

**APPROXIMATION OF COMMON FIXED
POINTS OF NONEXPANSIVE – TYPE
MAPPINGS**

BY

DOLAPO MUHAMMED OYETUNBI

A Thesis Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICS

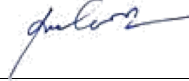
MAY 2020

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **DOLAPO MUHAMMED OYETUNBI** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN MATHEMATICS**.

Thesis Committee



Dr. Abdul Rahim Khan (Adviser)



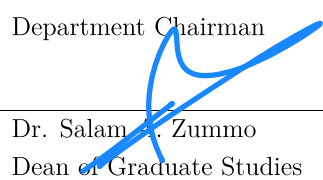
Dr. Monther Rashed Alfuraidean (Member)



Dr. Mohammed Alshahrani (Member)



Dr. Husain Salem Al-Attas
Department Chairman



Dr. Salam A. Zummo
Dean of Graduate Studies

23-4-2020
Date



Copyright: © Dolapo Muhammed Oyetunbi 2020

To my parents, my siblings and my wife

ACKNOWLEDGEMENTS

In the name of Allah, I begin. All praises and adorations are to Almighty Allah, the Lord of the universe, for his favours towards the completion of my M.SC. program.

I am indebted to my thesis advisor, Dr. Abdul Rahim Khan, for his guidance and discussions. His penchant for perfection, openness to new ideas and commitment help me in defining and achieving research goals for my M.Sc. thesis.

My appreciations go to Dr. Monther Alfuraidan and Dr. Mohammed Alshahrani for being an integral part of my career journey, as my course instructors and thesis committee members. Thanks for your thoughtful teaching, helpful suggestions and focused criticism.

My sincere gratitude goes to the King Fahd University of Petroleum and Minerals for giving me the needed platform to conduct work for this thesis. I thank the chairman, entire faculty and staff members of the Department of Mathematics and Statistics, KFUPM, for making my stay comfortable and worthwhile.

To my parents and siblings, I can not thank you enough for your selfless love and unwavering support that has molded me into what I am today. My heartfelt appreciation goes to my beloved wife for her sacrifice, patience, and understanding throughout the time apart.

My sincere thanks to my friends and colleagues who supported me when I needed it. To mention just a few, I thank Mr. Ibrahim Sarumi , Mr. Aliyu Mansur and the entire Nigerian community in the kingdom for their all-time guidance and support throughout my master program.

Contents

Acknowledgements	v
Abstract	viii
Arabic Abstract	ix
1 Introduction	1
1.1 Metric Space	1
1.2 Fixed Point Problem	2
1.2.1 Well-known Iteration Processes	2
1.3 Hyperbolic Space	3
1.3.1 Uniformly Convex Banach Space	4
1.3.2 CAT(0) Space	5
1.4 The $\triangle$ -convergence	7
1.5 Review of Nonexpansive-Type Mappings	8
1.5.1 Fundamentally Nonexpansive Mapping	8
1.5.2 α -nonexpansive Mapping	9
2 Common Fixed Point Results	12
2.1 Unique Common Fixed Point of Contractive-type Mappings	12
2.1.1 Introduction	12
2.1.2 Main Results	14
2.2 Kirk Type Iteration Process in CAT(0) Space	19

2.2.1	Introduction	19
2.2.2	Preliminaries and Notations	20
2.2.3	Approximation Results for Nonexpansive Mappings	22
2.2.4	Application to Convex Feasibility Problem	27
2.3	An Algorithm of Three Nonexpansive Mappings	32
2.3.1	Introduction and Preliminaries	32
2.3.2	Convergence Analysis	33
2.3.3	Application to Convex Minimization Problem	41
2.3.4	Numerical Example	42
3	Halpern's Type Algorithm for α-nonexpansive Semigroup	46
3.1	Introduction	46
3.2	Preliminaries in Banach and $CAT(0)$ spaces	47
3.3	Convergence Results in Banach space	50
3.4	Convergence Results in $CAT(0)$ space	58
4	Common Endpoints of Multivalued Mappings in Hyperbolic Space	63
4.1	Introduction and Preliminaries	63
4.2	Existence of Endpoint	68
4.3	Approximation of Common Endpoints	71
4.4	Numerical Experiment	80
4.5	Suggestions For Future Work	83

ABSTRACT

In this thesis, we study several algorithms for approximating common fixed points of nonexpansive-type mappings on both linear and nonlinear domains. We give an introduction of fundamentally nonexpansive mappings and α -nonexpansive mappings. Sufficient conditions for the existence of a unique common fixed point of contractive-type mappings are obtained in a metric space. We introduce some algorithms for approximating common fixed points of a finite number of nonexpansive mappings in a $CAT(0)$ space and a hyperbolic space. Applications of our results to convex feasibility and minimization problems are presented. Finally, we propose an algorithm for approximating common endpoints of two multivalued generalized nonexpansive mappings. Consequently, our results approximate common fixed point of fundamentally nonexpansive and α -nonexpansive single-valued mappings in hyperbolic spaces.

ملخص البحث

في هذه الأطروحة، ندرس العديد من الخوارزميات لتقريب النقاط الثابتة الشائعة للتعيينات من النوع غير الموسع على كلا من المجالات الخطية وغير الخطية. نعطي مقدمة بشكل أساسي عن التعيينات غير التوسعية وتعيينات ألفا-غير التوسعية. يتم الحصول على شروط كافية لوجود نقطة ثابتة مشتركة وفريدة من نوعها للتعينات من النوع الانقباضي في فضاء مترى. كما نقدم بعض الخوارزميات لتقريب النقاط الثابتة الشائعة لعدد محدود من التعيينات غير الموسعة في فضاء CAT (0) وفضاء زائدي. يتم عرض تطبيقات نتائجنا على دراسة جدوى محدبة ومسائل التقليل. أخيرًا، نقترح خوارزمية لتقريب نقاط النهاية الشائعة لاثنتين من التعيينات المعممة غير الموسعة ذات القيم المتعددة. وبالتالي، فإن نتائجنا تقارب نقطة ثابتة مشتركة بشكل أساسي من التعيينات أحادية القيمة غير الموسعة وتعيينات ألفا-غير الموسعة في الفضاءات الزائدة.

Chapter 1

Introduction

1.1 Metric Space

Definition 1.1.1. Let X be a nonempty set. A metric on the set X is a function $d : X \times X \rightarrow \mathbb{R}^+$ satisfying the following axioms:

1. $d(m, w) \geq 0$ for all $m, w \in X$.
2. $d(m, w) = 0$ for $m, w \in X$ if and only if $m = w$.
3. $d(m, w) = d(w, m)$ for all $m, w \in X$.
4. $d(m, w) \leq d(m, u) + d(u, w)$ for all $m, w, u \in X$ (triangle inequality).

X coupled with a metric d is called a metric space (X, d) .

Definition 1.1.2. Let X denote a linear vector space defined over a field of scalars $\mathbb{F}$.

A norm is a function $\|\cdot\| : X \rightarrow \mathbb{R}^+$ satisfying the following conditions:

1. $\|m\| \geq 0$ for all $m \in X$ and $\|m\| = 0$ if and only if $m = 0$.
2. $\|\alpha m\| = |\alpha| \|m\|$ for all $m \in X$ and $\alpha \in \mathbb{F}$.
3. $\|m + w\| \leq \|m\| + \|w\|$ for all $m, w \in X$ (triangle inequality).

The pair $(X, \|\cdot\|)$ is called a normed space.

Remark 1.1.3. Every normed space is a metric space i.e. if $(X, \|\cdot\|)$ is a normed space, then (X, d) is a metric space where $d(m, w) = \|m - w\|$ for all $m, w \in X$.

Definition 1.1.4. A sequence $\{w_n\}$ in a metric space (X, d) is called a Cauchy sequence if for each $\epsilon > 0$, there is an $N(\epsilon) \in \mathbb{N}$ such that $d(w_n, w_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.1.5. A sequence $\{w_n\}$ in a metric space (X, d) converges to $w \in X$ if and only if, for each $\epsilon > 0$, there is an $N(\epsilon) \in \mathbb{N}$ such that $d(w_n, w) < \epsilon$ for all $n \geq N$.

Definition 1.1.6. We say a metric space (X, d) is complete if every Cauchy sequence in X is convergent.

A complete normed space $(X, \|\cdot\|)$ is called a Banach space.

1.2 Fixed Point Problem

Suppose that X is a nonempty set and $A_1, A_2 \subseteq X$ with $A_1 \cap A_2 \neq \emptyset$. A point $a \in A_1$ is a fixed point of the mapping $G : A_1 \rightarrow A_2$ if $Ga = a$. The symbol $F(G)$ shall denote fixed point set of G . The fixed point problem entails setting appropriate conditions on X , the set A_1 and the mapping G in order to establish the following:

1. Existence and uniqueness of a fixed point
2. Iterative techniques for approximating the fixed points
3. Properties of the fixed point set.

Finding common fixed point of a family of mappings is an intrinsic part of fixed point theory. Let $G, H : A_1 \rightarrow A_2$ be two mappings. A point $a \in A_1$ such that $Ga = a = Ha$ is called common fixed point of G and H .

1.2.1 Well-known Iteration Processes

An iterative process is a mathematical technique using an initial guess to produce a sequence of elements with the n th step depending on the $(n-1)$ th step. Suppose that U

is a nonempty subset of a metric space (X, d) . Let $G : U \rightarrow U$ and $F(G) \neq \emptyset$. For a given $x_0 \in C$ and $n = 0, 1, \dots$, some well-known iteration processes for approximating fixed point of G are defined below:

1. Picard Iterations: $x_{n+1} = Gx_n$.

2. Mann iteration scheme: $x_{n+1} = \gamma_n Gx_n + (1 - \gamma_n)x_n$ for $0 \leq \gamma_n \leq 1$.

3. Ishikawa iteration scheme :

$$x_{n+1} = \gamma_n G y_n + (1 - \gamma_n)x_n$$

$$y_n = \lambda_n G x_n + (1 - \lambda_n)x_n$$

for $0 \leq \lambda_n, \gamma_n \leq 1$.

4. Halpern iterations: for any $u \in X$, $x_{n+1} = (1 - \beta_n)u + \beta_n Gx_n$ where $0 \leq \beta_n \leq 1$.

5. Noor three-step iteration scheme:

$$x_{n+1} = \gamma_n G y_n + (1 - \gamma_n)x_n$$

$$y_n = \lambda_n G z_n + (1 - \lambda_n)x_n$$

$$z_n = \mu_n G x_n + (1 - \mu_n)x_n$$

where $0 \leq \lambda_n, \mu_n, \gamma_n \leq 1$.

1.3 Hyperbolic Space

A metric space (X, d) coupled with $W : X^2 \times [0, 1] \rightarrow X$ fulfilling the following conditions.

$$(1) d(c, W(m, w, \tau)) \leq \tau d(c, m) + (1 - \tau)d(c, w)$$

$$(2) d(W(m, w, \lambda_1), W(m, w, \lambda_2)) = |\lambda_1 - \lambda_2| d(m, w)$$

$$(3) W(m, w, \tau) = W(w, m, 1 - \tau)$$

$$(4) d(W(m, v, \tau), W(w, z, \tau)) \leq \tau d(m, w) + (1 - \tau) d(v, z)$$

for all $m, c, v, w, z \in X$ and $\tau, \lambda_1, \lambda_2 \in [0, 1]$ is called a hyperbolic space.

For all $v, m, w \in X$, $\mu > 0$ and $0 < \xi \leq 2$, a hyperbolic space X [112] is uniformly convex if there is $0 < \beta \leq 1$ such that $d(W(m, w, \frac{1}{2}), v) \leq (1 - \beta)\mu$, whenever $d(m, v) \leq \mu$, $d(w, v) \leq \mu$ and $d(m, w) \geq \xi\mu$.

If $W(m, w, \tau) \in U$ for all $m, w \in U$ and $\tau \in [0, 1]$, then U is a convex set.

Here are two important examples of a uniformly convex hyperbolic space:

1.3.1 Uniformly Convex Banach Space

For sequences, we denote strong convergence, weak convergence and weak-star convergence by $\rightarrow$, $\rightharpoonup$, and $\rightharpoonup^*$, respectively. Let X^* be the dual of a Banach space X and

$$\psi(\mu) = \inf \left\{ 1 - \frac{1}{2} \|w + m\| : \|w\| \leq 1, \|m\| \leq 1, \|m - w\| \geq \mu \right\}.$$

X is uniformly convex if for each $0 < \mu \leq 2$, we have that $\psi(\mu) > 0$.

