REGULARITY AND RELATED INVARIANTS IN TENSOR PRODUCTS OF ALGEBRAS OVER A FIELD

BY

FAISAL SUWAYYID

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

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This thesis, written by **FAISAL SUWAYYID** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

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To my parents, professors and everyone who supported me

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TABLE OF CONTENTS

LIST OF FIGURES

THESIS ABSTRACT

NAME: Faisal Suwayyid

TITLE OF STUDY: Regularity and Related Invariants in Tensor Products of Algebras Over a Field

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It has been established, quite recently, that a tensor product of k-algebras, if Noetherian, it inherits the concepts of Cohen-Macaulay ring, Gorenstein ring, and complete intersection ring. However, it is well known that a tensor product of two extension fields is not necessarily regular. In 1965, Grothendieck showed that it is, in fact, regular if one of the two fields is separable and finitely generated.

Since then, many articles appeared in the literature featuring partial results on this topic. The problem on the transfer or defect of regularity in more general settings remains elusively open. This MS thesis tracks and studies some of these works which deal with this problem; precisely, we shed brighter light on the main results and examples published, chronologically, in [11], [40], [12], [3].

ملخص الرسالة

الاسم: فيصل سويد

عنوان الدراسة: الانتظام و الثوابت المتعلقة به في الضرب المؤتر للجبر الحقلي

التخصص: الرياضيات والاحصاء

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تم, في وقت قريب, إثبات ان حاصل الضرب المؤتر للجبر الحقلي, إذا كان من حلقات نويثر, يرث خصائص حلقة كوهن-ماكولاي, قورينستين, و التقاطع المكتمل منها. مع ذلك, فانه معلوم ان حاصل الضرب المؤتر لإمتدادين من الحقل ليس بالضرورة جبر منتظم. في عام ١٩٦٥, أثبت العالم قروثينديك, في الحقيقة, أن حاصل الضرب المؤتر لإمتدادين للحقل جبر منتظم إذا كان أحد الإمتدادين متعدد الحدود و قابل للفصل.

منذ ذلك الحين, ظهرت عدة مقالات تستعرض نتائج جزئية في هذا الموضوع. لزالت مشكلة نقل او تلف الإنتظام في ضوابط عامة بعيدة المنال. ستقوم رسالة الماجستير هذه بتعقب و دراسة بعض هذه المقالات التي تدرس هذه المشكلة منذ نشأتها, خاصة منذ مقالة العالم واتانابي في انتقال خصائص حلقات قورينستين. فعلا, سوف تقوم هذه الرسالة بتسليط الإضاءة على النتائج الأساسية و الأمثلة بترتيب زمني في المقالات [41], [11], [38], [12], [3].

INTRODUCTION

Through this thesis, all rings are commutative with unity and k will denote a field. Given a Noetherian local ring $(R, \mathfrak{m})$, the embedding dimension of R, denoted by embdim(R), is the dimension of $\mathfrak{m} / \mathfrak{m}^2$ as an $(R/\mathfrak{m})$-vector space; equivalently, the cardinality of a minimal generating set (called, minimal basis) for $\mathfrak{m}$. The ring R is regular if its embedding dimension coincides with its Krull dimension. Recall, from [3], that "the notion of regularity was initially introduced by Krull and became prominent when Zariski showed that a local regular ring corresponds to a smooth point on an algebraic variety. Later, Serre proved that a ring is regular if and only if it has finite global dimension. This allowed to see that regularity is stable under localization and then the definition got globalized." Then, R is called a complete intersection ring if its $\mathfrak{m}$-completion is equal to the quotient of a local regular ring modulo an ideal generated by a regular sequence; and R is called Gorenstein if its injective dimension is finite. Finally, recall that R is Cohen-Macaulay if the height of $\mathfrak{m}$ coincides of its grade. The definitions of all these notions carry over to localizations with respect to the prime ideals. For more details, we refer to the classic books [14], [15], [19], [21], [30], [32], [35], [44]. We have the following diagram of implications: It is, now,

$$\begin{array}{c}\text{Regular}\\ \Downarrow \\ \text{(Locally) Complete Intersection}\\ \Downarrow \\ \text{Gorenstein}\\ \Downarrow \\ \text{Cohen-Macaulay}\\ \Downarrow \\ \text{Noetherian}\end{array}$$

Figure 1: Implications of the invariants

established that a Noetherian tensor product of k-algebras inherits the notions of Cohen-Macaulay ring, Gorenstein ring, and complete intersection ring [11], [25], [36], [38], [40]. However, it is well known that a tensor product of two extension fields is not necessarily regular. Note that tensor products of k-algebras subject to the aforementioned geometric concepts were used to delimit or widen the scope of validity of some homological conjectures [26], [28].

In 1965, Grothendieck proved that, given a field k, "the tensor product of two extension fields (of k) is regular provided one of them is finitely generated and separable over k" [24]. In 1969, Watanabe et al. investigated the transfer of Gorensteiness and turned out that "the tensor product of two regular k-algebras, one of them being finitely generated, is a complete intersection [43]." Since then, it has been proved that "a Noetherian tensor product of k-algebras $A \otimes_k B$ inherits from A and B the notions of locally complete intersection ring, Gorenstein ring, and Cohen-Macaulay ring [11], [25], [36], [38], [40]." In particular, "$K \otimes_k L$ is a locally complete intersection ring, for any two extension fields K and L of k such that $K \otimes_k L$ is Noetherian [40]."

Recently, it has been proved that "a Noetherian tensor product of two k-algebras

A and B is regular if and only if so are A and B in the special case where k is perfect (i.e., every (algebraic) extension of k is separable) [40], [25]." Moreover, it is established in [12] that, under geometrical regularity (i.e., $A \otimes_k F$ is regular for every finite extension F of k), "$A \otimes_k B$ is regular if and only if B is regular and $A \otimes_k B$ is Noetherian", which allowed to establish transfer results for tensor product of two extension fields of k subject to separability conditions.

Invariants related to regularity, such as embedding dimension and codimension, of tensor products of algebras were investigated in the literature, yielding several results on defect or transfer of regularity in these constructions [3]. For any prime ideal P of $A \otimes_k B$ with $p := A \cap P$ and $q := B \cap P$, one of the recent results ([3, Theorem 5.1]) asserts that "if $\kappa_B(q)$ is a separable extension field of k, then

$$\begin{aligned}\operatorname{embdim}(A \otimes_k B)_P &= \operatorname{embdim}(A_p) + \operatorname{embdim}(B_q) \\ &\quad + \operatorname{embdim}\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{P(A_p \otimes_k B_q)}{pA_p \otimes_k B_q + A_p \otimes_k qB_q}}\right)\text{"}\end{aligned}$$

with a direct consequence on the (embedding) codimension:

$$\text{"}\operatorname{codim}(A \otimes_k B)_P = \operatorname{codim}(A_p) + \operatorname{codim}(B_q).\text{"}$$

The aforementioned works draw on basic results and use techniques and methods from prime spectra and dimension theory. For early and recent developments on these topics, we refer to [7], [9], [10], [12], [36], [37], [38], [41], [42] for special settings of tensor products of algebras, and to [5], [16], [23], [28], [30], [32], [33] for the general context of commutative rings. The problem on the transfer or defect of regularity in more general settings remains elusively open. This MS thesis tracks and studies some of the main works which deal with this problem; precisely, we shed brighter light on the main results and examples published, chronologically, in [11], [40], [12], [3].

CHAPTER 1

PRELIMINARIES

In order to make this thesis reasonably self-contained, this section recalls some notations and definitions as well as some basic relevant results, most of which deal particularly with various aspects of tensor products of algebras. These will be used through this thesis implicitly without specific mention.

Let R be a ring, S a multiplicatively closed subset of R, and M an R-module. It is customary to denote the localization of M with respect to S by $S^{-1}M$, and if S is a complement of a prime ideal P in R, the localization is denoted by M_P instead. If no ambiguity arises, we denote either localizations by M_S or M_P. If T is another multiplicatively closed subset and T' is its image in R_S and if S' is the image of S in R_T, then $(M_S)_{T'} \cong M_{ST} = M_{TS} = (M_T)_{S'}$ [2, Ex. 3, p. 43]. We usually drop the apostrophe, even though the meaning might be different, but the algebraic properties remain unchanged. Further, if S consists only of units in R, then obviously $R_S \cong R$ and, as a consequence, if $S \subseteq T$, then $R_T \cong (R_T)_S \cong (R_S)_T$.

It is known that the prime ideals of R_S have the form $\left\{\frac{x}{s} \mid x \in p, s \in S\right\}$, for some unique prime p in R [2, Proposition 3.11]. It is usually denoted by $S^{-1}p$ and if S is a complement of a prime q, then it is denoted by pR_q. We also use the notation p_S or p_q as in [30, Section 1.4]. Throughout, $(R, \mathfrak{m})$ or $(R, \mathfrak{m}, k)$ indicates that R is a local ring with maximal ideal $\mathfrak{m}$ and $k := R/\mathfrak{m}$ is its residue field. Frequently, it is denoted by $\kappa(\mathfrak{m})$ and, for any prime ideal p of R, R_p/pR_p is denoted by $\kappa_R(p)$.

The module M is flat if, for every injective R-map $f : E' \longrightarrow E$, the induced map $f \otimes_R 1 : E' \otimes_R M \longrightarrow E \otimes_R M$ is injective; and M is said to be faithfully flat if it is flat and whenever $f \otimes_R 1$ is injective (resp., surjective), f is injecitve (resp., surjective).

In the sequel, $\mathrm{Spec}(R)$ (resp., $\mathrm{Max}(R)$) will denote the set of all prime (resp., maximal) ideals of R.

Proposition 1.1 ([32, Theorem 7.2]) *"Let R be a ring and M an R-module. Then, the following statements are equivalent:*

(1) *M is faithfully flat;*

(2) *M is flat and, for every nonzero R-module N, $M \otimes_R N \neq 0$;*

(3) *M is flat and, for each $\mathfrak{m} \in \mathrm{Max}(R)$, $\mathfrak{m} M \neq M$."*

As a consequence of this proposition, free modules are faithfully flat, since if M is free, then $M \otimes_R N$ is just a sum of copies of N.

Let $f : A \longrightarrow B$ be a ring homomorphism and let $q \in \mathrm{Spec}(B)$ with $p := f^{-1}(q)$, which is usually (and, abusively, even if f is not one-to-one) denoted by $A \cap q$. Then, f induces two natural ring homomorphisms $g : A_p \longrightarrow B_q$, defined by $g(\frac{a}{s}) = \frac{f(a)}{f(s)}$, and $h : \dfrac{A}{p} \longrightarrow \dfrac{B}{q}$, defined by $h(\overline{a}) = \overline{f(a)}$. The extension of p to B is denoted by

$pB := f(p)B$. Notice that we always have the following natural isomorphisms

$$\frac{B_p}{pB_p} \cong \kappa_A(p) \otimes_A B$$

and hence

$$\frac{B_q}{pB_q} \cong \kappa(p_p) \otimes_{A_p} B_q$$

as, obviously, $f(A \setminus p) \subseteq B \setminus q$. The ring $\kappa_A(p) \otimes_A B$ is called the fiber ring over p. Fiber rings over prime ideals play a primordial role in the study of the embedding dimension and Krull dimension of tensor products of algebras. Now, if A, B are R-algebras and S, S' are multiplicatively closed subsets of A and B, respectively, then $T := \{s \otimes_R s' \mid s \in S,\ s' \in S'\}$ is a multiplicatively closed subset of $A \otimes_R B$ with

$$(A \otimes_R B)_T \cong A_S \otimes_R B_{S'}.$$

Let A, B be k-algebras, and I, J ideals of A and B, respectively. Then, by faithful flatness, we have

$$\frac{A \otimes_k B}{I \otimes_k B + A \otimes_k J} \cong \frac{A}{I} \otimes_k \frac{B}{J} \tag{1.0.1}$$

and if I is proper in A then so is $I \otimes_k B$ in $A \otimes_k B$. Also, throughout, we will identify A and B with their respective images in $A \otimes_k B$.

Finally, recall that $f : A \longrightarrow B$ is said to be flat if the A-algebra B is flat over A (i.e., actually over $f(A)$); and if $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are local rings, then f is said to be a local homomorphism if $f(\mathfrak{m}) \subseteq \mathfrak{n}$.

Proposition 1.2 ([32, Theorem 15.1]) *"Let $f : A \longrightarrow B$ be a flat homomorphism of Noetherian rings and let $q \in \operatorname{Spec}(B)$ with $p := A \cap q$. Then:*

$$\dim(B_q) = \dim(A_p) + \dim(\kappa_A(p) \otimes_A B_p)."$$

An element x of R is a zero-divisor on M if there exists a nonzero element m of M such that $xm = 0$. Equivalently, the R-map $f_x : m \longrightarrow xm$ is not injective. We denote the collection of zero-divisors of M by $\mathrm{Z}(M)$. Recall that $\mathrm{Z}(M)$ is a union of primes and this union is unique if it is taken over the maximal primes among them [30, p. 34]. A nonzero-divisor of M is also called a regular element of M. An associated prime of M is a prime p of R such that there exists a nonzero element m of M with $p = (0 : m) := \{r \in R \mid rm = 0\}$. If R is Noetherian and M is finitely generated over R, the maximal primes of M are finite and thus $\mathrm{Z}(M)$ is a unique finite union of maximal primes of M, each of which is an associated prime of M [30, Theorem 80].

A sequence $x_1, x_2, \ldots, x_n$ of elements of R is said to be *regular* on M or an M-*sequence*, provided

$$(x_1, x_2, \ldots, x_n)M \neq M$$

and

$$x_{i+1} \text{ is regular on } \frac{M}{(x_1, x_2, \ldots, x_i)M}, \text{ for each } i = 1, \ldots, n-1.$$

Lemma 1.1 ([30, Theorem 116]) *"Let R be a ring and M an R-module. Let $x_1, \ldots, x_n \in R$ and $1 \leq i \leq n-1$. Then, the following statements are equivalent:*

(1) *$x_1, \ldots, x_n$ is an M-sequence;*

(2) $x_1, \ldots, x_i$ *is an* M*-sequence and* $x_{i+1},\ \ldots,\ x_n$ *is an* $\dfrac{M}{(x_1, \ldots, x_i)M}$*-sequence."*

Note that a regular sequence $x_1, x_2, \ldots, x_n$ generates a strictly increasing sequence of ideals, $(x_1) \subsetneqq (x_1, x_2) \subsetneqq \cdots \subsetneqq (x_1, x_2, \ldots, x_n)$. Therefore, in Noetherian settings, regular sequences saturate. Northcott and Rees [34] proved that, given an Noetherian ring R, a finitely generated R-module M, and an ideal I of R with $IM \neq M$, then any two maximal M-sequences in I have the same length [30, Theorem 121]. Then, an M-sequence in I is maximal if and only if $I \subseteq \mathrm{Z}\left(\dfrac{M}{(x_1, x_2, \ldots, x_n)M}\right)$ (cf. [30, Appendix 3-1, p. 101]). This gives rise to the concept of grade.

