

BY

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DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

ON CERTAIN EXACT SOLUTIONS OF
NON-VACUUM EINSTEIN'S EQUATIONS

BY

ALI AL SAFWAN

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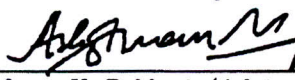
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This thesis, written by **ALI AL SAFWAN** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

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To my family and friends

for all their support and encouragement.

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“In the name of Allah, The Most Beneficent, The Most Merciful”.

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THESIS ABSTRACT

NAME: Ali Al Safwan

TITLE OF STUDY: On certain exact solutions of non-vacuum Einstein's
Equations

MAJOR FIELD: Mathematics

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The main aim of this thesis is to investigate some specific Bianchi type-I cosmological solutions of the Einstein field equations (EFEs). In particular, we study Bianchi-I type cosmological model with variable gravitational and cosmological constants in two cases, which allow or otherwise particle creation. Considering certain ansätze for scale factor $A(t)$ satisfying power t^m and exponential laws e^{mt} of evolution respectively, we provide certain cosmological solutions of the EFEs and discuss their physical implications. It is shown that the vacuum energy density Λ is positive and proportional to t^{-2} , and assuming particle creation we find that $G \sim t^{-1}$. The values of the dark-energy density parameter Ω_Λ and the matter density parameter Ω_m are calculated. It is observed that the best numerical values obtained are in good agreement with the 5-years WMAP observations. The physical and geometrical behavior of this model is also discussed.

ملخص الرسالة

الاسم: علي ال صفوان

عنوان الدراسة: عن بعض الحلول الدقيقة لمعادلة آينشتاين الافرغية

التخصص: رياضيات

تاريخ الدرجة العلمية: مايو 2020

الهدف من هذه الرسالة هو تحري نموذج كوني من نوع بيانكي-1 كل لمعادلات حقل آينشتاين. تحديداً، نقوم بدراسة نموذج بيانكي-1 الكوني بافتراض الثابت الكوني و الجذبي عوامل تعتمد على الوقت في حالتين: بافتراض و جود او عدم وجود الخلق المستمر للمادة.

باستخدام بعض الحلول التجريبية للعامل المقياسي $A(t)$ التي تحقق قانوني القوى t^m و الاسس e^{mt} كقياس لتوسع الكون. زيادة على ذلك، نوجد بعض الحلول الدقيقة لمعادلات آينشتاين وناقش تداعياتها الفيزيائية. حيث نحصل على نتيجة ان كثافة الطاقة الفرغية موجبة و متناسبة طردياً مع t^{-2} و بافتراض خلق المادة، نجد ان الثابت الكوني يتناسب طردياً مع t^{-1} . نقوم ايضاً بحساب قيمة عامل كثافة الطاقة المظلمة Ω_Λ و عامل كثافة المادة Ω_m ، و نلاحظ

ان القيم العددية التي حصلنا عليها توافق قياسات WMAP.

اخيراً، نقوم بمناقشة التداعيات الفيزيائية و الهندسية لهذا النموذج.

CHAPTER 1

PRELIMINARIES

1.1 Introduction and Literature review

The theory of general Relativity provides a most successful framework in describing effect of strong gravitational fields (those at which Newtons theory fails and we have to consider the curvature of spacetime, as in neutron stars or blackholes). Before the development of general theory of Relativity in 1915 by Albert Einstein [1], classical Newtonian mechanics was the framework to describe gravitational interaction. In Newtonian mechanics, the force is responsible for all motions via Newton's equation of motion. In general Relativity (GR), the Newtonian force is equivalent to the dynamics of geometry via the space-time curvature. This suggests that in GR, all motions take place due to the presence of curvature of space-time and not due to force. This relationship between curvature of space-time and gravity, had a very fruitful impact on mathematics via the newly conceived relationship between differential geometry of Riemann and tensor calculus of Ricci and Levi-Civita. It was due to this reason that the tensor calculus and Riemannian geometry underwent rapid development in

the twentieth century. One of the main goals of general theory of Relativity is to understand the evolution of universe and help analyze dynamics of the celestial bodies like planets, stars, and galaxies. In order to achieve this goal, it is necessary to understand the behavior and structure of their gravitational fields, which is achieved by studying the solutions of the EFEs. For good references discussing basic principles of general theory of Relativity, the reader is referred to Misner et al. [2], Carrol [3] or Wald [4].

In this thesis, our aim is to first present and discuss exact solutions of the EFEs which have physical and historical significance. Second, we present a derivation of an exact solution to the EFEs which describes the evolution of the universe as modeled via the well-known class of space-time metrics, generally known as Bianchi metrics [5]. We have many well established methods for finding exact solutions to the EFEs, ranging from symmetry analysis, to assuming ansatz for the gravitational or matter fields. A comprehensive review of the procedures and methods used in finding exact solutions are discussed in literature, for example, is provided by Stephani et al. [6] and Misner et al. [2]. There are numerous known solutions to the EFEs that can be found in the literature, among the most important solutions - in terms of their physical interpretation and historical significance - are the Schwarzschild [7], Reissner-Nordstrom [8] and Kerr [10] solutions. As for the Schwarzschild solution, it describes the gravitational field in the space-time outside a static, spherically symmetric uncharged object. The Reissner-Nordström solution of the EFEs describes the gravitational field outside a spherically-symmetric, non-rotating charged object. The Kerr metric describes the gravitational field outside of an uncharged, rotating, axially-symmetric object.

Modern physical cosmology begun hand in hand with general Relativity, when Albert Einstein found a static cosmological solution for his equations [11]. A major problem with the model proposed by Einstein was that it did not account for the expansion of the universe. The accepted model for the universe was later developed, independently, by Friedmann, Lemaître, Robertson and Walker (abbreviated as FRW or FLRW) model and described an exact solution of the EFEs, representing an isotropic, homogeneous, expanding universe [12], [13], [14], [15].

Isotropic cosmological models are considered the most acceptable models in the study of the evolution and large scale structure of the universe. However, it is believed that isotropy may have been violated in the early stages of the universe, this prediction leads to investigate anisotropic cosmological models [16]. The simplest generalization of FRW model which account for anisotropy is given by the Bianchi type-I cosmological model, where we relax the criterion of isotropy in the flat FRW metric. Moreover, current observations provide an evidence that the cosmological and gravitational constants may well be time-dependent if the correct answers are to be found to discuss the dark matter and energy [17]. Keeping these points in mind, it seems ideal to explore Bianchi type-I cosmological models with time-varying gravitational and cosmological parameters in the context of the present thesis.

There exists a wide body of literature on studies of such models. Beesham [18] considered a Bianchi-I model with variable gravitational and cosmological parameters. By assuming power law for the gravitational parameter G , he found a solution in which the cosmological parameter Λ varies proportionally to t^{-2} , he also showed that for a constant energy density ρ , Bianchi-I models cannot evolve with exponential form.

Yadav et al. [20] studied locally rotationally symmetric (LRS) Bianchi-I cosmological model in the presence of magnetized dark energy. By assuming that the metric potentials are proportional, they found the isotropic distribution of magnetized dark energy leads to the present accelerated expansion of the universe in Bianchi type-I models. Singh et al. [21] presented a class of solution of EFEs in the context of Bianchi-I model with variable gravitational and cosmological parameters, and found that in most cases G increases in a perpetually expanding universe.

In this thesis, we present an alternative interpretation of Bianchi type-I cosmological models with variable G and Λ , by assuming the possibility of matter creation processes in the early Universe by means and as a result of time dependency of G and Λ . Since the calculations can be extremely complicated for general values of the scale factor, we restrict ourselves to certain specific ansatz for the scale factor. With these ansatz, we provide solutions to the (generalized) EFEs and discuss their physical implications.

This thesis consists of the following chapters:

Chapter 1 In this chapter we provide an overview of necessary tools which are pivotal to the general theory of Relativity. For this matter, a brief overview of tensor calculus is presented.

Chapter 2 In this chapter we present a derivation of the EFEs in two ways, namely, tensorial and variation formulation via Einstein-Hilbert action. In the same chapter, we also provide a brief discussion of the physical implications of the EFEs equations.

Chapter 3 Here, we discuss some of the most important exact solutions to the Einstein field equations. In this context, we provide a detailed derivation of the Schwarzschild solution. Afterwards, we discuss the FRW cosmological models, and

solve the Friedman equations in simple cases. Finally, we briefly discuss the Reissner-Nordstrom and Kerr solutions.

Chapter 4 In this chapter, we present our original work, in which we provide some cosmological solutions to the EFEs working considering a spacetime metric (gravitational theory) describing Bianchi-I model by taking the gravitational and cosmological constants to be time-dependent parameters.

Chapter 5 In the last chapter, we summarize the main ideas of this thesis, and we finalize by pointing toward alternative theories to general Relativity.

1.2 Definitions, Formulas and Conventions

1.2.1 n-dimensional space

In three dimensional Cartesian space, the coordinates of a point are given by (x, y, z) . More conveniently we write (x^1, x^2, x^3) for (x, y, z) . In general, the coordinates of a point in n-dimensional space are given by $(x^1, x^2, \dots, x^n)$.

1.2.2 Subscript and superscript

In the symbol T_{kl}^{ij} the upper indices i, j are called contravariant indices and the lower indices k, l are called covariant indices.

1.2.3 Einstein's summation convention

The Einstein summation convention is used through out this thesis. According to this convention, the summation is taken over two repeated indices appearing in the sea-saw form given by:

$$a^i e_i = \sum_{i=1}^n a^i e_i = a^1 e_1 + a^2 e_2 + \cdots + a^n e_n .$$

1.2.4 Free and dummy indices

A free index is an index which occurs only once in an expression A dummy index is any index which repeats in the sea-saw form as below,

$$a_k^{ij} x^k .$$

Here k is a dummy index, while i, j are free indices.

1.2.5 Scalars, vectors and tensors

A scalar is a single value with no directional indicator which represent a quantity that does not change under coordinate transformation, e.g. density, energy.

A vector is a quantity whose specification requires n-numbers (in n-dimensional space) called the components of the vector (with respect to some basis) e.g. force, acceleration.

A tensor (of order n) is an object whose specification in any coordinate system

requires 3^n numbers, called the components of a tensor e.g. stress-energy tensor.

1.2.6 Transformation of coordinates

Let $(x^1, x^2, \dots, x^n)$ be the coordinates of a point in X-coordinate system, and let $(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n)$ be the coordinates of the same point in Y-coordinate system.

We can write $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$ as independent functions of $x^1, x^2, \dots, x^n$, so that,

$$\begin{aligned}\bar{x}^1 &= \bar{x}^1(x^1, x^2, \dots, x^n), \\ \bar{x}^2 &= \bar{x}^2(x^1, x^2, \dots, x^n), \\ &\vdots \\ \bar{x}^n &= \bar{x}^n(x^1, x^2, \dots, x^n).\end{aligned}$$

More concisely,

$$\bar{x}^i = \bar{x}^i(x^1, x^2, \dots, x^n), \quad i = 1, \dots, n. \quad (1.1)$$

Solving these equations and expressing x^i as functions of $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n$, we have,

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n). \quad (1.2)$$

Equations (1.1) and (1.2) are said to be transformation of the coordinates from one coordinate system to another.

Below are list of important formulas for tensor calculus, put together here for quick

reference. The details and derivations of these formulas are to be discussed later.

1.2.7 Summary of important formulae

In this section, we present a summary of important formulae used in this thesis for quick reference. These formulas are to derived later.

1- Christoffel symbol

$$\Gamma_{ab}^c = \frac{1}{2}g^{dc}(g_{da,b} + g_{bd,a} - g_{ab,d}) . \quad (1.3)$$

If the metric g_{ab} is diagonal, then

$$\Gamma_{ab}^a = \Gamma_{ba}^a = \frac{1}{2}g^{aa}g_{aa,b} , \quad (1.4)$$

$$\Gamma_{bb}^a = -\frac{1}{2}g^{aa}g_{bb,a} , \quad (1.5)$$

$$\Gamma_{bc}^a = 0 \quad \text{if } a \neq b \neq c . \quad (1.6)$$

2- Riemann curvature tensor

$$R_{abc}^s = \Gamma_{ac,b}^s - \Gamma_{ab,c}^s + \Gamma_{ac}^k \Gamma_{kb}^s - \Gamma_{ab}^k \Gamma_{kc}^s . \quad (1.7)$$

3- Ricci tensor

$$R_{ab} = \Gamma_{ab,s}^s - \Gamma_{as,b}^s + \Gamma_{ab}^k \Gamma_{sk}^s - \Gamma_{as}^k \Gamma_{kb}^s , \quad (1.8)$$

or

$$R_{ab} = \Gamma_{ab,c}^c - \Gamma_{ac}^d \Gamma_{bd}^c - \left(\frac{\partial}{\partial x^a} (\log \sqrt{|g|}) \right)_{;b}. \quad (1.9)$$

4- Ricci scalar

$$R = g^{ab} R_{ab}. \quad (1.10)$$

1.3 Tensor calculus

1.3.1 Contravariant, covariant, and mixed tensors

Let $T^1, T^2, \dots, T^n$ be functions of coordinates $x^1, x^2, \dots, x^n$ in the X-coordinate system. If the functions T^i are transformed to $\bar{T}^i$ in the Y-coordinate system according to the following transformation law:

$$\bar{T}^i = \frac{\partial \bar{x}^i}{\partial x^j} T^j, \quad (1.11)$$

then T^i are called component of contravariant vector.

Let $T_i, i = 1, \dots, n$ be n functions of the coordinates $x^1, x^2, \dots, x^n$ in the X-coordinate system. If the functions T_i are transformed to $\bar{T}_i$ in the Y-coordinate system according to the following transformation law:

$$\bar{T}_i = \frac{\partial x^j}{\partial \bar{x}^i} T_j, \quad (1.12)$$

then T_i are called components of covariant vector. The contravariant (or covariant) vectors are also called a contravariant (or covariant) tensor of rank one.

Note that a superscript is always used to indicate a contravariant components, whereas a subscript is used to indicate a covariant component.

Now, following the rules of transformation for contravariant and covariant tensors of rank one (vectors), we can define contravariant, covariant and mixed tensors of rank two, as well as the general rule for tensors of rank $m + n$:

A contravariant tensor of rank two transforms as:

$$\bar{T}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} T^{kl}.$$

A covariant tensor of rank two transforms as:

$$\bar{T}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} T_{kl}.$$

And a tensor of mixed tensor of rank two transforms as:

$$\bar{T}^i_j = \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^j} T^k_l.$$

Note that this is also called a tensor of type (1,1), where the first component indicates the rank of contravariant tensor and the second component indicates the rank of a covariant tensor. A contravariant tensor of rank m transforms as:

$$\bar{T}^{i_1 i_2 \dots i_m} = \frac{\partial \bar{x}^{i_1}}{\partial x^{j_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{j_2}} \dots \frac{\partial \bar{x}^{i_m}}{\partial x^{j_m}} T^{j_1 j_2 \dots j_m}.$$

A covariant tensor of rank n transforms as:

$$\bar{T}_{i_1 i_2 \dots i_n} = \frac{\partial x^{j_1}}{\partial \bar{x}^{i_1}} \frac{\partial x^{j_2}}{\partial \bar{x}^{i_2}} \cdots \frac{\partial x^{j_n}}{\partial \bar{x}^{i_n}} T_{j_1 j_2 \dots j_n} .$$

And a tensor of rank $m + n$ transforms as:

$$\bar{T}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}} \cdots \frac{\partial \bar{x}^{i_m}}{\partial x^{k_m}} \frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}} \frac{\partial x^{l_2}}{\partial \bar{x}^{j_2}} \cdots \frac{\partial x^{l_n}}{\partial \bar{x}^{j_n}} T_{l_1 l_2 \dots l_n}^{k_1 k_2 \dots k_m} . \quad (1.13)$$

We also say the above is a tensor of type (m, n) .

