DECAY RATES OF SOME WEAKLY DISSIPATIVE WAVE EQUATIONS,

THEORY AND NUMERICS

BY

KHALEEL ANAYA

A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fufillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

MAY 2021

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **KHALEEL ANAYA** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS**.

Dissertation Committee

Dr. Kassem Mustapha (Advisor)

Dr. Salim Messaoudi (Co-advisor)

Dr. Khaled Furati (Member)

Dr. Ashfaque Bokhari (Member)

Dr. Muhammad Yousuf (Member)

Dr. Hussain Al-Attas
Department Chairman

Dr. Suliman Al-Homidan
Dean of Graduate Studies

Date

To my father Ibrahim,

To my mother Lamees,

To my lovely wife Sawsan, my children Ibrahim and Reem.

ACKNOWLEDGMENTS

All praise is to Allah Who is the One, and His blessings and peace be upon whom there is no messenger after him.

The foremost acknowledgments are due to my beloved grandfathers, grandmothers, my parents and my wife for their support, encouragement and prayers.

I would like to thank King Fahd University of Petroleum and Minerals (KFUPM) for giving me this opportunity to work on my PhD program.

My deep appreciation goes to my thesis supervisor Dr. Kassem Mustapha and the co-advisor Dr. Salim Messaoudi for their significant roles, encouragements and advices during this research work. They left a high impact on me.

Moreover, my deep thanks go to the thesis committee members for all of their efforts. I am also grateful to all faculty members and staff for providing a suitable research environment.

Finally, I sincerely thank all my relatives, teachers and friends.

TABLE OF CONTENTS

LIST OF FIGURES

THESIS ABSTRACT

NAME:	Khaleel Anaya
TITLE OF STUDY:	Decay Rates of Some Weakly Dissipative Wave Equations, Theory and Numerics
MAJOR FIELD:	Mathematics
DATE OF DEGREE:	May 2021

In this thesis, we consider some weakly dissipative viscoelastic plate equations with nonlinear damping, general damping and infinite memory. General decay rates are proved for a wide class of relaxation functions. To support our theoretical finding, some numerical results are delivered.

ملخص الرسالة

الاسم: خليل عنايا

عنوان الدراسة: معدّلات الاضمحلال العام لبعض أنظمة المرونة اللزجة، نظري وعددي

التخصص: الرياضيات

تاريخ الدرجة العلمية: أيار ٢٠٢١

في هذه الرسالة ، نقوم ببحث السلوك التقاربي لبعض أنظمة اللزوجة المرنة لمعادلات الصفائح. ندرس الحالات عندما تكون الذاكرة منتهية مع حد غير خطي وعندما تكون الذاكرة غير منتهية. ونثبت بعض نتائج الاضمحلال الجديدة لطاقة الحلول لهذه المعادلات، وندعم نتائجنا النظرية بحلول عددية. فنتائجنا تعمّم وتحسّن جميع النتائج الموجودة ذات الصلة بموضوع دراسنا وتسمح باستعمال مجموعة أكبر من دوال الإسترخاء.

CHAPTER 1

PRELIMINARIES

In this chapter, we introduce some terminology to be used in this thesis, give some literature review and present our results and contributions.

1.1 Introduction and Literature review

The well-posedness and asymptotic behaviour of viscoelastic equations have been the subject of study of many mathematicians since the pioneer work of Dafermos [9, 10] in 1970. Therein, he considered a one-dimensional viscoelastic equation and proved that its solution decays to zero without giving explicitly the rate of decay. In 2002, Cavalcanti *et al*. [8] looked into the following problem

$$\begin{cases} u'' - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u' = 0, & \text{in } \ \Omega \times (0,\infty), \\ u = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$ and $a : \Omega \longrightarrow \mathbb{R}_+$ is bounded and satisfies

$$a(x) \geq a_0 \quad a.e. \quad \text{on} \quad \omega \subset \Omega.$$

They made the following assumptions on the relaxation function g:

$$g(0) > 0, \qquad \int_0^\infty g(\tau)d\tau < 1,$$

and there exist two positive constants ξ_1, ξ_2 such that

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \qquad \forall\, t \geq 0, \tag{1.2}$$

and established an exponential decay rate for the solution of (1.1) under some extra conditions on ω. Berrimi and Messaoudi [4] removed the condition imposed on ω in [8] and they still had the exponential decay of the solution of (1.1). Moreover, they pushed the result to the case where a source term is competing with a viscoelastic damping in [5].

Up to the year 2008, most of the studies of viscoelastic problems considered relaxation functions satisfying

$$g'(t) \leq -\alpha g^p(t), \qquad \forall\, t \geq 0, \tag{1.3}$$

where α is a positive constant and $1 \leq p < \frac{3}{2}$ which gave either exponential or polynomial decay only. In 2008, Messaoudi [21, 22] established a more general rate of decay from which the exponential and polynomial decay rates are special cases. Precisely, he studied the following problem

$$\begin{cases} u'' - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau = \delta|u|^{m-2}u, & \text{in } \ \Omega \times (0,\infty), \\ u = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \ \Omega, \end{cases} \tag{1.4}$$

with $\delta = 0$ or $\delta = 1$ and g satisfying

$$g'(t) \le -\xi(t)g(t), \qquad \forall\, t \ge 0, \tag{1.5}$$

where $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-increasing differentiable function and proved that the energy of the solution of (1.4) decays with the same rate as g. Motivated by these two works of Messaoudi, many general decay results using (1.5) have been later proved. (See Han and Wang [13], Liu [18, 19], Cao [7] and references therein).

In 2016, Messaoudi and Al-Khulafi [24] proved a general and optimal decay rate of the solution of (1.4), with $\delta = 0$, for a class of relaxation functions satisfying

$$g'(t) \le -\xi(t)g^p(t), \qquad \forall\, t \ge 0, \qquad 1 \le p < \frac{3}{2}. \tag{1.6}$$

In 2017, Mustafa [26] pushed the result of Messaoudi and Al-Khulafi for a wider class of relaxation functions satisfying

$$g'(t) \leq -\xi(t)G(g(t)), \qquad \forall\, t \geq 0, \tag{1.7}$$

where $G : [0, \infty) \to (0, \infty)$ is a $\mathcal{C}^1$ function which is strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0, r],\ r \leq g(0)$, with $G(0) = G'(0) = 0$.

In 2010, Revira et.al.[31] studied the class of viscoelastic equations of the form

$$\begin{cases} u''(t) + Au(t) - \displaystyle\int_0^\infty g(\tau)A^\alpha u(t-\tau)d\tau = 0, \quad t > 0, \\ u(x, -t) = u_0(x, t), \qquad u'(x, 0) = u_1(x), \end{cases} \tag{1.8}$$

for $\alpha \in (0, 1)$ and $A : \mathcal{D}(A) \subset H \longrightarrow H$ is a positive definite self-adjoint operator on a Hilbert space H. They also assumed that, for a positive constant δ,

$$g'(t) \leq -\delta g(t), \qquad \forall\, t \geq 0. \tag{1.9}$$

Hassan and Messaoudi [14] investigated the general decay rate for

$$\begin{cases} u''(t) + Au(t) - \displaystyle\int_0^t g(t-\tau)A^\alpha u(\tau)d\tau = 0, \quad t > 0, \\ u(x, 0) = u_0(x), \qquad u'(x, 0) = u_1(x), \end{cases} \tag{1.10}$$

assuming that the relaxation function g satisfies condition (1.7).

In the absent of memory term, but the presence of non-linear damping, Messaoudi [23] considered the following problem:

$$\begin{cases} u'' + \Delta^2 u + |u'|^{m-2}u' = |u|^{p-2}u, & \text{in } \ \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \ \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \ \Omega. \end{cases} \tag{1.11}$$

Some well-posedness results were proved. Recently, Al-Gharabli, Guesmia and Messaoudi [1] studied the decay rate of the following viscoelastic plate equation with a log-arithmic nonlinearity:

$$\begin{cases} u'' + \Delta^2 u + u - \displaystyle\int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = u \ln|u|, & \text{in } \ \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \ \partial\Omega \times [0,\infty), \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \ \Omega, \end{cases} \tag{1.12}$$

where the relaxation function g satisfies condition (1.6). For more studies on the energy decaying analysis of various types of viscoelastic equations, we refer the readers to [3, 8, 12, 13, 16, 17, 18, 30] and references therein.

1.2 Results Description and Contributions

The main objective of this thesis is proving some decay rates for weakly dissipative viscoelastic plate equations with a wider class of relaxation functions. Our results extend that of Messaoudi and Hassan [14] to the case of plate equations with finite memory, and also, generalize the result of Rivera [30]. Precisely, our work is as follows:

In Chapter 2, we consider the viscoelastic plate equation with nonlinear damping and finite memory of the form

$$\begin{cases} u'' + \Delta^2 u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + d\,|u'|^{m-2}u' = 0 & \text{in } \ \Omega \times (0, +\infty), \\ u = \Delta u = 0, & \text{on } \ \partial\Omega, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \ \Omega, \end{cases} \tag{1.13}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$, d is a positive constant, u_0, u_1 are given data, $m > 1$ and g is a given non-negative function satisfying some conditions to be specified later.

In Chapter 3, we study the viscoelastic plate equation with infinite memory of the form

$$\begin{cases} u'' + \Delta^2 u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)ds = 0 & \text{in } \ \Omega \times (0, +\infty), \\ u(x,-t) = u_0(x,t), \quad u'(x,0) = u_1(x), & \text{in } \ \Omega, \end{cases} \tag{1.14}$$

subject to the homogeneous boundary conditions $u = \Delta u = 0$ on $\partial\Omega$, where the physical domain $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$. The history condition $u(x,-t) = u_0(x,t)$ means that we are taking into account all the deformation the material has undergone before the instant $t = 0$. The initial velocity u_1 and the non-negative relaxation function g are given.

Chapter 4 is devoted to the study of the viscoelastic plate equation with general nonlinear damping and finite memory and a given non-negative relaxation function g. Precisely, we are concerned with the following problem

$$\begin{cases} u'' + \Delta^2 u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + h(u') = 0 & \text{in } \Omega \times (0, +\infty), \\ u = \Delta u = 0, & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.15}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$, h is a function satisfying some conditions, see (4.5) below, and the initial data u_0, u_1 are given.

1.3 Some Important Notations and Inequalities

In this section, we present some notations and inequalities that are frequently used throughout this work.

- $\mathbb{R}_+$ denotes the set of non-negative real numbers, i.e., $\mathbb{R}_+ = [0, +\infty)$.

- Let $\Omega \subset \mathbb{R}^n$ and m be a non-negative integer, then we have the following:

  $\mathcal{C}^m(\Omega)$ represents the space of $m-$times continuously differentiable functions defined on Ω,

  $\mathcal{C}_0^m(\Omega)$ represents the space of $m-$times continuously differentiable functions with compact support in Ω,

$$\mathcal{C}^\infty(\Omega) = \bigcap_{m\geq 0} \mathcal{C}^m(\Omega) \qquad \text{and} \qquad \mathcal{C}_0^\infty(\Omega) = \bigcap_{m\geq 0} \mathcal{C}_0^m(\Omega).$$

- Let $\Omega \subset \mathbb{R}^n$ be a domain, we denote by $L^p(\Omega)$, $1 \leq p < +\infty$, the Lebesgue space

$$L^p(\Omega) := \left\{ f : \Omega \longrightarrow \mathbb{R} \middle| f \ \text{ is measurable and } \ \int_\Omega |f|^p < +\infty \right\},$$

equipped with the norm

$$\|f\|_p = \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

The space $L^\infty(\Omega)$ denotes

$$L^\infty(\Omega) := \left\{ f : \Omega \longrightarrow \mathbb{R} \middle| f \ \text{ is measurable and } \ \exists M \geq 0 \ \text{ s.t } \ |f| \leq M \ \ a.e. \ \text{ on } \ \Omega \right\},$$

equipped with the norm

$$\|f\|_\infty = \|f\|_{L^\infty(\Omega)} = \inf\{M \geq 0 : |f(x)| \leq M \quad a.e. \quad \text{on} \quad \Omega\}.$$

- $L^p_{loc}(\Omega) = \big\{ f : \Omega \longleftarrow \mathbb{R} \big| \ f \ \text{ is measurable and } \ f \in L^p(K) \ \forall \, K \hookrightarrow\hookrightarrow \Omega \big\}$

- For $m \in \mathbb{N}$, $1 \leq p \leq \infty$, we define the Sobolev space

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \middle| \ D^\alpha u \in L^p(\Omega) \ \ \forall \, \alpha \in \mathbb{N}^n \ \text{ with } \ |\alpha| \leq m \right\}$$

equipped with the norm

$$\|u\|_{m,p} = \|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_p^p \right)^{1/p}.$$

$D^\alpha u$ is the $\alpha-$th "weak" partial derivative of u which is defined as a locally integrable function g satisfying

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g\varphi, \qquad \forall\, \varphi \in \mathcal{C}_0^\infty(\Omega),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \ge 0$ an integer,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \qquad \text{and} \qquad D^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_n} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The space $W^{m,2}(\Omega)$ is denoted by $H^m(\Omega)$, which is a Hilbert space.

- Let $H^s(\Omega)$ ($s \ge 0$) be the standard Sobolev space. For $s = 0$, $H^s(\Omega)$ reduces to the $L^2(\Omega)$ space. On this space, $\langle \cdot, \cdot \rangle$ denotes the usual inner product and $\| \cdot \|$ is the associated $L^2(\Omega)$-norm. For $p \ne 2$, the norm on $L^p(\Omega)$ is denoted by $\| \cdot \|_p$.

- The space $W_0^{m,p}(\Omega)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ with respect to $W^{m,p}(\Omega)$ norm. The space $W_0^{m,2}(\Omega)$ is denoted by $H_0^m(\Omega)$.

- $\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n} \right)$, for any $u \in W^{1,p}(\Omega)$

- $\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$, for any $u \in W^{2,p}(\Omega)$

- $u' = \dfrac{\partial u}{\partial t}, \quad u'' = \dfrac{\partial^2 u}{\partial t^2}$

Throughout this thesis, we use the symbols c and C to denote generic positive constants that vary from one occurrence to the next. We also use the following inequalities repeatedly.

1. **Young's inequality.** Let $1 < p, q < \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $\varepsilon > 0$, we have

$$ab \leq \varepsilon a^p + C_\varepsilon b^q \qquad \forall\, a, b > 0,$$

where $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{p/q}}$. For $p = q = 2$, we have

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2.$$

2. **Hölder's inequality.** Let $1 \leq p, q \leq \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$

$$\int_\Omega |uv| \leq \|u\|_p \|v\|_q.$$

By taking $p = q = 2$, we obtain **Cauchy-Schwarz inequality**.

3. **Poincaré's inequality.** Suppose $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then there is a positive constant C (depending only on Ω and p) such that

$$\|u\|_p \leq C \|\nabla u\|_p \qquad \forall\, u \in W_0^{1,p}(\Omega).$$

4. **Jensen's inequality.**

Lemma 1.1 *Let* $H : [a, b] \to \mathbb{R}$ *be a convex function. If* $v : \Omega \to [a, b]$ *and* $w : \Omega \to \mathbb{R}^+$ *are integrable functions with* $\langle w, 1\rangle = \gamma > 0$*, then,* $H\left(\gamma^{-1}\langle v, w\rangle\right) \leq \gamma^{-1}\langle H(v), w\rangle$.

5. **Green's formula.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a regular boundary $\partial\Omega$. Then

$$\int_{\Omega} u\Delta v dx + \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\partial\Omega} u \frac{\partial v}{\partial n} d\sigma, \qquad \forall\, u \in H^1(\Omega),\ v \in H^2(\Omega).$$

CHAPTER 2

GENERAL DECAY RATE OF A WEAKLY DISSIPATIVE VISCOELASTIC PLATE EQUATION WITH A NONLINEAR DAMPING

2.1 Introduction

The modeling of a generalized Kirchhoff viscoelastic plate, where a bending moment relation with memory is considered, can be described by the following nonlinear weakly dissipative viscoelastic equation: for $m > 1$ and for a given non-negative relaxation function g,

$$\begin{cases} u'' + \Delta^2 u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + d\,|u'|^{m-2}u' = 0 & \text{in } \Omega \times (0, +\infty), \\ u = \Delta u = 0, & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$, d is a positive constant, and the initial data u_0, u_1 are given. Due to the use of the embedding theorem, we assume that $m > 1$ if $n = 1, 2$, while, $1 < m < \frac{2n}{n-2}$ for $n > 2$.

The main focus of this chapter is on investigating the energy decay of problem (2.1). The presence of the weakly viscoelastic dissapative term forces us to deal with what so-called a modified (or second) energy to achieve our goals. This makes the analysis more technical and also delicate. The proposed approach suggests to deal with the damping exponent m by considering first the case $m \geq 2$, followed by the case $1 < m < 2$. This is due to the validity of some of the technical inequalities used in our analysis. For different values of m, to confirm our theoretical finding numerically, we provide graphical illustrations of the decay of the energy, where the solution of problem (2.1) is approximated via finite differences in time and finite elements in space.

2.2 Preliminaries

We introduce the Sobolev space $\mathcal{H}(\Omega) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$. An application of the Poincaré inequality and by using the elliptic regularity property, there exist two positive constants ω_0 and ω_1 such that

$$\omega_0^{-1} \|\nabla w\|^2 \leq \|\Delta w\|^2 \leq \omega_1 \, \|\nabla(\Delta w)\|^2, \ \forall w \in \mathcal{H}(\Omega) . \tag{2.2}$$

In the decaying energy analysis, the following hypothesis is imposed.

(**A1**) The relaxation function $g \in \mathcal{C}^1(0,\infty)$ is assumed to be non-increasing,

$$g(0) > 0, \quad 1 - \max\{\omega_0, \omega_1\} \int_0^\infty g(s)ds =: l > 0, \tag{2.3}$$

and there exists a function $G \in \mathcal{C}^1(0,\infty)$ which is strictly increasing, and strictly convex $\mathcal{C}^2$ function on $(0, g(0)]$, with $G(0) = G'(0) = 0$, such that

$$-c_0 g(t) < g'(t) \leq -\xi(t) G(g(t)), \quad \forall \ t \geq 0, \tag{2.4}$$

where ξ is a positive non-increasing $\mathcal{C}^1$ function, and c_0 is a positive constant.

Remark 2.1 *[27] From (**A1**);*

(1) We conclude that $\lim_{s\to\infty} g(s) = 0$*, also* $\exists \ t_0 > 0$ *such that* $g(t_0) = r$*, while,* $g(s) \leq r$ *for* $s \geq t_0$*. Since* g *is non-increasing,* $0 < g(t_0) \leq g(s) \leq g(0)$ *for* $s \in [0, t_0]$*. Continuity of* G *on* $[0, r]$ *yields* $a \leq G(g(s)) \leq b$ *on* $[0, t_0]$*, for*

some constants $a, b > 0$. Consequently, using (2.4) yields

$$g'(s) \le -a\xi(s) = -\frac{a}{g(0)}\xi(s)g(0) \le -\frac{a}{g(0)}\xi(s)g(s).$$

Hence,

$$\xi(s)g(s) \le -\frac{g(0)}{a}g'(s), \qquad \forall\, s \in [0, t_0]. \tag{2.5}$$

(2) *There is a non-negative function $\bar{G}$ that extends G and its properties. As an example,*

$$\bar{G}(t) := \frac{G''(r)}{2}t^2 + \big(G'(r) - G''(r)r\big)t + \left(G(r) + \frac{G''(r)}{2}r^2 - G'(r)r\right), \quad \text{for } t > r.$$

For later use, by (2.2) and the second inequality in (2.3), we have

$$\|\Delta u(t)\|^2 \ge \hat{g}(t)\|\nabla u(t)\|^2 \text{ and } \|\nabla(\Delta u(t))\|^2 \ge \hat{g}(t)\|\Delta u(t)\|^2, \tag{2.6}$$

with $\hat{g}(t) := \int_0^t g(s)\,ds$. For convenience, we introduce the following notations: for $t > 0$, and for $0 < \varepsilon < 1$,

$$(g \circ w)(t) := \int_0^t g(t-s)\|w(t) - w(s)\|^2 ds, \quad C_\varepsilon := \int_0^\infty \frac{g^2(s)}{h_\varepsilon(s)}ds, \text{ with } h_\varepsilon(t) := \varepsilon g(t) - g'(t).$$

The next two lemmas will be used in the forthcoming decaying analysis section.

Lemma 2.1 *[15] If (**A1**) holds, then for any* $v \in L^2_{loc}\big([0,\infty); L^2(\Omega)\big)$,

$$\int_\Omega \left(\int_0^t g(t-s)(v(t)-v(s))ds\right)^2 dx \le C_\varepsilon (h_\varepsilon \circ v)(t), \qquad \text{for} \quad t \ge 0.$$

Lemma 2.2 *Assume that (**A1**) holds true. Then for any* $w \in H^1\big([0,\infty); L^2(\Omega)\big)$,

$$\int_0^t g(t-s)\langle w(s), w'(t)\rangle ds = \frac{1}{2}\frac{d}{dt}\Big[\|w(t)\|^2\hat{g}(t) - (g\circ w)(t)\Big] - \frac{1}{2}g(t)\|w(t)\|^2 + \frac{1}{2}(g'\circ w)(t)\,.$$

Proof. The desired identity follows immediately from

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\Big[\hat{g}(t)\|w(t)\|^2 - (g\circ w)(t)\Big] \\
&= \frac{1}{2}\frac{d}{dt}\Big[\hat{g}(t)\|w(t)\|^2 - \int_0^t g(t-s)\|w(t)-w(s)\|^2 ds\Big] \\
&= \frac{1}{2}g(t)\|w(t)\|^2 + \hat{g}(t)\langle w(t), w'(t)\rangle - \frac{1}{2}\int_0^t g'(t-s)\|w(t)-w(s)\|^2 ds \\
&\quad - \int_0^t g(t-s)\langle w(t)-w(s), w'(t)\rangle ds \\
&= \frac{1}{2}g(t)\|w(t)\|^2 - \frac{1}{2}(g'\circ w)(t) + \int_0^t g(t-s)\langle w(s), w'(t)\rangle ds\,.
\end{aligned}$$

Therefore,

$$\int_0^t g(t-s)\langle w(s), w'(t)\rangle ds = \frac{1}{2}\frac{d}{dt}\Big[\hat{g}(t)\|w(t)\|^2 - (g\circ w)(t)\Big] - \frac{1}{2}g(t)\|w(t)\|^2 + \frac{1}{2}(g'\circ w)(t).$$

∎

2.3 Decay

In this section, we aim to find the best possible estimate of the energy functional of problem (2.1). First, taking the inner product of (2.1) with u' gives

$$\langle u'', u'\rangle + \langle \Delta^2 u, u'\rangle + \int_0^t g(t-s)\langle \Delta u(s), u'(t)\rangle ds + d\langle |u'|^m, 1\rangle = 0\,.$$

Applying Green's formula (twice for the second term and once for the third term) and using the fact that $u' = \Delta u' = 0$ on $\partial\Omega$, yield the following weak formulation of (2.1):

$$\langle u'', u'\rangle + \langle \Delta u, \Delta u'\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla u'(t)\rangle ds + d\langle |u'|^m, 1\rangle = 0\,.$$

Using Lemma 2.2 with $w = \nabla u$, this equation can be rewritten as:

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - d\langle |u'(t)|^m, 1\rangle \leq 0. \tag{2.7}$$

where E is the first energy functional defined by

$$E(t) := \frac{1}{2}\Big[\|u'(t)\|^2 + \|\Delta u(t)\|^2 - \hat{g}(t)\|\nabla u(t)\|^2 + (g \circ \nabla u)(t)\Big] \geq 0, \tag{2.8}$$

Noting that, we used (2.6) in the last inequality.