Definition 1.1 "Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal of R with $IM \neq M$. The length of a maximal M-sequence in I is called the grade of I on M, denoted $G(I, M)$."

If $I \subseteq \mathrm{Z}(M)$, we write $G(I, M) = 0$; and if $M = R$ we just write $G(I)$ instead. Some authors use the term 'depth' instead of 'grade' (cf. [32]).

Proposition 1.3 ([30, Theorem 132]) *"Let R be a Noetherian ring, M a finitely generated R-module, and $I \subseteq J$ proper ideals of R with $JM \neq M$. Then:*

(1) $G(I, M) \leq G(J, M)$.

(2) $G(I) \leq \mathrm{ht}(I)$."

Definition 1.2 ([30, Theorems 136]) "A Noetherian ring R is Cohen-Macaulay if it satisfies any one of the following equivalent conditions:

(1) $G(\mathfrak{m}) = \mathrm{ht}(\mathfrak{m})$, for each $\mathfrak{m} \in \mathrm{Max}(R)$;

(2) $G(I) = \mathrm{ht}(I)$, for each ideal I of R.

The Cohen-Macaulay property is stable under localization. In fact, it is a local property, as seen below.

Theorem 1.1 ([30, Theorems 139 & 140]) *"Let R be a Noetherian ring and let S be a multiplicatively closed subset S of R. Then:*

(1) *If R is Cohen-Macaulay, then so is R_S.*

(2) *R is Cohen-Macaulay if and only if $R_\mathfrak{m}$ is Cohen-Macaulay, for each $\mathfrak{m} \in \mathrm{Max}(R)$ if and only if R_p is Cohen-Macaulay, for each $p \in \mathrm{Spec}(R)$"*

Finally, it is worthwhile recalling that Cohen-Macaulay property is stable under polynomial and power series rings, as stated below.

Proposition 1.4 ([30, Theorems 151 & 157]) *"A Noetherian ring R is Cohen-Macaulay if and only if so is $R[x]$ (resp., $R[[x]]$)."*

If $(R, \mathfrak{m})$ is local, then we write $G(R) := G(\mathfrak{m})$ and is called the grade of R [30]. For local flat homomorphisms, fiber rings over prime ideals are involved as follows:

Proposition 1.5 ([32, Corollary, p. 181]) *"Let $f : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ be a local flat homomorphism of Noetherian rings. Then:*

(1) $G(B) = G(A) + G(\kappa(\mathfrak{m}) \otimes_A B)$.

(2) *A and $\kappa(\mathfrak{m}) \otimes_A B$ are Cohen-Macaulay if and only if so is B.*

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Hence, $\mathfrak{m}$ is finitely generated and so $\mathfrak{m} / \mathfrak{m}^2$ is a finite-dimensional vector space over k. The embedding dimension of R is defined as $\text{embdim}(R) := \dim_k(\mathfrak{m} / \mathfrak{m}^2)$; which is equal to the cardinality of a minimal generating set (called, minimal basis) for $\mathfrak{m}$. We always have

$$G(R) \leq \dim(R) \leq \text{embdim}(R).$$

Definition 1.3 "A Noetherian local ring R is regular if $\text{embdim}(R) = \dim(R)$.

A Noetherian ring R is regular, if $R_{\mathfrak{m}}$ is regular, for each $\mathfrak{m} \in \text{Max}(R)$."

The transfer of regularity through flat local homomorphisms is of great significance for the study of regularity in tensor products of algebras. The following three relevant basic results will be used most often in the sequel.

Theorem 1.2 ([32, Theorem 23.7]) *"Let $f : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ be a local flat homomorphism of Noetherian rings.*

(1) *If B is regular, then so is A.*

(2) *If A and $\kappa_A(\mathfrak{m}) \otimes_A B$ are regular, then so is B.*

(3) *Assume $\mathfrak{m} B = \mathfrak{n}$. Then, A is regular if and only if so is B."*

The first result, on the transfer of regularity to the tensor product of two extension fields, is due to Grothendieck (1965).

Lemma 1.2 ([24, Lemma 6.7.4.1]) *"Let K and L be two extension fields of k. If either K or L is finitely generated and K is separable over k, then $K \otimes_k L$ is regular."*

The injective dimension of an R-module M, denoted $\mathrm{id}_R(M)$, is defined as the smallest number n such that there exists an injective resolution

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_n$$

and we write $\mathrm{id}_R(M) = n$. If no such resolution exists, we define $\mathrm{id}_R(M) = \infty$. Obviously, M is injective if and only if $\mathrm{id}_R(M) = 0$. For more details on this homological invariant, we refer to [35].

Definition 1.4 "A Noetherian local ring R is Gorenstein if $\mathrm{id}_R(R) < \infty$.

A Noetherian ring R is Gorenstein, if $R_{\mathfrak{m}}$ is Gorenstein, for each $\mathfrak{m} \in \mathrm{Max}(R)$."

The $\mathfrak{m}$-adic completion of a local ring $(R, \mathfrak{m})$ is defined as the inverse limit of $\left\{R/\,\mathfrak{m}^n\right\}_{n\geq 0}$; that is,

$$\widehat{R} = \lim_{\longleftarrow} \frac{R}{\mathfrak{m}^n}.$$

If $(R, \mathfrak{m})$ is a Noetherian local ring, then so is $\widehat{R}$ with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\widehat{R}$ and the natural ring homomorphism $R \to \widehat{R}$ is flat. For instance, the power series ring $R[[x]]$ is the (x)-adic completion of the polynomial ring $R[x]$; that is, $\widehat{R[x]} = R[[x]]$. For more details on the $\mathfrak{m}$-adic topology, we refer to [2, Chap. 10].

Definition 1.5 "A Noetherian local ring $(R, \mathfrak{m})$ is a complete intersection if its $\mathfrak{m}$-completion $\widehat{R}$ is equal to the quotient of a local regular ring modulo an ideal generated by a regular sequence.

A Noetherian ring R is a locally complete intersection, if $R_{\mathfrak{m}}$ is a complete intersection, for each $\mathfrak{m} \in \mathrm{Max}(R)$."

We close this preliminary with some recalls on Gorenstein and complete intersection rings.

Proposition 1.6 ([43, Corollary 1]&[32, Theorem 23.5]) *"If a Noetherian local ring R is Gorenstein (resp., complete intersection), then so are $R[x]$ and $R[[x]]$."*

For the Gorenstein and complete intersection notions, the fiber rings are involved as follows:

Theorem 1.3 ([43, Theorem 1]&[19, Remark 2.3.5]) *"Let $f : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$ be a local flat homomorphism of Noetherian rings. Then:*

(1) *A and $\kappa(\mathfrak{m}) \otimes_A B$ are Gorenstein if and only if so is B.*

(2) *A and $\kappa(\mathfrak{m}) \otimes_A B$ are complete intersection if and only if so is B."*

Proposition 1.7 ([43, Corollary 2]&[32, Theorem 23.6]) *"Let A be a Gorenstein (resp., complete intersection) ring containing k and K a finitely generated extension field of k. Then, $A \otimes_k K$ is a Gorenstein (resp., complete intersection) ring."*

Theorem 1.4 ([43, Theorem 2]) *"Let B and C be A-algebras, with B being flat over A and C finitely generated over A. Then:*

(1) *If A, B and C are Gorenstein rings, then $B \otimes_A C$ is a Gorenstein ring.*

(2) *If B is a complete intersection ring and A and C are regular, then $B \otimes_A C$ is a complete intersection ring."*

CHAPTER 2

TENSOR PRODUCTS OF COHEN-MACAULAY K-ALGEBRAS

This section deals with the transfer of Cohen-Macaulayness to Noetherian tensor products of k-algebras. For this purpose, we first examine the grade and height of some special ideals which play a crucial role within the ideal structure of a tensor product of k-algebras.

The first main theorem investigates the grade in Noetherian tensor products.

Theorem 2.1 ([11, Theorem 1.1]) *"Let A and B be k-algebras such that $A \otimes_k B$ is Noetherian and let I and J be proper ideals of A and B, respectively. Then:*

(1) $G(I \otimes_k B) = G(I)$ *and, similarly,* $G(A \otimes_k J) = G(J)$.

(2) $G(I \otimes_k B + A \otimes_k J) = G(I) + G(J)$.

(3) $G(I \otimes_k J) = \mathrm{Min}(G(I), G(J))$."

The proof of this result draws on the following lemmas.

Lemma 2.1 *"Let A and B be k-algebras and let $x \in A$. Then, x is an A-sequence if and only if $x \otimes_k 1$ is an $(A \otimes_k B)$-sequence."*

Proof. Consider the linear map $f : A \to A$, $a \mapsto xa$ and note that B is faithfully flat (since a k-vector space). So, x is an A-sequence if and only if f is injective but not surjective if and only if $f \otimes_k 1$ is injective but not surjective if and only if $x \otimes_k 1$ is an $(A \otimes_k B)$-sequence. □

Recall that a regular sequence $x_1, \ldots, x_n$ is said to be permutable if any permutation of the x_i's is also regular.

Lemma 2.2 *"Let A and B be k-algebras, x an A-sequence, and y a B-sequence. Then, $x \otimes_k 1$, $1 \otimes_k y$ is a permutable $(A \otimes_k B)$-sequence."*

Proof. By Lemma 2.1, $x \otimes_k 1$ is an $(A \otimes_k B)$-sequence and $1 \otimes_k y$ is an $(\frac{A}{(x)} \otimes_k B)$-sequence. But, $\frac{A}{(x)} \otimes_k B \cong \frac{A \otimes_k B}{(x) \otimes_k B}$. Therefore, by Lemma 1.1, $x \otimes_k 1$, $1 \otimes_k y$ is an $(A \otimes_k B)$-sequence. Likewise, starting with y yields $1 \otimes_k y$, $x \otimes_k 1$ is an $(A \otimes_k B)$-sequence. □

Lemma 2.3 *"Let A and B be k-algebras. If $x_1, \ldots, x_n$ is an A-sequence, and $y_1, \ldots, y_m$ is a B-sequence, then $x_1 \otimes_k 1, \ldots, x_n \otimes_k 1$, $1 \otimes_k y_1, \ldots, 1 \otimes_k y_m$ is an $(A \otimes_k B)$-sequence. Moreover, if $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are permutable, so are their respective images."*

Proof. First, we show that $x_1 \otimes_k 1, \dots, x_n \otimes_k 1$ is an $(A \otimes_k B)$-sequence. Lemma 2.1 handles the case $n = 1$. We proceed by induction on n . Consider the ideal $I_{n-1} := (x_1, x_2, \dots, x_{n-1})$. Again, recall that $\dfrac{A}{I} \otimes_k B \cong \dfrac{A \otimes_k B}{I \otimes B}$, for any ideal I of A. Since $\bar{x}_n$ is an $\dfrac{A}{I_{n-1}}$-sequence, $\bar{x}_n \otimes_k 1$ is an $\dfrac{A \otimes_k B}{I_{n-1} \otimes_k B}$- sequence. It follows, by induction, that $x_1 \otimes_k 1, \dots, x_n \otimes_k 1$ is an $(A \otimes_k B)$-sequence. Similar arguments shows that $1 \otimes_k y_1, \dots, 1 \otimes_k y_m$ is an $\dfrac{A}{I_n} \otimes_k B$-sequence. By Lemma 1.1, $x_1 \otimes_k 1, \dots, x_n \otimes_k 1$, $1 \otimes_k y_1, \dots, 1 \otimes_k y_m$ is an $(A \otimes_k B)$-sequence. Permutability follows easily. □

Lemma 2.4 ([11, Lemma 1.2]) *"Let A and B be k-algebras and let $x_1, \dots, x_n$ be a permutable A-sequence and $y_1, \dots, y_n$ be a permutable B-sequence. Then, $x_1 \otimes y_1, \dots, x_n \otimes y_n$ is a permutable $(A \otimes_k B)$-sequence."*

Proof. Combine Lemma 2.3 with the general fact that if $x_1, x_2, ..., x_n$ is a permutable regular sequence, then so is $(x_1x_2), x_3, \dots, x_n$." □

Proof of Theorem 2.1.

(1) If $x_1, x_2, \dots, x_n$ is a maximal A-sequence in I, then by Lemma 2.3, $x_1 \otimes_k 1, \dots, x_n \otimes_k 1$ is an $A \otimes_k B$ sequence in $I \otimes_k B$. Since it is maximal, there exists a nonzero $\bar{a}$ in $\frac{A}{(x_1,x_2,\dots,x_n)}$ such that $I\bar{a} = 0$ by [30, Theorem 82]. Hence, $(I \otimes_k B)(\bar{a} \otimes_k 1) = I\bar{a} \otimes_k B = 0$ in $\frac{A}{(x_1,x_2,\dots,x_n)} \otimes_k B$. Since $\frac{A}{(x_1,x_2,\dots,x_n)}$ is a faithfully flat k-algebra, $\bar{a} \otimes_k 1 \neq 0$; i.e., $(\bar{a}) \otimes_k B \neq 0$. Therefore, $G(I \otimes_k B) = G(I)$. Similarly, $G(A \otimes_k J) = G(J)$.

(2) Let $x_1, x_2, \dots, x_n$ be a maximal A-sequence in I and $y_1, y_2, \dots, y_m$ be a maximal B-sequence in J. Then $x_1 \otimes_k 1, \dots, x_n \otimes_k 1$ is a maximal $(A \otimes_k B)$-sequence in $I \otimes_k B$, and $1 \otimes_k y_1, \dots, 1 \otimes_k y_m$ is a maximal $\frac{A}{(x_1,x_2,\dots,x_n)} \otimes_k B$-sequence in $\frac{A}{(x_1,x_2,\dots,x_n)} \otimes_k J$ by

similar steps in (1). Therefore, by Lemma 1.1, $x_1 \otimes_k 1, \ldots, x_n \otimes_k 1, 1 \otimes_k y_1, \ldots, 1 \otimes_k y_m$ is a maximal $(A \otimes_k B)$-sequence in $I \otimes_k B + A \otimes_k J$.

(3) Without loss of generality assume that $G(I) = n \leq G(J) = m$. By [30, Ex. 23, p. 104], there exists a maximal permutable A-sequence $x_1, x_2, \ldots, x_n$ in I and a maximal permutable B-sequence $y_1, y_2, \ldots, y_m$ in J. By Lemma 2.4, $x_1 \otimes_k y_1, \ldots, x_n \otimes_k y_n$ is a permutable $(A \otimes_k B)$-sequence in $I \otimes_k J$. Hence, $n \leq G(I \otimes_k J)$. Since $I \otimes_k J \subseteq I \otimes_k A$, $G(I \otimes_k J) \leq G(I \otimes_k A) = n$ by Proposition 1.3. Therefore, $G(I \otimes_k J) = n$. □

Now, we state below the second main theorem of this section.

