

**BOUNDARY-TYPE RITZ METHOD FOR THE
ANALYSIS OF POLYGONAL PLATES**

BY

MOHAMMAD HISHAM MOHAMMAD AL-JAWHARY

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

CIVIL ENGINEERING

APRIL 2020

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN- 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **MOHAMMAD HISHAM ALJAWHARY** under the direction of his thesis advisor and approved by his thesis committee, has been presented and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN CIVIL ENGINEERING**.



Dr. Salah U. Al-Dulaijan

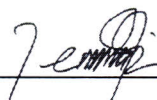
Department Chairman



Prof. Suliman Saleh Al-Homidan
Acting Dean of Graduate Studies



Prof. Husain J. Al-Gahtani
(Advisor)



Dr. Faisal M. Mukhtar
(Member)



Dr. Saheed K. Adekunle
(Member)

Date

© MOHAMMAD HISHAM ALJAWHARY

2020

*To my parents who always give me support and care, I dedicate this thesis work.
Without them, I would have never been where I stand today.*

ACKNOWLEDGMENTS

I am very thankful to my supervisor Dr. Husain J. Al-Gahtani, for his magnificent helpful guidance and support that could not be shortened in few words. Imparting his knowledge and expertise in this study is a memorable grand assistance.

A special gratitude to the committee members Dr. Faisal Mukhtar and Dr. Saheed Adekunle for their guidance and constructive comments.

I also want to thank all the teaching staff at the Civil Engineering Department for the time and great help and knowledge they offered.

I could not forget my colleagues who stood by me during my university life.

And all those who helped in completion of my Master thesis, thanks all.

TABLE OF CONTENTS

ACKNOWLEDGMENTS	V
TABLE OF CONTENTS	VI
LIST OF TABLES.....	IX
LIST OF FIGURES.....	X
LIST OF ABBREVIATIONS.....	XII
ABSTRACT	XIV
ملخص الرسالة.....	XV
CHAPTER 1 INTRODUCTION.....	16
1.1 General.....	16
1.2 Problem Statement.....	17
1.3 Significance of the Research	17
1.4 Objectives	18
1.5 Thesis Outline	19
CHAPTER 2 LITERATURE REVIEW.....	20
2.1 General.....	20
2.2 Solutions of Plates under Static Loading	20
2.3 Natural Vibration of Plates	39

2.4 Summary.....	57
CHAPTER 3 MATHEMATICAL FORMULATIONS	58
3.1 Bending of Thin Plates.....	58
3.2 Ritz Formulation for the Static Analysis of Plates.....	62
3.3 Rayleigh-Ritz Formulation for the Vibrational Analysis of Plates	66
3.3.1 The Governing Differential Equation of Plates Free Vibration	66
3.3.2 Formulation of the Rayleigh-Ritz Method	68
3.3.3 The Boundary Characteristic Orthogonal Polynomials (BCOPs)	71
3.4 Formulation of the Boundary-Type Ritz Method.....	73
CHAPTER 4 RESULTS AND DISCUSSION	77
4.1 Plates subjected to static loading.....	77
4.1.1 Rectangular plate	77
4.1.2 Five-side plate.....	80
4.1.3 Irregular plate	83
4.1.4 Discussion of section (4.1)	86
4.2 Free vibration of plates.....	86
4.2.1 Square plate.....	86
4.2.2 Triangular plate	90
4.2.3 Five-side Plate	93
4.2.4 Irregular plate	97

4.2.5 Discussion of section (4.2)	101
CHAPTER 5 CONCLUSIONS AND RECOMMENDATIONS.....	102
5.1 Conclusions	102
5.2 Recommendations	103
REFERENCES	104
VITAE.....	107

LIST OF TABLES

Table 2. 1: Natural frequencies for SSSS and CCCC square plates obtained by R. B. Bhat (1985) [14]	40
Table 4. 1: Comparison between the results of Levy's solution and the boundary-type Ritz method for an SSSS square plate	78
Table 4. 2: Comparison between the results of FEM and the boundary-type Ritz method of the plate in Figure (4. 3).....	80
Table 4. 3: Comparison between the results of FEM and the boundary-type Ritz method of the plate in Figure (4. 6), at three different points.....	83
Table 4. 4: The natural frequencies obtained from the boundary-type Ritz method and the FEM, compared to results from different references, for an SSSS square plate	87
Table 4. 5: The natural frequencies obtained from the boundary-type Ritz method, compared to the results from (Bhat, R., 1987)[15] and FEM, for the plate in Figure (4. 11)	90
Table 4. 6: The natural frequencies obtained from the boundary-type Ritz method and the FEM, for the plate in Figure (4. 14).....	93
Table 4. 7: The natural frequencies obtained from the boundary-type Ritz method and the FEM, for the plate in Figure (4. 18).....	97

LIST OF FIGURES

Figure 2. 1: Rectangular plate considered by Lee, S. and Ballesteros, P [1].....	21
Figure 2. 2: Rhombic orthotropic plate as assumed by [4]	25
Figure 2. 3: Mapping of arbitrary quadrilateral plates into natural coordinates [8]	31
Figure 2. 4: The three sub-problems whose superposition constitutes the main problem solved in [11]	36
Figure 4. 1: Rectangular plate clamped along $y=0$, and simply supported along the dashed line	77
Figure 4. 2: The first two columns from left are the plots of the Boundary-type Ritz solutions. The last column shows the corresponding Levy's solutions.....	79
Figure 4. 3: Pentagonal plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)	80
Figure 4. 4: Plots of the boundary-type Ritz solutions for the pentagonal plate in Figure (4. 3)	81
Figure 4. 5: Plots of the boundary-type Ritz solutions to the left, and for the FEM solutions to the right, for the pentagonal plate in Figure (4. 3)	82
Figure 4. 6: Arbitrarily shaped plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)	83
Figure 4. 7: Plots of the boundary-type Ritz solutions for the plate in Figure (4. 6)	84
Figure 4. 8: Plots of the boundary-type Ritz solutions to the left, and for the FEM solutions to the right, for the plate in Figure (4. 6)	85
Figure 4. 9: Modal shapes of modes 1 and 2, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for an SSSS square plate	88

Figure 4. 10: Modal shapes of modes 3 and 4, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for an SSSS square plate	89
Figure 4. 11: Triangular plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)	90
Figure 4. 12: Modal shapes of the first 4 modes, obtained from the boundary-type Ritz method for the plate in Figure (4. 11).....	91
Figure 4. 13: Modal shapes of the first 4 modes, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 11)	92
Figure 4. 14: Pentagonal plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)	93
Figure 4. 15: Modal shapes of the first four modes, obtained from the boundary-type Ritz method for the plate in Figure (4. 14)	94
Figure 4. 16: Modal shapes of modes 1 and 2, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 14)	95
Figure 4. 17: Modal shapes of modes 3 and 4, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 14)	96
Figure 4. 18: Arbitrarily shaped plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)	97
Figure 4. 19: Modal shapes of the first four modes, obtained from the boundary-type Ritz method for the plate in Figure (4. 18)	98
Figure 4. 20: Modal shapes of modes 1 and 2, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 18)	99
Figure 4. 21: Modal shapes of modes 3 and 4, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 18)	100

LIST OF ABBREVIATIONS

BTR	The Boundary-Type Ritz Method
FEM	The Finite Element Method
BCOPs	: The Boundary Characteristic Orthogonal Polynomials
NDIF	: Non-dimensional dynamic influence functions
SSSS	: Boundary conditions implying simply supported four edges of a rectangular plate
CCCC	: Boundary conditions implying clamped supported four edges of a rectangular plate
w	: Plate deflection
w_n	: Slope of the deflection function along a vector (n)
M_x	: Plate Bending moment along the x-axis per unit width
M_y	: Plate Bending moment along the y-axis per unit width
M_n	: Plate Bending moment along a vector (n) per unit width
V_x	: Plate effective shear force along the x-axis per unit width
V_y	: Plate effective shear force along the y-axis per unit width
V_n	: Plate effective shear force along a vector (n) per unit width

E	: Modulus of elasticity of the plate material
ν	: Poisson's ratio of the plate material
ρ	: Density of the plate material
D	: Rigidity of the plate
ω	: Angular natural frequency
Π	: Total potential energy of a mechanical system
Γ	: Plate boundary
Ω	: Plate domain
$J(\mathbf{r})$	: Bessel's functions of the first type
$I(\mathbf{r})$	: Modified Bessel's functions of the first type
$\mathbf{J}$	: The Jacobian matrix

]

ABSTRACT

Full Name : Mohammad Hisham Mohammad Aljawhary

Thesis Title : Boundary-Type Ritz Method for the Analysis of Polygonal Plates

Major Field : Civil Engineering - Structures

Date of Degree : April, 2020

The flexural behavior of thin plates is governed by a biharmonic differential equation, derived based on the “Kirchhoff plate theory”. Analytical solution of this equation has been obtained for plates with simple geometries and boundary conditions. However, plates of more complex geometries and boundary conditions exist often in reality. Therefore, this research presents an upgraded Ritz method capable of providing solutions to thin polygonal plates of regular and irregular shapes with various boundary conditions. Dealing with these domain shapes makes difficult the encountered integrations and may lead to convergence problems when resorting to numerical integration methods. The method presented herein is called the Boundary-Type Ritz (BTR) Method and it circumvents the complexity of the involved domains by transforming the encountered integrals to other forms that need to be evaluated only at corners. This leads to very fast computations with full preservation of the accuracy. The Divergence Theorem is implemented for this purpose. Following this method, the coordinates of the corners are solely needed to perform the integrations. The applicability of this method is proved by presenting various examples involving plates of different polygonal shapes with different boundary conditions. The results are compared to the counterpart FEM solutions with very good agreement observed.

ملخص الرسالة

الاسم الكامل: محمد هشام محمد الجوهري

عنوان الرسالة: طريقة "ريتز" المعتمدة على حدود المجال الحسابي لتحليل الصفائح الإنشائية مضلعة الشكل

التخصص: الهندسة المدنية - إنشاءات

تاريخ الدرجة العلمية: نيسان، 2020

يحكم السلوك الإنحنائي للصفائح الإنشائية الرقيقة معادلة تفاضلية بيهارمونية تم اشتقاقها بناءً على نظرية "كيرشوف للصفائح الإنشائية". تم التوصل سابقاً لحلول تحليلية لهذه المعادلة لصفائح ذات أشكال هندسية وظروف حدودية بسيطة. في الواقع، غالباً ما توجد صفائح ذات أشكال هندسية وظروف حدودية أكثر تعقيداً. لذلك يقدم هذا البحث تطويراً لطريقة ريتز قادراً على توفير حلول للصفائح الإنشائية الرقيقة، ذات أشكال مضلعة منتظمة وغير منتظمة، وبظروف حدودية مختلفة. إن التعامل مع أشكال المجال هذه يجعل عمليات التكامل في طريقة ريتز غير المطورة صعبة الحساب، وقد يؤدي حسابها بطرق التكامل العددية إلى مشاكل في التقارب العددي. تسمى الطريقة المقدمة في هذا البحث "طريقة ريتز المعتمدة على حدود المجال الحسابي" (Boundary-Type Ritz Method)، والتي تتجاوز عقبة تعقيد المجالات الحسابية عن طريق تحويل التكاملات إلى أشكال أخرى من التعابير الرياضية، والتي تحتاج إلى إيجاد قيمها عند زوايا الشكل فقط. يؤدي ذلك إلى حسابات سريعة للغاية بدون التأثير على دقة النتائج. لهذا الغرض فقد تمت الاستفادة من مبرهنة التباعد (Divergence Theorem). باتباع هذه الطريقة، فإن كل ما يلزم معرفته من معالم المجال الحسابي لإجراء التكاملات هو فقط احداثيات رؤوس المجال (أو الصفيحة). تم إثبات قابلية تطبيق هذه الطريقة من خلال تقديم أمثلة مختلفة تتضمن صفائح ذات أشكال مضلعة مختلفة بظروف حدودية مختلفة، وتمت مقارنة النتائج مع نظائرها الناتجة من حلول طريقة العناصر المحدودة (Finite Element Method)، مع ملاحظة اتفاق ممتاز في نتائج كلتي الطريقتين.

CHAPTER 1

INTRODUCTION

1.1 General

Plates are 2D elements, characterized by their small thickness compared to other dimensions. Flexural stresses are the main stresses considered in plates. Structural plates are encountered in many structural applications. Floor slabs, water tank bases, bridge decks, aircraft structures, solar panels and printed circuit boards are all typical applications where plates are used.

When it comes to the two cases of plates under static loading as well as plates undergoing free vibration, some analytical solutions are available, but limited to only specific shapes of plates, with specific boundary conditions. However, plates in reality are designed to fit different applications, and therefore complex and general shapes of plates may be encountered, along with general patterns of boundary conditions. By reviewing the literature, one can realize that analytical solutions are not available for such generality of plate configurations. Reasons may be referred to the difficulty of mathematically handling such cases. It thus becomes inevitable to resort to a more general approach capable of generating the solution in functional form. The energy-based methods are among the best options.

1.2 Problem Statement

The Ritz method is an example of the well-known energy-based approaches that has been employed for solving many boundary value problems in the area of engineering mechanics. However, its application is limited by the availability of the trial functions satisfying the complex boundary conditions. It is the aim of this research to utilize the capability of Mathematica in searching for these trial functions. In addition, in order to enhance the capability of the Ritz method to handle integrations over complex domain shapes, this research proposes the boundary-type Ritz method. This new method leads to very fast calculations, where complicated domains are skipped to only the corners of the plate. In this method the coordinates of the corners suffice to run the required integrations. Mathematica will also be utilized to automate the proposed method.

1.3 Significance of the Research

This research is significant in terms of theory and practicality. This is summarized in the following points:

- 1- Having this solution in functional form will serve two purposes:
 - a) It provides the design engineer with simple formulae that are more convenient for repeated calculations or optimization
 - b) It can be utilized for validation of developed numerical techniques.
- 2- Applying the boundary-type Ritz method will significantly reduce the running time for the codes that implement the Ritz method. This will enable the researchers

interested in the Ritz method, or any method that involve integration over complex domain shapes to accelerate the evaluation of these integrations.

- 3- The employment of the Ritz method and evaluating its accuracy for different problems will serve as a guide for future researchers and design engineers.

1.4 Objectives

This research aims at achieving the following objectives:

- 1- To formulate the Boundary-Type Ritz method for the problem of bending of polygonal plates.
- 2- To implement the formulation in Mathematica code capable of automating the application of the Boundary-Type Ritz method for an arbitrary number of trial functions.
- 3- To employ the developed code in deriving functional form solutions for the deflection of several regular polygonal plates (triangular, square, pentagonal, etc.) with various loading and boundary conditions.
- 4- To verify the accuracy of the obtained analytical solutions by comparison with very fine mesh FEM solutions.

1.5 Thesis Outline

This report starts with an introduction that provides a general background about structural plates, followed by a description of the problem of the main focus. The significance of this study as well as the objectives are outlined within the introductory part.

The second chapter lists previous studies considered of evident relevance to the current study, extracted from the literature. These studies are categorized based on the type of problems they sought to solve.

Chapter three is devoted to elaborate all mathematical formulations that contributed to the evolvement of the final mathematical solutions of the addressed problem. This consists of the formulations of the classical Ritz method for both static and dynamic conditions. The last part of this chapter shows the formulation of the (BTR) method which is the central achievement of this thesis.

Several and various examples that prove the applicability of the (BTR) method are presented in chapter four. Through these examples, analytical and numerical solutions led to results very close to those obtained by the (BTR) method.

The final chapter provides concluding remarks and recommended topics that open up a horizon for research in the future |

[CHAPTER 2]

LITERATURE REVIEW

2.1 General

Since the past few decades several researchers have been trying to propose analytical solutions for plates circumscribed by boundary conditions for which mathematical representation is considered hard. The following subsections summarize the works of past researchers, and introduce their attempts to solve different configurations of plates. The works related to solving problems of plates subjected to static loading are introduced first, followed by a group of studies targeting the free vibration of plates.

2.2 Solutions of Plates under Static Loading

Lee, S. and Ballesteros, P. (1960) [1]

Proposed an approximate solution for a uniformly loaded rectangular plate supported at the corners. A polynomial deflection function is assumed as follows:

$$w = C_1 + C_2x^2 + C_3y^2 + C_4x^4 + C_5x^2y^2 + C_6y^4 \quad (1)$$

Dimensions of the considered plate and its origin are selected as in Figure 2. 1).

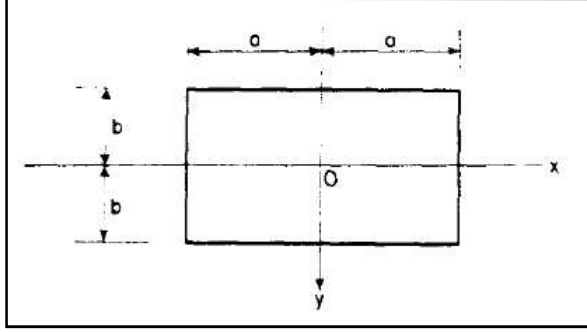


Figure 2. 1: Rectangular plate considered by Lee, S. and Ballesteros, P [1]

In order to find the six constants of the deflection function (w), it is substituted in the governing PDE of Kirchhoff plates, along with five boundary conditions as follows:

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \quad (2)$$

Boundary conditions:

$$\begin{aligned} [V_x]_{a,y} &= -D \left(\left[\frac{\partial^3 w}{\partial x^3} \right]_{a,y} + (2 - \nu) \left[\frac{\partial^3 w}{\partial x \partial y^2} \right]_{a,y} \right) = 0 \\ [V_y]_{x,b} &= -D \left(\left[\frac{\partial^3 w}{\partial y^3} \right]_{x,b} + (2 - \nu) \left[\frac{\partial^3 w}{\partial x^2 \partial y} \right]_{x,b} \right) = 0 \\ \int_0^b [M_x]_{a,y} dy &= -D \int_0^b \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right]_{a,y} dy = 0 \\ \int_0^a [M_y]_{x,b} dx &= -D \int_0^a \left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right]_{x,b} dx = 0 \\ [w]_{a,b} &= 0 \end{aligned} \quad (3)$$

After determination of the constants, moment and shear functions are derived as well as the formula of the corner reaction. The obtained results bring to the following conclusions:

1. For $(b/a < 0.4)$, the behavior of $[w]_{0,0}$ and $[M_x]_{0,0}$ approach those of the simply supported strip of unit width.
2. Due to the used approximation, errors of M_x and M_y are largest in the vicinity of the corners.

Leissa, A., Lo, C. and Nietenfuhr, F. (1965) [2]

Proposed a solution, based on the point matching method, for the deflection of thin regular polygonal shapes, under uniform load, with all edges either simply supported or clamped.

The solution form is assumed as in the following form:

$$w = \sum_{n=0}^N (A_n r^n + B_n r^{n+2}) \cos(n\theta) + \frac{q_0 r^4}{64D} \quad (4)$$

Where:

θ : angle measured from a reference axis bisecting one of the sides

r : radius whose origin is located at the plate centroid

q_0, D : the uniform pressure and flexural rigidity of the plate, respectively

n : equals 0, m, 2m, 3m ..., where m is the number of plate sides

A_n, B_n : Constants to be determined by enforcing boundary conditions of nine equally spaced points along a half of a side. The rest of side halves are similarly treated by symmetry.

The following remarks are concluded:

1. Bending moment is found to be maximum at the center of regular polygonal plates for any number of sides, except for the triangular plates, where maximum bending moment develops at $(x = -0.210 a), (y = 0)$ and takes the value of $(M_y = 0.1028 q_0 a^2)$ when plate is clamped, and develops at $(x = -0.388 a), (y = 0)$ and takes the value of $(M_y = 0.233 q_0 a^2)$ when plate is simply supported.
2. Uniformly loaded simply supported plates, of number of sides more than 4, have always less deflection compared to a simply supported circular plate that inscribes the considered polygon. This can be ascribed to the clamping effect resulting at corners as a result of zero boundary slope in the tangential direction. If sides are increased to more than 10, this effect almost vanishes as internal angles become large that the clamping effect is reduced. More interestingly, as number of sides approaches infinity, solution converges to that of a circular simply supported plate.

