

A Generalization of Ahlfors-Schwarz Lemma

BY

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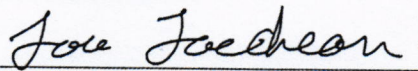
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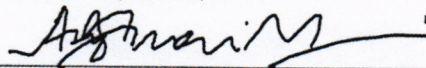
DEANSHIP OF GRADUATE STUDIES

This thesis, written by **MOHMMAD OTHMAN AL GANBER** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

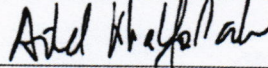
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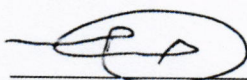
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For my beloved family

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THESIS ABSTRACT

NAME: Mohmmad Othman Al Ganber
TITLE OF STUDY: A generalization of Ahlfors-Schwarz Lemma
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Schwarz's lemma is a key tool in complex analysis and geometry that gives derivative estimates of holomorphic functions. In a recent paper by Xueyuan Wan, Schwarz's lemma was generalized to the case the domain is Riemann surface while the target is a complex Finsler manifold. In this diSSERTATION, we present the necessary background of complex manifold and Hermitian geometry to illustrate Wan's theorem in the case the domain is the unit disk of the complex plane.

ملخص الرسالة

الاسم: محمد عثمان آل قنبر

عنوان الدراسة: تعميم نظرية شوارز إلى متعددة الشعب من نوع فنسلر

التخصص: الرياضيات

تاريخ الدرجة العلمية: ربيع الثاني ١٤٤٢

برهان شواز اللمي يعتبر من الأدوات الأساسية في التحليل العقدي والهندسة الرياضية لتقدير مشتقة الدوال التامة الشكل. في دراسة حديثة قام الباحث وان بتعميم برهان شوارز إلى الدوال التي مجالها سطح ريمان ومجالها المقابل هو متعددة الشعب من نوع فنسلر. في هذا البحث نقوم بعرض الخلفية اللازمة عن المنتوعات العقدية وهندسة هيرميشيان لنشرح نظرية وان في حالة أن المجال هو القرص ذو مقدار واحد في المستوى العقدي بينما يكون المجال المقابل متعددة الشعب من نوع فنسلر.

CHAPTER 1

INTRODUCTION

1.1 Historical background and Motivation

Schwarz lemma is a well-known fact in complex analysis that gives an estimate for the derivatives of holomorphic functions from the unit disk to itself. Derivative estimates are used to investigate the convergence of sequences of maps, which is essential in solving analytic and geometric functional problems. One particular theorem in complex analysis which provides sufficient conditions for the convergence of a given family of functions is Montel's theorem asserting that a locally bounded family of holomorphic functions is normal. The condition of boundedness however is not preferred in geometry as it is not invariant under isometries e.g. the isometry between the unit disk and the upper half-plane. Rather, one would like to get an estimate based on geometric invariants such as the curvature, which is the subject of this paper.

In its classical form, Schwarz's lemma is given by

Theorem 1.1 (Schwarz's Lemma) *Let $f : D \rightarrow D$ be a holomorphic function from*

the unit disk to itself such that $f(0) = 0$. Then

- $|f(z)| \leq |z|$ for all $z \in D$.
- $|f'(0)| \leq 1$

Moreover, if $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \in D \setminus \{0\}$, then $f(z) = cz$ for some constant $c \in \mathbb{C}$ with $|c| = 1$.

The theorem was improved by Georg Pick essentially by dropping the condition that the function f fixes the origin.

Theorem 1.2 (Schwarz-Pick Lemma) *Let $f : D \rightarrow D$ be a holomorphic function from the unit disk to itself. Then*

$$|f(z)'| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

for every $z \in D$ and the equality holds if and only if f is an automorphism.

In 1938, Ahlfors [1] discovered that the derivative estimate in Schwarz's lemma is a consequence of the negativity of the curvature of the unit disk in Poincaré metric; his work shows that holomorphic functions from the unit disk into the complex plane are contractions with respect to the Poincaré metric, yielding a strong analytical constraint: equicontinuity. The Poincaré metric is given by

$$ds_z^2 = \lambda(z) dz \otimes d\bar{z},$$

where $\lambda(z) = \frac{1}{(1 - |z|^2)^2}$. The Gaussian curvature of D in this metric is

$$K(z) = \frac{-2}{\lambda(z)} \frac{\partial^2}{\partial z \partial \bar{z}} \log(\lambda(z)) = -4.$$

Given a holomorphic function $f : D \rightarrow D$, the pullback of ds_z^2 is defined by

$$f^* ds_z^2 = f^* \lambda(z) dz \otimes d\bar{z},$$

where $f^* \lambda(z) = \frac{|f'(z)|^2}{(1 - |f(z)|^2)^2}$. Hence, Schwarz-Pick lemma can be stated as $f^* \lambda(z) \leq \lambda(z)$. Ahlfors generalized this fact to Riemannian surfaces i.e. one-dimensional complex manifolds endowed with Hermitian metric.

Theorem 1.3 *Let $f : D \rightarrow M$ be a holomorphic map from the unit disk to a Riemannian surface M equipped with a Hermitian metric ds_M^2 such that its curvature is bounded by a negative number $-K$. Then*

$$f^* ds_M^2 \leq \frac{4}{K} ds_D^2,$$

where ds_D^2 is the Poincaré metric of the unit disk

This key discovery has led others to consider holomorphic maps in n -dimensional complex manifolds. It turned out that if the domain is a Riemannian surface then one can generalize Schwarz's lemma using the fact that the holomorphic sectional and bisectional curvatures do not exceed those of the ambient manifold. On the other hand, if the domain is of dimension two or higher new problems are presented. For

example, how do we compare the pullback of the Hermitian metric, which is multi-dimensional, to the canonical metric of the unit ball? By utilizing tools from Hermitian Geometry, S.S. Chern [2] and Y.C. Lu [3] resolved the case for maps from the complex n -dimensional unit ball into Hermitian manifolds. Additionally, Yau [4] and Royden [5] tackled the case of when the domain is a complete Kählerian manifolds.

Generalizations of Schwarz's lemma are still an ongoing effort. One particular direction for this field of research is to consider maps with a codomain that Finsler i.e. instead of having a inner product on the tangent space, a norm is given. Recent advancement has been achieved by Wan [6]. Wan tackled the case of the domain being a Riemannian surface and the target being a Finsler manifold. The goal of this dissertation is introduce the necessary background to prove the following simpler version of Wan's theorem.

Theorem 1.4 *Let (M, F) be a complex Finsler manifold with holomorphic sectional curvatures bounded above by a negative constant $-K$, D the unit disk, and $f : D \rightarrow M$ a holomorphic map. Let $\lambda(z)$ be the coefficient of the pulled tensor f^*F . Then*

$$\lambda(z) \leq \frac{2}{K(1 - |z|^2)^2}, \quad \text{for all } z \in D$$

1.2 Thesis Organization

In preparation to prove the simpler version of Wan's theorem, the paper is divided into four chapters.

Chapter 2 begins with a discussion of the basics of complex manifold theory. Def-

initions of complex manifolds, holomorphic maps on complex manifolds, and certain operators on complex manifolds are given. Standard examples are included for a better coherence of the concepts discussed.

Chapter 3 defines the Hermitian structure on complex manifolds. The first two sections focus on the case when the Hermitian structure is naturally related to the associated Riemannian structure i.e. the case when the Chern connection coincides with Levi-Citiva connection. The second part deals with the case when the Levi-Citiva connection is no longer valid to study the complex structures. In both cases, the notion of curvature, namely the Hermitian curvature is defined and formulas for computing it are given. Lastly, the chapter ends with proving the non-increasing property of holomorphic sectional curvature.

Chapter 4 mirrors the work done in Chapter 3 but for Finsler structure.

Chapter 5 gives the proof Wan's theorem in the case the domain is the unit Disk.

CHAPTER 2

COMPLEX STRUCTURE AND COMPLEX MANIFOLDS

2.1 Holomorphic Maps in $\mathbb{C}^n$

The space of complex numbers $\mathbb{C}$ is identified homeomorphically with the real plane $\mathbb{R}^2$ by sending the complex numbers $z = x + iy$ to the coordinates $(x, y) = (\operatorname{Re} z, \operatorname{Im} z)$. More generally, $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$ by viewing the point $(z_1, \dots, z_n) \in \mathbb{C}^n$, $z_j = x_j + iy_j$, as $(x_1, y_1, \dots, x_n, y_n) = (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_n, \operatorname{Im} z_n)$. With this identification, the definition of a function on $\mathbb{C}^n$ being holomorphic is extended as follow.

Definition 2.1 *A function $f : D \subseteq \mathbb{C}^n \rightarrow \mathbb{C}$, where D is open, is holomorphic if it is smooth as a map from $D \subseteq \mathbb{R}^{2n}$ into $\mathbb{R}^2$ and satisfies the following Cauchy-Riemann equations:*

$$\frac{\partial(\operatorname{Re} f)}{\partial x_j} = \frac{\partial(\operatorname{Im} f)}{\partial y_j}, \quad \frac{\partial(\operatorname{Im} f)}{\partial x_j} = -\frac{\partial(\operatorname{Re} f)}{\partial y_j},$$

for all $j = 1, \dots, n$.

Definition 2.2 A map $f : D \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^m$, where D is open, is holomorphic if it is smooth as a map from $D \subseteq \mathbb{R}^{2n}$ into $\mathbb{R}^{2m}$ and satisfies the following Cauchy-Riemann equations componentwisely:

$$\frac{\partial(\operatorname{Re} f_i)}{\partial x_j} = \frac{\partial(\operatorname{Im} f_i)}{\partial y_j}, \quad \frac{\partial(\operatorname{Im} f_i)}{\partial x_j} = -\frac{\partial(\operatorname{Re} f_i)}{\partial y_j},$$

for all $j = 1, \dots, n$ and $i = 1, \dots, m$.

A useful characterization of holomorphic maps is given by the concept of a complex structure. Consider the map $J_n =: \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $J_n(z_1, \dots, z_n) = (iz_1, \dots, iz_n)$. Notice that J_n viewed as a map from $\mathbb{R}^{2n}$ is represented by the matrix

$$J_n = \begin{bmatrix} J & & \\ & \ddots & \\ & & J \end{bmatrix},$$

where J is the matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Proposition 2.3 A smooth map $f : D \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^m$ is holomorphic if and only if $df \circ J_n = J_m \circ df$, where df is the differential of f i.e. the Jacobian of f .

Proof. Write $f = (f_1, \dots, f_m)$. Then, the differential df is represented by the matrix

$$df = \begin{bmatrix} \frac{\partial(\operatorname{Re} f_1)}{\partial x_1} & \frac{\partial(\operatorname{Re} f_1)}{\partial y_1} & \cdots & \frac{\partial(\operatorname{Re} f_1)}{\partial x_n} & \frac{\partial(\operatorname{Re} f_1)}{\partial y_n} \\ \frac{\partial(\operatorname{Im} f_1)}{\partial x_1} & \frac{\partial(\operatorname{Im} f_1)}{\partial y_1} & \cdots & \frac{\partial(\operatorname{Im} f_1)}{\partial x_n} & \frac{\partial(\operatorname{Im} f_1)}{\partial y_n} \\ \vdots & & & & \vdots \\ \frac{\partial(\operatorname{Re} f_n)}{\partial x_1} & \frac{\partial(\operatorname{Re} f_n)}{\partial y_1} & \cdots & \frac{\partial(\operatorname{Re} f_n)}{\partial x_n} & \frac{\partial(\operatorname{Re} f_n)}{\partial y_n} \\ \frac{\partial(\operatorname{Im} f_n)}{\partial x_1} & \frac{\partial(\operatorname{Im} f_n)}{\partial y_1} & \cdots & \frac{\partial(\operatorname{Im} f_n)}{\partial x_n} & \frac{\partial(\operatorname{Im} f_n)}{\partial y_n} \end{bmatrix}$$

Direct matrix multiplication gives

$$df \circ J_n = \begin{bmatrix} \frac{\partial(\operatorname{Re} f_1)}{\partial y_1} & \frac{-\partial(\operatorname{Re} f_1)}{\partial x_1} & \cdots & \frac{\partial(\operatorname{Re} f_1)}{\partial y_n} & \frac{-\partial(\operatorname{Re} f_1)}{\partial x_n} \\ \frac{\partial(\operatorname{Im} f_1)}{\partial y_1} & \frac{-\partial(\operatorname{Im} f_1)}{\partial x_1} & \cdots & \frac{\partial(\operatorname{Im} f_1)}{\partial y_n} & \frac{-\partial(\operatorname{Im} f_1)}{\partial x_n} \\ \vdots & & & \vdots & \\ \frac{\partial(\operatorname{Re} f_n)}{\partial y_1} & \frac{-\partial(\operatorname{Re} f_n)}{\partial x_1} & \cdots & \frac{\partial(\operatorname{Re} f_n)}{\partial y_n} & \frac{-\partial(\operatorname{Re} f_n)}{\partial x_n} \\ \frac{\partial(\operatorname{Im} f_n)}{\partial y_n} & \frac{-\partial(\operatorname{Im} f_n)}{\partial x_1} & \cdots & \frac{\partial(\operatorname{Im} f_n)}{\partial y_n} & \frac{-\partial(\operatorname{Im} f_n)}{\partial x_n} \end{bmatrix}$$

$$J_m \circ df = \begin{bmatrix} \frac{-\partial(\operatorname{Im} f_1)}{\partial x_1} & \frac{-\partial(\operatorname{Im} f_1)}{\partial y_1} & \cdots & \frac{-\partial(\operatorname{Im} f_1)}{\partial x_n} & \frac{-\partial(\operatorname{Im} f_1)}{\partial y_n} \\ \frac{\partial(\operatorname{Re} f_1)}{\partial x_1} & \frac{\partial(\operatorname{Re} f_1)}{\partial y_1} & \cdots & \frac{\partial(\operatorname{Re} f_1)}{\partial x_n} & \frac{\partial(\operatorname{Re} f_1)}{\partial y_n} \\ \vdots & & & \vdots & \\ \frac{-\partial(\operatorname{Im} f_n)}{\partial x_1} & \frac{-\partial(\operatorname{Im} f_n)}{\partial y_1} & \cdots & \frac{-\partial(\operatorname{Im} f_n)}{\partial x_n} & \frac{-\partial(\operatorname{Im} f_n)}{\partial y_n} \\ \frac{\partial(\operatorname{Re} f_n)}{\partial x_1} & \frac{\partial(\operatorname{Re} f_n)}{\partial y_1} & \cdots & \frac{\partial(\operatorname{Re} f_n)}{\partial x_n} & \frac{\partial(\operatorname{Re} f_n)}{\partial y_n} \end{bmatrix}$$

It is clear that equality of the two matrices is equivalent to Cauchy-Riemann equations. |

The map J_n is called the *standard complex structure* on $\mathbb{C}^n$ and it will be used in

the study of complex manifolds in the next section.

2.2 Complex Structures on Vector Spaces

2.2.1 Definitions and Properties

Let V be a finite-dimensional real vector space. A *complex structure* on V is a linear map $J : V \rightarrow V$ satisfying $J^2 = -I$, where I is the identity on V . Existence of such map places a restriction on the dimension of V .

Proposition 2.4 *If a none-empty vector space V admits a complex structure J , then the dimension of V is an even number.*

Proof. Since $J^2 = -I$, the minimal polynomial of J must divide $X^2 + 1$. Thus, J has no real eigenvalues. Let x_1 be a non-zero vector in V and set $y_1 = Jx_1$. The vectors x_1 and y_1 must be linearly independent; for if $x_1 = \lambda y_1, \lambda \in \mathbb{R}$, then $Jx_1 = -\lambda x_1$ which contradicts the fact that J has no real eigenvalues. Now suppose a list of linearly independent vectors $\{x_1, y_1, \dots, x_n, y_n\}$, where $y_i = Jx_i$ has been found and that $\dim(V) = k > 2n$. Let x_{n+1} be a vector in V but not in the span of $\{x_1, y_1, \dots, x_n, y_n\}$ and set $y_{n+1} = Jx_{n+1}$. We shall see that the vectors $\{x_1, y_1, \dots, x_{n+1}, y_{n+1}\}$ are linearly independent. Suppose for some real numbers $a_i, b_i \in \mathbb{R}, i = 1, \dots, n+1$ which are not all zero that

$$a_1x_1 + b_1y_1 + \dots + a_{n+1}x_{n+1} + b_{n+1}y_{n+1} = 0. \quad (2.1)$$

Because of the choice of x_{n+1} , the number b_{n+1} cannot be zero. By rearranging

and relabelling some terms, one gets

$$a_1x_1 + b_1y_1 + \dots + a_{n+1}x_{n+1} = y_{n+1}. \quad (2.2)$$

Note that a_{n+1} cannot be zero as that would imply $-x_{n+1} = Jy_{n+1}$ is in the span of $\{x_1, y_1, \dots, x_n, y_n\}$. Hence, one can write

$$a_1x_1 + b_1y_1 + \dots + x_{n+1} = \frac{1}{a_{n+1}}y_{n+1}. \quad (2.3)$$

Finally, applying the map J , one obtains

$$-a_1y_1 + b_1x_1 + \dots - \frac{1}{a_{n+1}}x_{n+1} = y_{n+1}. \quad (2.4)$$

Equations 2.2 and 2.3 imply that $a_{n+1}^2 = -1$, which is a contradiction. It follows that $\{x_1, y_1, \dots, x_{n+1}, y_{n+1}\}$ is a list of linearly independent vectors. This completes the proof by induction. ■

An example of complex structure is the action by a 90° rotation of the plane i.e. $V = \mathbb{R}^2$ and J is represented in the standard basis by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Given a real vector space V , its *complexification* is the complex vector space $V_{\mathbb{C}} := \mathbb{C} \otimes V$. Elements of $V_{\mathbb{C}}$ will be written as $X + iY := 1 \otimes X + i \otimes Y$, where $X, Y \in V$. Any complex structure on V extends uniquely to a linear structure $J_{\mathbb{C}}$ such that the

following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{J} & V \\ \downarrow & & \downarrow \\ V_{\mathbb{C}} & \xrightarrow{J_{\mathbb{C}}} & V_{\mathbb{C}} \end{array}$$

For proofs and further details refer to [7]. Essentially the map $J_{\mathbb{C}}$ sends the vector $X + iY$ to $JX + iJY$, and it can be seen that $J_{\mathbb{C}}^2 = -I_{V_{\mathbb{C}}}$, where $-I_{V_{\mathbb{C}}}$. This in turn implies that $J_{\mathbb{C}}$ has eigenvalues i and $-i$. For simplicity, $J_{\mathbb{C}}$ will be denoted by J as well.

