

PERSISTENT HOMOLOGY AND ITS APPLICATIONS

BY

HANEEN NADER BALLAN

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

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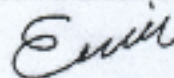
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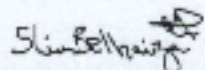
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This thesis, written by HANEEN NADER BALLAN under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICS.

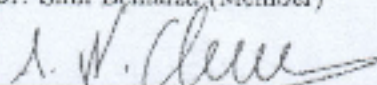
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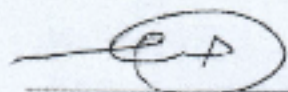
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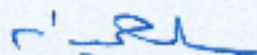
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Every challenging work needs diligence and self efforts as well as guidance of elders especially those who are very close to our hearts.

I dedicate my humble achievement to the apple of my eyes and my backbone my father and to the wonderful deeply missed mother, whose affection, love, encouragement, and prayers of day and night since years ago make me able to pursue my endeavors and get such honor.

Along with all hard working, respected, and professional instructors and professors.

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THESIS ABSTRACT

NAME: Haneen Nader Ballan
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One of the methods of topological data analysis is using persistence to differentiate the actual data from the noise. Finding persistent homology modules for simplicial complexes constructed out of a point cloud helps to specify the shape of the data. In this thesis, we keep track of the persistent homology modules by understanding and proving how for a given homological order the homology modules corresponding to different parameters are related. In addition, we show that persistent modules have the structure of a well known mathematical object which is a graded module over a polynomial ring when the coefficients ring is a PID. We also introduce pictorial representations of a persistent module with a brief discussion on using persistent homology in a real world application.

ملخص الرسالة

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يعتبر مبدأ الاستمرارية من إحدى الطرق المستخدمة في تحليل البيانات تبولوجيًا و فصل البيانات الحقيقية عن دونها بشكلٍ فعال، فحساب الموديولات المتعلقة به يساعد في الكشف عن شكل الفضاء الذي جُمعت منه البيانات. نتناول في هذه الرسالة دراسة استقصائية عن ذلك مع شيءٍ من التفصيل و البرهنة الجبرية لطبيعة تلك الموديولات، و كيف أنها تتكافأ مع الموديولات المتدرجة على حلقة كثيرات الحدود عندما تكون موديولات على حلقة مثاليات عظمى خالية من القواسم الصفرية. كما أننا سنقدم تمثيلات بيانية لتلك العناصر الجبرية، و من ثم نختم هذه الأوراق بعرض موجز حول واحد من التطبيقات المعاصرة المستخدمة في تحليل البيانات وفق هذا المنهج.

CHAPTER 1

INTRODUCTION

Topology is the mathematical discipline that studies shape of objects. Topological Data Analysis (TDA) is the application of this discipline to analyse complex data built on the ideologies “Data has a shape and that shape has a meaning”. TDA deploys mathematical, statistical, and algorithmic tools to explore the topological and the geometric structure of point clouds, i.e. finite set of points equipped with a distance function. TDA has two important branches Persistent Theory and Mapper. An unavoidable challenge in analyzing any data set is filtering out the meaningless data, known as the “noise”. For instance, improperly sampled data and outlier are noisy data. One of the helpful tools to isolate the actual data and features from the noise is the so called persistent homology, which we give a closer look throughout the thesis.

Similar to the other versions of homology, persistent homology compute the homological properties of a topological space and quantify the patterns such as connected components, loops, voids, and higher dimensional analogues of holes. Furthermore, it allows us to differentiate the real features from the noise by letting us deal with vari-

ous resolutions of the data set in a simultaneous way. To find the persistent homology modules, we should first create a filtration of the data. A filtration is a collection of simplicial complexes equipped with homomorphisms. There are many methods to create a filtration. Eventually, such a filtration turns to be the input of the algorithms used to calculate persistent homology modules.

This thesis is organized as follows. In Chapter 2, we remind some basic and important information from category theory and abstract and linear algebra. We also review some concepts from algebraic topology, such as simplicial complexes, filtrations, and simplicial homology. After that in Chapter 3, we present the notion of persistence and define the persistent homology. Moreover, we introduce persistence modules, a notion which is better suited to understand the persistent homology modules. We also give a detailed algebraic discussion of the persistence modules objects when the coefficients ring is a PID. Then, we talk about two popular pictorial presentations for the persistence modules. Finally, in Chapter 4, we use the persistent homology to shed light on the analysis of the brain arteries lengths of human objects.

CHAPTER 2

BACKGROUND INFORMATION

2.1 Categorical and Algebraic Background

Commutative diagrams, especially those indexed by posets, have an essential role in topological data analysis. Category theory provides a very appropriate language for this, and we will often be using that language in this thesis. In this section we cover the very basics of category theory adopted from [1], [2], and [3]. We also emphasize that throughout this thesis, and unless otherwise explicitly mentioned, R is a commutative ring with unity.

Definition 2.1 *A partially ordered set (or a poset for short) is a set P with a binary relation $\leq$ (the partial order) such that*

- $x \leq x$ (reflexivity),
- $x \leq y$ and $y \leq x$ implies $x = y$ (anti-symmetry),
- $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity).

Example 2.1 (i) We know that a poset with the property that any two elements are comparable is a totally ordered set. In fact every totally ordered set is a poset, thus $\mathbb{Z}, \mathbb{N}$, and $\mathbb{R}$ are all posets.

(ii) For a set A and B is any subset of A , $(B, \subseteq)$ is a poset. A concrete case of this which appears repeatedly in mathematics, in particular in TDA theory, is taking B to the set of open sets of a topological space A .

Definition 2.2 A category $\mathcal{C}$ consists of:

- a collection of objects $\text{Ob } \mathcal{C}$,
- a set of morphisms $\text{hom}(x, y)$ for every $x, y \in \text{Ob } \mathcal{C}$,
- a composition rule for morphisms, which for each $f \in \text{hom}(x, y)$ and $g \in \text{hom}(y, z)$, specifies a morphism $g \circ f \in \text{hom}(x, z)$,
- for each $x \in \text{Ob } \mathcal{C}$, a distinguished morphism $\text{Id}_x \in \text{hom}(x, x)$, called the identity.

These must satisfy the following axioms:

- composition is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$,
- $f \circ \text{Id}_x = f$ and $\text{Id}_y \circ g = g$ whenever such composition is defined.

Example 2.2 The following categories: **Set**, **Top**, **Grp**, **R-Mod**, and **K-Vect** have objects of all sets, all topological spaces, all groups, all R -modules, for some fixed ring R , and all K -vector spaces, for some fixed field K , respectively and morphisms of

all functions, all continuous maps, all group homomorphisms, all module homomorphisms, and all linear transformations respectively.

Definition 2.3 A morphism $f : x \rightarrow y$ in a category $\mathcal{C}$ is called an isomorphism if f has an inverse, i.e., a morphism $g : y \rightarrow x$ such that

$$g \circ f = \text{Id}_x \quad \text{and} \quad f \circ g = \text{Id}_y.$$

If there exists an isomorphism $f : x \rightarrow y$, we write $x \cong y$.

Definition 2.4 Given categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- An object $F(x) \in \text{Ob } \mathcal{D}$ for every $x \in \text{Ob } \mathcal{C}$,
- A morphism $F(\gamma) \in \text{hom}(F(x), F(y))$ for each $\gamma \in \text{hom}(x, y)$.

such that

- F respects the composition operation in $\mathcal{C}$ and $\mathcal{D}$, i.e., $F(f \circ g) = F(f) \circ F(g)$,
- F maps identity morphisms to identity morphisms, i.e., $F(\text{Id}_x) = \text{Id}_{F(x)}$ for all $x \in \text{Ob } \mathcal{C}$.

Example 2.3 (i) As we will be observing in the forthcoming sections, for any $i \geq 0$,

i -th singular homology with coefficients in $\mathbb{Z}$ is a functor $H_i : \mathbf{Top} \rightarrow \mathbf{Grp}$.

(ii) Homology with coefficients in a field K is a functor $\mathbf{Top} \rightarrow K\text{-Vect}$.

(iii) The so called “forgetful functor” $\mathcal{F}$, is a functor that sends a rich input to a poorer output, i.e., it drops or ignores some or all of the domain’s structure properties before sending it to the range. It is also called a “stripping functor”. For instance,

$\mathcal{F} : \mathbf{Grp} \rightarrow \mathbf{Set}$ is a forgetful functor, where it forgets the group structure and sends every group to its underlying set and to every group homomorphism assigns the same function considered as a function between the underlying sets. In a similar way we can define a forgetful functor traveling from commutative unitary rings to abelian groups, by dropping the multiplication and the unit element.

Definition 2.5 Given two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation

$N : F \rightarrow G$ is choice of morphism $N_x : F_x \rightarrow G_x$ for each $x \in \text{Ob } \mathcal{C}$, such that these morphisms commute with those $\mathcal{C}$ and $\mathcal{D}$. That is, for all morphisms $\gamma : x \rightarrow y$ in $\text{hom}(\mathcal{C})$, the following diagram commutes:

$$\begin{array}{ccc} F_x & \xrightarrow{F(\gamma)} & F_y \\ N_x \downarrow & & \downarrow N_y \\ G_x & \xrightarrow{G(\gamma)} & G_y \end{array}$$

Example 2.4 A natural transformation $N : F \rightarrow G$ of functors $F, G : \mathbb{N} \rightarrow \mathbf{Top}$ is exactly the data of continuous maps

$$\{N_i : F_i \rightarrow G_i\}_{i \in \mathbb{N}}$$

making the following diagram commute:

$$\begin{array}{ccccccc} F : & F_0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow \dots \\ & N_0 \downarrow & & N_1 \downarrow & & N_2 \downarrow & \\ G : & G_0 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow \dots \end{array}$$

Definition 2.6 A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence if there exists a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that

$$G \circ F \simeq \text{Id}_{\mathcal{C}} \quad \text{and} \quad F \circ G \simeq \text{Id}_{\mathcal{D}}$$

An isomorphism of categories is an invertible functor. One often would like to say that two categories have essentially the same structure. For this, the notion of isomorphism is sometimes too firm to be used. The above notion of category equivalence is weaker but more common.

Proposition 2.1 *A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ produces an isomorphism of categories if and only if it is bijective on the level of objects and on the set of morphisms.*

The acceptance of the above proposition comes from the ability of avoiding the construction of the inverse functor.

Definition 2.7 *A homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Continuous functions f and g are said to be homotopic if and only if there is a homotopy H taking f to g as described above. Furthermore, being homotopic is an equivalence relation on the set of all continuous functions from X to Y .*

Definition 2.8 *Given two topological spaces X and Y , a homotopy equivalence between X and Y is a pair of continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$, such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y . If such a pair exists, then X and Y are said to be homotopy equivalent, or of the same homotopy type.*

Informally, we say that two spaces X and Y are of the same homotopy type if we can transform one into another by shrinking, bending, and expanding operations. Spaces that are homotopy equivalent to a point are called *contractible*.

Definition 2.9 *A submonoid $\mathbb{A} \subseteq \mathbb{R}_+$ is said to be pure if given any r_0, r_1 , and r_2 in $\mathbb{R}_+$, with r_0 and r_2 in $\mathbb{A}$, so that $r_0 + r_1 = r_2$, then $r_1 \in \mathbb{A}$. In fact, $\mathbb{A}$ is a totally ordered set in its own right by restriction of the total order on $\mathbb{R}_+$.*

Example 2.5 $\mathbb{N}$ (the non-negative integers) and $\mathbb{Q}_+$ are both pure.

2.2 Algebra and the Structure Theorem

In this section, we give a brief review of some essential algebraic aspects necessarily for our later work in the thesis. We skip mentioning the algebraic basics such as, groups, rings, fields, modules, vector spaces, ideals, polynomial rings, integral domains, principle ideal domains (PIDs), since the reader is supposed to be familiar with. We will recall the structure of finitely generated modules over PIDs, from [4], provided with its proof after we introduce some needed notions as the graded rings and graded modules.

Definition 2.10 *A graded ring is a ring $\langle R, +, \otimes \rangle$ equipped with a direct sum decomposition of Abelian groups $R \cong \bigoplus_i R_i, i \in \mathbb{Z}$, so that the multiplication is defined by bilinear pairings $R_n \otimes R_m \rightarrow R_{n+m}$. Elements in a single R_i are homogeneous and have degree i . The direct sum decomposition is referred to as the grading of R . A graded ring is non-negatively graded if $R_i = 0$ for all $i < 0$.*

The graded module is defined in a similar way.

Definition 2.11 *A graded module M over a graded ring R is a module equipped with a direct sum decomposition of Abelian groups $M \cong \bigoplus_i M_i, i \in \mathbb{Z}$, so that the action of R on M is defined by bilinear pairings $R_n \otimes M_m \rightarrow M_{n+m}$. Each M_i is also considered as a module over R_0 . Elements in a single M_i are homogeneous and have degree i . The direct sum decomposition is referred to as the grading of M . A module is non-negatively graded if $M_i = 0$ for all $i < 0$.*

In fact, all graded modules over a graded ring R form a category denoted by **R -Gr-Mod**, where a morphism $f : M = \bigoplus_i M_i \rightarrow N = \bigoplus_i N_i$ of graded modules over R is a family of R -module homomorphisms $(f_i)_{i \geq 0}$, where each f_i is just the restriction of f to M_i and compatible with φ_i and ψ_i , the module homomorphisms in the collections $(M_i)_{i \geq 0}$ and $(N_i)_{i \geq 0}$ respectively, for all $i \geq 0$, so that the following diagram commutes.

$$\begin{array}{ccccccccc}
M_0 & \xrightarrow{\varphi_0} & M_1 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_{i-1}} & M_i & \xrightarrow{\varphi_i} & M_{i+1} & \xrightarrow{\varphi_{i+1}} & \cdots \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow & & \downarrow f_i & & \downarrow f_{i+1} & & \\
N_0 & \xrightarrow{\psi_0} & N_1 & \xrightarrow{\psi_1} & \cdots & \xrightarrow{\psi_i} & N_i & \xrightarrow{\psi_i} & N_{i+1} & \xrightarrow{\psi_{i+1}} & \cdots
\end{array}$$

By the definition of the direct sum, any $m \in M$ can be written as $m = \sum_{j=1}^l m_j$ for some homogeneous elements $m_1, m_2, \dots, m_l$. A homogeneous submodule of M is one generated by a set of homogeneous elements.

In general, the decomposition may not be finite. For our purposes we are only concerned with those gradings that are bounded from below.

Example 2.6 *Let $R[t]$ be a polynomial ring over R with indeterminate t . $R[t]$ can be*

graded non-negatively as $R[t] = \bigoplus_{n \geq 0} (t^n) = R \oplus Rt \oplus Rt^2 \oplus \dots \oplus Rt^n \oplus \dots$. This is called the standard grading for $R[t]$.

Definition 2.12 An ideal of the graded ring $R \cong \bigoplus_i R_i$ is a graded (homogeneous) ideal of R if and only if it is generated by homogeneous elements.

Definition 2.13 A graded ring $R \cong \bigoplus_i R_i$ is said to be a graded principal ideal domain if and only if every graded ideal is generated by a single homogeneous element.

To introduce our main theorem of this section (the Structure Theorem for Finitely Generated Modules over a Principal Ideal Domain), we have to give the following few definitions and facts.

Definition 2.14 A module M over a PID R is finitely generated if there exists a generating set $\{m_1, m_2, \dots, m_n\}$ in M such that for any $x \in M$, there exist scalars $r_1, r_2, \dots, r_n \in R$ in which $x = r_1 m_1 + r_2 m_2 + \dots + r_n m_n$.

Definition 2.15 An R -module M is free (or free on a subset A of M) if for all $x \in M$ which are nonzero, there exists unique $r_1, r_2, \dots, r_n \in R$, which are all nonzero, and there exists unique linearly independent elements $a_1, a_2, \dots, a_n \in A$ such that $x = r_1 a_1 + r_2 a_2 + \dots + r_n a_n$ holds for $n \in \mathbb{Z}^+$. A that forms a set of linearly independent generators in M is called a basis.

From linear algebra, we know that all bases of a vector space have the same number of elements. However, this property is not valid in general for modules unless if the coefficients ring is commutative, which is the situation throughout this thesis. The following proposition from [5] ensures this fact.

Proposition 2.2 *Let M be a free left R -module. If R is commutative, then all bases of M have the same number of elements. This number is called the rank of M .*

Free modules can have infinite basis as well. Discussing these types of modules is beyond the scope of this thesis, more details about them can be found in [5].

Definition 2.16 *Let M' be a submodule of M . The annihilator of M' in R is the set*

$$\text{Ann}_R(M') := \{r \in R \mid rn = 0 \text{ for all } n \in M'\}.$$

In particular, the annihilator of M' is a two-sided ideal in R .

Definition 2.17 *For an R -module M , the set of torsion elements in M is defined as*

$$\text{Tor}(M) := \{m \in M \mid \exists x \in R \setminus 0 \text{ s.t. } xm = 0_M\}.$$

Proposition 2.3 *If R is an integral domain, then $\text{Tor}(M)$ is a submodule of M .*

Proof. To show that $\text{Tor}(M)$ is a submodule of M , and by the submodule criterion, we have to show the following: (i) $\emptyset \neq \text{Tor}(M) \subseteq M$ and (ii) $m_1 + rm_2 \in \text{Tor}(M)$, for all $r \in R$ and all $m_1, m_2 \in \text{Tor}(M)$.