Wah, T. (1979) [3]

Wah solved the problem of elastic quadrilateral plates, bounded by any boundary conditions along edges except elastic boundaries. Load is considered uniform over one or both of the two triangles that result when plate is divided by a diagonal.

Wah used eigenfunctions to express the solution of this boundary value problem.

The proposed solution is divided into complementary and particular solutions added together in the following formula:

$$w = w_c + w_p \quad (5)$$

Where w_p is the particular solution, and w_c is the complementary solution, assumed in the following form:

$$w_c = r^{1+\lambda} F(\theta, \lambda) \quad (6)$$

$$F(\theta, \lambda) = C1 \sin(1 + \lambda) \theta + C2 \cos(1 + \lambda) \theta + C3 \sin(1 - \lambda) \theta + C4 \cos(1 - \lambda) \theta \quad (7)$$

Equation (5) is assumed twice, one for each triangular part of the plate.

Substituting equation (7) in the governing equation of plates along with the boundary conditions results in transcendental equations in terms of (λ) . The different solutions of (λ) correspond to eigenfunctions (equation (7)), for each are assigned multiplicative arbitrary constants. These constants are determined by the continuity conditions along the diagonal separating the two triangles, represented by continuity in deflection, slope, curvature and shear.

Wah showed evident agreement between his results and those already obtained by others for rectangular and parallelogram plates.

Shanmugam, et al (1988) [4]

Authors used an approximate method to predict the bending behavior of rhombic orthotropic plates supported at corners and subjected to uniformly distributed load, based on the minimization of the total potential energy. Dimensions and origin of axes are assumed as in Figure 2. 2).

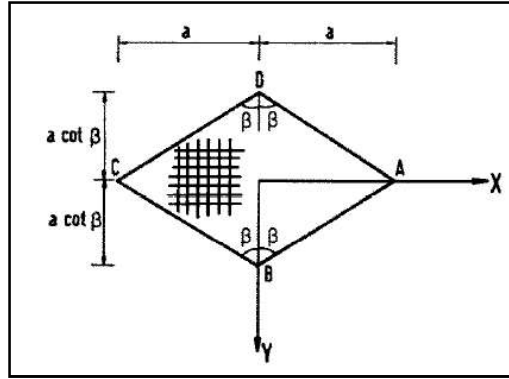


Figure 2. 2: Rhombic orthotropic plate as assumed by [4]

With regard to symmetry in x and y axes, the approximate solution can be a polynomial in the following form:

$$w = C_1 + C_2x^2 + C_3y^2 + C_4x^4 + C_5x^2y^2 + C_6y^4 \quad (8)$$

To find an expression for the total potential energy, stresses and strains have to first be derived from the assumed deflection function in equation (8). The total potential energy can then be expressed in terms of deflection as given by equation (9):

$$\pi = 2 \int_0^a \int_0^{(a-x) \cot \beta} \left[D_x \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_1 \left(\frac{\partial^2 w}{\partial x^2} \right) \left(\frac{\partial^2 w}{\partial y^2} \right) + D_y \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{xy} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - 2qw \right] dy dx \quad (9)$$

To find the constants in the assumed deflection function, the boundary conditions are applied at only two adjacent corners because of symmetry (see equation (10)), which results in two of the constants being expressed in terms of the other four. To determine the other four constants, the total potential energy function is derived four times with respect to the remaining four constants and derivatives are then equated to zero. This produces four equations that when solved yield the four constants.

$$\begin{aligned} [w]_{a,0} &= 0 \\ [w]_{0,a \cot \beta} &= 0 \end{aligned} \quad (10)$$

After comparing with the finite element solutions, the following conclusions are drawn:

- 1- The comparison of predicted deflections and bending moments obtained by using a polynomial deflection function with values obtained from the finite element method indicates reasonable correlation.
- 2- It is observed that the variation in flexural rigidity ratios has a negligible effect on deflections and moments in some cases.

Azarkhin, A. (1992) [5]

Proposed a solution for a rectangular thin plate point-supported at three corners, while the fourth corner kept free. All four edges are assumed free.

The proposed solution is given by the following form:

$$w = \sum_{i=1}^{\infty} A_i \sin\left(\frac{i\pi x}{a}\right) + \sum_{i=1}^{\infty} B_i \sin\left(\frac{i\pi y}{b}\right) + \sum_{i,j=1}^{\infty} D_{ij} \sin\left(\frac{i\pi x}{a}\right) \sin\left(\frac{j\pi y}{b}\right) + Cxy \quad (11)$$

The third term in equation (11) represents the solution for a totally simply supported rectangular plate. It also accounts for the force distribution applied to the plate. The first two terms ensure zero deflection at corners and nonzero deflection along plate edges. The fourth term allows the free corner to displace. The four constants (A_i, B_i, D_{ij}, C_i) are determined such that they minimize the potential energy of the plate.

Liew, K. (1992) [6]

This paper addresses arbitrarily shaped plates (Specifically flat, thin, isotropic, elastic plate of uniform thickness (h), with given support conditions along edges).

The analysis is performed using the principle of minimum potential energy with the admissible pb-2 Ritz functions.

The deflection function is assumed as the pb-2 Ritz function which consists of the product of a two-dimensional polynomial (p-2) and a basic function (b), as follows:

$$w(x, y) = \sum_{q=0}^p \sum_{j=0}^q c_r \phi_r(x, y) \quad (12)$$

In this function, p is the order of the polynomial and ϕ_r is a product of the polynomial function and the boundary function ϕ_1 , which serves as the basic function. These functions are given in equations (13) and (14).

$$\phi_r = (x^j y^{q-j}) \phi_1(x, y) \quad (13)$$

$$\phi_1(x, y) = \prod_{j=1}^{n_e} [\Gamma_i(x, y)]^{\lambda_i} \quad (14)$$

$\Gamma_i(x, y)$ is the boundary equation of the i^{th} supported edge. Values of λ_i are listed below:

Free i^{th} edge: $\lambda_i = 0$

Simply supported i^{th} edge: $\lambda_i = 1$

Clamped i^{th} edge: $\lambda_i = 2$

The coefficients c_r are determined such that they minimize the total energy functional (F), thus resulting in a set of equations represented by equation (15) to be solved for the constants c_i .

$$\frac{\partial F}{\partial c_i} = 0 \quad (15)$$

Wang, C. and Alwis, W. (1995) [7]

Wang and Alwis investigated a relationship that links deflection in polygonal simply supported Mindlin plates to that in its corresponding Kirchhoff plates. Using this relationship one can apply specific modifications on the solution of Kirchhoff plates, to account for the part of deflection caused by shear deformation, without going through complicated analysis of Mindlin plates.

The governing equation of Kirchhoff plate is given as:

$$\nabla^2 w_k = \frac{q}{D} \quad (16)$$

While for Mindlin Plate, the governing equation is given as:

$$\nabla^2 \Phi = \nabla^2 \left[\nabla^2 \left(-w_m + \frac{\mathcal{M}}{k^2 G h} \right) \right] = \frac{-q}{D} \quad (17)$$

Where:

w_k and w_m : Kirchhoff and Mindlin deflections, respectively.

q : Load function

D : Flexural rigidity of the plate

$\mathcal{M} = \frac{M_x + M_y}{1 + \nu}$: Marcus moment

k^2 : Shear correction factor

G : Shear modulus

h : Plate thickness

By comparing equations (16) and (17), the following relation can be deduced:

$$w_m = w_k + \frac{\mathcal{M}}{k^2 G h} \quad (18)$$

Which indicates that Mindlin deflection can be obtained by substituting only Kirschhoff deflection and Marcus moment. This result becomes advantageous when used in verifying numerical analysis results of Mindlin plates.

Saadatpour, M. and Azhari, M. (1998) [8]

They set a theoretical formulation for arbitrary quadrilateral shaped plates, based on the Galerkin method.

First of all, the plate boundaries are simplified by doing coordinates transformation. It is assumed that the plate domain, expressed in the natural coordinates and has the simple boundaries ($\xi = \pm 1$ and $\eta = \pm 1$), is mapped to the arbitrarily shaped quadrilateral plate under study, by the following equations:

$$x = \sum_{i=1}^N x_i N_i(\xi, \eta), \quad y = \sum_{i=1}^N y_i N_i(\xi, \eta) \quad (19)$$

Where x_i, y_i are the coordinates of N points on the boundary of the quadrilateral region and $N_i(\xi, \eta)$ are the interpolation functions for parabolic boundaries. This coordinate transformation is graphically shown in Figure 2. 3).

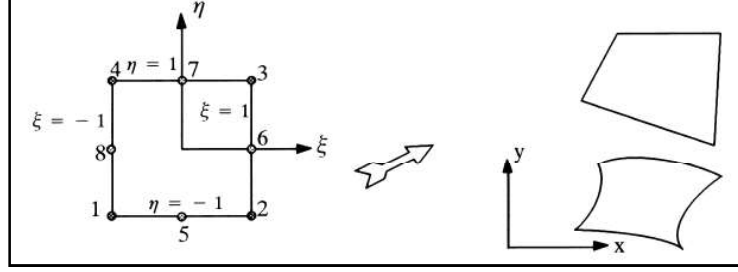


Figure 2. 3: Mapping of arbitrary quadrilateral plates into natural coordinates [8]

Applying the chain rule of differentiation, the first and second derivatives in the natural coordinates are related to those in the original coordinates by using the first and second order Jacobian matrix of mapping.

The considered plate governing PDE is given by equations (20) and (21).

$$\nabla^2 M = -p(x, y) \quad (20)$$

$$\nabla^2 w = -\frac{M}{D} \quad (21)$$

The differential equations (20) and (21) are solved for cases as follows:

1- For plates of straight edges:

For simply supported straight boundary conditions, solutions of equations (20) and (21) are assumed in the following expanded form:

$$M = \sum_{i=1}^N b_i \phi_i(\xi, \eta) \quad (22)$$

$$w = \sum_{i=1}^N a_i \phi_i(\xi, \eta) \quad (23)$$

In which $\phi_i(\xi, \eta)$ are the basic functions which satisfy at least the geometric boundary conditions. a_i and b_i are unknown coordinates or so called generalized coordinates and N is the number of basic functions.

In order to determine the constants in the assumed moment function and deflection functions (equations (22) and (23)), these functions are substituted into equations (20) and (21). After substitution, they are multiplied by the weighting functions, which are selected as the same of the basic functions $\phi_i(\xi, \eta)$, as the Galerkin method dictates, and then integrating over the whole region. This results in simultaneous, linear equations that can be solved to obtain the constants in the assumed moment and deflection functions.

2- For plates with curved edges:

The above procedure is not applicable to the plates of curved edges because moment sum on the curved edges is non-zero in general. The moment sum is given by:

$$\begin{aligned}
M = (1 - \xi^2)(1 - \eta^2) \sum_{i=1}^K b_i \bar{\phi}_i(\xi, \eta) + (1 - \xi^2)(1 - \eta) \sum_{i=1}^K B_i^1 \psi_i(\xi, \eta) \\
+ (1 - \xi^2)(1 + \eta) \sum_{i=1}^K B_i^2 \psi_i(\xi, \eta)
\end{aligned} \tag{24}$$

The first term in the RHS of equation (24) is the expansion of M for homogeneous boundary condition $M = 0$ at all boundaries and the second and third terms in the RHS of the equation express the non-homogenous boundary conditions at edges ($\eta = +1$ and $\eta = -1$), respectively.

The unknown coefficient vectors B_i^1 and B_i^2 are determined so that the moment boundary condition ($M = -D\nabla^2 w$) is satisfied at certain predetermined points of boundaries ($\eta = \pm 1$). The remaining procedure is the same as presented before.

Odishelidze, N., Criado-Aldeanueva, F., Criado, F. and Sanchez, J. (2014) [9]

Odishelidze et al. solved the problem of finding the deflection and full strength contours of a regular polygonal plate, with a circular central full-strength hole. The considered plate is isotropic elastic. Boundaries of the plate are attached to rigid bars subjected to concentrated moments at their midpoints. Edge of the hole is assumed free from force actions and free to displace. Solution of this problem is based on the complex analysis.

Li, R., Wang, B. and Li, P. (2014) [10]

Derived an analytical solution for thin rectangular plates, point-supported at one corner whose adjacent edges are free. The other edges are considered for three boundary cases – both clamped, both simply supported and one clamped and the other simply supported.

Work was conducted using the symplectic superposition method, utilizing the Hamiltonian variational principle. As a result, a set of simultaneous equations was generated for each of the mentioned three boundary cases, taking into account that each boundary case was split into three sub-problems, that when superposed they yield the solution of that boundary case. For example, the first boundary case (a point supported corner at $x = 0, y = 0$, its adjacent edges free along x, y axes, and the other edges are clamped along $x = a, y = b$), this case is split into three sub-problems as follows:

A. All edges are simply supported, plate is subjected to uniform loading.

B. A plate of two opposite simply supported edges (along $x = 0, x = a$), while the other edges are assigned a deflection distribution of $w = \sum_{n=1,2,3...}^{\infty} E_n \sin(\alpha_n x)$ along $y = 0$, and a moment distribution of $M_y = \sum_{n=1,2,3...}^{\infty} F_n \sin(\alpha_n x)$ along $y = b$.

C. A plate of two opposite simply supported edges (along $y = 0, y = a$), while the other edges are assigned a deflection distribution of $w = \sum_{n=1,2,3...}^{\infty} G_n \sin(\beta_n x)$ along $x = 0$, and a moment distribution of $M_x = \sum_{n=1,2,3...}^{\infty} H_n \sin(\beta_n x)$ along $x = b$.

In order to satisfy the boundary condition of the major boundary case, the summation of the effective shear from the three sub-problems must vanish along the free edges along $x = 0, y = 0$. On the other hand, summation of rotation from the three sub-problem must

also vanish along the clamped edges along $x = a$, $y = b$. This results in four equations that when solved yield the four constants (E_n, F_n, G_n, H_n) . Once these constants are obtained, they are substituted into the deflection solutions of the three sub-problems, and the three solutions are summed to obtain the final solution of the boundary case.

Results were validated by an FEM analysis conducted on the three case boundaries, using ABAQUS software. Results from the derived solutions and the FEM analysis were in excellent agreement, with the exception at the free edges, where FEM analysis failed to yield zero moments.

Li, R., Wang, B. and Li, G. (2015) [11]

In this paper, an analytical solution for a thin rectangular plate was derived. One of the edges was assumed clamped or simply supported with its opposite two corners point-supported. The other three edges, connected by the point supports, are considered free. The same approach followed in [10] is also employed in this problem. The problem is split into three sub-problems with their associated boundary conditions as shown in Figure 2. 4).

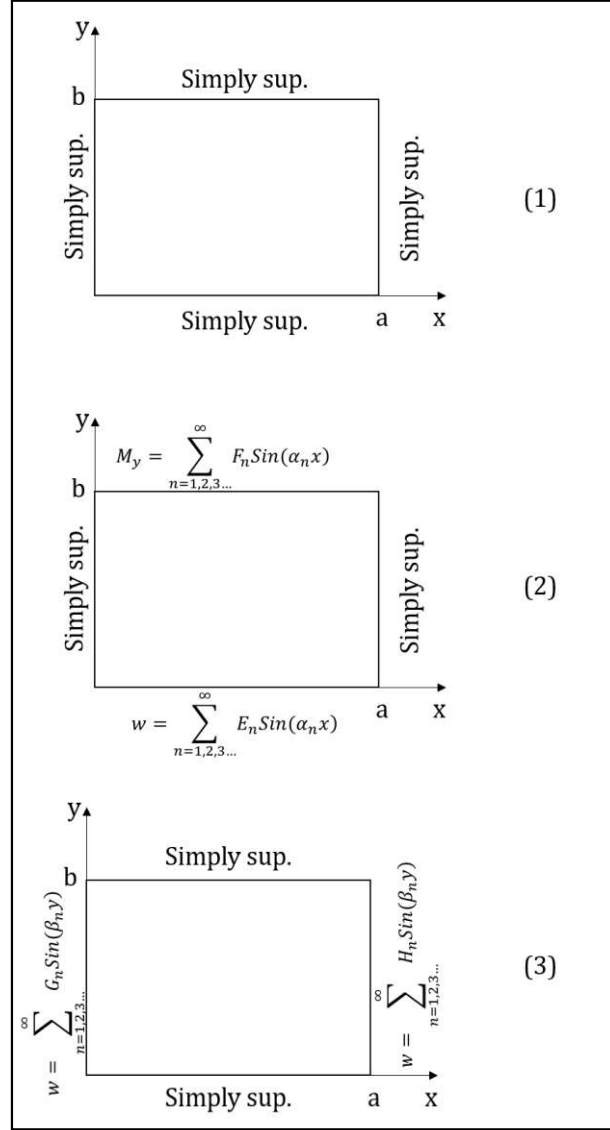


Figure 2. 4: The three sub-problems whose superposition constitutes the main problem solved in [11]

Results were validated by an FEM analysis, using ABAQUS software. Results from the derived solutions and the FEM analysis were in excellent agreement, with the exception at the free edges, where FEM analysis failed to yield zero moments.

Li, R., Wang, B., Lv, Y., Zhang, Q., Wang, H., Jin, F., Teng, F. and Wang, B. (2016)
[12]

Solved analytically the problem of rectangular thin plate with free edges and point-supported at three corners. The solution was obtained by first formulating Hamiltonian-system-based equations. Analytical solutions were then obtained for three sub-problems, that they, when superimposed, yield the analytical solution of the original problem.

Assuming the origin of the x-y plane at the free corner, and the plate has dimensions of $a \times b$, the three sub-problems are described as follows:

- 1- A plate subjected to a uniform load per unit area of magnitude (q), clamped but allowed to slide along $x = 0$ and $y = 0$, and simply supported along $x = a$ and $y = b$.
- 2- A plate clamped but allowed to slide along $x = 0$ and $y = 0$, and simply supported along $x = a$ and $y = b$. At the same time, the plate is given a slope distribution of $\theta|_{y=0} = \sum_{n=1,3,5...}^{\infty} E_n \cos(\frac{\alpha_n x}{2})$ along $y = 0$, and a deflection distribution of $w|_{y=b} = \sum_{n=1,3,5...}^{\infty} F_n \cos(\frac{\alpha_n x}{2})$ along $y = b$.
- 3- A plate clamped but allowed to slide along $x = 0$ and $y = 0$, and simply supported along $x = a$ and $y = b$. At the same time, the plate is given a slope distribution of $\psi_x|_{x=0} = \sum_{n=1,3,5...}^{\infty} G_n \cos(\frac{\beta_n y}{2})$ along $x = 0$, and a deflection distribution of $w|_{x=a} = \sum_{n=1,3,5...}^{\infty} H_n \cos(\frac{\beta_n y}{2})$ along $x = a$.

The obtained analytical solution was validated by comparing results of a square plate to their corresponding FEM results. Comparison showed excellent agreement.

Li, R., Tian, Y., Zheng, X., Wang, H., Xiong, S. and Wang, B. (2017) [13]

Reached an analytical solutions of thin rectangular plates, one of its edges has one end point-supported and the other free. The other edge is clamped or simply supported.

Authors employed the Hamiltonian system-based symplectic approach to solve analytically three sub-problems, which are superposed to yield the solution of the titled problem. The Hamiltonian system based governing equations are derived for the three sub-problems, then solved by the separation of variables and symplectic Eigen expansion.

In summary, the three sub-problems are described as follows:

- 1- Uniformly loaded plate, one of its edges is simply supported and the other edges are clamped, but allowed to slide.
- 2- Plate has two clamped but allowed to slide opposite edges, and the other edges are given slope and moment distributions.
- 3- Plate has two opposite edges slidingly clamped and simply supported, respectively.
The other edges are given slope distributions.

Final results were validated by using finite element analysis. The studied plate was modeled in ABAQUS 6.13 software. It was found that as number of series terms of the solution increases, quick convergence takes place.

In all of the above cited work, the solutions are either for simple geometries and boundary conditions or obtained in a complicated form that cannot be used for design purposes. The

purpose of the current study is to generate the solutions in simpler yet more accurate functional forms for polygonal plates.