Proposition 2.5 *Let V be a real vector space and J a complex structure on V . The eigenspaces of the complexified space $V_{\mathbb{C}}$ are given by*

$$V_{1,0} := \{x - iJx, x \in V\}, \quad V_{0,1} := \{x + iJx, x \in V\}$$

Proof. Let $x \in V$. Then $J(x - iJx) = Jx + ix = i(x - iJx)$ and $J(x + iJx) = Jx - ix = -i(x + iJx)$. Hence every vector in $V_{1,0}$ or $V_{0,1}$ is an eigenvector associated with i or $-i$, respectively. Notice that any vector $x + iy \in V_{\mathbb{C}}$ can be written as

$$\begin{aligned} x + iy &= \frac{1}{2}(x - iJx) + \frac{1}{2}(x + iJx) + \frac{i}{2}(y - iJy) + \frac{i}{2}(y + iJy). \\ &= \frac{1}{2}[x + Jy - iJ(x + Jy)] + \frac{1}{2}[x - Jy + iJ(x - Jy)] \in V_{1,0} + V_{0,1} \end{aligned}$$

The sets $V_{1,0}$ and $V_{0,1}$ are closed under addition and complex multiplication, and hence are subspaces of $V_{\mathbb{C}}$. Finally, if $x \in V_{1,0} \cap V_{0,1}$, then $ix = Jx = -ix$ which implies that $x = 0$. Therefore, $V = V_{1,0} \oplus V_{0,1}$. ■

The next proposition shows how to get a basis for the eigenspaces.

Proposition 2.6 *Let $\{x_1, y_1, \dots, x_n, y_n\}$, where $y_j = Jx_j$, be a basis for V . Then the set of vectors $\{z_1, \dots, z_n\}$, where $z_j = \frac{1}{2}(x_j - iy_j)$, forms a basis for $V_{1,0}$.*

Proof. To see that the set $\{z_1, \dots, z_n\}$ spans $V_{1,0}$, let $X \in V_{1,0}$. By definition, $X = x - iJx$ for some $x \in V$. Writing x in terms of the basis $\{x_1, y_1, \dots, x_n, y_n\}$, one obtains

$$x = a^j x_j + b^j y_j$$

Then

$$\begin{aligned} X &= a^j x_j + b^j y_j - i(a^j y_j - b^j x_j) \\ &= a^j (x_j - iy_j) + ib^j (x_j - iy_j) \\ &= 2(a^j + ib^j)z_j \end{aligned}$$

Now suppose $c^j z_j = 0$, where $c^j = a^j + ib^j$ are complex numbers. Then

$$\begin{aligned} 0 &= 2c^j z_j = (a^j + ib^j)(x_j - iy_j) \\ &= a^j x_j + b^j y_j + i(b^j x_j - a^j y_j) \end{aligned}$$

The linear independence of the vectors $\{x_1, y_1, \dots, x_n, y_n\}$ over $\mathbb{R}$ implies that $a^j = b^j = 0$ for $j = 1, \dots, n$. ■

Remark 2.7 A similar proof shows that the vectors $\bar{z}_j = \frac{1}{2}(x_j + iy_j)$ form a basis for $V_{0,1}$, and hence the set $\{z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n\}$ forms a basis for $V_{\mathbb{C}}$.

The concept of complex structure extends to the dual space V^* and the exterior

algebra $\Lambda(V)$. A brief treatment that is needed for this paper is given below.

Complexified Dual Space

As before, let V be a finite-dimensional real vector space, J a complex structure, and denote by J^* its dual operator. It is clear that J^* defines a complex structure on V^* since for any functional $f \in V^*$ and any vector $x \in V$, $J^*J^*f(x) = f(JJx) = -fx$; hence, $J^*J^*f = -f$ or, $J^*J^* = -I_{V^*}$. By proposition 2.5, the complexified dual space can be written as $(V^*)_{\mathbb{C}} = (V^*)_{1,0} \oplus (V^*)_{0,1}$, where $(V^*)_{1,0} = \{f - iJ^*f, f \in V^*\}$ and $(V^*)_{0,1} = \{f + iJ^*f, f \in V^*\}$.

Let $\{x_1, y_1, \dots, x_n, y_n\}$ be a basis for V where $y_j = Jx_j$. Let $\{\zeta^1, \eta^1, \dots, \zeta^n, \eta^n\}$ denotes the dual basis i.e.

$$\zeta^j(x_k) = \eta^j(y_k) = \delta_k^j, \quad \zeta^j(y_k) = \eta^j(x_k) = 0.$$

Notice that $J^*(\zeta^j) = -\eta^j$ and $J^*(\eta^j) = \zeta^j$. Let

$$\theta^j = \zeta^j + i\eta^j \tag{2.5}$$

Then a proof similar to the one in 2.6 shows that the set $\{\theta^1, \dots, \theta^n\}$ forms a basis for $(V^*)_{1,0}$ and its conjugate set $\{\bar{\theta}^1, \dots, \bar{\theta}^n\}$ i.e. $\bar{\theta}^j = \zeta^j - i\eta^j$, forms a basis for $(V^*)_{0,1}$. The maps θ^j and $\bar{\theta}^j$ are complex-values functionals on V but can be canonically extended to $V_{\mathbb{C}}$ by $\theta^j(a + ib) := \zeta^j(a) - \eta^j(b) + i(\zeta^j(b) + \eta^j(a))$, for each j and $a, b \in V$. This gives the identification $(V^*)_{\mathbb{C}} = (V_{\mathbb{C}})^*$, which can be seen in the next lemma.

Lemma 2.8 *The eigenspaces $(V^*)_{1,0}$ and $(V^*)_{0,1}$ are complex dual to $V_{1,0}$ and $V_{0,1}$,*

respectively.

Proof. Let $\{x_1, y_1, \dots, x_n, y_n\}$ and $\{z_1, \dots, z_n \bar{z}_1, \dots, \bar{z}_n\}$ be basis for V and $V_{\mathbb{C}}$ as in proposition 2.6 and remark 2.7. Let $\{\theta^1, \dots, \theta^n\}$ be basis for $(V^*)_{1,0}$, where θ^j 's are defined by equation 2.5. By direct computations,

$$\begin{aligned}\theta^j(z_k) &= (\zeta^j + i\eta^j)\left(\frac{1}{2}(x_k - iy_k)\right) \\ &= \frac{1}{2}(\zeta^j(x_k - iy_k) + \eta^j(ix_k + y_k)) \\ &= \delta_k^j\end{aligned}$$

$$\begin{aligned}\theta^j(\bar{z}_k) &= (\zeta^j + i\eta^j)\left(\frac{1}{2}(x_k + iy_k)\right) \\ &= \frac{1}{2}(\zeta^j(x_k + iy_k) + \eta^j(ix_k - y_k)) \\ &= 0.\end{aligned}$$

It follows that that $\{\theta^1, \dots, \theta^n\}$ is the dual basis for $\{z_1, \dots, z_n\}$. Similarly,

$$\begin{aligned}\bar{\theta}^j(\bar{z}_k) &= (\zeta^j - i\eta^j)\left(\frac{1}{2}(x_k + iy_k)\right) \\ &= \frac{1}{2}(\zeta^j(x_k + iy_k) - \eta^j(ix_k - y_k)) \\ &= \delta_k^j\end{aligned}$$

$$\begin{aligned}
\bar{\theta}^j(z_k) &= (\zeta^j - i\eta^j)\left(\frac{1}{2}(x_k - iy_k)\right) \\
&= \frac{1}{2}(\zeta^j(x_k - iy_k) - \eta^j(ix_k + y_k)) \\
&= 0,
\end{aligned}$$

which shows that $\{\bar{\theta}^1, \dots, \bar{\theta}^n\}$ is the dual basis for $\{\bar{z}_1, \dots, \bar{z}_n\}$. ■

For simplicity, the spaces $(V^*)_{1,0}$ and $(V^*)_{0,1}$ will be denoted by $V^{1,0}$ and $V^{0,1}$, respectively, and the space $(V^*)_{\mathbb{C}}$ by $V_{\mathbb{C}}^*$.

Complexified Covariant Tensors

Let V be a finite-dimensional vector space. A k -covariant tensor on V is a real-valued multilinear map of k vectors of V . Such tensor is called *alternating* if interchanging two of its arguments changes its sign. An example of an alternating tensor is the cross product in $\mathbb{R}^3$. The space of all k -covariant alternating tensors on V is denoted by $\Lambda^k(V^*)$. If α is a k -alternating tensor, and β is an l -alternating tensor, it is possible to define a new alternating tensor called the wedge product of α and β , denoted by $\alpha \wedge \beta$, which will be a $(k + l)$ -alternating tensor. For more details on alternating tensors and wedge product, refer to [8]. It follows from proposition 2.5 that

$$\Lambda^k(V_{\mathbb{C}}^*) = \bigoplus_{p+q=k} \Lambda^p(V^{1,0}) \wedge \Lambda^q(V^{0,1})$$

2.2.2 Hermitian and J-Invariant Inner Products

Let V be a real vector space equipped with a complex structure J . An inner product $\langle \cdot, \cdot \rangle$ on V is J -invariant if for any $X, Y \in V$, $\langle X, Y \rangle = \langle JX, JY \rangle$. The Euclidean

inner product on $\mathbb{R}^{2n}$ is an example a J -invariant inner product i.e. when J is the standard complex structure. To see this, let $\{x_1, y_1, \dots, x_n, y_n\}$ denotes the standard coordinates on $\mathbb{R}^{2n}$, and let $v = a^j x_j + b^k y_k$ and $w = c^s x_s + d^t y_t$ two vectors in $\mathbb{R}^{2n}$. Then

$$\begin{aligned}
\langle v, w \rangle &= \langle a^j x_j + b^k y_k, c^s x_s + d^t y_t \rangle \\
&= \sum_{j=1}^n a^j c^j + b^j d^j \\
&= \langle a^j y_j - b^k x_k, c^s y_s - d^t x_t \rangle \\
&= \langle Jv, Jw \rangle
\end{aligned}$$

In fact, any inner product $\langle \cdot, \cdot \rangle'$ on a vector space V with a complex structure J induces a J -invariant inner $\langle \cdot, \cdot \rangle$ product according to the equation

$$\langle X, Y \rangle = \langle X, Y \rangle' + \langle JX, JY \rangle'.$$

The next proposition defines a correspondance between J -invariant inner products on V and Hermitian inner products on $V_{1,0}$. The proof uses the following lemma.

Lemma 2.9 *Let V be a vector space equipped with a complex structure J . The map $\phi : X \rightarrow V_{1,0}$ defined by $\phi(X) = (X - iJX)/2$ is an isomorphism such that*

$$\phi(JX) = i\phi(X), \quad \phi^{-1}((a + ib)\phi(X)) = (a + bJ)X,$$

for any $X \in V$ and $a, b \in \mathbb{R}$.

Proof. Linearity and injectivity of ϕ are clear from the definition. Surjectivity can be seen from proposition 2.5. Hence, ϕ is an isomorphism. Notice that

$$\phi(JX) = (JX - iJ^2X)/2 = i(X - iJX)/2 = i\phi(X).$$

It follows that

$$\phi((a + bJ)X) = (a + ib)\phi(X).$$

Applying ϕ^{-1} to both sides, one gets

$$(a + bJ)X = \phi^{-1}((a + ib)\phi(X)).$$

■

Proposition 2.10 *There is a 1–1 correspondance between J -invariant inner products on V and Hermitian inner products on $V_{1,0}$.*

Proof. Let ϕ be as in lemma 2.9. Suppose $(.,.)$ is a Hermitian inner product on $V_{1,0}$. Let $\langle X, Y \rangle := \operatorname{Re}(\phi(X), \phi(Y))$. Then

- Bilinearity: the maps Re and ϕ are real-linear, and $(.,.)$ is real bilinear.
- Symmetry: $\operatorname{Re}(\phi(X), \phi(Y)) = \operatorname{Re}(\overline{\phi(Y)}, \phi(X)) = \operatorname{Re}(\phi(Y), \phi(X))$.
- Positive-definiteness: $\langle X, X \rangle = \operatorname{Re}(\phi(X), \phi(X)) = (\phi(X), \phi(X)) \geq 0$.
- J -invariance: from lemma 2.9

$$\langle JX, JY \rangle = \operatorname{Re}(i\phi(X), i\phi(Y)) = \operatorname{Re} i(-i)(\phi(X), \phi(Y)) = \langle X, Y \rangle.$$

On the other hand, suppose a J -invariant inner product $\langle \cdot, \cdot \rangle$ is given. Define $(\cdot, \cdot)$ by

$$(Z, W) := \langle \phi^{-1}(Z), \phi^{-1}(W) \rangle + i \langle \phi^{-1}(Z), J\phi^{-1}(W) \rangle.$$

Then

- Skew-symmetry:

$$\begin{aligned} (Z, W) &= \langle \phi^{-1}(Z), \phi^{-1}(W) \rangle + i \langle \phi^{-1}(Z), J\phi^{-1}(W) \rangle \\ &= \langle \phi^{-1}(W), \phi^{-1}(Z) \rangle + i \langle J\phi^{-1}(Z), J^2\phi^{-1}(W) \rangle \\ &= \langle \phi^{-1}(W), \phi^{-1}(Z) \rangle - i \langle \phi^{-1}(W), J\phi^{-1}(Z) \rangle \\ &= \overline{(W, Z)} \end{aligned}$$

- Positive-definiteness: this follows from the positive-definiteness of $\langle \cdot, \cdot \rangle$ and the fact that for all $X \in V$,

$$\langle X, JX \rangle = \langle JX, J^2X \rangle = -\langle X, JX \rangle = 0.$$

- Conjugate-linearity: linearity over addition follows from the linearity of $\langle \cdot, \cdot \rangle$ and ϕ^{-1} . Let a, b be real numbers and $X = \phi^{-1}(Z)$. Then

$$\begin{aligned}
((a + ib)Z, W) &= \langle \phi^{-1}((a + ib)Z), \phi^{-1}(W) \rangle + i \langle \phi^{-1}((a + ib)Z), J\phi^{-1}(W) \rangle \\
&= \langle \phi^{-1}((a + ib)\phi(X)), \phi^{-1}(W) \rangle + i \langle \phi^{-1}((a + ib)\phi(X)), J\phi^{-1}(W) \rangle \\
&= \langle (a + bJ)X, \phi^{-1}(W) \rangle + i \langle (a + bJ)X, J\phi^{-1}(W) \rangle \\
&= a(\langle \phi^{-1}(Z), \phi^{-1}(W) \rangle + i \langle \phi^{-1}(Z), J\phi^{-1}(W) \rangle) \\
&\quad + b(\langle J\phi^{-1}(Z), \phi^{-1}(W) \rangle + i \langle J\phi^{-1}(Z), J\phi^{-1}(W) \rangle) \\
&= a(Z, W) + ib(i \langle \phi^{-1}(Z), J\phi^{-1}(W) \rangle + \langle \phi^{-1}(Z), \phi^{-1}(W) \rangle) \\
&= (a + ib)(Z, W),
\end{aligned}$$

where in the third line lemma 2.9 was used.

■

A Hermitian inner product $(\cdot, \cdot)$ defined on $V_{1,0}$ can be extended to a bilinear form h on $V_{\mathbb{C}}$ by setting

$$h(Z, \bar{W}) = (Z, W), \text{ where } Z, W \in V_{1,0},$$

and

$$h(Z, W) = h(\bar{Z}, W) = h(\bar{Z}, \bar{W}) = 0, \text{ for all } Z, W \in V_{1,0}.$$

Proposition 2.11 *Let $\langle \cdot, \cdot \rangle$ be a J -invariant inner product on V , and let h be the associated bilinear form on $V_{\mathbb{C}}$. Then for all $X, Y \in V$, $\langle X, Y \rangle = \operatorname{Re} h(X, Y)$.*

Proof. Let $(\cdot, \cdot)$ denotes the Hermitian inner product on $V_{1,0}$ associated with $\langle \cdot, \cdot \rangle$.