Theorem 2.2 ([11, Theorem 2.1]) *"Let A and B be k-algebras such that $A \otimes_k B$ is Noetherian. Then, the following statements are equivalent:*

(1) *$A \otimes_k B$ is a Cohen-Macaulay ring;*

(2) *$G(I \otimes_k B + A \otimes_k J) = \mathrm{ht}(I \otimes_k B + A \otimes_k J)$, for all proper ideals I and J of A and B, respectively;*

(3) *$G(p \otimes_k B + A \otimes_k q) = \mathrm{ht}(p \otimes_k B + A \otimes_k q)$, for all prime ideals p and q of A and B, respectively;*

(4) *A and B are Cohen-Macaulay rings."*

Discussion 2.3 In general, even if A and B are Noetherian k-algebras, $A \otimes_k B$ does not have to be Noetherian [41]. But, if $A \otimes_k B$ is Noetherian, then so are A and B through faithfully flatness [32, Change of coefficient rings, p. 46 and Ex. 7.9, p. 53].

Moreover, if $A \otimes_k B$ is Noetherian, then the transcendence degree of A or B over k must be finite. Indeed, let $\mathrm{t.deg}_k(A)$ denotes the transcendence degree of A over k. Then, by [42, p. 392],

$$\mathrm{t.deg}_k(A) := \sup\Big\{\, \mathrm{t.deg}_k\big(\frac{A}{p}\big) \mid p \in \mathrm{Spec}(A)\Big\}.$$

Since A and B are Noetherian, they have a finite number of minimal prime ideals. Hence, there exist primes p in A and q in B such that $\mathrm{t.deg}_k(A) = \mathrm{t.deg}_k(\frac{A}{p})$ and $\mathrm{t.deg}_k(B) = \mathrm{t.deg}_k(\frac{B}{q})$. But, $\kappa_A(p) \otimes_k \kappa_B(q)$ is Noetherian, as a localization of the Noetherian ring $\dfrac{A}{p} \otimes_k \dfrac{B}{q} \cong \dfrac{A \otimes_k B}{p \otimes_k B + A \otimes_k q}$. Therefore, by [41, Corollary 10], $\mathrm{t.deg}_k(\kappa_A(p)) < \infty$ or $\mathrm{t.deg}_k(\kappa_B(q)) < \infty$.

To prove Theorem 2.2, we first establish the following results.

Lemma 2.5 ([11, Lemma 2.2]) *"Let T and V be two extension fields of k such that $T \otimes_k V$ is Noetherian. Then, $T \otimes_k V$ is a Cohen-Macaulay ring."*

Proof. Since $T \otimes_k V$ is Noetherian, then without loss of generality, we may assume that the transcendence degree of T is finite; say, $\mathrm{t.deg}_k(T) = n < \infty$, with $x_1, x_2, \ldots, x_n$ being algebraically independent over k. Observe that

$$T \otimes_k V \cong T \otimes_{k(x_1,x_2,\ldots,x_n)} S^{-1}V[x_1, x_2, \ldots, x_n]$$

where $S := k[x_1, x_2, \ldots, x_n] \setminus \{0\}$ [14, Proposition 2.6]. Now, T is algerbaic over $k(x_1, x_2, \ldots, x_n)$ and $S^{-1}V[x_1, x_2, \ldots, x_n]$ is a Cohen-Macaulay ring by Proposition

1.4 and Theorem 1.1. Therefore, we reduce the proof to the case of a tensor product of an algebraic extension and a Cohen-Macaulay ring.

Next, let T be an algebraic extension field of k such that $T \otimes_k A$ is Noetherian. Notice first that $T \otimes_k A$ is an integral extension of A; that is, faithfully flat over A since T is algebraic over k. Let Q be a prime ideal of $T \otimes_k A$ with $p := A \cap Q$. Then, since $T \otimes_k A$ is integral over A and $T \otimes_k p \subseteq Q$, by Proposition 1.3, Theorem 2.1 and [30, Theorem 48], we get

$$\mathrm{ht}(Q) \leq \mathrm{ht}(p) = G(p) = G(T \otimes_k p) \leq G(Q) \leq \mathrm{ht}(Q).$$

Therefore, $G(Q) = \mathrm{ht}(Q)$ and so $A \otimes_k B$ is Cohen-Macaulay by Theorem 1.1. □

Proposition 2.1 ([11, Proposition 2.3]) *"Let A and B be k-algebras such that $A \otimes_k B$ is Noetherian and let Q be a prime of $A \otimes_k B$ with $p := A \cap Q$ and $q := B \cap Q$. Then:*

(1) $\mathrm{ht}(Q) = \mathrm{ht}(p) + \mathrm{ht}(q) + \mathrm{ht}(\frac{Q}{p \otimes_k B + A \otimes_k q})$.

(2) $G(Q_Q) = G(p_p) + G(q_q) + \mathrm{ht}(\frac{Q}{p \otimes_k B + A \otimes_k q})$."

Proof. (1) Notice that $\mathrm{ht}(Q) = \mathrm{ht}(Q_Q) = \dim(A \otimes_k B)_Q$ and consider the flat local homomorphism $A_p \longrightarrow (A \otimes_k B)_Q$. By Proposition 1.2, we have that

$$\dim(A \otimes B)_Q = \dim(A_p) + \dim\big(\frac{A}{p} \otimes_k B\big)_{\frac{Q}{p \otimes_k B}}. \tag{2.0.1}$$

Now, consider the flat local homomorphism $B_q \longrightarrow (\frac{A}{p} \otimes_k B)_{\frac{Q}{p \otimes_k B}}$. Again by Proposition 1.2,

$$\dim \big(\frac{A}{p} \otimes_k B\big)_{\frac{Q}{p\otimes_k B}} = \dim(B_q) + \dim \big(\frac{A}{p} \otimes_k \frac{B}{q}\big)_{\frac{Q}{p\otimes_k B + A \otimes_k q}}.$$

Therefore

$$\dim((\frac{A}{p} \otimes_k B)_{\frac{Q}{p\otimes_k B}}) = \dim(B_q) + \mathrm{ht}(\frac{Q}{p \otimes_k B + A \otimes_k q}). \tag{2.0.2}$$

Combining (2.0.1) and (2.0.2) yields to (1)

$$\mathrm{ht}(Q) = \mathrm{ht}(p) + \mathrm{ht}(q) + \mathrm{ht}(\frac{Q}{p \otimes_k B + A \otimes_k q}).$$

(2) By similar steps and by using Proposition 1.5 and replacing $\mathrm{ht}(-)$ with $G(-)$, we get that

$$G(A \otimes B)_Q = G(A_p) + G(\frac{A}{p} \otimes_k B)_{\frac{Q}{p\otimes_k B}}, \tag{2.0.3}$$

$$G(\frac{A}{p} \otimes_k B)_{\frac{Q}{p\otimes_k B}} = G(B_q) + G(\frac{A}{p} \otimes_k \frac{B}{q})_{\frac{Q}{p\otimes_k B + A \otimes_k q}}. \tag{2.0.4}$$

Let $S := (A \setminus p) \otimes (B \setminus q)$ and $H := \dfrac{S^{-1}Q}{pA_p \otimes B_q + A_p \otimes qB_q}$. Then, we have the natural isomorphism

$$(\frac{A}{p} \otimes_k \frac{B}{q})_{\frac{Q}{p\otimes_k B + A \otimes_k q}} \cong (\kappa_A(p) \otimes_k \kappa_B(q))_H.$$

Further, Lemma 2.5 implies that $\kappa_A(p) \otimes_k \kappa_B(q)$ is a Cohen-Macaulay ring and therefore

$$
\begin{aligned}
G(\kappa_A(p) \otimes_k \kappa_B(q))_H &= \dim(\kappa_A(p) \otimes_k \kappa_B(q))_H \\
&= \mathrm{ht}(\frac{Q}{p \otimes_k B + A \otimes_k q}). \qquad (2.0.5)
\end{aligned}
$$

Hence, by combining (2.0.3), (2.0.4) and (2.0.5) we get (2). □

Proof of Theorem 2.2.

The implications (1)⇒(2)⇒(3) are obvious. Assume that (3) holds. Let p and q be prime ideals of A and B, respectively. Notice that flat homomorphisms satisfy GD (Going Down) [32, Theorem 15.1] and under GD, obviously, minimal primes contract to minimal primes. So, minimal primes of $\frac{A \otimes_k B}{p \otimes_k B + A \otimes_k q} \cong \frac{A}{p} \otimes \frac{B}{q}$ contract to the null ideal in the domains $\frac{A}{p}$ and $\frac{B}{q}$. That is, minimal primes of $A \otimes B$ over $p \otimes_k B + A \otimes_k q$ contract to p and q in A and B, respectively. Therefore, by Proposition 2.1, $\mathrm{ht}(p \otimes_k B + A \otimes_k q) = \mathrm{ht}(p) + \mathrm{ht}(q)$. By Theorem 2.1, we obtain

$$
\mathrm{ht}(p) + \mathrm{ht}(q) = \mathrm{ht}(p \otimes_k B + A \otimes_q) = G(p \otimes_k B + A \otimes_k q) = G(p) + G(q).
$$

It follows that

$$
\mathrm{ht}(p) - G(p) = G(q) - \mathrm{ht}(q).
$$

The left side is a non-negative quantity while the other side is a non-positive one by Proposition 1.3. Therefore, $G(p) = \mathrm{ht}(p)$ and $\mathrm{ht}(q) = G(q)$. Hence, A and B are Cohen-Macaulay rings by Theorem 1.1.

Assume now that (4) holds. Let Q be a prime ideal of $A \otimes_k B$ with $p := A \cap Q$ and $q := B \cap Q$. Since A and B are Cohen-Macaulay rings, $G(p) = G(p_p)$ and $G(q) = G(q_q)$. Therefore, by Proposition 2.1,

$$\begin{aligned}\dim((A \otimes_k B)_Q) = \mathrm{ht}(Q_Q) = \mathrm{ht}(p) + \mathrm{ht}(q) + \mathrm{ht}(\frac{Q}{p \otimes_k B + A \otimes_k q}) \\ = G(p) + G(q) + \mathrm{ht}(\frac{Q}{p \otimes_k B + A \otimes_k q}) = G((A \otimes_k B)_Q).\end{aligned}$$

Hence, $(A \otimes_k B)_Q$ is a Cohen-Macaulay ring, for every prime ideal Q. Therefore, by Theorem 1.1, $A \otimes_k B$ is a Cohen-Macaulay ring. □

CHAPTER 3

REGULARITY, GORENSTEINESS, AND COMPLETE INTERSECTION IN TENSOR PRODUCTS

This section deals with the transfer of regularity, Gorensteiness, complete intersection, and Serre's conditions to Noetherian tensor products of k-algebras.

Given a Noetherian ring A and a positive integer n, the so-called Serre's conditions are defined as follows:

(R_n) A_p is regular, for every $p \in \mathrm{Spec}(A)$ with $\mathrm{ht}(p) \leq n$.

(S_n) $G(A_p) \geq \mathrm{Min}(\dim(A_p), n)$, for each $p \in \mathrm{Spec}(A)$.

Notice that A satisfies (R_n) for each n is equivalent to A being a regular ring. Similarly, A satisfies (S_n) for each n is equivalent to A being a Cohen-Macaulay ring

[32, p. 183]. Serre's conditions transfer through flat homomorphisms of Noetherian rings, as shown below.

Proposition 3.1 ([19, Propositions 2.1.16 and 2.2.21]) *"Let $f : A \longrightarrow B$ be a flat homomorphism of Noetherian rings. Let $Q \in \mathrm{Spec}(B)$ and set $q := Q \cap A$. Then:*

(1) *If B_Q satisfies (S_n) (resp., (R_n)), then so does A_q.*

(2) *If A and $\kappa_A(p) \otimes_A B$, $p \in \mathrm{Spec}(A)$, satisfy (S_n) (resp., (R_n)), then so does B."*

I t is known that a Noetherian tensor product of two extension fields is Gorenstein [36]. This section features the extension of this result to a more general context of tensor products of k-algebras, in addition to the transfer of complete intersection and Serre's conditions. The main theorem reads as follows:

Theorem 3.1 ([40, Theorem 6]) *"Let A and B be two k-algebras such that $A \otimes_k B$ is Noetherian. Then:*

(1) *$A \otimes_k B$ is a locally complete intersection (resp., Gorenstein) ring if and only if so are A and B.*

(2) *$A \otimes_k B$ satisfies (S_n) if and only if so do A and B.*

(3) *If $A \otimes_k B$ is regular, then so are A and B.*

(4) *If $A \otimes_k B$ satisfies (R_n), then so do A and B.*

(5) *The converses of (3) and (4) hold if k is perfect."*

Recall that k is said to be perfect if every (algebraic) extension of k is separable; equivalently, either $\text{char}(k) = 0$ or $\text{char}(k) = p$ with $k = \{a^p \mid a \in k\}$ [14, § 4, Definition 2] and [14, § 7, Proposition 2 and Corollary]. The proof of the theorem requires the following preliminary results.

Lemma 3.1 ([40, Corollary 2]) *"Let $f : A \longrightarrow B$ be a flat homomorphism of Noetherian rings. Then:*

(1) *If A and the fibers $\kappa_A(p) \otimes_A B$, $p \in \text{Spec}(A)$, are regular (resp., locally complete intersection, Gorenstein) rings, then so is B.*

(2) *If B is a locally complete intersection (resp., Gorenstein) ring, then so are the fibers $\kappa_A(p) \otimes_A B$, $p \in \text{Spec}(A)$."*

Proof. (1) Let Q be a prime ideal of B and set $q := Q \cap A$. Consider the induced flat local homomorphism $A_q \longrightarrow B_Q$ and note that

$$\frac{B_Q}{qB_Q} \cong (\frac{B_q}{qB_q})_{\frac{Q_q}{qB_q}} \cong (\kappa_A(q) \otimes_A B)_{\frac{Q_q}{qB_q}}.$$

Since A and $\kappa_A(q) \otimes_A B$ are regular (resp., locally complete intersections, Gorenstein), A_q and $(\kappa_A(q) \otimes_A B)_{\frac{Q_q}{qB_q}}$ are regular (resp., complete intersections, Gorenstein). By Theorems 1.2 and 1.3, B_Q is regular (resp., a complete intersection, Gorenstein). Since Q is an arbitrary prime ideal of B, B is regular (resp., locally complete intersection, Gorenstein).

