

**NONLINEAR BUCKLING OF CYLINDRICAL SHELLS
UNDER AXIAL COMPRESSION**

BY

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In

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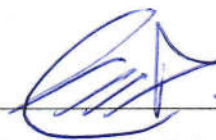
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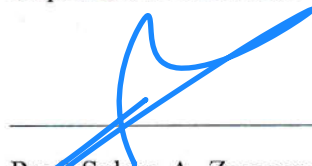
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To my wife, daughter, parents, brother and sisters.

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LIST OF ABBREVIATIONS

The following are the list of abbreviations used in this dissertation.

FEM	:	Finite Element Method
FE	:	Finite Element
LBA	:	Linear Buckling Analysis
GNA	:	Geometric Nonlinear Analysis
GMNA	:	Geometric and Material Nonlinear Analysis
GNIA	:	Geometric Nonlinear with Imperfection Analysis
GNMIA	:	Geometric and Material Nonlinear with Imperfection Analysis
KDF	:	Knock Down Factor
BSR	:	The buckling strength reduction.

LIST OF SYMBOLS

The following are the list of symbols used in this dissertation. Other symbols are defined within the text.

E	:	The elastic modulus.
ν	:	Poisson's ratio.
R	:	Radius of the shell.
t	:	Thickness of the shell.
L	:	Length of the shell.
D	:	Diameter of the shell.
u	:	The shell displacement in the longitudinal direction.
v	:	The shell displacement in the circumferential direction.
w	:	The shell displacement in the radial direction.
m	:	Number of halve sign waves in the longitudinal direction.
n	:	Number of halve sign waves in the circumferential direction.
F	:	Airy's stress function.
a	:	Halve length of the wave in the longitudinal direction.
b	:	Halve length of the wave in the circumferential direction.

N	:	The axially applied edge load with the units of force per unit circumferential distance
N_{cr}	:	The critical edge load that causes buckling.
σ_{cl}	:	The classical axisymmetric buckling stress.
σ_{cr}	:	The critical non-axisymmetric buckling stress.
σ_{max}	:	The axial applied stress the shell can carry.
P_{max}	:	The maximum load carrying capacity of the imperfect shell.
P_{cp}	:	The load carrying capacity of the perfect shell.
$\bar{P}$	:	The load ratio defined as P_{max} divided by P_{cp} .
N_c	:	Number of elements along the circumferential direction.
N_l	:	Number of elements along the longitudinal direction.
U_r	:	The out-of-roundness parameter.
w_d	:	The dent imperfection.
δ_d	:	The dent amplitude.
L_{dc}	:	The length of the dent along the circumferential direction.
L_{dl}	:	The length of the dent along the longitudinal direction

x_{dc} : The local coordinate of the dent along the circumferential direction.

x_{dl} : The local coordinate of the dent along the longitudinal direction.

ABSTRACT

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Buckling is the most common type of failure associated with cylindrical shells subjected to axial compressive forces. This is mainly due to large radius to thickness ratio of such shells making them very thin structures. This behavior has been discovered more than a century ago and undergone a considerable development from axisymmetric to non-axisymmetric mode, from small-deflection theory to large-deflection theory (geometrical nonlinearity), from an elastic linear material to a nonlinear material (material nonlinearity) and from a perfect shell to an imperfect one. Prior to the discovery of the geometrical and material nonlinearities, and introduction of imperfections, there was exist a great discrepancy between the experimental results and the theoretical ones, but these discoveries highly improved the theoretical results towards reducing the mismatching.

In this work, buckling and postbuckling behavior of axially compressed circular cylindrical shells are studied taking into account geometric and material nonlinearities of both perfect and imperfect shells. The symbolic computational capabilities of Mathematica software are utilized for cases of elastic perfect shells, which serves only as a reference for designers. The finite element software ABAQUS is utilized to have flexibility to simulate the real structure where there exists geometric and material nonlinearity in an imperfect shell. Different forms of imperfections in the shell geometry are tackled in this study such as

irregularity in shape of the cylinder (out-of-roundness) and localized damage in the form of a dent.

In the light of this investigation, capacity curves of the perfect shells are proposed for a large range of R/t ratios to be used as references for shell design. Guidelines and the key steps of FEM simulation are reported to make it easy for designers to adopt different forms of imperfections such as out-of-roundness and the localized dent. A parametric study on the effect of out-of-roundness imperfections is conducted. One of the available parametric studies on the effect of localized dent for small R/t ratios is critically examined and corrections provided. A parametric study on the effect of localized dent for shells with large R/t ratios is also conducted to study the effect of the dent: size, amplitude, and position on the buckling strength for case of thin shells.

ملخص الرسالة

الاسم الكامل: أبوبكر الفاتح صديق موسى

عنوان الرسالة: الانبعاج اللاخطي للقشريات الدائرية الأسطوانية تحت تأثير الحمل المحوري

التخصص: الهندسة المدنية

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الانبعاج هو أكثر أنواع الفشل شيوعاً لدى القشريات الأسطوانية المعرضة لقوى ضغط محورية. هذا يرجع أساساً إلى نصف القطر الكبير مقارنة مع سمك هذه القشريات مما يجعلها هياكل رقيقة جداً. لقد تم اكتشاف هذا السلوك منذ أكثر من قرن وخضع لتطور كبير من الانبعاج ذو الشكل المتناظر محورياً إلى الانبعاج ذو الشكل غير المتناظر محورياً، من نظرية الانحراف الصغير إلى نظرية الانحراف الكبير (اللاخطية الهندسية)، من مادة خطية مرنة إلى مادة غير خطية (اللاخطية المادية) ومن قشريات مثالية مستديرة إلى غير مثالية بتشوهات طفيفة نتيجة للتصنيع أو التجميع. قبل اكتشاف اللاخطيات الهندسية والمادية وإعتبار عيوب التصنيع أو التجميع في عملية التحليل، كان هناك تباين كبير بين النتائج التجريبية والنظرية، ولكن هذه الاكتشافات حسنت النتائج النظرية بشكل كبير و قللت من عدم التطابق.

في هذه الدراسة، تتم دراسة سلوك القشريات الدائرية الأسطوانية المعرضة لحمل محوري في مرحلتي الانبعاج و ما بعد الانبعاج مع الأخذ في الإعتبار اللاخطيات الهندسية والمادية وذلك لحالات القشريات المثالية والغير مثالية (نتيجة لتشوهات التصنيع). تُستخدم الإمكانات الحسابية الرمزية لبرنامج Mathematica لحالات القشريات المثالية المرنة، والتي تفيد للإستخدام فقط كمرجع للمصممين. يتم استخدام برنامج العناصر المحدودة ABAQUS للحصول على مرونة لمحاكاة المنشأة الحقيقية حيث توجد اللاخطيات الهندسية والمادية لقشريات غير مثالية بها العديد من تشوهات التصنيع والتجميع. يتم تناول أشكال مختلفة من العيوب في شكل القشريات في هذه الدراسة مثل عدم انتظام شكل الأسطوانة (الإنحراف عن الدوران) والتلف الموضعي في شكل صدمة خفيفة على سطح الأسطوانة.

بناءً على هذه الدراسة، تقترح منحنيات قدرة تحمل القشريات الأسطوانيات المثالية لمجموعة كبيرة من نسب القطر للسمك لإستخدامها كمرجع لتصميم هذا النوع من القشريات. تم شرح و تفصيل العديد من الإرشادات والخطوات الرئيسية لمحاكاة هذا النوع من القشريات في برامج العناصر المحدودة ليسهل على المصممين تبني أشكال مختلفة من العيوب مثل الإنحراف عن الدوران و التشوهات الموضعية على سطح الأسطوانة. أجريت دراسة حديثة على تأثير الإنحراف عن الدوران. إحدى الدراسات الحديثة المتوفرة حول تأثير الصدمة الموضعية الصغيرة على القشريات ذات

نسب قطر لسمك صغيرة تم فحصها بدقة و قدمت التصحيحات المناسبة. كما تم إجراء دراسة حدية حول تأثير التشوه الموضعي على القشريات ذات نسب قطر لسمك كبيرة لدراسة أثر كل من مساحة، موضع، و عمق التشوه على مقاومة الإنبعاج لهذا النوع من القشريات الرقيقة.

CHAPTER 1

INTRODUCTION

1.1 General

The use of cylindrical shells is very common in oil and gas industries (Figure 1), aerospace, and civil engineering applications, etc. Some specific examples are: fuel tanks, oil storage tanks, water tanks, silos, and columns of circular hollow cross-sections. These applications make a vital need to study the behavior of such structural members. Due to their geometrical properties of having a thin-walled property, their buckling behavior when subjected to compression attracts researchers worldwide. The buckling of cylindrical shells is not limited to axial compressive forces. Other reasons such as compression due to wind load, excessive temperature variations, and settlement of the foundation also lead to buckling behavior as shown in Figure 2.

In many cases, these tanks are utilized for construction of very sensitive and strategic structures. A failure of such kind of structures will have a negative economic, social, and environmental impact [1]. This makes buckling of shells a serious research problem attracts both the governmental and non-governmental organizations to fund such kind of research projects to run even full-scale tests as that done by NASA for an eight diameter ortho-grid stiffened cylinder [2].



Figure 1 Cylindrical tanks in an oil field [1]



a) Buckling of a tank due to fire



b) Buckling of a tank due to wind
load



c) Buckling of a tank due
foundation settlement

Figure 2 Buckling of vertical tank due to different conditions other than axial loads [1]

A cylinder subjected to axial compression force generally experiences buckling mechanism when this force approaches a critical value – i.e. the critical buckling load. The buckling mode depends on many factors such as geometry of the shell, geometric and material nonlinearity, and imperfections. For a perfect cylinder under uniform axial compressive force, there are two common types of buckling modes. The first one is axisymmetric in which the shell is assumed to buckle making waves along the longitudinal direction as in

Figure 3 a. The second one is a non-axisymmetric which is described by three displacements of the shell and lobes as shown in Figure 3 b. In the literature, this is also referred to as the wrinkling.

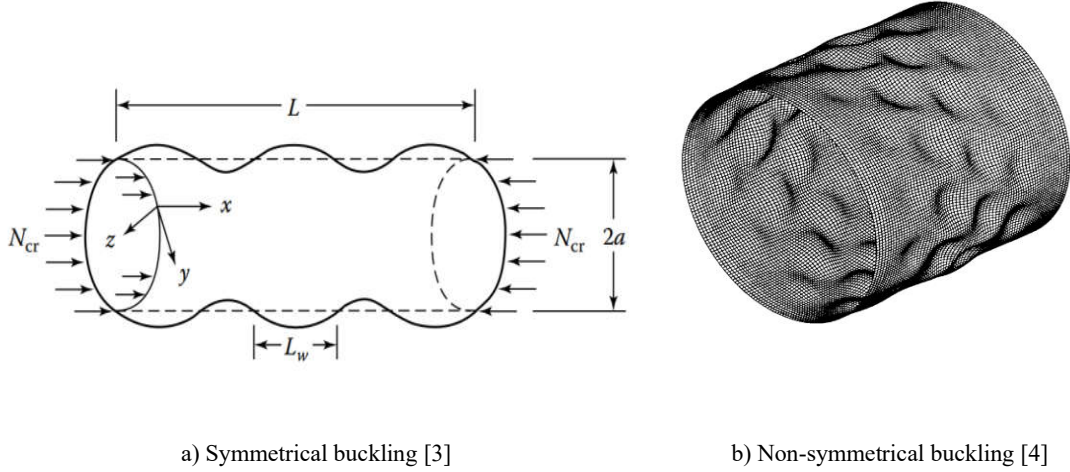


Figure 3 Buckling of a perfect cylinder under axial compression

As a general definition, the buckling phenomena is referred to the collapse of the shell at the maximum load in the axial load-end shortening curve or the bifurcation to another stable equilibrium path. An explanation of the former can be seen in Figure 4, in which the path OAC corresponds to the axisymmetric deformation and point A is the buckling limit load. If a bifurcation happens to be after the limit load (point B), it has no practical importance for engineers and the design will be based on the limit load. Figure 4 is valid for cases of thick shells and such axisymmetric buckling form of deformation is also referred to in the literature as “elephant foot buckling”. The most common case where the bifurcation occurs before the limit load as in Figure 5 where the bifurcation happens at point B and the path BD known as the postbuckling response. In this case, the buckling

behavior is described as non-axisymmetric buckling and the bifurcation load is the most critical parameter to be considered in the shell design. It can also be seen in Figure 5, that the imperfection highly reduces the load carrying capacity of the shell and the limit load is far less than that of the perfect shell depending on the severity of the imperfections.

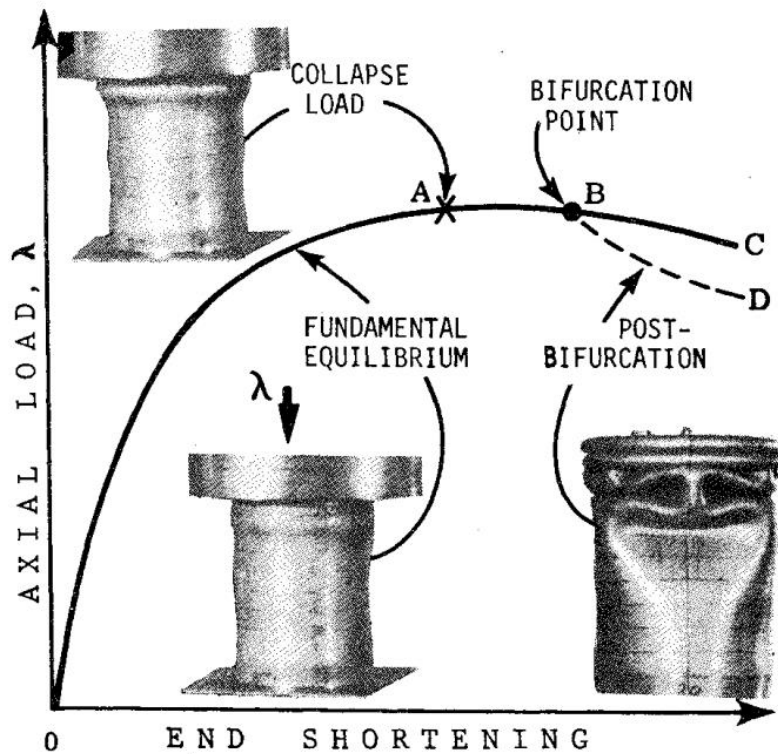


Figure 4 Load-deformation curve with bifurcation after limit load [5]

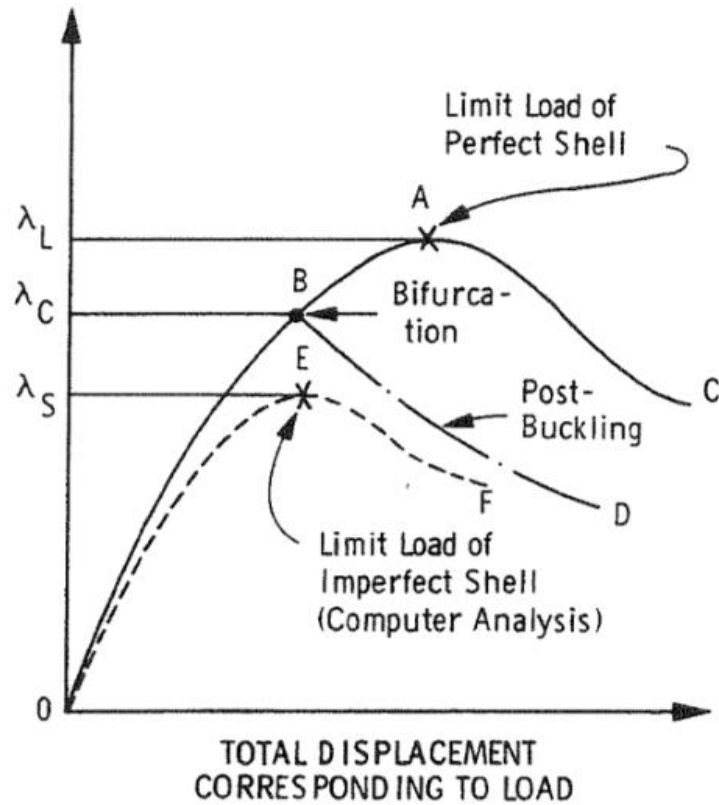


Figure 5 Load-deformation curve with bifurcation before limit load [5]

The axial load-end shortening curve is the most important curve as it is the key information that enables the design engineer to determine the load carrying capacity of the shell.

Determination of the load carrying capacity of the perfect and imperfect shells based on nonlinear buckling analysis is the main concern of this research. The nonlinearity here is related to both geometric and material. Different forms of imperfections in the shell geometry are tackled in this study such as irregularity in shape of the cylinder, weld depression, and localized damage in the form of a dent. The presence of localized dent gets more attention in this study and its most influential factors are studied. The interaction of different forms of imperfections is also studied.

1.2 Research Significance

In spite the development undergone for buckling of cylindrical shells, the available design guidelines are still complicated and difficult to be handled by designers. The most recent design code that adopts the state-of-the art in the topic is the European standard (EN 1993-1-6: 2007 E) [6], in which running a finite element (FEM) simulation for an imperfect shell considering the geometric and material nonlinearity is recommended to determine the load carrying capacity of the shell to be designed. While this recommendation is reasonable, it is quite a challenge for an average skilled designer to perform a nonlinear FE analysis, especially that the analysis involves different forms of imperfections (weld depression, irregularity in the shape of the cylinder, localized dents, etc.) and has to be performed for every shell to be designed. Instead, providing designers and practitioners in the field with simplified curves of the ultimate capacity based on a nonlinear buckling analysis of perfect shells will give them a base to verify their results. The capacity of the imperfect shell under design will, of course, be less than that of the proposed curves with a certain magnitude depending on the severity of the imperfection used in the design stage.

Studying the effect of different forms of imperfections have a profound impact on this research area. The presence of a localized dent, for example, will be of practical importance to those interested in rehabilitation of damaged shells. The available experimental and numerical studies are limited and most of them are based on small R/t ratios with a low strength carbon steel material, which could lead to a misleading conclusion about the effect of the dent due to the dominant plastic response of the shell. A typical study involving such erroneous approach has been critically investigated and corrected in this work. Suitable

geometric and material properties are then used to investigate the effect of the dent for cases of large R/t ratios (750 and 1250).

1.3 Research Objectives and Scope

The aim of this investigation is to study the nonlinear buckling of cylindrical shells more closely using the available symbolic and numerical tools. The results of this study will be utilized for providing the relevant industries with some design guidelines in terms of charts and simplified formulae. The specific objectives are:

- 1- Use of the available symbolic computational capabilities of Mathematica to obtain the non-axisymmetric buckling load of a perfect shell. This provides academic and industrial researchers, practitioners, and students with an efficient and accurate method that overcomes the mathematical difficulty encountered in computing the non-axisymmetric buckling stress of a circular cylindrical shell.
- 2- Derive and critically review the published analytical postbuckling solutions and clarify their limitations by utilizing the available symbolic computational capabilities of Mathematica.
- 3- Use of the available finite element software ABAQUS to simulate the buckling and postbuckling behavior of a perfect shell and running a suitable number of cases for parametric studies to determine the capacity of the perfect shell for different radius-to-thickness ratios and grades of carbon steel.
- 4- Utilize the obtained analytical and numerical results to generate capacity curves of the perfect shells in terms of formulae and charts that will be useful for designers.

- 5- Use of Mathematica and ABAQUS to generate the imperfect shell geometry and study the effects of different forms of imperfections on the buckling strength of the shell.
- 6- Critically examine the available studies on shells with small range of R/t ratios, highlight their weakness and provide possible improvements.
- 7- Study the effects of dent imperfections on shells with large R/t ratios.

Throughout this research, the following limitations apply:

- 1- No experimental work has been done. However, the experimental data available in the literature has been utilized.
- 2- The shell is subjected to a uniform axial compression load only.
- 3- The work is applicable to metallic circular cylindrical shells, in general, provided that the material is isotropic and homogeneous.

1.4 Research Methodology

To accomplish the above stated objectives, the following tasks were considered:

Task 1: literature review

A comprehensive literature was conducted about the buckling of cylindrical shells with more focus on the geometric and material nonlinear buckling analysis of perfect and imperfect shells.

Task 2: Selection of the suitable software packages

The basic software used in this work are Wolfram Mathematica [7] and ABAQUS. Mathematica was used for the symbolic computations, derivation of the analytical solutions

and fitting the data of the parametric study. Moreover, it was used for generating the FE nodes of the imperfect shell surface. ABAQUS was used to run the geometric and material nonlinear analysis of perfect and imperfect shells. Notepad software is used to facilitate importing the FE nodes of the imperfect shell generated in Mathematica to ABAQUS software.

Task 3: Analytical linear buckling analysis of perfect shells

The general non-axisymmetric linear buckling of cylindrical shells was first considered and Mathematica software was used to obtain an exact analytical formula for the minimum buckling stress and its corresponding mode for a given shell geometry. Then, a comprehensive parametric study was conducted leading to a regression-based simplified equation for predicting the non-axisymmetric buckling stress.

Task 4: Geometric nonlinear buckling analysis of perfect shells

Utilizing the capabilities of Mathematica for deriving one of the available postbuckling solutions and discussing its recent improvements. This is a base for critically examining the analytically obtained postbuckling solutions and their limitations.

Task 5: FE modeling of the geometric and material nonlinear analysis of perfect shells

Utilizing the available FE software ABAQUS for nonlinear buckling of perfect shells considering the geometric and material nonlinearity.

Task 6: Development of the capacity curves of the perfect shells

After providing general guidelines for perfect shell modeling considering geometric and material nonlinearity, this task was oriented to provide a reference capacity of the perfect shell to be used by designers and researchers for future verification. A large number of models for geometric and material nonlinear buckling analysis of perfect shells were analyzed considering different structural carbon steel grades and R/t ratios. The results of these were reported in what is referred to as “capacity curves of perfect shells” and simplified formulae for easier evaluation.

Task 7: Generation of the FE nodes coordinates of the imperfect shell

A Mathematica code was developed to generate the coordinates of the FE nodes representing the imperfect cylindrical surface. The code is capable of generating the nodes for two different types of imperfections: (a) irregularity in the shape of the cylinder (it is also referred to as “out-of-roundness”), and (b) local damage in the form of a dent.

Task 8: FE modeling of the geometric and material nonlinear analysis of imperfect shells

The developed Mathematica code for nodes generation was used to study the effect of different forms of imperfections on the buckling strength of circular cylindrical shell. In each case, a FE model for a perfect shell was first conducted to have the input file of the perfect shell. Then the FE nodes accommodating the required form of imperfections were

generated using the developed Mathematica code and the results were exported to be opened in a Notepad (txt file). These nodes coordinates were then copied and pasted in the input file of ABAQUS to replace the nodes of the perfect shell by the newly generated nodes of the imperfect shell. Based on this procedure, the behavior of imperfect cylindrical shells was studied and general guidelines for imperfect shell buckling analysis were provided.

Task 9: Utilization of the capacity curves for critical examination of an earlier work sample

Using the above discussed capacity curves, one of the available parametric study on the effect of a localized dent on shells with small R/t ratios is critically examined and corrected. A parametric study on the effect of dent considering large R/t ratios was then conducted.

1.5 Outline of the Dissertation

This dissertation is organized to have the introduction and general descriptions of the problem in Chapter 1. The literature review is covered in Chapter 2. The analytical solutions of the buckling and postbuckling of cylindrical shell are covered in Chapter 3 and 4, respectively. Chapter 5 covers the geometric and material nonlinear buckling analysis of perfect shells using FE modeling. Based on the perfect shell buckling analysis, capacity curves of perfect shells are obtained for a wide range of R/t ratios and different steel grades to be used as a reference for shell design. Modelling of shells with out-of-roundness imperfections and localized dent are given in Chapter 6. Parametric studies on the effect of

out-of-roundness imperfections and localized dent on the shell buckling strength is covered in Chapter 7. Finally, the conclusions and recommendations for future work are reported in Chapter 8.

CHAPTER 2

LITERATURE REVIEW

2.1 General

There is a huge number of reported research studies in the literature about buckling of cylindrical shells due to their applications in a wide range of industrial fields and disciplines. Different review papers are available providing directions to the newly introduced researchers to the field of shell buckling. This section is intended to provide more information about the available major resources in shell buckling and it will serve as a short review about the available resources in the literature.

Teng [8] presented a review concentrating on three major aspects of shell buckling: the imperfections, the buckling behavior under localized compressive forces, and the buckling analysis using computers. Research studies on nonlinear buckling analysis of shells were also reviewed. The nonlinearities considered are geometrical (large-deflection strains theory) and material. In a related work, Teng and Rotter [9] presented a book that reviews a wide range of studies in shell buckling including different forms of imperfections.

A general report reviewing the stability of steel shells was presented by Schmidt [10]. This review provided the state-of-the-art knowledge in the area of shell buckling. Descriptions of the types of shell analysis adopted by the European standards were also covered in this review as shown in Table 1.

Table 1 Types of shell analysis adopted by the European standards ENV 1993: 1-6 [10]

Abbreviations	Type of analysis	Shell theory	Material law	Shell geometry
÷	Membrane theory of shells	Membrane equilibrium	Not applicable	Perfect
LA	Linear elastic shell analysis	Linear bending and stretching	Linear	Perfect
GNA	Geometrically nonlinear elastic analysis	Nonlinear	Linear	Perfect
MNA	Materially nonlinear analysis	Linear	Nonlinear	Perfect
GMNA	Geometrically and materially nonlinear analysis	Nonlinear	Nonlinear	Perfect
GNIA	Geometrically nonlinear elastic analysis with imperfections	Nonlinear	Linear	Imperfect
GMNIA	Geometrically and materially nonlinear analysis with imperfections	Nonlinear	Nonlinear	Imperfect

A state-of-the-art and the associated challenges related to silos and tanks were proposed in 2009 by Rotter [11]. The paper reviewed the previous work related to tanks and silos and outlined the most influential factors related to the loading and the modes of failures of both tanks and silos as well as their design considerations. The work contained a useful material that could help the researcher to fully comprehend the structural behavior of these structures and serves as a good guideline for further research in the field. This work was proceeded by a study about buckling of silos under eccentric discharge by Sadowski and Rotter [12] and followed by investigating the buckling behavior of very slender silos under the same discharge conditions [13]. In the earlier work [12], the authors claimed that the behavior of shell structures under this eccentric loading conditions is not well covered in the literature.

In 2014, Elishakoff presented a review on the development in the area of shell buckling in the XX century [14]. He reported the works of different research groups in the field such as Harvard group, Group of Mosco Power Engineering, and other research groups of

international universities and research institutes worldwide. A brief historical background is reported in this text which can be a helpful material to introduce the readers about the topic of shell buckling.

A comprehensive literature review about the recent advances in the shell buckling was presented by Thompson in 2014 [15]. Presentation of the previous work from the early 1930s to his present days was provided. Nay more the earlier work of the same author (Thompson) with Heijden [16] about the role of static- dynamic analogy in comprehending the localized post-buckling behavior of shell-like structures was extended in this review.

A recent review paper on the recent developments in strength, stability, and dynamics of shells of revolution was done by Zingoni and published in 2015 [17]. The paper concentrated only on shells designed for storage of liquids. A complete section was devoted to review the buckling of vertical cylindrical tanks under different kinds of loads such as: compressive forces, external pressure, torsion and combined loading. This work was not limited to vertical tanks, but horizontal ones were also reviewed.

In 2015, Qian and Wang presented a review paper [18] on buckling of cylindrical shells under axial compressive forces and provided recommendations for future researchers in the field. The paper was concerned mainly about the effect of the geometrical imperfections, residual welding stresses and the internal pressure on the stability of an axially compressed cylinder.

A review paper on buckling of vertical cylindrical tanks [1] was presented by Godoy in the early 2016. Although this paper was oriented to study the buckling of oil storage tanks made of steel, it covered a wide range of the published work in the literature about buckling

of cylindrical shells in general. Buckling of these tanks under different types of loads was considered in this review. The loads considered were uniform pressure, wind loading, foundation settlement and temperature load due to fire.

The postbuckling behavior of compressed shells also get attention of the researchers in the field such as the review published in 1970 by Hutchinson and Koiter [19]. The review discussed the work of von Karman and Tsien and their followers in the field as well as Koiter's work. It also gave explanation for the loss of stability of structure followed by bifurcation into another state of equilibrium. According to this review, the bifurcation load was widely referred to as the classical buckling load.

Furthermore, Bushnell have a continuous work [20] aimed to list the up-to-date published works in the field of buckling, vibrations, and other structural behaviors of plates, shells, tubes, membranes and other thin-walled structural members. The work has been updated in November 2017 and has been made available online [21]. The work has paid more attention to shell buckling research due to large number of authors in this field. The work has been organized in different sections. Lists of books, and review papers in the area were also provided.

Moreover, an astonishingly rich website prepared by Bushnell and Bushnell about shell buckling is available [21]. The website provides lists of the famous people who have concretely contributed in the area since it was first initiated to the present day as well as their contributions. Images, papers, presentations, and computer programs are also available. This work could serve as a starting point to the newly introduced researchers in the field.

Generally, the considerable variations between the experimental results and the theoretically (analytically or numerically) obtained ones motivates the researchers in the field of mechanics to study this topic more fully. This is quite in accordance with a quote by J. Thompson [14]: *“To a mature, well-educated scientist a discrepancy means an opportunity. It shows there is something to be discovered”*.

2.2 Linear Buckling of Circular Cylindrical Shells

Analytical solution for buckling of a uniform compressed cylinder has been discussed in different books in the area of shells [3, 22-25]. In the axisymmetric buckling behavior, the shell is assumed to be deformed laterally and this radial displacement as a function of the longitudinal distance along the length of the shell. Although it is theoretically easy and simple, but the behavior found does not predict the real behavior and the tests show that the actual buckling load is far less than the theoretical value [25]. Another more general buckling mode is the non-axisymmetric buckling, in this case three displacements are assumed, and the buckling load is found. It is noteworthy to mention that these analytical solutions are applied to a perfect cylindrical shell.

In 2002, Kim and Kim proposed a simple formula to predict the linear buckling stress based on a parametric study of axially loaded cylindrical shells [26]. The parameters of the study were diameter to thickness ratio and the height to diameter ratio. The analysis was done using FE software ABAQUS. A simple design formula was provided in this study. The critical buckling load was also found to be highly reduced with the increase of the diameter to thickness ratio, but slightly reduced by the increase of the height to diameter ratio. Furthermore, the study tackled the nonlinear buckling of imperfect shell and concluded

that the increase in the amplitude of the imperfections severely reduced the critical buckling load of the shell.

An analytical and numerical study on the behavior of elliptical hollow sections was conducted in 2008 by Ruiz-Teran and Gardner [27]. The buckling load of an elliptical section is bounded between that of the circular hollow section and that of the plate buckling taking into account the aspect ratio of the ellipse. The buckling load was derived based on the axisymmetric buckling mode considering the shell as a system of longitudinal strips and other are transverse strips in the circumferential direction. The stiffness of these strips is used in the derivation of the buckling load. A formula was proposed for determination of the buckling load and it was verified with the available test results.

Two analytical investigations for scaling the critical buckling stress of axially compressed cylinder were presented between 2014 and 2015 by Grabovsky and Harutyunyan [28, 29]. The former was represented the scaling of the Korn constant while the later was due to load imperfections. The critical stress is defined by the well-known expression derived on the bases of axisymmetric buckling mode, i.e. $\sigma_{cr} = Eh/\sqrt{3(1-\nu^2)}$ where h is the ratio between the thickness of the shell and its outer radius. The results of these studies manifest that the shape imperfection requires the replacement of the h with $h^{\frac{3}{2}}$ while the load imperfections requires the use of $h^{5/4}$ as a replacement of h in the critical stress expression.

A statistical study [30] between 2000-2001 on the reliability analysis and reliability-based design optimization of a circular composite cylinder under uniform axial compression was conducted using average diameter of 15 inches. Monte Carlo simulation was used in this study with the First Order Reliability Method. The results of this study showed that the

reliability index is very sensitive to the applied axial compression, the Young's moduli of the material in the fiber direction, and the diameter of the cylinder.

2.3 Experimental Studies on Buckling of Cylindrical Shells

Since the early 1930s, it was proved experimentally [31, 32] that the buckling stress is far less than the classical buckling stress. The difference is getting more as the radius to thickness ratio increased. A collection of data from about two hundred experimental tests done by different researchers has been normalized with respect to the classical buckling stress as shown in Figure 6. The same data has been plotted in a double-logarithmic scale and normalized with the elastic modulus of the material and found to be fitted proportional to $(t/R)^{1.5}$, which is similar to the finding of Grabovsky and Harutyunyan [28, 29].

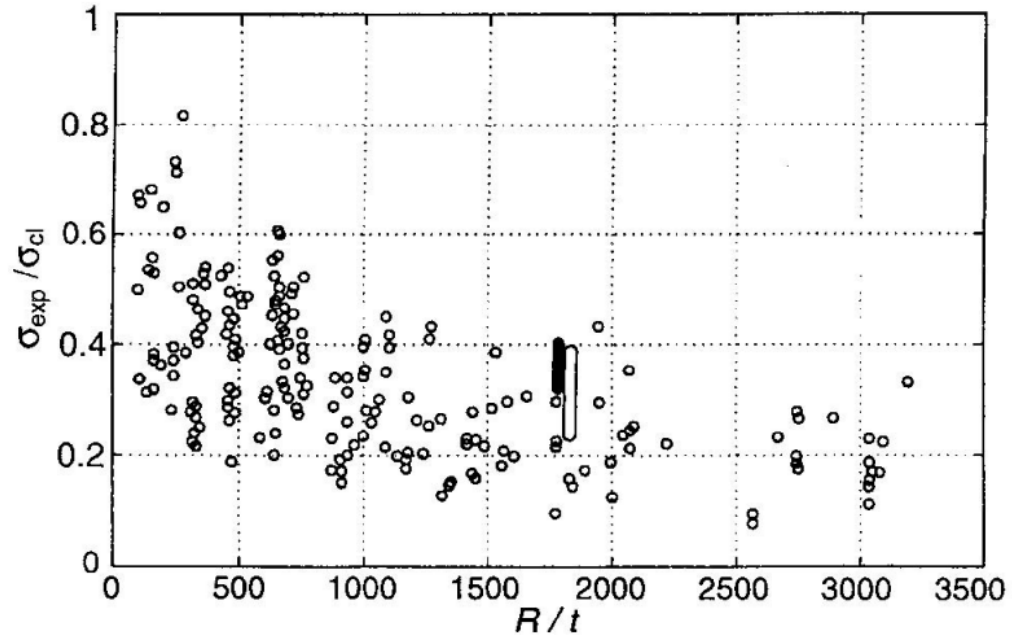
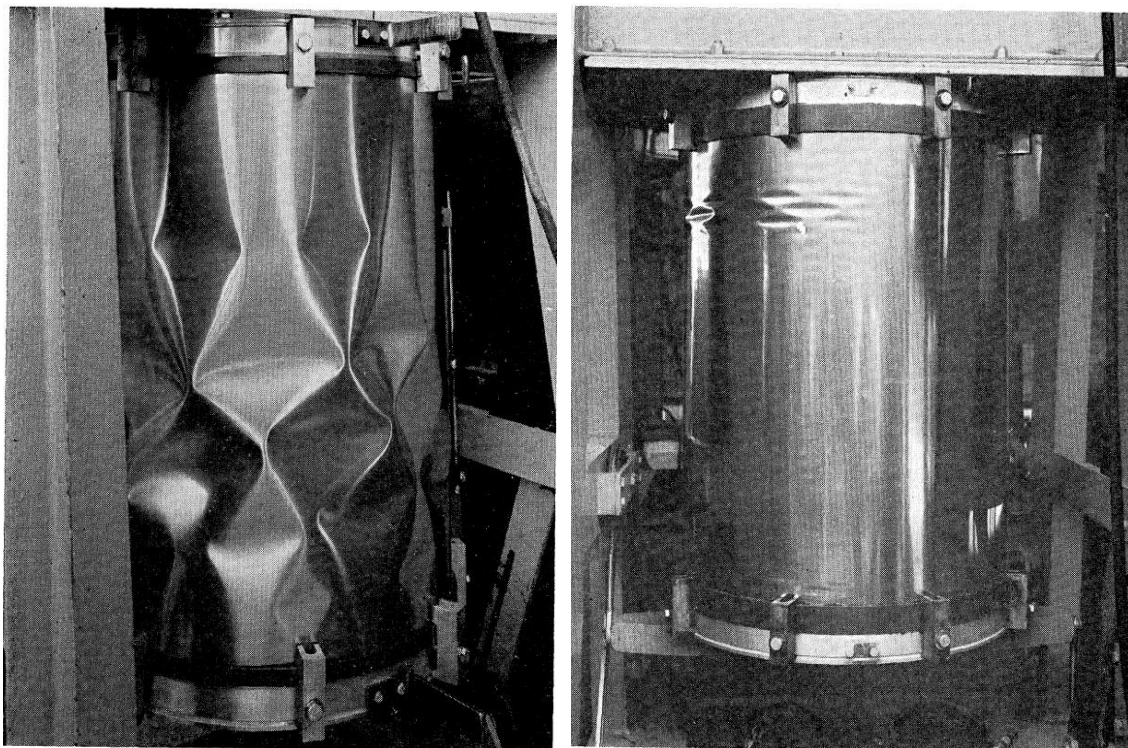


Figure 6 The ratio of the experimentally found critical stress to the theoretical obtained versus R/t [33]

Wilson and Newmark carried out a comprehensive experimental program [31] in 1933 to investigate the buckling of cylindrical shells and its influencing factors. The parameters of

this study were the material properties and the geometrical properties of the cylinder such as the radius to thickness ratio R/t , the shell length, and the end conditions. The radius to thickness ratio was varied from 28 to 991. As other previous investigations of that time, the experimentally obtained buckling stress was found to be far below the theoretical value. The most carefully prepared specimen was failed at stress equal to 60 percent of the theoretical value. Similar to the theoretical formula, the wrinkling stress was found to be varied with t/R up to a certain value and then remains constant. The value is called the critical value and was found to be 0.003 for thin specimens.

Suer *et al.* in 1958 conducted a series of experimental tests to investigate the effect of the internal pressure on the bending buckling behavior of cylindrical shells [34], which could be applicable for cases of compression due to lateral loading where there is a bending. The results of the study showed that the buckling mode changed considerably with application of the internal pressure and the shell become more stable under the same compression if compared with an unpressurized cylinder. Qualitatively, Figure 7 indicates the improvement of the shell strength with application of the internal pressure. The main outcome of this work was proposing design curves for pressurized and unpressurized cylindrical shells under bending based on statistical analysis of the obtained test data as well as the available in the literature of their time. A design curve for an unpressurized cylinder is shown in Figure 8, while for a pressurized cylinder is in Figure 9. Such design curves are referred to, here and in the literature, as semi-empirical design curves.



a) Buckling shape of unpressurized cylinder

Buckling shape of internally pressurized cylinder

Figure 7 Buckling shapes of axially compressed cylinder with and without internal pressure [34]

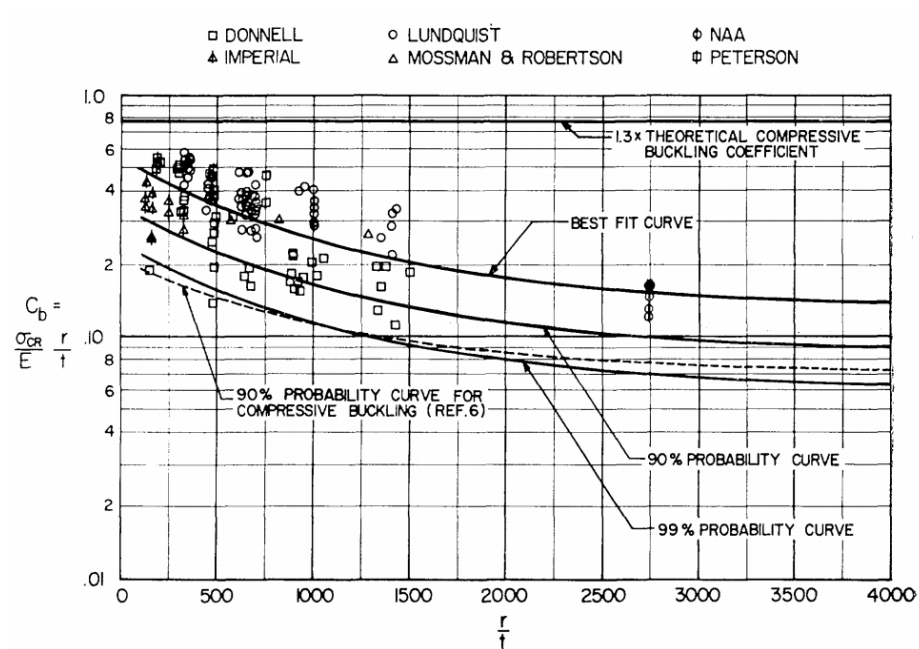


Figure 8 Semi-empirical design charts for axially compressed cylinder [34]

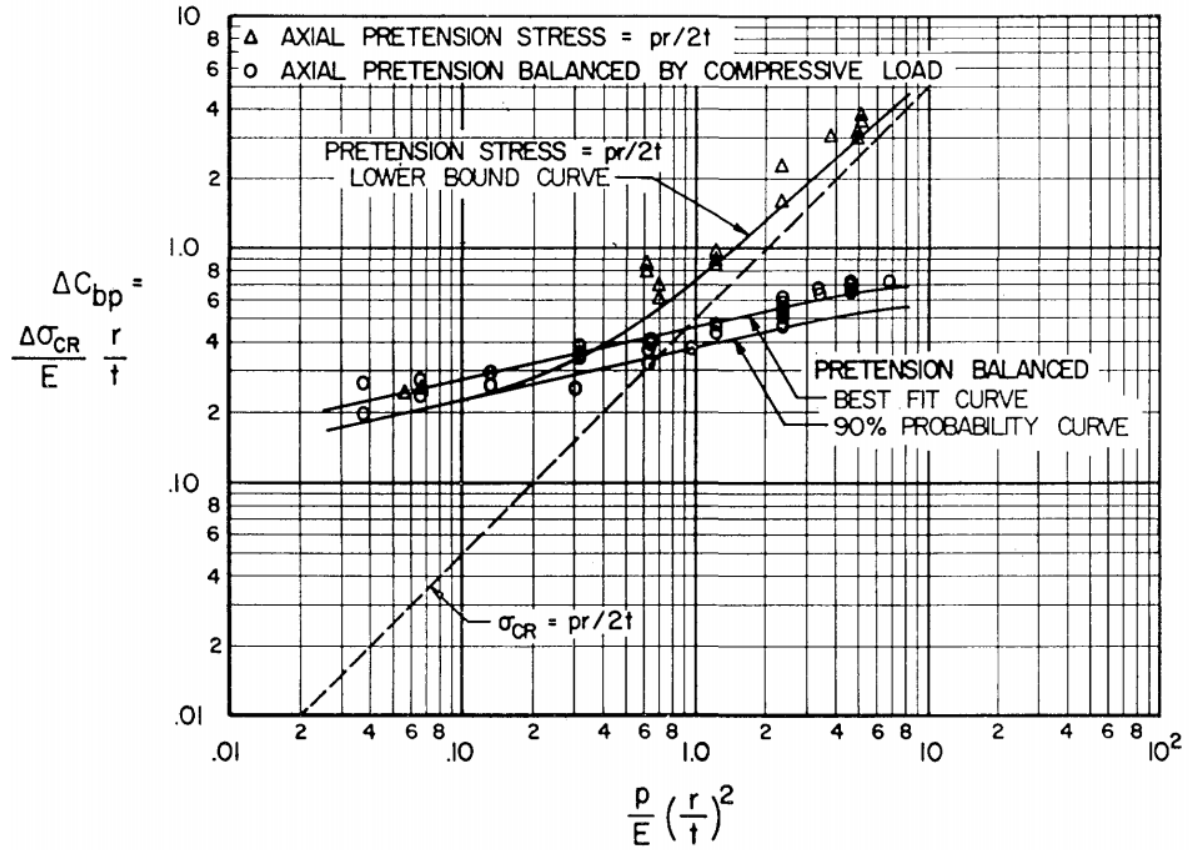


Figure 9 Semi-empirical design charts for internally pressurized axially compressed cylinders [34]

Available experimental data were also used to develop semi-empirical design chart (Figure 10) as reported and used in a book for design of flight vehicle structures [35]. This graph was originally generated by Harris *et al.* after conducting a series of experimental tests for cylindrical shells varied from short to long cylinders. A detailed explanation about the use of this design chart is also outlined by Himayat Ullah [36].

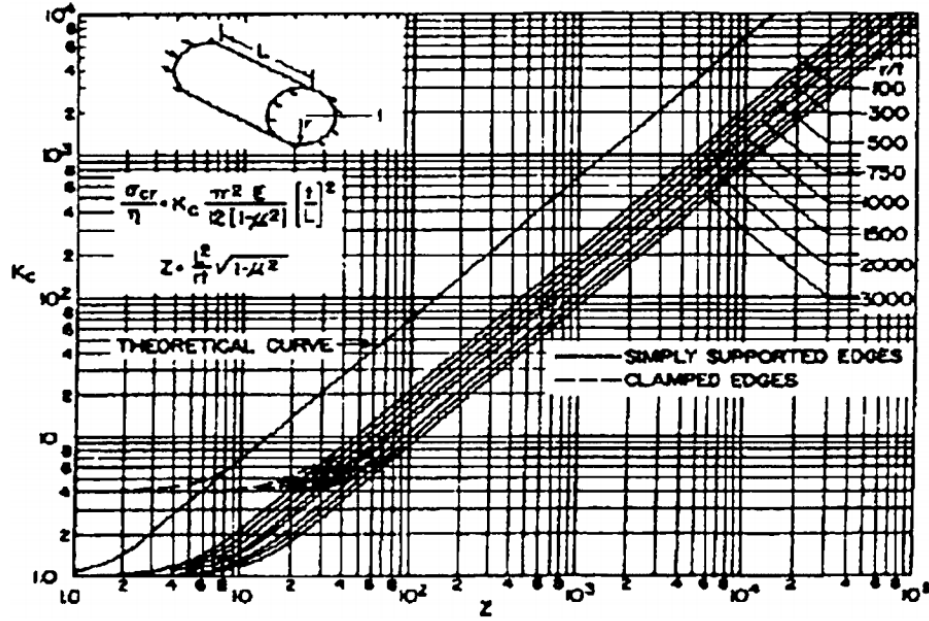


Figure 10 Semi-empirical design charts for axially compressed cylindrical shell [35]

A comprehensive experimental study on the postbuckling of circular cylindrical shell under uniform axial compression was presented by Yamaki *et al.* [37] since 1975. A geometrical parameter $Z = \sqrt{1 - \nu^2} L^2 / RL$ was used and taken to be varied from 20 to 1000. For short shells of Z value less than 200, the buckling behavior was found to be symmetrical about the middle of the cylinder, and the nonsymmetrical buckling behavior was not noticed. For larger values of Z , however, the behavior was found to be nonsymmetrical and the first buckling load was found to be at high number of half sign waves in the radial direction with sudden reduction in the load carrying capacity. After further increase of the shortening, successive buckling modes occur with reduction of the radial sign waves by one each time as shown in Figure 11.

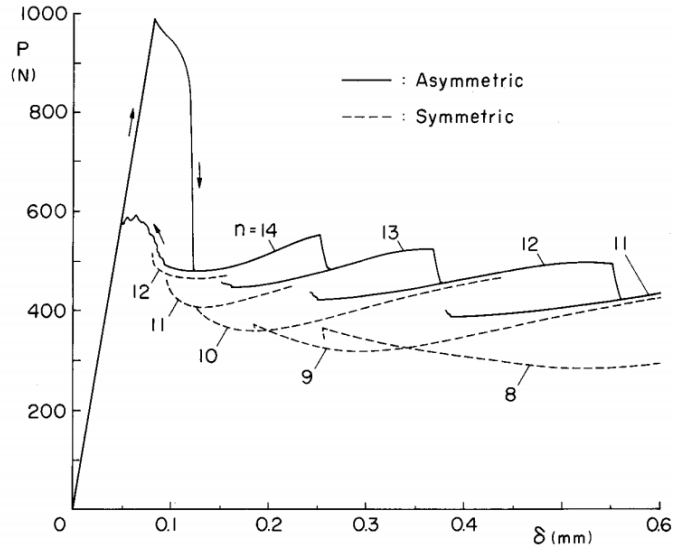


Figure 11 The axial load-end shortening curve for a tested specimen [37]

2.4 Postbuckling Buckling of Circular Cylindrical Shells

Large-deflection analysis were introduced first by von Karman [38] in 1910 for the bending of flat plates. This theory allows the interaction between the membrane and bending actions. It provides more accurate stress resultants when the deflection is considerable compare to the plate thickness.

Donnell [39] was the first to develop the large-deflection strains theory for buckling of thin cylindrical shells under axial compressive forces in 1934. The theory was developed to be used to justify the discrepancy between the experimentally obtained buckling load of cylindrical shells and the theoretically obtained one based on small-deflection theory. A differential equation relating Airy stress function (F) and the radial displacement (w) was developed in this study. Although there is analogy between the differential equation obtained in this work and the one obtained by von Karman for flat plate with one additional term, the author clearly claimed that he derived it independently before knowing von

Karman solution. Along with the obtained differential equation, an expression for the elastic strain energy is provided in this work to be used for buckling of cylindrical shells considering large-deflection strains theory. An approximate expression was assumed for the radial displacement with unknown parameters determined later from the principle of virtual work. It is noteworthy that the obtained theory in this work is limited to thin shells those can buckle under the elastic limit of the material and short enough to buckle globally as Euler column does.

von Karman and Tsien also published their work [40] in 1941 on buckling of cylindrical shells under axial compression considering the large-deflection strains theory. Certain additional terms involving the longitudinal and circumferential displacements in the curvature relations were neglected due to their negligible effects [41] and only the derivatives of the radial displacement were retained. Two differential equations relating the stress functions and the radial displacement were developed. The first one was exactly as the one developed by Donnell [39], while the second one was derived from the expression of the elastic strain energy. The exact solution, of course, can have been obtained by solving the two governing differential equations exactly under boundary conditions but that was found to be difficult at that time and they solved them approximately. An approximate radial displacement was assumed with unknown parameters. These parameters were then determined by minimizing the elastic strain energy. The axial stress-end shortening curve was generated (Figure 12) leading to the minimum postbuckling stress of $0.194 E t/R$. This work is well cited in the literature and became the base for postbuckling of cylindrical shell. It has also got different improvements over decades.

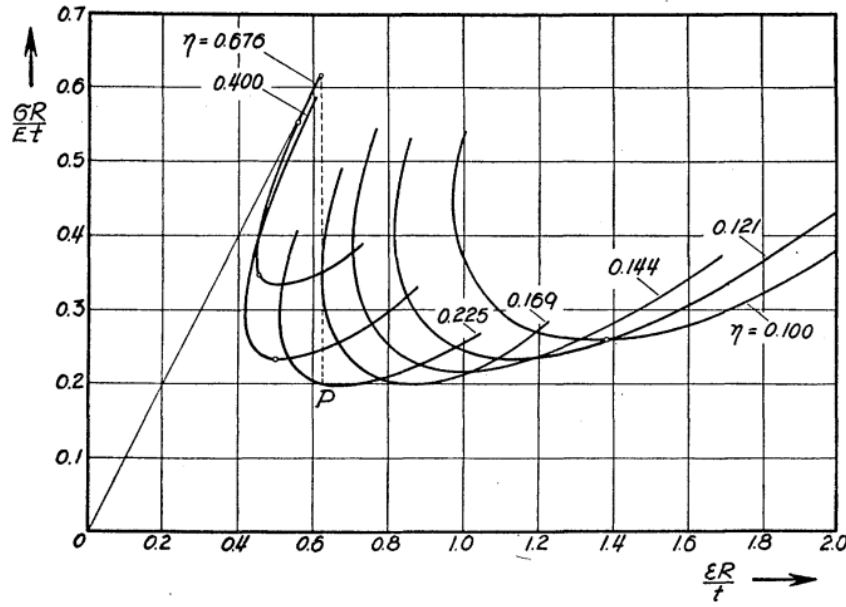


Figure 12 The axial stress-end shortening curve for different waves produced by von Karman and Tsien [40]

In 1948, the work of von Karman and Tsien in nonlinear buckling of cylindrical shells [40] was improved by Michielsen [42]. The improvement involved rearranging the equations in a simpler form by introducing new parameters, and minimizing the potential energy with respect two more parameters, namely, η , which is a function of the number of waves in the circumferential direction and the R/t ratio, and μ which is the ratio between the number of waves in the circumferential and longitudinal direction. More graphical representation was presented in this work and further illustration of the previous work was provided. The minimum postbuckling stress was found to be $0.195 Et/R$. Furthermore, the negative compressive stress obtained at small value of η by von Karman and Tsien was criticized.

In 1954, a further improvement for von Karman and Tsien's work was done by Kempner [43]. This work tackled cylindrical shells under axial compression of either applied as a dead load or through a rigid testing machine. In both cases, the minimum applied

compressive stress in the postbuckling state is found to be $0.182 Et/R$, which is slightly less than that obtained by von Karman and Tsien ($0.194 Et/R$). The compatibility equations used in this investigation contains additional equation which also relates the fourth order derivatives of the displacement w to Airy's stress function F . The displacement w of the classical linear theory was used as a first approximation to obtain the first approximation of the function F . Then, the obtained F was utilized to obtain the secondly approximated w leading to a secondly approximated F . The secondly approximated F and w were then used to evaluate the stress and the potential energy expression. The potential energy was then minimized with respect to the constants as the previous authors did.

In the early 1963, the postbuckling of cylindrical shells under axial compression was refined by Almroth [44]. Instead of dealing with a simple expression for the assumed displacement w , the author used Fourier's series to have algebraic sum of trigonometric functions. This led him, theoretically, to use as many terms as possible till convergence was obtained. The use of many terms led him to find that the minimum buckling stress under which the shell in equilibrium in the postbuckling state (Figure 13) is $0.0652 Et/R$, which is almost one third of the previous investigators [40, 42, 43]. The findings of this work were found to be in a good agreement with those obtained experimentally. Newton's Raphson method, as a powerful numerical tool, was utilized to solve the resulting nonlinear system of equations in this work. A further improvement to this work was provided by Jones [45, 46], the equilibrium postbuckling curves are shown in Figure 13 .

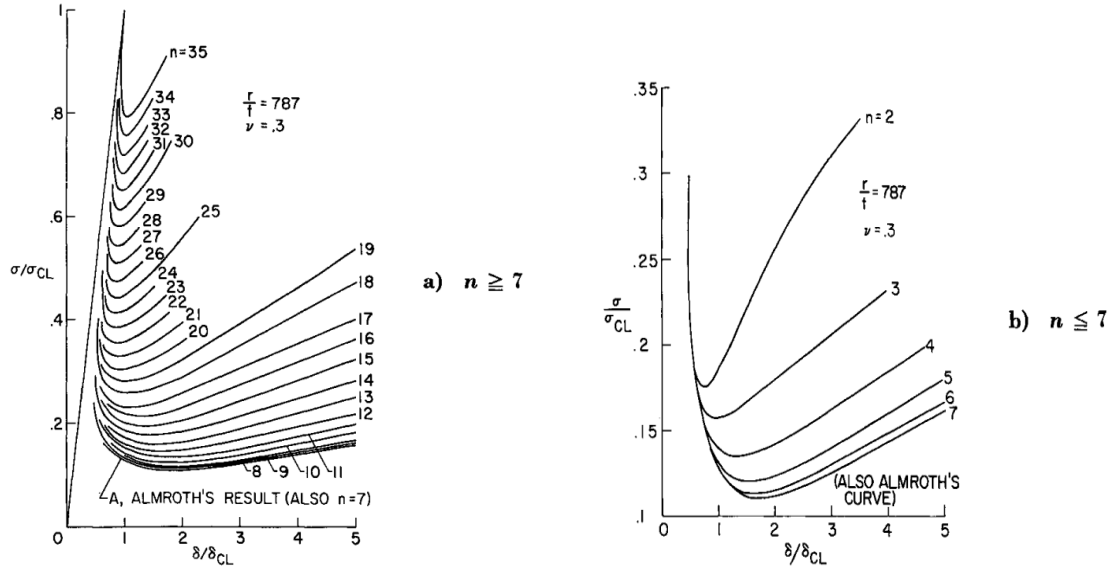


Figure 13 The normalized axial stress-end shortening curve for different waves produced by Jones [45]

Sobey [47] studied the buckling strength of an axially compressed cylindrical shells in 1964. This added a further improvement to the previous work of von Karman and Tsien. He first and foremost gave a brief discussion in selection of the approximation function of the radial displacement. An explanation of the variation of the potential energy during buckling was also provided. After selection of the assumed radial displacement, it was processed in the two well-known $w - F$ compatibility relations to come up with a refined displacement, which a practice followed by some of the previous authors. The outcomes of this work were found to be similar to the findings of Kempner [43].

A reasonably accurate theoretical investigation of the postbuckling behavior of circular cylindrical shell was proposed by Yamaki and Kodama [48] in 1976. Galerkin method was used for solution of the governing differential equations. Since this work was based on a perfect shell, comparison with a carefully prepared specimen tested by Yamaki *et al.* [37] were carried out as shown in Figure 14. The comparison showed that this solution is reliable

and can be used in future if generating the full postbuckling curve is required. Furthermore, the work of Yamaki and his research team is not limited to axially compressed cylindrical shells, he investigated the buckling problem under different loads such as torsion and internal pressure. Further details on Yamaki's work can be found on his book on elastic stability of cylindrical shells [49].

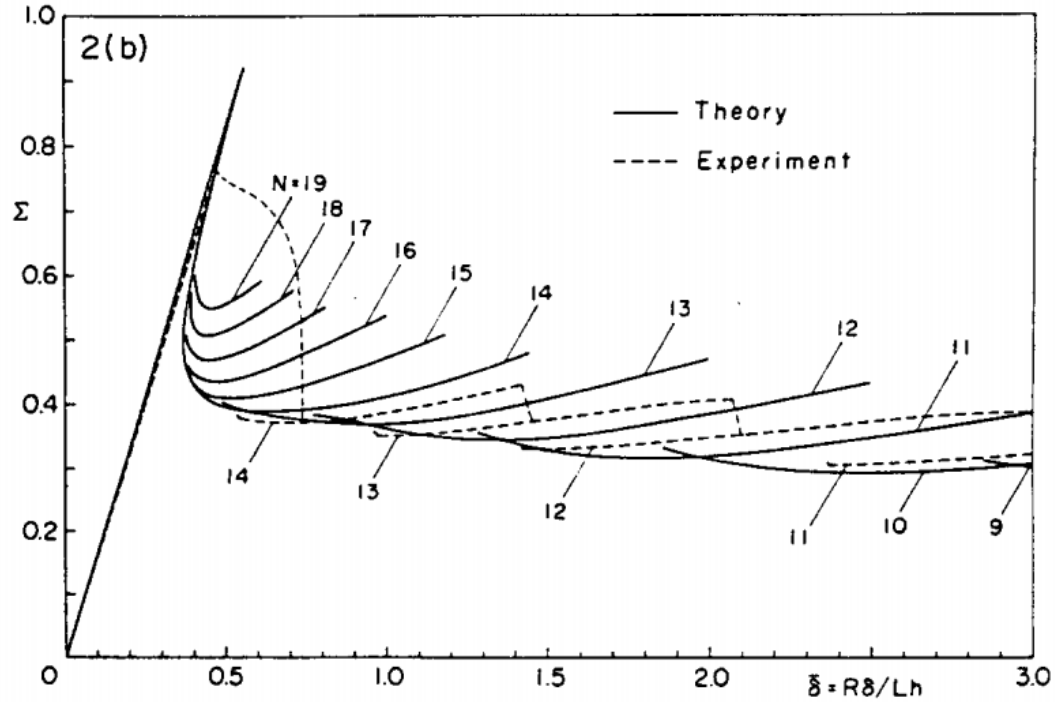


Figure 14 Comparison of experimental and theoretical postbuckling curves [48]

Hunt et al. [50] discussed the symmetry of the buckling modes using nonlinear buckling analysis in 1985. First, the Donnell strain equations were used in the analysis which are in terms of the radial displacement and Airy's stress function. The work was then extended to use the full large strain equations as a function of the three displacements (u , v , and w). Computer programs were recommended in this work for the manipulations of the equations in algebraic fashion. The effect of the imperfections was not included in this study.