By proposition 2.10,

$$\begin{aligned}
\operatorname{Re} h(X, Y) &= \operatorname{Re} h(\phi(X) + \overline{\phi(X)}, \phi(Y) + \overline{\phi(Y)}) \\
&= \operatorname{Re} (\phi(X), \phi(Y)) \\
&= \operatorname{Re} (\langle X, Y \rangle + i\langle X, JY \rangle) \\
&= \langle X, Y \rangle
\end{aligned}$$

■

Let $\{X_1, Y_1, \dots, X_n, Y_n\}$ be basis for V such that $Y_j = JX_j$ for $j = 1, \dots, n$, $\{Z_1, \dots, Z_n\}$ basis for $V_{1,0}$ as in proposition 2.6, and $\{\theta^1, \dots, \theta^n\}$ be the dual basis. Then the bilinear form h is given by

$$h = h_{j\bar{k}} \theta^j \otimes \theta^{\bar{k}},$$

where $h_{j\bar{k}} = h(Z_j, Z_{\bar{k}})$. Note that the bar in $h_{j\bar{k}}, Z_{\bar{k}}, \theta^{\bar{k}} \dots$ etc is put over the index rather than the symbol. This convention will be used from now on. Define the forms $g := h_{j\bar{k}}(\theta^j \otimes \theta^{\bar{k}} + \theta^{\bar{k}} \otimes \theta^j)/2$ and $\omega := ih_{j\bar{k}}(\theta^j \otimes \theta^{\bar{k}} - \theta^{\bar{k}} \otimes \theta^j)/2 = ih_{j\bar{k}}\theta^j \wedge \theta^{\bar{k}}$. Since $\bar{h}_{j\bar{k}} = \overline{h(Z_j, Z_{\bar{k}})} = (Z_k, Z_j) = h_{k\bar{j}}$, one has $g = \bar{g}$ and $\omega = \bar{\omega}$ i.e. both forms are real.

It is clear that

$$h = g - i\omega,$$

and that this decomposition does not depend on the choice of basis. Finally, from proposition 2.11, g is the J -invariant inner product on V associated with h . The form ω is called the associated $(1, 1)$ form of h .

2.3 Complex Manifolds

Definition 2.12 An n -dimensional *complex manifold* M is a $2n$ -dimensional smooth manifold that admits an atlas of charts (U_α, ϕ_α) compatible with the smooth structure and such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1} : \mathbb{R}^{2n} = \mathbb{C}^n \rightarrow \mathbb{R}^{2n} = \mathbb{C}^n$ are holomorphic. In terms of coordinates, a chart map ϕ_α has the form $\phi_\alpha = (z^1, \dots, z^n)$, where each z_i is a smooth map from M into $\mathbb{R}^2$.

As in the smooth manifold case, the collection of all charts that are compatible with a given atlas, i.e. the transition maps are holomorphic, is called a *complex structure*

EXAMPLES:

1. The standard complex structure on the complex plane is defined by the global chart map $\phi : \mathbb{C} \rightarrow \mathbb{C}$, $\phi(z) = z$.
2. A different complex structure on the complex plane is given by the chart map $\psi(z) = \bar{z}$. This is a different structure since the map $\psi \circ \phi^{-1}$ is not holomorphic.
3. The n -dimensional complex space $\mathbb{C}^n$ with the identity map is a complex manifold.
4. The n -dimensional complex unit ball $\mathbb{B}^n := \{(z^1, \dots, z^n) \in \mathbb{C}^n; \sum_{i=1}^n |z^i|^2 < 1\}$ with the inclusion map is a complex manifold.
5. The n -dimensional complex projective $\mathbb{CP}^n$ space is the set of all one-dimensional subspaces in $\mathbb{C}^{n+1}$. An element in $\mathbb{CP}^n$ can be represented by a point $[z^1, \dots, z^{n+1}]$, where $z^i \in \mathbb{C}$. Let U_i be the set of lines $[z^1, \dots, z^{n+1}]$ such that $z^i \neq 0$ and define

the map $\phi_i : U_i \rightarrow \mathbb{C}^n$ by

$$\phi_i([z^1, \dots, z^{n+1}]) = \left(\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^{n+1}}{z^i} \right).$$

The inverse of ϕ_i is given by

$$\phi_i^{-1}(z^1, \dots, z^n) = [z^1, \dots, z^{i-1}, 1, z^i, \dots, z^n].$$

Then for $i > j$,

$$\phi_i \circ \phi_j^{-1}(z^1, \dots, z^n) = \left(\frac{z^1}{z^i}, \dots, \frac{z^{j-1}}{z^i}, \frac{1}{z^i}, \frac{z^j}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots, \frac{z^n}{z^i} \right),$$

which is holomorphic on $\phi_j(U_i \cap U_j)$. The case for $j > i$ is similar and the collection of charts (U_i, ϕ_i) hence define a complex structure on $\mathbb{CP}^n$.

Definition 2.13 Let M be a complex manifold. A function $f : M \rightarrow \mathbb{C}$ is *holomorphic* if for each chart (U, ϕ) on M , the function $f \circ \phi^{-1}$ is holomorphic. Generally, a function $f : M \rightarrow N$ between two complex manifolds is holomorphic if for each chart (U, ϕ) on M , and each chart (V, ψ) on N , the map $\psi \circ f \circ \phi^{-1}$ is holomorphic.

Note that due to the compatibility between different charts, a function is holomorphic if when expressed in local coordinates, each component is holomorphic. Another useful characterization can be obtained from complex structures in a similar manner to what has been shown in section 2.1.

Let M be an n -dimensional complex manifold and $p \in M$. There is a natural com-

plex structure on $T_p M$ defined as follow: Let ϕ be any chart map on a neighborhood of p . Define $J = d\phi^{-1} \circ J_n \circ d\phi$.

Proposition 2.14 *The map J defined above is a complex structure on $T_p M$ and it independent of the choice of chart.*

Proof. The map J defines a complex structure since

$$\begin{aligned} J^2 &= d\phi^{-1} \circ J_n \circ d\phi \circ d\phi^{-1} \circ J_n \circ d\phi \\ &= d\phi^{-2} \circ J_n^2 \circ d\phi \\ &= d\phi^{-1} \circ (-I_{\mathbb{C}^n}) \circ d\phi \\ &= -I_{T_p M}. \end{aligned}$$

Suppose ψ is another chart at p . It suffices to show that

$$d\psi^{-1} \circ J_n \circ d\psi = d\phi^{-1} \circ J_n \circ d\phi,$$

or

$$J_n = d\psi \circ d\phi^{-1} \circ J_n \circ d\phi \circ d\psi^{-1}.$$

Using the fact that $d\psi \circ d\phi^{-1} = d(\psi \circ \phi^{-1})$ and $d\phi \circ d\psi^{-1} = d(\phi \circ \psi^{-1})$ are holomorphic, and J_n commutes with holomorphic maps as shown in proposition 2.3, one can see that J does not depend on the chosen chart. ■

Definition 2.15 Let M be a smooth manifold. An *almost complex sturcture* is a smooth $(1, 1)$ tensor J on M such that for each $p \in M$, $J_p : T_p M \rightarrow T_p M$ is a complex

structure i.e. $J_p^2 = -I_{T_p M}$.

It follows from the fact that chart maps are smooth and proposition 2.14 that a complex manifold M always admits an almost complex structure, denoted henceforth by J_M .

Proposition 2.16 *Let M and N be two complex manifolds. A map $f : M \rightarrow N$ is holomorphic if and only if $df \circ J_M = J_N \circ df$.*

Proof. Let $p \in M$ be an arbitrary point, ϕ a holomorphic coordinate map at p , and ψ a holomorphic coordinatemap at $f(p)$. Notice that

$$\begin{aligned}
df \circ J_M &= J_N \circ df \Leftrightarrow df = J_N \circ df \circ J_M^{-1} \\
&\Leftrightarrow df = d\psi^{-1} \circ J_n \circ d\psi \circ df \circ d\phi^{-1} \circ J_m^{-1} \circ d\phi \\
&\Leftrightarrow d\phi \circ df \circ d\psi^{-1} = d\phi \circ d\psi^{-1} \circ J_n \circ d\psi \circ df \circ d\phi^{-1} \circ J_m^{-1} \circ d\phi \circ d\psi^{-1} \\
&\Leftrightarrow d\phi \circ df \circ d\psi^{-1} \circ J_m = J_n \circ d\phi \circ df \circ d\psi^{-1},
\end{aligned}$$

where in the fourth line we used the fact that the standard complex structures J_n and J_m commutes with the holomorphic function $d\phi \circ d\psi^{-1}$, see proposition 2.3. The conclusion follows from 2.3 as well. ■

Similar to the case of a complex structure on a vector space, an almost complex structure on manifold gives a decomposition on the tangent space. Let M be a complex manifold, p a point in M , and $(x^1, y^1, \dots, x^n, y^n)$ be a holomorphic chart at p . Then

as in proposition 2.5, one has the following decompositions

$$T_p M_{\mathbb{C}} = T_p M_{1,0} \oplus T_p M_{0,1},$$

where $T_p M_{1,0} = \text{span}\{\frac{\partial}{\partial z^j}\}$, $\frac{\partial}{\partial z^j} := \frac{1}{2}(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j})$ and $T_p M_{0,1} = \text{span}\{\frac{\partial}{\partial \bar{z}^j}\}$, $\frac{\partial}{\partial \bar{z}^j} := \frac{1}{2}(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j})$.

$$T_p M_{\mathbb{C}}^* = T_p M^{1,0} \oplus T_p M^{0,1},$$

where $T_p M^{1,0} = \text{span}\{dz^j\}$, $dz^j := dx^j + idy^j$ and

$T_p M^{0,1} = \text{span}\{d\bar{z}^j\}$, $d\bar{z}^j := dx^j - idy^j$. The space $\Lambda^k(T_p M^{\mathbb{C}})$ has the following

decomposition

$$\Lambda^k(T_p M^{\mathbb{C}}) = \bigoplus_{m+n=k} \Lambda^m(T_p M^{1,0}) \wedge \Lambda^n(T_p M^{0,1}).$$

These decompositions induce the following bundle decompositions

$$TM_{\mathbb{C}} = TM_{1,0} \oplus TM_{0,1},$$

$$TM_{\mathbb{C}}^* = TM^{1,0} \oplus TM^{0,1},$$

$$\Lambda^k(TM^{\mathbb{C}}) = \bigoplus_{m+n=k} \Lambda^m(TM^{1,0}) \wedge \Lambda^n(TM^{0,1}).$$

For some final remarks on the relation between complex and almost complex structure, recall that a complex manifold has a naturally defined almost complex structure induced by the holomorphic coordinate. The opposite of this is not always true i.e. starting with an almost complex structure, one may not always find holomorphic

coordinates that are compatible with the almost complex structure. A famous example of such is given on S^6 in [9]. In fact, sufficient and necessary condition for the existence of holomorphic coordinates on an almost complex manifold is given by Newlander-Nirenberg theorem

Theorem 2.17 *An almost complex manifold (M, J) is a complex manifold if and only if the Lie bracket of any two $(1, 0)$ vector fields is $(1, 0)$ vector fields.*

A proof of this theorem can be found in [10]. In this paper, the focus will be on complex manifolds so it will always be assumed that holomorphic coordinates exist on any point.

2.3.1 ∂ and $\bar{\partial}$ Operators

Let M be a complex manifold. A smooth section of $\Lambda^k(TM^{\mathbb{C}})$ is called a k -form. The space of all k -forms is denoted by $E^k(M)$ and it can be decomposed as

$$E^k(M) = \oplus_{p+q=k} E^{p,q}(M),$$

where $E^{p,q}(M)$ is a section of $\Lambda^p(TM^{1,0}) \wedge \Lambda^q(TM^{0,1})$. Elements in $E^{p,q}(M)$ are (p, q) forms, or forms of type (p, q) . Let $(z^1, \dots, z^n)$ be a local basis and let I and J be a set of ordered indices $1 < i_1 < \dots < i_p < n$ and $1 < j_1 < \dots < j_q < n$. The term $dz^I \wedge d\bar{z}^J$ is defined as

$$dz^I \wedge d\bar{z}^J = dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}.$$

Using this expression, a form α of type (p, q) is written locally as

$$\alpha = \sum_{I,J} \alpha_{I,J} dz^I \wedge d\bar{z}^J,$$

where $\alpha_{I,J}$ are smooth complex-valued functions and the sum is taken over all ordered indices.

The exterior differentiation operator d defined on M as a smooth manifold extends to a complex linear operator acting on the space of sections $E^k(M)$. Notice that

$$d(dz^j) = d(dx^j) + id(dy^j) = 0,$$

and

$$d(d\bar{z}^j) = d(dx^j) - id(dy^j) = 0,$$

for all $j = 1, \dots, n$. It follows that the exterior differential of the (p, q) form α is given by

$$d\alpha = \sum_{I,J} \frac{\partial \alpha_{I,J}}{\partial z^k} dz_k \wedge dz^I \wedge d\bar{z}^J + \frac{\partial \alpha_{I,J}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^I \wedge d\bar{z}^J.$$

The first term on the right-hand side is a $(p+1, q)$ form while the second term is a $(p, q+1)$ form. This motivates the following definition.

Definition 2.18 The operators ∂ and $\bar{\partial}$ are defined to send a (p, q) form α to the $(p+1, q)$ and $(p, q+1)$ components of $d\alpha$, respectively.

Note that although the decomposition of $d\alpha$ into $(p+1, q)$ and $(p, q+1)$ components was given in coordinates, the definition is independent of the choice of coordinates since

the two components are linearly independent. Also, note that with this definition, one can write $d = \partial + \bar{\partial}$.

Lemma 2.19 *The following relations are true*

$$\partial^2 = \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Proof. This follows from the fact that $d^2 = 0$. If α is form of type (p, q) , then

$$0 = d^2\alpha = \partial^2\alpha + \bar{\partial}^2\alpha + (\partial\bar{\partial} + \bar{\partial}\partial)\alpha.$$

The term $\partial^2\alpha$ is of type $(p+2, q)$, $\bar{\partial}^2\alpha$ is of type $(p, q+2)$, and $(\partial\bar{\partial} + \bar{\partial}\partial)\alpha$ is of type $(p+1, q+1)$. Since these terms are linearly independent, they must all equal zero. **■**

A k -form α is called d -closed if $d\alpha = 0$. It is said to be d -exact if there exists a $(k-1)$ -form β such that $\alpha = d\beta$. Similarly, a (p, q) form α is called $\bar{\partial}$ -closed if $\bar{\partial}\alpha = 0$. It is said to be $\bar{\partial}$ -exact if there exists a $(p, q-1)$ form β such that $\alpha = \bar{\partial}\beta$. It follows from the previous lemma that a $\bar{\partial}$ -exact form is $\bar{\partial}$ -closed. The converse is true but locally which is given by Dolbeault-Grothendieck lemma. For a proof, see [10].

Theorem 2.20 (Dolbeault-Grothendieck lemma) *Let α be a (p, q) -form such that $q > 0$ and $\bar{\partial}\alpha = 0$. Then, locally there exists a $(p, q-1)$ form β such that $\alpha = \bar{\partial}\beta$.*

CHAPTER 3

HERMITIAN STRUCTURE ON COMPLEX MANIFOLDS

This chapter defines the Hermitian structure on a complex manifold, and discusses a special case of Hermitian manifolds, namely, Kähler manifolds. Characterizing properties and examples of Kähler manifolds will be presented. The chapter concludes with the definition and computations of the curvature tensor for Kähler manifolds.

3.1 Hermitian and Kähler Metrics

Let M be a complex manifold. A Hermitian metric on M is a $\mathbb{C}$ -valued smooth 2-covariant tensor h on $TM_{\mathbb{C}}$ such that for all $z \in M$ and $Z, W \in T_z M_{1,0}$

1. $h_z(Z, W) = h_z(\bar{Z}, W) = h_z(\bar{Z}, \bar{W}) = 0$
2. $h_z(Z, \bar{W}) = \overline{h_z(W, \bar{Z})}$.
3. $h_z(Z, \bar{Z}) \geq 0$ and $h_z(Z, \bar{Z}) = 0 \Leftrightarrow Z = 0$.

The results of section 2.2.2 shows that a Hermitian metric induces a J -invariant Riemannian metric $g = \operatorname{Re} h$ on M , and vice versa. Let $\{z_1, \dots, z_n\}$ be a local holomorphic coordiante. Then h can be expressed as

$$h = h_{j\bar{k}} dz^j \otimes dz^{\bar{k}},$$

where $h_{j\bar{k}} = h(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{k}}})$. The associated $(1, 1)$ -form of h given by $\omega = ih_{j\bar{k}} dz^j \wedge dz^{\bar{k}}$ is well-defined i.e. independent of the choice of coordinates. To see this, let $\{w_1, \dots, w_n\}$ be another holomorphic coordiantes and let $\frac{\partial}{\partial z^j} = a_j^k \frac{\partial}{\partial w^k}$ and $dz^j = b_k^j dw^k$. This implies that $a_l^k b_k^j = \delta_l^j$. Then

$$\begin{aligned} \omega &= ih(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{k}}}) dz^j \wedge dz^{\bar{k}} \\ &= ia_j^s a_k^t b_m^j b_n^{\bar{k}} h(\frac{\partial}{\partial w^s}, \frac{\partial}{\partial w^{\bar{t}}}) dw^m \wedge dw^{\bar{n}} \\ &= ih(\frac{\partial}{\partial w^s}, \frac{\partial}{\partial w^{\bar{t}}}) dw^s \wedge dw^{\bar{t}} \end{aligned}$$

Definition 3.1 A Hermitian metric h on M is called a *Kähler metric* if for each point in M there exists a coordinate chat $\{z_1, \dots, z_n\}$ and a real-valued function ψ defined on the chart such that

$$h_{j\bar{k}} = \frac{\partial^2 \psi}{\partial z^j \partial z^{\bar{k}}}$$

for $j, k = 1, \dots, n$ and all points in the chart. The function ψ is called a Kähler potential.

