_1} x_{i_2}] \\ \text{cum} [x_{i_1}, x_{i_2}, x_{i_3}] &= \mathbb{E} [x_{i_1} x_{i_2} x_{i_3}] \\ \text{cum} [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] &= \mathbb{E} [x_{i_1} x_{i_2} x_{i_3} x_{i_4}] - \mathbb{E} [x_{i_1} x_{i_2}] \mathbb{E} [x_{i_3} x_{i_4}] \\ &\quad - \mathbb{E} [x_{i_1} x_{i_2}] \mathbb{E} [x_{i_3} x_{i_4}] - \mathbb{E} [x_{i_1} x_{i_4}] \mathbb{E} [x_{i_2} x_{i_3}] \end{aligned} \quad (2.13)$$

Note that the second order and third order cumulants are identical to second order and third order moments. Important properties of cumulants are listed in appendix A.2. In practice, the higher order moments can be found using their empirical estimate.

For example, [16]

$$\mathbb{E} [x_{i_1} x_{i_2} x_{i_3} x_{i_4}] := \frac{1}{T} \sum_{t=1}^T x_{i_1}[t] x_{i_2}[t] x_{i_3}[t] x_{i_4}[t] \quad (2.14)$$

There are multiple ways to define fourth-order statistics estimate for use in DoA estimation. They have different computational complexities, and perform well when applied with particular estimation methods. First, the full fourth-order cumulants matrix definition is presented, then a simpler definition is examined.

Full Fourth-order Cumulants

The Full Fourth-order Cumulant is defined as

$$\mathbf{C}_f = \mathbb{E} [(\mathbf{x} \otimes \mathbf{x}^*)(\mathbf{x} \otimes \mathbf{x}^*)^H] - \mathbb{E} [\mathbf{x} \otimes \mathbf{x}^*] \mathbb{E} [(\mathbf{x} \otimes \mathbf{x}^*)^H] - \mathbb{E} [\mathbf{x} \mathbf{x}^H] \otimes \mathbb{E} [(\mathbf{x} \mathbf{x}^H)^*] \quad (2.15)$$

where $\otimes$ is the Kronecker product (which is explained in Appendix A.1), $*$ is the conjugate operator, and the dimensions of $\mathbf{C}_f$ are $N^2 \times N^2$.

The estimate of $\mathbf{C}_f$ can be found by [17]

$$\hat{\mathbf{C}}_{\mathbf{y}\mathbf{y}} = \frac{1}{N} \sum_{t=1}^N \mathbf{y}(t) \mathbf{y}^H(t) - \frac{1}{N^2} \sum_{t=1}^N \mathbf{y}(t) \sum_{t=1}^N \mathbf{y}^H(t) - (\hat{\mathbf{R}}_{\mathbf{x}\mathbf{x}} \otimes \hat{\mathbf{R}}_{\mathbf{x}\mathbf{x}}^*) \quad (2.16)$$

where

$$\mathbf{y}(t) = \mathbf{x}(t) \otimes \mathbf{x}^*(t) \quad (2.17)$$

$\mathbf{x}(t) \in \mathbb{C}^{N \times 1}$ is a single time snapshot at time t , $\mathbf{y}(t) \in \mathbb{C}^{N^2 \times 1}$, and T is the number

of snapshots.

A Simpler Fourth-order Cumulants Matrix

The fourth-order cumulants matrix defined in (2.15) is $N^2 \times N^2$, and this requires modifying the used estimation algorithm. For instance, the use of (2.15) with MUSIC is explained in Section 2.3.1. Here, a simpler fourth-order cumulants matrix is presented, which has the dimensions of $\mathbf{R}_{\mathbf{xx}} \in \mathbb{C}^{N \times N}$, thus, it can replace $\mathbf{R}_{\mathbf{xx}}$ in the estimation algorithm. The matrix is defined as [18]

$$\mathbf{C}_1 = \text{cum} \begin{bmatrix} \begin{bmatrix} x_1, x_1^*, x_1, x_1^* & \dots & x_1, x_1^*, x_1, x_N^* \\ x_2, x_2^*, x_2, x_1^* & \dots & x_2, x_2^*, x_2, x_N^* \\ \vdots & \ddots & \vdots \\ x_N, x_N^*, x_N, x_1^* & \dots & x_N, x_N^*, x_N, x_N^* \end{bmatrix} \end{bmatrix} \quad (2.18)$$

where cum is the fourth order cumulant operator (2.13), and $x_n, n = 1, 2, \dots, N$ is the output of the n th sensor. This simplified matrix can be used instead of $\mathbf{R}_{\mathbf{xx}}$ in (2.21). Next, a popular subspace parameter estimation method is presented.

2.3 Parameter Estimation

This section describes example algorithms for DoA estimation. This section describes the second block of Fig. 2.1. In particular, MUSIC algorithm is explored using the three statistical correlation matrices introduced in the previous Section. Namely, autocorrelation $\mathbf{R}_{xx}$, full fourth-order cumulants $\mathbf{C}_f$, and simpler fourth-order cumulants

$\mathbf{C}_1$.

2.3.1 MUSIC

MUltiple SIgnal Classification (MUSIC) is one of widely used algorithms for DoA with many variations. While it is sub-optimal, it is not computationally infeasible, and yields good results. The following text presents the use of MUSIC with second order statistics, and the two matrices defined earlier for fourth order statistics.

MUSIC using autocorrelation

Consider the array output in (2.3), and the corresponding autocorrelation matrix in (2.11) which can be decomposed as

$$\begin{aligned}
\mathbf{R}_{\mathbf{xx}} &= \mathbb{E} [\mathbf{x} \cdot \mathbf{x}^H] \\
&= \mathbb{E} [(\mathbf{A}\mathbf{s} + \mathbf{n}) (\mathbf{s}^H \mathbf{A}^H + \mathbf{n}^H)] \\
&= \mathbf{A} \mathbb{E} [\mathbf{s} \cdot \mathbf{s}^H] \mathbf{A}^H + \mathbb{E} [\mathbf{n} \cdot \mathbf{n}^H] \\
&= \mathbf{A} \mathbf{R}_{\mathbf{ss}} \mathbf{A}^H + \mathbf{R}_{\mathbf{nn}}
\end{aligned} \tag{2.19}$$

where $\mathbf{R}_{\mathbf{ss}}$ is the source correlation matrix ($K \times K$), and $\mathbf{R}_{\mathbf{nn}}$ is the noise correlation matrix ($N \times N$).

The autocorrelation $\mathbf{R}_{\mathbf{xx}}$ has N eigen-values ($\lambda_1, \lambda_2, \dots, \lambda_N$) and N eigen-vectors $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N]$. Sorting the eigen-values from small to large, the subspace of signal $\mathbf{V}_s$ and that of noise $\mathbf{V}_n$ are separated,

$$\mathbf{V} = [\mathbf{V}_n, \mathbf{V}_s] \tag{2.20}$$

where $\mathbf{V}_n$ is $N \times (N - K)$, and $\mathbf{V}_s$ is $N \times K$.

Utilizing the noise subspace, the MUSIC pseudo-spectrum is [19]

$$P(\theta) = \frac{1}{\mathbf{a}^H(\theta) \mathbf{V}_n \mathbf{V}_n^H \mathbf{a}(\theta)} \quad (2.21)$$

Again, in practice, the autocorrelation estimate in (2.10) is used.

MUSIC using the simple fourth-order cumulants

The matrix $\mathbf{C}_1$ (2.18) has the same dimensions as $\mathbf{R}_{\mathbf{xx}}$, and can be used in (2.21).

As an example, consider a ULA with $N = 6$ elements, $T = 100$ snapshots, $d = \lambda/2$, $\text{SNR} = 15\text{dB}$, and four independent, non-Gaussian sources impinging from angles $\theta_1 = 1^\circ, \theta_2 = 5^\circ, \theta_3 = 31^\circ$, and $\theta_4 = 45^\circ$. Fig. 2.3 shows the application of MUSIC estimate (2.21) with $\mathbf{R}_{\mathbf{xx}}$ (2.11) and $\mathbf{C}_1$ (2.18). At this configuration, Fig. 2.3a shows more accuracy for the estimated angles, however, the two closely-spaced angles are not resolved. On the other hand, Fig. 2.3b shows that the two closely-spaced angles could be identified as two angles, although the values are not very accurate.

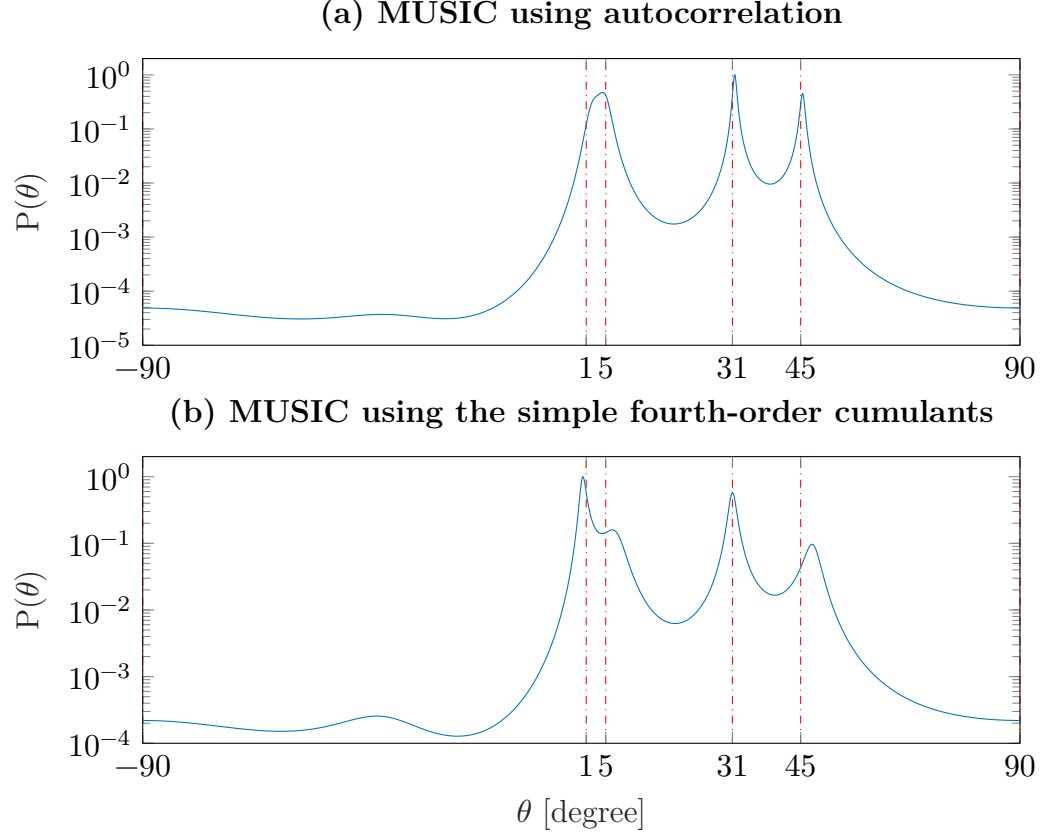


Figure 2.3: Estimation of four DoAs using (a) $\mathbf{R}_{\mathbf{xx}}$ (2.11) and (b) $\mathbf{C}_1$ (2.18).

MUSIC using full fourth-order cumulants

It is also possible to utilize the full fourth-order cumulants (2.16) for the MUSIC algorithm. To do that, the array cumulant matrix is given by

$$\mathbf{C}_f = \sum_{k=1}^K [\mathbf{a}(\theta_k) \otimes \mathbf{a}^*(\theta_k)] \mu_k [\mathbf{a}(\theta_k) \otimes \mathbf{a}^*(\theta_k)]^H \quad (2.22)$$

where μ_k is the fourth order cumulant of s_k , $k = 1, 2, \dots, K$. Let

$$\mathbf{b}(\theta_k) = \mathbf{a}(\theta_k) \otimes \mathbf{a}^*(\theta_k), \quad k = 1, 2, \dots, K, \text{ and} \quad (2.23)$$

$$\mathbf{D} = \text{diag}\{\mu_1, \mu_2, \dots, \mu_K\} \in \mathbb{C}^{K \times K}. \quad (2.24)$$

Then

$$\mathbf{B}(\theta) = [\mathbf{b}(\theta_1) \ \mathbf{b}(\theta_2) \ \dots \ \mathbf{b}(\theta_K)] \in \mathbb{C}^{N^2 \times K} \quad (2.25)$$

and the array cumulant matrix can be decomposed to

$$\mathbf{C}_f = \mathbf{B} \mathbf{D} \mathbf{B}^H. \quad (2.26)$$

This time, the matrix $\mathbf{C}_f$ has N^2 eigen-values $(\lambda_1, \lambda_2, \dots, \lambda_{N^2})$ and N^2 eigenvectors $[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{N^2}]$. Sorting the eigen-values from small to large separates the ‘signal’ ($\mathbf{Q}$) and ‘noise’ ($\mathbf{V}$) subspaces

$$\mathbf{V} = [\mathbf{V}_n, \mathbf{V}_s] \quad (2.27)$$

where $\mathbf{V}_n$ is $N^2 \times (N^2 - K)$, and $\mathbf{V}_s$ is $N^2 \times K$. Therefore, the MUSIC-like estimate is [17]

$$P(\theta) = \frac{1}{\mathbf{b}^H(\theta) \mathbf{V}_n \mathbf{V}_n^H \mathbf{b}(\theta)} \quad (2.28)$$

This chapter discussed detailed system model and estimation procedures for a ULA with N elements. Since the focus of this work is on 2D arrays, other linear arrays (LAs) are not described here. Another reason is that this chapter is meant as an easy introduction to the technical background, and not to review DoA estimation algorithms with all LAs. The next chapter extends the system model to 2D, and

describes useful performance metrics for 2D DoA estimation. Next, a comprehensive review of 2D sparse geometries for DoA estimation is presented in Chapter 4. Based on the reviewed geometries, new geometries are proposed in Chapter 5. In general, the statistical correlation methods described in Section 2.2 and the MUSIC algorithm discussed in Section 2.3 are applicable to 2D arrays with appropriate modifications.

CHAPTER 3

DIRECTION FINDING

ALGORITHMS IN 2D

3.1 Introduction

DoA estimation is an important application of array signal processing that has received increasing interest in the past decades. It is widely used in radio frequency and acoustics domains with similar array design and processing concepts. The performance and accuracy of the estimation algorithms are usually affected by some factors such as the coherence of sources, distribution of noise, signal-to-noise ratio (SNR), and the geometry of the sensor array. DoA estimation has proved useful in applications like sonar and radar [20]. It is also finding many applications in new generations of wireless communications and multiple input multiple output systems [21–23].

Various sensor arrays have been studied previously for DoA estimation like one-dimensional (1D) LAs, and 2D planar arrays. An essential property of a sensor array is its *resolution capacity*, i.e. the maximum number of resolvable waves impinging on the

array [24, 25]. For many years, the linear equispaced array (LES) [26], more commonly known as the ULA was paramount for the development of DoA estimation algorithms. For a ULA with N omnidirectional elements, and with the assumptions of narrow-band sources, having the same carrier frequency, with full-rank sample correlation matrix (of sources amplitudes), a full-rank correlation matrix of sensor outputs, and inter-sensor spacing less than or equal to half signal wavelength (λ), it was known that $N - 1$ sources can be resolved [25]. This limit, however, was exceeded using methods employing higher order statistics like the fourth order cumulants (FOC), which has the capability of eliminating Gaussian noise, but obviously, cannot work if the sources are also Gaussian [27, 28]. A different approach for increasing the resolution capacity of linear arrays focused on reducing the redundancy of inter-sensor¹ spacing, which opened the realm of sparse sensor arrays, i.e. arrays that have non-uniform inter-sensor spacings but are integer multiple of a fundamental distance, $d = \lambda/2$.

For linear arrays, it is known that only four arrays exhibit zero-redundancy, i.e. each inter-sensor spacing occurs once for all elements in the array, and the largest zero-redundancy array has only four sensors [29]. An easy way to list these four arrays is on a number line, thus the first zero-redundancy array has sensors at $\{0\}$, the second at $\{0, 1\}$, the third at $\{0, 1, 3\}$, and the fourth at $\{0, 1, 4, 6\}$. Due to the limited number of sensors with zero-redundancy, researchers focused on finding minimum redundancy arrays (MRAs), previously called linear minimum redundancy (LMR) arrays [29]. Finding and using MRAs for direction finding is a challenging task, since locations of antenna elements does not have a closed-form expression, and are found using complicated approaches [24, 30, 31]. Other sensor arrays were developed

over the last few decades including *Wichmann* [32], *coprime* [33, 34], *multi-level prime* [35], *nested* [36], *super nested* [37], and *cantor* [38] arrays, among others.

In terms of space and power budget, sparse arrays are valuable as they can reduce the cost, or increase the performance for the same cost. The array geometry plays a key role in the direction finding performance and capability. In the literature, 1D antenna arrays have been used extensively. A nice comparison for the three linear sparse arrays: *Wichmann* [32], *nested* [36], and *super nested* [37] arrays was published in 2017 by Rajamäki and Koivunen [39]. The article focused on deriving expressions for the maximum aperture, and comparing the performance of these arrays with the optimal MRA. Another work by Alawsh and Muqaibel [35] compared the *coprime* [33, 34], *Nested* [36], and *Super nested* [37] arrays with *multi-level prime array*, which is an extension of the coprime array, and they also compared with a *compressed multi-level prime array* in terms of DoA estimation performance under mutual coupling considering MUSIC algorithm and sparse reconstruction algorithms. However, the two studies above are limited to 1D arrays.

There is a growing interest in 2D and three-dimensional (3D) arrays that usually result in a much finer location resolution. Mazlout et al. [40] presented a comparison between the uniform rectangular array (URA) and the L-shaped array (made up of two orthogonal ULAs sharing a single sensor). Nonetheless, they considered only two arrays that are not sparse (refer to the discussion in Section 4.3.2). Another very thorough review for 1D and 2D-DoA estimation was published in 2010 by Gershman et al. [41]. It covered the fundamental concepts and focused on search-free techniques, and algorithms that work for arbitrary arrays, which include uniform and sparse arrays.

On the general topic of estimation in signal processing, Zoubir et al. [42] presented a tutorial which includes valuable information on DoA estimation, but does not describe specific array geometries. There are also less related comparison studies by Kiani and Pezeshk [43], Khedekar and Mukhopadhyay [44], Ahmed et al. [45], Adhikari and Drozdenko [46], and reviews in other areas related to DoA estimation [47, 48].

Most of the published work is generally concerned with DoA estimation of sources in the far-field. That is, sources far away from the Fresnel region of the antenna array. There is also 3D-DoA estimation, which is concerned with finding DoA and range for sources in the near-field of the array. In fact, the far-field regime can be considered as an extension of the near-field one, with infinite range.

This chapter aims to provide an overview of two-dimensional direction finding algorithms. In particular, the essentials of 2D-DoA estimation are explored in section 3.2. The next chapter aims to provide an overview of specific 2D sparse sensor geometries in the context of DoA estimation, i.e. 2D sparse sensor geometries that go in tandem with special DoA estimation algorithms.

Moreover, the following mathematical notation is used: For a matrix $\mathbf{A}$, $\mathbf{A}^*$ is its complex conjugate, $\mathbf{A}^T$ is its transpose, and $\mathbf{A}^H$ is its Hermitian. The operator $\mathbb{E}$ is the expectation operator, $\otimes$ is the Kronecker product, $\odot$ is the Khatri-Rao product, bold-face lower-case letters (e.g., $\mathbf{a}$) stand for vectors, bold-face upper-case letters (e.g., $\mathbf{A}$) stand for matrices, and the overhead hat (as in $\hat{\theta}$) denotes the estimated version of the variable below it.

3.2 Two-dimensional DoA Estimation Essentials

A two-dimensional DoA estimation problem can be decomposed into two separate 1D-DoA estimation problems, where some 1D algorithms can be utilized. When the 2D estimation problem is decomposed into two linear problems, automatic pairing of angles is an advantage, as this usually means less complexity and better estimates [6, 49]. In the research literature, several well-established algorithms for 1D direction finding are used such as MUSIC [19] and ESPRIT [50]. However, real-time analysis often requires faster algorithms that do not require either singular-value decomposition (SVD) or eigen-value decomposition (EVD). Usually, search-free algorithms are faster than subspace methods [41]. An example of an algorithm that does not require SVD or EVD is the propagator method (PM) which employs least-squares on the received covariance matrix [51]. The use of extended versions of these algorithms can be found in many works. For example, 2D MUSIC was reported in [52], 2D ESPRIT in [6, 53, 54], and the PM in [55–58]. The following general questions are important in characterizing any 2D-DoA estimation algorithm, some of them are borrowed from the 1D case:

- Can the method resolve more signals than the number of physical sensors? which is valuable in numerous cases [59].
- Are the available physical sensors fully utilized? That is, sensors are not under-utilized or non-essential [38, 60]. For instance, algorithms based on PM usually only resolve sources less than half the number of physical sensors [61].

- What are the ranges of angles the 2D algorithm successfully covers? For instance, some algorithms fail for high elevation angles [56].
- Can the sensor array perform well in real conditions? for example in presence of mutual coupling [62] and model mismatch [63, 64]. Also is it robust to sensor failure? [65, 66]

Generally, the aperture of the array is directly related to the maximum number of resolvable sources, and the inter-sensor spacing is related to possible aliasing of some angle estimates; more details in [46] and references therein. If the linear sparse array can provide the same main lobe beamwidth as a ULA, then it should be able to resolve the same number of sources as the ULA. Alternatively, if the linear sparse array has a continuous (hole-free) coarray ULA, then it should be able to resolve the same number of sources as a ULA similar to the coarray [46]. Similar ideas are used for 2D-DoA estimation, where it is preferred to get as close as possible to uniform rectangular coarray. The next subsections briefly mention the system model for 2D-DoA estimation, then explore the coarray concept, a useful mathematical property that is utilized in many DoA estimation algorithms.

3.2.1 2D System Model

The 2D system model describes the signals emerging from the sources to be localized, the antenna array, and the employed signal processing to extract features of interest. The signals can be in the far-field or the near-field, polarized or non-polarized, coherent or incoherent, correlated or uncorrelated. The sensor array could be 1D,

2D, or 3D, uniform or sparse or neither (see Section 4.3). The sensors can be omnidirectional or directional, polarized or non-polarized. The features to be estimated could be the 1D-DoA, or 2D-DoA, polarization, or range (distance to the source). The previous scenarios are certainly not comprehensive and extra considerations are required for practical implementations. For instance, mutual coupling, sensor failure, array calibration, and sensor characteristics mismatch are some physical hindrances for successful deployment of a functioning system.

An example of nine-element URA is shown in the x – y plane of Fig. 3.1 to illustrate important geometrical variables. The following symbols are used across this document, and the meaning mentioned here is assumed unless otherwise noted.

- T is the number of snapshots,
- K is the number of narrow-band impinging sources and k is used for the k th source (i.e. $k = 1, 2, \dots, K$),
- N_i and M_i are used for the number of antenna elements in the linear subarray i (the exact meaning is clarified for each antenna array geometry),
- N_t is the total number of physical sensors in the array,
- d_x is the fundamental spacing along the x -axis, d_y is the fundamental spacing along the y -axis, often $d_x = d_y$,
- $\phi_k \in [0, 2\pi]$ is the azimuth angle of the k th source (measured from the positive x -axis), $\theta_k \in [0, \pi]$ is the elevation angle (measured from the positive z -axis); these angles are illustrated in Fig. 3.1,

- $\alpha_k \in [0, \pi]$ is the angle between the line connecting k th source with the origin and the positive y -axis, $\beta_k \in [0, \pi]$ is the angle between the line connecting k th source with the origin and the positive x -axis.

In addition, the following *direction cosines* are used by some authors to simplify expressions

$$u_k := \cos(\beta_k) := \sin(\theta_k) \cos(\phi_k), \quad (3.1)$$

$$v_k := \cos(\alpha_k) := \sin(\theta_k) \sin(\phi_k), \quad (3.2)$$

where α_k and β_k are called electric angles, and they simplify many analytical expressions since they can be separated from each other. Other used geometrical variables are [62]

$$\bar{\theta}_k := \frac{d}{\lambda} \sin(\theta_k) \cos(\phi_k), \quad (3.3)$$

$$\bar{\phi}_k := \frac{d}{\lambda} \sin(\theta_k) \sin(\phi_k). \quad (3.4)$$

Apart from variables that depend on the sensor array geometry (N_i, M_i, d_x and d_y), the rest of the variables are shown in Fig. 3.1. While this work tries to unify the notation in the majority of existing works, several differences remain present, and Table 3.1 shows some differences comparing other works to the selected notation.