Since R is an integral domain, then $1 \neq 0$ and $1 \cdot 0 = 0$ and so, $\text{Tor}(M)$ is not empty subset of M .

Let $m_1, m_2 \in \text{Tor}(M)$, then there exist non-zero scalars $r_1, r_2 \in R$ such that $r_1 m_1 = r_2 m_2 = 0$. For any $r \in R$, we have that

$$r_1 r_2 (m_1 + r m_2) = r_2 (r_1 m_1) + r_1 r (r_2 m_2) = 0 + 0 = 0, \text{ hence } m_1 + r m_2 \in \text{Tor}(M).$$

Therefore, as we have verified conditions (i) and (ii), we get that $\text{Tor}(M)$ is a submodule of M . ■

Definition 2.18 *A module M is a torsion module if $\text{Tor}(M) = M$.*

Theorem 2.1 *Let R be a PID and M be a finitely generated free R -module of rank n . Then every submodule of M is also free of rank $\leq n$.*

Proof. Let $\{u_1, u_2, \dots, u_n\}$ be a basis of M and so, $M \cong R^n$. By induction on the rank n , we have the following:

- When $n = 1$, then we have that $M \cong R$. The submodules of R are its ideals and as R is a PID, every ideal is principle and so it is free of rank at most one. Hence, the theorem holds for $n = 1$.
- We assume that the theorem is valid for all submodules of rank $\leq n - 1$. Let M' be a submodule of M . When $M' = \{0\}$ the theorem holds immediately. If $M' \neq \{0\}$, let us consider the projection maps $\pi_i : M = R^n \rightarrow R$, for all $1 \leq i \leq n$. The standard basis B_i for $\ker \pi_i$ is $B_i = \{e_1, e_2, \dots, \hat{e}_i, \dots, e_n\}$, where $\hat{e}_i$ is omitted from the basis. We know that $\ker \pi_i$ is an R -module and it is clearly of rank $n - 1$. Since $M' \neq \{0\}$ then $\pi_i(M') \neq \{0\}$ in R for at least one i . So, $\pi_i(M')$ is a non-zero submodule (ideal) in the PID R , hence it is a free R -module of rank 1.

We have also that $\ker \pi_i \cap M'$ is a submodule of $\ker \pi_i$, thus by the induction assumption we have that, $\text{rank } \ker \pi_i \cap M' \leq \text{rank } \ker \pi_i = n - 1$.

- Now, we let $(m) = \pi_i(M')$, i.e. m is a generator of $\pi_i(M')$. Let $w \in M'$ such that $\pi_i(w) = m$. Next we can show that M' can be decomposed as

$M' = \ker \pi_i \cap M' \oplus wR$ and for this we refer to the proof of Lemma 2.1. Next, we suppose that $\{w_1, w_2, \dots, w_s\}$ is a basis of $\ker \pi_i \cap M'$, where $s \leq n-1$. Then, from the above decomposition $\{w_1, w_2, \dots, w_s, w = w_{s+1}\}$ is a basis of M' , and so this implies that

$\text{rank } M' = s+1 \leq n$. Therefore, the theorem holds for any n and this concludes the proof.

■

Corollary 2.1 *Let R be a PID, M is a finitely generated free R -module of rank n , and N is a submodule of M of rank m . Then $\text{rank } M/N = n - m$.*

Lemma 2.1 *Let R be a PID, let M be a free R -module of finite rank n and let M' be a submodule of M . Then, there exists a basis $\{y_1, y_2, \dots, y_n\}$ of M such that $\{a_1 y_1, a_2 y_2, \dots, a_m y_m\}$ is a basis of M' where $a_1, a_2, \dots, a_n$ are elements of R with the divisibility relation $a_1 | a_2 | \dots | a_n$. These bases are called aligned bases.*

We give a proof of the above Lemma as analogous to the one mentioned in [6].

Proof. We assume that $M' \neq 0$ and $\varphi : M \rightarrow R$ is an R -module homomorphism. Then, we know that this implies that $\varphi(M')$ is an R -submodule of $\varphi(M)$ and an ideal of R . Since, R is a PID and so, every ideal in it is principle, then there exist $a_\varphi \in R$ such that $\varphi(M') = (a_\varphi)$.

We define the following set $S := \{\varphi(M') = (a_\varphi) \mid \varphi \in \text{Hom}_R(M, R)\}$ to be a set

of certain ideals in R , more precisely S is the set all principal ideals of all R -module homomorphisms from M to R . As $0 \in S$ when φ is the zero homomorphism, $S \neq \emptyset$.

From a previous proposition in Algebra, we know that when R is a PID, every nonempty set of ideals of R has a maximal element. Hence, this holds in our case and so, let $(a_f) = f(M')$ be the maximal element of S , for some $f \in \text{Hom}_R(M, R)$. Let us denote this maximal a_f by a_1 , and since homomorphisms are determined by their generators, then there exist m_1 in the generating set of M' such that $f(m_1) = a_1$, i.e. a generator of M' is sent to a generator a_1 .

Next, we want to show that $a_1 \neq 0$. Let us consider $\{u_1, u_2, \dots, u_n\}$ to be any basis of M and we take any natural projection R -module homomorphism π_i such that $\pi_i(u_1, u_2, \dots, u_i, \dots, u_n) = u_i$. As we assumed that $M' \neq 0$, there should be at least one i such that $\pi_i(M') \neq 0$. Since a_1 is maximal and S has more than the trivial ideal, a_1 is nonzero.

Now, we want to show that for any $\psi \in \text{Hom}_R(M, R)$, $a_1 | \psi(m_1)$. We suppose $b := \psi(m_1) \in R$. Since R is a PID, then there exists $r \in R : (r) = (a_1, b)$ and so, there exist $x, y \in R$ such that $r = xa_1 + yb$, thus

$$r = xf(m_1) + y\psi(m_1) = (xf + y\psi)(m_1) \in \text{Im}(xf + y\psi) \text{ for any } m_1 \in M'. \text{ So,}$$

$$(a_1) \subset (r) \subset (xf + y\psi)(M') \in S. \text{ As } (a_1) \text{ is a maximal element of } S, (a_1) = (r) = (a_1, b)$$
implies that $b \in (a_1)$ and therefore, $a_1 | b = \psi(m_1)$.

We pick again $\{u_1, u_2, \dots, u_n\}$, any basis of M . Then there exist scalars $c_1, c_2, \dots, c_n \in R$ such that $m_1 = c_1u_1 + c_2u_2 + \dots + c_nu_n = \sum_{i=1}^n c_iu_i$. Now, we define the i -th coordinate function $\psi_i : M \rightarrow R$, such that $m \mapsto c_i$ (the i -th

coordinate in $\{u_1, u_2, \dots, u_n\}$ for any $m \in M$. Hence, from what we proved above we have that $a_1|\psi_i(m_1) = c_i$ and so, we can choose elements $b_1, b_2, \dots, b_n \in R$ such that $c_i = b_i a_1$. Then, we can substitute the new values of the c_i 's in the basis decomposition of m_1 and so, $m_1 = a_1(\sum_{i=1}^n b_i u_i)$. We let $y_1 := \sum_{i=1}^n b_i u_i$, it follows that $m_1 = a_1 y_1$. We realize that $a_1 f(y_1) = f(m_1) = a_1$, and since both sides are elements of a free rank one module R which is a domain so there are no zero divisors and thus, we can cancel a_1 out to get that $f(y_1) = 1$.

Next, in order to show that y_1 is a basis element of M and $a_1 y_1$ is a basis element of M' , we first show that:

$$(i) \quad M = Ry_1 \oplus \ker f,$$

$$(ii) \quad M' = Ra_1 y_1 \oplus M' \cap \ker f.$$

To prove (i), we have first to show that $M \subseteq Ry_1 + \ker f$, as the other inclusion holds directly since Ry_1 and $\ker f$ are submodules of M . So, we let $u \in M$ such that $u = f(u)y_1 + (u - y_1 f(u))$. We have that $f(u - y_1 f(u)) = f(u) - f(y_1)f(u) = f(u) - f(u) = 0$, since $f(y_1) = 1$, and so, $u - y_1 f(u) \in \ker f$. Hence, any element in M can be expressed as a sum of elements in Ry_1 and $\ker f$.

Second, we show that $Ry_1 \cap \ker f = \{0\}$. Let us assume that there is an element ry_1 of Ry_1 which is also in $\ker f$. Then $f(ry_1) = rf(y_1) = r.1 = 0$ implies that $r = 0$ and so, an element ry_1 will be in the intersection of Ry_1 and $\ker f$ when $r = 0$, which means that the only element in the intersection is the zero element. Therefore, part

(i) is verified.

Similarly we prove (ii). As we have assumed previously that $(a_1) = f(M')$, an element $f(u')$ is generated by a_1 for any $u' \in M'$. Thus, $f(u') = ca_1$ for some $c_1 \in R$.

Now, as we did before, we let $u' = f(u')y_1 + u' - f(u')y_1 = ca_1y_1 + (u' - ca_1y_1)$.

Since, $f(u' - ca_1y_1)$ is $f(u') - ca_1f(y_1) = ca_1 - ca_1(1) = 0$, then

$u' - ca_1y_1 \in M' \cap \ker f$.

Lastly, we show that $Ra_1y_1 \cap (M' \cap \ker f) = \{0\}$. As what we have done in the first claim, we suppose that an arbitrary element ra_1y_1 of Ra_1y_1 is also in $M' \cap \ker f$. Then, $f(ra_1y_1) = ra_1f(y_1) = ra_1 = 0$, and as we are working over a PID R and we showed that a_1 is non-zero, we have that $r = 0$.

Hence, part (ii) holds.

Since Ry_1 is a free module of rank 1, then by claim (i) and as $\ker f$ is a submodule of the free module M over a PID R , $\ker f$ must be free and of rank $n - 1$. We let $M'_1 = M' \cap \ker f$ and $M_1 = \ker f$ and so we get in the situation where $M'_1 \subset M_1$. Then, we can repeat all the previous process again and at some point we will arrive to the case $M'_m \subset M_m$ with $M'_m = 0$ because we assumed that M' is of rank m and when $M'_m = 0$ then we will already have an aligned basis since the basis of M'_m is empty. By induction argument, we can argue that M'_1 and M_1 have aligned bases and therefore, $Ra_1y_1 \oplus M'_1 \subset Ry_1 \oplus M_1$. So, these direct sums have also aligned bases by adding a_1y_1 and y_1 in front of the bases of M'_1 and M_1 respectively,

i.e. there exists a basis $\{y_2, y_3, \dots, y_n\} \in M_1$ such that $\{a_2y_2, a_3y_3, \dots, a_ny_n\}$ is a basis of M'_1 where $a_2, \dots, a_n \in R$ under the assumption that $a_2 | a_3 | \dots | a_n$. Hence, it follows from dimension counting of the direct sums in (i) and (ii) that $\{y_1, y_2, \dots, y_n\}$ is a basis for M and $\{a_1y_1, a_2y_2, \dots, a_ny_n\}$ is a basis of M' .

We conclude the proof by showing that $a_1 | a_2$. Let us assume that

$\varphi(y_1) = \varphi(y_2) = 1$ and $\varphi(y_i)$ is 0 for any y_i with $i > 2$ in the basis of M for a homomorphism $\varphi \in \text{Hom}_R(M, R)$.

Then we have that $\varphi(a_1y_1) = a_1\varphi(y_1) = a_1(1) = a_1$ so $a_1 \in \varphi(M')$. Similarly, we have that $\varphi(a_2y_2) = a_2\varphi(y_2) = a_2 \cdot 1 = a_2$ so, $a_2 \in \varphi(M')$. This implies, by the definition of a generator, that both of (a_1) and (a_2) are submodules of M' . As (a_1) is a maximal element of S , $(a_2) \subseteq (a_1)$ and hence, $a_2 \in (a_1)$ which follows that $a_1 | a_2$. ■

Lemma 2.2 *Let R be any ring, let $A_1, A_2, \dots, A_n$ be R -modules. In addition, suppose that B_i is a submodule of A_i , $1 \leq i \leq n$. Then*

$$(A_1 \oplus A_2 \oplus \dots \oplus A_n) / (B_1 \oplus B_2 \oplus \dots \oplus B_n) \cong (A_1/B_1) \oplus (A_2/B_2) \oplus \dots \oplus (A_n/B_n).$$

Note that finite direct sums are the same as finite direct products.

Proof. Let us define the following map

$$\Phi : (A_1 \oplus A_2 \oplus \dots \oplus A_n) / (B_1 \oplus B_2 \oplus \dots \oplus B_n) \rightarrow (A_1/B_1) \oplus (A_2/B_2) \oplus \dots \oplus (A_n/B_n),$$

such that $\Phi([a_1 + a_2 + \dots + a_n]) = [a_1] \oplus [a_2] \oplus \dots \oplus [a_n]$ and $[a_i] \in A_i/B_i$. In order to prove our lemma, we have to show that Φ is a well-defined isomorphism. Let us

suppose that $b_1 + b_2 + \dots + b_n = a_1 + a_2 + \dots + a_n - a_{n+1} + \dots + a_{2n}$, then by the uniqueness of the direct sum we have that $b_i = a_i - a_{n+i}$ for all $1 \leq i \leq n$. Thus, Φ is well-defined.

Now, we prove that Φ is an R -module homomorphism as follows:

- For any $\alpha, \beta \in (A_1 \oplus A_2 \oplus \dots \oplus A_n) / (B_1 \oplus B_2 \oplus \dots \oplus B_n)$, such that

$$\alpha = [\oplus_{i=1}^n a_i], \beta = [\oplus_{i=1}^n a'_i] \text{ where}$$

$$[\oplus_{i=1}^n a_i], [\oplus_{i=1}^n a'_i] \in (A_1 \oplus A_2 \oplus \dots \oplus A_n) / (B_1 \oplus B_2 \oplus \dots \oplus B_n), \text{ we have that}$$

$$\Phi(\alpha + \beta) = \Phi([\oplus_{i=1}^n (a_i + a'_i)]) = \oplus_{i=1}^n [a_i + a'_i]$$

$$= \oplus_{i=1}^n [a_i] + \oplus_{i=1}^n [a'_i] = \Phi([\oplus_{i=1}^n a_i]) + \Phi([\oplus_{i=1}^n a'_i]) = \Phi(\alpha) + \Phi(\beta).$$

- For any $c \in R$, $\alpha = [\oplus_{i=1}^n a_i] \in (A_1 \oplus A_2 \oplus \dots \oplus A_n) / (B_1 \oplus B_2 \oplus \dots \oplus B_n)$. We have that

$$\Phi(c\alpha) = \Phi([\oplus_{i=1}^n ca_i]) = c \oplus_{i=1}^n [a_i] \text{ from the multiplicative property of an equivalence class, we have that}$$

$$c \oplus_{i=1}^n [a_i] = c \Phi([\oplus_{i=1}^n a_i]) = c \Phi(\alpha).$$

Finally, we show that Φ is an R -module isomorphism by showing that it is a bijection.

It is clear that for any $y = [a_1] \oplus [a_2] \oplus \dots \oplus [a_n] \in \oplus_{i=1}^n A_i / B_i$, there exists

$x = [a_1 \oplus a_2 \oplus \dots \oplus a_n] \in \frac{\oplus_{i=1}^n A_i}{\oplus_{i=1}^n B_i}$ such that $\Phi(x) = y$ and hence, Φ is surjective. Also,

we can easily check that that this homomorphism has a kernel of $\oplus_{i=1}^n B_i$.

Next we prove that Φ is injective. Let $x = [\oplus_{i=1}^n a_i] \in \ker \Phi$, then $\Phi([\oplus_{i=1}^n a_i]) = [0]$ and by the definition of an equivalence class, we have that $\oplus_{i=1}^n a_i \in \oplus_{i=1}^n B_i$ and so, $[\oplus_{i=1}^n a_i] = [0]$. It follows that Φ is injective. Therefore, Φ is an R -module isomorphism and the lemma is proved. ■

Definition 2.19 *An R -module M is called cyclic if M can be generated by a single element, i.e. $M = (x) = Rx = \{rx \mid r \in R\}$ for some x in M .*

These prerequisites allow us to state the structure theorem for finitely generated modules over a principal ideal domain.

Theorem 2.2 (Structure Theorem) . *Let R be a principal ideal domain and let M be a finitely generated R -module. Then,*

(a) $M \cong (\text{free finite}) \oplus \text{Tor}(M)$. More precisely,

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m),$$

for some integer $r \geq 0$ and nonzero elements $a_1, a_2, \dots, a_m$ of R which are not units in R satisfy the divisibility relations $a_1 \mid a_2 \mid \cdots \mid a_{m-1} \mid a_m$.

(b) The decomposition in part (a) is unique, in the following sense: Suppose

$$M \cong R^\ell \oplus R/(b_1) \oplus R/(b_2) \oplus \cdots \oplus R/(b_t),$$

where the $b_1, b_2, \dots, b_t$ are nonzero, nonunit elements of R satisfy the divisibility relations $b_1 \mid b_2 \mid \cdots \mid b_{m-1} \mid b_m$. Then $m = t, r = \ell$ and $(a_i) = (b_i)$ for all i .