Theorem 3.2 *Let h be a Hermitian metric on a complex manifold M . Then the*

following are equivalent.

1. h is Kähler

2. For each point in M there exists a chart $\{z_1, \dots, z_n\}$ such that

$$\frac{\partial h_{j\bar{k}}}{\partial z^l} = \frac{\partial h_{l\bar{k}}}{\partial z^j}, \quad \frac{\partial h_{j\bar{k}}}{\partial z^{\bar{l}}} = \frac{\partial h_{j\bar{l}}}{\partial z^{\bar{k}}}$$

for all $j, k, l = 1, \dots, n$.

3. The associated $(1,1)$ -form ω is closed

Proof. (1) $\Rightarrow$ (2) Given a point in M , let $\{z_1, \dots, z_n\}$ be coordinate chart at that point with a Kähler potential function ψ . Then

$$\begin{aligned} \frac{\partial h_{j\bar{k}}}{\partial z^l} &= \frac{\partial^3 \psi}{\partial z^l \partial z^j \partial z^{\bar{k}}} \\ &= \frac{\partial^3 \psi}{\partial z^j \partial z^l \partial z^{\bar{k}}} \\ &= \frac{\partial h_{l\bar{k}}}{\partial z^j} \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial h_{j\bar{k}}}{\partial z^{\bar{l}}} &= \frac{\partial^3 \psi}{\partial z^{\bar{l}} \partial z^j \partial z^{\bar{k}}} \\ &= \frac{\partial^3 \psi}{\partial z^{\bar{k}} \partial z^j \partial z^{\bar{l}}} \\ &= \frac{\partial h_{j\bar{l}}}{\partial z^{\bar{k}}} \end{aligned}$$

(2) $\Rightarrow$ (3) Notice that

$$\begin{aligned}
\frac{\partial h_{j\bar{k}}}{\partial z^s} dz^s \wedge dz^j \wedge dz^{\bar{k}} &= \frac{1}{2} \frac{\partial h_{j\bar{k}}}{\partial z^s} dz^s \wedge dz^j \wedge dz^{\bar{k}} + \frac{1}{2} \frac{\partial h_{j\bar{k}}}{\partial z^s} dz^s \wedge dz^j \wedge dz^{\bar{k}} \\
&= \frac{1}{2} \frac{\partial h_{j\bar{k}}}{\partial z^s} dz^s \wedge dz^j \wedge dz^{\bar{k}} - \frac{1}{2} \frac{\partial h_{j\bar{k}}}{\partial z^s} dz^j \wedge dz^s \wedge dz^{\bar{k}} \\
&= \frac{1}{2} \frac{\partial h_{j\bar{k}}}{\partial z^s} dz^s \wedge dz^j \wedge dz^{\bar{k}} - \frac{1}{2} \frac{\partial h_{j\bar{k}}}{\partial z^s} dz^s \wedge dz^j \wedge dz^{\bar{k}} \\
&= 0,
\end{aligned}$$

where in the third line the indexes on the second term were swapped and the assumption in (2) was used. Similar computation shows that $\frac{\partial h_{j\bar{k}}}{\partial z^{\bar{s}}} dz^{\bar{s}} \wedge dz^j \wedge dz^{\bar{k}} = 0$. Hence, $d\omega = 0$.

(3) $\Rightarrow$ (1) By Poincaré local exactness lemma, there exists a real 1-form η defined locally such that $\omega = d\eta$. Let $\eta = \eta^{1,0} + \eta^{0,1}$ be its decomposition by type. Then

$$\omega = (\partial + \bar{\partial})(\eta^{1,0} + \eta^{0,1}) = \partial\eta^{1,0} + \bar{\partial}\eta^{0,1} + \partial\eta^{0,1} + \bar{\partial}\eta^{1,0}.$$

Since ω is of type $(1,1)$, $\partial\eta^{1,0} = \bar{\partial}\eta^{0,1} = 0$. It follows from Dolbeaut-Grothendieck lemma that locally there exists a function ϕ such that $\eta^{0,1} = \bar{\partial}\phi$. Since η is real, $\eta^{1,0} = \overline{\eta^{0,1}}$ and thus $\eta^{1,0} = \partial\bar{\phi}$. Therefore,

$$\omega = (\partial + \bar{\partial})(\partial\bar{\phi} + \bar{\partial}\phi) = \partial\bar{\partial}\phi + \bar{\partial}\partial\bar{\phi}.$$

Using the identity $\partial\bar{\partial} + \bar{\partial}\partial = 0$, one obtains

$$\omega = \partial\bar{\partial}(\phi - \bar{\phi}).$$

Let $\psi = -i(\phi - \bar{\phi})$ and notice that $h_{j\bar{k}} = \frac{\partial^2 \phi}{\partial z^j \partial \bar{z}^k}$ is equivalent to $\omega = i\partial\bar{\partial}\phi$. |

Corollary 3.3 *Every Hermitian metric h on a one-dimensional complex manifold M is a Kähler metric.*

Proof. Since at every $z \in M$ $T_z M_{\mathbb{C}}$ has complex dimension 2, the 3-form $d\omega_z$ must be zero i.e. ω is closed. |

EXAMPLES:

1. The Euclidean metric on $\mathbb{C}^n$ defined by $h_{j\bar{k}} = \delta_{j,k}$ is given by the Kähler potential function $\psi(z_1, \dots, z_n) = |z_1|^2 + \dots + |z_n|^2$.
2. The Poincaré metric on the unit disk $D := \{z \in \mathbb{C}; |z| < 1\}$ is defined by

$$h = \frac{dz \otimes d\bar{z}}{(1 - |z|^2)^2}.$$

Since D is a one-dimensional complex manifold, h is Kähler. It can be easily checked that $\psi(z) = -\log(1 - |z|^2)$ is a Kähler potential for h .

3. The Poincaré metric is a special case the metric on the n -dimensional unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n; |z| < 1\}$ defined by

$$h_{j\bar{k}} = \frac{\partial^2}{\partial z^j \partial \bar{z}^k} (-\log(1 - \|z\|^2)).$$

It is clear that such metric is Kähler if it was shown to be Hermitian. By direct computation,

$$h_{j\bar{k}} = \frac{\delta_{j,\bar{k}}(1 - \|z\|^2) + z_k z_{\bar{j}}}{(1 - \|z\|^2)^2}.$$

In matrix form this becomes

$$[h_{j\bar{k}}] = \frac{1}{(1 - \|z\|^2)^2} \begin{bmatrix} 1 - \|z\|^2 + \|z_1\|^2 & z_2 z_{\bar{1}} & \dots & z_n z_{\bar{1}} \\ z_1 z_{\bar{2}} & 1 - \|z\|^2 + \|z_2\|^2 & \dots & z_n z_{\bar{2}} \\ \vdots & \vdots & \ddots & \vdots \\ z_1 z_{\bar{n}} & z_2 z_{\bar{n}} & \dots & 1 - \|z\|^2 + \|z_n\|^2 \end{bmatrix}$$

which is Hermitian symmetric. Let $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ be a complex vector.

We will show that

$$(1 - \|z\|^2)^2 c^* [h_{j\bar{k}}] c > 0.$$

Indeed, let $\alpha_i = 1 - \|z\|^2 + \|z_i\|^2$. Then

$$(1 - \|z\|^2)^2 [h_{j\bar{k}}] c = \begin{bmatrix} c_1 \alpha_1 + z_{\bar{1}} (\sum_{i=1}^n c_i z_i - c_1 z_1) \\ c_2 \alpha_2 + z_{\bar{2}} (\sum_{i=1}^n c_i z_i - c_2 z_2) \\ \vdots \\ c_n \alpha_n + z_{\bar{n}} (\sum_{i=1}^n c_i z_i - c_n z_n) \end{bmatrix}$$

Multiplying by c^* we obtain

$$\begin{aligned}
(1 - \|z\|^2)^2 c^* [h_{j\bar{k}}] c &= \sum_{j=1}^n \|c_j\|^2 \alpha_j + c_{\bar{j}} z_{\bar{j}} \left(\sum_{i=1}^n c_i z_i - c_j z_j \right) \\
&= \sum_{j=1}^n \|c_j\|^2 \alpha_j + \sum_{i=1}^n c_i c_{\bar{j}} z_{\bar{j}} z_i - \|c_j\|^2 \|z_j\|^2 \\
&= \|c_j\|^2 (1 - \|z\|^2) + \|c \cdot z\|^2 > 0,
\end{aligned}$$

where $c \cdot z$ is the dot product of c and z . It follows that $[h_{j\bar{k}}]$ is positive definite and hence defines a Hermitian metric on B^n .

4. The Fubini-Study metric on the complex projective space is defined using the holomorphic coordinates defined in chapter 2 by

$$h_{j\bar{k}} = \frac{\partial^2}{\partial z^j \partial z^{\bar{k}}} \log(1 + \|z\|^2).$$

To see that this does not depend on the choice of coordinates, suppose a point lie in $U_1 \cap U_2$, where U_1 and U_2 are as defined in page 21. Let $(z^1, \dots, z^n)$ and $(w^1, \dots, w^n)$ denote the coordinates on U_1 and U_2 , respectively. Then

$$w^1 = \frac{1}{z^1}, \text{ and } w^j = \frac{z^j}{z^1} \text{ for } j = 2, \dots, n.$$

It follows that

$$\begin{aligned}
\|z\|^2 + 1 &= z^1 z^{\bar{1}} + \dots + z^n z^{\bar{n}} + 1 \\
&= \frac{1}{w^1 w^{\bar{1}}} + \frac{w^2 w^{\bar{2}}}{w^1 w^{\bar{1}}} + \dots + \frac{w^n w^{\bar{n}}}{w^1 w^{\bar{1}}} + 1 \\
&= \frac{1 + \|w\|^2}{w^1 w^{\bar{1}}},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial z^1} &= -w^1 w^j \frac{\partial}{\partial w^j} \\
\frac{\partial}{\partial z^j} &= w^1 \frac{\partial}{\partial w^j} \text{ for } j = 2, \dots, n \\
dz^1 &= -\frac{1}{(w^1)^2} dw^1 \\
dz^j &= \frac{1}{w^1} dw^j - \frac{w^j}{(w^1)^2} dw^1 \text{ for } j = 2, \dots, n.
\end{aligned}$$

Notice that for any $j, k = 1, \dots, n$,

$$\frac{\partial^2}{\partial w^j \partial w^{\bar{k}}} \log\left(\frac{1 + \|w\|^2}{w^1 w^{\bar{1}}}\right) = \frac{\partial^2}{\partial w^j \partial w^{\bar{k}}} \log(1 + \|w\|^2) := \tilde{h}_{j\bar{k}}.$$

Write

$$h = h_{1\bar{1}} dz^1 \otimes dz^{\bar{1}} + h_{1\bar{k}} dz^1 \otimes dz^{\bar{k}} + h_{j\bar{1}} dz^j \otimes dz^{\bar{1}} + h_{j\bar{k}} dz^j \otimes dz^{\bar{k}}$$

where the sum in indexes j and k is from 2 to n . Substituting the change of

variable equations above, one finds that

$$\begin{aligned}
h_{1\bar{1}}dz^1 \otimes dz^{\bar{1}} &= \tilde{h}_{s\bar{t}} \frac{w^s w^{\bar{t}}}{w^1 w^{\bar{1}}} dw^1 \otimes dw^{\bar{1}} \\
h_{1\bar{k}}dz^1 \otimes dz^{\bar{k}} &= \tilde{h}_{s\bar{k}} \left(\frac{w^s}{w^1} dw^1 \otimes dw^{\bar{k}} - \frac{w^s w^{\bar{k}}}{w^1 w^{\bar{1}}} dw^1 \otimes dw^{\bar{1}} \right) \\
h_{j\bar{1}}dz^j \otimes dz^{\bar{1}} &= \tilde{h}_{j\bar{t}} \left(\frac{w^{\bar{t}}}{w^{\bar{1}}} dw^j \otimes dw^{\bar{1}} - \frac{w^j w^{\bar{t}}}{w^1 w^{\bar{1}}} dw^1 \otimes dw^{\bar{1}} \right) \\
h_{j\bar{k}}dz^j \otimes dz^{\bar{k}} &= \tilde{h}_{j\bar{k}} \left(\frac{w^j w^{\bar{k}}}{w^1 w^{\bar{1}}} dw^1 \otimes dw^{\bar{1}} - \frac{w^j}{w^1} dw^1 \otimes dw^{\bar{k}} \right. \\
&\quad \left. - \frac{w^{\bar{k}}}{w^{\bar{1}}} dw^j \otimes dw^{\bar{1}} + dw^j \otimes dw^{\bar{k}} \right),
\end{aligned}$$

where s and t run from 1 to n . Summing these gives

$$h = \tilde{h}_{j\bar{k}} dw^j \otimes dw^{\bar{k}}.$$

Another characterization of Kähler metrics is given by linear connections. Given a Riemannian metric (M, g) , there exists a unique connection

$$D : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM),$$

written as $(X, Y) \rightarrow D_X Y$ which satisfies

1. Function linear in the first argument:

$$D_{f_1 X_1 + f_2 X_2} Y = f_1 D_{X_1} Y + f_2 D_{X_2} Y, \text{ for all } f_1, f_2 \in C^\infty(M).$$

2. Real linear in the second argument:

$$D_X(a_1Y_1 + a_2Y_2) = a_1D_XY_1 + a_2D_XY_2, \text{ for all } a_1, a_2 \in \mathbb{R}.$$

3. Product rule: $D_XfY = Xf + fD_XY$, for all $f \in C^\infty(M)$.

4. Metric compatibility: $D_Xg(Y, Z) = g(D_XY, Z) + g(Y, D_XZ)$.

5. Torsion-free: $D_XY - D_YX = [X, Y] = XY - YX$.

Recall that if h is a Hermitian metric on a complex manifold M , then the 2-form $g = \operatorname{Re} h$ is Riemannian metric, and hence one can talk about the Levi-Civita connection associated with h . Also, it can be seen that the properties of connections are still preserved if its domain is extended to complex vector fields i.e.

$$D : \Gamma(TM_{\mathbb{C}}) \times \Gamma(TM_{\mathbb{C}}) \rightarrow \Gamma(TM_{\mathbb{C}}).$$

Lemma 3.4 *Let M be a Hermitian manifold, h be its Hermitian metric, and J be its almost complex structure. Let g and ω be the associated Riemannian metric and $(1,1)$ -form of h . Then for any complex vectors Z, W ,*

$$\omega(Z, W) = g(JZ, W).$$

Proof. Let X, Y be real vectors. Then

$$\begin{aligned}
g(X, Y) - i\omega(X, Y) &= h(X, Y) \\
&= h(\phi(X) + \overline{\phi(X)}, \phi(Y) + \overline{\phi(Y)}) \\
&= h(\phi(X), \overline{\phi(Y)}) \\
&= -ih(\phi(JX), \overline{\phi(Y)}) && \text{(by lemma 2.9)} \\
&= -ig(JX, Y) + g(JX, JY) && \text{(by proposition 2.10)}
\end{aligned}$$

Therefore, $\omega(X, Y) = g(JX, Y)$. Now let $Z = X_1 + iY_1$ and $W = X_2 + iY_2$, where X_1, X_2, Y_1, Y_2 are real vectors. Then

$$\begin{aligned}
\omega(Z, W) &= \omega(X_1 + iY_1, X_2 + iY_2) \\
&= \omega(X_1, X_2) + i\omega(X_1, Y_2) - \omega(Y_1, Y_2) + i\omega(Y_1, X_2) \\
&= g(JX_1, X_2) + ig(JX_1, Y_2) - g(JY_1, Y_2) + ig(JY_1, X_2) \\
&= g(J(X_1 + iY_1), X_2 + iY_2) \\
&= g(JZ, W)
\end{aligned}$$

■

Theorem 3.5 *Let h be a Hermitian metric on a complex manifold M , and let J be its almost complex structure. Let D denotes the Levi-Citiva connection associated with h . Then the following are equivalent*

1. $D_Z W$ is of type $(1, 0)$ if W is of type $(1, 0)$ for any complex vector Z .

2. $D_X JY = JD_X Y$ for any complex vector fields X, Y

3. $D_X \omega(X_1, X_2) = \omega(D_X X_1, X_2) + \omega(X_1, D_X X_2)$ for any complex vector fields X, X_1, X_2

4. h is Kähler.