Turning now to the second energy functional of (2.1). Taking the inner product of problem (2.1) with $-\Delta u'$ and then, applying Green's formula to the first, second and fourth terms, we get

$$\langle \nabla u'', \nabla u' \rangle + \langle \nabla(\Delta u), \nabla(\Delta u') \rangle - \int_0^t g(t-s)\langle \Delta u(s), \Delta u'(t)\rangle ds + d\,\langle \nabla(|u'|^{m-2}u'), \nabla u' \rangle = 0.$$

Hence, by Lemma 2.2, the above equation can be rewritten as:

$$\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u'(t)\|^2 + \|\nabla(\Delta u)(t)\|^2 - \hat{g}(t)\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\Big] = \\ -\frac{1}{2}g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g' \circ \Delta u)(t) - d\langle \nabla(|u'(t)|^{m-2}u'(t)), \nabla u'(t)\rangle.$$

Therefore, using (2.6), the second energy functional of (2.1) is

$$\mathcal{E}(t) := \frac{1}{2}\Big[\|\nabla u'(t)\|^2 + \|\nabla(\Delta u(t))\|^2 - \hat{g}(t)\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\Big] \geq 0\,. \quad (2.9)$$

Moreover, since $\langle \nabla(|u'|^{m-2}u'), \nabla u' \rangle = (m-1)\langle |u'|^{m-2}\nabla u', \nabla u' \rangle \geq 0$,

$$\mathcal{E}'(t) = -\frac{1}{2}g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g' \circ \Delta u)(t) - d(m-1)\langle |u'(t)|^{m-2}\nabla u'(t), \nabla u'(t)\rangle \leq 0. \quad (2.10)$$

In the next two lemmas, assuming that (**A1**) holds, we estimate the time derivative of

$$I_1(t) = \langle u(t), u'(t)\rangle \quad \text{and} \quad I_2(t) = -\int_0^t g(t-s)\langle \eta_t(s), u'(t)\rangle ds,$$

with $\eta_t(s) = u(t) - u(s)\,$.

Lemma 2.3 *Along the solution of (2.1), and for $\delta > 0$, we have*

$$I_1'(t) \le \|u'(t)\|^2 + \Big(\delta - \frac{l}{2}\Big)\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon \circ \nabla u)(t) + \frac{c}{\delta}\Big[\langle |u'|^m, 1\rangle\Big]^{\min\{1, \frac{2m-2}{m}\}}.$$

Proof. Differentiating I_1 yields $I_1'(t) = \|u'(t)\|^2 + \langle u''(t), u(t)\rangle$, and hence, using (2.1) and applying Green's formula, we get

$$I_1'(t) = \|u'(t)\|^2 - \|\Delta u(t)\|^2 + \int_0^t g(t-s)\langle \nabla u(s), \nabla u(t)\rangle ds - d\langle u(t)|u'(t)|^{m-2}, u'(t)\rangle\,.$$

Young's inequality, Lemma 2.1, and the inequalities in (2.2) imply that the third term in the right-hand side equals

$$\begin{aligned}
&\Big\langle \nabla u(t), -\int_0^t g(t-s)\nabla\eta_t(s)ds\Big\rangle + \hat{g}(t)\|\nabla u(t)\|^2 \\
&\quad \le \frac{l}{2\omega_0}\|\nabla u(t)\|^2 + \frac{\omega_0}{2l}\int_\Omega \Big(\int_0^t g(t-s)\nabla\eta_t(s)ds\Big)^2 dx + \omega_0\,\hat{g}(t)^2\|\Delta u(t)\| \\
&\qquad \le \frac{l}{2}\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon \circ \nabla u)(t) + (1-l)\|\Delta u(t)\|^2,
\end{aligned}$$

and hence,

$$I_1'(t) \le \|u'(t)\|^2 - \frac{l}{2}\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon \circ \nabla u)(t) - d\langle u(t)|u'(t)|^{m-2}, u'(t)\rangle\,.$$

For $m \ge 2$, an application of Hölder's inequality yields

$$d|\langle u|u'|^{m-2}, u'\rangle| \le d\|u\|_m\,\||u'|^{m-1}\|_{\frac{m}{m-1}} \le \delta\|u\|_m^m + (c/\delta)\|u'\|_m^m\,.$$

Since $H_0^1(\Omega) \hookrightarrow L^m(\Omega)$,

$$\|u\|_m^m \le c\|\nabla u\|^m \le c\|\nabla u\|^{m-2}\|\nabla u\|^2 \le c\big(E(0)\big)^{\frac{m-2}{2}}\|\Delta u\|^2,$$

and consequently, by Young's inequality,

$$d|\langle u|u'|^{m-2}, u'\rangle| \le c\Big(\|\Delta u\|^2\Big)^{1/m} \||u'|^{m-1}\|_{\frac{m}{m-1}} \le \delta\|\Delta u\|^2 + \frac{c}{\delta}\|u'\|_m^m\,.$$

In a similar fashion, for $1 < m < 2$,

$$d|\langle u|u'|^{m-2}, u'\rangle| \le d\|u\|\||u'|^{m-1}\| \le \delta\|\Delta u\|^2 + \frac{c}{\delta}\|u'\|_m^{2m-2}\,.$$

Combining the above results completes the proof of the lemma. ∎

Lemma 2.4 *Along the solution of (2.1), and for* $\delta > 0$, *we have*

$$I_2'(t) \le \delta\|\Delta u\|^2 - \Big(\hat{g}(t) - \delta\Big)\|u'\|^2 + \frac{c}{\delta}(C_\varepsilon + 1)(h_\varepsilon \circ \Delta u)(t) + \frac{c}{\delta}\Big[\langle |u'|^m, 1\rangle\Big]^{\min\{1, \frac{2m-2}{m}\}}\,.$$

Proof. Differentiating I_2 and using the differential equation in (2.1), we get

$$I_2'(t) = I_{2,1}(t) + I_{2,2}(t) + I_{2,3}(t) + I_{2,4}(t), \qquad (2.11)$$

where

$$
\begin{aligned}
I_{2,1}(t) &= \int_0^t g(t-s)\langle \eta_t(s), \Delta^2 u(t)\rangle ds \\
I_{2,2}(t) &= \left\langle \int_0^t g(t-s)\Delta u(s)ds, \int_0^t g(t-s)\eta_t(s)ds \right\rangle \\
I_{2,3}(t) &= \left\langle \int_0^t g(t-s)\eta_t(s)\, ds, d|u'(t)|^{m-2}u'(t) \right\rangle \\
I_{2,4}(t) &= -\left\langle u'(t), \int_0^t \Big[g'(t-s)\eta_t(s) + g(t-s)u'(t) \Big] ds \right\rangle .
\end{aligned}
$$

We, now, estimate these four terms. By Green's formula, Young's inequality and Lemma 2.1, we have

$$
I_{2,1}(t) \le \|\Delta u(t)\| \int_0^t g(t-s)\|\Delta \eta_t(s)\|ds \le \frac{\delta}{2}\|\Delta u(t)\|^2 + \frac{c}{\delta} C_\varepsilon (h_\varepsilon \circ \Delta u)(t),
$$

and

$$
\begin{aligned}
I_{2,2}(t) &= \Big\| \int_0^t g(t-s)\nabla \eta_t(s)ds \Big\|^2 + \left\langle \nabla u(t) \int_0^t g(t-s)ds, \int_0^t g(t-s)\nabla\eta_t(s)ds \right\rangle \\
&\le \Big(\int_0^t g(t-s)\|\nabla \eta_t(s)\|ds \Big)^2 + c\|\nabla u(t)\| \int_0^t g(t-s)\|\nabla\eta_t(s)\|ds \\
&\le C_\varepsilon (h_\varepsilon \circ \nabla u)(t) + \frac{\delta}{2\omega_0}\|\nabla u(t)\|^2 + \frac{c}{\delta} C_\varepsilon (h_\varepsilon \circ \nabla u)(t) .
\end{aligned}
$$

To estimate $I_{2,3}(t)$, we first handle the case $m \ge 2$. Young's inequality gives

$$
I_{2,3}(t) \le \delta \int_\Omega \Big| \int_0^t g(t-s)\eta_t(s)ds \Big|^m dx + \frac{c}{\delta}\int_\Omega |u'(t)|^m dx.
$$

Since

$$
\Big|\int_0^t g(t-s)\eta_t(s)ds\Big| \le \int_0^t g^{\frac{m-1}{m}}(t-s)g^{\frac{1}{m}}(t-s)\big|\eta_t(s)\big|ds
$$

$$
\le \Big(\hat{g}(t)\Big)^{\frac{m-1}{m}}\Big(\int_0^t g(t-s)|\eta_t(s)|^m ds\Big)^{\frac{1}{m}},
$$

applications of Hölder's inequality, Lemma 2.1, and the embedding $H_0^1(\Omega) \hookrightarrow L^m(\Omega)$ give

$$
\begin{aligned}
\int_\Omega \Big|\int_0^t g(t-s)\eta_t(s)ds\Big|^m dx &\le \Big(\hat{g}(t)\Big)^{m-1}\int_0^t g(t-s)\|\eta_t(s)\|_m^m\, ds \\
&\le c\int_0^t g(t-s)\|\nabla\eta_t(s)\|^{m-2}\|\nabla\eta_t(s)\|^2\, ds \\
&\le c\big(E(0)\big)^{\frac{m-2}{2}}\int_0^t g(t-s)\|\nabla\eta_t(s)\|^2 ds \le c(g\circ\nabla u)(t)\,.
\end{aligned}
$$

Merge the above three estimates, we obtain

$$
I_{2,3}(t) \le c\,\delta(g\circ\nabla u)(t) + \frac{c}{\delta}\langle|u'(t)|^m, 1\rangle, \quad \text{for} \ \ m\ge 2\,.
$$

For the second case $1<m<2$, we follow similar derivations. So, we get

$$
\begin{aligned}
I_{2,3}(t) &\le \delta\int_\Omega\Big(\int_0^t g(t-s)\eta_t(s)ds\Big)^2 dx + \frac{c}{\delta}\int_\Omega |u'|^{2m-2}dx \\
&\le c\,\delta\, C_\varepsilon(h_\varepsilon\circ\Delta u)(t) + \frac{c}{\delta}\Big[\langle|u'|^m, 1\rangle\Big]^{\frac{2m-2}{m}}\,.
\end{aligned}
$$

To estimate the last term $I_{2,4}(t)$, we use again Young's inequality, the fact that

$|g'| = |\varepsilon g - h_\varepsilon| \leq \varepsilon g + h_\varepsilon$, and Lemma 2.1. So, we have

$$I_{2,4}(t) \leq \delta\|u'(t)\|^2 + \frac{c}{\delta}\Big(\int_0^t (\varepsilon g(t-s) + h_\varepsilon(t-s))\|\eta_t(s)\|ds\Big)^2 - \hat{g}(t)\|u'(t)\|^2$$
$$\leq \Big(\delta - \hat{g}(t)\Big)\|u'(t)\|^2 + \frac{c}{\delta}\Big(\varepsilon^2(g \circ u)(t) + (h_\varepsilon \circ u)(t)\Big)$$
$$\leq \Big(\delta - \hat{g}(t)\Big)\|u'(t)\|^2 + \frac{c}{\delta}C_\varepsilon(h_\varepsilon \circ u)(t)\,.$$

Inserting the obtained estimates of $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, and $I_{2,4}$ in (2.11) will complete the proof. ∎

The achieved convolution estimates in the next lemma will also be needed in our forthcoming analysis. For convenience, we introduce the following notations: with $f(t) := \int_t^\infty g(s)ds$, let

$$J_1(t) := \int_0^t f(t-s)\|\nabla u(s)\|^2 ds \quad \text{and} \quad J_2(t) := \int_0^t f(t-s)\|\Delta u(s)\|^2 ds\,.$$

Lemma 2.5 *Assume that (**A1**) holds, then for $t \geq 0$,*

$$J_1'(t) \leq \frac{3}{\omega_0}(1-l)\|\nabla u\|^2 - \frac{1}{2}(g \circ \nabla u)(t) \;\text{ and }\; J_2'(t) \leq \frac{3}{\omega_0}(1-l)\|\Delta u\|^2 - \frac{1}{2}(g \circ \Delta u)(t)\,.$$

Proof. We only show the first inequality, the proof of the second inequality is similar. Using Young's inequality, assumption (**A1**), and the inequality $f(t) \leq$

$f(0) = \frac{1-l}{\omega_0}$ for $t \geq 0$, we observe

$$
\begin{aligned}
J_1'(t) &= f(0)\left\|\nabla u(t)\right\|^2 - \int_0^t g(t-s)\left\|\nabla u(s)\right\|^2 ds \\
&= f(t)\left\|\nabla u(t)\right\|^2 - \int_0^t g(t-s)\Big(\|\nabla \eta_t(s)\|^2 - 2\langle \nabla \eta_t(s), \nabla u(t)\rangle\Big) ds \\
&\leq f(0)\left\|\nabla u(t)\right\|^2 - (g \circ \nabla u)(t) + \frac{2}{\omega_0}(1-l)\left\|\nabla u(t)\right\|^2 + \frac{1}{2}(g \circ \nabla u)(t) \\
&\leq \frac{3}{\omega_0}(1-l)\left\|\nabla u(t)\right\|^2 - \frac{1}{2}(g \circ \nabla u)(t).
\end{aligned}
$$

Lemma 2.6 *For N, ε_1, $\varepsilon_2 > 0$, the functional*

$$
\mathcal{L}(t) := N(E(t) + \mathcal{E}(t)) + \varepsilon_1 I_1(t) + \varepsilon_2 I_2(t)
$$

satisfies

$$
\mathcal{L} \sim E + \mathcal{E} \quad \text{for a sufficiently large } N. \tag{2.12}
$$

Moreover, for any $t \geq t_0$, we have

$$
\begin{aligned}
&\mathcal{L}'(t) \leq -(1-l)\left(4 + \frac{3}{2\omega_0}\right)\left(\frac{2}{l}\|u'(t)\|^2 + \|\Delta u(t)\|^2\right) \\
&+\frac{1}{4}\Big((g\circ\nabla u)(t)+(g\circ\Delta u)(t)\Big)+\begin{cases} -c_0E'(t), & \text{for} \quad m \geq 2 \\ c\Big(\frac{\varepsilon_1}{\delta} + \varepsilon_2\Big)\Big[\langle |u'(t)|^m, 1\rangle\Big]^{\frac{2m-2}{m}}, & \text{for} \quad 1 < m < 2\,. \end{cases}
\end{aligned} \tag{2.13}
$$

Proof. The proof of (2.12) is done in [21]. To show (2.13), differentiating and get

$$\mathcal{L}'(t) = N(E'(t) + \mathcal{E}'(t)) + \varepsilon_1 I_1'(t) + \varepsilon_2 I_2'(t)\,.$$

Use of (2.7), (2.10), and Lemmas 2.3 and 2.4 yields

$$\mathcal{L}'(t) \leq N\Big[\frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 - d\langle|u'|^m,1\rangle - \frac{1}{2}g(t)\|\Delta u\|^2 + \frac{1}{2}(g'\circ\Delta u)(t)$$
$$-d\int_\Omega(m-1)|u'|^{m-2}\|\nabla u'\|^2\Big] + \varepsilon_1\Big[\|u'\|^2 + (\delta - \frac{l}{2})\|\Delta u\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t) + \frac{c}{\delta}\langle|u'|^m,1\rangle\Big]$$
$$+\,\varepsilon_2\Big[\delta\|\Delta u\|^2 - \Big(\hat{g}(t) - \delta\Big)\|u'\|^2 + \frac{c}{\delta}(C_\varepsilon + 1)((h_\varepsilon\circ\Delta u)(t)) + \frac{c}{\delta}\langle|u'|^m,1\rangle\Big]\,.$$

Since the relaxation function $g > 0$, then

$$\mathcal{L}'(t) \leq -\big[\big(\hat{g}(t) - \delta\big)\varepsilon_2 - \varepsilon_1\big]\|u'\|^2 - ((\frac{l}{2} - \delta)\varepsilon_1 - \varepsilon_2\delta)\|\Delta u\|^2 + \frac{N}{2}\Big[(g\circ\nabla u)(t) + (g\circ\Delta u)(t)\Big]$$
$$+\,\varepsilon_1 c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t) + \frac{\varepsilon_2 c}{\delta}(C_\varepsilon + 1)(h_\varepsilon\circ\Delta u)(t) + \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2)\langle|u'|^m,1\rangle.$$

Using $g'(t) := \varepsilon g(t) - h_\varepsilon(t)$, and noting that $h_\varepsilon > 0$, we get

$$\mathcal{L}'(t) \leq -\big[\big(\hat{g}(t) - \delta\big)\varepsilon_2 - \varepsilon_1\big]\|u'\|^2 - ((\frac{l}{2} - \delta)\varepsilon_1 - \varepsilon_2\delta)\|\Delta u\|^2 + \frac{N\varepsilon}{2}\Big[(g\circ\nabla u)(t) + (g\circ\Delta u)(t)\Big]$$
$$-\Big[\frac{N}{2} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) - \frac{c}{\delta}C_\varepsilon(\varepsilon_1 + \varepsilon_2)\Big](h_\varepsilon\circ\Delta u)(t) + \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2)\langle|u'|^m,1\rangle. \quad (2.14)$$

Choose $\delta < \frac{l}{8}\hat{g}(t_0)$ and $\varepsilon_1 = \frac{3}{8}\hat{g}(t_0)\varepsilon_2$ with $\varepsilon_2 = \frac{16(1-l)}{l\hat{g}(t_0)}\big(4 + \frac{3}{2\omega_0}\big)$. Simple calculations show that

$$(\hat{g}(t_0)-\delta)\varepsilon_2-\varepsilon_1 > \frac{2}{l}(1-l)\left(4+\frac{3}{2\omega_0}\right) \text{ and } (\frac{l}{2}-\delta)\varepsilon_1-\delta\varepsilon_2 > (1-l)\left(4+\frac{3}{2\omega_0}\right). \tag{2.15}$$

From $\dfrac{\varepsilon g^2(s)}{\varepsilon g(s)-g'(s)} < g(s)$, and by using the Lebesgue dominated convergence theorem, we deduce that

$$\lim_{\varepsilon\to 0^+} \varepsilon C_\varepsilon = \lim_{\varepsilon\to 0^+} \int_0^\infty \frac{\varepsilon g^2(s)}{\varepsilon g(s)-g'(s)} ds = 0.$$

So, there exists $\varepsilon_0 \in (0,1)$, such that, $\varepsilon C_\varepsilon < \frac{1}{\frac{8c}{\delta}(\varepsilon_1+\varepsilon_2)}$, for any $\varepsilon < \varepsilon_0$. Putting $\varepsilon = \frac{1}{2N}$ and choosing $N > \frac{4c}{\delta}(\varepsilon_1+\varepsilon_2)$, we observe, for $N > \frac{1}{2\varepsilon_0}$, that

$$\frac{N}{2}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2)-\frac{c}{\delta}C_\varepsilon(\varepsilon_1+\varepsilon_2) > \frac{N}{2}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2)-\frac{1}{8\varepsilon} = \frac{N}{4}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2) > 0. \tag{2.16}$$

Moreover, we have $\langle |u'|^m, 1\rangle \leq -c_0 E'(t)$. Finally, using this and the results in (2.15), (2.16), the estimate (2.14) becomes

$$\mathcal{L}'(t) \leq -(1-l)\left(4+\frac{3}{2\omega_0}\right)\left(\frac{2}{l}\|u'\|^2+\|\Delta u\|^2\right)+\frac{1}{4}\Big((g\circ\nabla u)(t)+(g\circ\Delta u)(t)\Big)-c_0E'(t).$$

which is the desired result. Similarly, we can easily establish the result when $1 < m < 2$. ∎

We are ready now to show our energy decaying results. The next theorem focuses on the case $m \geq 2$. Our subsequent analysis makes a frequent use of the quadratic functional defined, for a purely time dependent function ϕ and for

$0 \leq t_1 \leq t_2 \leq t$,

$$\mathcal{I}(\phi, t_1, t_2, t) := \int_{t_1}^{t_2} \phi(s) \left(\|\nabla(u(t) - u(t-s))\|^2 + \|\Delta(u(t) - u(t-s))\|^2 \right) ds.$$

For convenience, if $t_2 = t$, we let $\mathcal{I}(\phi, t_1, t) := \mathcal{I}(\phi, t_1, t, t)$. For later use, from the definition of $\mathcal{I}$ and the estimates $E'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t)$ (by (2.7)) and $\mathcal{E}'(t) \leq \frac{1}{2}(g' \circ \Delta u)(t)$ (by (2.10)), we have

$$-\mathcal{I}(g', t_1, t) \leq -c\big(E'(t) + \mathcal{E}'(t)\big), \qquad \forall\, t \geq t_1, \tag{2.17}$$

Theorem 2.2 (The case: $\mathbf{m \geq 2}$) *Putting $G_0(t) = tG'(t)$ and assuming that hypothesis (**A1**) holds, and the initial data $u_0 \in \mathcal{H}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, there exist positive constants λ_1, λ_2 such that the energy functional associated to problem (2.1) satisfies the estimate*

$$E(t) \leq \lambda_2 G_0^{-1} \left(\frac{\lambda_1}{\int_{t_0}^{t} \xi(s) ds} \right), \qquad \forall\, t > t_0\,. \tag{2.18}$$

Proof. From the non-increasing property of ξ, (2.5), the inequality $\mathcal{I}(\xi\, g, 0, t_0, t) \geq \mathcal{I}(g', 0, t_0, t)$, and the estimate in (2.17), we easily check that

$$\mathcal{I}(g, 0, t_0, t) \leq \frac{1}{\xi(t_0)} \mathcal{I}(\xi\, g, 0, t_0, t) \leq -\frac{g(0)}{\xi(t_0) a} \mathcal{I}(g', 0, t_0, t) \leq -c\big(E'(t) + \mathcal{E}'(t)\big), \tag{2.19}$$

for any $t \geq t_0$. Substituting this in (2.13), we obtain, for some $m_0 > 0$,

$$\mathcal{L}'(t) \leq -m_0 E(t) - c[E'(t) + \mathcal{E}'(t)] + c\,\mathcal{I}(g, t_0, t), \quad \forall \;\; t \geq t_0. \tag{2.20}$$