Proof. We prove part (a). By assumption M is a finitely generated R -module, so let $\{x_1, x_2, \dots, x_n\}$ be our set of generators with minimal cardinality n , and R^n has a free R -module structure of rank n with basis vectors (the standard basis) $e_1, e_2, \dots, e_n$, where e_i has the unit element of R in the i -th component and zero elsewhere, for all $1 \leq i \leq n$. In order to show that M is isomorphic to the direct sum on the right hand side, we have to construct a homomorphism, say φ from R^n to M . To define φ , it is enough to define it on the generators, so we let $\varphi(e_i) = x_i$ for all $1 \leq i \leq n$. In fact the rule of φ does not depend on how we write v for any v in R^n , to check this let us assume that there exist two collections of scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\beta_1, \beta_2, \dots, \beta_n$ in R such that $v = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n = \beta_1 e_1 + \beta_2 e_2 + \dots + \beta_n e_n$. As R^n is free, $e_1, e_2, \dots, e_n$ are linearly independent, hence $\alpha_i - \beta_i = 0$ for all $1 \leq i \leq n$.

Therefore, $\varphi(\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n) = \alpha_1 \varphi(e_1) + \alpha_2 \varphi(e_2) + \dots + \alpha_n \varphi(e_n) = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$ and $\varphi(\beta_1 e_1 + \beta_2 e_2 + \dots + \beta_n e_n) = \beta_1 \varphi(e_1) + \beta_2 \varphi(e_2) + \dots + \beta_n \varphi(e_n) = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n$ coincide, and so φ is a well-defined R -module homomorphism.

Now, by the First Isomorphism Theorem of modules we have that

$R^n / \ker(\varphi) \cong \text{Im}(\varphi)$. However, since any element of M can be written as a linear combination of its generators, it follows that φ is surjective. Hence, $R^n / \ker(\varphi) \cong M$.

As $\ker(\varphi)$ is a submodule of R^n , then by Lemma 2.1, there exists a basis $u_1, u_2, \dots, u_n$ for R^n such that $a_1 u_1, a_2 u_2, \dots, a_n u_n$ is a basis for $\ker(\varphi)$ where $a_1, a_2, \dots, a_n$ are scalars in R such that $a_1 | a_2 | \dots | a_{n-1} | a_n$.

Since R^n can be written as a direct sum of modules of rank 1, i.e. $R^n = \bigoplus_{i=1}^n Ru_i$, and similarly $\ker(\varphi) = \bigoplus_{i=1}^n Ra_i u_i$. Then, the above isomorphism implies that

$$M \cong R^n / \ker(\varphi) \cong \frac{Ru_1 \oplus Ru_2 \oplus \cdots \oplus Ru_n}{Ra_1 u_1 \oplus Ra_2 u_2 \oplus \cdots \oplus Ra_m u_m}.$$

It follows that

$$M \cong \bigoplus_{i=1}^m \frac{Ru_i}{Ra_i u_i} \oplus Ru_{m+1} \oplus \cdots \oplus Ru_n \cong \bigoplus_{i=1}^m \frac{Ru_i}{Ra_i u_i} \oplus R^{n-m}.$$

Thus, R^{n-m} is the free module of the direct sum. And since the u_i 's generate R^n , then $\frac{Ru_i}{Ra_i u_i} \cong R/(a_i)$.

Therefore, we get that $M \cong R^r \oplus_{i=1}^m R/(a_i)$, where $r = n - m$ is a non-negative integer. We note that if any of the a_i 's are arbitrary units, then the ideal generated by any of those units is the whole ring R , hence $R/(a_i)$ is the trivial group, and so we can remove all of the terms where the a_i 's are units in the direct sum. ■

A proof of part (b) can be found in [7].

Proposition 2.4 *The integer r in the Structure Theorem is the rank of M .*

From the Structure Theorem, it follows that M is torsion free if and only if M is free. Also, from the definition of $\text{Tor}(M)$, realize that

$$\text{Tor}(M) \cong R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m).$$

Definition 2.20 *The elements $a_1, a_2, \dots, a_m \in R$ (defined up to multiplication by units in R) are the torsion coefficients of M .*

In Chapter 2, we will define the persistence module for a filtered simplicial complex K and discuss how to give it the structure of a graded module over the graded ring $R[t]$. The following theorem is a generalized version of Theorem 2.2.

Theorem 2.3 *Let M be graded module over a graded PID R . Then it decomposes uniquely to form*

$$M \cong \left(\bigoplus_{i=1}^n \Sigma^{\alpha_i} R \right) \oplus \left(\bigoplus_{j=1}^m \Sigma^{\gamma_j} R / (a_j) \right),$$

where $a_j \in R$ are homogeneous elements so that $a_j \mid a_{j+1}$, $\alpha_i, \gamma_j \in \mathbb{Z}$, and Σ^α denotes an α -shift upward in the grading.

As before, the first term is the free part, and the second term represents the torsional part, where in the graded case the torsional elements are also homogeneous.

2.3 Simplicial Complexes

Let $S = \{v_0, \dots, v_k\}$ be $k + 1$ points in $\mathbb{R}^d$, and let $\alpha_0, \dots, \alpha_k \in \mathbb{R}$.

Definition 2.21 *A point $x = \sum_{i=0}^k \alpha_i v_i$ is a linear combination of S . An affine combination of S is a linear combination with $\sum_{i=0}^k \alpha_i = 1$. A convex combination of S is an affine combination with $\alpha_i \geq 0$ for all i . The convex hull of S is the set of all convex combinations of S .*

Definition 2.22 *The points $v_0, \dots, v_k$ are affinely independent if no v_i is an affine combination of the other points in the set.*

Definition 2.23 *A k -simplex σ is the convex hull spanned by $S = \{v_0, \dots, v_k\}$, where $v_0, \dots, v_k$ are affinely independent. The points of S are the vertices of σ . Its dimension is $\dim \sigma = k$.*

We frequently use simplices to study shapes. For the first few dimensions, special names are used: a 0-simplex is referred to as a vertex, a 1-simplex is referred to as an edge, a 2-simplex is referred to as a triangle, and a 3-simplex is referred to as a tetrahedron and so on of higher dimensional versions of the triangle. Generally a k -simplex it is a $k + 1$ fully-connected data points.

To be consistent with Definition 2.23, we use (-1) -simplex to describe an empty set.

Definition 2.24 *Let σ be a k -simplex defined by its set of vertices $S = \{v_0, \dots, v_k\}$. If a simplex τ is defined by its set of vertices T , where $T \subseteq S$, then τ is a face of σ , denoted as $\tau \leq \sigma$. So, the faces of a 1-simplex (an edge given by 2 vertices) are the two endpoint vertices and, technically, the edge itself. The faces of a 2-simplex (a triangle given by 3 vertices) are the three vertices, the three edges, and the triangle itself. Furthermore, a 0-face is a vertex, a 1-face is an edge, and so on.*

Definition 2.25 *An orientation of a k -simplex $\sigma = \{v_0, \dots, v_k\}$ is an equivalence class of orderings for the vertices of σ where $(v_0, v_1, \dots, v_k) \sim (v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(k)})$ are equivalent ordering if they differ by an even order of the permutation π .*

Hence, any simplex has precisely two orientations. In fact, when we commute the order of two vertices, the orientation changes to the opposite orientation. For instance,

choosing an orientation of a 1-simplex means to decide an orientation between the two possible directions, and choosing an orientation of a 2-simplex means to decide which clockwise or counterclockwise direction should be considered. Throughout this thesis, if an orientation is not explicitly mentioned or visualized, a counterclockwise orientation is taken for the 2-simplices.

Usually an oriented k -simplex is denoted by a list form as $[v_0, \dots, v_k]$.

Now, we define an essential mathematical tool used in TDA to study the shapes, which is the so called “simplicial complex”. Intuitively, constructing a simplicial complex structure on a space is an expression of the space as a union of points, edges, or intervals, triangles, and higher dimensional versions of the triangle. They actually supply a specially simple combinatorial way to describe specific topological spaces. That is why people often attempt to approximate (in different meanings) topological spaces by simplicial complexes [8].

Definition 2.26 *A simplicial complex $\tilde{K}$ in $\mathbb{R}^d$ satisfies the following conditions:*

- (a) *If $\sigma \in \tilde{K}, \tau \leq \sigma$, then $\tau \in \tilde{K}$.*
- (b) *If $\sigma_1, \sigma_2 \in \tilde{K}$, then either $\sigma_1 \cap \sigma_2 = \emptyset$, or $\sigma_1 \cap \sigma_2 \subseteq \sigma_1$ and $\sigma_1 \cap \sigma_2 \subseteq \sigma_2$, i.e. $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2 .*

And a subcomplex $\tilde{L}$ of $\tilde{K}$ is a subset $\tilde{L} \subseteq \tilde{K}$ such that $\tilde{L}$ is also a simplicial complex.

In other words, $\tilde{K}$ is a finite set of simplices such that all faces of any simplex in the set are also in the set. For example, a 2-simplex cannot be in the simplicial

complex unless all its edges are also there. Some k -simplices and a simplicial complex are elaborated in Figure 2.1.

Definition 2.27 $\tilde{K}$ is an oriented simplicial complex if each simplex in $\tilde{K}$ is oriented.

In this thesis, and unless otherwise is explicitly mentioned, all the simplicial complexes are assumed to be oriented.

The dimension of $\tilde{K}$ is the highest dimension of its simplices, i.e.

$$\dim(\tilde{K}) = \max\{\dim(\sigma) : \sigma \in \tilde{K}\}.$$

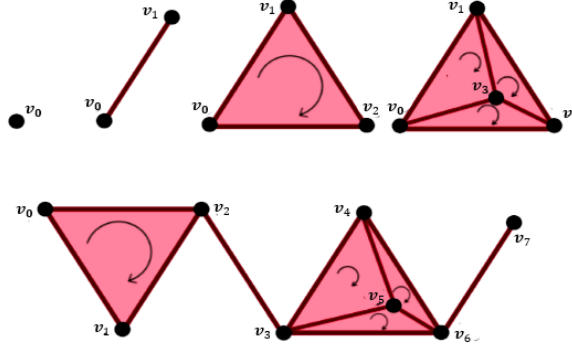


Figure 2.1: k -simplices for $k = 0, 1, 2$, and 3 (top). A simplicial complex (bottom).

Definition 2.28 Let $(X, \mathcal{T}_X)$ be a topological space and $Y \subseteq X$. The following topology $\mathcal{T}_Y = \{Y \cap O : O \in \mathcal{T}_X\}$ is called a relative topology, or a subspace topology of $\mathcal{T}_X$, to Y . The topological space $(Y, \mathcal{T}_Y)$ is called a subspace of $(X, \mathcal{T}_X)$.

The following definition describes how any simplicial complex carries within it the structure of a topological space that can be embedded gently in Euclidean space, and so simplices can be viewed from a geometric perspective.

Definition 2.29 *Let $\tilde{K}$ be a simplicial complex with a vertex set $\tilde{K}_0$ and φ is a map such that $\varphi : \tilde{K}_0 \rightarrow \mathbb{R}^d$. The geometric realization of $\tilde{K}$ with respect to φ is the union $|\tilde{K}|_\varphi = \cup_{\sigma \in \tilde{K}} |\sigma|_\varphi \subseteq \mathbb{R}^d$, where $|\sigma|_\varphi$ is the geometric simplex spanned by the φ -images of vertices contained in σ . Furthermore, $\varphi : \tilde{K}_0 \rightarrow \mathbb{R}^d$ is called an affine embedding (of $\tilde{K}$ in $\mathbb{R}^d$) if it is injective and $\varphi(\tilde{K}_0) \subseteq \mathbb{R}^d$ is affinely independent.*

Theorem 2.4 *The topology of $|\tilde{K}|_\varphi$ does not depend on φ as long as φ is an affine embedding.*

Definition 2.30 *The geometric realization of a simplicial complex $\tilde{K}$ is defined to be $|\tilde{K}|_\varphi$ for any affine embedding φ , call it $|\tilde{K}|$.*

The next definition gives the key method of the simplicial homology that we introduce later.

Definition 2.31 *A triangulation of a topological space X is a simplicial complex $\tilde{K}$ such that K is homeomorphic to X . Where K is the underlying topological space of $\tilde{K}$ and a homeomorphism is a continuous bijection preserving the topology.*

Example 2.7 *The underlying topological space K of a simplicial complex $\tilde{K} \subseteq \mathbb{R}^d$ is defined to be $K = \bigcup_{\sigma \in \tilde{K}} \sigma$, where a relative topology of the usual topology of $\mathbb{R}^d$ is given to K .*

We recall that the usual topology on $\mathbb{R}^d$ is defined as

$$\mathcal{T}_{\text{usual}} := \{U \subseteq \mathbb{R}^d : \forall x \in U, \exists \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subseteq U\},$$

where $B_\varepsilon(x)$ is a d -dimensional ball centered at x of radius ε .

Simplicial complexes are also known as triangulations. Essentially, the aim of topology is to find the homeomorphism type of a topological space. Informally, we would like to classify the spaces up to stretching and bending but not tearing and gluing. However, most often it is very difficult to get the homeomorphism type of a space. Thus, we need to consider other invariants such as: homotopy, homology, cohomology, and so on.

Simplicial complexes are very good tools for that, in fact they are used to extract the topological features of an object. For instance, in the Figure 2.2 a ring is represented as a simplicial complex. We can notice that the main topological features of the ring which are the one connected component and the central hole, are preserved as features of its triangulation, i.e., simplicial complex.

In this case, the ring and the simplicial complex are said to be homotopic, or homotopy equivalent, this relationship is denoted by $\simeq$. Those spaces that can be deformed continuously into one another without tearing or breaking are identified under homotopy equivalence. Furthermore, we can create a homotopically equivalent representation of any shape we are dealing with, this is due to the flexibility of simplicial complexes. Later we will see how we can specify the features of original objects from the algebraic treatments and computations done to their simplicial complexes.

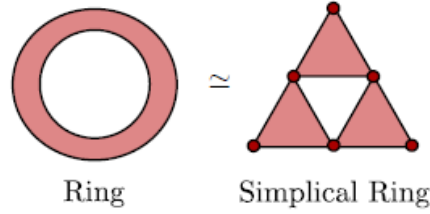


Figure 2.2: A ring and its triangulation using 2-simplices.

As a simplicial complex in $\mathbb{R}^d$ is specified entirely by its set of vertices and a family of subsets of those vertices related to each of the individual simplices of the complex, in line with this setting, a simplicial complex can be given a more purely combinatorial view as a combinatorial model, and hence we can introduce the following definition.

Definition 2.32 *An abstract simplicial complex is a pair $X = (V, S)$ where V is a finite set referred to the vertices of X and S is a set of non-empty subsets of the power set of V . Each element σ in S is called a simplex of X . It is important that the set S satisfy the following:*

- (a) *if $v \in V$, then $\{v\} \in S$,*
- (b) *if $\sigma \in S$ and $\tau \subset \sigma$ where $\tau \neq \emptyset$, then $\tau \in S$.*

We might be interested in knowing whether an abstract simplicial complex is isomorphic to a geometric simplex in some Euclidean space $\mathbb{R}^d$, this is known as a geometric realization. The following theorem shows this possibility in a Euclidean space of sufficient dimensions.

Theorem 2.5 *Every abstract simplicial complex of dimension n has a geometric realization in $\mathbb{R}^{2n+1}$.*

A proof of Theorem 2.5 can be found in [9].

Now, we move to our next task which is constructing a helpful simplicial complex corresponding to a given data set. In fact, this simplicial complex represents the data's structure and uses the original data set as the vertex set. As simplicial complexes are crucial objects for the algorithms established in this thesis, let us introduce the following notion of a filtered simplicial complex. We also should point to that from now on, we will denote a simplicial complex by K .

Definition 2.33 *A filtration $\mathcal{K}$ of a simplicial complex K is a nested subsequence of subcomplexes of K such that $\mathcal{K} : \emptyset \subseteq K_0 \subseteq K_1 \subseteq \dots \subseteq K_m = K$. We call K a filtered complex.*

2.4 Constructing Simplicial Complexes from Data

Through this section we clarify how filtration of simplicial complexes arises. We also introduce some methods in order to build a geometric filtered complex on top of data.

Definition 2.34 *A point cloud data is a finite set of points X from a topological subspace $\mathbb{X}$ of $\mathbb{R}^d$.*

We draw our attention to the following. If the given sampled data set is dense enough, then we shall be able to extract the topological features of the underlying space $\mathbb{X}$ directly from the point cloud. Otherwise, we have to topologize our data set [4].

Informally, we have to thicken the data set slightly so that it has a sort of geometric structure. We will be choosing some distance ϵ , which is our scale parameter or the

threshold, and then drawing an ϵ -neighborhood around each of the points (Figure 2.3). So, this gives some sort of topology on our data, then we are going to use this to build the simplicial complex that we want using either of the methods we introduce next. Later, in the fourth coming sections, we will be applying some algebraic gadgets from algebraic topology on those filtered simplicial complexes in order to find the topological features of $\mathbb{X}$.