2.3 Natural Vibration of Plates

R. B. Bhat (1985) [14]

In this research paper, the natural frequencies of plates of rectangular shapes are calculated using the Rayleigh-Ritz method. Mode shapes are assumed based on a group of orthogonal polynomials, which are generated based on the Gram-Schmidt procedure. The general form of the mode shapes is given by Equation (25).

$$W(x, y) = \sum_m \sum_n A_{mn} \phi_m(x) \psi_n(y) \quad (25)$$

Where (x) and (y) are normalized coordinates, with $(x = \xi/a)$ and $(y = \eta/b)$. In these coordinate ratios, (ξ) and (η) are the absolute coordinates along the two dimensions of the plate (a) and (b), respectively.

Examined cases include (SSSS), (CCCC), (SSFF) and (CCFF) plates, where S, C and F indicate simply supported edge, clamped edge and free edge, respectively. The obtained results are verified numerically. Part of the results are listed in

Table 2. 1).

This research shows that in addition to the simplification of analysis brought about by the orthogonal polynomials, they yield accurate results, especially at lower modes and in the

case of the existence of free edges. The paper also indicates that Poisson's ratio has no effect on the natural frequencies unless at least one edge is free.

Table 2. 1: Natural frequencies for SSSS and CCCC square plates obtained by R. B. Bhat (1985) [14]

Mode	SSSS		CCCC
	Exact $\sqrt{\rho h \omega^2 a^4 / D}$ (rad/s)	$\sqrt{\rho h \omega^2 a^4 / D}$ By Orthogonal Polynomials (rad/s)	$\sqrt{\rho h \omega^2 a^4 / D}$ By Orthogonal Polynomials (rad/s)
1	19.739	19.739	35.9855
2	49.348	49.348	73.395
3	49.348	49.348	73.395
4	78.957	78.957	108.22

R. B. Bhat (1987) [15]

The author applied the Rayleigh-Ritz method to examine the vibrational characteristics of polygonal plates. He made use of the orthogonal polynomials generated by the Gram-Schmidt procedure to find the first six mode shapes and their associated natural frequencies.

The deflection function was assumed taking the form in Equation (26):

$$W(x, y) = \sum_{i=1}^n c_i \varphi_i(x, y) \quad (26)$$

The first polynomial in Equation (26), φ_1 , is selected such that it satisfies at the least the geometrical boundary conditions, with the help of the guidance given in Equation (27).

$$\varphi_1(x, y) = \prod_{j=1}^k \psi_j(x, y) \quad (27)$$

$$\psi_j(x, y) = (y - m_j x - c_j)^l$$

Where:

k : the number of sides of the polygonal plate

ψ_j : a function that expresses the path of the boundary line j

l : equals 0, 1 or 2 for free, simply supported and clamped boundary, respectively.

The subsequent set of orthogonal functions are generated using the Gram-Schmidt procedure, outlined in Equation (28). This procedure benefits from the functions orthogonality principle to find the coefficients a_{ij} .

$$\varphi_i(x, y) = p_i + \sum_{j=1}^{i-1} a_{ij} \varphi_j(x, y)$$

$$p \text{ (set of monomials)} = \{1, x, y, x^2, xy, y^2, \dots, x^n, x^{n-1}y, \dots, x^{n-m}y^m\} \quad (28)$$

$$a_{ij} = \frac{\langle p_i, \varphi_j \rangle}{\langle \varphi_j, \varphi_j \rangle}$$

Where:

φ_i : the orthogonal polynomial i

φ_j : the orthogonal polynomials generated prior to φ_i

$\langle f, g \rangle$: the operator of inner product of given two functions f and g

Having the polynomials in Equation (28) determined, they are combined in the deflection function $W(x, y)$ in Equation (27), and substituted in the minimization of the Rayleigh-Quotient. The constants c_i in Equation (26), which represent essentially the eigenvectors of the problem, are determined to establish the mode shapes and the corresponding natural frequencies. Results of a triangular plate were almost identical to those found earlier in the literature, with the remark that the frequency associated with each mode was found converging to a specific value upon increasing the number of terms in $W(x, y)$.

Liew, K.M. and Lam, K.Y. (1991) [16]

The authors used the characteristic orthogonal polynomials suggested by the Gram-Schmidt procedure, and conducted an analysis following the same methodology as of [15], with the following differences:

- The analysis considered plates of regular domains, with various combinations of boundary conditions.
- Each added monomial p_i is multiplied by a function f to guarantee satisfaction of the geometrical boundary conditions by all polynomials used to represent the deflection function. The function f used to find the polynomial m in the deflection function is calculated as in Equation (29).

$$f_m(\xi, \eta) = \xi^s \eta^r \tag{29}$$

$$r = \lceil \sqrt{m-1} \rceil$$

$$s = \begin{cases} \frac{t}{2}; & 0 \leq s \leq r & \text{if } t \text{ is even} \\ \frac{t-1}{2}; & 1 \leq s \leq r-1 & \text{if } t \text{ is odd} \end{cases}, \quad t = (m-1) - r^2$$

Where:

ξ, η : the normalized coordinates that correspond to the absolute spatial coordinates x, y , respectively

m : the total number of terms used in the deflection function up to (and including) the term under current formulation

$\lceil \cdot \rceil$: denotes the greatest integer function

Results of this analysis were verified for hexagonal and octagonal plates with fully simply supported and fully clamped edges. The same polygonal plates were analyzed for other combinations of boundary conditions with no presented verification due to the unavailability of benchmark results in the literature. The convergence of solutions started to stabilize at a number of terms equal to 30 in the deflection function. Convergence proceeded to the 4th decimal place upon increasing the number of terms up to 60.

Kang, S.W. and Lee, J.M. (2001) [17]

This research investigates the free vibration of clamped plates of arbitrary shapes. The idea behind this research is based on using the non-dimensional dynamic influence functions (NDIF). A function of this type represents the displacement induced, at some point within an infinite domain, by an excitation caused at displacing another point (a source point) by

one-unit displacement. In case of plates, the NDIF is the exact solution to Equation (31), which results when substituting the harmonic motion function ($w(r, t) = W(r) e^{j\omega t}$) in the governing equation of plate free vibration Equation (30).

$$D \nabla^4 w + \rho \frac{\partial^2 w}{\partial t^2} = 0 \quad (30)$$

$$\nabla^4 W - \Lambda^4 W = 0 \quad (31)$$

Where:

w : the transverse deflection.

D : the flexural rigidity of the plate ($D = Eh^3/12(1 - \nu^2)$), where E is the modulus of elasticity, ν is the Poisson's ratio and h is the plate thickness.

ρ : the surface mass density.

Λ : a frequency parameter ($\Lambda = (\rho\omega^2/D)^{\frac{1}{4}}$).

The NDIF consists of linear addition of Bessel's functions. As the deflection at the source point is considered non-singular, the Bessel's function of the first type $J(r)$ and the modified Bessel's function of the first type $I(r)$, both of the order 0, appear in the general solution of Equation (31). The resulting NDIF is therefore expressed as in Equation (32).

$$W(r) = A_s J_0(\Lambda|r - r_s|) + B_s I_0(\Lambda|r - r_s|) \quad (32)$$

Where

A_s, B_s : coefficients that reflect how powerful the exciting source is.

$|r - r_s|$: the scalar distance between the source point (r_s) and the field point (r).

For an arbitrarily shaped plates, a number (N) of source points are assumed distributed along the boundary. The total response of a random field point will be the summation of the responses to all source points, represented by Equation (33).

$$W(r) = \sum_{n=1}^N \{A_k J_0(\Lambda|r - r_k|) + B_k I_0(\Lambda|r - r_k|)\} \quad (33)$$

The boundary conditions of a plate with clamped edges are described by Equation (34) and Equation (35). These equations can be alternatively expressed by a more compact form (Equation (36)).

$$\begin{aligned} W(r_i) &= \sum_{n=1}^N \{A_k J_0(\Lambda|r_i - r_k|) + B_k I_0(\Lambda|r_i - r_k|)\} = 0, \quad i \\ &= 1, 2, \dots, N \end{aligned} \quad (34)$$

$$\begin{aligned} \frac{\partial W(r_i)}{\partial n_i} &= \sum_{n=1}^N \{A_k \frac{\partial}{\partial n_i} J_0(\Lambda|r_i - r_k|) + B_k \frac{\partial}{\partial n_i} I_0(\Lambda|r_i - r_k|)\} = 0, \\ i &= 1, 2, \dots, N \end{aligned} \quad (35)$$

$$SM_{2N} C = 0 \quad (36)$$

$$SM_{2N} = \begin{bmatrix} SM^{(J)} & SM^{(I)} \\ SM_n^{(J)} & SM_n^{(I)} \end{bmatrix}, \quad C = \begin{Bmatrix} A \\ B \end{Bmatrix}$$

Where:

Subscript 2N: the order of the matrix SM .

$SM^{(J)}, SM^{(I)}$: the parts of Equation (34) that involve Bessel functions (J_0) and (I_0), respectively.

$SM_n^{(J)}, SM_n^{(I)}$: the parts of Equation (35) that involve the derivatives of Bessel functions (J_0) and (I_0), respectively.

A, B : the coefficients in Equation (34) and Equation (35), respectively.

The authors of this paper further reduced the matrix SM_{2N} by returning to Eq (11), and finding the coefficients B in terms of A from the first line of the matrix equation, then substituting B back in the second line of the matrix equation, resulting in a single equation in only the coefficients A (Equation (37)).

$$SM_N A = 0$$

(37)

$$SM_N = SM_n^{(J)} - SM_n^{(I)} SM^{(I)^{-1}} SM^{(J)}$$

The matrix SM_N in Equation (37) is called “the reduced system matrix”. If non-trivial solutions are sought from Equation (37), the determinant of SM_N should be equated to zero, resulting in “the reduced frequency equation” (Equation (38)). This equation is used to find the frequency parameters Λ that make the aforementioned determinant singular, i.e. the eigenvalues of the problem. The resulting values are substituted back in Equation (37) and

then in Equation (36) to find A and B for each Λ , which together form the eigenvectors of the problem.

$$\det[SM_N(\Lambda)] = 0 \quad (38)$$

The above proposed procedure was applied to clamped plates of circular, rectangular and arbitrary shapes. The results proved very good convergence to those obtained by numerical analyses, even when a few nodes are used along the plate boundary.

Liu, A.Q. and Chen, H.L. (2005) [18]

The authors proposed the “Bessel function method” in which they provided an exact solution to the governing equation of plate free vibration. The solution was set for rectangular plates. The proposed method initially followed the technique of separation of variables. The solution is therefore resolved into a product of a function of r and a function of θ , with the former consisting of a combination of Bessel functions of the first kind, as given by Equation (39)

$$W_n(r, \theta) = [A_n J_n(kr) + B_n I_n(kr)]_{\sin}^{\cos}(n\theta), \quad k^4 = \omega^2 \rho h / D \quad (39)$$

By taking advantage of the identities proposed by [19], the solution in Equation (39) can be expressed in the rectangular coordinates (Equation (40)).

$$W_n(x, y) = \sum_{m=-\infty}^{\infty} [A_n J_{n-m}(kx) J_m(ky) + B_n I_{n-m}(kx) I_m(ky)] \cos\left(\frac{m\pi}{2}\right) \quad (40)$$

The coefficients A_n and B_n in Equation (40) can be determined by considering the orthogonality of Bessel functions.

Rectangular plates under three different configurations were considered in this paper. The mode functions for each case are reconstructed to satisfy the boundary conditions of that case. Each case with its associated boundary conditions, mode functions and frequency equation are presented below:

1- Fully simply supported:

Boundary conditions: Deflection and curvature at all sides = 0

Mode Function:

$$\begin{aligned} W_{n,m} &= (A_n \{J_{n-m}(kx) + J_{n-m}[k(a-x)]\} \{J_m(ky) \\ &\quad + J_m[k(b-y)]\} \\ &\quad + B_n \{I_{n-m}(kx) + I_{n-m}[k(a-x)]\} \{I_m(ky) \\ &\quad + I_m[k(b-y)]\} \cos\left(\frac{m\pi}{2}\right), \quad m: \text{odd}, n: \text{even}, m \\ &= n/2 \end{aligned} \quad (41)$$

Frequency equation:

$$\det \begin{vmatrix} J_{n-m}(ka) & I_{n-m}(ka) \\ J_m''(kb) & I_m''(kb) \end{vmatrix} = 0 \quad (42)$$

2- Fully clamped edges:

Boundary conditions: Deflection and slope at all sides =0

Mode Function:

$$\begin{aligned}
 W_{n,m} = & (A_n \{J_{n-m}(kx) + J_{n-m}[k(a-x)]\} \{J_m(ky) \\
 & + J_m[k(b-y)]\} \\
 & + B_n \{I_{n-m}(kx) + I_{n-m}[k(a-x)]\} \{I_m(ky) \\
 & + I_m[k(b-y)]\} \cos\left(\frac{m\pi}{2}\right), \quad m: \text{even}, n: \text{even}, m \\
 & = n/2, m \neq 0
 \end{aligned} \tag{43}$$

Frequency equation:

$$\det \begin{vmatrix} J_{n-m}(ka) & I_{n-m}(ka) \\ J'_m(kb) & I'_m(kb) \end{vmatrix} = 0 \tag{44}$$

3- Two opposite sides are clamped; the other sides are simply supported:

Boundary conditions: Deflection and curvature at two opposite sides = 0.

Deflection and slope at the other sides =0.

Mode Function:

$$\begin{aligned}
W_{n,m} = & (A_n \{J_{n-m}(kx) + J_{n-m}[k(a-x)] - J_{n-m}(ka) \\
& - J_{n-m}(0)\} \{J_m(ky) + J_m[k(b-y)] - J_m(kb) \\
& - J_m(0)\} \\
& + B_n \{I_{n-m}(kx) + I_{n-m}[k(a-x)] - I_{n-m}(ka) \\
& - I_{n-m}(0)\} \{I_m(ky) + I_m[k(b-y)] - I_m(kb) \\
& - I_m(0)\} \exp\left(\frac{m\pi i}{2}\right), \quad n: \text{even}, m = n/2, m \\
& \neq 0, \pm 1, \pm 2
\end{aligned} \tag{45}$$

Frequency equation:

$$\det \begin{vmatrix} J''_{n-m}(ka) & I''_{n-m}(ka) \\ J'_m(kb) & I'_m(kb) \end{vmatrix} = 0 \tag{46}$$

The results of the above explained method were compared to Navier's solution in the case of fully simply supported plate, and compared with the results obtained by the Rayleigh-Ritz method and the FEM in the case of fully clamped plates. All comparisons proved very good agreement.

Catania, G. and Sorrentino, S. (2011) [20]

This paper provides a method for the analysis of the free vibration of generally shaped thin plates confined to different boundary conditions. Authors assume the applicability of this

method to plates exhibiting either elastic or viscoelastic behaviors, and to those plates with varying thickness and non-uniform distribution of stiffness and density.

The first step of the proposed method is to simplify the geometry of the domain by following a non-conformal mapping technique, by which the Cartesian coordinates (x, y) of an arbitrary shaped domain are used to map a square in natural coordinates (ξ, η) , by the help of appropriate interpolation functions P . The transformation of the coordinates is given by Equation (47), and that of the differential operators is performed using the Jacobin matrix ($\mathbf{J}$) as given by Equation (48). The operators of subsequent derivatives can be transformed following the same sense, as explained by Equation (49).

$$x(\xi, \eta) = \sum_{i=1}^p x_i P_i(\xi, \eta), \quad y(\xi, \eta) = \sum_{i=1}^p y_i P_i(\xi, \eta) \quad (47)$$

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \sum_{i=1}^p x_i \frac{\partial P_i}{\partial \xi} & \sum_{i=1}^p y_i \frac{\partial P_i}{\partial \xi} \\ \sum_{i=1}^p x_i \frac{\partial P_i}{\partial \eta} & \sum_{i=1}^p y_i \frac{\partial P_i}{\partial \eta} \end{bmatrix} \quad (48)$$

$$\begin{Bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x \partial y} \end{Bmatrix} = \mathbf{A} \begin{Bmatrix} \frac{\partial}{\partial \xi^2} \\ \frac{\partial}{\partial \eta^2} \\ \frac{\partial}{\partial \xi \partial \eta} \end{Bmatrix} - \mathbf{B} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \quad (49)$$

$$= \mathbf{C} \left\{ \frac{\partial}{\partial \xi^2} \quad \frac{\partial}{\partial \eta^2} \quad \frac{\partial}{\partial \xi \partial \eta} \quad \frac{\partial}{\partial \xi} \quad \frac{\partial}{\partial \eta} \right\}$$

$$\mathbf{C} = [\mathbf{A} | -\mathbf{AB}]$$

The method at hand considers the Rayleigh-Ritz method as the basis of analysis. The corresponding total potential energy functional can be expressed by Equations (50) to (53).

$$\Pi = U + V + \Delta V \quad (50)$$

$$U = \frac{1}{2} \int_S D (w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx} w_{yy} + 2(1-\nu) w_{xy}^2) dS, \quad dS(\xi, \eta) = \det[\mathbf{J}] d\xi d\eta \quad (51)$$

$$V = \int_S \rho_h w \ddot{w} dS \quad (52)$$

$$\Delta V = \frac{1}{2} \int_{\gamma} \kappa w^2 d\gamma \quad (53)$$

Where:

Π : the total potential energy in the plate.

U : strain energy in the plate.

V : the potential of all applied loads including the inertial forces.

ΔV : the potential energy stored in the lumped and distributed elastic constraints.

$w, \ddot{w}$: the transverse deflection of the plate and its second time derivative, respectively.

D : the rigidity of the plate.

$\kappa(x, y)$: the distributed constraint stiffness.

γ : the domain of the constraint which can take any geometrical spaces including surfaces, curved paths and points.

Inclusion of the ΔV term can be skipped provided that the assumed deflection functions are admissible. The functions used to estimate the deflection of the plate are products of the prismatic beam eigenfunctions as in Equation (54).

$$w = \sum_{n=1}^N q_n \phi_n(\xi, \eta) = \boldsymbol{\Phi}^T \mathbf{q}, \quad \mathbf{q} = \mathbf{z} e^{i\omega t} \quad (54)$$

Where:

$\phi_n(\xi, \eta)$: is the n^{th} term of the deflection function which represents a product of beam eigenfunctions $\phi_i \phi_j$.

$\mathbf{q}$: the generalized coordinate vector.

N : the number of the beam eigenfunctions defined in terms of the ξ coordinates times those defined in terms of the η coordinates.

The total potential energy functional is minimized with respect to $\mathbf{q}$ after substituting the deflection function given by Equation (54) into Equation (50). This process is expressed by Equation (55) and results in the eigenvalue problem given by Equations (56)-(59).

$$\frac{\partial \Pi}{\partial \mathbf{q}} = 0 \quad (55)$$

$$[(\mathbf{K} + \Delta \mathbf{K}) - \omega \mathbf{M}] \mathbf{z} = \mathbf{0}, \quad \omega = \frac{\lambda^2}{a^2} \sqrt{\frac{D}{\rho_h}} \quad (56)$$

$$\mathbf{M} = \int_S \rho_h (\boldsymbol{\Phi} \boldsymbol{\Phi}^T) dS \quad (57)$$

$$\mathbf{K} = \int_S D \hat{\boldsymbol{\Phi}} [\mathbf{L}] \hat{\boldsymbol{\Phi}}^T \det[\mathbf{J}] d\xi d\eta, \quad \hat{\boldsymbol{\Phi}} = \{\Phi_{\xi\xi} \quad \Phi_{\eta\eta} \quad \Phi_{\xi\eta} \quad \Phi_{\xi} \quad \Phi_{\eta}\} \quad (58)$$

$$\Delta \mathbf{K} = \int_S \kappa (\boldsymbol{\Phi} \boldsymbol{\Phi}^T) dS \quad (59)$$

Where:

λ^2 : a dimensionless frequency parameter.

a : a reference length parameter.

The matrix $\mathbf{L}$ includes combinations of Poisson's ratio with the elements of the matrix $\mathbf{C}$ given in Equation (49).

Following the method explained above, the frequency parameter λ^2 was determined for a certain number of modes along with the corresponding modal shapes for various plate

shapes either with straight line and curved boundaries. The coordinates in each case were mapped using appropriate interpolation functions P . The results showed good agreement with those found in the literature and obtained from the FEM analysis.