By Lemmas 2.5 and 2.6, the functional

$$\mathscr{L}(t) := \mathcal{L}(t) + J_1(t) + \frac{1}{2} J_2(t) + c_0 E(t)$$

is nonnegative and satisfies,

$$\mathscr{L}'(t) \leq -\frac{2}{l}(1-l)\left(4 + \frac{3}{2\omega_0}\right) \|u'\|^2 - (1-l)\,\|\Delta u\|^2 - \frac{1}{4}\,(g \circ \nabla u)\,(t) \leq -c_1 E(t),$$

for some $c_1 > 0$ and for any $t \geq t_0$. This estimate leads to the following bound

$$\int_0^\infty E(s) ds < \infty. \tag{2.21}$$

From the definition of E and the first inequality in (2.2), $E(t) \geq \frac{l}{2}\|\Delta u(t)\|^2 \geq \frac{l}{2\omega_0}\|\nabla u(t)\|^2$ for $t \geq 0$. Thus, for a positive constant γ, we obtain

$$\begin{aligned}\mathcal{I}(\gamma, t_0, t) &\leq 2\gamma \int_{t_0}^{t} \left(\|\nabla u(t)\|^2 + \|\nabla u(t-s)\|^2 + \|\Delta u(t)\|^2 + \|\Delta u(t-s)\|^2\right) ds \\ &\leq \frac{4\gamma}{l}(1+\omega_0) \int_{t_0}^{t} \big(E(t) + E(t-s)\big) ds. \end{aligned} \tag{2.22}$$

However, the last integral is finite due to (2.21) and the inequality $E(t) \geq 0$, and hence, for γ sufficiently small,

$$\mathcal{I}(\gamma, t_0, t) < 1, \ \forall t \geq t_0. \tag{2.23}$$

Since G is strictly convex and $G(0) = 0$, so we have

$$G(\zeta s) \leq \zeta G(s), \qquad \text{for} \quad 0 \leq \zeta \leq 1 \quad \text{and} \quad s \in (0, r]. \tag{2.24}$$

Combining (2.24) with (**A1**), Jensen's inequality and (2.23), we arrive at

$$\begin{aligned}
-\mathcal{I}(g', t_0, t) &= -\frac{1}{\mathcal{I}(\gamma, t_0, t)} \mathcal{I}(\mathcal{I}(\gamma, t_0, t)\, g', t_0, t) \\
&\geq \frac{1}{\mathcal{I}(\gamma, t_0, t)} \mathcal{I}(\mathcal{I}(\gamma, t_0, t)\, \xi\, G(g), t_0, t) \\
&\geq \frac{\xi(t)}{\mathcal{I}(\gamma, t_0, t)} \mathcal{I}(G(\mathcal{I}(\gamma, t_0, t)\, g), t_0, t) \\
&\geq \frac{\xi(t)}{\gamma} G\left(\gamma\, \mathcal{I}(g, t_0, t)\right) = \frac{\xi(t)}{\gamma} \bar{G}\left(\gamma\, \mathcal{I}(g, t_0, t)\right),
\end{aligned}$$

for any $t > t_0$, where $\bar{G}$ is introduced in Remark 2.1. Taking the inverse, we infer

$$\mathcal{I}(g, t_0, t) \leq \frac{1}{\gamma} \bar{G}^{-1}\Big(-\gamma \mathcal{I}\left(g', t_0, t\right) / \xi(t) \Big), \quad \text{for any} \ \ t \geq t_0.$$

Inserting this in (2.20), then with $\mathcal{F} := \mathcal{L} + cE + c\mathcal{E}$ (c is the constant occurred in (2.20)), yields

$$\mathcal{F}'(t) \leq -m_0 E(t) + \frac{c}{\gamma} \bar{G}^{-1}\left(-\gamma \mathcal{I}(g', t_0, t)/\xi(t)\right), \qquad \forall\, t \geq t_0\,. \tag{2.25}$$

Let $E_0(t) = E(t)/E(0)$. For $t \geq t_0$, define a functional $\mathcal{F}_1(t) := \bar{G}'\left(r_1 E_0(t)\right)\mathcal{F}(t)$. Differentiating, then using (2.25) and recalling $E_0' \leq 0$ and $G', G'' > 0$ lead to

$$\begin{aligned}
\mathcal{F}_1'(t) &= r_1 E_0'(t)\bar{G}''\left(r_1 E_0(t)\right)\mathcal{F}(t) + \bar{G}'\left(r_1 E_0(t)\right)\mathcal{F}'(t) \\
&\leq -m_0 E(t)\bar{G}'\left(r_1 E_0(t)\right) + \frac{c}{\gamma}\bar{G}'\left(r_1 E_0(t)\right)\bar{G}^{-1}\left(-\gamma\mathcal{I}(g', t_0, t)/\xi(t)\right), \quad \forall\, t \geq t_0.
\end{aligned} \tag{2.26}$$

If $\bar{G}^*$ is the convex conjugate function of $\bar{G}$ (see [2, pp. 61–64]), defined as

$$\bar{G}^*(s) = s\big(\bar{G}'\big)^{-1}(s) - \bar{G}\left(\big(\bar{G}'\big)^{-1}(s)\right) \tag{2.27}$$

and satisfies the generalized Young inequality

$$AB \leq \bar{G}^*(A) + \bar{G}(B), \tag{2.28}$$

with $A = \bar{G}'\left(r_1 E_0(t)\right)$ and $B = \bar{G}^{-1}\left(-\gamma\mathcal{I}(g', t_0, t)/\xi(t)\right)$, then combining (2.26) and (2.28) gives

$$\begin{aligned}
\mathcal{F}_1'(t) &\leq -m_0 E(t)\bar{G}'\left(r_1 E_0(t)\right) + c\gamma^{-1}\bar{G}^*\left[\bar{G}'\left(r_1 E_0(t)\right)\right] - c\mathcal{I}(g', t_0, t)/\xi(t) \\
&\leq -(m_0 E(0) - c r_1 \gamma^{-1})E_0(t)\bar{G}'\left(r_1 E_0(t)\right) - c\mathcal{I}(g', t_0, t)/\xi(t), \qquad \forall\, t \geq t_0.
\end{aligned}$$

Choosing $0 < r_1 < r$ such that $m_0 E(0) - c r_1 \gamma^{-1} = m_1$ for some $m_1 > 0$, then, multiplying both sides by $\xi(t)$, and using $r_1 E_0(t) < r$, and (2.17), we reach

$$\xi(t)\mathcal{F}_1'(t) \leq -m_1 E_0(t) G'\left(r_1 E_0(t)\right)\xi(t) - c\big(E'(t) + \mathcal{E}'(t)\big), \qquad \forall\, t \geq t_0.$$

Let $\mathcal{F}_2 = \xi\mathcal{F}_1 + c(E + \mathcal{E})$. Since $\xi' \leq 0$ and $\mathcal{F}_1 \geq 0$, the above equation leads to

$$m_1 E_0(t) G'\left(r_1 E_0(t)\right)\xi(t) \leq -\mathcal{F}_2'(t), \qquad \forall\, t \geq t_0. \tag{2.29}$$

However, the map $t \longmapsto E(t)G'\left(r_1 E_0(t)\right)$ is non-increasing (because G' is positive and increasing and E is non-increasing). Consequently, integrating (2.29) over the time interval (t_0, t) yields

$$\frac{m_1}{r_1} G_0\left(r_1 E_0(t)\right) \int_{t_0}^{t} \xi(s)ds \leq \frac{m_1}{r_1} \int_{t_0}^{t} G_0\left(r_1 E_0(s)\right)\xi(s)ds \leq \mathcal{F}_2(t_0) - \mathcal{F}_2(t) \leq \mathcal{F}_2(t_0).$$

Finally, using the fact that G_0 is strictly increasing, then the desired bound follows immediately. ▮

Example 2.3 *We give some examples of relaxation functions to illustrate our decay result (2.18).*

*(1) Choose $g(t) = ae^{-bt^{\nu}}$ with $\nu \in (0,1)$. Here, a and b are positive constants so that $1 - \max\{\omega_0, \omega_1\}\frac{a}{b} > 0$, i.e., (**A1**) is satisfied. Then, $g' = -\xi\, G(g)$ with $G(t) = t$ and $\xi(t) = \nu\, bt^{\nu-1}$. Theorem 2.2 concludes that, for a sufficiently large t,*

$$E(t) \leq c(t - t_0)^{-\nu}.$$

(2) *Choose* $g(t) = a(1+t)^{-\nu}$ *where* $\nu > 1$ *and* a *is chosen so that hypothesis* (**A1**) *remains valid. Here,* $g'(t) = -\xi(t)G(g(t))$ *with* $G(t) = t^{1+1/\nu}$ *and* $\xi(t) = b$*, where* b *is a fixed constant. Theorem 2.2 yields , for a sufficiently large* t*,*

$$E(t) \leq c(1+t)^{-\nu/(\nu+1)}.$$

Following the convention of the previous theorem, we show next the energy decaying rate for $1 < m < 2$. Indeed, some of the achieved estimates in Theorem 2.2 will be used in this case.

Theorem 2.4 (The case: $\mathbf{1 < m < 2}$) *Putting* $G_0(t) = t^{\frac{m}{2m-2}}G'(t)$*. Under the assumptions of the previous theorem, there exist positive constants* λ_1*,* λ_2 *such that*

$$E(t) \leq \lambda_2 (t-t_0)^{\frac{2m-2}{m}} G_0^{-1}\left(\frac{\lambda_1}{(t-t_0)\int_{t_0}^{t}\xi(s)ds}\right), \qquad \forall\, t > t_0 \tag{2.30}$$

Proof. From the first achieved estimate in the proof of the previous theorem, (2.19), and (2.13), we infer

$$\mathcal{L}'(t) \leq -m_0 E(t) - c[E'(t) + \mathcal{E}'(t)] + c\mathcal{I}(g, t_0, t) + c\Big[\langle |u'|^m, 1\rangle\Big]^{\frac{2m-2}{m}}, \quad \forall t \geq t_0. \tag{2.31}$$

By Lemmas 2.5 and 3.5, $\mathscr{L}(t) := \mathcal{L}(t) + J_1(t) + \frac{1}{2}J_2(t)$ is nonnegative and satisfies,

$$\mathscr{L}'(t) \leq -\frac{2}{l}(1-l)\left(4+\frac{3}{2\omega_0}\right)\|u'\|^2 - (1-l)\|\Delta u\|^2 - \frac{1}{4}(g \circ \nabla u)(t) + c\Big[\langle |u'|^m, 1\rangle\Big]^{\frac{2m-2}{m}}$$
$$\leq -c_0 E(t) + c_1\big(-E'(t)\big)^{\frac{2m-2}{m}}, \quad \forall t \geq t_0,$$

and for some $c_0,\ c_1 > 0$. Multiplying both sides by $E^q(t),\ q > 0$, and using Young's inequality, we have

$$E^q(t)\mathscr{L}'(t) \leq -c_0 E^{q+1}(t) + c_1 E^q(t)\big(-E'(t)\big)^{\frac{2m-2}{m}} \leq -c_0 E^{q+1}(t) + \varepsilon E^{\frac{qm}{2-m}}(t) + C_\varepsilon\big(-E'(t)\big).$$

Choosing $q = \frac{2-m}{2m-2}$, implies that, for ε small,

$$E^q\,\mathscr{L}' \leq -c_0 E^{q+1} - cE'$$

Consequently, (2.7) and (2.12) lead to

$$c_0 E^{q+1}(t) \leq -(E^q\mathscr{L} + cE)'(t). \tag{2.32}$$

However, the functional $E^q\mathscr{L} + cE \geq 0$, so $\int_0^\infty E^{q+1}(s)ds < \infty$. Hence, by Hölder's inequality, we obtain, $\forall t \geq t_0$,

$$\int_{t_0}^t E(s)ds \leq (t-t_0)^{\frac{q}{q+1}}\left[\int_{t_0}^t E^{q+1}(s)ds\right]^{\frac{1}{q+1}} \leq c(t-t_0)^{\frac{q}{q+1}} = c(t-t_0)^{\frac{2-m}{m}}. \tag{2.33}$$

Recalling (2.22) and using the above bound twice, we notice that, for a small positive constant γ and $\forall\, t \geq t_0$,

$$\mathcal{I}(\gamma, t_0, t) \leq \frac{4\gamma}{l}(1+\omega_0)\int_{t_0}^{t}[E(s)+E(t-s)]ds < c\gamma(t-t_0)^{\frac{2m-2}{m}} < (t-t_0)^{\frac{2m-2}{m}}. \tag{2.34}$$

Using (2.24), the hypothesis (**A1**), Jensen's inequality and (2.34), we obtain

$$\begin{aligned}
-\mathcal{I}(g', t_0, t) &= -\frac{(t-t_0)^{\frac{2m-2}{m}}}{\mathcal{I}(\gamma, t_0, t)}\mathcal{I}((t-t_0)^{\frac{2-2m}{m}}\mathcal{I}(\gamma, t_0, t)\, g', t_0, t) \\
&\geq \frac{(t-t_0)^{\frac{2m-2}{m}}}{\mathcal{I}(\gamma, t_0, t)}\mathcal{I}((t-t_0)^{\frac{2-2m}{m}}\mathcal{I}(\gamma, t_0, t)\, \xi\, G(g), t_0, t) \\
&\geq \frac{(t-t_0)^{\frac{2m-2}{m}}\xi(t)}{\mathcal{I}(\gamma, t_0, t)}\mathcal{I}(G((t-t_0)^{\frac{2-2m}{m}}\mathcal{I}(\gamma, t_0, t)\, g), t_0, t) \\
&\geq \gamma^{-1}(t-t_0)^{\frac{2m-2}{m}}\xi(t)G\Big(\gamma(t-t_0)^{\frac{2-2m}{m}}\mathcal{I}(g, t_0, t)\Big).
\end{aligned}$$

Replace G with $\bar{G}$ ($\bar{G}$ is introduced in Remark 2.1) and then, taking the inverse, we reach

$$\mathcal{I}(g, t_0, t) \leq \gamma^{-1}(t-t_0)^{\frac{2m-2}{m}}\bar{G}^{-1}\Big(-\frac{\gamma\mathcal{I}(g', t_0, t)}{(t-t_0)^{\frac{2m-2}{m}}\xi(t)}\Big), \quad \forall \;\; t \geq t_0,$$

and (2.31) becomes

$$\mathcal{F}'(t) \leq -m_0E(t) + \frac{(t-t_0)^{\frac{2m-2}{m}}}{\gamma}\bar{G}^{-1}\left(-\frac{\gamma\mathcal{I}(g', t_0, t)}{(t-t_0)^{\frac{2m-2}{m}}\xi(t)}\right) + c\Big[\langle|u'|^m, 1\rangle\Big]^{\frac{2m-2}{m}}, \tag{2.35}$$

for $t \geq t_0$, where $\mathcal{F} := \mathcal{L} + cE + c\mathcal{E}$.

Let $\mathcal{F}_1(t) := \bar{G}'\left(r_1E_0^m(t)\right)\mathcal{F}(t)$ for $t \geq t_0$, with $E_0^m(t) = (t-t_0)^{\frac{2-2m}{m}}E_0(t)$.

Since $E' \le 0$ and $G', G'' > 0$,

$$\begin{aligned}\mathcal{F}_1'(t) = &\Big[\frac{(m-2)r_1}{m(t-t_0)^{\frac{m+2}{m}}}E_0(t) + r_1(t-t_0)^{\frac{2-2m}{m}}E_0'(t)\Big]\bar{G}''\left(r_1 E_0^m(t)\right)\mathcal{F}(t)\\ &+\bar{G}'\left(r_1 E_0^m(t)\right)\mathcal{F}'(t)\\ &\le \bar{G}'\left(r_1 E_0^m(t)\right)\mathcal{F}'(t), \quad \forall\, t \ge t_0. \end{aligned} \tag{2.36}$$

For $t \ge t_0$, estimate (2.35) implies that

$$\begin{aligned}&\mathcal{F}_1'(t) \le -m_0 E(t)\bar{G}'\left(r_1 E_0^m(t)\right) + \gamma^{-1}(t-t_0)^{\frac{2m-2}{m}} \times\\ &\bar{G}^{-1}\left(-\gamma(t-t_0)^{\frac{2-2m}{m}}\frac{\mathcal{I}(g', t_0, t)}{\xi(t)}\right)\bar{G}'\left(r_1 E_0^m(t)\right) + c\bar{G}'\left(r_1 E_0^m(t)\right)\Big[\langle |u'|^m, 1\rangle\Big]^{\frac{2m-2}{m}}.\end{aligned}$$

Let $\bar{G}^*$ be defined as in (2.27) and satisfies (2.28). Setting

$$A = \bar{G}'\left(r_1 E_0^m(t)\right) \qquad \text{and} \qquad B = \bar{G}^{-1}\left(-\gamma(t-t_0)^{\frac{2-2m}{m}}\frac{\mathcal{I}(g', t_0, t)}{\xi(t)}\right),$$

then combining (2.28) and (2.36) gives

$$\begin{aligned}\mathcal{F}_1'(t) \le -m_0 E(t)\bar{G}'\left(r_1 E_0^m(t)\right) &+ c(t-t_0)^{\frac{2m-2}{m}}\bar{G}^*\left[\bar{G}'\left(r_1 E_0^m(t)\right)\right]\\ &- \frac{\gamma\mathcal{I}(g', t_0, t)}{\xi(t)} + c\bar{G}'\left(r_1 E_0^m(t)\right)\Big[\langle |u'|^m, 1\rangle\Big]^{\frac{2m-2}{m}}.\end{aligned}$$

Using the definition of $\bar{G}^*$ and $\bar{G} > 0$, the second term on the right-hand side is bounded by $cr_1E_0(t)\bar{G}'\left(r_1E_0^m(t)\right)$, and so, we get

$$\mathcal{F}_1'(t) \leq -m_1E_0(t)\bar{G}'\left(r_1E_0^m(t)\right) - \frac{\gamma\mathcal{I}(g',t_0,t)}{\xi(t)} + c\bar{G}'\left(r_1E_0^m(t)\right)\Big[\langle|u'|^m,1\rangle\Big]^{\frac{2m-2}{m}},$$

where $m_1 = m_0E(0) - cr_1$. Choosing $0 < r_1 < r$ (this implies that $r_1E_0^m(t) < r$ for a sufficiently large t) such that $m_1 > 0$, and then, multiplying both sides of the above equation by $\xi(t)E^{\frac{2-m}{2m-2}}(t)$, and using the achieved inequalities in (2.17) (for the middle term) and (2.7) (for the last term), we reach

$$\begin{aligned}\xi(t)E^{\frac{2-m}{2m-2}}(t)\mathcal{F}_1'(t) \leq & -\frac{m_1}{E(0)}\xi(t)E^{\frac{m}{2m-2}}(t)G'\left(r_1E_0^m(t)\right)\\ & - c\left(E'(t)+\mathcal{E}'(t)\right)E^{\frac{2-m}{2m-2}}(t) + c\,\xi(t)G'\left(r_1E_0^m(t)\right)E^{\frac{2-m}{2m-2}}(t)\Big(-E'(t)\Big)^{\frac{2m-2}{m}}.\end{aligned}$$

Since, $E^{\frac{2-m}{2m-2}}(t)\Big(-E'(t)\Big)^{\frac{2m-2}{m}} \leq \varepsilon E^{\frac{m}{2m-2}}(t) + \frac{1}{4\varepsilon}\Big(-E'(t)\Big)$, the above estimate takes the form

$$\begin{aligned}\xi(t)E^{\frac{2-m}{2m-2}}(t)\mathcal{F}_1'(t) \leq & \left(c\,\varepsilon - \frac{m_1}{E(0)}\right)\xi(t)E^{\frac{m}{2m-2}}(t)G'\left(r_1E_0^m(t)\right)\\ & - c\left(E'(t)+\mathcal{E}'(t)\right)E^{\frac{2-m}{2m-2}}(t) - \frac{c}{\varepsilon}\xi(t)G'\left(r_1E_0^m(t)\right)E'(t).\end{aligned}$$

Let $\mathcal{F}_2(t) = \xi(t)\mathcal{F}_1(t)E^{\frac{2-m}{2m-2}}(t) + c(E(t)+\mathcal{E}(t))E^{\frac{2-m}{2m-2}}(t) + \frac{c}{\varepsilon}\xi(t)G'\left(r_1E_0^m(t)\right)E(t)$. Since ξ, E and $\mathcal{E}$ are all non-increasing and since $1 < m < 2$, then simple calculations give, for ε small enough,

$$\mathcal{F}_2'(t) \leq -m_2\xi(t)E^{\frac{m}{2m-2}}(t)G'\left(r_1E_0^m(t)\right),$$

for some $m_2 > 0$. However, the map $t \longmapsto E^{\frac{m}{2m-2}}(t)\, G'\left(r_1 E_0^m(t)\right)$ is non-increasing (because G', $G'' > 0$ and $E' \leq 0$), and so, using the above inequality, we reach

$$m_2\, E^{\frac{m}{2m-2}}(t) G'\left(r_1 E_0^m(t)\right) \int_{t_0}^{t} \xi(s) ds \leq \int_{t_0}^{t} m_2 E^{\frac{m}{2m-2}}(s)\, G'\left(r_1 E_0^m(s)\right) \xi(s) ds$$
$$\leq - \int_{t_0}^{t} \mathcal{F}_2'(s) ds \leq \mathcal{F}_2(t_0)\,.$$

Multiplying both sides by $\frac{1}{t-t_0}$, and then, recalling that $G_0(s) = s^{\frac{m}{2m-2}} G'(s)$ is strictly increasing, the desired energy bound follows. ∎

Example 2.5 *We give some examples of relaxation functions to illustrate our decay result (2.30).*

(1) Choose $g(t) = ae^{-bt}$, *where* a *and* b *are positive constants such that* $1 - \max\{\omega_0, \omega_1\}\frac{a}{b} > 0$, *so, hypothesis* (**A1**) *is satisfied. Then,* $g'(t) = -\xi(t)G(g(t))$ *with* $G(t) = t$ *and* $\xi(t) = b$. *Thus, from Theorem 2.4, we conclude that*

$$E(t) \leq c(t-t_0)^{\frac{2-2m}{m}},\ \forall t > t_0.$$

(2) Choose $g(t) = a(1+t)^{-2}$ *for* $t \geq 0$, *and* a *is chosen so that hypothesis* (**A1**) *remains valid. Then* $g'(t) = -\xi(t)G(g(t))$ *with* $G(t) = t^{\frac{3}{2}}$ *and* $\xi(t) = b$, *is a constant function. So, under the assumptions of Theorem 2.4, we have*

$$E(t) \leq c(t-t_0)^{-\nu},\ \forall t > t_0,$$

where $\nu = \frac{6m-3m^2-6}{m(2m-1)} > 0$, *for* $1 < m < 1 + \frac{1}{\sqrt{3}}$.