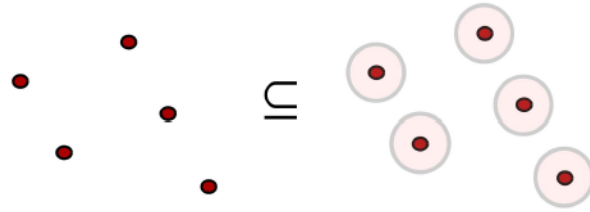


Figure 2.3: Point cloud data X (left). Thickened version of X defined by a collection of balls $B(x_i, \epsilon)$ expanded around each data point (right).

Before we give some methods of building a simplicial complex from a point cloud, we introduce this important concept of a nerve set followed by a fundamental theorem, known as the Nerve Theorem [9].

Definition 2.35 *Let F be a finite collection of sets. The nerve of F is the set of all non-empty sub collections whose sets have a non-empty common intersection. In other words,*

$$\text{Nrv } F = \{X \subseteq F \mid \bigcap X \neq \emptyset\}.$$

Theorem 2.6 (The Nerve Theorem) *Let F be a finite collection of closed, convex sets in Euclidean space. Then the nerve of F and the union of the sets in F have the same homotopy type, i.e. they are homotopic.*

Definition 2.36 (Čech Complex) *Let S be a finite data set in $\mathbb{R}^d$. For a radius ϵ , the Čech Complex $\mathcal{C}_\epsilon$ of S and ϵ , is defined as the nerve of this collection of balls $\{B_x(\epsilon) \mid x \in S\}$, but we substitute the center for each ball, that is,*

$$\mathcal{C}_\epsilon = \left\{ \sigma \subseteq S \mid \bigcap_{x \in \sigma} B_x(\epsilon) \neq \emptyset \right\}.$$

.

Informally, for the data set S , we define the thickened version of S by a collection of balls $B_x(\epsilon)$ expanded around each point of S . First we take all of our data points and consider them to be the vertex set, then for every time two balls intersect we include a 1-simplex (an edge), and for any three balls intersect we include a 2-simplex (a filled triangle), and so we include a 3-simplex (a tetrahedron) when four balls intersect, and so on.

According to the Nerve Theorem, the Čech complex and the union of this collection of balls are homotopic. Hence, the Čech complex records their topology. In fact, this is why people use Čech complex more in theory. However, constructing the Čech complex of a finite data set in $\mathbb{R}^d$ needs to examine all sub collections, which can be both complicated and computationally expensive.

There is another method for complex construction, the so called “Vietoris-Rips” complex, and it is often referred to in the literature as the Rips complex, that allows us to approximate the Čech complex by merely checking pairwise distances and then adding higher-dimensional simplices whenever all the edges are within the complex. Let us first give this definition.

Definition 2.37 For a finite set $S \subseteq \mathbb{R}^d$, the diameter of S is defined as

$$\text{diam } S := \sup_{x, y \in S} \|x - y\|$$

Definition 2.38 (Vietoris-Rips Complex) For a set S and radius ϵ , the Vietoris-Rips Complex $\mathcal{R}_\epsilon$ of S and ϵ is

$$\mathcal{R}_\epsilon = \{\sigma \subseteq S \mid \text{diam } \sigma \leq 2\epsilon\}.$$

So, for each pair of points in the data set, we include the edge between them if the distance between them is at most 2ϵ . Thus, we include a triangle between three points when the pairwise distance of them is no more than 2ϵ . Similarly for a higher dimensional simplex, we include the simplex if all possible edges are in the complex, i.e., every vertex of the simplex is within no more than 2ϵ distance of every other vertex.

Now, let us consider the following example in Figure 2.4 made of points in the plane $\mathbb{R}^2$ to find out their Čech and Vietoris-Rips complexes and the difference between them.

It is clear that the both constructions share the same edges set and that $\mathcal{C}_\epsilon \subseteq \mathcal{R}_\epsilon$. However, the only difference is that the 2-simplex abc is included in $\mathcal{R}_\epsilon$ but it is not a part of $\mathcal{C}_\epsilon$. This implies that $\mathcal{C}_\epsilon$ is homotopic to the unit circle and $\mathcal{R}_\epsilon$ is to a single point and so has the same homotopy type of a point. Having different homotopy types leads to the issue of that the Rips-complex does not very much approximate the Čech

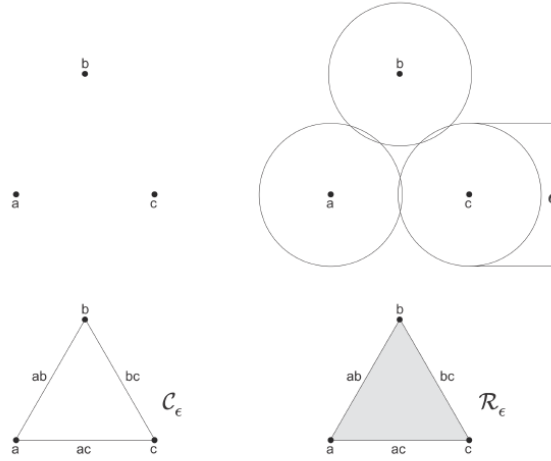


Figure 2.4: A fixed data set (upper left). Thickened version of the data set at radius ϵ (upper right). Čech complex of the data set (lower left). Vietoris-Rips complex of the data set (lower right).

complex.

Nevertheless, it turns out that this issue is resolvable when a sufficiently increased radius ϵ is considered. In fact, the Vietoris-Rips Lemma shows that the factor by which ϵ must be increased is $\sqrt{2}$, so that we can have the following reversed inclusion: $\mathcal{R}_\epsilon \subseteq \mathcal{C}_{\sqrt{2}\epsilon}$ as stated in the lemma [9]. Furthermore, when we choose a radius suitably small, the complex we get is a discrete set, i.e., just the sampled data set. When we choose it large enough we get a single high-dimensional simplicial complex [10].

For two Vietoris-Rips complexes, $\mathcal{R}_{\epsilon_1}$ and $\mathcal{R}_{\epsilon_2}$ of radii ϵ_1 , ϵ_2 respectively, let us assume that $\epsilon_1 \leq \epsilon_2$, then there is an obvious inclusion: $\mathcal{R}_{\epsilon_1} \subseteq \mathcal{R}_{\epsilon_2}$. Hence, for any increasing sequence of non-negative real radii, we obtain an increasing sequence of simplicial complexes representing the finite data set X from a topological space, i.e., we get a filtration of the simplicial complex representing X [4]. Most often in the literature this phenomenon is known as *growing a complex* (Figure 2.5).

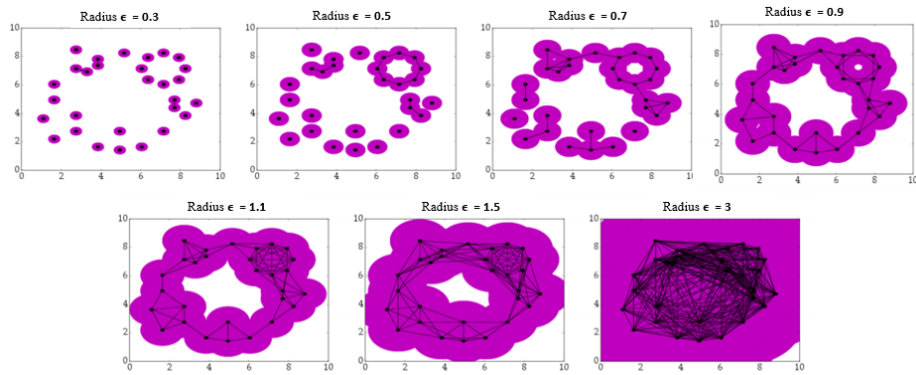


Figure 2.5: Growing a simplicial complex.

The Čech and Vietoris-Rips complexes are naturally of high dimensions. On the other hand, there are other methods for fitting simplicial complexes of lower dimensions, such as Alpha Complex, Witness Complex and several other methods. However, to be able to explain the usage of the other methods, some of them demand more advanced background, e.g., Morse Theory, and that is beyond than the framework of this thesis. For more detailed elaborations about the different methods of simplicial complex constructions, refer to [1], [9], or [10]. For our goals, it is sufficient to work with Vietoris-Rips complexes as they are computationally the most efficient.

So far we have seen how topology understands the spaces by creating their combinatorial presentation, i.e., their triangulation, to quantify different features such as connected components, loops, cavities, etc. Algebraic topology is the mathematical formalism for that, in the next section we will be introducing some essential tools and definitions from there, in particular the homology modules.

2.5 Simplicial Homology

Quantification of spaces is done with the help of homology. Even though there are several homology theories that can be used to that purpose, such as the *singular homology*, the *simplicial homology* is the most favored tool, especially for computational topologists, since it fits naturally in the simplicial complexes that represent topological spaces, also its combinatorial merit makes it easy to compute and fast to implement computer algorithms. By these reasons, simplicial homology precedes the singular homology. In fact, the singular homology theory despite being simple to define, has the disadvantage of being very difficult to be computed. Sophisticated tools such as Mayer-Vietoris sequence, excision theorem are needed to calculate them [11].

Definition 2.39 *For an oriented simplicial complex K and a ring R as before, the p -th chain module $C_p(K, R)$ of K over R , is the free R -module on a set of oriented p -simplices in K . An element c of $C_p(K, R)$ is called a p -chain. Furthermore, let $\{\sigma_1, \sigma_2, \dots, \sigma_{n_p}\}$ be a set of oriented p -simplices of K , c is written as $c = \lambda_1\sigma_1 + \lambda_2\sigma_2 + \dots + \lambda_{n_p}\sigma_{n_p}$, where λ_i is a scalar in R for all $i = 1, \dots, n_p$.*

The set of oriented p -simplices forms the standard basis for $C_p(K, R)$. In addition, We define any two similar simplices ω and τ with opposite orientation, as $\omega = -\tau$.

One example of p -chains over commutative unitary rings, is when $R = \mathbb{Z}$. Another important example is over polynomial ring, $R = \mathbb{Z}[t]$, and for this we give the following definition:

Definition 2.40 *A chain $c \in C_p(K, \mathbb{Z}[t])$ is monic if $c = \lambda_1\sigma_1 + \lambda_2\sigma_2 + \dots + \lambda_{n_p}\sigma_{n_p}$,*

where $\lambda_i \in \mathbb{Z}[t]$ for $i = 1, \dots, n_p$ and every nonzero λ_i is of the form $\pm t^{k_i}$ where $k_i \in \mathbb{Z}^* = \{0\} \cup \mathbb{N}$.

Definition 2.41 The boundary morphism $\partial_p : C_p(K, R) \rightarrow C_{p-1}(K, R)$ is an R -module homomorphism defined on any simplex $\sigma = [v_0, v_1, \dots, v_p]$ in $C_p(K, R)$ as

$$\partial_p(\sigma) = \sum_{i=0}^p (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_p],$$

where $\hat{v}_i$ is the standard notation to indicate that v_i is omitted from the list. This map is extended linearly to give an R -module homomorphism.

Let σ be a p -simplex in the chain $c \in C_p(K, R)$ such that τ is a face of σ of dimension $p-1$. Then the boundary of c is a boundary chain, $\partial(c)$, which is a $(p-1)$ -chain in $C_{p-1}(K, R)$, i.e., we find the boundary of p -dimensional objects in terms of $p-1$ -dimensional objects. For instance, Figure 2.6 visualizes the boundary operator ∂_2 of a 2-simplex.

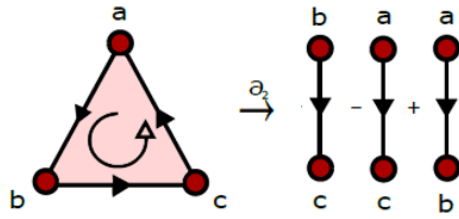


Figure 2.6: The boundary operator maps $[a, b, c]$ onto its bounding 1-simplices.

Definition 2.42 The boundary homomorphisms together with a family of chain modules on K implies the following so called chain complex $C_*(K, R)$:

$$\dots \xrightarrow{\partial_{p+2}} C_{p+1}(K, R) \xrightarrow{\partial_{p+1}} C_p(K, R) \xrightarrow{\partial_p} C_{p-1}(K, R) \xrightarrow{\partial_{p-1}} \dots$$

There are two very essential submodules of each chain module, $C_p(K, R)$, existing from the boundary homomorphism. The first, is called the *cycle module* denoted by Z_p , it is the submodule of all those p -chains having no boundaries, i.e. $Z_p = \ker \partial_p$. The second important chain submodule is the *boundary module* denoted by B_p , which represents the boundary of the $(p+1)$ -chains, i.e. $B_p = \text{Im } \partial_{p+1}$.

Now, we give a fundamental result about the boundary homomorphisms.

Lemma 2.3 *The composition $\partial_{p-1} \circ \partial_p$ is zero. (Often abbreviated $\partial_*^2 = 0$ suppressing subscripts).*

Proof. Let $\sigma = [v_0, v_1, \dots, v_p] \in C_p(K, R)$. Then we have that

$$\begin{aligned} \partial_{p-1} \circ \partial_p(\sigma) &= \sum_{i=0}^p (-1)^i \left[\sum_{j < i} (-1)^j [v_0, v_1, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_p] \right. \\ &\quad \left. + \sum_{i < j} (-1)^{j-1} [v_0, v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_p] \right] \\ &= 0 \end{aligned}$$

This is because after switching i and j in the second sum, the latter two summations are canceled. In fact, at each time the same face of σ shows up, one time it shows up with the sign $(-1)^i(-1)^j$ and the other time it shows up with the sign $(-1)^i(-1)^{j-1}$, but $(-1)^i(-1)^j + (-1)^i(-1)^{j-1} = 0$. So, each face appears exactly twice with opposite signs, hence we get the zero composition. █

We can show easily how this result implies that B_p and Z_p are nested submodules of the p -th chain module $C_p(K, R)$ with $B_p \subseteq Z_p \subseteq C_p(K, R)$. For instance, Figure 2.7 shows these nested submodules and how they are related under the boundary homomorphisms.

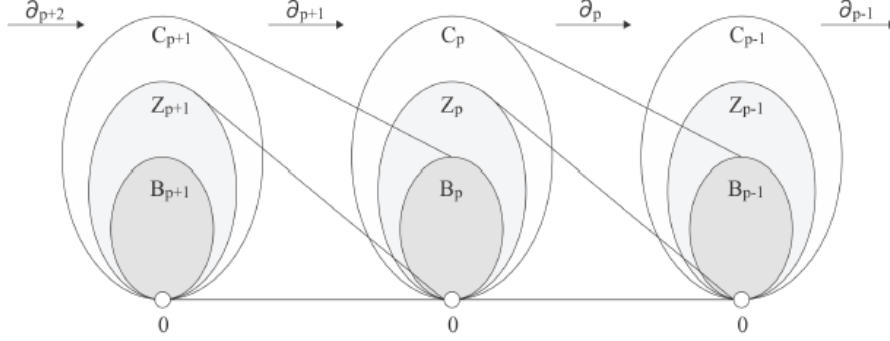


Figure 2.7: A chain complex C_* of chain modules with its cycles and boundaries submodules connected by boundary homomorphisms.

As the boundary submodules are also submodule of the cycle submodules, then taking a quotient makes sense and we can give the following definition.

Definition 2.43 *The p -th homology module $H_p(K, R)$ is the p -th cycle module of K quotient by the p -th boundary module, namely $H_p(K, R) = Z_p/B_p$. Each element of $H_p(K, R)$ is a homology class.*

Proposition 2.5 *Homology is a functor from the category of simplicial complexes to the category of R -modules.*

On the left side of Figure 2.8, we have representations of two simplicials, z and ω . The first one, z is a cycle but it is not a boundary for some higher dimensional simplex, thus $z \in Z_1$ but $z \notin B_1$. Whereas for the second simplicial ω , it is also a cycle

but this time it bounds a higher dimensional simplex (a 2-simplex) and so, $\omega \in Z_1$ and $\omega \in B_1$. We can observe that the simplicials z and $z + \omega$, on the right side of the figure, have the same 1-dimensional hole and they are described to be homologically equivalent. In fact, this concept is formally defined by the homology module H_p , that is if a cycle $z_1 = z_2 + B_p$ and $z_1, z_2 \in Z_p$, hence the two cycles differ by a boundary and so they are equivalent, $z_1 \simeq z_2$, which means that they are not independent elements of H_p . Furthermore, we avoid counting the same topological feature more than once by this fact, thus z and $z + \omega$ represent the same element of H_1 [12].



Figure 2.8: A representation of the homology module H_1 .

Definition 2.44 (p^{th} -Betti Number) *The p^{th} -Betti number β_p is the rank of H_p .*

When the coefficients ring of the p^{th} -homology module is a PID, then Theorem 2.1 and Corollary 2.1 can be applied. Hence, we have the following consequence.

Corollary 2.2 *The p^{th} -Betti number is given by:*

$$\beta_p = \text{rank}(H_p) = \text{rank}(Z_p) - \text{rank}(B_p).$$

Informally, the input stored in Z_p provides the cycles of dimension p within a given complex, whereas B_p provides whether a cycle is a boundary of some higher dimensional collection of simplices, or not. Furthermore, a p -cycle that is not a boundary is called a p -hole. This can be summarized further for any complex, by defining the p -th homology module H_p and the Betti number β_p . Roughly speaking, every H_p contains the unique p -holes living within a complex and β_p records the number of each of these unique p -holes.