Flugge's was perhaps the first to study the effect of internal pressure on the buckling load of shells in 1932 [51]. His investigation was prior to surfacing of the large-deflection strains theory and was based on small strains formulation. He concluded that the effect of the internal pressure is negligible on the critical buckling stress. A decade later, an experimental program [52] was carried out by the Langely Aeronautical Laboratory of the NACA to investigate the effect of the internal pressure on curved sheets. The experimental results revealed that the internal pressure increases the critical buckling load considerably. Although the experimental study was limited to curved shells and not specifically cylinders, it can be used to give an idea about the trend for cases of cylinders. The discrepancy between the experimental results and the Flugge's results motivated Lo *et al.* to investigate the effect of the internal pressure more closely in 1951 [53]. They used the large-deflection strains formulation to investigate the effect internal pressure on the critical buckling stress of a thin compressed cylinder [53]. An experimental program was also conducted to be used for comparison of the theoretically obtained results and fair comparison was obtained. Both the theoretical and the experimental results indicate the increase of the critical buckling stress with the increase of the internal pressure. This was contradicted with Flugge's findings and the reason, according to Lo *et al.* [53], was due to the use of small strains formulation in Flugge's work. The noted variation between the theoretical and the experimental results was claimed to be due to imperfections.

The experimental specimens on nearly perfect shells conducted by Yamaki *et al.* [37] became references for validation of the postbuckling solutions of the considered problem. In 2010, Kobayashi and Mihara [54] investigated the postbuckling analysis of a perfect shell using FE simulation in ABAQUS software. They used two different procedures for

producing the axial load-end shortening curve: explicit dynamic and artificial dynamic. In another similar study [55] with a third co-author, they discussed the use of Riks method. Although they were tackling perfect shells in these papers, a small amplitude of imperfections (1 % of the shell thickness) was used with a combination of 18 eigenmodes to guide the shell to buckle. These imperfections are very small and used as a tactic. The results so obtained were compared with Yamaki *et al.* [37] and the comparison showed that both explicit dynamic and artificial damping are the best two methods to generate the load end-shortening curve-i.e. to trace the postbuckling curve. However, in terms of the maximum axial load the shell can carry, all the three methods yield nearly the same results.

In 2009, Himayat Ullah [36] studied the buckling of cylindrical shells under axial compression using the classical analytical approach, the available semi-empirical models, and numerical simulation in ANSYS. Comparison of the obtained results led to conclude that the classical small-deflection theory provides almost similar results to the numerical simulation in ANSYS if linear analysis is used, but both results are far higher than the semi-empirically obtained ones. Nonlinear analysis involving large-deflection strains was then conducted using ANSYS and good improvements of the results were reported towards reducing the mismatching between the theoretically and the semi-empirically obtained buckling load. Introduction of the imperfections with different amplitudes (0.5, 5 and 50%) along with the geometrical nonlinearity improved the results further leading to a good agreement with the experimental results especially for case of high imperfection amplitude.

Finally, many of the available research works have been incorporated in the design standards. The European standards for design of steel structure, for example, have specific part about strength and stability of shell structures (EN 1993-1-6: 2007 E) [6]. More details

and information about design of shells structures are given in this part of the standards. Different analysis approaches for estimation of the buckling load are discussed with their limitations. These are:

- 1- Linear elastic analysis (LA) for estimation of the buckling load at which failure take place, specific interpretation is discussed for estimation of the buckling load using this procedure.
- 2- Linear elastic bifurcation analysis (LBA) to estimate the critical buckling stress.
- 3- Materially nonlinear analysis (MNA) for cases of nonlinear materials.
- 4- Geometrically nonlinear elastic analysis (GNA), which can be used for cases of perfect shells with elastic material and considering large-deflection strains.
- 5- Geometrically and materially nonlinear analysis (GMNA) to consider both nonlinearities related to geometry and material.
- 6- Geometrically and materially nonlinear with imperfections analysis (GMNIA) to consider both nonlinearities related to geometry and material in the presence of imperfections.

It is noteworthy to tell that the factor of safety is considered for all of the above-mentioned methods. Furthermore, the cyclic loading and fatigue of such steel shells are also discussed in this standard.

2.5 Buckling of Circular Cylindrical Shells with Imperfections

2.5.1 General

The shell is found to be very sensitive structure to imperfections and a large variation in the buckling load can be found upon the severity of the imperfections. This can be shown

in Figure 15, where the buckling load for an imperfect cylinder is far more less than that of the perfect cylinder. The imperfections are also one of the main factors responsible for the discrepancy between the experimental and theoretical buckling load. This motivated the researchers to pay attention and study the effects of the imperfection more closely.

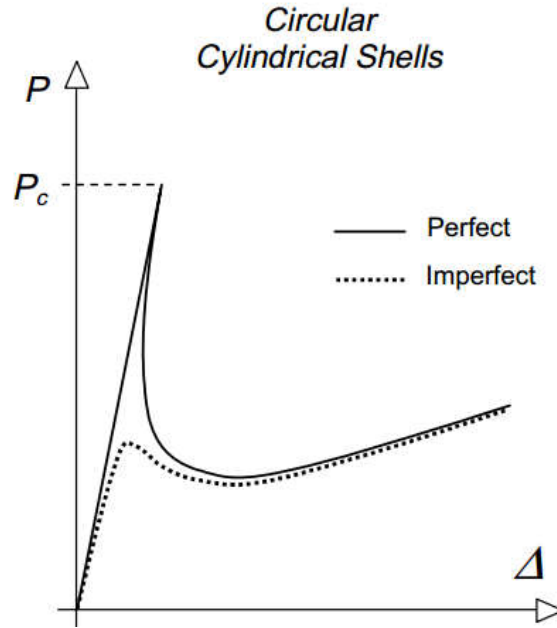


Figure 15 A comparison of a perfect and imperfect shell buckling behavior [56]

Generally, investigation of the stability of imperfect cylindrical shells get more attention among the research community after the pioneering work of Koiter in 1945 [57] (see also the translated English version [58]), who was the first to obtain an accurate prediction of a geometrically imperfect cylindrical shell. Different forms of geometric imperfections were then studied in the literature such as axisymmetric imperfections [59-62], weld depression [63-68], distributed and localized imperfections [69], the presence of nozzle [70], and more complicated shapes of imperfections such as a localized asymmetric damage, which has been discussed in Section 2.5.3.

There are so many reported studies in the literature discussed the effect of these forms of imperfections on the buckling strength of shells. More information about the works of other researchers in buckling of imperfect shells are available in the collection of papers provided by Bushnell [71]. This work was updated in 2011 and serves as an informative text highlighting the recent updates in buckling of imperfect shells. The work was divided to four main sections: shells with high imperfection sensitivity, shells with moderately imperfection sensitivity, shells with no imperfection sensitivity, and, finally, the unavoidable imperfections.

2.5.2 Factors Defining the Geometric Imperfections

The geometric imperfections are generally defined by two-fold factors, the *shape* and the *amplitude*. Knowing these two-fold factors will results in an accurate strength prediction of the shell to be designed. However, knowing them is not an easy task and it is still under development.

It is a customary practice nowadays to use the eigenmodes produced from the linear buckling analysis as modes describing the expected forms of imperfections. The use of the eigenmode imperfection scaled by a certain factor has widely been adopted in the literature, e.g. [54, 55, 72-74] and it also recognized by the European Standards [6] in the absence of a justified unfavorable form of imperfections. In Ref. [54, 55, 72-74], the scaled eigenmode (s) of the perfect shell was (were) used in the nonlinear buckling analysis by importing the imperfection file from the linear buckling analysis and scaling it by the maximum amplitude required as described in details by Castro et al. [73].

2.5.3 Effect of Localized Imperfections

The effect of a localized dent has got attention in the last two decades. In 2002, Gavrilenko [75] proposed a nonlinear procedure for stability of cylindrical shells with local and longitudinal dents. In 2003, he also proposed stability analysis of cylindrical shells with a localized dent imperfection [76] using two different approaches: the mesh method and the reduced stiffness method. He, and Krasovskii, used the former approach to compare experimental results of cylindrical shells with a local dent in another study [77]. They both extended their work to study the presence of periodic dents in the shell surface [78]. In another study [79], Gavrilenko went further to study the effect of local dents on ribbed shell. Prabu with his co-workers conducted different FE studies on buckling of axially compressed cylindrical shells with localized dents [80-83]. In the forgoing studies, they studied the influence of the size and inclination of the dent on the stability of cylindrical shells made of stainless steel [80] and carbon steel [81], the effect of the diameter and depth of circular dent [82], and neighborhood of two closely spaced dents [83]. Ghazijahani et al. [84] conducted experimental tests of cylindrical shells with localized sharpened dent on carbon steel cylinders. They later conducted a similar experimental program to investigate the effect of the dented tubes with $R/t = 55$ [85]. Cerik [86] also conducted experimental and FE study on stability of stiffened cylindrical shells.

Furthermore, studies went beyond isotropic axially loaded cylinders. In 2002, Shen and Li [87] studied the effect of localized dent in a shear deformable laminated cylindrical shell under a combined action of axial and temperature loads. Wullschleger and Meyer-Piening [88] conducted a FE analysis of CFRP cylinders with localized dents. Guggenberger [89] conducted an experimental and numerical study of externally pressurized cylindrical shells

with longitudinal localized dent and evenly distributed imperfections. He found that the effect of unintended imperfections tends to vanish in the presence of a dent with large amplitude. Hambly and Calladine [90] conducted an experimental study on buckling of cylindrical drink cans with a localized damage under eccentric loading.

Although there is a wealth of literature on the buckling of locally damage cylindrical shells, it has been noticed that most of the available experimental and numerical studies are for small R/t ratios. Table 2 shows a summary of different research studies. This small R/t ratios along with low yield strength materials are not appropriate to expect the behavior for large R/t ratios where the buckling happens under the elastic range for most parts of the shell other than the dent effective region.

2.5.4 Effect of Some Other Forms of Imperfections

In 1983, Sheinman and Simitses [91] studied the effects of geometrical imperfection on the post-buckling behavior of axially compressed cylindrical shells. Numerical examples of imperfections amplitude of up to $1.1 t$ was presented. The effect of different geometrical parameters was also studied such as radius to thickness ratio and length to radius ratio. One year later, Toda [92] studied the effect of the of circular hole on the critical buckling stress of axially compressed cylindrical shells. The study was performed experimentally on polyester shells with radius to thickness ratio of 100 and 400. The study concluded that for cases of small holes, the effect on the buckling stress is negligible but for large holes there is a considerable reduction in the critical buckling stress. When doublers were placed around the holes, however, there was no effect on the buckling stress.

Table 2 A summary of studies on locally dented cylindrical shells

Ref.	R/t	Type of dent	Type of load
Guggenberger [89]	200	U-shape and V-shape longitudinal dents.	External pressure
Hambly and Calladine [90]	350	Two locally damage region with different spaces between them	Eccentric axial load
Gavrilenko [75]	400	Rectangular in plan	Axial load
Gavrilenko [76]	150, 260, 360	Rectangular in plan	Axial load
Gavrilenko and Krasovskii [77]	150, 260, 360	Rectangular in plan	Axial load
Gavrilenko and Krasovskii [78]	360	Rectangular in plan	Axial load
Gavrilenko [79]	150, 260, 360	Rectangular in plan	Axial load
Parbu <i>et al.</i> [80]	280	Rectangular in plan	Axial compression
Parbu <i>et al.</i> [81]	117, 175, and 280	Rectangular in plan	Axial compression
Parbu <i>et al.</i> [82]	175, 280, and 388.9	Circular in plan	Axial compression
Parbu <i>et al.</i> [83]	280	Two closely spaced dents (Rectangular plan)	Axial compression
Ghazijahani <i>et al.</i> [84]	169.76, 303.54	Knife edge dent	Axial compression
Ghazijahani <i>et al.</i> [85]	55	Knife edge dent	Axial compression
Cerik [86]	120, 306.21-467.37	Knife edge dent	Axial compression

Pircher and Bridge studied the effect of weld-induced imperfections on buckling strength of silos and tanks in 2001 [93]. A FE analysis was used in this investigation. Calibration of the geometry with measurements of such imperfections from existing tanks and silos was performed. The residual stresses due to weld was found to have a negligible effect on the strength of the shell. Postbuckling analysis was also performed, and attention was paid to explain the load-carrying response after the first bifurcation was taken place.

Khamlichi *et al.* studied the effect of the initial axisymmetric imperfections on the critical buckling stress of cylindrical shells subjected to axial compression in 2004 [94]. The investigation was carried out analytically and the results were compared with those obtained in the literature. Unlike other imperfections, the axisymmetric ones were found not in a proportional relation with the critical stress. For practical applications, hence, the authors recommended to have measurements on the existing imperfections and correlate them with the discussed models in this study. In the absence of actual measurements, however, they recommended to conduct parametric study to determine the amplitude and the width of strip that results in the minimum buckling stress.

In 2004, Song *et al.* [68] conducted a numerical study on the behavior of imperfect cylindrical shells under symmetrically arranged partial axial compression loads. The imperfections considered in this study were such as linear and nonlinear buckling modes and weld effects. The load was taken to be partially distributed over two segments in symmetrical layout over the top loaded edge. Attention was given to imperfections amplitude and the mesh convergence.

In 2015, Cao *et al.* [95] studied the buckling behavior of uniformly compressed cylindrical shells with varied thickness imperfections with arbitrary arrangement. The imperfections were in symmetrical and non-symmetrical arrangement. The work started first by derivation of the governing differential equations those take into account the effect of thickness imperfections. Then the equations were solved by combining the perturbation method with Fourier series expansion to obtain the buckling load.

In 2016, Wang *et al.* [96] conducted an experimental and numerical investigation for prediction of the knock down factor KDF of imperfect stiffened cylindrical shells. They used ABAQUS in the numerical part of this study. Different types of imperfections were used in this study: NASA SP-8007, single perturbation load approach (SPLA), and the worst multiple perturbation load approach (WMPLA). The numerical results of the WMPLA imperfections approach were found to be similar to those obtained experimentally, while other imperfections were found to be conservative. The authors recommended the use of WMPLA for future predictions of the KDFs in design of Launch vehicles.

In 2018, Cheng *et al.* [97] studied buckling of axially compressed cylindrical shells subjected to differential settlement of their foundations. The behavior of the shell subjected to differential settlement was first been investigated. The resulting deformed shape was later been introduced to an axially compressed cylinder as initial imperfection. Shells having higher length to radius ratios and lower radius to thickness ration were found to be more stable as compared to their counterparts of other ratios.

2.6 The Recent Design Standards

There are different international standards that provide shell buckling scenarios in design of cylindrical shell structures. The most recent design codes that adopt the recent development in shell buckling are the European Standards (EN 1993-1-6: 2007 E [6]), in which different methods of analysis are considered as discussed in Table 1. Instead of developing a generalized knock down factor (KDF), as many standards and research bodies [98-100] do, the EN adopt doing a FE analysis for every case of a shell to be designed. The standards also adopt three different levels of construction: A, B, and C, when defining the tolerable limits of imperfections, which is a motivation for good construction- i.e. excellent construction can be designed based on a lower imperfection sensitivity.

According to the EN 1993-1-6: 2007 E [6], and based on a geometric and material nonlinear analysis of imperfect shell (GMNIA), the shell buckling resistance is defined based on four criteria as discussed below and shown in Figure 16 :

C1: the limit load on the load-end shortening curve

C2: The bifurcation load.

C3: The load corresponds to the maximum allowable displacement

C4: The load corresponding to the first yield.

Further details and discussions about this design code can be found in the literature [9, 101, 102].

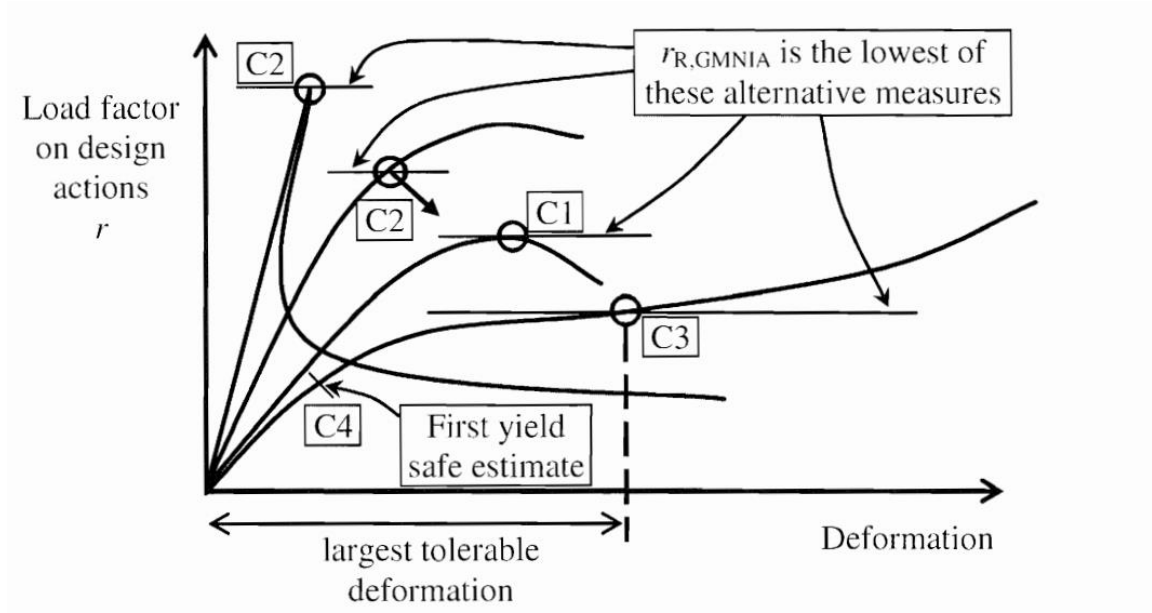


Figure 16 The shell buckling resistance according to EN 1993:1-6 [6]

2.7 Deficiency in the Literature

In spite of the wealth of literature on the stability of cylindrical shells, more research is needed regarding the following issues:

- 1- The non-axisymmetric buckling formula is too lengthy and not practical for design purpose. So, more simplified formula is needed.
- 2- Modeling of the imperfect shell is still under development, so providing general guidelines and computer codes for generation of the imperfect shell geometries is of practical importance. Furthermore, providing capacity curves with simple parameters will make it easy for shell designers and researchers to verify their work.
- 3- Most of the available studies on dented cylindrical shell have been done for small R/t ratios (< 500) and most of them are for low strength materials. It is universally accepted that the sensitivity of the shell to imperfections increases with the increase

of R/t ratio. So, it is necessary to study the effect of the dent on the strength of shells with large R/t ratios.

CHAPTER 3

LINEAR BUCKLING OF PERFECT CYLINDRICAL SHELLS

3.1 Axisymmetric Buckling

The first attempt in obtaining the buckling strength of circular cylindrical shells is dated to more than a century ago [24]. The obtained formula is based on axisymmetric buckling mode (Figure 17) and it is commonly now known as the classical buckling stress.

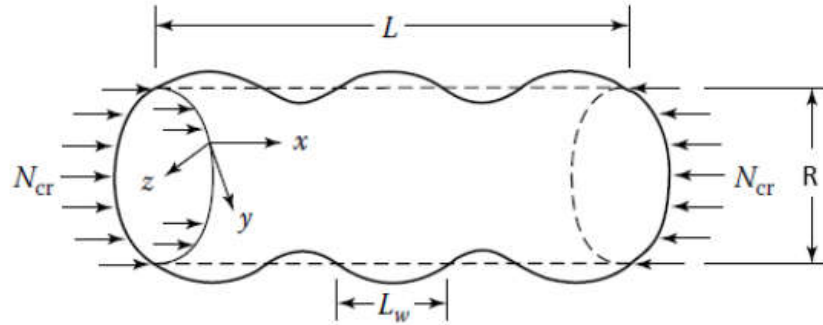


Figure 17 The axisymmetric buckling mode [3]

To derive the classical buckling formula, let us consider a buckled shell as shown in Figure

17. The governing differential equation of axisymmetric buckling is given by [103]:

$$\frac{E t^3}{12 (1 - \nu^2)} \frac{d^4 w}{dx^4} + N \frac{d^4 w}{dx^4} + \frac{E t}{R^2} w = 0 \quad (3.1)$$

Where: R , t , E , ν and w are the radius of the shell, shell thickness, elastic modulus of the material, Poisson's ratio, and the radial displacement, respectively. N is the axially applied edge load with the units of force per unit circumferential distance.

For the case of a simply supported shell, the following displacement expression satisfies the essential boundary conditions:

$$w = W_m \sin\left(\frac{m\pi x}{L}\right) \quad (3.2)$$

Where m is the number of half sign waves in the axial direction and W_m is the amplitude of the wave.

Substitution of Equation 3.2 into 3.1 and simplifying leads to determination of the axial load to be

$$N = \frac{E t (m^4 \pi^4 R^2 t^2 + 12 L^4 (1 - \nu^2))}{12 L^2 m^2 \pi^2 R^2 (1 - \nu^2)} \quad (3.3)$$

All the terms appearing in the above formula are known for a given shell except m which describe the buckling mode. Therefore, buckling load N for a given shell depends solely on the number of half sign waves m and the critical buckling load can then be obtained by finding m that makes N minimum, i.e.:

$$\frac{dN}{dm} = 0 \quad (3.4)$$

Solving the above equation for m , yields

$$m = \frac{\sqrt{2}L(3 - 3\nu^2)^{1/4}}{\pi\sqrt{R}t} \quad (3.5)$$

The number of half sign waves, m , given by Equation 3.5, represents the buckling mode that corresponds to the minimum load, i.e. the buckling load.

Substitution of Equation 3.5 into 3.3 leads to expressing the critical buckling edge load by

$$N_{cr} = \frac{E t^2}{R\sqrt{3(1-\nu^2)}}, \quad (3.6)$$

which depends only on the radius and thickness of the shell; and the elastic material properties. This is also referred to in the literature by the classical buckling load. The classical buckling stress can simply be obtained by dividing the critical load by the thickness of the shell to be

$$\sigma_{cl} = \frac{E}{\sqrt{3(1-\nu^2)}} \frac{t}{R}, \quad (3.7)$$

which was derived a long time ago and it is available in many standards textbooks in shell buckling [3, 22-25].

For Poisson's ratio $\nu = 0.3$, the classical buckling stress simplifies to

$$\sigma_{cl} \approx 0.605 E \frac{t}{R} \quad (3.8)$$

3.2 The Non-axisymmetric Buckling

3.2.1 General

In the early 1930s, it was proved experimentally [31, 32] that the classical buckling formula highly overestimates the buckling strength. The large discrepancy between the classical and experimental buckling strengths could be attributed to many factors such as: imperfections in load, geometry and boundary conditions; material and geometrical nonlinearities; and the presence of a non-axisymmetric buckling mode (Figure 18), which yields a critical stress less than that predicted by the axisymmetric-based classical formula.

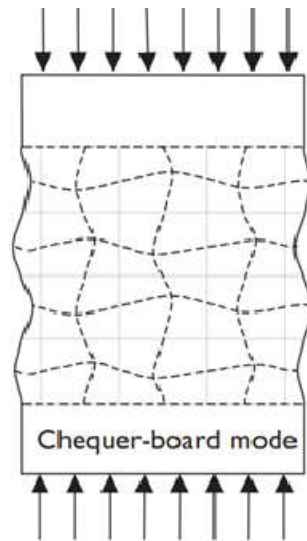


Figure 18 The non-axisymmetric buckling modes of a perfect elastic cylindrical shell [9]

The derivation of the non-axisymmetric buckling stress leads to a very complicated and length formula and involves the solution of a non-linear constrained optimization problem which is difficult to be used by practical engineers and average-skilled researchers. Thus,

this section aims to utilize the computational capabilities of the Wolfram Mathematica software [7] in providing a complete and efficient solution to the problem.

The shell equilibrium equations are used to form the buckling eigenvalue problem which leads to the derivation of the objective function, which is a highly nonlinear function of the shell geometric parameters and buckling mode numbers. The minimization of the derived objective function is then performed for different shell geometric parameters. The minimization process yields the buckling stress and modes. The obtained results are then utilized to study the effect of shell geometric parameters on the buckling stress and buckling modes of the shell.

The main findings of this section are overcoming the burdens of symbolic computations associated with the minimum buckling stress and the corresponding buckling mode. The obtained results were found to be in excellent match with literature and FEM. In addition, it aims to provide academic and industrial researchers, practitioners, and students with an efficient and accurate method that overcomes the difficulty encountered in computing the non-axisymmetric buckling stress of a circular cylindrical shell. Furthermore, the section is concluded with a parametric study on the effects of the shell geometrical parameters on the buckling stress/modes. The parametric study is used to generate a refined simplified formula for design purposes.

3.2.2 Derivation of the Non-axisymmetric Buckling Stress

For non-axisymmetric buckling of circular cylindrical shells, the governing differential equations are given by [3, 23,24]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2R} \frac{\partial^2 v}{\partial x \partial \theta} - \frac{\nu}{R} \frac{\partial w}{\partial x} + \frac{1-\nu}{2R^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (3.9)$$

$$\begin{aligned} \frac{1+\nu}{2R} \frac{\partial^2 u}{\partial x \partial \theta} + \frac{1-\nu}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{R^2} \frac{\partial^2 v}{\partial \theta^2} - \frac{1}{R^2} \frac{\partial w}{\partial \theta} - \frac{\sigma_{cr}(1-\nu^2)}{E} \frac{\partial^2 v}{\partial x^2} \\ + \frac{t^2}{12R^2} \left[\frac{1}{R^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{R^2} \frac{\partial^3 w}{\partial \theta^3} + \frac{\partial^3 w}{\partial x^2 \partial \theta} + (1-\nu) \frac{\partial^2 v}{\partial x^2} \right] = 0 \end{aligned} \quad (3.10)$$

$$\begin{aligned} \nu \frac{\partial u}{\partial x} + \frac{1}{R} \frac{\partial v}{\partial \theta} - \frac{w}{R} \\ - \frac{t^2}{12R^2} \left[\frac{1}{R} \frac{\partial^3 v}{\partial \theta^3} + (2-\nu)R \frac{\partial^3 v}{\partial x^2 \partial \theta} + R^3 \frac{\partial^4 w}{\partial x^4} + \frac{1}{R} \frac{\partial^4 w}{\partial \theta^4} \right. \\ \left. + 2R \frac{\partial^4 w}{\partial x^2 \partial \theta^2} \right] - \frac{\sigma_{cr}R(1-\nu^2)}{E} \frac{\partial^2 w}{\partial x^2} = 0 \end{aligned} \quad (3.11)$$

Where: R , t , E , and ν are the radius of the shell, shell thickness, elastic modulus of the material, and Poisson's ratio, respectively. u is the displacement along the length of the shell, v is the displacement along the circumferential direction, and w is the radial displacement.

For the origin at the edge of the cylinder, the three displacements u , v , and w represent the general solution of the above governing equations can be selected to appears as [24]

$$u = \frac{C_1}{\nu R} x + C_2 + \sum_m \sum_n A_{mn} \sin n\theta \cos \frac{m\pi x}{L} \quad (3.12)$$

$$v = \sum_m \sum_n B_{mn} \cos n\theta \sin \frac{m\pi x}{L} \quad (3.13)$$

$$w = C_1 + \sum_m^{\infty} \sum_n^{\infty} C_{mn} \sin n\theta \sin \frac{m\pi x}{L} \quad (3.14)$$

Where: L is the length of the cylinder; m is the number of the halve-sign waves in the axial direction; n is the number of halve-sign waves in the circumferential direction; and C_1 and C_2 are constants.

Substitution of the above displacements into the Equations (3.9 to 3.11), yields the following system:

$$[K]\{C\} = 0 \quad (3.15)$$

where $[K]$ is the coefficient matrix and $\{C\} = \{A_{mn}, B_{mn}, C_{mn}\}^T$. The system defined by Equation (3.15) is an eigen value problem and the non-trivial solution is found by setting the determinant of the coefficient matrix to zero. Using Wolfram Mathematica software (Appendix A), the following are the elements of the K matrix:

$$K[1, 1] = \frac{-2 m^2 \pi^2 + L R^2 n^2 (-1 + \nu)}{2 L R^2}$$

$$K[1, 2] = -\frac{L R m n \pi + L R m n \pi \nu}{2 L R^2}$$

$$K[1, 3] = -\frac{m \pi \nu}{L R}$$

$$K[2, 1] = -\frac{6 L R m n \pi + 6 L R m n \pi \nu}{12 L R^2}$$

$$K[2, 2] = -\frac{(L R^2 n^2 (1 + 12 R T^2) - m^2 \pi^2 (1 + 6 R T^2) (-1 + \nu)) + 12 m^2 \pi^2 R T^2 (-1 + \nu^2) \sigma c r}{12 L R^2 R T^2}$$

$$K[2, 3] = -\frac{L R^2 n^3 + m^2 n \pi^2 + 12 L R^2 n R T^2}{12 L R^2 R T^2}$$

$$K[3, 3] = -\frac{E(2LR^2m^2n^2\pi^2 + m^4\pi^4 + LR^4(n^4 + 12RT^2)) + 12LR^2m^2\pi^2RT^2(-1 + v^2)\sigma cr}{12ELR^4RT^2}$$

Where RT represents the ratio R/t and LR represents the ratio L/R .

Equating the determinate of the K matrix to zero results in a lengthy expression for the critical buckling stress, a screen shot of which is given in Figure 19.

$$\begin{aligned} & - \left((EE (LR^6 m^2 n^4 RT^2 + LR^6 m^2 n^6 RT^2 + 3 LR^4 m^4 n^2 \pi^2 RT^2 + 4 LR^4 m^4 n^4 \pi^2 RT^2 + 2 LR^2 m^6 \pi^4 RT^2 + 5 LR^2 m^6 n^2 \pi^4 RT^2 + 2 m^8 \pi^6 RT^2 + 12 LR^6 m^2 n^2 RT^4 + 12 LR^6 m^2 n^4 RT^4 + 24 LR^4 m^4 \pi^2 RT^4 + 24 LR^4 m^4 n^2 \pi^2 RT^4 + 12 LR^2 m^6 \pi^4 RT^4 + LR^4 m^4 n^2 \pi^2 RT^2 \vee + \right. \\ & 2 LR^4 m^4 n^4 \pi^2 RT^2 \vee + 4 LR^2 m^6 n^2 \pi^4 RT^2 \vee + 2 m^8 \pi^6 RT^2 \vee + 24 LR^4 m^4 \pi^2 RT^4 \vee - LR^6 m^2 n^4 RT^2 \vee^2 - LR^6 m^2 n^6 RT^2 \vee^2 - \\ & LR^4 m^4 n^2 \pi^2 RT^2 \vee^2 - 2 LR^4 m^4 n^4 \pi^2 RT^2 \vee^2 - 2 LR^2 m^6 \pi^4 RT^2 \vee^2 - LR^2 m^6 n^2 \pi^4 RT^2 \vee^2 - 12 LR^6 m^2 n^2 RT^4 \vee^2 - 12 LR^6 m^2 n^4 RT^4 \vee^2 - \\ & 24 LR^4 m^4 \pi^2 RT^4 \vee^2 - 24 LR^4 m^4 n^2 \pi^2 RT^4 \vee^2 - 12 LR^2 m^6 \pi^4 RT^4 \vee^3 + LR^4 m^4 n^2 \pi^2 RT^2 \vee^3 - 24 LR^4 m^4 \pi^2 RT^4 \vee^3 - \\ & \sqrt{(m^4 RT^4 (1 + \vee)^2 ((2 m^6 \pi^6 + LR^2 m^4 \pi^4 (-n^2 (-5 + \vee) - 2 (1 + 6 RT^2) (-1 + \vee)) - LR^6 n^2 (1 + n^2) (n^2 + 12 RT^2) (-1 + \vee) + \\ & LR^4 m^2 \pi^2 (-2 n^4 (-2 + \vee) - 24 RT^2 (-1 + \vee^2) + n^2 (3 - 24 RT^2 (-1 + \vee) - 2 \vee + \vee^2)))^2 - \\ & 4 LR^2 (-2 m^2 \pi^2 + LR^2 n^2 (-1 + \vee)) (-1 + \vee) (12 LR^8 n^4 (-1 + n^2)^2 RT^2 + 2 m^8 \pi^8 (1 + 6 RT^2) + \\ & 12 LR^6 m^2 n^2 (-1 + n^2) \pi^2 RT^2 (-3 + 4 n^2 - \vee) + LR^2 m^6 n^2 \pi^6 (3 + 48 RT^2 - \vee) + \\ & LR^4 m^4 \pi^4 (n^4 (1 + 72 RT^2 - \vee) - 24 RT^2 (1 + 6 RT^2) (-1 + \vee^2) + 12 n^2 RT^2 (-6 - \vee + \vee^2)))))) \Big) \Big) / \\ & (24 LR^2 m^4 \pi^2 RT^4 (2 m^2 \pi^2 - LR^2 n^2 (-1 + \vee)) (-1 + \vee) (1 + \vee)^2) \Big) \end{aligned}$$

Figure 19 The obtained non-axisymmetric buckling stress

In order to compare the obtained non-axisymmetric buckling stress with the classical axisymmetric buckling stress, let us define the parameter α as the ratio between the complicated non-axisymmetric buckling stress (Figure 19) and the classical buckling stress, which results in another lengthy formula for the non-dimensional parameter α (Figure 20).

For the sake of verification, three arbitrary cases are selected, Case1 has a geometrical property of $L/R=1$ and $R/t=500$, Case 2 with $L/R=5$ and $R/t=500$, and Case 3 with $L/R=2$ and $R/t=2000$. The first two cases are available in Ref. [24]. The three cases were also modeled in ABAQUS, which make it easy to visualize the buckling modes. The material properties used were $E=210\text{ GPa}$ and $\nu=0.3$. In all cases, the radius of the cylinder was selected to be 5 m while the other properties were adjusted according to the given L/R and R/t ratios.

In the preparation of the FE model for a simply supported cylindrical shell, half of the model was used with symmetric boundary condition at the top and a simply supported condition at the bottom edge. If the complete model is used, ABAQUS does not predict the buckling stress and mode accurately [26]. A convergence study on the mesh size for different types of element was conducted in the literature [26], it proved that S8R element with 20 elements on the axial direction (for half of the cylinder) and 64 elements in the circumferential direction predict the buckling stress and mode accurately. S8R is an eight-noded doubly curved shell element with six degrees of freedom per node (three displacements and three rotations). The findings of Ref. [26] have been adopted to model the shell in this chapter.

The analytical solutions for the three cases were obtained following the same procedure described previously and compared with the results of Wang *et al.* [24] and FEM as shown in Table 3. The comparison with the results of Wang *et al.* [24] shows an exact agreement in both the buckling stress and mode for each case. This because they are both based on exact solutions of the governing differential equations. The comparison with FEM, however, shows an excellent agreement in the buckling stress with a maximum error of

1.04 % and exact agreement in predicting the buckling modes. To visualize the buckling modes, the deformed shapes for the three cases are shown in Figure 21 for halves of the shells, as modeled.

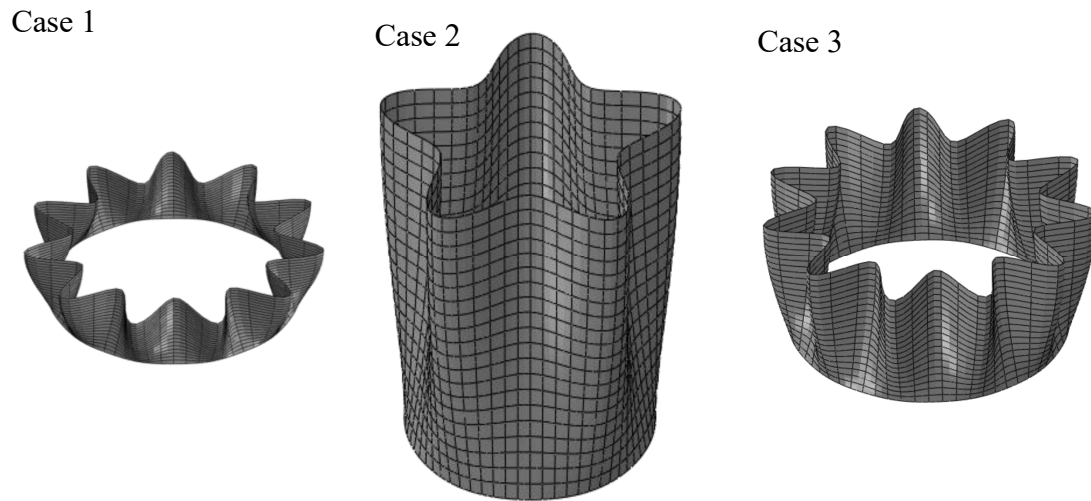


Figure 21 The deformed shapes of the buckled shells models in ABAQUS

Table 3 Comparison of the analytical long formula with FEM

The case	Present			Wang <i>et al.</i> [24]			FEM			% age difference between σ_{cr} and FEM results
	σ_{cr} (MPa)	m	n	σ_{cr} (MPa)	m	n	σ_{cr} (MPa)	m	n	
Case 1 (L/R=1, R/t= 500)	250.252	1	11	250.252	1	11	252.860	1	11	1.04
Case 2 (L/R=5, R/t= 500)	234.938	1	5	234.938	1	5	234.83	1	5	0.05
Case 3 (L/R=2, R/t= 2000)	62.684	1	11	-	-	-	63.256	1	11	0.91

It has now been verified that the derived non-axisymmetric buckling formula yields an exact agreement with the previously published results and excellent agreement with FEM in terms of buckling stress and mode. In Section 3.2.5, more comparisons with FEM will be shown to verify the obtained simplified formula.

3.2.4 Parametric Study

Since the geometrical parameters L/R and R/t are the sole parameters to describe a perfect shell of a given material, they are used to conduct a parametric study for a wide range of them. The range of the former is taken to be varied from 0.5 to 12.5 in step of 1 while of the later is taken from 300 to 3000 in step of 100. The Poisson's ratio is taken to be 0.30. In every case of L/R and R/t values, the minimization was carried out to yields the non-dimensional value of $\sigma_{cr} \times 10^3/E$ (Appendix A). Collecting the obtained minimum values

of $\sigma_{cr} \times 10^3/E$ was performed and plotted in the 3D Plot shown in Figure 22, from which the data are found to be fitted using the following formula [26]:

$$\frac{\sigma_{cr}}{E} \times 10^3 = \frac{c_1}{R/t} (L/R)^{c_2} \quad (3.16)$$

Where c_1 and c_2 are unknown parameters to be determined based on the best fit.

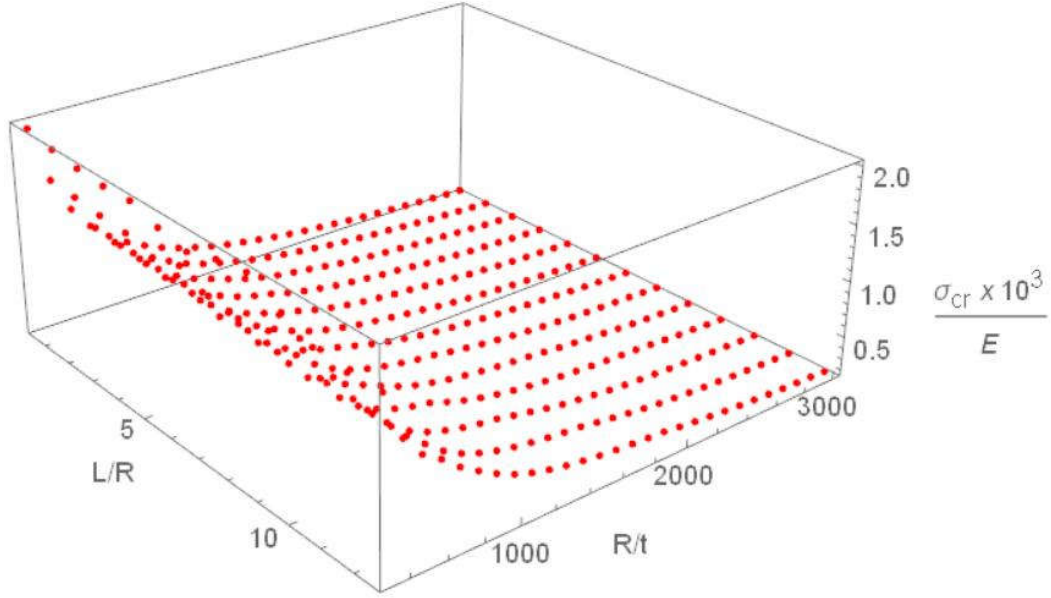


Figure 22 Variation of $\sigma_{cr} \times 10^3/E$ with the geometrical parameters

Utilizing Mathematica (Appendix A), the values of the parameters corresponding to the best fit are found to be $c_1 = 603$ and $c_2 = -0.043$. The simplified formula of the critical buckling stress of a perfect shell is then expressed by

$$\frac{\sigma_{cr}}{E} \times 10^3 = \frac{603}{R/t} (L/R)^{-0.043} \quad (3.17)$$

Or

$$\sigma_{cr} = \frac{0.603}{R/t} E (L/R)^{-0.043} \quad (3.18)$$

To compare the obtained simplified formula with the data points, Figure 23 was prepared. In this figure, the surface denotes the prediction by the developed fitting, while the dots are the generated data points computed using the lengthy exact formula.

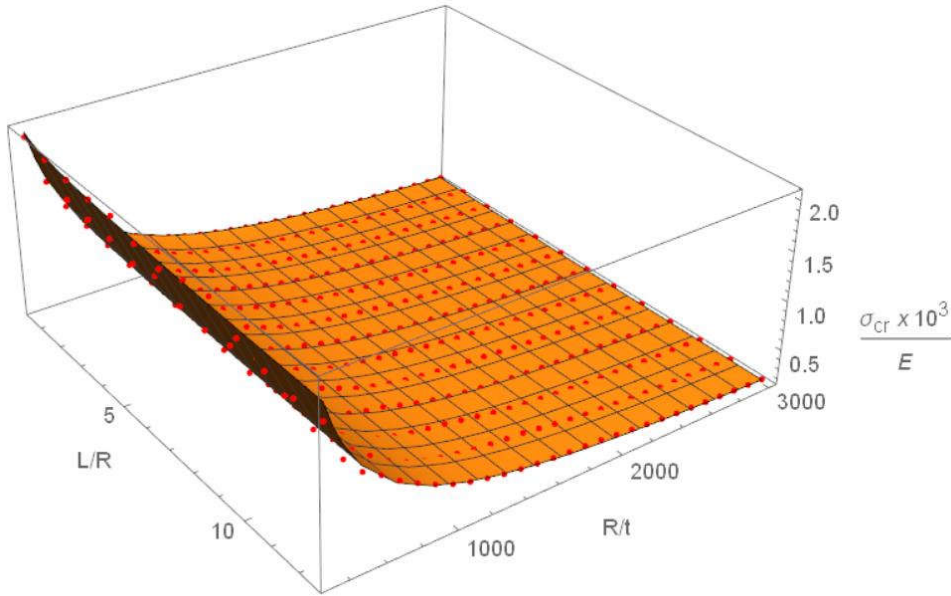


Figure 23 Comparison of the simplified formula and the discrete data of $\sigma_{cr} \times 10^3/E$

3.2.5 Verification of the Simplified Formula

3.2.5.1 Comparison with the Exact Formula and FEM

To compare the results of the proposed simplified formula with the exact one and FEM, different shells geometries have been selected as shown in Table 4. In all cases, the proposed simplified formula yields an excellent agreement with both the exact formula and FEM.

Table 4 The simplified formula of the buckling stress of a perfect cylindrical shell, $\sigma_{cr} \times 10^3/E$, compared to FEM and the long formula

R/t	L/R=1			L/R=4			L/R=6		
	FEM	Exact Formula	Simplified Formula	FEM	Exact Formula	Simplified Formula	FEM	Exact Formula	Simplified Formula
400	1.49	1.49	1.51	1.44	1.44	1.42	1.43	1.43	1.40
450	1.34	1.33	1.34	1.32	1.33	1.26	1.32	1.34	1.24
500	1.19	1.19	1.21	1.18	1.18	1.14	1.17	1.17	1.12
550	1.09	1.08	1.10	1.05	1.05	1.03	1.04	1.04	1.02
600	1	1.00	1.01	0.96	0.96	0.95	0.94	0.94	0.93
650	0.93	0.92	0.93	0.88	0.88	0.87	0.86	0.86	0.86
700	0.85	0.85	0.86	0.82	0.82	0.81	0.8	0.8	0.80
750	0.8	0.8	0.80	0.77	0.77	0.76	0.75	0.75	0.74
800	0.75	0.75	0.75	0.73	0.73	0.71	0.71	0.71	0.70
850	0.71	0.71	0.71	0.7	0.7	0.67	0.67	0.67	0.66
900	0.67	0.67	0.67	0.66	0.67	0.63	0.65	0.65	0.62
950	0.63	0.63	0.63	0.63	0.63	0.60	0.62	0.62	0.59
1000	0.6	0.60	0.60	0.59	0.59	0.57	0.6	0.59	0.56

NOTE: The above FEM results has been taken from Ref. [103].

3.2.5.2 Comparison with the Classical Axisymmetric Formula

To compare the results of the proposed simplified elastic formula for the non-axisymmetric buckling stress with that of the axisymmetric one (Equation 3.8), let us define the term β to be the ratio between the simplified elastic formula (Equation 3.18) and the classical axisymmetric formula (Equation 3.8). i.e.:

$$\beta = \frac{\sigma_{cr}}{\sigma_{cl}} = 0.997 \left(\frac{L}{R} \right)^{-0.043} \quad (3.19)$$

It can clearly be seen that β is only function of the length-to-radius ratio, which is not accounted for in case of axisymmetric buckling. The variation of β with the L/R is shown in Figure 24, which indicates that the reduction of the non-axisymmetric buckling stress from the classical axisymmetric one increases with the increase of L/R and it can be around 10 % for case of relatively long shells with $L/R = 11$.

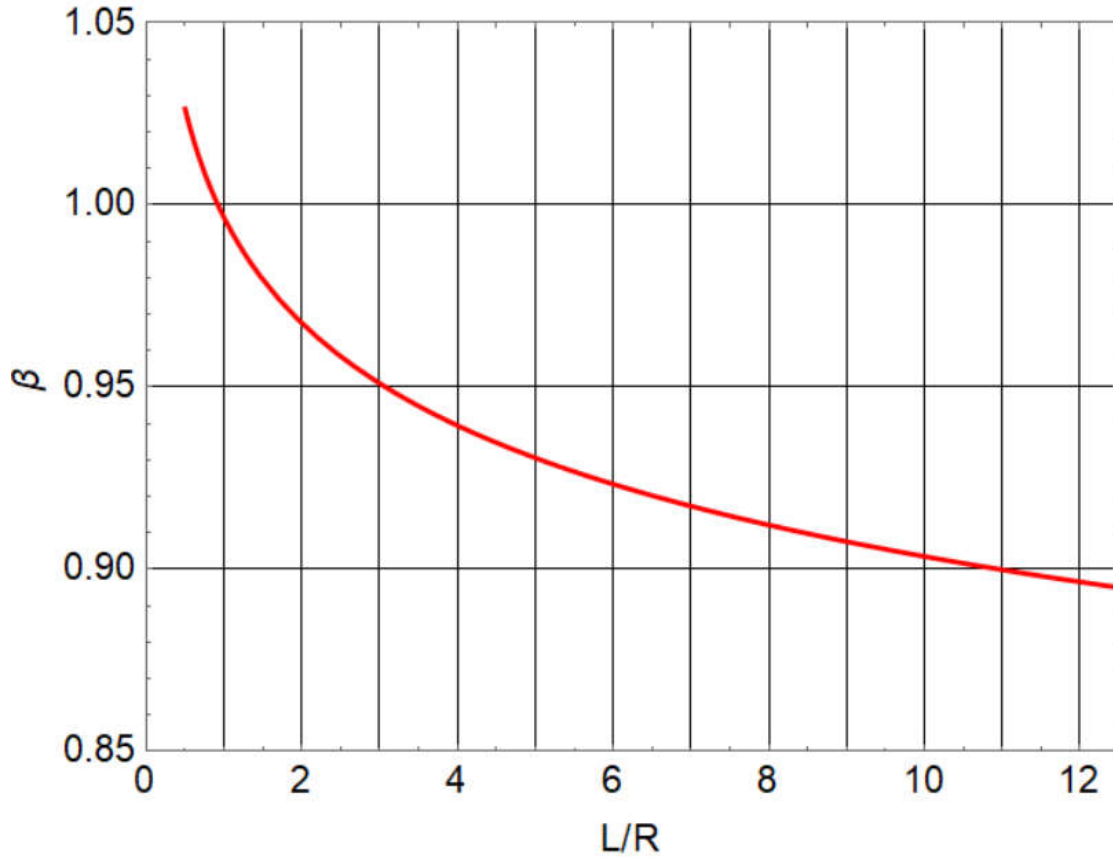


Figure 24 Variation of β with the L/R

Since the simplified non-axisymmetric buckling formula (Equation 3.17 or 3.18) has been verified against FEM and the long exact formula, it can now be used for future use whenever the non-axisymmetric buckling stress is required.

To sum up, both the classical (axisymmetric) and non-axisymmetric buckling have been derived in this chapter. In the non-axisymmetric buckling, the complete formula has been derived and utilized to obtain the buckling strength and the corresponding mode. The non-axisymmetric buckling stress is found to be less than that of the classical buckling for shells with $L/R > 1$. The reduction from the classical buckling stress can reach 10 % for shells with $L/R \geq 11$. Moreover, determination of the non-axisymmetric buckling modes is of practical importance since it gives a buckling shape similar to that obtained experimentally and can be used to describe the mode of imperfections [54, 55, 72-74] as allowed by the European standards [6] in the absence of a known severe form of imperfection in the design step.

CHAPTER 4

GEOMETRIC NONLINEAR BUCKLING OF PERFECT CYLINDRICAL SHELLS

4.1 General

During shell postbuckling, the end shortening as well as the radial displacement get larger, which limits the use of a small-deflection theory and requires the use of the nonlinear large-deflection theory. The first attempt in obtaining the complete postbuckling curve was done by von Karman and Tsien in 1941 [40]. It was found that after the shell buckles, there are different stable postbuckling equilibrium paths. Although this attempt has been followed by numerous improvements through decades, it is outlined in this chapter to provide a full understanding of the buckling and postbuckling mechanisms. The chapter is concluded by listing the improvements done in the work, limitations of the analytical solutions, and recent research trends and challenges.

The postbuckling involves lengthy mathematical manipulation. So, to avoid flooding the text with lengthy mathematical operations, only the key steps for derivation of the governing equations and their solution are outlined in this chapter. The detailed mathematical manipulation and step-by-step procedure are available in the Mathematica code provided in Appendix B.

4.2 Derivation of the Governing Equations

A typical buckled thin shell due to an axial compression is shown in Figure 25. The deformation of the shell mid-surface is represented by the three displacement components u , v , and w in the x , y and z , respectively. Such form of deformation generates membrane and bending strains. The membrane strains are related to the displacements by

$$\begin{Bmatrix} \epsilon_{x0} \\ \epsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \frac{\partial v}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 - \frac{w}{R} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{Bmatrix} \quad (4.1)$$

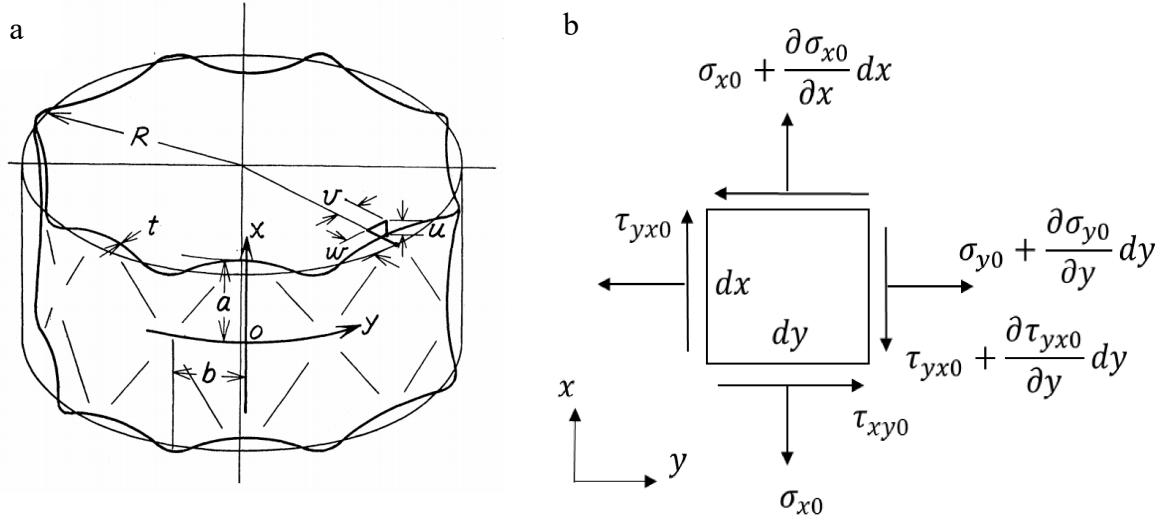


Figure 25 A buckled cylindrical shell: a) The deformed shape [40], and b) The in-plane internal stresses in a differential element

The membrane stresses (Figure 25 b) are also given by

$$\begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \tau_{xy0} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{Bmatrix} \epsilon_{x0} + \nu \epsilon_{y0} \\ \epsilon_{y0} + \nu \epsilon_{x0} \\ \frac{(1-\nu)}{2} \gamma_{xy0} \end{Bmatrix} \quad (4.2)$$

Substitution of Equations 4.1 into 4.2 yields

$$\begin{Bmatrix} \sigma_{x0} \\ \sigma_{y0} \\ \tau_{xy0} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \nu \left[\frac{\partial v}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 - \frac{w}{R} \right] \\ \frac{\partial v}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 - \frac{w}{R} + \nu \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \\ \frac{(1-\nu)}{2} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \end{Bmatrix} \quad (4.3)$$

The equilibrium in the in-plane directions of the shell surface (Figure 25 b) can be expressed by [40]:

$$\begin{aligned} \frac{\partial \sigma_{x0}}{\partial x} + \frac{\partial \tau_{xy0}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy0}}{\partial x} + \frac{\partial \sigma_{y0}}{\partial y} &= 0 \end{aligned} \quad (4.4)$$

Automatic satisfaction of the above pair of equations can be obtained by introducing Airy's stress function, $F(x, y)$, expressed by the following relations:

$$\sigma_{x0} = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_{y0} = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy0} = -\frac{\partial^2 F}{\partial x \partial y} \quad (4.5)$$

Substitution of Equations (4.5) into (4.3) and eliminating the in-plane displacements, u and v (Appendix B) results in relation between the radial displacement, w , and the Airy's stress function F , i.e.:

$$\nabla^4 F = E \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \frac{1}{R} \frac{\partial^2 w}{\partial x^2} \right] \quad (4.6)$$

Where

$$\nabla^4 = (\nabla^2)^2 = \left(\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right)$$

Equation 4.6 results from the equilibrium in the two in-plane directions while the equilibrium in the out-of-plane direction is given by [47]:

$$D \nabla^4 w = \frac{t}{R} \frac{\partial^2 F}{\partial x^2} + t \left[\frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \quad (4.7)$$

where $D \equiv \frac{Et^3}{(1-\nu^2)}$ is the flexural rigidity of the shell.

The exact solution of the problem can be obtained by solving the coupled two differential equations (Equations 4.6 and 4.7) with appropriate boundary conditions. However, the efforts in the literature proved that an analytical solution is hard to be obtained. Instead, approximate solutions were provided by von Karman and Tsien [40] and other researchers [42-49] as discussed next.

4.3 Von Karman and Tsien's Approximate Formulation

The solution is approximated by adopting the equilibrium equation of the in-plane stresses (Equation 4.6) and the equilibrium in the radial direction is obtained by minimization of the potential energy instead of dealing with Equation 4.7 directly. The formulation procedure is outlined here.

The formulation is based on one wave pattern of a size $2a \times 2b$ (Figure 26). The membrane strain energy is given by:

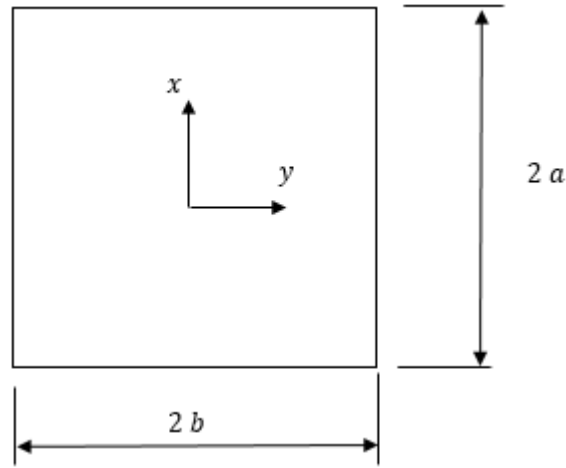


Figure 26 The size of one wave pattern

$$U_m = \frac{t}{2E} 4 \int_0^a \int_0^b \left[(\sigma_{x0} + \sigma_{y0})^2 - 2(1 + \nu)(\sigma_{x0}\sigma_{y0} - \tau_{xy0}^2) \right] dx dy, \quad (4.8)$$

where a and b are the half-sine waves in x and y directions, respectively.

Adopting the following curvatures relations

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}, \quad (4.9)$$

the bending strain energy becomes

$$U_b = \frac{1}{2D} 4 \int_0^a \int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) \right] dx dy \quad (4.10)$$

The work done by the applied axial load is given by

$$W = 4 t \int_0^a \int_0^b (\sigma_{x0})_{x=a} \frac{\partial u}{\partial x} dx dy \quad (4.11)$$

The potential energy is given by

$$\Pi = U_m + U_b - W \quad (4.12)$$

Minimization of the potential energy expressed by Equation 4.12 along with the satisfaction of Equation 4.6 for an assumed expression of the radial displacement w yields an approximate solution of the problem.

4.4 Von Karman and Tsien's Approximate Solution

As explained earlier, the solution of the problem is based on Equations 4.6 and 4.12. Due to the lengthy mathematical manipulation of Karman and Tsien's solution [40], only the main solution steps are outlined here while the detailed operations are given as part of the Mathematica code given in Appendix B.

Since it has been proven experimentally that the buckled surface of the shell takes a diamond shape, the radial deflection, w , can be represented by

$$\begin{aligned} \frac{w}{R} = & \left(f_0 + \frac{f_1}{4} \right) + \frac{f_1}{2} \left(\cos\left(\frac{mx}{R}\right) \cos\left(\frac{ny}{R}\right) + \frac{1}{4} \cos\left(\frac{2mx}{R}\right) + \frac{1}{4} \cos\left(\frac{2ny}{R}\right) \right) \\ & + \frac{f_2}{4} \left(\cos\left(\frac{2mx}{R}\right) + \cos\left(\frac{2ny}{R}\right) \right) \end{aligned} \quad (4.13)$$

Where f_0 , f_1 , and f_2 are unknown parameters to be determined; m and n are the number of waves in the axial and circumferential directions, respectively.