2.4 Numerical study

This section is devoted to illustrate numerically the achieved theoretical decaying results in Theorems 2.2 and 2.4 on a sample test problem of the form (2.1). To do so, we develop a numerical scheme for the nonlinear model problem (2.1) using finite differences for the time discretization combined with the $\mathcal{C}^2$ continuous bicubic Galerkin method in space [28]. To avoid solving any nonlinear algebraic systems of equations, the approximation of the damping term is based on an extrapolation technique.

To discretize in time, we truncate the interval $(0, \infty)$ and work instead on the finite interval $(0, T]$ where T is large enough. Divide $[0, T]$ uniformly into N subintervals with size τ each and nodes $\{t_n\}_{n=0}^{N}$, that is, $t_n = n\tau$ for $0 \le n \le N$, where $\tau = T/N$. For the grid function w^n, let

$$\delta_t w^n = \frac{w^n - w^{n-1}}{\tau}, \quad \delta_{tt} w^n = \frac{w^{n+1} - 2w^n + w^{n-1}}{\tau^2},$$
$$w^{n+\frac{1}{2}} = \frac{w^n + w^{n-1}}{2}, \quad w^{n+\frac{1}{4}} = \frac{w^{n+1} + 2w^n + w^{n-1}}{4}.$$

Turning into the spatial discretization, choose $\Omega = (a, b) \times (c, d)$ and then divide both (a, b) (in the x-direction) and (c, d) (in the y-direction) into a family of uniform (quasi-uniform) cells. To elaborate, let $x_i = i\, h_x$ for $0 \le i \le M_x$ with $h_x = (b-a)/M_x$ and let $y_j = j\, h_y$ for $0 \le j \le M_y$ with $h_y = (d-c)/M_y$. Then, the $\mathcal{C}^2$ Galerkin finite dimensional space $S_h := S_{h_x} \otimes S_{h_y}$, where

$$S_{h_x} = \{v \in H^3(a,b) : \; v|_{[x_{i-1},x_i]} \in P_3 \text{ for } 1 \le i \le N_x, \text{ with } v(x)|_{x=a,b} = v''(x)|_{x=a,b} = 0\},$$

where P_3 is the space of polynomials of degree at most 3 in x and S_{h_y} is defined similarly.

Usually, continuous Galerkin finite element schemes are motivated by the weak formulation of the model problem. So, we take the inner product of (2.1) with $\phi \in \mathcal{H}(\Omega)$ then use Green's formula. This leads to

$$\langle u'', \phi\rangle + \langle \Delta u, \Delta\phi\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla\phi\rangle \, ds + d\langle |u'|^{m-2}u', \phi\rangle = 0. \quad (2.37)$$

Consequently, for each $t > 0$, the semi-discrete finite element solution $u_h(t) \in S_h$ is defined by

$$\langle u_h'', \phi\rangle + \langle \Delta u_h, \Delta\phi\rangle - \int_0^t g(t-s)\,\langle \nabla u_h(s), \nabla\phi\rangle \; ds + d\langle |u_h'|^{m-2}u_h', \phi\rangle = 0, \;\; \forall\, \phi \in S_h.$$

Our fully-discrete numerical solution U_h approximates u (at the time nodes) and is defined by

$$\langle \delta_{tt} U_h^n, \phi\rangle + \langle \Delta U_h^{n+\frac{1}{4}}, \Delta\phi\rangle - \int_0^{t_{n+1}} g(t_{n+1}-s)\langle \nabla \bar{U}_h(s), \nabla\phi\rangle ds + d\langle |\delta_t U^n|^{m-2}\delta_t U_h^n, \phi\rangle = 0, \quad (2.38)$$

$\forall\phi \in S_h$, and for $1 \le n \le N-1$, where $\bar{U}_h(s) = U_h^{j+\frac{1}{2}}$ for $t_j < s < t_{j+1}$ with $0 \le j \le N-1$.

At each time level, the above scheme amounts to a square linear system (see the matrix form below). So the existence of the approximate solution U_h^{n+1} follows from its uniqueness. For uniqueness, we need to show that if

$$\frac{1}{\tau^2}\langle U_h^{n+1},\phi\rangle+\frac{1}{4}\langle\Delta U_h^{n+1},\Delta\phi\rangle-\frac{1}{2}\int_{t_n}^{t_{n+1}} g(t_{n+1}-s)\,ds\langle\nabla U_h^{n+1},\nabla\phi\rangle ds=0,\quad \phi\in S_h,$$

then $U_h^{n+1}\equiv 0$. Choose $\phi=U_h^{n+1}$ and then, use the first inequality in (2.2) in addition to the non-increasing and positivity properties of g, we observe

$$\frac{1}{\tau^2}\|U_h^{n+1}\|^2+\frac{1}{4}\|\Delta U_h^{n+1}\|^2<\frac{\omega_0}{2}g(0)\tau\|\Delta U_h^{n+1}\|^2\,.$$

Hence, for a sufficiently small τ, $(\frac{\omega_0}{2}g(0)\tau\ \le\ \frac{1}{4})$, $\frac{1}{\tau^2}\|U_h^{n+1}\|^2\ \le\ 0$ and thus, $\frac{1}{\tau^2}\|U_h^{n+1}\|\equiv 0$. This completes the proof of uniqueness, and consequently the existence, of the numerical solution. For computing purposes, we write our scheme in a matrix form. Let $\{\phi_p\}_{p=1}^{d_{hx}}$ and $\{\psi_p\}_{p=1}^{d_{hy}}$ denote the basis functions of S_{h_x} and S_{h_y}, respectively, with $d_{hx}:=\dim S_{hx}=N_x-1$ and $d_{hy}:=\dim S_{hy}=N_y-1$. So, U_h^n can be written in terms of the basis functions as:

$$U_h^n=\sum_{i=1}^{d_{hx}}\sum_{j=1}^{d_{hy}} b_{i,j}^n\phi_{p_i}\psi_{pj}.$$

We define the $d_{hx}\times d_{hx}$ matrices in the $x-$direction as

$$\boldsymbol{M}_x=\left[\int_a^b \phi_q\phi_p dx\right],\quad \boldsymbol{G}_x=\left[\int_a^b \phi_q'\phi_p' dx\right],\quad \text{and}\quad \boldsymbol{S}_x=\left[\int_a^b \phi_q''\phi_p'' dx\right],$$

and the $d_{hy} \times d_{hy}$ matrices in the $y-$direction are

$$\boldsymbol{M}_y = \left[\int_c^d \psi_q \psi_p dy \right], \quad \boldsymbol{G}_y = \left[\int_c^d \psi_q' \psi_p' dy \right], \quad \text{and} \quad \boldsymbol{S}_y = \left[\int_c^d \psi_q'' \psi_p'' dy \right].$$

The $(d_{hx} \times d_{hy})$-dimensional column vectors $\mathbf{b}^n$ and $\mathbf{F}^n$ are the transpose of the vectors

$$[b_{1,1}^n, b_{1,2}^n, \cdots, b_{1,d_{hy}}^n, \cdots, b_{d_{hx},1}^n, \cdots, b_{d_{hx},d_{hy}}^n],$$

and

$$[f_{1,1}^n, f_{1,2}^n, \cdots, f_{1,d_{hy}}^n, \cdots, f_{d_{hx},1}^n, \cdots, f_{d_{hx},d_{hy}}^n], \quad \text{with} \quad f_{i,j}^n := d\langle |\delta_t U_h^n|^{m-2} \delta_t U_h^n, \phi_i \psi_j \rangle,$$

respectively. Therefore, through tensor products of one-dimensional $\mathcal{C}^2$ splines, the fully-discrete scheme (2.38) has the following matrix representation:

$$\begin{aligned} \Big(\boldsymbol{M}_x \otimes \boldsymbol{M}_y \Big) \delta_{tt} \mathbf{b}^n + \Big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y \Big) \mathbf{b}^{n+\frac{1}{4}} \\ - \sum_{j=0}^{n} g_{n+1,j} \Big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y \Big) \mathbf{b}^{j+\frac{1}{2}} = -\mathbf{F}^n, \end{aligned}$$

with $g_{n+1,j} := \int_{t_j}^{t_{j+1}} g(t_{n+1} - s)\, ds$. Alternatively, this can be rewritten as:

$$
\begin{aligned}
&\Big(4\boldsymbol{M}_x \otimes \boldsymbol{M}_y + \tau^2\big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\big) \\
&\quad - 2\tau^2 g_{n+1}^n\big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\big)\Big)\mathbf{b}^{n+1} = 4\boldsymbol{M}_x \otimes \boldsymbol{M}_y(2\mathbf{b}^n - \mathbf{b}^{n-1}) \\
&\quad - \tau^2\Big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\Big)(2\mathbf{b}^n + \mathbf{b}^{n-1}) \\
&+2\tau^2 g_{n+1}^n\big(\boldsymbol{G}_x\otimes\boldsymbol{M}_y+\boldsymbol{M}_x\otimes\boldsymbol{G}_y\big)\mathbf{b}^n - 2\tau^2\big(\boldsymbol{G}_x\otimes\boldsymbol{M}_y+\boldsymbol{M}_x\otimes\boldsymbol{G}_y\big)\Big(\sum_{j=0}^{n-1} g_{n+1,j}(\mathbf{b}^{j+1}+\mathbf{b}^j)\Big) - \mathbf{F}^n,
\end{aligned}
$$

for $1 \leq n \leq N-1$. Therefore, at each time level t_{n+1}, we solve a finite square linear system, where the unknown is the column vector $\mathbf{b}^{n+1}$.

Furthermore, from the matrix form, it is clear that our scheme (2.38) is a three-time level scheme. That is, the approximate solutions U_h^0 and U_h^1 need to be determined first, and then U_h^j for $2 \leq j \leq N$ can be computed by solving the above linear system recursively. We choose $U_h^0 \in S_h$ to be the bicubic spline polynomial which interpolates u_0 at the interior nodal nodes. However, motivated by the Taylor series expansion of u about $t = 0$, we choose $U_h^1 \in S_h$ to be the bicubic spline polynomial that interpolates $u_0 + t_1 u_1$ at the interior nodal nodes.

For the computer implementation of the linear system, it is important to consider discretization of spatial Galerkin-type integrals in the scheme. To this end, on each cell of our two-dimensional partition, the integrals are approximated using 2-point Gauss quadrature rule in each direction (x and y).

In our test problem, we choose $\Omega = (0,1)\times(0,1)$, the time interval is $(0,80)$, the initial data $u_0(x,y) = 64^2[xy(1-x)(1-y)]^3$, $u_1(x,y) = 0$, the relaxation function

$g(t) = \mathrm{e}^{-t}$, and the damping coefficient $d = 1$. The spatial mesh consists of 400 (square) cells of equal areas, while the time domain consists of 80000 subintervals.

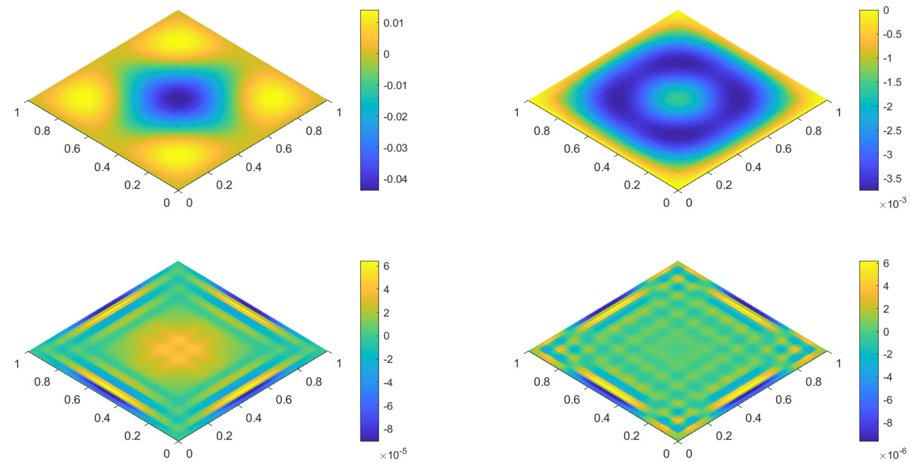

Figure 2.1: The numerical solutions when $m = 2$, and for $t = 5$ (top-left), $t = 10$ (top-right), $t = 20$ (bottom-left), and $t = 30$ (bottom-right).

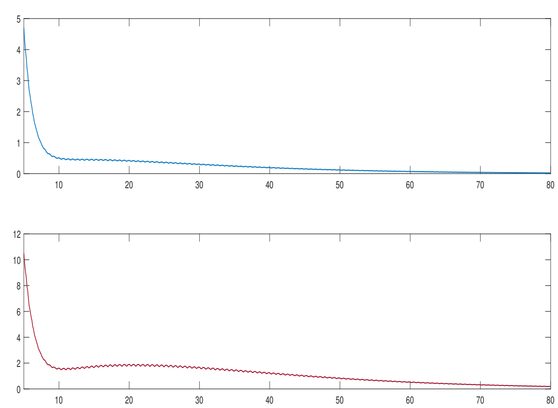

Figure 2.2: The numerical weighted energy plots against $t \in [5, 80]$, with $m = 2$. The top and the bottom are the approximations of $tE(t)$ and $t^{1.5}E(t)$, respectively.

Figures 2.1 and 2.3 show that the numerical solutions U_h converge to zero as the time t getting far away from 0, for different choices of damping exponent m. The graphical plots of the numerical approximations of the weighted energy in Figures 2.2 and 2.4 confirm that $tE(t) \leq 1$ for a sufficiently large t. This is compatible with the achieved theoretical results in Theorem 2.2 (see Example 2.3).

The next aim is to illustrate numerically the achieved theoretical results in Theorem 2.4 when the damping exponent $1 < m < 2$. For large t, we expect the energy functional to decay as $t^{\frac{2-2m}{m}}$, see Example 2.5. We choose $m = 1.2$, the numerical weighted energy plots in Figure 2.6 confirm this, furthermore, the graphical plots of the approximate solutions of U_h^n in Figure 2.5 show the zero decaying as t getting large.

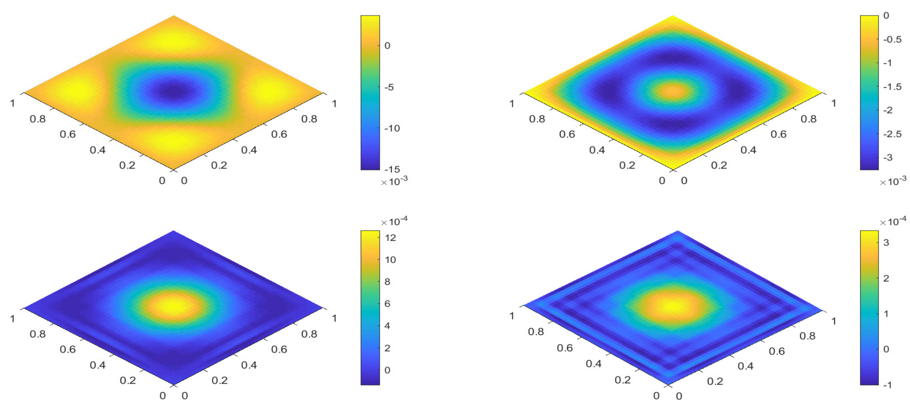

Figure 2.3: The numerical solutions when $m = 2.5$, and for $t = 5$ (top-left), $t = 10$ (top-right), $t = 20$ (bottom-left), and $t = 30$ (bottom-right).

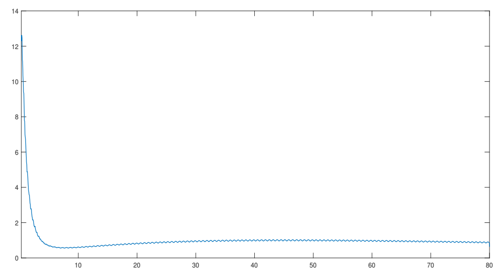

Figure 2.4: The graphical plot of the approximation of the weighted energy $tE(t)$ against t in the interval $[0.2, 80]$, with $m = 2.5$.

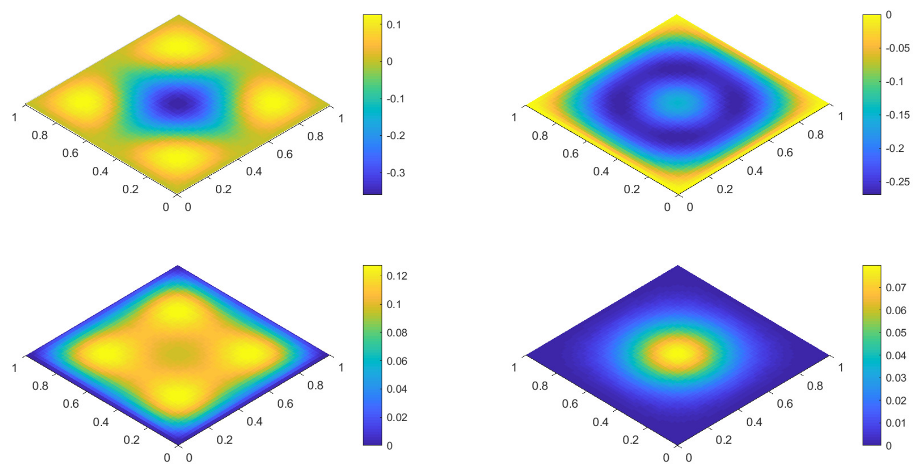

Figure 2.5: The numerical solutions when $m = 1.2$, and for $t = 5$ (top-left), $t = 10$ (top-right), $t = 15$ (bottom-left), and $t = 20$ (bottom-right).

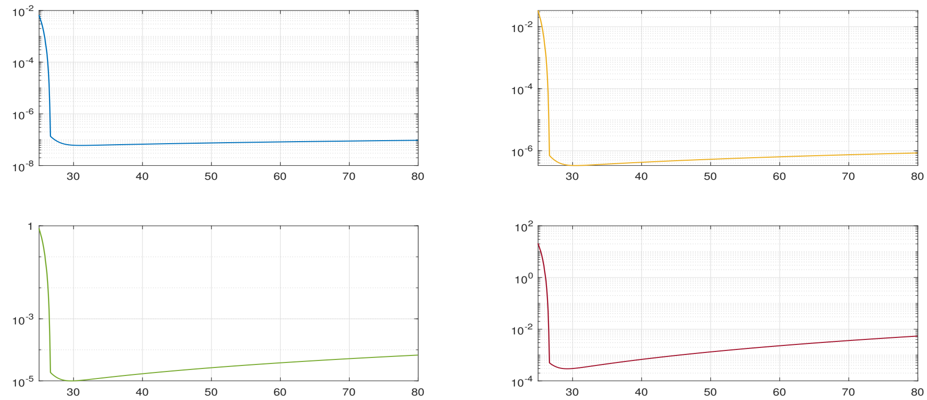

Figure 2.6: The numerical weighted energy plots (log-scale) against $t \in [25, 80]$, with $m = 1.2$. The top-left, the top-right, the bottom-left, and the bottom-right are the approximations of $t^{1/2}E(t)$, $tE(t)$, $t^2E(t)$, and $t^3E(t)$, respectively.

CHAPTER 3

GENERAL DECAY RATE OF A WEAKLY DISSIPATIVE VISCOELASTIC PLATE EQUATION WITH AN INFINITE MEMORY

3.1 Introduction

The modeling of a generalized Kirchhoff viscoelastic plate, where a bending moment relation with infinite memory is taken in account, can be described by the following weakly dissipative viscoelastic equation:

$$\begin{cases} u'' + \Delta^2 u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)ds = 0 \quad \text{in } \Omega \times (0, +\infty), \\ u(x, -t) = u_0(x, t), \quad u'(x, 0) = u_1(x), \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

subject to the homogeneous conditions $u = \Delta u = 0$ on $\partial\Omega$, where the physical domain $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$. The history condition $u(x, -t) = u_0(x, t)$ means that we are taking into account all the deformation the material has undergone before the instant $t = 0$. The initial velocity u_1 and the non-negative relaxation function g are given.

In the next section, we introduce some necessary notations and assumptions. For our decaying analysis, we state and prove a few technical lemmas. Section 3.3 is dedicated to show the decaying rates of the energy functional E, see Theorem 3.1. Having a weakly dissepative term in problem (3.1) leads us to introduce a second energy functional $\mathcal{E}$ (see (3.8)) to overcome the difficulties in proving the decay of E. For the sake of illustrating the theoretical decaying rate of E numerically, we develop a fully-discrete numerical method in Section 3.4. To avoid dealing with $\mathcal{C}^2$ numerical methods in the spatial variable (which is often not convenient on various physical domains Ω) due to the presence of the biharmonic operator in problem (3.1), we rewrite (3.1) as a coupled system that involves a second-order elliptic operator instead. Then, we apply the $\mathcal{C}^0$ Galerkin finite element method to discretize in space. In the time variable, a second central difference is used to handle the second time derivative, while the other terms are approximated appropriately. We show the decaying of both, the numerical solution of problem (3.1) and also the approximation of the energy functional E.

3.2 Preliminaries

An application of the Poincaré inequality and by using the elliptic regularity property, we have

$$\frac{1}{\omega_0}\|\nabla w\|^2 \leq \|\Delta w\|^2 \leq \omega_1 \ \|\nabla(\Delta w)\|^2, \ \forall w \in \mathcal{H}(\Omega), \tag{3.2}$$

for some positive constants ω_0 and ω_1, where $\mathcal{H}(\Omega) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$. In the decaying energy analysis (including Lemmas 3.1 and 3.2), we assume that the relaxation function $g \in \mathcal{C}^1(\mathbb{R}^+)$ and satisfies

$$g(0) > 0, \ \ 1-\max\{\omega_0, \omega_1\}\mu^0 =: l > 0, \text{ and } -c_0 g(t) < g'(t) \leq -\xi(t)g(t), \quad \forall\, t \geq 0, \tag{3.3}$$

where ξ is a positive non-increasing $\mathcal{C}^1$ function, $\mu^0 = \int_0^\infty g(s)ds$ and c_0 is a positive constant, see Example 3.2, for various choices of g.

For later use, by (3.2) and the second inequality in (3.3), we have, for $t \geq 0$,

$$\frac{1}{\mu^0}\|\nabla(\Delta u(t))\|^2 \geq \|\Delta u(t)\|^2 \geq \|\nabla u(t)\|^2 \mu^0\,. \tag{3.4}$$

For convenience, we introduce the following notations: for $t \geq 0$,

$$(g \circ u)(t) := \int_0^\infty g(s)\|\eta^t(s)\|^2 ds, \quad (g \circ [\nabla u, \Delta u])(t) := (g \circ \nabla u)(t) + (g \circ \Delta u)(t),$$

and for $0 < \varepsilon < 1$,

$$C_\varepsilon := \int_0^\infty \frac{g^2(s)}{h_\varepsilon(s)} ds \qquad \text{with} \qquad h_\varepsilon(t) := \varepsilon g(t) - g'(t),$$

where η^t is the relative history of u [11], defined as $\eta^t(s) = u(t) - u(t-s)$.