Consider a system for estimating the 2D-DoA of far-field non-polarized signals. When a single source is considered, the sampled output of the sensor array at a time

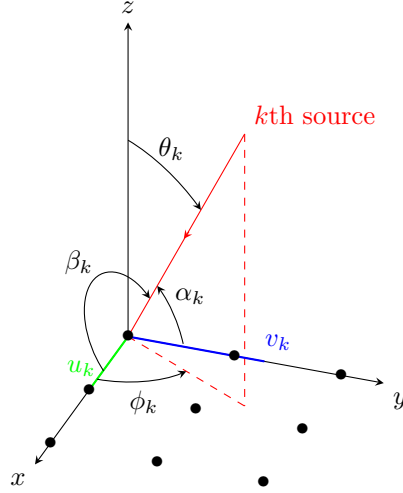


Figure 3.1: A nine-element URA example to illustrate important geometrical variables.

Table 3.1: Mapping between some symbols in cited papers and their equivalent in this document

Ref.	Symbol	Equiv. Symbol
[67]	ϕ, θ	θ, ϕ
[61, 68–74]	M, N	M_1, M_2
[68, 75]	$\mathbb{L}$	$\mathbb{S}$

instant t can be written as [19, 62, 76]

$$\mathbf{x}(t) = \mathbf{a}(\theta_k, \phi_k)s(t) + \mathbf{n}(t), \quad (3.5)$$

where $\mathbf{a}(\theta_k, \phi_k) \in \mathbb{C}^{N_t \times 1}$ is the array manifold (or steering vector) with entries $e^{j2\pi(\bar{\theta}_k p_x + \bar{\phi}_k p_y)}$ for all $(p_x, p_y) \in \mathbb{S}$ and $\mathbb{S}$ is the set of sensor locations, $s(t)$ is the transmitted signal, $\mathbf{x}(t) \in \mathbb{C}^{N_t \times 1}$ for N_t sensors, and $\mathbf{n}(t) \in \mathbb{C}^{N_t \times 1}$ is the additive noise vector, that is often assumed to be spatially and temporally white Gaussian noise and uncorrelated at each array element, that is $\mathbb{E}[\mathbf{n}(t)\mathbf{n}^H(t)] = \sigma^2 \mathbf{I}$, where σ^2 is the noise power and $\mathbf{I}$ is the identity matrix.

When more sources are considered, the model can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_{N_t}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{a}(\theta_1, \phi_1) & \dots & \mathbf{a}(\theta_K, \phi_K) \end{bmatrix} \times \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_K(t) \end{bmatrix} + \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ \vdots \\ n_{N_t}(t) \end{bmatrix} \quad (3.6)$$

Then taking T time samples (snapshots), the model can be written compactly as

$$\mathbf{X} = \mathbf{A}\mathbf{S} + \mathbf{N} \quad (3.7)$$

where $\mathbf{X} \in \mathbb{C}^{N_t \times T}$, $\mathbf{A} \in \mathbb{C}^{N_t \times K}$, $\mathbf{S} \in \mathbb{C}^{K \times T}$, $\mathbf{N} \in \mathbb{C}^{N_t \times T}$.

The source is considered in the far-field if the sources is beyond the Fraunhofer

distance $2D^2/\lambda$ where D is the aperture of the array, and λ is the wavelength of the source [77, 78]. If the source range is less than this limit, it is considered in the near-field.

3.2.2 Terminology of 2D Arrays

This section is a necessary extension to the previous one. It explains the set of array elements $\mathbb{S}$ and how mutual coupling is modeled. Since this work considers 2D sensor arrays which can be represented in a rectangular grid, it is convenient to describe the sensor locations as integers, where it is understood that the fundamental unit is d : the fundamental (minimum) distance between any pair of sensors, which is often chosen as $d = \lambda/2$. The set of sensor elements is denoted by $\mathbb{S}$ in this work. Note that other authors using this set might call it $\mathbb{L}$ for example [68, 75], as noted in Table 3.1.

Mutual coupling is usually modeled by a mutual coupling matrix (MCM) $\mathbf{C} \in \mathbb{C}^{N_t \times N_t}$ that is multiplied by the array manifold matrix (AMM) $\mathbf{A} \in \mathbb{C}^{N_t \times K}$. Technically, MCM should be specific to each set of antennas, taking into consideration the physical dimensions of the antennas and feeding points [63]. Usually, antenna spacing is the dominant factor. In fact, it was empirically observed that the behavior of the MCM is approximately a function of sensor separations only [37]. In two dimensions, the MCM can be assumed to be a B -banded symmetric Toeplitz matrix [62, 79, 80]

$$\langle \mathbf{C} \rangle_{\mathbf{p}_1, \mathbf{p}_2} = \begin{cases} c(\|\mathbf{p}_1 - \mathbf{p}_2\|_2), & \text{if } \|\mathbf{p}_1 - \mathbf{p}_2\|_2 < B \\ 0, & \text{otherwise} \end{cases} \quad (3.8)$$

where $\langle \mathbf{C} \rangle_{\mathbf{p}_1, \mathbf{p}_2}$ denotes the value of $\mathbf{C}$ at a sensor pair $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{S}$, $\|\cdot\|_2$ is the ℓ_2 -norm, B is the maximum separation between coupled sensors, and $c(\cdot)$ denotes the mutual coupling coefficient. In addition, it is assumed that

$$c(0) = 1, \quad \& \quad \left| \frac{c(k)}{c(\ell)} \right| = \frac{\ell}{k}, \quad \text{for } k, \ell > 0 \quad (3.9)$$

3.2.3 Coarray

An essential step employed in most DoA estimation algorithms with sparse arrays is finding the coarray of the physical array. For the case of non-coherent sources, the difference coarray is of interest, and is defined as follows [59].

Definition 3.2.1 (Difference Coarray). If a set $\mathbb{S}$ denotes the set of ordered pairs of points representing the coordinates of physical sensors in a sensor array, then the difference coarray set is given by

$$\mathbb{D} = \{\mathbf{m} | \mathbf{m} = \mathbf{p}_1 - \mathbf{p}_2, \forall \mathbf{p}_1, \mathbf{p}_2 \in \mathbb{S}\}. \quad (3.10)$$

Alternatively, $\mathbf{p}_1$ and $\mathbf{p}_2$ can be considered as vectors in $\mathbb{R}^{2 \times 1}$ denoting physical sensor locations. The maximum number of virtual sensor elements in the coarray is $N_t(N_t - 1) + 1$, regardless of array geometry [36]. The difference coarray sensors are symmetric with respect to the origin, and it is important since it appears naturally in the cross correlation between two sensors in the array, i.e. when second order statistics (SOS) are used, which is an equivalent representation of virtual sensors at the locations of the differences [81]. Another widely used term for the coarray is the

holes, which can be defined as follows

Definition 3.2.2 (Hole-free Coarray). If a set $\mathbb{D}$ denotes a coarray on a rectangular integer grid, and the minimum and maximum elements are denoted as $\min \mathbb{D}$ and $\max \mathbb{D}$. If all possible combinations of pairs in $[\min \mathbb{D}, \max \mathbb{D}]$ exist in the coarray, then the coarray is hole-free. Alternatively, if $\mathbb{U}$ denotes a URA which includes the origin, and $\mathbb{D} = \mathbb{U}$, then $\mathbb{D}$ is hole-free.

This definition implies that any element in $\mathbb{U}$ but not in $\mathbb{D}$ is a hole. It is often desirable to get a continuous (hole-free) coarray, i.e. a ULA for linear arrays or a URA for planar arrays. Reasons include: Better identifiability of DoAs using estimation algorithms like MUSIC with spacial smoothing [34, 36]; and $\mathcal{O}(N_t^2)$ uncorrelated DoAs can be identified with a coarray of $\mathcal{O}(N_t^2)$ sensors. In 2D, this holds only almost surely [62, 82]. In general, if the elements of a coarray are continuous on uniform rectangular grid, then the coarray is hole-free. It is possible to extend the hole-free definition to the 3D arrays [62]. An important definition for the coarray is the weight function, which gives an insight on the significance of mutual coupling on DoA estimation [37, 62].

Definition 3.2.3 (Weight Function). If a set $\mathbb{S}$ denotes the set of physical sensors' positions in a sensor array, and $\mathbb{D}$ denotes the difference coarray, then

$$w(\mathbf{m}) = w(m_x, m_y) = |\{(\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{S}^2 | \mathbf{p}_1 - \mathbf{p}_2 = \mathbf{m}\}| \quad (3.11)$$

denotes the number of sensor pairs with separation $\mathbf{m} \in \mathbb{D}$. Note that $\mathbf{m} = (m_x, m_y)$ is a two-component vector.

For example, the most significant weights are $w(0, 1)$, $w(1, 0)$, and then $w(1, 1)$, $w(1, -1)$, since the first two count the elements with unity inter-sensor spacing, and the other two for $\sqrt{2}$ inter-sensor spacing (of course scaling factors d_x and d_y are neglected). The first two mean the number of sensors having inter-element spacing of unity, which is the smallest possible inter-sensor spacing. The difference is just which axis this weight is calculated across. An array geometry with the less value for $w(0, 1)$, $w(1, 0)$, $w(1, 1)$, and $w(1, -1)$ is usually less susceptible to mutual coupling degradation in DoA performance.

3.2.4 Performance Metrics

Many performance metrics have been utilized to assess the performance of 2D-DoA estimation algorithms. This includes computing the deviation (error) from the known DoA; where the root mean square error (RMSE) is the most widely used metric [62, 67–75, 78, 79, 83–101]. However, other metrics can be used like the mean square angular error (MSAE) [102], mean square error (MSE) [103], and maximum root mean square error (MRMSE) [104]. In addition, when the 2D-DoA algorithm is expected to work in the underdetermined case, i.e. when estimating more sources than the available sensors, it is insightful to compare the degrees of freedom (DOF) offered by the 2D-DoA estimation algorithms, the coarray, and the aperture. Furthermore, the computational complexity or running time and probability measures for success (correct resolution) or failure can be employed. This subsection highlights some performance metrics that are commonly utilized.

Root Mean Square Error (RMSE)

For 2D-DoA estimation, the RMSE usually considers both azimuth and elevation angles [88, 91, 99, 100, 105]

$$\text{RMSE} = \sqrt{\frac{1}{HK} \sum_{h=1}^H \sum_{k=1}^K \left((\theta_k - \hat{\theta}_{k,h})^2 + (\phi_k - \hat{\phi}_{k,h})^2 \right)} \quad (3.12)$$

where H is the number of Monte-Carlo trials. However, some authors still compute separate errors for azimuth and elevation angles [70, 94]

$$\text{RMSE}_\theta = \sqrt{\frac{1}{HK} \sum_{h=1}^H \sum_{k=1}^K \left((\theta_k - \hat{\theta}_{k,h})^2 \right)} \quad (3.13)$$

$$\text{RMSE}_\phi = \sqrt{\frac{1}{HK} \sum_{h=1}^H \sum_{k=1}^K \left((\phi_k - \hat{\phi}_{k,h})^2 \right)} \quad (3.14)$$

The RMSE can also be calculated based on the electrical angles α_k and β_k . Note that many variations exist in the literature like dividing by K outside the square root instead of inside [75], dividing by $2HK$ instead of HK [73, 74, 93], keeping the averaging over sources outside the square root [89], or defining the RMSE keeping only the averaging over sources inside the square root and then average Monte-Carlo trials [62, 79].

Coarray Properties

As discussed earlier, the coarray is preferred to be hole-free, that is, all virtual elements in the coarray are contiguous. If this is not the case, then some properties are

considered like the number of unique lags, n_{ul}

$$n_{ul} = |\mathbb{D}| \quad (3.15)$$

where $|\mathbb{D}|$ is the cardinality of the set $\mathbb{D}$. In addition, the contiguous URA portion of a coarray can be determined by finding the longest contiguous ULA on the x -axis $\mathbb{D}_{x\text{ULA}}$ and the longest contiguous ULA on the y -axis $\mathbb{D}_{y\text{ULA}}$, then multiplying the number of elements in $\mathbb{D}_{x\text{ULA}}$ and $\mathbb{D}_{y\text{ULA}}$ to find the number of consecutive lags, n_{cl}

$$n_{cl} = |\mathbb{D}_{x\text{ULA}}| \times |\mathbb{D}_{y\text{ULA}}| \quad (3.16)$$

where $|\cdot|$ is the cardinality. Note that the two virtual ULAs $\mathbb{D}_{x\text{ULA}}$ and $\mathbb{D}_{y\text{ULA}}$ are symmetric about the origin since the coarray is. It is possible, however, to consider consecutive lags on other contiguous URAs (if they exist), but those are usually smaller than the consecutive URA that includes the origin. The weight function is another coarray property that is also considered for evaluating a sensor array.

Redundancy

When all elements in the coarray appear exactly once, the sensor array is said to have unity redundancy, i.e. $R = 1$. Usually, sensor arrays with contiguous (hole-free) coarrays have $R > 1$. The asymptotic redundancy $R_\infty = \lim_{N_t \rightarrow \infty}$ can be used to compare sparse sensor arrays [106].

Sparseness

Unlike the weight function, sparseness (S) counts the sensor pairs separated by a positive distance $\tilde{d}$ [106]

$$S(\tilde{d}) = \frac{1}{2} \sum_{\mathbf{m} \in \mathbb{D}} v_{\Delta}(\mathbf{m}) \cdot \mathbb{1}(\|\mathbf{m}\|_2 = \tilde{d}) \quad (3.17)$$

where $v_{\Delta}(\mathbf{m}) = \sum_{\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{S}} \mathbb{1}(\mathbf{m} = \mathbf{p}_1 - \mathbf{p}_2)$ is the multiplicity function (analogous to the weight function) for the difference coarray and $\mathbb{1}(\cdot)$ is the indicator function. For instance, for a sensor array on a uniform grid, $\tilde{d} \in \{1, \sqrt{2}, 2, \sqrt{5}, \sqrt{8}, 3, \dots\}$. Alternatively, sparseness can be viewed as the summation of essential weight function pairs with the same absolute distance. For example, $S(1) = w(1, 0) + w(0, 1)$, $S(\sqrt{2}) = w(1, 1) + w(1, -1)$, etc. In other words, sparseness abstracts dimensions-specific weight values.

Coupling Leakage

Coupling leakage, $L \in [0, 1]$ quantifies the mutual coupling of a sensor array [37]

$$L = \frac{\|\mathbf{C} - \text{diag}(\mathbf{C})\|_F}{\|\mathbf{C}\|_F} \quad (3.18)$$

where $\|\cdot\|_F$ is the Frobenius norm, and $[\text{diag}(\mathbf{C})]_{i,j} = [\mathbf{C}]_{i,j} \delta_{i,j}$. Note that lower L is often preferred to have lower the degradation effect on DoA estimation algorithms.

3.2.5 2D-DoA Estimation Algorithms

Many algorithms are used for 2D-DoA estimation with uniform or sparse planar arrays. For example, MUSIC, ESPRIT, PM, DoA matrix method (DMM) and others have been used with sparse planar arrays. This section gives a brief introduction to MUSIC with two variants and DMM.

MUSIC

MUSIC algorithm does not assume anything about the array geometry [19]. However, the maximum number of resolvable sources is limited by the number of physical sensors in the array, since it is required to find the noise subspace of the autocorrelation matrix. This algorithm is called direct-MUSIC, to distinguish it from other MUSIC variants discussed later in this section. For direct-MUSIC, if $\mathbf{x}(t)$ denotes the output of all N_t physical sensors, the covariance matrix is

$$\mathbf{R}_S = \mathbb{E} [\mathbf{x}(t)\mathbf{x}^H(t)] \in \mathbb{C}^{N_t \times N_t} \quad (3.19)$$

where the subscript S denotes the set of physical array elements. The next steps are finding the eigenvalue decomposition and searching for the peaks in the pseudospectrum [19, 41, 107]

$$P_{\text{MUSIC}}(\theta, \phi) = \frac{1}{\mathbf{a}^H(\theta, \phi) \mathbf{E}_N \mathbf{E}_N^H \mathbf{a}(\theta, \phi)} \quad (3.20)$$

which means minimizing the distance from $\mathbf{a}(\theta_k, \phi_k)$ to the signal subspace. Note that $\mathbf{a}(\theta_k, \phi_k)$ in (3.20) does not depend on the data, and $\mathbf{E}_N \in \mathbb{C}^{N_t \times (N_t - K)}$ denotes the

noise subspace. Again, $N_t > K$ is assumed.

DA-MUSIC and SS-MUSIC

Direct augmentation MUSIC (DA-MUSIC) and spatial smoothing MUSIC (SS-MUSIC) are two MUSIC variants that utilize the coarray model of the physical array. Therefore, DA-MUSIC and SS-MUSIC can, in many cases, resolve more sources than the number of physical sensors. Both DA-MUSIC and SS-MUSIC start from the covariance matrix in (3.19) and vectorize it (i.e. stack columns one below the other) to yield

$$\mathbf{r} = \text{vec}(\mathbf{R}_s) \quad (3.21)$$

Next, a new covariance matrix can be formed for each of DA-MUSIC and SS-MUSIC. Note that there are other possible ways to manipulate the vectorized covariance in (3.21) for use with DoA estimation algorithms (not necessarily MUSIC). For instance, correlating $\mathbf{r}$ with itself [78].

It is worth mentioning that *spatial smoothing* can be used to decorrelate correlated sources [108], or to exploit the difference coarray to achieve high DOF [33, 36, 98]. Many 2D-DoA estimation algorithms use it in the second regime, since this allows for resolving more sources than the number of sensors. However, spatial smoothing does not fully exploit all possible DOF.

DoA Matrix Method

Yin et al. [109] proposed the DMM for two parallel linear arrays. Yin et al. [110] also explained the relation of their method to ESPRIT, and how ESPRIT can be

considered a special case of the DoA matrix method. Later, Dai et al. [111] proposed an extended DMM (EDMM) which utilizes the sensors of both parallel linear arrays more effectively. This method first estimates the auto- and cross- correlations of both linear arrays, then performs eigenvalue decomposition to estimate the noise by averaging the $N_t - K$ smallest eigenvalues, then it forms the DoA matrix, which is used to get the azimuth and elevation estimates.

3.3 Chapter Summary

This chapter presented the essence of the vast work on 2D-DoA estimation algorithms. In particular, the 2D system model and basic terminology was introduced. In addition, many performance metrics were examined, and some differences in using them among various authors were highlighted. Finally, examples of 2D-DoA estimation algorithms were given to complete the picture. This chapter, however, did not explore sensor arrays which is the topic of the next chapter, where linear and planar arrays are introduced, then a comprehensive review of sparse planar arrays is presented.

CHAPTER 4

SPARSE PLANAR ARRAYS FOR DIRECTION FINDING: A COMPREHENSIVE REVIEW

The first known experiments of using multiple antennas were conducted around the year 1900 in the kHz band. Later, these experiments led to the development of radar, which was mechanically rotated to steer a transmitted beam, and capture reflected echos. World War II was a strong impetus for radar advancement. Nonetheless, engineers continued the development after the war with the idea of steering the beam by changing the phase of signals at antenna elements instead of mechanical rotation. Then, advances in electronics allowed for operating these phased antenna arrays at higher frequencies, and performing more complex tasks [112].

Since antennas can be used for transmitting and receiving EM waves, phased antenna arrays found many applications in passive receivers [20]. One application of

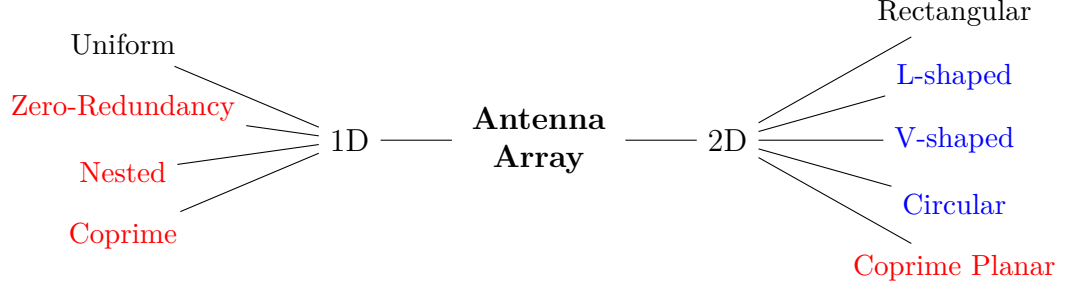


Figure 4.1: Overview of antenna arrays described in chapter 4. Arrays in **red** are sparse. Arrays in **blue** can be uniform or sparse.

interest here is using the concept of steering a beam (beamforming) in reverse to find the direction of arrival of signals impinging on a receiver.

This chapter discusses several important one-dimensional (1D) and two-dimensional (2D) arrays, i.e. linear and planar arrays to serve as an introduction to the review of sparse planar arrays in Section 4.3. The outline of the first two sections is depicted in Fig. 4.1 where the uniform linear array and three sparse linear arrays are described, in addition to the uniform rectangular array and four examples of planar arrays that can be sparse. Then, Section 4.3 focuses only on sparse planar arrays like the coprime planar array (CPA). The comprehensive review in Section 4.3 was motivated by the growing number of publications utilizing sparse planar arrays for direction finding, yet with no clear framework or structure for the published designs. Hence, by reviewing about 34 papers using sparse planar arrays and attempting to group them in a structured way, this review could serve as a starting point for interested researchers who want to pursue further research in this field. The chapter concludes by tables that summarize some important comparison categories like expressions of number of sensors and degrees of freedom (DOF).

4.1 Linear Arrays

When several antenna elements are arranged along a straight line, a LA is formed. It is also referred to as 1D Array. If the spacing between consecutive elements is uniform, the array is called uniform linear array (ULA). Otherwise, the array is a nonuniform linear array (NULA). If a NULA is formed by filling some elements on an equally-spaced grid, the array is called sparse array [9]. All NULAs described next and considered in this work are sparse arrays. This section describes the ULA and several NULAs.

Although the main focus of this work is 2D arrays, LAs are helpful in constructing 2D arrays. In addition, some useful array characteristics are easily explained with LAs, and then extended to 2D arrays.

4.1.1 Uniform Linear Array

ULAs have constant inter-element spacing, d . Mainlobe beamwidth, $\Delta\Phi$, is an important parameter of ULAs that is affected by d , and is given by [77]

$$\Delta\Phi_{3dB} = 0.886 \frac{\lambda}{D} \quad (4.1)$$

where the $3dB$ width is considered, λ is the wavelength of operation, and D is the effective aperture of the array given by

$$D = N_t d \quad (4.2)$$

where N_t is the number of elements in the ULA. Note that N_t is used for the total number of elements in an antenna array to avoid confusion. Fig 4.4a depicts a ULA with $N_t = 6$ elements, where each dot represents an antenna element. The same array is also used in Fig 4.2 with different inter-element spacings.

It is evident from (4.1) and Fig 4.2 that the mainlobe beamwidth can be reduced by increasing N_t or d . While this results in a finer resolution, it requires a larger aperture, i.e. larger spatial extent [9]. It also increases the cost of the array. Among the benefits of using NULAs is that they reduce the total number of antenna elements for a fixed aperture by removing some elements from the middle of the ULA, which reduces the implementation cost. The next subsections present three NULAs.

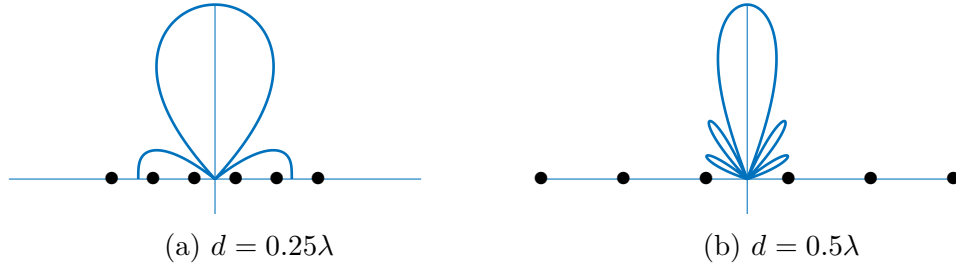


Figure 4.2: Normalized power gain of a ULA with $N_t = 6$ elements and different inter-element spacing d . • represents an antenna element.

4.1.2 Zero-redundancy Array

Fully-redundant array is the ULA [113] described earlier. NULAs, however, have less redundancy and, in special circumstances the redundancy is zero. This subsection explains redundancy and zero-redundancy arrays. Next subsections describe NULAs not having zero-redundancy.

A strong motivation for using a NULA is to reduce the number of sensors (antenna

elements) or to get better performance than a ULA with the same number of sensors. It is evident from (4.1) that increasing N_t for a ULA makes $\Delta\Phi_{3dB}$ narrower. However, some NULAs can achieve the same $\Delta\Phi_{3dB}$ with less N_t [114].

Redundancy refers to the repetition of virtual elements in the coarray (the weight function is a measure of redundancy, and is described in Section 3.2.3). The coarray is a virtual array constructed by mathematical operations on the physical locations of the real array. It is possible to identify whether a NULA is a zero-redundancy array, without finding the co-array, using the following test (assuming inter-element spacings are integer-multiples of a fundamental distance d): Every inter-element spacing appears only once, except the zero spacing (between the element and itself) and, the number of elements does not exceed 4 [29]. This statement means that only four zero-redundancy arrays are possible, which are depicted in Fig 4.3. It is proven that only these four LAs have zero-redundancy [9, 29].