More precisely, β_0 measures the connectivity, i.e. the number of connected components, β_1 measures the number of 1-dimensional holes or loops, and β_2 measures the number of 2-dimensional cavities or voids and so on. We also note that for an n -dimensional topological space, the homology module H_p terminates for any $p > n$, therefore the p^{th} Betti number is zero for any $p > n$. Figure 2.9 shows the Betti numbers for some topological spaces.

	β_0	β_1	β_2	β_3
	1	0	0	0
	1	1	0	0
	1	0	1	0
	1	2	1	0
	1	2	1	0

Figure 2.9: Betti numbers for different topological spaces (a point, a circle, a sphere, a 1-hole torus, and a Klein bottle).

We end this section with algebraic insights. When our ground ring is a PID, then the Structure Theorem for finitely generated modules over a PID (Theorem 2.2) fully characterises the homology modules $H_p(K, R)$ into the form

$$H_p(K, R) \cong R^{\beta_p} \oplus \left(\bigoplus_{i=1}^{m_p} R/d_i R \right),$$

where $d_i \in R, \beta \in \mathbb{Z}$, such that $d_i \mid d_{i+1}$. From Proposition 2.4, β_p is the p^{th} Betti number.

2.6 Homology Computation by Matrix Reduction

In this section we introduce a method known as *Smith normal form* [13] that uses elementary row and columns operations. It transforms the boundary matrix into a form which reveals its rank and properties and allows us to compute among other things, like the rank of the cycles and boundaries' modules and as a consequence the Betti numbers. Therefore, by using matrix reduction process we can compute the homology modules effectively. We explain the process on an example when $R = \mathbb{Z}$.

Definition 2.45 *Let G_1 and G_2 be free Abelian groups with bases $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_m$, respectively. If $f : G_1 \rightarrow G_2$ is a group homomorphism, then*

$$f(v_j) = \sum_{i=1}^m \lambda_{ij} w_i,$$

for the unique integers λ_{ij} . The matrix (λ_{ij}) is the matrix of f relative to the bases of G_1 and G_2 .

Theorem 2.7 *Let G_1 and G_2 be finitely generated free Abelian groups, and let $f : G_1 \rightarrow G_2$ be a homomorphism. Then there are bases for G_1 and G_2 such that, the matrix of f relative to these bases has the following Smith normal form*

$$B = \left[\begin{array}{cc|c} b_1 & 0 & \\ & \ddots & 0 \\ 0 & b_\ell & \\ \hline & 0 & 0 \end{array} \right] \quad (2.1)$$

where $b_i \geq 1$ and $b_1 | b_2 | \dots | b_\ell$. These non-zero ring elements are called the invariant factors.

In fact, the above matrix is uniquely determined by the homomorphism, however there are multiple bases of G_1 and G_2 , respectively, that produce the above form of the matrix of f relative to these bases [13]. Before we establish the proof of Theorem 2.7, let us recall elementary row and column operations which are utilized to reduce the matrix to its Smith normal form. We list the row operations with their column analogues as follows:

- (Op.1) exchange row (or column) i and row (or column) j ,
- (Op.2) multiply row (or column) i by -1 ,
- (Op.3) replace row (or column) i by $\alpha(\text{row } j) + (\text{row } i)$ (or $\alpha(\text{column } j) + (\text{column } i)$),
for any $\alpha \in \mathbb{Z}$ and $i \neq j$.

We know that any row or column operation performed on the matrix yields to

a change of basis in G_1 and G_2 respectively. The following proposition from [13] explains the effects of the row and column operations on the basis of G_1 and G_2 .

Proposition 2.6 *For elementary row and column operations performed on a boundary matrix, the basis elements that correspond to the columns and rows being modified are altered in the following way:*

- *Row (or column) operations of type Op.1 do not affect on the row (or column) basis elements.*
- *Row (or column) operations of type Op.2 yield to replacing w_i (or v_i) with $-w_i$ (or $-v_i$).*
- *Row (or column) operations of type Op.3 change the row (or column) basis elements in the following manner:*
 - (1) *If row i is replaced by $\alpha(\text{row } j) + (\text{row } i)$, then the change of basis observed is that w_j is replaced with $w_j - \alpha w_i$.*
 - (2) *If column i is replaced by $\alpha(\text{column } j) + (\text{column } i)$, then the change of basis observed is that v_i is replaced with $\alpha v_j + v_i$.*

Definition 2.46 *Let $A = (a_{ij})$ be a matrix of integer entries, not all zero. The entry $\beta(A) := \min \{|a_{ij}| \mid a_{ij} \neq 0\}$ is called the minimal entry of A .*

We introduce the following proposition which assures the existence of a minimal positive entry in the matrix that divides every other nonzero entry.

Proposition 2.7 *The value $\beta(A)$ fails to divide some nonzero entry of A if and only if it is possible to decrease the value of $\beta(A)$ by applying elementary row or column operations to A .*

Now, we give the proof of Theorem 2.7. It is the Smith normal form reduction algorithm and it is similar to Gaussian elimination process.

Proof. Phase #1: Specify a “pivot”. From Proposition 2.7, we can find a minimal positive entry in the matrix that divides all the other non-zero entries. By exchanging rows and columns (Op.1) we bring that entry to the upper left corner of the matrix. This entry is the so called *pivot*.

Phase #2: Zero out all the rest of the entries in the pivot’s column and row. Since the pivot element divides all the non-zero elements in its column and row, besides every other non-zero element of the matrix, then type (Op.3) of row and column operations can be applied to zero out all of those elements. We note that the pivot still divides all the new non-zero entries of the matrix.

Iterate: We next keep repeating phase #1 and phase #2 on every obtained block, i.e. sub matrix, by ignoring the first row and column of the previous matrix.

Final phase: The termination of the process occurs as follows. While we are repeating phases one and two, either we will end up with the zero sub matrix (a pivot can not be chosen), or an empty sub matrix, i.e. the matrix A has non-zero elements in the main diagonal.

We conclude that our matrix is now reduced to the form 2.1. To get a Smith normal form, we just need to verify that $b_1 | b_2 | \dots | b_\ell$, where b_i s are the diagonal

elements. Since we observed that after the first phase, every pivot divides all the other elements in the matrix to the right of it and below it, then when the process terminates we guarantee that the diagonal elements successively divide each other. ■

For our purpose, we would like to generalize what have been discussed in this section to the category of R -modules. In fact, since Abelian groups are modules over $\mathbb{Z}$, then Theorem 2.7 and Proposition 2.7 are valid for module homomorphisms between R -modules, where the ring R should be endowed with a division algorithm. In other words, the ground ring should be a Euclidean domain, which is an integral domain that can be equipped with at least one Euclidean function. More detailed information about this can be found in [6].

We now give a generalized version of Theorem 2.7.

Theorem 2.8 *Let R be a Euclidean domain and M_1 and M_2 be finitely generated R -modules. Also, let $f : M_1 \rightarrow M_2$ be a module homomorphism. Then there are bases for M_1 and M_2 such that the matrix of f relative to these bases has the following Smith normal form*

$$B = \left[\begin{array}{cc|c} b_1 & 0 & \\ & \ddots & 0 \\ 0 & b_\ell & \\ \hline & 0 & 0 \end{array} \right]$$

where $b_i \geq 1$ and $b_1 | b_2 | \dots | b_\ell$.

We apply Smith normal form to the boundary homomorphism

$\partial_p : C_p(K, R) \rightarrow C_{p-1}(K, R)$, where R is a Euclidean domain. In the rest, we are going to write C_p, C_{p-1} for $C_p(K, R)$ and $C_{p-1}(K, R)$, respectively.

Let us define A_p the matrix associated to the boundary homomorphism

$\partial_p : C_p \rightarrow C_{p-1}$ relative to the standard bases of C_p and C_{p-1} . We recall that the standard basis for any chain module C_p is the set of oriented p -simplices.

We let m_p and m_{p-1} to be the ranks of C_p and C_{p-1} respectively. Thus, the matrix A_p will be of size $m_{p-1} \times m_p$. By Definition 2.41, the entries of A_p are either $-1, 0$, or 1 .

To obtain the torsion coefficients and the Betti numbers of H_p , we first have to reduce the matrix A_p to its Smith normal form. By Proposition 2.6, the standard bases of the chain modules, C_p and C_{p-1} , will be altered to new bases. Let us denote these new bases by $\{e_1, e_2, \dots, e_{m_p}\}$ and $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{m_{p-1}}\}$. The Smith normal form of A_p , denoted by $\tilde{A}_p$, will be as follows:

$$\tilde{A}_p = \left[\begin{array}{cc|c} b_1 & 0 & \\ & \ddots & 0 \\ 0 & b_{l_p} & \\ \hline & 0 & 0 \end{array} \right],$$

where $b_i \mid b_{i+1}$ for all $1 \leq i < l_p$.

From linear algebra we know that to find the rank of a matrix, we just have to reduce it to its row echelon form and count the number of non-zero rows. This is since the number of non-zero rows in the echelon matrix is equal to the maximum number

of linearly independent columns in the matrix, which is the definition of the rank. For the above matrix reduced to its Smith normal form, the rank is obtained easier, we have that $\text{rank } \tilde{A}_p = l_p = \text{rank } A_p$.

After transforming every level p of the boundary matrix of a simplicial complex K to its Smith normal form, we are ready to employ Theorem 2.7 to find a complete description of the homology module $H_p(K, R)$. The characterization components can be computed as follows:

(i) The basis of Z_p is $\{e_i \mid l_p + 1 \leq i \leq m_p\}$. Hence, $\text{rank } Z_p = m_p - l_p$.

(ii) The rank of B_p is obtained from the Smith normal form of A_{p+1} , that is

$$\text{rank } B_p = \text{rank } \tilde{A}_{p+1} = l_{p+1}.$$

(iii) The diagonal elements b_i in the Smith normal form $\tilde{A}_p$ are the torsion coefficients of $H_p(K, R)$.

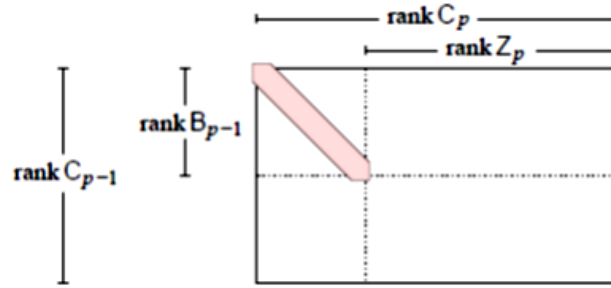


Figure 2.10: Structure of the Smith normal form matrix of the p -th boundary homomorphism. The shaded portion include diagonal entries of 1 and all other entries are 0.

From Proposition 2.4, we have that $\beta_p = m_p - l_p - l_{p+1} = \text{rank } Z_p - \text{rank } B_p$. In

terms of these notations, the decomposition of $H_p(K, R)$ is

$$H_p \cong R^{m_p - l_p - l_{p+1}} \oplus \left(\bigoplus_{i=1}^{l_{p+1}} R/b_{pi}R \right).$$

Therefore, after reducing the associated matrix of each boundary homomorphism to its Smith normal form, we can easily find the Betti numbers and thus describe every homology module $H_p(K, R)$.

Let us consider the following simple example.

Example 2.8 *For the simplicial complex K in Figure 2.6, we characterize the homology modules $H_0(K, \mathbb{Z})$ and $H_1(K, \mathbb{Z})$.*

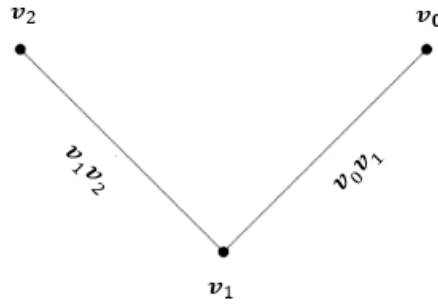


Figure 2.11: A simplicial complex K .

From a first look, it is clear that K is path connected, so

$H_0(K, \mathbb{Z}) = \mathbb{Z}^{\#\text{of connected components of } K}$. K is also contractible to a point, so it has no 1-dimensional holes and $H_1(K, \mathbb{Z}) = 0$. Hence, $\beta_0 = 1$ and $\beta_1 = 0$.

Now, let us find the boundary matrix A_1 and then reduce it to its Smith normal form to find the Betti numbers of H_0 and H_1 together with the torsion coefficients.

$$A_1 = \left[\begin{array}{c|cc} & v_0v_1 & v_1v_2 \\ \hline v_0 & -1 & 0 \\ v_1 & 1 & -1 \\ v_2 & 0 & 1 \end{array} \right]$$

In order to reduce our matrix to its Smith normal form, we have to do elementary row and column operations. We can start by, either of these ways, replacing the second row with the sum of the first two rows, or we can replace the second column with the sum of the first two columns. We will be performing both of the ways and observing the modifications happened to the bases elements after each.

The following matrix on the left is when we replace the second row with row 1 + row 2. The matrix on the right is the result of replacing the second column with column 1 + column 2:

$$A_1 = \left[\begin{array}{c|cc} & v_0v_1 & v_1v_2 \\ \hline v_0 - v_1 & -1 & 0 \\ v_1 & 0 & -1 \\ v_2 & 0 & 1 \end{array} \right], \quad A_1 = \left[\begin{array}{c|cc} & v_0v_1 & v_0v_1 + v_1v_2 \\ \hline v_0 & -1 & -1 \\ v_1 & 1 & 0 \\ v_2 & 0 & 1 \end{array} \right].$$

We can notice that the changes in the basis in the left matrix appear in the first row by having the basis simplex $(v_0 - v_1)$. Although we have replaced the second row, the boundary matrix successfully outputted the boundary of the basis simplex v_0v_1 as $-(v_0 - v_1) = v_1 - v_0$ as it should be. Similarly for the matrix on the right, the

modifications are in the second column by the basis simplex $v_0v_1 + v_1v_2$, but again the boundary matrix deals with that suitably and assigns $v_0v_1 + v_1v_2$ a boundary, in terms of the original basis simplices, of $-1(v_0) + 1(v_2) = v_2 - v_0$.

By converting either of the above matrices to the Smith normal form, we get

$$\tilde{A}_1 = \left[\begin{array}{c|cc} & -v_0v_1 & -v_1v_2 \\ \hline v_0 - v_1 & 1 & 0 \\ v_1 - v_2 & 0 & 1 \\ v_2 & 0 & 0 \end{array} \right].$$

Hence, we have that $\beta_0 = n_0 - l_0 - l_1 = 3 - 0 - 2 = 1$ and

$\beta_1 = n_1 - l_1 - l_2 = 2 - 2 - 0 = 0$, as it was observed.

CHAPTER 3

PERSISTENCE

3.1 Persistent Homology

The main focus of TDA, and especially computational topology, is the topological features of the underlying point cloud space are those who persist longer in a filtration. In this regard, the gadget used to measure the life span a feature existence is *persistence*. As homology counts the number of holes in each dimension, a technique known as *persistent homology*, originally proposed by Edelsbrunner, Letscher, and Zomorodian and later developed by many others, tracks the number of holes in each dimension as the scale changes. Moreover, it tracks how they are mapped to each other. Throughout this section, we shed light on the theory of persistence and persistent homology with relevant information and figures based on [9].

Let $\mathcal{K} : \emptyset \subseteq K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n = K$ be a filtration of a simplicial complex K . By the functoriality of homology modules (Proposition 2.5) the inclusion maps induce

homomorphisms between the associated homology modules, such that

$$f_p^{i,j} : H_p(K_i, R) \rightarrow H_p(K_j, R)$$

is a module homomorphism for every dimension p . Hence, there exists a sequence of homology modules connected by homomorphisms,

$$H_p(\mathcal{K}, R) : 0 \rightarrow H_p(K_0, R) \rightarrow H_p(K_1, R) \rightarrow \cdots \rightarrow H_p(K_n, R) = H_p(K, R).$$

So, $H_p(\mathcal{K}, R)$ is a persistence module as defined in Definition 3.6.

We give the following definition to describe the homology classes that appear at or before specific index in the filtration and those who terminate to exist at some stage of the filtration.

Definition 3.1 *The p -th persistent homology modules $H_p^{i,j}(\mathcal{K}, R)$ are the images of the homomorphisms $f_p^{i,j} : H_p(K_i, R) \rightarrow H_p(K_j, R)$ induced by inclusion $K_i \hookrightarrow K_j$, for all $0 \leq i \leq j \leq n$. There exist corresponding persistent Betti numbers, defined as $\beta_p^{i,j} = \text{rank } H_p^{i,j}(K, R)$.*

Intuitively, for two simplicial complexes K_i and K_{i+1} in a filtration, the new created simplices added to K_i that form the simplicial K_{i+1} lead to create new homology classes and damage some of the old ones. Destroying a p -th homology class may happen by adding $(p+1)$ -dimensional simplex in which it fills the hole of the previous p -th homology class and then latter becomes trivial.