1.3.2 Tensor addition and subtraction

Just like ordinary vectors, higher-order tensors can be added and subtracted by adding or subtracting their corresponding components, provided that the tensors to be added have the same contravariant and covariant rank. The result of tensor addition or subtraction is also a tensor that has the same covariant and contravariant rank of the tensors that are added or subtracted, e.g. for tensors of rank two we have:

$$\begin{aligned} A^{ij} \pm B^{ij} &= C^{ij} , \\ A_{ij} \pm B_{ij} &= C_{ij} , \\ A_j^i \pm B_j^i &= C_j^i . \end{aligned} \quad (1.14)$$

Theorem 1.1 *The sum (or difference) of two tensors which have the same number of covariant of contravariant and covariant rank is again a tensor of the same rank*

and type of the added (or subtracted) tensors

Proof. Consider two tensors of the same rank $A_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}$ and $B_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}$. Then,

according to the law of transformation (1.13):

$$\overline{A}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = \frac{\partial \overline{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{k_2}} \dots \frac{\partial \overline{x}^{i_m}}{\partial x^{k_m}} \frac{\partial x^{l_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{l_2}}{\partial \overline{x}^{j_2}} \dots \frac{\partial x^{l_n}}{\partial \overline{x}^{j_n}} A_{l_1 l_2 \dots l_n}^{k_1 k_2 \dots k_m},$$

and

$$\overline{B}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = \frac{\partial \overline{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{k_2}} \dots \frac{\partial \overline{x}^{i_m}}{\partial x^{k_m}} \frac{\partial x^{l_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{l_2}}{\partial \overline{x}^{j_2}} \dots \frac{\partial x^{l_n}}{\partial \overline{x}^{j_n}} B_{l_1 l_2 \dots l_n}^{k_1 k_2 \dots k_m}.$$

Then

$$\overline{A}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} \pm \overline{B}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = \frac{\partial \overline{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{k_2}} \dots \frac{\partial \overline{x}^{i_m}}{\partial x^{k_m}} \frac{\partial x^{l_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{l_2}}{\partial \overline{x}^{j_2}} \dots \frac{\partial x^{l_n}}{\partial \overline{x}^{j_n}} (A_{l_1 l_2 \dots l_n}^{k_1 k_2 \dots k_m} \pm B_{l_1 l_2 \dots l_n}^{k_1 k_2 \dots k_m}).$$

If we define

$$C_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = A_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} \pm B_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m},$$

then we have

$$\overline{C}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = \frac{\partial \overline{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{k_2}} \dots \frac{\partial \overline{x}^{i_m}}{\partial x^{k_m}} \frac{\partial x^{l_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{l_2}}{\partial \overline{x}^{j_2}} \dots \frac{\partial x^{l_n}}{\partial \overline{x}^{j_n}} C_{l_1 l_2 \dots l_n}^{k_1 k_2 \dots k_m}.$$

Hence our new tensor satisfies the transformation property of a tensor of rank $m + n$,

given by equation (1.13). ■

1.3.3 Tensor multiplication

The multiplication of tensors will result in a tensor whose rank will be the sum of the ranks of the multiplies tensors, as the following theorem shows.

Theorem 1.2 *The result of multiplication of two tensors is a tensor whose rank is the sum of the ranks of the two tensors to be multiplied.*

Proof. Consider two tensors, $A_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}$ which is a tensor of rank $m + n$, and $B_{k_1 k_2 \dots k_t}^{l_1 l_2 \dots l_s}$ which is a tensor of rank $s + t$. Then, according to the law of transformation (1.13):

$$\overline{A}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \cdots \frac{\partial \overline{x}^{i_m}}{\partial x^{p_m}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{q_n}}{\partial \overline{x}^{j_n}} A_{q_1 q_2 \dots q_n}^{p_1 p_2 \dots p_m},$$

and

$$\overline{B}_{k_1 k_2 \dots k_t}^{l_1 l_2 \dots l_s} = \frac{\partial \overline{x}^{l_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \overline{x}^{l_s}}{\partial x^{\alpha_s}} \frac{\partial x^{k_1}}{\partial \overline{x}^{\beta_1}} \cdots \frac{\partial x^{k_t}}{\partial \overline{x}^{\beta_t}} B_{\beta_1 \beta_2 \dots \beta_t}^{\alpha_1 \alpha_2 \dots \alpha_s}.$$

Their product will be

$$\begin{aligned} \overline{A}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} \overline{B}_{k_1 k_2 \dots k_t}^{l_1 l_2 \dots l_s} &= \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \cdots \frac{\partial \overline{x}^{i_m}}{\partial x^{p_m}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{q_n}}{\partial \overline{x}^{j_n}} \frac{\partial \overline{x}^{l_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \overline{x}^{l_s}}{\partial x^{\alpha_s}} \frac{\partial x^{k_1}}{\partial \overline{x}^{\beta_1}} \\ &\quad \cdots \frac{\partial x^{k_t}}{\partial \overline{x}^{\beta_t}} A_{q_1 q_2 \dots q_n}^{p_1 p_2 \dots p_m} B_{\beta_1 \beta_2 \dots \beta_t}^{\alpha_1 \alpha_2 \dots \alpha_s}. \end{aligned}$$

Now, if we define

$$\overline{C}_{j_1 j_2 \dots j_n \quad k_1 k_2 \dots k_t}^{i_1 i_2 \dots i_m \quad l_1 l_2 \dots l_s} = \overline{A}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m} \overline{B}_{k_1 k_2 \dots k_t}^{l_1 l_2 \dots l_s},$$

then we have

$$\begin{aligned} \overline{C}_{j_1 j_2 \dots j_n \quad k_1 k_2 \dots k_t}^{i_1 i_2 \dots i_m \quad l_1 l_2 \dots l_s} &= \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \cdots \frac{\partial \overline{x}^{i_m}}{\partial x^{p_m}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{q_n}}{\partial \overline{x}^{j_n}} \frac{\partial \overline{x}^{l_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \overline{x}^{l_s}}{\partial x^{\alpha_s}} \frac{\partial x^{\beta_1}}{\partial \overline{x}^{k_1}} \\ &\quad \cdots \frac{\partial x^{\beta_t}}{\partial \overline{x}^{k_t}} C_{q_1 q_2 \dots q_n \quad \beta_1 \beta_2 \dots \beta_t}^{p_1 p_2 \dots p_m \quad \alpha_1 \alpha_2 \dots \alpha_s}. \end{aligned}$$

This is the transformation law of a tensor of rank $(m + s) + (n + t)$ or of type $(m + s, n + t)$. ■

Example : We want to show that the product of two tensors A^{ij} and B_k is a tensor of rank 3. By the law of transformation of contravariant tensor of rank 2 (1.1):

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial \bar{x}^j}{\partial x^\beta} A^{\alpha\beta}$$

and the transformation law for a covariant tensor of rank 1:

$$\bar{B}_k = \frac{\partial x^\gamma}{\partial \bar{x}^k} B_\gamma$$

Multiplying the above two tensors we get:

$$\bar{A}^{ij} \bar{B}_k = \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial \bar{x}^j}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \bar{x}^k} A^{\alpha\beta} B_\gamma$$

This is the law of transformation of a tensor of rank 3. So, $A^{ij} B_k$ is indeed a tensor of rank 3.

1.3.4 Tensor contraction

To contract a tensor, we set one covariant index equal to a contravariant index, and then sum over the repeated index. This process leads to a tensor of rank two less than the rank of the tensor that we contracted.

Example Let T_k^{ij} be a tensor of rank 3 (in 3-D space), we contract this tensor in

the first and third indices to get a new tensor of rank 1:

$$T_i^{ij} = T_1^{1j} + T_2^{2j} + T_3^{3j} = A^j .$$

Theorem 1.3 *The contraction of a tensor of rank $m+n$ is a tensor of rank $m+n-2$*

Proof. For simplicity, we show this for a tensor of rank 2+1, the general case follows the same procedure. Let T_k^{ij} be a tensor of rank 2+1, this tensor obeys the transformation law:

$$\bar{T}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial \bar{x}^j}{\partial x^t} \frac{\partial x^s}{\partial \bar{x}^k} T_s^{lt} .$$

Contracting the first and third indices we get

$$\begin{aligned} T_i^{ij} &= \left(\frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial x^s}{\partial \bar{x}^i} \right) \frac{\partial \bar{x}^j}{\partial x^t} T_s^{lt} \\ &= \delta_l^s \frac{\partial \bar{x}^j}{\partial x^t} T_s^{lt} \\ &= \frac{\partial \bar{x}^j}{\partial x^t} T_l^{lt} \\ &= \frac{\partial \bar{x}^j}{\partial x^t} T^t . \end{aligned} \tag{1.15}$$

Where δ_l^s is the Kronecker delta function, defined as:

$$\frac{\partial x^s}{\partial x^l} = \delta_l^s = \begin{cases} 1, & \text{if } s = l, \\ 0, & \text{if } s \neq l. \end{cases}$$

Eq. (1.15) is the transformation rule for a contravariant tensor of rank 1 (1.1).

Hence, we showed that a contraction of rank 3 tensor is a tensor of rank $3 - 2 = 1$

tensor. The proof for the general case follows exactly the same procedure. I

1.3.5 The inner product of a tensor

Consider the tensors A^{ij} and B_{ls}^k , if we first take their outer product, we get a $2 + 3$ tensor $A^{ij}B_{ls}^k$, if we then contract a contravariant index with a covariant index of this product tensor, for example by setting $i = l$, we get the rank 3 tensor $A^{ij}B_{is}^k = C_s^{jk}$, this tensor is called the inner product of the given two tensors.

In summary, the inner product is formed by taking the product of two tensors then contracting the result.

Example Let V_k^{ij} be a mixed tensor of rank 3, and U_t^s be a mixed tensor of rank 2, we show that their inner product $V_s^{ij}U_t^s$ is a mixed tensor of rank 3.

V_k^{ij} Transforms as

$$\bar{V}_k^{ij} = \frac{\partial \bar{x}^i}{\partial x^q} \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^w}{\partial \bar{x}^k} V_w^{qp},$$

and U_t^s transforms as

$$\bar{U}_t^s = \frac{\partial \bar{x}^s}{\partial x^\alpha} \frac{\partial x^\beta}{\partial \bar{x}^t} U_\beta^\alpha.$$

Multiplying the above two tensors, and setting $k = s$ we get

$$\begin{aligned} \bar{V}_s^{ij} \bar{U}_t^s &= \frac{\partial \bar{x}^i}{\partial x^q} \frac{\partial \bar{x}^j}{\partial x^p} \left(\frac{\partial x^w}{\partial \bar{x}^s} \frac{\partial \bar{x}^s}{\partial x^\alpha} \right) \frac{\partial x^\beta}{\partial \bar{x}^t} V_w^{qp} U_\beta^\alpha \\ &= \frac{\partial \bar{x}^i}{\partial x^q} \frac{\partial \bar{x}^j}{\partial x^p} (\delta_\alpha^w) \frac{\partial x^\beta}{\partial \bar{x}^t} V_w^{qp} U_\beta^\alpha \\ &= \frac{\partial \bar{x}^i}{\partial x^q} \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^\beta}{\partial \bar{x}^t} V_\alpha^{qp} U_\beta^\alpha, \end{aligned}$$

which is the transformation law of a tensor of type (2,1).

1.3.6 The metric tensor

The metric tensor, also called the metric, plays a most central part in Riemannian geometry as well as general Relativity. In $\mathbb{R}^3$, the distance between two points (x, y, z) and $(x + dx, y + dy, z + dz)$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2.$$

In his revolutionary work on differential geometry, Bernhard Riemann generalized this definition to any n -dimensional space. For x^i and $x^i + dx^i$ two points in n -dimensional space X , the distance ds is given by the quadratic differential form

$$\begin{aligned} ds^2 = & g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + \cdots + g_{1n}dx^1dx^n \\ & + g_{21}dx^2dx^1 + g_{22}dx^2dx^2 + \cdots + g_{2n}dx^2dx^n \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad + \\ & + g_{n1}dx^ndx^1 + g_{n2}dx^ndx^2 + \cdots + g_{nn}dx^ndx^n. \end{aligned}$$

Or, using the Einstein summation convention

$$ds^2 = g_{ij}dx^i dx^j, \tag{1.16}$$

where the matrix elements $g_{ij} = g_{ij}(x^k)$, and the determinant of this matrix, denoted by g is non-zero

$$g = |g_{ij}| \neq 0.$$

The g_{ij} term in (1.16) defines a symmetric, non-degenerate, covariant tensor of rank two. Since this tensor relates the coordinate differentials $dx^i dx^j$ in various coordinates to the distance (squared) ds^2 , which is an invariant quantity over all coordinates, g_{ij} is usually called the *metric* or *fundamental tensor*.

All the geometric and causal structures of spacetime used to define physical notions such as distance, time, curvature, angle, volume, are captured in the metric.

Key properties of the metric tensor are:

(a) Raising and lowering indices

$$g_{ij}T^j = T_i ,$$

$$g^{ij}T_j = T^i ,$$

$$g_{ij}T_{lt}^{ik} = T_{jlt}^k .$$

(b) g_{ab} is a symmetric tensor ($g_{ab} = g_{ba}$), with n^2 components reduced to $\frac{n(n+1)}{2}$ components due to symmetry .

(c) $g^{ab} = \frac{1}{g_{ab}}$ and $g^{ab}g_{ab} = \delta_a^a = n$ in $\mathbb{R}^n$.

Examples

1- The Euclidean metric in polar coordinates : $(x^1, x^2) = (r, \theta)$

$$ds^2 = dr^2 + r^2 d\theta^2$$

2- The Euclidean metric in spherical coordinates : $(x^1, x^2, x^3) = (r, \theta, \phi)$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2.$$

2- The Minkowski metric in spherical coordinates : $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin(\theta)^2 d\phi^2.$$

1.3.7 Covariant differentiation and the Christoffel symbol

The idea of the covariant derivatives comes from the differentiation of the basis vectors.

In the case that the basis vectors are Cartesian, the differentiation of such basis is zero, thus reducing the covariant derivative to ordinary derivative. In general, the derivative of the general basis vector may not be zero. This extra contribution from differentiation of the basis vector is what distinguishes the covariant derivative and the partial derivatives. Consider a vector $A = (A^1, A^2, A^3)$ in $\mathbb{R}^3$, A can be written as

$$A = A^1 e_1 + A^2 e_2 + A^3 e_3,$$

where e_1, e_2 , and e_3 are orthonormal basis vectors. Taking the partial derivative of A w.r.t x^j , where $j = 1, 2, 3$ gives

$$\begin{aligned} \frac{\partial A}{\partial x^j} &= \frac{\partial(A^1 e_1 + A^2 e_2 + A^3 e_3)}{\partial x^j} \\ &= \frac{\partial(A^i e_i)}{\partial x^j} \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial A^i}{\partial x^j} e_i + A^i \frac{\partial e_i}{\partial x^j} \\
&= \frac{\partial A^i}{\partial x^j} e_i + A^i (\Gamma_{ij}^k e_k),
\end{aligned} \tag{1.17}$$

where Γ_{ij}^k are called the Christoffel symbols (of second kind), and they represent the weighting coefficients for the change in basis vectors.

The Christoffel symbols are defined as

$$\Gamma_{ij}^k e_k = \frac{\partial e_i}{\partial x^j}. \tag{1.18}$$

Using Christoffel symbols, we have a way of differentiating a vector or any higher rank tensor that takes into account the change of the basis vectors used to expand the tensor in a given coordinate system.