The next two lemmas will be used in the forthcoming decaying analysis section.

Lemma 3.1 *[15] For any $v \in L^2_{loc}\big([0, +\infty); L^2(\Omega)\big)$,*

$$\Big\| \int_0^\infty g(s)(v(t-s) - v(t)) ds \Big\|^2 \leq C_\varepsilon (h_\varepsilon \circ v)(t), \qquad \forall\, t \geq 0.$$

Lemma 3.2 *[29] For any $\varphi \in H^1\big([0, \infty); L^2(\Omega)\big)$,*

$$\int_0^\infty \langle g(s)\varphi(t-s), \varphi'(t)\rangle ds = \frac{1}{2}\frac{d}{dt}\Big[\mu^0\|\varphi(t)\|^2 - (g \circ \varphi)(t)\Big] + \frac{1}{2}(g' \circ \varphi)(t).$$

3.3 Decay

In this section, we find the energy functional E of problem (3.1). As a starting point, taking the inner product of (3.1) with u', and then, applying Green's formula (twice for the second term and once for the third term) and using the fact that $u' = \Delta u' = 0$ on $\partial\Omega$, yield the following weak formulation of (3.1):

$$\langle u''(t), u'(t)\rangle + \langle \Delta u(t), \Delta u'(t)\rangle - \int_0^\infty g(s)\langle \nabla u(t-s), \nabla u'(t)\rangle ds = 0\,.$$

Using Lemma 3.2 with $\varphi = \nabla u$, this equation can be rewritten as:

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \tag{3.5}$$

with E is the first energy functional given by

$$E(t) := \frac{1}{2}\Big[\|u'(t)\|^2 + \|\Delta u(t)\|^2 - \mu^0\|\nabla u(t)\|^2 + (g \circ \nabla u)(t)\Big] \geq 0, \tag{3.6}$$

where the non-negative property of E follows from (3.4).

Now, taking the inner product of (3.1) with $-\Delta u'$, then, following the above steps, we get

$$\mathcal{E}'(t) = \frac{1}{2}(g' \circ \Delta u)(t) \leq 0, \tag{3.7}$$

where, $\mathcal{E}$ is the second energy functional, defined as:

$$\mathcal{E}(t) := \frac{1}{2}\Big[\|\nabla u'(t)\|^2 + \|\nabla(\Delta u(t))\|^2 - \mu^0\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\Big] \geq 0\,, \tag{3.8}$$

and its non-negativity follows also from (3.4). In the remaining analysis, c is a generic positive constant. We estimate in the next two lemmas the time derivative of the functionals

$$I_1(t) = \langle u'(t), u(t)\rangle \quad \text{and} \quad I_2(t) = -\int_0^\infty \langle \eta^t(s), u'(t)\rangle g(s)ds\,.$$

Lemma 3.3 *Along the solution of (3.1), we have*

$$I_1'(t) \le \|u'(t)\|^2 - \frac{l}{2}\|\Delta u(t)\|^2 + c\; C_\varepsilon(h_\varepsilon \circ \nabla u)(t)\,. \tag{3.9}$$

Proof. Since $I_1'(t) = \|u'(t)\|^2 + \langle u''(t), u(t)\rangle$, using (3.1) and Green's formula, we get

$$I_1'(t) = \|u'(t)\|^2 - \|\Delta u(t)\|^2 + \mu^0\|\nabla u(t)\|^2 - \int_0^\infty g(s)\langle \nabla\eta^t(s), u(t)\rangle\, ds.$$

Young's inequality, Lemma 3.1, and the inequalities in (3.2) imply that the last two terms in the right hand side are estimated as follows:

$$\begin{aligned}
&\mu^0\|\nabla u(t)\|^2 - \int_0^\infty g(s)\langle \nabla\eta^t(s), u(t)\rangle\, ds \\
&\qquad \le \omega_0\mu^0\|\Delta u(t)\|^2 + \frac{l}{2\omega_0}\|\nabla u(t)\|^2 + \frac{\omega_0}{2l}\left\|\int_0^\infty \nabla\eta^t(s)g(s)ds\right\|^2 \\
&\qquad\qquad \le (1-l)\|\Delta u(t)\|^2 + \frac{l}{2}\|\Delta u(t)\|^2 + cC_\varepsilon(h_\varepsilon \circ \nabla u)(t)\,.
\end{aligned}$$

Combining the above results completes the proof. ∎

Lemma 3.4 *Along the solution of (3.1) and for $\delta > 0$, we have*

$$I_2'(t) \le \delta \|\Delta u(t)\|^2 - \left(\mu^0 - \delta\right) \|u'(t)\|^2 + \frac{c}{\delta}(C_\varepsilon + 1)(h_\varepsilon \circ \Delta u)(t). \tag{3.10}$$

Proof. Differentiating I_2 and using the differential equation in (3.1), we get:

$$I_2'(t) = I_{2,1}(t) + I_{2,2}(t) + I_{2,3}(t) - \mu^0 \|u'(t)\|^2, \tag{3.11}$$

where

$$I_{2,1}(t) = \int_0^\infty g(s)\langle \eta^t(s), \Delta^2 u(t)\rangle ds, \quad I_{2,2}(t) = \left\langle \int_0^\infty g(s)\Delta u(t-s)ds, \int_0^\infty g(s)\eta^t(s)ds \right\rangle,$$
$$I_{2,3}(t) = \left\langle \int_0^\infty g(s)\eta_s^t(s)\, ds, u'(t) \right\rangle.$$

By Green's formula, Young's inequality and Lemma 3.1, we have

$$I_{2,1}(t) \le \|\Delta u(t)\| \int_0^\infty g(s)\|\Delta \eta^t(s)\| ds \le \frac{\delta}{2}\|\Delta u(t)\|^2 + \frac{c}{\delta} C_\varepsilon (h_\varepsilon \circ \Delta u)(t),$$

and in addition, using $u(t-s) = u(t) - \eta^t(s)$, we get

$$\begin{aligned} I_{2,2}(t) &= \left\| \int_0^\infty g(s)\nabla \eta^t(s) ds \right\|^2 - \left\langle \int_0^\infty g(s)\nabla u(t) ds, \int_0^\infty g(s)\nabla \eta^t(s) ds \right\rangle \\ &\le C_\varepsilon (h_\varepsilon \circ \nabla u)(t) + c\|\nabla u(t)\| \int_0^\infty g(s)\|\nabla \eta^t(s)\| ds \\ &\le C_\varepsilon (h_\varepsilon \circ \nabla u)(t) + \frac{\delta}{2\omega_0}\|\nabla u(t)\|^2 + \frac{c}{\delta} C_\varepsilon (h_\varepsilon \circ \nabla u)(t)\,. \end{aligned}$$

To estimate $I_{2,3}(t)$, we perform integration by parts and then make use of Young's and Hölder's inequalities, the fact that $g' = \varepsilon g - h_\varepsilon$ and Lemma 3.1. So, we obtain

$$\begin{aligned}
I_{2,3}(t) &= -\Big\langle \int_0^\infty g'(s)\eta^t(s)ds, u'(t)\Big\rangle \\
&= -\varepsilon\Big\langle \int_0^\infty g(s)\eta^t(s)ds, u'(t)\Big\rangle + \Big\langle \int_0^\infty h_\varepsilon(s)\eta^t(s)ds, u'(t)\Big\rangle \\
&\leq \frac{\delta}{2}\|u'(t)\|^2 + \frac{\varepsilon^2}{2\delta}\Big\| \int_0^\infty g(s)\eta^t(s)ds\Big\|^2 + \frac{\delta}{2}\|u'(t)\|^2 + \frac{1}{2\delta}\Big\| \int_0^\infty h_\varepsilon(s)\eta^t(s)ds\Big\|^2 \\
&\leq \delta\|u'(t)\|^2 + \frac{c}{\delta}(h_\varepsilon \circ u)(t) + \frac{1}{2\delta}\Big(\int_0^\infty h_\varepsilon^{\frac{1}{2}}(s)h_\varepsilon^{\frac{1}{2}}(s)\|\eta^t(s)\|ds\Big)^2 .
\end{aligned}$$

Inserting the obtained estimates of $I_{2,1}(t)$, $I_{2,2}(t)$, and $I_{2,3}(t)$ in (3.11), and using

$$\Big(\int_0^\infty h_\varepsilon^{\frac{1}{2}}(s)h_\varepsilon^{\frac{1}{2}}(s)\|\eta^t(s)\|ds\Big)^2 \leq \int_0^\infty h_\varepsilon(s)ds \int_0^\infty h_\varepsilon(s)\|\eta^t(s)\|^2 ds \leq c(h_\varepsilon \circ u)(t),$$

(by Young's and Hölder's inequalities), in addition to the elliptic regularity property of the operator $-\Delta$, the desired bound follows. ▮

Lemma 3.5 *For N, ε_1, $\varepsilon_2 > 0$, the functional*

$$\mathcal{L}(t) := N(E(t) + \mathcal{E}(t)) + \varepsilon_1 I_1(t) + \varepsilon_2 I_2(t)$$

satisfies

$$\mathcal{L} \sim E + \mathcal{E} \quad \textit{for a sufficiently large } N. \tag{3.12}$$

Moreover, there exist positive constants α_1 and α_1, such that

$$\mathcal{L}'(t) \leq -\alpha_1 E(t) + \alpha_2 \int_0^\infty g(s)\left(\|\nabla\eta^t(s)\|^2 + \|\Delta\eta^t(s)\|^2\right) ds, \quad \forall t \geq 0 . \tag{3.13}$$

Proof. For the proof of (3.12), we refer to [20]. To establish the estimate (3.13), we differentiate $\mathcal{L}$ and use (3.5) and (3.7), in addition to Lemmas 3.3 and 3.4. Hence, we get

$$\mathcal{L}'(t) \leq \frac{N}{2}(g' \circ [\nabla u, \Delta u])(t) + \varepsilon_1 \Big[\|u'(t)\|^2 - \frac{l}{2}\|\Delta u(t)\|^2 + c\, C_\varepsilon (h_\varepsilon \circ \nabla u)(t) \Big] + \varepsilon_2 \Big[\delta \|\Delta u(t)\|^2 - \left(\mu^0 - \delta\right) \|u'(t)\|^2 + \frac{c}{\delta}(C_\varepsilon + 1)(h_\varepsilon \circ \Delta u)(t) \Big].$$

Recalling the identity $g'(t) = \varepsilon g(t) - h_\varepsilon(t)$, then rearranging the terms in the above equation and noting that $h_\varepsilon > 0$, we arrive at

$$\mathcal{L}'(t) \leq -\Big[\left(\mu^0 - \delta\right) \varepsilon_2 - \varepsilon_1 \Big] \|u'\|^2 - \big(\frac{l}{2}\varepsilon_1 - \varepsilon_2 \delta\big) \|\Delta u\|^2 + \frac{N\varepsilon}{2}(g \circ [\nabla u, \Delta u]) + \Big[\frac{c}{\delta}(1 + C_\varepsilon)(\varepsilon_1 + \varepsilon_2) - \frac{N}{2} \Big] (h_\varepsilon \circ \Delta u)(t).$$

Choosing $\varepsilon = \frac{1}{N}$. Since $\dfrac{g^2(s)}{\varepsilon g(s) - g'(s)} \leq \dfrac{g(s)}{\varepsilon}$, $C_\varepsilon \leq N\mu^0$. Hence, for $N \geq 2\frac{c}{\delta}(1 + N\mu^0)(\varepsilon_1 + \varepsilon_2)$ (which is valid for ε_1 and ε_2 are sufficiently small), we notice from the above equation that

$$\mathcal{L}'(t) \leq -\Big[\left(\mu^0 - \delta\right) \varepsilon_2 - \varepsilon_1 \Big] \|u'\|^2 - \big(\frac{l}{2}\varepsilon_1 - \varepsilon_2 \delta\big) \|\Delta u\|^2 + \frac{1}{2}(g \circ [\nabla u, \Delta u]).$$

Choosing $\delta < \frac{l}{8}\mu^0 < \frac{1}{8}\mu^0$ (because $0 < l < 1$) and $\varepsilon_1 = \frac{3}{8}\mu^0 \varepsilon_2$, after some calculations, we have

$$(\mu^0 - \delta)\varepsilon_2 - \varepsilon_1 > \frac{1}{2}\mu^0 \varepsilon_2, \; \frac{l}{2}\varepsilon_1 - \delta \varepsilon_2 > \frac{l}{16}\mu^0 \varepsilon_2,$$

and consequently,

$$\mathcal{L}'(t) \leq -\frac{1}{2}\mu^0\varepsilon_2\|u'\|^2 - \frac{l}{16}\mu^0\varepsilon_2\|\Delta u\|^2 + \frac{1}{2}(g \circ [\nabla u, \Delta u])(t).$$

Finally, recalling the definition of E and η^t, and using the above bound, we obtain the desired estimate. ∎

Now we state and finalize the proof of our main result.

Theorem 3.1 *Assume that* $u_0(.,0) \in \mathcal{H}(\Omega)$ *with* $\max\{\|\nabla u_0(.,s)\|, \|\Delta u_0(.,s)\|\} \leq m$ *(for some positive constant* m*), and* $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$*. Then, there exists a positive constant* $\delta > 0$ *such that the energy functional associated to problem (3.1) satisfies the estimate*

$$E(t) \leq \delta \left(\frac{1 + \int_0^t \xi(s) \int_s^\infty g(s) ds ds}{1 + \int_0^t \xi(s) ds} \right), \ \forall t \geq 0. \tag{3.14}$$

Proof. From the non-increasing property of ξ, the inequality $g' \leq -\xi\, g$, we have for $t \geq 0$;

$$\xi(t) \int_0^t g(s)\|\nabla\eta^t(s)\|^2 ds \leq \int_0^t \|\nabla\eta^t(s)\|^2 \xi(s) g(s) ds \leq -\int_0^t \|\nabla\eta^t(s)\|^2 g'(s) ds \leq -c_0 E'(t),$$

for some positive constant c_0. However, owing to the inequality,

$$\|\nabla u(t)\|^2 \leq \frac{2\omega_0}{l} E(t) \leq \frac{2\omega_0}{l} E(0),$$

we have for $s > t$,

$$\|\nabla\eta^t(s)\|^2 \leq 2\|\nabla u(t)\|^2 + 2\|\nabla u(t-s)\|^2 \leq \frac{4\omega_0}{l}E(0) + 2\sup_{\zeta>0}\|\nabla u_0(\zeta)\|^2 \leq \frac{4\omega_0}{l}E(0) + 2m^2,$$

Consequently, we get

$$\xi(t)\int_t^\infty g(s)\|\nabla\eta^t(s)\|^2 ds \leq \xi(t)\Big(\frac{4\omega_0}{l}E(0) + 2m^2\Big)\int_t^\infty g(s)ds\,.$$

Therefore, for $t \geq 0$, we obtain

$$\xi(t)\int_0^\infty g(s)\|\nabla\eta^t(s)\|^2 ds \leq -c_0E'(t) + \xi(t)\Big(\frac{4\omega_0}{l}E(0) + 2m^2\Big)\int_t^\infty g(s)ds. \quad (3.15)$$

Following similar arguments, and using the inequality $\|\Delta u(t)\|^2 \leq \frac{2}{l}E(t) \leq \frac{2}{l}E(0)$ instead of $\|\nabla u(t)\|^2 \leq \frac{2\omega_0}{l}E(t) \leq \frac{2\omega_0}{l}E(0)$, we obtain

$$\xi(t)\int_0^\infty g(s)\|\Delta\eta^t(s)\|^2 ds \leq -c_0\mathcal{E}'(t)) + \xi(t)\Big(\frac{4}{l}E(0) + 2m^2\Big)\int_t^\infty g(s)ds. \quad (3.16)$$

Multiplying (3.13) by $\xi(t)$, and using (3.15) and (3.16), we get

$$\xi(t)\mathcal{L}'(t) \leq -\alpha_1\xi(t)E(t) - \beta_1(E'(t) + \mathcal{E}'(t)) + \beta_2\,\xi(t)\int_t^\infty g(s)ds. \quad (3.17)$$

where $\beta_1 = c_0\alpha_2$ and $\beta_2 = \alpha_2\Big(\frac{4}{l}(1+\omega_0)E(0) + 2m^2\Big)$. Therefore, differentiating the functional $\mathscr{L}(t) := \xi(t)\mathcal{L}(t) + \beta_1(E(t) + \mathcal{E}(t))$ and using (3.17), we have

$$\mathscr{L}'(t) \leq \xi(t)\mathcal{L}'(t) + \beta_1(E'(t) + \mathcal{E}'(t)) \leq -\alpha_1\xi(t)E(t) + \beta_2\,\xi(t)\int_t^\infty g(s)ds.$$

Integrating the above equation over $(0, t)$ yields

$$\mathscr{L}(t) \leq \mathscr{L}(0) + \beta_2\int_0^t \xi(s)\int_s^\infty g(s)ds\,ds - \alpha_1\int_0^t \xi(s)E(s)ds\,.$$

However, $E(s) \geq E(t)$ for $s \leq t$ (by the non-increasing property of E), and $\beta_1 E(t) \leq \mathscr{L}(t)$ (from the definition of $\mathscr{L}$), so

$$\Big(\beta_1 + \alpha_1\int_0^t \xi(s)ds\Big)E(t) \leq \mathscr{L}(0) + \beta_2\int_0^t \xi(s)\int_s^\infty g(s)ds\,ds\,.$$

Choosing $\delta = \frac{\max\{\mathscr{L}(0),\beta_2\}}{\min\{\beta_1,\alpha_1\}}$, then the desired energy bound is obtained. ∎

Example 3.2 *We give some examples of relaxation functions to illustrate our decay result (3.14).*

(1) Let $g(t) = e^{-t}$. Then $g'(t) = -\xi(t)g(t)$ with $\xi(t) = 1$. Thus, Theorem 3.1, $E(t) \leq K/t$ for some positive constant K and for t large enough.

(2) Let $g(t) = \frac{1}{(1+t)^2}$. Then $g'(t) = -\xi(t)g(t)$ but with $\xi(t) = \frac{2}{1+t}$. Again, Theorem 3.1 yields that $E(t)$ decays logarithmically, i.e., $E(t) \leq K/\ln(1+t)$, for some positive constant K and for t large enough.

3.4 Numerical study

This section is devoted to illustrate numerically the achieved theoretical decaying result in Theorem 3.1 on a two-dimensional test problem of the form (3.1) with space variables x and y. To do so, we develop a numerical scheme for problem (3.1) using finite differences for the time discretization combined with the continuous Galerkin finite element method in space. Applying Galerkin method to problem (3.1) directly forces us to deal with $\mathcal{C}^2$ polynomial approximations, which is definitely not convenient owing to the complexity in constructing the basis functions on various physical domains, and also increase the cost of computations. To avoid this, we rewrite (3.1) as a coupled system of lower order elliptic problems. More precisely, the model problem (3.1) is equivalent to

$$\begin{cases} u'' - \Delta w + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = F & \text{in } \Omega \times (0,\infty), \\ w + \Delta u = 0 & \text{in } \Omega \times (0,\infty), \\ u(x,y,0) = u_0(x,y,0), \quad u'(x,y,0) = u_1(x,y), & \text{in } \Omega, \end{cases} \tag{3.18}$$

with $u = w = 0$ on $\partial\Omega$, where $F(x,y,t) = -\displaystyle\int_t^\infty g(s)\Delta u_0(x,y,s-t)ds$. Our numerical discretization in space is applicable on any bounded polygonal physical domain Ω. However, and for simplicity, we choose Ω to be the unit square $(0,1)\times(0,1)$. Let $\mathcal{T}_h$ be a family of uniform M^2-square mesh-cells of Ω with diagonal $h = \sqrt{2}/M$ each. Let $V_h \subset H_0^1(\Omega)$ denotes the usual space of continuous, piecewise-linear polynomials on $\mathcal{T}_h$ that vanish on $\partial\Omega$.

To discretize in time, we truncate the interval $(0,\infty)$ and work instead on the finite interval $(0,T]$ with T being sufficiently large. Divide $[0,T]$ uniformly into N subintervals each with size $\tau := T/N$ and nodes $\{t_n\}_{n=0}^N$. For a given grid function v^n, let

$$\delta_{tt}v^n = \frac{v^{n+1}-2v^n+v^{n-1}}{\tau^2}, \quad v^{n+\frac{1}{2}} = \frac{v^n+v^{n-1}}{2}, \quad v^{n+\frac{1}{4}} = \frac{v^{n+1}+2v^n+v^{n-1}}{4}.$$

Taking the inner product of the first two equations in (3.18) with ϕ, $\psi \in H_0^1(\Omega)$, respectively, then, applying Green's formula and using $u = w = 0$ on $\partial\Omega$. This implies

$$\begin{cases} \langle u'', \phi\rangle + \langle \nabla w, \nabla \phi\rangle - \displaystyle\int_0^t g(t-s)\langle \nabla u(s), \nabla\phi\rangle\, ds = \langle F, \phi\rangle, & \forall\ \phi \in H_0^1(\Omega), \\ \langle w, \psi\rangle - \langle \nabla u, \nabla \psi\rangle = 0, & \forall\ \psi \in H_0^1(\Omega)\,. \end{cases}$$

Motivated by the above weak formulation, our fully-discrete numerical scheme is defined by: Find U_h^{n+1}, $W_h^{n+1} \in V_h$ such that

$$\begin{cases} \langle \delta_{tt}U_h^n, \phi_h\rangle + \langle \nabla W_h^{n+\frac{1}{4}}, \nabla\phi_h\rangle - \displaystyle\int_0^{t_{n+1}} g(t_{n+1}-s)\langle \nabla \bar{U}_h(s), \nabla\phi_h\rangle ds = \langle F^{n+1}, \phi_h\rangle, \\ \langle W_h^{n+1}, \psi_h\rangle - \langle \nabla U_h^{n+1}, \nabla\psi_h\rangle = 0, \end{cases} \tag{3.19}$$

$\forall\ \phi_h$, $\psi_h \in V_h$, with $1 \le n \le N-1$, where the piecewise constant function (in time variable) $\bar{U}_h(s) = U_h^{j+\frac{1}{2}}$ for $t_j < s < t_{j+1}$ with $0 \le j \le N$, and $F^{n+1} = F(t_{n+1})$.