Let $S_X = \{w \in X : \|w\| = 1\}$. X is said to be smooth if the norm of X is Gateaux differentiable i.e. for each $m, w \in S_X$, $\lim_{t \rightarrow 0} \frac{\|m + tw\| - \|m\|}{t}$ exists.

For $f \in X^*$ and $w \in X$, the value of f at w is denoted by $\langle w, f \rangle$ and is called the duality pairing. By normalized duality mapping, we mean a mapping $J : X \rightarrow 2^{X^*}$ defined by

$$J(w) = \{f \in X^* : \langle w, f \rangle = \|w\|^2, \|w\| = \|f\|\}, \quad \forall w \in X.$$

By the Hahn-Banach theorem, we have that $J(w)$ is nonempty. If $\{w_n\} \subset X$ and $w_n \rightharpoonup w \in X$ imply that $J(w_n) \rightharpoonup^* J(w)$, we say J is weakly sequentially continuous. If X is a Hilbert space, Riesz representation theorem and the inner product properties of X imply that J is the identity mapping.

Moreover, X satisfies the Opial condition [48] if for any sequence $\{w_n\}$ with $w_n \rightharpoonup w$, we get $\limsup_{n \rightarrow \infty} \|w_n - w\| < \limsup_{n \rightarrow \infty} \|w_n - m\|$ for all $m \in X$ with $m \neq w$. l^p ($1 \leq p < \infty$), Hilbert spaces and finite dimensional Banach spaces satisfy the Opial condition. X is smooth and satisfies the Opial condition ([52], [117]) if X admits a weakly sequentially continuous duality mapping.

Lemma 1.3.1 ([120]). *For a $q > 0$ and uniformly convex Banach space X , there is a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\xi m + (1 - \xi)w\|^2 \leq \xi\|m\|^2 + (1 - \xi)\|w\|^2 - \xi(1 - \xi)g(\|m - w\|)$$

for any $\|m\|, \|w\| \leq q$ and $\xi \in [0, 1]$.

1.3.2 CAT(0) Space

Let (X, d) be a metric space and $\mathcal{I} = [0, 1] \subset \mathbb{R}$. A mapping $\varphi : \mathcal{I} \rightarrow X$ with the property that $\varphi(0) = a_1$, $\varphi(1) = a_2$, and $d(\varphi(s_1), \varphi(s_2)) = |s_1 - s_2|$ for all $0 \leq s_1, s_2 \leq 1$ is known as a geodesic path from a_1 to a_2 . The image of φ is the geodesic segment connecting a_1 and a_2 and $[a_1, a_2]$ represents the unique segment joining a_1 to a_2 . If a geodesic segment connects any two points of X , then (X, d) is called a geodesic space. Moreover, X is uniquely geodesic if such segment is unique for every pair of elements of X . A geodesic triangle $\triangle(m_1, m_2, m_3)$ comprises of three points m_1, m_2, m_3 in X as well as geodesic segments connecting any two of the points m_1, m_2, m_3 . For $\triangle(m_1, m_2, m_3)$ in (X, d) , a comparison triangle is defined as $\overline{\triangle}(m_1, m_2, m_3) := \triangle(\overline{m_1}, \overline{m_2}, \overline{m_3})$ in $\mathbb{R}^2$ with $d_{\mathbb{R}^2}(\overline{m_j}, \overline{m_k}) = d(m_j, m_k)$ for $j, k = 1, 2, 3$. A $CAT(0)$ space is a metric space such that any given two points are connected by a segment and for every $m, w \in \triangle(m_1, m_2, m_3)$ in X and every $\overline{m}, \overline{w} \in \overline{\triangle} := \triangle(\overline{m_1}, \overline{m_2}, \overline{m_3})$ in $\mathbb{R}^2$,

$$d(m, w) \leq d_{\mathbb{R}^2}(\overline{m}, \overline{w}) \text{ holds.}$$

By $u = (1 - s_1)m_1 \oplus s_1m_2$, we mean u is a point on the geodesic segment joining m_1 with m_2 where $d(m_2, u) = (1 - s_1)d(m_1, m_2)$ and $d(m_1, u) = s_1d(m_1, m_2)$. For any $\mu \in [0, 1]$, and w_1, w_2, w_3 in a $CAT(0)$ space X , we have the (CN) inequality [21]:

$$d^2((1 - \mu)w_1 \oplus \mu w_2, w_3) \leq (1 - \mu)d^2(w_1, w_3) + \mu d^2(w_2, w_3) - \mu(1 - \mu)d^2(w_1, w_2).$$

Simply connected Riemannian manifold, R-trees, Pre-Hilbert spaces, the complex Hilbert ball with a hyperbolic metric and Euclidean spaces ([71]) are some well-known examples of $CAT(0)$ space.

Lemma 1.3.2 ([32]). *Let X be a $CAT(0)$ space. For any $m, w, z, r \in X$ and $\zeta \in [0, 1]$. we have the following:*

1. $d(\zeta m \oplus (1 - \zeta)w, z) \leq \zeta d(m, z) + (1 - \zeta)d(w, z)$
2. $d(\zeta m \oplus (1 - \zeta)w, \xi m \oplus (1 - \xi)w) = |\zeta - \xi|d(m, w)$
3. $d(\zeta m \oplus (1 - \zeta)w, \zeta z \oplus (1 - \zeta)r) \leq \zeta d(m, z) + (1 - \zeta)d(w, r).$

Definition 1.3.3 ([14]). A pair $(m, w) \in X^2$ is called a vector and is denoted by $\overrightarrow{mw}$. For any $m, w, z, r \in X$, the quasilinearization is the map $\langle \cdot, \cdot \rangle : (X^2) \times (X^2) \rightarrow \mathbb{R}$ defined by

$$\langle \overrightarrow{mw}, \overrightarrow{zr} \rangle = \frac{1}{2} [d^2(m, r) + d^2(w, z) - d^2(m, z) - d^2(w, r)]. \quad (1.3.1)$$

We can easily show that for any $m, w, z, r, u \in X$,

$$\begin{aligned} \langle \overrightarrow{mw}, \overrightarrow{zr} \rangle &= \langle \overrightarrow{zr}, \overrightarrow{mw} \rangle, \quad \langle \overrightarrow{mw}, \overrightarrow{zr} \rangle = -\langle \overrightarrow{wm}, \overrightarrow{rz} \rangle \\ \text{and } \langle \overrightarrow{mw}, \overrightarrow{zr} \rangle + \langle \overrightarrow{ur}, \overrightarrow{zw} \rangle &= \langle \overrightarrow{mw}, \overrightarrow{zr} \rangle. \end{aligned} \quad (1.3.2)$$

A geodesic space is $CAT(0)$ space if and only if it satisfies the Cauchy-Schwartz inequality ([14]):

$$\langle \overrightarrow{mw}, \overrightarrow{zr} \rangle \leq d(m, w)d(z, r) \text{ for all } m, w, z, r \in X.$$

Definition 1.3.4. Let U be a nonempty convex subset of a $CAT(0)$ space X and $x \in X$.

A metric proection is the mapping $P_U : U \rightarrow 2^U$ defined by

$$P_U m = \{w \in U : d(m, w) = d(m, U)\}.$$

In a complete $CAT(0)$ space, a metric projection is a single-valued mapping [18]. Moreover, characterization of a metric projection in a complete $CAT(0)$ spaces X is given as follows:

Lemma 1.3.5 ([30]). *Let U be a nonempty closed and convex subset of a complete $CAT(0)$ space X and $P_U : U \rightarrow U$ be a metric projection . Let $z \in U$ and $m \in X$. Then,*

$$z = P_U m \text{ if and only if } \langle \overrightarrow{wz}, \overrightarrow{zm} \rangle \geq 0 \text{ for all } w \in U.$$

1.4 The $\triangle$ -convergence

The concept of $\triangle$ -convergence was introduced by Lim [84]. Kirk and Panyanak [73] showed that $\triangle$ -convergence in $CAT(0)$ space, and in general hyperbolic space, is similar to weak convergence in a Banach space.

Definition 1.4.1. Let X be a metric space. For a bounded sequence $\{w_n\} \subseteq X$ and $w \in X$, define $r(\cdot, \{w_n\}) : X \rightarrow [0, \infty)$ by

$$r(w, \{w_n\}) = \limsup_{n \rightarrow \infty} d(w, w_n).$$

Then:

(a) the asymptotic radius of $\{w_n\}$ in relation to $U \subset X$ is defined by

$$r_U(w_n) = \inf\{r(w, \{w_n\}) : w \in U\}$$

(b) for any $m \in U \subset X$, the asymptotic center of $\{w_n\}$ in relation to U is defined by

$$A_U(w_n) = \{w \in X : r(w, \{w_n\}) \leq r(m, \{w_n\})\}.$$

Definition 1.4.2. If every subsequence $\{w_{n_i}\}$ of $\{w_n\}$ has a unique asymptotic center $w \in X$, we say $\{w_n\} \subseteq X$, $\triangle$ -converges to w . Symbolically, $\triangle - \lim_{n \rightarrow \infty} w_n = w$.

Remark 1.4.3. $\{w_n\}$ is said to be regular relative to U if for every subsequence $\{w_{n_k}\}$ of $\{w_n\}$, we have $r_U(w_n) = r_U(w_{n_k})$. In a Banach space with Opial condition, weak convergence is the same as $\triangle$ -convergence (cf.[77]).

Lemma 1.4.4 ([69]). *Let (X, d, W) be a uniformly convex hyperbolic space and $w \in X$. Let $\{\alpha_n\} \in [c_1, c_2]$ for some $c_1, c_2 \in (0, 1)$ and $\{a_n\}, \{b_n\} \subseteq X$. If for some $\zeta > 0$, $\limsup_{n \rightarrow \infty} d(a_n, w) \leq \zeta$, $\limsup_{n \rightarrow \infty} d(b_n, w) \leq \zeta$ and $\lim_{n \rightarrow \infty} d(W(a_n, b_n, \alpha_n), w) = \zeta$. Then $\lim_{n \rightarrow \infty} d(a_n, b_n) = 0$.*

Lemma 1.4.5 ([82]). *In relation to any nonempty closed and convex subset U of a complete uniformly convex hyperbolic space X , the asymptotic center of every bounded sequence $\{w_n\}$ in X is unique.*

Lemma 1.4.6 ([22]). *Let $\{y_n\}$ be a subsequence of a bounded sequence $\{z_n\}$ of a complete uniformly convex hyperbolic space X . Suppose the asymptotic center of $\{z_n\}$ and $\{y_n\}$, respectively, are z and m . If $\{d(z_n, m)\}$ is convergent, then we have that $z = m$.*

1.5 Review of Nonexpansive-Type Mappings

Let U be a subset of a metric space (X, d) . A mapping $G : U \rightarrow X$ is called :

1. Nonexpansive if for every $m, w \in U$, $d(Gm, Gw) \leq d(m, w)$.

2. Mean nonexpansive if for $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 \leq 1$,

$$d(Gm, Gw) \leq \lambda_1 d(m, w) + \lambda_2 d(m, Gw) \text{ holds for all } m, w \in U.$$

3. Quasi-nonexpansive when $F(G) \neq \emptyset$ and $d(Gm, \bar{w}) \leq d(m, \bar{w})$ for all $m \in U, \bar{w} \in F(G)$.

1.5.1 Fundamentally Nonexpansive Mapping

A mapping $G : U \rightarrow X$ is said to be fundamentally nonexpansive [46] if for any $m, w \in U$, we have

$$d(G^2m, Gw) \leq d(Gm, w).$$

Lemma 1.5.1 ([89]). *If G is a fundamentally nonexpansive mapping on a subset U of a metric space X , then for all $m, w \in U$, the following holds:*

$$d(m, Gw) \leq 3d(m, Gm) + d(m, w).$$

Definition 1.5.2 ([44]). We say a mapping $G : U \rightarrow X$ satisfies condition (E) if for some $\mu \geq 1$, we have

$$d(m, Gw) \leq \mu d(m, Gm) + d(m, w) \text{ for all } m, w \in U.$$

By Lemma 1.5.1, the class of fundamentally nonexpansive mappings is a subclass of the set of mappings satisfying condition (E). Ghoncheh and Razani [46] introduced

fundamentally nonexpansive mappings in Banach space and Moosaei [88] showed that a commutative family of fundamentally nonexpansive mappings has common fixed point. In a complete $CAT(0)$ space, Salahifard et al. [110] studied fundamentally nonexpansive mappings while Suanoom and Klin-eam [114] extended the results to a hyperbolic space. Recently, Fukhar-ud-din and Khan [40] proved common fixed results of fundamentally nonexpansive mapping and α -nonexpansive mapping in uniformly convex hyperbolic space.