1- Covariant Derivative of a Contravariant Vector

Using the same analysis as in (1.17), we have

$$\begin{aligned}
\frac{\partial A}{\partial x^j} &= \frac{\partial A^i}{\partial x^j} e_i + A^i (\Gamma_{ij}^k e_k) \\
&= \left(\frac{\partial A^i}{\partial x^j} + A^k \Gamma_{kj}^i \right) e_i.
\end{aligned}$$

The covariant derivative is defined as the quantity inside the parentheses.

Common notation for the covariant derivative is to use a semicolon (;) in front of the index which the covariant derivative is to be taken. Thus, covariant derivative of a

contravariant vector defined by:

$$A^i{}_{;j} = \frac{\partial A^i}{\partial x^j} + A^k \Gamma_{kj}^i. \quad (1.19)$$

2- Covariant Derivative of a Contravariant Vector

A similar analysis for expanding the vector A in covariant coefficients leads to the covariant derivative of a covariant vector

$$A_{i;j} = \frac{\partial A_i}{\partial x^j} - A_k \Gamma_{ij}^k. \quad (1.20)$$

3- Covariant Derivative of Higher Order Tensors

The process of covariant differentiation can be also applied to higher order tensors, this simply requires the addition of a Christoffel symbol for each contravariant index and the subtraction of a Christoffel symbol for each covariant index.

$$T^{ij}{}_{;k} = \frac{\partial T^{ij}}{\partial x^k} + T^{lj} \Gamma_{lk}^i + T^{il} \Gamma_{lk}^j,$$

$$T_{ij;k} = \frac{\partial T_{ij}}{\partial x^k} - T_{lj} \Gamma_{ik}^l - T_{il} \Gamma_{jk}^l,$$

$$T^i{}_{j;k} = \frac{\partial T^i{}_{j}}{\partial x^k} + T_j^l \Gamma_{lk}^i - T_l^i \Gamma_{jk}^l.$$

And, for a tensor of type (m, n) we have

$$\begin{aligned} T_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}{}_{;k} &= \frac{\partial T_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m}}{\partial x^k} + T_{j_1 j_2 \dots j_n}^{\alpha i_2 \dots i_m} \Gamma_{\alpha k}^{i_1} + T_{j_1 j_2 \dots j_n}^{i_1 \alpha \dots i_m} \Gamma_{\alpha k}^{i_2} \dots + T_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots \alpha} \Gamma_{\alpha k}^{i_m} \\ &\quad - T_{\alpha j_2 \dots j_n}^{i_1 i_2 \dots i_m} \Gamma_{j_1 k}^{\alpha} - T_{j_1 \alpha \dots j_n}^{i_1 i_2 \dots i_m} \Gamma_{j_2 k}^{\alpha} - T_{j_1 j_2 \dots \alpha}^{i_1 i_2 \dots i_m} \Gamma_{j_n k}^{\alpha} . \end{aligned}$$

Common derivative notation

There are multiple notations for partial derivatives in the, such as:

$$\frac{\partial A^j}{\partial x^k} = \partial_{x^k} A^j = A^j_{,k} .$$

Common notation for the covariant derivative:

$$A^j{}_{;k} = \nabla_k A^j .$$

It is common in the context of tensor calculus to use the comma notation for partial derivatives $\frac{\partial A^j}{\partial x^k} = A^j_{,k}$, and semi-colon for covariant derivatives $A^j{}_{;k}$, henceforth we will stick to these notations throughout the rest of this thesis.

Properties of the Christoffel symbols

1- The Christoffel symbols are connected to the metric by the following important equation

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (g_{li,j} + g_{jl,i} - g_{ij,k}) . \quad (1.21)$$

2- Symmetry in the lower indices

$$\Gamma_{jk}^i = \Gamma_{kj}^i .$$

3- All Γ_{jk}^i vanish if all g_{ij} are constants.

In many cases in applications, we have a diagonal metric, and in this case we have a simpler way of calculating the Christoffel symbols:

4- Christoffel symbols for a diagonal metric

$$\Gamma_{ab}^a = \Gamma_{ba}^a = \frac{1}{2} g^{aa} g_{aa,b} ,$$

$$\Gamma_{bb}^a = -\frac{1}{2} g^{aa} g_{bb,a} ,$$

$$\Gamma_{bc}^a = 0 \quad \text{if } a \neq b \neq c .$$

5- In the special case when an upper and a lower index match

$$\Gamma_{ba}^a = (\log \sqrt{|g|})_{,b} , \tag{1.22}$$

where $g = |g_{ab}|$ is the determinant of the metric tensor.

Properties of the covariant derivative

1- The covariant derivative of a tensor of type (m, n) is a tensor of type $(m, n + 1)$

2- The covariant derivative of the metric is zero:

$$g_{ij;k} = 0 .$$

3- The covariant derivative distributes over summation of tensors

$$(A + B)_{;k} = A_{;k} + B_{;k} .$$

4- The covariant derivative obeys the product rule for derivatives

$$(AB)_{;k} = (A_{;k})B + A(B_{;k}) .$$

1.3.8 The divergence of a tensor

Given a covariant vector V^a , the divergence is defined by

$$V^a_{;a} = V^a_{,a} + V^k \Gamma^a_{ka} ,$$

which by (1.22) can be written as

$$V^a_{;a} = V^a_{,a} + V^k (\log \sqrt{|g|})_{,k} .$$

Following the same procedure, we can define the divergence of a rank two tensor as

$$\begin{aligned} T^a_{;b} &= T^a_{,b} + T^{kb} \Gamma^a_{kb} + T^{ak} (\log \sqrt{|g|})_{,k} , \\ T_{ab;b} &= T_{ab,b} - T_{kb} \Gamma^k_{ab} - T_{ak} \Gamma^k_{bb} \\ T^a_{b;a} &= T^a_{b,b} + T^k_b (\log \sqrt{|g|})_{,k} - T^a_k \Gamma^k_{ba} . \end{aligned} \tag{1.23}$$

1.4 The Curvature Tensors

The Riemann curvature tensor is used to express the curvature of Riemannian manifolds. It assigns a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of Euclidean space. In a coordinate basis, the Riemann curvature tensor is determined by transporting a covariant vector T_a along two different directions and then taking the difference. This transportation can be expressed as a covariant derivative. Therefore, starting with a covariant vector T_a and transporting the vector along x^b , we get $T_{a;b}$, we transport the above vector along the direction of x^c , to get

$$T_{a;b;c}.$$

Now we transport the vector T_a again, but this time along x^c then x^b , and we take the difference $T_{a;b;c} - T_{a;c;b}$.

To calculate $T_{a;b;c}$, we take the covariant derivative of T_a first w.r.t. x^b

$$T_{a;b} = T_{a,b} - T_k \Gamma_{ab}^k, \quad (1.24)$$

we call this result T_{ab} . Now we take the covariant derivative of T_{ab} w.r.t. x^c

$$T_{ab;c} = T_{ab,c} - T_{kb} \Gamma_{ac}^k - T_{ak} \Gamma_{bc}^k. \quad (1.25)$$

Using the expression from (1.24) into (1.25) we get

$$T_{a;b;c} = T_{a,bc} - T_k \Gamma_{ab,c}^k - T_{k,c} \Gamma_{ab}^k - (T_{k,b} - T_s \Gamma_{kb}^s) \Gamma_{ac}^k - (T_{a,k} - T_s \Gamma_{ak}^s) \Gamma_{bc}^k. \quad (1.26)$$

Similarly, we can find $T_{a;c;b}$

$$T_{a;b;c} = T_{a,cb} - T_k \Gamma_{ac,b}^k - T_{k,b} \Gamma_{ac}^k - (T_{k,c} - T_s \Gamma_{kc}^s) \Gamma_{ab}^k - (T_{a,k} - T_s \Gamma_{ak}^s) \Gamma_{cb}^k. \quad (1.27)$$

Now, we subtract (1.27) from (1.26), and canceling common terms, we get

$$\begin{aligned} T_{a;c;b} - T_{a;b;c} &= -T_k \Gamma_{ab,c}^k + T_k \Gamma_{ac,b}^k + T_s \Gamma_{kb}^s \Gamma_{ac}^k - T_s \Gamma_{kc}^s \Gamma_{ab}^k \\ &= (\Gamma_{ac,b}^s - \Gamma_{ab,c}^s + \Gamma_{kb}^s \Gamma_{ac}^k - \Gamma_{kc}^s \Gamma_{ab}^k) T_s. \end{aligned}$$

The Riemann curvature tensor is defined to be the term between the parentheses in the above equation

$$R_{abc}^s = \Gamma_{ac,b}^s - \Gamma_{ab,c}^s + \Gamma_{ac}^k \Gamma_{kb}^s - \Gamma_{ab}^k \Gamma_{kc}^s. \quad (1.28)$$

The Riemann tensor measures the curvature of the space; $R_{bcd}^a \equiv 0$ if and only if the space is flat.

In order to study the symmetry properties of the Riemann tensor, we define the fully covariant form

$$R_{abcd} = \frac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac}) - g^{ef}(\Gamma_{eac} \Gamma_{fbd} - \Gamma_{ead} \Gamma_{fbc}). \quad (1.29)$$

From above we notice that the Riemann tensor satisfies, but not limited to, the following properties:

First skew symmetry	$R_{abcd} = -R_{bacd}$
Second skew symmetry	$R_{abcd} = -R_{abdc}$
Second Bianchi identity	$R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} = 0$

1.4.1 Ricci and scalar curvature

The Ricci tensor is the unique (up to sign) non-trivial contraction of the Riemann tensor

$$R_{ab} = \Gamma^s_{ab,s} - \Gamma^s_{as,b} + \Gamma^k_{ab}\Gamma^s_{sk} - \Gamma^k_{as}\Gamma^s_{kb}, \quad (1.30)$$

The following formula is useful in calculations

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^d_{ac}\Gamma^c_{bd} - \left(\frac{\partial}{\partial x^a}(\log \sqrt{|g|})\right)_{;b}. \quad (1.31)$$

here $|g|$ is the determinant of the metric tensor g_{ab} .

Note: depending on the definition of Ricci tensor (either contracting the first and third indices or the first and forth indices of the Riemann tensor) we may introduce a negative sign to the EFEs.

The Ricci scalar or scalar tensor is the contraction of the Ricci tensor

$$R = g^{ab}R_{ab} = R^a_a = R^1_1 + R^2_2 + \dots R^n_n.$$

1.4.2 The Einstein tensor

Starting from the second Bianchi identity

$$R_{bcd;e}^a + R_{bec;d}^a + R_{bde;c}^a = 0,$$

contract a with c

$$R_{bad;e}^a + R_{bea;d}^a + R_{bde;a}^a = 0.$$

Now, note that $R_{bad;e}^a = R_{bd;e}$ and $R_{bea;d}^a = -R_{bae;d}^a = R_{be;d}$ by the second skew symmetry and the definition of the Ricci tensor. Thus we have

$$R_{bd;e} - R_{be;d} + R_{bde;a}^a = 0. \quad (1.32)$$

Now multiply (1.32) by g^{bd} to get

$$g^{bd}(R_{bd;e} - R_{be;d} + R_{bde;a}^a) = 0,$$

which raises the index d to a contravariant index

$$R_{d;e}^d - R_{e;d}^d + R_{de;a}^{ad} = 0.$$

Noting that R_d^d is the Ricci scalar, we get

$$R_{;e} - R_{e;a}^a - R_{e;a}^a = 0.$$

Moreover, noting that $R_{e;a} = \delta_e^a R_{;a}$, we can factor out the covariant derivative to get

$$(2R_e^a - \delta_e^a R)_{;a} = 0.$$

Now, multiply the quantity between the parenthesis by g_{ab} to get

$$g_{ab}(2R_e^a - \delta_e^a R)_{;a} = 0,$$

since $g_{ab;e} = 0$ we can move it inside the parentheses, hence

$$(2g_{ab}R_e^a - g_{ab}\delta_e^a R)_{;a} = 0.$$

Using the lowering property of the metric tensor, we get

$$(2R_{be} - g_{eb}R)_{;a} = 0.$$

Finally, we relabel the indices, so we have

$$(R_{ab} - \frac{1}{2}g_{ab}R)_{;e} = 0. \tag{1.33}$$

If we define the symmetric tensor of type (0,2)

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \tag{1.34}$$

we see that (1.33) is equivalent to

$$G_{ab;e} = 0. \tag{1.35}$$

Hence, the tensor G_{ab} is a divergence free, symmetric tensor of type (0,2), which is constructed from the metric and its first and second derivatives, as well as the Riemann tensor. This tensor is known as the Einstein tensor, and it is of central importance in general Relativity.

CHAPTER 2

THE EINSTEIN FIELD EQUATIONS

2.1 The Einstein field equations

Einstein's theory of gravity, or general relativity, is the modern theory of gravity. To formulate the Einstein field equations, we use the methods of Riemannian geometry and tensor calculus from the previous chapter.

The fundamental idea behind general relativity is that gravity is a consequence of curvature of spacetime. Mass and energy affects how spacetime curves and in turn the curvature of spacetime influences the movement of mass and energy.

Starting from the idea that

$$\text{gravity} \Leftrightarrow \text{curvature in spacetime},$$

the "derivation" of the EFEs is relatively easy.

One of the basic assumptions of physics is that spacetime is homogeneous, i.e. no point in space or time is special, so the same basic laws of physics should govern all of space at any moment of time. In a homogeneous spacetime, the laws of physics are invariant under translation in time and space, hence, it follows from Noether's theorem that both energy and momentum are conserved [22].

This conservation law is expressed in mathematical terms by saying that the covariant divergence of the stress-energy tensor vanishes

$$T_{ab;b} = 0, \tag{2.1}$$

where the tensor T_{ab} is called the *energy-momentum tensor* or the *stress-energy tensor*, and it is a divergence-free, symmetric tensor of type (0,2).

Now, we need to describe the curvature of spacetime mathematically. Using the ideas of Riemannian geometry, we know that the fundamental object in this geometry is the metric. We need to construct a tensorial object from the metric that is divergence free, symmetric tensor of type (0,2) that is function of the metric and at most its second derivative. It turns out that there is a unique tensor that satisfies these criteria, which is the Einstein tensor (1.34)

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R.$$

This interplay between curvature in space-time and energy-momentum is represented

by the famous Einstein field equations [3]

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{8\pi G}{c^2}T_{ab}, \quad (2.2)$$

where G is the gravitational constant and c is the speed of light in space.

The EFEs (2.2) are a set of 16 coupled, non-linear partial differential equations for the metric and its first and second derivatives, due to symmetry in the metric tensor are reduced to 10 [3]. On the left hand side, the Einstein tensor, which describes curvature of space-time, and on the right hand side, the energy-momentum tensor, which represents the matter content of the universe.

The Einstein field equations carry a lot of information that cannot be overemphasized. For instance, in 1915, Albert Einstein used his equations to correctly calculate the anomalous precession of Mercury [1], a riddle that confused physicists since Newton, as well as the calculating the deflection of light rays by the sun's gravitational field. Mathematicians and cosmologists used these equations to model and understand the expansion and dynamics of the universe. Moreover, in their weak field limit (where we have a weak gravitational source), one can derive Newton's inverse square law for gravitation from the EFEs [3]. Other important predictions and application of these equations include gravitational waves, tidal forces and GPS.

2.2 The Einstein-Hilbert action

2.2.1 Einstein's equations in vacuum

General theory of relativity is a field theory, hence we can derive the governing equations using variational principle starting from a suitable Lagrangian. The advantage of deriving the field equations using the variational approach is that it makes explicit the connection between symmetries and conservation laws: any symmetry of a theory corresponds to a conservation law.