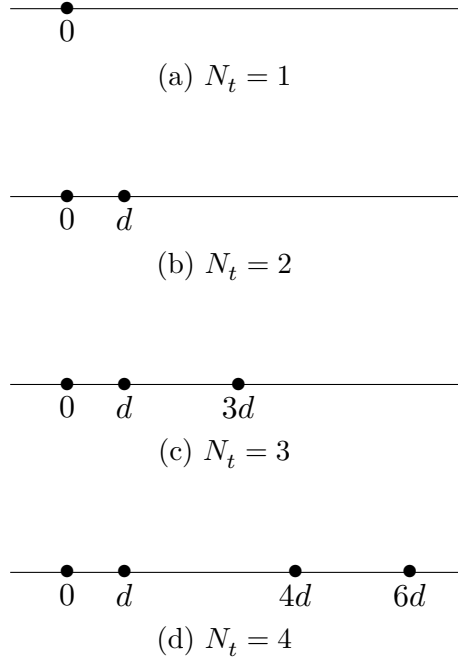


Figure 4.3: The four zero-redundancy linear arrays. Dots denote antenna elements.

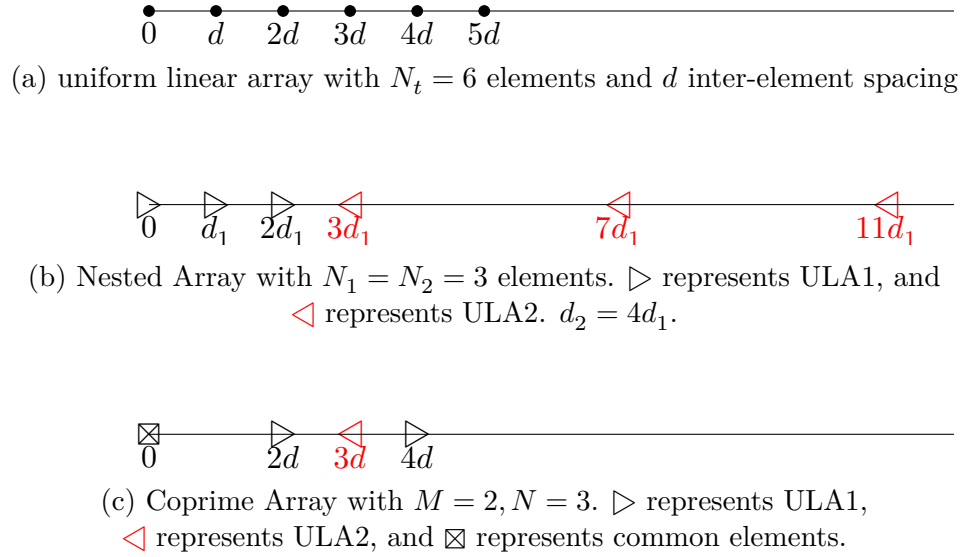


Figure 4.4: Examples of linear antenna arrays.

4.1.3 Nested Linear Array

Nested arrays are formed by stacking two (or more) ULAs, where a ULA fits within the inter-element spacing of the subsequent ULA. No elements are shared between subsequent ULAs. Therefore, the total number of elements N_t is the sum of number of elements of individual ULAs.

For a two-level nested array (two ULAs with N_1 and N_2 elements), the total number of nested array elements is $N_t = N_1 + N_2$, as shown in Fig 4.4b. The inter-element spacing of the first (dense or inner) array is d_1 and the inter-element spacing of the second (outer) array is $d_2 = (N_1 + 1)d_1$. Thus, the set of element positions for the first array is [36]

$$\mathbb{S}_1 = \{md_1, m = 1, 2, \dots, N_1\} \quad (4.3)$$

and for the second array is

$$\mathbb{S}_2 = \{nd_2, n = 1, 2, \dots, N_2\} = \{n(N_1 + 1)d_1, n = 1, 2, \dots, N_2\} \quad (4.4)$$

Hence, the set of elements of the nested array is $\mathbb{S}_1 \cup \mathbb{S}_2$.

Note that it is possible to define these two arrays differently, for example considering that there is a shared element between the two arrays. In addition, several variations of nested arrays exist like the super nested array [37], and the design in [115] which keeps a space between the inner and outer arrays. An example nested array is shown in Fig 4.4b. In this example, the first four elements are spaced by d_1 and the last three elements are spaced by quadruple that distance. The total number

of elements is 6, like the ULA in Fig. 4.4a. However, the nested array takes up more spatial space, i.e. has more effective aperture.

4.1.4 Coprime Linear Array

Coprime arrays are constructed by interleaving two ULAs, where the inter-element spacing of one array is coprime with respect to that of the other array. The first element is shared by the two arrays. If the two ULAs have M and N elements respectively, then the inter-element spacing of the first ULA is Nd and for the other is Md . Often $d = \frac{\lambda}{2}$ but not necessarily. Therefore, the set of elements of the coprime array is [116]

$$\mathbb{S} = \{nMd, n = 0, 1, \dots, N - 1\} \cup \{mNd, m = 0, 1, \dots, M - 1\} \quad (4.5)$$

Fig 4.4c shows an example coprime array, with three elements belonging to ULA1 and two elements in ULA2. One element is common.

4.2 Planar Arrays

Two-dimensional (2D) arrays (known as Planar Arrays) can be formed by combining two linear arrays. The combination can be parallel, or non-parallel. The latter can be perpendicular or at arbitrary angle. In addition, multiple linear arrays can be combined to form a two-dimensional array. Furthermore, it is possible to construct 2D arrays without utilizing linear arrays. An example is the circular array.

After introducing some important linear arrays in the previous section, this section explores several 2D arrays. Namely, rectangular arrays, L-shaped arrays, V-shaped arrays, circular arrays, and coprime planar arrays.

4.2.1 Rectangular Array

Arranging antenna elements in a 2D grid results in a rectangular array as depicted in Fig. 4.5a. Denote the number of ‘columns’ along the x direction by M with inter-element spacing d_x and the number of ‘rows’ along the y direction by N with inter-element spacing d_y , the total number of elements is MN . The set of elements can be written as [117]

$$\mathbb{S} = \{(md_x, nd_y), 0 \leq m \leq M-1, 0 \leq n \leq N-1\}. \quad (4.6)$$

Notable special cases are:

- Parallel linear array: When $N = 2$. In other words, two parallel ULAs. This structure is not commonly used compared to L-shaped array for example (Section 4.2.2).
- Uniform rectangular array: When $d_x = d_y$.
- Uniform square array: When $d_x = d_y$ and $M = N$. Fig. 4.5a shows an example of this case with $M = N = 5$.

4.2.2 L-shaped Array

Two linear arrays can be arranged orthogonally to form the L-shaped array, while sharing a single element [118]. Thus, if the first ULA has M elements, and the other has N elements, the total elements of the L-shaped array is $N_t = M + N - 1$. Each ULA can be used to estimate an angle, and the two estimated angles are paired to find the elevation and azimuth angles [6]. L-shaped array is generally made up of ULAs [6, 93, 119], but recently, sparse arrays are getting more attention [93, 94]. Fig 4.5b shows an L-shaped array built using two ULAs with $N = M = 5$; thus, the total elements of this L-shaped array is 9.

4.2.3 V-shaped Array

A more general way to combine two non-parallel linear arrays is at an angle less than 90° , which results in a V-shaped array. It was reported that such arrays when built using sparse linear arrays achieve better DoA estimation performance compared to the coprime planar array (CPA) (Section 4.2.4) and it should be better than L-shaped arrays (Section 4.2.2) since it does not require coupling of elevation and azimuth estimation [74]. An example V-shaped array is shown in Fig. 4.5c made up of 2 ULAs each with $N = M = 5$ elements, and inter-element spacing d . Similar to the L-shaped array in Fig. 4.5b, the total number of elements is $N_t = N + M - 1$.

4.2.4 Coprime Planar Array

Similar to the treatment of coprime linear array (CLA) in Section 4.1.4, CPA can be formed by interleaving two uniform square arrays; the first is $N \times N$ with inter-element spacing Md , and the other is $M \times M$ with inter-element spacing Nd . The integers M, N are coprime and the set of array elements is [120]

$$\mathbb{S} = \mathbb{S}_1 \cup \mathbb{S}_2 \quad (4.7)$$

where

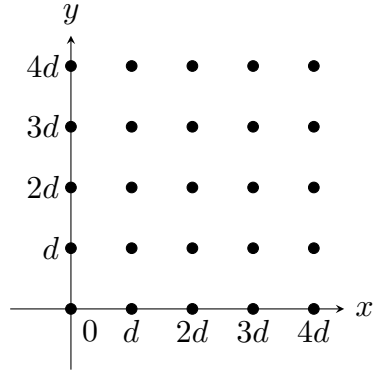
$$\begin{aligned} \mathbb{S}_1 &= \{(n_1 Md, n_2 Md), 0 \leq n_1, n_2 \leq N - 1\}, \\ \mathbb{S}_2 &= \{(m_1 Nd, m_2 Nd), 0 \leq m_1, m_2 \leq M - 1\}. \end{aligned} \quad (4.8)$$

Fig 4.5d shows a CPA. Furthermore, a generalization of the CPA was proposed where the constituent arrays are not square, and it provided higher performance than a square CPA [96].

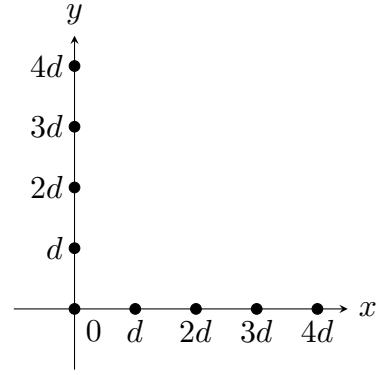
4.2.5 Circular Array

Antenna elements placed on the circumference of a circle produce a circular array. Usually, the angular separation between antenna elements is uniform [121]. Fig. 4.5e shows an example of a uniform circular array with $N_t = 8$ elements and inter-element spacing d .

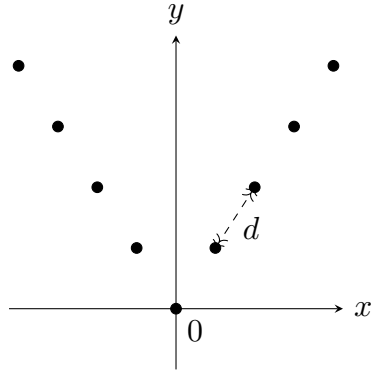
Another circular array has one element at the center of the circle and multiple elements on the circumference. Examples of using this array in the literature include [7] and [122].



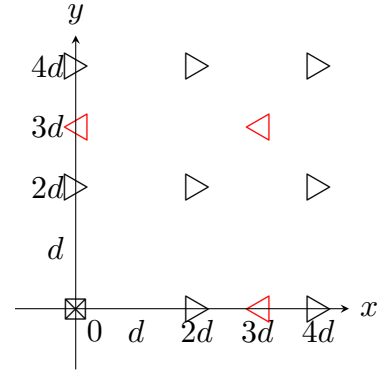
(a) Uniform Rectangular array with $M = N = 5$.



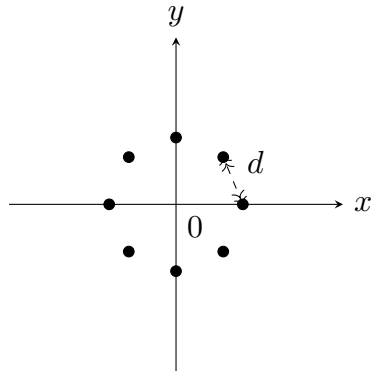
(b) Uniform L-shaped array with $M = N = 5$. The element at the origin is common between the two perpendicular ULAs.



(c) V-shaped Array. The element at the origin is common between the two non-perpendicular ULAs.



(d) Coprime Planar Array with $M = 2, N = 3$. $\boxtimes$ represent the common element.



(e) Uniform Circular Array with $N_t = 8$ elements.

Legend:
 • denotes an antenna element.
 $\triangleright$ represents the first uniform square array.
 $\triangleleft$ represents the second uniform square array.

Figure 4.5: Examples of planar antenna arrays.

4.3 Sparse Planar Arrays

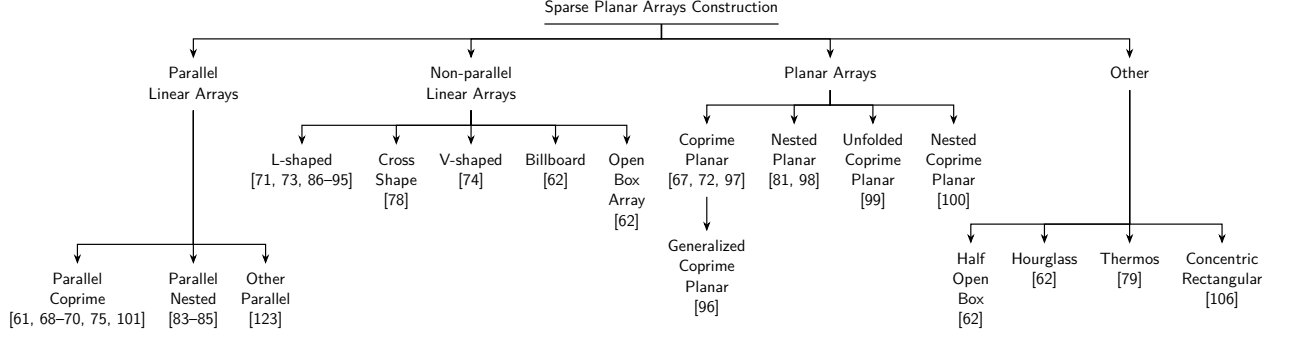


Figure 4.6: Classification of sparse 2D geometries based on the constituent arrays and the associated literature.

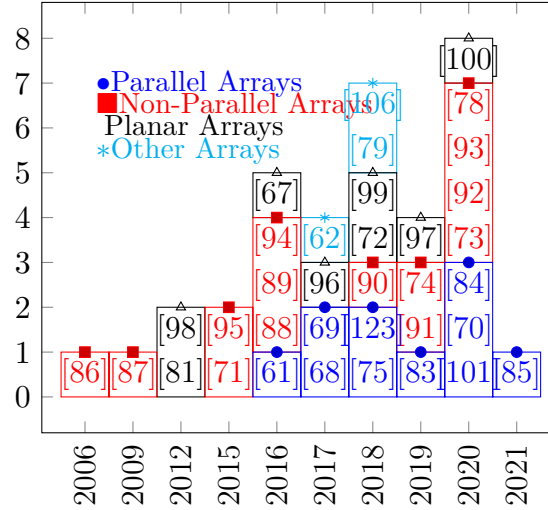


Figure 4.7: Timeline of papers using 2D sparse array geometries [61, 62, 67–75, 78, 79, 81, 83–101, 106, 123].

This section presents a comprehensive overview of the sparse array geometries employed for 2D-DoA estimation. The geometries are classified, as shown in Fig. 4.6, into four broad categories: 1) arrays constructed using parallel linear arrays, 2) arrays constructed using non-parallel linear arrays, 3) arrays constructed using planar arrays, and 4) other geometries. In addition, a timeline of the usage of arrays from each category is shown in Fig. 4.7. Note that some papers may deal with more than one

geometry such as the work of Liu and Vaidyanathan [62]. Another important note is that some of these geometries are named slightly differently by some authors. For instance, the parallel coprime array (PCA) is called coprime parallel array by two papers [68, 75], yet the rest of the authors use the name used in this work [61, 69, 70, 101]. This name also avoids confusing the PCA with the CPA. The knowledge of sparse linear arrays, especially the *nested array* (NA) and the *coprime array* (CA) is important since many 2D sensor arrangements are constructed using a combination of these and possibly the *uniform linear array* (ULA).

This section describes the sensor arrays shown in Fig. 4.6 from left to right. The adopted classification categories based on the building blocks of the 2D array are:

1. Parallel linear arrays, which include PCA, parallel nested array (PNA), and other parallel structures. These are discussed in Section 4.3.1.
2. Non-parallel linear arrays, which include various sparse L-shaped arrays, sparse cross-shaped array, sparse V-shaped arrays, billboard array, and open box array (OBA). These are discussed in Section 4.3.2.
3. Planar arrays, which include CPA, generalized coprime planar array (GCPA), nested planar array (NPA), unfolded coprime planar array (UCPA), nested coprime planar array (NCPA). These are discussed in Section 4.3.3.
4. Other arrays, which include half open box array with two layers (HOBA-2), hourglass array, thermos array, and concentric rectangular array. These are discussed in Section 4.3.4.

Since this work deals with *sparse* array geometries, it is useful to mention that the term *sparse* can have different appearances in the context of DoA estimation:

1. Sparse sensor array: where sensors are placed on *some* intersections of a uniform grid. This is equivalent to having *some* sensors with inter-element spacing of unity and the rest greater than unity (assuming a normalized fundamental inter-sensor spacing d). Put another way: the inter-element spacing is a positive integer (normalized). Another name that is sometimes used for this meaning is *thinned array*. This meaning is used throughout this manuscript.
2. Uniform sparse sensor array: where the spacing of all sensor elements are *identical* and greater than unity. This is a special case of the previous one, and essentially the resultant array is a uniform array. In case of linear arrays, that resultant array is referred to as *sparse ULA*, some examples of this usage are [124, 125].
3. Sparse recovery: a method from the compressive sensing framework used in some DoA estimation algorithms.

The next subsection starts with parallel arrays as building blocks for 2D arrays. Then, non-parallel, planar, and other arrays follow.

4.3.1 Parallel Arrays

This section describes 2D arrays made of two, or more, parallel arrays. In particular, the parallel coprime array (PCA), the three parallel coprime array (TPCA), and the parallel nested array (PNA) are considered. In addition, other reported parallel arrays

that does not fall under any of these categories are also mentioned. Note that 2D arrays made of two parallel ULAs are not considered in this survey.

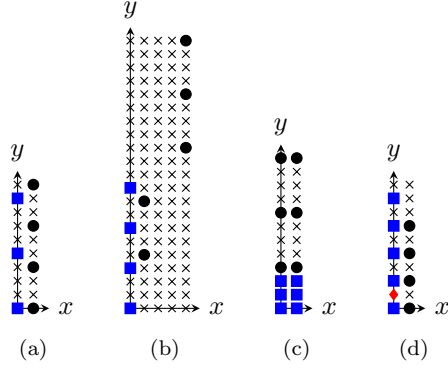


Figure 4.8: Examples of 2D sparse arrays constructing using parallel linear arrays. **(a)** PCA. **(b)** Three parallel coprime array. **(c)** PNA. **(d)** Two parallel sparse ULAs with extra sensor (shown by a red diamond).

Parallel Coprime Array (PCA)

This section presents an overview of parallel coprime array (PCA) geometry [61, 68, 75, 101]. The structure of PCA is shown in Fig. 4.8a, where two uniform linear arrays (ULAs) are in parallel (spaced by d_x), one of them has M_1 elements with $M_2 d_y$ inter-element spacing (shown in blue boxes), the other has M_2 elements with $M_1 d_y$ inter-element spacing (shown in black circles), M_1 and M_2 are coprime integers, and the example of Fig. 4.8a assumes $M_1 = 3$ and $M_2 = 4$. Note that often $M_1 < M_2$ is assumed without loss of generality. The set of array elements is [75]

$$\begin{aligned} \mathbb{S} = & \{(0, m M_2 d_y) | 0 \leq m \leq M_1 - 1\} \\ & \cup \{(d_x, n M_1 d_y) | 0 \leq n \leq M_2 - 1\}. \end{aligned} \quad (4.9)$$

The work in [101] finds the cross covariance between the two arrays as a first

step in estimating 2D-DoA, then decouples the angles α_k and β_k , finds α_k using least spectral search (1D-MUSIC over each subarray), and presents a least squares solution for automatic pairing of β_k . A variant of PCA using two symmetric parallel coprime arrays was also reported [68].

Three Parallel Coprime Array (TPCA)

This section presents an extension to the PCA of the previous section using a third parallel linear array (LA) [69, 70]. A modified version of the PCA is the three parallel coprime structure presented in [70], where the third LA starts after the end of the other two LAs as shown in Fig. 4.8b, where the first ULA has M_2 elements with inter-element spacing $M_1 d_y$ (shown in blue boxes) and the other two ULAs have inter-element spacing $M_2 d_y$ (shown in black circles). The example shown in Fig. 4.8b assumes $M_1 = 3$, $M_2 = 4$, and $L = 3$. In addition, the second ULA is spaced by d_x from the y -axis and has M_1 elements, whereas the third is spaced by Ld_x , $L > d_x$ from the second ULA and has M_1 elements. The set of elements is [70]

$$\begin{aligned} \mathbb{S} = & \{(0, mM_1 d_y) | 0 \leq m \leq M_2 - 1\} \\ & \cup \{(d_x, nM_2 d_y) | 1 \leq n \leq M_1 - 1\} \\ & \cup \{(Ld_x + d_x, pM_2 d_y + M_1 M_2 d_y) | 0 \leq p \leq M_1 - 1\}. \end{aligned} \quad (4.10)$$

Another work presented three parallel coprime array where an extra LA can be added on the negative x -axis of the PCA in (4.9) to form the three parallel coprime array [69]. The middle array has $2M_1$ elements with inter-element spacing of $M_2 d_y$,

and the outer coprime LAs has M_2 elements with inter-element spacing of $M_1 d_y$, where $d_y \leq \lambda/2$, M_1 and M_2 are coprime integers with $M_1 < M_2$. In addition, the LAs are spaced by $d_x \leq \lambda/2$.

Parallel Nested Array (PNA)

Parallel nested arrays (PNAs) were reported in [83–85]. All three papers utilized two identical parallel nested linear arrays (NLAs) [36] (shown in Fig. 4.8c with $N_1 = N_2 = 3$). Assuming each linear array has N elements, the total number of elements is $2N$. The set of antenna elements is

$$\begin{aligned} \mathbb{S} = \{ (0, m d_y) | m \in \mathfrak{g} \} \cup \{ (d_x, m d_y) | m \in \mathfrak{g} \}, \mathfrak{g} = \{0, 1, \\ \dots, N_1 - 1, N_1, 2(N_1 + 1) - 1, \dots, N_2(N_1 + 1) - 1\} \end{aligned} \quad (4.11)$$

Li et al. [83] utilized a DMM (see Section 3.2.5) which allows getting paired estimates for u_k and v_k . He et al. [85] proposed an algorithm using FOC for enhancing DOF, and reported a maximum DOF of $6N_2(N_1 + 1) - 3$.

Other Parallel Arrays

Other than the PCA or the PNA, Zheng et al. [123] proposed two parallel ULAs with extra sensor. The two ULAs are spaced by $\lambda/2$, and each of the two ULAs has inter-sensor spacing of λ , and the extra sensor is placed at $\lambda/2$ from the start of the first ULA, as shown in Fig. 4.8d, where the first ULA (denoted subarray Y) has $N + 1$ elements and is represented by blue squares, the extra sensor by a red square, and the

second ULA (denoted subarray Z) has N elements and is represented by black circles. The first linear array that is made up of subarray Y and the extra sensor is denoted Subarray X. Therefore, the set of array elements can be written as

$$\begin{aligned} \mathbb{S} = & \{(0, 2md_y) | 0 \leq m \leq N\} \cup \{(0, d_y)\} \\ & \cup \{(d_x, 2nd_y) | 0 \leq n \leq N-1\}. \end{aligned} \quad (4.12)$$

where $d_y = d_x = \lambda/2$.

Zheng et al. [123] proposed this array assuming the number of sources is less than the number of elements in subarray Z. They estimate a propagator matrix, which yields estimates of u_k and v_k . An important step in the algorithm is to exclude the ambiguous estimates of u_k which relies on the fact that the unambiguous u_k will be orthogonal to the noise subspace of subarray X.

4.3.2 Non-Parallel Arrays

This section describes 2D arrays made of non-parallel linear arrays, namely: the L-shaped and cross-shaped arrays. In addition, the billboard and open box array (OBA) are also mentioned for a more complete big picture.

L-shaped Array

The L-shaped array is arguably the most straightforward extension from the 1D case to 2D, since it requires two orthogonal ULAs. Many works were published utilizing this structure like Hua et al. [118] and Liang and Liu [6]. However, the focus here

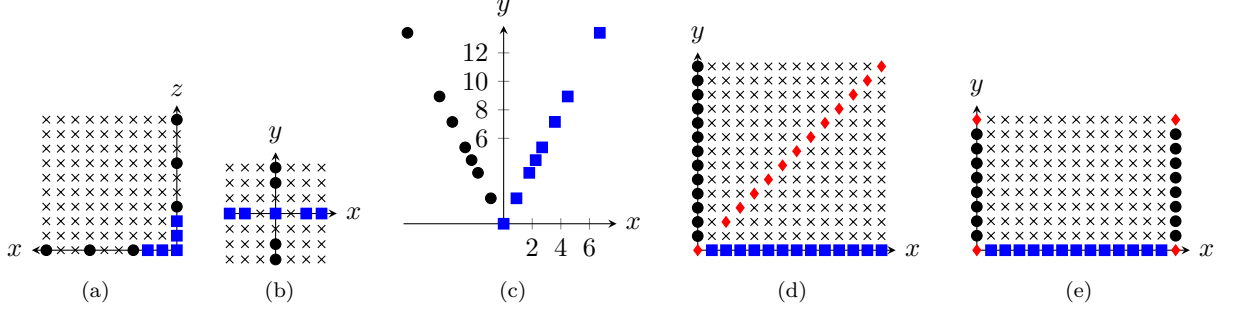


Figure 4.9: Examples of 2D sparse arrays constructed using non-parallel linear arrays. (a) L-shaped sparse array using two ULAs with an extra sensor. (b) Cross-shaped sparse array. (c) V-shaped coprime array. (d) Billboard array. (e) Open box array (OBA).

is on works utilizing orthogonal sparse linear arrays like [71, 73, 86–95]. It can be argued that the L-shaped array is sparse when compared to the URA, but since it is composed of ULAs, they can be treated as non-sparse in this survey. Note that many works on L-shaped and cross-shaped arrays use the $y - z$ plane for the placement of the sensor array, and this is hinted by showing an example in Fig. 4.9a.