1.5.2 α -nonexpansive Mapping

In 2011, Aoyama and Kohsaka [10] initiated the study of α -nonexpansive mapping. For $\alpha < 1$ and $m, w \in U$, $G : U \rightarrow X$ is a α -nonexpansive mapping if

$$d^2(Gm, Gw) \leq \alpha d^2(Gm, w) + \alpha d^2(Gw, m) + (1 - 2\alpha)d^2(m, w) \text{ holds.} \quad (1.5.1)$$

Ariza et al. [12] showed that for $\alpha < 0$, T becomes the identity mapping. Among various generalizations of nonexpansive mapping, this class is pertinent because it contains several important nonlinear mappings with application to minimization problem, variational inequality and zeros of maximal operators ([107], [27], [34]). Recently, Song et al. [113] defined G to be α -nonexpansive mapping if it satisfies

$$d(Gm, Gw) \leq \alpha d(Gm, w) + \alpha d(Gw, m) + (1 - 2\alpha)d(m, w) \text{ for all } m, w \in C. \quad (1.5.2)$$

Using convexity of the mapping $t \rightarrow t^2$, it is easy to see that (1.5.2) implies (1.5.1). We give an example to show that the class of mappings satisfying (1.5.2) is indeed a proper subclass of the class of mappings satisfying (1.5.1). Define a mapping $G : [0, 3] \rightarrow [0, 3]$ by

$$Gm = \begin{cases} 0, & m \neq 3 \\ 2, & m = 3. \end{cases}$$

It is easy to see that $2\|Gm - Gw\|^2 \leq \|Gm - w\|^2 + \|m - Gw\|^2$ (See [33]). However, there is no $\alpha < 1$ such that for $m, w \in [0, 3]$, $\|Gm - Gw\| \leq \alpha\|Gm - w\| + \alpha\|Gw - m\| +$

$(1 - 2\alpha)\|m - w\|$ holds. Indeed, for $m = 2, w = 3$

$$\|Gm - Gw\| = 2, \|m - w\| = 1, \|m - Gw\| = 0, \|w - Gm\| = 3$$

and $\|Gm - Gw\| \leq \alpha\|Gm - w\| + \alpha\|Gw - m\| + (1 - 2\alpha)\|m - w\|$ holds only if $\alpha \geq 1$.

In this thesis, we consider α -nonexpansive mapping in the sense of Aoyama and Kohsaka [10] i.e. mapping for which (1.5.1) holds.

Muangchoo-in et al. [91] have studied common fixed point approximation of two α -nonexpansive mappings via Ishikawa iterations. Moreover, Mongkolkeha et al. [87] proved strong and weak convergence of a new iteration involving two nonspreading mappings and an α -nonexpansive mapping to a common fixed point. Very Recently, Fukhar-uddin [37] has approximated common fixed points of mappings satisfying condition(C) and α -nonexpansive mapping through a one step iterative scheme.

In Banach spaces with normal structure, Zhang [124] proved that his newly introduced mean nonexpansive mapping possesses unique fixed point. Consequently, this mapping was further studied in linear spaces by many authors including Wu and Zhang [119], Gu [53]. Bose [15] showed that two mean nonexpansive mappings have common fixed point and gave Picard-like iterations for estimating their common fixed point. Later, Gu and Li [54] proved that the Ishikawa iteration associated with two mean nonexpansive mappings converges to a common fixed point of both mappings. Fixed point theory of a mean nonexpansive mapping was extended to $CAT(0)$ space by Zhou and Cui [125], where they established the non-emptiness of its fixed point set and convergence of its associated Ishikawa iterations.

Lemma 1.5.3 ([55]). *Let U be a nonempty subset of a metric space (X, d) . If $G : U \rightarrow X$ is a mean nonexpansive mappings, then G is an α -nonexpansive mapping.*

Lemma 1.5.4 ([40]). *Let U be a nonempty subset of a metric space (X, d) . Let $G : U \rightarrow X$ be an α -nonexpansive mapping for some $0 \leq \alpha < 1$. Then, for all $m, w \in U$*

$$d^2(m, Gw) \leq \frac{1 + \alpha}{1 - \alpha} d^2(m, Gm) + \frac{2}{1 - \alpha} [\alpha d(m, w) + d(Gm, Gw)] d(m, Gm) + d^2(m, w).$$

Lemma 1.5.5 ([38]). *Let U be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space (X, d, W) . Let $G : U \rightarrow U$ be a nonexpansive mapping. If $\{w_n\} \subset U$ is such that $\lim_{n \rightarrow \infty} d(w_n, Gw_n) = 0$ and $\{w_n\}$, Δ -converges to w , then $w \in F(G)$.*

Definition 1.5.6. Let U be a nonempty subset of a metric space (X, d) . A mapping $G : U \rightarrow X$ is :

- (i) Nonspreading if $2d^2(Gm, Gw) \leq d^2(m, Gw) + d^2(w, Gm) \quad \forall m, w \in U$.
- (ii) Hybird if $3d^2(Gm, Gw) \leq d^2(m, Gw) + d^2(w, Gm) + d^2(m, w) \quad \forall m, w \in U$.

Remark 1.5.7.

1. Nonspreading and hybird mappings are examples of α -nonexpansive mappings.
2. Set of quasi-nonexpansive mappings is a proper subset of the set of fundamental and α -nonexpansive mappings. It is easy to see that nonexpansive mappings are mean nonexpansive and α -nonepansive mappings. Moreover, unlike nonexpansive mappings, mean nonexpansive and α -nonepansive mappings need not be continuous.

Chapter 2

Common Fixed Point Results

2.1 Unique Common Fixed Point of Contractive-type Mappings

In this section, we extend and complement recent main result of Górnicki [50] and Bisht [16] for a pair of asymptotically regular mappings. These results simultaneously extend the famous Banach contraction principle and Kannan fixed point result for two mappings.

2.1.1 Introduction

Browder and Petryshyn [19] introduced the notion of asymptotic regularity. A self-mapping T of a metric space (X, d) is said to be asymptotically regular if $\lim_{n \rightarrow \infty} d(T^n x, T^{n+1} x) = 0$ for all $x \in X$.

Recently, Górnicki [50] has obtained the following generalization of a result of Reich [104]:

Theorem 2.1.1. *Suppose that (X, d) is a complete metric space and T is a continuous asymptotically regular self-mapping on X satisfying the following condition:*

$$d(Tx, Ty) \leq Md(x, y) + K\{d(x, Tx) + d(y, Ty)\} \text{ for all } x, y \in X, \quad (2.1.1)$$

where $M \in [0, 1)$ and $K \in [0, \infty)$. Then T has a unique fixed point $p \in X$ and $T^n x \rightarrow p$ for each $x \in X$.

We recall some weaker forms of continuity:

Definition 2.1.2 ([26]). If T is a self-mapping of a metric space (X, d) , then the set $O(T, x) = \{T^n x : n = 0, 1, 2, \dots\}$ is called the orbit of T at x . T is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(T, x)$ for some $x \in X$, $\lim_{n \rightarrow \infty} x_n = z$ implies $\lim_{n \rightarrow \infty} T x_n = T z$.

Clearly, continuity of T implies that T is orbital continuous. The example below shows that the converse is not true.

Example 1. Let $X = [-1, 1]$ be endowed with the usual metric and $T : X \rightarrow X$ be defined by :

$$Tx = \begin{cases} \frac{2x}{3}, & x \geq 0 \\ 1, & x < 0 \end{cases}.$$

Then T is orbitally continuous but discontinuous at $x = 0$.

Definition 2.1.3 ([97]). A self-mapping T of a metric space (X, d) is called k -continuous, $k \in \mathbb{N}$ if $\lim_{n \rightarrow \infty} T^{k-1} x_n = z$ implies $\lim_{n \rightarrow \infty} T^k x_n = T z$.

Note that 1-continuity is equivalent to continuity and for any $k \in \mathbb{N}$, k -continuity implies $(k+1)$ -continuity while the converse is not true. Moreover, continuity of T^k and k -continuity of T are independent conditions when $k > 1$ (see [97], Examples 1.2-1.5).

Bisht [16] improved Theorem 2.1.1 as follows :

Theorem 2.1.4. Suppose that (X, d) is a complete metric space and $T : X \rightarrow X$ is an asymptotically regular mapping satisfying (2.1.1). Then T has a unique fixed point $p \in X$ provided T is either k -continuous for some $k \geq 1$ or orbitally continuous. Moreover, $T^n x \rightarrow p$ for each $x \in X$.

Recently, Bisht and Singh [17] extended Theorem 2.1.4 for a pair of self-mappings S and T satisfying the following Lipschitz-Kannan type condition:

$$d(Tx, Ty) \leq Md(Sx, Sy) + K\{d(Tx, Sx) + d(Ty, Sy)\} \text{ for all } x, y \in X.$$

In this section, we extend Theorem 2.1.4 for a pair of asymptotically regular mappings satisfying a contractive condition of type (2.1.1). We also establish results for mappings satisfying (2.1.1) but are not asymptotically regular.

2.1.2 Main Results

We need the following technical result:

Lemma 2.1.5 ([122]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n$, $n \geq 0$ where $\alpha_n \in (0, 1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that*

$$(i) \sum_{n=0}^{\infty} \alpha_n = \infty \quad (ii) \limsup_{n \rightarrow \infty} \beta_n \leq 0 \quad \text{or} \quad \sum_{n=0}^{\infty} |\alpha_n \beta_n| = \infty.$$

Then $\{a_n\}$ converges to zero as $n \rightarrow \infty$.

We now present our main result by following the arguments of Geraghty [45] and Boyd and Wong [20]:

Theorem 2.1.6. *Suppose that (X, d) is a complete metric space, T and S are asymptotically regular self-mappings on X satisfying the following condition*

$$d(Tx, Sy) \leq Md(x, y) + K\{d(x, Tx) + d(y, Sy)\} \text{ for all } x, y \in X, \quad (2.1.2)$$

where $0 \leq M < 1$ and $0 \leq K < \infty$. Suppose further that T and S are either k -continuous for some $k \geq 1$ or orbitally continuous. Then T and S have a unique common fixed point p . Additionally, $\lim_{n \rightarrow \infty} T^n x = p = \lim_{n \rightarrow \infty} S^n x$ for any $x \in X$.

Proof. We prove the result in three steps:

Step 1: For any $x \in X$, we prove that $\lim_{n \rightarrow \infty} d(T^n x, S^n x) = 0$.

For $S = T$, the result is trivial. Suppose $S \neq T$ and $M = 0$. Then (2.1.2) becomes

$$d(Tx, Sy) \leq K\{d(x, Tx) + d(y, Sy)\} \text{ for all } x, y \in X.$$

For any $x \in X$, let $x_1 = T^n x$ and $y_1 = S^n x_1$. Then,

$$d(T^{n+1}x, S^{n+1}x) \leq K\{d(T^n x, T^{n+1}x) + d(S^n x, S^{n+1}x)\}.$$

Taking limit on both sides as $n \rightarrow \infty$, it follows from the asymptotic regularity of S and T that $\lim_{n \rightarrow \infty} d(T^{n+1}x, S^{n+1}x) = 0$. Moreover,

$$d(T^n x, S^n x) \leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, S^{n+1} x) + d(S^n x, S^{n+1} x). \quad (2.1.3)$$

Applying asymptotic regularity of S and T to (2.1.3), we have that

$$\lim_{n \rightarrow \infty} d(T^n x, S^n x) = 0.$$

Next, suppose that $S \neq T$ and $M \neq 0$. Define $x_1 = T^n x$ and $y_1 = S^n x$ for any $x \in X$.

Then (2.1.2) becomes

$$d(T^{n+1} x, S^{n+1} x) \leq M d(T^n x, S^n x) + K \{d(T^n x, T^{n+1} x) + d(S^n x, S^{n+1} x)\}.$$

Let $a_n = d(T^n x, S^n x)$, $\alpha_n = 1 - M$ and

$$\beta_n = \frac{K}{1 - M} d(T^n x, T^{n+1} x) + \frac{K}{1 - M} d(S^n x, S^{n+1} x).$$

By asymptotic regularity of T and S , we get that $\lim_{n \rightarrow \infty} \beta_n = 0$. Moreover, $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Hence by Lemma 2.1.5, we have that $\lim_{n \rightarrow \infty} d(T^n x, S^n x) = 0$ for any $x \in X$.