In 1915, the famous mathematician David Hilbert proposed the action that yields the Einstein field equations, known as the Einstein-Hilbert action [3]

$$I^{EH} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R, \quad (2.3)$$

where g is the determinant of the metric tensor g_{ab} and R is the Ricci scalar. This action describes gravitational fields in the absence of matter.

We shall now derive the Einstein field equations in vacuum ($T_{ab} = 0$) by varying the action (2.3) with respect to the metric tensor g_{ab} .

Varying (2.3) and requiring that $\delta I^{EH} = 0$

$$\delta I^{EH} = \frac{c^3}{16\pi G} \int d^4x \delta(\sqrt{-g} R).$$

Using the product rule, we get

$$\delta I^{EH} = \frac{c^3}{16\pi G} \int d^4x (\delta\sqrt{-g} R + \sqrt{-g} \delta R) . \quad (2.4)$$

So, in order to find δI^{EH} we need to calculate $\delta\sqrt{-g}$, and δR as $g_{ab} \rightarrow g_{ab} + \delta g_{ab}$
(to first order terms in δg_{ab})

1- Evaluation of $\delta\sqrt{-g}$

Since $g = g(g_{ab})$, by the chain rule

$$\delta g = \frac{\partial g}{\partial g_{ab}} \delta g_{ab} . \quad (2.5)$$

It can be established that [3]

$$\frac{\partial g}{\partial g_{ab}} = g g^{ab} . \quad (2.6)$$

Thus (2.5) becomes

$$\delta g = g g^{ab} \delta g_{ab} . \quad (2.7)$$

Furthermore, by the properties of the metric tensor

$$g_{ab} g^{ab} = \delta_a^a = n, \quad 0 \leq a \leq n-1$$

so that

$$\delta(g_{ab} g^{ab}) = \delta n = 0 .$$

However, by the product rule

$$\delta(g_{ab}g^{ab}) = \delta g_{ab}g^{ab} + g_{ab}\delta g^{ab} ,$$

hence we get

$$g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab} . \quad (2.8)$$

Substituting by (2.8) into (2.7), we get

$$\delta g = -g g_{ab}\delta g^{ab} . \quad (2.9)$$

Finally, by the chain rule

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g \\ &= -\frac{1}{2}\frac{1}{\sqrt{-g}}(-g g_{ab}\delta g^{ab}) \\ &= -\frac{1}{2}\sqrt{-g} g_{ab} \delta g^{ab} . \end{aligned} \quad (2.10)$$

2- Evaluation of δR

To calculate δR , we start from the definition of the Ricci scalar

$$R = g^{ab}R_{ab} ,$$

thus, by the chain Rule

$$\delta R = \delta g^{ab}R_{ab} + g^{ab}\delta R_{ab} .$$

For the second term we use the *Platini Identity* [3]:

$$\delta R_{ab} = (\delta \Gamma_{ab}^c)_{;c} - (\Gamma_{ac}^c)_{;b}, \quad (2.11)$$

so that

$$\begin{aligned} g^{ab} \delta R_{ab} &= g^{ab} [(\delta \Gamma_{ab}^c)_{;c} - (\Gamma_{ac}^c)_{;b}] = (g^{ab} \delta \Gamma_{ab}^c)_{;c} - (g^{ab} \Gamma_{ac}^c)_{;b} \\ &= (g^{ab} \delta \Gamma_{ab}^c - g^{ae} \delta \Gamma_{ac}^c)_{;e}, \end{aligned} \quad (2.12)$$

where we used the property $g_{ab;c} = 0$ in the last equation.

Now, the term $(g^{ab} \delta \Gamma_{ab}^c - g^{ae} \delta \Gamma_{ac}^c)_{;e}$ is a covariant divergence of a vector; it can be shown using *Gauss's theorem* that this term vanishes when integrated over the 4-volume $\sqrt{-g} d^4x$. Consequently, we have

$$\delta R = \delta g^{ab} R_{ab} + \text{surface terms}. \quad (2.13)$$

Now, using (2.10) and (2.13) in (2.4),

$$\begin{aligned} \delta I^{EH} &= \frac{c^3}{16\pi G} \int d^4x (\delta \sqrt{-g} R + \sqrt{-g} \delta R) \\ &= \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} (R_{ab} - \frac{1}{2} g_{ab} R) \delta g^{ab}. \end{aligned} \quad (2.14)$$

By imposing

$$\delta I^{EH} = 0, \quad \forall \delta g^{ab}$$

we get the Einstein field equations in vacuum

$$R_{ab} - \frac{1}{2}g_{ab}R = 0.$$

We can further simplify these by contracting the above equation by g^{ab} to get

$$g^{ab}(R_{ab} - \frac{1}{2}g_{ab}R) = R_a^a - \frac{1}{2}\delta_a^a R = 0.$$

Since $\delta_a^a = \delta_0^0 + \dots + \delta_3^3 = 4$ we get

$$R - 2R = -R = 0. \tag{2.15}$$

Thus, in vacuum $R = 0$. Therefore, the Einstein equations in vacuum can be written as

$$R_{ab} = 0. \tag{2.16}$$

2.2.2 Einstein's equation with source

In the presence of a source of a gravitational field such as a matter field or electromagnetic field, we modify the action (2.3) to accommodate for this source, such that

$$I^{total} = I^{EH} + I^{fields}$$

The corresponding equations can be found by varying the total action with respect to the metric tensor $\mathbf{g}$

$$\delta I^{total} = \delta I^{EH} + \delta I^{fields} = 0, \quad \forall \delta g^{ab}$$

We calculated the first term in the previous section to be

$$\delta I^{EH} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} (R_{ab} - \frac{1}{2} g_{ab} R) \delta g^{ab}. \quad (2.17)$$

It can be shown that the variation of the matter field is given by [3]:

$$\delta I^{fields} = \int \sqrt{-g} \left[\frac{\partial \mathcal{L}}{\partial g^{ab}} - \frac{1}{2} \mathcal{L} g_{ab} \right] \delta g^{ab}, \quad (2.18)$$

where $\mathcal{L}$ is the Lagrangian density.

If we define the stress-energy tensor to be

$$T_{ab} = -2c \left[\frac{\partial \mathcal{L}}{\partial g^{ab}} - \frac{1}{2} \mathcal{L} g_{ab} \right], \quad (2.19)$$

then the variation of the total action can be written as

$$\delta I^{total} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[R_{ab} - \frac{1}{2} g_{ab} R - \frac{8\pi G}{c^4} T_{ab} \right] \delta g^{ab} = 0, \quad (2.20)$$

from which the Einstein field equations

$$R_{ab} - \frac{1}{2}g_{ab} R = \frac{8\pi G}{c^4}T_{ab}$$

immediately follow.

CHAPTER 3

SOME EXACT SOLUTIONS OF THE EINSTEIN EQUATIONS

3.1 The Schwarzschild solution

Is it possible to solve the Einstein field equations for any metric g_{ab} ? If one examines how g_{ab} enters Ricci and Einstein tensors, one could readily see the high degree of non-linearity possessed by these equations. Thus, any solution of the Einstein field equations will not be easy to obtain. However, this problem becomes simpler if one assumes that the spacetime admits certain symmetries. Karl Schwarzschild obtained the first exact solution of the Einstein field equation in 1916. The Schwarzschild solution is obtained under the assumptions that:

- 1- Spacetime is spherically symmetric: invariant under rotations.
- 2- Spacetime is static: the metric components are independent of time.
- 3- Spacetime is empty: we are solving the field equations with $T_{ab} = 0$.
- 4- The signature of the metric is $(+,-,-,-)$.

5- Spacetime is asymptotically flat: it approaches the Minkowski metric of special relativity in the limit $r \rightarrow \infty$.

To derive the Schwarzschild solution, we start with a general metric

$$ds^2 = g_{ab} dx^a dx^b, \quad 0 \leq a, b \leq 3$$

Writing the metric tensor g_{ab} explicitly

$$g_{ab} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}.$$

The metric is a symmetric tensor, that is $g_{ab} = g_{ba}$

$$g_{ab} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}.$$

Since the metric is spherically symmetric, it is convenient to write it in spherical coordinates

$$(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi).$$

And it is unchanged under θ and ϕ reversal ($\theta \rightarrow -\theta, \phi \rightarrow -\phi$), hence

$$g_{a2} = -g_{a2} \Rightarrow g_{a2} = 0, \quad a \neq 2$$

$$g_{a3} = -g_{a3} \Rightarrow g_{a3} = 0, \quad a \neq 3$$

Thus

$$g_{ab} = \begin{pmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{01} & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}.$$

Next, we impose that the metric is static and symmetric under time reversal

a. static metric $\frac{\partial g_{ab}}{\partial t} = 0$, components of g_{ab} are time-independent.

b. symmetric under time reversal ($t \rightarrow -t$)

$$g_{a0} = -g_{a0} \Rightarrow g_{a0} = 0, \quad a \neq 0$$

Thus

$$g_{ab} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}.$$

Now, the sought metric has the form

$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 .$$

From the fact that spherical surfaces with constant t and r have line element

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

Now, for constant t, θ and ϕ (i.e. on the radial line) the metric should only depend on the radial coordinate r ; to preserve the spherical symmetry, which means that the metric components for time and radius, g_{11} and g_{22} , must be functions of r only.

This simplifies the metric even further to:

$$ds^2 = U(r)dt^2 + V(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 .$$

Since we want the metric to be invariant, and asymptotically reduce to the Minkowski metric, we write it as

$$ds^2 = U(r)dt^2 - V(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2 . \quad (3.1)$$

Now, we want to solve the Einstein field equations in vacuum

$$R_{ab} - \frac{1}{2}g_{ab} R = 0$$

for the metric (3.1), written in diagonal form as

$$g_{ab} = \text{diag}(U(r), -V(r), -r^2, -r^2 \sin^2(\theta)) .$$

Since it is a diagonal matrix, its inverse

$$g^{ab} = \frac{1}{g_{ab}} = \text{diag}\left(\frac{1}{U(r)}, \frac{-1}{V(r)}, \frac{-1}{r^2}, \frac{-1}{r^2 \sin^2(\theta)}\right) .$$

To find the values of the Ricci tensor R_{ab} , we first need to calculate the Christoffel symbols for our metric: The non-zero Christoffel symbols are given by (primes denote $\frac{\partial}{\partial r}$):

$$\begin{aligned} \Gamma_{10}^0 &= \Gamma_{01}^0 = \frac{1}{2} g^{00} g_{00,1} = \frac{U'}{2U} , \\ \Gamma_{00}^1 &= -\frac{1}{2} g^{11} g_{00,1} = \frac{U'}{2V} , \\ \Gamma_{11}^1 &= \frac{1}{2} g^{11} g_{11,1} = \frac{V'}{2V} , \\ \Gamma_{22}^1 &= -\frac{1}{2} g^{11} g_{22,1} = -\frac{r}{V} , \\ \Gamma_{33}^1 &= -\frac{1}{2} g^{11} g_{33,1} = -\frac{r}{V} \sin^2(\theta) , \\ \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{2} g^{22} g_{22,1} = \frac{1}{r} , \\ \Gamma_{33}^2 &= -\frac{1}{2} g^{22} g_{33,2} = -\cos(\theta) \sin(\theta) , \\ \Gamma_{31}^3 &= \Gamma_{13}^3 = \frac{1}{2} g^{33} g_{33,1} = \frac{1}{r} , \\ \Gamma_{32}^3 &= \Gamma_{23}^3 = \frac{1}{2} g^{33} g_{33,2} = \frac{\cos(\theta)}{\sin(\theta)} . \end{aligned}$$

From which we calculate the Ricci tensor (1.8)

$$R_{ab} = \Gamma_{ab,s}^s - \Gamma_{as,b}^s + \Gamma_{ab}^k \Gamma_{sk}^s - \Gamma_{as}^k \Gamma_{kb}^s.$$

The non-vanishing Ricci tensors turn out to be

$$R_{00} = -\frac{U''}{2V} + \frac{U'V'}{4V^2} + \frac{(U')^2}{4UV} - \frac{U'}{rV},$$

$$R_{11} = \frac{U''}{2U} - \frac{(U')^2}{4U^2} - \frac{U'V'}{4UV} - \frac{V'}{Vr},$$

$$R_{22} = \frac{rU'}{2UV} + \frac{1}{V} - \frac{rV'}{2V^2} - 1,$$

$$R_{33} = \sin^2(\theta)R_{22}.$$

Thus, the Ricci scalar R is the contraction of the Ricci tensor

$$\begin{aligned} R &= g^{ab}R_{ab} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \\ &= \frac{1}{U}R_{00} - \frac{1}{V}R_{11} - \frac{1}{r^2}R_{22} - \frac{1}{r^2 \sin^2(\theta)} \sin^2(\theta)R_{22} \\ &= \frac{1}{U}R_{00} - \frac{1}{V}R_{11} - \frac{2}{r^2}R_{22} \\ &= -\frac{U''}{UV} + \frac{U'V'}{2UV^2} + \frac{(U')^2}{2U^2V} - \frac{2U'}{rUV} + \frac{2V'}{rV^2} + \frac{2}{r^2}\left(1 - \frac{1}{V}\right). \end{aligned}$$

Now that we have calculated the values of the Ricci tensors and scalar for our metric

(3.1), we can substitute them into the vacuum equations

$$R_{ab} - \frac{1}{2}g_{ab} R = 0. \quad 0 \leq a, b \leq 3 \quad (3.2)$$

First we note that since both R_{ab} and g_{ab} vanish for $a \neq b$, then the only surviving equations are the diagonal equations, that is when $a = b$. And since the $(a, b) = (3, 3)$ equation is a multiple of the $(2, 2)$ equation, they are linearly dependent, hence we only need one of them.

Thus, after simplification, the vacuum equations (3.2) for the components

$(a, b) = \{(0, 0), (1, 1), (2, 2)\}$ are given by

$$\frac{V'}{rV^2} + \frac{1}{r^2}\left(1 - \frac{1}{V}\right) = 0, \quad (3.3)$$

$$-\frac{U'}{rUV} + \frac{1}{r^2}\left(1 - \frac{1}{V}\right) = 0, \quad (3.4)$$

$$-\frac{U'}{U} + \frac{V'}{V} - \frac{rU''}{U} + \frac{rU'V'}{2UV} + \frac{r(U')^2}{2U^2} = 0. \quad (3.5)$$

Now, we want to find the value of $V(r)$ and $U(r)$.

From (3.3)

$$-\frac{dr}{r} = \frac{dV}{V(V-1)},$$

integrating both sides

$$-\ln r + C = \ln \frac{(V-1)}{V},$$

where C is an integration constant. Simplifying the above expression, we get

$$\frac{C}{r} = \frac{V - 1}{V},$$

from which we can solve for

$$V(r) = \frac{1}{1 - \frac{C}{r}}. \quad (3.6)$$

Substituting by the value of $V(r)$ into (3.4)

$$\frac{U'}{U} \left(1 - \frac{C}{r}\right) - \frac{C}{r^2} = 0,$$

which can be written in an integrable form

$$\frac{dU}{U} = \frac{C}{r^2 - Cr} dr,$$

integrating both sides, we have

$$\ln U = \ln \frac{r - C}{r},$$

which we can solve for

$$U(r) = 1 - \frac{C}{r}. \quad (3.7)$$

Substituting by the values of U and V into (3.1), and Identifying $C = \frac{2GM}{c^2}$, the

Schwarzschild metric becomes

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(\frac{1}{1 - \frac{2GM}{c^2 r}}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2. \quad (3.8)$$

The Schwarzschild metric (3.8) describes the geometry around a point mass. The equation shows that there are two singular points, the first at $r = 0$, and the second at $r = \frac{c^2}{2GM}$. The former is called an essential singularity, and physically, it represents the center of a black hole. Whereas at the second singularity $r_s = 2GM/c^2$ there is a discontinuity in spacetime. This radius r_s known as the *Schwarzschild radius*, and the spherical shell defined by this radius is called the event horizon. The Schwarzschild radius divides the black hole into two connected regions. The region within the black hole $0 < r < r_s$, and the region outside the black hole $r_s < r < \infty$.