Substitution of the assumed expression of the radial displacement w in the right-hand side of Equation 4.6 results in:

$$\begin{aligned}
\nabla^4 F = -E \mu^2 \left(\frac{n}{R} \right)^2 & \left(A \cos \left[\frac{2mx}{R} \right] + B \cos \left[\frac{2ny}{R} \right] + C \cos \left[\frac{mx}{R} \right] \cos \left[\frac{ny}{R} \right] \right. \\
& + D \cos \left[\frac{3mx}{R} \right] \cos \left[\frac{ny}{R} \right] + G \cos \left[\frac{mx}{R} \right] \cos \left[\frac{3ny}{R} \right] \\
& \left. + H \cos \left[\frac{2mx}{R} \right] \cos \left[\frac{2ny}{R} \right] \right)
\end{aligned} \tag{4.14}$$

where:

$$\mu = m/n$$

$$A = \frac{1}{8} f_1^2 n^2 - \left(\frac{1}{2} f_1 + f_2 \right)$$

$$B = \frac{1}{8} f_1^2 n^2$$

$$C = \frac{1}{2} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right) - \frac{1}{2} f_1$$

$$D = \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right)$$

$$G = \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right)$$

$$H = n^2 \left(\frac{1}{2} f_1 + f_2 \right)^2$$

Solving the differential equation (Equation 4.14) yields an expression for the Airy's stress function, F . i.e.:

$$\begin{aligned}
F = -E \mu^2 \left(\frac{n}{R} \right)^2 & \left(\frac{R^4}{16m^4} A \cos \left[\frac{2mx}{R} \right] + \frac{R^4}{16n^4} B \cos \left[\frac{2ny}{R} \right] \right. \\
& + \frac{R^4}{(m^2 + n^2)^2} C \cos \left[\frac{mx}{R} \right] \cos \left[\frac{ny}{R} \right] \\
& + \frac{R^4}{(9m^2 + n^2)^2} D \cos \left[\frac{3mx}{R} \right] \cos \left[\frac{ny}{R} \right] \\
& + \frac{R^4}{(m^2 + 9n^2)^2} G \cos \left[\frac{mx}{R} \right] \cos \left[\frac{3ny}{R} \right] \\
& \left. + \frac{R^4}{16(m^2 + n^2)^2} H \cos \left[\frac{2mx}{R} \right] \cos \left[\frac{2ny}{R} \right] \right) + \alpha x^2/2 + \beta y^2/2
\end{aligned} \tag{4.15}$$

where the last two terms represent the particular solution with constants (α and β) to be determined from the boundary conditions.

Once the expressions for both w and F are obtained, all the remaining steps to derive the potential energy expression (Equation 4.12) can easily be performed since all the stresses and the in-plane displacements are expressed in terms of F and w .

The substitution of w and F expressions in Equation 4.5 yields:

$$\begin{aligned}
\sigma_{x0} = \beta + & \frac{D E \mu^2 \cos \left[\frac{3mx}{R} \right] \cos \left[\frac{ny}{R} \right]}{(1 + 9\mu^2)^2} + \frac{1}{4} B E \mu^2 \cos \left[\frac{2ny}{R} \right] \\
& + \frac{E H \mu^2 \cos \left[\frac{2mx}{R} \right] \cos \left[\frac{2ny}{R} \right]}{4(1 + \mu^2)^2} \\
& + E \mu^2 \cos \left[\frac{mx}{R} \right] \left(\frac{C \cos \left[\frac{ny}{R} \right]}{(1 + \mu^2)^2} + \frac{9 G \cos \left[\frac{3ny}{R} \right]}{(9 + \mu^2)^2} \right)
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
\sigma_{y0} = \alpha &+ \frac{9 D E m^2 \mu^2 \cos \left[\frac{3mx}{R} \right] \cos \left[\frac{ny}{R} \right]}{(n + 9n\mu^2)^2} \\
&+ \frac{E m^2 \cos \left[\frac{2mx}{R} \right] \left(A (1 + \mu^2)^2 + H \mu^4 \cos \left[\frac{2ny}{R} \right] \right)}{4n^2 \mu^2 (1 + \mu^2)^2} \\
&+ E \frac{m^2}{n^2} \mu^2 \cos \left[\frac{mx}{R} \right] \left(\frac{C \cos \left[\frac{ny}{R} \right]}{(1 + \mu^2)^2} + \frac{G \cos \left[\frac{3ny}{R} \right]}{(9 + \mu^2)^2} \right)
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\tau_{xy0} = \frac{1}{4n} E m \mu^2 &\left(\frac{12D \sin \left[\frac{3mx}{R} \right] \sin \left[\frac{ny}{R} \right]}{(1 + 9\mu^2)^2} + \frac{H \sin \left[\frac{2mx}{R} \right] \sin \left[\frac{2ny}{R} \right]}{(1 + \mu^2)^2} \right. \\
&\left. + \sin \left[\frac{mx}{R} \right] \left(\frac{4 C \sin \left[\frac{ny}{R} \right]}{(1 + \mu^2)^2} + \frac{12 G \sin \left[\frac{3ny}{R} \right]}{(9 + \mu^2)^2} \right) \right)
\end{aligned} \tag{4.18}$$

The stress σ_{x0} at $x = a$ is given by

$$\begin{aligned}
\sigma_{x0} = \beta &+ \frac{E D \mu^2 \cos \left[\frac{3am}{R} \right] \cos \left[\frac{ny}{R} \right]}{(1 + 9\mu^2)^2} + \frac{1}{4} E B \mu^2 \cos \left[\frac{2ny}{R} \right] \\
&+ \frac{E H \mu^2 \cos \left[\frac{2am}{R} \right] \cos \left[\frac{2ny}{R} \right]}{4(1 + \mu^2)^2} \\
&+ E \mu^2 \cos \left[\frac{am}{R} \right] \left(\frac{C \cos \left[\frac{ny}{R} \right]}{(1 + \mu^2)^2} + \frac{9 G \cos \left[\frac{3ny}{R} \right]}{(9 + \mu^2)^2} \right)
\end{aligned} \tag{4.19}$$

Since the axial applied stress is uniform (equal to the average applied compressive stress σ) as commonly performed experimentally, all the terms depend on y in the above expression should vanish leading to:

$$\beta = \sigma \quad (4.20)$$

It has been shown in Appendix B that the use of Equations 4.3 and 4.5 leads to the following relations:

$$\frac{\partial u}{\partial x} = \frac{1}{E} \left(\frac{\partial^2 F}{\partial y^2} - \nu \frac{\partial^2 F}{\partial x^2} - \frac{E}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right) \quad (4.21)$$

$$\frac{\partial v}{\partial y} = \frac{1}{E} \left(\frac{\partial^2 F}{\partial x^2} + E \frac{w}{R} - \frac{E}{2} \left(\frac{\partial w}{\partial y} \right)^2 - \nu \frac{\partial^2 F}{\partial y^2} \right) \quad (4.22)$$

The substitution of w and F expressions in Equation 4.22 yields:

$$\begin{aligned} \frac{\partial v}{\partial y} = f_0 + \frac{f_1}{4} - \frac{3f_1^2 n^2}{64} - \frac{1}{16} f_1 f_2 n^2 - \frac{f_2^2 n^2}{16} + \frac{\alpha}{E} + \frac{\nu \sigma}{E} \\ + \text{Terms of periodic functions in } x \text{ and } y \end{aligned} \quad (4.23)$$

Since y is measured along the circumference of the shell, v must be a periodic function in y ; therefore, the constant term in $\frac{\partial v}{\partial y}$ must be equal to zero. This condition is used to determine α to be:

$$\alpha = -E \left(f_0 + \frac{f_1}{4} - \frac{3f_1^2 n^2}{64} - \frac{1}{16} f_1 f_2 n^2 - \frac{f_2^2 n^2}{16} + \frac{\nu \sigma}{E} \right) \quad (4.24)$$

Since the two constants α and β appear in the Airy's stress function F (Equation 4.15) have been determined, the only unknown parameters in both w and F are f_0 , f_1 , and f_2 which are to be determined from minimization of the potential energy.

Using the stresses and derivatives of displacement defined in the foregoing, the potential energy expression has easily been derived utilizing Equation 4.12. The potential energy is then minimized with respect to the three unknown parameters f_0 , f_1 , and f_2 .

After determination of the unknown parameters, the unit end shortening of the shell due to the axially applied stress σ is given by

$$\varepsilon = -\frac{1}{a b} \int_0^a \int_0^b \frac{\partial u}{\partial x} dx dy, \quad (4.25)$$

which is the total axial shortening in one wave divided by the wave of the length in the axial direction. Evaluation of the above equation and relating the unit elastic shortening to the axial stress yields

$$\frac{\varepsilon R}{t} = \frac{\sigma R}{E t} + \frac{1}{16} \eta \mu^2 \xi^2 \left(\rho^2 + \rho + \frac{3}{4} \right) \quad (4.26)$$

Where ξ is a parameter defining the wave amplitude by the thickness of the shell, η is a parameter given by $\eta = n^2 \frac{t}{R}$, and ρ is a parameter representing a lengthy nonlinear function of geometrical and material parameters given by

$$\begin{aligned} \rho = & (-48 (-1 - 2\mu^2 + 3\mu^4) (9 + 82\mu^2 + 9\mu^4)^2 (-1 + \nu^2) + 24\eta (9 + 82\mu^2 + 9\mu^4)^2 (1 + 2\mu^2 + 17\mu^4) (-1 + \nu^2) \xi + \\ & \eta^2 (-16 (3 - 2\mu^2 + 3\mu^4) (9 + 91\mu^2 + 91\mu^4 + 9\mu^6)^2 - \\ & 3 (81 + 1638\mu^2 + 10652\mu^4 + 25066\mu^6 + 48326\mu^8 + 25066\mu^{10} + 10652\mu^{12} + 1638\mu^{14} + 81\mu^{16}) (-1 + \nu^2) \xi^2) + \\ & \sqrt{(-48\eta (-1 + \nu^2) \xi (- (9 + 82\mu^2 + 9\mu^4)^2 (1 + 2\mu^2 + 17\mu^4) + \\ & 3\eta\mu^4 (325 + 4628\mu^2 + 20894\mu^4 + 4628\mu^6 + 325\mu^8) \xi) \\ & (16 (9 + \mu^2)^2 (1 + 9\mu^2)^2 (\eta^2 (1 + \mu^2)^2 (3 - 2\mu^2 + 3\mu^4) + 3 (-1 - 2\mu^2 + 3\mu^4) (-1 + \nu^2)) + \\ & 3\eta^2 (81 + 1638\mu^2 + 8052\mu^4 - 11958\mu^6 - 118826\mu^8 - 11958\mu^{10} + 8052\mu^{12} + 1638\mu^{14} + 81\mu^{16}) \\ & (-1 + \nu^2) \xi^2) + \\ & (16 (9 + \mu^2)^2 (1 + 9\mu^2)^2 (\eta^2 (1 + \mu^2)^2 (3 - 2\mu^2 + 3\mu^4) + 3 (-1 - 2\mu^2 + 3\mu^4) (-1 + \nu^2)) - \\ & 24\eta (9 + 82\mu^2 + 9\mu^4)^2 (1 + 2\mu^2 + 17\mu^4) (-1 + \nu^2) \xi + \\ & 3\eta^2 (81 + 1638\mu^2 + 10652\mu^4 + 25066\mu^6 + 48326\mu^8 + 25066\mu^{10} + 10652\mu^{12} + 1638\mu^{14} + 81\mu^{16}) \\ & (-1 + \nu^2) \xi^2)) / \\ & (48\eta (-1 + \nu^2) \xi (- (9 + 82\mu^2 + 9\mu^4)^2 (1 + 2\mu^2 + 17\mu^4) + 3\eta\mu^4 (325 + 4628\mu^2 + 20894\mu^4 + 4628\mu^6 + 325\mu^8) \xi)) \end{aligned} \quad (4.27)$$

Equation 4.26 was used to construct the full axial stress-unit end-shortening curve. The second part of the right-hand side implies that if the amplitude of the buckling wave is small $\xi \rightarrow 0$ (no buckling), the equation becomes

$$\varepsilon = \frac{\sigma}{E} \quad (4.28)$$

which corresponds to the pre-buckling stress. The nonlinearity, however, appears after the buckling happens and the associated buckling waves start to appear on the surface of the shell. Once the buckling waves are developed, the effect of the nonlinearity increases with the increase of the wave amplitude (higher ξ) as can be noticed from Equation 4.26.

It should be noted that the end effects are neglected here, and the work is valid for relatively long shells with an effective length equal to the shell parameter $2\pi R$. Thus, the wave length in the axial and circumferential directions can be found by $2\pi R/m$ and $2\pi R/n$, respectively. Therefore, the parameter $\mu = m/n$ can be used to define the wave pattern. If $\mu = 1$, the wave is square in its projection (Figure 27 a), but If $\mu < 1$, the wave is rectangular in its projection and its longer side in the axial direction (Figure 27 b).

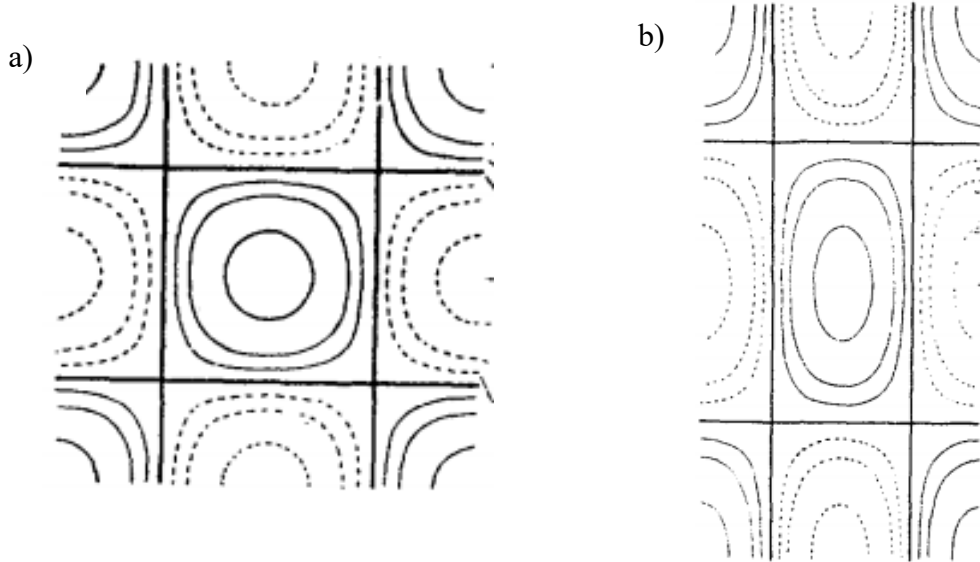


Figure 27 The wave patterns: a) square wave $\mu = 1$. b) rectangular wave $\mu < 1$ [40]

For the case of a shell with $R/t = 1000$, Poisson's ratio $\nu = 0.3$, and a square wave pattern ($\mu = 1$), the nondimensional axial stress versus the unit axial shortening (Equation 4.26) is shown in Figure 28 for different number of waves in the circumferential direction. Figure 29 is shown for the case of a rectangular wave with $\mu = 0.5$, in which a negative stress (for $n = 10$) is observed which indicates that the specimen will have some tension at a stage during the post-buckling response. Experimental evidence proved that this is not a possible scenario. Hence, it can be counted as drawback of this solution. Experiments also proved that in most cases the buckling waves are more like round squares.

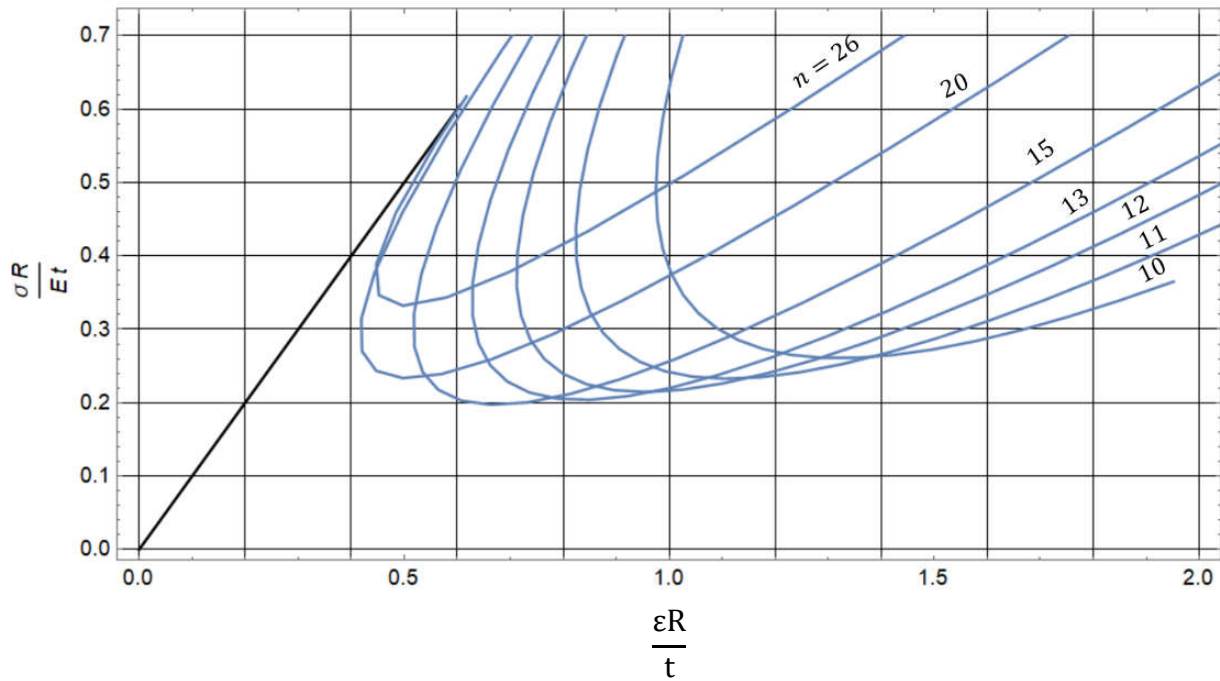


Figure 28 The nondimensional axial stress against the unit axial-unit end shortening for different number of waves in the circumferential direction for $\mu = 1$.

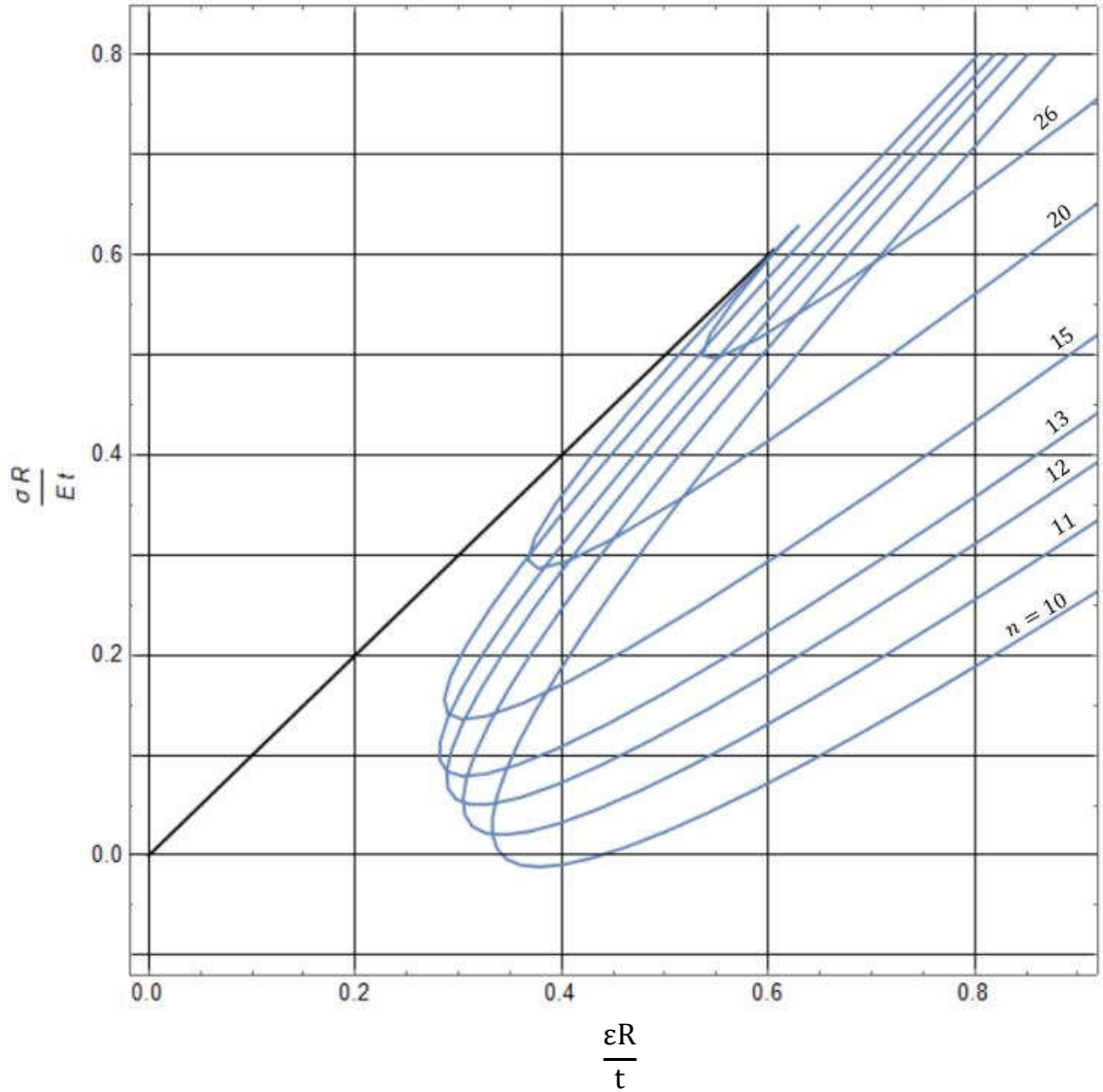


Figure 29 The nondimensional axial stress against the unit axial-unit end shortening for different number of waves in the circumferential direction for $\mu = 0.5$.

4.5 Improvements of von Karman and Tsien's Approximate Solution

As discussed in Section 2.4, the work of von Karman and Tsien has been improved by different researchers in the years that followed. Most of the early attempts [42-47] to improve the work were based on assuming new forms for the radial displacements w different from Equation 4.13. The goal was to obtain the axial stress-unit end shortening

relation, from which the minimum postbuckling stress (the minimum axial stress in the postbuckling region at which the shell is in equilibrium) can be obtained.

In the work of von Karman and Tsien [40], the minimum postbuckling stress was found to be $0.197 Et/R$, which is around 32 % of the classical solution (Equation 3.8). In the improvements that followed [42-47], less postbuckling stress was obtained. The work of Almroth [44], for example, yielded the minimum postbuckling stress to be $0.0652 Et/R$, which is around 11 % of the classical solution.

A reasonably accurate theoretical investigation of the postbuckling behavior of circular cylindrical shells was proposed by Yamaki and Kodama [48] in 1976. Instead of using energy method to replace the equilibrium equation in the radial direction, the two $w - F$ governing differential equations (Equation 4.6 and 4.7) were used. Galerkin method was used for solution of the governing differential equations. The results were found to be matched with a carefully prepared experimental specimen tested by Yamaki *et al.* [37] as shown previously in Figure 14.

4.6 Limitations of the Analytical Postbuckling Solutions

Despite the improvements undergone the analytical solution of the postbuckling of cylindrical shells and their role in describing the behavior of the shell under the testing machine, the obtained most accurate solution is limited and is of less practical importance if design of such shells is concerned. The limitations of such solutions are as follow:

- 1- These solutions are based on a perfect shell geometry and no initial imperfections are considered. The shell is found to be very sensitive to initial imperfections. In real practice, regardless of the quality of construction, obtaining a perfect shell

geometry is not possible especially the presence of the inevitable imperfections such as weld depression, irregularity in the shape of the cylinder, and the presence of nozzles.

- 2- These solutions are based on a linearly elastic material, which is not representative near the locations of nozzles, weld depressions, and any sort of local damage (if any).
- 3- They are not applicable for shells with low R/t ratios or shells made of low yield stress as these shells experience elastoplastic behavior before buckling.

In summary, the available analytical postbuckling solutions are impractical to be used for design purposes. However, they provide a better understanding of the shell buckling behavior and serve as a reference for a perfect shell buckling resistance.

CHAPTER 5

FE MODELING OF THE GEOMETRIC AND MATERIAL NONLINEAR BUCKLING OF PERFECT CYLINDRICAL SHELLS

5.1 General

As it has been discussed in the previous chapter that it is necessary to include the material nonlinearity in the buckling analysis of cylindrical shells, it has been covered in this chapter along with geometric nonlinearity. However, it is difficult to obtain analytical solutions for the case of geometric and material nonlinearity and one has to resort to numerical methods. The FEM is the best available tool for studying the considered problem. It allows tackling different forms of nonlinearities (geometric and material), and the imperfections in geometry, load, etc. In short, it is the best tool for accurately simulating both the lab and real-life environment.

This chapter is aimed at studying the geometric and material nonlinear buckling analysis of perfect cylindrical shells. It is concluded with a section providing capacity curves for perfect shells including both geometrical and material nonlinearities. These curves are generated to be used as a reference for designers to determine the upper limits of the shells to be designed.

5.2 The Material Model

Thin perfect shells are usually buckled within the elastic range of the material. The buckling load gets influenced with plasticity of the material as the shell get thicker (small radius to thickness (R/t) ratio). For the case of imperfect thin or thick shell, however, the plastic material property should be taken into account whenever local depressions or dent are there in the shell surface. These local depressions or damages results in localized plastic deformations, which highly reduces the buckling load of the shell.

The material used hereinafter are different grades of carbon steel. In the European Design Recommendation [102] (as well as most of the design standards) a bilinear (elastic perfectly plastic) model is adopted to represents the stress-strain curve of carbon steel. However, the preliminary investigations of shells with localized dent proves that the shell is stable and can carry relatively additional load after a small localized deformation took place, which justify the inclusion of the strain hardening of the material and, hence, adopting a more accurate model to represent the stress-strain curve of carbon steel. A recently proposed nonlinear stress-strain curve of steel by Yun and Gardner [104] is adopted in this work, which is accurately predicting the stress-stain curve of carbon steel as shown in Figure 30.

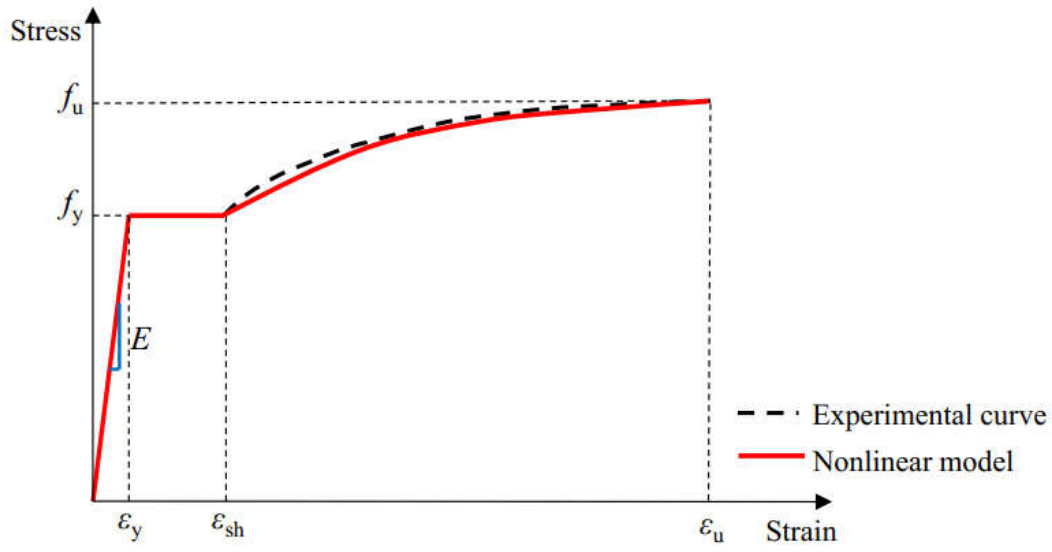


Figure 30 Bilinear plus nonlinear stress-strain curve of carbon steel [104]

This form of stress-strain curve can be expressed mathematically by [104]

$$f(\varepsilon) = \begin{cases} E \varepsilon, & \varepsilon \leq \varepsilon_y \\ f_y, & \varepsilon_y < \varepsilon \leq \varepsilon_{sh} \\ f_y + (f_u - f_y) \left\{ 0.4 \left(\frac{\varepsilon - \varepsilon_{sh}}{\varepsilon_u - \varepsilon_{sh}} \right) + 2 \left(\frac{\varepsilon - \varepsilon_{sh}}{\varepsilon_u - \varepsilon_{sh}} \right) / \left[1 + 400 \left(\frac{\varepsilon - \varepsilon_{sh}}{\varepsilon_u - \varepsilon_{sh}} \right)^5 \right]^{1/5} \right\}, & \varepsilon_{sh} < \varepsilon \leq \varepsilon_u \end{cases} \quad (5.1)$$

where the key parameters involved in the above model can be taken from Table 5 for different grades of standard structural steel [105].

Table 5 The key parameters of different grades of standard structural steel [104]

Steel grade	E N/mm^2	f_y N/mm^2	f_u N/mm^2	ε_y %	ε_{sh} %	ε_u %
S 235	210000	235	360	0.11	1.50	20.83
S 275	210000	275	430	0.13	1.50	21.63
S 355	210000	355	490	0.17	1.74	16.53
S 450	210000	440	550	0.21	2.50	12.00

5.3 The Method of Solution in ABAQUS

The load-end shortening curve of the buckling and postbuckling behavior can exhibit the form shown in Figure 31, in which the load and/or displacement can decrease with the progress of the solution. In this case, neither the incremental load approach nor the incremental displacement approach can converge to the correct solution. One of the powerful methods for solving such kind of problems was proposed by E. Riks in 1979 [106], which is commonly now known in the literature by Riks method or the Arc-length method.

Riks method treats both the load and displacement as unknowns to be solved for simultaneously. The progress of the solution is measured by the so called “arc-length”.

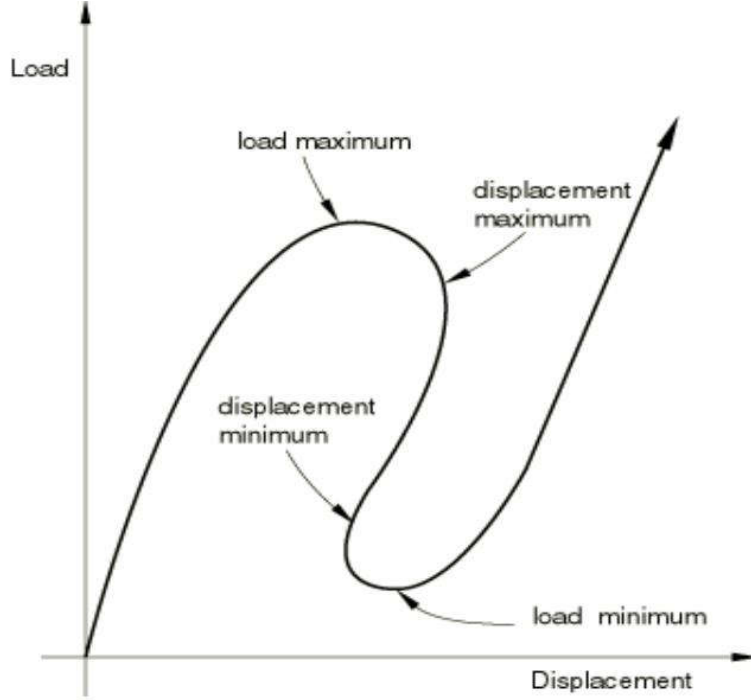


Figure 31 A typical equilibrium path of a shell buckling behavior [107]

For the sake of explanation, let us consider the variation of the load proportionality factor λ with the displacement u as shown in Figure 32. Suppose that the point (u_i, λ_i) is such to satisfy the system of equations and thus belongs to the “equilibrium path” that we are trying to identify. To progress in the solution and reach to the point (u_{i+1}, λ_{i+1}) , it is assumed that this point is located at the intersection of an arc of a radius Δl with the equilibrium path.

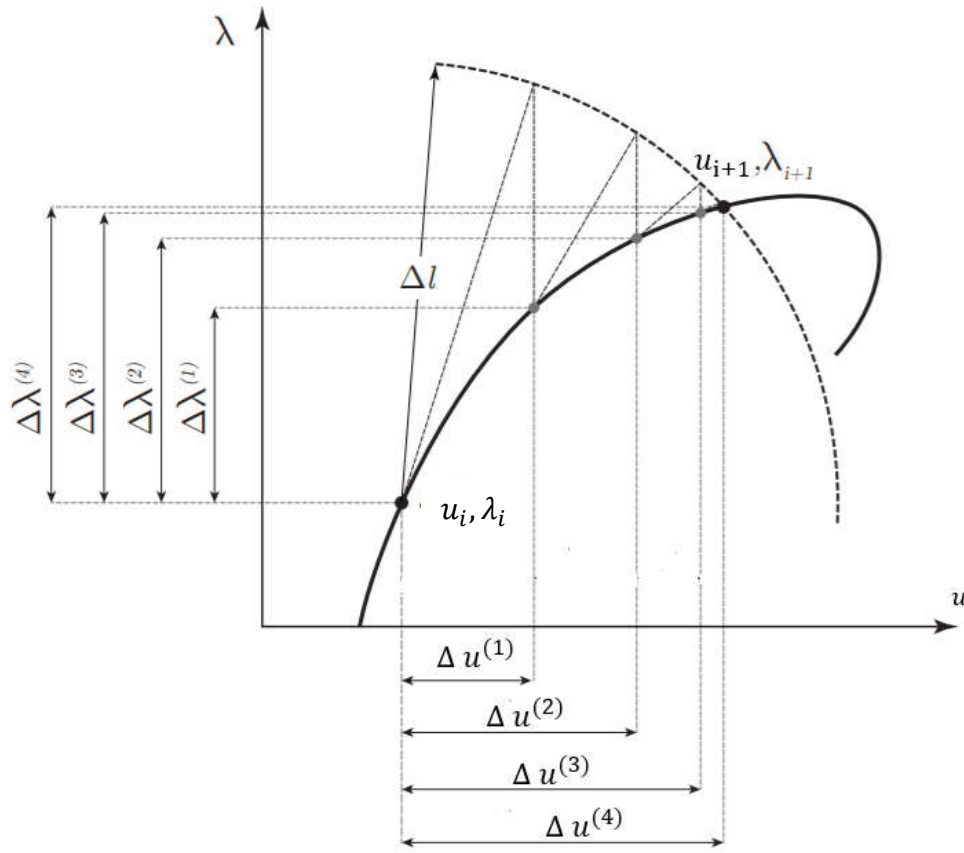


Figure 32 A schematic representation of the Arc-length method iterations [108]

To utilize this method in ABAQUS, the user is requested to specify the initial increment in the arc length Δl_{in} , the total arc length l_{period} (typically set to 1), the maximum and minimum arc length increment. Fixed incrementation is also available in which only the total arc length and the number of iterations are required.

Since both the load and displacement magnitudes are part of the solution, it is required to specify a procedure to stop the analysis when the step is complete. This can be done by either specifying a maximum load proportionality factor, or the maximum displacement at

a specific degree of freedom. Otherwise, the solution continues the analysis for the number of increments specified in the step definition.

5.4 Numerical Example of a Perfect Shell

To illustrate the modeling process, let us consider the following geometry of the perfect shell model:

Radius (R) = 7500 mm

Length (L) = 15000 mm

Thickness (t) = 10 mm

The material used is S 275 with the stress-strain curve and mechanical properties discussed in Section 5.2 and shown again in Figure 33.

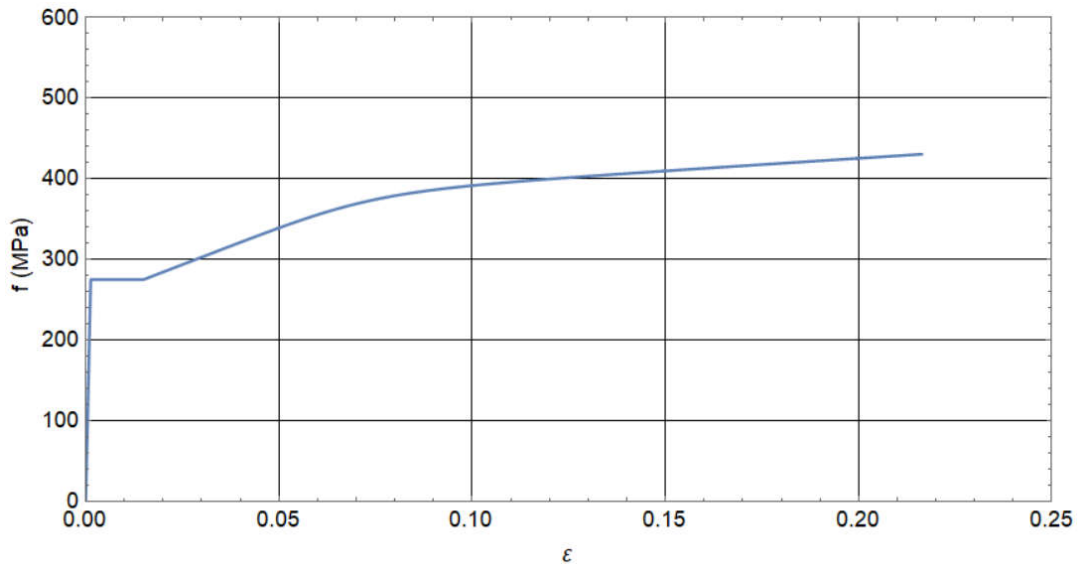


Figure 33 The stress-strain curve of the standard steel grade S275

To use the stress-strain curve in the FE model in ABAQUS, it is required to transform the engineering stress-strain curve to the true one using the following two equations [103]:

$$f_{true} = f_{Eng}(1 + \varepsilon_{Eng}) \quad (5.2)$$

$$\varepsilon_{true} = Ln(1 + \varepsilon_{Eng}) \quad (5.3)$$

The engineering and the true stress-strain curves are both shown in Figure 34.

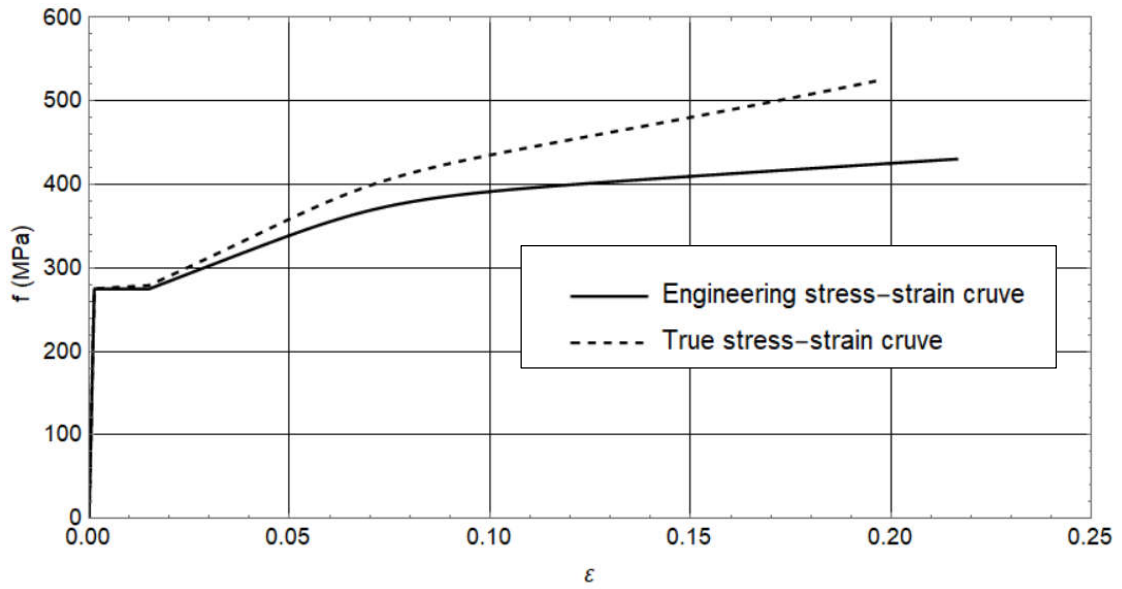


Figure 34 The engineering and true stress-strain curves of S275

In the preparation of the FE models, a clamped perfect shell was modeled. The four-noded shell element S4R was used with a fine mesh composed of 300 divisions in the circumferential direction and 100 divisions in the axial direction leading to a fine mesh (Figure 35) with a total number of 30300 elements.

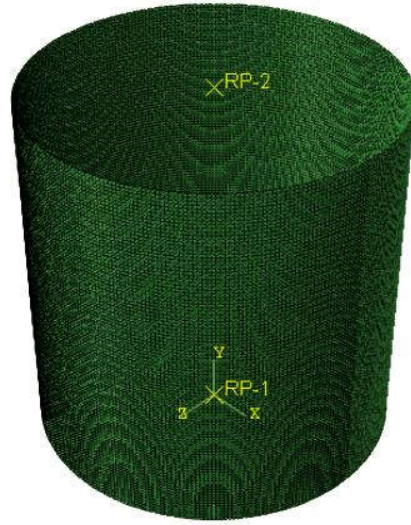


Figure 35 The mesh of the perfect shell model in ABAQUS

Riks method (Section 5.3) was adopted to solve the problem in ABAQUS. To show different modes through the buckling process, the axial load against the arc length is shown in Figure 36, which shows different buckling forms of deformations starting from axisymmetric to non-axisymmetric one. It is also clear that after the first buckling is achieved at Point B, the axial load capacity gets reduced till reaching a minimum load at Point C. The interesting part of the shell buckling problem is after the first buckling is achieved and the capacity drops, the shell regains part of the capacity and starts carrying more load than the minimum one as shown in Point D. However, the recent design standards do not consider the additional gained capacity and the design is based on the maximum load (Point B).

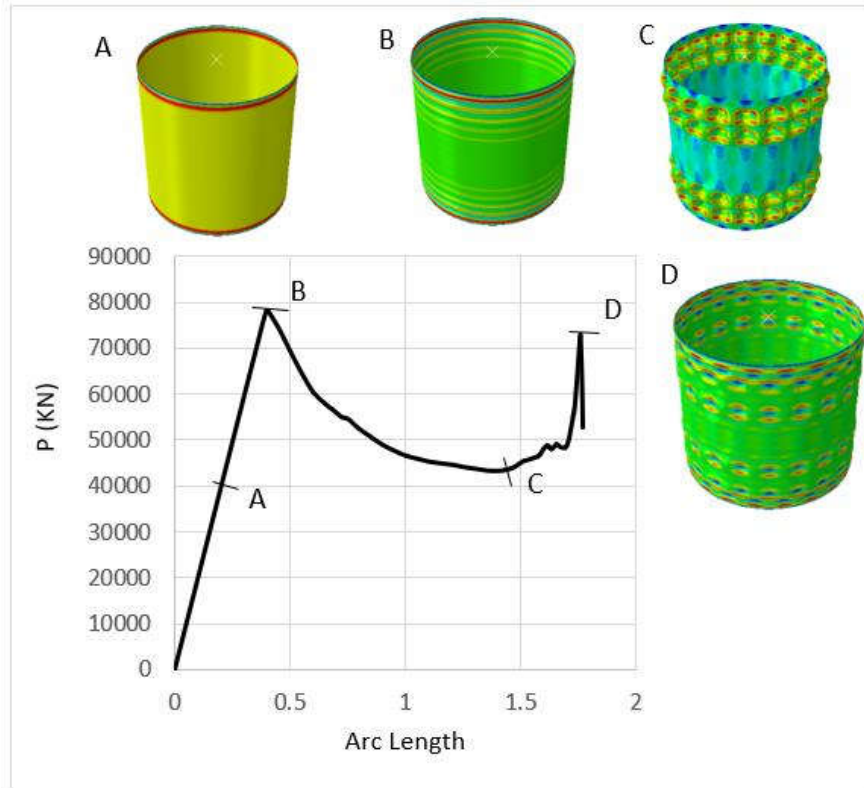


Figure 36 The variation of the axial load with the arch length for the perfect shell

The axial load-end shortening curve is also shown in Figure 37 with some selected deformation forms at some points throughout the curve. From Figures 36 and 37, one can consider the maximum load based on geometric nonlinear analysis of the perfect shell is at point B. This load might control the capacity of the perfect shell discussed in the next section.

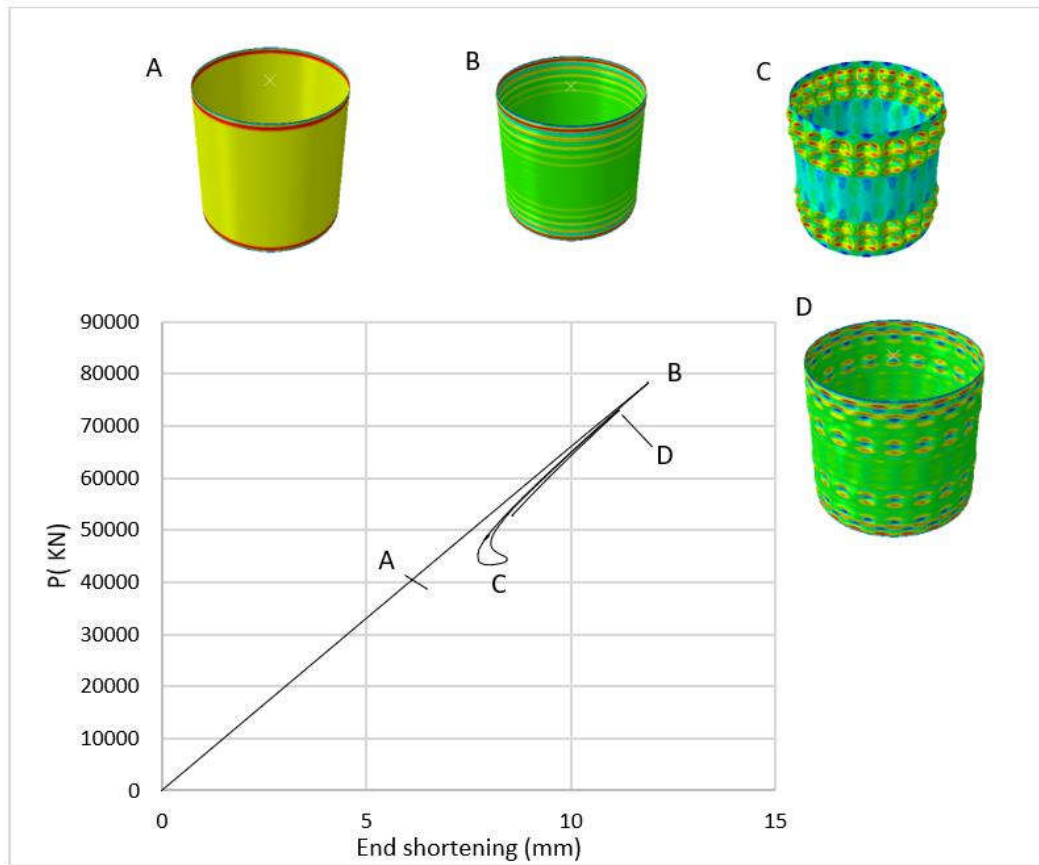


Figure 37 The axial load-end shortening curve for the perfect shell

5.5 The Capacity Curves of Perfect Shells

The literature proved that perfect shells cannot be produced in real construction practice. This is the main reason why most shells fail far below the classical buckling stress when tested in the lab, despite the fact that the small size of the lab specimen offers a better opportunity of having a good quality of the prepared specimen. Other reasons of the strength reduction can be related to geometric and material nonlinearity.

To safely design a shell, a FE analysis considering geometric and material nonlinearity and all possible forms of imperfections is recommended. The problem, however, poses a

challenging task for average skill designers. It has also been found, in the literature, that the imperfections cannot be generalized so that a research group can come with up a certain formula or curve that relieve the burdens of running FE analysis for every shell to be designed.

The key factors responsible for the strength reduction are: (a) the imperfections and (b) the geometric and material nonlinearity. The literature and the current study, as will be shown in the next chapter, proved that the former one cannot be generalized. So, it will be of practical importance to generalize the later and provide the shell designers with the upper limits of shell buckling capacity.

It has been found that perfect shells with similar L/R and R/t ratios have the same buckling stress. It has also been found that the effect of L/R is small compared to R/t . Consequently, only R/t and the grades of steel are considered here for generating the capacity curves of the perfect shells.

The four standard grades of structural steel shown in Table 5 are considered with a wide range of R/t from 25 to 2000 (from a thick tube to thin shells). A geometric and material nonlinear analysis were performed for different combinations of steel grades and R/t . The results obtained proved that there is an interaction between the elastic and plastic buckling at intermediate values of R/t .

In the results of the steel grade S 275 shown in Figure 38, for example, if the shell is having a large R/t ratio (very thin), it will buckle far below the yield limit (275 MPa) and the proposed simplified non-axisymmetric buckling formula (Equation 3.18) controls the buckling strength. On the contrary, if the shell is having a small R/t ratio, the elastic

buckling formula overestimates the buckling stress and the material yielding will dominate the load carrying capacity of the shell. There is a range of R/t where neither the elastic non-axisymmetric buckling nor the yield will control. This is governed by the geometric and material nonlinear analysis (GMNA) for the perfect shells.

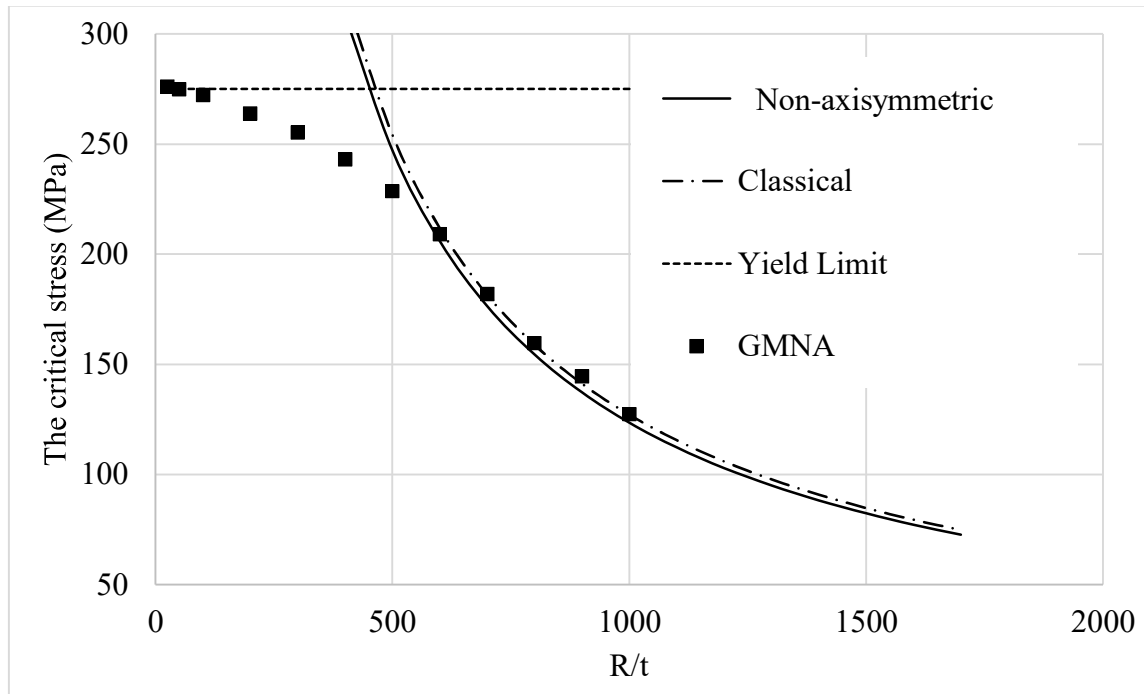


Figure 38 Variation of the critical stress with different R/t ratios of perfect shells

Removing the non-controlling parts of the curves from Figure 38, the capacity curve can be given as shown in shown in Figure 39. The part of the GMNA can be fitted by

$$\sigma_{max} = f_y - \beta \left(\frac{R}{t} \right)^\alpha \quad (5.4)$$

Using Mathematica, the values of β and α corresponding to the best fit were found and the updated form of Equation 5.4 becomes

$$\sigma_{max} = 275 - 0.00101 \left(\frac{R}{t} \right)^{1.73114} \text{ for } R/t \leq 600 \quad (5.5)$$

Comparison of the of the proposed formula given by Equation 5.5 with the discrete data of GMNA is also shown in Figure 39 where good fitting is noticed.

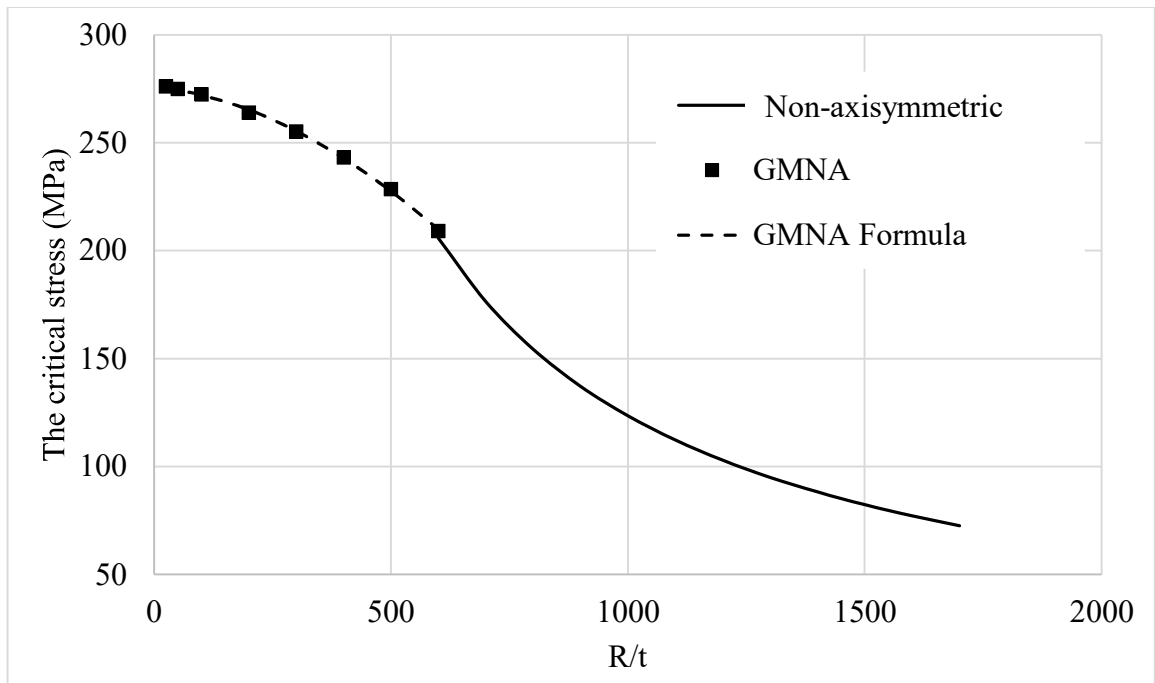


Figure 39 The capacity curve of the perfect shell made of steel grade S 275

Thus, the final form of the capacity curve for a shell made of the standard structural grade S 275 can be approximated by

$$\sigma_{max} = \begin{cases} 275 - 0.00101 \left(\frac{R}{t}\right)^{1.73114} & \frac{R}{t} \leq 600 \\ \frac{0.603}{R/t} E (L/R)^{-0.043} & \frac{R}{t} > 600 \end{cases} \quad (5.6)$$

Following the same approach for shells made with other steel grades defined in Table 5 yields the capacity curves shown in Figure 40. The simplified equations approximating the capacity curves of the perfect shells made with all considered steel grades are given in Table 6.

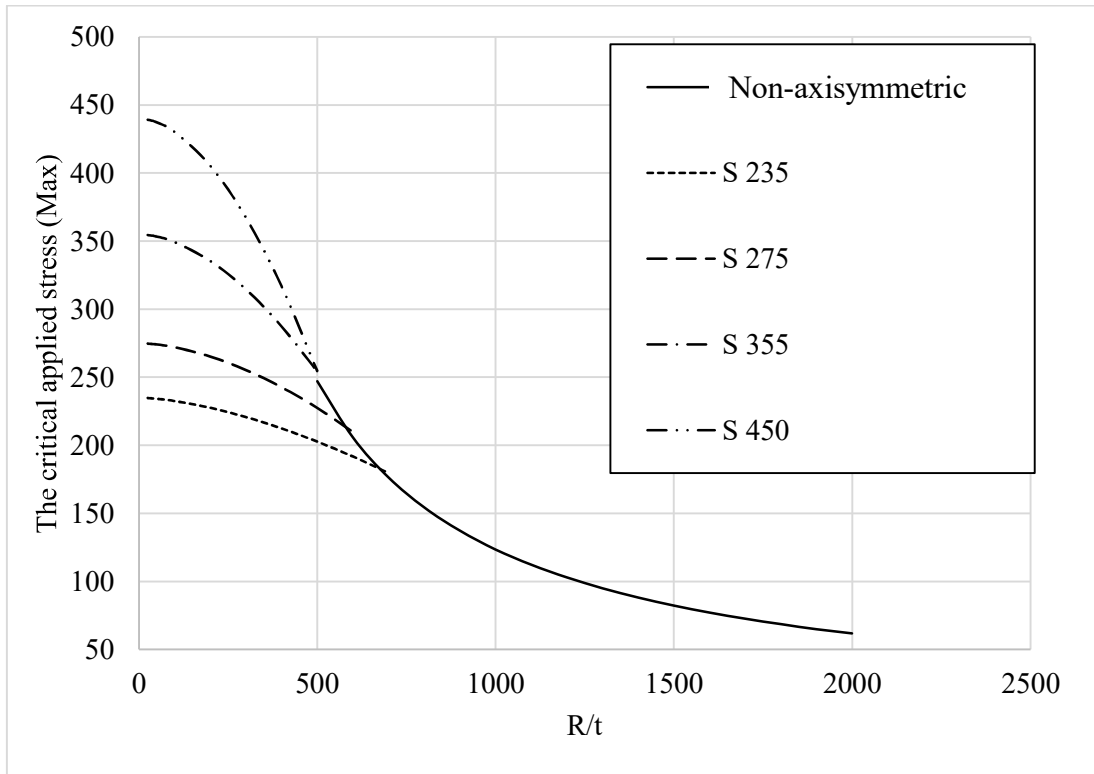


Figure 40 The capacity curves of the perfect shell made of different steel grades

Table 6 Buckling capacity equations of perfect circular cylindrical shells

Steel grade	Formula (MPa)	Equation
S 235	$\sigma_{max} = \begin{cases} 235 - 0.001544 \left(\frac{R}{t}\right)^{1.60059} & \frac{R}{t} \leq 700 \\ \frac{0.603}{R/t} E (L/R)^{-0.043} & \frac{R}{t} > 700 \end{cases}$	5.7
S 275	$\sigma_{max} = \begin{cases} 275 - 0.00101 \left(\frac{R}{t}\right)^{1.73114} & \frac{R}{t} \leq 600 \\ \frac{0.603}{R/t} E (L/R)^{-0.043} & \frac{R}{t} > 600 \end{cases}$	5.6
S 355	$\sigma_{max} = \begin{cases} 355 - 0.001468 \left(\frac{R}{t}\right)^{1.79203} & \frac{R}{t} \leq 500 \\ \frac{0.603}{R/t} E (L/R)^{-0.043} & \frac{R}{t} > 500 \end{cases}$	5.8
S 450	$\sigma_{max} = \begin{cases} 440 - 0.002068 \left(\frac{R}{t}\right)^{1.83497} & \frac{R}{t} \leq 500 \\ \frac{0.603}{R/t} E (L/R)^{-0.043} & \frac{R}{t} > 500 \end{cases}$	5.9

Finally, it should be noted that manufacturers usually produce steel of strength more than the nominal values shown in Table 5. For design purposes, it is recommended to use the nominal values unlike in research where the actual values are recommended. So, in case the one wants to use a carbon steel material with a yield stress, f_y , not equal to one of the nominal values shown in Table 5, the *generalized capacity equation* of the perfect shell can be used, which is expressed by the lesser of the following two expressions:

$$\sigma_{max} = f_y - \beta \left(\frac{R}{t} \right)^\alpha \quad (5.10)$$

$$\sigma_{max} = \frac{0.603}{R/t} E (L/R)^{-0.043} \quad (5.11)$$

where

$$\beta = 5 \times 10^{-8} \times f_y^2 - 3 \times 10^{-5} \times f_y + 0.0057$$

$$\alpha = -7 \times 10^{-6} \times f_y^2 + 0.0056 \times f_y + 0.6669$$

The above expression of β and α were found by fitting their corresponding values for different steel grades shown in Table 6.