(2) Let p be a prime ideal of A. If p is not a contraction of any prime of B then $pB = B$ (since f satisfies GD) and hence there is nothing to show. Next, suppose

that $p = A \cap q$, for some prime ideal q of B. We show that $T := \kappa_A(p) \otimes_A B \cong \dfrac{B_p}{pB_p}$ is a locally complete intersection (resp., Gorenstein) ring. Let $\frac{q'_p}{pB_p}$ be a prime ideal of T and notice that $q'_p \cap A_p = p_p$; that is, $q' \cap A = p$. Consider the induced homomorphism $A_p \longrightarrow B_{q'}$. Since B is locally complete intersection (resp., Gorenstein), then so is $B_{q'}$. By Theorem 1.3, $\dfrac{B_{q'}}{pB_{q'}} \cong (\dfrac{B_p}{pB_p})_{\frac{q'_p}{pB_p}}$ is a complete intersection (resp., Gorenstein) ring. Since this holds for arbitrary prime ideals of T, so T is a locally complete intersection (resp., Gorenstein) ring, proving (2). □

Lemma 3.2 *"Let $f : A \longrightarrow B$ be a faithfully flat homomorphism of Noetherian rings. If B is locally complete intersection (resp., Gorenstein), then so is A."*

Proof. Assume that B is locally complete intersection (resp., Gorenstein). Since f is faithfully flat, $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective [2, Ex. 16, p. 45]. Let p be a prime ideal of A and let q a prime ideal of B such that $p = A \cap q$. Then, B_q is a complete intersection (resp., Gorenstein) ring. Consider the induced flat local homomorphism $A_p \longrightarrow B_q$. By Theorem 1.3, A_p is a complete intersection (resp., Gorenstein) ring, completing the proof. □

The following lemma handles the converse of Theorem 1.2 and Proposition 3.1.

Lemma 3.3 ([40, Corollary 4]) *"Let $f : A \longrightarrow B$ be a faithfully flat homomorphism of Noetherian rings. Then:*

(1) *If B is regular (resp., a complete intersection, Gorenstein), then so is A.*

(2) *If B satisfies (S_n) (resp., (R_n)), then so does A."*

Proof. The proof steps are similar to those in Lemma 3.1. □

Lemma 3.4 *"Let $f : A \longrightarrow B$ be a flat homomorphism of Noetherian rings. Let $q \in \mathrm{Spec}(B)$ and set $p := A \cap q$. Assume that $pB = q$. Then B_q is a Gorenstein (resp., complete intersection) ring if and only if so is A_p."*

Proof. Consider the flat local homomorphism $A_p \longrightarrow B_q$ and observe that $\dfrac{B_q}{pB_q} \cong \dfrac{B_q}{q_q} = \kappa_B(q)$, as $pB = q$. Therefore, Theorem 1.3 leads to the conclusion. □

Lemma 3.5 ([36, Theorem 2.2]) *"Let K and L be two extension fields of k such that $K \otimes_k L$ is Noetherian. Then, $K \otimes_k L$ is Gorenstein."*

Proof. Let q be a prime ideal of $K \otimes_k L$. Since $K \otimes_k L$ is Noetherian, q is finitely generated; that is, there exist $m_i \in K \otimes_k L$ for $i = 1, 2, \ldots, n$ for some n such that $q = (m_1, \ldots, m_n)$. Each $m_i = \sum_{j=1}^{n_i} \alpha_{j_i} \otimes_k \beta_{j_i}$ for some $\alpha_{j_i} \in K$ and $\beta_{j_i} \in L$. Let $S := \bigcup_{i=1}^{n} \{\alpha_{1_i}, \ldots, \alpha_{n_{i_i}}\}$ and let $V := k(S)$ the extension field of k generated by S. Notice that $V \subseteq K$ and, since L is flat over k, $V \otimes_k L \subseteq K \otimes_k L$. Let $p := (V \otimes_k L) \cap q$ and notice that $q = p(K \otimes_k L)$ as q's generators are in V. Since K is faithfully flat over V, $K \otimes_k L \cong K \otimes_V (V \otimes_k L)$ is faithfully flat over $V \otimes_k L$ by [32, p. 46]. By Lemma 3.4, $(K \otimes_k L)_q$ is Gorenstein if and only if so is $(V \otimes_k L)_p$. Further, by Corollary 1.7, since V is finitely generated over k and L is Gorenstein, $V \otimes_k L$ is Gorenstein, and so is $(V \otimes_k L)_p$. Hence $(K \otimes_k L)_q$ is Gorenstein. Since q is arbitrary, we showed that $K \otimes_k L$ is Gorenstein. □

Let us recall again Grothendieck's result that "the tensor product of two extension fields of k is regular provided one of them is finitely generated and separable over k" [24].

Lemma 3.6 ([40, Proposition 5]) *"Let K and L be two extension fields of k such that $K \otimes_k L$ is Noetherian. Then:*

(1) *$K \otimes_k L$ is a locally complete intersection ring.*

(2) *If k is perfect, then $K \otimes_k L$ is regular."*

Proof. (1) Mimic Lemma 3.5 after substituting '(locally) complete intersection' for 'Gorenstein'.

(2) Here too mimic Lemma 3.5 after substituting 'regular' for 'Gorenstein' and use Grothendieck's aforementioned result. $\square$

Proof of Theorem 3.1. (1) Let Q be a prime of $A \otimes_k B$ with $p := A \cap Q$ and $q := B \cap Q$. By similar steps of Proposition 2.1 and using Theorem 1.3, one can see that $(A \otimes_k B)_Q$ is a complete intersection (resp., Gorenstein) ring if and only if A_p, B_q and $(\kappa_A(p) \otimes_k \kappa_B(q))_H$ are complete intersection (resp., Gorenstein) rings, where $S := (A \setminus p) \otimes (B \setminus q)$ and $H := \dfrac{S^{-1}Q}{pA_p \otimes B_q + A_p \otimes qB_q}$. By Lemma 3.6 (resp., 3.5), $\kappa_A(p) \otimes_k \kappa_B(q)$ is a locally complete intersection (resp., Gorenstein) ring and so is $(\kappa_A(p) \otimes_k \kappa_B(q))_H$. It follows that $(A \otimes_k B)_Q$ is a complete intersection (resp., Gorenstein) ring if and only if so are A_p and B_q. So, if A and B are locally complete intersection (resp., Gorenstein) rings, so is $A \otimes_k B$. Conversely, since $A \otimes_k B$ is faithfully flat over A (resp., B), then the transfer of locally complete intersection and Gorensteiness to A and B is guaranteed by Lemma 3.2.

(2) If $A \otimes_k B$ satisfies (S_n), then$A$ and B satisfy (S_n) by Lemma 3.3. Conversely, following similar steps of (1), we have $\kappa_A(p) \otimes_k \kappa_B(q)$ is Cohen-Macaulay by Theorem 2.2 and therefore it satisfies (S_n). Further, since A and B satisfy (S_n), so do A_p and B_q. Finally, by Proposition 3.1, $A \otimes_k B$ satisfies (S_n).

(3) and (4) Similar arguments of (1) lead to the conclusion via Lemma 3.3.

(5) Suppose k is perfect; i.e., every extension field of k is separable. Let Q be a prime ideal of $A \otimes_k B$ with $p := A \cap Q$ and $q := B \cap Q$. Since A and B are regular, A_p and B_q are regular. By Lemma 3.6, since k is perfect and $\kappa_A(p) \otimes_k \kappa_B(q)$ is Noetherian, $\kappa_A(p) \otimes_k \kappa_B(q)$ is regular. Therefore, $(\kappa_A(p) \otimes_k \kappa_B(q))_H$ is regular. By similar steps of Proposition 2.1 and Lemma 3.1, it is easily seen that since A_p, B_q and $(\kappa_A(p) \otimes_k \kappa_B(q))_H$ are regular, $(A \otimes_k B)_Q$ is regular, completing the proof of the theorem. $\square$

A counter-example to the converse of Theorem 3.1(3) is given below. At this point, recall that a regular local ring is an integral domain [30, Theorem 164].

Example 3.2 ([40, Remark 7]) Let k be an imperfect field with $\mathrm{char}(k) = 3$. Hence, there exists $a \in k$ such that $x^3 - a$ has no root in k. Then, $K := \dfrac{k[x]}{(x^3 - a)} = k(\beta)$, for $\beta := \bar{x}$, is a splitting field of $x^3 - a = (x - \beta)^3$. Therefore, $K \otimes_k K \cong \dfrac{K[x]}{(x^3 - a)}$ is not a regular ring , since the localization with respect to $(x - \beta)$ is not an integral domain (e.g., , $(x - \beta)^3 = 0$).

CHAPTER 4

REGULARITY OF TENSOR PRODUCTS OF EXTENSION FIELDS AND APPLICATIONS

This section deals with the transfer of regularity to tensor products of extension fields. Precisely, it establishes necessary and sufficient conditions for the tensor product of two extension fields of k to inherit regularity under various assumptions of separability. Then, among the applications, the results are extended to residually separable k-algebras.

Recall that, in contrast to the notions of Cohen-Macaulay, Gorenstein, and complete intersection rings, the tensor product of two extension fields of k need not be regular, in general (see Example 3.2). Several cases have been covered regarding the transfer of regularity to tensor products. Grothendieck proved that the tensor product of two extension fields K and L is regular provided that either K or L is finitely generated over k and if K is separable over k (see Lemma 1.2). In [40], the authors showed that for two k-algebras A and B, if $A \otimes_k B$ is Noetherian, then $A \otimes_k B$ is regular if and only if A and B are regular (Theorem 3.1) in the special case where k is perfect (i.e., every (algebraic) extension of k is separable). Later, [12] tackled more general settings; including, the cases where one of the extension fields K or L is purely inseparable, normal, or algebraic over k, or a product of the separable closure and the purely inseparable closure of k. The case where, for instance, K is purely inseparable or normal over k, or product of the separable closure and the purely inseparable closure of k is handled in Theorem 4.1. The case where K is algebraic is established in Corollary 4.5.

Recall that a Noetherian k-algebra A is said to be *geometrically regular* over k if $A \otimes_k F$ is regular for every finite extension field F of k; in particular, A is regular. The authors of [12] defined a new class of k-algebras called residually separable. It turns out that regular residually separable k-algebras are geometrically regular. The converse, in general, is not true as illustrated by Example 4.7. With this class, new cases have been investigated in Theorem 4.6 which generalizes Theorem 3.1.

Suppose k has characteristic p. Let K and L be extension fields of k such that K is (algebraic) purely inseparable over k; i.e., for every $x \in K$ there exists an integer n such that $x^{p^n} \in k$ (cf. [14, p. V.24]). Then, by [14, Proposition 5, p. V.26], K can be embedded in an algebraic closure of L via a unique homomorphism u that acts as an identity map on k. Further, $u(K)$ is purely inseparable. Therefore, in such a case, we can assume that K and L are contained in a common field. We denote the separable closure and the purely inseparable closure of k in K by K_s and K_i, respectively. Recall further, if K and L are subfields of a common field Ω, then K and L are linearly disjoint if every free subset of K (resp., L) over k is also free over L (resp., K); equivalently, if $K \otimes_k L$ is a domain (cf. [14, p. V.13]).

We now announce the first main theorem of this section.

Theorem 4.1 ([12, Theorem 2.4]) *"Let K and L be two extension fields of k such that $K \otimes_k L$ is Noetherian. Assume that $K = K_i K_s$ and let $K_i = k(S)$ for some generating subset S of K_i. Then, the following assertions are equivalent:*

(1) *$K \otimes_k L$ is regular;*

(2) *$K_i \otimes_k L$ is a domain;*

(3) *$K_i \otimes_k L$ is a field;*

(4) *$[k(S') : k] = [L(S') : L]$, for each finite subset S' of S;*

(5) *$K_i \cap L(S') = k(S')$, for each finite subset S' of S."*

For the proof of this theorem, we need the following lemmas.

Lemma 4.1 ([12, Lemma 2.1]) *"Let A be a geometrically regular k-algebra and K an extension field of k such that $A \otimes_k K$ is Noetherian. Then, $A \otimes_k K$ is regular."*

Proof. Let Δ denote the set of all finitely generated extension fields of k contained in K and let

$$R := A \otimes_k K = \varinjlim_{E \in \Delta} R(E)$$

where $R(E) := A \otimes_k E$, for each $E \in \Delta$. Next, let $P \in \mathrm{Spec}(R)$ with $P_E := P \cap R(E)$, for each $E \in \Delta$.

Step 1: We show that "*if $F \in \Delta$ such that $P_E R = P_F R$ for each $E \in \Delta$ containing F, then $P = P_F R$.*" Indeed, let $F \in \Delta$ such that $P_E R = P_F R$ for each $E \in \Delta$ containing F. Let $x \in P$. Then, there exists $E' \in \Delta$ such that $x \in R(E')$, and thus $x \in P_{E'} R$. Whence, $x \in P_{F(E')} = P_F R$, where $F(E')$ denotes the composite field of F and E' in K. It follows that $P = P_F R$, proving the claim.

Step 2: We claim that "*there exists $E \in \Delta$ with $P = P_E R$.*" Indeed, assume, for contradiction, that $P_E R \subsetneqq P$ for any $E \in \Delta$ (notice that under this hypothesis K is necessarily infinitely generated over k; i.e., $K \notin \Delta$). Choose $E_1 \in \Delta$. By Step 1, there exists $E_2 \in \Delta$ containing E_1 such that $P_{E_1} R \subsetneqq P_{E_2} R$. Iterating this process yields the following infinite chain of ideals in R $P_{E_1} R \subsetneqq ... \subsetneqq P_{E_n} R \subsetneqq ... \subsetneqq P$, where the $E_j' s \in \Delta$. This is absurd, as R is Noetherian.

Step 3: We show that "*PR_P is generated by an R_P-regular sequence.*" By Step 2, $P = P_E R$ for some $E \in \Delta$. Note that $R_P := (A \otimes_k K)_P \cong (R(E)_{P_E} \otimes_E K)_P$ and $PR_P \cong (P_E R(E)_{P_E} \otimes_E K) R_P$ with $P_E R(E)_{P_E}$ being the maximal ideal of $R(E)_{P_E}$. As E is finitely generated over k, $R(E)$ is regular (recall that A is geometrically regular). Hence $R(E)_{P_E}$ is a regular local ring. By [30, Theorem 169], $P_E R(E)_{P_E}$ is generated by an $R(E)_{P_E}$-regular sequence $x_1, ..., x_r$. Further, it is easily seen that $x_1 \otimes_k 1, ..., x_r \otimes_E 1$ is an $R(E)_{P_E} \otimes_E K$-regular sequence of $P_E R(E)_{P_E} \otimes_E K$. As $\big(P_E R(E)_{P_E} \otimes_E K\big) R_P \cong PR_P$, $\frac{x_1 \otimes_E 1}{1}, ..., \frac{x_n \otimes_E 1}{1}$ is an R_P-regular sequence of PR_P. Now, since $P_E R(E)_{P_E} = (x_1, ..., x_n) R(E)_{P_E}$, we get $PR_P = \big(\frac{x_1 \otimes_E 1}{1}, ..., \frac{x_n \otimes_E 1}{1}\big) R_P$, establishing the last step.