One approach of constructing sparse L-shaped arrays [88, 89] (shown in Fig. 4.9a) used two orthogonal two-level nested arrays [36] along x - and z -axes. As shown in Fig. 4.9a, the first (dense) ULA of each nested array has inter-element spacing $d_1 = d_x = d_y$ and the other ULA has the wider $d_2 = (N_1 + 1)d_1$ inter-element spacing. Dong et al. [88] used signal subspace joint diagonalization (SSJD) for automatic pairing of azimuth and elevation angles.

Another approach uses a ULA along the x -axis with inter-element spacing $d_x = \lambda/2$, and a simple sparse linear array along the z -axis made up of a ULA with $d_z = \lambda$ with an extra sensor at a distance $\lambda/2$ from the origin [94]. After computing the cross correlation between the two subarrays (along x and z axes), the ambiguous elevation estimates are obtained. Next, the ambiguity is removed, and then the estimated

elevation angles are utilized to estimate the azimuth [94].

Another work utilized sparse LAs along each axis. Each sparse LA is made up of two interleaved ULAs [93], and the largest contiguous ULA in the difference coarray is used for estimation. The estimation of azimuth and elevation are done separately using MUSIC, and cross covariance is used to pair the azimuth and elevation estimates. Co-prime arrays were also utilized in the legs of the L-shaped array [71, 73]. For instance, the work by Elbir [73] showed the possibility of resolving $M_1 M_2$ sources while having $2M_1 + M_2 - 1$ sensors in each leg. According to Elbir [73], this requirement of $4M_1 + 2M_2 - 3$ sensors is less than CPA and GCPA which are discussed in Section 4.3.3.

Cross-Shaped Array

Wu and Zhu [78] considered a uniform cross array, and a sparse symmetric cross array for the estimation of three parameters: azimuth, elevation, and range of a single source in the near-field or far-field. For the uniform cross array, two orthogonal ULAs are used with $N_x := 2M_x + 1$ and $N_y := 2M_y + 1$ elements on the x - and y -axes, respectively. The set of array elements can be written as

$$\begin{aligned} \mathbb{S} = \{ & (md, 0) \mid -M_x \leq m \leq M_x \} \\ & \cup \{ (0, nd) \mid -M_y \leq n \leq M_y \}, \end{aligned} \quad (4.13)$$

where d is the fundamental spacing. To construct the sparse symmetric cross array, each ULA is replaced by a symmetric sparse linear array. In this case, the set of

elements becomes

$$\begin{aligned} \mathbb{S} = \{ & (md, 0) | m \in \mathbb{G}_x \} \\ & \cup \{ (0, nd) | n \in \mathbb{G}_y \}, \end{aligned} \quad (4.14)$$

where $\mathbb{G}_x$ is a symmetric set (about the origin) and is a subset of $\{-M_x, -M_x + 1, \dots, -1, 0, 1, \dots, M_x - 1, M_x\}$ and $\mathbb{G}_y$ is a subset of $\{-M_y, -M_y + 1, \dots, -1, 0, 1, \dots, M_y - 1, M_y\}$. An example of this structure when $M_x = M_y = 3$ and $\mathbb{G}_x = \mathbb{G}_y = \{-3, -2, 0, 2, 3\}$ is shown in Fig. 4.9b where blue boxes resemble $\mathbb{G}_x$ and black circles resemble $\mathbb{G}_y$. Note that the two linear arrays share a common element at the origin, the blue boxes denote the first linear array, the black circles denote the second linear array, and the crosses denote empty locations.

The algorithm proposed by Wu and Zhu [78] has two main steps. The first step is finding the DoA by computing the cross correlation matrix, vectorizing it, and correlating the resultant vectors. The authors choose to apply atomic norm minimization to get the angle estimates, and justify why this is better than some other methods like 2D-MUSIC, and the fact that they avoid the grid mismatch issue. They further mentioned why their method can work for sparse linear arrays without spatial smoothing which causes aperture loss as was done in oblique projection (OP)-MUSIC [126]. While this suffices for DoA estimation in the far-field, the authors also describe the second step for the near-field case, which is estimating the range.

V-shaped Array

A generalization to the L-shaped array is the V-shaped array, where the ninety-degrees angle between the two LAs of the L-shaped array becomes $\Omega < 90$ deg. Elbir [74] proposed V-shaped coprime array (VCA) and V-shaped nested array (VNA). An example of VCA is shown in Fig. 4.9c with $M_1 = 2$, $M_2 = 5$, and $\Omega = 53.28$ deg. The value of Ω is selected such that the estimation of azimuth and elevation is uncoupled [74]. The author reports that the V-shaped array can resolve the same number of sources with less number of sensors when compared to CPA or GCPA (discussed in Section 4.3.3).

Billboard Array

The billboard array is constructed using an L-shaped array with an extra linear array at 45 degrees from both legs of the L-shaped array. An example of such array is shown in Fig. 4.9d. This array enjoys a hole-free difference coarray [62].

Open Box Array

The open box array (OBA) is also formed using three LAs, orthogonal to each other, which are the three edges of rectangle. The OBA can also be considered as two L-shaped arrays sharing a LA. An example OBA when $N_x = 15$ and $N_y = 10$ is shown in Fig. 4.9e where blue boxes represent $\mathbb{G}$, black circles represent $\mathbb{H}_1$ and $\mathbb{H}_2$, and red diamonds represent $\mathbb{F}$. For two integers N_x and N_y , the set of sensor locations for the OBA can be written as [62]

$$\mathbb{S} = \mathbb{G}_1 \cup \mathbb{H}_1 \cup \mathbb{H}_2 \cup \mathbb{F} \quad (4.15)$$

where

$$\mathbb{G}_1 = \{(l_x, 0) | l_x \in \mathfrak{g}_1\}, \mathfrak{g}_1 = \{1, 2, \dots, N_x - 2\} \quad (4.16)$$

$$\mathbb{H}_1 = \{(0, l_y) | l_y \in \mathfrak{h}_1\}, \mathfrak{h}_1 = \{1, 2, \dots, N_y - 2\} \quad (4.17)$$

$$\mathbb{H}_2 = \{(N_x - 1, l_y) | l_y \in \mathfrak{h}_2\}, \mathfrak{h}_2 = \{1, 2, \dots, N_y - 2\} \quad (4.18)$$

$$\mathbb{F} = \{(0, 0), (N_x - 1, 0), (0, N_y - 1), (N_x - 1, N_y - 1)\} \quad (4.19)$$

The difference coarray is given by

$$\begin{aligned} \mathbb{D} = & \{(m_x, m_y) \in \mathbb{Z}^2 | \\ & -N_x + 1 \leq m_x \leq N_x - 1, -N_y + 1 \leq m_y \leq N_y - 1\} \end{aligned} \quad (4.20)$$

which is a uniform rectangular array. Generalizations of OBA are discussed in Section 4.3.4. In particular, the generalizations will ensure the same continuous difference coarray in (4.20) while increasing inter-sensor spacing, thus reducing mutual coupling effects.

The two arrays (billboard, and OBA) are constructed using ULAs, and the performance of both is generally exceeded by three newly-developed arrays which are: HOBA-2 [62], the hourglass array [62], and the thermos array [79], discussed in Sections 4.3.4, 4.3.4, and 4.3.4, respectively. Therefore, billboard and OBAs are not discussed further.

4.3.3 Planar Arrays

This section describes 2D arrays made by interleaving two planar arrays like the coprime planar array (CPA), and the unfolded coprime planar array (UCPA). In addition, other planar structures like the nested planar array (NPA) or the nested coprime planar array (NCPA) are examined.

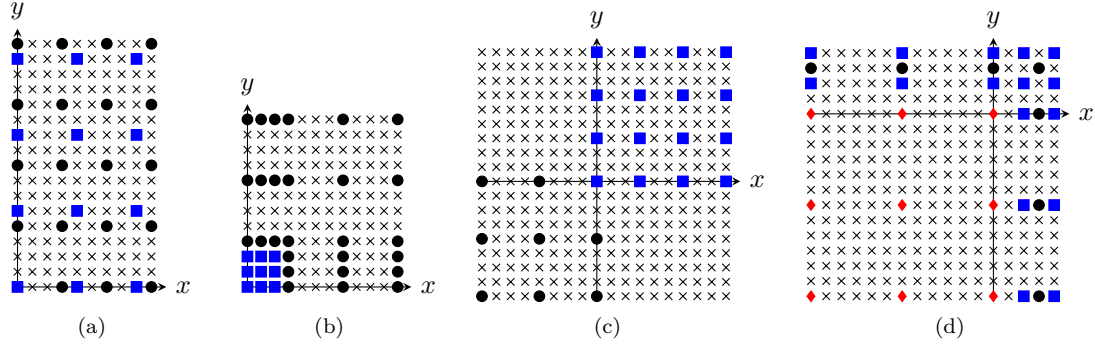


Figure 4.10: Examples of sparse arrays constructed using planar subarrays. Blue squares represent subarray 1. Black circles represent subarray 2. Red diamonds represent common elements. (a) Generalized coprime planar array (GCPA); CPA is a special case. (b) Nested planar array (NPA). (c) Unfolded coprime planar array (UCPA). (d) Nested coprime planar array (NCPA).

Coprime Planar Array (CPA)

Coprime planar array (CPA) results from interleaving two uniform planar arrays, which are denoted subarrays. The CPA [67, 72, 97] is described first. Then, the general way of interleaving the subarrays, the generalized coprime planar array (GCPA) [96] is described. The latter is just a generalization of the former where the number of elements along the x - and y -axes need not be equal. Next, some details of DoA estimation algorithms using these arrays are outlined.

A CPA is constructed using two interleaved uniform square arrays. Denoting each uniform square array as a subarray, there are $M_1 \times M_1$ and $M_2 \times M_2$ elements in

subarrays 1 and 2 respectively. Since the two subarrays share a common element at the origin, the total number of elements is $M_1^2 + M_2^2 - 1$. The set of antenna elements is [67, 72, 97]

$$\begin{aligned} \mathbb{S} = \{ & (md_1, nd_1) | 0 \leq m, n \leq M_1 - 1 \} \\ & \cup \{ (pd_2, qd_2) | 0 \leq p, q \leq M_2 - 1 \} \end{aligned} \quad (4.21)$$

where $d_1 = M_2d$, $d_2 = M_1d$, and $d = \lambda/2$.

The GCPA structure, proposed by Zheng et al. [96], has more DOF and yields better DoA estimates for the same number of sensors as CPA. GCPA relaxes the condition of using uniform square subarrays, and allows uniform rectangular subarrays. An example of GCPA is shown in Fig. 4.10a. GCPA is made up using two interleaved URAs having the dimensions of $M_1 \times N_1$ and $M_2 \times N_2$ whereas CPA is constructed from two interleaved uniform square arrays. Here, M_1 (M_2) is the number of elements of the first (second) URA along the x -axis, N_1 (N_2) is the number of elements of the first (second) URA along the y -axis, and $N_1 = M_1$, $N_2 = M_2$ in the case of restricting URAs to become uniform square arrays. In the case of GCPA, the set of array elements become

$$\begin{aligned} \mathbb{S} = \{ & (mM_2d_x, nN_2d_y) | 0 \leq m \leq M_1 - 1, 0 \leq n \leq N_1 - 1 \} \\ & \cup \{ (pM_1d_x, qN_1d_y) | 0 \leq p \leq M_2 - 1, 0 \leq q \leq N_2 - 1 \} \end{aligned} \quad (4.22)$$

where d_x (d_y) is the fundamental distance along the x - (y -) axis, and is often chosen

to be $\lambda/2$.

Algorithms for arbitrary arrays do not utilize the uniform nature of subarrays, and they tend to be more complex than specially tailored algorithms. Therefore, a special efficient method for CPAs was developed that utilizes uniformity of subarrays [67]. The proposed method essentially limits the 2D-MUSIC search requirement to a small subset; hence, the method is named the partial spectral search (PSS). The first two steps are to estimate the covariance matrix of each subarray, and detail how to choose an arbitrary small sector to find some ambiguous peaks, which they prove are related to ambiguous peaks in other sectors. To pick the true peaks, defining the difference between the two subarrays allows for selecting the K minimum values which yield the estimate $(\hat{u}_k, \hat{v}_k)$ and consequently $(\hat{\phi}_k, \hat{\theta}_k)$.

Another work utilized the signal subspace in addition to the noise subspace to avoid any spectral search [72]; rather, a double polynomial rooting algorithm is utilized. Furthermore, a study on the separation of coherent and uncorrelated signals was reported [97], where unitary ESPRIT is used for the planar geometry and root-MUSIC for a linear coprime array.

Nested Planar Array (NPA)

The nested planar array (NPA) was proposed by Pal and Vaidyanathan [81, 98] as an extension to the 1D nested array, and it can provide $\mathcal{O}(MN)$ virtual sensors continuously in the difference coarray while only having $\mathcal{O}(M + N)$ sensors. Here M denotes the number of sensors in a dense grid, and N is the physical sensors in a sparse grid. For instance, using two linear nested arrays with $N_1 = N_2 = 3$, the nested planar

array looks like Fig. 4.10b, where N_1 is the number of elements in the dense LA, and N_2 is the number of elements in the sparse LA. The authors also show that with effective selection of M and N , $\mathcal{O}(N_t^2)$ sources can be identified using the difference coarray. To achieve that, they explain how to build a covariance-like matrix of dimensions $\mathcal{O}(N_t^2) \times \mathcal{O}(N_t^2)$ which correspond to virtual sensors from only the estimated covariance of size $N_t \times N_t$.

Unfolded Coprime Planar Array (UCPA)

Instead of interleaving two uniform square arrays (subarrays) to construct the CPA in Section 4.3.3, one of the uniform square arrays is *unfolded*, that is, flipped across the x -axis or the y -axis. Therefore, the two subarrays lie in different quadrants. Fixing one subarray at the first quadrant, Zheng et al. [99] investigated the performance of 2D-DoA estimation when the other subarray is in the three other quadrants. They also proposed an ambiguity-free MUSIC algorithm, and further utilized a successive method of implementing it to relief the computational burden. They concluded that when one subarray is flipped across the line $y = -x$ (so it lies in the third quadrant), the performance is better than other flipped structures and the traditional CPA. Fig. 4.10c shows this case where the first (second) subarray has $M_1 \times M_1$ ($M_2 \times M_2$) sensor elements, and the set of array elements is

$$\begin{aligned} \mathbb{S} = & \{(md_1, nd_1) | 0 \leq m, n \leq M_1 - 1\} \\ & \cup \{(pd_2, qd_2) | -(M_2 - 1) \leq p, q \leq 0\} \end{aligned} \quad (4.23)$$

where $d_1 = M_2 d$, $d_2 = M_1 d$, $d = \lambda/2$, and m, n, p, q are integers.

Nested Coprime Planar Array (NCPA)

The structure of nested coprime planar array (NCPA) is quite involved to describe, despite being simple in design. Among the motivations reported by Si et al. [100] is that their NCPA can outperform all other CPAs, namely GCPA [96], CPA [67, 72, 97], and UCPA [99] in terms of being able to detect more sources than the number of sensor elements. The NCPA also produces two virtual coprime planar arrays that are used to obtain two sets of DoA estimation that are linked together to get the unique directions of arrival. Note that Si et al. [100] did not use the *NCPA* term.

The NCPA is constructed from two sparse planar arrays, unlike the other types: GCPA, CPA, and UCPA which are constructed from uniform planar arrays. Each sparse planar array is made up of a nested planar array (NPA). Each NPA is composed of a dense subarray and a sparse subarray. The example of this structure is shown in Fig. 4.10d. In this example, $M_1 = 2$, and $M_2 = L = 3$. The NPA is constructed from the elements of two orthogonal nested linear arrays (NLAs) and the elements at the intersections of lines, in the first quadrant, emerging from each sensor element and orthogonal to the nested linear array. Mathematically, the set of array elements is

$$\mathbb{S} = \{(m, n) | m, n \in \mathbb{S}_{\text{NPA1}}\} \cup \{(m, n) | m, n \in \mathbb{S}_{\text{NPA2}}\}. \quad (4.24)$$

This equation just describe the fact that the subarrays of the NCPA are made up of NPAs. Now the NPAs are made up of two orthogonal NLAs and other sensors, all

of them are described by

$$\begin{aligned} \mathbb{S}_{\text{NPA1}} &= \{-L+1, -L+2, \dots, 0\} \times M_1 M_2 d \\ &\cup \{0, 1, \dots, M_1-1\} \times M_2 d, \end{aligned} \quad (4.25)$$

$$\begin{aligned} \mathbb{S}_{\text{NPA2}} &= \{-L+1, -L+2, \dots, 0\} \times M_1 M_2 d \\ &\cup \{0, 1, \dots, M_2-1\} \times M_1 d, \end{aligned} \quad (4.26)$$

where M_1, M_2 are coprime positive integers, and L is an integer.

4.3.4 Other 2D Arrays

This section describes four more 2D sparse arrays that does not readily fall under any of the previous three categories. In particular, the half open box array with two layers (HOBA-2), the hourglass array, the thermos array, and the concentric rectangular array are considered.

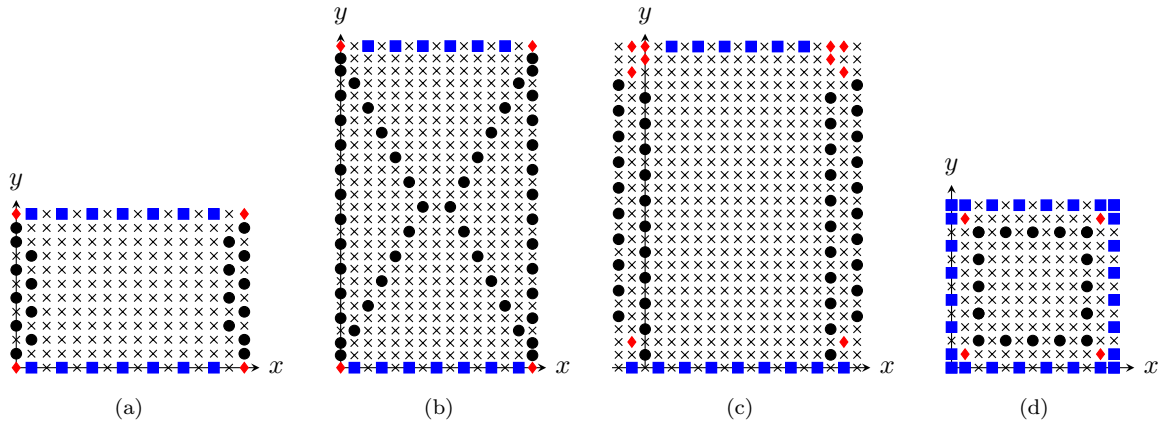


Figure 4.11: Additional examples of 2D sparse arrays. (a) Half open box array with two layers (HOBA-2). (b) Hourglass array. (c) Thermos array. (d) Concentric rectangular array.

Half Open Box Array-2 (HOBA-2)

The half open box array with two layers (HOBA-2) is an extension of the half open box array (HOBA) which is a special case of partial open box array (POBA). The partial open box array (POBA) is a systematic redistribution of elements in the OBA (Section 4.3.2) to reduce closely-spaced elements which reduces mutual coupling. An example of this array with $N_x = 16$ and $N_y = 12$ is shown in Fig. 4.11a, where black circles stand for $\mathbb{H}_1, \mathbb{H}_2$, blue squares stand for $\mathbb{G}_1 \cup \mathbb{G}_2$, and red diamonds stand for $\mathbb{F}$. If some elements of $\mathbb{G}_1$ of the OBA (4.15) are redistributed on the opposite (empty) side of the OBA, the elements of POBA can be written as [62]

$$\mathbb{S} = \mathbb{G}_1 \cup \mathbb{G}_2 \cup \mathbb{H}_1 \cup \mathbb{H}_2 \cup \mathbb{F} \quad (4.27)$$

where

$$\mathbb{G}_1 = \{(\ell_x, 0) | \ell_x \in \mathfrak{g}_1\}, \mathfrak{g}_1 = \{1, 2, \dots, N_x - 2\} \quad (4.28)$$

$$\mathbb{G}_2 = \{(\ell_x, N_y - 1) | \ell_x \in \mathfrak{g}_2\}, \mathfrak{g}_2 = \{1, 2, \dots, N_x - 2\} \quad (4.29)$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subsets of $\{1, 2, \dots, N_x - 2\}$ with $|\mathfrak{g}_1| + |\mathfrak{g}_2| = N_x - 2$. Note that POBA is still defined using N_x and N_y as the OBA, and $\mathbb{H}_1, \mathbb{H}_2$, and $\mathbb{F}$ are as defined in (4.17), (4.18), and (4.19), respectively.

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are chosen to be of the following form, the array is called half open

box array (HOBA)

$$\mathfrak{g}_1 = \{1 + 2\ell \mid 0 \leq \ell \leq \lfloor (N_x - 3)/2 \rfloor\}, \quad (4.30)$$

$$\mathfrak{g}_2 = \{N_x - 1 - 2\ell \mid 1 \leq \ell \leq \lfloor (N_x - 2)/2 \rfloor\}. \quad (4.31)$$

Similar argument can be applied to the other ULAs (on the right and left) of the OBA, which results in the partial open box array with L levels (POBA-L)

$$\mathbb{S} = \mathbb{G}_1 \cup \mathbb{G}_2 \cup \left(\bigcup_{\ell=1}^L \mathbb{H}_{1,\ell} \cup \mathbb{H}_{2,\ell} \right) \cup \mathbb{F} \quad (4.32)$$

where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a partition of $\{1, 2, \dots, N_x - 2\}$, $\{\mathfrak{h}_{1,\ell}\}_{\ell=1}^L$ is a partition of $\{1, 2, \dots, N_y - 2\}$, and $\mathfrak{h}_{2,\ell} = N_y - 1 - \mathfrak{h}_{1,\ell}$ for $\ell = 1, \dots, L$. Note that a third positive integer $L \leq N_x/2$ is required to design partial open box array with L levels (POBA-L). If L is chosen to be 2, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as in (4.30), (4.31), and

$$\mathfrak{h}_{1,1} = \{1 + 2\ell \mid 0 \leq \ell \leq \lfloor (N_y - 3)/2 \rfloor\} \cup \{N_y - 2\}, \quad (4.33)$$

$$\mathfrak{h}_{1,2} = \{2\ell \mid 1 \leq \ell \leq \lfloor (N_y - 3)/2 \rfloor\}. \quad (4.34)$$

the array is called half open box array with two layers (HOBA-2). The next subsection describes the hourglass array, which is a POBA-L array.

Hourglass Array

Hourglass array is a special case of POBA, which was presented by Liu and Vaidyanathan [62]. The virtues of hourglass arrays include 1) having the same coar-

ray as the OBA (using the same aperture), 2) having a hole-free difference co-array (based on the previous virtue), and 3) reduced mutual coupling compared with the OBA. Following the notation in Section 4.3.4, and for positive integers N_x and N_y , and L layers, the element locations of the array can be described by

$$\mathbf{g}_1 = \{1 + 2p \mid 0 \leq p \leq \lfloor (N_x - 3)/2 \rfloor\}, \quad (4.35)$$

$$\mathbf{g}_2 = \{N_x - 1 - 2p \mid 1 \leq p \leq \lfloor (N_x - 2)/2 \rfloor\}. \quad (4.36)$$

and $\mathbf{h}_{1,\ell}$ as in (4.37), where

$$\mathbf{h}_{1,\ell} = \begin{cases} \{2p, N_y - 1 - 2p \mid 1 \leq p \leq \lfloor (N_y - 1)/4 \rfloor\} \cup \{1, N_y - 2\}, & \text{if } \ell = 1, \\ \{2\ell - 1, N_y - 2\ell\}, & \text{if } N_y \text{ is odd and } 0 \leq \ell \leq L, \\ \{2\ell - 1, 2\lfloor N_y/4 \rfloor - 2\ell + 3, 2\lceil N_y/4 \rceil + 2\ell - 4, N_y - 2\ell\}, & \text{if } N_y \text{ is even and } 2 \leq \ell \leq L, \end{cases} \quad (4.37)$$

$$L = \begin{cases} \lfloor (N_y + 1)/4 \rfloor, & \text{if } N_y \text{ is odd,} \\ \lfloor N_y/8 + 1 \rfloor, & \text{if } N_y \text{ is even.} \end{cases} \quad (4.38)$$

An example for hourglass array is shown in Fig. 4.11b with $N_x = 15$ and $N_y = 27$.