Step 2: Let $x_n = T^n x$ for any $x \in X$. Then, we prove that $\{x_n\}$ is Cauchy and thus converges to $p \in X$. Moreover, $S^n x$ converges to $p \in X$.

Suppose $\{T^n x\}$ is not Cauchy. Then there exists an $\epsilon > 0$ and sequences of integers $\{m(k)\}$, $\{n(k)\}$ with $m(k) > n(k) \geq k$ such that for $k = 1, 2, \dots$, we have

$$d(T^{m(k)} x, T^{n(k)} x) \geq \epsilon \quad (2.1.4)$$

By choosing $m(k)$ to be the smallest number exceeding $n(k)$ for which (2.1.4) holds, we may assume that $d(T^{m(k)-1} x, T^{n(k)} x) < \epsilon$. Now,

$$\begin{aligned} \epsilon &\leq d(T^{m(k)} x, T^{n(k)} x) \leq d(T^{m(k)} x, T^{m(k)-1} x) + d(T^{m(k)-1} x, T^{n(k)} x) \\ &< d(T^{m(k)} x, T^{m(k)-1} x) + \epsilon. \end{aligned} \quad (2.1.5)$$

Letting $k \rightarrow \infty$ in (2.1.5), it follows by asymptotic regularity of T that

$$\lim_{k \rightarrow \infty} d(T^{m(k)} x, T^{n(k)} x) = \epsilon. \quad (2.1.6)$$

Moreover, asymptotic regularity of T and the inequality

$$\begin{aligned} d(T^{m(k)-1}x, T^{n(k)-1}x) &\leq d(T^{m(k)-1}x, T^{m(k)}x) + d(T^{m(k)}x, T^{n(k)}x) \\ &\quad + d(T^{m(k)}x, T^{n(k)-1}x) \end{aligned}$$

imply

$$\lim_{k \rightarrow \infty} d(T^{m(k)-1}x, T^{n(k)-1}x) = \epsilon. \quad (2.1.7)$$

Now, (2.1.2) implies

$$\begin{aligned} d(T^{m(k)}x, T^{n(k)}x) &\leq d(T^{m(k)}x, S^{n(k)}x) + d(S^{n(k)}x, T^{n(k)}x) \\ &\leq d(S^{n(k)}x, T^{n(k)}x) + M[d(T^{m(k)-1}x, T^{n(k)-1}x) + d(T^{n(k)-1}x, S^{n(k)-1}x)] \\ &\quad + K[d(T^{m(k)-1}x, T^{m(k)}x) + d(S^{n(k)-1}x, S^{n(k)}x)]. \end{aligned}$$

Letting $k \rightarrow \infty$, it follows by (2.1.6), (2.1.7) and asymptotic regularity of T and S that $\epsilon \leq M\epsilon$. This is a contradiction. Hence $\{x_n\}$ is Cauchy. Since X is complete, $\{x_n\}$ converges to $p \in X$. Moreover,

$$d(S^n x, p) \leq d(S^n x, T^n x) + d(T^n x, p).$$

It follows from Step 1 and convergence of $\{x_n\}$ to $p \in X$ that $S^n x$ converges to p .

Step 3: p is the unique common fixed point of S and T .

Suppose that T is k -continuous. Since $\lim_{n \rightarrow \infty} T^{k-1}x_n = p$, k -continuity of T implies that $\lim_{n \rightarrow \infty} T^k x_n = Tp$. By uniqueness of the limit, $Tp = p$. Analogically, suppose that T is orbitally continuous. Since $\lim_{n \rightarrow \infty} x_n = p$, orbital continuity implies that $\lim_{n \rightarrow \infty} Tx_n = Tp$. This yields $Tp = p$.

Similarly, since $\lim_{n \rightarrow \infty} S^n x = p$, we have that $Sp = p$ whenever S is k -continuous or orbitally continuous. Hence $p \in F(T) \cap F(S)$. Suppose that there is $q \neq p$ in $F(T) \cap F(S)$. Let $x = p$ and $y = q$. Then (2.1.2) becomes $d(p, q) \leq Md(p, q)$. Hence a contradiction. In conclusion, p is the unique common fixed point of S and T . $\square$

We give an example to illustrate Theorem 2.1.6.

Example 2. Let $X = [-1, 1]$ be endowed with the usual metric. Suppose that $T, S : X \rightarrow$

X are mappings defined by :

$$Tx = \begin{cases} \frac{2x}{3}, & x \geq 0 \\ 1, & x < 0 \end{cases} \quad Sx = \begin{cases} 0, & x \leq 0 \\ -1, & x > 0. \end{cases}$$

It is easy to verify that $|Tx - Sy| \leq \frac{1}{6}|x - y| + |x - Tx| + |y - Sy|$. Moreover, T and S are asymptotically regular, orbitally continuous and discontinuous at 0. All the assumptions of Theorem 2.1.6 hold, so T and S have 0 as their unique common fixed point. Also, $\lim_{n \rightarrow \infty} T^n x = 0 = \lim_{n \rightarrow \infty} S^n x$ for any $x \in X$.

As a special case of our result, we have the following generalization of Theorem 1.4 of Górnicki [50].

Corollary 2.1.7. *Suppose that (X, d) is a complete metric space, T and S are self-mappings on X for which T^m and S^q are asymptotically regular for some positive integers m and q , respectively. Suppose that the following condition holds:*

$$d(T^m x, S^q y) \leq Md(x, y) + K\{d(x, T^m x) + d(y, S^q y)\} \text{ for all } x, y \in X, \quad (2.1.8)$$

where $M \in [0, 1)$ and $K \in [0, \infty)$. Then T and S have a unique common fixed point $p \in X$ provided both T^m and S^q are either k -continuous for some $k \geq 1$ or orbitally continuous.

Proof. Take $f = T^m$ and $g = S^q$. Then for all $x, y \in X$, (2.1.8) becomes

$$d(fx, gy) \leq Md(x, y) + K[d(x, fx) + d(y, gy)]. \quad (2.1.9)$$

By Theorem 2.1.6, f and g have a unique common fixed point p . Now, $f(Tp) = T^m(Tp) = T^{m+1}p = T(fp) = Tp$ implies that Tp is a fixed point of f . Similarly, we have that Sp is also a fixed point of g . Using (2.1.9), we have that $Sp = Tp$. By the uniqueness of common fixed point of f and g , it follows that $Sp = Tp = p$. Suppose that z is a common fixed point of S and T . Then $fz = gz = z$. Uniqueness of common fixed point of f and g implies $p = z$. $\square$

Remark 2.1.8. For $M + 2K < 1$, asymptotic regularity and continuity are not necessary for the existence of unique common fixed point of S and T (See [62]).

Next, we consider mappings that are not necessarily asymptotically regular but satisfy (2.1.1).

Theorem 2.1.9. *Suppose that (X, d) is a complete metric space and $T : X \rightarrow X$ is a continuous mapping satisfying (2.1.1). Suppose that T has an approximate fixed point sequence i.e. there exists a sequence $\{x_n\} \subset X$ such that $d(x_n, Tx_n) \rightarrow 0$ as $n \rightarrow \infty$. Then T has a unique fixed point p . In particular, $x_n \rightarrow p$ as $n \rightarrow \infty$.*

Proof. Suppose that $m > n$. Then using triangle inequality and (2.1.1), we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, Tx_n) + d(Tx_n, Tx_m) + d(Tx_m, x_m) \\ &\leq d(x_n, Tx_n) + Md(x_n, x_m) + K\{d(x_n, Tx_n) + d(x_m, Tx_m)\} \\ &\quad + d(Tx_m, x_m). \end{aligned}$$

Now $(1-M)d(x_n, x_m) \leq (1+K)\{d(x_n, Tx_n) + d(x_m, Tx_m)\}$ implies that $\lim_{n,m \rightarrow \infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is Cauchy. By completeness of X , $\{x_n\}$ converges to $p \in X$. Since $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$, continuity of T implies p is a fixed point of T . The uniqueness of the fixed point follows from the inequality (2.1.1). $\square$

Example 3. Let $X = [0, 1]$ be endowed with the usual metric and $T : X \rightarrow X$ be defined by $Tx = 1 - x$, for all $x \in [0, 1]$. It is easy to verify that

$$|Tx - Ty| \leq (1 - 2a)|x - y| + a\{|x - Tx| + |y - Ty|\} \text{ for all } x, y \in X,$$

where $a \in [0, \frac{1}{2})$. Theorem 2.1.4 fails since T is not asymptotically regular and for any $x \neq \frac{1}{2}$, $\lim_{n \rightarrow \infty} T^n x \nrightarrow \frac{1}{2}$, the unique fixed point of T . However, there is a sequence $x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n$, where $0 < \lambda < 1$ and $n \geq 0$ such that $|x_n - Tx_n| \rightarrow 0$ as $n \rightarrow \infty$. By Theorem 2.1.9, T has a unique fixed point $\frac{1}{2}$ and x_n converges to $\frac{1}{2}$.

The following common fixed point theorem can be established easily on the lines of proof of Theorem 2.1.9.

Theorem 2.1.10. *Suppose that (X, d) is a complete metric space, T and S are continuous self-mappings on X satisfying the following condition:*

$$d(Tx, Sy) \leq Md(x, y) + K\{d(x, Tx) + d(y, Sy)\} \text{ for all } x, y \in X, \quad (2.1.10)$$

where $M \in [0, 1)$ and $K \in [0, \infty)$. Suppose that T and S have a common approximate fixed point sequence i.e. there exists a sequence $\{x_n\} \subset X$ such that $d(x_n, Tx_n) \rightarrow 0$ and $d(x_n, Sx_n) \rightarrow 0$ as $n \rightarrow \infty$. Then T and S have a unique common fixed point p . In particular, $x_n \rightarrow p$ as $n \rightarrow \infty$.

Remark 2.1.11. Our work in this section improves Theorem 1 of Kannan [65] and Corollary 3.3 of Jo [62] in the sense that the mappings may be discontinuous at their unique common fixed point.

2.2 Kirk Type Iteration Process in $CAT(0)$ Space

In this section, we establish Δ –convergence and strong convergence of Kirk-type iteration process in $CAT(0)$ spaces. As an application, we use our algorithm to solve consistent and inconsistent convex feasibility problems in $CAT(0)$ spaces.

2.2.1 Introduction

Kirk [74] introduced an iteration process for estimating fixed point of a nonexpansive mapping in a uniformly convex Banach space and proved:

Theorem 2.2.1 ([74]). *Let K be a subset of a uniformly convex Banach space X and T be a compact (i.e. T maps bounded subsets of X into precompact subsets of X) and nonexpansive mapping of K into K with at least one fixed point. Define $U : X \rightarrow X$ by*

$$Ux = \alpha_0 x + \alpha_1 T^1 x + \alpha_2 T^2 x + \dots + \alpha_k T^k x = \sum_{i=0}^k \alpha_i T^i x \quad (2.2.1)$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^k \alpha_i = 1$. Let $x_{n+1} = Ux_n$. Then for each $x_0 \in X$, the sequence $\{x_n\}$ converges to a fixed point of T .

Recently, Dehici [31] studied generalized Kirk process in order to approximate common fixed point of a finite number of nonexpansive mappings. To do that, he introduced the notion of punctual inequality defined below in (2.2.3). Let $T_i : K \rightarrow K$ ($i = 1, 2, \dots, k$)

be mappings and $S : K \rightarrow K$ be defined as

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_k T_k = \sum_{i=0}^k \alpha_i T_i$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$, $\sum_{i=0}^k \alpha_i = 1$ and T_0 is the identity mapping on K . For $x_0 \in K$, define $\{x_n\}$ by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T_1 x_n + \alpha_2 T_2 x_n + \cdots + \alpha_k T_k x_n = S x_n \quad (2.2.2)$$

Then, $\{x_n\}$ is called generalized Kirk iteration process associated with the mappings T_i .