Proof. (1) $\Leftrightarrow$ (2) Assume $D_X W$ is of type $(1, 0)$ if W is of type $(1, 0)$ for any complex vector X . Notice that since $\overline{D_X W} = D_{\bar{X}} \bar{W}$, D sends a $(0, 1)$ field to a $(0, 1)$. Let $(z^1, \dots, z^n)$ be holomorphic coordinates and let X be any complex vector. Then $D_X J \frac{\partial}{\partial z^j} = D_X i \frac{\partial}{\partial z^j} = i D_X \frac{\partial}{\partial z^j} = J D_X \frac{\partial}{\partial z^j}$ since $D_X \frac{\partial}{\partial z^j}$ is of type $(1, 0)$. Similarly, one can show that $D_X J \frac{\partial}{\partial \bar{z}^j} = J D_X \frac{\partial}{\partial \bar{z}^j}$. It follows by the linearity of D over $\mathbb{R}$ that $D_X JY = JD_X Y$ for any vector field X and Y . On the other hand, if (2) is satisfied, W is a vector field of type $(1, 0)$ and X is any complex vector, then $JD_X W = D_X JW = i D_X W$. This implies that $D_X W$ is of type $(1, 0)$.

(2) $\Leftrightarrow$ (3) Assume (2) holds and let X, X_1, X_2 be any complex vector fields on M .

Using 3.4 and metric compatability,

$$\begin{aligned}
 D_X \omega(X_1, X_2) &= Xg(JX_1, X_2) \\
 &= g(D_X JX_1, X_2) + g(JX_1, D_X X_2) \\
 &= g(JD_X X_1, X_2) + g(JX_1, D_X X_2) \\
 &= \omega(D_X X_1, X_2) + \omega(X_1, D_X X_2)
 \end{aligned}$$

On the other hand, if (3) holds and X, Y, Z are any real vector fields, then the above

computation shows that

$$g(D_X JY, Z) = g(JD_X Y, Z)$$

Since Z is arbitrary, it follows that $D_X JY = JD_X Y$. By complex linearity of D and J , the result extends to complex vector fields.

(3) $\Rightarrow$ (4) Assume (3) holds and X_0, X_1, X_2 are any complex vectors on M . Using the invariant formula of the exterior derivative (see Lee 14.32) and the torsion-free condition of the connection D ,

$$\begin{aligned} d\omega(X_0, X_1, X_2) &= X_0\omega(X_1, X_2) - X_1\omega(X_0, X_2) + X_2\omega(X_0, X_1) \\ &\quad - \omega([X_0, X_1], X_2) + \omega([X_0, X_2], X_1) - \omega([X_1, X_2], X_0) \\ &= \omega(D_{X_0}X_1, X_2) + \omega(X_1, D_{X_0}X_2) \\ &\quad - \omega(D_{X_1}X_0, X_2) + \omega(X_0, D_{X_1}X_2) \\ &\quad + \omega(D_{X_2}X_0, X_1) + \omega(X_0, D_{X_2}X_1) \\ &\quad - \omega(D_{X_0}X_1 - D_{X_1}X_0, X_2) \\ &\quad + \omega(D_{X_0}X_2 - D_{X_2}X_0, X_1) \\ &\quad - \omega(D_{X_1}X_2 - D_{X_2}X_1, X_0) \\ &= 0 \end{aligned}$$

This implies that ω is a closed form and it follows from theorem 3.2 that h is Kähler.

(4) $\Rightarrow$ (2) Suppose h is Kähler. It suffice to show that for any complex vector fields X, Y, Z ,

$$g(D_X JY - JD_X Y, Z) = 0.$$

Notice that the identity on the left is function linear, and hence it can be assumed that X, Y, Z are coordinate vectors fields i.e. commute. Then one immediatly obtains,

$$d\omega(X, Y, Z) = X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y).$$

By Koszul formula (See Do' Carmo), J -invariance of g , and lemma 3.4,

$$\begin{aligned} 2g(D_X JY, Z) &= Xg(JY, Z) + JYg(X, Z) - Zg(X, JY) \\ &= X\omega(Y, Z) + JY\omega(JZ, X) + Z\omega(X, Y) \end{aligned}$$

and

$$\begin{aligned} 2g(D_X Y, JZ) &= Xg(Y, JZ) + Yg(X, JZ) - JZg(X, Y) \\ &= -X\omega(JY, JZ) - Y\omega(X, Z) - JZ\omega(X, JY) \end{aligned}$$

Then,

$$\begin{aligned} 2g(D_X JY - JD_X Y, Z) &= 2g(D_X JY, Z) - 2g(JD_X Y, Z) \\ &= 2g(D_X JY, Z) + 2g(D_X Y, JZ) \\ &= X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ &\quad - X\omega(JY, JZ) - JZ\omega(X, JY) + JY\omega(JZ, X) \\ &= d\omega(X, Y, Z) - d\omega(X, JY, JZ) \\ &= 0 \end{aligned}$$

Remark 3.6 Notice that if h is kähler, it compatible with Levi-Citiva connection i.e.

If X, Y, Z are complex vector fields, then

$$\begin{aligned}
 D_X h(Y, Z) &= D_X(g(Y, Z) - i\omega(Y, Z)) \\
 &= g(D_X Y, Z) + g(Y, D_X Z) - i\omega(D_X Y, Z) - i\omega(Y, D_X Z) \\
 &= h(D_X Y, Z) + h(Y, D_X Z)
 \end{aligned}$$

3.2 Curvature of Kähler Metric

Let h be a Kähler metric on a complex manifold M and let X, Y, Z, W be complex vectors in $T_p M_{\mathbb{C}}$ for some p in M . The curvature tensors of h denoted by $R_{XY}Z$ and $R(Z, W, X, Y)$ are defined by

$$R_{XY}Z := D_X D_Y Z - D_Y D_X Z - D_{[X, Y]}Z,$$

and

$$R(Z, W, X, Y) = h(R_{XY}Z, W).$$

where D is the Levi-citva connection associated with h . Let $(z^1, \dots, z^n)$ be holomorphic coordinates at p . The curvature tensor coefficients in these coordinates are defined as

$$R_{sjkl} = h(R_{\frac{\partial}{\partial z^k} \frac{\partial}{\partial z^l}} \frac{\partial}{\partial z^s}, \frac{\partial}{\partial z^j}),$$

where $s, j, k, l = 1, \dots, n, \bar{1}, \dots, \bar{n}$. These definitions parallel the ones in Riemann geometry and it is easy to verify that they are function linear. In this section, symmetry relations on the tensor coefficients and formulas to compute them in a given coordinate will be presented.

To simplify the notation, the symbols ∂_j and D_j will be used to denote $\frac{\partial}{\partial z^j}$ and $D \frac{\partial}{\partial z^j}$, respectively. Moreover, the alphabets $j, k, l \dots$ etc will be used to denote both $(1, 0)$ and $(0, 1)$ vectors while the greek letters $\alpha, \beta, \gamma \dots$ etc will be used to denote $(1, 0)$ only. For example, ∂_α is of type $(1, 0)$, $\partial_{\bar{\alpha}}$ is of type $(0, 1)$, and ∂_j can be of either types.

The *Christoffel symbols* of h in a given coordinates $(z^1, \dots, z^n)$ are defined by

$$D_j \partial_k = \Gamma_{jk}^l \partial_l.$$

Proposition 3.7 *If h is Kähler, then*

1. *All Christoffel symbols vanish except possibly $\Gamma_{\alpha\beta}^\gamma$ and $\Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$.*
2. *$\Gamma_{\alpha\beta}^\gamma = h^{\bar{\sigma}\gamma} \partial_\alpha h_{\beta\bar{\sigma}}$, where $h^{\bar{\sigma}\gamma}$ is the inverse of $h_{\gamma\bar{\sigma}}$.*
3. *$\Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = h^{\gamma\sigma} \partial_{\bar{\alpha}} h_{\sigma\bar{\beta}}$.*

Proof. By theorem 3.5, the Levi-Civita connection preserves the type. Hence, all Christoffel symbols of the form $\Gamma_{j\alpha}^{\bar{\beta}}$ and $\Gamma_{j\bar{\alpha}}^\beta$ vanish. From the torsion free condition of D ,

$$D_{\bar{\alpha}} \partial_\beta - D_\beta \partial_{\bar{\alpha}} = [\bar{\partial}_\alpha, \partial_\beta] = 0,$$

which implies that

$$\Gamma_{\bar{\alpha}\beta}^{\gamma}\partial_{\gamma} = \Gamma_{\beta\bar{\alpha}}^{\bar{\gamma}}\partial_{\bar{\gamma}}.$$

Since each side is of different type, one must have

$$\Gamma_{\bar{\alpha}\beta}^{\gamma} = \Gamma_{\beta\bar{\alpha}}^{\bar{\gamma}} = 0.$$

This completes (1). To show (2) notice that

$$\begin{aligned}\partial_{\alpha}h_{\beta\bar{\sigma}} &= h(D_{\alpha}\partial_{\beta}, \partial_{\bar{\sigma}}) + h(\partial_{\beta}, D_{\alpha}\partial_{\bar{\sigma}}) \\ &= h(\Gamma_{\alpha\beta}^{\nu}\partial_{\nu}, \partial_{\bar{\sigma}}) \\ &= \Gamma_{\alpha\beta}^{\nu}h_{\nu\bar{\sigma}}\end{aligned}$$

Multiplying both sides by $h^{\bar{\sigma}\gamma}$, one obtains

$$\begin{aligned}h^{\bar{\sigma}\gamma}\partial_{\alpha}h_{\beta\bar{\sigma}} &= h^{\bar{\sigma}\gamma}\Gamma_{\alpha\beta}^{\nu}h_{\nu\bar{\sigma}} \\ &= \Gamma_{\alpha\beta}^{\nu}\delta_{\nu}^{\gamma} \\ &= \Gamma_{\alpha\beta}^{\gamma}\end{aligned}$$

Part (3) follows directly from the fact $D_{\bar{\alpha}}\partial_{\bar{\beta}} = \overline{D_{\alpha}\partial_{\beta}}$. |

Corollary 3.8 $\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma}$.

Proof. Direct consequence of Kählerian property $\partial_{\alpha}h_{\alpha\bar{\sigma}} = \partial_{\beta}h_{\beta\bar{\sigma}}$. |

So far in the treatment of this paper, coordinate charts were only taken to be holo-

morphic. It turns out that one can always find certain holomorphic coordinates that simplify the expression of the Hermitian metric. Such coordinates will be particularly useful when dealing with pointwise information such as tensors.

Proposition 3.9 *Let M be a complex manifold, h a Hermitian metric on M , and p a point in M . Then there exists a holomorphic coordinate system $z = \{z^1, \dots, z^n\}$ such that*

$$1. \ z(p) = 0$$

$$2. \ h_{j\bar{k}}(p) = \delta_{j,\bar{k}}$$

$$3. \ \partial_l h_{j\bar{k}}(p) = -\partial_{\bar{j}} h_{l\bar{k}}(p)$$

Proof. Starting with an arbitrary chart, it will be shown that such system exists by applying suitable coordinate changes. Hence, let $\phi = (z^1, \dots, z^n)$ be a holomorphic coordinates at p . The first condition is achieved by applying a linear translation i.e. let $\phi(z) := \phi(z) - \phi(p)$. Let B denotes the Hermitian matrix $[h_{j\bar{k}}(p)]$ in these new coordinates. Note that being Hermitian, B admits a decomposition $B = UDU^*$, where D is a diagonal real matrix and U is a unitary matrix i.e. $UU^* = I_n$. Define the coordinates $\psi = (w^1, \dots, w^n) = D^{1/2}\bar{U} \circ \phi$. The map ψ is holomorphic since it is only a linear transform of ϕ , and it satisfies $\psi(p) = D^{1/2}U\phi(p) = 0$. To see that ψ satisfies the second condition, write

$$U = [U_{jk}], \quad D = [D_{jk}], \quad \text{and} \quad B = [B_{jk}].$$

The matrix equation $UBU^* = D$ implies that

$$\sum_{s,t} U_{js} B_{st} \bar{U}_{kt} = D_{jk}.$$

Also, it follows from $\psi = D^{1/2} \bar{U} \circ \phi$ that

$$z^s = \sum_k U_{ks} D_{kk}^{-1/2} w^k.$$

Then

$$\begin{aligned} h\left(\frac{\partial}{\partial w^j}, \frac{\partial}{\partial w^{\bar{k}}}\right) &= h\left(\frac{\partial z^s}{\partial w^j} \frac{\partial}{\partial z^s}, \frac{\partial z^{\bar{t}}}{\partial w^{\bar{k}}} \frac{\partial}{\partial z^{\bar{t}}}\right) \\ &= D_{kk}^{-1/2} D_{jj}^{-1/2} \sum_{s,t} U_{js} B_{st} \bar{U}_{kt} \\ &= D_{kk}^{-1/2} D_{jj}^{-1/2} D_{jk} \\ &= \delta_{jk} \end{aligned}$$

So one can assume that $(z^1, \dots, z^n)$ satisfies the first two conditions. The last condition is obtained by applying the quadratic change $w^j = z^j + A_{kl}^j z^k z^l$ for suitable complex numbers A_{kl}^j . Notice that $w^j(p) = 0$ and $\frac{\partial}{\partial w^j} \Big|_p = \frac{\partial}{\partial z^j} \Big|_p$ for $j = 1, \dots, n$. Hence condition 1 and 2 are preserved by these coordinate changes. Let

$$h_{j\bar{k}} = h\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{k}}}\right) \qquad \tilde{h}_{j\bar{k}} = h\left(\frac{\partial}{\partial w^j}, \frac{\partial}{\partial w^{\bar{k}}}\right).$$

It follows that

$$\begin{aligned} h_{j\bar{k}} &= \frac{\partial w^s}{\partial z^j} \frac{\partial w^{\bar{t}}}{\partial z^{\bar{k}}} \tilde{h}_{s,\bar{t}} \\ &= (\delta_j^s + 2A_{jq}^s z^q)(\delta_k^{\bar{t}} + 2A_{k\bar{q}}^{\bar{t}} z^{\bar{q}}) \tilde{h}_{s,\bar{t}} \end{aligned}$$

Then

$$\frac{\partial h_{j\bar{k}}}{\partial z^r} = 2\delta_k^{\bar{t}} A_{jr}^s \tilde{h}_{s,\bar{t}} + (\delta_j^s + 2A_{jq}^s z^q)(\delta_k^{\bar{t}} + 2A_{k\bar{q}}^{\bar{t}} z^{\bar{q}}) \frac{\partial \tilde{h}_{s,\bar{t}}}{\partial z^r}$$

At $z = p$, $z^q = 0$, $\tilde{h}_{s,\bar{t}} = \delta_{s,\bar{t}}$, and $\frac{\partial}{\partial w^j} = \frac{\partial}{\partial z^j}$. Hence,

$$\left. \frac{\partial h_{j\bar{k}}}{\partial z^r} \right|_p = 2A_{jr}^k + \left. \frac{\partial \tilde{h}_{j\bar{k}}}{\partial w^r} \right|_p.$$

The desired complex numbers are then given by

$$A_{jr}^k = \frac{1}{4} \left(\left. \frac{\partial \tilde{h}_{j\bar{k}}}{\partial z^r} \right|_p + \left. \frac{\partial \tilde{h}_{r,\bar{k}}}{\partial z^j} \right|_p \right),$$

as can be seen in

$$\left. \frac{\partial \tilde{h}_{j\bar{k}}}{\partial w^r} \right|_p + \left. \frac{\partial \tilde{h}_{r,\bar{k}}}{\partial w^j} \right|_p = \left. \frac{\partial \tilde{h}_{j\bar{k}}}{\partial z^r} \right|_p + \left. \frac{\partial \tilde{h}_{r,\bar{k}}}{\partial z^j} \right|_p - 2A_{jr}^k - 2A_{rj}^k = 0.$$

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Coordinates which satisfies the conditions in proposition 3.9 are called *Hermitian normal coordinates*. Notice that by taking the conjugate of the second condition, one

obtains

$$\partial_{\bar{l}}\partial h_{k\bar{j}}(p) = -\partial_{\bar{j}}\partial h_{k\bar{l}}(p)$$

and the next two corollaries follow immediately.

Corollary 3.10 *Let h be a Kähler metric and $(z^1, \dots, z^n)$ be Hermitian normal coordinates at p . Then*

$$dh_{j\bar{k}}(p) = 0 \text{ for all } j, k = 1, \dots, n.$$

Corollary 3.11 *All of Christoffel symbols vanish in Hermitian normal coordinates.*

Using Hermitian normal coordinates gives a short proof of the following theorem about the curvature tensors.