To conclude, a FE modeling of the buckling of perfect cylindrical shell was explained. Simplified capacity curves of the perfect shells were also presented for different grades of structural carbon steel. Due to the absence of the imperfections, these curves represent the upper limits of the capacity of the shells to be designed. Design engineers who cannot afford carrying out the FE analysis of the imperfect shells can use these capacity curves with proper knock down factors.

CHAPTER 6

FE MODELING OF GEOMETRIC AND MATERIAL

NONLINEAR BUCKLING OF IMPERFECT

CYLINDRICAL SHELLS

6.1 General

The maximum load in the axial load-end shortening curve is highly sensitive to imperfections. Since this maximum load is used to determine the design load of the shell [6], considering different possible forms of imperfections is a vital step towards accurate prediction of the design load.

There are different forms of imperfections that can negatively affect the capacity of the shell. Some of these forms are:

- 1- Irregularity in the shape of the circular cylinder, which includes axisymmetric imperfections or non-axisymmetric ones like the out-of-roundness.
- 2- Localized damage.
- 3- Weld depression.
- 4- The presence of nozzles.
- 5- Non-uniform thickness.
- 6- Non-uniform axial compression or eccentric load.

In this chapter, geometric and material nonlinear with imperfection analysis (GMNIA) is outlined, which is the best type of analysis to simulate the real-life shells. The modeling process of two types of imperfections is explained. These are the out-of-roundness in the shape of the cylinder and the presence of localized damage, in the form of a dent, on the shell surface.

6.2 The Load Ratio

Before discussing the details of the modeling process of imperfect cylindrical shells, let us now define a term that will be recalled frequently in the rest of this dissertation. That is the load ratio, $\bar{P}$, which is defined by:

$$\bar{P} = \frac{P_{max}}{P_{cp}} \quad (6.1)$$

where P_{max} is the minimum of: a) the maximum load in the load-end shortening curve as shown in Figure 41 or 16, b) the bifurcation load (Figure 16). P_{cp} is the load carrying capacity of the perfect shell, which can be calculated by the multiplication of the maximum critical stress according to the capacity curves proposed in Section 5.5 by the edge area. i. e:

$$P_{cp} = \sigma_{max} (2\pi R t) \quad (6.2)$$

where σ_{max} is to be calculated as in Table 6 depending on the grade of steel or the generalized capacity curves (Equations 10 and 11).

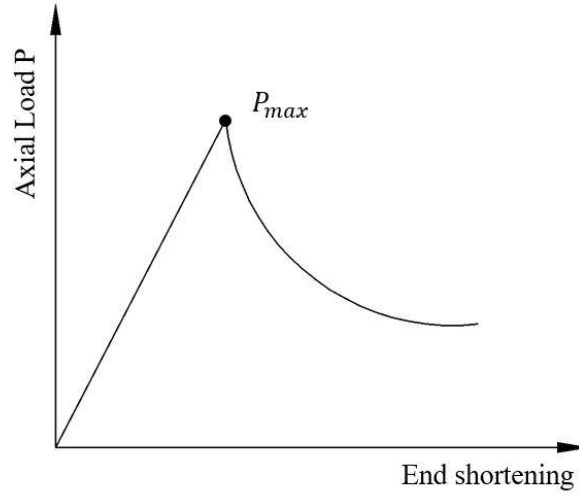


Figure 41 A typical axial load-end shortening curve

6.3 The Selected Forms of Imperfections

6.3.1 The Out-of-roundness Imperfections

The out-of-roundness is one form of the irregularities in the shape of the circular cylindrical shell. This form of the imperfections (Figure 42) is adopted by the European Standards EN 2007: 1-6 [6]. A parameter U_r is used to define the out-of-roundness and given by

$$U_r = \frac{d_{max} - d_{min}}{d_{nom}} \quad (6.3)$$

which should be less than the recommended tolerable limits shown in Table 7 for different classes of fabrication.

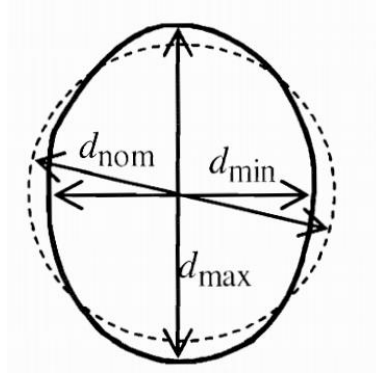


Figure 42 The out-of-roundness imperfections [6]

Table 7 The tolerable limits of the out-of-roundness imperfections [6].

Fabrication tolerance quality class	Diameter	$D \leq 0.50 \text{ m}$	$0.50 \text{ m} < D < 1.25 \text{ m}$	$D \geq 1.25 \text{ m}$
	Description	$U_{r,max}$		
Class A	Excellent	0.014	$0.007 + 0.0093 (1.25 - D)$	0.007
Class B	High	0.020	$0.010 + 0.0133 (1.25 - D)$	0.010
Class C	Normal	0.030	$0.015 + 0.0200 (1.25 - D)$	0.015

In this study, the tolerable limits are tested for two different possible forms of the out-of-roundness:

- 1- **Form A:** The out-of-roundness is in the shape of an ovality, where the shape of the cylinder is defined by extrusion of an ellipse along the longitudinal axis of the cylinder.

- 2- **Form B:** The out-of-roundness is in the shape of a partial ovality, where the shape of the cylinder has a circular cross-section at the top and bottom and an elliptical cross-section at the middle. This type takes into account the high level of accuracy of obtaining a circular cross-section at the base and the top because of the ground work at the base and the stiff cover at the top.

The two selected forms of imperfections (Form A and B) are sketched as shown in Figure 43.

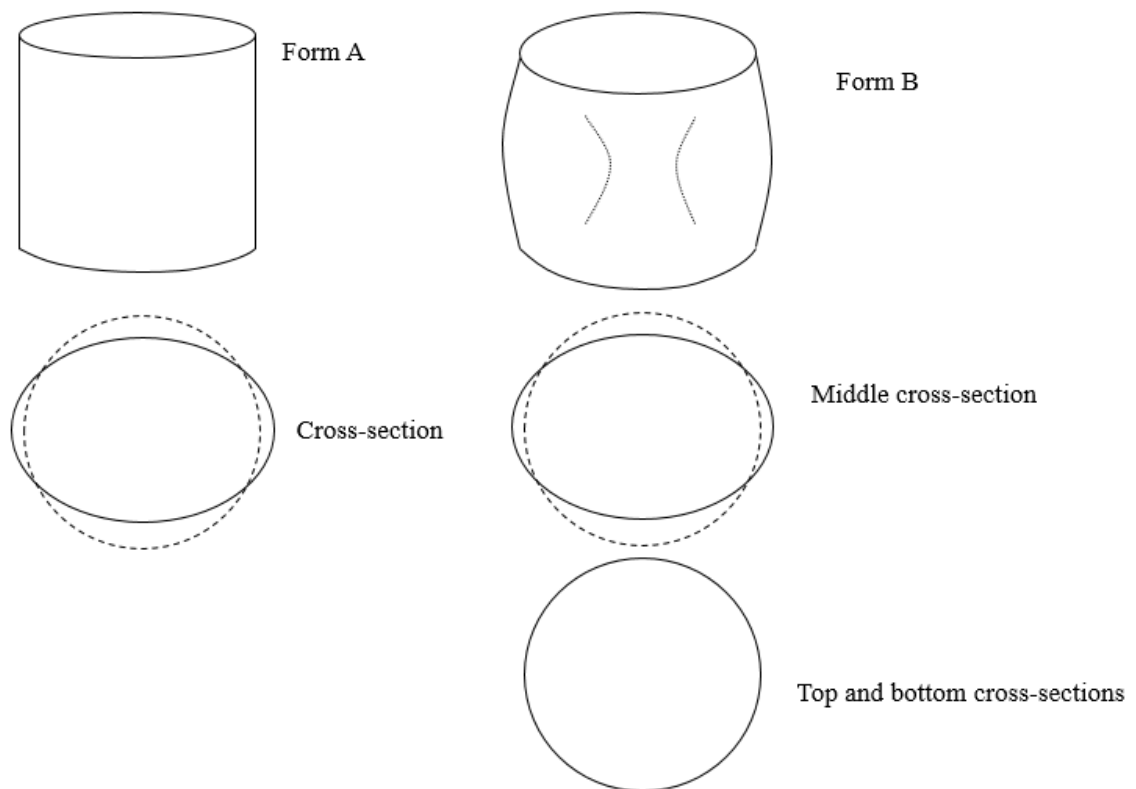


Figure 43 Sketches for the selected forms of out-of-roundness and their cross-sections

6.3.2 The Localized Dent Imperfections

The presence of local damage in the form of a dent has also been addressed in this chapter.

This form of damage is commonly found in real-life cylindrical shells as in Figure 44.



Figure 44 A tank with local damage [84]

The following mathematical expression is adopted here to reflect the damage in the shell surface

$$w_d = \delta_d e^{-\left(\frac{4x_{dc}}{L_{dc}}\right)^2} e^{-\left(\frac{4x_{dl}}{L_{dl}}\right)^2} \quad (6.4)$$

$$\text{for } -\frac{L_{dc}}{2} \leq x_{dc} \leq \frac{L_{dc}}{2} \text{ and } -\frac{L_{dl}}{2} \leq x_{dl} \leq \frac{L_{dl}}{2}$$

Where w_d is the dent imperfections; δ_d is the amplitude of the dent; x_{dc} , x_{dl} , are the local coordinates of the dent along the circumferential and longitudinal directions, respectively, with origin at the center of the dent; L_{dc} , L_{dl} are the length of the dent in the circumferential and longitudinal directions, respectively.

Modeling of dent or out-of-roundness imperfections in the finite element software ABAQUS requires, as a priori, generating a shell geometry to reflect the shape of the imperfections considered.

6.4 Generation of the Imperfect Shell Geometry

To generate an imperfect shell geometry in ABAQUS, a perfect shell model was first performed as discussed in the previous chapter. Then, the geometric imperfections were imposed through replacement of the finite element nodes coordinates, available in the input file of the perfect shell model, by a newly created ones with consideration of the geometric imperfections. The new nodes coordinates of the imperfect shell were created with appropriate mathematical expressions using Wolfram Mathematica software [7] for coding and automating the process as shown in Appendix C. Then, the generated nodes coordinates were imported to ABAQUS by means of a suitable text editor (NOTEPAD).

This process requires, as priori construction, a clear idea about the default coordinate system and labeling of the finite element nodes in ABAQUS. Figure 45 shows the coordinate system and default labeling of the finite element nodes adopted in ABAQUS. It can easily be noted that the labeling starts from the bottom and rounds counter clockwise. This can easily be obtained without the burden of transferring to the cylindrical coordinate system. The coordinates (x, y, z) of the node at an angle $\Delta\theta$ i and a position along y equal to ΔL j can be computed by

$$(R \cos(\Delta\theta \ i), \Delta L (j - 1), -R \sin(\Delta\theta \ i)), \text{ for } i, 1, \dots, N_c, j, 1, \dots, N_t + 1$$

Where $\Delta\theta$ is equal to $2\pi/N_c$ and $\Delta L = L/N_l$; and N_c and N_l are the number of divisions along the circumferential and longitudinal directions, respectively.

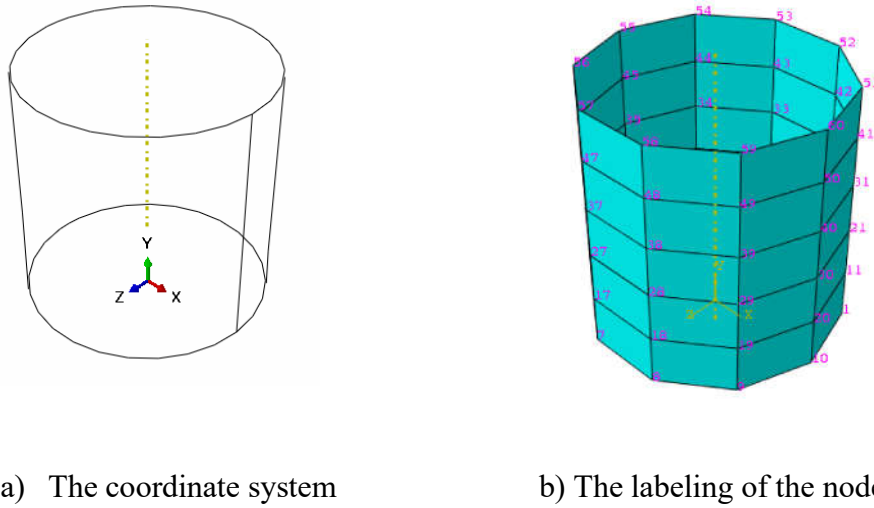


Figure 45 The coordinate system and labeling of the FE nodes in ABAQUS

It is noteworthy that there are two options in ABAQUS to generate the cylinder either using Extrusion or Revolution. The two options may result in two different coordinate systems. Revolution option has been adopted in this work to have the option of having different mesh in the circumferential and longitudinal directions.

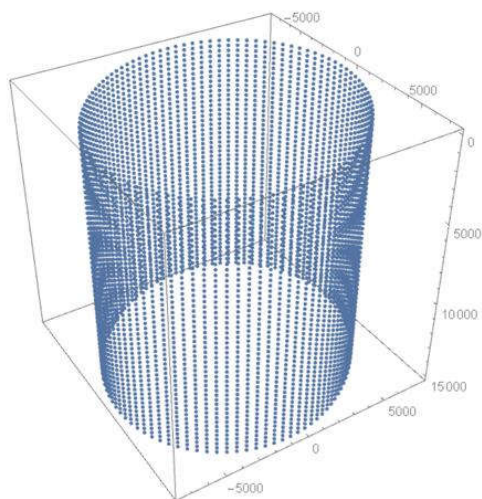
6.5 Modeling of the Shell with Out-of-roundness Imperfections

In this section, both forms of out-of-roundness imperfections discussed in the foregoing were modeled for a selected value of U_r . The purpose of this section is to explain an easy approach to model the out-of-roundness imperfections.

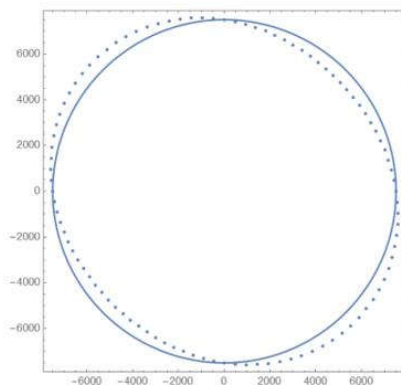
To explain the modeling process, the perfect shell discussed in Section 5.4 is used here with two different forms of out-of-roundness (Form A and Form B as defined in Section 6.2.1). The degree of the out-of-roundness U_r is taken to be 0.015 corresponding to the fabrication class C as in Table 7.

As discussed in Section 6.4, the nodes of the imperfect shell geometry were generated using a short Mathematica code (Appendix C) and were used to replace the FE nodes of the perfect shell in the input file. In doing so, the input file of the perfect shell is no longer valid for a perfect one and is transformed to an input file of an imperfect shell model.

For the sake of illustration, a coarse mesh of the perfect shell geometry is used for generation of the imperfect shell geometries of both forms of out-of-roundness. The nodes of the imperfect shell were generated in Mathematica (Appendix C) for both forms of out-of-roundness. The imperfect shell nodes for Form A are shown in Figure 46, while the eventual geometry in ABAQUS is shown in Figure 47. For Form B out-of-roundness, the imperfect shell nodes are shown in Figure 48, which shows that the geometry is of a perfect circular cross-section at the top and bottom boundaries while the imperfection gets higher amplitude as going towards the middle of the shell. The final geometry generated in ABAQUS is shown in Figure 49.

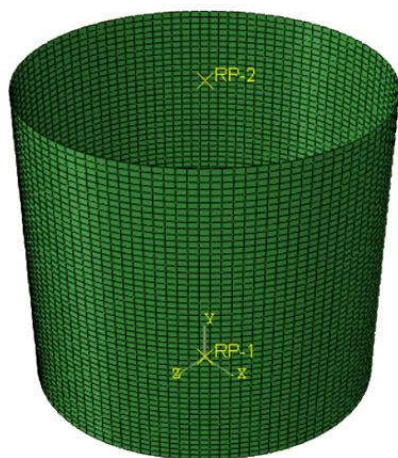


a) 3D

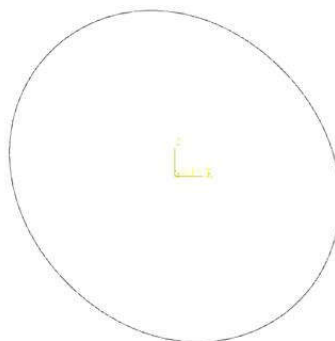


b) 2D

Figure 46 The nodes of the shell with out-of-roundness Form A ($\times 5$)



a) 3D



b) 2D

Figure 47 The imperfect shell geometry with out-of-roundness Form A ($\times 5$)

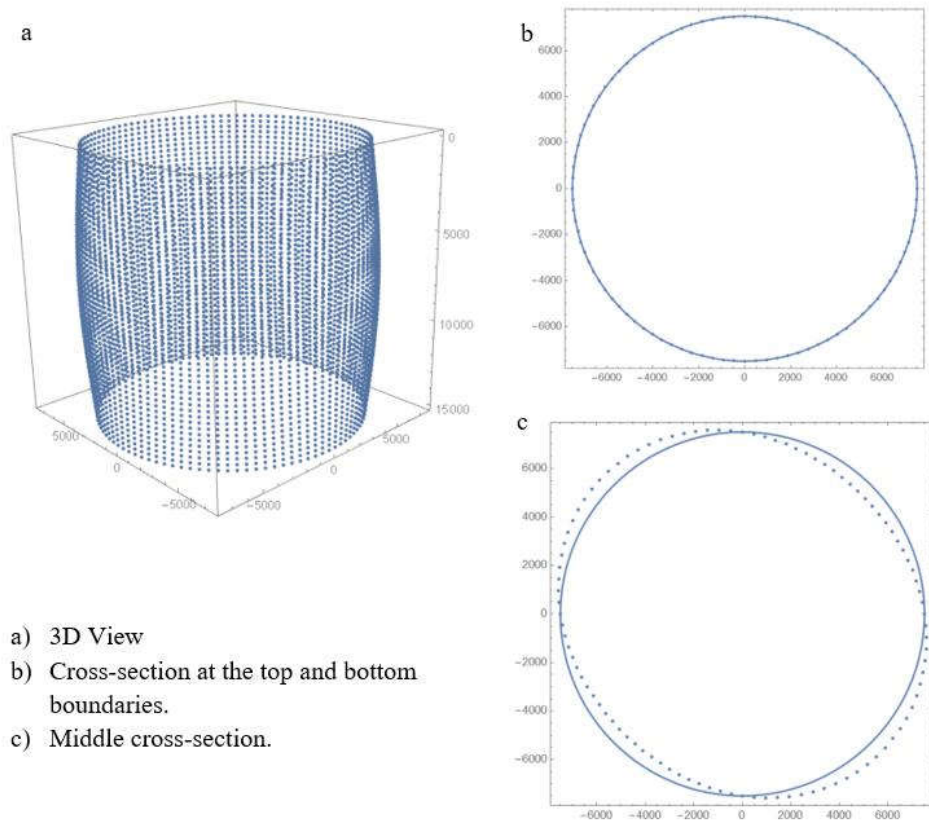


Figure 48 The nodes of the shell with out-of-roundness Form B ($\times 5$)

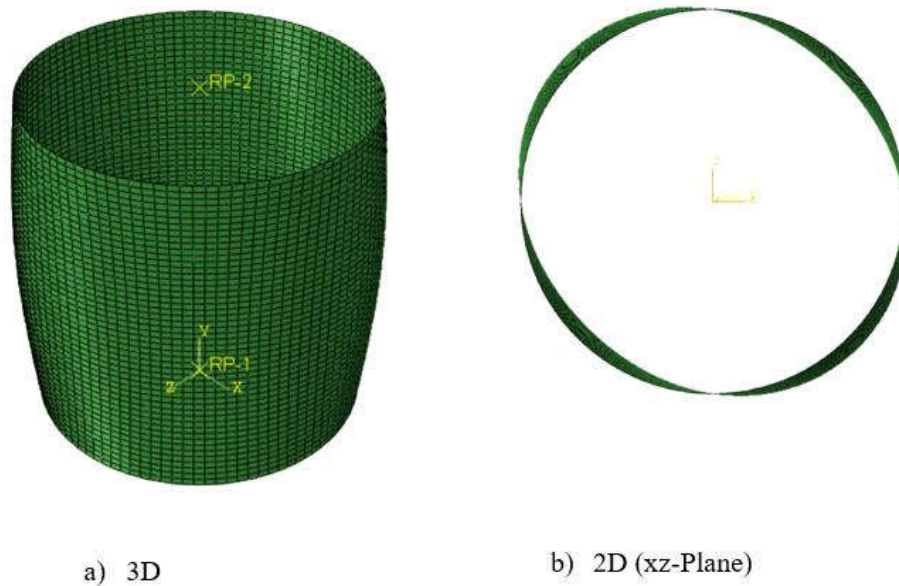


Figure 49 The imperfect shell geometry with out-of-roundness Form B ($\times 5$)

Running the prepared imperfect shell models were then performed and the results were extracted. For Form A, the maximum load is found to be 75726 KN while maximum load the perfect shell can carry is $P_{cp} = 77\,228$ KN (Equation 6.2). Thus, the load ratio $\bar{P}$ (Equation 6.1) for this form and magnitude of out-of-roundness is 0.981, which means 1.9 % reduction on the buckling strength of the shell. Such slight reduction indicates the tolerable limit for the fabrication class C is adequate and no considerable buckling strength reduction is noticed. For Form B, however, the maximum axial load is found to be 55078 KN leading to a load ratio $\bar{P} = 0.713$ which is 28.7 % reduction on the buckling strength. This indicates that the tolerable limit cannot be adopted for such form of out-of-roundness.

Since this section is oriented to explain the modeling process of a shell with out-of-roundness imperfections, only one value of the out-roundness has been considered. Next, in Chapter 7, a parametric study on the effect of different degrees of out-of-roundness is presented.

6.6 Modeling of the Shell with a Localized Dent Imperfection

The presence of the localized damage severely affects the strength capacity of the shell. To show this, let us consider again the perfect shell discussed in Section 5.4 but with a single dent of a length equal to 1000 mm in the circumferential and longitudinal directions ($L_{dc} = L_{dl} = 1000\text{ mm}$). The amplitude of the dent is taken to be equal to the shell thickness, $\delta_d = 10\text{ mm}$, leading to dent amplitude to radius ratio equal to $1/750$.

As discussed in Section 6.4, the nodes of the imperfect shell geometry were generated in Mathematica (Appendix C) with the adoption of Equation 6.4 representing the dent

geometry. Then, the nodes of the perfect shell, in the input file, were replaced with the FE nodes of the imperfect shell to have the input file of the shell with the required dent imperfection.

With a relatively course mesh, the generated nodes in Mathematica are shown in Figure 50, while the final imperfect geometry in ABAQUS is shown in Figure 51.

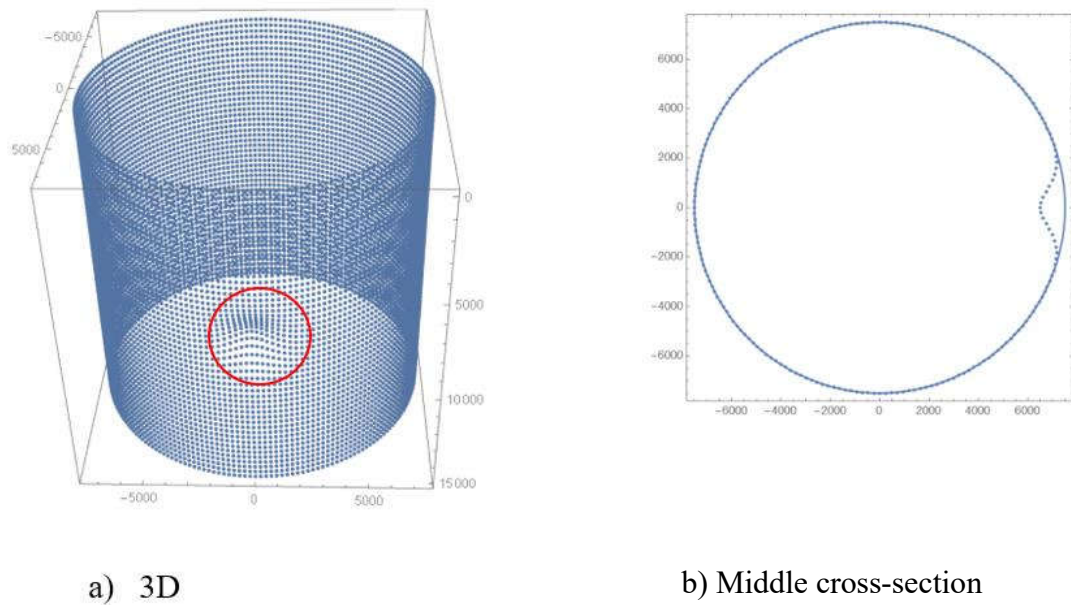


Figure 50 The nodes of the shell with a dent at the middle ($\times 100$ *Out of plane* and $\times 5$ *in plane*)

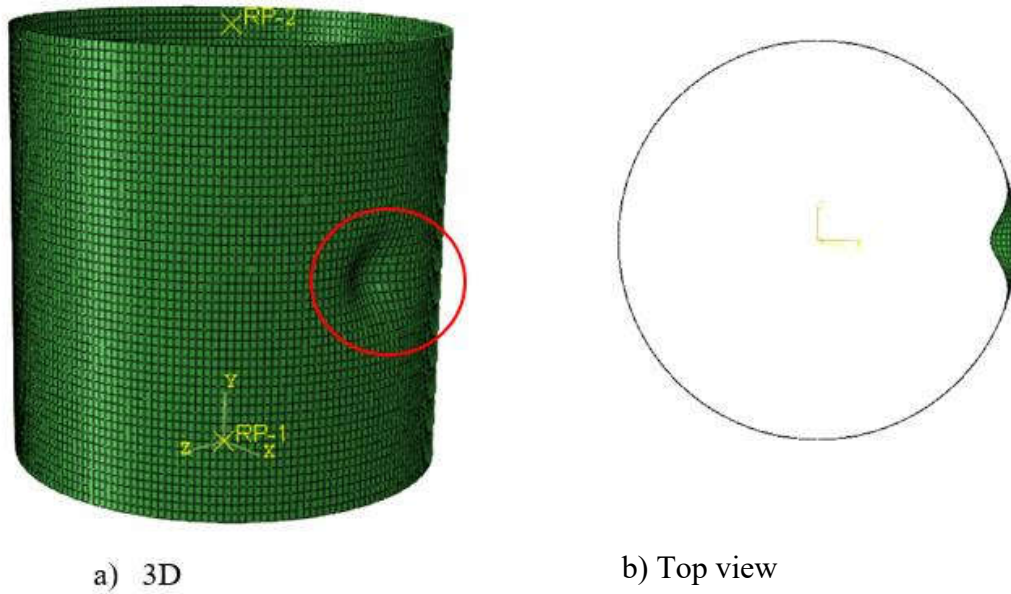


Figure 51 The final geometry of the shell with a dent at the middle ($\times 100$ *Out of plane* and $\times 5$ *in plane*)

Running the prepared input file of the imperfect shell was then performed and the load-end shortening curve is shown Figure 52. The curve is bifurcated at point A and the shell buckling capacity reduces to point B before increasing again to point C. In this form of load-end shortening curve, the bifurcation load is more important in design and, as defined by Equation 6.1, P_{max} takes the value of the load at point A which is 42630 KN. Thus, the load ratio $\bar{P}$ resulting from this very small dent of one shell thickness amplitude is 0.55. This means that as high as 45 % reduction in the shell buckling capacity is due to the effect of a single dent. A detailed discussion and a parametric study on the influence of the amplitude and size of the dent on the shell buckling capacity is given in the next chapter.

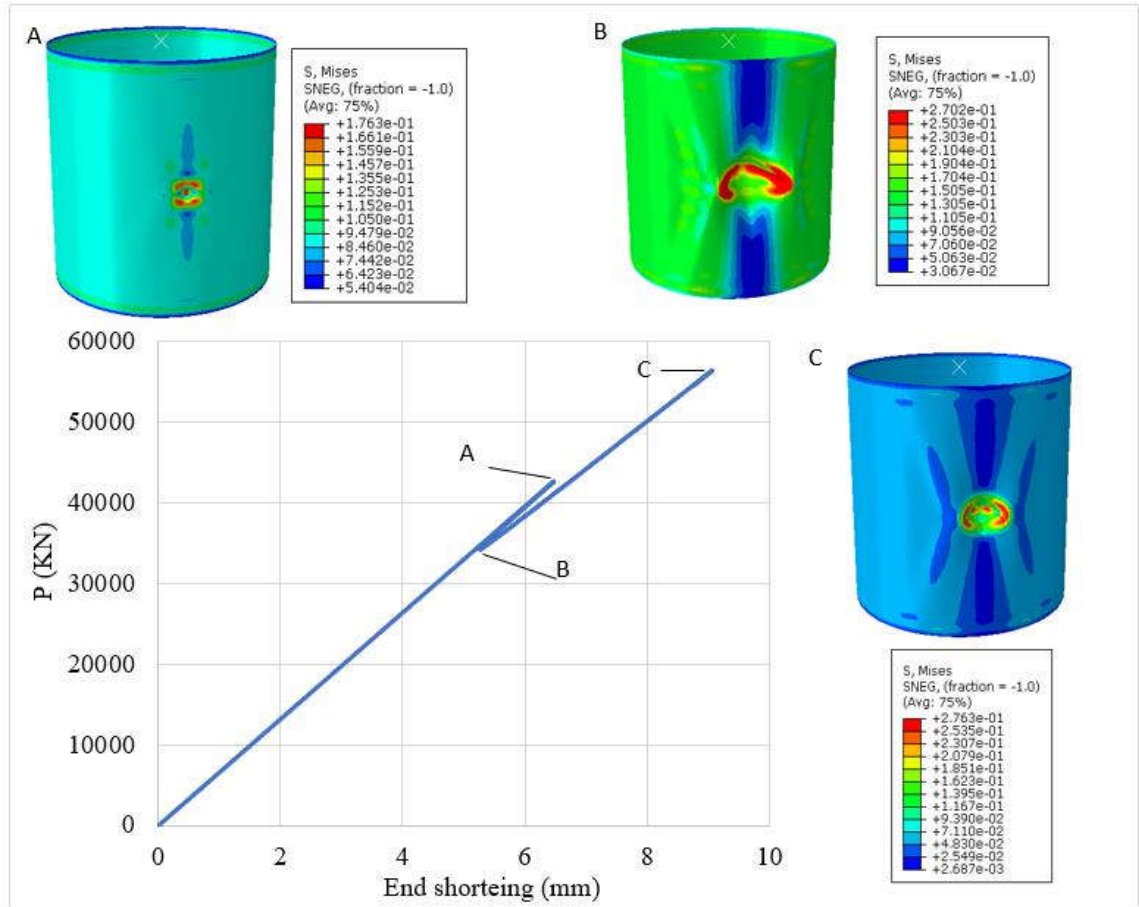


Figure 52 The axial load-end shortening curve and the von Misses stresses (in GPa) for the shell with dent

The variation of von Mises stresses on the deformed shell surface is shown in Figure 52. A very low stresses can be noted on the strip above and below the location of the dent, which means that the presence of the dent reduces the in-plane load carrying capacity on a longitudinal strip passing through the dent due to the excessive out of plane deformation and, consequently, reduces the load carrying capacity of the whole shell.

To conclude, general guidelines for addressing different form of geometric imperfections using the nodes translation techniques are presented. Using the prescribed procedure, two different forms of out-of-roundness imperfection have been tested. The dent imperfection

has also been studied in this chapter and it has been proved that there is a significant buckling strength reduction due to the presence of a localized dent. In the following chapter, more information will be provided on the effect of out-of-roundness and localized dent imperfections through parametric studies.

CHAPTER 7

PARAMETRIC STUDIES ON THE EFFECTS OF IMPERFECTIONS ON THE BUCKLING STRENGTH

7.1 General

In this chapter, two main different parametric studies are presented. The first one oriented to study effect of the degree out-of-roundness imperfections on the shell buckling strength while the second one is devoted to study the effect of different dent imperfection parameters on the buckling strength of the shell.

7.2 The Effect of the Out-of-roundness Imperfections

The purpose of this section is to investigate (1) the effect of the degree of out-of-roundness U_r on the buckling strength of the shell and (2) the effect of partial out-of-roundness (Form B) as compared to the cylindrical out-of-roundness (Form A).

Six levels for the out-of-roundness parameter U_r were selected including the corresponding to the three fabrication classes (A, B, and C) as shown in Table 8. In the table, the corresponding equivalent value of the amplitude of the out-of-roundness divided by the shell thickness is also provided since it is common in the literature to express the imperfections in terms of their amplitudes.

Table 8 The effect of different values of out-of-roundness U_r

U_r	Amplitude/ t	Standard class [6]	P ($\times 1000$ KN)		Load Ratio, $\bar{P}$		Strength reduction (%)	
			From A	Form B	From A	Form B	From A	Form B
0.00267	2	-	77.976	75.725	1.010	0.981	-1.0	1.9
0.00533	4	-	78.814	76.275	1.021	0.988	-2.1	1.2
0.007	5.25	A	77.048	74.883	0.998	0.970	0.2	3.0
0.01	7.5	B	76.978	67.893	0.997	0.879	0.3	12.1
0.015	11.25	C	75.726	55.078	0.981	0.713	1.9	28.7
0.03	22.5	-	71.24	37.655	0.922	0.488	7.8	51.2

Preparing the imperfect shell models and performing the analysis, as discussed in Section 6.5, yielded the results tabulated in Table 8 and shown also in Figure 53. The table and the figure proved that the buckling strength reduces with the increase of the magnitude of out-of-roundness parameter U_r . They also show that Form A out-of-roundness has a little effect on the buckling strength as compared to Form B. The tolerable limits of out-roundness imperfections for the three fabrication classes (A, B, and C) give very little strength reduction for case of Form A out-of-roundness, that is about 1.9 % for the worst fabrication class (Class C). For case of Form B, however, the tolerable limits lead to a considerable reduction in the shell capacity, which restricts the use of the defined tolerable limits and proves the needs for further refinement of the tolerable limits in case of such form of out-of-roundness.

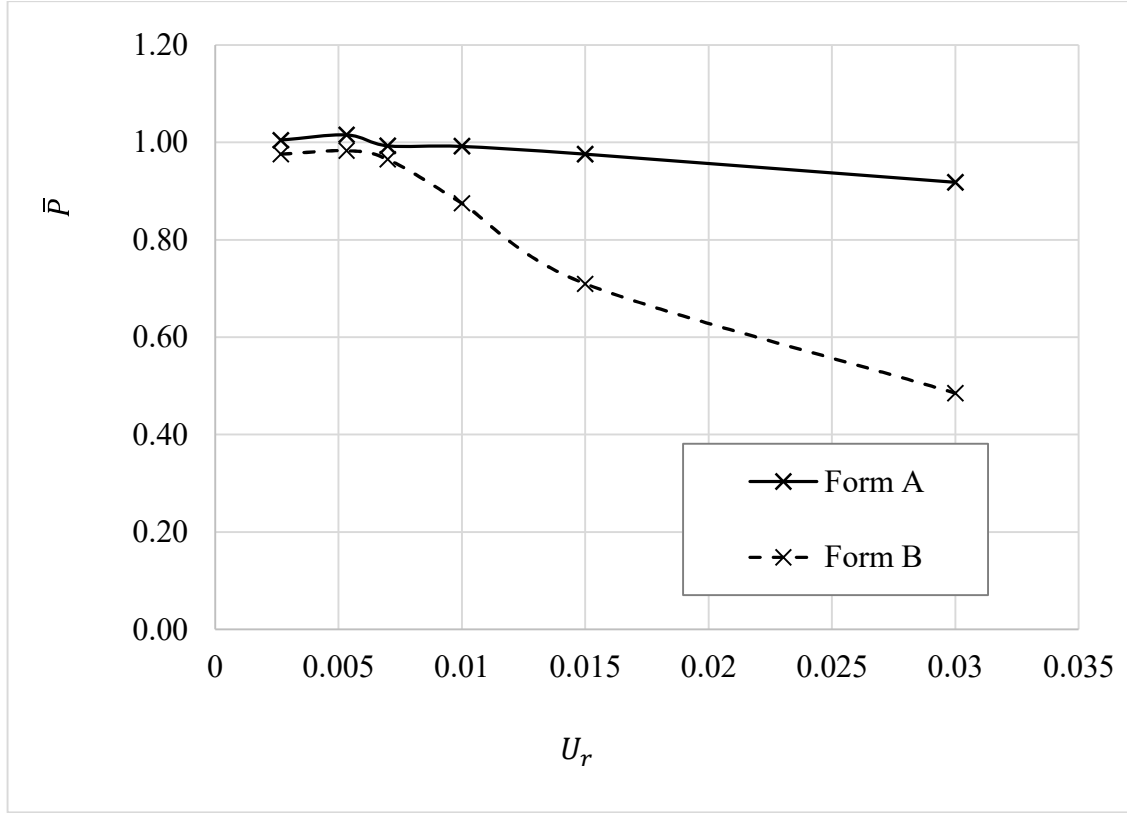


Figure 53 The effect of different values of out-of-roundness U_r .

To conclude, the degree of out-of-roundness has been tested for two different forms of out-of-roundness imperfection. The first one its surface defined by extrusion of an ellipse along a straight line and the other has circular cross-sections at the bottom and top and elliptical cross-section as moving towards the middle of the shell. The former one results in a maximum strength reduction of 1.9 % for the worst fabrication class while the later affect the buckling strength significantly, which imply that the code tolerable limits of the out-of-roundness are not valid for such possible form of out-roundness.

7.3 The Effect of the Dent Imperfections

The aim of this section is to study the effect of the localized dent imperfections for the case of small and large R/t ratios. In the literature, there is a comprehensive parametric study [81] in which different parameters of the dent (inclination, size, and amplitude) have been investigated for case of small R/t ratios. Their investigation, however, led to a misleading conclusion, which requires to be examined and corrected as will be shown in the rest of this section. Since there are no reported studies on the effect of the localized dent on shells with large R/t ratios, a parametric study is also carried out for large R/t ratios. The dent parameters considered in this part of study are the: location, size, and amplitude.

7.3.1 Dented Shells with Small R/t Ratios

In Chapter 2, it has been shown that most of the previous studies investigated the effect of localized dent for shells with small R/t ratios. In case the material used is elastic or has a high offset yield stress such as that considered by Gavrilenko and Krasovskii [77], the effect of the localized dent on the buckling strength reduction can easily be recognized. However, the same cannot be said for materials of relatively low yield stress as some standards grades of structural carbon steel. In these materials, the plasticity has a higher effect on the buckling strength more than that caused by a localized dent. The severity of the effect of plastic deformation is proportional to the decrease in the R/t ratio.

Although, several studies have addressed the effect of the dent on shells with small R/t ratios, the results of some of these studies are erroneous due to the wrongly defined buckling strength ratio. A typical study [81] falling in this category of studies is selected, critically examined and corrected

A comprehensive parametric study on the effect of localized dent on short carbon steel cylindrical shell was proposed by Parbu et al. [81]. The study included different parameters, namely: the size, amplitude, and angle of inclination of the dent as well as the thickness of the shell. They considered three levels of the shell thickness, but all of them are leading to the ratios of R/t to be 280, 175, and 116.7. As will be seen later, such low R/t values, with the low strength carbon steel considered, will result in more influence of the material nonlinearity and a negligible effect of the localized dent.

The study assumed a bilinear stress-strain model (Figure 54) of a carbon steel material with yield stress $f_y = 240 \text{ MPa}$, elastic modulus $E = 210 \text{ GPa}$, and Poisson's ratio $\nu = 0.3$.

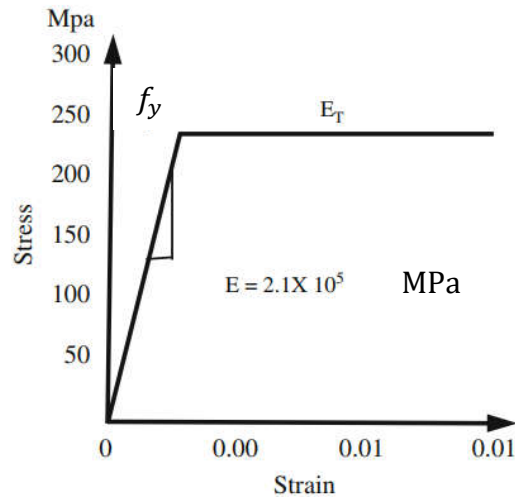


Figure 54 Stress-strain curve of the adopted carbon steel [81]

The dimensions of the shells used are:

Radius (R) = 350 mm

Length (L) = 340 mm

Thickness (t) = 1.25, 2, and 3 mm

The above three thicknesses yield the ratios of R/t to be 280, 175, and 116.7, respectively.

To keep a consistent notation throughout this dissertation, the geometry of the dent adopted by Parbu et al. [81] is re-written here using the notations adopted in the previous chapter, i.e.

$$w_d = \delta_d e^{-\left(\frac{4x_{dc}}{L_{dc}}\right)^2} \cos\left(\frac{x_{dl}}{L_{dl}}\right) \quad (7.1)$$

$$\text{for } -\frac{L_{dc}}{2} \leq x_{dc} \leq \frac{L_{dc}}{2} \text{ and } -\frac{L_{dl}}{2} \leq x_{dl} \leq \frac{L_{dl}}{2}$$

Again, w_d is the dent imperfections; δ_d is the amplitude of the dent; x_{dc} , x_{dl} , are the local coordinates of the dent along the circumferential and longitudinal directions, respectively, with origin at the center of the dent; L_{dc} , L_{dl} are the lengths of the dent in the circumferential and longitudinal directions, respectively.

All the results in Ref. [81] are reported in terms of a newly defined term called the *buckling strength reduction* (BSR) which is defined by the results of the nonlinear analysis of the imperfect shell (GMNIA) divided by the results of the linear non-axisymmetric buckling analysis. For cases of large R/t ratios, the shell undergoes elastic buckling and it is adequate to use the non-axisymmetric linear buckling stress as a reference capacity for the perfect shells as discussed in the capacity curves in Section 5.5. In this case, the BSR is similar to the load ratio $\bar{P}$ (Equation 6.1) defined earlier. For the case of small R/t , however, the capacity of the perfect shell is controlled by the geometric and material nonlinear analysis due to the dominant effect of the material nonlinearity and the results of the linear buckling analysis should not be used as a reference for the perfect shell. The misleading results due

to the use of the defined BSR are examined, in this section, and corrected to reflect the effect of dent imperfections for case of small R/t ratios.

The results of the linear buckling analysis performed in Ref. [81] are first verified here using the exact non-axisymmetric formula (Figure 19) derived in Chapter 3. The comparison given in Table 9 proved that their calculations for the linear buckling of the perfect shell are accurate. It should be noted that the stress appearing in Table 9 is the applied axial stress which is calculated as the axial load divided by the cross-sectional area of the shell ($2\pi Rt$).

Table 9 The results of the linear buckling analysis of perfect shells

t (mm)	R/t	m	n	Analytical (Figure 19)		FE		Error %
				P (KN)	Stress (MPa)	P (KN)[81]	Stress (MPa)	
1.25	280	1	9	1232.89	448.50	1242.98	452.17	0.82
2	175	2	11	3158.88	718.22	3140.87	714.12	0.57
3	116.67	1	7	6990.01	1059.52	7050.99	1068.76	0.87

The main issue, however, in utilizing the linear buckling strength as a reference to evaluate the buckling strength ratio (BSR) of the dented shells with the considered small R/t ratios. The analysis leading to the numerator of the BSR based on nonlinear buckling of imperfect shell (GMNIA), which contains two factors: (a) the nonlinearity in both material and geometry; and (b) the imperfections in the form of a dent. Despite the influence of these two factors, the work of Parbu and his group [81] attributed the whole buckling strength reduction to the second factor (the dent) alone and neglected the influence of the first factor completely. This is not realistic especially for the case of small R/t where the first factor is expected to have more influence especially for the considered low yield strength material.

Having two factors influencing a behavior make it a difficult task to judge on the influence of one of them. So, there is a need to reduce these two factors to one as explained below.

The denominator in the BSR [81] is based on the non-axisymmetric buckling stress as shown in Figure 55, which leads to stresses far higher than the yield limit and the severity increase with the decrease in R/t ratio. Figure 55 shows the capacity curve of a perfect shell as discussed in Section 5.5, which has been generated for the case of $f_y = 240 \text{ MPa}$ using Equations 5.10 and 5.11. It can be noted that the non-axisymmetric buckling is not a possible scenario for the case of small R/t (thick shell) where the geometric and material nonlinear analysis (GMNA) is far less due to dominant effect of material yielding.

In order to correctly identify the sole effect of the localized dent on the reduction in buckling strength, the load ratio $\bar{P}$ defined by Equation 6.1 should be used which yields, for this case of small R/t ratios, the results of *nonlinear* buckling strength of the imperfect (with dent) shell (GMNIA) divided by the *nonlinear* buckling strength of the perfect shell (GMNA). $\bar{P}$ is more general and can be used for both small and large R/t ratios. Its denominator/reference is the non-axisymmetric buckling for cases of large R/t ratios and the results obtained from GMNA for cases of small R/t ratios. Unlike the BSR [81] which, from its definition, converges to $\bar{P}$ for cases of large R/t and diverges for cases of small R/t ratios.

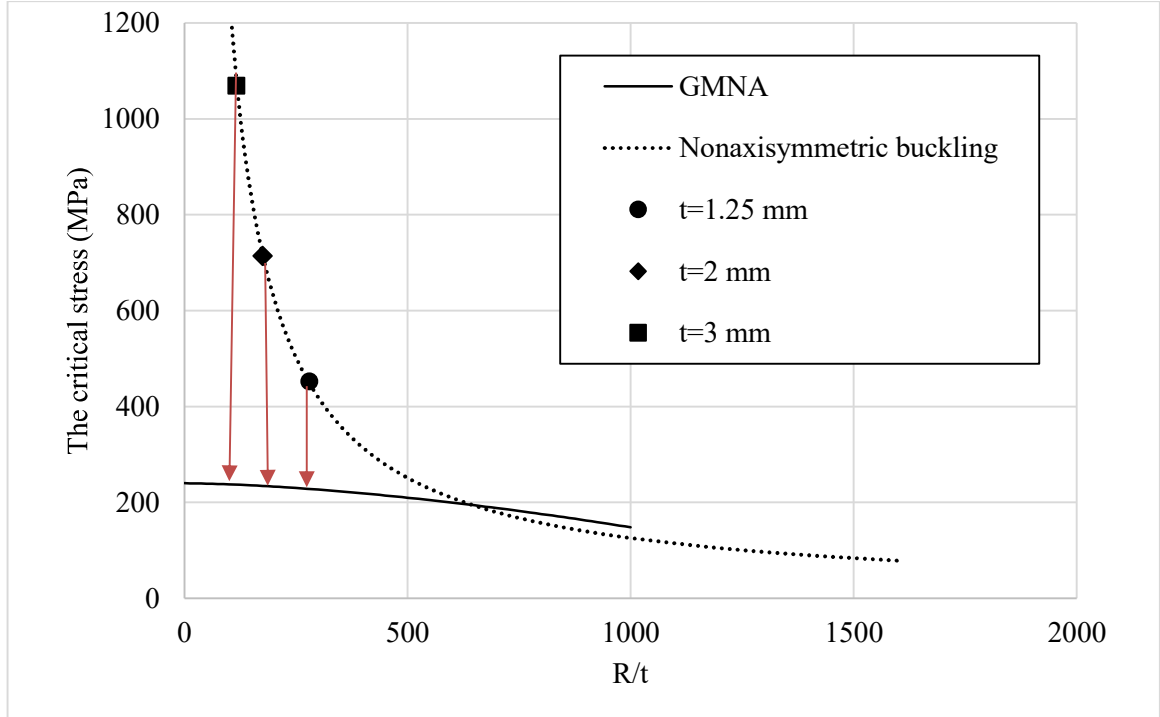


Figure 55 The capacity curve of the perfect shell for $f_y = 240 \text{ MPa}$

To accurately calculate the buckling strength of the perfect shell considering the geometric and material nonlinearity (GMNA), there are two alternatives:

- Using Equations 5.10 and 5.11 as appears in Figures 55.
- Running a GMNA and take the maximum applied axial stress.

Although the first option is easier, but its equations were prepared based on a nonlinear material model considering the strain hardening zone and some of their coefficients were determined based on the best fit. Thus, selecting the first option might lead to a difference as far as the comparison is going to be with results generated based on a bilinear model. Consequently, the second option is adopted hereinafter while the first option was used to generate Figures 55 for the purpose of illustration.

Before running a GMNA for the perfect shells and as it is always preferred in numerical modeling to have a reference for comparison, a selected model from the work Parbu et al. [81] was first modeled to ensure the accuracy of the present modeling and the mesh convergence. The modeled shell had 1.25 mm thickness and a dent of 3.75 amplitude (δ_d), 175 mm along the length of the shell (L_{dl}), and 87.5 along the circumferential direction (L_{dc}).

Although it has been discussed in the previous chapter that a mesh with 300 elements in the circumferential direction and 100 elements in the axial direction is sufficient to model a shell with a localized dent, a mesh convergence study was conducted here again for further clarification. In this study, the number of elements in the axial direction was kept 100 and the number of elements in the circumferential direction was varied as 200, 300 and 400. Taking the maximum applied axial stress to evaluate the mesh, the two courser mesh sizes were found to be in an excellent agreement with a maximum difference of 0.5 % from the finer mesh. Figure 56 shows the agreement among the curves of the results for the three mesh sizes. It can also be seen from the figure that the maximum stress is around 200 MPa, which is less than the yield stress of the material. The mesh size of 400 elements in the circumferential direction and 100 elements in the longitudinal direction was considered hereinafter.

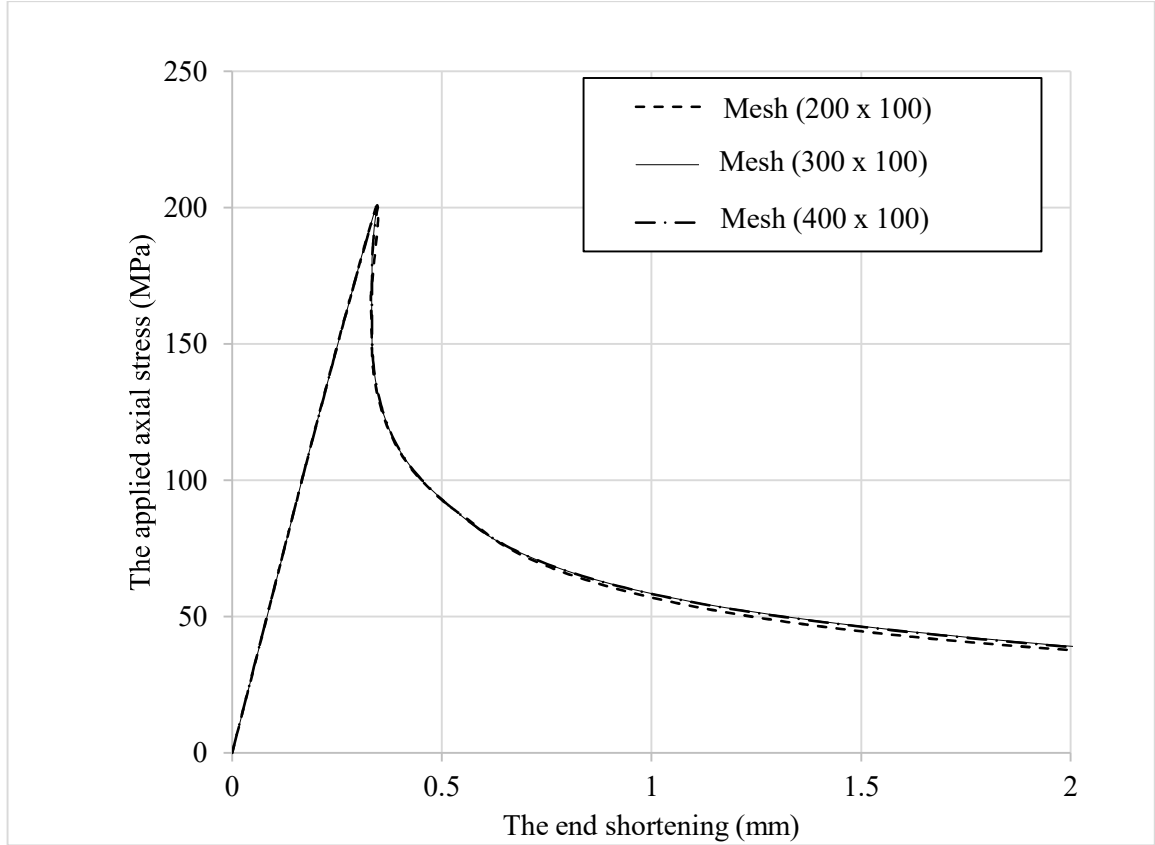


Figure 56 Comparison of the considered three mesh sizes

To compare the present shell model with the work of Parbu et al. [81], the BSR defined by them was used just to have the same reference for the sake of comparison only. Later, the load ratio $\bar{P}$ will be used, which is based on the nonlinear buckling analysis of the perfect shell in the denominator for cases of small R/t ratios. Figure 57 shows the comparison between the present work and that of Parbu et al. [81], the comparison shows a good agreement between the two with a different of 1.6 % in the maximum BSR value.

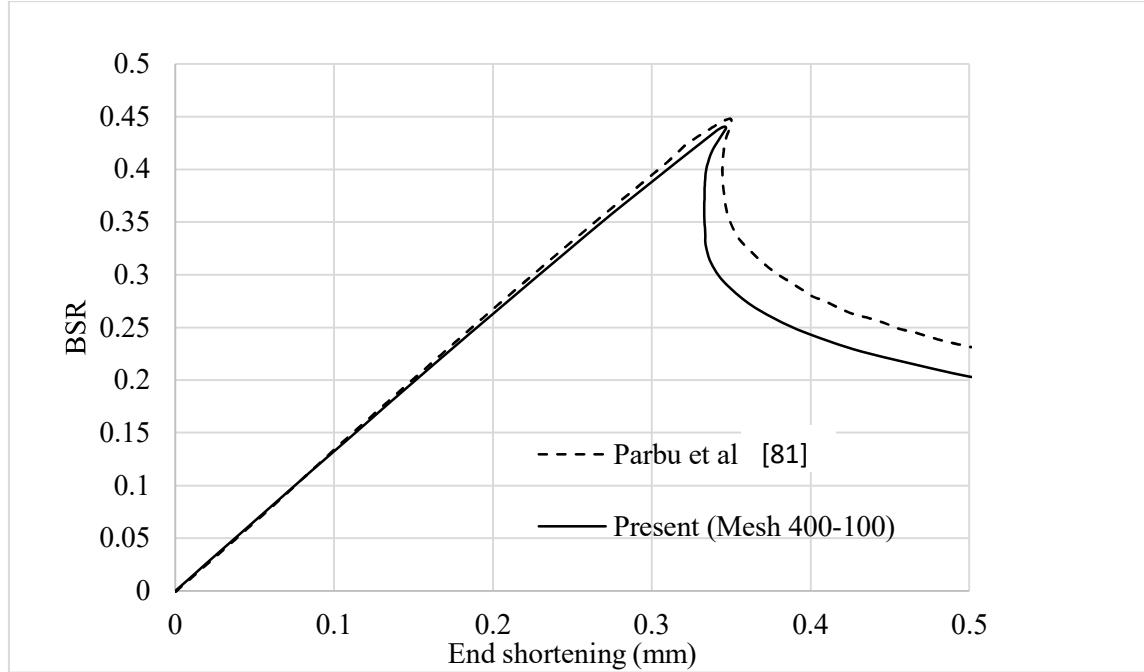


Figure 57 Comparison of the present analysis with the work of Parbu *et al.* [81]

After verification of the present modeling, the buckling strength of the perfect shells based on geometric and material nonlinear analysis is presented next. Figure 58 shows the applied axial stress against the end shortening curve. The precise numerical values are tabulated in Table 10 which shows a great different between the maximum applied stresses used by the Parbu and his group [81] and the ones should be used (based on GMNA). It can also be seen from the table that all the applied axial stresses used as references by Parbu *et al.* [81] are at least twice the yield stress. As discussed in the foregoing, the maximum stress results from GMNA should be used to single out the effect of the localized dent.

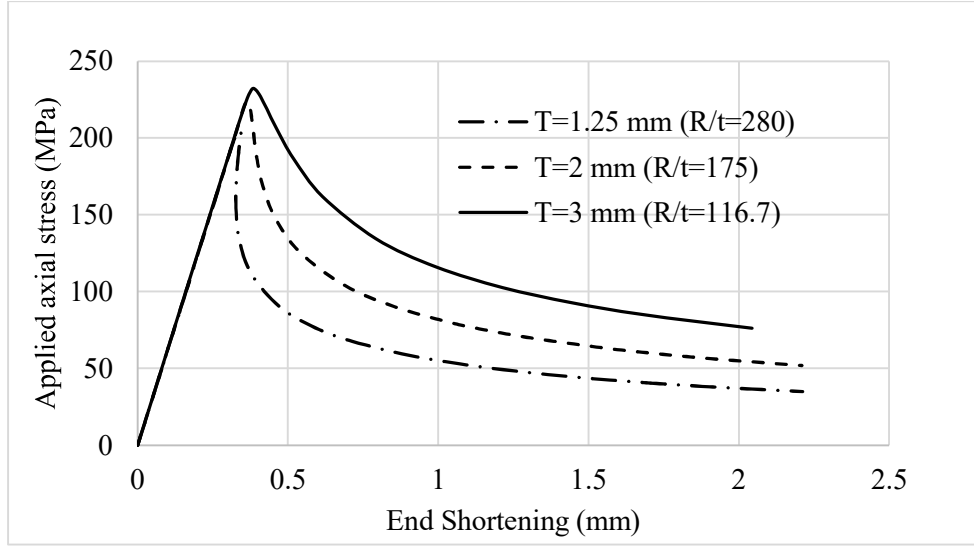


Figure 58 The applied axial stress against the end shortening of GMNA for perfect shells (without a dent)

To calculate the load ratio $\bar{P}$ from the BSR values, a correction factor was calculated and shown in Table 10 to be multiplied by all the BSR values appearing in the work of Parbu et al. [81].