Eftekhari, S.A. and Jafari, A.A. (2011) [21]

The objective fulfilled by this paper is to overcome the difficulty of using the Ritz method in solving boundary value problems with boundaries given along lines and at corners; and for irregular plate geometry. Authors developed a novel version of the Ritz method and proved its accuracy in solving the problems of rectangular and skew plates. The proposed method does not require the use of special admissible functions.

The deflection function of the plate is assumed as in Equation (60) which shows a summation of orthonormal functions $\phi_j(x)$, each multiplied by a Ritz coefficient $\Psi_j(y)$. This summation is introduced into Equation (61), which represents the governing non-dimensional vibration equation of thin plates.

$$W(x, y) = \sum_{j=1}^n \Psi_j(y) \phi_j(x) \quad (60)$$

$$W_{,xxxx} + 2\lambda^2 W_{,xxyy} + W_{,yyyy} = \Omega^2 W \quad (61)$$

$$\lambda = a/b \quad (62)$$

$$\Omega^2 = \frac{\omega^2 a^4 \rho h}{D} \quad (63)$$

The resulting equation is then multiplied by $\phi_j(x)$, and integrating both sides of the equation along the normalized length of the plate in the x-axis ($0 \leq x \leq 1$) to benefit from the orthonormality of the trial functions $\phi_j(x)$. This process results in a system of ordinary differential equations (Equation (62)).

$$[A]\{\Psi\} + 2\lambda^2[C]\{\Psi_{yy}\} + \lambda^4\{\Psi_{yyyy}\} = \Omega^2\{\Psi\}$$

$$A_{ij} = \int_0^1 \phi_i \phi_{j,xxxx} dx \quad (64)$$

$$C_{ij} = \int_0^1 \phi_i \phi_{j,xx} dx$$

The next step is to decompose the function $\Psi_i(y)$ into a summation of the orthonormal functions $\psi_j(y)$, each is multiplied by the Ritz coefficient β_j^i (Equation (63)).

$$\Psi_i(y) = \sum_{j=1}^m \beta_j^i \psi_j(y), \quad i = 1, 2, \dots, n \quad (65)$$

Both sides of the system of equations Eq (37) are multiplied by $\psi_j(y)$. To take the advantage of the orthonormality of the functions $\psi_j(y)$, the resulting equations are integrated over the normalized width of the plate in the y-direction ($0 \leq y \leq 1$). The resulting system of equations takes the eigenvalue problem form given in Equation (64), in which the frequencies Ω are the square roots of the eigenvalues of $[\tilde{K}]$.

$$[\tilde{K}]\{\tilde{\beta}\} = \Omega^2\{\tilde{\beta}\} \quad (66)$$

$$[\tilde{K}_{ij}] = A_{ij}[\bar{I}] + 2\lambda^2 C_{ij}[\bar{C}] + \lambda^4 I_{ij}[\bar{A}], \quad i, j = 1, 2, \dots, n$$

$$\{\tilde{\beta}\} = [\{\beta^1\}^T \quad \{\beta^2\}^T \quad \dots \quad \{\beta^n\}^T]^T$$

The boundary conditions equations are integrated in the system of equations (Equation (64)) after being treated the same way as that of the governing differential equation (Equation (61)). The discussed method is highly convergent and it guarantees stability in the numerical results.

2.4 Summary

Reviewing the research works introduced above, researchers tried to solve the plates using either analytical or approximate solutions. The analytical solutions were restricted to simple shapes with typical boundary conditions. On the other hand, the approximate solutions addressed general shapes of plates with general boundary conditions. However, these methods did not resolve the issue of complex domain shapes and its accompanying problem of complicated calculations, and the needed long running time for the codes prepared to obtain the solutions. Moreover, some of these methods begin with preparatory formulation steps in order to simplify the domain shapes or the approximating functions.

The proposed new boundary-type Ritz method utilizes the same ordinary Ritz formulations, yet enhances the calculation capability. Beside its role in accelerating the involved calculations, the method proved its applicability to generally shaped plates with any combination of boundary conditions.

[CHAPTER 3 MATHEMATICAL FORMULATIONS]

3.1 Bending of Thin Plates

Bending of thin plates subjected to a certain distribution of force $q(x, y)$ is governed by the biharmonic differential equation given by Equation (68).

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x, y)}{D} \quad (67)$$

Where $D = Eh^3/(12(1 - \nu^2))$ is the flexural rigidity of the plate, (E) and (ν) are the modulus of elasticity and the Poisson's ratio of the plate material, respectively, and h is the thickness of the plate.

The bending moments M_x and M_y , as well as the twisting moment M_{xy} can be determined based on the deflection function w by referring to equations (68) – (70).

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (68)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (69)$$

$$M_{xy} = -D(1 - \nu) \left(\frac{\partial^2 w}{\partial x \partial y} \right) \quad (70)$$

Shear forces Q_x and Q_y in the thin plate theory are combined with the shearing effect of the twisting moment in what is called the effective shear forces V_x and V_y (equations (71)

and (72)). This combination reduces the number of boundary conditions needed to consider for each edge to two conditions, which is consistent with the governing equation of thin plates.

$$\begin{aligned}
 V_x &= -D \frac{\partial}{\partial x} (\nabla w) - \frac{\partial M_{xy}}{\partial y} \\
 &= -D \left(\frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right)
 \end{aligned} \tag{71}$$

$$\begin{aligned}
 V_y &= -D \frac{\partial}{\partial y} (\nabla w) - \frac{\partial M_{xy}}{\partial x} \\
 &= -D \left(\frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial y \partial x^2} \right)
 \end{aligned} \tag{72}$$

Referring to Figure (3.1), bending moment, twisting moment and effective shear values along an arbitrary direction indicated by a unit vector $\mathbf{n}$ are denoted by M_n, M_{nt} and V_n , respectively. Equations (73) – (83) explain the derivations of these forces.

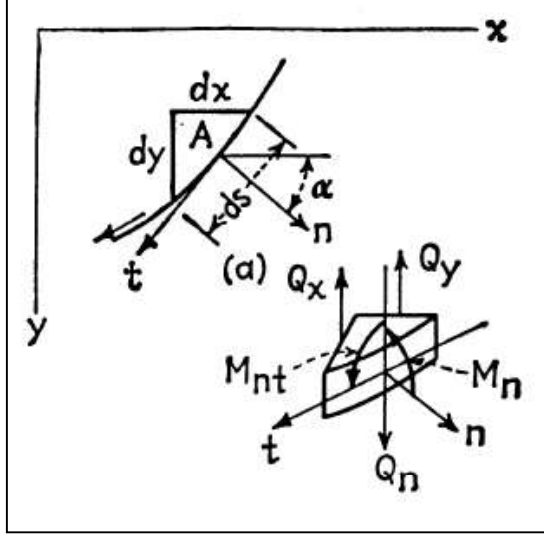


Figure 3. 1: Bending moment, twisting moment and shear forces along an inclined line whose direction is indicated by the orthogonal vectors $\mathbf{n}$ and $\mathbf{t}$

$$M_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_n z \, dz \quad (73)$$

$$\sigma_n = \sigma_x n_x^2 + \sigma_y n_y^2 + 2 \tau_{xy} n_x n_y \quad (74)$$

Where σ_n is the normal stress at an edge facing the normal vector $\mathbf{n}$ which has the components n_x and n_y . Likewise, σ_x and σ_y are the normal stress at edges facing the x and y directions, respectively.

$$M_n = M_x n_x^2 + M_y n_y^2 - 2 M_{xy} n_x n_y \quad (75)$$

$$M_{nt} = - \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{nt} z \, dz \quad (76)$$

$$\tau_{nt} = \tau_{xy} (n_x^2 - n_y^2) + (\sigma_y - \sigma_x) n_x n_y \quad (77)$$

Where τ_{nt} is the shear stress at an edge facing the normal vector $\mathbf{n}$. Likewise, τ_{xy} is the shear stress at edges facing the x or y directions.

$$M_{nt} = M_{xy}(n_x^2 - n_y^2) + (M_x - M_y) n_x n_y \quad (78)$$

$$V_n = Q_n - \frac{\partial M_{nt}}{\partial s} \quad (79)$$

$$Q_n = Q_x n_x + Q_y n_y \quad (80)$$

$$Q_x = -D \frac{\partial}{\partial x} (\nabla w) \quad (81)$$

$$Q_y = -D \frac{\partial}{\partial y} (\nabla w) \quad (82)$$

Where Q_x , Q_y and Q_n are the shear forces acting at edges facing the x-axis, y-axis and the normal vector $\mathbf{n}$, respectively.

$$V_n = Q_n - \left(\frac{\partial M_{nt}}{\partial x} n_y - \frac{\partial M_{nt}}{\partial y} n_x \right) \quad (83)$$

The geometrical boundary conditions are sufficient to be satisfied before the implementation of the Ritz method in the subsequent sections. Equations (84) – (87) present the mathematical expressions related to the various types of geometrical boundary conditions.

Corner-supported with all edges free

$$w|_{corner} = 0 \quad (84)$$

Simply supported edge

$$w|_{along\ edge} = 0 \quad (85)$$

Clamped edge

$$w = 0 \quad (86)$$

$$\frac{\partial w}{\partial n} = 0 \quad (87)$$

Where $(\partial w / \partial n)$ is the slope of the plate in the direction normal to the edge.

3.2 Ritz Formulation for the Static Analysis of Plates

The Ritz method is based on the principle of minimizing the total potential energy (Π). The functional (Π) is the sum of the elastic strain energy stored in the deformed plate (U) and the work of the applied forces (W).

The strain energy stored in an infinitesimal element is the sum of the products of the stresses and the corresponding strains at the element faces. Integrating this sum over the domain of the plate gives us the total strain energy stored in the plate. The derivation of (U) is explained in equations (88) – (91).

$$U = \frac{1}{2} \iiint_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy}) dx dy dz \quad (88)$$

Where σ_x and σ_y are the normal stresses acting at the elements edges facing the x and y directions, respectively, and τ_{xy} is the shear stress at any edge of the element. The quantities ε_x , ε_y and γ_{xy} are the strains corresponding to the aforementioned stresses.

The strain energy in equation (88) can be expressed in terms of the deflection function w by using the stress-strain relations given in equation (89), and the strain-displacement relations given by equation (90).

$$\begin{aligned}\sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \\ \tau_{xy} &= G\gamma_{xy}\end{aligned}\tag{89}$$

$$\begin{aligned}\varepsilon_x &= -z \frac{\partial^2 w}{\partial x^2} \\ \varepsilon_y &= -z \frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= -2z \frac{\partial^2 w}{\partial x \partial y}\end{aligned}\tag{90}$$

Using (90) in (89), then in (88) and simplifying, the strain energy (U) in terms of the plate deflection function becomes:

$$U = \frac{D}{2} \iint_A \left((\nabla w)^2 - 2(1-\nu) \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right) dx dy \tag{91}$$

The total potential energy can subsequently be introduced in terms of the plate deflection function in equation (92).

$$\Pi = U + W \tag{92}$$

$$\Pi = \frac{D}{2} \int_{\Omega} \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\nu) \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right] d\Omega - \int_{\Omega} w q d\Omega$$

The deflection function is approximated as a series of polynomials of powers up to (N) as follows

$$w = \sum_{i,j=1}^N C_{ij} (x^i y^j) \quad (93)$$

Where (C_{ij}) are constant coefficients to be determined such that they minimize the functional in Equation (92).

Each term in the assumed deflection function (Equation (93)) should satisfy the edge geometrical boundary conditions by multiplying the deflection function by the product (ϕ_b) which is expressed in Equation (94).

$$\phi_b = \prod_{i=1}^{s_n} [\Gamma_i(x, y)]^{\Omega_i} \quad (94)$$

Where $\Gamma_i(x, y)$ are the equations of the boundary lines that count to (s_n) which is the number of plate edges, and (Ω_i) equals 0 in case of free edge, 1 for simply supported edge and 2 for clamped edge. If displacement is restrained at corners located at the intersection of free edges, then some constants of the (C_{ij}) are expressed in terms of others such that deflection vanishes at those corners. This results in more than one term multiplied by the same constant. Collecting each group of terms with common constants together the resulting deflection function will have the form in Equation (95).

$$w = \sum_{i=1}^n C_i \phi_i(x, y) \quad (95)$$

Upon reaching this step, each function $\phi_i(x, y)$ certainly satisfies the geometrical boundary conditions of the plate. Equation (95) is substituted into Equation (92), followed by minimizing the total potential energy functional (Π) with respect to the constants (Equation (96)), which produces a system of linear algebraic equations indicated by Equation (97). that can be solved for C_i .

$$\frac{\partial \Pi}{\partial C_i} = 0, \quad i = 1, 2, \dots, N \quad (96)$$

$$\begin{aligned} C_j \int_{\Omega} \left[\nabla^2 \phi_i \nabla^2 \phi_j \right. \\ \left. - (1 - \nu) \left(\frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial y^2} + \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} \right. \right. \\ \left. \left. - 2 \frac{\partial^2 \phi_i}{\partial x \partial y} \frac{\partial^2 \phi_j}{\partial x \partial y} \right) \right] d\Omega = \int_{\Omega} \frac{q}{D} \phi_i d\Omega \end{aligned} \quad (97)$$

Equation (97) can be cast in the following matrix equation

$$[K]\{C\} = [F] \quad (98)$$

Where

$$\begin{aligned}
K_{ij} = \int_{\Omega} & \left[\nabla^2 \phi_i \nabla^2 \phi_j \right. \\
& - (1 - \nu) \left(\frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial y^2} + \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} \right. \\
& \left. \left. - 2 \frac{\partial^2 \phi_i}{\partial x \partial y} \frac{\partial^2 \phi_j}{\partial x \partial y} \right) \right] d\Omega
\end{aligned} \tag{99}$$

$$F_i = \int_{\Omega} \frac{q}{D} \phi_i d\Omega \tag{100}$$

$$\{C\} = \{C_1, C_2, \dots, C_N\} \tag{101}$$

3.3 Rayleigh-Ritz Formulation for the Vibrational Analysis of Plates

3.3.1 The Governing Differential Equation of Plates Free Vibration

Considering a differential element of a plate undergoing free vibrational motion, by identifying the forces applied to this element, the forces equilibrium equation can be established in the z-direction based on Newton's second law. The internal forces in this equation can be expressed in terms of the deflection function, which results in the final form of the free vibrational motion of the thin isotropic plates with constant thickness (Equation (102)).

$$D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \tag{102}$$

Where:

w : the transverse deflection.

D : the flexural rigidity of the plate ($D = Eh^3/12(1 - \nu^2)$), where E is the modulus of elasticity, ν is the Poisson's ratio and h is the plate thickness.

ρ : the mass density.

h : the thickness of the plate.

The solutions of Equation (102) can be expressed as a product of two functions. The first function is called the modal shape of the plate and is dependent on the spatial coordinates $W(x, y)$. The second is a periodic function of time, a sinusoidal function, that sets the modal shape in a vibrational motion. Therefore, the solution can be generally expressed as in Equation (103).

$$w(x, y, t) = W(x, y)e^{i\omega t} \quad (103)$$

Where:

ω : the natural frequency of motion of the plate.

t : the time

Substituting Equation (103) into Equation (102) results in a PDE eigenvalue problem (Equation (104)). As this equation describes a continuous domain, it yields an infinity number of eigenfunction solutions (modal shapes) depending on the value of the eigenvalues ($\rho h \omega^2 / D$) which take only specific values.

$$\nabla^4 W(x, y) + \left(\frac{\rho h \omega^2}{D} \right) W(x, y) = 0 \quad (104)$$

3.3.2 Formulation of the Rayleigh-Ritz Method

The Rayleigh-Ritz method is used to estimate the natural frequencies as well as the modal shapes of a vibrating system. It is viewed as an extension to the Rayleigh method in the sense that it can generate higher natural frequencies, thus completing the method of Rayleigh which only estimates the fundamental natural frequency.

The basic premise behind the Rayleigh-Ritz method is that an upper-bound estimation for the fundamental frequency can be reached by equating the maximum kinetic energy and the maximum potential energy a system, or a plate in our case, would have during vibration. Consequently, a weak form of Equation (102) is formulated. This method needs an approximate solution (mode shape function) to be predefined as a linear combination of several admissible functions with unknown coefficients as in Equation (105). The admissible functions (ϕ_i) are those functions that do not violate the geometrical boundary conditions. These functions are assumed as 2D polynomials. It should be noted that the accuracy of the solution depends on how close our assumed functions are to the exact ones.

$$W(x, y) = \sum_{i=1}^N C_i \phi_i(x, y) \quad (105)$$

The maximum kinetic energy of a vibrating plate (T_m) is given by Equation (106). The maximum potential energy (V_m) is attained by the plate when it reaches the highest amplitude. At this instant the time-dependent periodical function ($e^{i\omega t}$) in Equation (103)

evaluates to 1. The maximum potential energy in a vibrating plate is given by Equation (107).

$$T_m = \frac{1}{2} \rho h \int_{\Omega} \left(\frac{\partial w}{\partial t} \right)^2 d\Omega = \omega^2 \left(\frac{1}{2} \rho h \int_{\Omega} W^2 d\Omega \right) = \omega^2 \psi_m \quad (106)$$

$$\psi_m = \frac{1}{2} \rho h \int_{\Omega} W^2 d\Omega$$

$$V_m = \frac{D}{2} \int_{\Omega} \left[(\nabla^2 W)^2 - 2(1-\nu) \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} - \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \right) \right] d\Omega \quad (107)$$

By equating (T_m) with (V_m), and keeping (ω^2) alone in the left side, the result is called the “Rayleigh quotient” (Equation (108)). As this quotient provides an upper-bound estimation for the natural frequency, the quotient should be minimized with respect to the coefficients (C_i) of the modal shape functions. This results in an eigenvalue problem, in which multiple eigenvectors $\{C_i\}$ are evaluated, with each vector associated with an eigenvalue (ω^2).

$$\omega^2 = \frac{\psi_m}{V_m} \quad (108)$$

$$\frac{\partial \omega^2}{\partial C_i} = 0 \quad (109)$$

$$\frac{\psi_m \frac{\partial V_m}{\partial C_i} - V_m \frac{\partial \psi_m}{\partial C_i}}{\psi_m^2} = 0 \quad (110)$$

$$\frac{\psi_m \frac{\partial V_m}{\partial C_i} - \left(\frac{V_m}{\psi_m}\right) \frac{\partial \psi_m}{\partial C_i}}{\psi_m} = 0 \quad (111)$$

$$\psi_m \frac{\partial V_m}{\partial C_i} - (\omega^2) \frac{\partial \psi_m}{\partial C_i} = 0 \quad (112)$$

The maximum potential function and the (ψ_m) function should be expressed in terms of the assumed modal shape function (Equation(105)), and differentiated with respect to (C_i) .

The results of these differentiations are shown in Equations (113) – (115).

$$\begin{aligned} \frac{\partial V_m}{\partial C_i} = \frac{D}{2} \int_{\Omega} & \left[(\nabla^2 \phi_i \nabla^2 \phi_j) \right. \\ & - (1 - \nu) \left(\frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial y^2} + \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} \right. \\ & \left. \left. - 2 \frac{\partial^2 \phi_i}{\partial x \partial y} \frac{\partial^2 \phi_j}{\partial x \partial y} \right) \right] d\Omega \quad C_j \end{aligned} \quad (113)$$

$$\frac{\partial V_m}{\partial C_i} = [K_{ij}] \{C_j\}$$

$$\begin{aligned} [K_{ij}] = \frac{D}{2} \int_{\Omega} & \left[(\nabla^2 \phi_i \nabla^2 \phi_j) \right. \\ & - (1 - \nu) \left(\frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial y^2} + \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} \right. \\ & \left. \left. - 2 \frac{\partial^2 \phi_i}{\partial x \partial y} \frac{\partial^2 \phi_j}{\partial x \partial y} \right) \right] d\Omega \end{aligned} \quad (114)$$

$$\frac{\partial \psi_m}{\partial C_i} = \frac{1}{2} \rho h \int_{\Omega} (\phi_i \phi_j) d\Omega C_j$$

$$\frac{\partial \psi_m}{\partial C_i} = [M_{ij}] \{C_j\}$$
(115)

Substituting Equation (113) and Equation (115) into Equation (112), the eigenvalue problem results, shown by Equation (106).

$$[K_{ij}] \{C_j\} - \omega^2 [M_{ij}] \{C_j\} = 0$$
(116)

3.3.3 The Boundary Characteristic Orthogonal Polynomials (BCOPs)

If (ϕ_i) and (ϕ_j) are selected to be orthonormal with respect to the weighting function (ρh) , then the generalized eigenvalue problem indicated by Equation (116) turns into a standard eigenvalue problem. This is attributed to the fact that the matrix $[M_{ij}]$ becomes an identity matrix and it can be omitted from the equation. This makes the problem more simplified and the solution easily obtainable.