It follows, by [30, Theorem 160], that R_P is regular, completing the proof. □

Lemma 4.2 ([12, Lemma 2.1]) *"Let A and B be k-algebras such that A is geometrically regular. Then, $A \otimes_k B$ is regular if and only if B is regular and $A \otimes_k B$ is Noetherian."*

Proof. Necessity holds by Theorem 3.1(3). For sufficiency, let $P \in \text{Spec}(A \otimes_k B)$ with $q := B \cap P$. We show that $(A \otimes_k B)_P$ is regular. Notice that we always have the induced flat local homomorphism $B_q \to (A \otimes_k B)_P$. Since B is regular, then so is B_q. Hence, it is enough to show that $(A \otimes_k \kappa_B(q))_{\frac{P_q}{A \otimes_k q_q}}$ is regular which would imply that $(A \otimes_k B)_P$ is regular by Theorem 1.2. But, this is a localization of a Noetherian tensor product of A with an extension field of k. So, it is regular by Lemma 4.1. $\square$

In particular, if K is a separable extension field of k; i.e., K is geometrically regular by Lemma 1.2 (cf. [32, p. 198]), the above lemma asserts that "$K \otimes_k A$ *is regular if and only if* A *is regular and* $K \otimes_k A$ *is Noetherian.*"

Proof of Theorem 4.1. Without loss of generality we may assume that $\text{char}(k) = p \neq 0$ and that k is imperfect by Lemma 3.6. We reduce the proof to the case where K is purely inseparable by the following steps. Since K_s is a separable extension field of k, $K_s \otimes_k K_i$ is reduced [44, Chap. III, §15, Theorem 39]. Moreover, since K_i is algebraic, $\dim(K_s \otimes_k K_i) = 0$ by Discussion 2.3 and therefore it has one unique minimal prime ideal by [41, Proposition 2(c)]. Hence, $K_s \otimes_k K_i$ is local. But since it is reduced, i.e., $\text{Nil}(K_s \otimes_k K_i) = 0$, it is a field. Next, consider the homomorphism $\phi : K_s \otimes_k K_i \longrightarrow K$ defined by $a \otimes_k b \mapsto ab$ using the bilinear map $(a, b) \to ab$. Since $K = K_s K_i$, ϕ is surjective, and since it is not a zero map and $K_s \otimes_k K_i$ is a field, it is injective. Therefore, $K \cong K_s \otimes_k K_i$. Now, $K \otimes_k L$ is regular if and only if

$K_s \otimes_k (K_i \otimes_k L)$ is regular. Since K_s is separable, this holds if and only if $K_i \otimes_k L$ is regular by Lemma 4.2. Hence, without loss of generality, we may assume K is purely inseparable over k. Therefore, by similar argument as above, $K \otimes_k L$ is an Artin local ring. It follows then (1) $\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(1) by [30, Theorem 164] (which states that a regular local ring is a domain). Furthermore, (2)$\Leftrightarrow$(4) by [14, Proposition 5(a), p. V.13] and that $K \otimes_k L \cong K \otimes_{k(S')} (k(S') \otimes_k L) \cong K \otimes_{k(S')} L(S')$ since the fact "$K \otimes_k L$ is a domain" is equivalent to "K and L are linearly disjoint over k". Also, by a similar argument, (2)$\Rightarrow$(5).

(5)$\Rightarrow$(3) Let Δ be the collection of all finite subsets of S and notice that $K = \varinjlim_{S' \in \Delta} k(S')$. Since $K \otimes_k L \cong \varinjlim_{S' \in \Delta} (k(S') \otimes_k L)$ by [2, Ex. 20, p. 33], it is enough to show that $k(S') \otimes_k L$ is a field for every finite subset S' of S. Let $x \in S$ and, without loss of generality, assume that $x \notin k$. Since K is purely inseparable over k and $\mathrm{char}(k) = p \neq 0$, $[k(x) : k] = p^m$ for some $m \geq 1$ and x has a minimal polynomial $y^{p^m} - a$, where $a = x^{p^m}$. Observe that $x \in S \subseteq K$, and by minimality of the polynomial and (5), $x^{p^r} \notin K \cap L = k$, for each integer $r < m$. Notice also that $X^{p^m} - a = (X^{p^r} - x^{p^r})^{p^{m-r}}$, for each integer $r < m$, and so it is irreducible over L. Therefore, $k(x) \otimes_k L \cong \frac{L[X]}{(X^{p^m} - a)} \cong L(x)$ is a field. We proceed by induction on the length of S'. Let $S' = \{x_1, x_2, \dots, x_n\}$ and notice that

$$k(x_1, x_2, \dots, x_n) \otimes_k L \cong k(x_1, x_2, \dots, x_n) \otimes_{k(x_1, x_2, \dots, x_{n-1})} (k(x_1, x_2, \dots, x_{n-1}) \otimes_k L).$$

By induction,

$$k(x_1, x_2, \dots, x_{n-1}) \otimes_k L \cong L(x_1, x_2, \dots, x_{n-1}).$$

Further, by (5) we have

$$k(x_1, x_2, \dots, x_{n-1}) \subseteq k(x_1, x_2, \dots, x_n) \cap L(x_1, x_2, \dots, x_{n-1})$$

$$\subseteq K \cap L(x_1, x_2, \dots, x_{n-1}) = k(x_1, x_2, \dots, x_{n-1}).$$

We obtain

$$k(x_1, x_2, \dots, x_{n-1})(x_n) \cap L(x_1, x_2, \dots, x_{n-1}) = k(x_1, x_2, \dots, x_{n-1}).$$

Therefore by similar steps replacing k by $k(x_1, x_2, \dots, x_{n-1})$, we get that

$$k(x_1, x_2, \dots, x_n) \otimes_k L \cong k(x_1, x_2, \dots, x_{n-1})(x_n) \otimes_{k(x_1,x_2,\dots,x_{n-1})} L(x_1, x_2, \dots, x_{n-1})$$

$$\cong L(x_1, x_2, \dots, x_n)$$

completing the proof of the theorem. □

Next, we provide a simple illustrative example of the previous theorem.

Example 4.2 (See also [12, Example 2.5]) Let p be a prime integer, x and y two indeterminates over $Z_p := \dfrac{Z}{pZ}$. Let $k := Z_p(x^p)$, $K := k(x)$ and $L := k(y)$. Then, K and L are both finitely generated extension over k and hence $K \otimes_k L$ is Noetherian. Notice also that K is purely inseparable over k, and that $K \cap L = k$ and $K \cap L(x) = k(x)$. Therefore, by Theorem 4.1(5), $K \otimes_k L$ is regular.

It is also possible to construct examples of tensor products, of two extension fields of k, that are locally complete intersection but not regular by a simple violation of

Theorem 4.1(5); i.e., when $k \subsetneqq K \subseteq L$ such that K is purely inseparable over k.

Example 4.3 (See [12, Example 2.7]) Let x be an indeterminate over Z_p, $k := Z_p(x^p)$, and $K := L := Z_p(x)$. Then, $K \otimes_k L$ is Noetherian, and therefore it is locally complete intersection by Lemma 3.6. Since $K \cap L \neq k$, $K \otimes_k L$ is not regular by Theorem 4.1(5).

In [40], the authors provided an example of a tensor product of two fields in the form $K \otimes_k K$ for which it is locally complete intersection but not regular (Example 3.2). The next corollary provides necessary and sufficient conditions when such a form of tensor products is regular.

Corollary 4.4 ([12, Corollary 2.6]) *"Let K be an extension field of k. The following assertions are equivalent:*

(1) *$K \otimes_k K$ is regular;*

(2) *$K \otimes_k K$ is Noetherian and K is separable over k;*

(3) *K is finitely generated and separable over k;*

(4) *$K \otimes_k L$ is regular, for each extension field L of k;*

(5) *K is finitely generated over k and a projective $K \otimes_k K$-module."*

Proof. (2)⇒(3) follows from Discussion 2.3, (3)⇒(4) is handled by Lemma 1.2, (4)⇒(1) is trivial (with $L = K$), and (3)⇔(5) is a particular case of [20, Theorem 7.10, p. 179].

(1)⇒(2) Assume that $K \otimes_k K$ is regular. Then, it is Noetherian and therefore by Discussion 2.3, it is a finitely generated extension field of k. We show that K is separable over k. First, let E be an extension field of k in K and notice that

$$\begin{aligned} K \otimes_k K &\cong K \otimes_E (E \otimes_k K) \\ &\cong K \otimes_E (K \otimes_k E) \\ &\cong (K \otimes_E K) \otimes_k E. \end{aligned}$$

It follows, by Theorem 3.1(3), that $K \otimes_E K$ is regular. Let B be a finite transcendence basis of K over k and let E be the algebraic separable closure of $k(B)$. Hence, by [14, Proposition 13, p. V.44], K is purely inseparable over E. By Theorem 4.1(5), with $L = K$ and $K_i = K$, $K = K_i \cap L(S) = E(S) = E$. So, K is separable over k. □

The investigation of the case when K or L is algebraic over k is handled below below, and it generalizes slightly [41, Proposition 8], since if K is separable over k, then $K \otimes_k L$ is reduced [44, Chap. III, §15, Theorem 39].

Corollary 4.5 ([12, Corollary 2.8]) *"Let K and L be two extension fields of k such that $K \otimes_k L$ is Noetherian. Assume that K is algebraic over k. Then, the following assertions are equivalent:*

(1) *$K \otimes_k L$ is (von Neumann) regular;*

(2) *$K \otimes_k L$ is reduced;*

(3) *$K \otimes_k L$ is a finite product of fields.*

If, in addition, K is separable and L is Galois over k such that K and L are contained in an algebraic closure of k, then the above statements are equivalent to:

(4) $n := [K \cap L : k] < \infty$.

Moreover, $K \otimes_k L$ is isomorphic to the product of n copies of the field $K(L)$."

Proof. Recall that a regular ring is reduced by a combination of [30, Theorem 164] and [30, Theorem 168]. On the other hand, a reduced zero-dimensional ring is von Neumann regular by [30, Ex. 22, p. 64]. Further, by [41, Lemma 0], if a zero-dimensional von Neumann ring is Noetherian, it is a finite product of fields. By [19, Corollary 2.2.20], a finite product of regular rings is regular. Therefore, on a zero-dimensional Noetherian ring, the above notions are equivalent. Now, if K and L are extension fields of k such that K is algebraic over K and $K \otimes_k L$ is Noetherian, then $\dim(K \otimes_k L) = 0$ by [37, Theorem 3.1], and therefore (1)⇔(2)⇔(3). Further, if the last assertion holds, then by [41, Proposition 8], we get (3)⇔(4). □

Definition 4.1 A k-algebra R is said to be residually separable, if $\kappa_R(p)$ is separable over k, for each $p \in \mathrm{Spec}(R)$.

Several examples of residually separable k-algebras can be easily constructed. For instance, let K be a separable extension field of k and let $A := K[x]_{(x)}$. Then, A is local (in fact, valuation) ring with only two prime ideals; namely, 0 and xA. Their residue fields are $K(x)$ and K, respectively, and so both are separable over k.

In order to emphasize the importance of this class of k-algebras, next we show that, in this class, regularity coincides with geometrical regularity.

Lemma 4.3 *Let A be a residually separable k-algebra. Then A is regular if and only if A is geometrically regular.*

Proof. As shown in the introduction of this section, if A is geometrically regular, then it is regular. Now, assume that A is regular and let K be a finitely generated extension field of k. Hence, $A \otimes_k K$ is Noetherian. Let P be a prime ideal of $A \otimes_k K$ and set $p := A \cap P$. Since $\kappa_A(p)$ is separable and K is finitely generated, $\kappa_A(p) \otimes_k K$ is regular Lemma 1.2. Since A_p is regular and $(\kappa_A(p) \otimes_k K)_H$ is regular, where $H = \frac{P_p}{p_p \otimes_k K}$, $(A \otimes_k K)_P$ is regular by Theorem 1.2. Since P is arbitrary, $A \otimes_k K$ is regular. Further, since K is arbitrary, A is geometrically regular. □

Next, we announce the second theorem of this section.

Theorem 4.6 ([12, Theorem 2.11]) *"Let A and B be k-algebras such that $A \otimes_k B$ is Noetherian. Consider the following assertions:*

(1) *A, B and $\kappa_A(p) \otimes_k \kappa_B(q)$ are regular, for each $(p, q) \in \mathrm{Spec}(A) \times \mathrm{Spec}(B)$;*

(2) *B and $A \otimes_k \kappa_B(q)$ are regular, for each $q \in \mathrm{Spec}(B)$;*

(3) *A and $\kappa_A(p) \otimes_k B$ are regular, for each $p \in \mathrm{Spec}(A)$;*

(4) *$A \otimes_k B$ is regular;*

(5) *A and B are regular.*

Then (1)⇒(2) (resp., (3)) ⇒ (4) ⇒ (5). If A (or B) is residually separable, then all assertions are equivalent"

Proof. The first statement is handled by a combination of Lemma 3.1, Lemma 3.3 and Theorem 3.1. Next, assume that A and B are regular and A is residually separable. Let $(p, q) \in \mathrm{Spec}(A) \times \mathrm{Spec}(B)$. Then, $\kappa_A(p) \otimes_k \kappa_B(q)$ is Noetherian (through quotient and localization) and hence it is regular by Lemma 4.2, which also make $\kappa_A(p) \otimes_k B$ and $A \otimes_k \kappa_B(q)$ regular by the first statement. Therefore, the assertions in the theorem fall down to "$A \otimes_k B$ is regular if and only if A and B are regular." □

The following example shows that the above implications are not reversible in general. It also shows that Lemma 4.2 does not hold, in general, for purely inseparable extensions. It also provides an example of a geometrically regular k-algebra that is not residually separable.