We deduce that the persistent homology modules consist of p -cycles that appear in

K_i and continued to appear until at least K_j . Since we are considering those p -cycles in K_i that are not boundaries of some higher dimensional simplices in K_j , i.e. the p -cycles in K_i modded by the boundaries of the same of them existing in K_j , it makes sense to give the following substitutional definition for the p -th persistent homology modules $H_p^{i,j}(K, R)$ as $Z_p(K_i, R) / (B_p(K_j, R) \cap Z_p(K_i, R))$. This definition is well passed. In fact, the intersection of the two modules in the denominator is a submodule of the numerator, furthermore, both of the modules in the denominator are actually submodules of the chain module $C_p(K_j, R)$ and so is their intersection. In particular, when $i = j$ the persistent homology module is $H_p^{i,i} = H_p(K_i, R)$.

Let us give a more concrete description for what may happen to homology classes as they are travelling in the filtration. As presented in Figure 3.1, we assume that λ is a homology class in $H_p(K_i, R)$. If K_i is the first simplicial complex in the filtration where λ appears, that is $\lambda \notin H_p^{i-1,i}$, we can declare that λ is *born* at K_i . On the other hand, we declare that λ is *death* when it enters the simplicial complex K_j , if the first time the image of λ falls into the image of $H_p(K_{i-1}, R)$ is at K_j , i.e. $f_p^{i,j-1}(\lambda) \notin H_p^{i-1,j-1}$ however, $f_p^{i,j}(\lambda) \in H_p^{i-1,j}$.

The grey zones in Figure 3.1 except the first one, represent the image of $H_p(K_{i-1}, R)$ at subsequenceant K'_s s, for all $i \leq s \leq j$.

Definition 3.2 *A simplex ω whose addition to a simplicial complex produces the construction of a homology class λ , i.e. p -dimensional hole, is called a creator or a positive simplex. A simplex z whose addition to a simplicial complex produces the removal of a homology class is called a destroyer or a negative simplex.*

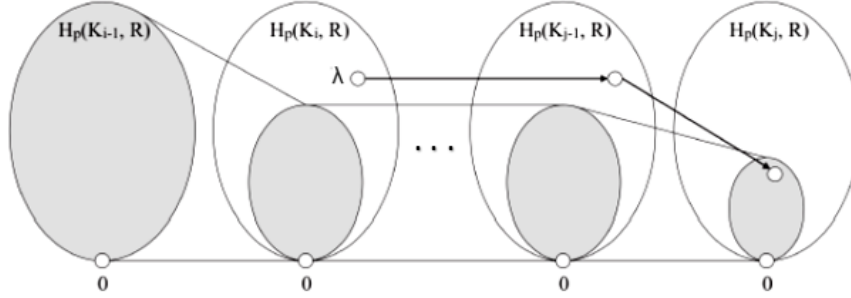


Figure 3.1: A homology class λ that is born at K_i and is death at K_j .

We note that, every time, a p -simplex is added to the complex either a p -cycle is born or a $(p - 1)$ -cycle is dead. Hence, every simplex is either a positive or a negative simplex.

For example, consider a filtered simplicial complex K in which a 0-simplex v_1 is added first to K , then another 0-simplex v_2 is added, and eventually the 1-simplex v_1v_2 is added. So, here v_1v_2 is treated to be a destroyer for the 0-dimensional homology class constructed by v_2 and so, v_1v_2 is a negative simplex.

There is a 1 – 1 correspondence between a filtration and a non-decreasing function. Let K be a simplicial complex and $f : K \rightarrow \mathbb{R}$ be a non-decreasing function used to make the filtration, such that $f^{-1}((-\infty, \epsilon_i]) = K_i$ and $f^{-1}((-\infty, \epsilon_j]) = K_j$. Next, we define the measure of persistence.

Definition 3.3 *For a homology class λ that was created at K_i and destroyed at K_j , the persistence of λ , denoted by $\text{pers}(\lambda)$, is the difference in the function f values, i.e.*

$\text{pers}(\lambda) = \epsilon_j - \epsilon_i$. If $\epsilon_j = \infty$ then we say that λ never dies and $\text{pers}(\lambda) = \infty$. The persistence index of λ is $j - i$.

3.2 The Persistence Module

After we have handled what persistence is, we turn our attention to look at it from a different perspective. The aim of this view is to combine all of the persistent homology modules for every simplicial complex in the filtration into a single algebraic structure. To be more precise, the persistent homology of a filtered simplicial complex is merely the standard homology of a particular graded module over a polynomial ring.[4] Even though one can understand the persistent homology groups from the context of spectral sequences, the method does not provide an algorithm, but a scheme that must be fitted for each problem independently. That is why we are following a better suited approach which is the persistence modules. In line with the above we first introduce the following essential definitions taken from [4], [14].

Definition 3.4 *Let $\mathcal{D}$ denote any category. A persistence object in $\mathcal{D}$ is a functor $\mathbb{R}_+ \rightarrow \mathcal{D}$, where $\mathbb{R}_+$ denotes the ordered set of non-negative real numbers, regarded as a category, so that there is a unique morphism $r_0 \rightarrow r_1$ whenever $r_0 \leq r_1$. More generally, if $\mathbb{A} \subseteq \mathbb{R}_+$ is any pure submonoid, an $\mathbb{A}$ -parametrized persistence object in $\mathcal{D}$ we will mean a functor from the ordered set $\mathbb{A}$ to $\mathcal{D}$.*

The $\mathbb{A}$ -parametrized persistence objects in $\mathcal{D}$ form a category in their own right, where the morphisms are the natural transformations of functors. These objects are landmarks of the persistence theory described throughout this section.

Definition 3.5 A persistence complex $\mathcal{C}$ is a family of chain complexes $\{C_*^i\}_{i \geq 0}$ over R , together with chain maps $f^i : C_*^i \rightarrow C_*^{i+1}$, so that we have the following diagram

$$C_*^0 \xrightarrow{f^0} C_*^1 \xrightarrow{f^1} C_*^2 \xrightarrow{f^2} \dots$$

The diagram below is a portion of a persistence complex, with the chain complexes expanded.

$$\begin{array}{ccccccc} & \downarrow \partial_3 & & \downarrow \partial_3 & & \downarrow \partial_3 & \\ C_2^0 & \xrightarrow{f^0} & C_2^1 & \xrightarrow{f^1} & C_2^2 & \xrightarrow{f^2} & \dots \\ & \downarrow \partial_2 & & \downarrow \partial_2 & & \downarrow \partial_2 & \\ C_1^0 & \xrightarrow{f^0} & C_1^1 & \xrightarrow{f^1} & C_1^2 & \xrightarrow{f^2} & \dots \\ & \downarrow \partial_1 & & \downarrow \partial_1 & & \downarrow \partial_1 & \\ C_0^0 & \xrightarrow{f^0} & C_0^1 & \xrightarrow{f^1} & C_0^2 & \xrightarrow{f^2} & \dots \end{array}$$

Definition 3.6 A persistence module $\mathcal{M}$ is a tuple $\mathcal{M} = ((M_i)_{i \in \mathbb{N}}, (\varphi_{i,j})_{i \leq j \in \mathbb{N}})$, such that M_i is an R module, $\varphi_{i,j} : M_i \rightarrow M_j$ is a module morphism, $\varphi_{i,i} = 1_{M_i}$ and $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ for all $i \leq j \leq k$.

$$\mathcal{M} : M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} \dots$$

We note that, categorically, a persistence module $\mathcal{M}$ is simply a functor $F : \mathcal{J} \rightarrow R\text{-}\mathbf{Mod}$, where $\mathcal{J}$ can be one of the indexing categories $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{R}$ and $R\text{-}\mathbf{Mod}$ denotes the category of R -modules.

In addition, the category of persistence modules $\text{Fun}(\mathcal{J}, R\text{-}\mathbf{Mod})$ over R is denoted by $\underline{\text{Pers}}(R)$ and the morphism between two persistence modules, $\mathcal{M}$ and $\mathcal{M}'$, is a natural transformation given by maps $\alpha_i : \mathcal{M}_i \rightarrow \mathcal{M}'_i$ compatible with the module homomorphisms φ_i and φ'_i as follows:

$$\begin{array}{ccccccccccc}
\mathcal{M} : & M_0 & \xrightarrow{\varphi_0} & M_1 & \xrightarrow{\varphi_1} & \dots & \xrightarrow{\varphi_{i-1}} & M_i & \xrightarrow{\varphi_i} & M_{i+1} & \xrightarrow{\varphi_{i+1}} & \dots \\
& \downarrow \alpha_0 & & \downarrow \alpha_1 & & \downarrow & & \downarrow \alpha_i & & \downarrow \alpha_{i+1} & & \\
\mathcal{M}' : & M'_0 & \xrightarrow{\varphi'_0} & M'_1 & \xrightarrow{\varphi'_1} & \dots & \xrightarrow{\varphi'_{i-1}} & M'_i & \xrightarrow{\varphi'_i} & M'_{i+1} & \xrightarrow{\varphi'_{i+1}} & \dots
\end{array}$$

Definition 3.7 A persistence module $\mathcal{M} = \left((M_i)_{i \in \mathbb{N}}, (\varphi_{i,j})_{i \leq j \in \mathbb{N}} \right)$ is of finite type if there is a $D \in \mathbb{N}$ such that for all $D \leq i \leq j$ the map $\varphi_{i,j}$ is an isomorphism. $\mathcal{M}$ is called of finitely generated type if it is of finite type and M_i is finitely generated as an R -module for all $i \in \mathbb{N}$.

Example 3.1 Let K be a finite dimensional complex and

$K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_m = K$ be a filtered simplicial complex. We can get a persistence complex $\{C_*^i, f^i\}$, for $i = 0, 1, \dots, m$, from the family of chain complexes $\{C_*(K_i, R)\}$, where

$$\{C_*(K_i, R)\} = \left\{ C_{p_i}(K_i, R) \xrightarrow{\partial_{p_i}} C_{p_i-1}(K_i, R) \xrightarrow{\partial_{p_i-1}} \dots \xrightarrow{\partial_1} C_0(K_i, R) \right\},$$

for all $0 \leq i \leq m$, $p_i = \dim K_i$ and taking $f^i : C_*(K_i, R) \rightarrow C_*(K_{i+1}, R)$ to be the homomorphism induced by the inclusion $K_i \hookrightarrow K_{i+1}$. Then, we get a persistence module by taking the homology of each chain complex and setting

$M_i = \bigoplus_{p \geq 0} H_p(K_i, R)$. The homomorphisms $\varphi_i : M_i \rightarrow M_{i+1}$ are also induced by the inclusion $K_i \hookrightarrow K_{i+1}$. Since K is finite dimensional, then the persistence complex $\{C_*^i, f^i\}$ and its persistent homology module $\{M_*^i, \varphi^i\}$ are both of finite type.

Under the consideration of Definition 3.6., Zomorodian and Carlsson [4] remarked that a persistence module can be exhibited as a single object which is a module over the polynomial ring $R[t]$.

3.3 The Persistence Module Over R as a Graded Module Over $R[t]$

Assume that $\mathcal{M} = \{M_i, \varphi_i\}_{i \geq 0}$ is a persistence module over a ring R . Let $R[t]$ be equipped with the standard grading, and define a graded module over $R[t]$ by the correspondence α as

$$\alpha(\mathcal{M}) := \bigoplus_{i=0}^{\infty} M_i$$

where the R -module structure is simply the direct sum of the structures on the individual components, and where the action of t is given by

$$t \cdot (m_0, m_1, m_2, \dots) = (0, \varphi_0(m_0), \varphi_1(m_1), \varphi_2(m_2), \dots).$$

That is, t simply shifts elements of the module up the gradation.

Before we give the crucial theorem of this part (Theorem 3.1) which is most often seen in the literature as ZC-Representation Theorem [14], let us introduce the following proposition.

Proposition 3.1 (Carlsson, Zomorodian 2006) . *The category $\underline{\text{Pers}}(R)$ of persistence modules is equivalent (in fact isomorphic) to the category $R[t]\text{-Gr-Mod}$ of graded R -modules.*

Proof. We assume that $\alpha : \underline{\text{Pers}}(R) \rightarrow R[t]\text{-Gr-Mod}$ is a functor from the category of persistence modules over R to the category of graded R -modules, with the action of $R[t]$ defined as above. Now, we check that α is a functor. We have the following:

(i) Let $\mathcal{M}$ and $\mathcal{N}$ be two persistence modules such that $\mathcal{M} = \{M_i, \varphi_i\}$ and $\mathcal{N} = \{N_i, \psi_i\}$, and let $\sigma = (\sigma_i : M_i \rightarrow N_i)_{i \geq 0}$ be a family of module homomorphisms seen as the natural transformations between the two functors $\mathcal{M}$ and $\mathcal{N}$.

We define the following map in the target category, $\alpha(\sigma) : \bigoplus_{i \geq 0} M_i \rightarrow \bigoplus_{i \geq 0} N_i$ component wise as $(m_i)_{i \geq 0} \mapsto (\sigma_i(m_i))_{i \geq 0}$, let us denote $\alpha(\sigma)$ by F . Now, we want to show that $F \in R[t]\text{-Gr-Mod}$, i.e. F is a graded module morphism. First, it is clear that F is a group homomorphism since every σ_i is a group homomorphism. Then we check the compatibility with $R[t]$, let $m = (m_0, m_1, \dots) \in \bigoplus_{i \geq 0} M_i$, we have the following

$$\begin{aligned}
F(tm) &= F(t(m_0, m_1, \dots)) \\
&= F(0, \varphi_0(m_0), \varphi_1(m_1), \dots) \\
&= F(0, tm_0, tm_1, \dots) \\
&= (0, \sigma_0(tm_0), \sigma_1(tm_1), \dots) \\
&= (0, t\sigma_0(m_0), t\sigma_1(m_1), \dots) \\
&= t(0, \sigma_0(m_0), \sigma_1(m_1), \dots) \\
&= t(\sigma_i(m_i))_{i \geq 0} = tF(m).
\end{aligned}$$

Hence, F is a graded module morphism and so, $\alpha(\mathcal{M})$ and $\alpha(\mathcal{N})$ are connected in the target category.

(ii) Let $\mathcal{M}, \mathcal{N}, \mathcal{W} \in \text{Ob}(\underline{Pers}(R))$, such that $\sigma : \mathcal{M} \rightarrow \mathcal{N}$, $\lambda : \mathcal{N} \rightarrow \mathcal{W}$, and so

$\lambda \circ \sigma : \mathcal{M} \rightarrow \mathcal{W}$. Let m be as previously, we find that

$$(\alpha(\lambda \circ \sigma))(m) = ((\lambda_i \circ \sigma_i)(m_i))_{i \geq 0} = (\lambda_i(\sigma_i(m_i)))_{i \geq 0} = \alpha(\lambda)(\sigma_i(m_i))_{i \geq 0} = (\alpha(\lambda) \circ \alpha(\sigma))(m).$$

Therefore, α preserves the composition.

(iii) $id_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{M} \Rightarrow \alpha(id_{\mathcal{M}}) : \alpha(\mathcal{M}) \rightarrow \alpha(\mathcal{M})$, so $\alpha(id_{\mathcal{M}}) = id_{\alpha(\mathcal{M})}$ which is the identity of the corresponding graded modules category.

We conclude that α is a functor.

Conversely, we construct a functor β that travels in the opposite orientation such that $\beta : R[t]\text{-Gr-Mod} \rightarrow \underline{\text{Pers}}(R)$. For $M = \bigoplus_{i \geq 0} M_i \in \text{Ob}(R[t]\text{-Gr-Mod})$, we define $\beta(M)$ by: $\beta(M)(i) = M_i$, $i \in \mathbb{Z}^*$ and for any $i, j \in \mathbb{Z}^*$, where $\mathbb{Z}^*$ is the non-negative integers, $\beta(M)(i \leq j)$ is the restriction of the action of t^{j-i} to the component M_i , i.e., the module homomorphisms in the persistence module structure, $\varphi_i : M_i \rightarrow M_{i+1}$ is a multiplication by t . So, $\beta(\bigoplus_{i \geq 0} M_i) := M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \rightarrow \dots$, where $\varphi_i(m_i) := t \cdot m_i$. The functoriality of β can be verified similarly to what have been done above. We have the following

(i) Let $\bigoplus_{i \geq 0} M_i, \bigoplus_{i \geq 0} N_i \in \text{Ob}(R[t]\text{-Gr-Mod})$ such that $\rho : \bigoplus_{i \geq 0} M_i \rightarrow \bigoplus_{i \geq 0} N_i$, and let $\beta(\bigoplus_{i \geq 0} M_i) = \mathcal{M}$, $\beta(\bigoplus_{i \geq 0} N_i) = \mathcal{N}$. We want to show that $\mathcal{M}$ and $\mathcal{N}$ are connected in $\underline{\text{Pers}}(R)$. The graded morphism ρ gives a family of maps $(\sigma_i)_{i \geq 0}$, where each σ_i is just the restriction of ρ to M_i . We define $\beta(\rho)$ to be this family. Now let us show that $\beta(\rho)$ is a persistence module morphism. It is clear that β implies that each $\sigma_i : M_i \rightarrow N_i$ is a group homomorphism. Also, we have that, for all $r \in R$ and $m_i \in M_i$, $\sigma_i(rm_i) = \rho(rm_i) = r\rho(m_i) = r\sigma_i(m_i)$, hence σ_i are R -

module homomorphisms. As the module homomorphisms in $\mathcal{M}$ and $\mathcal{N}$ are induced by multiplication of t , we have that $\sigma_{i+1}(tm_i) = \rho(tm_i) = t\rho(m_i) = t\sigma_i(m_i)$. Thus, $\beta(\sigma)$ is a persistence module homomorphism, meaning that $\mathcal{M}$ and $\mathcal{N}$ are connected in $\underline{\text{Pers}}(R)$.