Let $\{\phi_p\}_{p=1}^{d_h}$ be the two-dimensional hat basis functions of V_h, where $d_h =$

$(M-1)^2$ is the number of the interior nodes. Then, U_h^n and W_h^n can be written in term of the basis functions as:

$$U_h^n = \sum_{i=1}^{d_h} b_i^n \phi_{p_i} \quad \text{and} \quad W_h^n = \sum_{i=1}^{d_h} c_i^n \phi_{p_i}\,.$$

We define $d_h \times d_h$ matrices: $\boldsymbol{G} = [\langle \nabla \phi_q, \nabla \phi_p \rangle]$ and $\boldsymbol{M} = [\langle \phi_q, \phi_p \rangle]\,.$ The d_h-dimensional constant column vectors $\mathbf{b}^n$, $\mathbf{c}^n$, and $\boldsymbol{F}^{n+1}$ are respectively, the transpose of the vectors

$$[b_1^n, b_2^n, \cdots, b_{d_h}^n], \quad [c_1^n, c_2^n, \cdots, c_{d_h}^n], \quad \text{and} \quad [\langle F^{n+1}, \phi_1 \rangle, \langle F^{n+1}, \phi_2 \rangle, \cdots, \langle F^{n+1}, \phi_{d_h} \rangle].$$

Therefore, the fully-discrete scheme (3.19) has the following matrix representation:

$$\boldsymbol{M}\delta_{tt}\mathbf{b}^n + \boldsymbol{G}\mathbf{c}^{n+\frac{1}{4}} - g_{n+1,n}\boldsymbol{G}\mathbf{b}^{n+\frac{1}{2}} = \sum_{j=0}^{n-1} g_{n+1,j}\boldsymbol{G}\mathbf{b}^{j+\frac{1}{2}} + \boldsymbol{F}^{n+1}, \quad \boldsymbol{M}\mathbf{c}^{n+1} - \boldsymbol{G}\mathbf{b}^{n+1} = \mathbf{0},$$

with $g_{n+1,j} := \int_{t_j}^{t_{j+1}} g(t_{n+1}-s)\,ds$. From the second equation, we notice that $\boldsymbol{G}\mathbf{c}^{n+1} = \boldsymbol{S}\mathbf{b}^{n+1}$ where $\boldsymbol{S} = \boldsymbol{G}\,\boldsymbol{M}^{-1}\boldsymbol{G}$. Substitute this in the first equation and rearranging the terms, we obtain

$$\begin{aligned}
[4\boldsymbol{M} + \tau^2\boldsymbol{S} - 2\tau^2 g_{n+1,n}\boldsymbol{G}]\mathbf{b}^{n+1} &= \boldsymbol{M}(8\mathbf{b}^n - 4\mathbf{b}^{n-1}) - \tau^2\boldsymbol{S}(2\mathbf{b}^n + \mathbf{b}^{n-1}) \\
&\quad + 2\tau^2 g_{n+1,n}\boldsymbol{G}\mathbf{b}^n + 2\tau^2 \sum_{j=0}^{n-1} g_{n+1,j}\boldsymbol{G}[\mathbf{b}^{j+1} + \mathbf{b}^j] + \boldsymbol{F}^{n+1}\,. \qquad (3.20)
\end{aligned}$$

Therefore, at each time level t_{n+1}, the numerical coupled system (3.19) reduces to a finite square linear system, where the unknown is the column vector $\mathbf{b}^{n+1}$.

So the existence of the approximate solution U_h^{n+1} follows from its uniqueness. The latter follows from the fact that the matrices $\boldsymbol{M}$ and $\boldsymbol{S}$ (because $\boldsymbol{G}$ and $\boldsymbol{M}$ are symmetric and positive definite) are positive definite. Whence the coefficient vector $\mathbf{b}^{n+1}$ is computed, $\mathbf{c}^{n+1}$ can be determined by solving $\boldsymbol{M}\mathbf{c}^{n+1} = \boldsymbol{G}\mathbf{b}^{n+1}$, that is, $\mathbf{c}^{n+1} = \boldsymbol{M}^{-1}\boldsymbol{G}\mathbf{b}^{n+1}$.

To be able to solve linear system in (3.20) for $\mathbf{b}^{n+1}$, successively, the column vectors $\mathbf{b}^0$ and $\mathbf{b}^1$ need to be determined first. In other words, we have to compute the approximate solutions U_h^0 and U_h^1 first. We consider U_h^0 to be the Ritz projection of u_0 on the finite dimensional space V_h. However, motivated by the Taylor series expansion of u in time about $t = 0$, we choose U_h^1 to be the Ritz projection $u_0 + t_1 u_1$ on V_h.

For sake of comupting efficiently the spatial integrals in the linear system in (3.20), on each cell of our two-dimensional partition, the associated integral is approximated using 4-point (that is, 2-point in each direction) Gauss cubature (quadrature) rule.

In our numerical example, choose $T = 150$, $g(t) = \mathrm{e}^{-t}$, $u_0(x, y, t) = t^2 \sin(\pi x)\sin(\pi y)$ and $u_1(x, y) = 0$. We run our computer program with $M = 20$ (that is, 400 cells in space) and $N = 120000$. From the theoretical contribution in Theorem 3.1 (see Example 3.2), we expect our energy to decay monomially, that is, $tE(t) \leq c$ for a sufficiently large t. This is confirmed in Figure 3.2. In addition, the plot of the numerical solution U_h in Figure 3.1 shows its convergence to zero as the time t is getting far away from 0.

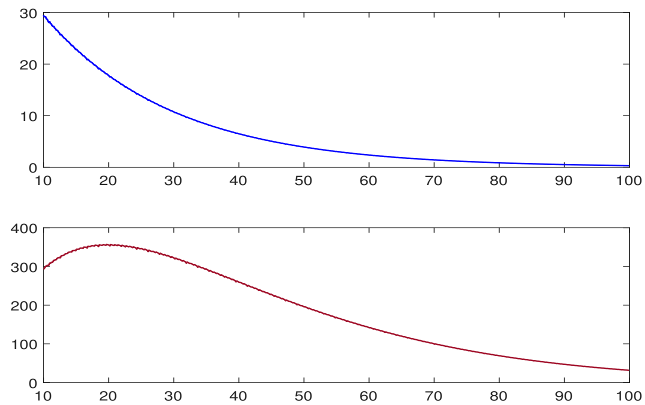

Figure 3.1: The graphical plots of the numerical approximations of the energy $E(t)$ (top) and the weighted energy $tE(t)$ (bottom) the against t.

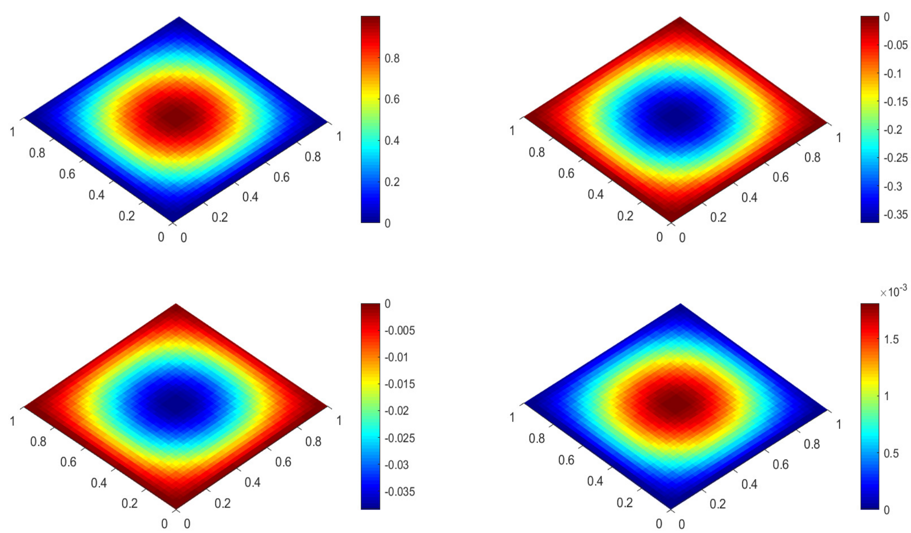

Figure 3.2: The numerical solution plots for $t = 0$ (top-left), $t = 10$ (top-right), $t = 50$ (bottom-left), and $t = 100$ (bottom-right).

CHAPTER 4

GENERAL DECAY RATE OF A WEAKLY DISSIPATIVE VISCOELASTIC PLATE EQUATION WITH A GENERAL DAMPING

4.1 Introduction

The modeling of a generalized Kirchhoff viscoelastic plate, where a bending moment relation with memory is considered and in the presence of a nonlinear damping, can be described by the following nonlinear weakly dissipative viscoelastic equation; for a given non-negative relaxation function g,

$$
\begin{cases}
u'' + \Delta^2 u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + h(u') = 0 & \text{in } \ \Omega \times (0, +\infty), \\
u = \Delta u = 0, & \text{on } \ \partial\Omega, \\
u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \ \Omega,
\end{cases}
\tag{4.1}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$, h is a function satisfying some specific conditions, see (4.5) below, and the initial data u_0, u_1 are given.

The main focus of this chapter is on investigating the decay of the energy of the above viscoelastic problem. The presence of the weakly viscoelastic dissapative term obliges us to deal with what so-called a modified (or second) energy to achieve our goals. Indeed, this will make the analysis more technical and also delicate. To confirm our theoretical finding numerically, we provide graphical illustrations of the decay of the energy, where the solution of problem (4.1) is approximated via finite differences in time and finite elements in space.

In the next section, we introduce some necessary notations and assumptions, and state and prove a few technical lemmas that will be used in the forthcoming decaying analysis. Section 4.3 is dedicated to show the decay rates of the energy functional E (see (4.9)). Having a weakly dissepative in problem (4.1) leads us to introduce a second energy functional $\mathcal{E}$ (see (4.11) below) to overcome the difficulties in proving the decay of E. For sake of illustrating the theoretical decaying rate of E numerically, we develop a fully-discrete numerical method in

Section 4.4. Owing to the presence of the biharmonic operator in problem (4.1), we use the $\mathcal{C}^2$ Galerkin finite element method for the spatial discretization. In the time variable and to avoid solving any nonlinear systems, our scheme is based on an appropriate combination between backward-forward Euler and second central differences. We show the decay of both, the numerical solution of problem (4.1) and the approximation of the energy functional E.

4.2 Preliminaries

Let's introduce the Sobolev space $\mathcal{H}(\Omega) = \{u \in H^3(\Omega) : u = \Delta u = 0 \text{ on } \partial\Omega\}$.

An application of the Poincaré inequality and by using the elliptic regularity property, there exist two positive constants ω_0 and ω_1 such that

$$\|\nabla w\|^2 \leq \omega_0 \, \|\Delta w\|^2 \quad \text{and} \quad \|\Delta w\|^2 \leq \omega_1 \, \|\nabla(\Delta w)\|^2, \; \forall w \in \mathcal{H}(\Omega)\,. \tag{4.2}$$

In the decay energy analysis, the following hypothesis is imposed.

(**A1**) The relaxation function $g \in \mathcal{C}^1(\mathbb{R}^+)$ is assumed to be non-increasing,

$$g(0) > 0, \quad 1 - \max\{\omega_0, \omega_1\} \int_0^\infty g(s)ds =: l > 0, \tag{4.3}$$

and there exists a $\mathcal{C}^1$ function $G : (0, \infty) \to (0, \infty)$ which is strictly increasing, and strictly convex $\mathcal{C}^2$ function on $(0, g(0)]$, with $G(0) = G'(0) = 0$, such that

$$-c_0 g(t) < g'(t) \leq -\xi(t) G(g(t)), \quad \forall \; t \geq 0, \tag{4.4}$$

where ξ is a positive non-increasing $\mathcal{C}^1$ function and c_0 is a positive constant. Moreover, the function $h \in \mathcal{C}^1(\mathbb{R})$ is assumed to be non-decreasing,

$$h(0) = 0, \qquad sh(s) \geq 0 \qquad \text{and} \qquad \alpha_1|s| \leq |h(s)| \leq \alpha_2|s|, \ \forall s \in \mathbb{R}, \tag{4.5}$$

for some positive contants α_1 and α_2.

Remark 4.1 *As in Mustafa [27], we present the following:*

*(1) From assumption (**A1**), we deduce that* $\lim_{t\to\infty} g(t) = 0$, *and there exists* $t_0 > 0$ *such that* $g(t_0) = r$, *while,* $g(t) \leq r$ *for* $t \geq t_0$. *The non-increasing property of* g *implies that*

$$0 < g(t_0) \leq g(t) \leq g(0), \qquad \forall\, t \in [0, t_0].$$

Continuity of G *on* $[0, r]$ *yields* $a \leq G(g(t)) \leq b$ *on* $[0, t_0]$, *for some constants* $a, b > 0$. *Consequently, for any* $t \in [0, t_0]$, *we have*

$$g'(t) \leq -\xi(t)G(g(t)) \leq -a\xi(t) = -\frac{a}{g(0)}\xi(t)g(0) \leq -\frac{a}{g(0)}\xi(t)g(t),$$

and hence,

$$\xi(t)g(t) \leq -\frac{g(0)}{a}g'(t), \qquad \forall\, t \in [0, t_0]. \tag{4.6}$$

(2) If G *is a strictly increasing and strictly convex* $\mathcal{C}^2$ *function on* $(0, r]$, *with* $G(0) = G'(0) = 0$, *then there is a function* $\bar{G} : [0, +\infty) \longrightarrow [0, +\infty)$ *that extends* G *and its properties. For instance, we can define* $\bar{G}$, *for any* $t > r$,

by

$$\bar{G}(t) := \frac{G''(r)}{2}t^2 + \big(G'(r) - G''(r)r\big)t + \left(G(r) + \frac{G''(r)}{2}r^2 - G'(r)r\right).$$

For later use, by (4.2) and the second inequality in (4.3), we have

$$\|\Delta u(t)\|^2 - \int_0^t g(s)ds\|\nabla u(t)\|^2 \geq 0 \ \text{ and } \ \|\nabla(\Delta u(t))\|^2 - \int_0^t g(s)ds\|\Delta u(t)\|^2 \geq 0\,. \tag{4.7}$$

For convenience, we introduce the following notations: for $t > 0$,

$$(g \circ w)(t) := \int_0^t g(t-s)\|w(t) - w(s)\|^2 ds,$$

and for $0 < \varepsilon < 1$, we put

$$C_\varepsilon := \int_0^\infty \frac{g^2(s)}{h_\varepsilon(s)} ds \qquad \text{with} \qquad h_\varepsilon(t) := \varepsilon g(t) - g'(t).$$

The next three lemmas will be used in the forthcoming decay analysis section.

Lemma 4.1 *[15] Assume that (**A1**) holds true. Then for any*

$$v \in L^2_{loc}\big([0, +\infty); L^2(\Omega)\big),$$

$$\int_\Omega \left(\int_0^t g(t-s)(v(t) - v(s))ds\right)^2 dx \leq C_\varepsilon (h_\varepsilon \circ v)(t), \qquad \text{for} \quad t \geq 0. \tag{4.8}$$

Lemma 4.2 *[29] Assume that* (**A1**) *holds true. Then for any*

$$w \in H^1\big([0,\infty); L^2(\Omega)\big),$$

$$\int_0^t g(t-s)\langle w(s), w'(t)\rangle ds = \frac{1}{2}\frac{d}{dt}\Big[\int_0^t g(s)ds\|w(t)\|^2 - (g\circ w)(t)\Big] - \frac{1}{2}g(t)\|w(t)\|^2 + \frac{1}{2}(g'\circ w)(t)\,.$$

4.3 Decay

In this section, our goal is to find the best possible estimate of the energy functional of problem (4.1). First, taking the inner product of (4.1) with u' gives

$$\langle u'', u'\rangle + \langle \Delta^2 u, u'\rangle + \int_0^t g(t-s)\langle \Delta u(s), u'(t)\rangle ds + \langle h(u'), u'\rangle = 0\,.$$

Applying Green's formula (twice for the second term and once for the third term) and using the fact that $u' = \Delta u' = 0$ on $\partial\Omega$, yield the following weak formulation of (4.1):

$$\langle u'', u'\rangle + \langle \Delta u, \Delta u'\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla u'(t)\rangle ds + \langle h(u'), u'\rangle = 0\,.$$

Using Lemma 4.2 with $w = \nabla u$, this equation yields

$$\frac{1}{2}\frac{d}{dt}\Big[\|u'(t)\|^2 + \|\Delta u(t)\|^2 - \int_0^t g(s)ds\|\nabla u(t)\|^2 + (g\circ\nabla u)(t)\Big]$$
$$= \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - \langle h(u'), u'\rangle\,.$$

Introducing the first energy functional

$$E(t) := \frac{1}{2}\Big[\|u'(t)\|^2 + \|\Delta u(t)\|^2 - \int_0^t g(s)ds\|\nabla u(t)\|^2 + (g \circ \nabla u)(t)\Big] \geq 0, \quad (4.9)$$

where, we used (4.7) in the last inequality. Hence, we infer

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - \langle h(u'(t)), u'(t)\rangle \leq 0. \qquad (4.10)$$

Similar to the previous chapters, taking the inner product of problem (4.1) with $-\Delta u'$ and then, applying Green's formula to the first, second and fourth terms, we get

$$\langle \nabla u'', \nabla u'\rangle + \langle \nabla(\Delta u), \nabla(\Delta u')\rangle - \int_0^t g(t-s)\langle \Delta u(s), \Delta u'(t)\rangle ds + \langle \nabla h(u'), \nabla u'\rangle = 0.$$

Hence, by Lemma 4.2, the above equation leads to

$$\frac{1}{2}\frac{d}{dt}\Big[\|\nabla u'(t)\|^2 + \|\nabla(\Delta u)(t)\|^2 - \int_0^t g(s)ds\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\Big] = \\ -\frac{1}{2}g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g' \circ \Delta u)(t) - \langle \nabla h(u'), \nabla u'\rangle.$$

Therefore, using (4.7), the second energy functional of (4.1) is defined by

$$\mathcal{E}(t) := \frac{1}{2}\Big[\|\nabla u'(t)\|^2 + \|\nabla(\Delta u(t))\|^2 - \int_0^t g(s)ds\|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\Big] \geq 0\,. \qquad (4.11)$$

Moreover, since $\langle \nabla h(u'), \nabla u' \rangle = \langle h'(u')\nabla u', \nabla u' \rangle \geq 0$, we easily check that

$$\mathcal{E}'(t) = -\frac{1}{2}g(t)\|\Delta u(t)\|^2 + \frac{1}{2}(g' \circ \Delta u)(t) - \langle h'(u'(t))\nabla u'(t), \nabla u'(t) \rangle \leq 0. \quad (4.12)$$

In the next two lemmas, assuming that (**A1**) holds, we estimate the time derivative of the functionals

$$I_1(t) = \langle u(t), u'(t) \rangle \quad \text{and} \quad I_2(t) = -\int_0^t g(t-s)\langle u(t) - u(s), u'(t) \rangle ds\,.$$

Lemma 4.3 *Along the solution of (4.1), and for $\delta > 0$, we have*

$$I_1'(t) \leq \|u'(t)\|^2 + \Big(\delta - \frac{l}{2}\Big)\|\Delta u(t)\|^2 + c\, C_\varepsilon (h_\varepsilon \circ \nabla u)(t) - cE'(t).$$

Proof. Differentiating I_1 and exploiting the differential equation in (4.1), we get:

$$\begin{aligned} I_1'(t) &= \|u'(t)\|^2 - \langle u(t), \Delta^2 u(t) \rangle - \int_0^t g(t-s)\langle \Delta u(s), u(t) \rangle ds - \langle h(u'(t)), u(t) \rangle \\ &= \|u'(t)\|^2 - \|\Delta u(t)\|^2 + \int_0^t g(t-s)\langle \nabla u(s), \nabla u(t) \rangle ds - \langle h(u'(t)), u(t) \rangle\,. \end{aligned}$$

Young's inequality, Lemma 4.1, and the inequalities in (4.2) imply that the third term in the right-hand side equals

$$\begin{aligned}
&\Big\langle \nabla u(t), \int_0^t g(t-s)\nabla\big(u(s)-u(t)\big)ds\Big\rangle + \int_0^t g(s)ds\|\nabla u(t)\|^2 \\
&\leq \frac{l}{2\omega_0}\|\nabla u(t)\|^2 + \frac{\omega_0}{2l}\int_\Omega\Big(\int_0^t g(t-s)\nabla\big(u(s)-u(t)\big)ds\Big)^2dx + \int_0^t g(s)ds\|\nabla u(t)\|^2 \\
&\leq \frac{l}{2\omega_0}\|\nabla u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t) + \omega_0\int_0^t g(s)ds\|\Delta u(t)\|^2 \\
&\leq \frac{l}{2}\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t) + (1-l)\|\Delta u(t)\|^2 \\
&\leq (1-\frac{l}{2})\|\Delta u(t)\|^2 + c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t).
\end{aligned}$$

Again, the forth term is handled, using Cauchy–Schwarz, Young's inequalities and (4.5), as follows:

$$\begin{aligned}
\langle h(u'(t)), u(t)\rangle &\leq \|u(t)\|\|h(u'(t))\| \\
&\leq c\|\Delta u(t)\|\|h(u'(t))\| \\
&\leq \delta\|\Delta u(t)\|^2 + \frac{c}{\delta}\langle|h(u'(t))|^2, 1\rangle \\
&\leq \delta\|\Delta u(t)\|^2 + \frac{c\alpha_2}{\delta}\langle h(u'(t)), u'(t)\rangle \\
&\leq \delta\|\Delta u(t)\|^2 - cE'(t).
\end{aligned}$$

Combining the above results completes the proof of the lemma. ∎

Lemma 4.4 *Along the solution of (4.1), and for $\delta > 0$, we have*

$$I_2'(t) \leq \delta\|\Delta u\|^2 - \Big(\int_0^t g(s)ds - \delta\Big)\|u'\|^2 + \frac{c}{\delta}(C_\varepsilon+1)(h_\varepsilon\circ\Delta u)(t) - cE'(t).$$

Proof. Differentiating I_2 and exploiting the differential equation in (4.1), we get

$$I_2'(t) = I_{2,1}(t) + I_{2,2}(t) + I_{2,3}(t) + I_{2,4}(t), \tag{4.13}$$

where

$$\begin{aligned}
I_{2,1}(t) &= \int_0^t g(t-s)\langle u(t)-u(s), \Delta^2 u(t)\rangle ds \\
I_{2,2}(t) &= \Big\langle \int_0^t g(t-s)\Delta u(s)ds, \int_0^t g(t-s)\big(u(t)-u(s)\big)ds \Big\rangle \\
I_{2,3}(t) &= \Big\langle \int_0^t g(t-s)(u(t)-u(s))\, ds, h(u'(t)) \Big\rangle \\
I_{2,4}(t) &= -\Big\langle u'(t), \int_0^t \Big[g'(t-s)\big(u(t)-u(s)\big) + g(t-s)u'(t)\Big] ds \Big\rangle .
\end{aligned}$$