To demonstrate the robustness of HOBA-2 and hourglass arrays in presence of mutual coupling, the authors compared the arrays with billboard, OBA, and URA using 2D unitary ESPRIT without modifications to account for mutual coupling. It was shown that hourglass performs best compared to other arrays, especially with low snapshots [62].

Thermos Array

Sun et al. [79] proposed the thermos array as an improvement to the hourglass array in terms of reducing mutual coupling. Since the vertical inter-element spacings of d were about five times larger than the horizontal ones, they proposed the thermos array which reduces these vertical spacings. Two parameters are enough to design a thermos array: $N_x \in \mathbb{Z}^+$ and $N_y \in \mathbb{Z}^+$ which yield the total number of sensors as $N_x + 2N_y - 2$. An example of a thermos array is shown in Fig. 4.11c with $N_x = 15$ and $N_y = 27$. In Fig. 4.11c, black circles represent $\mathbb{S}_1, \mathbb{S}_2, \mathbb{R}_1, \mathbb{R}_2$, blue boxes represent $\mathbb{B}, \mathbb{T}$, and red diamonds represent $\mathbb{F}$. The thermos array has a rectangular shape and is designed using six ULAs with inter-sensor spacing of $2d$ plus four or ten more elements at the corners depending on the number of elements in the y direction, N_y . If N_y is even, ten elements are needed, and four if N_y is odd. The set of array elements $\mathbb{S}$ can be written as a union of a top, bottom, two left, and two right ULAs plus the extra sensors at the corner denoted by $\mathbb{F}$. Thus,

$$\mathbb{S} = \mathbb{B} \cup \mathbb{T} \cup \mathbb{S}_1 \cup \mathbb{S}_2 \cup \mathbb{R}_1 \cup \mathbb{R}_2 \cup \mathbb{F} \quad (4.39)$$

where

$$\mathbb{B} = \{(-1 + 2\ell, 0) | 0 \leq \ell \leq (N_x + N_x \bmod 2)/2\} \quad (4.40)$$

$$\begin{aligned} \mathbb{T} = & \{(2\ell + 1 + N_x \bmod 2, N_y - 2) | \\ & 0 \leq \ell \leq (N_x - N_x \bmod 2)/2 - 2\} \end{aligned} \quad (4.41)$$

$$\mathbb{S}_1 = \{(-2, 2\ell) | 1 \leq \ell \leq (N_y - N_y \bmod 2)/2 - 2\} \quad (4.42)$$

$$\begin{aligned} \mathbb{S}_2 = & \{(0, 1 + 2\ell) | \\ & 1 \leq \ell \leq (N_y - N_y \bmod 2)/2 - 1 - N_y \bmod 2\} \end{aligned} \quad (4.43)$$

$$\begin{aligned} \mathbb{R}_1 = & \{(N_x - 1, -1 + 2\ell) | \\ & 1 \leq \ell \leq (N_y - N_y \bmod 2)/2 - 1 - N_y \bmod 2\} \end{aligned} \quad (4.44)$$

$$\mathbb{R}_2 = \{(N_x + 1, 2\ell) | 1 \leq \ell \leq (N_y - N_y \bmod 2)/2 - 2\} \quad (4.45)$$

and

$$\mathbb{F} = \{(-1, N_y - 2), (0, N_y - 2), (N_x - 1, N_y - 2), (N_x, N_y - 2)\} \quad (4.46)$$

when N_y is even, or

$$\begin{aligned} \mathbb{F} = & \{(-1, N_y - 2), (0, N_y - 2), (N_x - 1, N_y - 2), \\ & (N_x, N_y - 2), (-1, 2), (N_x, 2), (-1, N_y - 4), \\ & (0, N_y - 3), (N_x - 1, N_y - 3), (N_x, N_y - 4)\} \end{aligned} \quad (4.47)$$

when N_y is odd. Note that $N_x \bmod 2 = 0$ if N_x is even, and $N_x \bmod 2 = 1$ if N_x is odd.

Concentric Rectangular Array (CcRA)

Rajamäki and Koivunen [106] proposed the concentric rectangular array (CcRA) which can be thought of as a modification of the boundary array. An example of CcRA is shown in Fig. 4.11d with $N_x = N_y = 12$, blue squares for $\mathbb{G}_o$, and black circles for $\mathbb{G}_i$. The total number of sensors is $2(N_x + N_y)$. Although the paper does not directly implement this array for DoA estimation, it is cited as it proposes a sparse 2D array. In general, for even $N_x, N_y \geq 2$, the CcRA is given by

$$\mathbb{S} = \mathbb{G}_o \cup \mathbb{G}_m \cup \mathbb{G}_i$$

where

$$\begin{aligned} \mathbb{G}_o = & \{(p_x, p_y) | p_x \in \mathbb{P}_0(N_x), p_y \in \{0, N_y\}\} \cup \\ & \{(p_x, p_y) | p_x \in \{0, N_x\}, p_y \in \mathbb{P}_0(N_y)\} \end{aligned} \quad (4.48)$$

$$\begin{aligned} \mathbb{G}_m = & \{(p_x, p_y) | p_x \in \mathbb{P}_1(N_x), p_y \in \{1, N_y - 1\}\} \cup \\ & \{(p_x, p_y) | p_x \in \{1, N_x - 1\}, p_y \in \mathbb{P}_1(N_y)\} \end{aligned} \quad (4.49)$$

$$\begin{aligned} \mathbb{G}_i = & \{(p_x, p_y) | p_x \in \mathbb{P}_2(N_x), p_y \in \{2, N_y - 2\}\} \cup \\ & \{(p_x, p_y) | p_x \in \{2, N_x - 2\}, p_y \in \mathbb{P}_2(N_y)\} \end{aligned} \quad (4.50)$$

and

$$\mathbb{P}_0(N) = \{0, N\} \cup \{1 : 2 : N - 1\}$$

$$\mathbb{P}_1(N) = \{0, 1, N - 1, N\}$$

$$\mathbb{P}_2(N) = \{2 : 2 : N - 2\}$$

where $\{1 : 2 : N - 1\} = \{1, 1 + 2, 1 + 4, \dots, N - 1\}$ is an interval of integers with a step size of 2. The hourglass array is sparser than the CcRA, and both have a contiguous difference coarray. However, the CcRA has a contiguous sum coarray, unlike the hourglass array [106]. In addition, all the elements in the CcRA are essential [38].

4.4 Comparative Evaluation of Sparse Planar Arrays

This section presents a general comparison of the mentioned arrays and highlights some important differences. This should be useful in predicting the estimation performance and required complexity of a sensor geometry; thus, aiding the decision of selecting a sensor geometry for DoA estimation. It can also serve as a starting point for researchers interested in exploring or designing new sparse planar arrays in more detail. The comparative evaluation presented in this section is organized in a table split over two pages due to limited space. In other words, Table 4.1 and Table 4.2 share the same rows, the same ‘Array’ and ‘Ref.’ columns, but have the rest of columns different.

Table 4.1 lists the number of elements in each array geometry and the achievable DOF. For instance, the three papers [83–85] that used parallel nested arrays are easily

Table 4.1: Comparison between physical sensors and achievable degrees of freedom for sparse array geometries

Array	Ref.	Number of Sensors, N_t	Max DOF
Parallel	Coprime	[61]	$2M_1 + M_2$
		[68]	$4M_1 + 2M_2 - 2$
		[75]	$M_1 + M_2$
		[101]	$M_1 + M_2$
		[69]	$2M_1 + 2M_2$
		[70]	$2M_1 + M_2 - 1$
	Nested	[83]	$N_2(N_1 + 1) - 1$
		[84]	$2N_2(N_1 + 1) - 2$
		[85]	$6N_2(N_1 + 1) - 4$
	Other	[123]	$M + M + 2$
non-Parallel	L MRA	[86]	$M + N - 1$
	L 2	[87]	$4N + 1$
	L coprime	[71]	$2M_1 + M_2 - 1$
		[73]	$4M_1 + 2M_2 - 3$
	L Nested	[88]	$2N - 1, N := N_1 + N_2$
		[89]	$\begin{cases} 3N^2/2 - 2, & N \text{ even} \\ 3N^2/2 - N - 1/2, & N \text{ odd} \end{cases}$
		[90]	$N^2/4 + N/2 - 1$
		[91]	$(2N_{x1}N_{x2} + 2N_{x2} - 1) \times (2N_{y2}N_{y1} + 2N_{y2} + 2N_{y1} + 1)$
		[92]	$4N_1^2 - 2$
	IA	[93]	$2M + 2N - 1$
	L sULA+	[94]	$(M + 1)(2M + 2N - 1 - M - \lfloor M/2 \rfloor)$
	L sULA+	[94]	$M + 1 + M - 1$
	L Other	[95]	$M_x + M_z + 1$
	VCA	[74]	$2(2M_1 + M_2 - 1) - 1$
	VNA	[74]	$2N$
	Cross	[78]	$2M_x + 2M_y + 1$
	Billboard	[62]	$3(N_x - 1)$
	OBA	[62]	$N_x + 2N_y - 2$
Planar	CPA	[67]	$M_1^2 + M_2^2$
	GCPA	[96]	$M_1N_1 + M_2N_2$
	CPA	[72]	$M_1^2 + M_2^2 - 1$
	CPA	[97]	$M_1^2 + M_2^2 - 1$
	Nested	[81, 98]	$(N_1 + N_2)^2$
	Unfolded	[99]	$M_1^2 + M_2^2 - 1$
	NCPA	[100]	$(L + M_1 - 1)^2 + (L + M_2 - 1)^2 - L^2$
Other	HOBA-2	[62]	$N_x + 2N_y - 2$
	Hourglass	[62]	$N_x + 2N_y - 2$
	Thermos	[79]	$N_x + 2N_y - 2$
	Concentric	[106]	$2(N_x + N_y)$
-: a dash mark signifies that the value is not mentioned directly in the paper, or not easily verifiable.			

Table 4.2: Comparison between array geometries in terms of computational complexity, mutual coupling, and aperture

Array	Ref.	PS ¹	FOC	MC-1 ²	Aperture, x	Aperture, y	
Parallel	Coprime	[61]	✗	✗	Low	1	$M_2(2M_1 - 1)$
		[68]	✗	✗	Low	1	$2M_2(2M_1 - 1)$
		[75]	✓	✗	Low	1	$M_1(M_2 - 1)$
		[101]	1D	✗	Low	1	$M_1(M_2 - 1)$
		[69]	✓	✗	Low	1	$M_2(2M_1 - 1)$
		[70]	1D	✗	Low	$1 + L$	$M_2(2M_1 - 1)$
	Nested	[83]	✗	✗	High	1	$N_2(N_1 + 1) - 1$
		[84]	✗	✗	High	1	$N_2(N_1 + 1) - 1$
		[85]	✗	✓	High	1	$N_2(N_1 + 1) - 1$
	Other	[123]	✗	✗	High	1	$M + 1$
non-Parallel	L MRA	[86]	✗	✓	Low	-	-
	L 2	[87]	✗	✗	High	$N - 1$	$N - 1$
	L coprime	[71]	✗	✗	Low	$M_2(2M_1 - 1)$	$M_1(M_2 - 1)$
		[73]	1D	✗	Low	$M_2(2M_1 - 1)$	$M_2(2M_1 - 1)$
	L Nested	[88]	✗	✗	High	$N_2(N_1 + 1) - 1$	$N_2(N_1 + 1) - 1$
		[89]	✗	✗	High	$N_2(N_1 + 1) - 1$	$N_2(N_1 + 1) - 1$
		[90]	✗	✓	High	$N_{x2}(N_{x1} + 1) - 1$	$2N_{y2}(N_{y1} + 1) - 2$
		[91]	✗	✗	High	$2N_1 - 1$	$2N_1 - 1$
		[92]	2D	✗	High	$N_2(N_1 + 1) - 1$	$N_2(N_1 + 1) - 1$
	IA	[93]	1D	✗	Medium	$(M + N - \lfloor M/2 \rfloor - 2)(M + 1)M$	
	L sULA+	[94]	✗	✗	Medium	$M - 1$	M
	L Other	[95]	✗	✗	Medium	M_x	M_y
	VCA	[74]	1D	✗	Low	$2M_1(M_2 - 1)\sin(\Omega/2)$	$M_1(M_2 - 1)\cos(\Omega/2)$
	VNA	[74]	1D	✗	High	$2(N_2(N_1 + 1) - 1)\sin(\Omega/2)$	$(N_2(N_1 + 1) - 1)\cos(\Omega/2)$
Cross	[78]	✗	✗	Medium	$2(M_x - 1)$	$2(M_y - 1)$	
Billboard	[62]	✗	✗	High	$N_x - 1$	$N_y - 1$	
OBA	[62]	✗	✗	High	$N_x - 1$	$N_y - 1$	
Planar	CPA	[67]	2D-PSS ³	✗	Low	$M_1(M_2 - 1)$	$M_1(M_2 - 1)$
	GCPA	[96]	2D	✗	Low	$M_1(M_2 - 1)$	$N_1(N_2 - 1)$
	CPA	[72]	✗	✗	Low	$M_1(M_2 - 1)$	$M_1(M_2 - 1)$
	CPA	[97]	✗	⋈ ⁴	Low	$M_1(M_2 - 1)$	$M_1(M_2 - 1)$
	Nested	[81, 98]	2D	✗	High	$N_2(N_1 + 1) - 1$	$N_2(N_1 + 1) - 1$
	Unfolded	[99]	1D	✗	Low	$M_1(M_2 - 1) + M_2(M_1 - 1)$	$M_1(M_2 - 1) + M_2(M_1 - 1)$
	NCPA	[100]	2D	✗	Medium	$M_1(M_2 - 1) + M_1M_2(L - 1)$	$M_1(M_2 - 1) + M_1M_2(L - 1)$
	Other	HOBA-2	[62]	✗	✗	Low	$N_x - 1$
Hourglass		[62]	✗	✗	Low	$N_x - 1$	$N_y - 1$
Thermos		[79]	2D	✗	Low	$N_x + 3$	$N_y - 2$
Concentric		[106]	-	-	Low	N_x	N_y

¹PS: Peak Search. ²MC-1:Mutual Coupling approximate classification based on number of elements with unity inter-sensor spacing

³PSS: Partial Spectral Search. ⁴Means the paper does not propose a method based on FOC, but compares results with a method based on FOC.

¹PS: Peak Search. ²MC-1: Mutual Coupling approximate classification based on number of elements with unity inter-sensor spacing

³PSS: Partial Spectral Search. ⁴Means the paper does not propose a method based on FOC, but compares results with a method based on FOC.

comparable, and the latest one [85] using fourth order cumulants (FOC) has higher DOF as expected. It is not straight forward to compare all types of arrays since many sparse planar arrays are based on coprime, nested, or other linear arrays. In addition, some array geometries have different DOF when the parameters specifying the array are even or odd. In Table 4.1, a dash signifies that the value is not mentioned directly in the paper, or not easily verifiable. For example, Jian et al. [86] uses MRAs which do not have a closed form expression for the physical sensor locations; yet the expression shown for the number of sensors is a mere summation of the elements in the two subarrays (linear arrays). For other works, this could indicate that the work is focused on reducing computational complexity or other improvements apart from increasing DOF.

The arrays are listed in Table 4.1 based on the classification that was presented in Section 4.3, then chronologically. As such, it is generally expected that, within each category, later papers achieve higher DOF than older ones. As an example, Zheng et al. [99] state that their UCPA can exceed the DOF of the earlier works by Wu et al. [67] and Zheng et al. [96]. Similarly, Si et al. [100] show that their NCPA can exceed the DOF of the previous CPA variants: CPA [67], GCPA [96] and UCPA [99]. Another example is the thermos array [79] which can exceed the DOF of the hourglass array when $N_y > N_x + 1$. Furthermore, another interesting note might be the equal DOF of the arrays: OBA, HOBA-2, and hourglass [62]. In this case, this is an advantage for the newer arrays (HOBA-2 and Hourglass), since these arrays retain the hole-free difference coarray of the OBA, yet they reduce the mutual coupling.

Table 4.2 shows a comparison of the same arrays in Table 4.1, but now shows the

aperture, the estimated mutual coupling sensitivity based on the number of sensors with close inter-sensor spacing (MC-1), and whether the paper uses peak search (PS) or FOC which give an indication of the computational complexity. Only three papers considered FOC in the algorithm [85, 86, 90]. This can be explained by the computational complexity of FOC, and the condition that the sources must not be Gaussian. Despite the fact that it theoretically eliminates the Gaussian noise, and usually leads to much larger DOF compared to methods based on second order statistics (SOS). However, one paper showed that it is possible to exceed the performance of some FOC-based methods [97]. Further, it can be noted that, in general, array geometries utilizing nested arrays have higher number of sensors closely-spaced, and as such, are more susceptible to mutual coupling degradation. In addition, some of the recent arrays like hourglass or thermos arrays have low mutual coupling sensitivity. Other notes on Table 4.2 include:

- The column peak search (PS) only describes the existence of a step in the cited paper. For instance, Qin et al. [70] only employ 1D search in the case of having sources less than sensors, while exploiting sparse learning techniques when the number of sources exceeds the physical sensors.
- It is worth mentioning that comparing the DOF offered by a method using FOC to another method using SOS may not be fair. However, since not all arrays are expected to work well with FOC, and because of the low number of papers using FOC, the papers using FOC are also listed in the same table.
- Not having PS does not mean having lower computational time. For example,

Wu and Zhu [78] use atomic norm minimization (ANM) for DoA estimation, yet they report relatively higher computational time than some other methods. They still, however, employ a MUSIC-like method for range estimation.

- In many cases, it is possible to use different estimation methods. Hence, for some of the works that utilize 2D peak search, it could be possible to try a less complicated method. However, the use of such 2D peak searching methods can be justified when the main contribution of the paper is in a different direction, and a baseline comparison is sought. For example, Sun et al. [79] presented the thermos array trying mainly to reduce mutual coupling by spacing some sensors further apart from each other, based on ideas from the hourglass array proposed earlier by Liu and Vaidyanathan [62]. Another example is the work of Zheng et al. [96] which mainly presents the GCPA structure that is a generalization of the CPA.
- Using 2D peak search is not always the worst in terms of computational complexity. For example, Wu et al. [67] used a limited 2D spectral search that is having less complexity than the full 2D search.
- The last two columns show the aperture of the array in x and y directions. While in general, arrays based on coprime linear arrays for example have the same aperture. Some differences exist if authors use different definitions of coprime linear array, or if the linear array on the x and y directions are different. Similar arguments apply to other linear arrays. Note that it is not easy to write an expression for the aperture if the linear array (LA) is a minimum redundancy

array (MRA). In addition, sometimes $N_x - 1$ is the aperture of the array by design like hourglass and HOBA-2. In addition, if ULAs are used, the table shows the number of elements in the ULA on that axis. In many cases, the aperture is square and the same expression appear in both columns. However, for the work by Wu et al. [93], the expression is made to span both columns to save space.

4.4.1 Concluding Remarks

The increased number of location-based services and applications has led to increased interest in 2D sparse arrays which enable localization with minimum number of sensors. Published research efforts are not fully aligned and do not fall under clear framework. In this paper, we presented a comprehensive and structured literature overview of sparse arrays for 2D-DoA estimation. Popular designs like L-shaped, V-shaped, hourglass, thermos, nested planer, and coprime planer were classified into parallel arrays, non-parallel arrays, and other planer arrays. Existing designs were compared in terms of the required number of sensors, DOF, algorithm used, complexity and aperture size. A fair comparison should not overlook the aperture size associated with the improved estimation performance. L-shaped sparse arrays received the largest attention in the literature. However, other new structures seem promising. Arrays with rectangular aperture tend to show large DOF, like the hourglass or thermos arrays. Also, GCPA show large DOF compared to CPA. Arrays with relatively more complicated closed-form expressions seem to enjoy more DOF, like the hourglass or thermos arrays.

While the emphasis was on 2D the sparse arrays' structure, some designs require modified DoA estimation algorithms. Many algorithms for 2D-DoA estimation employ vectorization of cross correlation matrix, and then progress by various minimization methods. Some do spatial smoothing, and others avoid it to evade reduction in available DOF. Performance metrics of 2D-DoA estimation were outlined and existing solutions to alleviate problems were discussed. The presented structured review should help in predicting the DoA estimation performance and required complexity; thus, facilitating design and selection of sensor array geometry for 2D-DoA estimation.

CHAPTER 5

DESIGN OF SPARSE PLANAR ARRAYS MAXIMIZING LAGS

This chapter explores the design of sparse planar arrays with different objectives, namely: maximizing the contiguous URA in the fourth order difference coarray (4-DC), and exploring the best array geometry with limited number of sensors in terms of large number of lags in the second order difference coarray (2-DC) and the fourth order difference coarray (4-DC).

5.1 Arrays with contiguous virtual elements in the 4-DC

Many reported algorithms utilize the difference coarray, which can be referred to as the second order difference coarray (2-DC), since it is defined as the set of subtracting the locations of physical sensors from each other (3.10). This coarray is significant because

it appears naturally when vectorizing the covariance matrix of the output of the sensor array. If that 2-DC is contiguous, that is, hole-free, then DoA estimation algorithms that work on the URA like 2D-Unitary ESPRIT [127] can be readily applied [62]. This, however, is not the case for many sparse arrays, and getting a hole-free 2-DC is quite challenging while optimizing for some other parameter at the same time. Recently developed arrays that have low mutual coupling, high degrees of freedom, and hole-free 2-DC include the hourglass array that was reported by Liu and Vaidyanathan [62]. Next, Sun et al. [79] proposed the thermos array with less mutual coupling than the hourglass array, more degrees of freedom, but at the expense of losing the hole-free 2-DC property.

One possible direction is to consider higher order difference coarrays, including the 4-DC. These were explored in the literature for 1D arrays like nested array [128], multiple level nested arrays [129], coprime array [130], and also for augmented L-shaped nested array [90], and others [131, 132]. The 4-DC, denoted $\mathbb{D}_4$, is computed by applying the definition of the 2-DC (in Section 3.2.3 on page 31) again on the 2-DC, i.e.

$$\mathbb{D}_4 = \{\mathbf{m} | \mathbf{m} = \mathbf{p}_1 - \mathbf{p}_2, \forall \mathbf{p}_1, \mathbf{p}_2 \in \mathbb{D}\}. \quad (5.1)$$

This section explores the design of sparse planar arrays having high number of consecutive lags in the 4-DC. The remainder of this section introduces the proposed arrays, evaluates their 4-DC, and examines the estimation performance of selected examples.

5.1.1 Proposed Arrays

The proposed arrays fall under two categories: the billboard and the T-shaped. The former is an L-shaped array with an extra linear array at 45 degrees from both legs of the L-shaped array. The latter is made up of three linear arrays that form a (flipped) T-shape. Both classes are formed using three linear arrays, and the billboard array can be thought of as T-shaped array but with reduced angles between neighbor LAs from 90 degrees to 45 degrees. The billboard array with uniform linear subarrays is known in the literature [62]. However, to the best of the author's knowledge, using sparse linear arrays to construct T-shaped or billboard arrays was not considered before.

The objective is to design arrays having large 4-DC while keeping low mutual coupling. Since the array is made up of three linear arrays, one might expect that starting with good linear arrays would yield good billboard and T-shaped arrays, at least in terms of mutual coupling. The considered linear arrays are:

- coprime array,
- nested array,
- and super nested array.

The coprime array was introduced in Section 4.1.4, and the nested array was also introduced in Section 4.1.3. Using these three linear sparse arrays with the billboard and T-shaped classes lead to six possible geometries. Examples are shown in Fig. 5.1 for all the six combinations. Note that the total number of elements for all the examples in Fig. 5.1 is $N_t = 16$. Mathematical expressions for sensor locations of

T-shaped and billboard arrays are given next.