The Gram-Schmidt procedure is used to generate orthogonal polynomials (ϕ_i) starting from a set of functions defined in Equation (117). The function $g(x, y)$ satisfies the geometrical boundary conditions of the plate. The functions $f_i(x, y)$ are linearly independent monomials that follow the sequence indicated by Equation (118). The procedure begins with the first element (ϕ_1) taking the first element of the set in Equation (117). The rest of the steps are followed recursively based on Equations (120) and (121).

$$\{g(x, y) f_i(x, y)\}$$
(117)

$$\{f_i(x, y)\} = \{1, x, y, x^2, xy, y^2, \dots, x^i, x^{i-1}, y, \dots, x^{i-k} y^k\} \quad (118)$$

$$\phi_1 = gf_1 \quad (119)$$

$$\phi_i = gf_i - \sum_{j=1}^{i-1} a_{ij} \phi_j \quad i = 2, 3, 4, \dots \quad (120)$$

$$a_{ij} = \frac{\langle gf_i, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} \quad (121)$$

Where the mathematical symbol $\langle A(x), B(x) \rangle$ indicates the inner product of some functions $(A(x), B(x))$ with respect to some weighting function. The inner products in Equation (121) are explained in Equation (122) where the weighting function in the current case is (ρh) .

$$\begin{aligned} \langle gf_i, \phi_j \rangle &= \int_{\Omega} (\rho h) (gf_i \phi_j) d\Omega \\ \langle \phi_j, \phi_j \rangle &= \int_{\Omega} (\rho h) (\phi_j \phi_j) d\Omega \end{aligned} \quad (122)$$

Next, the orthogonal polynomials (ϕ) are divided by their norms to be converted to orthonormal polynomials $(\hat{\phi})$ as explained by Equations (123) and (124).

$$\hat{\phi}_i = \frac{\phi_i}{\|\phi_i\|} \quad (123)$$

$$\|\phi_i\| = \langle \phi_i, \phi_i \rangle^{\frac{1}{2}} = \left(\int_{\Omega} \rho h \phi_i^2 d\Omega \right)^{\frac{1}{2}} \quad (124)$$

3.4 Formulation of the Boundary-Type Ritz Method

For plates of regular and irregular polygonal domains, they complicate the analytical evaluation of the integrals in Equations (99), (100) and (114). In order to circumvent this issue the Divergence Theorem is incorporated to transform these integrals from 2D-continuous-domain-based integrals into an equivalent corner-based summation. The Divergence theorem, in general, states that the integration of the divergence of a function $\mathbf{p}$ over a domain equals the net flux of that function across the boundaries, represented mathematically as

$$\int_{\Omega} \nabla \cdot \mathbf{p} \, d\Omega = \int_{\Gamma} \mathbf{p} \cdot \mathbf{n} \, d\Gamma \quad (125)$$

Where $\mathbf{n}$ is a unit vector normal to the boundary and projected outward. Equation (125) can be split into two equations as follows

$$\int_{\Omega} \frac{\partial p_x}{\partial x} \, d\Omega = \int_{\Gamma} p_x \cdot n_x \, d\Gamma \quad (126)$$

$$\int_{\Omega} \frac{\partial p_y}{\partial y} \, d\Omega = \int_{\Gamma} p_y \cdot n_y \, d\Gamma \quad (127)$$

In Equation (127) the function (p_y) is the antiderivative of ($\frac{\partial p_y}{\partial y}$). For convenience, we can rewrite Equation (127) in the following form:

$$\int_{\Omega} P(x, y) d\Omega = \int_{\Gamma} P_y(x, y) n_y d\Gamma \quad (128)$$

$$P_y(x, y) = \int P(x, y) dy \quad (129)$$

Equation (128) is a transformation of the integral domain from the internal region inscribed by the plate boundaries to the external boundary lines. In the case of polygonal domains, the boundary consists of a group of straight lines of various slopes connected at the corners. Therefore, if we take Equation (128), the right side can be expressed as a summation of a number of boundary integrals equal to the number of the plate sides, resulting in rewriting Equation (128) in the following form:

$$\int_{\Gamma} P_y(x, y) n_y d\Gamma = \sum_{m=1}^M \int_{\Gamma^m} P_y(x, y) n_y^m d\Gamma^m \quad (130)$$

Where M is the number of plate sides. By noting that P_y is a function of x and y , the equations of boundary lines $y_m(x)$ can be substituted into P_y , turning it to be a function of x only. The following step is to conduct the integration in equation (130) over x . The integration limits are x_m and x_{m+1} for the m^{th} line. In order to conduct this integration, the following substitutions should be made:

$$\begin{aligned} P_y(x, y) &\rightarrow P_y|_{(y=y_m)} \\ ny &= -\frac{dx}{d\Gamma} \end{aligned} \quad (131)$$

Therefore, the integration over the m^{th} side of the plate can be rewritten as in equation (132).

$$\int_{\Gamma} P_y(x, y) n_y d\Gamma = \sum_{m=1}^M - \int_{\Gamma^m} P_y|_{(y=y_m)} dx^m \quad (132)$$

The integration in the right-hand side of equation (132) can be converted to a difference of its integrand's antiderivative P_{yx} values at the terminal corners m and $m + 1$, written mathematically as follows:

$$\sum_{m=1}^M - \int_{\Gamma^m} P_y|_{(y=y_m)} dx^m = \sum_{m=1}^M -P_{yx}(x, y)|_{x^m}^{x^{m+1}} \quad (133)$$

$$P_{yx}(x, y) = \int P_y|_{(y=y_m)} dx$$

Finally, by following the above steps, the domain-based integration introduced earlier in equation (128) can be reduced to the following simple summation:

$$\int_{\Omega} P(x, y) d\Omega = \sum_{m=1}^M -P_{yx}(x, y)|_{x^m}^{x^{m+1}} \quad (134)$$

Again in the above equation P_{yx} is the antiderivative of $P_y|_{(y=y_m)}$.

Equation (134) represents the pivotal result of the proposed Boundary-Type Ritz Method. It transforms a 2D-domain-based integral into a summation involving the values of the function P_{yx} evaluated at the corners of the plate. The computational effort is thus reduced

enormously from computing an integral over a complicated domain shape to computing values at a few countable points at the boundaries, with full preservation of the accuracy of the analytical domain-based integration. By the same token, the domain-based integrals in Equations (99) and (100) can be transformed as follows

$$\int_{\Omega} K_{ij}(x, y) d\Omega = \sum_{m=1}^M -(K_{yx})_{ij}(x, y) \Big|_{x^m}^{x^{m+1}}$$

Where $(K_{yx})_{ij}$

$$\begin{aligned} &= \int \left(\int \left[\nabla^2 \phi_i \nabla^2 \phi_j \right. \right. \\ &\quad - (1 - \nu) \left(\frac{\partial^2 \phi_i}{\partial x^2} \frac{\partial^2 \phi_j}{\partial y^2} + \frac{\partial^2 \phi_j}{\partial x^2} \frac{\partial^2 \phi_i}{\partial y^2} \right. \\ &\quad \left. \left. - 2 \frac{\partial^2 \phi_i}{\partial x \partial y} \frac{\partial^2 \phi_j}{\partial x \partial y} \right) \right] dy \Big|_{(y=y_m)} \right) dx \end{aligned} \quad (128)$$

$$\int_{\Omega} F_i(x, y) d\Omega = \sum_{m=1}^M -(F_{yx})_i(x, y) \Big|_{x^m}^{x^{m+1}} \quad (129)$$

$$\text{Where } (F_{yx})_i = \int \left(\int \left[\frac{q}{D} \phi_i \right] dy \Big|_{(y=y_m)} \right) dx$$

CHAPTER 4

RESULTS AND DISCUSSION

The results of different cases are presented herein for static analysis and free vibration analysis. Cases include regular polygonal plates of various number of sides as well as one irregular polygonal plate. Results of the boundary-type Ritz method are compared with analytical and FEM results for verification.

4.1 Plates subjected to static loading

4.1.1 Rectangular plate

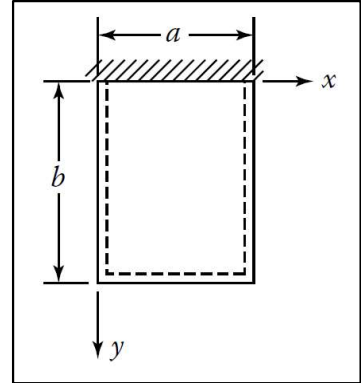
Analysis is performed for a rectangular plate, shown in Figure 4. 1), with the following properties:

Dimensions: $a \times b$

Boundary conditions: clamped ($y = 0$), simply supported at all other edges

Load: uniform

The levy's solution for this rectangular plate is:



[Figure 4. 1: Rectangular plate clamped along $y=0$, and simply supported along the dashed line]

$$w = \sum_{m=1,3,\dots}^{\infty} \left(A_m \sinh \left[\frac{m\pi y}{a} \right] + B_m \cosh \left[\frac{m\pi y}{a} \right] + C_m y \sinh \left[\frac{m\pi y}{a} \right] + D_m y \cosh \left[\frac{m\pi y}{a} \right] + \frac{4qa^4}{m^5\pi^5 D} \right) \sin \left[\frac{m\pi x}{a} \right] \quad (130)$$

$$\begin{aligned}
A_m &= -\frac{a}{m\pi} D_m \\
&= \frac{2qa^4}{m^5\pi^5 D} \frac{2 \cosh^2[\beta_m] - 2 \cosh[\beta_m] - \beta_m \sinh[\beta_m]}{\cosh[\beta_m] \sinh[\beta_m] - \beta_m} \\
B_m &= -\frac{4qa^4}{m^5\pi^5 D} \\
C_m &= \frac{2qa^4}{m^5\pi^5 D} \cdot \left(\frac{\beta_m}{b}\right) \cdot \frac{2 \sinh[\beta_m] \cosh[\beta_m] - \sinh[\beta_m] - \beta_m \cosh[\beta_m]}{\cosh[\beta_m] \sinh[\beta_m] - \beta_m} \\
\beta_m &= \frac{m\pi b}{a}
\end{aligned}$$

It was noted that the running time of the code was reduced from 28 min in the case of implementing the traditional Ritz method to only 3 min in the case of implementing the (BTR) method.

Values from the Ritz solution are compared to Levy's solution at some selected points, as shown in Table 4. 1). The plots of both Ritz and Levy's solutions are listed together in Figure 4. 2).

Table 4. 1: Comparison between the results of Levy's solution and the boundary-type Ritz method for an SSSS square plate

Method	$W_{(0.5,0.5)}$	$Mx_{(0.5,0.5)}$	$My_{(0.5,0.5)}$
Boundary-type Ritz	0.00278549	0.0338857	0.0391816
Levy	0.00278547	0.0338504	0.0391674
Relative Error (%)	0.0	0.1	0.0

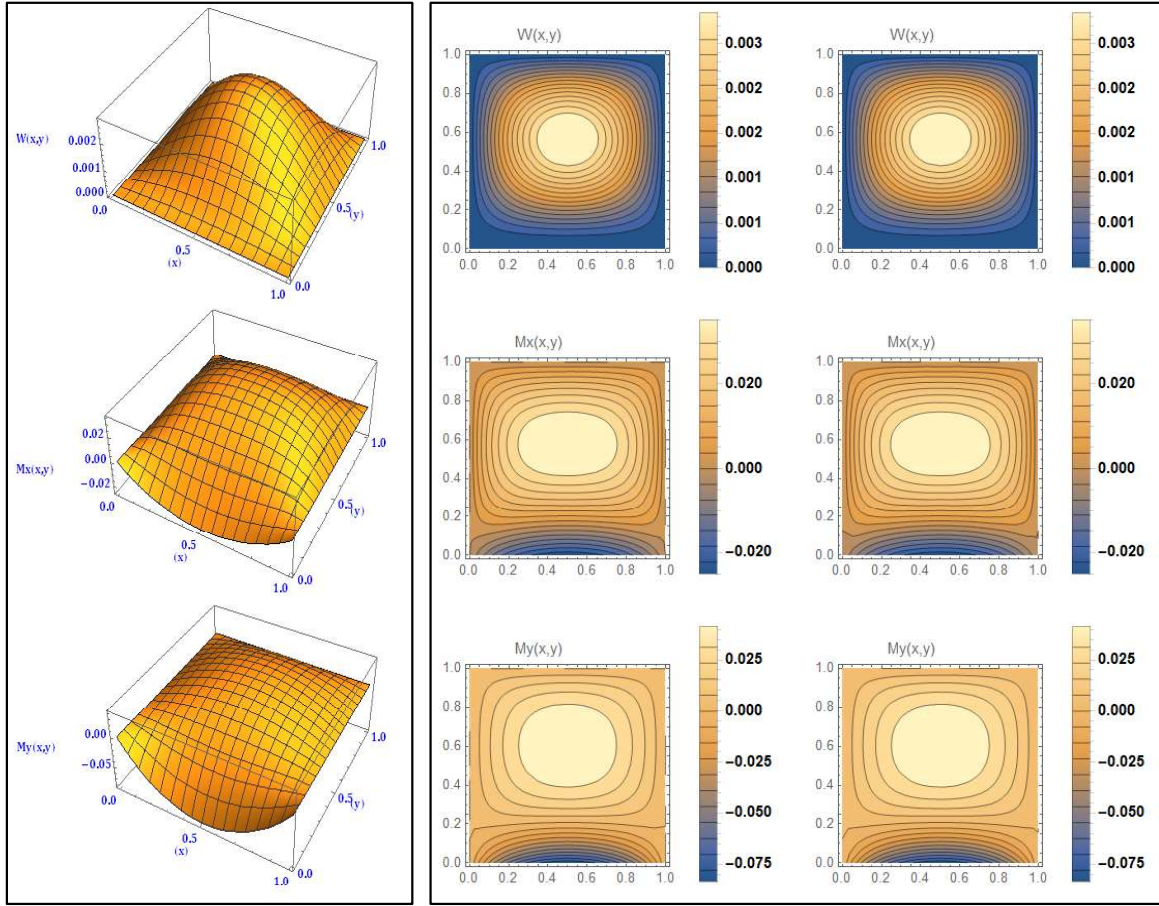


Figure 4. 2: The first two columns from left are the plots of the Boundary-type Ritz solutions. The last column shows the corresponding Levy's solutions

4.1.2 Five-side plate

Analysis is performed for a pentagonal plate, shown in Figure 4. 3) with the following properties:

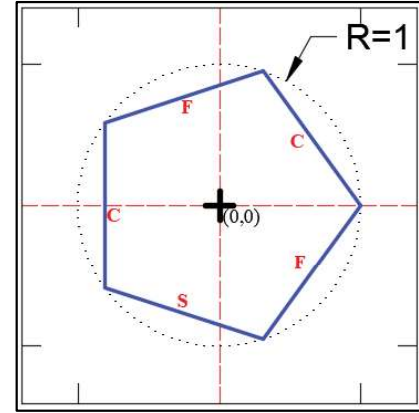
Dimensions: circumscribed within a unit circle.

Boundary conditions: indicated in Figure 4. 3).

Load: uniform

Properties of the FEM model:

Mesh size: $0.02 r$, r is the radius of the outer circle.



[Figure 4. 3: Pentagonal plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)]

It was noted that the running time of the code was reduced from 6 hrs in the case of implementing the traditional Ritz method to only 3 min in the case of implementing the (BTR) method.

Values from the Ritz solution are compared to the FEM results at the center of the plate, as shown in Table 4. 2). The plots of both Ritz and FEM solutions are listed together in Figure 4. 4) and Figure 4. 5).

Table 4. 2: Comparison between the results of FEM and the boundary-type Ritz method of the plate in Figure 4. 3)

Method	$W_{(0,0)}$	$Mx_{(0,0)}$	$My_{(0,0)}$
Boundary-type Ritz	-0.0161	0.088247	0.06355
FEM	-0.0162	-0.08838	-0.0634
Relative Error (%)	0.3	0.1	0.3

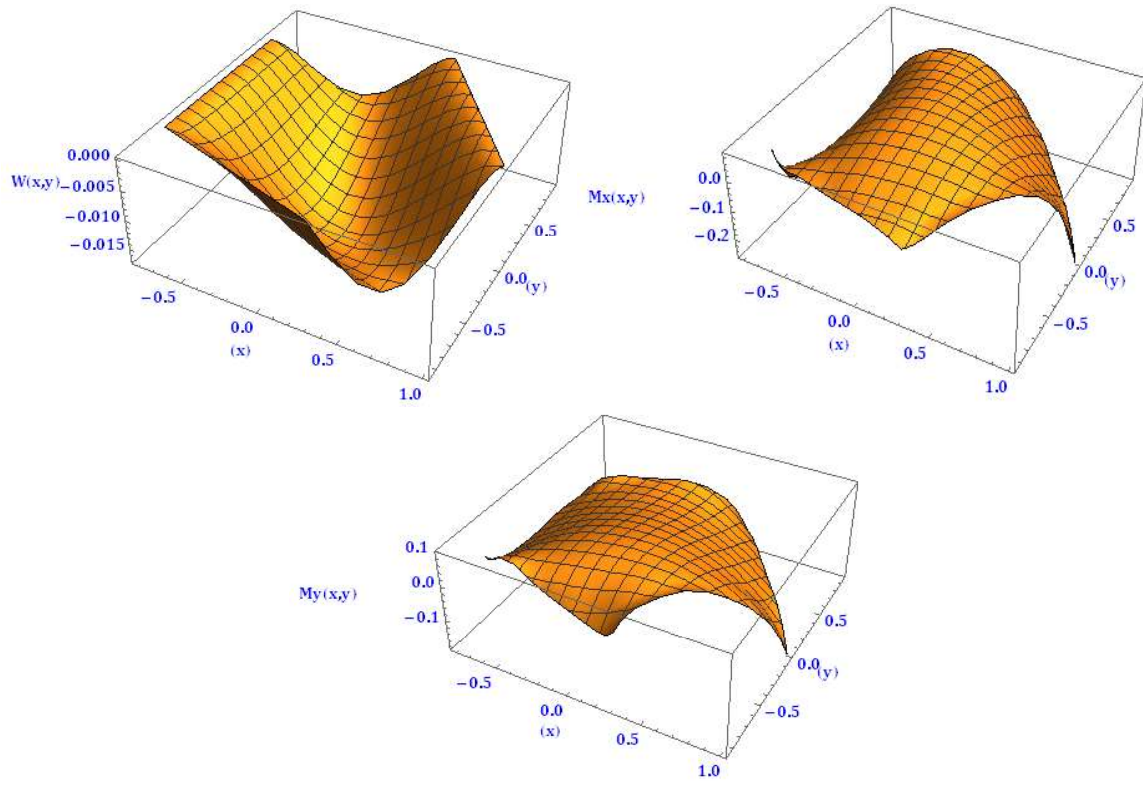


Figure 4. 4: Plots of the boundary-type Ritz solutions for the pentagonal plate in Figure 4. 3)