Theorem 2.2.2 ([31]). *Let C be a convex subset of a Banach space X and T_i ($i = 1, 2, \dots, k$), $k \geq 2$, be self-mappings on C satisfying the punctual inequality (i.e. for all $x \in C$ and for all integers $i, j = 1, 2, \dots, k$, ($i < j$), there exists an integer $1 < n(x) < k$ such that*

$$\|T_i(x) - T_j(x)\| \leq \|x - T_{n(x)}\|. \quad (2.2.3)$$

Set S as above. Then,

$$F(S) = \bigcap_{i=1}^k F(T_i).$$

Theorem 2.2.3 ([31]). *Let X be a uniformly convex Banach space and let T_i ($i = 1, 2, \dots, k$), $k \geq 2$, be nonexpansive and compact self-mappings on X satisfying the assumption (2.2.3). Define the mapping S as above. If $\bigcap_{i=1}^k F(T_i) \neq \emptyset$, then for each $x_0 \in X$, the Picard sequence $\{S^n x_0\}$ converges to a common fixed point of the mappings T_i .*

2.2.2 Preliminaries and Notations

We first introduce Kirk type iteration process for a finite number of mappings in $CAT(0)$ spaces.

Definition 2.2.4 ([94]). Let K be a nonempty closed and convex subset of a $CAT(0)$ space X and $T_i : K \rightarrow X$ ($i = 1, 2, \dots, k$) be mappings on K , $k \geq 2$. Let $\{\alpha_i\}_{i=1}^k \subseteq (0, 1)$ and $\sum_{i=1}^k \alpha_i = 1$. Set $a_j = \sum_{i=j}^k \alpha_i$, ($j = 1, 2, \dots, k$). Define the mapping $S_0 : K \rightarrow X$

as $S_0 = T_k$. If $k \geq 3$, for $i = 1, 2, \dots, k-2$, define $S_i : K \rightarrow X$ by

$$S_i = \frac{\alpha_{k-i}}{a_{k-i}} T_{k-i} \oplus \frac{a_{k-i+1}}{a_{k-i}} S_{i-1}.$$

Then, the mapping $T : K \rightarrow X$ defined by

$$T = (1 - a_2)T_1 \oplus a_2 S_{k-2} \quad (2.2.4)$$

is a convex combination of mappings T_i , ($i = 1, 2, \dots, k$).

Remark 2.2.5. In the case of linear domain, the mapping T in (2.2.4) becomes $T = \sum_{i=1}^k \alpha_i T_i$.

Definition 2.2.6. Let K be a nonempty closed and convex subset of a $CAT(0)$ space X and $T_i : K \rightarrow K$ ($i = 1, 2, \dots, k$) be mappings on K , $k \geq 2$, with $T_1 = I$, the identity mapping on K . Let T be as defined above. For $x_0 \in K$, define $\{x_n\}$ by

$$x_{n+1} = T x_n. \quad (2.2.5)$$

Then $\{x_n\}$ defined by (2.2.5) is called Kirk-type iteration process associated with the mappings T_i defined on X .

The following known lemmas are needed to prove our main results.

Lemma 2.2.7 ([82]). *Let K be a subset of a $CAT(0)$ space X and $\{x_n\}$ be a bounded sequence in X with asymptotic center $A_K(\{x_n\}) = \{c\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences such that $\alpha_n \geq 0$ for all $n \in \mathbb{N}$, $\limsup_n \alpha_n \leq 1$ and $\limsup_n \beta_n \leq 0$. If for each $y \in K$, there exist p and $N \in \mathbb{N}$ such that*

$$d(y, x_{n+p}) \leq \alpha_n d(c, x_n) + \beta_n \text{ for all } n \geq N,$$

then $y = c$.

Lemma 2.2.8 ([94]). *Let K be a nonempty and convex subset of a $CAT(0)$ space X . Suppose that $T : K \rightarrow K$ is a nonexpansive mapping and $S = (1 - \lambda)I \oplus \lambda T$ where I is the identity mapping and $0 < \lambda < 1$. Then for $x, y \in K$, we have*

$$d(Sx, Sy) \leq (1 - \lambda)d(Sx, y) + \lambda(1 - \lambda)d(x, Sy) + (1 - \lambda)^2 d(y, Sy) + \lambda^2 d(x, y).$$

Lemma 2.2.9 ([11]). *Let K be a nonempty closed and convex subset of a complete $CAT(0)$ space X and $T : K \rightarrow X$ a nonexpansive mapping with $F(T) \neq \emptyset$. If T is asymptotically regular at $x \in K$, then the Picard iterates $\{T^n(x)\}$, Δ -converge to a fixed point of T .*

2.2.3 Approximation Results for Nonexpansive Mappings

In this subsection, we establish Δ -convergence and strong convergence of the Kirk-type process in $CAT(0)$ spaces.

Lemma 2.2.10 ([109]). *Let K be a nonempty convex subset of a $CAT(0)$ space X . Suppose that $T_1, T_2 : K \rightarrow X$ are nonexpansive mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Then for any $0 < t < 1$, the mapping $T : K \rightarrow X$ defined by $T = (1 - t)T_1 \oplus tT_2$ is a nonexpansive mapping and $F(T) = F(T_1) \cap F(T_2)$.*

We prove an important proposition to develop our main result.

Proposition 2.2.11. *Let K be a nonempty convex subset of a $CAT(0)$ space X and T_i ($i = 1, 2, \dots, k$) be nonexpansive self-mappings on K . Let T be as defined in (2.2.4). Then*

(i) *For each $i = 0, 1, \dots, k - 2$, S_i and T are nonexpansive mappings.*

(ii) *If $\bigcap_{i=1}^k F(T_i) \neq \emptyset$, then $F(T) = \bigcap_{i=1}^k F(T_i)$.*

Proof. For $k = 2$, by Definition 2.2.4, $S_{k-2} = S_0 = T_k = T_2$. Thus, S_0 is nonexpansive and $T = (1 - a_2)T_1 \oplus a_2S_0 = (1 - a_2)T_1 \oplus a_2T_2$. So, it follows by Lemma 2.2.10 that T is nonexpansive. Moreover, if $F(T_1) \cap F(T_2) \neq \emptyset$, we have $F(T) = F(T_1) \cap F(T_2)$. By successive application of Lemma 2.2.10, we have the desired result for $k \geq 3$. Indeed,

for $k \geq 3$, $i = 1, 2, \dots, k-2$, we compute:

$$\begin{aligned}
d(S_i x, S_i y) &= d\left(\frac{\alpha_{k-i}}{a_{k-i}} T_{k-i} x \oplus \frac{a_{k-i+1}}{a_{k-i}} S_{i-1} x, \frac{\alpha_{k-i}}{a_{k-i}} T_{k-i} y \oplus \frac{a_{k-i+1}}{a_{k-i}} S_{i-1} y\right) \\
&\leq \frac{\alpha_{k-i}}{a_{k-i}} d(T_{k-i} x, T_{k-i} y) + \frac{\alpha_{k-i+1}}{a_{k-i}} d(S_{i-1} x, S_{i-1} y) \\
&\leq \frac{\alpha_{k-i}}{a_{k-i}} d(x, y) \\
&+ \frac{\alpha_{k-i+1}}{a_{k-i}} d\left(\frac{\alpha_{k-i+1}}{a_{k-i+1}} T_{k-i+1} x \oplus \frac{a_{k-i+2}}{a_{k-i+1}} S_{i-2} x, \frac{\alpha_{k-i+1}}{a_{k-i+1}} T_{k-i+1} y \oplus \frac{a_{k-i+2}}{a_{k-i+1}} S_{i-2} y\right) \\
&\leq \frac{\alpha_{k-i}}{a_{k-i}} d(x, y) + \frac{a_{k-i+1}}{a_{k-i}} \left[\frac{\alpha_{k-i+1}}{a_{k-i+1}} d(x, y) + \frac{a_{k-i+2}}{a_{k-i+1}} d(S_{i-2} x, S_{i-2} y) \right] \\
&\vdots \\
&\leq \frac{\alpha_{k-i}}{a_{k-i}} d(x, y) + \frac{\alpha_{k-i+1}}{a_{k-i}} d(x, y) + \frac{\alpha_{k-i+2}}{a_{k-i}} d(x, y) + \dots + \frac{\alpha_{k-1}}{a_{k-i}} d(x, y) \\
&+ \frac{\alpha_k}{a_{k-i}} d(S_0 x, S_0 y) \\
&\leq \frac{\alpha_{k-i}}{a_{k-i}} d(x, y) + \frac{\alpha_{k-i+1}}{a_{k-i}} d(x, y) + \frac{\alpha_{k-i+2}}{a_{k-i}} d(x, y) + \dots + \frac{\alpha_{k-1}}{a_{k-i}} d(x, y) \\
&+ \frac{\alpha_k}{a_{k-i}} d(x, y) \\
&= d(x, y).
\end{aligned}$$

That is, $d(S_i x, S_i y) \leq d(x, y)$ for $i = 1, 2, \dots, k-2$. Hence, S_i are nonexpansive for $i = 0, 1, 2, \dots, k-2$. In particular, S_{k-2} is nonexpansive and by Lemma 2.2.10, T is nonexpansive. Also, by Lemma 2.2.10, $F(S_1) = F(T_k) \cap F(T_{k-1})$, $F(S_2) = F(T_{k-2}) \cap F(S_1) = F(T_{k-2}) \cap F(T_k) \cap F(T_{k-1})$, $\dots$, $F(S_{k-2}) = \bigcap_{i=2}^k F(T_i)$ and hence, it follows that $F(T) = \bigcap_{i=1}^k F(T_i)$. $\square$

Lemma 2.2.12 ([11]). *Let X be a $CAT(0)$ space and K be as above. Assume that T is nonexpansive and $x \in K$. Then:*

- (i) *For all $k \geq 1$, $R_k = \lim_{n \rightarrow \infty} d(T^{n+k} x, T^n x)$ exists and $R_k \leq k R_1$*
- (ii) *$L = \lim_{n \rightarrow \infty} \frac{d(T^n x, x)}{n}$ exists and is independent of x*
- (iii) *$L \leq r_K(T) \leq R_1$, where $r_K(T) = \inf\{d(x, Tx) : x \in K\}$ is the minimal displacement.*

Next, we give a detailed proof of a result stated in [94].

Theorem 2.2.13 ([94]). *Let K be a nonempty convex subset of a $CAT(0)$ space X . Suppose that $T : K \rightarrow K$ is nonexpansive. Define $S : K \rightarrow K$ by $S = (1 - \lambda)I \oplus \lambda T$, $\lambda \in (0, 1)$. Then for each $x \in X$ and $k \in \mathbb{N}$, we have*

$$\lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x) = \frac{1}{k} \lim_{n \rightarrow \infty} d(S^{n+k}x, S^n x) = \lim_{n \rightarrow \infty} \frac{d(S^n x, x)}{n} = r_K(S).$$

Proof. By Lemma 2.2.10, S is nonexpansive. Also, by Lemma 2.2.12, we have

$$\lim_{n \rightarrow \infty} d(S^{n+k}, S^n x) \leq k \lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x).$$

For reverse inequality, we use induction on k , that is,

$$\lim_{n \rightarrow \infty} d(S^{n+k}x, S^n x) \geq k \lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x). \quad (2.2.6)$$

For $k = 1$, it is trivially true. Assume (2.2.6) holds for $k = j$, that is,

$$\lim_{n \rightarrow \infty} d(S^{n+j}x, S^n x) \geq j \lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x). \quad (2.2.7)$$

Since $\lim_{n \rightarrow \infty} d(S^{n+k}x, S^n x)$ exists, therefore we get

$$\lim_{n \rightarrow \infty} d(S^{n+k+1}x, S^{n+1}x) = \lim_{n \rightarrow \infty} d(S^{n+k}x, S^n x).$$

By Lemma 2.2.8,

$$\begin{aligned} d(S^{n+j+1}x, S^{n+1}x) &= d(S(S^{n+j}x), S(S^{n+1}x)) \\ &\leq (1 - \lambda)d(S^{n+j}x, S^{n+1}x) + \lambda(1 - \lambda)d(S^n x, S^{n+j+1}x) \\ &\quad + (1 - \lambda)^2 d(S^{n+j}x, S^{n+j+1}x) + \lambda^2 d(S^n x, S^{n+j}x). \end{aligned}$$

Thus,

$$\begin{aligned} \lambda(1 - \lambda)d(S^{n+j+1}x, S^n x) &\geq d(S^{n+j+1}x, S^{n+1}x) \\ &\quad - (1 - \lambda)d(S^{n+j}x, S^{n+1}x) \\ &\quad - (1 - \lambda)^2 d(S^{n+j}x, S^{n+j+1}x) \\ &\quad - \lambda^2 d(S^{n+j}x, S^n x). \end{aligned} \quad (2.2.8)$$