The current task is to estimate these four terms. By Green's formula, Young's inequality and Lemma 4.1, we have

$$I_{2,1}(t) \le \|\Delta u(t)\| \int_0^t g(t-s)\|\Delta\big(u(t)-u(s)\big)\|ds \le \frac{\delta}{2}\|\Delta u(t)\|^2 + \frac{c}{\delta}C_\varepsilon(h_\varepsilon \circ \Delta u)(t),$$

and

$$\begin{aligned}
I_{2,2}(t) = &\Big\| \int_0^t g(t-s)\nabla\big(u(t)-u(s)\big)ds\Big\|^2 \\
&+ \Big\langle \int_0^t g(t-s)\nabla u(t) ds, \int_0^t g(t-s)\nabla\big(u(t)-u(s)\big)ds \Big\rangle \\
\le &\Big(\int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|ds\Big)^2 + c\|\nabla u(t)\| \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|ds \\
&\le C_\varepsilon(h_\varepsilon \circ \nabla u)(t) + \frac{\delta}{2\omega_0}\|\nabla u(t)\|^2 + \frac{c}{\delta}C_\varepsilon(h_\varepsilon \circ \nabla u)(t) .
\end{aligned}$$

For $I_{2,3}(t)$, Young's inequality gives

$$I_{2,3}(t) \le \delta \int_\Omega \Big| \int_0^t g(t-s)\big(u(t)-u(s)\big)ds\Big|^2 dx + \frac{c}{\delta}\langle |h(u'(t))|^2, 1\rangle.$$

Since

$$\begin{aligned}\Big|\int_0^t g(t-s)\big(u(t)-u(s)\big)ds\Big| &\le \int_0^t g^{\frac{1}{2}}(t-s)g^{\frac{1}{2}}(t-s)\big|u(t)-u(s)\big|ds \\ &\le \Big(\int_0^t g(s)ds\Big)^{\frac{1}{2}}\Big(\int_0^t g(t-s)|(u(t)-u(s)|^2 ds\Big)^{\frac{1}{2}},\end{aligned}$$

applications of Hölder's inequality and Lemma 4.1, yield

$$\begin{aligned}\int_\Omega \Big|\int_0^t g(t-s)\big(u(t)-u(s)\big)ds\Big|^2 dx &\le \Big(\int_0^t g(s)ds\Big)^2 \int_0^t g(t-s)\|u(t)-u(s)\|^2\, ds \\ &\le c(g\circ\nabla u)(t)\,.\end{aligned}$$

Merge the above three equations and with the help of (4.5), we obtain

$$\begin{aligned}I_{2,3}(t) &\le c\,\delta(g\circ\nabla u)(t) + \frac{c}{\delta}\langle |h(u'(t))|^2, 1\rangle \\ &\le c\,\delta(g\circ\nabla u)(t) + \frac{c\alpha_2}{\delta}\langle h(u'(t)), u'(t)\rangle \\ &\le c\,\delta(g\circ\nabla u)(t) - cE'(t)\end{aligned}$$

To estimate the last term $I_{2,4}(t)$, we use again Young's inequality, the fact that

$|g'| = |\varepsilon g - h_\varepsilon| \leq \varepsilon g + h_\varepsilon$, and Lemma 4.1. So, we have

$$I_{2,4}(t) \leq \delta \|u'(t)\|^2 + \frac{c}{\delta}\Big(\int_0^t (\varepsilon g(t-s) + h_\varepsilon(t-s))\|u(t) - u(s)\| ds\Big)^2 - \|u'(t)\|^2 \int_0^t g(s)\, ds$$
$$\leq \Big(\delta - \int_0^t g(s)ds\Big)\|u'(t)\|^2 + \frac{c}{\delta}\Big(\varepsilon^2 (g \circ u)(t) + (h_\varepsilon \circ u)(t)\Big)$$
$$\leq \Big(\delta - \int_0^t g(s)ds\Big)\|u'(t)\|^2 + \frac{c}{\delta} C_\varepsilon (h_\varepsilon \circ u)(t)\,.$$

Inserting the obtained estimates of $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, and $I_{2,4}$ in (4.13), the proof is complete. ∎

The achieved convolution estimates in the next lemma will also be needed in our forthcoming analysis. For convenience, we introduce the following notations. With $f(t) := \int_t^\infty g(s)ds$, let

$$J_1(t) := \int_0^t f(t-s)\|\nabla u(s)\|^2 ds \quad \text{and} \quad J_2(t) := \int_0^t f(t-s)\|\Delta u(s)\|^2 ds\,.$$

Lemma 4.5 *[14] Assume that (**A1**) holds, then for $t \geq 0$,*

$$J_1'(t) \leq \frac{3}{\omega_0}(1-l)\|\nabla u\|^2 - \frac{1}{2}(g \circ \nabla u)(t) \;\text{ and }\; J_2'(t) \leq \frac{3}{\omega_0}(1-l)\|\Delta u\|^2 - \frac{1}{2}(g \circ \Delta u)(t)\,.$$

Lemma 4.6 *For $N, \varepsilon_1, \varepsilon_2 > 0$, the functional*

$$\mathcal{L}(t) := N(E(t) + \mathcal{E}(t)) + \varepsilon_1 I_1(t) + \varepsilon_2 I_2(t)$$

satisfies

$$\mathcal{L} \sim E + \mathcal{E} \quad \textit{for a sufficiently large } N. \tag{4.14}$$

Moreover, for any $t \geq t_0$*, with* t_0 *being introduced in Remark 4.1, we have*

$$\mathcal{L}'(t) \leq -(1-l)\left(4+\frac{3}{2\omega_0}\right)\Big(\frac{2}{l}\|u'(t)\|^2+\|\Delta u(t)\|^2\Big)+\frac{1}{4}\Big((g\circ\nabla u)(t)+(g\circ\Delta u)(t)\Big)-cE'(t). \tag{4.15}$$

Proof. The proof of (4.14) is done in [21]. To show (4.15), differentiating and get

$$\mathcal{L}'(t) = N(E'(t) + \mathcal{E}'(t)) + \varepsilon_1 I_1'(t) + \varepsilon_2 I_2'(t)\,.$$

Using (4.10), (4.12), and Lemmas 4.3 and 4.4, yield

$$\begin{aligned}\mathcal{L}'(t) \leq N\Big[&\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u\|^2-\langle h(u'(t)),u'(t)\rangle-\frac{1}{2}g(t)\|\Delta u\|^2+\frac{1}{2}(g'\circ\Delta u)(t)\\ &-\langle h'(u'(t))\nabla u'(t),\nabla u'(t)\rangle\Big]+\varepsilon_1\Big[\|u'\|^2+(\delta-\frac{l}{2})\|\Delta u\|^2+c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t)-cE'(t)\Big]\\ &+\varepsilon_2\Big[\delta\|\Delta u\|^2-\Big(\int_0^t g(s)ds-\delta\Big)\|u'\|^2+\frac{c}{\delta}(C_\varepsilon+1)((h_\varepsilon\circ\Delta u)(t))-cE'(t)\Big]\,.\end{aligned}$$

Since the relaxation function $g > 0$, then

$$\begin{aligned}\mathcal{L}'(t) \leq &-\Big[\Big(\int_0^t g(s)ds-\delta\Big)\varepsilon_2-\varepsilon_1\Big]\|u'\|^2-\left((\frac{l}{2}-\delta)\varepsilon_1-\varepsilon_2\delta\right)\|\Delta u\|^2\\ &+\frac{N}{2}(g'\circ\nabla u)(t)+\frac{N}{2}(g'\circ\Delta u)(t)+\varepsilon_1 c\,C_\varepsilon(h_\varepsilon\circ\nabla u)(t)+\frac{\varepsilon_2 c}{\delta}(C_\varepsilon+1)(h_\varepsilon\circ\Delta u)(t)\\ &-c(\varepsilon_1+\varepsilon_2)E'(t).\end{aligned} \tag{4.16}$$

Using $g'(t) := \varepsilon g(t) - h_\varepsilon(t)$, and noting that $h_\varepsilon > 0$, we observe

$$\begin{aligned}\mathcal{L}'(t) \leq -\Big[\Big(\int_0^t g(s)ds - \delta\Big)\varepsilon_2 - \varepsilon_1\Big]\|u'\|^2 - ((\frac{l}{2} - \delta)\varepsilon_1 - \varepsilon_2\delta)\|\Delta u\|^2 \\ + \frac{N\varepsilon}{2}\Big[(g \circ \nabla u)(t) + (g \circ \Delta u)(t)\Big] \\ - \Big[\frac{N}{2} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) - \frac{c}{\delta}C_\varepsilon(\varepsilon_1 + \varepsilon_2)\Big](h_\varepsilon \circ \Delta u)(t) - c(\varepsilon_1 + \varepsilon_2)E'(t). \quad (4.17)\end{aligned}$$

Now choose $\delta < \frac{l}{8}g_0$ with $g_0 = \int_0^{t_0} g(s)ds$. So, for $\varepsilon_1 = \frac{3}{8}g_0\varepsilon_2$ with $\varepsilon_2 = \frac{16(1-l)}{lg_0}\big(4 + \frac{3}{2\omega_0}\big)$, simple calculations show that

$$(g_0 - \delta)\varepsilon_2 - \varepsilon_1 > \frac{2}{l}(1-l)\left(4 + \frac{3}{2\omega_0}\right) \quad \text{and} \quad (\frac{l}{2} - \delta)\varepsilon_1 - \delta\varepsilon_2 > (1-l)\left(4 + \frac{3}{2\omega_0}\right). \tag{4.18}$$

From $\dfrac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)} < g(s)$, and by the Lebesgue dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0^+} \varepsilon C_\varepsilon = \lim_{\varepsilon \to 0^+} \int_0^\infty \frac{\varepsilon g^2(s)}{\varepsilon g(s) - g'(s)} ds = 0.$$

So, there exists $0 < \varepsilon_0 < 1$ such that if $\varepsilon < \varepsilon_0$, then

$$\varepsilon C_\varepsilon < \frac{1}{\frac{8c}{\delta}(\varepsilon_1 + \varepsilon_2)}.$$

Putting $\varepsilon = \frac{1}{2N}$ and choosing $N > \max\left\{\frac{4c}{\delta}(\varepsilon_1 + \varepsilon_2),\ \frac{1}{2\varepsilon_0}\right\}$, then

$$\frac{N}{4} - \frac{c}{\delta}(\varepsilon_1 + \varepsilon_2) > 0 \qquad \text{and} \qquad \varepsilon < \varepsilon_0.$$

From the above two equations, one can easily check that

$$\frac{N}{2}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2)-\frac{c}{\delta}C_\varepsilon(\varepsilon_1+\varepsilon_2) > \frac{N}{2}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2)-\frac{1}{8\varepsilon} = \frac{N}{4}-\frac{c}{\delta}(\varepsilon_1+\varepsilon_2) > 0. \quad (4.19)$$

Finally, substituting (4.18) and (4.19) in the estimate (4.17) gives the desired result. ∎

We are ready now to show our energy decay result. Our subsequent analysis makes a frequent use of the quadratic functional defined, for a purely time dependent function ϕ and $0 \leq t_1 \leq t_2 \leq t$,

$$\mathcal{I}(\phi, t_1, t_2, t) := \int_{t_1}^{t_2} \phi(s)\left(\|\nabla(u(t)-u(t-s))\|^2 + \|\Delta(u(t)-u(t-s))\|^2\right) ds.$$

For convenience, if $t_2 = t$, we let $\mathcal{I}(\phi, t_1, t) := \mathcal{I}(\phi, t_1, t, t)$.

Theorem 4.2 *Putting $G_0(t) = tG'(t)$. Assume that hypothesis (**A1**) holds and the initial data $u_0 \in \mathcal{H}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, there exist positive constants λ_1, λ_2 such that the energy functional associated to problem (4.1) satisfies the estimate*

$$E(t) \leq \lambda_2 G_0^{-1}\left(\frac{\lambda_1}{\int_{t_0}^{t}\xi(s)ds}\right), \qquad \forall\, t > t_0\,. \quad (4.20)$$

Proof. From the non-increasing property of ξ and the inequalities (4.10), (4.12) and (4.6), we get

$$\mathcal{I}(g,0,t_0,t) \leq \frac{1}{\xi(t_0)}\mathcal{I}(\xi\, g,0,t_0,t) \leq -\frac{g(0)}{a\,\xi(t_0)}\mathcal{I}(g',0,t_0,t) \leq -\frac{g(0)}{a\,\xi(t_0)}\mathcal{I}(g',0,t)$$
$$\leq -c\big(E'(t)+\mathcal{E}'(t)\big)$$

for any $t \geq t_0$. Inserting this estimate in (4.15), we obtain

$$\mathcal{L}'(t) \leq -mE(t) - c\big(E'(t)+\mathcal{E}'(t)\big) + c\,\mathcal{I}(g,t_0,t), \quad \forall \;\; t \geq t_0. \tag{4.21}$$

By Lemmas 4.5 and 4.6, the functional $\mathscr{L}(t) := \mathcal{L}(t)+J_1(t)+\frac{1}{2}J_2(t)$ is nonnegative and satisfies,

$$\mathscr{L}'(t) \leq -\frac{2}{l}(1-l)\left(4+\frac{3}{2\omega_0}\right)\|u'\|^2 - (1-l)\,\|\Delta u\|^2 - \frac{1}{4}\left(g\circ\nabla u\right)(t) \leq -c_0E(t),$$

for some $c_0 > 0$ and for any $t \geq t_0$. This estimate leads to the following bound

$$\int_0^\infty E(s)ds < +\infty. \tag{4.22}$$

From the first inequality in (4.2), we easily see that

$$E(t) \geq \frac{l}{2}\,\|\Delta u(t)\|^2 \qquad \text{and} \qquad E(t) \geq \frac{l}{2\omega_0}\,\|\nabla u(t)\|^2\,, \qquad \forall\, t \geq 0. \tag{4.23}$$

For a positive constant γ, these last estimates together with (4.22) yield

$$\begin{aligned}\mathcal{I}(\gamma,t_0,t) &\leq 2\gamma\int_{t_0}^{t}\left(\|\nabla u(t)\|^2+\|\nabla u(t-s)\|^2+\|\Delta u(t)\|^2+\|\Delta u(t-s)\|^2\right)ds\\ &\leq \frac{4\gamma}{l}(1+\omega_0)\int_{t_0}^{t}\big(E(t)+E(t-s)\big)ds. \quad (4.24)\end{aligned}$$

However, the last integral is finite (due to (4.22) and the inequality $E(t)\geq 0$), and hence, we choose $0<\gamma<1$ such that

$$\mathcal{I}(\gamma,t_0,t)<1, \qquad \forall\, t\geq t_0. \tag{4.25}$$

The strict convexity of G and the fact that $G(0)=0$ gives that

$$G(s\tau)\leq sG(\tau), \qquad \text{for} \quad 0\leq s\leq 1 \quad \text{and} \quad \tau\in(0,r]. \tag{4.26}$$

Combining this with the hypothesis (**A1**), Jensen's inequality and (4.25), we obtain

$$\begin{aligned}-\mathcal{I}(g',t_0,t) &= -\frac{1}{\mathcal{I}(\gamma,t_0,t)}\mathcal{I}(\mathcal{I}(\gamma,t_0,t)\,g',t_0,t)\\ &\geq \frac{1}{\mathcal{I}(\gamma,t_0,t)}\mathcal{I}(\mathcal{I}(\gamma,t_0,t)\,\xi\,G(g),t_0,t)\\ &\geq \frac{\xi(t)}{\mathcal{I}(\gamma,t_0,t)}\mathcal{I}(G(\mathcal{I}(\gamma,t_0,t)\,g),t_0,t)\\ &\geq \frac{\xi(t)}{\gamma}G\left(\gamma\,\mathcal{I}(g,t_0,t)\right)\\ &= \frac{\xi(t)}{\gamma}\bar{G}\left(\gamma\,\mathcal{I}(g,t_0,t)\right),\end{aligned}$$

for any $t > t_0$, where $\bar{G}$ is introduced in Remark 4.1. Thus,

$$\mathcal{I}(g, t_0, t) \leq \frac{1}{\gamma}\bar{G}^{-1}\left(-\frac{\gamma\mathcal{I}(g', t_0, t)}{\xi(t)}\right), \quad \text{for any} \quad t \geq t_0,$$

and with $\mathcal{F} := \mathcal{L} + cE + c\mathcal{E}$ (the constant c here is the one occurred in (4.21)), (4.21) becomes

$$\mathcal{F}'(t) \leq -mE(t) + \frac{c}{\gamma}\bar{G}^{-1}\left(-\frac{\gamma\mathcal{I}(g', t_0, t)}{\xi(t)}\right), \qquad \forall\, t \geq t_0\,. \tag{4.27}$$

Let $0 < r_1 < r$, then define a functional $\mathcal{F}_1$ by

$$\mathcal{F}_1(t) := \bar{G}'\left(r_1 E_0(t)\right)\mathcal{F}(t), \qquad \forall\, t \geq t_0, \quad \text{with} \quad E_0(t) = \frac{E(t)}{E(0)}.$$

and so,

$$\mathcal{F}_1'(t) = r_1 E_0'(t)\bar{G}''\left(r_1 E_0(t)\right)\mathcal{F}(t) + \bar{G}'\left(r_1 E_0(t)\right)\mathcal{F}'(t)\,.$$

Then, estimate (4.27) together with the facts that $E_0' \leq 0$, $G' > 0$ and $G'' > 0$ lead to

$$\mathcal{F}_1'(t) \leq -mE(t)\bar{G}'\left(r_1 E_0(t)\right) + \frac{c}{\gamma}\bar{G}'\left(r_1 E_0(t)\right)\bar{G}^{-1}\left(-\frac{\gamma\mathcal{I}(g', t_0, t)}{\xi(t)}\right), \quad \forall\, t \geq t_0. \tag{4.28}$$

Let $\bar{G}^*$ be the convex conjugate of $\bar{G}$ in the sense of Young (see [2, pp. 61–64]), given by

$$\bar{G}^*(s) = s\big(\bar{G}'\big)^{-1}(s) - \bar{G}\left[\big(\bar{G}'\big)^{-1}(s)\right] \tag{4.29}$$

and satisfies the following generalized Young inequality

$$AB \leq \bar{G}^{*}(A) + \bar{G}(B). \tag{4.30}$$

Set $A = \bar{G}'\left(r_1 E_0(t)\right)$ and $B = \bar{G}^{-1}\left(-\gamma \mathcal{I}(g', t_0, t)/\xi(t)\right),$ then it follows from a combination of (4.28) and (4.30) that

$$\begin{aligned} \mathcal{F}_1'(t) &\leq -mE(t)\bar{G}'\left(r_1 E_0(t)\right) + c\gamma^{-1}\bar{G}^{*}\left[\bar{G}'\left(r_1 E_0(t)\right)\right] - c\mathcal{I}(g', t_0, t)/\xi(t) \\ &\leq -m(E(0) - cr_1)E_0(t)\bar{G}'\left(r_1 E_0(t)\right) - c\mathcal{I}(g', t_0, t)/\xi(t), \qquad \forall\, t \geq t_0. \end{aligned}$$

After fixing r_1 so that $E(0) - cr_1 = m_1 > 0$, we arrive at

$$\mathcal{F}_1'(t) \leq -m_1 E_0(t)\bar{G}'\left(r_1 E_0(t)\right) - c\mathcal{I}(g', t_0, t)/\xi(t), \qquad \forall\, t \geq t_0.$$

Hence, multiplying both sides by $\xi(t)$, and using $r_1 E_0(t) < r$ and the inequality

$$-\mathcal{I}(g', t_0, t) \leq -c\big(E'(t) + \mathcal{E}'(t)\big), \qquad \forall\, t \geq t_0. \tag{4.31}$$

By using the definition of $\mathcal{I}$ and the estimates $E'(t) \leq \frac{1}{2}(g' \circ \nabla u)(t)$ (by (4.10)) and $\mathcal{E}'(t) \leq \frac{1}{2}(g' \circ \Delta u)(t)$ (by (4.12)), we reach

$$\begin{aligned} \xi(t)\mathcal{F}_1'(t) &\leq -m_1 E_0(t) G'\left(r_1 E_0(t)\right)\xi(t) - c\mathcal{I}(g', t_0, t) \\ &\leq -m_1 E_0(t) G'\left(r_1 E_0(t)\right)\xi(t) - c\big(E'(t) + \mathcal{E}'(t)\big), \qquad \forall\, t \geq t_0. \end{aligned}$$

Let $\mathcal{F}_2 = \xi\mathcal{F}_1 + c(E+\mathcal{E})$, we obtain, from the non-increasing property of ξ that

$$m_1 E_0(t) G'\left(r_1 E_0(t)\right) \xi(t) \leq -\mathcal{F}_2'(t), \qquad \forall\, t \geq t_0. \tag{4.32}$$

Since $G'' > 0$ and by the non-increasing property of E, the map $t \longmapsto E(t)G'\left(\varepsilon_1 E_0(t)\right)$ is non-increasing. Consequently, an integration of (4.32) over (t_0, t) yields

$$\begin{aligned} m_1 E_0(t) G'\left(r_1 E_0(t)\right) \int_{t_0}^{t} \xi(s) ds \leq \int_{t_0}^{t} m_1 E_0(s) G'\left(r_1 E_0(s)\right) \xi(s) ds &\leq \mathcal{F}_2(t_0) - \mathcal{F}_2(t) \\ &\leq \mathcal{F}_2(t_0) \end{aligned}$$

Since $G_0(\tau) = \tau G'(\tau)$ is strictly increasing, then the desired bound follows immediately. ∎

Example 4.3 *We give some examples of relaxation functions to illustrate our decay result (4.20).*

*(1) Choose $g(t) = ae^{-bt^{\nu}}$ with $0 < \nu < 1$, where a and b are positive constants such that $1 - \max\{\omega_0, \omega_1\}\frac{a}{b} > 0$ and $h(s) = s$ so, hypothesis (**A1**) is satisfied. Then, $g'(t) = -\xi(t)G(g(t))$ with $G(t) = t$, and $\xi(t) = \nu\, bt^{\nu-1}$. By Theorem 4.2, we conclude that*

$$E(t) \leq c(t - t_0)^{-\nu}$$

for a sufficiently large t.

*(2) Choose $g(t) = a(1+t)^{-\nu}$ where $\nu > 1$ and a is chosen so that hypothesis (**A1**)*

remains valid and $h(s) = s$*. Here,* $g'(t) = -\xi(t)G(g(t))$ *with* $G(t) = t^{1+1/\nu}$ *and* $\xi(t) = b$*, where* b *is a fixed constant. By Theorem 4.2,*

$$E(t) \leq c(1+t)^{-\nu/(\nu+1)}$$

for a sufficiently large t*.*

(3) *Choose* $g(t) = e^{-t}$ *and* $h(s) = s$ *so, hypothesis (***A1***) is satisfied. Then,* $g'(t) = -\xi(t)G(g(t))$ *with* $G(t) = t$*, and* $\xi(t) = 1$*. By Theorem 4.2, we conclude that*

$$E(t) \leq \frac{c}{(t-t_0)}$$

for a sufficiently large t*.*

4.4 Numerical study

This section is devoted to illustrate numerically the achieved theoretical decaying result in Theorem 4.2 on a sample test problem of the form (4.1). To do so, we develop a numerical scheme for the nonlinear model problem (4.1) using finite differences for the time discretization combined with the $\mathcal{C}^2$ continuous bicubic Galerkin method in space [28]. To avoid solving any nonlinear algebraic systems of equations, the approximation of the damping term is based on an extrapolation technique.