(ii) For $\bigoplus_{i \geq 0} M_i, \bigoplus_{i \geq 0} N_i, \bigoplus_{i \geq 0} W_i \in \text{Ob}(R[t] - \text{Gr} - \text{Mod})$ such that $\rho : \bigoplus_{i \geq 0} M_i \rightarrow \bigoplus_{i \geq 0} N_i$, $\mu : \bigoplus_{i \geq 0} N_i \rightarrow \bigoplus_{i \geq 0} W_i$ and so $\mu \circ \rho : \bigoplus_{i \geq 0} M_i \rightarrow \bigoplus_{i \geq 0} W_i$. For any sequence $(m_i)_{i \geq 0}$ where $m_i \in M_i$, we have the follows:

$$(\beta(\mu \circ \rho))(m_i)_{i \geq 0} = (\mu(\rho(m_i)))_{i \geq 0} = \beta(\mu)(\rho(m_i))_{i \geq 0} = (\beta(\mu) \circ \beta(\rho))(m_i)_{i \geq 0}.$$

(iii) As what we did for α , it is again clear that β maps the graded $R[t]$ -module identity to the identity of the corresponding persistence modules.

Finally, it is also clear that $(\beta \circ \alpha)(\mathcal{M}) = \beta(\alpha(\mathcal{M})) = \beta(\bigoplus_{i \geq 0} M_i) = \mathcal{M}$ implies that $\beta \circ \alpha = \text{id}_{\underline{\text{Pers}}(R)}$. Similarly, we can get that $\alpha \circ \beta = \text{id}_{R[t] - \text{Gr} - \text{Mod}}$. In fact, this is a canonical isomorphism. Therefore the two categories are isomorphic. ■

Theorem 3.1 *Let R be a PID, the correspondence α , as defined in Proposition 3.1, defines an equivalence of categories between the category of persistence modules of finite generation type over R and the category of finitely generated non-negatively graded modules over $R[t]$.*

The proof of the above theorem is based on Artin-Rees Lemma which can be found in (Eisenbud, 1995). As Artin-Rees Lemma is valid for commutative Noetherian rings, we can prove Theorem 3.1 using Artin-Rees Lemma by assuming R to be a PID.

The theorem also holds over rings which are not Noetherian with “stronger” finiteness conditions on modules (see [14]).

Before we give a sketch of the proof in the light of the Artin-Rees lemma, based on [14], we present some needed information to that end. First, we consider the following notion of filtrations of modules:

Definition 3.8 *Let R be a PID, $I \subseteq R$ be an ideal, and M be an R -module. An I -filtration of M is a collection $(M_n)_{n \in \mathbb{N}}$ such that $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ and $IM_n \subseteq M_{n+1}$ for all $n \in \mathbb{N}$. An I -filtration is called I -stable if there is an $n_0 \in \mathbb{N}$ such that $IM_n = M_{n+1}$ for all $n \geq n_0$.*

Lemma 3.1 (Artin-Rees) *Let R be a Noetherian ring, let $I \subset R$ be an ideal, and let $M' \subset M$ be finitely generated R -modules. If $M = M_0 \supset M_1 \supset \dots$ is an I -stable filtration, then the induced filtration $M' \supset M' \cap M_1 \supset M' \cap M_2 \supset \dots$ is also I -stable. That is, there exists a number n such that for all $i \geq 0$, $M' \cap M_{i+n} = I^i(M' \cap M_n)$.*

Lemma 3.2 (Criterion for stability) *Let R be a commutative Noetherian ring with unity, $I \subseteq R$ an ideal. Let M be a finitely generated R -module, $(M_n)_{n \in \mathbb{N}}$ an I -filtration of M . Then the followings are equivalent:*

- (a) $\bar{M}$ is a finitely generated $\bar{R}$ -module, where $\bar{R} := \bigoplus_{i \in \mathbb{N}} I^i$, $\bar{M} := \bigoplus_{i \in \mathbb{N}} M_i$.
- (b) $(M_n)_{n \in \mathbb{N}}$ is I -stable.

Theorem 3.2 (Hilbert Basis Theorem) *If a ring R is Noetherian, then the polynomial ring $R[x]$ is Noetherian.*

Corollary 3.1 *If R and $R[t]$ are Noetherian rings with unity, the category of finitely generated graded $R[t]$ -modules is isomorphic to the category of persistence modules of finitely generated type.*

Now we are ready to establish the proof of Theorem 3.1.

Proof. The functoriality of the correspondence α and its inverse β have been shown in the proof of Proposition 3.1. What is remaining is to show the finite generation. Since R is a PID then R is a Noetherian ring and, by Theorem 3.2, so is $R[t]$. Let $I = (t)$, the ideal of R generated by t . We let $\mathcal{M}$ to be a persistence module over R of finitely generated type, such that $\alpha(\mathcal{M}) = \bigoplus_{i \geq 0} M_i$. Let us consider the filtration $(\widetilde{M}_n)_{n \in \mathbb{N}}$ defined by $\widetilde{M}_0 = \bigoplus_{i \in \mathbb{N}} M_i$ and $\widetilde{M}_n = (t^n) \widetilde{M}_0$ for $n > 0$. By finite generation type assumption, there exists an $n_0 \in \mathbb{N}$ such that $(t) \widetilde{M}_n = \widetilde{M}_{n+1}$ for all $n \geq n_0$.

In order to use the criterion for stability (Lemma 3.2) we should first show that $\bigoplus_{i \geq 0} M_i$ is a finitely generated $R[t]$ -module.

Since $\mathcal{M}$ is of finitely generated type, then by definition there exists $D \in \mathbb{N}$ such that for all $D \leq i \leq j$, $\varphi_{i,j} : M_i \rightarrow M_j$ is an isomorphism, and every M_i is finitely generated. So we consider $\mathcal{G}_i$ to be a generating set for M_i for all $i \geq 0$.

Let us claim that $\bigcup_{i=1}^D \mathcal{G}_i$ generates $\alpha(\mathcal{M}) = \bigoplus_{i \geq 0} M_i$. To prove our aim, we have to show that every homogeneous element in $\alpha(\mathcal{M})$ can be written as a linear

combination of elements in the union of $\mathcal{G}_i$, i.e. every homogeneous element is generated by that union.

Let m_k be a homogeneous element of degree k for some $k \geq 0$. Then, we have the following two cases.

If $k \leq D$, then m_k is generated by the elements of $\mathcal{G}_k$ by construction. If $k > D$, we have that $\varphi_{D,k} : M_D \rightarrow M_k$ is an isomorphism, so it is invertible. Hence, $m_D := \varphi_{D,k}^{-1}(m_k)$. Since, m_D is generated by the set $\mathcal{G}_D$, then m_k is generated by $\varphi_{D,k}(\mathcal{G}_D)$.

By the definition of α , $\varphi_{D,k}(\mathcal{G}_D) = t^{k-D}\mathcal{G}_D$. Since t^{k-D} is a ring element in the ring $R[t]$, then $t^{k-D}\mathcal{G}_D \in M_D$, but M_D is generated by $\mathcal{G}_D$, hence m_k is generated by $\mathcal{G}_D$. This shows that any homogeneous element in $\alpha(M)$ is generated by the union of $\mathcal{G}_i$, therefore $\alpha(\mathcal{M})$ is finitely generated.

Now, conversely and using the criterion, finite generation of the $R[t]$ -module, $\alpha(\mathcal{M}) = \bigoplus_{i \geq 0} M_i$ implies the (t) -stability of the filtration $\left(\widetilde{M}_n\right)_{n \in \mathbb{N}}$. Thus, the correspondence $\beta\left(\bigoplus_{i \in \mathbb{N}} M_i\right)$ is of finite type. We still need to show that each M_i is a finitely generated R -module. Let $\mathcal{G} := \{g^{(1)}, \dots, g^{(n)}\}$ be the generating set of $\bigoplus_{i \in \mathbb{N}} M_i$. We assume that each $g^{(i)}$ is homogeneous with respect to the grading of the corresponding module. Let $d_j := \deg(g^{(j)})$, for $1 \leq j \leq n$, and let n_i be the number of elements in $\mathcal{G}$ of degree at most i . Now, we define the map $\mu_i : R^{n_i} \rightarrow M_i$ as $\mu_i(e_i^{(j)}) = t^{i-d_j}g^{(j)}$, where $e_i^{(j)}$ is the j -th generator of R^{n_i} . Then, it is clear that this map is surjective and so M_i is finitely generated. ■

The grandness of this equivalence basically comes from the applicable cases, especially when the ground ring is a field. Before we discuss this regard in more details, the

previous theorem worth some explanations in a somewhat informal language about its treatment to the persistence homology modules of a filtered simplicial complex.

Example 3.2 *Going back to the persistence module we set for the filtration in Example 3.1, we find that $M_i = \bigoplus_{p \geq 0} H_p(K_i, R)$ and $\{M_i, \varphi_i\}$ is a persistence module let it be $\mathcal{M}$. Then, in this case $\alpha(\mathcal{M}) = \bigoplus_{i \geq 0} M_i = \bigoplus_{i \geq 0} (\bigoplus_{p \geq 0} H_p(K_i, R))$ is a single algebraic object, which is a graded $R[t]$ -module, that eventually presents the homology of the whole filtered simplicial complex. In fact, this can be formulated at the chain level by taking the direct sum of chain complexes $\bigoplus_{i=0}^n C^*(K_i, R)$, finding the correspondence persistence module and then applying an $R[t]$ -grading based on the filtration ordering.*

Furthermore, each simplex has the time that it entered the filtration, recorded by multiplying that simplex by a polynomial of the form t^n , where n is the filtration index that the simplex first appeared in. For instance, if a simplex ω enters the filtration in K_0 , then we write that simplex as $t^0 \cdot \omega = \omega$. Now if we want to shift that simplex upwards in the grading, we multiply ω by t . Hence, that same simplex ω that exists in the next simplicial complex K^1 is $t^1 \cdot \omega = (t)\omega$, and the simplex ω that exists in the simplicial complex K^2 in the filtration is $t^2 \cdot \omega = (t^2)\omega$ and so on.

Therefore, we see that the filtration index of the simplex is encoded by the degree of the polynomial coefficient. This is the principle of the work of Zomorodian and Carlsson. [4]

3.4 $\mathcal{P}$ -intervals

Let $\mathcal{M}$ be a persistence module. When our coefficients ring R is some field F , $\alpha(\mathcal{M})$ has a simple decomposition. As F is a field, it is a PID and then the graded ring $F[t]$ is necessarily a PID and its ideals are on the form (t^n) . On this account, the structure of the $F[t]$ -module is entirely characterized by the Structure Theorem 2.3. Hence, $\alpha(\mathcal{M})$ is isomorphic to that decomposition, i.e.

$$\alpha(\mathcal{M}) \cong \left(\bigoplus_{i=1}^n \Sigma^{\alpha_i} F[t] \right) \oplus \left(\bigoplus_{j=1}^m \Sigma^{\gamma_j} F[t]/(t^{n_j}) \right).$$

Definition 3.9 A $\mathcal{P}$ -interval is an ordered pair (i, j) with $0 \leq i < j \in \mathbb{Z}^\infty = \mathbb{Z} \cup \{\infty\}$.

In order to make a link between the $\mathcal{P}$ -intervals and persistence modules, we define a relation between the $\mathcal{P}$ -intervals and a graded $F[t]$ -module.

There exists a natural correspondence between a set, say, $\mathcal{S}$ of $\mathcal{P}$ -intervals and a graded $F[t]$ -module. Let $\mathcal{Q}$ be a bijection defined on $\mathcal{S}$ for that purpose, to associate a graded $F[t]$ -module to a set $\mathcal{S}$, by the following

$$\mathcal{Q}(i, j) = \begin{cases} \Sigma^{\gamma_i} F[t]/(t^{j-i}), & \text{for a finite } \mathcal{P}\text{-interval } (i, j) \\ \Sigma^{\alpha_i} F[t], & \text{for an infinite } \mathcal{P}\text{-interval } (i, +\infty) \end{cases}$$

For a set of $\mathcal{P}$ -intervals $\mathcal{S} = \{(i_1, j_1), (i_2, j_2), \dots, (i_n, j_n)\}$, we define

$$\mathcal{Q}(\mathcal{S}) = \bigoplus_{\ell=1}^n \mathcal{Q}(i_\ell, j_\ell)$$

Proposition 3.2 The correspondence $\mathcal{S} \rightarrow \mathcal{Q}(\mathcal{S})$ defines a bijection between the finite sets of $\mathcal{P}$ -intervals and the finitely generated graded modules over the graded ring $F[t]$.

This leads to the following restating of Theorem 3.1 in terms of the $\mathcal{P}$ -intervals.

Corollary 3.2 *The isomorphism classes of persistence modules of finite type over F are in bijective correspondence with the finite sets of $\mathcal{P}$ -intervals.*

We finalize this section by making a clear connection between the $\mathcal{P}$ -intervals and persistence that we discussed previously in Section 3.2. Realize that in the case when our ground ring is some field F , then the persistent homology groups that can be viewed as F -modules are also vector spaces over F . Moreover, when we compose the persistence module that corresponds to a filtered complex K over F , the persistence module is a collection of vector spaces, each of which is described fully by its rank (the Betti numbers).

We have deduced from Theorem 3.1, that this direct sum $\bigoplus_{i \geq 0} \bigoplus_{p \geq 0} H_p(K_i, F)$ of these vector spaces is a graded $F[t]$ -module, which is also characterized by a set of $\mathcal{P}$ -intervals. In fact, each interval of the $\mathcal{P}$ -intervals describes a basis element of one of these vector spaces. More concretely, an interval (i, j) forms an element in the basis of $H_p^{i,j}(K, F)$ where it corresponds to the p -hole (p -cycle) that is created in the i -th indexed simplicial complex of the filtration and lives as a p -hole until the j -th index of the filtration. There, at time j , it either becomes the boundary of some $(p+1)$ -dimensional simplex or is not a p -hole any more. Hence, this basis element has a life span, i.e. a persistence, of $j - i$.

We conclude that finding the $\mathcal{P}$ -intervals of a persistence module corresponding to a filtered complex is sufficient to compute its persistent homology over a field.

3.5 Barcodes and Persistence Diagrams

One of the aims of TDA and computational topology is to find a way to isolate data from noise. One way of achieving this is persistence. Furthermore, persistence gives us the chance to know which topological features of the data are meaningful and should be considered and which should be eliminated. In this section, we introduce barcodes and persistence diagrams. The main references are [10] and [15].

A *barcode* is a pictorial presentation of $H_p(\mathcal{K}, R)$, where

$\mathcal{K} : \emptyset \subseteq K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K$. We create this pictorial presentation in several steps. First, we have found in the previous sections that $H_p(\mathcal{K}, R)$ is a persistence module. Then, Theorem 3.1 tells that this persistence module is a single algebraic object which is a non-negatively graded $R[t]$ -module. Hence, by the generalized version of the Structured Theorem (Theorem 2.3) $H_p(\mathcal{K}, R)$ decomposes uniquely to the form:

$$H_p(\mathcal{K}, R) \simeq \left(\bigoplus_{i=1}^n \Sigma^{\alpha_i} R[t] \right) \oplus \left(\bigoplus_{j=1}^m \Sigma^{\gamma_j} R[t] / t^{a_j} R[t] \right).$$

The barcode representation arises from the parameter intervals in the above decomposition form. The grading gives the shifts, so Σ^{α_i} means shift over to the right by some amount α_i , and similarly for Σ^{γ_j} . Moreover, let λ be one of the homology generators of the free parts, i.e. $\lambda \in \Sigma^{\alpha_i} R[t]$ and so, λ born at parameter α_i and never dies. Let μ be one of the homology generators of the torsion parts, i.e. $\mu \in \Sigma^{\gamma_j} R[t] / t^{a_j} R[t]$ then, μ born at parameter γ_j and dies at $\gamma_j + a_j$, since a_j presents the length of the finite bars. The long bars in a barcode tell us the real information

whereas short barcodes is the noise.

For instance, let us assume that a homology class of a 1-dimensional hole appears somewhere in the filtration at time 2 (radius = 2) and lasts until time 5 (radius = 5), hence this homology class has a persistence of 3 units. Now, if we would like to interpret the torsion, which can be equivalently written in the ideal form as (t^{a_j}) , we start with an element from the persistence module at time 2. As applying t maps us forward in time, we apply t once to end up at time 3 in the next simplicial complex which has a 1-dimensional hole, then applying t again moves us to the simplicial complex at time 4. Finally, we apply t for a third time to get the zero homology class which means that the bar continues no further and so, the feature has been destroyed. So, it is merely like taking quotient by t^3 in this module to get the zero homology class.