Taking limit of (2.2.8) as $n \rightarrow \infty$ and using induction hypothesis (2.2.7) and Proposition 2.2.11(i), we get

$$\lim_{n \rightarrow \infty} d(S^{n+j+1}x, S^n x) \geq (j + 1) \lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x).$$

Hence (2.2.6) holds for $k = j + 1$. So by induction, (2.2.6) holds for all $k \geq 1$ and hence

$$\lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x) = \frac{1}{k} \lim_{n \rightarrow \infty} d(S^{n+k}x, S^n x).$$

As $d(S^{n+k}x, S^n x) \leq d(S^k x, x)$ for all $n, k \geq 1$, so we have

$$\lim_{n \rightarrow \infty} \frac{d(S^{n+k}x, S^n x)}{k} \leq \frac{d(S^k x, x)}{k} \text{ for all } k \geq 1$$

So, $\lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x) = \frac{1}{k} \lim_{n \rightarrow \infty} d(S^{n+k}x, S^n x)$ implies

$$\lim_{n \rightarrow \infty} d(S^{n+1}x, S^n x) \leq \lim_{n \rightarrow \infty} \frac{d(S^n x, x)}{n}$$

Now by Lemma 2.2.12(iii), we have the desired result. $\square$

Theorem 2.2.14. *Let K be a nonempty convex subset of a $CAT(0)$ space X . Let $T_i : K \rightarrow K$ ($i = 1, 2, \dots, k$) $k \geq 2$, be nonexpansive mappings with $T_1 = I$, the identity mapping on K . Suppose further that $\bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let T be as defined in (2.2.4). Then T is asymptotically regular on K .*

Proof. By Definition 2.2.4, $T = (1 - a_2)I \oplus a_2 S_{r-2}$ where

$$\begin{aligned} S_i &= \frac{\alpha_{k-i}}{a_{k-i}} T_{k-i} \oplus \frac{a_{k-i+1}}{a_{k-i}} S_{i-1} \text{ for } i = 1, 2, \dots, k-2, \quad k \geq 3, \\ S_0 &= T_k, \end{aligned}$$

$$(\alpha_i)_{i=1}^k \subseteq (0, 1), \sum_{i=1}^k \alpha_i = 1 \text{ and } a_j = \sum_{i=j}^k \alpha_i, \quad j = 1, 2, \dots, k.$$

By Proposition 2.2.11, S_{r-2} and T are nonexpansive mappings and

$$F(T) = \bigcap_{i=1}^k F(T_i) \neq \emptyset. \text{ By Theorem 2.2.13,}$$

$$\lim_{n \rightarrow \infty} d(T^{n+1}x, T^n x) = r_K(T).$$

Since $F(T) \neq \emptyset$, there exists $x \in K$ such that $d(x, Tx) = 0$. Hence $r_K(T) = 0$ and consequently,

$$\lim_{n \rightarrow \infty} d(T^{n+1}x, T^n x) = 0.$$

So T is asymptotically regular on K . $\square$

Theorem 2.2.15. *Let K be a nonempty convex subset of a complete $CAT(0)$ space X . Let $T_i : K \rightarrow K$ ($i = 1, 2, \dots, k$), $k \geq 2$, be nonexpansive mappings with $T_1 = I$, the identity mapping on K . Suppose further that $\bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let T be as defined*

in (2.2.4). Then $\{x_n\}$ in (2.2.5), Δ -converges to a common fixed point of T_i , ($i = 2, 3, \dots, k$).

Proof. For each $n \geq 1$, $x_n = T^n x_0$. It follows from Theorem 2.2.14 and Proposition 2.2.11 that T is asymptotically regular nonexpansive mapping and $F(T) \neq \emptyset$. By Lemma 2.2.9, we have $\{T^n x_0\}$ Δ -converges to a fixed point of T . By Proposition 2.2.11, $F(T) = \bigcap_{i=1}^k F(T_i)$. Hence, we have the desired result. $\square$

Definition 2.2.16 (Condition (A)). A finite family $\{T_i, i = 1, 2, \dots, k\}$ of self-mappings on a nonempty convex set K is said to satisfy condition (A) if there is a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Tx) \geq f(d(x, F)) \text{ for all } x \in K,$$

where T is as defined in (2.2.4) and $F = \bigcap_{i=1}^k F(T_i)$.

Now, we prove a strong convergence result.

Theorem 2.2.17. Let K be a nonempty convex subset of a complete $CAT(0)$ space X . Let $T_i : K \rightarrow K$ ($i = 1, 2, \dots, k$) $k \geq 2$, be nonexpansive mappings with $T_1 = I$, the identity mapping on K . Suppose further that $\bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let T be as defined in (2.2.4). If T_i satisfies condition (A), then the sequence $\{x_n\}$ in (2.2.5), converges to a common fixed point of the mappings T_i ($i = 2, \dots, k$).

Proof. For any $x_0 \in K$, $x_{n+1} = T^{n+1} x_0$. By Proposition 2.2.11, T is nonexpansive and $F(T) \neq \emptyset$. Thus $d(x_{n+1}, p) = d(Tx_n, p) \leq d(x_n, p)$ for any $p \in F(T)$. This implies that $\{d(x_n, F(T))\}$ is decreasing and bounded below. Hence its limit exists.

$T = (1 - a_2)I + a_2 S_{k-2}$, so we have $d(T^n x_0, T^{n+1} x_0) = a_2 d(x_n, S_{k-2} x_n)$. Thus

$$\begin{aligned} d(x_n, Tx_n) &= d(x_n, (1 - a_2)x_n \oplus a_2 S_{k-2} x_n) \\ &= a_2 d(T^n x_0, T^{n+1} x_0). \end{aligned}$$

Since T is asymptotically regular (i.e. $\lim_{n \rightarrow \infty} d(T^n x_0, T^{n+1} x_0) = 0$), $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. As T_i satisfy Condition (A), we have $d(x_n, Tx_n) \geq f(d(x_n, F(T)))$.

Taking limit on both sides, we have $\lim_{n \rightarrow \infty} f(d(x_n, F(T))) \leq 0$.

Now $f(0) = 0$ and f is nondecreasing imply that

$$\lim_{n \rightarrow \infty} d(x_n, F(T)) = 0. \quad (2.2.9)$$

By Proposition 2.2.11(ii), we have the desired result. $\square$

2.2.4 Application to Convex Feasibility Problem

In this subsection, we study image recovery problem in the setting of $CAT(0)$ spaces. The image recovery problem may be stated as follows: Let $C_1, C_2, \dots, C_k$ be nonempty closed and convex subsets of a Hilbert space H such that $C_0 = \bigcap_{i=1}^k C_i \neq \emptyset$; the original unknown image z is a known priori belonging to C_0 ; given only the metric projection P_i of H onto C_i , $i = 1, 2, \dots, k$, recover z by an iteration process.

In 1991, Crombez [29] proved the following: Let $T = \alpha_0 I + \sum_{i=1}^k \alpha_i T_i$ with $T_i = (1 - \lambda_i)I + \lambda_i P_i$ for all $i = 1, 2, \dots, k$, $0 < \lambda_i < 2$, $\alpha_i > 0$ and $\sum_{i=1}^k \alpha_i = 1$ where each P_i is the metric projection of H onto C_i and $C_0 \neq \emptyset$. Then for any $x \in H$, the sequence $\{T^n x\}$ converges to an element of C_0 . Takahashi and Tamura [116] extended results in [29] to uniformly convex Banach space setting and the case of P_i being nonexpansive retraction of C onto C_i . Takahashi and Tamura [116] also considered the inconsistent feasibility problem i.e. $C_0 = \bigcap_{i=1}^k C_i = \emptyset$.

Definition 2.2.18. Let C be a closed and convex subset of a complete $CAT(0)$ space X , $C_i \subseteq C$.

- (i) A mapping $P : C \rightarrow C_i$ is called a nonexpansive retraction of C onto C_i if P is nonexpansive and for all $x \in C_i$, $Px = x$.
- (ii) C_i is called a nonexpansive retract of C if there exists a nonexpansive retraction of C onto C_i .
- (iii) P is said to be sunny if $P((1 - t)x \oplus tPx) = Px$ whenever $(1 - t)x \oplus tPx \in C$ for $x \in C$ and $t \geq 0$.

Theorem 2.2.19. *Let X be a complete $CAT(0)$ space and C a nonempty closed and convex subset of X . Let C_i , ($i = 1, 2, \dots, k$) be nonexpansive retracts of C with $C_0 \neq \emptyset$. Define $T_i : C \rightarrow C$ by $T_1 = I$, the identity mapping on C and for $i = 2, 3, \dots, k$, $T_i = (1 - \lambda_i)I \oplus \lambda_i P_i$ where $\lambda_i \in (0, 1)$ and P_i is a nonexpansive retraction of C onto C_i . Let T be as defined in (2.2.4). Then for each $x_0 \in C$, $\{x_n\}$ defined in (2.2.5), Δ -converges to an element of C_0 .*

Proof. For $i = 2, 3, \dots, k$, P_i is a retraction of C onto C_i . Thus, $F(T_i) = C_i$, ($i = 2, 3, \dots, k$). By Lemma 2.2.10, T_i , ($i = 2, 3, \dots, k$) are nonexpansive and $F(T_i) = F(P_i) = C_i$. Hence $\bigcap_{i=1}^k F(T_i) = \bigcap_{i=1}^k C_i \neq \emptyset$. By Theorem 2.2.15, we have the desired result. $\square$

We establish some results on inconsistent feasibility problem in $CAT(0)$ spaces by following the line of action of Takahashi and Tamura [116].

Lemma 2.2.20. *Let X be a complete $CAT(0)$ space and C a nonempty closed and convex subset of X . Let C_i , $i = 1, 2, \dots, k$, $k \geq 2$, be nonempty bounded nonexpansive retracts of C . Let $T_i = (1 - \lambda_i)I \oplus \lambda_i P_i$ where $\lambda_i \in (0, 1)$ and P_i is nonexpansive retraction of C onto C_i , $i = 1, 2, \dots, k$. Let T be as defined in (2.2.4). Then $F(T)$ is nonempty. Moreover, if $C_0 = \emptyset$, then $F(T) \cap C_i = \emptyset$ for some i .*

Proof. Let $x \in C$. Consider a closed ball $B_R[x]$ of center x and radius R containing all the sets C_i , ($i = 1, 2, \dots, k$). Then we have $\{T^n x\} \subset B_R[x] \cap C$. This implies that $\{T^n x\}$ is bounded. By Proposition 2.2.11, T is nonexpansive. Let $u \in C$ be the unique asymptotic center of $\{T^n x\}$. Then for all $n \in \mathbb{N}$, $d(Tu, T^{n+1}x) \leq d(u, T^n x)$. Applying Lemma 2.2.7 with $y = Tu$, $x_n = T^n x$, $p = 1$, $N = 0$, $\alpha_n = 1$, $\beta_n = 0$, we have $Tu = u$. Thus T has a fixed point.