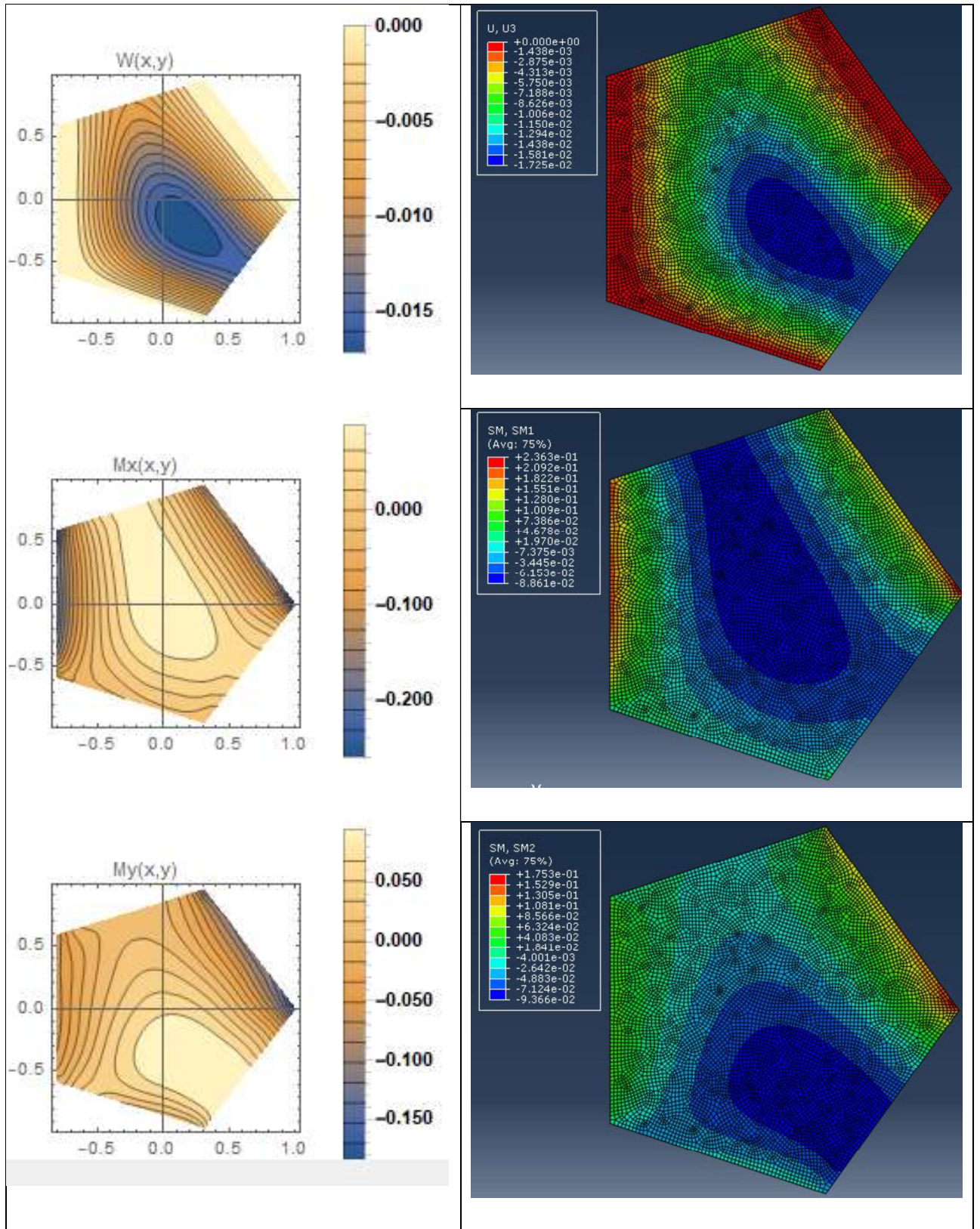


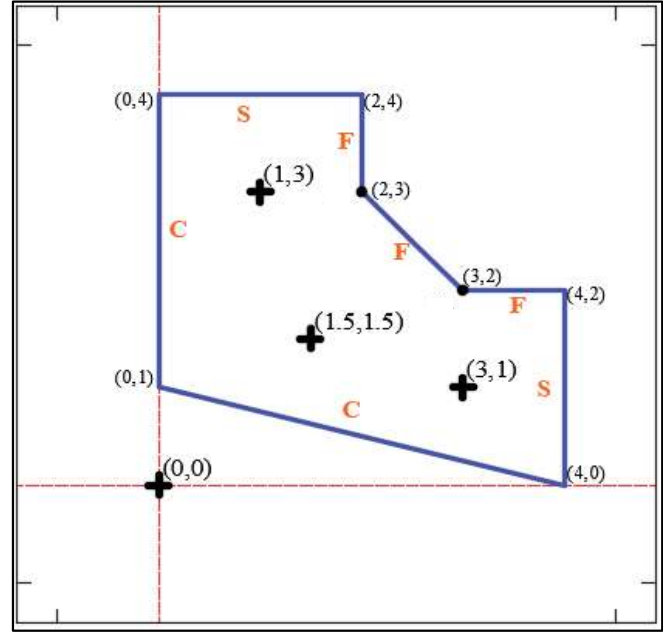
Figure 4. 5: Plots of the boundary-type Ritz solutions to the left, and for the FEM solutions to the right, for the pentagonal plate in Figure 4. 3)

4.1.3 Irregular plate

Analysis is performed for a plate of irregular shape, shown in Figure 4. 6) with dimensions and boundary conditions indicated in Figure 4. 6). The load is assumed uniformly distributed.

Properties of the FEM model:
Mesh size: 0.05 m.

It was noted that the running time of the code was reduced from 11 hrs in the case of implementing the traditional Ritz method to only 13 min in the case of implementing the (BTR) method.



[Figure 4. 6: Arbitrarily shaped plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)]

Values from the Ritz solution are compared to the FEM results at some selected points within the plate, as shown in Table 4. 3). The plots of both Ritz and FEM solutions are listed together in Figure 4. 7) and Figure 4. 8).

Table 4. 3: Comparison between the results of FEM and the boundary-type Ritz method of the plate in Figure 4. 6), at three different points

	P1 (3,1)			P2 (1.5,1.5)			P3 (1,3)		
	Boundary-Type Ritz	FEM	Er. (%)	Boundary-Type Ritz	FEM	Er. (%)	Boundary-Type Ritz	FEM	Er. (%)
$W(x,y)$	-0.03197	-0.031828	0.4	-0.07306	-0.073848	1.1	-0.06442	0.0646871	0.4
$Mx(x,y)$	0.054163	-0.055111	1.7	0.122355	-0.120291	1.7	0.199648	-0.198839	0.4
$My(x,y)$	0.104132	-0.109317	4.7	0.144002	-0.146817	1.9	0.098297	-0.090336	8.8

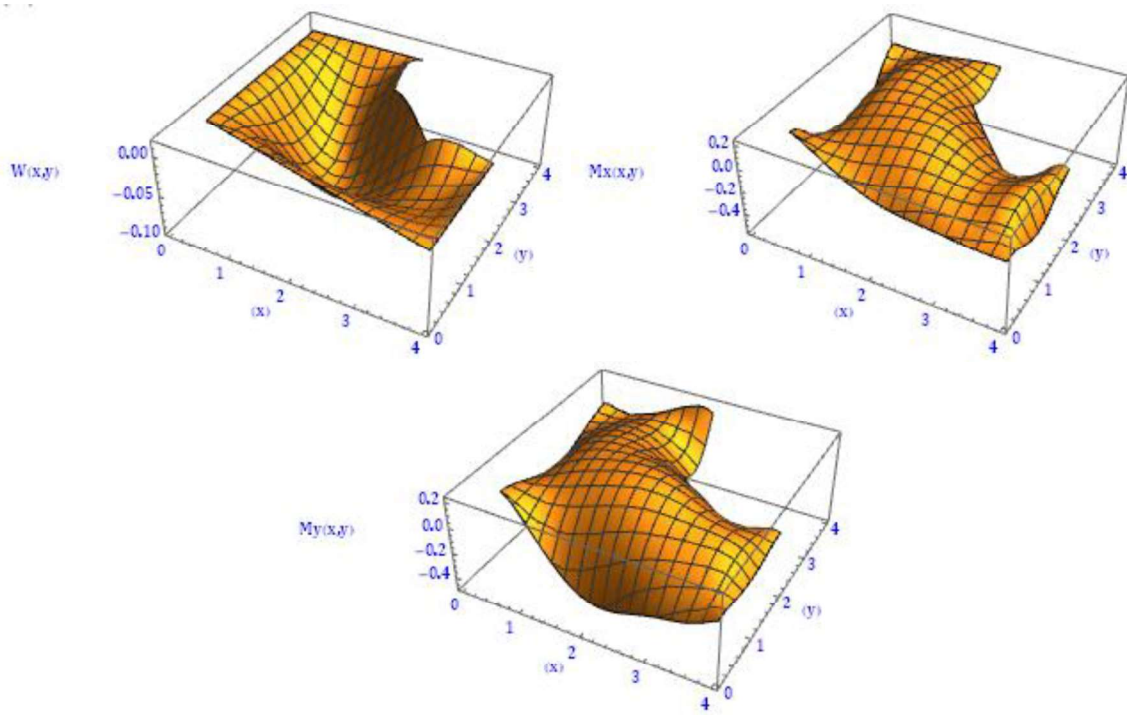


Figure 4. 7: Plots of the boundary-type Ritz solutions for the plate in Figure 4. 6)

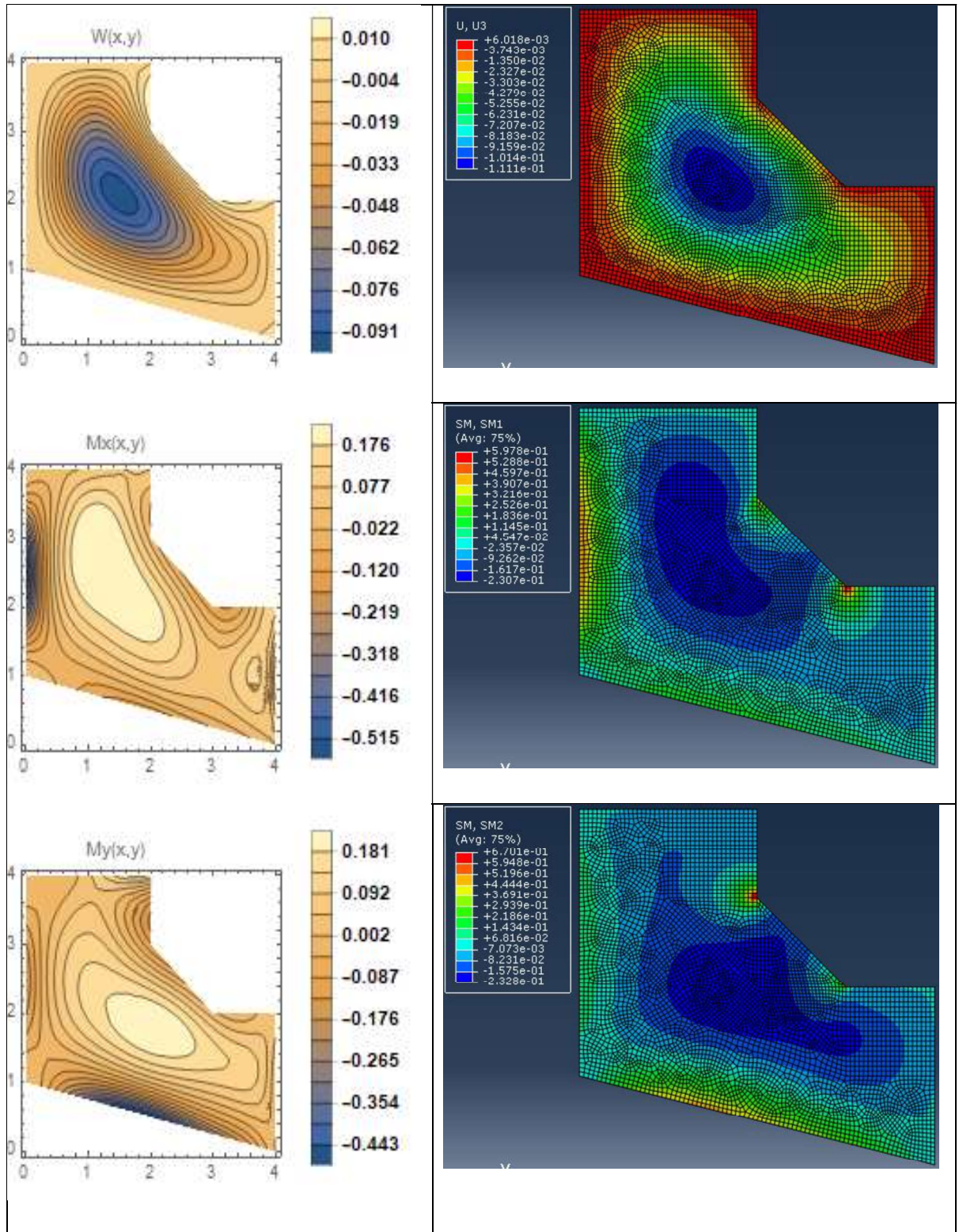


Figure 4. 8: Plots of the boundary-type Ritz solutions to the left, and for the FEM solutions to the right, for the plate in Figure 4. 6)

4.1.4 Discussion of section (4.1)

Results of the rectangular plate simply supported along all edges show very good agreement with the analytical solution (Levy's solution), with the values of deflection and moments almost vanishing at the plate center.

As for the pentagonal and the irregular plates, values of the deflection and moments at the center of the plate are consistent with the corresponding FEM results.

In all of the previous examples, a huge reduction in running time was achieved by applying the proposed (BTR) method, which drops from several hours to a few minutes.

In general, the results introduced previously confirm the reliability of the proposed method (the boundary-type Ritz method), and the applicability of the method to various polygonal plate shapes with any arrangement of boundary conditions. The remarkably fast calculations achieved by this method adds a great advantage to its use.

4.2 Free vibration of plates

4.2.1 Square plate

Analysis is performed for a square plate with the following properties:

Dimensions: $a \times a$

Boundary conditions: simply supported along all edges

Properties of the FEM model:

Mesh size: $0.01 a$

It was noted that the running time of the code was reduced from 2.4 min in the case of implementing the traditional Rayleigh-Ritz method to only 16 sec in the case of implementing the (BTR) method.

Values of the natural frequencies from the Rayleigh-Ritz solution are compared to the available analytical solutions for the first four modes. The FEM results are also validated using the analytical solutions in order to prove the reliability of the FEM analysis for the next tested plate shapes. All of these comparisons are shown in Table 4. 4).

The contour plots of the first four modal shapes are shown in Figure 4. 9) and Figure 4. 10) as obtained from the boundary-type Rayleigh-Ritz method. Adjacent to them are listed the FEM plots.

Table 4. 4: The natural frequencies obtained from the boundary-type Ritz method and the FEM, compared to results from different references, for an SSSS square plate

Mode	Analytical ($\omega a^2 \sqrt{\rho h}$)			FEM		Boundary-Type Ritz (rad/s)
	(BHAT, 1985)[14] (rad/s)	(Snehashish, 2009)[22] (rad/s)	(Leissa, 1973)[23] (rad/s)	Hz	rad/s	
1	19.739	19.739	19.7392	3.1281	19.654	19.7392
2	49.348	49.348	49.348	7.8295	49.194	49.5132
3	49.348	49.348	49.348	7.8295	49.194	49.5132
4	78.957	78.957	78.9568	12.502	78.552	79.4007

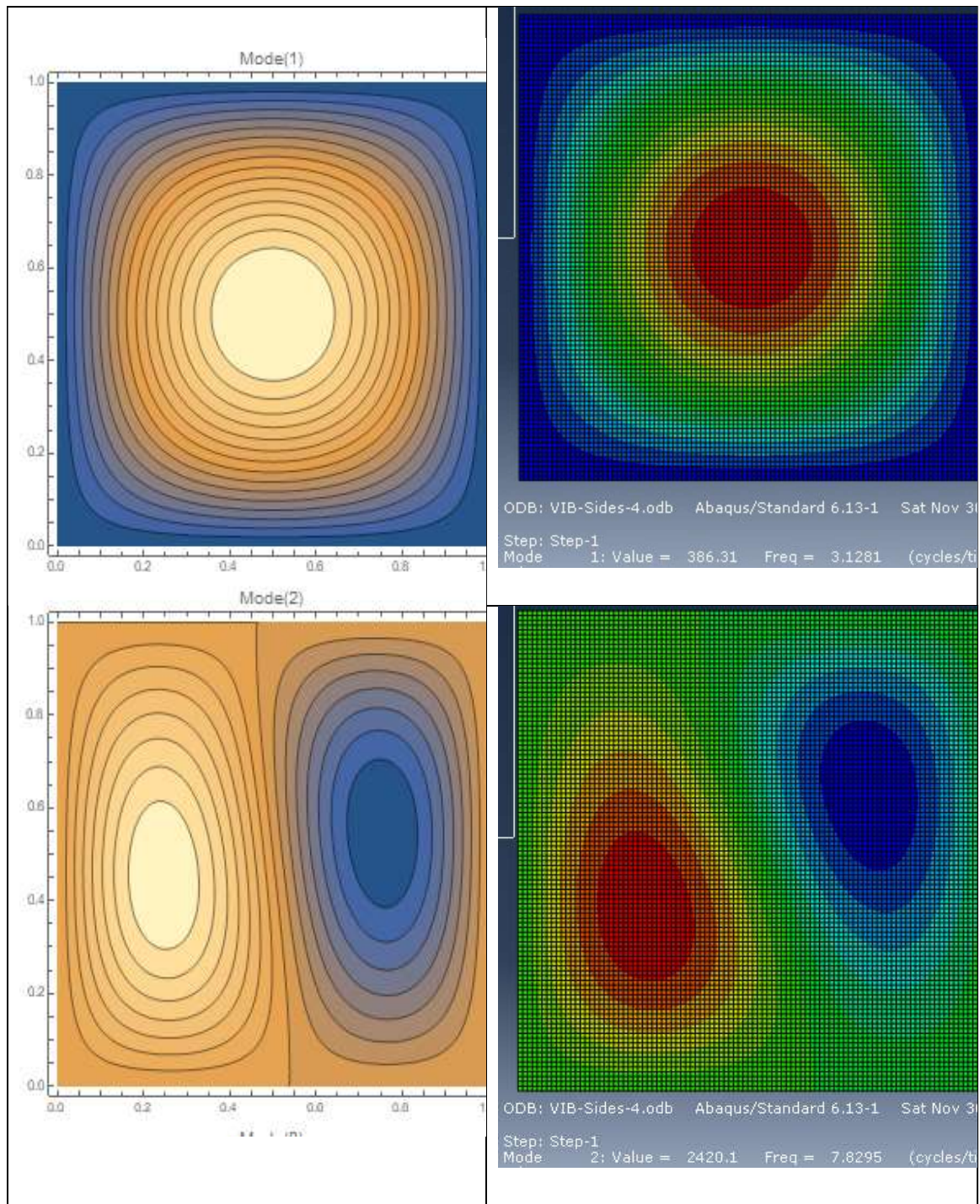


Figure 4. 9: Modal shapes of modes 1 and 2, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for an SSSS square plate