Table 10 The results of the GMNA analysis and the correction factor

t (mm)	R/t	Maximum Load (KN)		Maximum (reference) stress (MPa)		Correction factor
		Parbu et al. [81]	Current GMNA	Parbu <i>et al.</i> [81]	Current GMNA	
1.25	280	1242.98	563.36	452.1747	204.94	2.206
2	175	3140.87	984.75	714.1214	223.9	3.190
3	116.67	7050.99	1530.4	1068.762	231.97	4.607

The variation of the BSR values defined by Parbu *et al.* [81] as changing some parameters while fixing others are shown in Tables 11 to 14. All of these tables are reported to study the influence of the dent. Taking a closer look at these tables, one can summarize the ranges of the BSR values as tabulated in Table 15. As discussed in the foregoing, these BSR values

were evaluated using a misleading reference capacity. Multiplications of the BSR ranges by the correction factor defined in Table 10 yields the $\bar{P}$ values as shown in Table 15, which consider the nonlinear buckling analysis of the perfect shell as a reference for these cases of small R/t ratios. It can be shown that the ranges of all $\bar{P}$ values are close to the unity which clearly indicates that the effect of the localized dent on the buckling strength of shells made of low yield strength materials and having small R/t ratios is negligible.

This is, of course, completely contradicting the conclusion drawn in [81], that the influence of the dent on reducing the buckling strength is higher when the shell gets thicker. Their conclusion [81] also contradicts the common trend in shell buckling which implies that the severity to imperfections is higher for large R/t ratios.

A more realistic parametric study on the effect of the localized dents on the buckling strength of shells should consider high R/t ratios. Shells with high R/t ratios are more sensitive to imperfection. Furthermore, they usually fail at stress levels much lower than their yield stresses. The following section presents the results of the proposed parametric study.

Table 11 The change of the BSR with the angle of inclination and the width (L_{dl}) keeping $L_{dt} = 175 \text{ mm}$ and $\delta_d = 3.75 \text{ mm}$ constant [81]

Angle, degrees	For $t = 1.25 \text{ mm}$			For $t = 2 \text{ mm}$			For $t = 3 \text{ mm}$		
	Width of dent, mm			Width of dent, mm			Width of dent, mm		
	25	50	87.5	25	50	87.5	25	50	87.5
0	0.44942	0.44749	0.44775	0.30556	0.30514	0.30528	0.21283	0.21266	0.21201
15	0.44709	0.44616	0.44646	0.30634	0.30586	0.30539	0.21371	0.21316	0.21318
30	0.4453	0.44543	0.44629	0.30883	0.30741	0.30724	0.21521	0.21427	0.21385
45	0.44885	0.44724	0.44644	0.31029	0.3092	0.30875	0.21617	0.21548	0.21485
60	0.45877	0.45331	0.44985	0.31509	0.3147	0.31105	0.2182	0.21742	0.21524
75	0.47062	0.46444	0.45773	0.31745	0.31581	0.31356	0.21884	0.21814	0.21695
90	0.47345	0.47043	0.46602	0.31777	0.3165	0.31348	0.21886	0.21854	0.21725

Table 12 The change of the BSR with the angle of inclination and dent amplitude (δ_d) keeping $L_{dt} = 175 \text{ mm}$ and $L_{dc} = 87.5 \text{ mm}$ constant [81]

Angle, degrees	For $t = 1.25 \text{ mm}$			For $t = 2 \text{ mm}$			For $t = 3 \text{ mm}$		
	Depth of dent, mm			Depth of dent, mm			Depth of dent, mm		
	1.25	2.5	3.75	1.25	2.5	3.75	1.25	2.5	3.75
0	0.45338	0.44998	0.44775	0.31201	0.30792	0.30528	0.21725	0.21439	0.21201
15	0.45315	0.44962	0.44646	0.31233	0.30933	0.30539	0.21746	0.21461	0.21318
30	0.45264	0.44858	0.44629	0.31332	0.30901	0.30724	0.21780	0.21539	0.21385
45	0.45397	0.44942	0.44644	0.31497	0.31063	0.30875	0.21829	0.21655	0.21485
60	0.45894	0.45287	0.44985	0.31649	0.31275	0.31105	0.21863	0.21727	0.21524
75	0.46457	0.46092	0.45773	0.31734	0.31463	0.31356	0.21870	0.21799	0.21695
90	0.4691	0.46701	0.46602	0.31751	0.31503	0.31348	0.21874	0.21823	0.21725

Table 13 The change of the BSR with the angle of inclination and length of the dent (L_{dl}) keeping $L_{dc} = 25\text{ mm}$ and $\delta_d = 3.75\text{ mm}$ constant [81]

Angle, degrees	For $t=1.25\text{ mm}$			For $t=2\text{ mm}$			For $t=3\text{ mm}$		
	Length of dent, mm			Length of dent, mm			Length of dent, mm		
	50	112.5	175	50	112.5	175	50	112.5	175
0	0.46582	0.45459	0.44942	0.31586	0.30967	0.30556	0.21779	0.21369	0.21283
15	0.46582	0.45447	0.44709	0.31568	0.30981	0.30634	0.21783	0.21515	0.21371
30	0.46772	0.45440	0.4453	0.31828	0.31086	0.30883	0.21806	0.21608	0.21521
45	0.46852	0.45672	0.44885	0.31665	0.31362	0.31029	0.21828	0.21721	0.21617
60	0.46952	0.46354	0.45877	0.31719	0.31588	0.31509	0.21858	0.21833	0.2182
75	0.47364	0.47146	0.47062	0.31753	0.31738	0.31745	0.21864	0.21871	0.21884
90	0.47364	0.47311	0.47345	0.31760	0.31762	0.31777	0.21865	0.21862	0.21886

Table 14 The change of the BSR with the angle of inclination and dent amplitude (δ_d) keeping $L_{dl} = 50\text{ mm}$ and $L_{dc} = 25\text{ mm}$ constant [81]

Angle, degrees	For $t=1.25\text{ mm}$			For $t=2\text{ mm}$			For $t=3\text{ mm}$		
	Depth of dent, mm			Depth of dent, mm			Depth of dent, mm		
	1.25	2.5	3.75	1.25	2.5	3.75	1.25	2.5	3.75
0	0.46996	0.46652	0.46582	0.31737	0.31627	0.31586	0.21865	0.21818	0.21779
15	0.4706	0.46702	0.46582	0.31734	0.31635	0.31568	0.21863	0.21830	0.21783
30	0.4714	0.46804	0.46772	0.31760	0.31677	0.31828	0.21878	0.21834	0.21806
45	0.47294	0.46964	0.46852	0.31769	0.31712	0.31665	0.21865	0.21915	0.21828
60	0.47344	0.47104	0.46952	0.31781	0.31751	0.31719	0.21882	0.21871	0.21858
75	0.47382	0.47321	0.47364	0.31782	0.31770	0.31753	0.21884	0.21890	0.21864
90	0.47412	0.4737	0.47364	0.30147	0.31770	0.31760	0.21872	0.21890	0.21865

Table 15 The range of the BSR values and the corrected $\bar{P}$

	$t = 1.25 \text{ mm}$	$t = 2 \text{ mm}$	$t = 3 \text{ mm}$
BSR range	0.44530-0.47412	0.30528-0.31777	0.21201-0.21890
$\bar{P}$ range	0.98233-1.04590	0.97384-1.01369	0.97673-1.00847

7.3.2 Parametric Study for Dented Shells with Large R/t Ratios

To single out the effect of the dent on the buckling strength of cylindrical shells, the geometric and material properties were selected in a way to have neglected influence of the plasticity of the material especially away from the dent effective region.

7.3.2.1 Geometry and Material of the Shell

Large R/t ratios; and a structural carbon steel material S 275 with parameters defined in Table 5 and a bilinear plus nonlinear stress-strain curve as shown in Figure 30 are adopted.

The dimensions of the shells are:

Radius (R) = 7500 mm

Length (L) = 10000 mm

Thickness (t) = 6 and 10 mm

The considered dimensions result in R/t ratios to be 1250 and 750, respectively.

7.3.2.2 Geometry of the Dent

The adopted model of the dent can be expressed mathematically by [77]

$$w_d = \delta_d \cos \left(\frac{x_{dc}}{L_{dc}} \right) \cos \left(\frac{x_{dl}}{L_{dl}} \right) \quad (7.2)$$

$$\text{for } -\frac{L_{dc}}{2} \leq x_{dc} \leq \frac{L_{dc}}{2} \text{ and } -\frac{L_{dl}}{2} \leq x_{dl} \leq \frac{L_{dl}}{2}$$

All the involved parameters are as defined in the foregoing. This has been used in the computer program prepared in Mathematica (Appendix D) for generation of the imperfect shell geometry.

7.3.2.3 The Selected Parameters

The parameters involved in the parametric study along with their levels are listed in Table 16. The first two parameters were selected to have the option of investigating whether the size of the dent in the longitudinal direction (the direction of the load) is more severe than its size in the circumferential direction or not. Only two thicknesses corresponding to R/t equal to 1250 and 750 were selected because the preliminary results proved that the effect of R/t is negligible provided that the selected R/t ratio ensures no plastic deformation in the shell at locations away from the dent effective region.

Table 16 The parameters adopted in the parametric study

Parameter	Levels	No. of levels
Length of the dent along the circumferential direction L_{dc}	500, 2500, and 5000 mm	3
Length of the dent along the longitudinal direction L_{dl}	500, 2500, and 5000 mm	3
The amplitude of the dent δ_d	10, 40, and 80 mm	3
The position of the center of the dent along the longitudinal direction.	0.25 L, 0.50 L, and 0.75 L	3
Thickness of the shell	6 and 10 mm	2

7.3.2.4 Results and discussion

The combinations of all parameters listed in Table 16 yield a total of 162 models, the results of which are shown in Tables 18 and 19. The results were evaluated in the form of the load ratio $\bar{P}$ which considers the least of the nonlinear buckling of the perfect shell and the non-axisymmetric buckling of the perfect shell as a reference capacity (the capacity curves defined in Chapter 5).

Running the perfect shell models considering a perfect shell geometry yields the values tabulated in Table 17. It can be clearly seen that the results of the GMNA are slightly higher than that of Equation 5.6. The reason is that the considered equation has two parts, one controlled by GMNA results and the other by the non-axisymmetric buckling which is for

higher values of R/t (>600). Eventually, the lowest of these were used as a reference (shown in *italics*) to express the capacity of the perfect shell.

Table 17 The reference capacities of the perfect shells

t (mm)	R/t	Maximum applied axial stress (MPa)	
		Equation 5.6	GMNA
10	750	<i>166.76</i>	169.82
6	1250	<i>100.06</i>	101.96

Table 18 The $\bar{P}$ values for different combinations of the adopted parameters for $R/t=750$

δ_d (mm)	L_{dc} (mm)	γ_{dent}								
		0.25 L			0.5 L			0.75 L		
		L_{dl} (mm)			L_{dl} (mm)			L_{dl} (mm)		
		500	2500	5000	500	2500	5000	500	2500	5000
10	500	0.610	0.598	0.800	0.636	0.535	0.601	0.610	0.598	0.800
	2500	0.647	0.616	0.756	0.679	0.642	0.700	0.647	0.616	0.756
	5000	0.585	0.796	0.865	0.646	0.560	0.842	0.591	0.796	0.865
40	500	0.716	0.701	0.672	0.690	0.647	0.665	0.717	0.699	0.672
	2500	0.678	0.636	0.657	0.688	0.653	0.666	0.679	0.628	0.657
	5000	0.664	0.570	0.486	0.651	0.546	0.615	0.663	0.569	0.487
80	500	0.760	0.781	0.764	0.725	0.720	0.752	0.760	0.781	0.764
	2500	0.659	0.678	0.691	0.714	0.690	0.729	0.659	0.678	0.691
	5000	0.629	0.558	0.592	0.670	0.540	0.553	0.635	0.558	0.592

Table 19 The $\bar{P}$ values for different combinations of the adopted parameters for $R/t=1250$

δ_d (mm)	L_{dc} (mm)	y_{dent}								
		0.25 L			0.5 L			0.75 L		
		L_{dl} (mm)			L_{dl} (mm)			L_{dl} (mm)		
		500	2500	5000	500	2500	5000	500	2500	5000
10	500	0.600	0.569	0.601	0.591	0.527	0.614	0.616	0.569	0.601
	2500	0.628	0.590	0.710	0.682	0.630	0.647	0.627	0.590	0.710
	5000	0.589	0.571	0.860	0.535	0.714	0.887	0.589	0.571	0.860
40	500	0.694	0.733	0.696	0.663	0.669	0.702	0.694	0.733	0.696
	2500	0.608	0.611	0.663	0.669	0.657	0.729	0.657	0.611	0.663
	5000	0.632	0.545	0.475	0.637	0.518	0.591	0.632	0.544	0.475
80	500	0.737	0.783	0.782	0.697	0.729	0.759	0.737	0.783	0.782
	2500	0.646	0.666	0.736	0.642	0.683	0.713	0.646	0.666	0.702
	5000	0.597	0.564	0.564	0.662	0.503	0.547	0.597	0.564	0.564

To discuss the effect of considered parameters carefully, let us start by the two selected R/t ratios. Let us define the term $\aleph$ as the relative percentage difference between the $\bar{P}$ values evaluated at $R/t=750$ and those at $R/t = 1250$ (Equation 7.3). The values of $\aleph$ are calculated and listed in Table 20 for different levels of the studied parameters. In general, the table shows 26 negative values out of 81, which means that shells with large R/t can possibly have higher $\bar{P}$ values compared to those with small R/t and one cannot say that the sensitivity of the shell due to dent imperfections is more with the increase in R/t ratio.

$$\aleph = \left(\frac{\bar{P}_{750} - \bar{P}_{1250}}{\bar{P}_{750}} \right) \times 100 \quad (7.3)$$

Table 20 The relative percentage difference δ between the $\bar{P}$ evaluated for $R/t=750$ and $R/t=1250$

δ_d (mm)	L_{dc} (mm)	y_{dent}								
		0.25 L			0.5 L			0.75 L		
		L_{dl} (mm)			L_{dl} (mm)			L_{dl} (mm)		
		500	2500	5000	500	2500	5000	500	2500	5000
10	500	1.7	4.8	24.8	7.1	1.5	-2.1	-1.0	4.8	24.8
	2500	2.9	4.2	6.1	-0.6	1.8	7.6	3.1	4.1	6.1
	5000	-0.5	28.2	0.5	17.2	-27.6	-5.4	0.4	28.2	0.5
40	500	3.1	-4.5	-3.6	3.9	-3.4	-5.5	3.1	-4.8	-3.6
	2500	10.3	3.9	-0.8	2.7	-0.7	-9.4	3.2	2.7	-0.8
	5000	4.8	4.5	2.3	2.3	5.1	3.9	4.6	4.5	2.3
80	500	3.0	-0.3	-2.4	3.8	-1.2	-0.9	3.0	-0.3	-2.4
	2500	1.9	1.8	-6.6	10.2	1.0	2.1	1.9	1.8	-1.6
	5000	5.1	-1.1	4.8	1.1	6.8	1.0	5.9	-1.1	4.7

Considering Tables 18 to 20, and Figures 59 through 64, the following conclusions can be drawn:

- 1- The minimum and maximum $\bar{P}$ values for the case of $R/t=750$ are 0.486 and 0.865, respectively, whereas for the case of $R/t=1250$, the minimum and maximum $\bar{P}$ values are 0.475 and 0.887, respectively.
- 2- For high levels of R/t (high enough to prevent plastic deformations in locations away from the dent effective region), the change of R/t has no considerable influence on $\bar{P}$.
- 3- Position of the dent at 0.25 L and 0.75 L yields almost similar results, which means that the position of the dent above or below the middle of the shell gives similar results provided that the offset distance is same. This is expected due to the symmetry of the geometry, loading and boundary conditions about the middle of the shell. This point justifies limiting the figures 59 to 64 to the positions 0.25 L and 0.5 L only.

- 4- It can be seen that, on average, there is no remarkable trend on the effect of the position of the dent along the longitudinal direction. It has been noted that for some cases the position $0.50 L$ yields higher $\bar{P}$ values while lower ones for other cases.
- 5- For the case of a square dent in plan (Figure 59 and 60), the increase of the size of the dent found to reduce the $\bar{P}$ values except for the case of a small dent amplitude where the reverse is true. This could be manifested by the fact that if the dent amplitude is small enough, the increase of the size of the dent reduces the change in curvature making it insignificant to reduce the $\bar{P}$ value.
- 6- For the case of a rectangular dent in plan (Figure 61 and 63/ Figure 62 and 64), it can be seen that, on average, there is no much difference if the dent is placed with its longer side being parallel to the load or perpendicular to it.

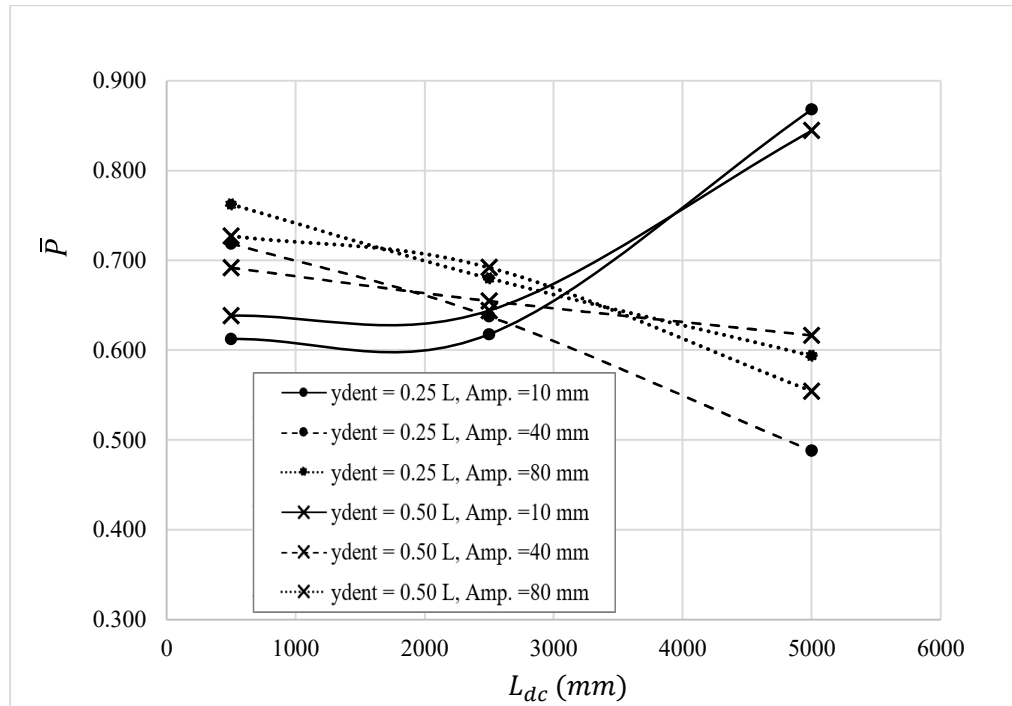


Figure 59 Variation of the $\bar{P}$ values with the size of square dent ($L_{dc} = L_{dl}$) at $R/t=750$

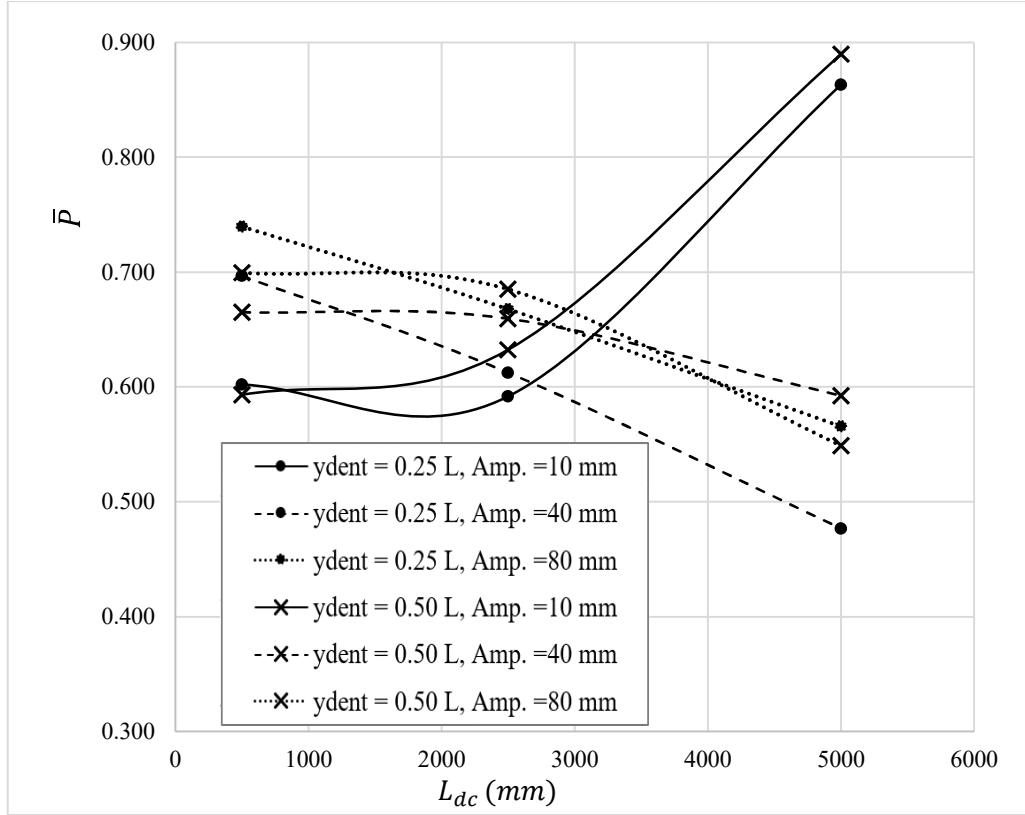


Figure 60 Variation of the $\bar{P}$ values with the size of square dent ($L_{dc} = L_{dl}$) at $R/t=1250$

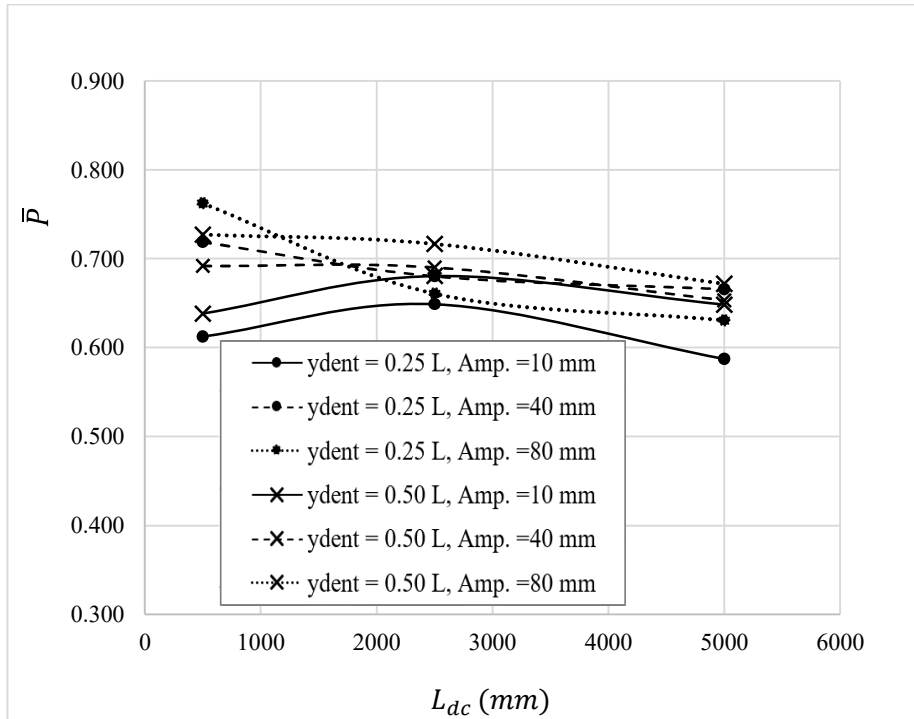


Figure 61 Variation of the $\bar{P}$ for the case of a rectangular dent with $L_{dl} = 500$ mm and different values of L_{dc} at $R/t=750$

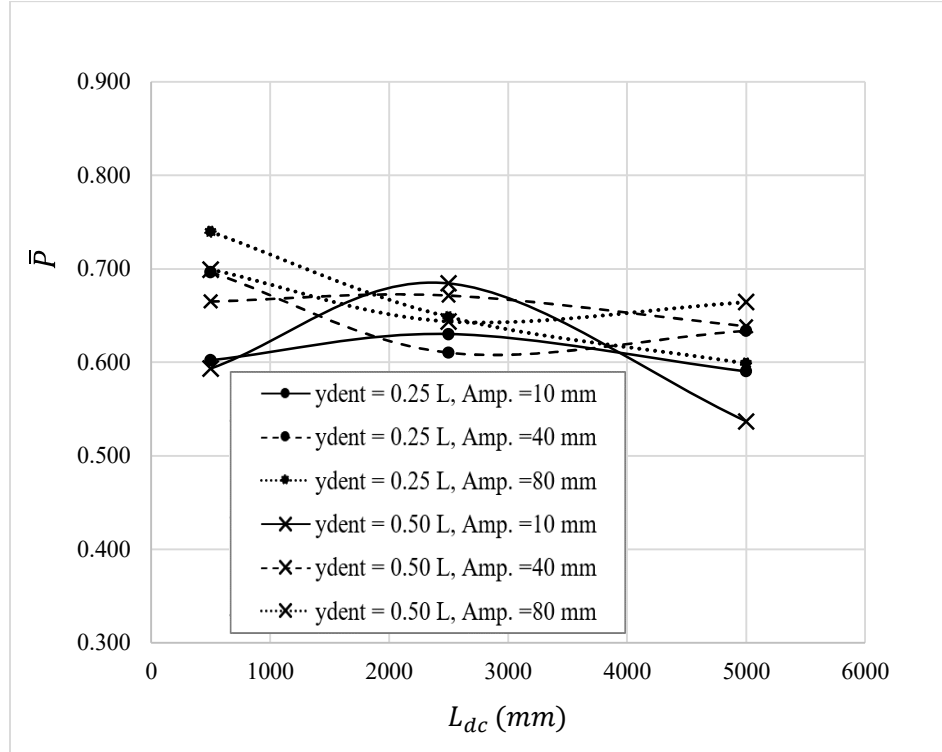


Figure 62 Variation of the $\bar{P}$ for the case of a rectangular dent with $L_{d1} = 500$ mm and different values of L_{dc} at $R/t=1250$

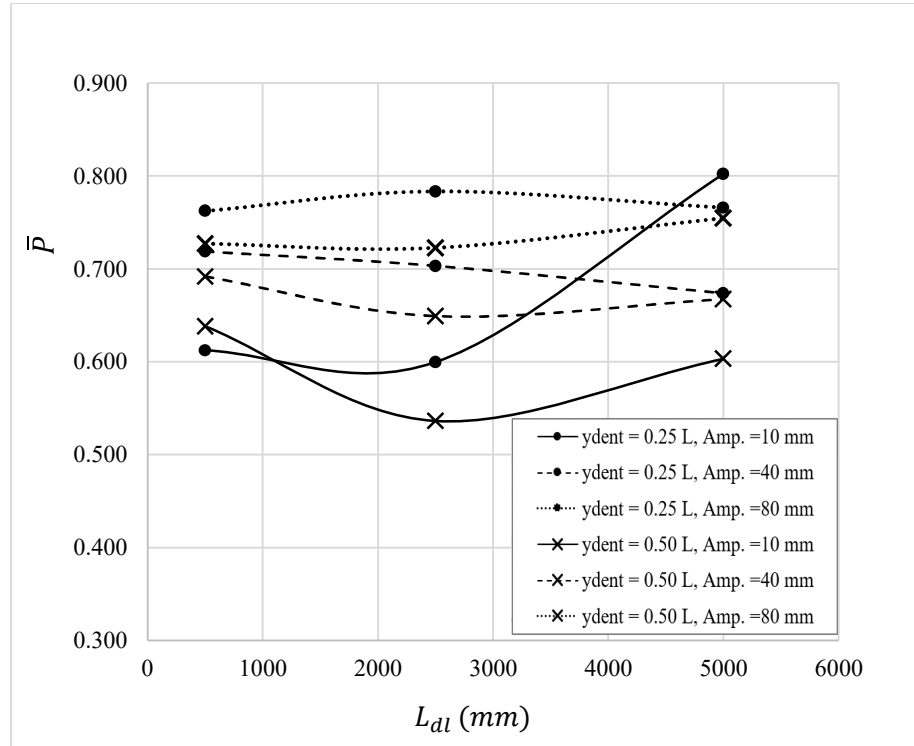


Figure 63 Variation of the $\bar{P}$ for the case of a rectangular dent with $L_{dc} = 500$ mm and different values of L_{d1} at $R/t=750$

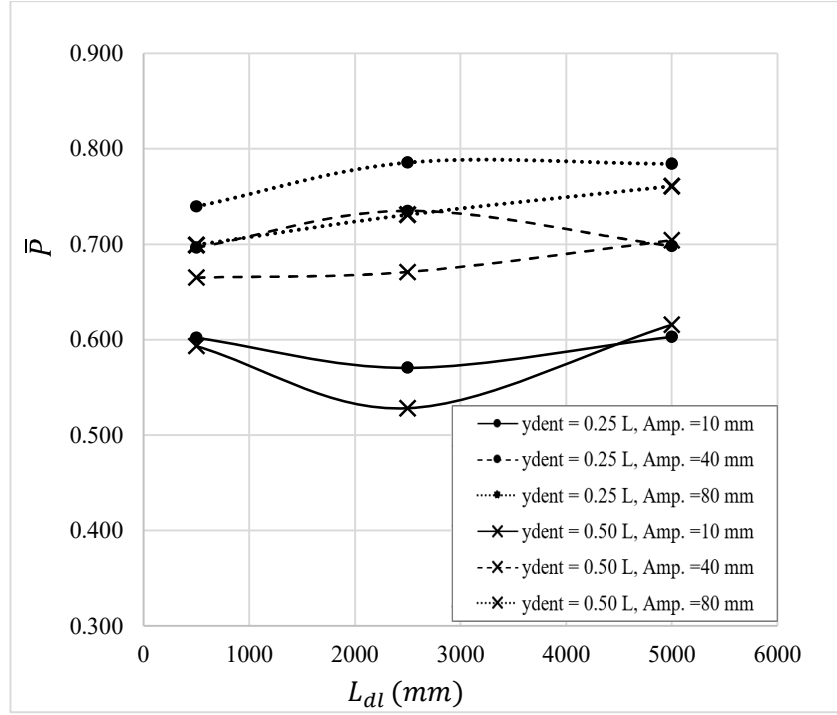


Figure 64 Variation of the $\bar{P}$ for the case of a rectangular dent with $L_{dc} = 500$ mm and different values of L_{dl} at $R/t=1250$

As a general conclusion, it can be seen the presence of a dent affects the buckling strength of the shell by around 11 to 53 % with an average of 34 % strength reduction. To be on the safer side, a 53 % reduction can be applied if there is a possibility for the shell to get a localized damage in the form of a dent.

CHAPTER 8

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

This investigation has studied the nonlinear buckling behavior of axially compressed circular cylindrical shells more closely using the available symbolic and numerical tools. This chapter summarizes the main accomplishments of the conducted research and outlines the conclusions drawn from the obtained results. It also provides some future research directions towards accurately modeling the behavior of axially compressed cylindrical shells.

8.1 Summary

The behavior of axially compressed circular cylindrical shells has been studied through the following types of analyses:

a) Linear buckling of perfect cylindrical shells:

The classical buckling formulae based on axisymmetric and the non-axisymmetric buckling were derived. Although, the axisymmetric buckling formula is well documented in textbooks, it has been presented here for the sake of comparison with the non-axisymmetric-based buckling formula. The obtained non-axisymmetric-based buckling formula was utilized by solving a constrained nonlinear optimization analysis leading to the minimum buckling stress and mode.

The obtained formula was lengthy and therefore it was further simplified by running a parametric study for a large range of geometric parameters L/R and R/t and fitting the resulted data with a simple exponential formula. The simplified formula was verified against FEM and found to be very accurate.

This type of analysis does not give good prediction of the shell postbuckling behavior which involves large strains. Furthermore, it does not account for the inevitable shell imperfections. However, it provides an upper limit of the buckling load. It also serves as a benchmark solution for the verification of the FEM model

b) Geometric nonlinear buckling of perfect cylindrical shells:

This analysis was based on the geometric nonlinearity, but a linear elastic material was retained. A step-by-step and computer implementation of von Kraman and Tsien's analytical solution has been illustrated. The obtained solution was very useful in describing the postbuckling behavior of the shell. However, as the previous linear analysis, it neglects the vital factor of shell imperfection.

c) Material nonlinear buckling of perfect cylindrical shells:

This type of analysis is too difficult to be handled analytically and therefore it was modeled by FEM. This analysis was found to be necessary for shells made of low strength grades and having small R/t ratios as these shells tend to fail at stress levels near their yield strength. However, again, this type of analysis shares the same disadvantage of the previous analyses in ignoring the shell imperfection. The results of the geometric and material nonlinear buckling of perfect shells were utilized to

generate the capacity formulae and curves for perfect cylindrical shells made of different standard steel grades and having different R/t ratios.

d) Geometric and material nonlinear buckling of imperfect cylindrical shells:

This is the most comprehensive type of analysis that took into account the shell imperfection which is the key factor responsible for the large deviation between experimental and theoretical buckling loads. Two types of imperfections were considered: out of roundness and localized dents. The localized dent imperfection was given more emphasis due to its greater effect compared to other types of imperfection. A parametric study on the presence of localized dent has been conducted. The key parameters were: the dent amplitude, size of the dent, and position of the dent.

8.2 Conclusions

Based on the obtained results of the above analyses, the following can be concluded:

- 1- The derived non-axisymmetric buckling formula produces a buckling strength which is less than the classical buckling strength. The derived formula is lengthy and impractical to be used for design purposes and therefore has been simplified further.
- 2- The short classical (axisymmetric) buckling formula is much simpler than the lengthy non-axisymmetric buckling formula. However, the corresponding buckling

mode of the later is more realistic and closer to the shape of experimentally buckled shell.

- 3- It is hard to come up with a general buckling formula for imperfect shells due to the high variability in the type and geometry of the imperfections. It is recommended to perform a geometric and material nonlinear analysis with imperfections (GMNIA) for every shell to be designed. The developed capacity curves and their corresponding simplified formulae can be used as upper limits for the buckling capacity of the imperfect shells to be designed.
- 4- The out-of-roundness in terms in partial ovality (Form B) has much more significant effect on the reduction of shell buckling strength as compared to the out-of-roundness in terms of prismatic ovality (Form A).
- 5- The effect of the localized dent on the shell buckling strength is negligible for shells with small R/t ratios and having low yield strength.
- 6- For the case of shells with large R/t ratios, a considerable buckling strength reduction can be resulted from the presence of the localized dent on the shell surface. The amount of the reduction can reach up to 53% regardless of the size, amplitude, or position of the dent.
- 7- For accurate prediction of the buckling strength of imperfect shells, different shapes and amplitudes of the imperfections should be tested.

8.3 Recommendations for Future Work

Finally, the behavior of axially compressed imperfect cylindrical shells requires more investigations to be comprehended. This is because the shell is very sensitive to imperfections and the imperfections cannot simply be generalized. Following are some future research directions which can be of importance towards fully comprehending the behavior:

- 1- The tolerable limits of out-of-roundness provided by the European Standards EN 1993-1-6 (2007) should be evaluated for more possible forms of out-of-roundness and, hence, be refined.
- 2- The effect of the presence of nozzles on the buckling strength should be investigated to reflect the real-life applications.
- 3- The vast majority of conducted experimental work on the buckling of cylindrical shells address shells with small R/t ratios. There is a need for more experimental data for shells with large R/t ratios which are more sensitive to imperfection.
- 4- The considered load in this study, and most of the literature, is uniform axial compression. The study of imperfect cylindrical shells under non-uniform axial compression and compression due to bending will have an invaluable addition to the pool of knowledge in shell buckling.

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Appendix A: The Non-axisymmetric Buckling Using Mathematica Software

Appendix A : The Non-axisymmetric Buckling Using Mathematica Software

A.I Derivation of the Non-axisymmetric buckling Formula Using the Equilibrium Method

```

Clear["Global`*"]
(* Approximate the three components of
   the shell displacements (Equations 3.6 to 3.8): *)
u =  $\frac{C1}{\nu R} x + C2 + Amn \sin[n \theta] \cos\left[\frac{m \pi x}{L}\right]$ ;
v = Bmn Cos[n  $\theta$ ] Sin $\left[\frac{m \pi x}{L}\right]$ ;
w = C1 + Cmn Sin[n  $\theta$ ] Sin $\left[\frac{m \pi x}{L}\right]$ ;
(* Find all derivatives appearing in the
   equilibrium equations (Equations 3.3 to 3.5) *)
ux = D[u, x];
uxx = D[ux, x];
uxxx = D[u, {x, 3}];
ux $\theta$  = D[ux,  $\theta$ ];
u $\theta$  = D[u,  $\theta$ ];
u $\theta\theta$  = D[u $\theta$ ,  $\theta$ ];
vx = D[v, x]; vxx = D[vx, x];
v $\theta$  = D[v,  $\theta$ ];
v $\theta\theta$  = D[v $\theta$ ,  $\theta$ ];
v $\theta\theta\theta$  = D[v, { $\theta$ , 3}];
vx $\theta$  = D[vx,  $\theta$ ];
vxx $\theta$  = D[vxx,  $\theta$ ];
wx = D[w, x]; wxx = D[w, {x, 2}]; wxxxx = D[wxx, {x, 2}];
w $\theta$  = D[w,  $\theta$ ];
w $\theta\theta$  = D[w $\theta$ ,  $\theta$ ];
wxx $\theta$  = D[wxx,  $\theta$ ];
w $\theta\theta\theta$  = D[w, { $\theta$ , 3}];
wxx $\theta\theta$  = D[w $\theta\theta$ , {x, 2}];
w $\theta\theta\theta\theta$  = D[w, { $\theta$ , 4}];
(* Obtaining the left hand side of Equation 3.3(LHS1): *)
LHS1 = uxx +  $\frac{(1 + \nu)}{2 R} vx\theta - \frac{\nu}{R} wx + \frac{1 - \nu}{2 R^2} u\theta\theta$ ;
LHS1 = Coefficient[LHS1, Cos $\left[\frac{m \pi x}{L}\right]$  Sin[n  $\theta$ ]]
-  $\frac{Amn m^2 \pi^2}{L^2} - \frac{Amn n^2 (1 - \nu)}{2 R^2} - \frac{Cmn m \pi \nu}{L R} - \frac{Bmn m n \pi (1 + \nu)}{2 L R}$ 

```

(* Express LHS1 in terms of the geometric parameters L/R and R/T *)

LHS1 = Simplify[R² * LHS1 /. L → LR * R /. R → RT * t]

$$\frac{1}{2 LR^2} \left(Amn \left(-2 m^2 \pi^2 + LR^2 n^2 (-1 + \nu) \right) - LR m \pi \left(2 Cmn \nu + Bmn n (1 + \nu) \right) \right)$$

(* Similarly, the LHS2 and LHS3 for Equation 3.4 and 3.5, respectively: *)

$$LHS2 = \frac{1 + \nu}{2 R} ux\theta + \frac{1 - \nu}{2} vxx + \frac{1}{R^2} v\theta\theta - \frac{1}{R^2} w\theta +$$

$$\frac{t^2}{12 R^2} \left(\frac{1}{R^2} v\theta\theta + \frac{1}{R^2} w\theta\theta\theta + wxx\theta + (1 - \nu) vxx \right) - \frac{\sigma cr (1 - \nu^2)}{EE} vxx;$$

$$LHS2 = \text{Coefficient}[LHS2, \text{Cos}[n \theta] \text{Sin}\left[\frac{m \pi x}{L}\right]];$$

$$LHS3 = \nu ux + \frac{1}{R} v\theta - \frac{w}{R} -$$

$$\frac{t^2}{12 R^2} \left(\frac{1}{R} v\theta\theta\theta + R (2 - \nu) vxx\theta + R^3 wxxxx + \frac{1}{R} w\theta\theta\theta\theta + 2 R wxx\theta\theta \right) - \frac{\sigma cr R (1 - \nu^2)}{EE} wxx;$$

$$LHS3 = \text{Coefficient}[LHS3, \text{Sin}\left[\frac{m \pi x}{L}\right] \text{Sin}[n \theta]];$$

LHS2 = Simplify[R² * LHS2 /. L → LR * R /. R → RT * t]

LHS3 = Simplify[R * LHS3 /. L → LR * R /. R → RT * t]

$$- \frac{1}{12 EE LR^2 RT^2} \left(EE n \left(Cmn m^2 \pi^2 + Cmn LR^2 \left(n^2 + 12 RT^2 \right) + 6 Amn LR m \pi RT^2 (1 + \nu) \right) + \right. \\ \left. Bmn \left(EE LR^2 n^2 \left(1 + 12 RT^2 \right) - EE m^2 \pi^2 \left(1 + 6 RT^2 \right) (-1 + \nu) + 12 m^2 \pi^2 RT^2 (-1 + \nu^2) \sigma cr \right) \right)$$

$$- \frac{1}{12 EE LR^4 RT^2} \left(Cmn EE \left(2 LR^2 m^2 n^2 \pi^2 + m^4 \pi^4 + LR^4 \left(n^4 + 12 RT^2 \right) \right) + \right. \\ \left. Bmn EE LR^2 n \left(LR^2 \left(n^2 + 12 RT^2 \right) - m^2 \pi^2 (-2 + \nu) \right) + \right. \\ \left. 12 Amn EE LR^3 m \pi RT^2 \nu + 12 Cmn LR^2 m^2 \pi^2 RT^2 (-1 + \nu^2) \sigma cr \right)$$

(*The matix of the coefficents (as appears in Equation 3.9): *)

$$K = \begin{pmatrix} \text{Coefficient}[LHS1, Amn] & \text{Coefficient}[LHS1, Bmn] & \text{Coefficient}[LHS1, Cmn] \\ \text{Coefficient}[LHS2, Amn] & \text{Coefficient}[LHS2, Bmn] & \text{Coefficient}[LHS2, Cmn] \\ \text{Coefficient}[LHS3, Amn] & \text{Coefficient}[LHS3, Bmn] & \text{Coefficient}[LHS3, Cmn] \end{pmatrix};$$

Do[Print["K[" , i, ", ", j, "]=", K[[i, j]]], {i, 1, 3}, {j, 1, 3}]

$$K[1,1] = \frac{-2 m^2 \pi^2 + LR^2 n^2 (-1 + \nu)}{2 LR^2}$$

$$K[1,2] = \frac{-LR m n \pi - LR m n \pi \nu}{2 LR^2}$$

$$K[1,3] = -\frac{m \pi \nu}{LR}$$

$$K[2,1] = -\left((6 EE LR m n \pi RT^2 + 6 EE LR m n \pi RT^2 \nu) / (12 EE LR^2 RT^2) \right)$$

$$K[2,2] = -\frac{1}{12 EE LR^2 RT^2} \left(EE LR^2 n^2 (1 + 12 RT^2) - EE m^2 \pi^2 (1 + 6 RT^2) (-1 + \nu) + 12 m^2 \pi^2 RT^2 (-1 + \nu^2) \sigma_{cr} \right)$$

$$K[2,3] = -\left((EE LR^2 n^3 + EE m^2 n \pi^2 + 12 EE LR^2 n RT^2) / (12 EE LR^2 RT^2) \right)$$

$$K[3,1] = -\frac{m \pi \nu}{LR}$$

$$K[3,2] = -\left((n (LR^2 (n^2 + 12 RT^2) - m^2 \pi^2 (-2 + \nu))) / (12 LR^2 RT^2) \right)$$

$$K[3,3] = -\frac{1}{12 EE LR^4 RT^2} \left(EE (2 LR^2 m^2 n^2 \pi^2 + m^4 \pi^4 + LR^4 (n^4 + 12 RT^2)) + 12 LR^2 m^2 \pi^2 RT^2 (-1 + \nu^2) \sigma_{cr} \right)$$

(* Obtaining the critical buckling stress: *)

sol = Simplify[Solve[Det[K] == 0, σ_{cr}]];

$\sigma_{cr} = \sigma_{cr} /. sol[[1]]$

$$\begin{aligned} & - \left((EE LR^6 m^2 n^4 RT^2 + EE LR^6 m^2 n^6 RT^2 + 3 EE LR^4 m^4 n^2 \pi^2 RT^2 + 4 EE LR^4 m^4 n^4 \pi^2 RT^2 + \right. \\ & \quad 2 EE LR^2 m^6 \pi^4 RT^2 + 5 EE LR^2 m^6 n^2 \pi^4 RT^2 + 2 EE m^8 \pi^6 RT^2 + 12 EE LR^6 m^2 n^2 RT^4 + \\ & \quad 12 EE LR^6 m^2 n^4 RT^4 + 24 EE LR^4 m^4 \pi^2 RT^4 + 24 EE LR^4 m^4 n^2 \pi^2 RT^4 + 12 EE LR^2 m^6 \pi^4 RT^4 + \\ & \quad EE LR^4 m^4 n^2 \pi^2 RT^2 \nu + 2 EE LR^4 m^4 n^4 \pi^2 RT^2 \nu + 4 EE LR^2 m^6 n^2 \pi^4 RT^2 \nu + 2 EE m^8 \pi^6 RT^2 \nu + \\ & \quad 24 EE LR^4 m^4 \pi^2 RT^4 \nu - EE LR^6 m^2 n^4 RT^2 \nu^2 - EE LR^6 m^2 n^6 RT^2 \nu^2 - EE LR^4 m^4 n^2 \pi^2 RT^2 \nu^2 - \\ & \quad 2 EE LR^4 m^4 n^4 \pi^2 RT^2 \nu^2 - 2 EE LR^2 m^6 \pi^4 RT^2 \nu^2 - EE LR^2 m^6 n^2 \pi^4 RT^2 \nu^2 - 12 EE LR^6 m^2 n^2 RT^4 \nu^2 - \\ & \quad 12 EE LR^6 m^2 n^4 RT^4 \nu^2 - 24 EE LR^4 m^4 \pi^2 RT^4 \nu^2 - 24 EE LR^4 m^4 n^2 \pi^2 RT^4 \nu^2 - \\ & \quad 12 EE LR^2 m^6 \pi^4 RT^4 \nu^2 + EE LR^4 m^4 n^2 \pi^2 RT^2 \nu^3 - 24 EE LR^4 m^4 \pi^2 RT^4 \nu^3 - \\ & \quad \left. \sqrt{(EE^2 m^4 RT^4 (1 + \nu)^2 ((2 m^6 \pi^6 + LR^2 m^4 \pi^4 (-n^2 (-5 + \nu) - 2 (1 + 6 RT^2) (-1 + \nu)) - LR^6 n^2 \right. \right. \\ & \quad (1 + n^2) (n^2 + 12 RT^2) (-1 + \nu) + LR^4 m^2 \pi^2 (-2 n^4 (-2 + \nu) - 24 RT^2 (-1 + \nu^2)) + \\ & \quad n^2 (3 - 24 RT^2 (-1 + \nu) - 2 \nu + \nu^2)))^2 - 4 LR^2 (-2 m^2 \pi^2 + LR^2 n^2 (-1 + \nu))} \\ & \quad (-1 + \nu) (12 LR^8 n^4 (-1 + n^2)^2 RT^2 + 2 m^8 \pi^8 (1 + 6 RT^2) + 12 LR^6 m^2 n^2 (-1 + n^2) \pi^2 \\ & \quad RT^2 (-3 + 4 n^2 - \nu) + LR^2 m^6 n^2 \pi^6 (3 + 48 RT^2 - \nu) + LR^4 m^4 \pi^4 (n^4 (1 + 72 RT^2 - \nu) - \\ & \quad 24 RT^2 (1 + 6 RT^2) (-1 + \nu^2) + 12 n^2 RT^2 (-6 - \nu + \nu^2)))))) / \\ & \quad \left. (24 LR^2 m^4 \pi^2 RT^4 (2 m^2 \pi^2 - LR^2 n^2 (-1 + \nu)) (-1 + \nu) (1 + \nu)^2) \right) \end{aligned}$$

(* The classical buckling stress is given by*)

$$\sigma_{CL} = \frac{EE}{\sqrt{3(1-\nu^2)}} \frac{1}{RT};$$

(* Let us define α as the ratio between σ_{cr}/σ_{CL} (as in Figure 19): *)

$\alpha = \text{Simplify}[\sigma_{cr} / \sigma_{CL}, EE > 0]$

$$\begin{aligned} & \left(\sqrt{1-\nu^2} \left(-LR^6 m^2 n^4 RT^2 - LR^6 m^2 n^6 RT^2 - 3 LR^4 m^4 n^2 \pi^2 RT^2 - 4 LR^4 m^4 n^4 \pi^2 RT^2 - 2 LR^2 m^6 \pi^4 RT^2 - \right. \right. \\ & 5 LR^2 m^6 n^2 \pi^4 RT^2 - 2 m^8 \pi^6 RT^2 - 12 LR^6 m^2 n^2 RT^4 - 12 LR^6 m^2 n^4 RT^4 - 24 LR^4 m^4 \pi^2 RT^4 - \\ & 24 LR^4 m^4 n^2 \pi^2 RT^4 - 12 LR^2 m^6 \pi^4 RT^4 - LR^4 m^4 n^2 \pi^2 RT^2 \nu - 2 LR^4 m^4 n^4 \pi^2 RT^2 \nu - \\ & 4 LR^2 m^6 n^2 \pi^4 RT^2 \nu - 2 m^8 \pi^6 RT^2 \nu - 24 LR^4 m^4 \pi^2 RT^4 \nu + LR^6 m^2 n^4 RT^2 \nu^2 + \\ & LR^6 m^2 n^6 RT^2 \nu^2 + LR^4 m^4 n^2 \pi^2 RT^2 \nu^2 + 2 LR^4 m^4 n^4 \pi^2 RT^2 \nu^2 + 2 LR^2 m^6 \pi^4 RT^2 \nu^2 + \\ & LR^2 m^6 n^2 \pi^4 RT^2 \nu^2 + 12 LR^6 m^2 n^2 RT^4 \nu^2 + 12 LR^6 m^2 n^4 RT^4 \nu^2 + 24 LR^4 m^4 \pi^2 RT^4 \nu^2 + \\ & 24 LR^4 m^4 n^2 \pi^2 RT^4 \nu^2 + 12 LR^2 m^6 \pi^4 RT^4 \nu^2 - LR^4 m^4 n^2 \pi^2 RT^2 \nu^3 + 24 LR^4 m^4 \pi^2 RT^4 \nu^3 + \\ & \left. \sqrt{\left(m^4 RT^4 (1+\nu)^2 \left(\left(2 m^6 \pi^6 + LR^2 m^4 \pi^4 \left(-n^2 (-5+\nu) - 2 (1+6 RT^2) (-1+\nu) \right) - LR^6 n^2 \right. \right. \right. \right. \right. \\ & \left. \left. \left(1+n^2 \right) \left(n^2 + 12 RT^2 \right) (-1+\nu) + LR^4 m^2 \pi^2 \left(-2 n^4 (-2+\nu) - 24 RT^2 (-1+\nu^2) \right) + \right. \right. \\ & \left. \left. n^2 \left(3 - 24 RT^2 (-1+\nu) - 2 \nu + \nu^2 \right) \right) \right)^2 - 4 LR^2 \left(-2 m^2 \pi^2 + LR^2 n^2 (-1+\nu) \right) \right. \\ & \left. (-1+\nu) \left(12 LR^8 n^4 (-1+n^2) RT^2 + 2 m^8 \pi^8 (1+6 RT^2) + 12 LR^6 m^2 n^2 (-1+n^2) \pi^2 \right. \right. \\ & \left. \left. RT^2 (-3+4 n^2 - \nu) + LR^2 m^6 n^2 \pi^6 (3+48 RT^2 - \nu) + LR^4 m^4 \pi^4 \left(n^4 (1+72 RT^2 - \nu) - \right. \right. \right. \\ & \left. \left. \left. 24 RT^2 (1+6 RT^2) (-1+\nu^2) + 12 n^2 RT^2 (-6-\nu+\nu^2) \right) \right) \right) \right) \right) \right) \Bigg/ \\ & \left(8 \sqrt{3} LR^2 m^4 \pi^2 RT^3 \left(2 m^2 \pi^2 - LR^2 n^2 (-1+\nu) \right) (-1+\nu) \right. \\ & \left. (1+\nu)^2 \right) \end{aligned}$$

A.2 Minimization Examples

An example of a shell with $L/R = 5$, $R/t = 200$, and Poisson's ratio $\nu=0.3$ is performed below:

Minimize[

$\{(\alpha /. \{LR \rightarrow 5, RT \rightarrow 200, \nu \rightarrow 0.3\}), m > 0, n > 0, m \in \text{Integers}, n \in \text{Integers}\}, \{m, n\}$

$\{0.882746, \{m \rightarrow 1, n \rightarrow 4\}\}$

Following the same approach, the three cases shown in Table 3 were done as follows:

```

Minimize [ { (ocr /. {LR → 1, RT → 500, v → 0.3, EE → 210 × 103}) ,
  m > 0, n > 0, m ∈ Integers, n ∈ Integers} , {m, n} ]
Minimize [ { (ocr /. {LR → 5, RT → 500, v → 0.3, EE → 210 × 103}) ,
  m > 0, n > 0, m ∈ Integers, n ∈ Integers} , {m, n} ]
Minimize [ { (ocr /. {LR → 2, RT → 2000, v → 0.3, EE → 210 × 103}) ,
  m > 0, n > 0, m ∈ Integers, n ∈ Integers} , {m, n} ]

{250.252, {m → 1, n → 11} }
{234.938, {m → 1, n → 5} }
{62.6844, {m → 1, n → 11} }

```

A.3 Parametric Study for Different L/R and R/T Ratios

$$\begin{aligned} \sigma_{cr} = & - \left(\left(EE LR^6 m^2 n^4 RT^2 + EE LR^6 m^2 n^6 RT^2 + 3 EE LR^4 m^4 n^2 \pi^2 RT^2 + 4 EE LR^4 m^4 n^4 \pi^2 RT^2 + 2 EE LR^2 m^6 \pi^4 \right. \right. \\ & RT^2 + 5 EE LR^2 m^6 n^2 \pi^4 RT^2 + 2 EE m^8 \pi^6 RT^2 + 12 EE LR^6 m^2 n^2 RT^4 + 12 EE LR^6 m^2 n^4 RT^4 + \\ & 24 EE LR^4 m^4 \pi^2 RT^4 + 24 EE LR^4 m^4 n^2 \pi^2 RT^4 + 12 EE LR^2 m^6 \pi^4 RT^4 + EE LR^4 m^4 n^2 \pi^2 RT^2 \vee + \\ & 2 EE LR^4 m^4 n^4 \pi^2 RT^2 \vee + 4 EE LR^2 m^6 n^2 \pi^4 RT^2 \vee + 2 EE m^8 \pi^6 RT^2 \vee + 24 EE LR^4 m^4 \pi^2 RT^4 \vee - \\ & EE LR^6 m^2 n^4 RT^2 \vee^2 - EE LR^6 m^2 n^6 RT^2 \vee^2 - EE LR^4 m^4 n^2 \pi^2 RT^2 \vee^2 - 2 EE LR^4 m^4 n^4 \pi^2 RT^2 \vee^2 - \\ & 2 EE LR^2 m^6 \pi^4 RT^2 \vee^2 - EE LR^2 m^6 n^2 \pi^4 RT^2 \vee^2 - 12 EE LR^6 m^2 n^2 RT^4 \vee^2 - \\ & 12 EE LR^6 m^2 n^4 RT^4 \vee^2 - 24 EE LR^4 m^4 \pi^2 RT^4 \vee^2 - 24 EE LR^4 m^4 n^2 \pi^2 RT^4 \vee^2 - \\ & 12 EE LR^2 m^6 \pi^4 RT^4 \vee^2 + EE LR^4 m^4 n^2 \pi^2 RT^2 \vee^3 - 24 EE LR^4 m^4 \pi^2 RT^4 \vee^3 - \\ & \left. \sqrt{\left(EE^2 m^4 RT^4 (1 + \vee)^2 \left((2 m^6 \pi^6 + LR^2 m^4 \pi^4 (-n^2 (-5 + \vee) - 2 (1 + 6 RT^2) (-1 + \vee)) - \right. \right. \right. \right. \\ & LR^6 n^2 (1 + n^2) (n^2 + 12 RT^2) (-1 + \vee) + LR^4 m^2 \pi^2 \\ & (-2 n^4 (-2 + \vee) - 24 RT^2 (-1 + \vee^2) + n^2 (3 - 24 RT^2 (-1 + \vee) - 2 \vee + \vee^2)) \left. \right)^2 - \\ & 4 LR^2 (-2 m^2 \pi^2 + LR^2 n^2 (-1 + \vee)) (-1 + \vee) \left(12 LR^8 n^4 (-1 + n^2)^2 RT^2 + 2 \right. \\ & m^8 \pi^8 (1 + 6 RT^2) + 12 LR^6 m^2 n^2 (-1 + n^2) \pi^2 RT^2 (-3 + 4 n^2 - \vee) + LR^2 \\ & m^6 n^2 \pi^6 (3 + 48 RT^2 - \vee) + LR^4 m^4 \pi^4 (n^4 (1 + 72 RT^2 - \vee) - \\ & \left. \left. \left. \left. \left. 24 RT^2 (1 + 6 RT^2) (-1 + \vee^2) + 12 n^2 RT^2 (-6 - \vee + \vee^2) \right) \right) \right) \right) \right) \right) \Bigg/ \\ & \left. \left(24 LR^2 m^4 \pi^2 RT^4 (2 m^2 \pi^2 - LR^2 n^2 (-1 + \vee)) (-1 + \vee) (1 + \vee)^2 \right) \right); \end{aligned}$$

```

ocr1000ByE = Simplify[ocr * 1000 / EE, EE > 0];
LRdata = Range[0.5, 12.5, 1];
NoLR = Length[LRdata];
RTdata = Range[300, 3000, 100];
NoRT = Length[RTdata];
Do[
  sol[i, j] = Minimize[{(ocr1000ByE /. {LR → LRdata[[i]], RT → RTdata[[j]]}, v → 0.3)},
    m > 0, n > 0, m ∈ Integers, n ∈ Integers}, {m, n}];
  If[j == 1, Print[" ***** Results for L/R= ",
    LRdata[[i]], " *****"];
  Print[sol[i, j], "    L/R=", LRdata[[i]], ", R/T=", RTdata[[j]],
    {i, 1, NoLR}, {j, 1, NoRT}]

  ***** Results for L/R= 0.5 *****
{2.01737, {m → 5, n → 2}}    L/R=0.5, R/T=300
{1.5172, {m → 6, n → 1}}    L/R=0.5, R/T=400
{1.21007, {m → 6, n → 11}}   L/R=0.5, R/T=500
{1.00866, {m → 7, n → 5}}    L/R=0.5, R/T=600

NMinimize: Failed to converge to the requested accuracy or precision within 100 iterations.
{0.863497, {m → 5, n → 23}}   L/R=0.5, R/T=700
{0.756484, {m → 8, n → 8}}    L/R=0.5, R/T=800
{0.674274, {m → 9, n → 1}}    L/R=0.5, R/T=900
{0.605206, {m → 9, n → 7}}    L/R=0.5, R/T=1000
{0.550048, {m → 8, n → 22}}   L/R=0.5, R/T=1100

NMinimize: Failed to converge to the requested accuracy or precision within 100 iterations.
{0.504354, {m → 10, n → 3}}   L/R=0.5, R/T=1200
{0.465522, {m → 10, n → 13}}  L/R=0.5, R/T=1300
{0.432533, {m → 11, n → 1}}   L/R=0.5, R/T=1400
{0.403475, {m → 11, n → 10}}  L/R=0.5, R/T=1500
{0.378238, {m → 11, n → 16}}  L/R=0.5, R/T=1600
{0.356043, {m → 12, n → 1}}   L/R=0.5, R/T=1700
{0.336229, {m → 12, n → 11}}  L/R=0.5, R/T=1800
{0.31852, {m → 12, n → 17}}   L/R=0.5, R/T=1900
{0.302628, {m → 13, n → 1}}   L/R=0.5, R/T=2000
{0.288198, {m → 13, n → 12}}  L/R=0.5, R/T=2100

NMinimize: Failed to converge to the requested accuracy or precision within 100 iterations.
General: Further output of NMinimize::cvmit will be suppressed during this calculation.
{0.275067, {m → 11, n → 34}}   L/R=0.5, R/T=2200