Theorem 3.12 *Let h be a Kähler metric and $(z^1, \dots, z^n)$ be any holomorphic coordinates. Then*

1. *All curvature coefficient vanish except possibly $R_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ and $R_{\alpha\bar{\beta}\bar{\gamma}\sigma}$.*
2. *The tensor coefficients $R_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ completely determines the curvature tensor.*
3. *The curvature tensor coefficients are symmetric with respect to switching indexes of the same type i.e. $R_{\alpha\bar{\beta}\gamma\bar{\sigma}} = R_{\gamma\bar{\beta}\alpha\bar{\sigma}} = R_{\alpha\bar{\sigma}\gamma\bar{\beta}} \dots$ etc.*
4. *The curvature tensor coefficients are conjugate symmetric when all indexes of different types are switched i.e. $\overline{R_{\alpha\bar{\beta}\gamma\bar{\sigma}}} = R_{\sigma\bar{\gamma}\beta\bar{\alpha}}$.*

Proof. Recall that

$$R_{sjkl} = h(R_{\partial_k\partial_l}\partial_s, \partial_j),$$

and

$$R_{\partial_k \partial_l} \partial_s = D_k D_l \partial_s - D_l D_k \partial_s - D_{[\partial_k, \partial_l]} \partial_s = D_k D_l \partial_s - D_l D_k \partial_s.$$

It is clear from the definition of a Hermitian metric and the fact that the Levi-Civita connection preserves the type of vector fields that all curvature tensor coefficients vanish except possibly $R_{\alpha \bar{\beta} k l}$. It remains to show that $R_{\alpha \bar{\beta} \sigma \gamma} = R_{\alpha \bar{\beta} \bar{\sigma} \bar{\gamma}} = 0$. Notice that since the tensors are function-linear, it can be assumed the coordinates $(z^1, \dots, z^n)$ are Hermitian normal. Then corollary 3.11 gives

$$\begin{aligned} R_{\partial_\sigma \partial_\gamma} \partial_\alpha &= (D_\sigma D_\gamma - D_\gamma D_\sigma) \partial_\alpha \\ &= D_\sigma (\Gamma_{\gamma\alpha}^\nu \partial_\nu) - D_\gamma (\Gamma_{\sigma\alpha}^\nu \partial_\nu) \\ &= (\partial_\sigma \Gamma_{\gamma\alpha}^\nu - \partial_\gamma \Gamma_{\sigma\alpha}^\nu) \partial_\nu \end{aligned}$$

From proposition 3.7 and corollary 3.10,

$$\partial_\sigma \Gamma_{\gamma\alpha}^\nu = \partial_\sigma (h^{\bar{\mu}\nu} \partial_\gamma h_{\alpha\bar{\mu}}) = \partial_\sigma \partial_\gamma h_{\alpha\bar{\nu}}.$$

Similarly,

$$\partial_\gamma \Gamma_{\sigma\alpha}^\nu = \partial_\gamma \partial_\sigma h_{\alpha\bar{\nu}}.$$

This shows that $R_{\partial_\sigma \partial_\gamma} \partial_\alpha = 0$. One can also show in the same way that $R_{\partial_{\bar{\sigma}} \partial_{\bar{\gamma}}} \partial_\alpha = 0$.

This completes the proof of (1). Part (2) follows from the function linearity of the tensor and the fact that $R_{\alpha \bar{\beta} \gamma \bar{\sigma}} = -R_{\alpha \bar{\beta} \bar{\sigma} \gamma}$. To prove (3) and (4), it can again be

assumed that the coordiantes are Hermitian normal. Then

$$\begin{aligned}
R_{\alpha\bar{\beta}\gamma\bar{\sigma}} &= h((D_\gamma D_{\bar{\sigma}} - D_{\bar{\sigma}} D_\gamma)\partial_\alpha, \partial_{\bar{\beta}}) \\
&= -h(D_{\bar{\sigma}} D_\gamma \partial_\alpha, \partial_{\bar{\beta}}) & (D_{\bar{\sigma}} \partial_\alpha = 0) \\
&= -h(D_{\bar{\sigma}}(\Gamma_{\gamma\alpha}^\nu \partial_\nu), \partial_{\bar{\beta}}) \\
&= -h(\partial_{\bar{\sigma}}(h^{\bar{\mu}\nu} \partial_\gamma h_{\alpha\bar{\mu}}) \partial_\nu, \partial_{\bar{\beta}}) \\
&= -h^{\bar{\mu}\nu} h_{\nu\bar{\beta}} \partial_{\bar{\sigma}} \partial_\gamma h_{\alpha\bar{\mu}} & (\beta = \mu = \nu) \\
&= -\partial_{\bar{\sigma}} \partial_\gamma h_{\alpha\bar{\beta}} & (\beta = \mu = \nu)
\end{aligned}$$

The conclusion now follows from theorem 3.2 and the Hermitian symmetry of h . ■

Let (M, g) be a Riemannian manifold, and let $u, v \in T_p M$ be real vectors at some point p in M . The Riemannian sectional curvature of the section spanned by u and v is given by

$$K(u, v) = \frac{g(D_u D_v v - D_v D_u v - D_{[u,v]} v, u)}{\|u\|^2 \|v\|^2}.$$

The analogous definition for complex manifold is given below.

Definition 3.13 Let (M, h) be a Hermitian manifold and let X be a vector of type $(1, 0)$. The *holomorphic sectional curvature* in the section spanned by X is defined as

$$H(X) := \frac{R(X, \bar{X}, X, \bar{X})}{h(X, \bar{X})^2}.$$

Given any holomorphic coordiantes $(z^1, \dots, z^n)$, the vector X can be written as $X =$

$X^\alpha \partial_\alpha$. Then

$$\begin{aligned}
\overline{R(X, \bar{X}, X, \bar{X})} &= \overline{R_{\alpha\bar{\beta}\gamma\bar{\sigma}} X^\alpha X^{\bar{\beta}} X^\gamma X^{\bar{\sigma}}} \\
&= R_{\beta\bar{\alpha}\sigma\bar{\gamma}} X^{\bar{\alpha}} X^\beta X^{\bar{\gamma}} X^\sigma \\
&= R(X, \bar{X}, X, \bar{X})
\end{aligned}$$

Therefore, $H(X)$ is real. Note that for any non-zero complex number λ , $H(\lambda X) = H(X)$ i.e. it suffice to consider vectors with norm equal to 1 to define the holomorphic sectional curvature.

Proposition 3.14 *Let (M, h) be a Hermitian manifold, $u \in TM$ a real vector, and $X = (u - iJu)/2$. Then*

$$K(u, Ju) = 2H(X).$$

Proof. Let g denote the J -invariant metric associated with h . Then by definition,

$$K(u, Ju) = \frac{g(R_{uJu}Ju, u)}{g(u, u)g(Ju, Ju)}.$$

Notice that $h(X, \bar{X}) = h(X + \bar{X}, X + \bar{X}) = h(u, u) = g(u, u)$ since the associated $(1, 1)$ form satisfies $\omega(u, u) = 0$. Also, $g(u, u) = g(Ju, Ju)$. It follows that

$$K(u, Ju) = \frac{g(R_{uJu}Ju, u)}{h(X, \bar{X})^2}.$$

Using proposition 2.11,

$$\begin{aligned}
g(R_u J_u J u, u) &= \operatorname{Re} h(R_u J_u J u, u) \\
&= \operatorname{Re} R(J u, u, u, J u) \\
&= \operatorname{Re} R(i(X - \bar{X}), X + \bar{X}, X + \bar{X}, i(X - \bar{X})) \\
&= \operatorname{Re} R(X, \bar{X}, X, \bar{X}) - \operatorname{Re} R(X, \bar{X}, \bar{X}, X) \\
&= 2\operatorname{Re} R(X, \bar{X}, X, \bar{X}) \\
&= 2R(X, \bar{X}, X, \bar{X})
\end{aligned}$$

Therefore,

$$K(u, J u) = \frac{2R(X, \bar{X}, X, \bar{X})}{h(X, \bar{X})^2} = 2H(X).$$

■

3.3 Hermitian Canonical Connection

Let (M, h) be a Hermitian manifold. It was shown in the previous section that for a $\mathbb{C}$ -linear affine connection ∇ to exist on $TM_{\mathbb{C}}$, which preserves the type of vector fields and satisfies both metric compatibility and torsion-freeness, h must be Kähler. For a general Hermitian metric, one of the conditions that ∇ satisfies has to be dropped, and it is generally agreed that it is the torsion-free condition that is not assumed.

Let T be the torsion tensor of ∇ i.e.

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Lemma 3.15 *Suppose T satisfies $T(X, \bar{X}) = 0$ for any $(1, 0)$ type vector. Then*

$$T(X, \bar{Y}) = 0,$$

for every $(1, 0)$ type vectors X, Y .

Proof. Given X, Y in $TM_{1,0}$, one has

$$0 = T(X + Y, \bar{X} + \bar{Y}) = T(X, \bar{Y}) + T(Y, \bar{X}),$$

which implies $T(X, \bar{Y}) = T(\bar{X}, Y)$. Substituting iY in place of Y , one obtains

$-T(X, \bar{Y}) = T(\bar{X}, Y)$. Therefore,

$$T(X, \bar{Y}) = -T(X, \bar{Y}) = 0.$$

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It turned out that this weaker torsion-free condition is recoverable for a general Hermitian metric as will be demonstrated in the next theorem.

Theorem 3.16 *Let (M, h) be a Hermitian manifold. Then there exists a unique $\mathbb{C}$ -linear affine connection ∇ which satisfies*

1. ∇ preserves the type of vector fields.
2. ∇ is compatible with the metric.
3. The torsion tensor T of ∇ satisfies $T(X, \bar{X}) = 0$ for any $X \in TM_{1,0}$.

Proof. Let X and Y be type $(1, 0)$ vectors and define

$$\nabla_{\bar{X}}Y = [\bar{X}, Y]_{1,0}, \quad \nabla_X\bar{Y} = [X, \bar{Y}]_{0,1}, \quad (3.1)$$

where the $[\bar{X}, Y]_{1,0}$ and $[X, \bar{Y}]_{0,1}$ indicate the $(1, 0)$ and $(0, 1)$ components of $[\bar{X}, Y]$ and $[X, \bar{Y}]$, respectively. Note that this implies the torsion tensor satisfies condition (3) i.e.

$$T(X, \bar{X}) = \nabla_X\bar{X} - \nabla_{\bar{X}}X - [X, \bar{X}] = [X, \bar{X}]_{0,1} + [X, \bar{X}]_{1,0} - [X, \bar{X}] = 0.$$

The vector fields ∇_XY and $\nabla_{\bar{X}}\bar{Y}$ are defined by the equations

$$\begin{aligned} h(\nabla_XY, \bar{Z}) &= Xh(Y, \bar{Z}) - h(Y, [X, \bar{Z}]_{0,1}) \\ h(Z, \nabla_XY) &= 0 \\ h(Z, \nabla_{\bar{X}}\bar{Y}) &= \bar{X}h(Z, \bar{Y}) - h([X, Z]_{1,0}, \bar{Y}) \\ h(\nabla_{\bar{X}}\bar{Y}, \bar{Z}) &= 0, \end{aligned} \quad (3.2)$$

where Z is any arbitrary $(1, 0)$ vector field. By setting

$$\nabla_XY = \nabla_{X_{1,0}}Y_{1,0} + \nabla_{X_{0,1}}Y_{1,0} + \nabla_{X_{1,0}}Y_{0,1} + \nabla_{X_{0,1}}Y_{0,1}$$

for arbitrary complex vector fields X, Y , it is clear that ∇ is a $\mathbb{C}$ -linear affine connection on $TM_{\mathbb{C}}$. It is also clear from (3.1) and (3.2) that ∇ preserves the type of vector fields

and satisfies metric compatibility. To show uniqueness, suppose $\tilde{\nabla}$ is another such connection. Then lemma 3.15 implies that for vector fields X and Y of type $(1, 0)$

$$0 = T(X, \bar{Y}) = \tilde{\nabla}_X \bar{Y} - \tilde{\nabla}_{\bar{Y}} X - [X, \bar{Y}].$$

Since $\tilde{\nabla}$ preserves the type of vector fields, one must have

$$\tilde{\nabla}_X \bar{Y} = [X, \bar{Y}]_{0,1}$$

$$\tilde{\nabla}_{\bar{Y}} X = -[X, \bar{Y}]_{1,0}$$

Finally, metric compatibility implies that

$$Xh(Y, \bar{Z}) = h(\tilde{\nabla}_X Y, \bar{Z}) + h(Y, \tilde{\nabla}_X \bar{Z}),$$

or

$$h(\tilde{\nabla}_X Y, \bar{Z}) = Xh(Y, \bar{Z}) - h(Y, [X, \bar{Z}]_{0,1})$$

Condition (1) implies that $h(Z, \tilde{\nabla}_X Y) = 0$. Similar reasoning will show that

$$h(Z, \tilde{\nabla}_{\bar{X}} \bar{Y}) = \bar{X}h(Z, \bar{Y}) - h([X, Z]_{1,0}, \bar{Y})$$

$$h(\tilde{\nabla}_{\bar{X}} \bar{Y}, \bar{Z}) = 0.$$

Therefore, the connection $\tilde{\nabla}$ must coincide with ∇ . ■

The unique connection in theorem 3.16 is called the *Hermitian canonical connection*

of h . It is also sometimes called the *Chern connection* of h .

Let $(z^1, \dots, z^n)$ be holomorphic coordinates. As in the Kähler case, the Christoffel symbols of a Hermitian connection will be denoted by Γ_{jk}^l . It follows from theorem 3.16 that all Christoffel symbols vanish except possibly $\Gamma_{\alpha\beta}^\gamma$ and $\Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$. Also, the same computation that has been done for the Kähler case shows that

$$\begin{aligned}\Gamma_{\alpha\beta}^\gamma &= h^{\bar{\sigma}\gamma} \partial_\alpha h_{\beta\bar{\sigma}} \\ \Gamma_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} &= h^{\bar{\gamma}\sigma} \partial_{\bar{\alpha}} h_{\sigma\bar{\beta}}.\end{aligned}\tag{3.3}$$

The curvature tensors and the tensor coefficients are defined as in the Kähler case.

Then

Theorem 3.17 *Let h be a Hermitian metric and $(z^1, \dots, z^n)$ be any holomorphic coordinates. Then*

1. *All curvature coefficients vanish except possibly $R_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ and $R_{\alpha\bar{\beta}\bar{\gamma}\sigma}$.*
2. *The tensor coefficients $R_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ completely determine the curvature tensor.*
3. $R_{\alpha\bar{\beta}\gamma\bar{\sigma}} = -\partial_{\bar{\sigma}} \partial_\gamma h_{\alpha\bar{\beta}} + h^{\bar{\nu}\mu} (\partial_{\bar{\sigma}} h_{\mu\bar{\beta}}) (\partial_\gamma h_{\alpha\bar{\nu}}).$

Proof. Since the Chern connection preserves the type of vector fields, it is clear that all tensor coefficients vanish except those of the form $R_{\alpha\bar{\beta}jk}$. Assertion (1) is proved

if it is shown that $R_{\partial_\alpha \partial_\beta} \partial_\gamma = R_{\partial_{\bar{\alpha}} \partial_{\bar{\beta}}} \partial_{\bar{\gamma}} = 0$. By definition,

$$\begin{aligned}
R_{\partial_\alpha \partial_\beta} \partial_\gamma &= \nabla_\alpha \nabla_\beta \partial_\gamma - \nabla_\beta \nabla_\alpha \partial_\gamma \\
&= \nabla_\alpha (\Gamma_{\beta\gamma}^\sigma \partial_\sigma) - \nabla_\beta (\Gamma_{\alpha\gamma}^\sigma \partial_\sigma) \\
&= (\partial_\alpha \Gamma_{\beta\gamma}^\sigma + \Gamma_{\beta\gamma}^\mu \Gamma_{\alpha\mu}^\sigma - \partial_\beta \Gamma_{\alpha\gamma}^\sigma - \Gamma_{\alpha\gamma}^\mu \Gamma_{\beta\mu}^\sigma) \partial_\sigma.
\end{aligned} \tag{3.4}$$

Using equation 3.3,

$$\partial_\alpha \Gamma_{\beta\gamma}^\sigma = \partial_\alpha (h^{\bar{\mu}\sigma} \partial_\beta h_{\gamma\bar{\mu}}) = (\partial_\alpha h^{\bar{\mu}\sigma}) (\partial_\beta h_{\gamma\bar{\mu}}) + h^{\bar{\mu}\sigma} \partial_\alpha \partial_\beta h_{\gamma\bar{\mu}} \tag{3.5}$$

Similarly,

$$\partial_\beta \Gamma_{\alpha\gamma}^\sigma = (\partial_\beta h^{\bar{\mu}\sigma}) (\partial_\alpha h_{\gamma\bar{\mu}}) + h^{\bar{\mu}\sigma} \partial_\beta \partial_\alpha h_{\gamma\bar{\mu}} \tag{3.6}$$

Notice that $h_{\mu\bar{\eta}} h^{\bar{\nu}\mu} = \delta_\eta^\nu$, implies $h_{\mu\bar{\eta}} \partial_\alpha h^{\bar{\nu}\mu} = -h^{\bar{\nu}\mu} \partial_\alpha h_{\mu\bar{\eta}}$. Then

$$\begin{aligned}
\Gamma_{\beta\gamma}^\mu \Gamma_{\alpha\mu}^\sigma &= h^{\bar{\nu}\mu} (\partial_\beta h_{\gamma\bar{\nu}}) h^{\bar{\eta}\sigma} (\partial_\alpha h_{\mu\bar{\eta}}) \\
&= -h_{\mu\bar{\eta}} h^{\bar{\eta}\sigma} (\partial_\alpha h^{\bar{\nu}\mu}) (\partial_\beta h_{\gamma\bar{\nu}}) \\
&= -(\partial_\alpha h^{\bar{\mu}\sigma}) (\partial_\beta h_{\gamma\bar{\mu}})
\end{aligned} \tag{3.7}$$

Swapping α and β , one gets

$$\Gamma_{\alpha\gamma}^\mu \Gamma_{\beta\mu}^\sigma = -(\partial_\beta h^{\bar{\mu}\sigma}) (\partial_\alpha h_{\gamma\bar{\mu}}) \tag{3.8}$$

Substituting (3.5), (3.6), (3.7), (3.8) in (3.4) gives that

$$R_{\partial_\alpha \partial_\beta} \partial_\gamma = 0.$$

A similar argumet shows that $R_{\partial_{\bar{\alpha}} \partial_{\bar{\beta}}} \partial_{\bar{\gamma}} = 0$. Assertion (2) follows from the fact that

$R_{\alpha \bar{\beta} \gamma \bar{\sigma}} = -R_{\alpha \bar{\beta} \bar{\sigma} \gamma}$. Assertion (3) is proved by the following direction computation.