Example 4.7 ([12, Example 2.12]) Let k be an imperfect field of characteristic p and K a purely inseparable extension field of k. Let $u \in K$ such that $[k(u) : k] = p^e$ for some $e \geq 2$. Then, $a := u^{p^e} \in k$. Let $r \in \{1, \ldots, e-1\}$ and $A := k[x]_{(x^{p^{e-r}} - a)}$. Notice that $x^{p^{e-r}} - a$ is irreducible by similar argument in the last part of Theorem 4.1 and therefore $(x^{p^{e-r}} - a)$ is a maximal ideal since $k[x]$. Hence, A is local regular with maximal ideal $\mathfrak{m} := (x^{p^{e-r}} - a)A$ since $k[x]$ is regular. Further, $k(u) \otimes_k A \cong k(u) \otimes_k k[x] \otimes_{k[x]} (k[x]_{(x^{p^{e-r}} - a)}) \cong S^{-1}k(u)[x]$ is regular by [30, Theorem 171 and Ex. 9, p. 121], where $S := k[x] \setminus (x^{p^{e-r}} - a)$. Moreover, $\dfrac{A}{\mathfrak{m}} \cong \dfrac{k[x]}{(x^{p^{e-r}} - a)} \cong k(u^{p^r})$, which is finitely generated and purely inseparable over k. This shows that A is not residually separable since the only nonzero prime ideal of A is $\mathfrak{m}$. Since $k(u) \cap k(u^{p^r}) = k(u^{p^r}) \neq k$, $k(u) \otimes_k \dfrac{A}{\mathfrak{m}}$ is not regular by Theorem 4.1(5). Also, for any finitely generated extension field L of k, $A \otimes_k L \cong S^{-1}L[x]$ is regular. Therefore A is geometrically regular that is not residually separable.

CHAPTER 5

EMBEDDING DIMENSION AND CODIMENSION OF TENSOR PRODUCTS OF K-ALGEBRAS

Let A and B be two k-algebras and let Q be a prime ideal of $A \otimes_k B$ with $p := Q \cap A$ and $q := Q \cap B$. This section deals with the correlation between the embedding dimensions of $(A \otimes_k B)_Q$ and A_p, B_q. Two main cases are in order. The first one is when $A := K$ is a separable extension field of k. The second one is when A (or B) is residually separable. For this purpose, a new invariant is introduced and studied, which relates the embedding dimension of a local ring to the embedding dimension of its fibers. Further, the special case of polynomial rings is also examined, as it will play a primordial role in the investigation of tensor products.

Let $f : (A, \mathfrak{m},) \longrightarrow (B, \mathfrak{n})$ be a local homomorphism of Noetherian rings. Given an ideal I of A, define

$$\mu_A(I) := \dim_{\kappa_A(\mathfrak{m})} \left(\frac{I + \mathfrak{m}^2}{\mathfrak{m}^2} \right)$$

and

$$\mu_B(IB) := \dim_{\kappa_B(\mathfrak{n})} \left(\frac{IB + \mathfrak{n}^2}{\mathfrak{n}^2} \right).$$

Notice that $(I + \mathfrak{m}^2)/(\mathfrak{m}^2)$ (resp., $(IB + \mathfrak{n}^2)/(\mathfrak{n}^2)$) is a subspace of $\mathfrak{m} / \mathfrak{m}^2$ (resp., $\mathfrak{n} / \mathfrak{n}^2$). Therefore, $\mu_A(I)$ (resp., $\mu_B(IB)$) represents the maximum number of elements in I (resp., IB) that are part of a minimal basis for $\mathfrak{m}$ (resp., $\mathfrak{n}$). Also, if $x_1, \ldots, x_r \in \mathfrak{m}$ such that $f(x_1), \ldots, f(x_r)$ are part of a minimal basis of $\mathfrak{n}$, then $x_1, \ldots, x_r$ are part of a minimal basis of $\mathfrak{m}$ (see Remark 5.1). It follows that

$$0 \leq \mu_B(IB) \leq \mu_A(I) \leq \operatorname{embdim}(A) = \mu_A(\mathfrak{m}).$$

By the generalized principal ideal theorem [30, Theorem 152], we always have $\operatorname{embdim}(A) \geq \dim(A)$ and, hence, the (embedding) codimension of A is given by

$$\operatorname{codim}(A) := \operatorname{embdim}(A) - \dim(A).$$

In particular, A is regular if and only if $\operatorname{codim}(A) = 0$. Next, we state preparatory results that will simplify the investigation, later.

Proposition 5.1 ([3, Proposition 2.1]) *"Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homo-*

morphism and let I be an ideal of A. Then:

$$\operatorname{embdim}(B) = \mu_B(IB) + \operatorname{embdim}(\frac{B}{IB}). \tag{5.0.1}$$

In particular,

$$\operatorname{embdim}(A) = \mu_A(I) + \operatorname{embdim}(\frac{A}{I}).\text{''} \tag{5.0.2}$$

Proof. Follows from the facts that $\dfrac{IB+\mathfrak{n}^2}{\mathfrak{n}^2}$ is a subspace of $\dfrac{\mathfrak{n}}{\mathfrak{n}^2}$ and $\dfrac{\mathfrak{n}}{IB+\mathfrak{n}^2}$ is isomorphic to $\dfrac{\mathfrak{n}/IB}{(\mathfrak{n}/IB)^2}$. □

Remark 5.1 If $x_1, \ldots, x_r \in \mathfrak{m}$ such that $f(x_1), \ldots, f(x_r)$ are part of a minimal basis of $\mathfrak{n}$, then $x_1, \ldots, x_r$ are part of a minimal basis of $\mathfrak{m}$. Indeed, if $\sum_{i=1}^r a_i x_i \in \mathfrak{m}^2$ for some $a_1, \ldots, a_r \in A$, then $\sum_{i=1}^r f(a_i)f(x_i) \in \mathfrak{n}^2$. Since $f(x_1), \ldots, f(x_r)$ are part of a minimal basis of $\mathfrak{n}$, $f(a_i) \in \mathfrak{n}$ for each i, and hence $a_i \in \mathfrak{m}$ for each i (since A is local).

Corollary 5.2 ([3, Corollary 2.2]) *"Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism and let I be an ideal of A. Then:*

$$\operatorname{embdim}(B) \leq \operatorname{embdim}(A) - \operatorname{embdim}(\frac{A}{I}) + \operatorname{embdim}(\frac{B}{IB}). \tag{5.0.3}$$

In particular,

$$\operatorname{embdim}(B) \leq \operatorname{embdim}(A) + \operatorname{embdim}(\frac{B}{\mathfrak{m}B}).\text{''} \tag{5.0.4}$$

Proof. Recall that $\mu_B(IB) \leq \mu_A(I)$. By Proposition 5.1, $\mu_A(I) = \operatorname{embdim}(A) - \operatorname{embdim}(\frac{A}{I})$. By these observations, we obtain the first equation. In particular, if $I = \mathfrak{m}$, we obtain the second one. □

The next corollary generalizes Theorem 1.2 further, and provides a sufficient condition for the transfer of regularity.

Corollary 5.3 ([3, Corollary 2.3]) *"Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism that satisfies GD. Then:*

(1) $\operatorname{codim}(B) = (\mu_B(\mathfrak{m} B) - \dim(A)) + \operatorname{codim}(\dfrac{B}{\mathfrak{m} B})$.

(2) $\operatorname{codim}(B) + (\operatorname{embdim}(A) - \mu_B(\mathfrak{m} B)) = \operatorname{codim}(A) + \operatorname{codim}(\dfrac{B}{\mathfrak{m} B})$.

(3) *B is regular and $\mu_B(\mathfrak{m} B) = \operatorname{embdim}(A)$ if and only if A and $\dfrac{B}{\mathfrak{m} B}$ are regular."*

Proof. Since f satisfies GD, by [32, Theorem 15.1], $\dim(B) = \dim(A) + \dim(\dfrac{B}{\mathfrak{m} B})$. Subtracting this from Equation (5.0.1) yields (1). Adding $\operatorname{embdim}(A)$ to both sides and subtracting $\mu_B(\mathfrak{m} B)$ from both sides yields (2). Observe that $\operatorname{embdim}(A) = \mu_A(\mathfrak{m}) \geq \mu_B(\mathfrak{m} B)$, and that the codimensions are non-negative quantities. If one side is null, then so is the other side, forcing all the quantities to be null. Hence, we obtain (3). □

Corollary 5.4 ([3, Corollary 2.4]) *"Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism that satisfies GD. Then:*

(1) $\operatorname{codim}(B) \leq \operatorname{codim}(A) + \operatorname{codim}(\dfrac{B}{\mathfrak{m} B})$.

(2) *If $\dfrac{B}{\mathfrak{m} B}$ is regular, then* $\operatorname{codim}(B) \leq \operatorname{codim}(A)$*."*

Proof. We obtain (1) by Corollary 5.3 and observing that $\operatorname{embdim}(A) \geq \mu_B(\mathfrak{m} B)$. Assertion (2) is a special case of (1), with $\operatorname{codim}(\dfrac{B}{\mathfrak{m} B}) = 0$. □

Next, we announce and prove the first main theorem of this section.

Theorem 5.5 ([3, Theorem 3.1]) *"Let R be a Noetherian ring and let $X_1, \ldots, X_n$ be indeterminates over R. Let Q be a prime ideal of $R[X_1, \ldots, X_n]$ with $q := Q \cap R$. Then:*

$$\begin{aligned}\operatorname{embdim}(R[X_1,\ldots,X_n])_Q &= \operatorname{embdim}(R_q) + \operatorname{embdim}\left((\kappa_R(q)[X_1,\ldots,X_n])_{\frac{Q_q}{qR_q[X_1,\ldots,X_n]}}\right)\\ &= \operatorname{embdim}(R_q) + \operatorname{ht}(\frac{Q}{q[X_1,\ldots,X_n]}).\text{''}\end{aligned}$$

Proof. Notice first, since $\kappa_R(q)[X_1, \ldots, X_n]$ is regular,

$$\begin{aligned}\operatorname{ht}(\frac{Q_q}{qR_q[X_1,\ldots,X_n]}) &= \operatorname{ht}(\frac{Q}{q[X_1,\ldots,X_n]})\\ &= \dim\left((\kappa_R(q)[X_1,\ldots,X_n])_{\frac{Q_q}{qR_q[X_1,\ldots,X_n]}}\right)\\ &= \operatorname{embdim}\left((\kappa_R(q)[X_1,\ldots,X_n])_{\frac{Q_q}{qR_q[X_1,\ldots,X_n]}}\right).\end{aligned}$$

Thus, it is enough to show the second equality. Recall that if R is a local ring with maximal ideal $\mathfrak{m}$, and if $a \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $\operatorname{embdim}(R) = \operatorname{embdim}(\frac{R}{(a)}) + 1$ [30, Theorem 159]. Also, notice that $R[X_1, \ldots, X_n]_Q \cong R_q[X_1, \ldots, X_n]_{Q_q}$. Therefore, we reduce the problem to a Noetherian local ring R with a maximal ideal $\mathfrak{m}$ such that $Q \cap R = \mathfrak{m}$. Let $\mathfrak{m} = (a_1, \ldots, a_r)$, where $r = \operatorname{embdim}(R)$. We proceed by induction on n. Assume $n = 1$. We have two cases; namely, $Q = (a_1, \ldots, a_r)$ or $Q = (a_1, \ldots, a_r, f)$, where f is irreducible over $\kappa(\mathfrak{m})$ by [30, Theorem 28]. In both cases, we proceed by induction on r.

Case 1: "$Q = (a_1, \ldots, a_r)$." First, notice that the map $R \longrightarrow R[X_1]_Q$ defined by $a \mapsto \frac{a}{1}$ is injective since if $\frac{a}{1} = 0$, there exists $g \in R[X_1] \setminus Q$ such that $ag = 0$. Since

$g \notin Q \supseteq \mathfrak{m}[X_1]$, then one of its coefficients is a unit in R. Therefore, $a = 0$. Hence, we may identify $\frac{a}{1}$ as a in $R[X_1]_Q$. Now, if $r = 0$, then $\mathfrak{m} = 0$ and so $Q = 0$. So, assume $r \geq 1$, and notice that $a_i \neq 0$ for all i as seen above. Hence, $\text{embdim}(R[X_1]_Q) \geq 1$. Therefore, without loss of generality, assume $a_1 \notin Q_Q^2$. Whence, by [30, Theorem 159], $\text{embdim}(R[X_1]_Q) = 1 + \text{embdim}((\frac{R}{a_1})[X_1])_{\frac{Q}{a_1R[X_1]}}$. By induction on r, we have

$$\text{embdim}((\frac{R}{a_1})[X_1])_{\frac{Q}{a_1R[X_1]}} = \text{embdim}(\frac{R}{(a_1)}) = r - 1.$$

Hence, $\text{embdim}(R[X_1]_Q) = r$.

Case 2: "$Q = (a_1, \dots, a_r, f)$." Note that $\frac{Q}{\mathfrak{m}[X_1]} = (\bar{f}) \neq 0$ in $\kappa_R(\mathfrak{m})[X_1]$. Therefore, $\text{ht}(\dfrac{Q}{\mathfrak{m}[X_1]}) = 1$. If $r = 0$, then R is a field and therefore $R[X_1]$ is regular. Hence, $\text{embdim}(R[X_1]_Q) = 1$. Next, assume $r \geq 1$. We claim that $\mathfrak{m}[X_1]_Q \nsubseteq Q_Q^2$. Deny. We get that $Q_Q^2 = \mathfrak{m}[X_1] + f^2R[X_1]_Q$. This implies that $\dfrac{Q_Q}{Q_Q^2} \cong fR[X_1]_Q$. By [30, Theorem 158], $Q_Q = fR[X_1]_Q$. Notice now that, if $a \in \mathfrak{m}$, then $a \in fR[X_1]_Q$. Therefore, there exist $s, t \in R[X_1] \setminus Q$ and $g \in R[X_1]$ such that $fgt = sta \in \mathfrak{m}[X_1]$. As $f \notin \mathfrak{m}[X_1]$, we must have $gt \in \mathfrak{m}[X_1] \subseteq Q$. Hence, $a \in Q_Q^2$. Iterating the process eventually yields $a \in \cap Q^n = 0$ since $R[X_1]_Q$ is a Noetherian local ring by [30, Theorem 79]. We obtain that $\mathfrak{m}[X_1]_Q = 0$. Since the map $R \longrightarrow R[X_1]_Q$ is injective, this yields to $\mathfrak{m} = 0$, a contradiction. Therefore, assume without loss of generality, that $a := a_1 \notin Q_Q^2$. By induction and similar arguments of Case 1, we get that $\text{embdim}(R[X_1]_Q) = 1 + r$.