```

{0.263186, {m → 14, n → 1}} L/R=0.5, R/T=2300
 {0.252175, {m → 14, n → 10}} L/R=0.5, R/T=2400
 {0.242083, {m → 14, n → 16}} L/R=0.5, R/T=2500
 {0.232715, {m → 10, n → 43}} L/R=0.5, R/T=2600
 {0.224158, {m → 15, n → 5}} L/R=0.5, R/T=2700
 {0.216149, {m → 15, n → 14}} L/R=0.5, R/T=2800
 {0.208694, {m → 15, n → 19}} L/R=0.5, R/T=2900
 {0.201781, {m → 16, n → 1}} L/R=0.5, R/T=3000

***** Results for L/R= 1.5 *****

{1.95866, {m → 1, n → 8}} L/R=1.5, R/T=300
 {1.50067, {m → 3, n → 14}} L/R=1.5, R/T=400
 {1.18094, {m → 1, n → 9}} L/R=1.5, R/T=500
 {0.996949, {m → 2, n → 13}} L/R=1.5, R/T=600
 {0.849068, {m → 1, n → 10}} L/R=1.5, R/T=700
 {0.743464, {m → 1, n → 10}} L/R=1.5, R/T=800
 {0.671062, {m → 1, n → 10}} L/R=1.5, R/T=900
 {0.596879, {m → 1, n → 11}} L/R=1.5, R/T=1000
 {0.541307, {m → 1, n → 11}} L/R=1.5, R/T=1100
 {0.49904, {m → 1, n → 11}} L/R=1.5, R/T=1200
 {0.46422, {m → 1, n → 12}} L/R=1.5, R/T=1300
 {0.427522, {m → 1, n → 12}} L/R=1.5, R/T=1400
 {0.397917, {m → 1, n → 12}} L/R=1.5, R/T=1500
 {0.373687, {m → 1, n → 12}} L/R=1.5, R/T=1600
 {0.353605, {m → 1, n → 12}} L/R=1.5, R/T=1700
 {0.335344, {m → 1, n → 13}} L/R=1.5, R/T=1800
 {0.315839, {m → 1, n → 13}} L/R=1.5, R/T=1900
 {0.299186, {m → 1, n → 13}} L/R=1.5, R/T=2000
 {0.284855, {m → 1, n → 13}} L/R=1.5, R/T=2100
 {0.272434, {m → 1, n → 13}} L/R=1.5, R/T=2200
 {0.261597, {m → 1, n → 13}} L/R=1.5, R/T=2300
 {0.252007, {m → 1, n → 14}} L/R=1.5, R/T=2400
 {0.24077, {m → 1, n → 14}} L/R=1.5, R/T=2500
 {0.230805, {m → 1, n → 14}} L/R=1.5, R/T=2600
 {0.221926, {m → 1, n → 14}} L/R=1.5, R/T=2700
 {0.213981, {m → 1, n → 14}} L/R=1.5, R/T=2800
 {0.206844, {m → 1, n → 14}} L/R=1.5, R/T=2900

{0.200409, {m → 1, n → 14}} L/R=1.5, R/T=3000

***** Results for L/R= 2.5 *****

{1.9246, {m → 1, n → 6}} L/R=2.5, R/T=300

{1.47687, {m → 1, n → 7}} L/R=2.5, R/T=400

{1.1629, {m → 1, n → 7}} L/R=2.5, R/T=500

{0.992346, {m → 1, n → 7}} L/R=2.5, R/T=600

{0.847118, {m → 1, n → 8}} L/R=2.5, R/T=700

{0.733293, {m → 1, n → 8}} L/R=2.5, R/T=800

{0.655256, {m → 1, n → 8}} L/R=2.5, R/T=900

{0.599436, {m → 1, n → 8}} L/R=2.5, R/T=1000

{0.543648, {m → 1, n → 9}} L/R=2.5, R/T=1100

{0.493346, {m → 1, n → 9}} L/R=2.5, R/T=1200

{0.454199, {m → 1, n → 9}} L/R=2.5, R/T=1300

{0.423138, {m → 1, n → 9}} L/R=2.5, R/T=1400

{0.398079, {m → 1, n → 9}} L/R=2.5, R/T=1500

{0.37757, {m → 1, n → 9}} L/R=2.5, R/T=1600

{0.352438, {m → 1, n → 10}} L/R=2.5, R/T=1700

{0.330733, {m → 1, n → 10}} L/R=2.5, R/T=1800

{0.312364, {m → 1, n → 10}} L/R=2.5, R/T=1900

{0.296681, {m → 1, n → 10}} L/R=2.5, R/T=2000

{0.283185, {m → 1, n → 10}} L/R=2.5, R/T=2100

{0.271487, {m → 1, n → 10}} L/R=2.5, R/T=2200

{0.261281, {m → 1, n → 10}} L/R=2.5, R/T=2300

{0.251797, {m → 1, n → 11}} L/R=2.5, R/T=2400

{0.240228, {m → 1, n → 11}} L/R=2.5, R/T=2500

{0.229967, {m → 1, n → 11}} L/R=2.5, R/T=2600

{0.220826, {m → 1, n → 11}} L/R=2.5, R/T=2700

{0.212646, {m → 1, n → 11}} L/R=2.5, R/T=2800

{0.205298, {m → 1, n → 11}} L/R=2.5, R/T=2900

{0.198672, {m → 1, n → 11}} L/R=2.5, R/T=3000

***** Results for L/R= 3.5 *****

{1.90817, {m → 1, n → 5}} L/R=3.5, R/T=300

{1.46366, {m → 1, n → 6}} L/R=3.5, R/T=400

{1.145, {m → 1, n → 6}} L/R=3.5, R/T=500

{0.971897, {m → 1, n → 6}} L/R=3.5, R/T=600

{0.859397, {m → 1, n → 7}} L/R=3.5, R/T=700

{0.732563, {m → 1, n → 7}}	L/R=3.5, R/T=800
{0.645606, {m → 1, n → 7}}	L/R=3.5, R/T=900
{0.583406, {m → 1, n → 7}}	L/R=3.5, R/T=1000
{0.537386, {m → 1, n → 7}}	L/R=3.5, R/T=1100
{0.502383, {m → 1, n → 7}}	L/R=3.5, R/T=1200
{0.458367, {m → 1, n → 8}}	L/R=3.5, R/T=1300
{0.421242, {m → 1, n → 8}}	L/R=3.5, R/T=1400
{0.391291, {m → 1, n → 8}}	L/R=3.5, R/T=1500
{0.366779, {m → 1, n → 8}}	L/R=3.5, R/T=1600
{0.346464, {m → 1, n → 8}}	L/R=3.5, R/T=1700
{0.329439, {m → 1, n → 8}}	L/R=3.5, R/T=1800
{0.315032, {m → 1, n → 8}}	L/R=3.5, R/T=1900
{0.302151, {m → 1, n → 9}}	L/R=3.5, R/T=2000
{0.285116, {m → 1, n → 9}}	L/R=3.5, R/T=2100
{0.27035, {m → 1, n → 9}}	L/R=3.5, R/T=2200
{0.257468, {m → 1, n → 9}}	L/R=3.5, R/T=2300
{0.246163, {m → 1, n → 9}}	L/R=3.5, R/T=2400
{0.236186, {m → 1, n → 9}}	L/R=3.5, R/T=2500
{0.227339, {m → 1, n → 9}}	L/R=3.5, R/T=2600
{0.219456, {m → 1, n → 9}}	L/R=3.5, R/T=2700
{0.212403, {m → 1, n → 9}}	L/R=3.5, R/T=2800
{0.206067, {m → 1, n → 9}}	L/R=3.5, R/T=2900
{0.200354, {m → 1, n → 9}}	L/R=3.5, R/T=3000

***** Results for L/R= 4.5 *****

{1.92314, {m → 1, n → 5}}	L/R=4.5, R/T=300
{1.39726, {m → 1, n → 5}}	L/R=4.5, R/T=400
{1.15386, {m → 1, n → 5}}	L/R=4.5, R/T=500
{0.982756, {m → 2, n → 8}}	L/R=4.5, R/T=600
{0.825649, {m → 1, n → 6}}	L/R=4.5, R/T=700
{0.715604, {m → 1, n → 6}}	L/R=4.5, R/T=800
{0.640158, {m → 1, n → 6}}	L/R=4.5, R/T=900
{0.586191, {m → 1, n → 6}}	L/R=4.5, R/T=1000
{0.537235, {m → 2, n → 9}}	L/R=4.5, R/T=1100
{0.495525, {m → 1, n → 7}}	L/R=4.5, R/T=1200
{0.45107, {m → 1, n → 7}}	L/R=4.5, R/T=1300
{0.415797, {m → 1, n → 7}}	L/R=4.5, R/T=1400

$\{0.38734, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=4.5, R/T=1500$
 $\{0.364051, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=4.5, R/T=1600$
 $\{0.344749, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=4.5, R/T=1700$
 $\{0.328573, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=4.5, R/T=1800$
 $\{0.314884, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=4.5, R/T=1900$
 $\{0.299186, \{m \rightarrow 3, n \rightarrow 13\}\}$ $L/R=4.5, R/T=2000$
 $\{0.284436, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2100$
 $\{0.269418, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2200$
 $\{0.256315, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2300$
 $\{0.244816, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2400$
 $\{0.234669, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2500$
 $\{0.22567, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2600$
 $\{0.217652, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2700$
 $\{0.210478, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2800$
 $\{0.204033, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=2900$
 $\{0.198222, \{m \rightarrow 1, n \rightarrow 8\}\}$ $L/R=4.5, R/T=3000$

***** Results for $L/R = 5.5$ *****

$\{1.83918, \{m \rightarrow 1, n \rightarrow 4\}\}$ $L/R=5.5, R/T=300$
 $\{1.47347, \{m \rightarrow 2, n \rightarrow 6\}\}$ $L/R=5.5, R/T=400$
 $\{1.12718, \{m \rightarrow 1, n \rightarrow 5\}\}$ $L/R=5.5, R/T=500$
 $\{0.932167, \{m \rightarrow 1, n \rightarrow 5\}\}$ $L/R=5.5, R/T=600$
 $\{0.814583, \{m \rightarrow 1, n \rightarrow 5\}\}$ $L/R=5.5, R/T=700$
 $\{0.738266, \{m \rightarrow 1, n \rightarrow 5\}\}$ $L/R=5.5, R/T=800$
 $\{0.653765, \{m \rightarrow 2, n \rightarrow 8\}\}$ $L/R=5.5, R/T=900$
 $\{0.581173, \{m \rightarrow 1, n \rightarrow 6\}\}$ $L/R=5.5, R/T=1000$
 $\{0.522053, \{m \rightarrow 1, n \rightarrow 6\}\}$ $L/R=5.5, R/T=1100$
 $\{0.477088, \{m \rightarrow 1, n \rightarrow 6\}\}$ $L/R=5.5, R/T=1200$
 $\{0.442095, \{m \rightarrow 1, n \rightarrow 6\}\}$ $L/R=5.5, R/T=1300$
 $\{0.414328, \{m \rightarrow 1, n \rightarrow 6\}\}$ $L/R=5.5, R/T=1400$
 $\{0.391928, \{m \rightarrow 1, n \rightarrow 6\}\}$ $L/R=5.5, R/T=1500$
 $\{0.369204, \{m \rightarrow 2, n \rightarrow 9\}\}$ $L/R=5.5, R/T=1600$
 $\{0.348774, \{m \rightarrow 2, n \rightarrow 9\}\}$ $L/R=5.5, R/T=1700$
 $\{0.329621, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=5.5, R/T=1800$
 $\{0.309306, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=5.5, R/T=1900$
 $\{0.29196, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=5.5, R/T=2000$
 $\{0.277033, \{m \rightarrow 1, n \rightarrow 7\}\}$ $L/R=5.5, R/T=2100$

{0.264095, {m → 1, n → 7}} L/R=5.5, R/T=2200
 {0.252808, {m → 1, n → 7}} L/R=5.5, R/T=2300
 {0.242902, {m → 1, n → 7}} L/R=5.5, R/T=2400
 {0.23416, {m → 1, n → 7}} L/R=5.5, R/T=2500
 {0.226408, {m → 1, n → 7}} L/R=5.5, R/T=2600
 {0.219501, {m → 1, n → 7}} L/R=5.5, R/T=2700
 {0.213321, {m → 1, n → 7}} L/R=5.5, R/T=2800
 {0.207769, {m → 1, n → 7}} L/R=5.5, R/T=2900
 {0.199375, {m → 3, n → 13}} L/R=5.5, R/T=3000

***** Results for L/R= 6.5 *****

{1.78364, {m → 1, n → 4}} L/R=6.5, R/T=300
 {1.36812, {m → 1, n → 4}} L/R=6.5, R/T=400
 {1.1758, {m → 1, n → 4}} L/R=6.5, R/T=500
 {0.967162, {m → 1, n → 5}} L/R=6.5, R/T=600
 {0.804139, {m → 1, n → 5}} L/R=6.5, R/T=700
 {0.69833, {m → 1, n → 5}} L/R=6.5, R/T=800
 {0.625788, {m → 1, n → 5}} L/R=6.5, R/T=900
 {0.573899, {m → 1, n → 5}} L/R=6.5, R/T=1000
 {0.535508, {m → 1, n → 5}} L/R=6.5, R/T=1100
 {0.491974, {m → 2, n → 8}} L/R=6.5, R/T=1200
 {0.451471, {m → 2, n → 8}} L/R=6.5, R/T=1300
 {0.414604, {m → 1, n → 6}} L/R=6.5, R/T=1400
 {0.383477, {m → 1, n → 6}} L/R=6.5, R/T=1500
 {0.358002, {m → 1, n → 6}} L/R=6.5, R/T=1600
 {0.336889, {m → 1, n → 6}} L/R=6.5, R/T=1700
 {0.319196, {m → 1, n → 6}} L/R=6.5, R/T=1800
 {0.304223, {m → 1, n → 6}} L/R=6.5, R/T=1900
 {0.291439, {m → 1, n → 6}} L/R=6.5, R/T=2000
 {0.280437, {m → 1, n → 6}} L/R=6.5, R/T=2100
 {0.268465, {m → 2, n → 9}} L/R=6.5, R/T=2200
 {0.257323, {m → 2, n → 9}} L/R=6.5, R/T=2300
 {0.247544, {m → 2, n → 9}} L/R=6.5, R/T=2400
 {0.237417, {m → 1, n → 7}} L/R=6.5, R/T=2500
 {0.226631, {m → 1, n → 7}} L/R=6.5, R/T=2600
 {0.21702, {m → 1, n → 7}} L/R=6.5, R/T=2700
 {0.208421, {m → 1, n → 7}} L/R=6.5, R/T=2800

{0.200695, {m → 1, n → 7}} L/R=6.5, R/T=2900

{0.19373, {m → 1, n → 7}} L/R=6.5, R/T=3000

***** Results for L/R= 7.5 *****

{1.86997, {m → 2, n → 5}} L/R=7.5, R/T=300

{1.33704, {m → 1, n → 4}} L/R=7.5, R/T=400

{1.08284, {m → 1, n → 4}} L/R=7.5, R/T=500

{0.944755, {m → 1, n → 4}} L/R=7.5, R/T=600

{0.861493, {m → 1, n → 4}} L/R=7.5, R/T=700

{0.724191, {m → 1, n → 5}} L/R=7.5, R/T=800

{0.628058, {m → 1, n → 5}} L/R=7.5, R/T=900

{0.559295, {m → 1, n → 5}} L/R=7.5, R/T=1000

{0.508418, {m → 1, n → 5}} L/R=7.5, R/T=1100

{0.469722, {m → 1, n → 5}} L/R=7.5, R/T=1200

{0.439607, {m → 1, n → 5}} L/R=7.5, R/T=1300

{0.415712, {m → 1, n → 5}} L/R=7.5, R/T=1400

{0.396435, {m → 1, n → 5}} L/R=7.5, R/T=1500

{0.368635, {m → 2, n → 8}} L/R=7.5, R/T=1600

{0.345389, {m → 2, n → 8}} L/R=7.5, R/T=1700

{0.324318, {m → 1, n → 6}} L/R=7.5, R/T=1800

{0.304447, {m → 1, n → 6}} L/R=7.5, R/T=1900

{0.287482, {m → 1, n → 6}} L/R=7.5, R/T=2000

{0.272881, {m → 1, n → 6}} L/R=7.5, R/T=2100

{0.260227, {m → 1, n → 6}} L/R=7.5, R/T=2200

{0.249186, {m → 1, n → 6}} L/R=7.5, R/T=2300

{0.239497, {m → 1, n → 6}} L/R=7.5, R/T=2400

{0.230947, {m → 1, n → 6}} L/R=7.5, R/T=2500

{0.223365, {m → 1, n → 6}} L/R=7.5, R/T=2600

{0.216609, {m → 1, n → 6}} L/R=7.5, R/T=2700

{0.210564, {m → 1, n → 6}} L/R=7.5, R/T=2800

{0.203645, {m → 2, n → 9}} L/R=7.5, R/T=2900

{0.197103, {m → 2, n → 9}} L/R=7.5, R/T=3000

***** Results for L/R= 8.5 *****

{1.91394, {m → 1, n → 3}} L/R=8.5, R/T=300

{1.39619, {m → 1, n → 4}} L/R=8.5, R/T=400

{1.07126, {m → 1, n → 4}} L/R=8.5, R/T=500

{0.894757, {m → 1, n → 4}} L/R=8.5, R/T=600

{0.78833, {m → 1, n → 4}} L/R=8.5, R/T=700
 {0.719255, {m → 1, n → 4}} L/R=8.5, R/T=800
 {0.65213, {m → 2, n → 6}} L/R=8.5, R/T=900
 {0.583222, {m → 1, n → 5}} L/R=8.5, R/T=1000
 {0.518076, {m → 1, n → 5}} L/R=8.5, R/T=1100
 {0.468527, {m → 1, n → 5}} L/R=8.5, R/T=1200
 {0.429966, {m → 1, n → 5}} L/R=8.5, R/T=1300
 {0.399369, {m → 1, n → 5}} L/R=8.5, R/T=1400
 {0.374685, {m → 1, n → 5}} L/R=8.5, R/T=1500
 {0.354483, {m → 1, n → 5}} L/R=8.5, R/T=1600
 {0.33774, {m → 1, n → 5}} L/R=8.5, R/T=1700
 {0.32371, {m → 1, n → 5}} L/R=8.5, R/T=1800
 {0.311835, {m → 1, n → 5}} L/R=8.5, R/T=1900
 {0.295673, {m → 2, n → 8}} L/R=8.5, R/T=2000
 {0.280189, {m → 2, n → 8}} L/R=8.5, R/T=2100
 {0.266768, {m → 2, n → 8}} L/R=8.5, R/T=2200
 {0.253968, {m → 1, n → 6}} L/R=8.5, R/T=2300
 {0.241549, {m → 1, n → 6}} L/R=8.5, R/T=2400
 {0.230591, {m → 1, n → 6}} L/R=8.5, R/T=2500
 {0.220873, {m → 1, n → 6}} L/R=8.5, R/T=2600
 {0.212214, {m → 1, n → 6}} L/R=8.5, R/T=2700
 {0.204466, {m → 1, n → 6}} L/R=8.5, R/T=2800
 {0.197506, {m → 1, n → 6}} L/R=8.5, R/T=2900
 {0.191231, {m → 1, n → 6}} L/R=8.5, R/T=3000

***** Results for L/R= 9.5 *****

{1.73455, {m → 1, n → 3}} L/R=9.5, R/T=300
 {1.49424, {m → 1, n → 3}} L/R=9.5, R/T=400
 {1.11563, {m → 1, n → 4}} L/R=9.5, R/T=500
 {0.895903, {m → 1, n → 4}} L/R=9.5, R/T=600
 {0.763414, {m → 1, n → 4}} L/R=9.5, R/T=700
 {0.677424, {m → 1, n → 4}} L/R=9.5, R/T=800
 {0.618469, {m → 1, n → 4}} L/R=9.5, R/T=900
 {0.576299, {m → 1, n → 4}} L/R=9.5, R/T=1000
 {0.545097, {m → 1, n → 4}} L/R=9.5, R/T=1100
 {0.49167, {m → 1, n → 5}} L/R=9.5, R/T=1200
 {0.443607, {m → 1, n → 5}} L/R=9.5, R/T=1300

{0.405471, {m → 1, n → 5}} L/R=9.5, R/T=1400
 {0.374704, {m → 1, n → 5}} L/R=9.5, R/T=1500
 {0.349524, {m → 1, n → 5}} L/R=9.5, R/T=1600
 {0.328655, {m → 1, n → 5}} L/R=9.5, R/T=1700
 {0.311167, {m → 1, n → 5}} L/R=9.5, R/T=1800
 {0.296367, {m → 1, n → 5}} L/R=9.5, R/T=1900
 {0.283731, {m → 1, n → 5}} L/R=9.5, R/T=2000
 {0.272856, {m → 1, n → 5}} L/R=9.5, R/T=2100
 {0.263431, {m → 1, n → 5}} L/R=9.5, R/T=2200
 {0.255208, {m → 1, n → 5}} L/R=9.5, R/T=2300
 {0.247707, {m → 2, n → 8}} L/R=9.5, R/T=2400
 {0.236419, {m → 2, n → 8}} L/R=9.5, R/T=2500
 {0.226408, {m → 2, n → 8}} L/R=9.5, R/T=2600
 {0.217488, {m → 2, n → 8}} L/R=9.5, R/T=2700
 {0.209507, {m → 2, n → 8}} L/R=9.5, R/T=2800
 {0.201001, {m → 1, n → 6}} L/R=9.5, R/T=2900
 {0.193174, {m → 1, n → 6}} L/R=9.5, R/T=3000

***** Results for L/R= 10.5 *****

{1.64262, {m → 1, n → 3}} L/R=10.5, R/T=300
 {1.35037, {m → 1, n → 3}} L/R=10.5, R/T=400
 {1.11826, {m → 2, n → 5}} L/R=10.5, R/T=500
 {0.933939, {m → 1, n → 4}} L/R=10.5, R/T=600
 {0.772491, {m → 1, n → 4}} L/R=10.5, R/T=700
 {0.667705, {m → 1, n → 4}} L/R=10.5, R/T=800
 {0.595864, {m → 1, n → 4}} L/R=10.5, R/T=900
 {0.544477, {m → 1, n → 4}} L/R=10.5, R/T=1000
 {0.506456, {m → 1, n → 4}} L/R=10.5, R/T=1100
 {0.477538, {m → 1, n → 4}} L/R=10.5, R/T=1200
 {0.455033, {m → 1, n → 4}} L/R=10.5, R/T=1300
 {0.422154, {m → 2, n → 6}} L/R=10.5, R/T=1400
 {0.390348, {m → 1, n → 5}} L/R=10.5, R/T=1500
 {0.359637, {m → 1, n → 5}} L/R=10.5, R/T=1600
 {0.334184, {m → 1, n → 5}} L/R=10.5, R/T=1700
 {0.312854, {m → 1, n → 5}} L/R=10.5, R/T=1800
 {0.294803, {m → 1, n → 5}} L/R=10.5, R/T=1900
 {0.279391, {m → 1, n → 5}} L/R=10.5, R/T=2000

{0.266128, {m → 1, n → 5}} L/R=10.5, R/T=2100
 {0.254632, {m → 1, n → 5}} L/R=10.5, R/T=2200
 {0.244602, {m → 1, n → 5}} L/R=10.5, R/T=2300
 {0.2358, {m → 1, n → 5}} L/R=10.5, R/T=2400
 {0.228033, {m → 1, n → 5}} L/R=10.5, R/T=2500
 {0.221145, {m → 1, n → 5}} L/R=10.5, R/T=2600
 {0.215008, {m → 1, n → 5}} L/R=10.5, R/T=2700
 {0.209517, {m → 1, n → 5}} L/R=10.5, R/T=2800
 {0.204584, {m → 1, n → 5}} L/R=10.5, R/T=2900
 {0.197411, {m → 2, n → 8}} L/R=10.5, R/T=3000

***** Results for L/R= 11.5 *****

{1.61386, {m → 1, n → 3}} L/R=11.5, R/T=300
 {1.26446, {m → 1, n → 3}} L/R=11.5, R/T=400
 {1.10274, {m → 1, n → 3}} L/R=11.5, R/T=500
 {0.931705, {m → 2, n → 5}} L/R=11.5, R/T=600
 {0.803471, {m → 2, n → 5}} L/R=11.5, R/T=700
 {0.681624, {m → 1, n → 4}} L/R=11.5, R/T=800
 {0.595609, {m → 1, n → 4}} L/R=11.5, R/T=900
 {0.534082, {m → 1, n → 4}} L/R=11.5, R/T=1000
 {0.488559, {m → 1, n → 4}} L/R=11.5, R/T=1100
 {0.453935, {m → 1, n → 4}} L/R=11.5, R/T=1200
 {0.42699, {m → 1, n → 4}} L/R=11.5, R/T=1300
 {0.405609, {m → 1, n → 4}} L/R=11.5, R/T=1400
 {0.388361, {m → 1, n → 4}} L/R=11.5, R/T=1500
 {0.374244, {m → 1, n → 4}} L/R=11.5, R/T=1600
 {0.34904, {m → 2, n → 6}} L/R=11.5, R/T=1700
 {0.325139, {m → 1, n → 5}} L/R=11.5, R/T=1800
 {0.303511, {m → 1, n → 5}} L/R=11.5, R/T=1900
 {0.285046, {m → 1, n → 5}} L/R=11.5, R/T=2000
 {0.269155, {m → 1, n → 5}} L/R=11.5, R/T=2100
 {0.255381, {m → 1, n → 5}} L/R=11.5, R/T=2200
 {0.243365, {m → 1, n → 5}} L/R=11.5, R/T=2300
 {0.232819, {m → 1, n → 5}} L/R=11.5, R/T=2400
 {0.223513, {m → 1, n → 5}} L/R=11.5, R/T=2500
 {0.215261, {m → 1, n → 5}} L/R=11.5, R/T=2600
 {0.207908, {m → 1, n → 5}} L/R=11.5, R/T=2700

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{0.201328, {m → 1, n → 5}}    L/R=11.5, R/T=2800
{0.195418, {m → 1, n → 5}}    L/R=11.5, R/T=2900
{0.190088, {m → 1, n → 5}}    L/R=11.5, R/T=3000
***** Results for L/R= 12.5 *****
{1.63294, {m → 1, n → 3}}    L/R=12.5, R/T=300
{1.22119, {m → 1, n → 3}}    L/R=12.5, R/T=400
{1.03062, {m → 1, n → 3}}    L/R=12.5, R/T=500
{0.927091, {m → 1, n → 3}}    L/R=12.5, R/T=600
{0.798911, {m → 2, n → 5}}    L/R=12.5, R/T=700
{0.700937, {m → 2, n → 5}}    L/R=12.5, R/T=800
{0.612413, {m → 1, n → 4}}    L/R=12.5, R/T=900
{0.539825, {m → 1, n → 4}}    L/R=12.5, R/T=1000
{0.486118, {m → 1, n → 4}}    L/R=12.5, R/T=1100
{0.44527, {m → 1, n → 4}}    L/R=12.5, R/T=1200
{0.41348, {m → 1, n → 4}}    L/R=12.5, R/T=1300
{0.388256, {m → 1, n → 4}}    L/R=12.5, R/T=1400
{0.367907, {m → 1, n → 4}}    L/R=12.5, R/T=1500
{0.351252, {m → 1, n → 4}}    L/R=12.5, R/T=1600
{0.337449, {m → 1, n → 4}}    L/R=12.5, R/T=1700
{0.325882, {m → 1, n → 4}}    L/R=12.5, R/T=1800
{0.316093, {m → 1, n → 4}}    L/R=12.5, R/T=1900
{0.296554, {m → 2, n → 6}}    L/R=12.5, R/T=2000
{0.279681, {m → 1, n → 5}}    L/R=12.5, R/T=2100
{0.263423, {m → 1, n → 5}}    L/R=12.5, R/T=2200
{0.249239, {m → 1, n → 5}}    L/R=12.5, R/T=2300
{0.236791, {m → 1, n → 5}}    L/R=12.5, R/T=2400
{0.225807, {m → 1, n → 5}}    L/R=12.5, R/T=2500
{0.216065, {m → 1, n → 5}}    L/R=12.5, R/T=2600
{0.207386, {m → 1, n → 5}}    L/R=12.5, R/T=2700
{0.199619, {m → 1, n → 5}}    L/R=12.5, R/T=2800
{0.192643, {m → 1, n → 5}}    L/R=12.5, R/T=2900
{0.186352, {m → 1, n → 5}}    L/R=12.5, R/T=3000

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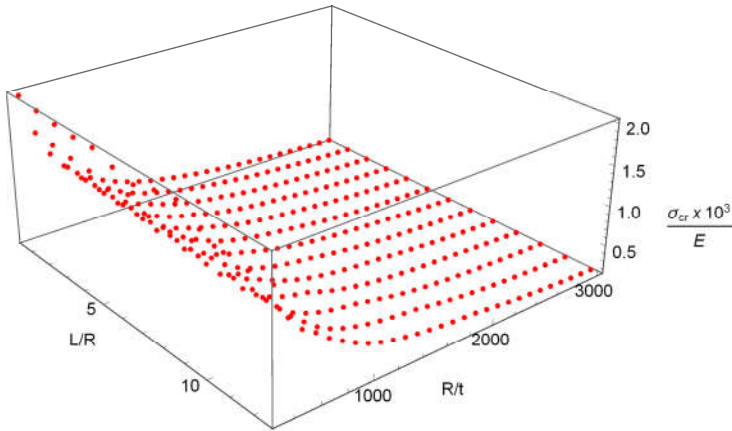
(* Collecting the above data : *)

$\sigma_{cr}data =$

Table[{LRdata[[i]], RTdata[[j]], sol[i, j][[1]]}, {i, 1, NoLR}, {j, 1, NoRT}];

(* Plotting the data in a 3D as in Figure 21 : *)

$\sigma_{cr}DataPlot = ListPointPlot3D[\sigma_{cr}data, PlotRange \rightarrow All,$
 $PlotStyle \rightarrow \{Red\}, AxesLabel \rightarrow \{ "L/R", "R/t", " \frac{\sigma_{cr} \times 10^3}{E} " \}]$



(* Fitting the data to the form of Equation 3.10: *)

$$\sigma_{cr1000ByE} = \frac{c1}{RT} LR^{c2};$$

$\sigma data = Flatten[\sigma_{cr}data, 1];$

$soln = FindFit[\sigma data, \sigma_{cr1000ByE}, \{c1, c2\}, \{LR, RT\}]$

{c1 $\rightarrow$ 603.132, c2 $\rightarrow$ -0.0425389}

(* Substituting the above values of the constants yields: *)

$\sigma_{cr1000ByE} = \sigma_{cr1000ByE} /. soln$

$$\frac{603.132}{LR^{0.0425389} RT}$$

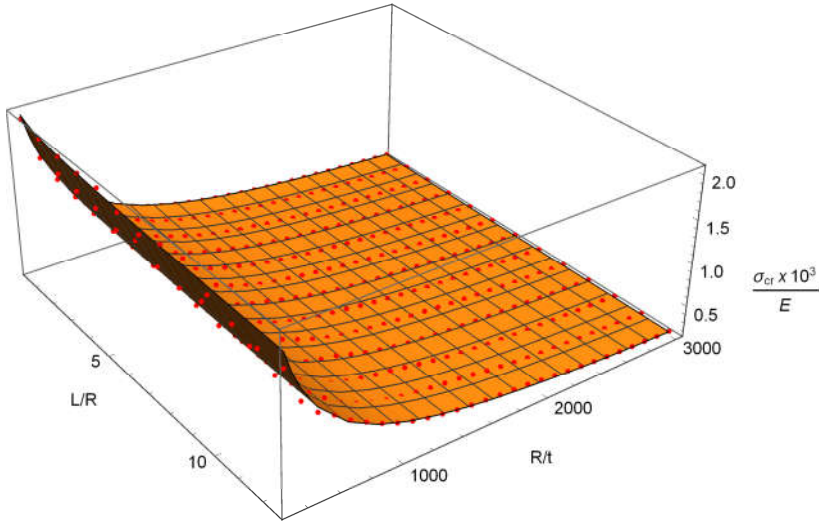
(*which is similar to Equation 3.11*)

(* It can also be put to appear as in the form of Equation 3.12. i.e: *)

$\sigma_{ByE} = \sigma_{cr1000ByE} / 1000$

$$\frac{0.603132}{LR^{0.0425389} RT}$$

```
(* To show the data and the simplified formula: *)
σExpPlot = Plot3D[σByE * 1000, {LR, 0.5, 12.5}, {RT, 300, 3000}, PlotRange → All];
Show[{σcrDataPlot, σExpPlot}, PlotRange → All]
```



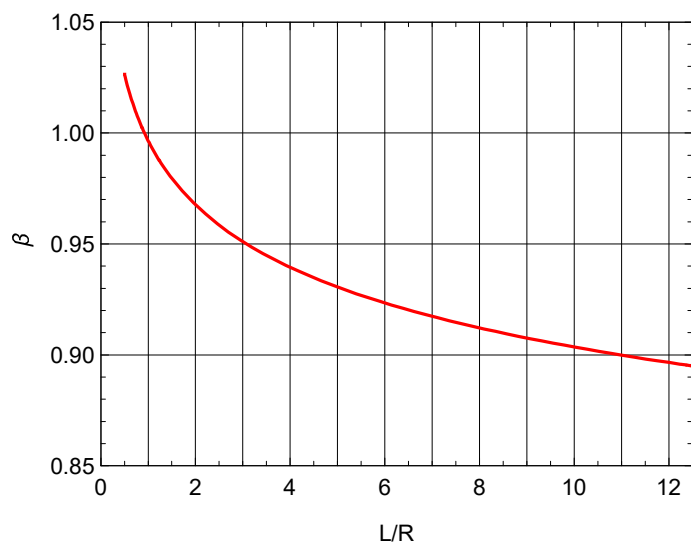
(*Comparison with the classical solution*)

$$\sigma_{cl} = \frac{1}{\sqrt{3(1-\nu^2)}} \frac{EE}{RT} \quad / . \nu \rightarrow 0.3;$$

$$\beta = \frac{\sigma_{ByE} * EE}{\sigma_{cl}}$$

$$\frac{0.996538}{LR^{0.0425389}}$$

```
Plot[ $\beta$ , {LR, 0.5, 12.5}, PlotStyle -> Red, AxesOrigin -> {0, 0},
  PlotRange -> {{0, 12.5}, {0.85, 1.05}}, AspectRatio -> 0.75,
  AxesStyle -> {Directive[Black, 10], Directive[Black, 10]},
  GridLines -> {Range[0, 12.5, 1], Range[0.85, 1.05, 0.05]},
  GridLinesStyle -> Directive[Black, Thin], AxesOrigin -> {0, 0}, Frame -> True,
  FrameLabel -> {"L/R", " $\beta$ "}, LabelStyle -> Directive[Black, 12]]
```



Appendix B: Reproduction of von Karman and Tsien's Work

Appendix B : Reproduction of von Karman and Tsien's Work [40]

B.1 Derivation of the Governing Differential Equations (F-VW Formulation)

Below is the derivation of the governing differential equation (Equation 4.6) resulting from the equilibrium in the in - plane directions.

```

Clear["Global`*"]
(* Definitions of the strains (Equations 4.1):*)

$$\epsilon_x = D[u[x, y], x] + \frac{1}{2} (D[w[x, y], x])^2;$$


$$\epsilon_y = D[v[x, y], y] + \frac{1}{2} (D[w[x, y], y])^2 - \frac{w[x, y]}{R};$$


$$\gamma_{xy} = D[u[x, y], y] + D[v[x, y], x] + D[w[x, y], x] * D[w[x, y], y];$$


(* Stress-strain relations (Equations 4.2):*)

$$\sigma_x = \frac{EE}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y);$$


$$\sigma_y = \frac{EE}{1 - \nu^2} (\epsilon_y + \nu \epsilon_x);$$


$$\tau_{xy} = \frac{EE}{2(1 + \nu)} \gamma_{xy};$$


rule = {u(1,0)[x, y] → ux, v(1,0)[x, y] → vx, u(0,1)[x, y] → uy,
  v(0,1)[x, y] → vy, u(0,2)[x, y] → uyy, u(2,0)[x, y] → uxx, v(1,1)[x, y] → vxy,
  v(2,0)[x, y] → vxx, v(0,2)[x, y] → vyy, u(1,1)[x, y] → uxy, w(0,1)[x, y] → wy,
  w(1,1)[x, y] → wxy, w(1,0)[x, y] → wx, w(2,0)[x, y] → wxx, w(0,2)[x, y] → wyy,
  w(0,0)[x, y] → w, w(1,2)[x, y] → wxyy, w(2,1)[x, y] → wxxy};


$$\sigma_x = \text{Simplify}[\sigma_x /. \text{rule}]$$


$$\sigma_y = \text{Simplify}[\sigma_y /. \text{rule}]$$


$$\tau_{xy} = \text{Simplify}[\tau_{xy} /. \text{rule}]$$


$$\frac{1}{1 - \nu^2} EE \left( ux + \frac{1}{2} \left( wx^2 + \left( 2 vy - \frac{2 w}{R} + wy^2 \right) \nu \right) \right)$$


$$\frac{1}{1 - \nu^2} EE \left( vy + \frac{1}{2} \left( - \frac{2 w}{R} + wy^2 + 2 ux \nu + wx^2 \nu \right) \right)$$


$$\frac{EE (uy + vx + wx wy)}{2 (1 + \nu)}$$


(*Introduction of the stress function, F (Equation 4.5)*)
sol1 = Solve[{ $\sigma_x == F_{yy}$ ,  $\sigma_y == F_{xx}$ }, {ux, vy}]

$$\left\{ \left\{ \begin{aligned} ux &\rightarrow -\frac{1}{2 EE} \left( -2 F_{yy} + EE wx^2 + 2 F_{xx} \nu \right), \\ vy &\rightarrow -\frac{1}{2 EE R} \left( -2 F_{xx} R - 2 EE w + EE R wy^2 + 2 F_{yy} R \nu \right) \end{aligned} \right\} \right\}$$


```

We know (From Equation 4.5) that $\tau_{xy} = -F_{xy}$ (the mixed derivative of F)

(*Differentiate both sides with x and y*)

(*First, let us start by the third term of τ_{xy} *)

ThirdTermOf $\tau_{xy} = (D[W[x, y], x] * D[W[x, y], y]);$

MixedDerOfTheThirdTermOf $\tau_{xy} = D[D[ThirdTermOf\tau_{xy}, x], y] /. rule$

$$w_{xy}^2 + w_x w_{xyy} + w_{xxy} w_y + w_{xx} w_{yy}$$

(*Thus, F_{xxyy} can be written as: *)

$$F_{xxyy} = - \frac{EE}{2(1+\nu)} (u_{xyy} + v_{xxy} + \text{MixedDerOfTheThirdTermOf}\tau_{xy})$$

$$- \left(EE (u_{xyy} + v_{xxy} + w_{xy}^2 + w_x w_{xyy} + w_{xxy} w_y + w_{xx} w_{yy}) \right) / (2(1+\nu))$$

u_x and u_y are already expressed previously in terms of F and w. So, to get u_{xyy} and v_{xxy} appears in the above expression, the following is required:

(* Double derivative of w_x^2 (appears in u_x) with respect to (wrt) y*)

DiffWxSqWRTyy = D[(D[W[x, y], x])^2, {y, 2}] /. rule;

(* Double derivative of w_y^2 (appears in v_y) wrt x *)

DiffWySqWRTxx = D[(D[W[x, y], y])^2, {x, 2}] /. rule;

F $_{xxyy} = .$

(* Then, u_{xyy} and v_{xxy} can be obtained:*)

$$u_{xyy} = - \frac{1}{2EE} (-2F_{yyyy} + EE \text{DiffWxSqWRTyy} + 2F_{xxyy} \nu)$$

$$v_{xxy} = - \frac{1}{2EE R} (-2F_{xxxx} R - 2EE w_{xx} + EE R \text{DiffWySqWRTxx} + 2F_{xxyy} R \nu)$$

$$- \frac{1}{2EE} (-2F_{yyyy} + EE (2w_{xy}^2 + 2w_x w_{xyy}) + 2F_{xxyy} \nu)$$

$$- \frac{1}{2EE R} (-2F_{xxxx} R - 2EE w_{xx} + EE R (2w_{xy}^2 + 2w_{xxy} w_y) + 2F_{xxyy} R \nu)$$

(*Substituting back into F_{xxyy} *)

eq1 =

$$\text{Simplify}[F_{xxyy} = \left(- \left(EE (u_{xyy} + v_{xxy} + w_{xy}^2 + w_x w_{xyy} + w_{xxy} w_y + w_{xx} w_{yy}) \right) / (2(1+\nu)) \right)]$$

$$\frac{1}{R + R \nu} (F_{xxxx} R + 2F_{xxyy} R + F_{yyyy} R + EE w_{xx} - EE R w_{xy}^2 + EE R w_{xx} w_{yy}) = 0$$

(*Organisation of the above equation yields :*)

$$\text{Del4F} = EE \left(w_{xy}^2 - w_{xx} w_{yy} - \frac{1}{R} w_{xx} \right);$$

(* which is Equation 4.6 *)

B.2 von Karman and Tsien's [40] Approximate Solution

Clear["Global`*"]

(*The first governing differential equation (Equation 4.6) is:*)

$$\text{Del4F} = \text{EE} \left(wxy^2 - wxx wyy - \frac{1}{R} wxx \right);$$

(*The solution starts first by assuming the expression w (Equation 4.14):*)

$$w = R \left(\left(f_0 + \frac{f_1}{4} \right) + \frac{f_1}{2} \left(\cos \left[\frac{mx}{R} \right] \cos \left[\frac{ny}{R} \right] + \frac{1}{4} \cos \left[\frac{2mx}{R} \right] + \frac{1}{4} \cos \left[\frac{2ny}{R} \right] \right) + \frac{f_2}{4} \left(\cos \left[\frac{2mx}{R} \right] + \cos \left[\frac{2ny}{R} \right] \right) \right);$$

$$wx = D[w, x];$$

$$wxx = D[wx, x];$$

$$wxy = D[wx, y];$$

$$wy = D[w, y];$$

$$wyy = D[w, \{y, 2\}];$$

$$\text{Del4F} = \text{Simplify} \left[\text{EE} \left(wxy^2 - wxx wyy - \frac{1}{R} wxx \right) \right]$$

$$- \frac{1}{4 R^2} \text{EE} m^2 \left(\left(\frac{1}{2} (f_1 + 2 f_2) \cos \left[\frac{2mx}{R} \right] \left(-4 + 2 f_1 n^2 \cos \left[\frac{mx}{R} \right] \cos \left[\frac{ny}{R} \right] + 2 (f_1 + 2 f_2) n^2 \cos \left[\frac{2ny}{R} \right] \right) + f_1 \left(f_1 n^2 \cos \left[\frac{mx}{R} \right]^2 \cos \left[\frac{ny}{R} \right]^2 + \cos \left[\frac{mx}{R} \right] \cos \left[\frac{ny}{R} \right] \left(-2 + (f_1 + 2 f_2) n^2 \cos \left[\frac{2ny}{R} \right] - f_1 n^2 \sin \left[\frac{mx}{R} \right]^2 \sin \left[\frac{ny}{R} \right]^2 \right) \right) \right)$$

(*Arranging Del4F: *)

$$\text{Del4F1} = \text{TrigFactor}[\text{Del4F}]$$

$$- \frac{1}{16 R^2} \text{EE} m^2 \left((-8 f_1 - 16 f_2 + 2 f_1^2 n^2) \cos \left[\frac{2mx}{R} \right] + 2 f_1^2 n^2 \cos \left[\frac{2ny}{R} \right] + (f_1^2 n^2 + 2 f_1 f_2 n^2) \cos \left[\frac{mx}{R} - \frac{3ny}{R} \right] + (2 f_1^2 n^2 + 8 f_1 f_2 n^2 + 8 f_2^2 n^2) \cos \left[\frac{2mx}{R} - \frac{2ny}{R} \right] + (-4 f_1 + 2 f_1^2 n^2 + 4 f_1 f_2 n^2) \cos \left[\frac{mx}{R} - \frac{ny}{R} \right] + (f_1^2 n^2 + 2 f_1 f_2 n^2) \cos \left[\frac{3mx}{R} - \frac{ny}{R} \right] + (-4 f_1 + 2 f_1^2 n^2 + 4 f_1 f_2 n^2) \cos \left[\frac{mx}{R} + \frac{ny}{R} \right] + (f_1^2 n^2 + 2 f_1 f_2 n^2) \cos \left[\frac{3mx}{R} + \frac{ny}{R} \right] + (2 f_1^2 n^2 + 8 f_1 f_2 n^2 + 8 f_2^2 n^2) \cos \left[\frac{2mx}{R} + \frac{2ny}{R} \right] + (f_1^2 n^2 + 2 f_1 f_2 n^2) \cos \left[\frac{mx}{R} + \frac{3ny}{R} \right] \right)$$

Then simplifying the above expression using the well know trigonometric expressions:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

This implies that the term $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$ which appears several times in the expression under consideration.

The expression referred to as "Del4F1" can then be written to be:

$$\begin{aligned} \text{Del4F2} = & -\frac{1}{16 R^2} \text{EE } m^2 \left((-8 f_1 - 16 f_2 + 2 f_1^2 n^2) \cos\left[\frac{2 m x}{R}\right] + \right. \\ & 2 f_1^2 n^2 \cos\left[\frac{2 n y}{R}\right] + 4 f_1 (-2 + (f_1 + 2 f_2) n^2) \cos\left[\frac{m x}{R}\right] \cos\left[\frac{n y}{R}\right] + \\ & 2 f_1 (f_1 + 2 f_2) n^2 \cos\left[\frac{3 m x}{R}\right] \cos\left[\frac{n y}{R}\right] + 2 (f_1^2 n^2 + 2 f_1 f_2 n^2) \cos\left[\frac{m x}{R}\right] \cos\left[\frac{3 n y}{R}\right] + \\ & \left. 2 (2 f_1^2 n^2 + 8 f_1 f_2 n^2 + 8 f_2^2 n^2) \cos\left[\frac{2 m x}{R}\right] \cos\left[\frac{2 n y}{R}\right] \right); \end{aligned}$$

(*To verify the above: *)

Simplify[Del4F1 - Del4F2]

0

(* Therefore the simplified Del4F can be written by :*)

$$\begin{aligned} \text{Del4F} = & -\frac{1}{16 R^2} \text{EE } m^2 \left((-8 f_1 - 16 f_2 + 2 f_1^2 n^2) \cos\left[\frac{2 m x}{R}\right] + \right. \\ & 2 f_1^2 n^2 \cos\left[\frac{2 n y}{R}\right] + 4 f_1 (-2 + (f_1 + 2 f_2) n^2) \cos\left[\frac{m x}{R}\right] \cos\left[\frac{n y}{R}\right] + \\ & 2 f_1 (f_1 + 2 f_2) n^2 \cos\left[\frac{3 m x}{R}\right] \cos\left[\frac{n y}{R}\right] + 2 (f_1^2 n^2 + 2 f_1 f_2 n^2) \cos\left[\frac{m x}{R}\right] \cos\left[\frac{3 n y}{R}\right] + \\ & \left. 2 (2 f_1^2 n^2 + 8 f_1 f_2 n^2 + 8 f_2^2 n^2) \cos\left[\frac{2 m x}{R}\right] \cos\left[\frac{2 n y}{R}\right] \right); \end{aligned}$$

(*Let*)

$$\mu = m / n;$$

$$A = \frac{1}{8} f_1^2 n^2 - \left(\frac{1}{2} f_1 + f_2\right);$$

$$B = \frac{1}{8} f_1^2 n^2;$$

$$CC = \frac{1}{2} f_1 n^2 \left(\frac{1}{2} f_1 + f_2\right) - \frac{1}{2} f_1;$$

$$DD = \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2\right);$$

$$G = \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2\right);$$

$$H = n^2 \left(\frac{1}{2} f_1 + f_2\right)^2;$$

(*Del4F can then be formed to be: *)

Clear[μ, A, B, CC, DD, G, H]

$$\begin{aligned} \text{Del4F} = & -\text{EE } \mu^2 \left(\frac{n}{R}\right)^2 \left(A \cos\left[\frac{2 m x}{R}\right] + B \cos\left[\frac{2 n y}{R}\right] + CC \cos\left[\frac{m x}{R}\right] \cos\left[\frac{n y}{R}\right] + \right. \\ & \left. DD \cos\left[\frac{3 m x}{R}\right] \cos\left[\frac{n y}{R}\right] + G \cos\left[\frac{m x}{R}\right] \cos\left[\frac{3 n y}{R}\right] + H \cos\left[\frac{2 m x}{R}\right] \cos\left[\frac{2 n y}{R}\right] \right); \end{aligned}$$

(*Let us now solve the above differential equation *)

(*By inspection, the particular solution can be obtained to be *)

$$Fp = -EE \mu^2 \left(\frac{n}{R} \right)^2 \left(\frac{R^4}{16 m^4} A \cos \left[\frac{2 m x}{R} \right] + \frac{R^4}{16 n^4} B \cos \left[\frac{2 n y}{R} \right] + \right. \\ \left. \frac{R^4}{(m^2 + n^2)^2} CC \cos \left[\frac{m x}{R} \right] \cos \left[\frac{n y}{R} \right] + \frac{R^4}{(9 m^2 + n^2)^2} DD \cos \left[\frac{3 m x}{R} \right] \cos \left[\frac{n y}{R} \right] + \right. \\ \left. \frac{R^4}{(m^2 + 9 n^2)^2} G \cos \left[\frac{m x}{R} \right] \cos \left[\frac{3 n y}{R} \right] + \frac{R^4}{16 (m^2 + n^2)^2} H \cos \left[\frac{2 m x}{R} \right] \cos \left[\frac{2 n y}{R} \right] \right);$$

(* To verify the particular solution: *)

Simplify[Simplify[D[Fp, {x, 4}] + 2 D[D[Fp, {x, 2}], {y, 2}] + D[Fp, {y, 4}]] - Del4F]

0

(* The homogenous solution is selected to be:*)

Fh = $\alpha x^2 / 2 + \beta y^2 / 2$;

Simplify[D[Fh, {x, 4}] + 2 D[D[Fh, {x, 2}], {y, 2}] + D[Fh, {y, 4}]]

0

(* The F can then be written, after rearrangement, by : *)

$$F = -EE \mu^2 \left(\frac{R}{n} \right)^2 \left(\frac{A}{16 \mu^4} \cos \left[\frac{2 m x}{R} \right] + \frac{B}{16} \cos \left[\frac{2 n y}{R} \right] + \frac{CC}{(\mu^2 + 1)^2} \cos \left[\frac{m x}{R} \right] \cos \left[\frac{n y}{R} \right] + \right. \\ \left. \frac{DD}{(9 \mu^2 + 1)^2} \cos \left[\frac{3 m x}{R} \right] \cos \left[\frac{n y}{R} \right] + \frac{G}{(\mu^2 + 9)^2} \cos \left[\frac{m x}{R} \right] \cos \left[\frac{3 n y}{R} \right] + \right. \\ \left. \frac{H}{16 (\mu^2 + 1)^2} \cos \left[\frac{2 m x}{R} \right] \cos \left[\frac{2 n y}{R} \right] \right) + (\alpha x^2 / 2 + \beta y^2 / 2);$$

(*Calculation of the stresses: *)

$$F_x = D[F, x]; F_{xx} = D[F, \{x, 2\}]; F_{xy} = D[F_x, y]; F_{yy} = D[F, \{y, 2\}];$$

$$u_x = - \frac{-2 F_{yy} + EE w x^2 + 2 F_{xx} v}{2 EE};$$

$$v_y = - \frac{1}{2 EE R} (-2 F_{xx} R - 2 EE w + EE R w y^2 + 2 F_{yy} R v);$$

$$\sigma_x = \text{Simplify}[F_{yy}]$$

$$\sigma_y = \text{Simplify}[F_{xx}]$$

$$\tau_{xy} = \text{Simplify}[-F_{xy}]$$

$$\beta + \left(DD EE \mu^2 \cos \left[\frac{3 m x}{R} \right] \cos \left[\frac{n y}{R} \right] \right) / (1 + 9 \mu^2)^2 + \\ \frac{1}{4} B EE \mu^2 \cos \left[\frac{2 n y}{R} \right] + \left(EE H \mu^2 \cos \left[\frac{2 m x}{R} \right] \cos \left[\frac{2 n y}{R} \right] \right) / (4 (1 + \mu^2)^2) + \\ EE \mu^2 \cos \left[\frac{m x}{R} \right] \left(\frac{CC \cos \left[\frac{n y}{R} \right]}{(1 + \mu^2)^2} + \frac{9 G \cos \left[\frac{3 n y}{R} \right]}{(9 + \mu^2)^2} \right)$$

$$\alpha + \left(9 DD EE m^2 \mu^2 \cos \left[\frac{3 m x}{R} \right] \cos \left[\frac{n y}{R} \right] \right) / (n + 9 n \mu^2)^2 + \\ \left(EE m^2 \cos \left[\frac{2 m x}{R} \right] \left(A (1 + \mu^2)^2 + H \mu^4 \cos \left[\frac{2 n y}{R} \right] \right) \right) / (4 n^2 \mu^2 (1 + \mu^2)^2) + \\ \frac{1}{n^2} EE m^2 \mu^2 \cos \left[\frac{m x}{R} \right] \left(\frac{CC \cos \left[\frac{n y}{R} \right]}{(1 + \mu^2)^2} + \frac{G \cos \left[\frac{3 n y}{R} \right]}{(9 + \mu^2)^2} \right)$$

$$\frac{1}{4 n} EE m \mu^2 \left(\left(12 DD \sin \left[\frac{3 m x}{R} \right] \sin \left[\frac{n y}{R} \right] \right) / (1 + 9 \mu^2)^2 + \right. \\ \left. \frac{H \sin \left[\frac{2 m x}{R} \right] \sin \left[\frac{2 n y}{R} \right]}{(1 + \mu^2)^2} + \sin \left[\frac{m x}{R} \right] \left(\frac{4 CC \sin \left[\frac{n y}{R} \right]}{(1 + \mu^2)^2} + \frac{12 G \sin \left[\frac{3 n y}{R} \right]}{(9 + \mu^2)^2} \right) \right)$$

(*Satisfying the boundary conditions to get α and β :*)

Simplify[ox /. x → a]

$$\beta + \left(DD \, EE \, \mu^2 \cos \left[\frac{3 a m}{R} \right] \cos \left[\frac{n y}{R} \right] \right) / (1 + 9 \mu^2)^2 +$$

$$\frac{1}{4} B \, EE \, \mu^2 \cos \left[\frac{2 n y}{R} \right] + \left(EE \, H \, \mu^2 \cos \left[\frac{2 a m}{R} \right] \cos \left[\frac{2 n y}{R} \right] \right) / (4 (1 + \mu^2)^2) +$$

$$EE \, \mu^2 \cos \left[\frac{a m}{R} \right] \left(\frac{CC \cos \left[\frac{n y}{R} \right]}{(1 + \mu^2)^2} + \frac{9 G \cos \left[\frac{3 n y}{R} \right]}{(9 + \mu^2)^2} \right)$$

(*Since the axial applied stress is uniform

(equal to the average compressive stress) as commonly performed experimentally,
all terms depends on y should vanish leading to *)

$$\beta = -\sigma;$$

(*Calculating the the displacements v: *)

vy = Simplify[vy]

$$-\frac{1}{8 EE}$$

$$\left(-EE \left(8 f_0 + 2 f_1 + (f_1 + 2 f_2) \cos \left[\frac{2 m x}{R} \right] + 4 f_1 \cos \left[\frac{m x}{R} \right] \cos \left[\frac{n y}{R} \right] + f_1 \cos \left[\frac{2 n y}{R} \right] + 2 f_2 \right. \right.$$

$$\left. \cos \left[\frac{2 n y}{R} \right] \right) - 8 \left(\alpha + \left(9 DD \, EE \, m^2 \, \mu^2 \cos \left[\frac{3 m x}{R} \right] \cos \left[\frac{n y}{R} \right] \right) / (n + 9 n \mu^2)^2 + \right.$$

$$\left. \left(EE \, m^2 \cos \left[\frac{2 m x}{R} \right] \left(A (1 + \mu^2)^2 + H \mu^4 \cos \left[\frac{2 n y}{R} \right] \right) \right) / (4 n^2 \mu^2 (1 + \mu^2)^2) + \right.$$

$$\left. \frac{1}{n^2} EE \, m^2 \, \mu^2 \cos \left[\frac{m x}{R} \right] \left(\frac{CC \cos \left[\frac{n y}{R} \right]}{(1 + \mu^2)^2} + \frac{G \cos \left[\frac{3 n y}{R} \right]}{(9 + \mu^2)^2} \right) \right) +$$

$$2 \vee \left(-4 \sigma + \left(4 DD \, EE \, \mu^2 \cos \left[\frac{3 m x}{R} \right] \cos \left[\frac{n y}{R} \right] \right) / (1 + 9 \mu^2)^2 + B \, EE \, \mu^2 \cos \left[\frac{2 n y}{R} \right] + \right.$$

$$\left. \left(EE \, H \, \mu^2 \cos \left[\frac{2 m x}{R} \right] \cos \left[\frac{2 n y}{R} \right] \right) / (1 + \mu^2)^2 + \right.$$

$$\left. 4 \, EE \, \mu^2 \cos \left[\frac{m x}{R} \right] \left(\frac{CC \cos \left[\frac{n y}{R} \right]}{(1 + \mu^2)^2} + \frac{9 G \cos \left[\frac{3 n y}{R} \right]}{(9 + \mu^2)^2} \right) \right) +$$

$$EE \, n^2 \left(f_1 \cos \left[\frac{m x}{R} \right] + (f_1 + 2 f_2) \cos \left[\frac{n y}{R} \right] \right)^2 \sin \left[\frac{n y}{R} \right]^2 \Big)$$

(*Simplifying the term vy*)