Since $\nabla_{\bar{\beta}} \gamma = 0$,

$$R_{\partial_\gamma \partial_{\bar{\sigma}}} \partial_\alpha = -\nabla_{\bar{\sigma}} \nabla_\gamma \partial_\alpha = -\partial_{\bar{\sigma}} \Gamma_{\gamma \alpha}^\mu \partial_\mu.$$

Then

$$\begin{aligned} R_{\alpha \bar{\beta} \gamma \bar{\sigma}} &= h(R_{\partial_\gamma \partial_{\bar{\sigma}}} \partial_\alpha, \bar{\beta}) \\ &= -h_{\mu \bar{\beta}} \partial_{\bar{\sigma}} \Gamma_{\gamma \alpha}^\mu \\ &= -h_{\mu \bar{\beta}} \partial_{\bar{\sigma}} (h^{\bar{\nu} \mu} \partial_\gamma h_{\alpha \bar{\nu}}) \\ &= -\partial_{\bar{\sigma}} \partial_\gamma h_{\alpha \bar{\beta}} - h_{\mu \bar{\beta}} (\partial_{\bar{\sigma}} h^{\bar{\nu} \mu}) (\partial_\gamma h_{\alpha \bar{\nu}}) \\ &= -\partial_{\bar{\sigma}} \partial_\gamma h_{\alpha \bar{\beta}} + h^{\bar{\nu} \mu} (\partial_{\bar{\sigma}} h_{\mu \bar{\beta}}) (\partial_\gamma h_{\alpha \bar{\nu}}) \end{aligned}$$

■

The holomorphic sectional curvature is also defined as in the Kähler case i.e. for a vector of type $(1, 0)$ the holomorphic sectional curvature in the section spanned by X is given by

$$H(X) = \frac{R(X, \bar{X}, X, \bar{X})}{h(X, \bar{X})^2}.$$

3.4 Curvature of Submanifold in the Hermitian Case

The goal of this section is to present the decreasing property of holomorphic sectional curvatures i.e. the holomorphic sectional curvature of a submanifold cannot exceed that of the ambient manifold. Let (M, h) be a Hermitian manifold of dimension $m+n$, and let M' be a submanifold of M of dimension n . The induced Hermitian metric on M' will be denoted by h' . Let ∇ and ∇' denote the Chern connection associated with h and h' , respectively.

Proposition 3.18 *For any vector fields X and Y on M' , the vector field $\nabla_X Y$ is well-defined i.e. it does not depend on the extension of X or Y in M .*

Essentially the proof goes on to show that the value of $\nabla_X Y$ at a point $p \in M$ depends only on the value X_p and the values of Y along any curve $\lambda(t)$ in M' such that $\left. \frac{d\lambda}{dt} \right|_p = Y_p$. The proof of these facts can be found in [11].

Definition 3.19 Let $\Gamma(TM'_\mathbb{C})$ denote the space of sections of $TM'_\mathbb{C}$. The *second fundamental form* of M' in M is the bilinear map $\Pi: \Gamma(TM'_\mathbb{C}) \times \Gamma(TM'_\mathbb{C}) \rightarrow \Gamma(TM_\mathbb{C})$ given by

$$\Pi(X, Y) = \nabla_X Y - \nabla'_X Y.$$

It is clear that Π is tensorial in both arguments i.e. for smooth complex-valued functions f and g on M' , one has

$$\Pi(fX, gY) = fg\Pi(X, Y).$$

It follows then from theorem 3.16 that

$$\Pi(X, \bar{Y}) = \Pi(\bar{X}, Y) = 0,$$

for any vectors X and Y of type $(1, 0)$.

Proposition 3.20 *For any vectors X, Y and Z of type $(1, 0)$ on M' ,*

$$h(\Pi(X, Y), \bar{Z}) = 0.$$

Proof. By direct computation,

$$\begin{aligned} h(\Pi(X, Y), \bar{Z}) &= h(\nabla_X Y, \bar{Z}) - h(\nabla'_X Y, \bar{Z}) \\ &= h(\nabla_X Y, \bar{Z}) - h'(\nabla'_X Y, \bar{Z}) \\ &= Xh(Y, \bar{Z}) - h(Y, \nabla_X \bar{Z}) - Xh'(Y, \bar{Z}) + h'(Y, \nabla'_X \bar{Z}) \\ &= h(Y, \nabla'_X \bar{Z}) - h(Y, \nabla_X \bar{Z}) \\ &= -h(Y, \Pi(X, \bar{Z})) \\ &= 0 \end{aligned}$$

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Let R and R' denote the curvature tensor of M and M' , respectively. Then

Theorem 3.21 *For any vectors X, Y, Z, W of type $(1, 0)$ on M ,*

$$R(X, \bar{Y}, Z, \bar{W}) = R'(X, \bar{Y}, Z, \bar{W}) + h(\Pi(Z, X), \Pi(\bar{W}, \bar{Y})).$$

Proof. since the R , R' and h are tensors, it is enough to prove the theorem in a coordinates. Let $(z^1, \dots, z^n, \dots, z^{n+m})$ be slice coordinates on M i.e. $(z^1, \dots, z^n)$ form a coordinate system on M' . Denote by $\Pi_{\alpha\beta}$ the vector field $\Pi(\partial_\alpha, \partial_\beta)$. Then by defitnion,

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\sigma}} &= h(R_{\partial_\gamma\partial_{\bar{\sigma}}}\partial_\alpha, \bar{\beta}) \\ &= -h(\nabla_{\bar{\sigma}}\nabla_\gamma\partial_\alpha, \partial_{\bar{\beta}}) \\ &= -\partial_{\bar{\sigma}}h(\nabla_\gamma\partial_\alpha, \partial_{\bar{\beta}}) + h(\nabla_\gamma\partial_\alpha, \nabla_{\bar{\sigma}}\partial_{\bar{\beta}}), \end{aligned}$$

where $\alpha, \beta, \gamma, \sigma = 1, \dots, n$. Substituting $\nabla_\gamma\partial_\alpha = \nabla'_\gamma\partial_\alpha + \Pi_{\gamma\alpha}$ and $\nabla_{\bar{\sigma}}\partial_{\bar{\beta}} = \nabla'_{\bar{\sigma}}\partial_{\bar{\beta}} + \Pi_{\bar{\sigma}\bar{\beta}}$ gives

$$R_{\alpha\bar{\beta}\gamma\bar{\sigma}} = -\partial_{\bar{\sigma}}h(\nabla'_\gamma\partial_\alpha + \Pi_{\gamma\alpha}, \partial_{\bar{\beta}}) + h(\nabla'_\gamma\partial_\alpha + \Pi_{\gamma\alpha}, \nabla'_{\bar{\sigma}}\partial_{\bar{\beta}} + \Pi_{\bar{\sigma}\bar{\beta}}).$$

It follows from proposition 3.20 and metric compatibility that

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\sigma}} &= -\partial_{\bar{\sigma}}h(\nabla'_\gamma\partial_\alpha, \partial_{\bar{\beta}}) + h(\nabla'_\gamma\partial_\alpha, \nabla'_{\bar{\sigma}}\partial_{\bar{\beta}}) + h(\Pi_{\gamma\alpha}, \Pi_{\bar{\sigma}\bar{\beta}}) \\ &= -\partial_{\bar{\sigma}}h'(\nabla'_\gamma\partial_\alpha, \partial_{\bar{\beta}}) + h'(\nabla'_\gamma\partial_\alpha, \nabla'_{\bar{\sigma}}\partial_{\bar{\beta}}) + h(\Pi_{\gamma\alpha}, \Pi_{\bar{\sigma}\bar{\beta}}) \\ &= -h'(\nabla'_{\bar{\sigma}}\nabla'_\gamma\partial_\alpha, \partial_{\bar{\beta}}) + h(\Pi_{\gamma\alpha}, \Pi_{\bar{\sigma}\bar{\beta}}) \\ &= R'_{\alpha\bar{\beta}\gamma\bar{\sigma}} + h(\Pi_{\gamma\alpha}, \Pi_{\bar{\sigma}\bar{\beta}}) \end{aligned}$$

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For a given vector X of type $(1, 0)$, let $H(X)$ and $H'(X)$ denote the holomorphic sectional curvature in M and M' , respectively. Then a direct consequence of the

previous theorem is

Corollary 3.22 *For any vector X of type $(1, 0)$ on M ,*

$$H(X) = H'(X) + \frac{\|\Pi(X, X)\|^2}{\|X\|^4}.$$

In particular $H \geq H'$.

CHAPTER 4

FINSLER STRUCTURE ON COMPLEX MANIFOLDS

Let M be a complex manifold and denote a point in TM by (z, ζ) , where z is the base coordinate and ζ is the fiber coordinate. A *complex Finsler structure* on M is a function $F : TM \rightarrow \mathbb{R}$ which satisfies

1. F is smooth outside the zero section of TM
2. $F(z, \zeta) \geq 0$ for all $(z, \zeta) \in TM$ and $F(z, \zeta) = 0 \Leftrightarrow \zeta = 0$.
3. $F(z, \lambda\zeta) = |\lambda|^2 F(z, \zeta)$, for all $(z, \zeta) \in TM$ and $\lambda \in \mathbb{C}$.

In this chapter, it will be shown how a complex Finsler structure gives rise to a Hermitian structure on the projectivization of the tangent bundle, and how one can define the holomorphic curvature for this case. The original work can be found in [12].

4.1 Projective and Pullback Bundle

Definition 4.1 A *holomorphic vector bundle* E over a complex manifold M is a vector bundle with a projection map $\pi : E \rightarrow M$ which satisfies

1. The fibers of E are complex vector spaces.
2. The total space E is a complex manifold.
3. The map π is holomorphic.

For each $p \in M$, the fiber $\pi^{-1}(p)$ will be denoted by E_p . The rank of E is the dimension of its fibers. Let $\mathbb{P}E_p$ denote the projectivization of E_p .

Definition 4.2 Let E be a holomorphic vector bundle. The *projective bundle* associated with E , denoted by $\mathbb{P}E$, is the collection of all projectivized fibers i.e.

$$\mathbb{P}E = \cup_{p \in M} \mathbb{P}E_p.$$

It follows that if E has rank r , then the rank of $\mathbb{P}E$ is $r - 1$. One should think of a point in $\mathbb{P}E$ over $p \in M$ as a complex line in E_p . Then there is a natural projection map $\pi^P : \mathbb{P}E \rightarrow M$ which sends $x \in \mathbb{P}E$ to $p \in M$ if $x \in \mathbb{P}E_p$. This projection in turn induces a bundle over $\mathbb{P}E$ called the *pullback bundle* which is defined as follow. Let $\tilde{E} = \{(x, v) \in \mathbb{P}E \times E; \pi^P(x) = \pi(v)\}$ equipped with the subspace topology and the projection map $\tilde{\pi} : \tilde{E} \rightarrow \mathbb{P}E$ given by $\tilde{\pi}(x, v) = x$. Let $e : \tilde{E} \rightarrow E$ be the projection

onto the second factor i.e. $e(x, v) = v$. Then one has the following commuting diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{e} & E \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ \mathbb{P}E & \xrightarrow{\pi^P} & M \end{array}$$

It is useful to think of $\tilde{E}$ in terms of its fibers. If x is a point in $\mathbb{P}E$, which itself represents a line in E_p over some point p in M , then an element in $\tilde{E}_x$ is a vector in E_p attached to the line x . By taking all the vectors in $\tilde{E}_x$ which lie on the line x , and then taking this collection for all $x \in \mathbb{P}E$, one obtains the tautological line subbundle $L \subset \tilde{E}$.

To get a local expression for the above constructions, let $(z^1, \dots, z^n)$ be a local holomorphic coordinates on M . Let $(s_1, \dots, s_r)$ be a local frame for E and $(\zeta^1, \dots, \zeta^r)$ be its dual frame so that $(z^1, \dots, z^n, \zeta^1, \dots, \zeta^r)$ are holomorphic coordinates for E . By setting

$$Z^j = \zeta^j \circ e,$$

one obtains the local coordinates $(z^1, \dots, z^n, \zeta^1, \dots, \zeta^r, Z^1, \dots, Z^r)$ for $\tilde{E}$. Then the tautological line bundle is defined by

$$Z^j = \zeta^j.$$

Remark 4.3 1. it is important to note that the coordinates $(\zeta^1, \dots, \zeta^n)$ are homogenous in the sense that any quantity defined on $\mathbb{P}E$ or $\tilde{E}$ involving them has to be constant with respect to scaling by a complex number.

2. It follows that for non-zero complex number λ , the equations $Z^j = \lambda \zeta^j$ still

defines the tautological line bundle.

4.2 Finsler Metric, Connection, and Curvature

Let M be a complex manifold, $\mathbb{P}(TM)$ the projectivied tangent bundle, and $\widetilde{TM}$ the induced pullback bundle with $(z^1, \dots, z^n, \zeta^1, \dots, \zeta^n, Z^1, \dots, Z^n)$ as local coordinates. Let F be a Finsler structure on M . The Roman letters $j, k, s, \dots$ will be used to indicate the derivative with respect to z while the Greek letters $\alpha, \beta, \gamma, \dots$ will be used for derivatives with respect to ζ . Moreover, the following notations will be used.

$$\partial_\alpha = \frac{\partial}{\partial \zeta^\alpha}, \quad \partial_j = \frac{\partial}{\partial z^j},$$

$$F_\alpha = \partial_\alpha F, \quad F_j = \partial_j F, \quad F_{\alpha\bar{j}} = \partial_\alpha \partial_{\bar{j}} F \dots \text{ and so on.}$$

Recall that, by definition, $F(z, \lambda\zeta) = |\lambda|^2 F(z, \zeta)$ for any complex number λ . Taking the derivative of both sides with respect to λ , one obtains

$$F_\alpha(z, \lambda\zeta)\zeta^\alpha = \bar{\lambda}F(z, \zeta).$$

Differentiating both sides again with respect to $\bar{\lambda}$ and letting $\lambda = 1$ gives

$$F_{\alpha\bar{\beta}}(z, \zeta)\zeta^\alpha\zeta^{\bar{\beta}} = F(z, \zeta). \tag{4.1}$$

On the other hand, differentiating $F(z, \lambda\zeta) = |\lambda|^2 F(z, \zeta)$ with respect to ζ^α and then

with $\zeta^{\bar{\beta}}$ when $\lambda \neq 0$ yields

$$F_{\alpha\bar{\beta}}(z, \lambda\zeta) = F_{\alpha\bar{\beta}}(z, \zeta). \quad (4.2)$$

These equations allows us to define a bilinear form $\widetilde{F}$ on $\widetilde{TM}$ which in local coordiantes is given by

$$\widetilde{F}_{(z,\zeta)} = F_{\alpha\bar{\beta}}(z, \zeta) Z^\alpha \otimes Z^{\bar{\beta}} \quad (4.3)$$

Notice that equation (4.2) implies that $\widetilde{F}$ is well-defined for complex lines in $\mathbb{P}(TM)$. Moreover, by equation (4.1),

$$\widetilde{F}_{z,\zeta}(\zeta, \bar{\zeta}) = F(z, \zeta), \quad (4.4)$$

i.e. restricting $\widetilde{F}$ on the tautological line bundle, one can recover information about the Finsler metric F . If the matrix $(F_{\alpha\bar{\beta}})$ is Hermitian, then $\widetilde{F}$ is a Hermitian metric on $\widetilde{TM}$ and the theory of chapter 3 can be applied; in this case, the Finsler structure is said to be *convex*, which will be assumed from now on.