However, by carefully studying at the line element (3.8), it turns out that $r = \frac{2GM}{c^2}$ is not a singularity of the spacetime, but rather a *coordinate singularity*, which is not a real physical singularity but rather an artifact that appears due to our choice of coordinates. One method used to determine whether we have a coordinate singularity for a given metric is to calculate the *curvature invariants* for that metric. Curvature invariants are a set of scalars formed from the Riemann and Ricci tensors, such as R , $R_{ab}R^{ab}$ and $R_{abcd}R^{abcd}$. Since these are scalars, they are not affected by our choice of coordinate, therefore coordinate singularities do not appear on them.

For the Schwarzschild metric (3.8), these invariants turn out to be [3]

$$R = 0, \quad (3.9)$$

$$R_{ab}R^{ab} = 0, \quad (3.10)$$

$$R_{abcd}R^{abcd} = \frac{48G^2M^2}{c^4r^6}. \quad (3.11)$$

Since neither of the invariants (3.9), (3.10) and (3.11) has $r_s = \frac{2GM}{c^2}$ as a singularity, it is indeed a coordinate singularity.

Figure 3.1 shows a sketch of Schwarzschild and its singularities.

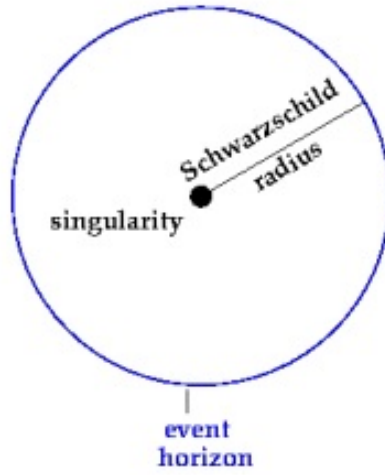


Figure 3.1: Schwarzschild blackhole

3.2 The Friedmann-Robertson-Walker cosmological models

The Friedmann-Robertson-Walker (FRW) metric is defined as [3]

$$ds^2 = dt^2 - a(t)^2 \left(\frac{1}{1 - kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right). \quad (3.12)$$

We can see that the FRW metric (3.12) describes a homogeneous universe because of the spherical symmetry of the metric, and it describes an isotropic universe, since it does not have cross terms between time and space variables, there is no privileged direction.

The factor $a(t)$ is called the *scale factor* [3], and it tells us how the expansion of the universe depends on time.

The number $k \in \{-1, 0, 1\}$, is known as the *curvature constant*. It takes the value $k = -1$ for negatively curved space, $k = +1$ for positively curved space, and $k = 0$ for flat space.

We want to solve the Einstein equations taking into account the effect of *dark energy*, a relativistic energy density that carries negative pressure and is responsible for the accelerated expansion of the universe, and it is expressed in the form of cosmological constant Λ [23]. In this case, the EFEs become [3]

$$R_{ab} - \frac{1}{2}g_{ab}R - \Lambda g_{ab} = 8\pi G T_{ab} , \quad (3.13)$$

for the FRW metric (3.12) and a stress-energy tensor [3]

$$T_{ab} = (\rho + p)u_a u_b - p g_{ab} . \quad (3.14)$$

The tensor (3.14) represents a perfect fluid model with energy density ρ and matter density p , where $u_a = (u_0, u_1, u_2, u_3) = (1, 0, 0, 0)$ is the macroscopic speed of the fluid.

From the line element (3.12), we get the metric tensor

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a(t)^2 \frac{1}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & -a(t)^2 r^2 \sin^2(\theta) \end{pmatrix} \quad (3.15)$$

$$= \text{diag}(1, \frac{-a(t)^2}{1-kr^2}, -a(t)^2 r^2, -a(t)^2 r^2 \sin^2(\theta)),$$

where *diag* means the diagonal elements of the matrix.

Since our metric tensor is represented by a diagonal matrix, its inverse $g^{ab} = \frac{1}{g_{ab}}$

$$g^{ab} = \text{diag}(1, \frac{kr^2 - 1}{a(t)^2}, \frac{-1}{a(t)^2 r^2}, \frac{-1}{a(t)^2 r^2 \sin^2(\theta)}). \quad (3.16)$$

Since the metric tensor is diagonal, the determinant g can be found by multiplying the elements of the diagonal

$$g = |g_{ab}| = \frac{-a^6 r^4 \sin^2 \theta}{1 - kr^2}. \quad (3.17)$$

By substituting the values of g_{ab} into (3.14), we find the non-vanishing components of the stress-energy tensor to be

$$T_{00} = (\rho + p)u_0^2 - pg_{00} = \rho,$$

$$T_{11} = (\rho + p)u_1^2 - pg_{11} = \frac{a(t)^2}{1 - kr^2} p,$$

$$T_{22} = (\rho + p)u_1^2 - pg_{11} = a(t)^2 r^2 p,$$

$$T_{33} = (\rho + p)u_1^2 - pg_{11} = a(t)^2 r^2 \sin^2(\theta) p.$$

And the non-vanishing components of the fully contravariant form $T^{ab} = g^{ac} g^{bd} T_{cd}$ are

$$T^{00} = (g^{00})^2 T_{00} = \rho,$$

$$T^{11} = (g^{11})^2 T_{11} = \frac{1 - kr^2}{a^2} p,$$

$$T^{22} = (g^{22})^2 T_{22} = \frac{1}{a^2 r^2} p,$$

$$T^{33} = (g^{33})^2 T_{33} = \frac{1}{a^2 r^2 \sin^2 \theta} p.$$

Writing T_{ab} and T^{ab} in a more compact form,

$$T_{ab} = \text{diag}(\rho, \frac{a(t)^2}{1 - kr^2} p, a(t)^2 r^2 p, a(t)^2 r^2 \sin^2(\theta) p), \quad (3.18)$$

and

$$T^{ab} = \text{diag}(\rho, \frac{1 - kr^2}{a^2} p, \frac{1}{a^2 r^2} p, \frac{1}{a^2 r^2 \sin^2 \theta} p). \quad (3.19)$$

To find the values of the Ricci tensor R_{ab} , we first need to calculate the Christoffel symbols for our metric (3.15).

The non-zero Christoffel symbols are given by

$$\begin{aligned}
\Gamma_{11}^0 &= -\frac{1}{2}g^{00}g_{11,0} = \frac{a\dot{a}}{1-kr^2}, \\
\Gamma_{22}^0 &= -\frac{1}{2}g^{00}g_{22,0} = a\dot{a}r^2, \\
\Gamma_{33}^0 &= -\frac{1}{2}g^{00}g_{33,0} = a\dot{a}r^2\sin^2(\theta), \\
\Gamma_{01}^1 &= \Gamma_{10}^1 = \frac{1}{2}g^{11}g_{11,0} = \frac{\dot{a}}{a}, \\
\Gamma_{11}^1 &= \frac{1}{2}g^{11}g_{11,1} = \frac{kr}{1-kr^2}, \\
\Gamma_{22}^1 &= -\frac{1}{2}g^{11}g_{22,1} = \frac{\dot{a}}{a}, \\
\Gamma_{33}^1 &= -\frac{1}{2}g^{11}g_{33,1} = \frac{\dot{a}}{a}, \\
\Gamma_{02}^2 &= \Gamma_{20}^2 = \frac{1}{2}g^{22}g_{22,0} = \frac{\dot{a}}{a}, \\
\Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{2}g^{22}g_{22,1} = \frac{1}{r}, \\
\Gamma_{33}^2 &= -\frac{1}{2}g^{22}g_{33,2} = -\sin\theta\cos\theta, \\
\Gamma_{03}^3 &= \Gamma_{30}^3 = \frac{1}{2}g^{33}g_{33,0} = \frac{\dot{a}}{a}, \\
\Gamma_{13}^3 &= \Gamma_{31}^3 = \frac{1}{2}g^{33}g_{33,1} = \frac{1}{r}, \\
\Gamma_{23}^3 &= \Gamma_{32}^3 = \frac{1}{2}g^{33}g_{33,2} = \cot\theta,
\end{aligned}$$

from which we calculate the Ricci tensor (1.8)

$$R_{ab} = \Gamma_{ab,s}^s - \Gamma_{as,b}^s + \Gamma_{ab}^k \Gamma_{sk}^s - \Gamma_{as}^k \Gamma_{kb}^s,$$

we find the non-vanishing Ricci tensors to be

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a}, \\
R_{11} &= \frac{2k + 2\dot{a}^2 + a\ddot{a}}{1 - kr^2}, \\
R_{22} &= \ddot{a}ar^2 + 2\dot{a}^2r^2 + 2kr^2, \\
R_{33} &= \sin^2\theta R_{22}.
\end{aligned}$$

The Ricci scalar R is the contraction of the Ricci tensor

$$\begin{aligned}
R &= g^{ab}R_{ab} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \\
&= -6\left(\frac{\ddot{a}a + \dot{a}^2 + k}{a^2}\right).
\end{aligned}$$

After calculating R_{ab} and R , we can write the set of EFEs.

For $(a,b) = \{(0,0),(1,1),(2,2)\}$,

$$R_{ab} - \frac{1}{2}g_{ab}R - \Lambda g_{ab} = 8\pi GT_{ab},$$

gives the following set of equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}, \quad (3.20)$$

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi Gp + \frac{\Lambda}{2} - \frac{1}{2}\frac{k}{a^2}, \quad (3.21)$$

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi Gp + \frac{\Lambda}{2} - \frac{1}{2}\frac{k}{a^2}. \quad (3.22)$$

Since (3.22) is that same as (3.21), hence we only have two independent equations.

Finally, we have two independent equations (3.20) and (3.21) with four unknowns:

a, ρ, p and k . Therefore, we need two more equations to be able to solve our system of equations.

We can get one more equation from the conservation of energy-momentum $T_{;b}^{ab} = 0$.

It follows from (1.23) that the divergence of the stress-energy tensor

$$T_{;b}^{ab} = T_{,b}^{ab} + T^{kb}\Gamma_{kb}^a + T^{ak}(\log \sqrt{-g})_{,k} = 0.$$

For $x^b = x^0 = t$ the formula reads

$$T_{;0}^{a0} = T_{,0}^{a0} + T^{k0}\Gamma_{k0}^a + T^{ak}(\log \sqrt{-g})_{,k} = 0.$$

Since T^{ab} is diagonal, the only surviving terms will be

$$T_{,0}^{00} + T^{nn}\Gamma_{nn}^0 + T^{00}(\log \sqrt{-g})_{,0} = 0, \quad (3.23)$$

where we employed the Einstein summation convention on the second term.

The first part in this equation is

$$T_{,0}^{00} = \frac{\partial}{\partial t}\rho = \dot{\rho}.$$

As for the second part

$$\begin{aligned}
T^{nn}\Gamma_{nn}^0 &= T^{00}\Gamma_{00}^0 + T^{11}\Gamma_{11}^0 + T^{22}\Gamma_{22}^0 + T^{33}\Gamma_{33}^0 \\
&= \frac{(1-kr^2)p}{a^2} \frac{a\dot{a}}{1-kr^2} + \frac{p}{a^2 r^2} a\dot{a}r^2 + \frac{p}{a^2 r^2 \sin^2 \theta} a\dot{a}r^2 \sin^2(\theta) \\
&= 3\frac{\dot{a}}{a}p.
\end{aligned}$$

Moreover

$$\begin{aligned}
T^{00}(\log \sqrt{-g})_{,0} &= \rho \frac{\partial}{\partial t} \left(\log \sqrt{\frac{a^6 r^4 \sin^2 \theta}{1-kr^2}} \right) \\
&= \rho \frac{\partial}{\partial t} \left(3 \log a + 2 \log r + \log \sin \theta - \frac{1}{2} \log(1-kr^2) \right) \\
&= 3\frac{\dot{a}}{a}\rho.
\end{aligned}$$

Consequently (3.23) becomes

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0. \quad (3.24)$$

This equation is known as *the fluid equation*.

Equations (3.20), (3.21) and (3.24), called the *Friedman equations*, which describe the evolution and expansion (or contraction) of the universe, under the assumption that the universe is isotropic and homogeneous.

Now, we need to define the behavior of energy density ρ for any given mass/energy component. This behavior is generally expressed through the *equation of state*, which

expresses the relation between pressure and energy density

$$p = \omega \rho, \quad (3.25)$$

where ω is a constant that takes the following values for each given component

- For non-relativistic matter like cosmic dust or planets : $\omega = 0$.
- For radiation: $\omega = \frac{1}{3}$.
- For dark energy: $\omega = -1$.

In reality, the universe contains a mix of each of these components. However, it is known that there are epochs in the evolution of the universe where one of these components was dominant. Having this in mind, solving the Friedman equation for each of these components lets us understand the evolution of the universe through that certain epoch.

Substituting by (3.25) into the fluid equation (3.24)

$$\dot{\rho} + 3(1 + \omega)\rho \frac{\dot{a}}{a} = 0,$$

separating the variables, we get

$$\frac{d\rho}{\rho} = -3(1 + \omega) \frac{da}{a},$$

integrating both sides

$$\rho_{\omega}(a) = \rho_{\omega,0} a^{-3(1+\omega)}, \quad (3.26)$$

where $\rho_{\omega,0}$ is the energy density of a component with equation of state parameter ω at the current time.

We can derive certain information about the evolution of the universe from the evolution of the energy density (3.26).

- For a universe filled with matter only ($\omega = 0$): $\rho \propto a^{-3}$.
- For a universe filled with radiation only ($\omega = 1/3$): $\rho \propto a^{-4}$.
- For a universe filled with dark energy only ($\omega = 2$): $\rho = \text{constant}$.

We can see that radiation decreases most rapidly in time. It follows that going back in time, it must increase the fastest, that is, at its early stage, the universe was dominated by radiation. As we go forward in time, radiation will dilute the fastest as matter becomes the dominant component. Finally, as we continue in time, all matter and radiation will dilute yet the dark energy stays constant, dominating the content of the universe.

Now we are ready to present few solutions to the Friedman equation (3.20) in simple scenarios.

1- Flat, matter dominated universe

In this case, $k = 0$ and $\omega = 0$. Friedmann equation (3.20) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.$$

Substituting by (3.26) into the above equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{m,0} a^{-3},$$

where $\rho_{m,0}$ is the current energy density of matter.

Which we write as

$$\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3} \rho_{m,0} a^{-1}.$$

Taking the square root of both sides, and separating the variables

$$a^{1/2} dt = \pm \sqrt{\frac{8\pi G}{3} \rho_{m,0}} dt.$$

Integrating both sides

$$\frac{2}{3} a^{3/2} = \pm \sqrt{\frac{8\pi G}{3} \rho_{m,0}} t.$$

Therefore, for a flat, matter dominated universe, the scale factor grows as

$$a(t) \propto t^{3/2}.$$

2- Flat, radiation dominated universe

In this case, $k = 0$ and $\omega = 1/3$. Friedmann equation (3.20) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho.$$

Substituting by (3.26)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{r,0} a^{-4},$$

where $\rho_{r,0}$ is the current energy density of radiation.