TrigExpand[vy]

$$\begin{aligned}
& f_0 + \frac{f_1}{4} - \frac{3 f_1^2 n^2}{64} - \frac{1}{16} f_1 f_2 n^2 - \frac{f_2^2 n^2}{16} + \frac{\alpha}{EE} + \frac{\gamma \sigma}{EE} + \frac{1}{8} f_1 \cos\left[\frac{mx}{R}\right]^2 + \\
& \frac{1}{4} f_2 \cos\left[\frac{mx}{R}\right]^2 - \frac{1}{32} f_1^2 n^2 \cos\left[\frac{mx}{R}\right]^2 + \frac{A m^2 \cos\left[\frac{mx}{R}\right]^2}{2 n^2 (1 + \mu^2)^2} + \frac{A m^2 \cos\left[\frac{mx}{R}\right]^2}{4 n^2 \mu^2 (1 + \mu^2)^2} + \\
& \frac{A m^2 \mu^2 \cos\left[\frac{mx}{R}\right]^2}{4 n^2 (1 + \mu^2)^2} + \frac{1}{2} f_1 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] - \frac{1}{16} f_1^2 n^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] - \\
& \frac{1}{8} f_1 f_2 n^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] + \frac{CC m^2 \mu^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right]}{n^2 (1 + \mu^2)^2} - \frac{CC \mu^2 \gamma \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right]}{(1 + \mu^2)^2} + \\
& \left(9 DD m^2 \mu^2 \cos\left[\frac{mx}{R}\right]^3 \cos\left[\frac{ny}{R}\right]\right) / (n + 9 n \mu^2)^2 - \frac{DD \mu^2 \gamma \cos\left[\frac{mx}{R}\right]^3 \cos\left[\frac{ny}{R}\right]}{(1 + 9 \mu^2)^2} + \\
& \frac{1}{8} f_1 \cos\left[\frac{ny}{R}\right]^2 + \frac{1}{4} f_2 \cos\left[\frac{ny}{R}\right]^2 + \frac{1}{32} f_1^2 n^2 \cos\left[\frac{ny}{R}\right]^2 - \frac{1}{4} B \mu^2 \gamma \cos\left[\frac{ny}{R}\right]^2 + \\
& \frac{1}{32} f_1^2 n^2 \cos\left[\frac{mx}{R}\right]^2 \cos\left[\frac{ny}{R}\right]^2 + \frac{H m^2 \mu^2 \cos\left[\frac{mx}{R}\right]^2 \cos\left[\frac{ny}{R}\right]^2}{4 n^2 (1 + \mu^2)^2} - \frac{H \mu^2 \gamma \cos\left[\frac{mx}{R}\right]^2 \cos\left[\frac{ny}{R}\right]^2}{4 (1 + \mu^2)^2} + \\
& \frac{1}{16} f_1^2 n^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right]^3 + \frac{1}{8} f_1 f_2 n^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right]^3 + \\
& \frac{G m^2 \mu^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right]^3}{n^2 (9 + \mu^2)^2} - \frac{9 G \mu^2 \gamma \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right]^3}{(9 + \mu^2)^2} + \frac{1}{64} f_1^2 n^2 \cos\left[\frac{ny}{R}\right]^4 + \\
& \frac{1}{16} f_1 f_2 n^2 \cos\left[\frac{ny}{R}\right]^4 + \frac{1}{16} f_2^2 n^2 \cos\left[\frac{ny}{R}\right]^4 - \frac{1}{8} f_1 \sin\left[\frac{mx}{R}\right]^2 - \\
& \frac{1}{4} f_2 \sin\left[\frac{mx}{R}\right]^2 + \frac{1}{32} f_1^2 n^2 \sin\left[\frac{mx}{R}\right]^2 - \frac{A m^2 \sin\left[\frac{mx}{R}\right]^2}{2 n^2 (1 + \mu^2)^2} - \frac{A m^2 \sin\left[\frac{mx}{R}\right]^2}{4 n^2 \mu^2 (1 + \mu^2)^2} - \\
& \frac{A m^2 \mu^2 \sin\left[\frac{mx}{R}\right]^2}{4 n^2 (1 + \mu^2)^2} - \left(27 DD m^2 \mu^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] \sin\left[\frac{mx}{R}\right]^2\right) / (n + 9 n \mu^2)^2 + \\
& \left(3 DD \mu^2 \gamma \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] \sin\left[\frac{mx}{R}\right]^2\right) / (1 + 9 \mu^2)^2 - \frac{1}{32} f_1^2 n^2 \cos\left[\frac{ny}{R}\right]^2 \sin\left[\frac{mx}{R}\right]^2 - \\
& \frac{H m^2 \mu^2 \cos\left[\frac{ny}{R}\right]^2 \sin\left[\frac{mx}{R}\right]^2}{4 n^2 (1 + \mu^2)^2} + \frac{H \mu^2 \gamma \cos\left[\frac{ny}{R}\right]^2 \sin\left[\frac{mx}{R}\right]^2}{4 (1 + \mu^2)^2} - \frac{1}{8} f_1 \sin\left[\frac{ny}{R}\right]^2 - \\
& \frac{1}{4} f_2 \sin\left[\frac{ny}{R}\right]^2 - \frac{1}{32} f_1^2 n^2 \sin\left[\frac{ny}{R}\right]^2 + \frac{1}{4} B \mu^2 \gamma \sin\left[\frac{ny}{R}\right]^2 - \\
& \frac{1}{32} f_1^2 n^2 \cos\left[\frac{mx}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2 - \frac{H m^2 \mu^2 \cos\left[\frac{mx}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2}{4 n^2 (1 + \mu^2)^2} + \frac{H \mu^2 \gamma \cos\left[\frac{mx}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2}{4 (1 + \mu^2)^2} - \\
& \frac{3}{16} f_1^2 n^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] \sin\left[\frac{ny}{R}\right]^2 - \frac{3}{8} f_1 f_2 n^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] \sin\left[\frac{ny}{R}\right]^2 - \\
& \left(3 G m^2 \mu^2 \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] \sin\left[\frac{ny}{R}\right]^2\right) / (n^2 (9 + \mu^2)^2) + \\
& \left(27 G \mu^2 \gamma \cos\left[\frac{mx}{R}\right] \cos\left[\frac{ny}{R}\right] \sin\left[\frac{ny}{R}\right]^2\right) / (9 + \mu^2)^2 - \frac{3}{32} f_1^2 n^2 \cos\left[\frac{ny}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2 - \\
& \frac{3}{8} f_1 f_2 n^2 \cos\left[\frac{ny}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2 - \frac{3}{8} f_2^2 n^2 \cos\left[\frac{ny}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2 + \\
& \frac{1}{32} f_1^2 n^2 \sin\left[\frac{mx}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2 + \frac{H m^2 \mu^2 \sin\left[\frac{mx}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2}{4 n^2 (1 + \mu^2)^2} - \frac{H \mu^2 \gamma \sin\left[\frac{mx}{R}\right]^2 \sin\left[\frac{ny}{R}\right]^2}{4 (1 + \mu^2)^2} + \\
& \frac{1}{64} f_1^2 n^2 \sin\left[\frac{ny}{R}\right]^4 + \frac{1}{16} f_1 f_2 n^2 \sin\left[\frac{ny}{R}\right]^4 + \frac{1}{16} f_2^2 n^2 \sin\left[\frac{ny}{R}\right]^4
\end{aligned}$$

In short, v_y can be written as :

$v_y =$

$$f_0 + \frac{f_1}{4} - \frac{3 f_1^2 n^2}{64} - \frac{1}{16} f_1 f_2 n^2 - \frac{f_2^2 n^2}{16} + \frac{\alpha}{EE} + \frac{\nu \sigma}{EE} + \text{Terms of periodic functions in } x \text{ and } y.$$

Since y is measured along the circumference of the shell, -

v must be a periodic function in y ; therefore, -

the constant term in v_y must be equal to zero. Or

$$f_0 + \frac{f_1}{4} - \frac{3 f_1^2 n^2}{64} - \frac{1}{16} f_1 f_2 n^2 - \frac{f_2^2 n^2}{16} + \frac{\alpha}{EE} + \frac{\nu \sigma}{EE} = 0$$

This conditions determine α .

$$\text{sol1} = \text{Solve}\left[f_0 + \frac{f_1}{4} - \frac{3 f_1^2 n^2}{64} - \frac{1}{16} f_1 f_2 n^2 - \frac{f_2^2 n^2}{16} + \frac{\alpha}{EE} + \frac{\nu \sigma}{EE} == 0, \alpha\right]$$

$$\left\{ \left\{ \alpha \rightarrow \frac{1}{64} \left(-64 EE f_0 - 16 EE f_1 + 3 EE f_1^2 n^2 + 4 EE f_1 f_2 n^2 + 4 EE f_2^2 n^2 - 64 \nu \sigma \right) \right\} \right\}$$

(*Updating σ_y considering the obtained expression for α : *)

$\sigma_y = \text{Simplify}[\sigma_y /. \text{sol1}[[1]]]$

$$\begin{aligned} & \frac{1}{64} \left(EE \left(-64 f_0 + 3 f_1^2 n^2 + 4 f_2^2 n^2 + 4 f_1 \left(-4 + f_2 n^2 \right) \right) - 64 \nu \sigma \right) + \\ & \left(9 DD EE m^2 \mu^2 \cos\left[\frac{3 m x}{R}\right] \cos\left[\frac{n y}{R}\right] \right) / \left(n + 9 n \mu^2 \right)^2 + \\ & \left(EE m^2 \cos\left[\frac{2 m x}{R}\right] \left(A \left(1 + \mu^2 \right)^2 + H \mu^4 \cos\left[\frac{2 n y}{R}\right] \right) \right) / \left(4 n^2 \mu^2 \left(1 + \mu^2 \right)^2 \right) + \\ & \frac{1}{n^2} EE m^2 \mu^2 \cos\left[\frac{m x}{R}\right] \left(\frac{CC \cos\left[\frac{n y}{R}\right]}{\left(1 + \mu^2 \right)^2} + \frac{G \cos\left[\frac{3 n y}{R}\right]}{\left(9 + \mu^2 \right)^2} \right) \end{aligned}$$

(*Calculation of u_x *)

$u_x = \text{Simplify}[u_x /. \text{sol1}[[1]]];$

(*Calculaing the membrane strain energy (Equation 4.9):*)

$$\text{rule} = \left\{ x \rightarrow \frac{\pi R x}{a}, y \rightarrow \frac{\pi R y}{b} \right\};$$

$$u_m = (\sigma_x + \sigma_y)^2 - 2 (1 + \nu) (\sigma_x \sigma_y - \tau_{xy}^2) /. \text{rule};$$

(* Expanding the term um to integrate

its part and summing them (this is faster): *)

um = Expand[um];

(*Length of the parts in um: *)

Lum = Length[um];

(* Summing the integrals of components of

um is faster than integrating the full expression *)

$$U_m = \frac{t}{2 EE} 4 \text{Sum}\left[\text{Simplify}\left[\int_0^b \int_0^a \text{um}[[i]] dx dy, \{m, n\} \in \text{Integers}\right], \{i, 1, \text{Lum}\}\right]$$

$$\frac{1}{EE} 2 t \left(a b EE^2 f_0^2 + \frac{1}{2} a b EE^2 f_0 f_1 + \frac{1}{16} a b EE^2 f_1^2 - \frac{3}{32} a b EE^2 f_0 f_1^2 n^2 - \right.$$

$$\frac{3}{128} a b EE^2 f_1^3 n^2 - \frac{1}{8} a b EE^2 f_0 f_1 f_2 n^2 - \frac{1}{32} a b EE^2 f_1^2 f_2 n^2 - \frac{1}{8} a b EE^2 f_0 f_2^2 n^2 -$$

$$\frac{1}{32} a b EE^2 f_1 f_2^2 n^2 + \frac{9 a b EE^2 f_1^4 n^4}{4096} + \frac{3}{512} a b EE^2 f_1^3 f_2 n^4 + \frac{5}{512} a b EE^2 f_1^2 f_2^2 n^4 +$$

$$\frac{1}{128} a b EE^2 f_1 f_2^3 n^4 + \frac{1}{256} a b EE^2 f_2^4 n^4 + \frac{1}{32} a b B^2 EE^2 \mu^4 + \frac{3 a A^2 b EE^2 m^4}{16 n^4 (1 + \mu^2)^4} +$$

$$\frac{a A^2 b EE^2 m^4}{32 n^4 \mu^4 (1 + \mu^2)^4} + \frac{a A^2 b EE^2 m^4}{8 n^4 \mu^2 (1 + \mu^2)^4} + \frac{a A^2 b EE^2 m^4 \mu^2}{8 n^4 (1 + \mu^2)^4} + \frac{a b CC^2 EE^2 \mu^4}{4 (1 + \mu^2)^4} +$$

$$\frac{a b EE^2 H^2 \mu^4}{64 (1 + \mu^2)^4} + \frac{a A^2 b EE^2 m^4 \mu^4}{32 n^4 (1 + \mu^2)^4} + \frac{a b CC^2 EE^2 m^4 \mu^4}{4 n^4 (1 + \mu^2)^4} + \frac{a b EE^2 H^2 m^4 \mu^4}{64 n^4 (1 + \mu^2)^4} + \frac{a b CC^2 EE^2 m^2 \mu^4}{2 n^2 (1 + \mu^2)^4} +$$

$$\frac{a b EE^2 H^2 m^2 \mu^4}{32 n^2 (1 + \mu^2)^4} + \frac{81 a b EE^2 G^2 \mu^4}{4 (9 + \mu^2)^4} + \frac{a b EE^2 G^2 m^4 \mu^4}{4 n^4 (9 + \mu^2)^4} + \frac{9 a b EE^2 G^2 m^2 \mu^4}{2 n^2 (9 + \mu^2)^4} +$$

$$\frac{a b DD^2 EE^2 \mu^4}{4 (1 + 9 \mu^2)^4} + \frac{9 a b DD^2 EE^2 m^2 \mu^4}{2 n^2 (1 + 9 \mu^2)^4} + \frac{81 a b DD^2 EE^2 m^4 \mu^4}{4 (n + 9 n \mu^2)^4} + a b \sigma^2 - a b \nu^2 \sigma^2 \Big)$$

(* Defining the term $\mu=m/n$: *)

Um = Simplify[Um /. m $\rightarrow$ μ n]

$$\frac{1}{2048 EE} a b t \left(EE^2 (4096 f_0^2 + 9 f_1^4 n^4 + 24 f_1^3 n^2 (-4 + f_2 n^2) + \right.$$

$$32 f_1 f_2^2 n^2 (-4 + f_2 n^2) + 8 f_1^2 (32 - 16 f_2 n^2 + 5 f_2^2 n^4) -$$

$$128 f_0 (3 f_1^2 n^2 + 4 f_2^2 n^2 + 4 f_1 (-4 + f_2 n^2)) + (1 / ((1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2))$$

$$16 (8 A^2 (9 + 91 \mu^2 + 91 \mu^4 + 9 \mu^6)^2 + f_2^4 n^4 (9 + 91 \mu^2 + 91 \mu^4 + 9 \mu^6)^2 +$$

$$4 \mu^4 (1296 DD^2 + 16 G^2 + 81 H^2 + 2880 DD^2 \mu^2 + 320 G^2 \mu^2 + 1476 H^2 \mu^2 +$$

$$1888 DD^2 \mu^4 + 1888 G^2 \mu^4 + 6886 H^2 \mu^4 + 320 DD^2 \mu^6 + 2880 G^2 \mu^6 +$$

$$1476 H^2 \mu^6 + 16 DD^2 \mu^8 + 1296 G^2 \mu^8 + 81 H^2 \mu^8 + 16 CC^2 (9 + 82 \mu^2 + 9 \mu^4)^2 +$$

$$2 B^2 (9 + 91 \mu^2 + 91 \mu^4 + 9 \mu^6)^2) - 4096 (-1 + \nu^2) \sigma^2 \Big)$$

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(* Calculating the bending strain energy (Equation 4.11): *)
ub = ((wxx + wyy)^2 - 2 (1 - ν) (wxx wyy - wxy^2)) /. rule;
(* Expanding the terms to separate it to summed parts: *)
ub = Expand[ub];
(*Length of ub :*)
Lub = Length[ub];
Ub = 
$$\frac{t^3 EE}{24 (1 - \nu^2)} 4 \text{Sum}\left[\text{Simplify}\left[\int_0^b \int_0^a \text{ub}[[i]] \, dx \, dy, \{m, n\} \in \text{Integers}\right], \{i, 1, \text{Lub}\}\right]$$


$$\left(EE \left(\frac{3 a b f_1^2 m^4}{16 R^2} + \frac{a b f_1 f_2 m^4}{2 R^2} + \frac{a b f_2^2 m^4}{2 R^2} + \frac{a b f_1^2 m^2 n^2}{8 R^2} + \frac{3 a b f_1^2 n^4}{16 R^2} + \frac{a b f_1 f_2 n^4}{2 R^2} + \frac{a b f_2^2 n^4}{2 R^2}\right) t^3\right) / (6 (1 - \nu^2))$$

(* Defining the term  $\mu=m/n$ : *)
Ub = Simplify[Ub /. m → μ n]
- ((a b EE n^4 t^3 (8 f1 f2 (1 + μ^4) + 8 f2^2 (1 + μ^4) + f1^2 (3 + 2 μ^2 + 3 μ^4))) / (96 R^2 (-1 + ν^2)))

(* Calculating the work done by the applied axial stress (Equation 4.12): *)
σATa = (σx /. {x →  $\frac{\pi R a}{a}$ , y →  $\frac{\pi R y}{b}$ });
DuDx = ux /. rule;
work = Expand[σATa * DuDx];
(*Length of the work done by the applied axial stress :*)
Lwork = Length[work];
W = 4 t Sum[Simplify[ $\int_0^b \int_0^a \text{work}[[i]] \, dx \, dy, \{m, n\} \in \text{Integers}\}], \{i, 1, \text{Lwork}\}]$ 

$$4 t \left( -\frac{1}{256} a b B EE f_1^2 m^2 \mu^2 + \frac{1}{32} a b B^2 EE \mu^4 - \frac{a b EE f_1^2 H m^2 \mu^2}{256 (1 + \mu^2)^2} + \frac{a b B EE H \mu^4}{32 (1 + \mu^2)^2} + \frac{3}{64} a b f_1^2 m^2 \sigma + \frac{1}{16} a b f_1 f_2 m^2 \sigma + \frac{1}{16} a b f_2^2 m^2 \sigma - a b f_0 \nu \sigma - \frac{1}{4} a b f_1 \nu \sigma + \frac{3}{64} a b f_1^2 n^2 \nu \sigma + \frac{1}{16} a b f_1 f_2 n^2 \nu \sigma + \frac{1}{16} a b f_2^2 n^2 \nu \sigma + \frac{a b \sigma^2}{EE} - \frac{a b \nu^2 \sigma^2}{EE} \right)$$

W = W /. m → μ n

$$4 t \left( \frac{1}{32} a b B^2 EE \mu^4 - \frac{1}{256} a b B EE f_1^2 n^2 \mu^4 + \frac{a b B EE H \mu^4}{32 (1 + \mu^2)^2} - \frac{a b EE f_1^2 H n^2 \mu^4}{256 (1 + \mu^2)^2} + \frac{3}{64} a b f_1^2 n^2 \mu^2 \sigma + \frac{1}{16} a b f_1 f_2 n^2 \mu^2 \sigma + \frac{1}{16} a b f_2^2 n^2 \mu^2 \sigma - a b f_0 \nu \sigma - \frac{1}{4} a b f_1 \nu \sigma + \frac{3}{64} a b f_1^2 n^2 \nu \sigma + \frac{1}{16} a b f_1 f_2 n^2 \nu \sigma + \frac{1}{16} a b f_2^2 n^2 \nu \sigma + \frac{a b \sigma^2}{EE} - \frac{a b \nu^2 \sigma^2}{EE} \right)$$


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(* Re-substituting the constants: *)

$$\text{constants} = \left\{ A \rightarrow \frac{1}{8} f_1^2 n^2 - \left(\frac{1}{2} f_1 + f_2 \right), B \rightarrow \frac{1}{8} f_1^2 n^2, CC \rightarrow \frac{1}{2} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right) - \frac{1}{2} f_1, \right. \\ \left. DD \rightarrow \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right), G \rightarrow \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right), H \rightarrow n^2 \left(\frac{1}{2} f_1 + f_2 \right)^2 \right\};$$

(* The potential energy (Equation 4.13): *)

PE = Simplify[(Um + Ub - W) /. constants]

$$\frac{1}{6144 EE} a b t \\ \left(- \left((64 EE^2 n^4 t^2 (8 f_1 f_2 (1 + \mu^4) + 8 f_2^2 (1 + \mu^4) + f_1^2 (3 + 2 \mu^2 + 3 \mu^4))) / (R^2 (-1 + v^2)) \right) - \right. \\ 384 \sigma (EE (3 f_1^2 n^2 (\mu^2 + v) + 4 f_1 (-4 v + f_2 n^2 (\mu^2 + v)) + 4 (-16 f_0 v + f_2^2 n^2 (\mu^2 + v))) - \\ 64 (-1 + v^2) \sigma) + \\ 3 \left(EE^2 (4096 f_0^2 + 9 f_1^4 n^4 + 24 f_1^3 n^2 (-4 + f_2 n^2) + 32 f_1 f_2^2 n^2 (-4 + f_2 n^2) + \right. \\ 8 f_1^2 (32 - 16 f_2 n^2 + 5 f_2^2 n^4) - 128 f_0 (3 f_1^2 n^2 + 4 f_2^2 n^2 + 4 f_1 (-4 + f_2 n^2))) + \\ (1 / ((1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2)) 16 (f_2^4 n^4 (9 + 91 \mu^2 + 91 \mu^4 + 9 \mu^6)^2 + \\ \frac{1}{8} (4 f_1 + 8 f_2 - f_1^2 n^2)^2 (9 + 91 \mu^2 + 91 \mu^4 + 9 \mu^6)^2 + \\ 4 \mu^4 \left(81 \left(\frac{f_1}{2} + f_2 \right)^4 n^4 + \frac{41}{2} f_1^2 (f_1 + 2 f_2)^2 n^4 + 50 f_1^2 (f_1 + 2 f_2)^2 n^4 \mu^2 + \right. \\ \frac{369}{4} (f_1 + 2 f_2)^4 n^4 \mu^2 + 59 f_1^2 (f_1 + 2 f_2)^2 n^4 \mu^4 + \frac{3443}{8} (f_1 + 2 f_2)^4 n^4 \mu^4 + \\ 50 f_1^2 (f_1 + 2 f_2)^2 n^4 \mu^6 + \frac{369}{4} (f_1 + 2 f_2)^4 n^4 \mu^6 + 81 \left(\frac{f_1}{2} + f_2 \right)^4 n^4 \mu^8 + \\ \frac{41}{2} f_1^2 (f_1 + 2 f_2)^2 n^4 \mu^8 + f_1^2 (-2 + f_1 n^2 + 2 f_2 n^2)^2 (9 + 82 \mu^2 + 9 \mu^4)^2 + \\ \left. \left. \left. \frac{1}{32} f_1^4 n^4 (9 + 91 \mu^2 + 91 \mu^4 + 9 \mu^6)^2 \right) \right) \right) - 4096 (-1 + v^2) \sigma^2 \right)$$

(* Minimization of the potential energy wrt the three constants

appears in the assumed expression of the radial displacement w*)

$$\text{eq1} = D[\text{PE}, f_0];$$

$$\text{eq2} = D[\text{PE}, f_1];$$

$$\text{eq3} = D[\text{PE}, f_2];$$

(*Solution of eq1*)

$$\text{soleq1} = \text{Expand}[\text{Solve}[\text{eq1} == 0, f_0]]$$

$$\left\{ \left\{ f_0 \rightarrow -\frac{f_1}{4} + \frac{3 f_1^2 n^2}{64} + \frac{1}{16} f_1 f_2 n^2 + \frac{f_2^2 n^2}{16} - \frac{v \sigma}{EE} \right\} \right\}$$

(*The above solution implies that $\alpha =$

0 if it has been compared with the equation produced previously

from equating the constant terms of v_y to zero. This as below*)

$$f0 + \frac{\alpha}{EE} = -\frac{f1}{4} + \frac{3 f1^2 n^2}{64} + \frac{1}{16} f1 f2 n^2 + \frac{f2^2 n^2}{16} - \frac{\sqrt{\sigma}}{EE}$$

(*simplify eq2*)

eqn2 =

$$\text{Expand}[\text{Simplify}[\text{eq2} /. \text{soleq1}[[1]]]] /. \left\{ f2 \rightarrow \rho f1, t \rightarrow \eta \frac{R}{n^2}, f1 \rightarrow \frac{t \xi}{R} \right\} /. f1 \rightarrow \frac{t \xi}{R} /.$$

$$t \rightarrow \eta \frac{R}{n^2}$$

$$\begin{aligned} & (81 a b E E R \eta^2 \xi) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (819 a b E E R \eta^2 \mu^2 \xi) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (10567 a b E E R \eta^2 \mu^4 \xi) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (7133 a b E E R \eta^2 \mu^6 \xi) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (65007 a b E E R \eta^2 \mu^8 \xi) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (6723 a b E E R \eta^2 \mu^{10} \xi) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (729 a b E E R \eta^2 \mu^{12} \xi) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\ & \frac{a b E E R \eta^4 \xi}{16 n^4 (-1 + \sqrt{2})} - \frac{a b E E R \eta^4 \mu^2 \xi}{24 n^4 (-1 + \sqrt{2})} - \frac{a b E E R \eta^4 \mu^4 \xi}{16 n^4 (-1 + \sqrt{2})} - \\ & (243 a b E E R \eta^3 \xi^2) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\ & (2457 a b E E R \eta^3 \mu^2 \xi^2) / (64 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\ & (33645 a b E E R \eta^3 \mu^4 \xi^2) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\ & (30255 a b E E R \eta^3 \mu^6 \xi^2) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\ & (360285 a b E E R \eta^3 \mu^8 \xi^2) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\ & (37881 a b E E R \eta^3 \mu^{10} \xi^2) / (64 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\ & (4131 a b E E R \eta^3 \mu^{12} \xi^2) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (81 a b E E R \eta^4 \xi^3) / (256 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (819 a b E E R \eta^4 \mu^2 \xi^3) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (6705 a b E E R \eta^4 \mu^4 \xi^3) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (35073 a b E E R \eta^4 \mu^6 \xi^3) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (127925 a b E E R \eta^4 \mu^8 \xi^3) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (35073 a b E E R \eta^4 \mu^{10} \xi^3) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (6705 a b E E R \eta^4 \mu^{12} \xi^3) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (819 a b E E R \eta^4 \mu^{14} \xi^3) / (128 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (81 a b E E R \eta^4 \mu^{16} \xi^3) / (256 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (81 a b E E R \eta^2 \xi \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (819 a b E E R \eta^2 \mu^2 \xi \rho) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (9919 a b E E R \eta^2 \mu^4 \xi \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (4181 a b E E R \eta^2 \mu^6 \xi \rho) / (4 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (9919 a b E E R \eta^2 \mu^8 \xi \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\ & (819 a b E E R \eta^2 \mu^{10} \xi \rho) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \end{aligned}$$

$$\begin{aligned}
& (81 \text{ a b EE R } \eta^2 \mu^{12} \xi \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \frac{\text{a b EE R } \eta^4 \xi \rho}{12 n^4 (-1 + \nu^2)} - \\
& \frac{\text{a b EE R } \eta^4 \mu^4 \xi \rho}{12 n^4 (-1 + \nu^2)} - (81 \text{ a b EE R } \eta^3 \xi^2 \rho) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (819 \text{ a b EE R } \eta^3 \mu^2 \xi^2 \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (11215 \text{ a b EE R } \eta^3 \mu^4 \xi^2 \rho) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (10085 \text{ a b EE R } \eta^3 \mu^6 \xi^2 \rho) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (120095 \text{ a b EE R } \eta^3 \mu^8 \xi^2 \rho) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (12627 \text{ a b EE R } \eta^3 \mu^{10} \xi^2 \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (1377 \text{ a b EE R } \eta^3 \mu^{12} \xi^2 \rho) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (2679 \text{ a b EE R } \eta^4 \mu^4 \xi^3 \rho) / (64 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (10263 \text{ a b EE R } \eta^4 \mu^6 \xi^3 \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (93669 \text{ a b EE R } \eta^4 \mu^8 \xi^3 \rho) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (10263 \text{ a b EE R } \eta^4 \mu^{10} \xi^3 \rho) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (2679 \text{ a b EE R } \eta^4 \mu^{12} \xi^3 \rho) / (64 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (1055 \text{ a b EE R } \eta^4 \mu^4 \xi^3 \rho^2) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (4159 \text{ a b EE R } \eta^4 \mu^6 \xi^3 \rho^2) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (38109 \text{ a b EE R } \eta^4 \mu^8 \xi^3 \rho^2) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (4159 \text{ a b EE R } \eta^4 \mu^{10} \xi^3 \rho^2) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (1055 \text{ a b EE R } \eta^4 \mu^{12} \xi^3 \rho^2) / (32 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (81 \text{ a b EE R } \eta^4 \mu^4 \xi^3 \rho^3) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (369 \text{ a b EE R } \eta^4 \mu^6 \xi^3 \rho^3) / (4 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (3443 \text{ a b EE R } \eta^4 \mu^8 \xi^3 \rho^3) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (369 \text{ a b EE R } \eta^4 \mu^{10} \xi^3 \rho^3) / (4 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (81 \text{ a b EE R } \eta^4 \mu^{12} \xi^3 \rho^3) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& \frac{\text{a b R } \eta \nu \sigma}{n^2} - (81 \text{ a b R } \eta \nu \sigma) / (n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (1638 \text{ a b R } \eta \mu^2 \nu \sigma) / (n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (9919 \text{ a b R } \eta \mu^4 \nu \sigma) / (n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (16724 \text{ a b R } \eta \mu^6 \nu \sigma) / (n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (9919 \text{ a b R } \eta \mu^8 \nu \sigma) / (n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (1638 \text{ a b R } \eta \mu^{10} \nu \sigma) / (n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \\
& (81 \text{ a b R } \eta \mu^{12} \nu \sigma) / (n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \frac{3 \text{ a b R } \eta^2 \mu^2 \xi \sigma}{8 n^2} - \\
& \frac{3 \text{ a b R } \eta^2 \nu \xi \sigma}{8 n^2} + (243 \text{ a b R } \eta^2 \nu \xi \sigma) / (8 n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (2457 \text{ a b R } \eta^2 \mu^2 \nu \xi \sigma) / (4 n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (29757 \text{ a b R } \eta^2 \mu^4 \nu \xi \sigma) / (8 n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (12543 \text{ a b R } \eta^2 \mu^6 \nu \xi \sigma) / (2 n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (29757 \text{ a b R } \eta^2 \mu^8 \nu \xi \sigma) / (8 n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (2457 \text{ a b R } \eta^2 \mu^{10} \nu \xi \sigma) / (4 n^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) +
\end{aligned}$$

$$\begin{aligned}
& (243 \, a \, b \, R \, \eta^2 \, \mu^{12} \, \nu \, \xi \, \rho \, \sigma) / (8 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \frac{a \, b \, R \, \eta^2 \, \mu^2 \, \xi \, \rho \, \sigma}{4 \, n^2} - \\
& \frac{a \, b \, R \, \eta^2 \, \nu \, \xi \, \rho \, \sigma}{4 \, n^2} + (81 \, a \, b \, R \, \eta^2 \, \nu \, \xi \, \rho \, \sigma) / (4 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (819 \, a \, b \, R \, \eta^2 \, \mu^2 \, \nu \, \xi \, \rho \, \sigma) / (2 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (9919 \, a \, b \, R \, \eta^2 \, \mu^4 \, \nu \, \xi \, \rho \, \sigma) / (4 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (4181 \, a \, b \, R \, \eta^2 \, \mu^6 \, \nu \, \xi \, \rho \, \sigma) / (n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (9919 \, a \, b \, R \, \eta^2 \, \mu^8 \, \nu \, \xi \, \rho \, \sigma) / (4 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (819 \, a \, b \, R \, \eta^2 \, \mu^{10} \, \nu \, \xi \, \rho \, \sigma) / (2 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (81 \, a \, b \, R \, \eta^2 \, \mu^{12} \, \nu \, \xi \, \rho \, \sigma) / (4 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2)
\end{aligned}$$

(*Further simplification*)

$$\begin{aligned}
& \text{eqn2} = \text{Expand}\left[\text{Simplify}\left[\text{eqn2} * \frac{n^4}{a \, b \, \epsilon \, \epsilon \, R}\right]\right] \\
& - \left((81 \, \eta^2 \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) \right) - \\
& (81 \, \eta^4 \, \xi) / (16 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (819 \, \eta^2 \, \mu^2 \, \xi) / (16 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (423 \, \eta^4 \, \mu^2 \, \xi) / (4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (10567 \, \eta^2 \, \mu^4 \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (2773 \, \eta^4 \, \mu^4 \, \xi) / (4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (7133 \, \eta^2 \, \mu^6 \, \xi) / (8 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (18731 \, \eta^4 \, \mu^6 \, \xi) / (12 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (65007 \, \eta^2 \, \mu^8 \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (46481 \, \eta^4 \, \mu^8 \, \xi) / (24 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (6723 \, \eta^2 \, \mu^{10} \, \xi) / (16 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (18731 \, \eta^4 \, \mu^{10} \, \xi) / (12 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (729 \, \eta^2 \, \mu^{12} \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (2773 \, \eta^4 \, \mu^{12} \, \xi) / (4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (423 \, \eta^4 \, \mu^{14} \, \xi) / (4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) - \\
& (81 \, \eta^4 \, \mu^{16} \, \xi) / (16 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (81 \, \eta^2 \, \nu^2 \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (819 \, \eta^2 \, \mu^2 \, \nu^2 \, \xi) / (16 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (10567 \, \eta^2 \, \mu^4 \, \nu^2 \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (7133 \, \eta^2 \, \mu^6 \, \nu^2 \, \xi) / (8 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (65007 \, \eta^2 \, \mu^8 \, \nu^2 \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (6723 \, \eta^2 \, \mu^{10} \, \nu^2 \, \xi) / (16 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (729 \, \eta^2 \, \mu^{12} \, \nu^2 \, \xi) / (32 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (243 \, \eta^3 \, \xi^2) / (128 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (2457 \, \eta^3 \, \mu^2 \, \xi^2) / (64 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) + \\
& (33645 \, \eta^3 \, \mu^4 \, \xi^2) / (128 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2 \, (-1 + \nu^2)) +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \\
& (30\,255\,\eta^3\,\mu^6\,\xi^2) / (32\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (360\,285\,\eta^3\,\mu^8\,\xi^2) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (37\,881\,\eta^3\,\mu^{10}\,\xi^2) / (64\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (4131\,\eta^3\,\mu^{12}\,\xi^2) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (243\,\eta^3\,\nu^2\,\xi^2) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (2457\,\eta^3\,\mu^2\,\nu^2\,\xi^2) / (64\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (33\,645\,\eta^3\,\mu^4\,\nu^2\,\xi^2) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (30\,255\,\eta^3\,\mu^6\,\nu^2\,\xi^2) / (32\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (360\,285\,\eta^3\,\mu^8\,\nu^2\,\xi^2) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (37\,881\,\eta^3\,\mu^{10}\,\nu^2\,\xi^2) / (64\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (4131\,\eta^3\,\mu^{12}\,\nu^2\,\xi^2) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (81\,\eta^4\,\xi^3) / (256\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (819\,\eta^4\,\mu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (6705\,\eta^4\,\mu^4\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (35\,073\,\eta^4\,\mu^6\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (127\,925\,\eta^4\,\mu^8\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (35\,073\,\eta^4\,\mu^{10}\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (6705\,\eta^4\,\mu^{12}\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (819\,\eta^4\,\mu^{14}\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (81\,\eta^4\,\mu^{16}\,\xi^3) / (256\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (81\,\eta^4\,\nu^2\,\xi^3) / (256\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (819\,\eta^4\,\mu^2\,\nu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (6705\,\eta^4\,\mu^4\,\nu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (35\,073\,\eta^4\,\mu^6\,\nu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (127\,925\,\eta^4\,\mu^8\,\nu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (35\,073\,\eta^4\,\mu^{10}\,\nu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (6705\,\eta^4\,\mu^{12}\,\nu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (819\,\eta^4\,\mu^{14}\,\nu^2\,\xi^3) / (128\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) + \\
& (81\,\eta^4\,\mu^{16}\,\nu^2\,\xi^3) / (256\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (81\,\eta^2\,\xi\,\rho) / (16\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (27\,\eta^4\,\xi\,\rho) / (4\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (819\,\eta^2\,\mu^2\,\xi\,\rho) / (8\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (273\,\eta^4\,\mu^2\,\xi\,\rho) / (2\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (9919\,\eta^2\,\mu^4\,\xi\,\rho) / (16\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (2500\,\eta^4\,\mu^4\,\xi\,\rho) / (3\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (4181\,\eta^2\,\mu^6\,\xi\,\rho) / (4\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (9181\,\eta^4\,\mu^6\,\xi\,\rho) / (6\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) - \\
& (9919\,\eta^2\,\mu^8\,\xi\,\rho) / (16\,(1+\mu^2)^2\,(9+\mu^2)^2\,(1+9\,\mu^2)^2\,(-1+\nu^2)) -
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (9919 \eta^4 \mu^8 \xi \rho) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^2 \mu^{10} \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9181 \eta^4 \mu^{10} \xi \rho) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^2 \mu^{12} \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2500 \eta^4 \mu^{12} \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (273 \eta^4 \mu^{14} \xi \rho) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (27 \eta^4 \mu^{16} \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^2 v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^2 \mu^2 v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (9919 \eta^2 \mu^4 v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (4181 \eta^2 \mu^6 v^2 \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (9919 \eta^2 \mu^8 v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^2 \mu^{10} v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^2 \mu^{12} v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^3 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^3 \mu^2 \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (11215 \eta^3 \mu^4 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (10085 \eta^3 \mu^6 \xi^2 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (120095 \eta^3 \mu^8 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (12627 \eta^3 \mu^{10} \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (1377 \eta^3 \mu^{12} \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^3 v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^3 \mu^2 v^2 \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (11215 \eta^3 \mu^4 v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10085 \eta^3 \mu^6 v^2 \xi^2 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (120095 \eta^3 \mu^8 v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (12627 \eta^3 \mu^{10} v^2 \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (1377 \eta^3 \mu^{12} v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2679 \eta^4 \mu^4 \xi^3 \rho) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10263 \eta^4 \mu^6 \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (93669 \eta^4 \mu^8 \xi^3 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10263 \eta^4 \mu^{10} \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2679 \eta^4 \mu^{12} \xi^3 \rho) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (2679 \eta^4 \mu^4 v^2 \xi^3 \rho) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (10263 \eta^4 \mu^6 v^2 \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (93669 \eta^4 \mu^8 v^2 \xi^3 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (10263 \eta^4 \mu^{10} v^2 \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (2679 \eta^4 \mu^{12} v^2 \xi^3 \rho) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) -
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (1055 \eta^4 \mu^4 \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (4159 \eta^4 \mu^6 \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (38109 \eta^4 \mu^8 \xi^3 \rho^2) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (4159 \eta^4 \mu^{10} \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (1055 \eta^4 \mu^{12} \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (1055 \eta^4 \mu^4 v^2 \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (4159 \eta^4 \mu^6 v^2 \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (38109 \eta^4 \mu^8 v^2 \xi^3 \rho^2) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (4159 \eta^4 \mu^{10} v^2 \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (1055 \eta^4 \mu^{12} v^2 \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (81 \eta^4 \mu^4 \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (369 \eta^4 \mu^6 \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (3443 \eta^4 \mu^8 \xi^3 \rho^3) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (369 \eta^4 \mu^{10} \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (81 \eta^4 \mu^{12} \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (81 \eta^4 \mu^4 v^2 \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (369 \eta^4 \mu^6 v^2 \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (3443 \eta^4 \mu^8 v^2 \xi^3 \rho^3) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (369 \eta^4 \mu^{10} v^2 \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (81 \eta^4 \mu^{12} v^2 \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (243 n^2 \eta^2 \mu^2 \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (2457 n^2 \eta^2 \mu^4 \xi \sigma) / (4 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (29757 n^2 \eta^2 \mu^6 \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (12543 n^2 \eta^2 \mu^8 \xi \sigma) / (2 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (29757 n^2 \eta^2 \mu^{10} \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (2457 n^2 \eta^2 \mu^{12} \xi \sigma) / (4 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (243 n^2 \eta^2 \mu^{14} \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (243 n^2 \eta^2 \mu^2 v^2 \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (2457 n^2 \eta^2 \mu^4 v^2 \xi \sigma) / (4 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (29757 n^2 \eta^2 \mu^6 v^2 \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (12543 n^2 \eta^2 \mu^8 v^2 \xi \sigma) / (2 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (29757 n^2 \eta^2 \mu^{10} v^2 \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (2457 n^2 \eta^2 \mu^{12} v^2 \xi \sigma) / (4 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (243 n^2 \eta^2 \mu^{14} v^2 \xi \sigma) / (8 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (81 n^2 \eta^2 \mu^2 \xi \rho \sigma) / (4 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (819 n^2 \eta^2 \mu^4 \xi \rho \sigma) / (2 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (9919 n^2 \eta^2 \mu^6 \xi \rho \sigma) / (4 EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (4181 n^2 \eta^2 \mu^8 \xi \rho \sigma) / (EE (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (9919 n^2 \eta^2 \mu^{10} \xi \rho \sigma) / (4 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 n^2 \eta^2 \mu^{12} \xi \rho \sigma) / (2 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 n^2 \eta^2 \mu^{14} \xi \rho \sigma) / (4 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 n^2 \eta^2 \mu^2 v^2 \xi \rho \sigma) / (4 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 n^2 \eta^2 \mu^4 v^2 \xi \rho \sigma) / (2 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9919 n^2 \eta^2 \mu^6 v^2 \xi \rho \sigma) / (4 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (4181 n^2 \eta^2 \mu^8 v^2 \xi \rho \sigma) / (EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9919 n^2 \eta^2 \mu^{10} v^2 \xi \rho \sigma) / (4 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 n^2 \eta^2 \mu^{12} v^2 \xi \rho \sigma) / (2 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 n^2 \eta^2 \mu^{14} v^2 \xi \rho \sigma) / (4 EE (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))
\end{aligned}$$

(*Please note: to get $\frac{\sigma R}{EE t}$ we can replace $\left(\frac{\sigma n^2}{EE}\right) \rightarrow$

$\sigma_{RET} \eta$ where σ_{RET} refers to $\left(\frac{\sigma R}{EE t}\right) *$)

$$eqn2 = \text{Expand} \left[eqn2 /. \left(\frac{\sigma n^2}{EE} \right) \rightarrow \sigma_{RET} \eta \right]$$

$$\begin{aligned}
& - \left((81 \eta^2 \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) \right) - \\
& (81 \eta^4 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^2 \mu^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (423 \eta^4 \mu^2 \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10567 \eta^2 \mu^4 \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2773 \eta^4 \mu^4 \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (7133 \eta^2 \mu^6 \xi) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (18731 \eta^4 \mu^6 \xi) / (12 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (65007 \eta^2 \mu^8 \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (46481 \eta^4 \mu^8 \xi) / (24 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (6723 \eta^2 \mu^{10} \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (18731 \eta^4 \mu^{10} \xi) / (12 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (729 \eta^2 \mu^{12} \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2773 \eta^4 \mu^{12} \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (423 \eta^4 \mu^{14} \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^4 \mu^{16} \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^2 v^2 \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^2 \mu^2 v^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (10567 \eta^2 \mu^4 v^2 \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (7133 \eta^2 \mu^6 v^2 \xi) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (65007 \eta^2 \mu^8 v^2 \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (6723 \eta^2 \mu^{10} v^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (729 \eta^2 \mu^{12} v^2 \xi) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (243 \eta^3 \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (2457 \eta^3 \mu^2 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (33645 \eta^3 \mu^4 \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (30255 \eta^3 \mu^6 \xi^2) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (360285 \eta^3 \mu^8 \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (37881 \eta^3 \mu^{10} \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (4131 \eta^3 \mu^{12} \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (243 \eta^3 v^2 \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2457 \eta^3 \mu^2 v^2 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (33645 \eta^3 \mu^4 v^2 \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (30255 \eta^3 \mu^6 v^2 \xi^2) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (360285 \eta^3 \mu^8 v^2 \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (37881 \eta^3 \mu^{10} v^2 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (4131 \eta^3 \mu^{12} v^2 \xi^2) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^4 \xi^3) / (256 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^4 \mu^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (6705 \eta^4 \mu^4 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (35073 \eta^4 \mu^6 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (127925 \eta^4 \mu^8 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (35073 \eta^4 \mu^{10} \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (6705 \eta^4 \mu^{12} \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^4 \mu^{14} \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^4 \mu^{16} \xi^3) / (256 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^4 v^2 \xi^3) / (256 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^4 \mu^2 v^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (6705 \eta^4 \mu^4 v^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (35073 \eta^4 \mu^6 v^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (127925 \eta^4 \mu^8 v^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (35073 \eta^4 \mu^{10} v^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (6705 \eta^4 \mu^{12} v^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^4 \mu^{14} v^2 \xi^3) / (128 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^4 \mu^{16} v^2 \xi^3) / (256 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (27 \eta^4 \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^2 \mu^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (273 \eta^4 \mu^2 \xi \rho) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9919 \eta^2 \mu^4 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2500 \eta^4 \mu^4 \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) -
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (4181 \eta^2 \mu^6 \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9181 \eta^4 \mu^6 \xi \rho) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9919 \eta^2 \mu^8 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9919 \eta^4 \mu^8 \xi \rho) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^2 \mu^{10} \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (9181 \eta^4 \mu^{10} \xi \rho) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^2 \mu^{12} \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2500 \eta^4 \mu^{12} \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (273 \eta^4 \mu^{14} \xi \rho) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (27 \eta^4 \mu^{16} \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^2 v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^2 \mu^2 v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (9919 \eta^2 \mu^4 v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (4181 \eta^2 \mu^6 v^2 \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (9919 \eta^2 \mu^8 v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^2 \mu^{10} v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^2 \mu^{12} v^2 \xi \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^3 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^3 \mu^2 \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (11215 \eta^3 \mu^4 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (10085 \eta^3 \mu^6 \xi^2 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (120095 \eta^3 \mu^8 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (12627 \eta^3 \mu^{10} \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (1377 \eta^3 \mu^{12} \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^3 v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (819 \eta^3 \mu^2 v^2 \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (11215 \eta^3 \mu^4 v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10085 \eta^3 \mu^6 v^2 \xi^2 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (120095 \eta^3 \mu^8 v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (12627 \eta^3 \mu^{10} v^2 \xi^2 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (1377 \eta^3 \mu^{12} v^2 \xi^2 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2679 \eta^4 \mu^4 \xi^3 \rho) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10263 \eta^4 \mu^6 \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (93669 \eta^4 \mu^8 \xi^3 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10263 \eta^4 \mu^{10} \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (2679 \eta^4 \mu^{12} \xi^3 \rho) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (2679 \eta^4 \mu^4 v^2 \xi^3 \rho) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (10263 \eta^4 \mu^6 v^2 \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (93669 \eta^4 \mu^8 v^2 \xi^3 \rho) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (10263 \eta^4 \mu^{10} v^2 \xi^3 \rho) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (2679 \eta^4 \mu^{12} v^2 \xi^3 \rho) / (64 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (1055 \eta^4 \mu^4 \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (4159 \eta^4 \mu^6 \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (38109 \eta^4 \mu^8 \xi^3 \rho^2) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (4159 \eta^4 \mu^{10} \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (1055 \eta^4 \mu^{12} \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (1055 \eta^4 \mu^4 v^2 \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (4159 \eta^4 \mu^6 v^2 \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (38109 \eta^4 \mu^8 v^2 \xi^3 \rho^2) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (4159 \eta^4 \mu^{10} v^2 \xi^3 \rho^2) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (1055 \eta^4 \mu^{12} v^2 \xi^3 \rho^2) / (32 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (81 \eta^4 \mu^4 \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (369 \eta^4 \mu^6 \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (3443 \eta^4 \mu^8 \xi^3 \rho^3) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (369 \eta^4 \mu^{10} \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (81 \eta^4 \mu^{12} \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (81 \eta^4 \mu^4 v^2 \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (369 \eta^4 \mu^6 v^2 \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (3443 \eta^4 \mu^8 v^2 \xi^3 \rho^3) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (369 \eta^4 \mu^{10} v^2 \xi^3 \rho^3) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (81 \eta^4 \mu^{12} v^2 \xi^3 \rho^3) / (16 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (243 \eta^3 \mu^2 \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (2457 \eta^3 \mu^4 \xi \sigma_{\text{RET}}) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (29757 \eta^3 \mu^6 \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (12543 \eta^3 \mu^8 \xi \sigma_{\text{RET}}) / (2 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (29757 \eta^3 \mu^{10} \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (2457 \eta^3 \mu^{12} \xi \sigma_{\text{RET}}) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (243 \eta^3 \mu^{14} \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (243 \eta^3 \mu^2 v^2 \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (2457 \eta^3 \mu^4 v^2 \xi \sigma_{\text{RET}}) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (29757 \eta^3 \mu^6 v^2 \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (12543 \eta^3 \mu^8 v^2 \xi \sigma_{\text{RET}}) / (2 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (29757 \eta^3 \mu^{10} v^2 \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (2457 \eta^3 \mu^{12} v^2 \xi \sigma_{\text{RET}}) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) - \\
& (243 \eta^3 \mu^{14} v^2 \xi \sigma_{\text{RET}}) / (8 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) + \\
& (81 \eta^3 \mu^2 \xi \rho \sigma_{\text{RET}}) / (4 (1+\mu^2)^2 (9+\mu^2)^2 (1+9\mu^2)^2 (-1+v^2)) +
\end{aligned}$$

$$\begin{aligned}
 & \left(\frac{(819 \eta^3 \mu^4 \xi \rho \sigma \text{RET})}{(2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} + \right. \\
 & \left(\frac{9919 \eta^3 \mu^6 \xi \rho \sigma \text{RET}}{(4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} + \right. \\
 & \left(\frac{4181 \eta^3 \mu^8 \xi \rho \sigma \text{RET}}{((1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} + \right. \\
 & \left(\frac{9919 \eta^3 \mu^{10} \xi \rho \sigma \text{RET}}{(4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} + \right. \\
 & \left(\frac{819 \eta^3 \mu^{12} \xi \rho \sigma \text{RET}}{(2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} + \right. \\
 & \left(\frac{81 \eta^3 \mu^{14} \xi \rho \sigma \text{RET}}{(4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} - \right. \\
 & \left(\frac{81 \eta^3 \mu^2 v^2 \xi \rho \sigma \text{RET}}{(4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} - \right. \\
 & \left(\frac{819 \eta^3 \mu^4 v^2 \xi \rho \sigma \text{RET}}{(2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} - \right. \\
 & \left(\frac{9919 \eta^3 \mu^6 v^2 \xi \rho \sigma \text{RET}}{(4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} - \right. \\
 & \left(\frac{4181 \eta^3 \mu^8 v^2 \xi \rho \sigma \text{RET}}{((1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} - \right. \\
 & \left(\frac{9919 \eta^3 \mu^{10} v^2 \xi \rho \sigma \text{RET}}{(4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} - \right. \\
 & \left(\frac{819 \eta^3 \mu^{12} v^2 \xi \rho \sigma \text{RET}}{(2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} - \right. \\
 & \left. \left. \left(\frac{81 \eta^3 \mu^{14} v^2 \xi \rho \sigma \text{RET}}{(4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2))} \right) \right) \right) \right) \right) \right) \right) \right) \right)
 \end{aligned}$$

(*Solving for σRET and the results will be function of ρ : *)

`soleq2 = Simplify[Solve[eqn2 == 0,  $\sigma \text{RET}$ ]]`

$$\begin{aligned}
 & \{ \{ \sigma \text{RET} \rightarrow (-6 \eta (9 + 82 \mu^2 + 9 \mu^4)^2 (1 + 2 \mu^2 + 17 \mu^4) (-1 + v^2) \xi (3 + 4 \rho) + \\
 & 24 (9 + 82 \mu^2 + 9 \mu^4)^2 (-1 + v^2) (1 + 2 \rho + \mu^4 (9 + 2 \rho) + \mu^2 (2 + 4 \rho)) + \\
 & \eta^2 (81 (3 (-1 + v^2) \xi^2 - 16 (3 + 4 \rho)) + 81 \mu^{16} (3 (-1 + v^2) \xi^2 - 16 (3 + 4 \rho)) + 18 \mu^2 \\
 & (273 (-1 + v^2) \xi^2 - 32 (141 + 182 \rho)) + 18 \mu^{14} (273 (-1 + v^2) \xi^2 - 32 (141 + 182 \rho)) + \\
 & \mu^4 (-64 (8319 + 10000 \rho) + 6 (-1 + v^2) \xi^2 (6705 + 5358 \rho + 4220 \rho^2 + 648 \rho^3)) + \\
 & \mu^{12} (-64 (8319 + 10000 \rho) + 6 (-1 + v^2) \xi^2 (6705 + 5358 \rho + 4220 \rho^2 + 648 \rho^3)) + \\
 & 2 \mu^6 (-32 (18731 + 18362 \rho) + 3 (-1 + v^2) \xi^2 (35073 + 82104 \rho + 66544 \\
 & \rho^2 + 11808 \rho^3)) + 2 \mu^{10} (-32 (18731 + 18362 \rho) + 3 (-1 + v^2) \xi^2 \\
 & (35073 + 82104 \rho + 66544 \rho^2 + 11808 \rho^3)) + 2 \mu^8 (-16 (46481 + 39676 \rho) + \\
 & 3 (-1 + v^2) \xi^2 (127925 + 374676 \rho + 304872 \rho^2 + 55088 \rho^3)) \} \} / \\
 & (96 \eta \mu^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2) (3 + 2 \rho)) \} \}
 \end{aligned}$$

(* Simplify eq3 :*)

`eqn3 =`

$$\text{Expand}[\text{Simplify}[eq3 /. \text{soleq1}[[1]]]] /. \left\{ f2 \rightarrow \rho f1, t \rightarrow \eta \frac{R}{n^2}, f1 \rightarrow \frac{t \xi}{R} \right\} /. f1 \rightarrow \frac{t \xi}{R} /.$$

$$t \rightarrow \eta \frac{R}{n^2}$$

$$\begin{aligned}
 & (81 a b E E R \eta^2 \xi) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
 & (819 a b E E R \eta^2 \mu^2 \xi) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
 & (9919 a b E E R \eta^2 \mu^4 \xi) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
 & (4181 a b E E R \eta^2 \mu^6 \xi) / (4 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
 & (9919 a b E E R \eta^2 \mu^8 \xi) / (16 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
 & (819 a b E E R \eta^2 \mu^{10} \xi) / (8 n^4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) +
 \end{aligned}$$

$$\begin{aligned}
& (81 \, a \, b \, E E \, R \, \eta^2 \, \mu^{12} \, \xi) / (16 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \frac{a \, b \, E E \, R \, \eta^4 \, \xi}{12 \, n^4 \, (-1 + \sqrt{2})} - \\
& \frac{a \, b \, E E \, R \, \eta^4 \, \mu^4 \, \xi}{12 \, n^4 \, (-1 + \sqrt{2})} - (81 \, a \, b \, E E \, R \, \eta^3 \, \xi^2) / (64 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \\
& (819 \, a \, b \, E E \, R \, \eta^3 \, \mu^2 \, \xi^2) / (32 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \\
& (11 \, 215 \, a \, b \, E E \, R \, \eta^3 \, \mu^4 \, \xi^2) / (64 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \\
& (10 \, 085 \, a \, b \, E E \, R \, \eta^3 \, \mu^6 \, \xi^2) / (16 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \\
& (120 \, 095 \, a \, b \, E E \, R \, \eta^3 \, \mu^8 \, \xi^2) / (64 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \\
& (12 \, 627 \, a \, b \, E E \, R \, \eta^3 \, \mu^{10} \, \xi^2) / (32 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \\
& (1377 \, a \, b \, E E \, R \, \eta^3 \, \mu^{12} \, \xi^2) / (64 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (893 \, a \, b \, E E \, R \, \eta^4 \, \mu^4 \, \xi^3) / (64 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (3421 \, a \, b \, E E \, R \, \eta^4 \, \mu^6 \, \xi^3) / (16 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (31 \, 223 \, a \, b \, E E \, R \, \eta^4 \, \mu^8 \, \xi^3) / (32 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (3421 \, a \, b \, E E \, R \, \eta^4 \, \mu^{10} \, \xi^3) / (16 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (893 \, a \, b \, E E \, R \, \eta^4 \, \mu^{12} \, \xi^3) / (64 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (81 \, a \, b \, E E \, R \, \eta^2 \, \xi \, \rho) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (819 \, a \, b \, E E \, R \, \eta^2 \, \mu^2 \, \xi \, \rho) / (4 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (9919 \, a \, b \, E E \, R \, \eta^2 \, \mu^4 \, \xi \, \rho) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (4181 \, a \, b \, E E \, R \, \eta^2 \, \mu^6 \, \xi \, \rho) / (2 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (9919 \, a \, b \, E E \, R \, \eta^2 \, \mu^8 \, \xi \, \rho) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (819 \, a \, b \, E E \, R \, \eta^2 \, \mu^{10} \, \xi \, \rho) / (4 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (81 \, a \, b \, E E \, R \, \eta^2 \, \mu^{12} \, \xi \, \rho) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \frac{a \, b \, E E \, R \, \eta^4 \, \xi \, \rho}{6 \, n^4 \, (-1 + \sqrt{2})} - \\
& \frac{a \, b \, E E \, R \, \eta^4 \, \mu^4 \, \xi \, \rho}{6 \, n^4 \, (-1 + \sqrt{2})} + (1055 \, a \, b \, E E \, R \, \eta^4 \, \mu^4 \, \xi^3 \, \rho) / (32 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (4159 \, a \, b \, E E \, R \, \eta^4 \, \mu^6 \, \xi^3 \, \rho) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (38 \, 109 \, a \, b \, E E \, R \, \eta^4 \, \mu^8 \, \xi^3 \, \rho) / (16 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (4159 \, a \, b \, E E \, R \, \eta^4 \, \mu^{10} \, \xi^3 \, \rho) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (1055 \, a \, b \, E E \, R \, \eta^4 \, \mu^{12} \, \xi^3 \, \rho) / (32 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (243 \, a \, b \, E E \, R \, \eta^4 \, \mu^4 \, \xi^3 \, \rho^2) / (16 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (1107 \, a \, b \, E E \, R \, \eta^4 \, \mu^6 \, \xi^3 \, \rho^2) / (4 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (10 \, 329 \, a \, b \, E E \, R \, \eta^4 \, \mu^8 \, \xi^3 \, \rho^2) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (1107 \, a \, b \, E E \, R \, \eta^4 \, \mu^{10} \, \xi^3 \, \rho^2) / (4 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (243 \, a \, b \, E E \, R \, \eta^4 \, \mu^{12} \, \xi^3 \, \rho^2) / (16 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (81 \, a \, b \, E E \, R \, \eta^4 \, \mu^4 \, \xi^3 \, \rho^3) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (369 \, a \, b \, E E \, R \, \eta^4 \, \mu^6 \, \xi^3 \, \rho^3) / (2 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (3443 \, a \, b \, E E \, R \, \eta^4 \, \mu^8 \, \xi^3 \, \rho^3) / (4 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (369 \, a \, b \, E E \, R \, \eta^4 \, \mu^{10} \, \xi^3 \, \rho^3) / (2 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) + \\
& (81 \, a \, b \, E E \, R \, \eta^4 \, \mu^{12} \, \xi^3 \, \rho^3) / (8 \, n^4 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) - \frac{a \, b \, R \, \eta^2 \, \mu^2 \, \xi \, \sigma}{4 \, n^2} - \\
& \frac{a \, b \, R \, \eta^2 \, \nu \, \xi \, \sigma}{4 \, n^2} + (81 \, a \, b \, R \, \eta^2 \, \nu \, \xi \, \sigma) / (4 \, n^2 \, (1 + \mu^2)^2 \, (9 + \mu^2)^2 \, (1 + 9 \, \mu^2)^2) +
\end{aligned}$$

$$\begin{aligned}
& (819 \text{ a b R } \eta^2 \mu^2 \vee \xi \sigma) / (2 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (9919 \text{ a b R } \eta^2 \mu^4 \vee \xi \sigma) / (4 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (4181 \text{ a b R } \eta^2 \mu^6 \vee \xi \sigma) / (\text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (9919 \text{ a b R } \eta^2 \mu^8 \vee \xi \sigma) / (4 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (819 \text{ a b R } \eta^2 \mu^{10} \vee \xi \sigma) / (2 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (81 \text{ a b R } \eta^2 \mu^{12} \vee \xi \sigma) / (4 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) - \frac{\text{ a b R } \eta^2 \mu^2 \xi \rho \sigma}{2 \text{ n}^2} - \\
& \frac{\text{ a b R } \eta^2 \vee \xi \rho \sigma}{2 \text{ n}^2} + (81 \text{ a b R } \eta^2 \vee \xi \rho \sigma) / (2 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (819 \text{ a b R } \eta^2 \mu^2 \vee \xi \rho \sigma) / (\text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (9919 \text{ a b R } \eta^2 \mu^4 \vee \xi \rho \sigma) / (2 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (8362 \text{ a b R } \eta^2 \mu^6 \vee \xi \rho \sigma) / (\text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (9919 \text{ a b R } \eta^2 \mu^8 \vee \xi \rho \sigma) / (2 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (819 \text{ a b R } \eta^2 \mu^{10} \vee \xi \rho \sigma) / (\text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2) + \\
& (81 \text{ a b R } \eta^2 \mu^{12} \vee \xi \rho \sigma) / (2 \text{ n}^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2)
\end{aligned}$$