Let ∇ denote the Chern connection associated with $\widetilde{F}$ and let s and t be sections in $\widetilde{TM}$. For any vector fields X and Y on $\mathbb{P}TM$, the curvature tensors $R_{XY}s$ and $R(s, t, X, Y)$ is given by

$$R_{XY}s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s,$$

$$R(s, t, X, Y) = \widetilde{F}(R_{XY}s, t).$$

Note that the frame $(s_1, \dots, s_n)$ associated with $(\zeta^1, \dots, \zeta^n)$ can naturally be used as a

frame on $\widetilde{TM}$ associated with $(Z^1, \dots, Z^n)$. Then there are four types of curvature tensor coefficients, which are given by

$$R_{\alpha\bar{\beta}\gamma\bar{\sigma}} = \widetilde{F}(R_{\gamma\bar{\sigma}}s_\alpha, s_{\bar{\beta}})$$

$$R_{\alpha\bar{\beta}\gamma\bar{k}} = \widetilde{F}(R_{\gamma\bar{k}}s_\alpha, s_{\bar{\beta}})$$

$$R_{\alpha\bar{\beta}j\bar{k}} = \widetilde{F}(R_{j\bar{k}}s_\alpha, s_{\bar{\beta}})$$

$$R_{\alpha\bar{\beta}j\bar{\sigma}} = \widetilde{F}(R_{j\bar{\sigma}}s_\alpha, s_{\bar{\beta}})$$

It follows from theorem 3.17 that

$$\begin{aligned} R_{\alpha\bar{\beta}\gamma\bar{\sigma}} &= -F_{\alpha\bar{\beta}\gamma\bar{\sigma}} + F^{\bar{\nu}\mu}F_{\alpha\bar{\nu}\gamma}F_{\mu\bar{\beta}\bar{\sigma}} \\ R_{\alpha\bar{\beta}\gamma\bar{k}} &= -F_{\alpha\bar{\beta}\gamma\bar{k}} + F^{\bar{\nu}\mu}F_{\alpha\bar{\nu}\gamma}F_{\mu\bar{\beta}\bar{k}} \\ R_{\alpha\bar{\beta}j\bar{k}} &= -F_{\alpha\bar{\beta}j\bar{k}} + F^{\bar{\nu}\mu}F_{\alpha\bar{\nu}j}F_{\mu\bar{\beta}\bar{k}} \\ R_{\alpha\bar{\beta}j\bar{\sigma}} &= -F_{\alpha\bar{\beta}j\bar{\sigma}} + F^{\bar{\nu}\mu}F_{\alpha\bar{\nu}j}F_{\mu\bar{\beta}\bar{\sigma}} \end{aligned} \tag{4.5}$$

Given a non-zero vector $X \in TM_z$, the holomorphic sectional curvature in the section spanned by X is defined by

$$H(X) = \frac{\widetilde{F}_{z\zeta}(R_{X\bar{X}}\widetilde{X}, \overline{\widetilde{X}})}{\widetilde{F}_{z,\zeta}(\widetilde{X}, \overline{\widetilde{X}})F(X)},$$

where $\widetilde{X}$ is any non-zero vector in $L_{(z,[X])}$. If $X = \zeta^j\partial_j$, then $\widetilde{X} = \zeta^js_j$. It follows

from (4.4) and (4.5) that

$$H(X) = \frac{1}{F(z, \zeta)^2} R_{\alpha\bar{\beta}j\bar{k}} \zeta^\alpha \zeta^{\bar{\beta}} \zeta^j \zeta^{\bar{k}}. \quad (4.6)$$

4.3 Curvature of Submanifold in the Finsler Case

In chapter 3, it was shown that the holomorphic sectional curvature of a Hermitian manifold cannot be less than the curvature of any submanifold. The same is true for Finsler manifolds as will be shown in this section.

Let (M, F) be a Finsler manifold of dimension $m + n$, M' a submanifold of dimension n , and $E = TM_{\mathbb{C}}$ and $E' = TM'_{\mathbb{C}}$ the complexified tangent bundle. Restricting E to M' , E' can be seen as a subbundle of E , and hence the Finsler structure F restricts to a Finsler structure on M' . Let $(s_1, \dots, s_n, \dots, s_{m+n})$ be a holomorphic frame for E such that $(s_1, \dots, s_n)$ is a frame for E' . Then clearly if $(F_{j\bar{k}})$, $1 \leq j \leq k \leq m + n$ is Hermitian, the submatrix $(F_{j\bar{k}})$, $1 \leq j \leq k \leq n$ is Hermitian. That is, if F is convex, then its restriction, denoted henceforth by F' , is also convex.

Let $\widetilde{E}$ and $\widetilde{E}'$ denote the pullback bundle over the projectivized bundles $\mathbb{P}E$ and $\mathbb{P}E'$, and also let ∇ and ∇' denote the Chern connection associated with the metric induced by the Finsler structure as defined in the previous section. Then one can define the second fundamental form of M' in M as a bilinear map $\Pi: \Gamma(T(\mathbb{P}E')_{\mathbb{C}}) \times \Gamma(\widetilde{E}') \rightarrow \Gamma(\widetilde{E}')$ given by

$$\Pi(X, s) = \nabla_X s - \nabla'_X s.$$

It follows from the definition of a Chern connection that for a vector X tangent to

the projectivied bundle $\mathbb{P}E'$ and a vector s in E' both of type $(1, 0)$,

$$\Pi(X, \bar{s}) = \Pi(\bar{X}, s) = 0.$$

Proposition 4.4 *Let X be a tangent vector in $\mathbb{P}E'$ of type $(1, 0)$ and s, t be vectors in E' of type $(1, 0)$. Then*

$$\tilde{F}(\Pi(X, s), \bar{t}) = 0.$$

Proof. The argument is similar to the one given for proposition 3.20 as can be seen in the following direct computation.

$$\begin{aligned} \tilde{F}(\Pi(X, s), \bar{t}) &= \tilde{F}(\nabla_X s, \bar{t}) - \tilde{F}(\nabla'_X s, \bar{t}) \\ &= \tilde{F}(\nabla_X s, \bar{t}) - \widetilde{F'}(\nabla'_X s, \bar{t}) \\ &= X\tilde{F}(s, \bar{t}) - \tilde{F}(s, \nabla_X \bar{t}) - X\widetilde{F'}(s, \bar{t}) + \widetilde{F'}(s, \nabla'_X \bar{t}) \\ &= \tilde{F}(s, \nabla'_X \bar{t}) - \tilde{F}(s, \nabla_X \bar{t}) \\ &= -\tilde{F}(s, \Pi(X, \bar{t})) \\ &= 0 \end{aligned}$$

■

As in section 3.4, let R and R' denote the curvatures of M and M' , respectively.

Theorem 4.5 *For any vectors X and Y of type $(1, 0)$ on $\mathbb{P}E'$ and any sections s and*

t of $\widetilde{E}'$ of type $(1, 0)$ one has

$$R(s, \bar{t}, X, \bar{Y}) = R'(s, \bar{t}, X, \bar{Y}) + \widetilde{F}(\Pi(X, s), \Pi(\bar{Y}, \bar{t})).$$

Proof. Since the R , R' and $\widetilde{F}$ are tensors, it is enough to prove the theorem in a coordinates. Let $(s_1, \dots, s_n, \dots, s_{m+n})$ be a local frame on $\widetilde{E}$ such that $(s_1, \dots, s_n)$ is a local frame on $\widetilde{E}'$. Denote by $\Pi_{j\alpha}$ the vector field $\Pi(\partial_j, s_\alpha)$. Then for $\alpha, \beta = 1, \dots, p$ and $j, k = 1, \dots, n$,

$$\begin{aligned} R_{\alpha\bar{\beta}j\bar{k}} &= \widetilde{F}(R_{\partial_j\partial_{\bar{k}}}s_\alpha, s_{\bar{\beta}}) \\ &= -\widetilde{F}(\nabla_{\partial_{\bar{k}}} \nabla_{\partial_j} s_\alpha, s_{\bar{\beta}}) \\ &= -\partial_{\bar{k}} \widetilde{F}(\nabla_{\partial_j} s_\alpha, s_{\bar{\beta}}) + \widetilde{F}(\nabla_{\partial_j} s_\alpha, \nabla_{\partial_{\bar{k}}} s_{\bar{\beta}}) \\ &= -\partial_{\bar{k}} \widetilde{F}(\nabla'_{\partial_j} s_\alpha, s_{\bar{\beta}}) + \widetilde{F}(\nabla'_{\partial_j} s_\alpha, \nabla'_{\partial_{\bar{k}}} s_{\bar{\beta}}) + \widetilde{F}(\Pi_{j\alpha}, \Pi_{\bar{k}\bar{\beta}}) \\ &= -\partial_{\bar{k}} \widetilde{F}'(\nabla'_{\partial_j} s_\alpha, s_{\bar{\beta}}) + \widetilde{F}'(\nabla'_{\partial_j} s_\alpha, \nabla'_{\partial_{\bar{k}}} s_{\bar{\beta}}) + \widetilde{F}(\Pi_{j\alpha}, \Pi_{\bar{k}\bar{\beta}}) \\ &= R'_{\alpha\bar{\beta}j\bar{k}} + \widetilde{F}(\Pi_{j\alpha}, \Pi_{\bar{k}\bar{\beta}}) \end{aligned}$$

■

Let H and H' denote the holomorphic sectional curvatures of M and M' . Given a vector $X = \zeta^j \partial_j$ tangent to M' at the point (z) , one has

$$\begin{aligned}
H(X) &= \frac{1}{F(z, \zeta)^2} R_{\alpha \bar{\beta} j \bar{k}} \zeta^\alpha \zeta^{\bar{\beta}} \zeta^j \zeta^{\bar{k}} \\
&= \frac{1}{F(z, \zeta)^2} (R'_{\alpha \bar{\beta} j \bar{k}} + \tilde{F}(\Pi_{j\alpha}, \Pi_{\bar{k}\bar{\beta}})) \zeta^\alpha \zeta^{\bar{\beta}} \zeta^j \zeta^{\bar{k}} \\
&= H'(X) + \frac{\tilde{F}(\Pi(X, \tilde{X}), \Pi(\bar{X}, \overline{\tilde{X}}))}{\tilde{F}(\tilde{X}, \overline{\tilde{X}})}
\end{aligned}$$

Therefore, it can be concluded that the holomorphic sectional curvature of a submanifold cannot exceed that of the ambient.

CHAPTER 5

SCHWARZ'S LEMMA FOR FINSLER MANIFOLDS

Let (M, F) be a complex Finsler manifold and let $D \subset \mathbb{C}$ be the unit disk. A holomorphic map $f : D \rightarrow M$ induces a $(1, 1)$ covariant tensor f^*F on D given as follow. Let (z) be the standard holomorphic coordinate on D . Then for any $z_0 \in D$,

$$f^*F_{z_0} = \lambda(z_0)dz \otimes d\bar{z}, \quad (5.1)$$

where $\lambda(z_0) = F \left(df \frac{\partial}{\partial z} \Big|_{z_0} \right)$. It follows from the definition of a Finsler metric that if the differential of f does not vanish at any point, then f^*F defines a Kähler structure on D .

Theorem 5.1 (Wan 2019, [6]) *Let (M, F) be a complex Finsler manifold with holomorphic sectional curvatures bounded above by a negative constant $-K$, D the unit*

disk, and $f : D \rightarrow M$ a holomorphic map. Then

$$\lambda(z) \leq \frac{2}{K(1 - |z|^2)^2}, \quad \text{for all } z \in D$$

Proof. Let $z \in D$ be given and assume that $df|_z \neq 0$. Then locally f is an embedding i.e. there exists a neighborhood U around z such that $f(U)$ is an embedded 1-dimensional submanifold of M . This implies that $f : (U, f^*F) \rightarrow (f(U), F)$ is an isometry and hence the holomorphic sectional curvature of D at z , denoted by $H_D(z)$ is the same as the holomorphic sectional curvature of $f(U)$ at $f(z)$. It follows from the decreasing property of holomorphic sectional curvature (refer to section 4.3) that $H_D(z)$ is not greater than that of M at $f(z)$ i.e. $H_D(z) \leq -K$.

Let $u(z) = (1 - |z|^2)^2 \lambda(z)$. Without loss of generality, it can be assumed that $u(z)$ is not identically zero. The proof then is broken into two cases.

Case 1: the function u attains its maximum at a point $z_0 \in D$.

Since $u(z_0) \neq 0$, one can locally define the function $\log(u(z))$, which will also attains its maximum at z_0 . It follows from the second partial derivative test that

$$\left. \frac{\partial^2}{\partial z \partial \bar{z}} \log(u(z)) \right|_{z_0} \leq 0$$

Hence,

$$0 \geq \frac{\partial^2}{\partial z \partial \bar{z}} \log((1 - z\bar{z})^2) \Big|_{z_0} + \frac{\partial^2}{\partial z \partial \bar{z}} \log(\lambda) \Big|_{z_0} \quad (5.2)$$

By direct computation,

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log((1 - |z|^2)^2) = \frac{-2}{(1 - |z_0|^2)^2}.$$

On the other hand,

$$H_D(z_0) = -\frac{1}{\lambda(z_0)} \frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda|_{z_0},$$

which implies

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log(\lambda)|_{z_0} = -\lambda(z_0)H_D \geq \lambda(z_0)K.$$

Substituting in (5.2) gives

$$0 \geq \frac{-2}{(1 - |z_0|^2)} + \lambda(z_0)K,$$

and therefore

$$\frac{2}{K} \geq u(z_0) \geq u(z),$$

as $u(z_0)$ is the maximum of $u(z)$.

Case 2: *the function u does not attain its maximum in D .*

Fix $z_0 \in D$. It will be shown that $\frac{2}{K} \geq u(z_0)$. Without loss of generality, assume $u(z_0) \neq 0$. Choose a constant r such that $1 > r > |z_0|$ and let $D_r = \{z \in D; |z| < r\}$. Define $u_r(z) = r^{-2}(r^2 - |z|^2)^2 \lambda$. Since u_r vanish at the boundary of D_r , it must attains its maximum inside D_r , say at z_1 . Then as before,

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log(u_r(z))|_{z_1} \leq 0.$$

Following the same line of calculations as in the previous case gives

$$\frac{2}{K} \geq u_r(z_1) \geq u_r(z_0).$$

Since r is arbitrarily chosen, the inequality

$$\frac{2}{K} \geq u_r(z_0)$$

holds for all r and therefore as $r \rightarrow 1$, one obtains

$$\frac{2}{K} \geq u(z_0).$$

Finally, since z_0 is arbitrary, the statement follows for all z in D . ■

It is worth reiterating that the original work by Wan considered the case of maps originating from a Riemann surface rather than the unit disk. Moreover, the equality case of the estimate for maps from the unit disk to a Finsler manifold has been studied by Royden in [13].

REFERENCES

- [1] L. V. Ahlfors, “An extension of Schwarz’s lemma,” *Trans. Am. Math. Soc.*, vol. 43, pp. 359–364, 1938.
- [2] S. shen Chern, “On holomorphic mappings of hermitian manifolds of the same dimension,” *Entire Funct. and Relat. Parts of Anal.*, La Jolla, Calif. 1966, Proc. Symp. Pure Math. 11, 157-170 (1968)., 1968.
- [3] Y. c. Lu, “Holomorphic of complex manifolds,” *J. Differ. Geom.*, vol. 2, pp. 299–312, 1968.
- [4] S. Yau, “A general schwarz lemma for kähler manifolds,” *Amer. J. Math.*, vol. 100, pp. 197–203, 1978.
- [5] H. L. Royden, “The Ahlfors-Schwarz lemma in several complex variables,” *Comment. Math. Helv.*, vol. 55, pp. 547–558, 1980.
- [6] X. Wan, “Holomorphic sectional curvature of complex Finsler manifolds,” *J. Geom. Anal.*, vol. 29, no. 1, pp. 194–216, 2019.
- [7] W. C. Brown, *A second course in linear algebra.* Wiley, 1988.
- [8] J. M. Lee, *Introduction to smooth manifolds.* Springer, 2013.

- [9] W. Ballmann, *Lectures on Kähler manifolds*. European Mathematical Society, 2006.
- [10] L. Schneps and C. Voisin, *Hodge Theory and Complex Algebraic Geometry. 1* (*Cambridge studies in advanced mathematics ; 76*). Cambridge University Press, 2002.
- [11] J. M. Lee, *Riemannian manifolds*. Springer, 1997.
- [12] S. Kobayashi, “Negative vector bundles and complex Finsler structures,” *Nagoya Math. J.*, vol. 57, pp. 153–166, 1975.
- [13] H. L. Royden, “The Ahlfors-Schwarz lemma: The case of equality,” *J. Anal. Math.*, vol. 46, p. 261–270, 1986.

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Undergraduate Math Events & Experience

Research Experience for Undergraduates Program :	Numerical Method for Rare Events Transition	(2013)
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Research Experience for Undergraduates Program :	Game-Theoretical Model of Debt Management	(2014)

Undergraduate Course Highlights

Numerical Analysis (Independent Study Course)

- Polynomial interpolation, methods in numerical integration, and least squares approximations.
- Numerical solution for nonlinear equations, system of linear equations, and differential equations.
- Error estimation and convergence analysis.

Real Analysis (Honor Student Course)

- Dedekind's cuts and functions on the real line (continuity, differentiability, and integrability)
- Metric spaces and homeomorphisms.
- Uniform convergence in function spaces and Weierstrass Approximation Theorem.

Introduction to Dynamical Systems (A course in MASS program in Penn State University)

- Fixed points, periodic points, and recurrent points of a dynamical system.
- Minimal, topologically transitive and topologically mixing dynamical systems
- Topological entropy and Poincaré Recurrence Theorem

Master's Course Highlights

Introduction To Smooth Manifolds (Informal Seminar)

- Definition and properties of smooth manifolds, tangents spaces, and bundles.
- Integral curves, distributions, and Frobenius theorem.
- Tensors, tensor fields, and differential forms.

A Generalization of Ahlfors-Schwarz Lemma (Reading and Research)

- Complex and almost complex structure
- Hermitian manifolds and holomorphic sectional curvature
- Ahlfors-Schwarz Lemma in Finsler Manifolds

Winner of M5M 2018 Contest (5 minutes math speech to laymen) in KFUPM

Employment

Employer: Saudi Aramco Hire Date: July 2016 End Date: March 2019 Job Title: Design Engineer	Summary of Employment Experience: • Led equipment failure investigation • Calculation of minimal wall thickness for piping design • Nominated to deliver a speech to the CEO
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