Furthermore, as in Figure 3.2, most often the horizontal axes of a barcode represents the chosen radius (or the scale parameter) and the vertical axes corresponds to a random distribution of the generators of different homology modules. For a simplicial complex K_i in the filtration, the rank of every $H_p(K_i, R)$ can be found by counting the appearing intervals in the corresponding stage of $H_p(K, R)$ which overlap with the vertical line $\epsilon = \epsilon_i$. Roughly speaking, from the barcode we can recover the number of holes in each dimension at any scale by just slicing the barcode.

Each Betti number β_p is considered as a numerical mirror of its corresponding H_p and thus, as with the Betti number, the barcode of H_p gives a continuously parameterized rank rather than structural information about the homology. In contrast, and as we mentioned before, the capability of finding out the meaningful topological

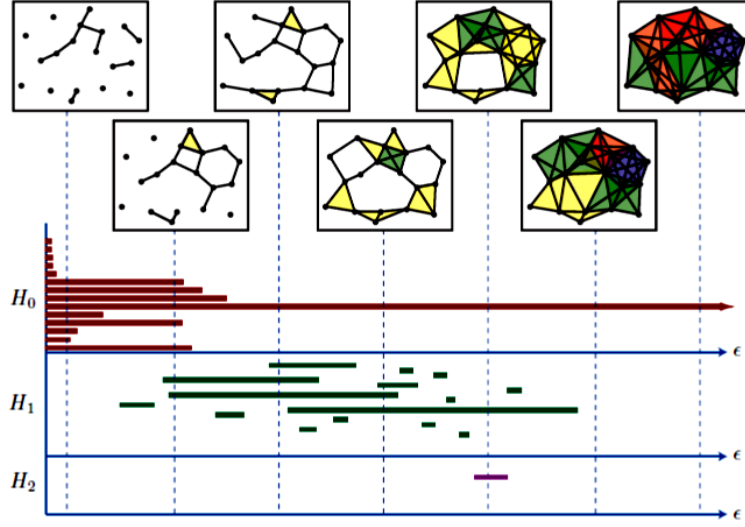


Figure 3.2: A filtration of a simplicial complex (top). Barcode representation for the ranks of the first three homology modules of each simplicial complex in the filtration (bottom).

features of a complex and filtering out the noisy ones, is provided by the brilliance of a barcode representation. For instance, we can observe that the data set in Figure 3.2 is probably sampled from an object that is connected and has one or two meaningful 1-dimensional holes and no other meaningful higher dimensional holes.

Lastly, we close this discussion on barcodes by giving this fundamental characterization of them as a result of the decomposition form.

Theorem 3.3 *The rank of the persistent homology module $H_p^{i,j}(K, R)$ is equal to the number of intervals in the barcode of $H_p(K, R)$ spanning the parameter interval $[i, j]$. In particular, $H_p(K_i, R)$ is equal to the number of intervals which contain i .*

We can visualize the same information in a slightly different way. For every barcode, let x represent its birth and y represent its death. The plot of points (x, y) in

$\mathbb{R}^2$ from all the barcodes is called a *persistence diagram*. In addition, the life span, or the persistence, of a topological feature is given by $y - x$.

The features that appear then disappear short after, i.e. those with low persistence, can be considered as noise and they are located near and on the diagonal line $y = x$. The following figure, Figure 3.3, is a persistence diagram of a given filtration $\mathcal{K}$ of a simplicial complex. It shows the birth and death times for the connected components and 1-dimensional holes of K , i.e. it represents the birth and death of the features of the homology modules $H_0(\mathcal{K}, R)$ and $H_1(\mathcal{K}, R)$ [12].

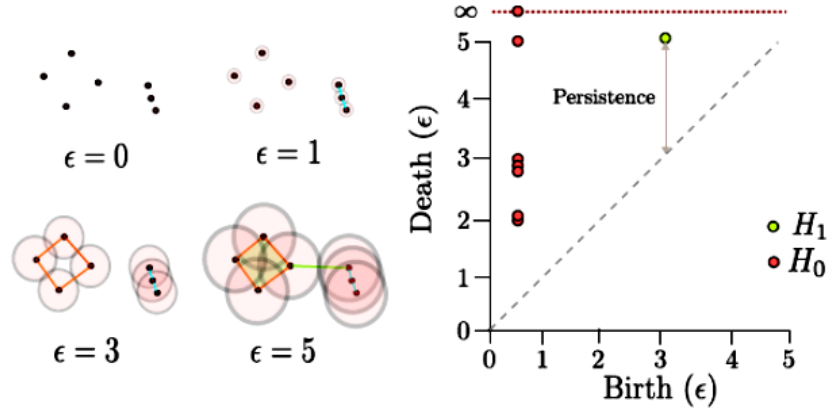


Figure 3.3: A filtration of a simplicial complex (left). The corresponding persistence diagram (right).

Moreover, each feature has an associated *persistence interval*, which is the vertical line segment between the persistence point and the diagonal. For instance, in Figure 3.3, the 1-dimensional hole (the cycle) is born at $\epsilon = 3$ ($x = 3$) and dies at $\epsilon = 5$ ($y = 5$) when it is destroyed and filled in, and so the persistence is $5 - 3 = 2$, and the persistence interval is $[3, 5]$.

CHAPTER 4

APPLICATION OF PERSISTENT HOMOLOGY: BRAIN ARTERY LENGTH

Topological Data Analysis (TDA) is a new thriving field providing a set of new topological and geometric tools to infer relevant features for possibly large and complex data. In this chapter, we apply persistent homology, as a TDA method, to quantify the bending of brain arteries through space. To be more precise, we show that the number of persistent number of 1-D holes correlates strongly with the artery length. Persistent homology analysis has proven to be useful on the study of brain arteries. In [16], it is shown that persistent homology analysis can be used to create new features that correlate with subjects' age and gender.

4.1 Dataset

The data set consists of brain artery tree information of 98 people with ages from 18 to 72 years old. The information is extracted from 3-D MRA (Magnetic Resonance Angiography) images with the help of the tube-tracking vessel segmentation algorithm followed by manual and automatic assembly. The details of this process can be found in [17]. The following auxiliary information is also retained on each subject: age, gender, artery length, and handedness (right handed, left handed, ambidextrous, right ambidextrous, or left ambidextrous). A 2-D visualization of a brain arteries tree is shown in Figure 4.1.

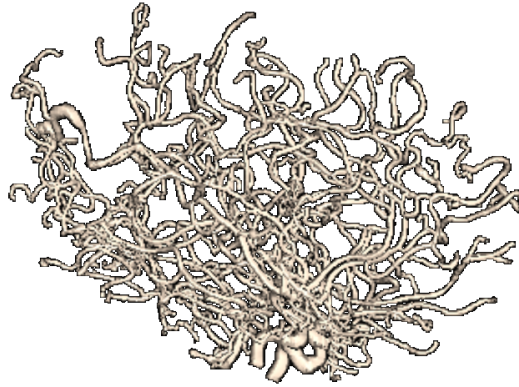


Figure 4.1: One data object of brain arteries tree of a person.

4.2 Preprocessing

For the purpose of this thesis, the tree structure of the brain arteries is going to be ignored and only the vertices of the underlying tree are retained. These vertices are the 3-dimensional cartesian coordinates of the points inside the arteries determined by

the tube-tracking algorithm. There are around 120K vertices per tree. However, due to lack of enough computational power, 10K vertices per tree are randomly selected for the analysis.

4.3 Methodology

We describe in short the methodology of the analysis using persistent homology implemented by the GUDHI Library [18], in Python. We start by creating a 2-dimensional Vietoris-Rips complex with *RipsComplex* function. Then, we create a filtration on the Rips complex with *create_simplex* method. After that we calculate the persistence on the filtration with the *persistence* method and extract the persistence intervals for 0 and 1 dimensions. Finally, we plot the intervals as barcodes using *plot_persistence_barcode* function and as persistence diagram (scatter plot) using *plot_persistence_diagram* function.

4.4 Results and Interpretation

Figures (4.2, 4.3, and 4.4) show the zero and one dimensional topological features, i.e. the connected components and the loops. We notice, as to our expectation, that only one connected component survives (a free element) and all the other connected components are torsional elements, hence have finite persistences. On the other hand, the loops are all torsional. This is also an expected fact, since the parameter ϵ of the VR complex increases, all the loops are eventually filled and therefore are death. In our experiment, we assumed maximum value of ϵ is 15. We call all the loops that still

exist beyond $\epsilon = 15$, high persistent loops. For further interpretation, we carry out some statistical analysis and study the correlation between the number of loops with high persistence, total artery length, and age. We find that the correlation between the total artery length and the number of loops with high persistence is approximately 0.7 which is an indication of a strong positive correlation. On the other hand, the correlation between the total artery length and the age is around -0.6 . Therefore, we can conclude that with persistence analysis, we have created a feature which has stronger correlation with the total artery length than the age. We should emphasize here that this is an original approach different than the approaches of other papers and is done for the first time only in this thesis.

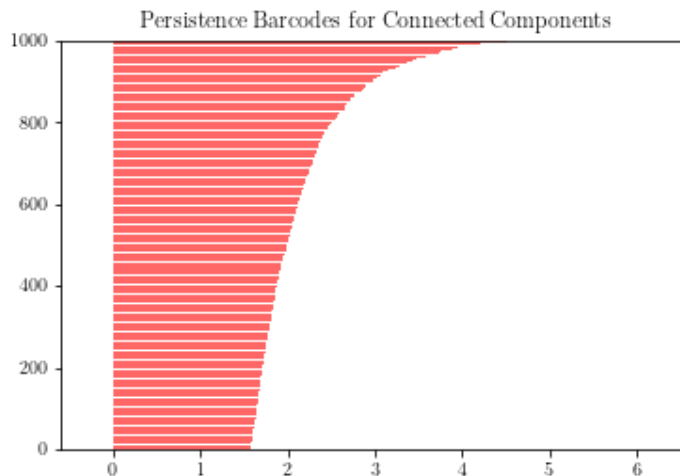


Figure 4.2: Zero dimensional persistence barcodes.

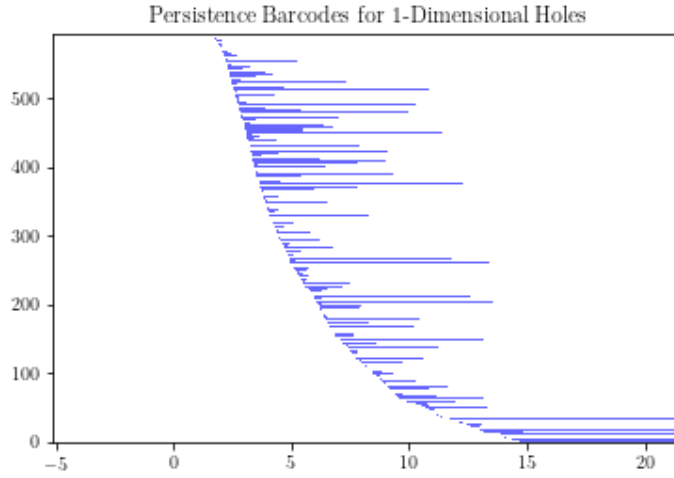


Figure 4.3: One dimensional persistence barcodes.

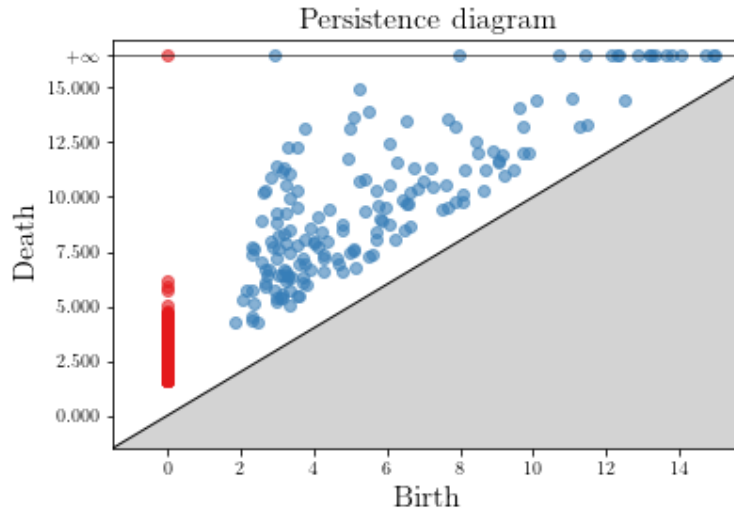


Figure 4.4: Persistence diagram of zero and one dimensional persistent homology features of the brain arteries of a person. Note that, the $+\infty$ horizontal line is above the maximum value of ϵ which is 15. The red point on the $+\infty$ line represents the free connected component. The blue points on the $+\infty$ line represent the high persistent loops.

We conclude this study with some health insights. As we have observed that the brains of old people are gradually decaying and the arteries are shrinking, we would like to introduce a brief explanation of that through the lens of medicine. Researchers

have presented a model that explains why memory deteriorates as the body ages. With age, the brain receives an increased load from the heart's beating as the body's large arteries stiffen over the years, causing damage to the smallest blood vessels in the brain. Moreover, the fact that human memory is declining with increasing age is something that most people experience sooner or later, even among those who avoid diseases such as Alzheimer's or a related dementia. However, researchers are studying many methods to prevent or delay Alzheimer's. Some focus on drugs, some on lifestyle or other changes. We take a look at the most promising interventions to date. A recent review of the paper [19], surveyed carefully at the evidence on approaches to prevent or delay Alzheimer's dementia or age-related reduction of cognitive. The paper found motivating but unsettled evidence for three types of interventions, as follows:

- Increased physical activity.
- Hypertension (blood pressure control for people with high blood pressure).
- Cognitive training which includes structured activities designed to enhance memory, reasoning, and speed of processing.

The verification for other preventions such as diet and medications was not as strong. However, scientists are proceeding to explore these and other possible interventions. For more details about the aforementioned interventions, refer to [19].

CHAPTER 5

CONCLUSION

Our central objective in this thesis is analysing the mathematical formalism behind TDA and its methods, in particular persistent homology. We have fulfilled this goal by accomplishing the following. First, we divided the topic into three important parts which are: the category of persistence objects, persistent homology, and computing persistent homology modules. Second, we proved important theorems that give an alternative view of a persistence module as a single algebraic object. Third, we demonstrated the connection among the parts. Finally, we discussed a real world application of TDA using persistent homology with a Python implementation.

REFERENCES

- [1] G. Carlsson, “Persistent Homology and Applied Homotopy Theory,” *arXiv*, pp. 1–42, 2020.
- [2] M. Lesnick, “Lecture notes for Math 840 : Multiparameter Persistence,” pp. 1–82, 2019.
- [3] “Homotopy.” [Online]. Available: <https://en.wikipedia.org/wiki/Homotopy>
- [4] A. Zomorodian and G. Carlsson, “Computing persistent homology,” *Discrete & Computational Geometry*, vol. 33, no. 2, pp. 249–274, 2005.
- [5] P. A. Grillet, *Abstract algebra*. Springer Science & Business Media, 2007, vol. 242.
- [6] D. S. Dummit and R. M. Foote, *Abstract algebra*. Wiley Hoboken, 2004, vol. 3.
- [7] F. Goodman, *Algebra: Abstract and Concrete, edition 2.6*. SemiSimple Press (Frederick Goodman), 2014.
- [8] G. Carlsson, *Topology and data*, 2009, vol. 46, no. 2.
- [9] H. Edelsbrunner and J. Harer, *Computational topology: an introduction*. American Mathematical Soc., 2010.

- [10] R. Ghrist, “Barcodes: the persistent topology of data,” *Bulletin of the American Mathematical Society*, vol. 45, no. 1, pp. 61–75, 2008.
- [11] A. J. Zomorodian, *Topology for computing*. Cambridge university press, 2005, vol. 16.
- [12] A. D. Smith, P. Dłotko, and V. M. Zavala, “Topological data analysis: Concepts, computation, and applications in chemical engineering,” *Computers and Chemical Engineering*, vol. 146, 2021.
- [13] J. R. Munkres, *Elements of algebraic topology*. CRC press, 2018.
- [14] R. Corbet and M. Kerber, “The representation theorem of persistence revisited and generalized,” *arXiv*, pp. 1–32, 2017.
- [15] N. Otter, M. A. Porter, U. Tillmann, P. Grindrod, and H. A. Harrington, “Open Access A roadmap for the computation of persistent homology,” *EPJ Data Sci.*, 2017. [Online]. Available: <http://dx.doi.org/10.1140/epjds/s13688-017-0109-5>
- [16] P. Bendich, J. S. Marron, E. Miller, A. Pieloch, and S. Skwerer, “Persistent homology analysis of brain artery trees,” *The annals of applied statistics*, vol. 10, no. 1, p. 198, 2016.
- [17] S. R. Aylward and E. Bullitt, “Initialization, noise, singularities, and scale in height ridge traversal for tubular object centerline extraction,” *IEEE transactions on medical imaging*, vol. 21, no. 2, pp. 61–75, 2002.
- [18] “Gudhi library.” [Online]. Available: <https://gudhi.inria.fr/python/latest/>

- [19] N. A. o. Sciences, “Preventing cognitive decline and dementia: A way forward,” Jun 2017. [Online]. Available: <https://www.nap.edu/catalog/24782/preventing-cognitive-decline-and-dementia-a-way-forward>

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