(*More simplifying of eqn 3*)

$$\begin{aligned}
& \text{eqn3} = \text{Expand} \left[\text{Simplify} \left[\text{eqn3} * \frac{\text{n}^4}{\text{a b EE R}} \right] \right] /. \left(\frac{\sigma \text{n}^2}{\text{EE}} \right) \rightarrow \sigma \text{RET } \eta \\
& - ((81 \eta^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2))) - \\
& (27 \eta^4 \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (819 \eta^2 \mu^2 \xi) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (273 \eta^4 \mu^2 \xi) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9919 \eta^2 \mu^4 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (2500 \eta^4 \mu^4 \xi) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (4181 \eta^2 \mu^6 \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9181 \eta^4 \mu^6 \xi) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9919 \eta^2 \mu^8 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9919 \eta^4 \mu^8 \xi) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (819 \eta^2 \mu^{10} \xi) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9181 \eta^4 \mu^{10} \xi) / (6 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (81 \eta^2 \mu^{12} \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (2500 \eta^4 \mu^{12} \xi) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (273 \eta^4 \mu^{14} \xi) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (27 \eta^4 \mu^{16} \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (81 \eta^2 \nu^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (819 \eta^2 \mu^2 \nu^2 \xi) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (9919 \eta^2 \mu^4 \nu^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (4181 \eta^2 \mu^6 \nu^2 \xi) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (9919 \eta^2 \mu^8 \nu^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) +
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (819 \eta^2 \mu^{10} \nu^2 \xi) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (81 \eta^2 \mu^{12} \nu^2 \xi) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (81 \eta^3 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (819 \eta^3 \mu^2 \xi^2) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (11215 \eta^3 \mu^4 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (10085 \eta^3 \mu^6 \xi^2) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (120095 \eta^3 \mu^8 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (12627 \eta^3 \mu^{10} \xi^2) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (1377 \eta^3 \mu^{12} \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (81 \eta^3 \nu^2 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (819 \eta^3 \mu^2 \nu^2 \xi^2) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (11215 \eta^3 \mu^4 \nu^2 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (10085 \eta^3 \mu^6 \nu^2 \xi^2) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (120095 \eta^3 \mu^8 \nu^2 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (12627 \eta^3 \mu^{10} \nu^2 \xi^2) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (1377 \eta^3 \mu^{12} \nu^2 \xi^2) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (893 \eta^4 \mu^4 \xi^3) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (3421 \eta^4 \mu^6 \xi^3) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (31223 \eta^4 \mu^8 \xi^3) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (3421 \eta^4 \mu^{10} \xi^3) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (893 \eta^4 \mu^{12} \xi^3) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (893 \eta^4 \mu^4 \nu^2 \xi^3) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (3421 \eta^4 \mu^6 \nu^2 \xi^3) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (31223 \eta^4 \mu^8 \nu^2 \xi^3) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (3421 \eta^4 \mu^{10} \nu^2 \xi^3) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) + \\
& (893 \eta^4 \mu^{12} \nu^2 \xi^3) / (64 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (81 \eta^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (27 \eta^4 \xi \rho) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (819 \eta^2 \mu^2 \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (273 \eta^4 \mu^2 \xi \rho) / ((1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9919 \eta^2 \mu^4 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (5000 \eta^4 \mu^4 \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (4181 \eta^2 \mu^6 \xi \rho) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9181 \eta^4 \mu^6 \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9919 \eta^2 \mu^8 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9919 \eta^4 \mu^8 \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (819 \eta^2 \mu^{10} \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) - \\
& (9181 \eta^4 \mu^{10} \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + \nu^2)) -
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{\quad}{\quad} \right) / \left(\left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \left(\frac{\quad}{\quad} \right) \right) \\
& (81 \eta^2 \mu^{12} \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (5000 \eta^4 \mu^{12} \xi \rho) / (3 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (273 \eta^4 \mu^{14} \xi \rho) / ((1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (27 \eta^4 \mu^{16} \xi \rho) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^2 v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^2 \mu^2 v^2 \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (9919 \eta^2 \mu^4 v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (4181 \eta^2 \mu^6 v^2 \xi \rho) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (9919 \eta^2 \mu^8 v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (819 \eta^2 \mu^{10} v^2 \xi \rho) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^2 \mu^{12} v^2 \xi \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (1055 \eta^4 \mu^4 \xi^3 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (4159 \eta^4 \mu^6 \xi^3 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (38109 \eta^4 \mu^8 \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (4159 \eta^4 \mu^{10} \xi^3 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (1055 \eta^4 \mu^{12} \xi^3 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (1055 \eta^4 \mu^4 v^2 \xi^3 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (4159 \eta^4 \mu^6 v^2 \xi^3 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (38109 \eta^4 \mu^8 v^2 \xi^3 \rho) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (4159 \eta^4 \mu^{10} v^2 \xi^3 \rho) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (1055 \eta^4 \mu^{12} v^2 \xi^3 \rho) / (32 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (243 \eta^4 \mu^4 \xi^3 \rho^2) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (1107 \eta^4 \mu^6 \xi^3 \rho^2) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (10329 \eta^4 \mu^8 \xi^3 \rho^2) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (1107 \eta^4 \mu^{10} \xi^3 \rho^2) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (243 \eta^4 \mu^{12} \xi^3 \rho^2) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (243 \eta^4 \mu^4 v^2 \xi^3 \rho^2) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (1107 \eta^4 \mu^6 v^2 \xi^3 \rho^2) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (10329 \eta^4 \mu^8 v^2 \xi^3 \rho^2) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (1107 \eta^4 \mu^{10} v^2 \xi^3 \rho^2) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (243 \eta^4 \mu^{12} v^2 \xi^3 \rho^2) / (16 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^4 \mu^4 \xi^3 \rho^3) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (369 \eta^4 \mu^6 \xi^3 \rho^3) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (3443 \eta^4 \mu^8 \xi^3 \rho^3) / (4 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (369 \eta^4 \mu^{10} \xi^3 \rho^3) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) - \\
& (81 \eta^4 \mu^{12} \xi^3 \rho^3) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (81 \eta^4 \mu^4 v^2 \xi^3 \rho^3) / (8 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) + \\
& (369 \eta^4 \mu^6 v^2 \xi^3 \rho^3) / (2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (-1 + v^2)) +
\end{aligned}$$

[illegible]

(*Replacing σ_{RET} by its expression found above (as a function of ρ_*)

```
eqn3 = Simplify[eqn3 /. soleq2[[1]]]
```

$$\begin{aligned}
& - \left(\eta^2 \xi \left(-48 \eta \left(9 + 82 \mu^2 + 9 \mu^4 \right)^2 \left(1 + 2 \mu^2 + 17 \mu^4 \right) \left(-1 + v^2 \right) \xi \rho \left(1 + \rho \right) + \right. \right. \\
& \quad 48 \left(-1 - 2 \mu^2 + 3 \mu^4 \right) \left(9 + 82 \mu^2 + 9 \mu^4 \right)^2 \left(-1 + v^2 \right) \left(1 + 2 \rho \right) + \\
& \quad \eta^2 \left(1 + 2 \rho \right) \left(243 \left(16 + \left(-1 + v^2 \right) \xi^2 \right) + 243 \mu^{16} \left(16 + \left(-1 + v^2 \right) \xi^2 \right) + \right. \\
& \quad \quad 54 \mu^2 \left(1408 + 91 \left(-1 + v^2 \right) \xi^2 \right) + 54 \mu^{14} \left(1408 + 91 \left(-1 + v^2 \right) \xi^2 \right) + \\
& \quad \quad 12 \mu^4 \left(35632 + \left(-1 + v^2 \right) \xi^2 \left(2013 + 1300 \rho + 1300 \rho^2 \right) \right) + \\
& \quad \quad 12 \mu^{12} \left(35632 + \left(-1 + v^2 \right) \xi^2 \left(2013 + 1300 \rho + 1300 \rho^2 \right) \right) + \\
& \quad \quad \mu^6 \left(563968 + 6 \left(-1 + v^2 \right) \xi^2 \left(-5979 + 37024 \rho + 37024 \rho^2 \right) \right) + \\
& \quad \quad \mu^{10} \left(563968 + 6 \left(-1 + v^2 \right) \xi^2 \left(-5979 + 37024 \rho + 37024 \rho^2 \right) \right) + \\
& \quad \quad \left. \left. 2 \mu^8 \left(208528 + 3 \left(-1 + v^2 \right) \xi^2 \left(-59413 + 167152 \rho + 167152 \rho^2 \right) \right) \right) \right) \right) / \\
& \quad \left(384 \left(1 + \mu^2 \right)^2 \left(9 + \mu^2 \right)^2 \left(1 + 9 \mu^2 \right)^2 \left(-1 + v^2 \right) \left(3 + 2 \rho \right) \right)
\end{aligned}$$

(* A simplified form of eqn3 can be obtained by setting $\rho^3 = 0$ (negligible effect) *)

eqn3SIM = Simplify[Expand[eqn3] /. { $\rho^3 \rightarrow 0$ }];

(*Solving the above equation to get ρ : *)

soleq3 = Simplify[Solve[eqn3SIM == 0, ρ]]

$$\begin{aligned}
& \left\{ \left\{ \rho \rightarrow \right. \right. \\
& \quad - \left(\left(3888 + 3888 \eta^2 + 78624 \mu^2 + 76032 \eta^2 \mu^2 + 460560 \mu^4 + 427584 \eta^2 \mu^4 + 519360 \mu^6 + 563968 \right. \right. \\
& \quad \quad \eta^2 \mu^6 - 846000 \mu^8 + 417056 \eta^2 \mu^8 - 204768 \mu^{10} + 563968 \eta^2 \mu^{10} - 11664 \mu^{12} + 427584 \eta^2 \\
& \quad \quad \mu^{12} + 76032 \eta^2 \mu^{14} + 3888 \eta^2 \mu^{16} - 3888 v^2 - 78624 \mu^2 v^2 - 460560 \mu^4 v^2 - 519360 \mu^6 v^2 + \\
& \quad \quad 846000 \mu^8 v^2 + 204768 \mu^{10} v^2 + 11664 \mu^{12} v^2 + 1944 \eta \xi + 39312 \eta \mu^2 \xi + \\
& \quad \quad 269160 \eta \mu^4 \xi + 968160 \eta \mu^6 \xi + 2882280 \eta \mu^8 \xi + 606096 \eta \mu^{10} \xi + 33048 \eta \mu^{12} \xi - \\
& \quad \quad 1944 \eta v^2 \xi - 39312 \eta \mu^2 v^2 \xi - 269160 \eta \mu^4 v^2 \xi - 968160 \eta \mu^6 v^2 \xi - 2882280 \eta \mu^8 v^2 \xi - \\
& \quad \quad 606096 \eta \mu^{10} v^2 \xi - 33048 \eta \mu^{12} v^2 \xi - 243 \eta^2 \xi^2 - 4914 \eta^2 \mu^2 \xi^2 - 31956 \eta^2 \mu^4 \xi^2 - \\
& \quad \quad 75198 \eta^2 \mu^6 \xi^2 - 144978 \eta^2 \mu^8 \xi^2 - 75198 \eta^2 \mu^{10} \xi^2 - 31956 \eta^2 \mu^{12} \xi^2 - 4914 \eta^2 \mu^{14} \xi^2 - \\
& \quad \quad 243 \eta^2 \mu^{16} \xi^2 + 243 \eta^2 v^2 \xi^2 + 4914 \eta^2 \mu^2 v^2 \xi^2 + 31956 \eta^2 \mu^4 v^2 \xi^2 + 75198 \eta^2 \mu^6 v^2 \xi^2 + \\
& \quad \quad 144978 \eta^2 \mu^8 v^2 \xi^2 + 75198 \eta^2 \mu^{10} v^2 \xi^2 + 31956 \eta^2 \mu^{12} v^2 \xi^2 + 4914 \eta^2 \mu^{14} v^2 \xi^2 + \\
& \quad \quad 243 \eta^2 \mu^{16} v^2 \xi^2 - \sqrt{\left(-48 \eta \left(-1 + v^2 \right) \xi \left(-81 - 1638 \mu^2 + 5 \mu^4 \left(-2243 + 195 \eta \xi \right) + \right. \right. \\
& \quad \quad 3 \mu^{12} \left(-459 + 325 \eta \xi \right) + 6 \mu^{10} \left(-4209 + 2314 \eta \xi \right) + 4 \mu^6 \left(-10085 + 3471 \eta \xi \right) + \\
& \quad \quad \mu^8 \left(-120095 + 62682 \eta \xi \right) \left(48 \left(-1 - 2 \mu^2 + 3 \mu^4 \right) \left(9 + 82 \mu^2 + 9 \mu^4 \right)^2 \left(-1 + v^2 \right) + \right. \\
& \quad \quad \eta^2 \left(\mu^8 \left(417056 - 356478 \left(-1 + v^2 \right) \xi^2 \right) + \mu^6 \left(563968 - 35874 \left(-1 + v^2 \right) \xi^2 \right) + \right. \\
& \quad \quad \mu^{10} \left(563968 - 35874 \left(-1 + v^2 \right) \xi^2 \right) + 243 \left(16 + \left(-1 + v^2 \right) \xi^2 \right) + \\
& \quad \quad 243 \mu^{16} \left(16 + \left(-1 + v^2 \right) \xi^2 \right) + 54 \mu^2 \left(1408 + 91 \left(-1 + v^2 \right) \xi^2 \right) + \\
& \quad \quad 54 \mu^{14} \left(1408 + 91 \left(-1 + v^2 \right) \xi^2 \right) + 12 \mu^4 \left(35632 + 2013 \left(-1 + v^2 \right) \xi^2 \right) + \\
& \quad \quad \left. \left. 12 \mu^{12} \left(35632 + 2013 \left(-1 + v^2 \right) \xi^2 \right) \right) \right) \right) + \left(48 \left(-1 - 2 \mu^2 + 3 \mu^4 \right) \right. \\
& \quad \quad \left(9 + 82 \mu^2 + 9 \mu^4 \right)^2 \left(-1 + v^2 \right) - 24 \eta \left(9 + 82 \mu^2 + 9 \mu^4 \right)^2 \left(1 + 2 \mu^2 + 17 \mu^4 \right) \\
& \quad \quad \left(-1 + v^2 \right) \xi + \eta^2 \left(243 \left(16 + \left(-1 + v^2 \right) \xi^2 \right) + 243 \mu^{16} \left(16 + \left(-1 + v^2 \right) \xi^2 \right) + \right. \\
& \quad \quad 54 \mu^2 \left(1408 + 91 \left(-1 + v^2 \right) \xi^2 \right) + 54 \mu^{14} \left(1408 + 91 \left(-1 + v^2 \right) \xi^2 \right) + \\
& \quad \quad 12 \mu^4 \left(35632 + 2663 \left(-1 + v^2 \right) \xi^2 \right) + 12 \mu^{12} \left(35632 + 2663 \left(-1 + v^2 \right) \xi^2 \right) + \\
& \quad \quad \left. \left. 2 \mu^8 \left(208528 + 72489 \left(-1 + v^2 \right) \xi^2 \right) + \mu^6 \left(563968 + 75198 \left(-1 + v^2 \right) \xi^2 \right) + \right. \right.
\end{aligned}$$

(* The above gives two possible solutions for ρ . After cheking with suitable numerical values, the second solution is rejected*)

`σRET = Simplify[σRET /. soleq2[[1]] /. soleq3[[1]]]`

$$-\left(\frac{\dots 1 \dots}{\left(4 \mu^2 (1 + \mu^2)^2 (9 + \mu^2)^2 (1 + 9 \mu^2)^2 (3888 + \dots 79 \dots)\right)}\right)$$

large output

show less

show more

show all

set size limit...

⋮ **\$OutputSizeLimit**: This output can only be updated in the same kernel session that generated it.

(*Calculation of the elastic shortening ε (Equation 4.15)*)

`DuDx = Expand[DuDx];`

`LDuDx = Length[DuDx];`

$$\varepsilon = \frac{-1}{b a} \text{Sum}\left[\text{Simplify}\left[\int_0^b \int_0^a \text{DuDx}[[i]] \, dx \, dy, \{m, n\} \in \text{Integers}\right], \{i, 1, \text{LDuDx}\}\right]$$

$$-\frac{1}{a b} \left(-\frac{3}{64} a b f_1^2 m^2 - \frac{1}{16} a b f_1 f_2 m^2 - \frac{1}{16} a b f_2^2 m^2 + a b f_0 v + \right.$$

$$\left. \frac{1}{4} a b f_1 v - \frac{3}{64} a b f_1^2 n^2 v - \frac{1}{16} a b f_1 f_2 n^2 v - \frac{1}{16} a b f_2^2 n^2 v - \frac{a b \sigma}{E E} + \frac{a b v^2 \sigma}{E E} \right)$$

(* Calaculation of $\frac{\varepsilon R}{t}$ (Equation 4.16)*)

`εRt = Expand[Simplify[ε  $\frac{R}{t}$ ]] /. soleq1[[1]] (*Substitution of  $f_0$ *)`

$$\frac{3 f_1^2 m^2 R}{64 t} + \frac{f_1 f_2 m^2 R}{16 t} + \frac{f_2^2 m^2 R}{16 t} - \frac{f_1 R v}{4 t} + \frac{3 f_1^2 n^2 R v}{64 t} + \frac{f_1 f_2 n^2 R v}{16 t} +$$

$$\frac{f_2^2 n^2 R v}{16 t} + \frac{R \sigma}{E E t} - \frac{R v^2 \sigma}{E E t} - \frac{1}{t} R v \left(-\frac{f_1}{4} + \frac{3 f_1^2 n^2}{64} + \frac{1}{16} f_1 f_2 n^2 + \frac{f_2^2 n^2}{16} - \frac{v \sigma}{E E} \right)$$

(*Updating $\varepsilon R t$ in a short form and rearrangement: *)

`εRt = Expand[Simplify[εRt /. {f2 → ρ f1, t → η $\frac{R}{n^2}$, f1 → $\frac{t \xi}{R}$ } /. f1 → $\frac{t \xi}{R}$ /. t → η $\frac{R}{n^2}$ /.
m → μ n]] /. σ → σRETexp $\frac{E E \eta}{n^2}$`

$$\frac{3}{64} \eta \mu^2 \xi^2 + \frac{1}{16} \eta \mu^2 \xi^2 \rho + \frac{1}{16} \eta \mu^2 \xi^2 \rho^2 + \sigma \text{RETexp}$$

(* which is matching with Equation 4.16*)

(*Substitution of ρ from soleq3: *)

(*It is clear now that $\varepsilon R t$ is a function of the same variables (η, μ, v, ξ) as $\sigma R E t$ *)

`εRETexp = Simplify[εRt /. soleq3[[1]]]`

$$\frac{3}{64} \eta \mu^2 \xi^2 -$$

$$\left(\mu^2 \xi \left(3888 + 3888 \eta^2 + 78624 \mu^2 + 76032 \eta^2 \mu^2 + 460560 \mu^4 + 427584 \eta^2 \mu^4 + 519360 \mu^6 + \right. \right.$$

$$563968 \eta^2 \mu^6 - 846000 \mu^8 + 417056 \eta^2 \mu^8 - 204768 \mu^{10} + 563968 \eta^2 \mu^{10} -$$

$$11664 \mu^{12} + 427584 \eta^2 \mu^{12} + 76032 \eta^2 \mu^{14} + 3888 \eta^2 \mu^{16} - 3888 v^2 - 78624 \mu^2 v^2 -$$

$$\begin{aligned}
& 460\,560\,\mu^4\,\nu^2 - 519\,360\,\mu^6\,\nu^2 + 846\,000\,\mu^8\,\nu^2 + 204\,768\,\mu^{10}\,\nu^2 + 11\,664\,\mu^{12}\,\nu^2 + \\
& 1944\,\eta\,\xi + 39\,312\,\eta\,\mu^2\,\xi + 269\,160\,\eta\,\mu^4\,\xi + 968\,160\,\eta\,\mu^6\,\xi + 2\,882\,280\,\eta\,\mu^8\,\xi + \\
& 606\,096\,\eta\,\mu^{10}\,\xi + 33\,048\,\eta\,\mu^{12}\,\xi - 1944\,\eta\,\nu^2\,\xi - 39\,312\,\eta\,\mu^2\,\nu^2\,\xi - 269\,160\,\eta\,\mu^4\,\nu^2\,\xi - \\
& 968\,160\,\eta\,\mu^6\,\nu^2\,\xi - 2\,882\,280\,\eta\,\mu^8\,\nu^2\,\xi - 606\,096\,\eta\,\mu^{10}\,\nu^2\,\xi - 33\,048\,\eta\,\mu^{12}\,\nu^2\,\xi - \\
& 243\,\eta^2\,\xi^2 - 4914\,\eta^2\,\mu^2\,\xi^2 - 31\,956\,\eta^2\,\mu^4\,\xi^2 - 75\,198\,\eta^2\,\mu^6\,\xi^2 - 144\,978\,\eta^2\,\mu^8\,\xi^2 - \\
& 75\,198\,\eta^2\,\mu^{10}\,\xi^2 - 31\,956\,\eta^2\,\mu^{12}\,\xi^2 - 4914\,\eta^2\,\mu^{14}\,\xi^2 - 243\,\eta^2\,\mu^{16}\,\xi^2 + 243\,\eta^2\,\nu^2\,\xi^2 + \\
& 4914\,\eta^2\,\mu^2\,\nu^2\,\xi^2 + 31\,956\,\eta^2\,\mu^4\,\nu^2\,\xi^2 + 75\,198\,\eta^2\,\mu^6\,\nu^2\,\xi^2 + 144\,978\,\eta^2\,\mu^8\,\nu^2\,\xi^2 + \\
& 75\,198\,\eta^2\,\mu^{10}\,\nu^2\,\xi^2 + 31\,956\,\eta^2\,\mu^{12}\,\nu^2\,\xi^2 + 4914\,\eta^2\,\mu^{14}\,\nu^2\,\xi^2 + 243\,\eta^2\,\mu^{16}\,\nu^2\,\xi^2 - \\
& \sqrt{\left(-48\,\eta\,(-1+\nu^2)\,\xi\,(-81-1638\,\mu^2+5\,\mu^4\,(-2243+195\,\eta\,\xi))+3\,\mu^{12}\,(-459+325\,\eta\,\xi)+\right. \\
& \quad 6\,\mu^{10}\,(-4209+2314\,\eta\,\xi)+4\,\mu^6\,(-10\,085+3471\,\eta\,\xi)+\mu^8\,(-120\,095+62\,682\,\eta\,\xi)) \\
& \quad \left.(48\,(-1-2\,\mu^2+3\,\mu^4)\,(9+82\,\mu^2+9\,\mu^4)^2\,(-1+\nu^2)+\eta^2\,(\mu^8\,(417\,056-356\,478\right. \\
& \quad \quad (-1+\nu^2)\,\xi^2)+\mu^6\,(563\,968-35\,874\,(-1+\nu^2)\,\xi^2)+\mu^{10}\,(563\,968-35\,874 \\
& \quad \quad (-1+\nu^2)\,\xi^2)+243\,(16+(-1+\nu^2)\,\xi^2)+243\,\mu^{16}\,(16+(-1+\nu^2)\,\xi^2)+54 \\
& \quad \quad \mu^2\,(1408+91\,(-1+\nu^2)\,\xi^2)+54\,\mu^{14}\,(1408+91\,(-1+\nu^2)\,\xi^2)+12 \\
& \quad \quad \mu^4\,(35\,632+2013\,(-1+\nu^2)\,\xi^2)+12\,\mu^{12}\,(35\,632+2013\,(-1+\nu^2)\,\xi^2))\Big)+ \\
& \quad \left.(48\,(-1-2\,\mu^2+3\,\mu^4)\,(9+82\,\mu^2+9\,\mu^4)^2\,(-1+\nu^2)-24\,\eta\,(9+82\,\mu^2+9\,\mu^4)^2\,(1+2\,\mu^2+ \right. \\
& \quad \quad 17\,\mu^4)\,(-1+\nu^2)\,\xi+\eta^2\,(243\,(16+(-1+\nu^2)\,\xi^2)+243\,\mu^{16}\,(16+(-1+\nu^2)\,\xi^2) \\
& \quad \quad \xi^2)+54\,\mu^2\,(1408+91\,(-1+\nu^2)\,\xi^2)+54\,\mu^{14}\,(1408+91\,(-1+\nu^2)\,\xi^2)+ \\
& \quad \quad 12\,\mu^4\,(35\,632+2663\,(-1+\nu^2)\,\xi^2)+12\,\mu^{12}\,(35\,632+2663\,(-1+\nu^2)\,\xi^2)+ \\
& \quad \quad 2\,\mu^8\,(208\,528+72\,489\,(-1+\nu^2)\,\xi^2)+\mu^6\,(563\,968+75\,198\,(-1+\nu^2)\,\xi^2)+ \\
& \quad \quad \left.\mu^{10}\,(563\,968+75\,198\,(-1+\nu^2)\,\xi^2))\right)^2\Big)\Big)/ \\
& (768\,(-1+\nu^2)\,(-81-1638\,\mu^2+5\,\mu^4\,(-2243+195\,\eta\,\xi))+3\,\mu^{12}\,(-459+325\,\eta\,\xi)+ \\
& \quad 6\,\mu^{10}\,(-4209+2314\,\eta\,\xi)+ \\
& \quad 4\,\mu^6\,(-10\,085+3471\,\eta\,\xi)+ \\
& \quad \mu^8\,(-120\,095+62\,682\,\eta\,\xi)) + \\
& \left(\mu^2\,(3888+3888\,\eta^2+78\,624\,\mu^2+76\,032\,\eta^2\,\mu^2+460\,560\,\mu^4+427\,584\,\eta^2\,\mu^4+519\,360\,\mu^6+ \right. \\
& \quad 563\,968\,\eta^2\,\mu^6-846\,000\,\mu^8+417\,056\,\eta^2\,\mu^8-204\,768\,\mu^{10}+563\,968\,\eta^2\,\mu^{10}- \\
& \quad 11\,664\,\mu^{12}+427\,584\,\eta^2\,\mu^{12}+76\,032\,\eta^2\,\mu^{14}+3888\,\eta^2\,\mu^{16}-3888\,\nu^2-78\,624\,\mu^2\,\nu^2- \\
& \quad 460\,560\,\mu^4\,\nu^2-519\,360\,\mu^6\,\nu^2+846\,000\,\mu^8\,\nu^2+204\,768\,\mu^{10}\,\nu^2+11\,664\,\mu^{12}\,\nu^2+ \\
& \quad 1944\,\eta\,\xi+39\,312\,\eta\,\mu^2\,\xi+269\,160\,\eta\,\mu^4\,\xi+968\,160\,\eta\,\mu^6\,\xi+2\,882\,280\,\eta\,\mu^8\,\xi+ \\
& \quad 606\,096\,\eta\,\mu^{10}\,\xi+33\,048\,\eta\,\mu^{12}\,\xi-1944\,\eta\,\nu^2\,\xi-39\,312\,\eta\,\mu^2\,\nu^2\,\xi-269\,160\,\eta\,\mu^4\,\nu^2\,\xi- \\
& \quad 968\,160\,\eta\,\mu^6\,\nu^2\,\xi-2\,882\,280\,\eta\,\mu^8\,\nu^2\,\xi-606\,096\,\eta\,\mu^{10}\,\nu^2\,\xi-33\,048\,\eta\,\mu^{12}\,\nu^2\,\xi- \\
& \quad 243\,\eta^2\,\xi^2-4914\,\eta^2\,\mu^2\,\xi^2-31\,956\,\eta^2\,\mu^4\,\xi^2-75\,198\,\eta^2\,\mu^6\,\xi^2-144\,978\,\eta^2\,\mu^8\,\xi^2- \\
& \quad 75\,198\,\eta^2\,\mu^{10}\,\xi^2-31\,956\,\eta^2\,\mu^{12}\,\xi^2-4914\,\eta^2\,\mu^{14}\,\xi^2-243\,\eta^2\,\mu^{16}\,\xi^2+243\,\eta^2\,\nu^2\,\xi^2+ \\
& \quad 4914\,\eta^2\,\mu^2\,\nu^2\,\xi^2+31\,956\,\eta^2\,\mu^4\,\nu^2\,\xi^2+75\,198\,\eta^2\,\mu^6\,\nu^2\,\xi^2+144\,978\,\eta^2\,\mu^8\,\nu^2\,\xi^2+ \\
& \quad 75\,198\,\eta^2\,\mu^{10}\,\nu^2\,\xi^2+31\,956\,\eta^2\,\mu^{12}\,\nu^2\,\xi^2+4914\,\eta^2\,\mu^{14}\,\nu^2\,\xi^2+243\,\eta^2\,\mu^{16}\,\nu^2\,\xi^2- \\
& \quad \left.\sqrt{\left(-48\,\eta\,(-1+\nu^2)\,\xi\,(-81-1638\,\mu^2+5\,\mu^4\,(-2243+195\,\eta\,\xi))+\right. \right. \\
& \quad \quad 3\,\mu^{12}\,(-459+325\,\eta\,\xi)+6\,\mu^{10}\,(-4209+2314\,\eta\,\xi)+4\,\mu^6\,(-10\,085+3471\,\eta\,\xi)+ \\
& \quad \quad \left.\mu^8\,(-120\,095+62\,682\,\eta\,\xi))\,(48\,(-1-2\,\mu^2+3\,\mu^4)\,(9+82\,\mu^2+9\,\mu^4)^2\,(-1+\nu^2)+\right. \\
& \quad \quad \left.\eta^2\,(\mu^8\,(417\,056-356\,478\,(-1+\nu^2)\,\xi^2)+\mu^6\,(563\,968-35\,874\,(-1+\nu^2)\,\xi^2)+\right.
\end{aligned}$$

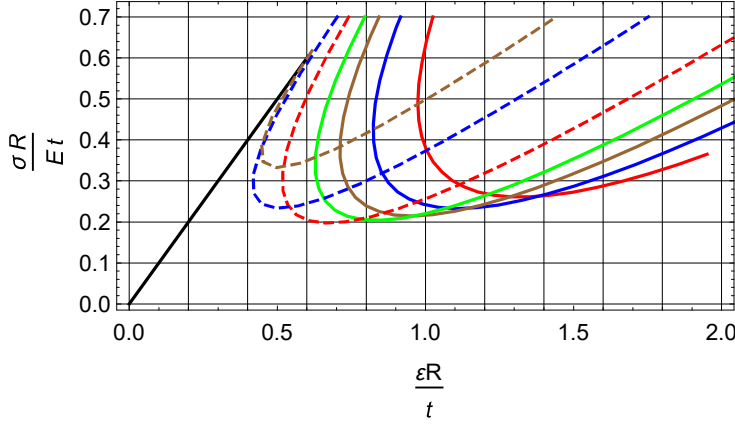
To plot $\frac{\sigma R}{ET}$ versus $\frac{\varepsilon R}{t}$ we have two possible ways : (1) Solving for $\frac{\sigma R}{ET}$ to express it in terms of $\frac{\varepsilon R}{t}$, but this way is difficult (2) calculate the values of $\frac{\sigma R}{ET}$ and $\frac{\varepsilon R}{t}$ for different given values of (η, μ, ν, ξ) , which has been adopted in this work.

```

(*Update  $\varepsilon_{RT}$  after substituting  $\sigma_{RETexp}$  *)
 $\varepsilon_{RTfinal} = \varepsilon_{RETexp} / . \sigma_{RETexp} \rightarrow \sigma_{RET}$ ;
(* Generating the axial stress-end shortening curves*)
(* First, for case of square waves  $\mu=1$  (Figure 24) *)
(* The linear part of the plot (classical plot) *)
clclplot = ListPlot[{{0, 0}, {0.605, 0.605}}, Joined → True, PlotStyle → {Black}];
(* The postbuckling curve *)
 $\xi$ values = Range[0.0001, 20, 0.5];
L $\xi$  = Length[ $\xi$ values];
 $\eta$ values =  $\frac{1}{1000} \{10^2, 11^2, 12^2, 13^2, 15^2, 20^2, 26^2\}$  // N;
plotstyle =
  {Red, Blue, Brown, Green, {Red, Dashed}, {Blue, Dashed}, {Brown, Dashed}};
Do[
  (*The numerical data *)
  Ndata = { $v \rightarrow 0.3$ ,  $\mu \rightarrow 1$ ,  $\eta \rightarrow \eta$ values[[j]]};
  (*The values of  $\delta_{RT}$  and  $\sigma_{RET}$  evaluated at given numerical data *)
   $\varepsilon_{RTdata}$  = Table[ $\varepsilon_{RTfinal} / . Ndata / . \xi \rightarrow \xi$ values[[i]], {i, 1, L $\xi$ }];
   $\sigma_{RETdata}$  = Table[ $\sigma_{RET} / . Ndata / . \xi \rightarrow \xi$ values[[i]], {i, 1, L $\xi$ }];
  (*Combining  $\varepsilon_{RT}$  and  $\sigma_{RET}$  data for plotting*)
  pData = Table[{ $\varepsilon_{RTdata}$ [[i]],  $\sigma_{RETdata}$ [[i]]}, {i, 1, L $\xi$ }];

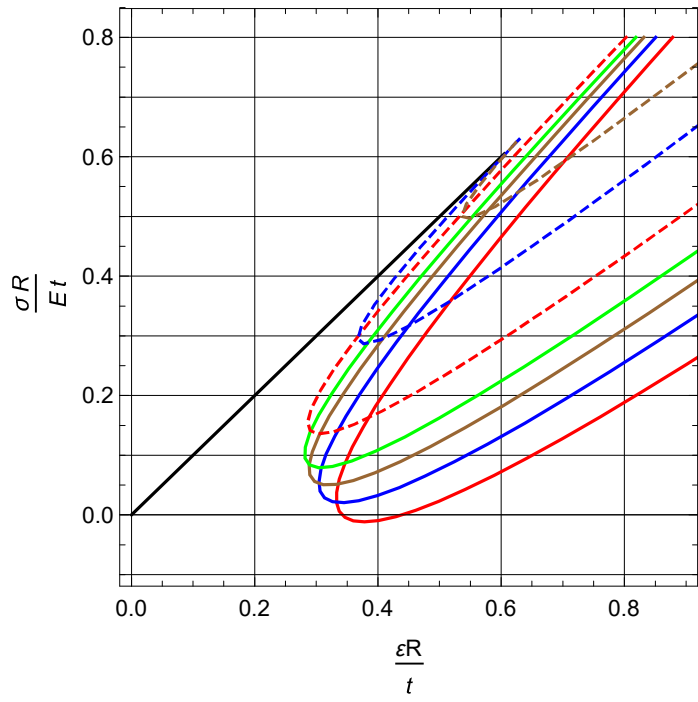
  p[j] =
    ListPlot[pData, AxesOrigin → {0, 0}, Joined → True, PlotStyle → plotstyle[[j]],
      PlotRange → {{0, 2}, {0, 0.7}}, {j, 1, Length[ $\eta$ values]}]
Show[Join[{clclplot}, Table[p[i], {i, 1, Length[ $\eta$ values]}]],
  AxesOrigin → {0, 0}, PlotRange → {{0, 2}, {0, 0.7}}, AspectRatio → 1 / 2,
  AxesStyle → {Directive[Black, 10], Directive[Black, 10]},
  GridLines → {Range[0, 2, 0.2], Range[0, 0.7, 0.1]},
  GridLinesStyle → Directive[Black, Thin], AxesOrigin → {0, 0.0}, Frame → True,
  FrameLabel → {"  $\frac{\varepsilon_R}{t}$ ", "  $\frac{\sigma_R}{E t}$ "}, LabelStyle → Directive[Black, 12]]

```



```
(* Secondly, for case of rectangular waves  $\mu=0.5$  (Figure 25) *)
(* The linear part of the plot (classical plot) *)
clclplot = ListPlot[{{0, 0}, {0.605, 0.605}}, Joined → True, PlotStyle → {Black}];
(* The postbuckling curve *)
ξvalues = Range[0.0001, 30, 0.5];
Lξ = Length[ξvalues];
ηvalues =  $\frac{1}{1000}$  {102, 112, 122, 132, 152, 202, 262} // N;
plotstyle =
  {Red, Blue, Brown, Green, {Red, Dashed}, {Blue, Dashed}, {Brown, Dashed}};
Do[
  (*The numerical data *)
  Ndata = {ν → 0.3, μ → 0.5, η → ηvalues[[j]]};
  (*The values of  $\delta RT$  and  $\sigma RET$  evaluated at given numerical data *)
  δRTdata = Table[εRTfinal /. Ndata /. ξ → ξvalues[[i]], {i, 1, Lξ}];
  σRETdata = Table[σRET /. Ndata /. ξ → ξvalues[[i]], {i, 1, Lξ}];
  (*Combining  $\delta RT$  and  $\sigma RET$  data for plotting*)
  pData = Table[{δRTdata[[i]], σRETdata[[i]]}, {i, 1, Lξ}];

  p[j] =
    ListPlot[pData, AxesOrigin → {0, 0}, Joined → True, PlotStyle → plotstyle[[j]],
      PlotRange → {{0, 0.9}, {-0.1, 0.8}}, {j, 1, Length[ηvalues]}]
Show[Join[{clclplot}, Table[p[i], {i, 1, Length[ηvalues]}]],
  AxesOrigin → {0, 0}, PlotRange → {{0, 0.9}, {-0.1, 0.8}}, AspectRatio → 1,
  AxesStyle → {Directive[Black, 10], Directive[Black, 10]},
  GridLines → {Range[0, 2, 0.2], Range[-0.1, 0.8, 0.1]},
  GridLinesStyle → Directive[Black, Thin], AxesOrigin → {0, -0.1},
  Frame → True, FrameLabel → {" $\frac{\epsilon R}{t}$ ", " $\frac{\sigma R}{E t}$ "}, LabelStyle → Directive[Black, 12]]
```



Appendix C: Generation of the Out-of-roundness and Dent

Imperfections

Appendix C : Generation of the Out-of-roundness and Dent Imperfections

C.1 Important Notes

Since the out-of-roundness and the dent imperfections are possibly be combined in real-life applications, a generalized code that can combine both of them is presented here. If one would like to model a shell with out-of-roundness alone, he/she can simply set the amplitude of the dent imperfections to zero. The same can be done for the opposite case. The code provided in this appendix should be read in accordance with forms of imperfections discussed in Chapter 6.

It is also recommended to exercise a high degree of caution while inputting the geometry of the shell and the parameters those defined the imperfection because, in most cases, it can not easily be visualized unless a scale factor is used for the sake of visualization.

C.2 The Out-of-roundness (Form A) and Localized Dent

```

Clear["Global`*"]
(*Dimensions of the shell*)
L = 15000;
R = 7500;
t = 10;
(* The dimensions of the dent (if any)*)
Ldl = 1000;
Ldc = 1000;
ydent = 0.5 L; (*Center of the dent along the length of the cylinder*)
(*The mesh size*)
Nc = 100; (*No. of elements in the circumfrential direction*)
NL = 50; (*No. of elements in the longitudinal direction*)
δθ = 2 π / Nc;
δL = L / NL;
m = 0; (*No. of halve-sine waveswaves in the axial direction*)
n = 2; (*No. of halve-sine waveswaves circumfrential direction*)

OFRAmp = 11.25 * t * 5; (* The amplitude of the out-of-roundness*)
(*It should be noted that the above amplitude
is scaled by 5 for the sake of visualization only*)
Rij = R + OFRAmp Sin[n i δθ] Cos[ $\frac{m \pi y_j}{L}$ ];

δd = 0.5 * t * 0; (* The amplitude of the dent*)
θdent = ArcSin[ $\frac{Ldc / 2}{R}$ ];
θcomp[θ_] = If[ $0 \leq \theta \leq \frac{\pi}{2}$ , θ, If[ $\frac{3\pi}{2} \leq \theta \leq 2\pi$ , (2 π - θ), θ]];

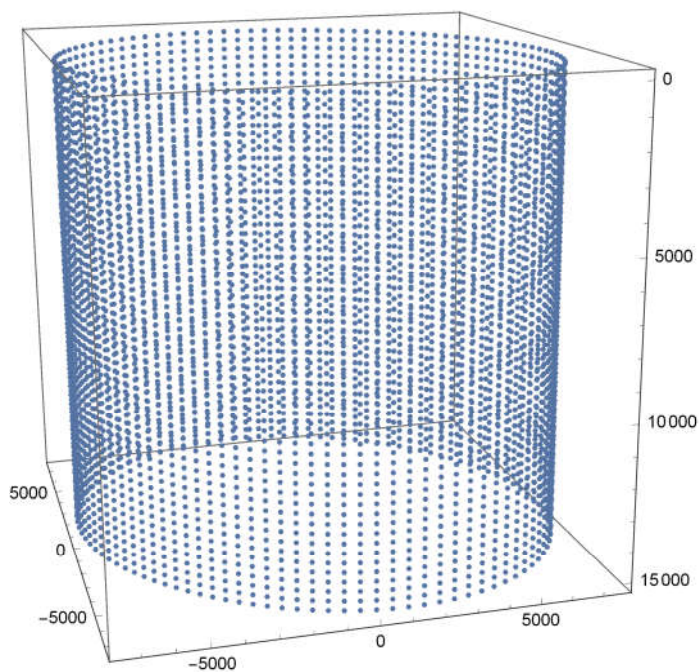
```

```

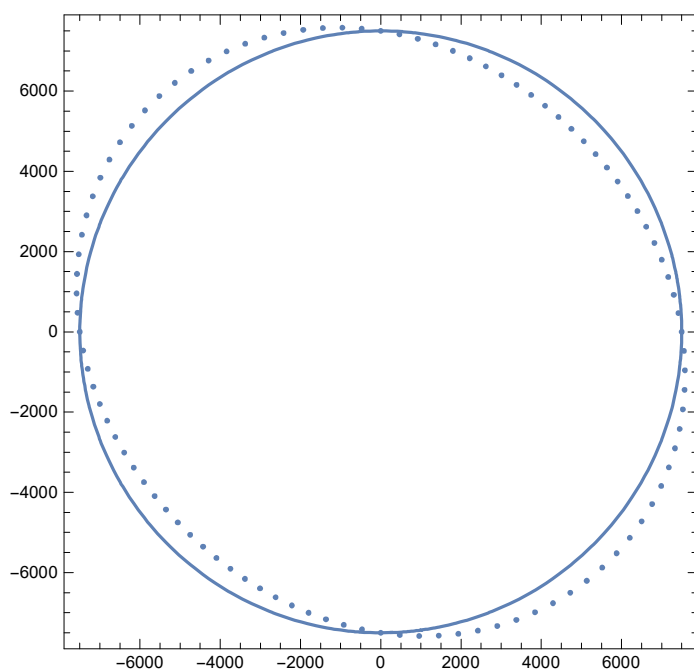
k = 0;
Do[k = k + 1;
  yj = (j - 1) δL;
  δdij = If[ $\left(\left(y_{dent} - \frac{L_{d1}}{2}\right) \leq y_j \leq \left(y_{dent} + \frac{L_{d1}}{2}\right)\right) \&\& \theta_{comp}[i \delta\theta] (*\leq*) < \theta_{dent},$ 
    δd  $\left(\text{Exp}\left[-1 * \left(\frac{4 * (y_j - y_{dent})}{L_{d1}}\right)^2\right] * \right.$ 
     $\left.\text{Exp}\left[-1 * \left((4 * (\theta_{comp}[i \delta\theta] * R)) / (2 * \theta_{dent} * R)\right)^2\right]\right], 0];$ 

  xi = (Rij - δdij) Cos[i δθ];
  zi = - (Rij - δdij) Sin[i δθ];
  (*Below are the data in 2D *)
  data2D[k] = {xi, zi};
  (*Below is the required mesh*)
  mesh[k] = {xi, yj, zi}, {j, 1, NL + 1}, {i, 1, Nc}]
meshData = Table[Join[{m}, mesh[m]], {m, 1, k}];
(*The following line is if you are interested to see the data in a table form*)
meshData // TableForm;
(* Just for the purpose of plotting, the following has been generated*)
PlottingData = Table[mesh[m], {m, 1, k}];
ListPointPlot3D[PlottingData, BoxRatios → {1, 1, 1}]

```



```
(* To check the plot in a 2D view*)
data2Dbottom = Table[data2D[i], {i, 1, Nc}];
ImpFct = ListPlot[data2Dbottom];
Pfct = ContourPlot[x^2 + y^2 == R^2, {x, -R, R}, {y, -R, R}];
Show[{Pfct, ImpFct}, PlotRange -> All, AspectRatio -> 1]
```



(*To export the data*)
 (*To export the data, the suitable path should be specified below is an example*)
 Export[
 "E:\\Ovality-Dent and Out-of-Roundness\\Manual Mesh\\FormA-Fig.txt", meshData]
 E:\\Ovality-Dent and Out-of-Roundness\\Manual Mesh\\FormA-Fig.txt
 (*Open the specified path and you find the file there. Remove
 the brackets by using FindAll and Replace by an empty space. Then,
 copy the data into the input file of ABAQUS*)

C.3 The Out-of-roundness (Form B) and Localized Dent

```

Clear["Global`*"]
(*Dimensions of the shell*)
L = 15000;
R = 7500;
t = 10;
(* The dimensions of the dent (if any)*)
Ld1 = 1000;
Ldc = 1000;
ydent = 0.5 L; (*Center of the dent along the length of the cylinder*)
(*The mesh size*)
Nc = 100; (*No. of elements in the perimeter*)
NL = 50; (*No. of elements in the longitudinal direction*)
δθ = 2 π / Nc;
δL = L / NL;
m = 1; (*No. of halve-sine waveswaves in the axial direction*)
n = 2; (*No. of halve-sine waveswaves circumfrential direction*)

OFRamp = 11.25 * t * 5; (* The amplitude of the out-of-roundness*)
(*It should be noted that the above amplitude
is scaled by 5 for the sake of visualization only*)
Rij = R + OFRamp Sin[n i δθ] Sin[ $\frac{\pi yj}{L}$ ];

δd = 2 * t * 0; (* The amplitude of the dent*)
θdent = ArcSin[ $\frac{Ldc / 2}{R}$ ];
θcomp[θ_] = If[ $0 \leq \theta \leq \frac{\pi}{2}$ , θ, If[ $\frac{3\pi}{2} \leq \theta \leq 2\pi$ , (2 π - θ), θ]];
k = 0;
Do[k = k + 1;

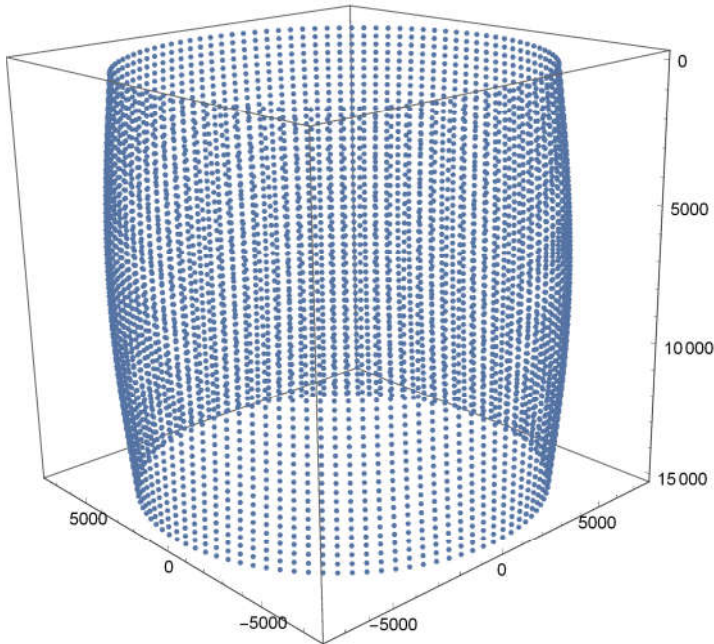
```

```

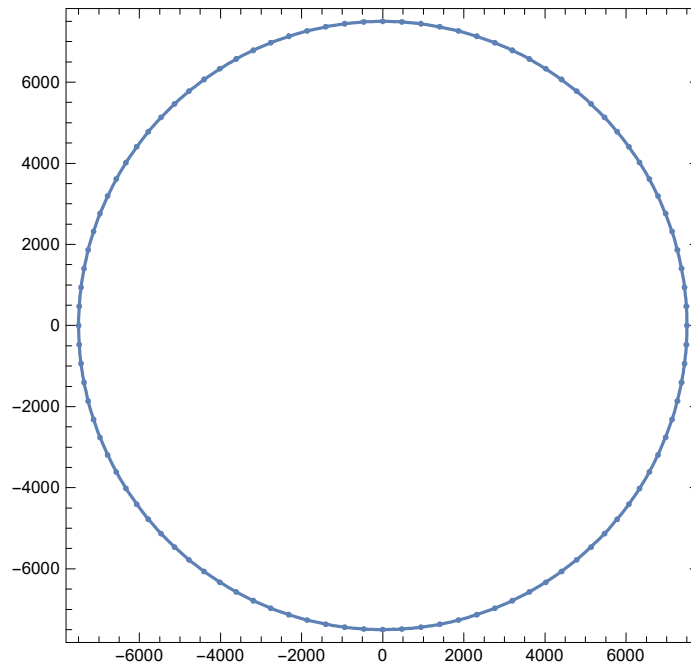
yj = (j - 1) δL;
δdij = If[ ((ydent -  $\frac{Ld1}{2}$ ) ≤ yj ≤ (ydent +  $\frac{Ld1}{2}$ )) && θcomp[i δθ] (*≤*) < θdent,
  δd (Exp[-1 * ( $\frac{4 * (yj - ydent)}{Ld1}$ )2] *
    Exp[-1 * ((4 * (θcomp[i δθ] * R)) / (2 * θdent * R))2]), 0];

xi = (Rij - δdij) Cos[i δθ];
zi = - (Rij - δdij) Sin[i δθ];
(*Below are the data in 2D *)
data2D[k] = {xi, zi};
(*Below is the required mesh*)
mesh[k] = {xi, yj, zi}, {j, 1, NL + 1}, {i, 1, Nc}]
meshData = Table[Join[{m}, mesh[m]], {m, 1, k}];
(*The following line is if you are interested to see the data in a table form*)
meshData // TableForm;
(* Just for the purpose of plotting, the following has been generated*)
PlottingData = Table[mesh[m], {m, 1, k}];
ListPointPlot3D[PlottingData, BoxRatios → {1, 1, 1}]

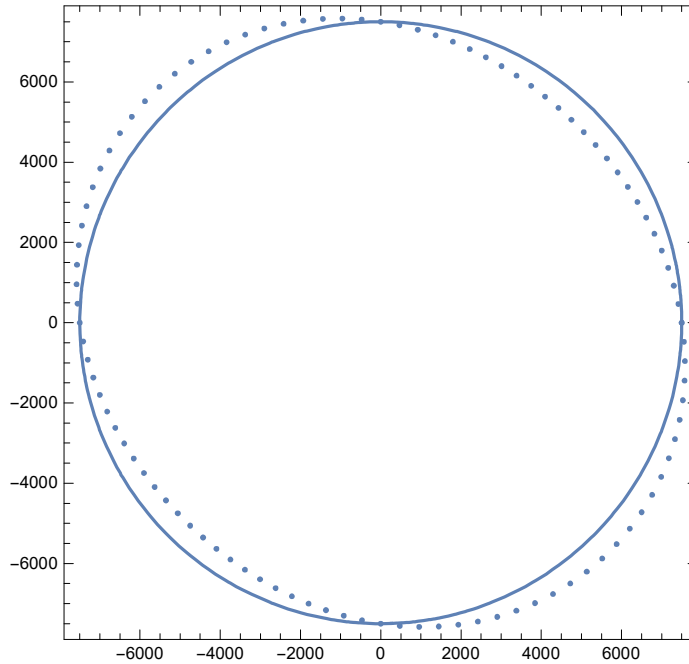
```



```
(* To check the plot in a 2D view*)  
data2Dbottom = Table[data2D[i], {i, 1, Nc}];  
Impfct = ListPlot[data2Dbottom];  
Pfct = ContourPlot[x2 + y2 == R2, {x, -R, R}, {y, -R, R}];  
Show[{Pfct, Impfct}, PlotRange → All, AspectRatio → 1]
```



```
(* To check the cross-section in the middle in a 2D view*)
data2Dbottom = Table[data2D[i], {i, 2500, 2500 + Nc}];
ImpFct = ListPlot[data2Dbottom];
Pfct = ContourPlot[x^2 + y^2 == R^2, {x, -R, R}, {y, -R, R}];
Show[{Pfct, ImpFct}, PlotRange -> All, AspectRatio -> 1]
```



(*To export the data*)

(*To export the data, the suitable path should be specified below is an example*)

```
Export[
  "E:\\Ovality-Dent and Out-of-Roundness\\Manual Mesh\\FormB-Fig.txt", meshData]
E:\\Ovality-Dent and Out-of-Roundness\\Manual Mesh\\FormA-Fig.txt
```

(*Open the specified path and you find the file there. Remove the brackets by using FindAll and Replace by an empty space. Then, copy the data into the input file of ABAQUS*)

Appendix D: Generation of Dent Imperfections for the Parametric Study

Appendix D : Generation of Dent Imperfections for the Parametric Study

D.1 Important Notes

This appendix is prepared to adopt the dent imperfection presented by Equation 7.2 in Chapter 7. All the parameters discussed in Table 16 can be modeled using the following code.

It is also recommended to exercise a high degree of caution while inputting the geometry of the shell and the parameters those defined the dent imperfection because in most cases it can not easily be visualized unless a scale factor is used for the sake of visualization.

D.2 Generation of the Imperfect Shell Geometry

```

Clear["Global`*"]
(*Dimensions of the shell*)
L = 10000.;
R = 7500.;
t = 10;
(* The dimensions of the dent*)
ydent = 0.75 L; (*Center of the dent along the length of the cylinder*)
Ldc = 5000;
Ld1 = 5000;

(*The mesh size*)
Nc = 200; (*No. of elements in the circumferential direction*)
NL = 100; (*No. of elements in the longitudinal direction*)
δθ = 2 π / Nc;
δL = L / NL;

δd = 80.0 * 20; (* The umplitude of the dent*)
(*in the above, the number 20 is scaling factor*)
θdent = ArcSin[ $\frac{L_{dc}}{R}$ ];
θcomp[θ_] = If[ $0 \leq \theta \leq \frac{\pi}{2}$ , θ, If[ $\frac{3\pi}{2} \leq \theta \leq 2\pi$ , (2 π - θ), θ]];
k = 0;
Do[k = k + 1;
  yj = (j - 1) δL;
  δdij = If[ $\left(y_{dent} - \frac{L_{d1}}{2}\right) \leq y_j \leq \left(y_{dent} + \frac{L_{d1}}{2}\right)$  && θcomp[i δθ] (*≤*) < θdent,

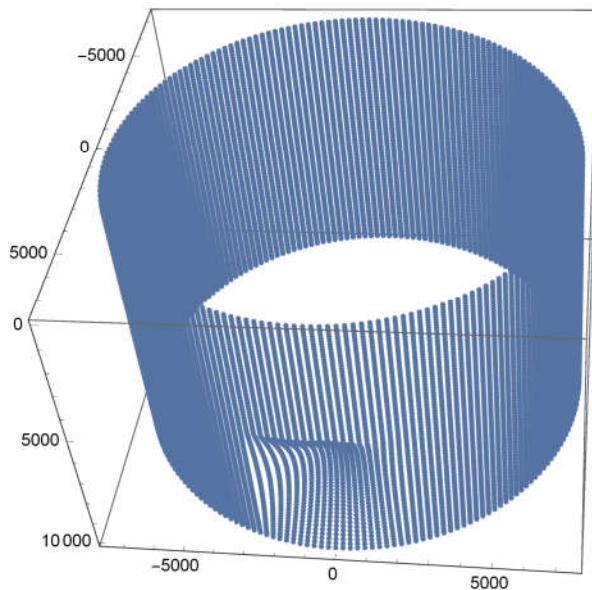
```

$$\delta d \left(\cos \left[\left(\frac{\pi * (y_j - y_{dent})}{L_{d1}} \right) \right] * \cos \left[\left(\left(\pi * (\theta_{comp}[i \delta \theta] * R) \right) / (2 * \theta_{dent} * R) \right) \right] \right), \theta \right];$$

```

xi = (R -  $\delta$ dij) Cos[i  $\delta$  $\theta$ ];
zi = - (R -  $\delta$ dij) Sin[i  $\delta$  $\theta$ ];
(*Below are the data in 2D *)
data2D[k] = {xi, zi};
(*Below is the required mesh*)
mesh[k] = {xi, yj, zi}, {j, 1, NL + 1}, {i, 1, Nc}]
meshData = Table[Join[{m}, mesh[m]], {m, 1, k}];
(*The following line is if you are interested to see the data in a table form*)
meshData // TableForm;
(* Just for the purpose of plotting, the following has been generated*)
PlottingData = Table[mesh[m], {m, 1, k}];
ListPointPlot3D[PlottingData, BoxRatios -> Automatic]

```



```

(*To export the data*)
(*To export the data, the suitable path should be specified below is an example*)
Export["E:\\Parametric Study-Dent\\Manual Mesh\\C81.txt", meshData]
E:\\Parametric Study-Dent\\Manual Mesh\\C81.txt

```

(*Open the specified path and you find the file there. Remove the brackets by using FindAll and Replace by an empty space. Then, copy the data into the input file of ABAQUS*)

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3. University of Khartoum Award for the best academic performance in Civil Engineering department, fifth year, for academic year 2012-2013.
4. Award of the National Corporation of Electricity for the best Academic Performance in Civil Engineering department at fifth year, for academic year 2012-2013.
5. Award of the National Corporation of Roads and Bridges for the best performance in highways at the fifth year.
6. Award of Sigheroon Elzain Sigheroon for the best academic performance in Civil Engineering Department at the fifth year.

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