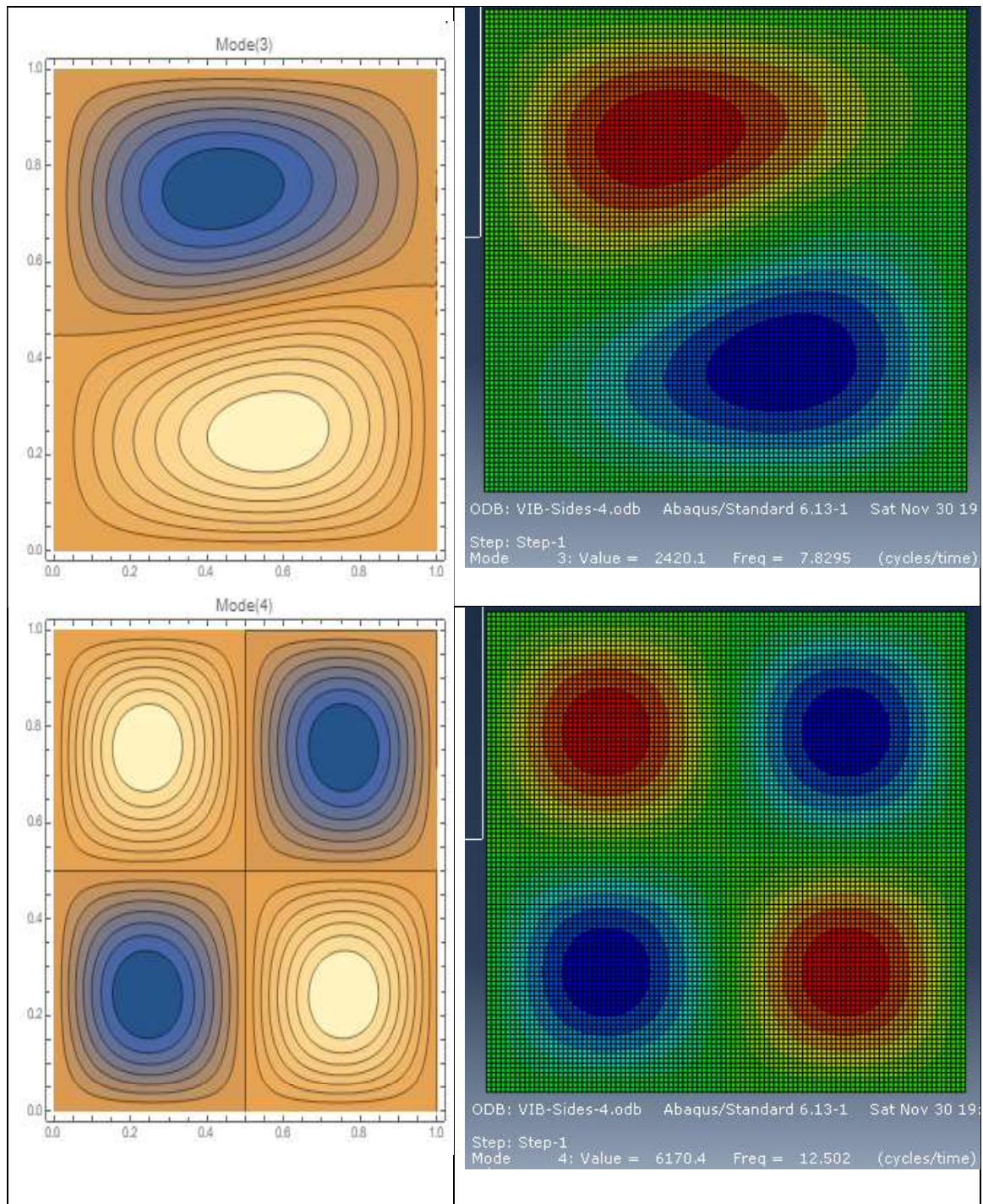


Figure 4. 10: Modal shapes of modes 3 and 4, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for an SSSS square plate

4.2.2 Triangular plate

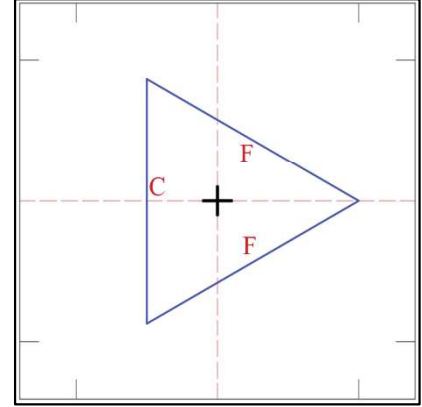
Analysis is performed for a triangular plate, shown in Figure 4. 11), with the following properties:

Dimensions: all sides are of one-unit length.

Boundary conditions: as indicated in Figure 4. 11).

Properties of the FEM model:

Mesh size: 0.01 of the side length.



[Figure 4. 11: Triangular plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)]

It was noted that the running time of the code was reduced from nearly 1 min in the case of implementing the traditional Rayleigh-Ritz method to only 7 sec in the case of implementing the (BTR) method.

Values of the natural frequencies from the Rayleigh-Ritz solution are compared to the FEM results and to the results obtained by Bhat, R. (1987) [15], as shown in Table (4.5). Plots of the first four modal shapes obtained from the Rayleigh-Ritz method are shown in Figure 4. 12). The contour plots of the first four modal shapes are shown in Figure 4. 13) as obtained from the Rayleigh-Ritz method. Adjacent to them are listed the FEM plots.

Table 4. 5: The natural frequencies obtained from the boundary-type Ritz method, compared to the results from (Bhat, R., 1987)[15] and FEM, for the plate in Figure 4. 11)

Mode	FEM		(Bhat, R., 1987) [15]	Boundary-Type Ritz (rad/s)
	Hz	rad/s		
1	1.4179	8.909	8.922	8.922
2	5.5671	34.979	35.132	35.131
3	6.1081	38.378	38.505	38.503
4	14.213	89.303	90.591	90.586

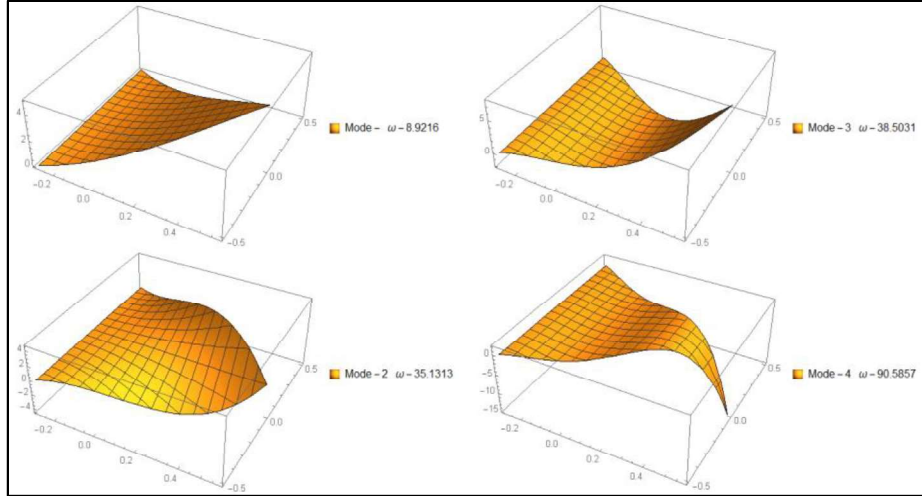


Figure 4. 12: Modal shapes of the first 4 modes, obtained from the boundary-type Ritz method for the plate in Figure 4. 11)

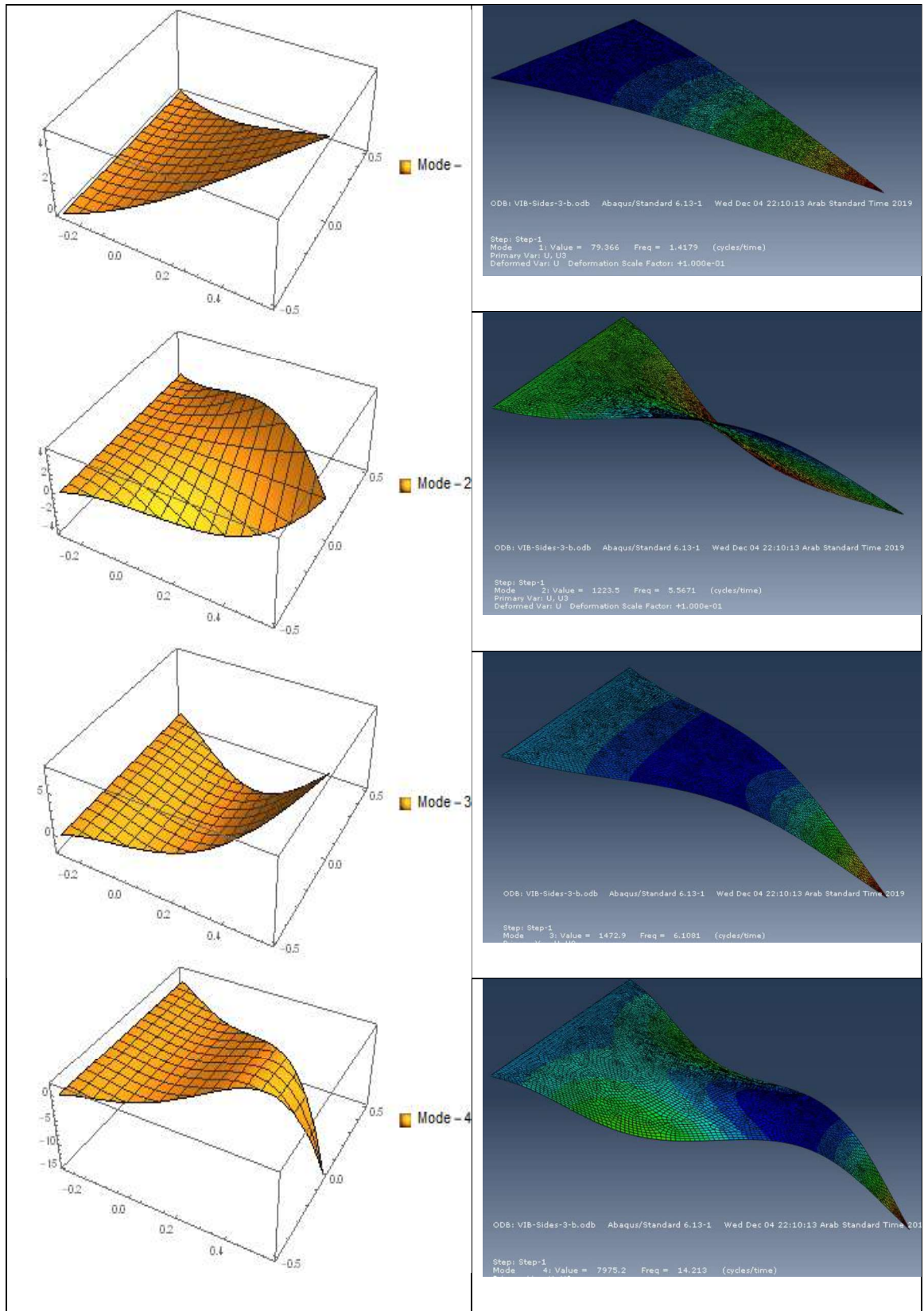


Figure 4. 13: Modal shapes of the first 4 modes, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure 4. 11)

4.2.3 Five-side Plate

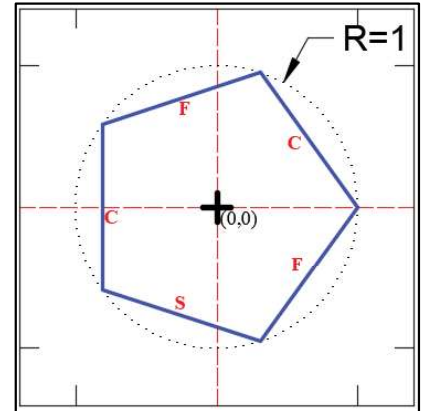
Analysis is performed for a pentagonal plate, shown in Figure 4. 14), with the following properties:

Dimensions: circumscribed within a unit circle

Boundary conditions: indicated in Figure 4. 14).

Properties of the FEM model:

Mesh size: $0.02 r$, r is the radius of the outer circle.



[Figure 4. 14: Pentagonal plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)]

It was noted that the running time of the code was reduced from 15 min in the case of implementing the traditional Rayleigh-Ritz method to only 1.6 min in the case of implementing the (BTR) method.

Values of the natural frequencies from the Rayleigh-Ritz solution are compared to the FEM results as shown in table (4.6). Plots of the first four modal shapes obtained from the Rayleigh-Ritz method are shown in Figure 4. 15). The contour plots of the first four modal shapes are shown Figure 4. 16) and Figure 4. 17) as obtained from the Rayleigh-Ritz method. Adjacent to them are listed the FEM plots.

Table 4. 6: The natural frequencies obtained from the boundary-type Ritz method and the FEM, for the plate in Figure 4. 14)

Mode	FEM		Boundary-Type Ritz (rad/s)
	Hz	rad/s	
1	1.4623	9.188	9.267
2	2.2196	13.946	14.381
3	3.0623	19.241	19.961
4	4.1157	25.860	26.658

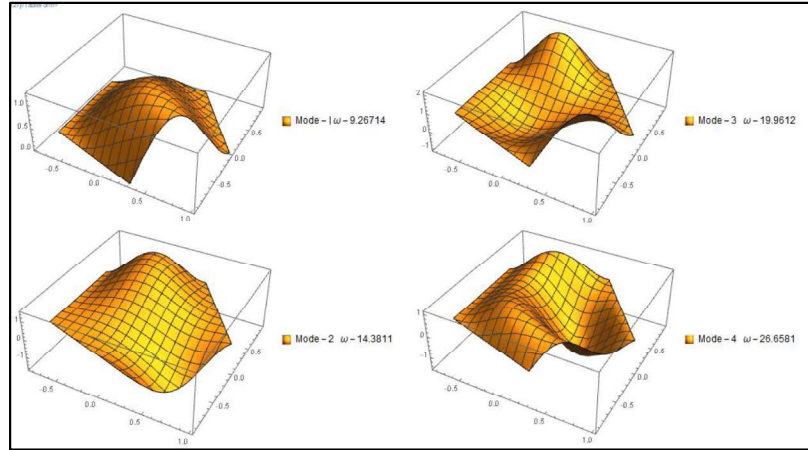


Figure 4. 15: Modal shapes of the first four modes, obtained from the boundary-type Ritz method for the plate in Figure 4. 14)

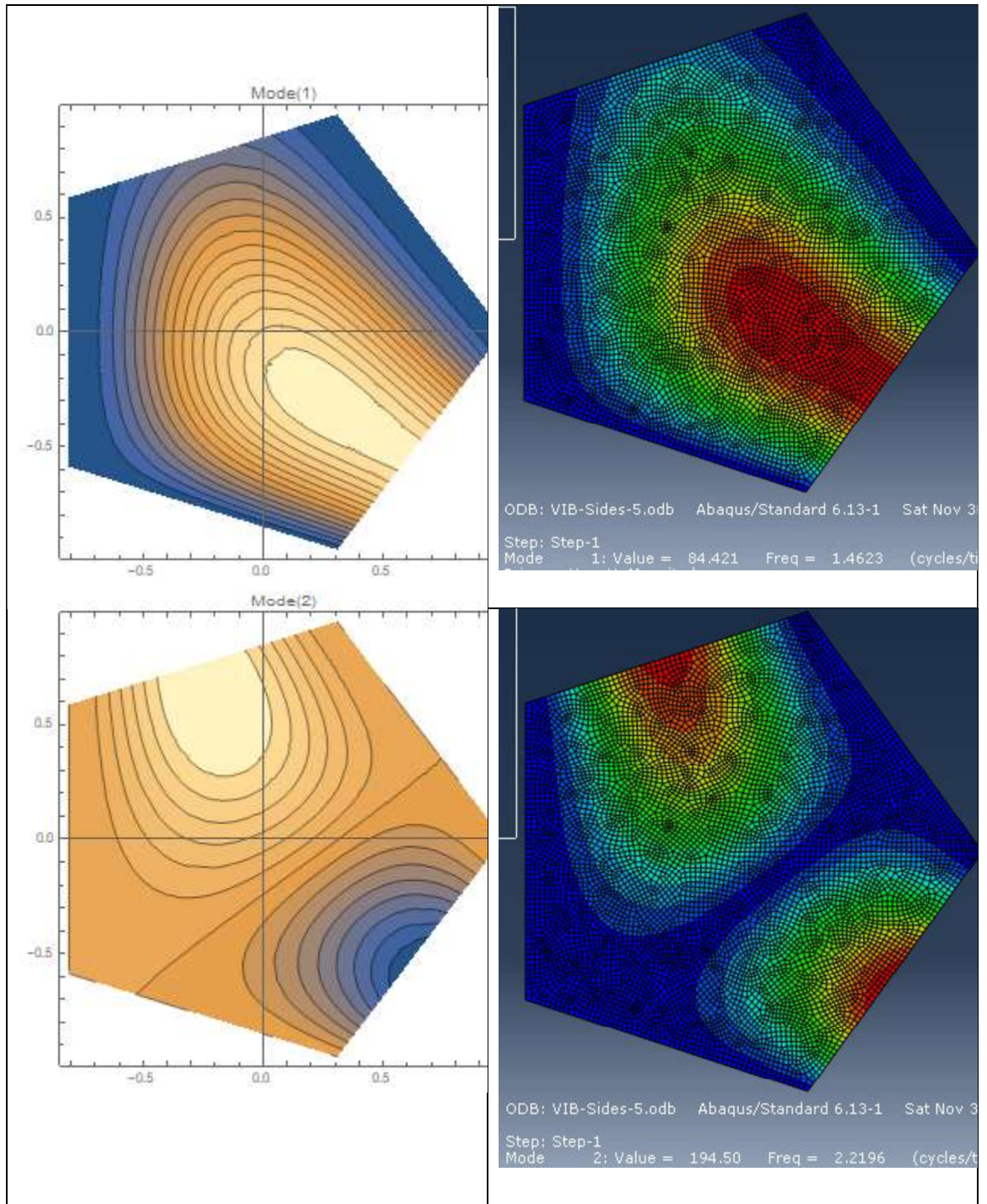


Figure 4. 16: Modal shapes of modes 1 and 2, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure 4. 14)

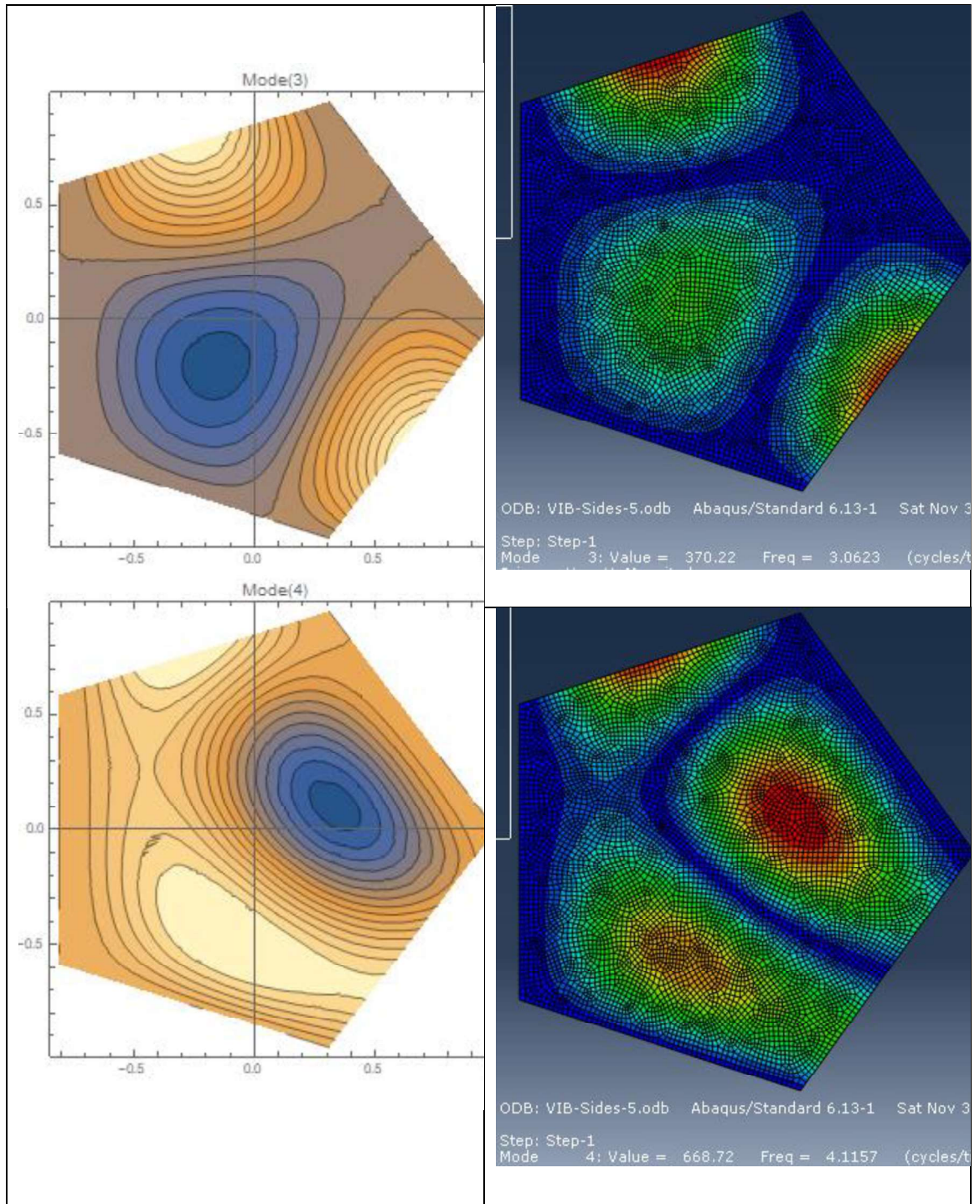


Figure 4. 17: Modal shapes of modes 3 and 4, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure 4. 14)

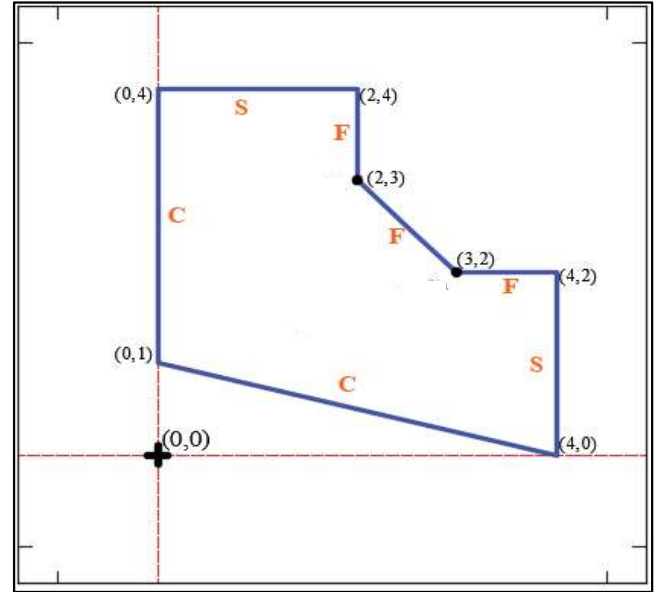
4.2.4 Irregular plate

Analysis is performed for an irregular shape, with dimensions and boundary conditions shown in Figure 4. 18).