Next, assume that $n \geq 2$ and $Q \cap R = \mathfrak{m}$, and set $R[t] := R[X_1, \dots, X_t]$ and $\mathfrak{m}[t] := \mathfrak{m}[X_1, \dots, X_t]$ for $t \in \{1, \dots, n\}$. Let $Q' := Q \cap R[n-1]$. By induction, we have

$$\operatorname{embdim}(R[n]_Q) = \operatorname{embdim}(R[n-1]_{Q'}) + \operatorname{ht}(\frac{Q}{Q'[X_n]})$$

and

$$\operatorname{embdim}(R[n-1]_{Q'}) = \operatorname{embdim}(R) + \operatorname{ht}(\frac{Q'}{\mathfrak{m}[n-1]}).$$

Then, by [32, Theorem 17.9], $\kappa(\mathfrak{m})[n]$ is catenary and, therefore,

$$\operatorname{ht}\left(\frac{Q}{\mathfrak{m}[n]}\right) = \operatorname{ht}\left(\frac{Q'[X_n]}{\mathfrak{m}[n]}\right) + \operatorname{ht}\left(\frac{Q}{Q'[X_n]}\right).$$

Further, by [30, Theorem 148], since $\kappa(\mathfrak{m})[n-1]$ is Noetherian, we have

$$\operatorname{ht}(\frac{Q'[X_n]}{\mathfrak{m}[n]}) = \operatorname{ht}(\frac{Q'}{\mathfrak{m}[n-1]})$$

and so

$$\begin{aligned}\operatorname{embdim}(R[n]_Q) &= \operatorname{embdim}(R) + \operatorname{ht}(\frac{Q'}{\mathfrak{m}[n-1]}) + \operatorname{ht}(\frac{Q}{Q'[X_n]}) \\ &= \operatorname{embdim}(R) + \operatorname{ht}(\frac{Q}{\mathfrak{m}[n]})\end{aligned}$$

completing the proof of the theorem. □

The following corollary is a simple application of Theorem 5.5 which recovers the classical result on the transfer of regularity to polynomial rings.

Corollary 5.6 ([3, Corollary 3.2]) *"Let R be a Noetherian ring and let $X_1, \ldots, X_n$ be indeterminates over R. Let Q be a prime ideal of $R[X_1, \ldots, X_n]$ with $q = Q \cap R$.*

Then:

$$\text{codim}(R[X_1, \ldots, X_n]_Q) = \text{codim}(R_q). \tag{5.0.5}$$

In particular, $R[X_1, \ldots, X_n]$ *is regular if and only if so is* R*."*

Proof. $R[X_1, \ldots, X_n]_Q$ is a flat local extension of R_q. Hence, by [32, Theorem 15.1], $\dim(R[X_1, \ldots, X_n]_Q) = \dim(R_q) + \text{ht}(\frac{Q}{q[X_1, \ldots, X_n]})$. Now, combine this fact with Theorem 5.5. □

Let R be a ring and S a multiplicatively closed subset of R. Let P be a prime ideal of R such that $P \cap S = \emptyset$. Then, $(R_S)_{P_S} \cong R_P$. With this observation and Theorem 5.5, the next corollary characterizes the regularity of localizations of polynomial rings of R. Further, it relates the regularity of R to two famous classes of localizations of polynomial rings; namely, Nagata rings and Serre's conjecture rings. Let $X_1, \ldots, X_n$ be indeterminate over R. The ring $R(X_1, \ldots, X_n) := R[X_1, \ldots, X_n]_S$, where S is the set of all polynomials in $R[X_1, \ldots, X_n]$ for which their coefficients generate R, is called Nagata ring. Let $R\langle X\rangle := R[X]_U$, where U be the set of all monic polynomials of $R[X]$, and $R\langle X_1, \ldots, X_n\rangle := R\langle X_1, \ldots, X_{n-1}\rangle\langle X_n\rangle$. Then, $R\langle X_1, \ldots, X_n\rangle$ is called Serre's conjecture ring.

Corollary 5.7 ([3, Corollary 3.3]) *"Let* R *be a Noetherian ring,* $X_1, \ldots, X_n$ *indeterminates over* R*, and* S *a multiplicatively closed subset of* $R[X_1, \ldots, X_n]$*. Then:*

(1) $R[X_1, \ldots, X_n]_S$ *is regular if and only if* R_p *is regular for each prime ideal* p *of* R *such that* $p[X_1, \ldots, X_n] \cap S = \emptyset$*.*

(2) *In particular,* $R(X_1, \ldots, X_n)$*,* $R\langle X_1, \ldots, X_n\rangle$*,* $R[X_1, \ldots, X_n]$ *and* R *are regular if and only if one of them is regular."*

Proof. (1) Let Q be a prime ideal of $R[X_1, \ldots, X_n]_S$ and let $Q = P_S$ for some (unique) prim ideal P of $R[X_1, \ldots, X_n]$. We have

$$(R[X_1, \ldots, X_n]_S)_{P_S} \cong R[X_1, \ldots, X_n]_P$$

so that they share the same Krull and embedding dimensions. Let $p := P \cap R$. Then, $p[X_1, \ldots, X_n] \cap S \subseteq P \cap S = \emptyset$. Further, we have

$$\frac{Q}{p[X_1, \ldots, X_n]_S} \cong (\frac{P}{p[X_1, \ldots, X_n]})_{S'}$$

where S' is the image of S in $\frac{R}{p}[X_1, \ldots, X_n]$. Hence,

$$\mathrm{ht}(\frac{Q}{p[X_1, \ldots, X_n]_S}) = \mathrm{ht}((\frac{P}{p[X_1, \ldots, X_n]})_{S'}) = \mathrm{ht}(\frac{P}{p[X_1, \ldots, X_n]}).$$

Hence, by Corollary 5.6, we get

$$\mathrm{codim}(R[X_1, \ldots, X_n]_{P_S}) = \mathrm{codim}(R[X_1, \ldots, X_n]_P) = \mathrm{codim}(R_p).$$

(2) Follows from Assertion (1) and the fact that the multiplicative sets of each of Nagata's and Serre's conjecture rings do not intersect with the extensions of prime ideals of R. $\square$

Throughout, given two k-algebras A and B such that $A \otimes_k B$ is Noetherian and Q

a prime ideal of $A \otimes_k B$ with $p := A \cap Q$ and $q = B \cap Q$, we use the following notation for the invariant μ to simplify the expressions in the coming results:

$$\mu_Q(p_p) := \mu_{(A\otimes_k B)_Q}(p_p(A \otimes_k B)_Q)$$

$$\mu_Q(q_q) := \mu_{(A\otimes_k B)_Q}(q_q(A \otimes_k B)_Q).$$

Next, we state the second main theorem of this section.

Theorem 5.8 ([3, Theorem 4.2]) *"Let K be a separable extension field of k and A a k-algebra such that $K \otimes_k A$ is Noetherian. Let Q be a prime of $K \otimes_k A$ with $p = Q \cap A$. Then:*

$$\operatorname{embdim}\Big((K \otimes_k A)_Q\Big) = \operatorname{embdim}(A_p) + \operatorname{ht}(\frac{Q}{K \otimes_k p}) \tag{5.0.6}$$

$$= \operatorname{embdim}(A_p) + \operatorname{embdim}\Big((K \otimes_k \kappa_A(p))_{\frac{Q_p}{K\otimes_k p_p}}\Big).\text{''} \tag{5.0.7}$$

For the proof of this theorem, we first establish the following results.

Proposition 5.2 ([3, Proposition 4.1]) *"Let A and B be two k-algebras such that $A \otimes_k B$ is Noetherian and let Q be a prime ideal of $A \otimes_k B$ with $p = Q \cap A$ and $q = Q \cap B$. Then:*

(1) $\operatorname{embdim}(A \otimes_k B)_Q = \mu_Q(p_p) + \operatorname{embdim}(\kappa_A(p) \otimes_k B)_{\frac{Q_p}{p_p\otimes_k B}}$.

(2) $\operatorname{codim}(A \otimes_k B)_Q + (\operatorname{embdim}(A_p) - \mu_Q(p_p)) = \operatorname{codim}(A_p) + \operatorname{codim}(\kappa_A(p) \otimes_k B)_{\frac{Q_p}{p_p\otimes_k B}}$.

(3) *$(A \otimes_k B)_Q$ is regular and $\mu_Q(p_p) = \operatorname{embdim}(A_p)$ if and only if A_p and $(\kappa_A(p) \otimes_k B)_{\frac{Q_p}{p_p\otimes_k B}}$ are regular."*

Proof. Direct application of Proposition 5.1 and Corollary 5.3. □

Lemma 5.1 ([3, Lemma 4.3]) *"Let A and B be two k-algebras such that $A \otimes_k B$ is Noetherian and let Q be a prime ideal of $A \otimes_k B$ with $p := Q \cap A$ and $q := Q \cap B$. Assume that $Q_Q = (p \otimes_k B + A \otimes_k q)_Q$. Then:*

(1) $\mu_Q(p_p) = \operatorname{embdim}(A_p)$ *and, similarly,* $\mu_Q(q_q) = \operatorname{embdim}(B_q)$.

(2) $\operatorname{embdim}(A \otimes_k B)_Q = \operatorname{embdim}(A_p) + \operatorname{embdim}(B_q)$."

Proof. We proceed through two steps.

Step 1: Assume that $K := B$ is a field extension of k. Hence, $q = 0$ and $Q_Q = (p \otimes_k K)_Q$. Let $r := \operatorname{embdim}(A)$ and let $a_1, \ldots, a_r \in p$ such that $p_p = (\frac{a_1}{1}, \ldots, \frac{a_r}{1})$. We argue by induction on r. If $r = 0$, then $p_p = 0$ and therefore $Q_Q = 0$. Therefore, $\operatorname{embdim}(A \otimes_k K)_Q = 0 = r$, and $\mu_Q(p_p) = 0$. Now, assume that $r \geq 1$. We claim that $\operatorname{embdim}(A \otimes_k K)_Q \geq 1$. Deny. Then, $(A \otimes_k K)_Q$ is regular and therefore A_p is regular by Theorem 1.2. Also, $r = \dim(A_p) = 0$ since $\dim(A \otimes_k K)_Q = 0$ by 1.2. Whence, $p_p = 0$ a contradiction. Therefore, $\operatorname{embdim}(A \otimes_k K)_Q \geq 1$. Assume, without loss of generality, $\frac{a_1}{1} \notin Q_Q \setminus Q_Q^2$. Then, $\frac{Q}{(a_1) \otimes_k K}$ contracts to $\frac{p}{(a_1)}$ and

$$\Big(\frac{Q}{(a_1) \otimes_k K}\Big)_{\frac{Q}{(a_1) \otimes_k K}} \cong \Big(\frac{p}{(a_1)} \otimes_k K\Big)_{\frac{Q}{(a_1) \otimes_k K}}$$

so that, by induction and via [30, Theorem 159], we obtain

$$\operatorname{embdim}\Big(\big(\frac{A}{(a_1)} \otimes_k K\big)_{\frac{Q}{(a_1) \otimes_k K}}\Big) = \operatorname{embdim}\Big(\big(\frac{A}{(a_1)}\big)_{\frac{p}{(a_1)}}\Big) = r - 1.$$

Therefore, by [30, Theorem 159], $\operatorname{embdim}(A \otimes_k K)_Q = r = \operatorname{embdim}(A)$. Then, since $Q_Q = (p \otimes_k K)_Q$, $(\kappa_A(p) \otimes_k K)_{\frac{Q_p}{p_p \otimes_k K}}$ is a field. By Proposition 5.2(1), $\mu_Q(p_p) = \operatorname{embdim}(A \otimes_k K)_Q = \operatorname{embdim}(A_p)$.

Step 2: Suppose that B is an arbitrary k-algebra. Since $Q_Q = (p \otimes_k B + A \otimes_k q)_Q$,

$$(\kappa_A(p) \otimes_k \kappa_B(q))_{\frac{Q_S}{p_p \otimes_k B + A \otimes_k q_q}}$$

is a field, where $S = \{a \otimes_k b \mid a \in A \backslash p,\ b \in B \backslash q\}$. Therefore, by applying Proposition 5.2(1) twice and similar arguments as above lead to

$$\operatorname{embdim}(A \otimes_k B)_Q = \mu_Q(p_p) + \mu_{\frac{Q_p}{p_p \otimes_k B}}(q_q) = \mu_Q(q_q) + \mu_{\frac{Q_q}{A \otimes_k q_q}}(p_p).$$

Notice further that

$$\left(\frac{Q}{A \otimes_k q}\right)_{\frac{Q}{A \otimes_k q}} \cong \frac{Q_Q}{(A \otimes_k q)_Q} \cong \left(p \otimes_k \frac{B}{q}\right)_{\frac{Q}{A \otimes_k q}} \cong (p \otimes_k \kappa_B(q))_{\frac{Q_q}{A \otimes_k q_q}}.$$

By Step 1, we have

$$\mu_{\frac{Q_p}{p_p \otimes_k B}}(q_q) = \operatorname{embdim}(\kappa_A(p) \otimes_k B)_{\frac{Q_p}{p_p \otimes_k B}} = \operatorname{embdim}(B_q)$$

and $\mu_{\frac{Q_q}{A \otimes_k q_q}}(p_p) = \operatorname{embdim}(A_p)$. By Remark 5.1,

$$\mu_{\frac{Q_q}{A \otimes_k q_q}}(p_p) = \mu_{\frac{Q}{A \otimes_k q}}(p_p) \leq \mu_Q(p_p) \leq \operatorname{embdim}(A_p).$$

Hence, $\mu_Q(p_p) = \text{embdim}(A_p)$. Similarly, $\mu_Q(q_q) = \text{embdim}(B_q)$, as desired. $\square$

Lemma 5.2 ([3, Lemma 4.4 and Step 3 of Theorem 4.2]) *"Let K be an extension field of k and A a k-algebra such that $K \otimes_k A$ is Noetherian. Let P be a prime ideal of $K \otimes_k A$ with $p := P \cap A$. Then, there exists a finitely generated extension field E of k contained in K such that, for each intermediate field F between E and K and $Q := P \cap (F \otimes_k A)$, we have*

(1) $\text{embdim}(K \otimes_k A)_P = \text{embdim}(F \otimes_k A)_Q$.

(2) $\text{ht}(\dfrac{P}{K \otimes_k p}) = \text{ht}(\dfrac{Q}{F \otimes_k p})$."

Proof. Let $K \otimes_k p \subseteq P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n = P$ be a maximal chain of distinct prime ideals of $K \otimes_k A$. Similar arguments as in the proof of Lemma 3.5, show that, for each i, there exists a finitely generated extension field E_i of k contained in K such that $P_i = Q_i(K \otimes_k A) = K \otimes_{E_i} Q_i$, where $Q_i = P_i \cap (E_i \otimes_k A)$.