Taking the square root of both sides, and separating the variables

$$a da = \pm \sqrt{\frac{8\pi G}{3} \rho_{r,0}} dt.$$

Integrating both sides

$$\frac{1}{2}a^2 = \pm \sqrt{\frac{8\pi G}{3} \rho_{r,0}} t.$$

We see that for a flat, matter dominated universe, the scale factor grows as

$$a(t) \propto t^{1/2}.$$

3- Flat, dark energy dominated universe

In this case, $k = 0$ and $\omega = -1$. Friedmann equation (3.20) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho.$$

Substituting by (3.26)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{\Lambda,0},$$

where the constant $\rho_{\Lambda,0}$ is the current dark energy density.

Simplifying the above expression

$$\frac{da}{a} = \pm \sqrt{\frac{8\pi G}{3} \rho_{\Lambda,0}} dt .$$

Integrating both sides

$$\log a = \pm \sqrt{\frac{8\pi G}{3} \rho_{\Lambda,0}} dt .$$

For a flat universe dominated by dark energy, the scale factor grows as

$$a(t) \propto e^{kt} .$$

3.3 Other solutions of physical interest

In this section, we will briefly discuss other important solutions to EFE without going into the details of the derivations.

3.3.1 The Reissner-Nordstrom solution

Reissner-Nordstrom metric represents one of the most important exact solutions of the coupled Einstein equations with a stress-energy-momentum tensor due to a point charge. This Reissner-Nordstrom line element describes the geometry of the spacetime surrounding a static charged object admitting spherical symmetry, and is given by [3]

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2(\theta)d\phi^2, \quad (3.27)$$

where m is the mass and Q is the electric charge of the resulting black hole.

The Reissner-Nordstrom metric features two roots of singularity, namely, it is singular at

$$r^2 - 2mr + Q^2 = 0,$$

implying that

$$r_{1,2} = m \pm \sqrt{m^2 - Q^2}.$$

This suggests that we have three possibilities for $m^2 - Q^2 < 0$, $m^2 - Q^2 = 0$ or $m^2 - Q^2 > 0$. At $m = Q$, the concentric event horizons corresponds to an extremal black hole. The concept of an extremal black hole is theoretical, and none have thus

far been observed in nature. However, many theories are based on their existence.

Black holes with $m < Q$ are believed not to exist in nature because they would contain a naked singularity, one that is not hidden behind an event horizon. Their appearance would contradict Roger Penrose's cosmic censorship hypothesis which is generally believed to be true [3].

Charged black holes with $m > Q$ are similar to Schwarzschild black holes, they have two horizons: the event horizon and an internal horizon known as Cauchy horizon.

Figure 3.2 shows a sketch of Reissner-Nordström and its singularities



Figure 3.2: Reissner-Nordström blackhole

3.3.2 The Kerr solution

The Schwarzschild solution (3.8) represents a non-rotating black hole, the endpoint of stellar collapse. But from observations, we know that stars and planets rotate around their axes. However, the problem for an uncharged, rotating body of mass M and angular momentum J , turns out to be a much more complicated, and it was unsolved for 48 years, until the solution was found in 1963 by mathematician Roy Kerr [10].

The line element for the Kerr metric is given by [24]

$$\begin{aligned} ds^2 = & - \left(1 - \frac{2mr}{r^2 + a^2 \cos^2 \theta}\right) dt^2 - \frac{4mra \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dt d\phi \\ & + \left(\frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2}\right) dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 \\ & + \left(r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta}\right) \sin^2 \theta d\phi^2. \end{aligned} \quad (3.28)$$

The metric above, is given in the Boyer–Lindquist coordinates, related to Cartesian coordinates by

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \phi,$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin \phi,$$

$$z = r \cos \theta,$$

and

$$a = \frac{J}{M}.$$

Note that in the limit $a \rightarrow 0$ (i.e. when we have no rotation), the Kerr metric (3.28) reduces to the Schwarzschild metric (3.8).

The Kerr metric has singularities either when

$$\begin{aligned} r^2 + a^2 \cos^2 \theta &= 0, \\ \Rightarrow r &= 0 \quad \text{and} \quad \theta = \frac{\pi}{2} \end{aligned} \tag{3.29}$$

or when

$$\begin{aligned} r^2 - 2mr + a^2 &= 0, \\ \Rightarrow r = r_{\pm} &= m \pm \sqrt{m^2 - a^2} \end{aligned} \tag{3.30}$$

To check if these are real or coordinate singularities, we look at the invariant quantity [24]

$$R_{abcd}R^{abcd} = \frac{48m^2(r^2 - a^2 \cos^2 \theta)[(r^2 + a^2 \cos^2 \theta)^2 - 16r^2a^2 \cos^2 \theta]}{(r^2 + a^2 \cos^2 \theta)^6}.$$

We see that the only real curvature singularity is the first case (3.29), at $r = 0$ and $\theta = \frac{\pi}{2}$ since $R_{abcd}R^{abcd} \rightarrow \infty$ there. As for (3.30), it is a coordinate singularity that shows up because of our choice of coordinates. Moreover, in the Schwarzschild limit as $a \rightarrow 0$ we have $r_{\pm} \rightarrow 2m$. This is the location of the horizon in the Schwarzschild metric, we will therefore identify $r_{\pm}$ as the outer and inner horizons. Figure 3.3 shows a sketch of Kerr blackhole and its singularities.

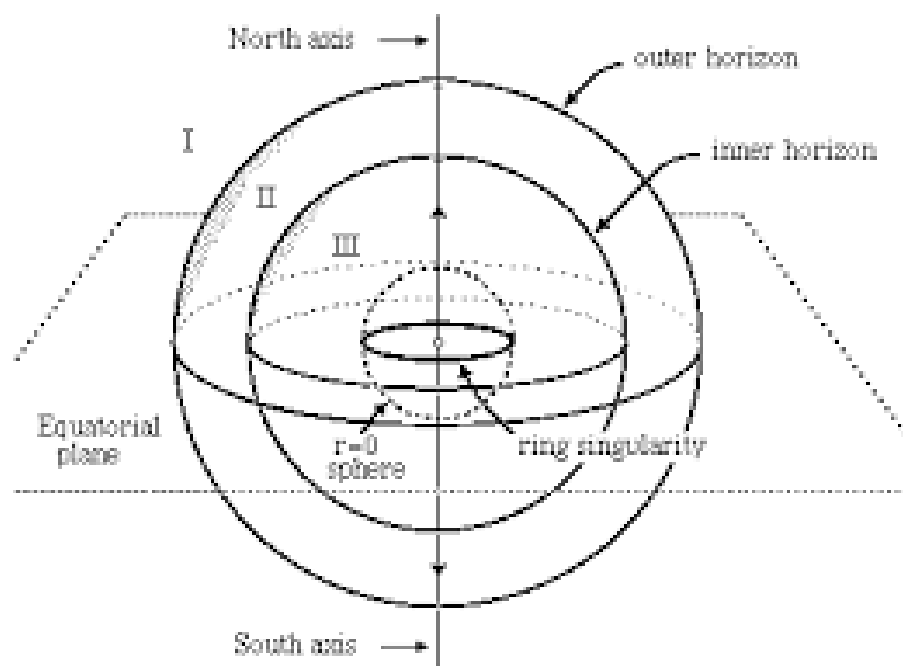


Figure 3.3: Kerr blackhole

CHAPTER 4

BIANCHI TYPE I

COSMOLOGICAL MODEL WITH

TIME-DEPENDENT

GRAVITATIONAL AND

COSMOLOGICAL CONSTANTS

4.1 Introduction

Motivated by the occurrence of the large dimensionless numbers, which are ratios of some physical constants, Dirac proposed a the large number hypothesis, allowing the gravitational constant G to be time-dependent [25]. This suggestion lead researchers to contemplate several arguments, speculations and new ideas in cosmology by allow-

ing G and Λ to be taken as functions of time. This consideration became a topic of great interest, and since then several investigations have been carried out such as referred in [26], [27], [28], [29], [30], [31], [32], [33], [34], [35]. In one of the recent papers, it is argued that the accelerated expansion of the universe is due to dark energy as revealed by type Ia supernovae observations [36], [37], [38]. In their work, Pradhan et al. [38] pointed out that present observations strongly reveal a small and positive value of the effective cosmological constant at the timeline of cosmological epochs and subsequent evolution of the Universe from the Big Bang to the present epoch. The Wilkinson Microwave Anisotropy Probe [39], [40] combined with more accurate SN Ia data [41] indicates that the Universe is almost spatially flat and the dark energy accounts to about 70% of the total content of the Universe. In their paper Chen and Wu [33] show that, in Friedman Robertson Walker (FRW) model, the effective cosmological model $\Lambda(t)$ should vary as a^{-2} (where $a(t)$ is the scale factor). Carvalho et al.[32] generalized their work by taking $\Lambda = \alpha a^{-2} + \beta H^2$ where H is the Hubble parameter and α, β are adjustable constants. Bertolami [42] was the first to consider cosmological model where Λ varies as t^{-2} . Al-Rawaf and Taha [43], Al-Rawaf [44], Arbab [45], [46], Overdin and Cooperstock [47], investigated cosmological models with $\Lambda = \beta \frac{\ddot{a}}{a}$, where β is constant.

Bianchi I cosmological models with variable gravitational and cosmological constants were studied by Beesham [18] showing that there is no internal inflationary solution for purely exponential form. Sistero [19] investigated cosmological models, where the cosmological and gravitational constants are treated as coupling scalars, and $G \propto a^n$. Generalized field equations with time-varying G and Λ were first proposed by Lau

[48] and Lau and Prokhovnick [49] in their attempt to reconcile the general theory of Relativity with the large number hypothesis. A detailed analysis of FRW cosmological models in several scalar-tensor theories of gravity have been carried out by Barrow and Parsons [30], as well as Damour, Gibbons and Taylor [34]; obtaining limits on the variability of G using binary-pulsar data. Matter creation in cosmology and its phenomenology also attracted many researchers lately [50], [32], [51], [52]; these classes of models considered the effect of matter creation by introducing additional pressure (or bulk viscosity) terms in the expression of the stress-energy tensor $T_{ab} = T_{ab}^{(m)} + T_{ab}^{(s)}$ with $T_{ab}^{(m)}$ the stress-energy tensor of matter and $T_{ab}^{(s)}$ describing matter creation, hence the stress-energy tensor of matter is no longer a conserved quantity $T_{a;b}^{(m)} \neq 0$ (the total stress-energy tensor is still a conserved quantity $T_{a;b}^b = 0$), where $a ;$ represents the covariant derivative of the metric. The fact that the divergence of the matter stress-energy tensor $T_{ab}^{(m)}$ is also non-zero in models with time-varying G and Λ leads to the possibility that the matter dominated phase in the evolution of the universe was by large non energy-conserving event caused by a time-varying gravitational and cosmological "constants". DerSarkissian [53] developed in a systematic way this point of view in a cosmological model that is consistent with the principle of absolute dark matter confinement, which suggested a new mechanism for matter creation as irreversible, localized and solitary vacuum fluctuations with finite, maximum hadronic mass density and temperature. Rahman [54] and Beesham [55] re-discussed and corrected the model proposed by DerSarkissian as well as the problem of energy conservation. Ozer and Taha [56] investigated a cosmological model with a cosmological constant but without applying the conservation of the matter stress-energy tensor tentatively to

solve the entropy problem.

Following Dirac in [57], many authors were in favor of $G(t) \propto t^{-1}$, but it lost interest because of the Viking-lander-range data on the orbit of Planet Mars [35], this idea regain interest again after Massa resolved this issue in [35] by means of introducing matter creation process inside the source, and by following Zel'dovich's Λ term and assuming Hubble parameter $H \propto t^{-1}$, he found that $\Lambda \propto t^{-2}$ and $G(t) \propto t^{-1}$ [58]. The possibility that in the early universe entropy generation processes and particle creation were the result of false vacuum decay in the case of variable gravitational and cosmological constants have been studied in detail by Harko and Mark [51], [59]. Recently, Singh [60] and Ullah et al. [61] analyzed LRS Bianchi-V cosmological model with time-varying gravitational and cosmological "constants" in the presence and absence of particle creation, where the particle creation rate is uniquely determined by the variation of cosmological and gravitational constants.

In this chapter, we present our study on the evolution and dynamics of perfect fluid Bianchi I cosmological model with time-varying gravitational and cosmological constants, taking into account the phenomenological description of matter creation by means and as a result of the time-dependence of G and Λ . We obtain exact solutions to the Einstein field equations in the case of particle creation and in the absence of particle creation.

4.2 Einstein Field equations

We consider the Bianchi type I metric which is given by the line element [65],

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2. \quad (4.1)$$

The Einstein field equations with time-dependent gravitational and cosmological constants are [65],

$$R_{ab} - \frac{1}{2}g_{ab}R = -8\pi G(t)T_{ab} + \Lambda(t)g_{ab}, \quad (4.2)$$

where R_{ab} , T_{ab} and R respectively represent Ricci & stress tensor and Ricci scalar.

Here we have taken the speed of light $c = 1$.

By (4.1), the components of the metric tensors are

$$g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & A(t)^2 & 0 & 0 \\ 0 & 0 & B(t)^2 & 0 \\ 0 & 0 & 0 & C(t)^2 \end{pmatrix}, \quad (4.3)$$

$$= \text{diag}(-1, A(t)^2, B(t)^2, C(t)^2).$$

The stress-energy tensor T_{ab} is chosen to represent a perfect fluid, given by [65],

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab}, \quad (4.4)$$

where ρ and p respectively denote energy density and pressure, and $u_a = (u_0, u_1, u_2, u_3) = (-1, 0, 0, 0)$ is the four-velocity vector.

Using (4.3) into (4.4), we find the non-vanishing components of the stress-energy tensors to be

$$T_{00} = (\rho + p)u_0^2 + pg_{00} = \rho ,$$

$$T_{11} = (\rho + p)u_1^2 + pg_{11} = p A(t)^2 ,$$

$$T_{22} = (\rho + p)u_2^2 + pg_{22} = p B(t)^2 ,$$

$$T_{33} = (\rho + p)u_3^2 + pg_{33} = p C(t)^2 .$$

The Christoffel symbols for Bianchi type I metric (4.1) are given by

$$\Gamma_{11}^0 = -\frac{1}{2}g^{00}g_{11,0} = A(t)\dot{A}(t) ,$$

$$\Gamma_{22}^0 = -\frac{1}{2}g^{00}g_{22,0} = B(t)\dot{B}(t) ,$$

$$\Gamma_{33}^0 = -\frac{1}{2}g^{00}g_{33,0} = C(t)\dot{C}(t) ,$$

$$\Gamma_{01}^1 = \Gamma_{10}^1 = \frac{1}{2}g^{11}g_{11,0} = \frac{\dot{A}(t)}{A(t)} ,$$

$$\Gamma_{20}^2 = \Gamma_{02}^2 = \frac{1}{2}g^{22}g_{22,0} = \frac{\dot{B}(t)}{B(t)} ,$$

$$\Gamma_{30}^3 = \Gamma_{03}^3 = \frac{1}{2}g^{33}g_{33,0} = \frac{\dot{C}(t)}{C(t)} .$$

From which we calculate the Ricci tensors (Note: In this chapter, we contract the

first and forth indices of Riemann tensor to form the Ricci tensor, which introduce a negative sign to (1.8))

$$R_{ab} = \Gamma_{as,b}^s - \Gamma_{ab,s}^s + \Gamma_{as}^k \Gamma_{kb}^s - \Gamma_{ab}^k \Gamma_{sk}^s,$$

which turn out to be

$$\begin{aligned} R_{00} &= \frac{\ddot{A}(t)}{A(t)} + \frac{\ddot{B}(t)}{B(t)} + \frac{\ddot{C}(t)}{C(t)}, \\ R_{11} &= -(A(t)\ddot{A}(t) + \frac{A(t)\dot{A}(t)\dot{B}(t)}{B(t)} + \frac{A(t)\dot{A}(t)\dot{C}(t)}{C(t)}), \\ R_{22} &= -(\frac{B(t)\dot{A}(t)\dot{B}(t)}{A(t)} + B(t)\ddot{B}(t) + \frac{B(t)\dot{B}(t)\dot{C}(t)}{C(t)}), \\ R_{33} &= -(\frac{C(t)\dot{A}(t)\dot{C}(t)}{A(t)} + \frac{C(t)\dot{B}(t)\dot{C}(t)}{B(t)} + C(t)\ddot{C}(t)). \end{aligned}$$

Ricci scalar is simply the summation of all Ricci tensors above

$$R = -(\frac{2\ddot{A}(t)}{A(t)} + \frac{2\dot{A}(t)\dot{B}(t)}{A(t)B(t)} + \frac{2\dot{A}(t)\dot{C}(t)}{A(t)C(t)} + \frac{2\ddot{B}(t)}{B(t)} + \frac{2\dot{B}(t)\dot{C}(t)}{B(t)C(t)} + \frac{2\ddot{C}(t)}{C(t)}).$$

Finally, the field equations (4.2) for Bianchi type I metric (4.1), with stress tensor given by (4.4), take the form

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} = 8\pi G(t)\rho + \Lambda(t), \quad (4.5)$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = -8\pi G(t)p + \Lambda(t), \quad (4.6)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{C}}{AC} = -8\pi G(t)p + \Lambda(t), \quad (4.7)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -8\pi G(t)p + \Lambda(t), \quad (4.8)$$

where a single and double dots respectively represent the first and second derivatives with respect to time.