Properties of the FEM model:

Mesh size: 0.05 m.

It was noted that the running time of the code was reduced from 20 min in the case of implementing the traditional Rayleigh-Ritz method to only 2 min in the case of implementing the (BTR) method.



[Figure 4. 18: Arbitrarily shaped plate with different boundary conditions (C: clamped, S: simply supported, F: free edge)]

Values of the natural frequencies from the

Rayleigh-Ritz solution are compared to the FEM results as shown in **Error! Reference source not found.** Plots of the first four modal shapes obtained from the Rayleigh-Ritz method are shown in Figure 4. 19). The contour plots of the first four modal shapes are shown in Figure 4. 20: Modal shapes of modes 1 and 2, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 18) and Figure 4. 21) as obtained from the Rayleigh-Ritz method. Adjacent to them are listed the FEM plots.

Table 4. 7: The natural frequencies obtained from the boundary-type Ritz method and the FEM, for the plate in Figure (4. 18)

Mode	FEM		Boundary-Type Ritz (rad/s)
	Hz	rad/s	
1	0.29048	1.825	1.880
2	0.59039	3.710	3.885
3	0.86877	5.459	5.628

4	1.042	6.547	7.300
---	-------	-------	-------

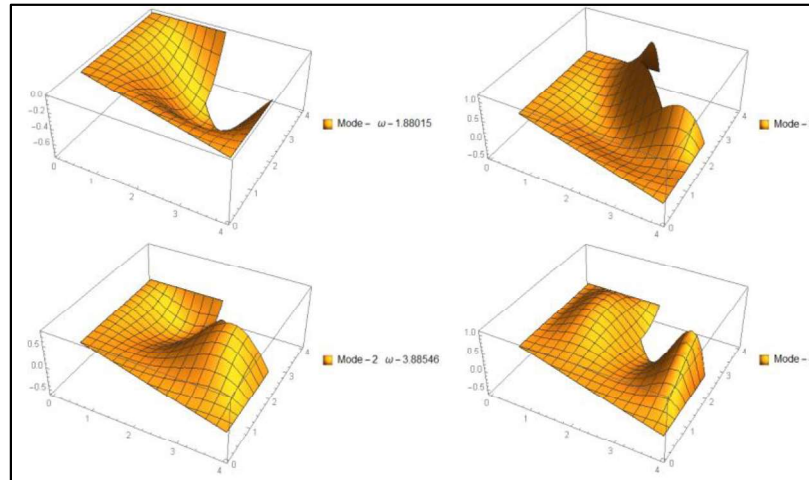


Figure 4. 19: Modal shapes of the first four modes, obtained from the boundary-type Ritz method for the plate in Figure (4. 18)

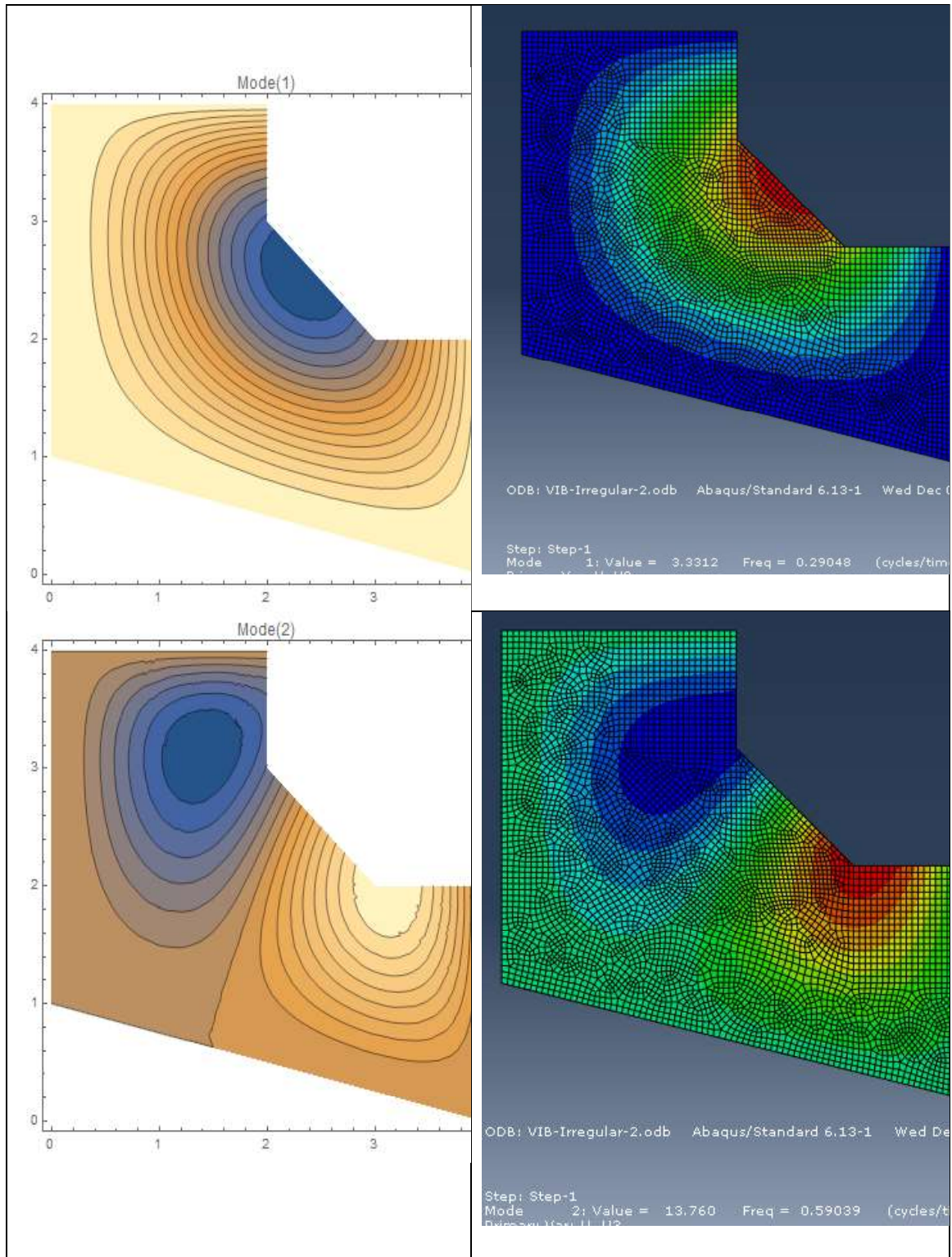


Figure 4. 20: Modal shapes of modes 1 and 2, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 18)

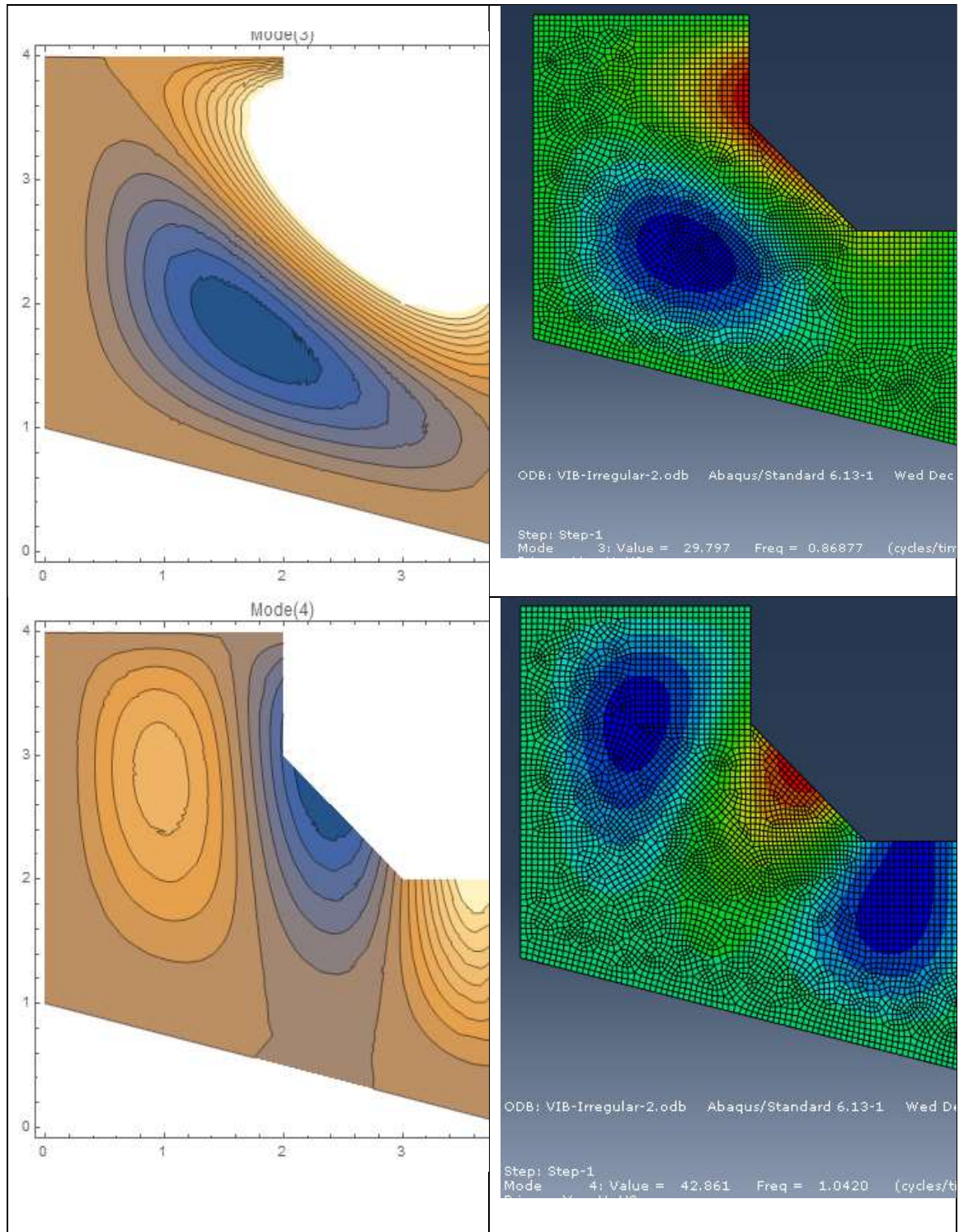


Figure 4. 21: Modal shapes of modes 3 and 4, obtained from the boundary-type Ritz method to the left, and from the FEM to the right, for the plate in Figure (4. 18)

4.2.5 Discussion of section (4.2)

It can be clearly noticed from the previous results that the natural frequencies predicted using the boundary-type Ritz method are very close to those obtained by the FEM analysis. This method makes very easy the analysis of free vibration of plates of any polygonal shape with various arrangement of boundary conditions.

With regard to the square plate, the results are compared to both the available analytical solutions and the FEM solution. The results are almost identical. In addition to verifying the proposed boundary-type Ritz method, it also confirmed the reliability of the FEM results.

In all of the previous examples, a huge reduction in running time was achieved by applying the proposed (BTR) method, which in most cases drops from several minutes to a few seconds.

In conclusion, the boundary-type Ritz method proved its applicability for a wide range of polygonal shapes of plates. It provides a very fast method for calculating the natural frequencies of the higher modes and the associated modal shapes, without compromising the accuracy of the results.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

The Ritz and Rayleigh-Ritz methods are powerful methods because they yield solutions in the form of simple formulae that are more convenient for repeated calculations or optimization. The presented method provides an enhancement to the performance of the conventional Ritz and Rayleigh-Ritz methods. The results of this study leads to the following conclusions:

- 1- The core advantage of the proposed BTR method is that it eliminates the need for the integrations over the plate domain when trying to calculate the elements of the $[K]$ matrix in both the static and vibrational analysis of plates. The method reduces this integration into simple summation at the plate corners.
- 2- The proposed method gives freedom as to the selection of the plate shape. It is observed that no convergence problems arise when shapes become too complicated.
- 3- Based on the formulation of the proposed method, there is no restriction on the type of function to be assumed as the trial deflection function, but self-evidently, polynomial functions are the simplest to deal with.
- 4- Mathematica provides an appropriate environment for coding efficient routines to implement the (BTR) method and display results in high quality.

5.2 Recommendations

In light of the outcomes and the challenges experienced during this research, the following ideas are recommended for further investigation in the subject:

- 1- The convergence of the Ritz solutions depends on the assumed deflection function. The polynomial functions were chosen in this research for both the static and free vibration analyses. Searching for more appropriate functions may result in faster convergence, even with small number of terms.
- 2- Research can be devoted to improve the Ritz method to be capable of treating the singularity at the boundaries, which result from the abrupt change in the boundary conditions (e.g. from simply supported to free edge at corners)
- 3- Corner supported plates are likely to undergo large deflection and therefore, the method should be modified to address the case of large deflection

References

- [1] Lee, S. and Ballesteros, P. (1960). Uniformly loaded rectangular plate supported at the corners. *International Journal of Mechanical Sciences*, 2(3), pp.206-211.
- [2] Leissa, A., Lo, C. and Nietenfuhr, F. (1965). Uniformly loaded plates of regular polygonal shape. *AIAA Journal*, 3(3), pp.566-567.
- [3] Wah, T. (1979). Elastic quadrilateral plates. *Computers & Structures*, 10(3), pp.457-466.
- [4] Shanmugam, N., Huang, R., Yu, C. and Lee, S. (1988). Uniformly loaded rhombic orthotropic plates supported at corners. *Computers & Structures*, 30(5), pp.1037-1045.
- [5] Azarkhin, A. (1992). Bending of Thin Plate with Three-Point Support. *Journal of Structural Engineering*, 118(5), pp.1416-1419.
- [6] Liew, K. (1992). Response of Plates of Arbitrary Shape Subject to Static Loading. *Journal of Engineering Mechanics*, 118(9), pp.1783-1794.
- [7] Wang, C. and Alwis, W. (1995). Simply Supported Polygonal Mindlin Plate Deflections Using Kirchhoff Plates. *Journal of Engineering Mechanics*, 121(12), pp.1383-1385.
- [8] Saadatpour, M. and Azhari, M. (1998). The Galerkin method for static analysis of simply supported plates of general shape. *Computers & Structures*, 69(1), pp.1-9.
- [9] Odishelidze, N., Criado-Aldeanueva, F., Criado, F. and Sanchez, J. (2014). General case of contact problems for a regular polygon weakened with full-strength hole. *Zeitschrift für angewandte Mathematik und Physik*, 66(3), pp.1187-1206.
- [10] Li, R., Wang, B. and Li, P. (2014). Hamiltonian system-based benchmark bending solutions of rectangular thin plates with a corner point-supported. *International Journal of Mechanical Sciences*, 85, pp.212-218.

- [11] Li, R., Wang, B. and Li, G. (2015). Analytic solutions for the free vibration of rectangular thin plates with two adjacent corners point-supported. *Archive of Applied Mechanics*, 85(12), pp.1815-1824.
- [12] Li, R., Wang, B., Lv, Y., Zhang, Q., Wang, H., Jin, F., Teng, F. and Wang, B. (2016). New analytic solutions for static problems of rectangular thin plates point-supported at three corners. *Meccanica*, 52(7), pp.1593-1600.
- [13] Li, R., Tian, Y., Zheng, X., Wang, H., Xiong, S. and Wang, B. (2017). New analytic bending solutions of rectangular thin plates with a corner point-supported and its adjacent corner free. *European Journal of Mechanics - A/Solids*, 66, pp.103-113.
- [14] Bhat, R. (1985). Natural frequencies of rectangular plates using characteristic orthogonal polynomials in rayleigh-ritz method. *Journal of Sound and Vibration*, 102(4), pp.493-499.
- [15] Bhat, R. (1987). Flexural vibration of polygonal plates using characteristic orthogonal polynomials in two variables. *Journal of Sound and Vibration*, 114(1), pp.65-71.
- [16] Liew, K. and Lam, K. (1991). A Set of Orthogonal Plate Functions for Flexural Vibration of Regular Polygonal Plates. *Journal of Vibration and Acoustics*, 113(2), pp.182-186.
- [17] KANG, S. and LEE, J. (2001). FREE VIBRATION ANALYSIS OF ARBITRARILY SHAPED PLATES WITH CLAMPED EDGES USING WAVE-TYPE FUNCTIONS. *Journal of Sound and Vibration*, 242(1), pp.9-26.
- [18] Wu, J., Liu, A. and Chen, H. (2005). Exact Solutions for Free-Vibration Analysis of Rectangular Plates Using Bessel Functions. *Journal of Applied Mechanics*, 74(6), pp.1247-1251.
- [19] Wang, Z. X., and Guo, D. R., 1989, *Special Functions*, World Scientific, Singapore, pp. 345–455.

- [20] Catania, G. and Sorrentino, S. (2011). Spectral modeling of vibrating plates with general shape and general boundary conditions. *Journal of Vibration and Control*, 18(11), pp.1607-1623.
- [21] Eftekhari, S. and Jafari, A. (2012). A Novel and Accurate Ritz Formulation for Free Vibration of Rectangular and Skew Plates. *Journal of Applied Mechanics*, 79(6).
- [22] Chakraverty, S. (2009). *Vibration of plates*. Boca Raton: CRC Press.
- [23] Leissa, A. (1973). The free vibration of rectangular plates. *Journal of Sound and Vibration*, 31(3), pp.257-293.

Vitae

Name	: [Mohammad Hisham Mohammad Aljawhary]
Nationality	: [Jordanian]
Date of Birth	: [10/31/1992]
Email	: [aljawhary.mo@gmail.com]
Address	: [KFUPM, Dhahran, Saudi Arabia. Zip code: 31261] Mobile: +966590328430
Academic Background	BS Degree (2015): Excellent (GPA 3.70/4.00) (Major: Building Engineering), An- Najah National University, Nablus, West Bank, Palestine. MS Degree (2020): First honors (GPA 3.89/4.00) (Major: Civil Engineering – Structures), KFUPM, Dhahran, Saudi Arabia Received the “Excellence Scholarship” in the undergraduate level from An-Najah National University for the excellent accumulative average (2011-2015). Received a scholarship from King Fahd University for Petroleum and Minerals (2018-2020) to pursue a Master program in Civil Engineering.
Work Experience	Assigned as a Research and Teaching Assistant at An-Najah National University in Palestine (2015-2018), teaching undergraduate Civil Engineering labs, and assisting the department in academic research.