Suppose $C_0 = \emptyset$ and let $x, y \in F(T)$. First, we shall show that

$$a_2 d(S_{k-2}x, y) \leq \sum_{i=2}^k \alpha_i d(T_i x, y).$$

If $k = 2$, the above inequality holds trivially in view of $S_{k-2}x = T_2x$. For $k \geq 3, i = 1, \dots, k-2$, we have

$$\begin{aligned} a_{k-i}d(S_ix, y) &= a_{k-i}d\left(\frac{\alpha_{k-i}}{a_{k-i}}T_{k-i}x \oplus \frac{a_{k-i+1}}{a_{k-i}}S_{i-1}x, y\right) \\ &\leq \alpha_{k-i}d(T_{k-i}x, y) + a_{k-i+1}d(S_{i-1}x, y). \end{aligned}$$

Setting $i = k-2$, we have

$$\begin{aligned} a_2d(S_{k-2}x, y) &\leq \alpha_2d(T_2x, y) + a_3d(S_{k-3}x, y) \\ &\leq \sum_{i=2}^k \alpha_2d(T_ix, y). \end{aligned}$$

Recall that $x, y \in F(T)$. Hence, it follows that

$$\begin{aligned} d(x, y) = d(Tx, y) &= d((1 - a_2)T_1x \oplus a_2S_{k-2}x, y) \\ &\leq (1 - a_2)d(T_1x, y) + a_2d(S_{k-2}x, y) \\ &\leq \sum_{i=1}^k \alpha_i d(T_ix, y) = \sum_{i=1}^k \alpha_i d((1 - \lambda_i)x \oplus \lambda_i P_ix, y) \\ &\leq \sum_{i=1}^k \alpha_i ((1 - \lambda_i)d(x, y) + \lambda_i d(P_ix, y)). \end{aligned}$$

That is,

$$d(x, y) \leq \sum_{i=1}^k \alpha_i [(1 - \lambda_i)d(x, y) + \lambda_i d(P_ix, y)]. \quad (2.2.10)$$

Similarly, by $y \in F(T)$, it follows that

$$d(P_ix, y) = d(P_ix, Ty) \leq \sum_{i=1}^k \alpha_i [(1 - \lambda_i)d(P_ix, y) + \lambda_i d(P_ix, P_iy)].$$

Hence,

$$\sum_{i=1}^k \alpha_i \lambda_i d(P_ix, y) \leq \sum_{i=1}^k [(\alpha_i - 1)d(P_ix, y) + \alpha_i \lambda_i d(P_ix, P_iy)]. \quad (2.2.11)$$

Using (2.2.11) in (2.2.10), we have

$$\begin{aligned}
d(x, y) &\leq \sum_{i=1}^k [\alpha_i(1 - \lambda_i)d(x, y) + (\alpha_i - 1)d(P_i x, y) + \alpha_i \lambda_i d(P_i x, P_i y)] \\
&\leq \sum_{i=1}^k \alpha_i [(1 - \lambda_i)d(x, y) + \lambda_i d(P_i x, P_i y)] \quad \text{Since } 0 < \alpha_i < 1 \\
&\leq \sum_{i=1}^k \alpha_i [(1 - \lambda_i)d(x, y) + \lambda_i d(x, y)] \\
&= \sum_{i=1}^k \alpha_i d(x, y) = d(x, y).
\end{aligned}$$

Thus,

$$d(x, y) = d((1 - \lambda_i)x \oplus \lambda_i P_i x, y) = d(P_i x, P_i y).$$

Suppose $y \in F(T) \cap C_i$ for all i , Then $P_i y = y$ and we have

$$d(x, y) = d((1 - \lambda_i)x \oplus \lambda_i P_i x, y) = d(P_i x, y).$$

Since X is strictly convex, we get $x = P_i x$ for all i . Hence $F(T) \subseteq \bigcap_{i=1}^k C_i$. This contradicts the fact that the fixed point set of T is nonempty, as $\bigcap_{i=1}^k C_i = \emptyset$. Hence $F(T) \cap C_i = \emptyset$ for some i . $\square$

Let C and D be nonempty convex subsets of a complete $CAT(0)$ space. We set

$$(i) \quad i_C D = \{z \in D : \text{for any } x \in C, \exists \lambda \in (0, 1) \text{ with } \lambda x \oplus (1 - \lambda)z \in D\}$$

$$(ii) \quad \partial_C D = \{z \in D : \text{for any } \lambda \in (0, 1), \exists x \in C \text{ with } \lambda x \oplus (1 - \lambda)z \notin D\}.$$

Theorem 2.2.21. *Let X be a complete $CAT(0)$ space and C be a nonempty closed and convex subset of X . Let C_i , ($i = 2, \dots, k$) be nonempty bounded sunny nonexpansive retracts of C such that for each i , an element of $\partial_C C_i$ is an extreme point(i.e. for $x_i, y_i \in \partial_C C_i$, $t \in (0, 1)$, $tx_i \oplus (1 - t)y_i \notin \partial_C C_i$). Let $T_1 = I$, the identity mapping on C and for $i = 2, \dots, k$, $T_i = (1 - \lambda_i)I \oplus \lambda_i P_i$ where $\lambda_i \in (0, 1)$ and P_i nonexpansive retraction of C onto C_i . Let T as defined in (2.2.4). If $\bigcap_{i=2}^k C_i = \emptyset$, then, for any $x_0 \in C$, $\{x_n\}$ defined in (2.2.5), Δ -converges to a unique fixed point of T .*

Proof. By Definition of T in (2.2.4), we have $T = (1 - a_2)I \oplus a_2S_{k-2}$. Now, it follows from Lemma 2.2.20 and the fact that S_{k-2} is nonexpansive, that $F(S_{k-2})$ is a nonempty closed and convex subset of C . Moreover, $F(S_{k-2}) \cap C_j = \emptyset$ for some $j = 2, 3, \dots, k$. As shown in the proof of Lemma 2.2.20, for any $x, y \in F(S_{k-2})$, $d(x, y) = d(P_jx, P_jy)$ and by convexity of $F(S_{k-2})$, $\lambda x \oplus (1 - \lambda)y \in F(S_{k-2})$, for $\lambda \in (0, 1)$. Hence,

$$\begin{aligned}
d^2(P_j(\lambda x \oplus (1 - \lambda)y), \lambda P_jx \oplus (1 - \lambda)P_jy) &\leq \lambda d^2(P_j(\lambda x \oplus (1 - \lambda)y), P_jx) \\
&\quad + (1 - \lambda)d^2(P_j(\lambda x \oplus (1 - \lambda)y), P_jy) \\
&\quad - \lambda(1 - \lambda)d^2(P_j, xP_jy) \\
&= \lambda d^2(\lambda x \oplus (1 - \lambda)y, x) \\
&\quad + (1 - \lambda)d^2(\lambda x \oplus (1 - \lambda)y, y) \\
&\quad - \lambda(1 - \lambda)d^2(x, y) \\
&\leq \lambda(1 - \lambda)^2d^2(x, y) \\
&\quad + (1 - \lambda)\lambda^2d^2(x, y) \\
&\quad - \lambda(1 - \lambda)d^2(x, y) \\
&= 0.
\end{aligned}$$

So, $d^2(P_j(\lambda x \oplus (1 - \lambda)y), \lambda P_jx \oplus (1 - \lambda)P_jy) = 0$ and

$$P_j(\lambda x \oplus (1 - \lambda)y) = \lambda P_jx \oplus (1 - \lambda)P_jy. \quad (2.2.12)$$

That is, P_j is a one-to-one affine mapping of C onto C_j . Moreover, for any $x \in F(S_{k-2})$, $j = 2, 3, \dots, k$, $P_jx \in \partial_C C_j$. Suppose otherwise; then there exists $\lambda \in (0, 1)$ such that $\lambda x \oplus (1 - \lambda)P_jx \in C_j$. Since P_j is a retraction of C onto C_j , we have $P_j(\lambda x \oplus (1 - \lambda)P_jx) = \lambda x \oplus (1 - \lambda)P_jx$. Moreover, P_j is sunny, thus, $P_j(\lambda x \oplus (1 - \lambda)P_jx) = P_jx$. It follows that ,

$$\lambda x \oplus (1 - \lambda)P_jx = P_j(\lambda x \oplus (1 - \lambda)P_jx) = P_jx.$$

Hence, $x = P_jx$ for all $j = 2, 3, \dots, k$. This is a contradiction in view of

$F(S_{k-2}) \cap C_j = \emptyset$ for some $j = 2, 3, \dots, k$. Hence, $P_jx \in \partial_C C_j$ for any $x \in F(S_{k-2})$, $j = 2, 3, \dots, k$. Let $x, y \in F(S_{k-2})$ with $x \neq y$. Then, $P_jx \neq P_jy$. Since $x, y \in F(S_{k-2})$,

we have that $\lambda x \oplus (1 - \lambda)y \in F(S_{k-2})$ and $P_j(\lambda x \oplus (1 - \lambda)y) \in \partial_C C_j$. By (2.2.12), $\lambda P_j x \oplus (1 - \lambda)P_j y \in \partial_C C_j$. This contradicts elements of $\partial_C C_j$ being an extreme point of C_j . Hence S_{k-2} has a unique fixed point. Consequently, T has a unique fixed point. Using Theorem 2.2.14, it is easy to show that T is asymptotically regular. Hence by Lemma 2.2.9, we have the desired result. $\square$

Remark 2.2.22.

1. Proposition 2.2.11(ii) extends Theorem 2.2.2 in a $CAT(0)$ space without the punctual inequality.
2. Our Theorem 2.2.17 is an analogue of Theorem 2.2.3 with a weaker Condition (A) and free of the punctual inequality.
3. Our results generalize the corresponding results of Khan et al. [67], Kuhfittig [78] and Rhoades [106] from Banach spaces to $CAT(0)$ spaces.
4. Our Theorem 2.2.19, Lemma 2.2.20 and Theorem 2.2.21 generalize Theorem 3.3, Lemma 4.1 and Theorem 4.2 of Takahashi and Tamura [116], respectively, on a nonlinear domain.

2.3 An Algorithm of Three Nonexpansive Mappings

In this section, we introduce a three-step iterative algorithm for three nonexpansive mappings in a hyperbolic space and establish $\triangle$ -convergence and strong convergence results for it. We apply our results to solve a common minimization problem in $CAT(0)$ spaces. An example is presented to validate the new findings.

2.3.1 Introduction and Preliminaries

Glubudom [47] introduced a three-step iterative algorithm for three nonexpansive mappings in a Banach space as follows:

Let C be a nonempty convex subset of a real Banach space X and let $T_i : C \rightarrow C$ be given mappings, where $i \in I = \{1, 2, 3\}$. Then for a given $x_1 \in C$, define

$$\begin{aligned} z_n &= a_n T_1 x_n + (1 - a_n) x_n \\ y_n &= b_n T_2 z_n + c_n T_1 x_n + (1 - b_n - c_n) x_n \\ x_{n+1} &= \alpha_n T_3 y_n + \beta_n T_2 z_n + \gamma_n T_1 x_n + (1 - \alpha_n - \beta_n - \gamma_n) x_n \end{aligned} \quad (2.3.1)$$

where $0 \leq a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n \leq 1$.

It was shown that this algorithm generalizes Ishikawa iterations involving three nonexpansive mappings, and its weak and strong convergence to a common fixed point of the mappings was established. We extend the iterative algorithm in (2.3.1) to a uniformly convex hyperbolic space as under:

Let C be a nonempty convex subset of a uniformly convex hyperbolic space X and T_i ($i \in I$) be given self-mappings on C . For a given $x_1 \in C$, we define

$$\begin{aligned} z_n &= W(T_1 x_n, x_n, a_n) \\ y_n &= W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), b_n\right) \\ x_{n+1} &= W\left(T_3 y_n, W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right), \alpha_n\right) \quad n \geq 1 \end{aligned} \quad (2.3.2)$$

where $0 \leq a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, b_n + c_n, \alpha_n + \beta_n, \alpha_n + \beta_n + \gamma_n \leq 1$.

2.3.2 Convergence Analysis

In what follows, we assume $F = F(T_1) \cap F(T_2) \cap F(T_3)$. We begin with the following technical result.