We can get one more important equation from the conservation of the Einstein tensor (1.33). Multiplying (4.2) by g^{ac}

$$g^{ac}R_{ab} - \frac{1}{2}g^{ac}g_{ab}R = -8\pi G(t)g^{ac}T_{ab} + \Lambda(t)g^{ac}g_{ab}, \quad (4.9)$$

Using the raising property of the metric tensor and $g^{ac}g_{ab} = \delta_b^c$, where δ_b^c is Kronecker's delta

$$R_b^c - \frac{1}{2}\delta_b^c R = -8\pi G(t)T_b^c + \Lambda(t)\delta_b^c. \quad (4.10)$$

Since the Einstein tensor is divergence free, it follows that the L.H.S. of (4.10) is also divergence free by $g^{ac}(G_{ab;c}) = (g^{ac}G_{ab})_{;c} = 0$, hence so is the R.H.S. of (4.10), so that

$$(-8\pi G(t)T_b^c + \Lambda(t)\delta_b^c)_{;c} = 0.$$

Distributing the covariant derivative

$$-8\pi G(t)_{;c}T_b^c - 8\pi G(t)T_{b;c}^c + \Lambda(t)_{;c}\delta_b^c = 0,$$

hence

$$T_{b;c}^c = \frac{-1}{8\pi G(t)}(8\pi G(t)_{;c}T_b^c - \Lambda(t)_{;b}). \quad (4.11)$$

For $x^b = x^0 = t$, we have $\Lambda(t)_{,b} = \dot{\Lambda}(t)$ and

$$G(t)_{,c}T_0^c = \sum_{c=0}^3 G(t)_{,c}T_0^c = \dot{G}(t)T_0^0 = -\dot{G}(t)\rho,$$

since $G(t)$ is a function of time only and $T_0^0 = g^{00}T_{00} = -\rho$.

Thus the R.H.S of (4.11) becomes

$$\frac{-1}{8\pi G(t)}(8\pi G(t)_{,c}T_0^c - \Lambda(t)_{,0}) = \left(\frac{\dot{G}(t)}{G(t)}\rho + \frac{\dot{\Lambda}(t)}{8\pi G(t)}\right) \quad (4.12)$$

As for the L.H.S. of (4.11), it follows from (1.23) that

$$T_{0;c}^c = T_{0,c}^c + T_0^k(\log \sqrt{|g|})_{,k} - T_k^c \Gamma_{0c}^k$$

Since all the components of T depends on $x^0 = t$ only, then

$$T_{0,c}^c = T_{0,0}^0 = (g^{00}T_{00})_{,0} = -\dot{\rho}$$

as for

$$\begin{aligned} T_0^k(\log \sqrt{g})_{,k} &= T_0^0(\log \sqrt{|g|})_{,0} = -\rho \frac{\partial}{\partial t}(\log(ABC)) \\ &= -\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{B}}{B}\right)\rho \end{aligned}$$

For the last term, $k = c$ since the stress tensor is diagonal, so that

$$\begin{aligned}
T_k^c \Gamma_{0c}^k &= T_0^0 \Gamma_{00}^0 + T_1^1 \Gamma_{01}^1 + T_2^2 \Gamma_{02}^2 + T_3^3 \Gamma_{03}^3 \\
&= g^{11} T_{11} \frac{\dot{A}}{A} + g^{22} T_{22} \frac{\dot{B}}{B} + g^{33} T_{33} \frac{\dot{C}}{C} \\
&= \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) p
\end{aligned}$$

Hence L.H.S of (4.11) becomes

$$\begin{aligned}
T_{0;c}^c &= T_{0,c}^c + T_0^k (\log \sqrt{|g|})_{,k} - T_k^c \Gamma_{0c}^k \\
&= -\dot{\rho} - (\rho + p) \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right)
\end{aligned} \tag{4.13}$$

By (4.12) and (4.13), the conservation of Einstein tensor leads to

$$\dot{\rho} + (\rho + p) \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = - \left(\frac{\dot{G}(t)}{G(t)} \rho + \frac{\dot{\Lambda}(t)}{8\pi G(t)} \right) \tag{4.14}$$

For the metric given in (4.1), we define the generalized Hubble parameter to be [65]

$$H = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right). \tag{4.15}$$

Using (4.15) into (4.14), we obtain

$$\dot{\rho} + 3H(\rho + p) = - \left(\frac{\dot{G}}{G} \rho + \frac{\dot{\Lambda}}{8\pi G} \right). \tag{4.16}$$

In the early universe, the particle content is formed by a non-interacting co-moving

fluid with number density $n(t)$ and obeys the equation of state given by [61],

$$n = n_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{1}{1+\alpha}}, \quad (4.17)$$

$$p = \alpha \rho, \quad (4.18)$$

where n_0 and ρ_0 are constants and $0 \leq \alpha \leq 1$. Now, from [61], the particle balance equation is given by

$$\dot{n} + 3Hn = \Gamma(t)n, \quad (4.19)$$

where $\Gamma(t)$ is the particle production rate. Dividing both sides by n

$$\frac{\dot{n}}{n} + 3H = \Gamma(t). \quad (4.20)$$

On the other hand, from (4.17)

$$\frac{\dot{n}}{n} = \frac{1}{1+\alpha} \frac{\dot{\rho}}{\rho},$$

and from (4.16)

$$3H = \frac{1}{(1+\alpha)\rho} \left[-\left(\frac{\dot{G}}{G} p + \frac{\dot{\Lambda}}{8\pi G} \right) - \dot{\rho} \right].$$

Using the above two equations into (4.20), we get the equation for particle production rate as

$$\Gamma(t) = -\frac{1}{(1+\alpha)\rho} \left(\frac{\dot{G}}{G} \rho + \frac{\dot{\Lambda}}{8\pi G} \right). \quad (4.21)$$

In the case of particle creation, we have $\Gamma(t) \geq 0$, which in presence of (4.21) requires

that

$$\frac{\dot{G}}{G}\rho + \frac{\dot{\Lambda}}{8\pi G} \leq 0. \quad (4.22)$$

Whereas in the case of no particle creation, $\Gamma(t) = 0$ leads to particle conservation law in cosmology,

$$\dot{\Lambda} + 8\pi\dot{G}\rho = 0, \quad (4.23)$$

and energy-momentum conservation

$$\dot{\rho} + 3H(1 + \alpha)\rho = 0. \quad (4.24)$$

The matter and the dark-energy density parameters given respectively by $\Omega_m = \frac{8\pi G}{3H^2}\rho$, and $\Omega_\Lambda = \frac{\Lambda}{3H^2}$ are useful tools for comparing different cosmological models. Recent measurements show that the values of these parameters are $\Omega_m = 0.279$ and $\Omega_\Lambda = 0.726$ [63].

4.3 Solutions of the Field equations

Subtracting (4.6) From (4.7) we obtain,

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\frac{\dot{C}}{C} = 0. \quad (4.25)$$

Similarly, subtracting (4.6) from (4.8) and subtracting (4.7) from (4.8) respectively, we get

$$\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} + \left(\frac{\dot{A}}{A} - \frac{\dot{C}}{C}\right)\frac{\dot{B}}{B} = 0, \quad (4.26)$$

and

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)\frac{\dot{A}}{A} = 0. \quad (4.27)$$

Form the symmetry of the above three equations, we can exchange between A, B and C , it follows that

$$B = k_1 A, \quad (4.28)$$

and

$$C = k_2 A, \quad (4.29)$$

where k_1 and k_2 are constants.

Substituting (4.28) and (4.29) into (4.5) we get

$$3\frac{\dot{A}^2}{A^2} = 8\pi G\rho + \Lambda. \quad (4.30)$$

Similarly, from (4.6) we find that

$$-2\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} = 8\pi Gp - \Lambda. \quad (4.31)$$

Adding the above two equations

$$\frac{\dot{A}^2}{A^2} - \frac{\ddot{A}}{A} = 4\pi G(1 + \alpha)\rho. \quad (4.32)$$

At this stage, we deal with the problem in two different cases, one by one, (I) Absence of particle creation and (II) Presence of particle creation.

4.3.1 Case (I): Absence of Particle creation

In this case, $\Gamma = 0$. By substituting (4.28) and (4.29) into (4.15), we obtain,

$$H = \frac{\dot{A}}{A}. \quad (4.33)$$

From the property of conservation law (4.24) we get,

$$\dot{\rho} + 3\frac{\dot{A}}{A}(1 + \alpha)\rho = 0. \quad (4.34)$$

Solving above equation in ρ we obtain,

$$\rho = A^{-3(1+\alpha)}, \quad (4.35)$$

For convenience we took the integration constant to be 1. Substituting (4.35) into (4.32) and solving for G we get,

$$G = \frac{A^{1+3\alpha}(-\ddot{A}A + \dot{A}^2)}{4\pi(1 + \alpha)}. \quad (4.36)$$

Finally, using the above two equations into (4.31) gives us

$$\Lambda = \frac{1 + 3\alpha}{1 + \alpha} \frac{\dot{A}^2}{A^2} + \frac{2}{1 + \alpha} \frac{\ddot{A}}{A}. \quad (4.37)$$

Thus we have our dynamical variables B , C , ρ , G and Λ all functions of $A(t)$. We now describe the models resulting from different dynamical laws in two cases using ansatz in $A(t)$.

Case (Ia, Power Law): $A = \beta t^m$

In this case β and m are arbitrary positive constants. From (4.28) and (4.29) we obtain:

$B(t) = k_1 \beta t^m$, and $C(t) = k_2 \beta t^m$, thus we have

$$ds^2 = -dt^2 + \beta^2 t^{2m} (dx^2 + k_1 dy^2 + k_2 dz^2). \quad (4.38)$$

For this metric, the Hubble parameter (4.15)

$$H = \frac{m}{t}, \quad (4.39)$$

while, using (4.35), (4.36), and (4.37) we obtain,

$$\rho = (\beta t^m)^{-3(1+\alpha)}, \quad (4.40)$$

$$G = \frac{m \beta^{3(1+\alpha)}}{4\pi(1+\alpha)} t^{3(1+\alpha)m-2}, \quad (4.41)$$

$$\Lambda = \frac{3(1+\alpha)m^2 - 2m}{1+\alpha} \frac{1}{t^2}, \quad (4.42)$$

Computing the dark-energy and matter density parameters for this model, it is found

that they are given respectively by,

$$\Omega_{\Lambda} = 1 - \frac{2}{3(1+\alpha)m}, \quad (4.43)$$

and

$$\Omega_m = \frac{2}{3(1+\alpha)m}. \quad (4.44)$$

We note that both the energy density Fig.4.1, 4.3 and the cosmological parameter Fig.4.5 starts at infinite value at the big-bang at $t = 0$, and then approaches to small positive values later as $t \rightarrow \infty$. Moreover, it can also be noted that in this case $\Lambda \propto t^{-2}$ which, in time-dependent cosmology, enjoys a broad consensus [62]. To determine the value of the constant m , we find the best fit for the values of Ω_{Λ} and Ω_m given above against those found by WMAP in [63], where $\Omega_{\Lambda} = 0.726$ and $\Omega_m = 0.279$, assuming stiff fluid ($\alpha = 1$) and matter dominated universe ($\alpha = 0$). For the model given by (4.38), we find that the two best fits matching the data are,

α	m	Ω_{Λ}	Ω_m	$(\Omega_{\Lambda})_{WMAP}$	$(\Omega_m)_{WMAP}$
1	1.07	0.724	0.263	0.726	0.279
0	2.4	0.722	0.278		

Table 4.1: Comparing best fit theoretical results with experimental values.

(1): Non particle creation

We note that for these values of m and α , The gravitational constant G increases rapidly with time Fig.4.2, 4.4. This problematic case, as we will see later, can be fixed by assuming particle creation.

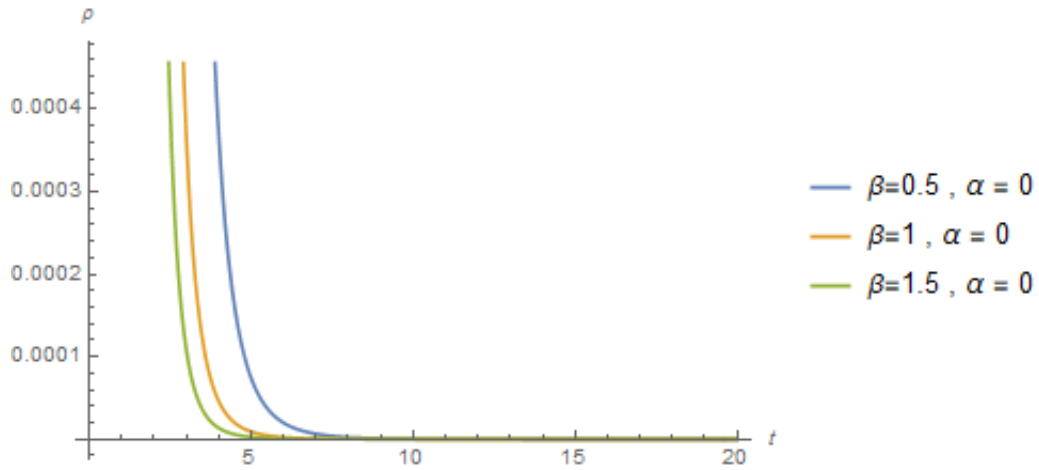


Figure 4.1: The variation of ρ vs. t . ($\alpha = 0$)
Case I: Non particle creation