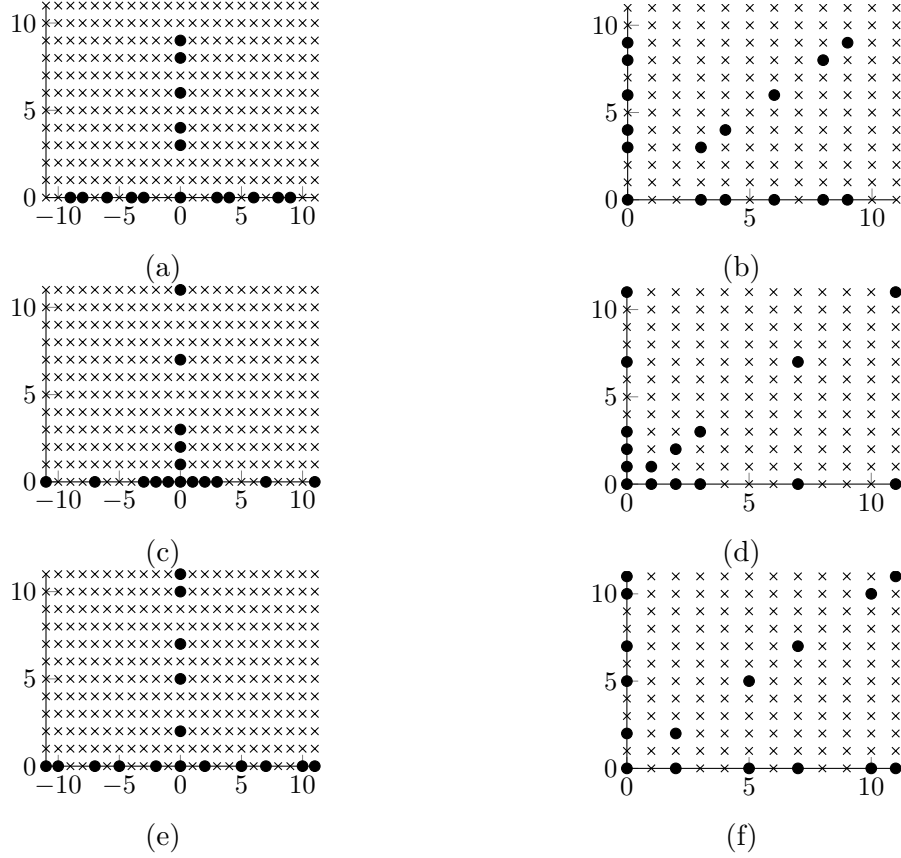


Figure 5.1: Examples of proposed sparse arrays constructed using three linear arrays. **(a)** T-shaped coprime array with $M = 3$, $N = 4$. **(b)** Billboard coprime array with $N_1 = N_2 = 3$. **(c)** T-shaped nested array with $M = 3$, $N = 4$. **(d)** Billboard nested array with $N_1 = N_2 = 3$. **(e)** T-shaped super nested array with $M = 3$, $N = 4$. **(f)** Billboard super nested array with $N_1 = N_2 = 3$.

Mathematically, the set of elements of the T-shaped array can be written as

$$\mathbb{S}_T = \mathbb{G}_x \cup \mathbb{G}_y \cup \mathbb{G}_{-x} \quad (5.2)$$

where

$$\begin{aligned}\mathbb{G}_x &= \{(nd, 0) | n \in \mathfrak{g}\}, \\ \mathbb{G}_y &= \{(0, nd) | n \in \mathfrak{g}\}, \\ \mathbb{G}_{-x} &= \{(-nd, 0) | n \in \mathfrak{g}\}.\end{aligned}\tag{5.3}$$

The set $\mathfrak{g}$ describes the linear array used for constructing the 2-D array. In case of T-shaped coprime array

$$\mathfrak{g} = \mathbb{P}_c,\tag{5.4}$$

$$\mathbb{P}_c = \{N_1 n | 0 \leq n \leq N_2 - 1\} \cup \{N_2 m | 0 \leq m \leq N_1 - 1\}.\tag{5.5}$$

Similarly, if nested linear arrays are used, the T-shaped nested array is formed where

$$\mathfrak{g} = \mathbb{P}_n,\tag{5.6}$$

$$\mathbb{P}_n = \{n | 0 \leq n \leq N_1 - 1\} \cup \{m(N_1 + 1) + N_1 | 0 \leq m \leq N_2 - 1\}.\tag{5.7}$$

Finally, super nested linear arrays can also be used to define the T-shaped super nested array with

$$\mathfrak{g} = \mathbb{P}_{\text{sn}}.\tag{5.8}$$

Note that the definition of the set $\mathbb{P}_{\text{sn}}$ is eliminated for brevity, and the interested reader is referred to [37].

The set of elements for the billboard arrays is

$$\mathbb{S}_B = \mathbb{G}_x \cup \mathbb{G}_y \cup \mathbb{G}_{xy}\tag{5.9}$$

where

$$\mathbb{G}_{xy} = \{(nd, nd) | n \in \mathbf{g}\}. \quad (5.10)$$

and $\mathbf{g}$ can be any of the three definitions in (5.4), (5.6), or (5.8). This results in three arrays: Bbillboard coprime, billboard nested, and billboard supernested arrays.

5.1.2 Evaluating the 4-DC

This section examines the 4-DC of the six proposed arrays. In addition to the aperture shown in Fig. 5.5, two variables are of interest: 1) the number of unique lags in the 4-DC, and 2) the number of consecutive lags in the 4-DC. To illustrate these two variables, three plots are given: The first is the number of unique lags vs the number of sensors, the second is the number of consecutive lags versus the number of sensors, and the third is the ratio (number of consecutive lags over number of unique lags) versus the number of sensors. Since not all values of total number of sensors are possible for all arrays, it is assumed that $M = N - 1 = N_1 = N_2$, which should be good enough to reveal the trend for each array, and to compare with other arrays. Before discussing the three plots, an example 4-DC is shown in Fig. 5.2 which shows 1891 unique lags, and 625 consecutive lags. Recall that the 4-DC is computed by applying the definition (in Section 3.2.3 on page 31) of the 2-DC again on the 2-DC, as shown in (5.1). In addition, the estimation is carried out using ESPRIT on the 4-DC, and the code accompanying the paper of Liu and Vaidyanathan [62] is used after repeating the 2-DC step, and removing virtual elements from the 4-DC that are not in the hole-free contiguous segment.

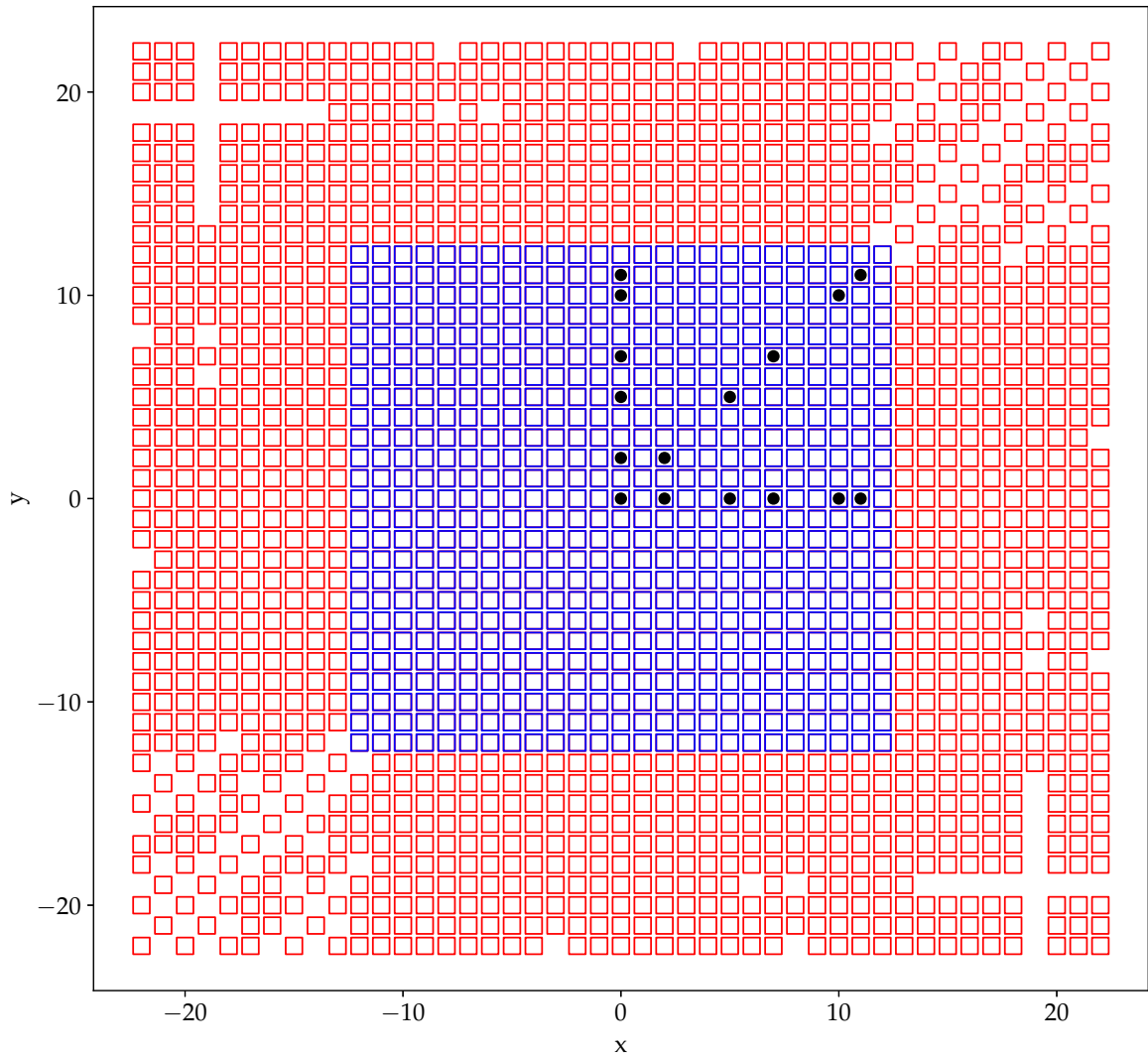


Figure 5.2: Example of the 4-DC of the array in Fig. 5.1f. The black circles are physical sensors, the red and blue squares are virtual lags in the 4-DC, and the blue squares resemble the central contiguous URA part of the 4-DC coarray.

Now, the plots of the lags of the 4-DC are examined. First, Fig. 5.3 shows that the unique lags are less than the consecutive lags, as expected. This can be seen by comparing the vertical values of subfigures a and b. Next, Fig. 5.3a shows that the billboard super nested array achieves the highest number of unique lags for most number of physical sensors. It is also interesting to note that coprime-based arrays start with low number of unique lags then increase, compared to T-shaped nested and super nested arrays which exhibit the opposite trend. Fig. 5.3b shows that despite the variation in which array gets the highest number of unique lags as the number of sensors is changed; the arrays retain fixed relative positions in terms of number of consecutive lags. This leads to the next figure which examines the ratio of consecutive lags over unique lags.

Second, Fig. 5.4 shows the ratio of the number of consecutive lags to that of unique lags. Usually, the higher the ratio, the better the array in terms of efficiency of utilizing the physical sensors. In this figure, T nested and T super nested arrays get higher ratio as the number of sensors increase. However, surprisingly, the billboard super nested has the lowest ratio, despite maintaining the highest number of unique lags. The trend of T-shaped coprime array is opposite to other T-shaped arrays.

It is also useful to note that the aperture is the same for arrays using nested or super nested arrays as shown in Fig. 5.5, and these arrays have higher aperture than the coprime-based arrays. In addition, arrays with coprime linear arrays have the same aperture. Note that the aperture on the x axis is double that of the aperture on the y axis for T-shaped arrays, and it is the same as the aperture on the y axis for billboard arrays. Hence, only the aperture on the y axis is shown in Fig. 5.5.

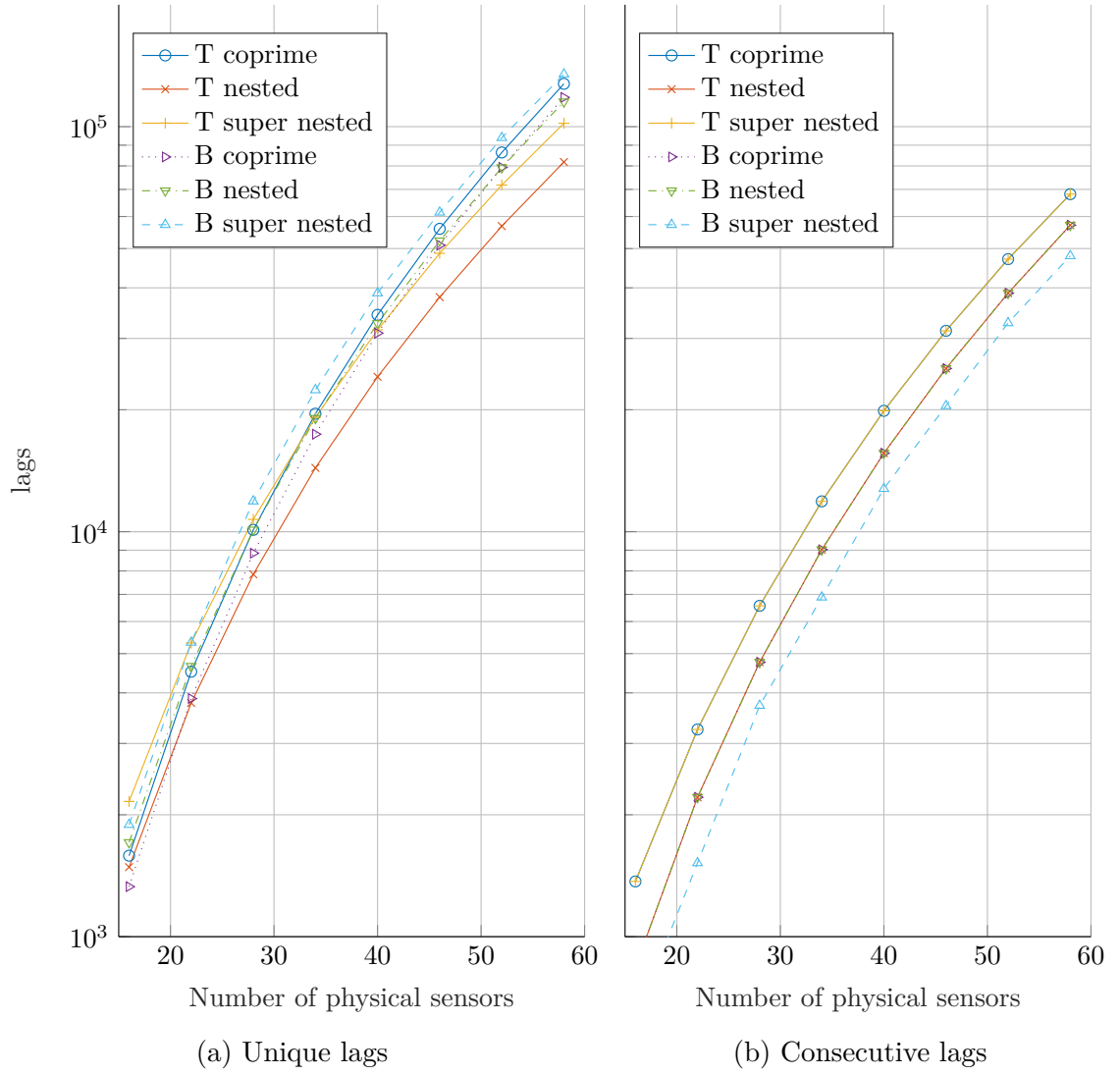


Figure 5.3: Unique and consecutive lags vs number of physical sensors.

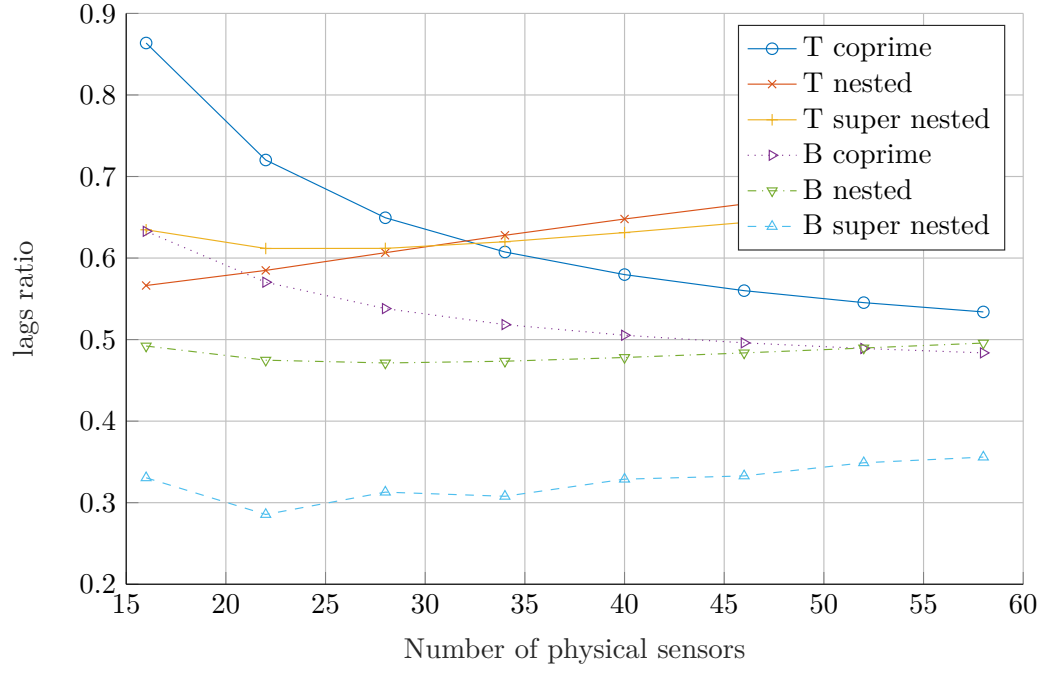


Figure 5.4: Ratio of consecutive lags to unique lags for the arrays in Fig. 5.3.

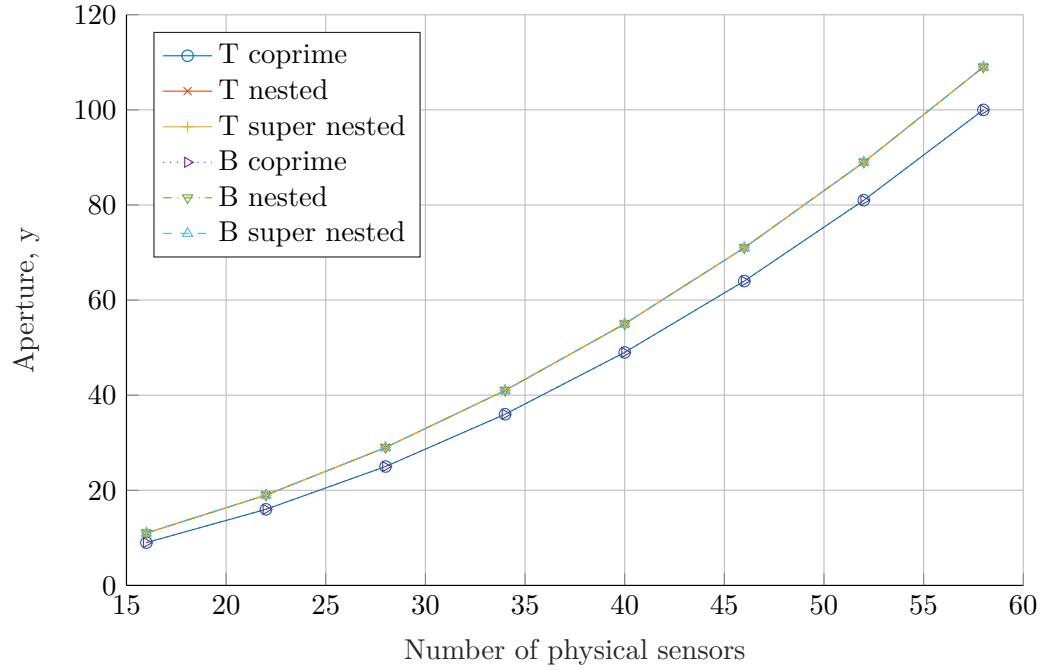


Figure 5.5: Aperture (on the y axis) of the arrays in Fig. 5.3. The aperture on the x -axis is double the shown value for T-shaped arrays.

5.1.3 Evaluating the Estimation Performance

This section examines the performance of the proposed arrays for 2D-DoA estimation. Since the proposed arrays are classes of arrays, example parameters are chosen for this performance comparison section. Specifically, the six chosen arrays have 16 physical elements each, and are shown in Fig. 5.1. The performance will be compared for variable number of snapshots and variable SNR. In addition, a 16-element URA is also included for comparison. The URA is used with the proposed method, and with ESPRIT on the 2-DC. For Fig. 5.6 and Fig. 5.7, two sources uniformly distributed between -0.4 and 0.4 normalized angles are assumed for each Monte-Carlo trial.

Fig. 5.6 shows the estimation performance in the absence of mutual coupling. In this case, B nested and T nested arrays seem to show the best performance, contrary to T super nested and B super nested arrays. In addition, all of the proposed arrays are almost always better than the URA using the proposed 4-DC, and are also better than the URA with 2-DC in most cases.

Fig. 5.7 shows that when mutual coupling is present, the dominance of B nested and T nested arrays is no longer the case. In fact, the T nested array is now the second worst after the URA with 4-DC, as these arrays have many closely-spaced elements that are affected by mutual coupling. The B nested array is better than the T nested array, but it is the second worst after the URA with 2-DC. Giving the fifth position to the T coprime array leaves the competition for the best performance for the other three arrays: T super nested, B coprime, and B super nested arrays. Some other notes on Fig. 5.6 and Fig. 5.7 are:

1. In general, except the T nested array, all other arrays are still robust in presence of mutual coupling.
2. Interestingly, the B nested array performs better than the T nested array in case of mutual coupling; despite the lower aperture of the B nested array.
3. Note that the URA using ESPRIT on the 2-DC, for example, can yield better estimates compared to the URA with the proposed 4-DC method. However, the URA is also used with the proposed 4-DC method to show the performance of the proposed method on various arrays.

The next section examines arrays with limited number of sensors. In particular, the performance of 4-sensor arrays is investigated.

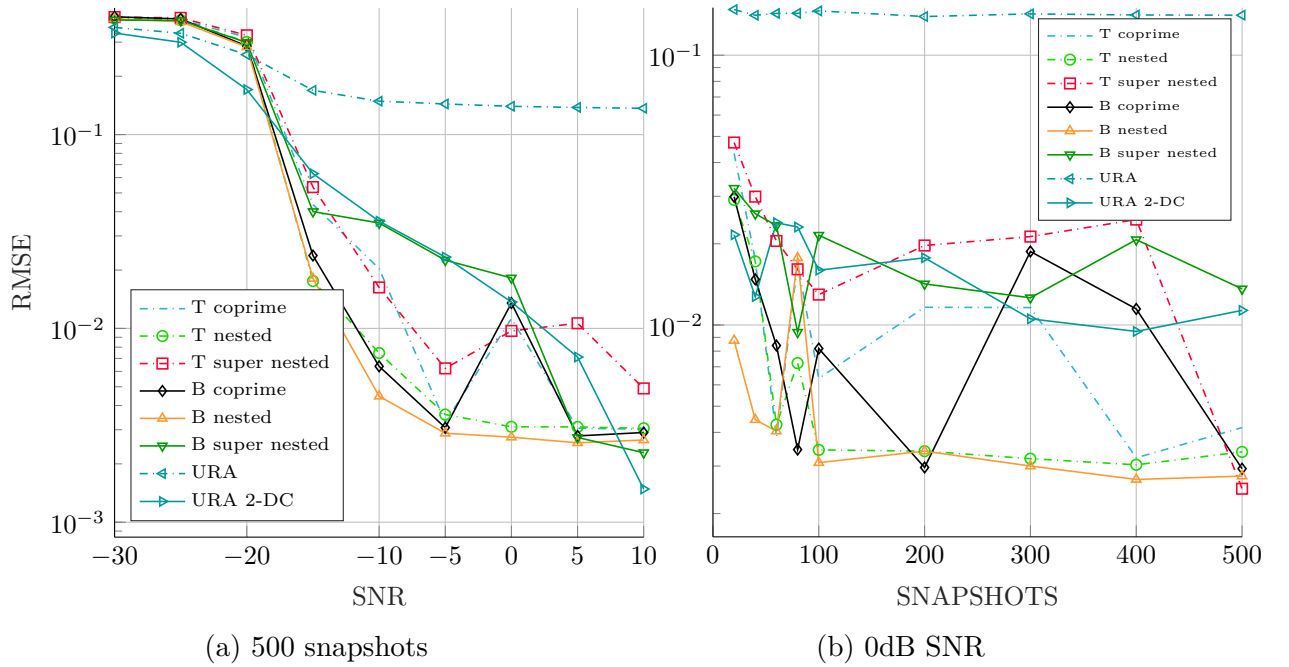


Figure 5.6: Performance of the six arrays in Fig. 5.1 using ESPRIT on 4-DC without mutual coupling with 1000 Monte-Carlo trials and

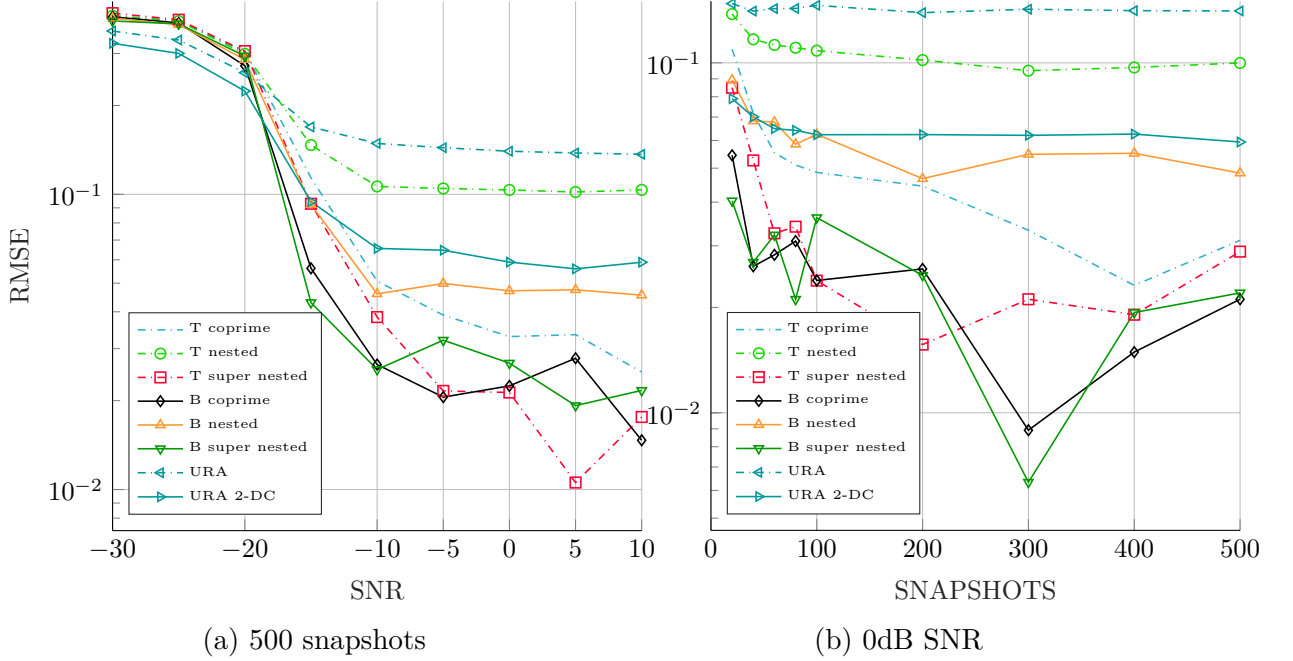


Figure 5.7: Performance of the six arrays in Fig. 5.1 using ESPRIT on 4-DC with mutual coupling with 1000 Monte-Carlo trials and

5.2 Arrays with constrained number of sensors

Several published works consider the design of sparse planar arrays by combining linear arrays [61, 68–71, 73–75, 78, 83–95, 101, 105, 123]. These linear arrays could be arranged parallel to each other [61, 68–70, 75, 83–85, 101, 123] or otherwise [71, 73, 74, 78, 86–95, 105]. In addition, some designs combine planar arrays [67, 72, 81, 96–100]. Furthermore, other works systematically develop one of the known arrays to enhance some property, like reducing mutual coupling sensitivity [62, 79]. These arrays enjoy closed-form expressions of sensor locations, but they may not provide the most optimal geometry of sensor array.