Let $E := k(\cup_{i=1}^{n} E_i)$. Then, E is a finitely generated extension field of k contained in K such that $P_i = Q'_i(K \otimes_k A) = K \otimes_E Q'_i$, for each i, where $Q'_i = P_i \cap (E \otimes_k A)$. Since $P_i \subsetneqq P_{i+1}$ for each $i = 0, \ldots, n-1$, then $Q'_i \subsetneqq Q'_{i+1}$. Set $Q' := Q'_n$. Hence, $\text{ht}(\dfrac{P}{K \otimes_k p}) = n \leq \text{ht}(\dfrac{Q'}{E \otimes_k p})$. Further, since $E \otimes_k A \longrightarrow K \otimes_k A$ is a flat extension, it satisfies GD and so $\text{ht}(\dfrac{Q'}{E \otimes_k p}) \leq \text{ht}(\dfrac{P}{K \otimes_k p})$. Therefore, $\text{ht}(\dfrac{P}{K \otimes_k p}) = \text{ht}(\dfrac{Q'}{E \otimes_k p})$. Now apply Lemma 5.1 to $K \otimes_k A \cong K \otimes_E (E \otimes_k A)$ to get $\text{embdim}(K \otimes_k A)_P = \text{embdim}(E \otimes_k A)_{Q'}$. Likewise, the above arguments hold for any extension field between E and K, completing the proof. $\square$

Proof of Theorem 5.8. We prove the theorem in three steps.

Step 1: Assume that K is algebraic over k. We show that $Q_Q = (K \otimes_k p)_Q$. Notice that

$$\frac{(K \otimes_k A)_Q}{(K \otimes_k p)_Q} \cong (K \otimes_k \frac{A}{p})_{\frac{Q}{K \otimes_k p}} \cong (K \otimes_k \kappa_A(p))_{\frac{Q_p}{K \otimes_k p_p}}.$$

Since K is algebraic separable over k and $K \otimes_k A$ is Noetherian, $K \otimes_k \kappa_A(p)$ is regular. By Corollary 4.5, $K \otimes_k \kappa_A(p)$ is finite product of fields and therefore $(K \otimes_k \kappa_A(p))_{\frac{Q_p}{K \otimes_k p_p}}$ is a field. Hence, the ideal $(K \otimes_k p)_Q$ is maximal in the local ring $(K \otimes_k A)_Q$, and so it must be Q_Q. So, $\text{embdim}(K \otimes_k A)_Q = \text{embdim}(A_p)$ by Lemma 5.1.

Step 2: Assume that K is finitey generated over k and let $V := \{v_1, \dots, v_n\}$ be a separating transcendence basis over k [14, Chap V 5, §16, Theorem 5]. Notice then that $K \otimes_k A \cong K \otimes_{k(V)} A[V]_S$, where $S := k[V] \setminus 0$. Let $Q'_S := Q \cap A[V]_S$ and notice that $p = Q' \cap A$. So, by Step 1,

$$\text{embdim}(K \otimes_k A)_Q = \text{embdim}((A[V]_S)_{Q'_S}) = \text{embdim}(A[V]_{Q'}).$$

By Theorem 5.5, $\text{embdim}(A[V]_{Q'}) = \text{embdim}(A_p) + \text{ht}(\frac{Q'}{p[V]})$. Since K is algebraic over $k(V)$, then $K \otimes_{k(V)} \frac{A[V]_S}{p[V]_S}$ is a flat integral extension of $\frac{A[V]_S}{p[V]_S}$. Since

$$\frac{Q}{K \otimes_k p} \cap \frac{A[V]_S}{p[V]_S} = \frac{Q'_S}{p[V]_S},$$

we have

$$\text{ht}(\frac{Q}{K \otimes_k p}) = \text{ht}(\frac{Q'_S}{p[V]_S}) = \text{ht}(\frac{Q'}{p[V]}).$$

So, $\text{embdim}(K \otimes_k A)_Q = \text{embdim}(A_p) + \text{ht}(\frac{Q}{K \otimes_k p})$.

Step 3: Assume K is an arbitrary separable extension field of k. Then, by Lemma 5.2, there exists a finitely generated extension field E of k contained in K such that $\operatorname{embdim}(K \otimes_k A)_Q = \operatorname{embdim}(E \otimes_k A)_P$, and $\operatorname{ht}(\frac{Q}{K \otimes_k p}) = \operatorname{ht}(\frac{P}{E \otimes_k p})$, where $P := Q \cap (E \otimes_k A)$ and $p = P \cap A$. Since a sub-extension of a separable field is separable [14, Chap P, §7, Proposition 1] and by Step 2, we have

$$\operatorname{embdim}(E \otimes_k A)_P = \operatorname{embdim}(A_p) + \operatorname{ht}(\frac{P}{E \otimes_k p}).$$

Hence, $\operatorname{embdim}(K \otimes_k A)_Q = \operatorname{embdim}(A_p) + \operatorname{ht}(\frac{Q}{K \otimes_k p})$. Now, since K is separable and $K \otimes_k A$ is Noetherian, $K \otimes_k \kappa_A(p)$ is regular and therefore

$$\begin{aligned}\operatorname{embdim}\left((K \otimes_k \kappa_A(p))_{\frac{Q_p}{K \otimes_k p_p}}\right) &= \dim\left((K \otimes_k \kappa_A(p))_{\frac{Q_p}{K \otimes_k p_p}}\right) \\ &= \operatorname{ht}(\frac{Q_p}{K \otimes_k p_p}) \\ &= \operatorname{ht}(\frac{Q}{K \otimes_k p})\end{aligned}$$

completing the proof of the theorem. □

Corollary 5.9 ([3, Corollary 4.5]) *"Let K be a separable extension field of k and A a k-algebra such that $K \otimes_k A$ is Noetherian. Let Q be a prime ideal of $K \otimes_k A$ with $q := Q \cap A$. Then:*

$$\operatorname{codim}(K \otimes_k A)_Q = \operatorname{codim}(A_q).$$

In particular, $K \otimes_k A$ is regular if and only if A is regular."

Proof. Combine Theorem 5.8 with Equation 2.0.1. □

Next, we announce and prove the last main theorem of this section.

Theorem 5.10 ([3, Theorem 5.1]) *"Let A and B be two k-algebras such that $A \otimes_k B$ is Noetherian and let Q be a prime ideal of $A \otimes_k B$ with $p := Q \cap A$ and $q := Q \cap B$. Assume $\kappa_B(q)$ is separable over k. Then:*

$$\begin{aligned}
\operatorname{embdim}(A \otimes_k B)_Q &= \operatorname{embdim}(A_p) + \operatorname{embdim}(B_q) + \operatorname{ht}\left(\frac{Q}{p \otimes_k B + A \otimes_k q}\right) \\
&= \operatorname{embdim}(A_p) + \operatorname{embdim}(B_q) \\
&\qquad + \operatorname{embdim}\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{Q_S}{p_p \otimes_k B_q + A_p \otimes_k q_q}}\right)
\end{aligned}$$

where $S = \{a \otimes_k b \mid a \in A \setminus p, b \in B \setminus q\}$."

Proof. Observe that

$$\operatorname{ht}\left(\frac{Q}{p \otimes_k B + A \otimes_k q}\right) = \operatorname{ht}\left(\frac{Q_S}{p_p \otimes_k B_q + A_p \otimes_k q_q}\right)$$

and hence

$$\operatorname{embdim}\left(\left(\kappa_A(p) \otimes_k \kappa_B(q)\right)_{\frac{Q_S}{p_p \otimes_k B_q + A_p \otimes_k q_q}}\right) = \operatorname{ht}\left(\frac{Q}{p \otimes_k B + A \otimes_k q}\right)$$

as $\kappa_A(p) \otimes_k \kappa_B(q)$ is regular by the facts that $A \otimes_k B$ is Noetherian and $\kappa_B(q)$ is separable. So, there is no harm to assume that $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are local k-algebras such $A \otimes_k B$ is Noetherian, $\kappa_B(\mathfrak{n})$ is a separable extension field of k, and Q is a prime ideal of $A \otimes_k B$ with $Q \cap A = \mathfrak{m}$ and $Q \cap B = \mathfrak{n}$; and, as seen above, it is enough to show the first equality.

Notice first that $\dfrac{Q}{A\otimes_k \mathfrak{n}}$ is a prime of $\dfrac{A\otimes_k B}{A\otimes_k \mathfrak{n}} \cong A\otimes_k \kappa_B(\mathfrak{n})$. By Lemma 5.2, there exists a finitely generated extension field E of k contained in $\kappa_B(\mathfrak{n})$ such that $\frac{Q}{A\otimes_k\mathfrak{n}} = P(A\otimes_k \kappa_B(\mathfrak{n}))$, where $P = \frac{Q}{A\otimes_k\mathfrak{n}} \cap (A\otimes_k E)$. Let $\bar{v}_1,\ldots,\bar{v}_t$ be a separating transcendence basis of $E \subseteq k(\mathfrak{n}) = \frac{B}{\mathfrak{n}}$ over k, for some $v_1,\ldots,v_t \in B$. Since $\bar{v}_1,\ldots,\bar{v}_t$ are algebraically independent over k in $\kappa_B(\mathfrak{n})$, $v_1,\ldots,v_t$ are algebraically independent over k in B. Set $T = \{v_1,\ldots,v_t\}$ and $\bar{T} := \{\bar{v}_1,\ldots,\bar{v}_t\}$. Therefore, $k[T] \cong k[\bar{T}]$, $k(T) \cong k(\bar{T})$ and $A[T] \cong A[\bar{T}]$. Next, notice that $A\otimes_k E \cong A[T]_S \otimes_{k(T)} E$, where $S = k[T]\setminus 0$. Now, $P'_S = \frac{Q}{A\otimes_k\mathfrak{n}} \cap A[T]_S = P\cap A[T]_S$ for some prime ideal P' of $A[T]$. Evidently $(\mathfrak{m}\otimes_k \kappa_B(\mathfrak{n}))\cap A[T]_S = \mathfrak{m}[T]_S$ and that $P'\cap A = \mathfrak{m}$. Notice further that E is an algebraic separable extension field of $k(T)$. Whence, by similar arguments of Step 1 of the proof of Theorem 5.8, $P_P = (P'_S \otimes_{k(T)} E)_P$. Further,

$$\left(\frac{Q}{A\otimes_k \mathfrak{n}}\right)_{\frac{Q}{A\otimes_k\mathfrak{n}}} = \left(\frac{Q_S}{A[T]_S\otimes_{k(T)}\mathfrak{n}_S}\right)_{\frac{Q_S}{A[T]_S\otimes_{k(T)}\mathfrak{n}_S}}$$

and

$$\begin{aligned}\left(\frac{Q_S}{A[T]_S\otimes_{k(T)}\mathfrak{n}_S}\right)_{\frac{Q_S}{A[T]_S\otimes_{k(T)}\mathfrak{n}_S}} &= (P_P(A\otimes_k \kappa_B(\mathfrak{n}))_{\frac{Q}{A\otimes_k\mathfrak{n}}})_S \\ &= P_P\Big(A[T]_S\otimes_{k(T)}\kappa_B(\mathfrak{n})\Big)_{\frac{Q_S}{A[T]_S\otimes_{k(T)}\mathfrak{n}_S}}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{Q_S}{A[T]_S \otimes_{k(T)} \mathfrak{n}_S} &= P'_S(A[T]_S \otimes_{k(T)} \kappa_B(\mathfrak{n}))_{\frac{Q_S}{A[T]_S \otimes_{k(T)} \mathfrak{n}_S}} \\ &= (P'_S \otimes_{k(T)} \kappa_B(\mathfrak{n}))_{\frac{Q_S}{A[T]_S \otimes_{k(T)} \mathfrak{n}_S}} \\ &\cong (P'_S \otimes_{k(T)} \kappa_{B_S}(\mathfrak{n}_S))_{\frac{Q_S}{A[T]_S \otimes_{k(T)} \mathfrak{n}_S}}.\end{aligned}$$

Hence, $Q_{S Q_S} = (P'_S \otimes_{k(T)} B_S + A[T]_S \otimes_{k(T)} \mathfrak{n}_S)_{Q_S}$. By Lemma 5.1,

$$\begin{aligned}\operatorname{embdim}(A \otimes_k B)_Q &= \operatorname{embdim}(A[T]_S \otimes_{k(T)} B_S)_{Q_S} \\ &= \operatorname{embdim}(A[T]_S)_{P'_S} + \operatorname{embdim}(B_S)_{\mathfrak{n}_S} \\ &= \operatorname{embdim}(A[T]_{P'}) + \operatorname{embdim}(B).\end{aligned}$$

By Theorem 5.5, $\operatorname{embdim}(A[T]_{P'}) = \operatorname{embdim}(A) + \operatorname{ht}(\frac{P'}{\mathfrak{m}[T]})$. Further, by the way of selecting E by Lemma 5.2,

$$\operatorname{ht}(\frac{Q}{\mathfrak{m} \otimes_k B + A \otimes_k \mathfrak{n}}) = \operatorname{ht}(\frac{\frac{Q_S}{A[T]_S \otimes_{k(T)} \mathfrak{n}_S}}{\mathfrak{m}[T] \otimes_{k(T)} \kappa_B(\mathfrak{n})}) = \operatorname{ht}(\frac{P}{\mathfrak{m}[T]_S \otimes_{k(T)} E}).$$

Moreover, as

$$\frac{A[T]_S}{\mathfrak{m}[T]_S} \longrightarrow (\frac{A[T]_S}{p[T]_S} \otimes_{k(T)} E)$$

is a flat integral extension (since E is algebraic over $k(T)$), $\operatorname{ht}(\frac{P}{\mathfrak{m}[T]_S \otimes_{k(T)} E}) = \operatorname{ht}(\frac{P'}{\mathfrak{m}[T]})$. Hence, finally

$$\operatorname{embdim}(A \otimes_k B)_Q = \operatorname{embdim}(A) + \operatorname{embdim}(B) + \operatorname{ht}(\frac{Q}{\mathfrak{m} \otimes_k B + A \otimes_k \mathfrak{n}}).$$

completing the proof. □

Corollary 5.11 ([3, Corollary 5.2]) *"Let A and B be two k-algebras such that $A \otimes_k B$ is Noetherian and let Q be a prime ideal of $A \otimes_k B$ with $p := Q \cap A$ and $q := Q \cap B$. Assume $\kappa_B(q)$ is separable over k. Then:*

$$\operatorname{codim}(A \otimes_k B)_Q = \operatorname{codim}(A_p) + \operatorname{codim}(B_q).$$

In particular, if B is residually separable, then $A \otimes_k B$ is regular if and only if so are A and B."

Proof. Combine Proposition 2.1 with Theorem 5.10. □

Corollary 5.12 ([3, Corollary 5.4]) *"Let k be an algebraically closed field, A a finitely generated k-algebra, $\mathfrak{m}$ a maximal ideal of A, and B an arbitrary k-algebra. Let Q be a prime ideal of $A \otimes_k B$ such that $\mathfrak{m} = Q \cap A$ and set $q := Q \cap B$. Then:*

$$\operatorname{codim}(A \otimes_k B)_Q = \operatorname{codim}(A_{\mathfrak{m}}) + \operatorname{codim}(B_q)."$$

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