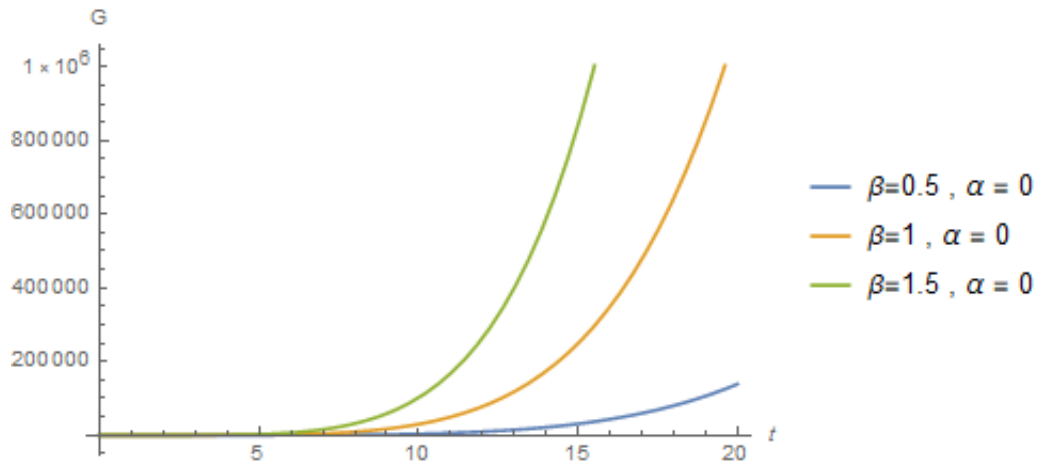


Figure 4.2: The variation of G vs. t . ($\alpha = 0$)
Case I: Non particle creation

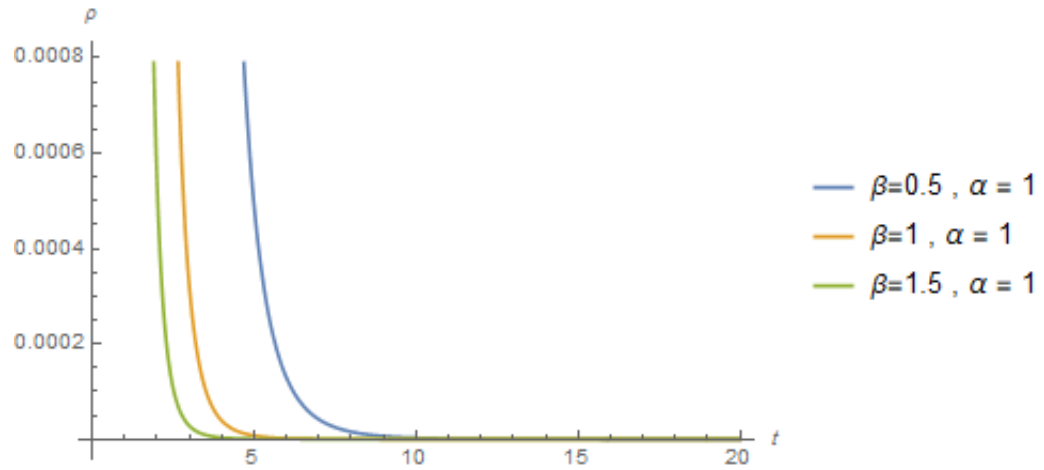


Figure 4.3: The variation of ρ vs. t . ($\alpha = 1$)
Case I: Non particle creation

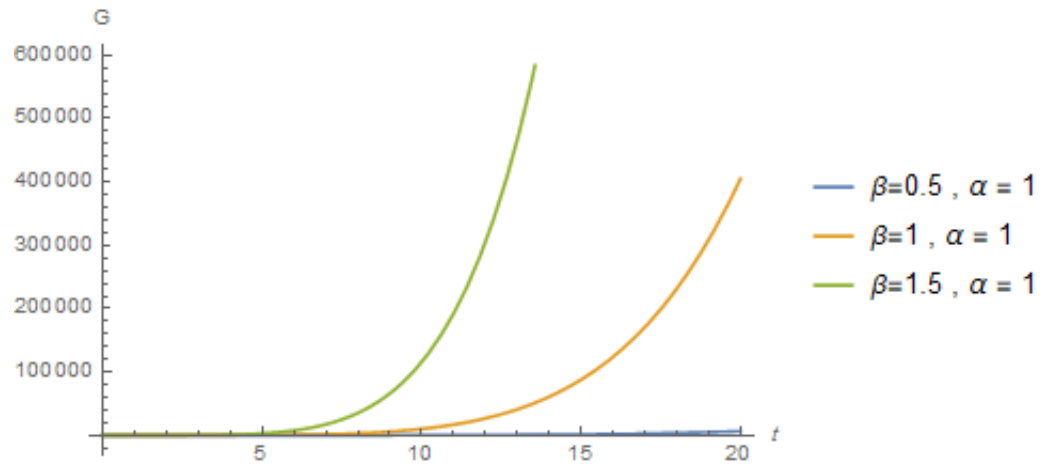


Figure 4.4: The variation of G vs. t . ($\alpha = 1$)
Case I: Non particle creation

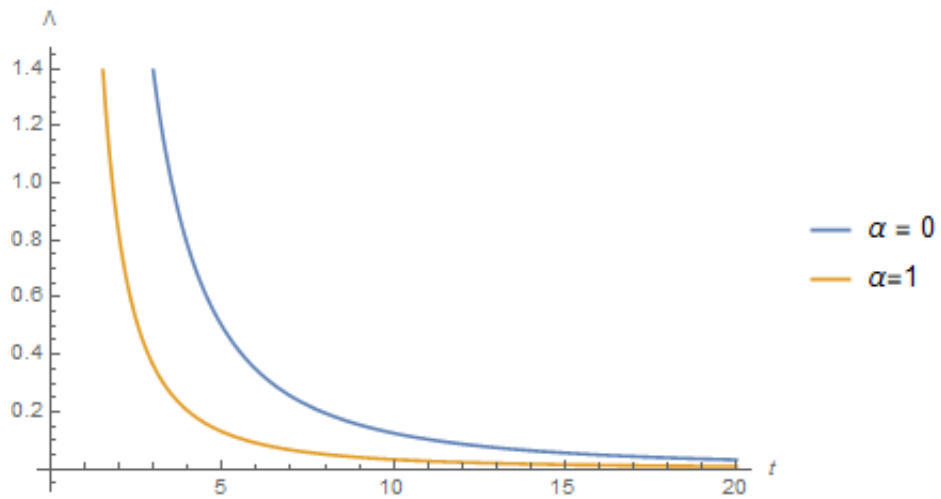


Figure 4.5: The variation of Λ vs. t .
Case I: Non particle creation

Case (Ib, Exponential Law): $A = \beta e^{mt}$

Here β and m are arbitrary positive constants. From (4.28) and (4.29) we obtain,
 $B(t) = k_1 \beta e^{mt}$ and $C(t) = k_2 \beta e^{mt}$ For this model the metric becomes,

$$ds^2 = -dt^2 + \beta^2 e^{2mt} (dx^2 + k_1 dy^2 + k_2 dz^2), \quad (4.45)$$

The Hubble parameter (4.15) is now a constant, as expected for a de-sitter type universe

$$H = m,$$

while from (4.35), (4.36), and (4.37) we get,

$$\rho = c_1 \beta e^{-3m(1+\alpha)t}, \quad (4.46)$$

$$G = 0, \quad (4.47)$$

$$\Lambda = 3m^2. \quad (4.48)$$

The matter and dark-energy density parameters for this model are ,

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = 1,$$

$$\Omega_m = \frac{8\pi G}{3H^2} \rho = 0.$$

In this model, there is no gravitational coupling, as $G = 0$, hence there is no effect of gravitational force. The matter energy density ρ decreases with time eventually reaching zero as $t \rightarrow \infty$, where we have a flat universe that is totally occupied by dark-energy in the form of cosmological constant. This may not be a viable model for the evolution of the universe, but rather for the fate of the universe, where it ends up in a big-freeze scenario.

4.3.2 Case (II): Presence of Particle creation

In this case, $\Gamma(t) \geq 0$. Here we have three independent equations (4.30), (4.32) and (4.21) in five unknowns A , G , ρ , Λ , Γ . For a deterministic solution, we need two more conditions. As ansatzes, we will take A and ρ in two cases, like in the previous part, first, as power law, and second, as exponential law.

Case (IIa, Power Law): $A = bt^m \quad \rho = kt^n$

The Hubble parameter (4.15) for this line element is

$$H = \frac{m}{t}.$$

Using the values of A and ρ in equation (4.32), we obtain

$$G(t) = \frac{mt^{-(2+n)}}{4\pi k(1+\alpha)}. \quad (4.49)$$

Following [57] and [58], we assume that $n = -1$ to get $G(t) \propto t^{-1}$.

Using the above equation with (4.30)

$$\Lambda(t) = \frac{3(1 + \alpha)m^2 - 2m}{(1 + \alpha)t^2}. \quad (4.50)$$

From (4.21), we find the particle production rate

$$\Gamma(t) = \frac{3(1 + \alpha)m - 1}{(1 + \alpha)t}. \quad (4.51)$$

For the matter and dark-energy density parameters,

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = 1 - \frac{2}{3(1 + \alpha)m}, \quad (4.52)$$

$$\Omega_m = \frac{8\pi G}{3H^2}\rho = \frac{2}{3(1 + \alpha)m}. \quad (4.53)$$

Note that $\Omega_\Lambda + \Omega_m = 1$, which is the case of flat universe dominated by matter and dark energy in the form of Λ . To ensure that the particle creation criterion holds ($\Gamma(t) \geq 0$), the value of m must satisfy $m \geq \frac{1}{3}$ when $\alpha = 0$ or $m \geq \frac{1}{6}$ when $\alpha = 1$.

Now, to determine the value of the constant m , we find best fit for Ω_Λ against those found by WMAP in [63] where $\Omega_\Lambda = 0.726$ and $\Omega_m = 0.279$. assuming stiff fluid ($\alpha = 1$) and matter dominated universe ($\alpha = 0$) and for brevity we considered $k = 1$.

We see from Fig.4.6, The particle creation rate decreases as the universe expands, and as a result, the dark-energy Λ and the density ρ decrease as well, approaching small positive values in the present epoch. Moreover, assuming particle creation, we

α	m	Ω_Λ	Ω_m	$(\Omega_\Lambda)_{WMAP}$	$(\Omega_m)_{WMAP}$
0	2.4	0.722	0.278	0.726	0.279
1	1.22	0.727	0.273		

Table 4.2: Comparing best fit theoretical results with experimental values.
(2): Particle creation

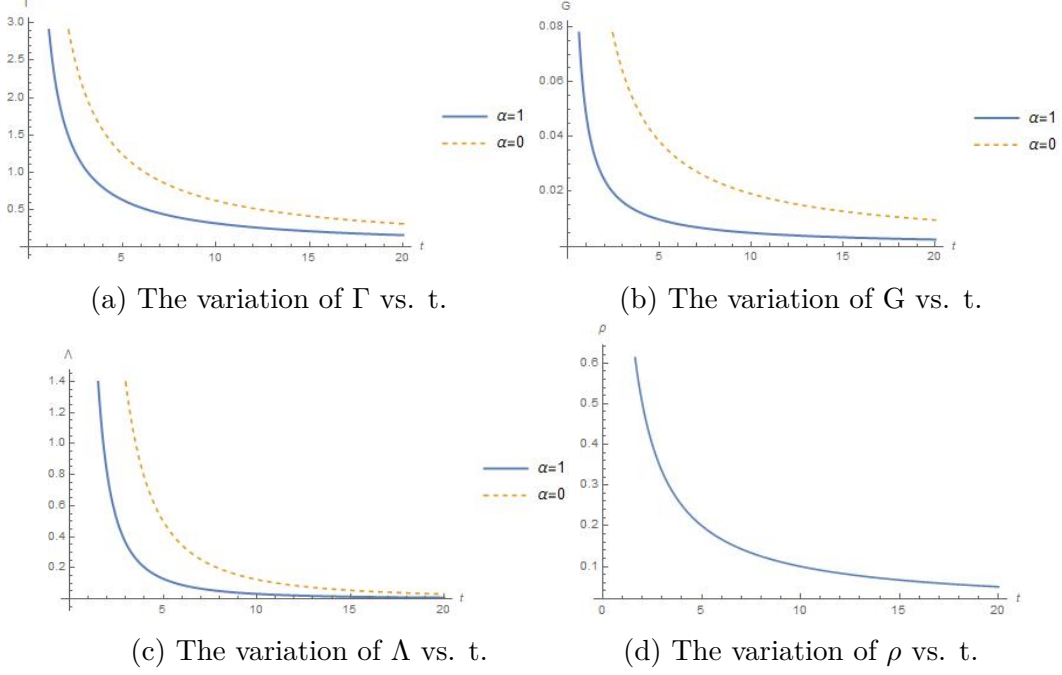


Figure 4.6: The variations of Γ , G , Λ & ρ vs. t
Case II: Particle Creation

get $G \propto t^{-1}$, which is widely supported in time-dependent gravity [35], [58].

Case (IIb, Exponential Law): $A = be^{mt}$ $\rho = ke^{nt}$

where b , k , m and n are constants. From (4.32), we find the value of G to be,

$$G(t) = 0. \quad (4.54)$$

Since $\Gamma \propto G^{-1}$, it follows that $\Gamma \rightarrow \infty$.

For the cosmological constant, (4.30) gives

$$\Lambda(t) = 3m^2. \quad (4.55)$$

As before, the Hubble parameter (4.15)

$$H = m.$$

Consequently, the matter and dark-energy density parameters,

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} = 1, \quad (4.56)$$

and

$$\Omega_m = \frac{8\pi G}{3H^2}\rho = 0. \quad (4.57)$$

In this case, we see that even assuming particle creation, exponentially evolving universe will be free of gravitational attraction, since the gravitational coupling $G = 0$,

and totally dominated by dark energy, suffering the same consequences as in case (Ib).

CHAPTER 5

CONCLUSION AND RECOMMENDATION

In this work, we reviewed the necessary tensor calculus in order to understand the theory of General Relativity. Then we derived the Einstein field equations using two methods, first by appealing to physical intuition and the relation of geometry of space-time to gravity, then by the variational formulation using the Einstein-Hilbert action. Furthermore, we have reviewed the most important exact solution to the EFEs, including a detailed derivation of the Schwarzschild metric, and a light discussion of the Reissner-Nordstrom and Kerr solutions. Moreover, we have derived the Friedman equations of cosmology and reviewed simple Friedman-Robertson-Walker cosmological models.

The standard models of cosmology, those based on the FRW metric, are built on the basis of the *cosmological principle*, the assumption that the universe is homogeneous and isotropic on large scales. However, recent observations from the *Planck Mission* [67]

have called this view into question. These observations lead to investigate anisotropic cosmological models. The simplest generalization of FRW model which accounts for anisotropy is given by the Bianchi type-I cosmological model. Therefore, it seems ideal to explore Bianchi type-I models of the universe as an alternative to the standard FRW models of cosmology.

In the last chapter, we have presented a discussion of the evolution and dynamics of Bianchi I cosmological model under variable gravitational and cosmological constants. In this process, we have obtained a class of exact solutions of the EFE in two cases; (i) considering a universe with no particle creation in which the particle production rate $\Gamma = 0$, and (ii) introducing the process of particle creation with $\Gamma \geq 0$. In both cases, we considered situations in which both the scale factor A and the matter density ρ obey the power ($A \propto t^m$) and exponential ($A \propto e^{mt}$) laws. By calculating the dark-energy density parameter Ω_Λ and the matter density parameter Ω_m , we have obtained the best numerical values for the constants involved in the ansatz of A and ρ , and which are in a good agreement with the results obtained from WMAP [63].

Recently, many authors have questioned the validity of General Relativity as the correct theory of gravity, and questioned the mysterious dark matter term Λ due to difficulty of experimental confirmation. This has lead to consider modification of the general theory of relativity. Among the interesting alternatives are $f(R)$ gravity, in which the higher curvature terms are added to the gravitational action [68], and $f(R, T)$ gravity, in which we introduce a coupling between geometry and energy-momentum [69]. In both theories Einstein-Hilbert action is modified to give a more general equations than EFEs, and it is hoped that it may provide new understanding

of dark matter or even inflation [70].

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