To discretize in time, we truncate the interval $(0, \infty)$ and work instead on the finite interval $(0, T]$, where T is large enough. Divide $[0, T]$ uniformly into N

subintervals with size τ each and nodes $\{t_n\}_{n=0}^{N}$, that is, $t_n = n\tau$ for $0 \le n \le N$, where $\tau = T/N$. For the grid function w^n, let

$$\delta_t w^n = \frac{w^n - w^{n-1}}{\tau}, \quad \delta_{tt} w^n = \frac{w^{n+1} - 2w^n + w^{n-1}}{\tau^2},$$
$$w^{n+\frac{1}{2}} = \frac{w^n + w^{n-1}}{2}, \quad w^{n+\frac{1}{4}} = \frac{w^{n+1} + 2w^n + w^{n-1}}{4}.$$

For the spatial discretization, choose $\Omega = (a,b) \times (c,d)$ and then divide both (a,b) (in the x-direction) and (c,d) (in the y-direction) into a family of uniform (quasi-uniform) cells. To elaborate, let $x_i = i\,h_x$ for $0 \le i \le M_x$ with $h_x = (b-a)/M_x$ and let $y_j = j\,h_y$ for $0 \le j \le M_y$ with $h_y = (d-c)/M_y$. Then, the $\mathcal{C}^2$ Galerkin finite dimensional space $S_h := S_{h_x} \otimes S_{h_y}$, where

$$S_{h_x} = \{v \in H^3(a,b) :\ v|_{[x_{i-1},x_i]} \in P_3 \text{ for } 1 \le i \le N_x, \text{ with } v(x)|_{x=a,b} = v''(x)|_{x=a,b} = 0\},$$
$$S_{h_y} = \{v \in H^3(c,d) :\ v|_{[y_{i-1},x_i]} \in P_3 \text{ for } 1 \le i \le N_y, \text{ with } v(y)|_{y=c,d} = v''(y)|_{y=c,d} = 0\}.$$

Here, P_3 is the space of polynomials of degree ≤ 3 in each direction x or y.

Usually, continuous Galerkin finite element schemes are motivated by the weak formulation of the model problem. So, we take the inner product of (4.1) with $\phi \in \mathcal{H}(\Omega)$ then use Green's formula. This leads to

$$\langle u'', \phi\rangle + \langle \Delta u, \Delta\phi\rangle - \int_0^t g(t-s)\langle \nabla u(s), \nabla\phi\rangle\, ds + \langle h(u'), \phi\rangle = 0. \qquad (4.33)$$

Consequently, for each $t > 0$, the semi-discrete finite element solution $u_h(t) \in S_h$

is defined by

$$\langle u_h'', \phi\rangle + \langle \Delta u_h, \Delta\phi\rangle - \int_0^t g(t-s)\,\langle \nabla u_h(s), \nabla\phi\rangle\; ds + \langle h(u_h'), \phi\rangle = 0, \quad \forall\; \phi \in S_h.$$

Our fully-discrete numerical solution U_h^n approximates $u(t_n)$ and is defined by

$$\langle \delta_{tt} U_h^n, \phi\rangle + \langle \Delta U_h^{n+\frac{1}{4}}, \Delta\phi\rangle - \int_0^{t_{n+1}} g(t_{n+1}-s)\langle \nabla \bar{U}_h(s), \nabla\phi\rangle ds + \langle h(\delta_t U_h^n), \phi\rangle = 0, \tag{4.34}$$

$\forall \phi \in S_h$, and for $1 \le n \le N-1$, where the piecewise constant function $\bar{U}_h(s) = U_h^{j+\frac{1}{2}}$ for $t_j < s < t_{j+1}$ with $1 \le j \le N-1$.

At each time level, the above scheme amounts to a square linear system (see the matrix form below). So the existence of the approximate solution U_h^{n+1} follows from its uniqueness. For uniqueness, we need to show that if

$$\frac{1}{\tau^2}\langle U_h^{n+1}, \phi\rangle + \frac{1}{4}\langle \Delta U_h^{n+1}, \Delta\phi\rangle - \frac{1}{2}\int_{t_n}^{t_{n+1}} g(t_{n+1}-s)\, ds \langle \nabla U_h^{n+1}, \nabla\phi\rangle ds = 0, \quad \phi \in S_h,$$

then $U_h^{n+1} \equiv 0$. Choose $\phi = U_h^{n+1}$ and, then, use the first inequality in (4.2) in addition to the non-increasing and positivity properties of g, to get

$$\frac{1}{\tau^2}\|U_h^{n+1}\|^2 + \frac{1}{4}\|\Delta U_h^{n+1}\|^2 < \frac{\omega_0}{2} g(0)\tau \|\Delta U_h^{n+1}\|^2\,.$$

Hence, for a sufficiently small τ, $(\frac{\omega_0}{2} g(0)\tau \le \frac{1}{4})$, $\frac{1}{\tau^2}\|U_h^{n+1}\|^2 \le 0$ and thus, $\frac{1}{\tau^2}\|U_h^{n+1}\| \equiv 0$. This completes the proof of uniqueness, and consequently the existence, of the numerical solution.

To write this numerical scheme in a matrix form, Let $\{\phi_p\}_{p=1}^{d_{hx}}$ and $\{\psi_p\}_{p=1}^{d_{hy}}$ denote the basis functions of S_{h_x} and S_{h_y}, respectively, with $d_{hx} := \dim S_{hx} = N_x - 1$ and $d_{hy} := \dim S_{hy} = N_y - 1$. So, U_h^n can be written in term of the basis functions as:

$$U_h^n = \sum_{i=1}^{d_{hx}} \sum_{j=1}^{d_{hy}} b_{i,j}^n \phi_{p_i} \psi_{pj}.$$

We define the $d_{hx} \times d_{hx}$ matrices in the $x-$direction as

$$\boldsymbol{M}_x = \left[\int_a^b \phi_q \phi_p dx \right], \quad \boldsymbol{G}_x = \left[\int_a^b \phi_q' \phi_p' dx \right], \quad \text{and} \quad \boldsymbol{S}_x = \left[\int_a^b \phi_q'' \phi_p'' dx \right],$$

and the $d_{hy} \times d_{hy}$ matrices in the $y-$direction are

$$\boldsymbol{M}_y = \left[\int_c^d \psi_q \psi_p dy \right], \quad \boldsymbol{G}_y = \left[\int_c^d \psi_q' \psi_p' dy \right], \quad \text{and} \quad \boldsymbol{S}_y = \left[\int_c^d \psi_q'' \psi_p'' dy \right].$$

The $(d_{hx} \times d_{hy})$-dimensional column vectors $\mathbf{b}^n$ and $\mathbf{F}^n$ are the transpose of the vectors

$$[b_{1,1}^n, b_{1,2}^n, \cdots, b_{1,d_{hy}}^n, \cdots, b_{d_{hx},1}^n, \cdots, b_{d_{hx},d_{hy}}^n],$$

and

$$[f_{1,1}^n, f_{1,2}^n, \cdots, f_{1,d_{hy}}^n, \cdots, f_{d_{hx},1}^n, \cdots, f_{d_{hx},d_{hy}}^n], \quad \text{with} \quad f_{i,j}^n := \langle h(\delta_t U_h^n), \phi_i \psi_j \rangle,$$

respectively. Therefore, through tensor products of one-dimensional $\mathcal{C}^2$ splines, the fully-discrete scheme (4.34) has the following matrix representation:

$$\Big(\boldsymbol{M}_x \otimes \boldsymbol{M}_y\Big)\delta_{tt}\mathbf{b}^n + \Big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\Big)\mathbf{b}^{n+\frac{1}{4}} - \sum_{j=0}^{n} g_{n+1}^{j}\Big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\Big)\mathbf{b}^{j+\frac{1}{2}} = -\mathbf{F}^n,$$

with $g_{n+1}^{j} := \int_{t_j}^{t_{j+1}} g(t_{n+1} - s)\,ds$. Alternatively, this can be rewritten as:

$$\begin{aligned}\Big(4\boldsymbol{M}_x \otimes \boldsymbol{M}_y + \tau^2\big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\big) \\ - 2\tau^2 g_{n+1}^{n}\big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\big)\Big)\mathbf{b}^{n+1} &= 4\boldsymbol{M}_x \otimes \boldsymbol{M}_y(2\mathbf{b}^n - \mathbf{b}^{n-1}) \\ - \tau^2\Big(\boldsymbol{S}_x \otimes \boldsymbol{M}_y + 2\boldsymbol{G}_x \otimes \boldsymbol{G}_y + \boldsymbol{M}_x \otimes \boldsymbol{S}_y\Big)(2\mathbf{b}^n + \mathbf{b}^{n-1}) \\ +2\tau^2 g_{n+1}^{n}\big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\big)\mathbf{b}^n - 2\tau^2\big(\boldsymbol{G}_x \otimes \boldsymbol{M}_y + \boldsymbol{M}_x \otimes \boldsymbol{G}_y\big)\Big(\sum_{j=0}^{n-1} g_{n+1}^{j}(\mathbf{b}^{j+1} + \mathbf{b}^{j})\Big) - \mathbf{F}^n,\end{aligned}$$

for $1 \le n \le N - 1$. Therefore, at each time level t_{n+1}, we solve a finite square linear system, where the unknown is the column vector $\mathbf{b}^{n+1}$.

Furthermore, from the above matrix form, it is clear that our scheme (4.34) is a three-time level scheme. That is, the approximate solutions U_h^0 and U_h^1 need to be determined first, and then U_h^j for $2 \le j \le N$ can be computed by solving the above linear system recursively. We choose $U_h^0 \in S_h$ to be the bicubic spline polynomial interpolating u_0 at the interior nodal nodes. However, motivated by the Taylor series expansion of u about $t = 0$, we choose $U_h^1 \in S_h$ to be the bicubic spline polynomial interpolating $u_0 + t_1 u_1$ at the interior nodal nodes.

For the computer implementation of the linear system, it is important to consider discretization of spatial Galerkin-type integrals in the scheme. To this end, on each cell of our two-dimensional partition, the integrals are approximated using 2-point Gauss quadrature rule in each direction (x and y).

In our test problem, we choose $\Omega = (0,1)\times(0,1)$, the time interval is $(0,80)$, the initial data $u_0(x,y) = 64^2[xy(1-x)(1-y)]^3$, $u_1(x,y) = 0$, the relaxation function $g(t) = \mathrm{e}^{-t}$, and the damping function $h(s) = s$. The spatial mesh consists of 400 (square) cells of equal areas, while the time domain consists of 80000 subintervals.

Figure 4.1 shows that the numerical solution U_h converge to zero as the time t gets far away from 0. The graphical plots of the numerical approximations of the weighted energy in Figure 4.2 confirms that $tE(t) \leq 1$ for a sufficiently large t. This is compatible with the achieved theoretical result in Theorem 4.2 (see Example 4.3).

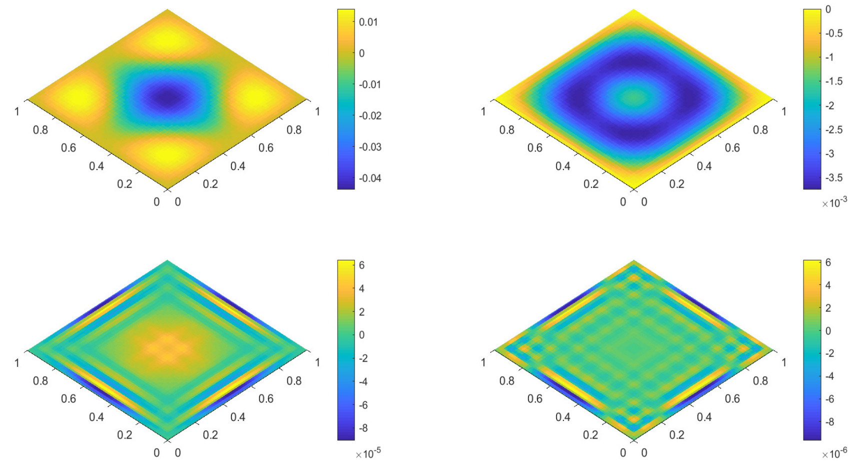

Figure 4.1: The numerical solutions for $t = 5$ (top-left), $t = 10$ (top-right), $t = 20$ (bottom-left), and $t = 30$ (bottom-right).

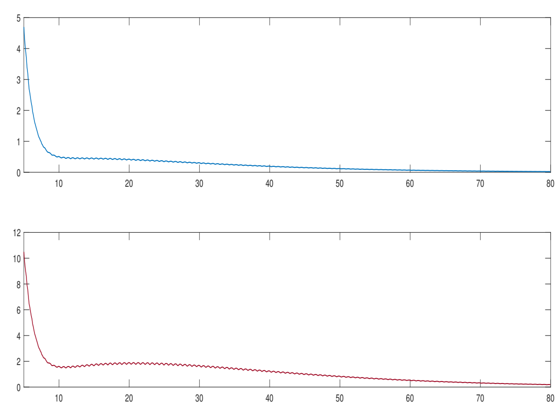

Figure 4.2: The numerical weighted energy plots against $t \in [5, 80]$. The top and the bottom are the approximations of $tE(t)$ and $t^{1.5}E(t)$, respectively.

CHAPTER 5

CONCLUSIONS AND FUTURE WORK

5.1 Conclusions

In this thesis we studied the general decay rates for weakly dissipative viscoelastic plate equations with a large class of relaxation functions. We proved several energy decay results for such equations with finite memory and nonlinear damping under the following condition on the relaxation function,

$$g'(t) \leq -\xi(t)G(g(t)), \qquad \forall\, t \geq 0.$$

Our results are the extension of that of Messaoudi and Hassan [14].

For the case of infinite memory, our result is the generalization of the work of

Revira [30]. His relaxation function satisfies the condition

$$g'(t) \leq -\delta g(t),\ \forall\, t \geq 0,\ \text{for a positive constant } \delta.$$

However, in our work, we used a more general condition

$$g'(t) \leq -\xi(t)g(t), \qquad \forall\, t \geq 0.$$

5.2 Future Work

- Investigating the viscoelastic plate equation with nonlinear damping and finite memory of the form

$$\begin{cases} u'' + \Delta^2 u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + d\,|u'|^{m-2}u' = 0 & \text{in } \Omega \times (0, +\infty), \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$, d is a positive constant, and the initial data u_0, u_1 are given and for $m > 1$ and a given non-negative relaxation function g satisfying the condition

$$g'(t) \leq -\xi(t)G(g(t)), \qquad \forall\, t \geq 0.$$

- Studying the viscoelastic plate equation with infinite memory of the form

$$\begin{cases} u'' + \Delta^2 u + \displaystyle\int_0^\infty g(s)\Delta u(t-s)ds = 0 \quad \text{in } \Omega \times (0,+\infty), \\ u(x,-t) = u_0(x,t), \quad u'(x,0) = u_1(x), \quad \text{in } \Omega, \end{cases} \tag{5.2}$$

  subject to the homogeneous conditions $u = \Delta u = 0$ on $\partial\Omega$, where the physical domain $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth (or piecewise smooth) boundary $\partial\Omega$. The initial velocity u_1 are given and the non-negative relaxation function g satisfies the condition

$$g'(t) \leq -\xi(t)G(g(t)), \qquad \forall\, t \geq 0.$$

REFERENCES

[1] Al-Gharabli M., Guesmia A., and Messaoudi S., Existence and a general decay results for a viscoelastic plate equation with a logarithmic nonlinearity, Communicat. Pure Appl. Anal., 18 (2019), 159–175.

[2] Arnold V., Mathematical Methods of Classical Mechanics, vol. 60 of Graduate Texts in Mathematics. Springer New York, NY, (1989).

[3] Belhannache F., Al-Gharabli M., and Messaoudi S., Asymptotic stability for a viscoelastic equation with nonlinear damping and very general type of relaxation functions, J. Dyn. Cont. Sys., 26 (2019), 45–67.

[4] Berrimi S. and Messaoudi S., Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping. Electronic Journal of Differential Equations [electronic only] (2004), Paper-No.

[5] Berrimi S., and Messaoudi S., Existence and decay of solutions of a viscoelastic equation with a nonlinear source. Nonlinear Analysis: Theory, Methods & Applications 64.10 (2006), 2314–2331.

[6] Brezis H., Functional analysis, Sobolev spaces and partial differential equations, Springer Science & Business Media, (2010).

[7] Cao X., Energy decay of solutions for a variable-coefficient viscoelastic wave equation with a weak nonlinear dissipation, Journal of Mathematical Physics 57.2 (2016), 021509.

[8] Cavalcanti M., Cavalcanti V., and Soriano J., Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping. Electronic Journal of Differential Equations, (2002), 54–65.

[9] Dafermos M., Asymptotic stability in viscoelasticity. Archive for rational mechanics and analysis 37.4 (1970), 297–308.

[10] Dafermos M., An abstract Volterra equation with applications to linear viscoelasticity, Journal of Differential Equations, 7.3 (1970), 554–569.

[11] Dafermos M., Asymptotic stability in viscoelasticity, Archive for rational mechanics and analysis, 37 (1970), 297–308.

[12] Guesmia A., Asymptotic stability of abstract dissipative systems with infinite memory, Journal of Mathematical Analysis and Applications, 382 (2011), 748–760.

[13] Han X. and Wang M., General decay of energy for a viscoelastic equation with nonlinear damping, Mathematical Methods in the Applied Sciences, 32 (2009), 346–358.

[14] Hassan J. and Messaoudi S., General decay rate for a class of weakly dissipative second-order systems with memory, Mathematical Methods in the Applied Sciences, 42 (2019), 2842–2853.

[15] Jin K., Liang J., and Xiao T., Coupled second order evolution equations with fading memory, Optimal energy decay rate, Journal of Differential Equations, 257 (2014), 1501–1528.

[16] Komornik V., On the nonlinear boundary stabilization of Kirchhoff plates, Nonlinear Differential Equations and Applications, (1994), 323–337.

[17] Lagnese J., Asymptotic energy estimates for Kirchhoff plates subject to weak viscoelastic damping, International series of numerical mathematics, 91 (1989), 211–236.

[18] Liu W., General decay rate estimate for a viscoelastic equation with weakly nonlinear time-dependent dissipation and source terms, Journal of Mathematical Physics, 50.11 (2009), 113–130.

[19] Liu W., General decay of solutions to a viscoelastic wave equation with nonlinear localized damping, Annales Academiæ Scientiarum Fennicæ. Mathematica 34.1 (2009), 291–302.

[20] Messaoudi S. and Al-Gharabli M., A general stability result for a nonlinear wave equation with infinite memory. Applied Mathematics Letters 26, no. 11 (2013), 1082–1086.

[21] Messaoudi S., General decay of solutions of a viscoelastic equation, Journal of Mathematical Analysis and Applications, 341 (2008), 1457–1467.

[22] Messaoudi S., General decay of the solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Analysis: Theory, Methods & Applications 69.8 (2008), 2589–2598.

[23] Messaoudi S., Global existence and nonexistence in a system of Petrovsky, Journal of Mathematical Analysis and Applications, 265 (2002), 296–308.

[24] Messaoudi S. and Al-Khulaifi W., General and optimal decay for a quasilinear viscoelastic equation, Applied Mathematics Letters 66 (2017), 16–22.

[25] Messaoudi S. and Tatar N., Exponential and polynomial decay for a quasilinear viscoelastic equation, Nonlinear Analysis: Theory, Methods & Applications 68 (2008), 785–793.

[26] Mustafa M., Optimal decay rates for the viscoelastic wave equation, Mathematical Methods in the Applied Sciences, 41 (2018), 192–204.

[27] Mustafa M., General decay result for nonlinear viscoelastic equations, Journal of Mathematical Analysis and Applications, 457 (2018), 134–152.

[28] Mustapha K. and Mustapha H., A quadrature finite element method for semilinear second-order hyperbolic problems, Numerical Methods for Partial Differential Equations, 24 (2008), 350–367.

[29] Rivera J. and Reinhard R., Magneto-thermo-elasticity-large-time behavior for linear systems, Advances in Differential Equations 6, 3 (2001), 359–384.

[30] Rivera, J., Eugenio C., and Rioco B., Decay rates for viscoelastic plates with memory, Journal of elasticity, 44 (1996), 61–87.

[31] Rivera J. and Maria N., Optimal energy decay rate for a class of weakly dissipative second-order systems with memory, Applied Mathematics Letters 23 (2010), 743–746.

[32] Vittorino P., Stability and exponential stability in linear viscoelasticity, Milan journal of mathematics 77 (2009), 333–340.

Vitae

- Name: Khaleel Ibrahim Khaleel Anaya
- Nationality: Palestinian
- Date of Birth: Dec 1, 1992
- Email: khalil.anaya1@gmail.com
- Present Adress: KFUPM, P.O. Box 7722, Dhahran 31261, SA.
- Permanent Address: Nablus, Palestine.

Qualifications:

- PhD in Mathematics Statistics from KFUPM, Dhahran in 2021.
- Master Degree in Mathematics from KFUPM in 2017.
- BSc Degree in Mathematics from An-Najah National University in 2013.

Practical Experiences:

- September 2013- June 2014: Math teacher at Abdullah Ibn Abbas schools in Palestine.
- September 2014- January 2015: Math teacher at the ministry of education in Palestine.

Publications:

- Anaya, Khaleel, Salim A. Messaoudi, and Kassem Mustapha. "Decay rate of a weakly dissipative viscoelastic plate equation with infinite memory." Arabian Journal of Mathematics (2020): 1-9.

- Anaya, Khaleel, and Salim A. Messaoudi. "General decay rate of a weakly dissipative viscoelastic equation with a general damping." Opuscula Mathematica 40, no. 6 (2020): 647-666.

- Anaya K., Messaoudi S., On decay rate of a weakly dissipative viscoelastic equation with general damping. Submitted.

References:

- Dr. Kassem Mustapha (kassem@kfupm.edu.sa)

  Professor

  Department of Mathematics and Statistics

  KFUPM

- Dr. Messaoudi, S. A. (smessaoudi@sharjah.ac.ae)

  Distinguished Professor

  Department of Mathematics and Statistics

  University of Sharjah

- Dr. Khaled Furati (kmfurati@kfupm.edu.sa)

  Professor

  Department of Mathematics and Statistics

  KFUPM