The other way to develop sensor arrays is to numerically investigate the possible geometries satisfying a certain constraint, like number of sensors. This could be useful, for example, for low number of sensors as some of the applications require a minimum

number of sensors. However, the main hindrances to this numerical approach is the complexity of search for array geometries, and the difficulty of generalizing the discovered geometry to a variable number of sensor elements, for example.

This section focuses on sensor arrays with four elements; since it is arguable where the sparse planar arrays start to offer interesting characteristics like larger coarray aperture. This is because two elements can only form a linear array, three elements have limited possibilities of arranging them in a planar array (while maintaining desired characteristics).

5.2.1 Four-element sensor arrays

This section examines the design of a four-element array maximizing the number of coarray elements and maximizing the number of consecutive lags at the same time. The number of consecutive lags is considered to be the largest uniform rectangular array (URA) including the origin. Since the focus is on sparse arrays, cases like that in Fig. 5.8a should not be considered because it is essentially a URA. However, it is shown for comparison with other arrays. In fact, if the total number of elements in a planar array can be factored as the multiplication of two integers, then the elements can be arranged as a URA. For instance, a planar array with four sensors have $2 \times 2 = 4$ sensors, thus, a URA geometry is possible as shown in Fig. 5.8a, where the black circles denote the physical sensors, and the red squares denote the elements of the difference coarray. Another representation of the difference coarray is shown in Fig. 5.9 where each square is an element in the difference coarray, and the number is the weight for that difference (See definition 3.2.3 on page 32). Fig. 5.9 shows that cases c and d

achieve the lowest possible non-negative weights.

Searching over different arrangement possibilities of the four sensors, only the geometries in Fig. 5.8c and Fig. 5.8d offer the highest coarray length while maintaining large number of consecutive lags. Thus, for four-element planar arrays, the maximum possible consecutive lags and coarray length are 9 and 13, respectively.

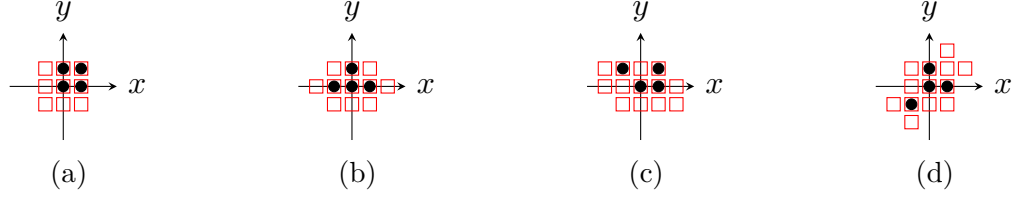


Figure 5.8: Examples of 2D sparse arrays with four elements and nine consecutive lags. **(a)** URA. **(b)** T-shaped. **(c)** two parallel linear arrays. **(d)** L-shaped with extra element. Black dots are sensors, and red boxes denote locations of difference coarray elements.

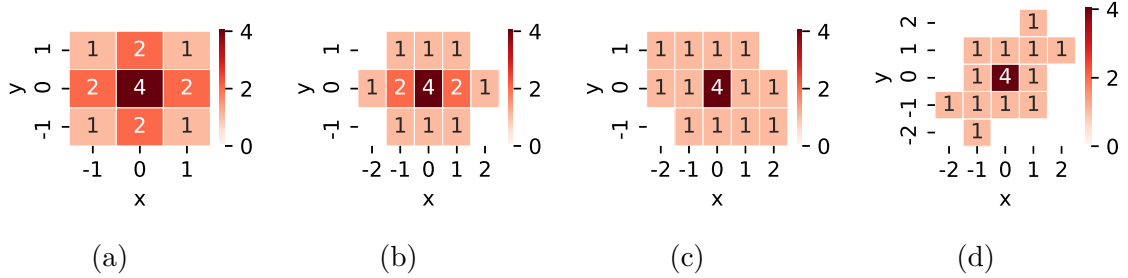


Figure 5.9: Weight function of the difference coarray of the arrays in Fig. 5.8

5.2.2 DoA Performance Simulation Results

This section presents simulation results for the four-element arrays in Fig. 5.8. The RMSE of azimuth and elevation estimates versus SNR is shown in Fig. 5.10, where 1000 Monte-Carlo trials were conducted, 500 snapshots are utilized for each trial, and two uniformly distributed sources, between normalized angles -0.4 and 0.4 are considered. The estimation is performed using two methods: 1) applying ESPRIT on

the contiguous part of the 2-DC, and 2) the proposed method, applying ESPRIT on the contiguous part of the 4-DC. The dotted traces are used for the former method, and the solid traces are used for the latter. It can be seen that the proposed method yields lower RMSE for the SNR values below 0dB if there is no mutual coupling, and this is also true below -8dB if there is mutual coupling. Note, however, that in case of mutual coupling, the difference in performance is not large, and the saturation in performance for almost all arrays happen at around 0.15 RMSE compared to lower values for the case where mutual coupling is absent. Another interesting note is that the URA performs better at high SNR values (with and without mutual coupling), and this can be attributed to the redundancy in the structure of the URA that help average the estimation. In addition, the other arrays in Fig. 5.8 are not that sparse compared to the URA of Fig. 5.8a.

Fig. 5.11 shows the RMSE of azimuth and elevation estimates versus snapshots, with 1000 Monte-Carlo trials at 0dB SNR. This figure shows that the improvement with increasing the number of snapshots is not large, especially in the case of mutual coupling. The case of the URA described earlier is again clear in Fig. 5.11. Furthermore, the array of Fig. 5.8c with the proposed 4-DC method seem to have the most stable performance for different values of SNR and snapshots. Unlike the array of Fig. 5.8d which sometimes produces higher errors at higher values of SNR or snapshots, especially when mutual coupling is absent.

The next chapter concludes the thesis by restating some important findings, and by mentioning possible direction of improvements over the current work.

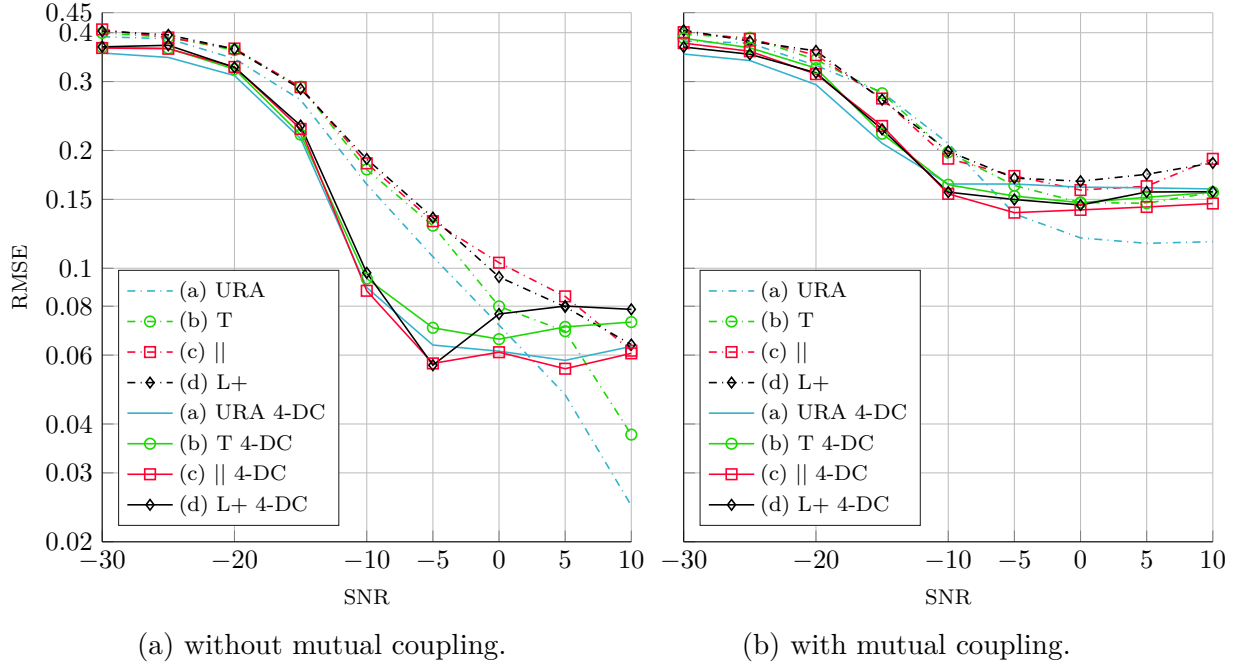


Figure 5.10: Performance of the four arrays in Fig. 5.8 using ESPRIT on 2-DC (dotted) and 4-DC (solid) with 1000 Monte-Carlo trials and 500 snapshots.

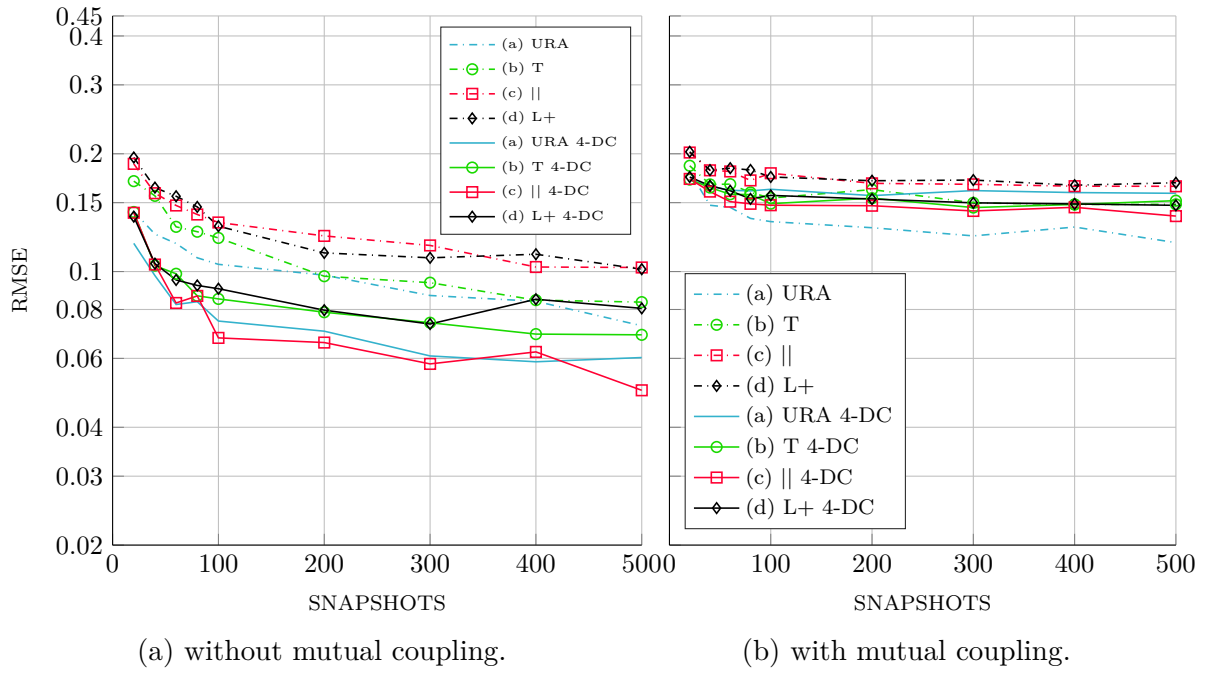


Figure 5.11: Performance of the four arrays in Fig. 5.8 using ESPRIT on 2-DC (dotted) and 4-DC (solid) with 1000 Monte-Carlo trials at 0dB SNR.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

DoA estimation can be a useful step in many applications like navigators and radars. The wide range of applications and the changing requirements of every situation makes DoA estimation a very actively researched topic for the past decades. And that is expected to increase in the coming years as new applications like Massive MIMO employ it as well. In many occasions, it is desired to have a functional system without a manual operator, and with the least mechanical actuation. This makes antenna arrays an attractive option for direction finding.

After briefly introducing the basics of linear arrays for direction finding, this work summarized the main planar arrays in the literature that was used for two-dimensional (azimuth and elevation) DoA estimation. Next, a very detailed review of sparse planar arrays was conducted, which revealed useful array characteristics that different authors

aim to achieve. For example, a very strong direction is to design arrays that reduce the effect of mutual coupling; thus, reducing the possible degradation of the array performance in practice due to mutual coupling. This example is an excellent case for the use of sparse arrays in DoA estimation, since mutual coupling is primarily linked to the distance between physical sensors, and sparse arrays inherently increase that distance, thereby reducing mutual coupling. Another example is the desire to design arrays having hole-free coarrays, like the difference coarray. Although this is only possible in a limited set of sensor arrays, those arrays that exhibit this feature can readily utilize DoA estimation algorithms that was applicable on the uniform rectangular array.

The presented comprehensive review of more than 30 papers using sparse planar arrays is hopefully useful for researchers aiming at finding new array designs. In addition, two developed tables that compare key selected array characteristics are presented at the end of this review. Among the notes from the review is that L-shaped arrays received the largest attention in terms of number of publications. Nonetheless, other new structures with relatively complicated expressions for sensor positions seem promising like the hourglass and the thermos arrays.

Another conclusion is the difficulty to use higher order statistics in 2D-DoA estimation. This is partly because the complexity increases with more sensors in the planar array, and also the requirement of having non-Gaussian statistical distribution of the sources' signals. In fact, only three works out of the 34 reviewed works utilized fourth order cumulants (a popular higher order statistic for direction finding) in the estimation algorithm. This does not deny the finer resolution capabilities of higher

order statistics, but suggests that their application range is probably niche.

Finally, this thesis proposed a method based on the 4-DC for 2D-DoA estimation, and it was tested on the two proposed array classes: billboard and T-shaped, in addition to four-sensor arrays. A URA is used for comparison in both cases, and it was found that most of the proposed billboard and T-shaped arrays perform better than the URA. Furthermore, the proposed method yield better estimates under some low values of SNR for four-element arrays compared to ESPRIT on the 2-DC.

6.2 Future Work

The area of DoA estimation is very broad. Several possible directions exist to amend this work, including:

1. A detailed, tutorial-like review of 2D-DoA estimation algorithms, and their special cases and assumptions. The main classes of algorithms are limited, but assumptions and extensions are very common. Such a work should explain why each extension of each algorithm works under the new scenarios. Despite the attempts of many authors to link their work to some previous works, the perspective of each paper is still limited to some selected algorithms. Therefore, a coherent presentation of all the algorithms would probably be very useful for beginners, and also for professional practitioners who might find it useful for identifying possible new extensions of current algorithms.
2. There is a vast literature on 1D-DoA estimation, both using uniform and sparse arrays. There is less, but increasing number of publications on 2D-DoA esti-

mation also for uniform and sparse planar arrays. However, the **experimental** evaluation for planar arrays is much less than linear arrays. Partly, this is due to the requirements of extra sensors on the plane compared to the linear case. The extra number of sensors require supported hardware for synchronizing all the channels. There has been some attempts to develop DoA algorithms for partially calibrated sensors, but that is not very common yet.

3. Another aspect of 2D-DoA estimation that is not yet as mature as the 1D case, is the evaluation studies on the effect of failed sensors, and misaligned sensors. Especially, if the coarray domain is utilized in the estimation algorithm, which might cause the co-array elements to be off the rectangular grid they were supposed to be on, and that might negatively affect the estimation performance.
4. Another less explored case is the possibility of exploiting non-rectangular structures in the coarray for ESPRIT. In some arrays, there is a possibility of finding shifted subarrays in the coarray that are not URAs, which could enhance the DOF and increase the utilization of the virtual sensors in the coarray.

APPENDIX A

SUPPLEMENTARY MATERIAL

A.1 Useful Matrix Operations

A.1.1 The Kronecker Product

Also known as Direct product, denoted by $\otimes$. It is found by multiplying each element of the first matrix by the whole other matrix. This applies to vectors as well [133, 134].

For instance, if $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ ($m \times n$), and $B = \begin{bmatrix} e & g \\ f & h \end{bmatrix}$ ($q \times r$), then,

$$A \otimes B = \begin{bmatrix} aB & cB \\ bB & dB \end{bmatrix} = \begin{bmatrix} ae & ag & ce & cg \\ af & ah & cf & ch \\ be & bg & de & dg \\ bf & bh & df & dh \end{bmatrix}$$

Note that the dimensions of $A \otimes B$ is $(mq \times nr)$.

A.1.2 The vec Operator

$\text{vec}(A)$ stacks columns of the matrix A ($m \times n$) below each other to form a vector with

dimensions $(mn \times 1)$ [133–135]. For example, if $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ then, $\text{vec}(A) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

A.2 Properties of Cumulants

Important properties of cumulants are [14, 15]

- (P1) For random variables $x_i, i = 1, 2, \dots, q$, and constants $\alpha_i, i = 1, 2, \dots, q$,

$$\text{cum}(\alpha_1 x_1, \dots, \alpha_q x_q) = \left(\prod_{i=1}^q \alpha_i \right) \text{cum}(x_1, \dots, x_q)$$

- (P2) Cumulants are symmetric in their arguments,

$$\text{cum}(x_1, \dots, x_q) = \text{cum}(x_{j+1}, \dots, x_q, x_1, \dots, x_j)$$

- (P3) Cumulants are additive in their arguments,

$$\text{cum}(x_0 + y_0, z_1, \dots, z_q) = \text{cum}(x_0, z_1, \dots, z_q) + \text{cum}(y_0, z_1, \dots, z_q)$$

- (P4) An added constant α has no effect

$$\text{cum}(\alpha + z_1, \dots, z_q) = \text{cum}(z_1, \dots, z_q)$$

- (P5) For random variables x_i that are independent of y_i ($i = 1, \dots, q$),

$$\text{cum}(x_1 + y_1, \dots, x_q + y_q) = \text{cum}(x_1, \dots, x_q) + \text{cum}(y_1, \dots, y_q)$$

- (P6) If a subset of random variables x_i is independent of the rest,

$$\text{cum}(x_1, \dots, x_q) = 0$$

- For Gaussian processes, cumulants of order > 2 are identically zero. This is easily proven from the cumulant generating function.

REFERENCES

- [1] Han Yi, Wu Chu-na, A new moving sound source localization method based on the time difference of arrival, in: 2010 International Conference on Image Analysis and Signal Processing, 2010, pp. 118–122.
- [2] S. He, H. Chen, Closed-form doa estimation using first-order differential microphone arrays via joint temporal-spectral-spatial processing, IEEE Sensors Journal 17 (2017) 1046–1060.
- [3] J. Yang, Z. Liu, H. Qi, S. Qi, Direction of arrival estimation method based on first-order differential microphone array, in: 2019 IEEE 21st International Conference on High Performance Computing and Communications; IEEE 17th International Conference on Smart City; IEEE 5th International Conference on Data Science and Systems (HPCC/SmartCity/DSS), 2019, pp. 2451–2454.
- [4] H. Wang, P. Chu, Voice source localization for automatic camera pointing system in videoconferencing, in: 1997 IEEE International Conference on Acoustics, Speech, and Signal Processing, volume 1, 1997, pp. 187–190 vol.1.
- [5] B. Friedlander, Chapter 1 - wireless direction-finding fundamentals, in: T. E.

- Tuncer, B. Friedlander (Eds.), Classical and Modern Direction-of-Arrival Estimation, Academic Press, Boston, 2009, pp. 1 – 51.
- [6] J. Liang, D. Liu, Joint elevation and azimuth direction finding using l-shaped array, *IEEE Transactions on Antennas and Propagation* 58 (2010) 2136–2141.
 - [7] A. Osman, M. M. E. Moussa, M. Tamazin, M. J. Korenberg, A. Noureldin, Doa elevation and azimuth angles estimation of gps jamming signals using fast orthogonal search, *IEEE Transactions on Aerospace and Electronic Systems* 56 (2020) 3812–3821.
 - [8] R. Zekavat, R. M. Buehrer, *Wireless Positioning Systems: Operation, Application, and Comparison*, IEEE, 2019, pp. 3–23.
 - [9] D. H. Johnson, D. E. Dudgeon, *Array Signal Processing: Concepts and Techniques*, Prentice Hall, 1993.
 - [10] Y. Albagory, A. Ashour, Music 2d-doa estimation using split vertical linear and circular arrays, *International Journal of Computer Network and Information Security* 5 (2013) 12.
 - [11] J. J. Shynk, *Probability, Random Variables, and Random Processes: Theory and Signal Processing Applications*, John Wiley & Sons, 2013.
 - [12] A. Papoulis, S. U. Pillai, *Probability, Random Variables, and Stochastic Processes*, McGraw Hill, 2002.
 - [13] P. Chevalier, A. Ferréol, L. Albera, High-resolution direction finding from higher

- order statistics: The 2q-music algorithm, *IEEE Transactions on signal processing* 54 (2006) 2986–2997.
- [14] P. Chevalier, A. Ferréol, L. Albera, Chapter 8 - high-resolution doa estimation with higher-order statistics, in: T. E. Tuncer, B. Friedlander (Eds.), *Classical and Modern Direction-of-Arrival Estimation*, Academic Press, Boston, 2009, pp. 289 – 341.
 - [15] A. Swami, System identification using cumulants, Ph.D. thesis, University of Southern California, 1989.
 - [16] P. Chevalier, L. Albera, A. Ferréol, P. Comon, On the virtual array concept for higher order array processing, *IEEE Transactions on Signal Processing* 53 (2005) 1254–1271.
 - [17] P. Palanisamy, N. Rao, Direction of arrival estimation based on fourth-order cumulant using propagator method, *Progress In Electromagnetics Research* 18 (2009) 83–99.
 - [18] A. Ahmed, M. F. Khan, M. Tufail, Multiple invariance cumulant esprit for doa estimation, in: *2014 International Conference on Robotics and Emerging Allied Technologies in Engineering (iCREATE)*, 2014, pp. 157–159.
 - [19] R. Schmidt, Multiple emitter location and signal parameter estimation, *IEEE Transactions on Antennas and Propagation* 34 (1986) 276–280.
 - [20] H. Krim, M. Viberg, Two decades of array signal processing research: the parametric approach, *IEEE Signal Processing Magazine* 13 (1996) 67–94.

- [21] M. Xiao, S. Mumtaz, Y. Huang, L. Dai, Y. Li, M. Matthaiou, G. K. Karagiannis, E. Björnson, K. Yang, C. I, A. Ghosh, Millimeter wave communications for future mobile networks, *IEEE Journal on Selected Areas in Communications* 35 (2017) 1909–1935.
- [22] E. Björnson, L. Sanguinetti, H. Wymeersch, J. Hoydis, T. L. Marzetta, Massive mimo is a reality—what is next?: Five promising research directions for antenna arrays, *Digital Signal Processing* 94 (2019) 3–20. Special Issue on Source Localization in Massive MIMO.
- [23] A. Fascista, A. Coluccia, H. Wymeersch, G. Seco-Granados, Millimeter-wave downlink positioning with a single-antenna receiver, *IEEE Transactions on Wireless Communications* 18 (2019) 4479–4490.
- [24] S. DeGraaf, D. Johnson, Optimal linear arrays for narrow-band beamforming, in: *ICASSP '84. IEEE International Conference on Acoustics, Speech, and Signal Processing*, volume 9, 1984, pp. 214–217.
- [25] Y. Bresler, A. Macovski, On the number of signals resolvable by a uniform linear array, *IEEE Transactions on Acoustics, Speech, and Signal Processing* 34 (1986) 1361–1375.
- [26] M. P. Wylie, S. Roy, H. Messer, Joint doa estimation and phase calibration of linear equispaced (les) arrays, *IEEE Transactions on Signal Processing* 42 (1994) 3449–3459.

- [27] B. Porat, B. Friedlander, Direction finding algorithms based on high-order statistics, *IEEE Transactions on Signal Processing* 39 (1991) 2016–2024.
- [28] A. Swami, J. M. Mendel, Cumulant-based approach to harmonic retrieval and related problems, *IEEE Transactions on Signal Processing* 39 (1991) 1099–1109.
- [29] A. Moffet, Minimum-redundancy linear arrays, *IEEE Transactions on antennas and propagation* 16 (1968) 172–175.
- [30] D. A. Linebarger, A fast method for computing the coarray of sparse linear arrays, *IEEE Transactions on Antennas and Propagation* 40 (1992) 1109–1112.
- [31] S. D. Bedrosian, Nonuniform linear arrays: Graph-theoretic approach to minimum redundancy, *Proceedings of the IEEE* 74 (1986) 1040–1043.
- [32] D. A. Linebarger, I. H. Sudborough, I. G. Tollis, Difference bases and sparse sensor arrays, *IEEE Transactions on Information Theory* 39 (1993) 716–721.
- [33] P. Pal, P. P. Vaidyanathan, Coprime sampling and the music algorithm, in: 2011 Digital Signal Processing and Signal Processing Education Meeting (DSP/SPE), 2011, pp. 289–294.
- [34] S. Qin, Y. D. Zhang, M. G. Amin, Generalized coprime array configurations for direction-of-arrival estimation, *IEEE Transactions on Signal Processing* 63 (2015) 1377–1390.
- [35] S. A. Alawsh, A. H. Muqaibel, Sparse doa estimation based on multi-level prime array with compression, *IEEE Access* 7 (2019) 70828–70841.

- [36] P. Pal, P. P. Vaidyanathan, Nested arrays: A novel approach to array processing with enhanced degrees of freedom, *IEEE Transactions on Signal Processing* 58 (2010) 4167–4181.
- [37] C.-L. Liu, P. Vaidyanathan, Super nested arrays: Linear sparse arrays with reduced mutual coupling—part i: Fundamentals, *IEEE Transactions on Signal Processing* 64 (2016) 3997–4012.
- [38] C. Liu, P. P. Vaidyanathan, Maximally economic sparse arrays and cantor arrays, in: 2017 IEEE 7th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP), 2017, pp. 1–5.
- [39] R. Rajamäki, V. Koivunen, Comparison of sparse sensor array configurations with constrained aperture for passive sensing, in: 2017 IEEE Radar Conference (RadarConf), 2017, pp. 0797–0802.
- [40] S. Mazlout, M. B. Ben Salah, A. Samet, Comparative study of two array configurations for 2d-doa estimation in ls-mimo systems, in: 2017 Sixth International Conference on Communications and Networking (ComNet), 2017, pp. 1–6.
- [41] A. B. Gershman, M. Rübsamen, M. Pesavento, One- and two-dimensional direction-of-arrival estimation: An overview of search-free techniques, *Signal Processing* 90 (2010) 1338 – 1349. Special Section on Statistical Signal & Array Processing.
- [42] A. M. Zoubir, V. Koivunen, Y. Chakhchoukh, M. Muma, Robust estimation

- in signal processing: A tutorial-style treatment of fundamental concepts, *IEEE Signal Processing Magazine* 29 (2012) 61–80.
- [43] S. Kiani, A. M. Pezeshk, A comparative study of several array geometries for 2d doa estimation, *Procedia Computer Science* 58 (2015) 18–25. Second International Symposium on Computer Vision and the Internet (VisionNet’15).
- [44] S. Khedekar, M. Mukhopadhyay, Analysis of estimation of direction of arrival by comparative study, *Materials Today: Proceedings* 5 (2018) 1696–1703. International Conference on Processing of Materials, Minerals and Energy (July 29th – 30th) 2016, Ongole, Andhra Pradesh, India.
- [45] A. Ahmed, S. A. Hamza, M. Tufail, Comparative analysis of fourth order cumulant based esprit algorithms, in: 2014 12th International Conference on Frontiers of Information Technology, 2014, pp. 133–138.
- [46] K. Adhikari, B. Drozdenko, Comparison of music variants for sparse arrays, in: 2019 IEEE National Aerospace and Electronics Conference (NAECON), 2019, pp. 398–405. doi:10.1109/NAECON46414.2019.9058169.
- [47] Q. Shen, W. Liu, W. Cui, S. Wu, Underdetermined doa estimation under the compressive sensing framework: A review, *IEEE Access* 4 (2016) 8865–8878.
- [48] Y. Liang, W. Liu, Q. Shen, W. Cui, S. Wu, A review of closed-form cramér-rao bounds for doa estimation in the presence of gaussian noise under a unified framework, *IEEE Access* 8 (2020) 175101–175124.

- [49] A. J. van der Veen, P. B. Ober, E. F. Deprettere, Azimuth and elevation computation in high resolution doa estimation, *IEEE Transactions on Signal Processing* 40 (1992) 1828–1832.
- [50] R. Roy, T. Kailath, Esprit-estimation of signal parameters via rotational invariance techniques, *IEEE Transactions on Acoustics, Speech, and Signal Processing* 37 (1989) 984–995.
- [51] S. Marcos, A. Marsal, M. Benidir, The propagator method for source bearing estimation, *Signal Processing* 42 (1995) 121–138.
- [52] Z. Ye, C. Liu, 2-d doa estimation in the presence of mutual coupling, *IEEE Transactions on Antennas and Propagation* 56 (2008) 3150–3158.
- [53] H. Chen, C. Hou, W.-P. Zhu, W. Liu, Y.-Y. Dong, Z. Peng, Q. Wang, Esprit-like two-dimensional direction finding for mixed circular and strictly noncircular sources based on joint diagonalization, *Signal Processing* 141 (2017) 48 – 56.
- [54] C.-H. Lin, W.-H. Fang, J.-D. Lin, K.-H. Wu, A fast algorithm for joint two-dimensional direction of arrival and frequency estimation via hierarchical space–time decomposition, *Signal Processing* 90 (2010) 207 – 216.
- [55] Y. Wu, G. Liao, H. So, A fast algorithm for 2-d direction-of-arrival estimation, *Signal Processing* 83 (2003) 1827 – 1831.
- [56] J. Li, X. Zhang, H. Chen, Improved two-dimensional doa estimation algorithm for two-parallel uniform linear arrays using propagator method, *Signal Processing* 92 (2012) 3032 – 3038.

- [57] P. Li, B. Yu, J. Sun, A new method for two-dimensional array signal processing in unknown noise environments, *Signal Processing* 47 (1995) 319–327.
- [58] Y. Zhang, X. Xu, Y. A. Sheikh, Z. Ye, A rank-reduction based 2-d doa estimation algorithm for three parallel uniform linear arrays, *Signal Processing* 120 (2016) 305–310.
- [59] R. T. Hoor, S. A. Kassam, The unifying role of the coarray in aperture synthesis for coherent and incoherent imaging, *Proceedings of the IEEE* 78 (1990) 735–752.
- [60] C. Zhu, W. Wang, H. Chen, H. C. So, Impaired sensor diagnosis, beamforming, and doa estimation with difference co-array processing, *IEEE Sensors Journal* 15 (2015) 3773–3780.
- [61] S. Qin, Y. D. Zhang, M. G. Amin, Two-dimensional doa estimation using parallel coprime subarrays, in: *2016 IEEE Sensor Array and Multichannel Signal Processing Workshop (SAM)*, 2016, pp. 1–4.
- [62] C. Liu, P. P. Vaidyanathan, Hourglass arrays and other novel 2-d sparse arrays with reduced mutual coupling, *IEEE Transactions on Signal Processing* 65 (2017) 3369–3383.
- [63] B. Friedlander, Antenna array manifolds for high-resolution direction finding, *IEEE Transactions on Signal Processing* 66 (2018) 923–932.
- [64] H. Huang, M. Fauß, A. M. Zoubir, Block sparsity-based doa estimation with

- sensor gain and phase uncertainties, in: 2019 27th European Signal Processing Conference (EUSIPCO), 2019, pp. 1–5.
- [65] C. Liu, P. P. Vaidyanathan, Robustness of difference coarrays of sparse arrays to sensor failures—part i: A theory motivated by coarray music, *IEEE Transactions on Signal Processing* 67 (2019) 3213–3226.
- [66] C. Liu, P. P. Vaidyanathan, Robustness of difference coarrays of sparse arrays to sensor failures—part ii: Array geometries, *IEEE Transactions on Signal Processing* 67 (2019) 3227–3242.
- [67] Q. Wu, F. Sun, P. Lan, G. Ding, X. Zhang, Two-dimensional direction-of-arrival estimation for co-prime planar arrays: A partial spectral search approach, *IEEE Sensors Journal* 16 (2016) 5660–5670.
- [68] J. Shi, G. Hu, X. Zhang, P. Gong, Sum and difference coarrays based 2-d doa estimation with co-prime parallel arrays, in: 2017 9th International Conference on Wireless Communications and Signal Processing (WCSP), 2017, pp. 1–4.
- [69] P. Gong, X. Zhang, J. Shi, W. Zheng, Three-parallel co-prime array configuration for two-dimensional doa estimation, in: 2017 9th International Conference on Wireless Communications and Signal Processing (WCSP), 2017, pp. 1–5.
- [70] S. Qin, Y. D. Zhang, M. G. Amin, Improved two-dimensional doa estimation using parallel coprime arrays, *Signal Processing* 172 (2020) 107428.
- [71] Q. Liu, X. Yi, L. Jin, W. Chen, Two dimensional direction of arrival estimation

- for co-prime l-shaped array using sparse reconstruction, in: 2015 8th International Congress on Image and Signal Processing (CISP), 2015, pp. 1499–1503.
- [72] D. Zhang, Y. Zhang, G. Zheng, B. Deng, C. Feng, J. Tang, Two-dimensional direction of arrival estimation for coprime planar arrays via polynomial root finding technique, *IEEE Access* 6 (2018) 19540–19549.
- [73] A. M. Elbir, L-shaped coprime array structures for doa estimation, *Multidimensional Systems and Signal Processing* 31 (2020) 205–219.
- [74] A. M. Elbir, V-shaped sparse arrays for 2-d doa estimation, *Circuits, Systems, and Signal Processing* 38 (2019) 2792–2809.
- [75] F. Sun, P. Lan, B. Gao, G. Zhang, An efficient dictionary learning-based 2-d doa estimation without pair matching for co-prime parallel arrays, *IEEE Access* 6 (2018) 8510–8518.
- [76] Y. I. Abramovich, D. A. Gray, A. Y. Gorokhov, N. K. Spencer, Positive-definite toeplitz completion in doa estimation for nonuniform linear antenna arrays. i. fully augmentable arrays, *IEEE Transactions on Signal Processing* 46 (1998) 2458–2471.
- [77] S. Orfanidis, *Electromagnetic Waves and Antennas*, 2016.
- [78] X. Wu, W. P. Zhu, Single far-field or near-field source localization with sparse or uniform cross array, *IEEE Transactions on Vehicular Technology* 69 (2020) 9135–9139.

- [79] L. Sun, M. Yang, B. Chen, Thermos array: Two-dimensional sparse array with reduced mutual coupling, *International Journal of Antennas and Propagation* 2018 (2018).
- [80] B. Friedlander, A. J. Weiss, Direction finding in the presence of mutual coupling, *IEEE Transactions on Antennas and Propagation* 39 (1991) 273–284.
- [81] P. Pal, P. P. Vaidyanathan, Nested arrays in two dimensions, part i: Geometrical considerations, *IEEE Transactions on Signal Processing* 60 (2012) 4694–4705.
- [82] T. Jiang, N. D. Sidiropoulos, J. M. F. ten Berge, Almost-sure identifiability of multidimensional harmonic retrieval, *IEEE Transactions on Signal Processing* 49 (2001) 1849–1859.
- [83] X. Li, W. Zhang, T. Shu, J. He, Two-dimensional direction finding with parallel nested arrays using doa matrix method, *IEEE Sensors Letters* 3 (2019) 1–4.
- [84] Z. Zheng, S. Mu, Two-dimensional doa estimation using two parallel nested arrays, *IEEE Communications Letters* 24 (2020) 568–571.
- [85] J. He, L. Li, T. Shu, 2-d direction finding using parallel nested arrays with full co-array aperture extension, *Signal Processing* 178 (2021) 107795.
- [86] C. Jian, S. Wang, L. Lin, 2-d doa estimation by minimum-redundancy linear array, in: 2006 8th international Conference on Signal Processing, volume 1, 2006.

- [87] J. He, Z. Liu, Extended aperture 2-d direction finding with a two-parallel-shape-array using propagator method, *IEEE Antennas and Wireless Propagation Letters* 8 (2009) 323–327.
- [88] Y.-Y. Dong, C.-X. Dong, Y.-T. Zhu, G.-Q. Zhao, S.-Y. Liu, Two-dimensional doa estimation for l-shaped array with nested subarrays without pair matching, *IET Signal Processing* 10 (2016) 1112–1117(5).
- [89] C. Niu, Y. Zhang, J. Guo, Interlaced double-precision 2-d angle estimation algorithm using l-shaped nested arrays, *IEEE Signal Processing Letters* 23 (2016) 522–526.
- [90] X. Li, S. Ren, J. Liu, W. Wang, Augmented l-shaped nested array based on the fourth-order difference co-array concept, in: *2018 IEEE 10th Sensor Array and Multichannel Signal Processing Workshop (SAM)*, 2018, pp. 31–35.
- [91] X. Gao, X. Hao, P. Li, G. Li, An improved two-dimensional direction-of-arrival estimation algorithm for l-shaped nested arrays with small sample sizes, *Sensors* 19 (2019).
- [92] Y. Yang, X. Mao, Y. Hou, G. Jiang, 2-d doa estimation via correlation matrix reconstruction for nested l-shaped array, *Digital Signal Processing* 98 (2020) 102623.
- [93] F. Wu, F. Cao, X. Ni, C. Chen, Y. Zhang, J. Xu, L-shaped sparse array structure for 2-d doa estimation, *IEEE Access* 8 (2020) 140030–140037.

- [94] Yuxuan Yang, Zhi Zheng, Haifen Yang, Jiao Yang, Yan Ge, A novel 2-d doa estimation method via sparse l-shaped array, in: 2016 2nd IEEE International Conference on Computer and Communications (ICCC), 2016, pp. 1865–1869.
- [95] J. Gu, W. Zhu, M. N. S. Swamy, Joint 2-d doa estimation via sparse l-shaped array, *IEEE Transactions on Signal Processing* 63 (2015) 1171–1182.
- [96] W. Zheng, X. Zhang, H. Zhai, Generalized coprime planar array geometry for 2-d doa estimation, *IEEE Communications Letters* 21 (2017) 1075–1078.
- [97] H. Xu, D. Wang, B. Ba, W. Cui, Y. Zhang, Direction-of-arrival estimation for both uncorrelated and coherent signals in coprime array, *IEEE Access* 7 (2019) 18590–18600.
- [98] P. Pal, P. P. Vaidyanathan, Nested arrays in two dimensions, part ii: Application in two dimensional array processing, *IEEE Transactions on Signal Processing* 60 (2012) 4706–4718.
- [99] W. Zheng, X. Zhang, L. Xu, J. Zhou, Unfolded coprime planar array for 2d direction of arrival estimation: An aperture-augmented perspective, *IEEE Access* 6 (2018) 22744–22753.
- [100] W. Si, F. Zeng, Z. Qu, Z. Peng, Two-dimensional doa estimation via a novel sparse array consisting of coprime and nested subarrays, *IEEE Communications Letters* 24 (2020) 1266–1270.
- [101] F. Sun, S. Ouyang, P. Lan, F. Li, Reduced dimensional 2-d doa estimation via least partial search with automatic pairing for parallel co-prime arrays*, in:

- 2020 IEEE 11th Sensor Array and Multichannel Signal Processing Workshop (SAM), 2020, pp. 1–5.
- [102] A. Agarwal, M. Agrawal, A. Kumar, Higher-order-statistics-based direction-of-arrival estimation of multiple wideband sources with single acoustic vector sensor, *IEEE Journal of Oceanic Engineering* 45 (2020) 1439–1449.
 - [103] K. Adhikari, B. Drozdenko, Symmetry-imposed rectangular coprime and nested arrays for direction of arrival estimation with multiple signal classification, *IEEE Access* 7 (2019) 153217–153229.
 - [104] H. Chen, C. Hou, Q. Wang, L. Huang, W. Yan, Cumulants-based toeplitz matrices reconstruction method for 2-d coherent doa estimation, *IEEE Sensors Journal* 14 (2014) 2824–2832.
 - [105] S. Liu, L. S. Yang, D. C. Wu, J. H. Huang, et al., Two-dimensional doa estimation using a co-prime symmetric cross array, *Progress In Electromagnetics Research* 54 (2014) 67–74.
 - [106] R. Rajamäki, V. Koivunen, Sparse active rectangular array with few closely spaced elements, *IEEE Signal Processing Letters* 25 (2018) 1820–1824.
 - [107] S. Sekizawa, Estimation of arrival directions using music algorithm with a planar array, in: *ICUPC '98. IEEE 1998 International Conference on Universal Personal Communications. Conference Proceedings (Cat. No.98TH8384)*, volume 1, 1998, pp. 555–559 vol.1.

- [108] Tie-Jun Shan, M. Wax, T. Kailath, On spatial smoothing for direction-of-arrival estimation of coherent signals, *IEEE Transactions on Acoustics, Speech, and Signal Processing* 33 (1985) 806–811.
- [109] Q. Y. Yin, R. W. Newcomb, L. H. Zou, Estimating 2-d angles of arrival via two parallel linear arrays, in: *International Conference on Acoustics, Speech, and Signal Processing*, 1989, pp. 2803–2806 vol.4. doi:10.1109/ICASSP.1989.267051.
- [110] Q. . Yin, R. W. Newcomb, S. Munjal, L. . Zou, Relation between the doa matrix method and the esprit method, in: *IEEE International Symposium on Circuits and Systems*, 1990, pp. 1561–1564 vol.2. doi:10.1109/ISCAS.1990.112432.
- [111] X. Dai, X. Zhang, Y. Wang, Extended doa-matrix method for doa estimation via two parallel linear arrays, *IEEE Communications Letters* 23 (2019) 1981–1984.
- [112] R. L. Haupt, *Antenna Array Basics*, IEEE, 2010, pp. 1–44.
- [113] M. Dogan, J. Mendel, APPLICATIONS OF CUMULANTS TO AWAY PROCESSING .1. APERTURE EXTENSION AND ARRAY CALIBRATION, *IEEE TRANSACTIONS ON SIGNAL PROCESSING* 43 (1995) 1200–1216.
- [114] T. E. Tuncer, T. K. Yasar, B. Friedlander, Chapter 4 - narrowband and wide-band doa estimation for uniform and nonuniform linear arrays, in: T. E. Tuncer, B. Friedlander (Eds.), *Classical and Modern Direction-of-Arrival Estimation*, Academic Press, Boston, 2009, pp. 125 – 160.

- [115] A. Ahmed, Y. D. Zhang, B. Himed, Effective nested array design for fourth-order cumulant-based doa estimation, in: 2017 IEEE Radar Conference (RadarConf), 2017, pp. 0998–1002.
- [116] Y. D. Zhang, M. G. Amin, B. Himed, Sparsity-based doa estimation using coprime arrays, in: 2013 IEEE International Conference on Acoustics, Speech and Signal Processing, 2013, pp. 3967–3971.
- [117] P. Ioannides, C. A. Balanis, Uniform circular and rectangular arrays for adaptive beamforming applications, *IEEE Antennas and Wireless Propagation Letters* 4 (2005) 351–354.
- [118] Y. Hua, T. K. Sarkar, D. D. Weiner, An l-shaped array for estimating 2-d directions of wave arrival, *IEEE Transactions on Antennas and Propagation* 39 (1991) 143–146.
- [119] N. Tayem, K. Majeed, A. A. Hussain, Two-dimensional doa estimation using cross-correlation matrix with l-shaped array, *IEEE Antennas and Wireless Propagation Letters* 15 (2016) 1077–1080.
- [120] X. Yang, Y. Wang, P. Chargé, Hole locations and a filling method for coprime planar arrays for doa estimation, *IEEE Communications Letters* (2020) 1–1.
- [121] M. Nazaroff, G. Byun, H. Choo, J. Shin, Y. Kim, 2-d direction-of-arrival estimation system using circular array with mutually coupled reference signal, *IEEE Sensors J.* 18 (2018) 9763–9769.

- [122] M. Burtowy, M. Rzymowski, L. Kulas, Low-profile ESPAR antenna for RSS-based DoA estimation in IoT applications, *IEEE Access* 7 (2019) 17403–17411.
- [123] Z. Zheng, Y. Yang, W.-Q. Wang, S. Zhang, Two-dimensional direction estimation of multiple signals using two parallel sparse linear arrays, *Signal Processing* 143 (2018) 112–121.
- [124] P. P. Vaidyanathan, P. Pal, Sparse sensing with coprime arrays, in: 2010 Conference Record of the Forty Fourth Asilomar Conference on Signals, Systems and Computers, 2010, pp. 1405–1409. doi:10.1109/ACSSC.2010.5757766.
- [125] P. P. Vaidyanathan, P. Pal, Direct-music on sparse arrays, in: 2012 International Conference on Signal Processing and Communications (SPCOM), 2012, pp. 1–5. doi:10.1109/SPCOM.2012.6289992.
- [126] J. He, M. N. S. Swamy, M. O. Ahmad, Efficient application of music algorithm under the coexistence of far-field and near-field sources, *IEEE Transactions on Signal Processing* 60 (2012) 2066–2070.
- [127] M. D. Zoltowski, M. Haardt, C. P. Mathews, Closed-form 2-d angle estimation with rectangular arrays in element space or beamspace via unitary esprit, *IEEE Transactions on Signal Processing* 44 (1996) 316–328.
- [128] Q. Shen, W. Liu, W. Cui, S. Wu, Extension of nested arrays with the fourth-order difference co-array enhancement, in: 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), 2016, pp. 2991–2995.

- [129] Q. Shen, W. Liu, W. Cui, S. Wu, P. Pal, Simplified and enhanced multiple level nested arrays exploiting high-order difference co-arrays, *IEEE Transactions on Signal Processing* 67 (2019) 3502–3515.
- [130] Z. Fu, P. Chargé, Y. Wang, Fourth order cumulant based active direction of arrival estimation using coprime arrays, in: *ICASSP 2020 - 2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2020, pp. 4547–4551.
- [131] J. Cai, W. Liu, R. Zong, Q. Shen, An expanding and shift scheme for constructing fourth-order difference coarrays, *IEEE Signal Processing Letters* 24 (2017) 480–484.
- [132] Z. Fu, P. Charge, Y. Wang, A virtual nested mimo array exploiting fourth order difference coarray, *IEEE Signal Processing Letters* 27 (2020) 1140–1144.
- [133] J. Brewer, Kronecker products and matrix calculus in system theory, *IEEE Transactions on circuits and systems* 25 (1978) 772–781.
- [134] S. R. Searle, *Matrix algebra*, Wiley StatsRef: Statistics Reference Online (2014).
- [135] H. Neudecker, Some theorems on matrix differentiation with special reference to kronecker matrix products, *Journal of the American Statistical Association* 64 (1969) 953–963.

VITAE

Name: Ibrahim Aboumahmoud

Email: *ibrahim3immm@gmail.com*

Address: *P.O. Box 8357 Dhahran 31261.*

Education

M.Sc. in Electrical Engineering, KFUPM, 2021.

B.Sc. in Electrical Engineering, KFUPM, 2019.

Related Publications

I. Aboumahmoud, A. Muqaibel, M. Alhassoun and S. Alawsh, “A Review of Sparse Sensor Arrays for Two-Dimensional Direction-of-Arrival Estimation,” in IEEE Access, vol. 9, pp. 92999-93017, 2021, doi: 10.1109/ACCESS.2021.3092529.