Lemma 2.3.1. *Let (X, d, W) be a complete uniformly convex hyperbolic space and C be a nonempty closed and convex subset of X . Let $T_i : C \rightarrow C$ be nonexpansive mappings. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be as defined in (2.3.2). If $F \neq \emptyset$, then $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in F$.*

Proof. Clearly,

$$\begin{aligned}
d(z_n, p) &= d(W(T_1x_n, x_n, a_n), p) \\
&\leq a_nd(T_1x_n, p) + (1 - a_n)d(x_n, p) \\
&\leq d(x_n, p)
\end{aligned} \tag{2.3.3}$$

and

$$\begin{aligned}
d(y_n, p) &= d\left(W\left(T_2z_n, W\left(T_1x_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), p\right) \\
&\leq b_nd(T_2z_n, p) + (1 - b_n)d\left(W\left(T_1x_n, x_n, \frac{c_n}{1-b_n}\right), p\right) \\
&\leq b_nd(z_n, p) + (1 - b_n)\left[\frac{c_n}{1-b_n}d(T_1x_n, p) + \frac{1-b_n-c_n}{1-b_n}d(x_n, p)\right] \\
&\leq b_nd(x_n, p) + c_nd(x_n, p) + (1 - b_n - c_n)d(x_n, p) \\
&= d(x_n, p).
\end{aligned}$$

That is,

$$d(y_n, p) \leq d(x_n, p). \tag{2.3.4}$$

With the help of (2.3.3) and (2.3.4), we have

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(W\left(T_3y_n, W\left(T_2z_n, W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\
&\leq \alpha_nd(T_3y_n, p) \\
&+ (1 - \alpha_n)d\left(W\left(T_2z_n, W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\leq \alpha_nd(y_n, p) \\
&+ (1 - \alpha_n)\left[\frac{\beta_n}{1-\alpha_n}d(T_2z_n, p)\right] \\
&+ (1 - \alpha_n)\left[\frac{1-\alpha_n-\beta_n}{1-\alpha_n}d\left(W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), p\right)\right] \\
&\leq \alpha_nd(x_n, p) + \beta_nd(T_2z_n, p) \\
&+ (1 - \alpha_n - \beta_n)\left[\frac{\gamma_n}{1-\alpha_n-\beta_n}d(T_1x_n, p) + \frac{1-\alpha_n-\beta_n-\gamma_n}{1-\alpha_n-\beta_n}d(x_n, p)\right] \\
&\leq \alpha_nd(x_n, p) + \beta_nd(x_n, p) + \gamma_nd(x_n, p) + (1 - \alpha_n - \beta_n - \gamma_n)d(x_n, p) \\
&= d(x_n, p).
\end{aligned}$$

Thus, $d(x_n, p)$ is a decreasing and bounded sequence of real numbers. Hence, $\lim_{n \rightarrow \infty} d(x_n, p)$

exists. □

Lemma 2.3.2. *Let (X, d, W) be a complete uniformly convex hyperbolic space and C a nonempty closed and convex subset of X . Let $T_i : C \rightarrow C$ be nonexpansive mappings. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be as defined in (2.3.2). Assume that $F \neq \emptyset$.*

(i) *If $\alpha_n \in [a, b]$ and $\limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$ where $0 < a \leq b < 1$, then*

$$\lim_{n \rightarrow \infty} d \left(T_3 y_n, W \left(T_2 z_n, W \left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n} \right), \frac{\beta_n}{1 - \alpha_n} \right) \right) = 0.$$

(ii) *If $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$ and $b_n \in [a, b]$ where $0 < a \leq b < 1$, then*

$$\lim_{n \rightarrow \infty} d \left(T_2 z_n, W \left(T_1 x_n, x_n, \frac{c_n}{1 - b_n} \right) \right) = 0.$$

(iii) *If $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$ and, $a_n, b_n \in [a, b]$ where $0 < a \leq b < 1$, then $\lim_{n \rightarrow \infty} d(T_1 x_n, x_n) = 0$.*

Proof. (i) Assume $\alpha_n \in [a, b]$ and $\limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$ where $0 < a \leq b < 1$. By Lemma 2.3.1, $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in F$. Let $\lim_{n \rightarrow \infty} d(x_n, p) = c$ for some $c \geq 0$. For $c = 0$, we are done. Assume $c > 0$. Using (2.3.3) and (2.3.4), we have

$$d(T_3 y_n, p) \leq d(y_n, p) \leq d(x_n, p), \quad \text{that is} \quad \limsup_{n \rightarrow \infty} d(T_3 y_n, p) \leq c. \quad (2.3.5)$$

Similarly,

$$d(T_2 z_n, p) \leq d(x_n, p) \quad \text{gives that} \quad \limsup_{n \rightarrow \infty} d(T_2 z_n, p) \leq c. \quad (2.3.6)$$

Since $\limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$, we have

$$\begin{aligned} d \left(W \left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n} \right), p \right) &\leq \frac{\gamma_n}{1 - \alpha_n - \beta_n} d(T_1 x_n, p) \\ &\quad + \frac{1 - \alpha_n - \beta_n - \gamma_n}{1 - \alpha_n - \beta_n} d(x_n, p) \\ &\leq d(x_n, p) \end{aligned} \quad (2.3.7)$$

that is,

$$\limsup_{n \rightarrow \infty} d \left(W \left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n} \right), p \right) \leq c. \quad (2.3.8)$$

Using (2.3.7) and (2.3.8) and the fact that $\limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$, we have

$$\begin{aligned} d\left(W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right), p\right) &\leq \frac{\beta_n}{1 - \alpha_n} d(x_n, p) \\ &\quad + \left(1 - \frac{\beta_n}{1 - \alpha_n}\right) d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} d\left(W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right), p\right) \leq c. \quad (2.3.9)$$

Using Lemma 2.3.1, it follows from (2.3.5), (2.3.9) and $\lim_{n \rightarrow \infty} d(x_n, p) = c$ that

$$\lim_{n \rightarrow \infty} d\left(T_3 y_n, W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right)\right) = 0. \quad (2.3.10)$$

(ii) Suppose that $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$ and $b_n \in [a, b]$.

Note that

$$\begin{aligned} d\left(W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), p\right) &\leq \frac{c_n}{1 - b_n} d(T_1 x_n, p) + \left(1 - \frac{c_n}{1 - b_n}\right) d(x_n, p) \\ &\leq d(x_n, p). \end{aligned} \quad (2.3.11)$$

(2.3.11) implies that

$$\limsup_{n \rightarrow \infty} d\left(W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), p\right) \leq c. \quad (2.3.12)$$

Now,

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(T_3 y_n, W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right), \alpha_n\right), p\right) \\ &\leq \alpha_n d(T_3 y_n, p) \\ &\quad + (1 - \alpha_n) d\left(W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right), p\right) \\ &\leq d(T_3 y_n, p) \\ &\quad + (1 - \alpha_n) d\left(W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right), T_3 y_n\right) \\ &\quad + (1 - \alpha_n) d(T_3 y_n, p) \\ &\leq d(y_n, p) \\ &\quad + (1 - \alpha_n) d\left(W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}\right), \frac{\beta_n}{1 - \alpha_n}\right), T_3 y_n\right). \end{aligned}$$

Taking $\liminf$ over both sides of the above inequality and using (2.3.10), we have $c \leq$

$\liminf_{n \rightarrow \infty} d(y_n, p)$. Also from (2.3.6), we have $c \leq \liminf_{n \rightarrow \infty} d(y_n, p) \leq \limsup_{n \rightarrow \infty} d(y_n, p) \leq c$.

Hence

$$\lim_{n \rightarrow \infty} d(y_n, p) = c. \quad (2.3.13)$$

By Lemma 1.4.3, (2.3.7), (2.3.12) and (2.3.13), we have

$$\lim_{n \rightarrow \infty} d\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right)\right) = 0. \quad (2.3.14)$$

(iii) Suppose $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1$ and , $a_n, b_n \in [a, b]$.

$$\begin{aligned} d(y_n, p) &= d\left(W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), b_n\right), p\right) \\ &\leq b_n d(T_2 z_n, p) + (1 - b_n) d\left(W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), p\right) \\ &\leq b_n d(T_2 z_n, p) + (1 - b_n) d\left(W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), T_2 z_n\right) \\ &\quad + (1 - b_n) d(T_2 z_n, p) \\ &\leq d(z_n, p) + (1 - b_n) d\left(W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), T_2 z_n\right). \end{aligned}$$

Taking $\liminf$ in the above inequality and using (2.3.13) and (2.3.14) , we have $c \leq$

$\liminf_{n \rightarrow \infty} d(z_n, p) \leq \limsup_{n \rightarrow \infty} d(z_n, p) \leq c$. Hence

$$\lim_{n \rightarrow \infty} d(z_n, p) = c. \quad (2.3.15)$$

By Lemma 2.3.1, it follows that $\lim_{n \rightarrow \infty} d(T_1 x_n, x_n) = 0$. □

Lemma 2.3.3. *Let (X, d, W) be a complete uniformly convex hyperbolic space and C a nonempty closed and convex subset of X . Let $T_i : C \rightarrow C$ be nonexpansive mappings. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be as defined in (2.3.2). Assume that $F \neq \emptyset$. If the following conditions hold :*

$$(i) \quad 0 < 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1,$$

$$(ii) \quad b_n, a_n \in [a, b] \text{ where } 0 < a \leq b < 1,$$

then

$$\lim_{n \rightarrow \infty} d(T_1 x_n, x_n) = \lim_{n \rightarrow \infty} d(T_2 x_n, x_n) = \lim_{n \rightarrow \infty} d(T_3 x_n, x_n) = 0.$$

Proof. By Lemma 2.3.2(iii), we have $\lim_{n \rightarrow \infty} d(T_1 x_n, x_n) = 0$. Proof of the other conclusions of this lemma go as follows:

$$\begin{aligned} d(x_n, z_n) &= d(x_n, W(T_1 x_n, x_n, a_n)) \\ &\leq a_n d(T_1 x_n, x_n) + (1 - a_n) d(x_n, x_n) = a_n d(T_1 x_n, x_n) \end{aligned} \quad (2.3.16)$$

and,

$$\begin{aligned} d(T_2 z_n, x_n) &\leq d(T_2 z_n, T_2 x_n) + d(T_2 x_n, x_n) \\ &\leq a_n d(T_1 x_n, x_n) + d(T_2 x_n, x_n). \end{aligned} \quad (2.3.17)$$

Note that

$$\begin{aligned} d(W(T_1 x_n, x_n, \frac{c_n}{1 - b_n}), x_n) &\leq \frac{c_n}{1 - b_n} d(T_1 x_n, x_n) \\ d(W(T_1 x_n, x_n, \frac{\gamma_n}{1 - \alpha_n - \beta_n}), x_n) &\leq \frac{\gamma_n}{1 - \alpha_n - \beta_n} d(T_1 x_n, x_n). \end{aligned} \quad (2.3.18)$$

It follows from (2.3.16) and (2.3.18) that

$$\begin{aligned} d(T_2 x_n, x_n) &\leq d(T_2 x_n, T_2 z_n) + d(T_2 z_n, W(T_1 x_n, x_n, \frac{c_n}{1 - b_n})) \\ &\quad + d(W(T_1 x_n, x_n, \frac{c_n}{1 - b_n}), x_n) \\ &\leq a_n d(T_1 x_n, x_n) + d(T_2 z_n, W(T_1 x_n, x_n, \frac{c_n}{1 - b_n})) \\ &\quad + \frac{c_n}{1 - b_n} d(T_1 x_n, x_n). \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using Lemma 2.3.2((ii), (iii)), we have

$$\lim_{n \rightarrow \infty} d(T_2 x_n, x_n) = 0. \quad (2.3.19)$$

By (2.3.17) and (2.3.18), we have

$$\begin{aligned} d(x_n, y_n) &= d\left(x_n, W\left(T_2 z_n, W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), b_n\right)\right) \\ &\leq b_n d(T_2 z_n, x_n) \\ &\quad + (1 - b_n) d\left(W\left(T_1 x_n, x_n, \frac{c_n}{1 - b_n}\right), x_n\right) \\ &\leq a_n b_n d(T_1 x_n, x_n) + a_n d(T_2 x_n, x_n) + a_n d(T_2 x_n, x_n). \end{aligned} \quad (2.3.20)$$

Finally, with the help of (2.3.17), (2.3.18) and (2.3.20), we get

$$\begin{aligned}
d(T_3x_n, x_n) &\leq d(T_3x_n, T_3y_n) \\
&+ d\left(T_3y_n, W\left(T_2z_n, W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), \frac{\beta_n}{1-\alpha_n}\right)\right) \\
&+ d\left(W\left(T_2z_n, W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \\
&\leq d(x_n, y_n) + d\left(T_3y_n, W\left(T_2z_n, W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), \frac{\beta_n}{1-\alpha_n}\right)\right) \\
&+ \frac{\beta_n}{1-\alpha_n}d(T_2z_n, x_n) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right)d\left(W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), x_n\right) \\
&\leq a_nb_nd(T_1x_n, x_n) + a_nd(T_2x_n, x_n) \\
&+ d\left(T_3y_n, W\left(T_2z_n, W\left(T_1x_n, x_n, \frac{\gamma_n}{1-\alpha_n-\beta_n}\right), \frac{\beta_n}{1-\alpha_n}\right)\right) \\
&+ \frac{a_n\beta_n}{1-\alpha_n}d(T_1x_n, x_n) + \frac{\beta_n}{1-\alpha_n}d(T_2x_n, x_n) + \frac{\gamma_n}{1-\alpha_n-\beta_n}d(T_1x_n, x_n).
\end{aligned}$$

Appealing to Lemma 2.3.2(i), (iii)) and using (2.3.19), we have $\lim_{n \rightarrow \infty} d(T_3x_n, x_n) = 0$. $\square$

Let (X, d) be a metric space, $C \subset X$ and $T : C \rightarrow C$ be a mapping. Then, T is completely continuous if for a bounded sequence $\{x_n\}$ in C , the sequence $\{Tx_n\}$ has a convergent subsequence in C .

Theorem 2.3.4. *Let (X, d, W) be a complete uniformly convex hyperbolic space and C be a nonempty closed and convex subset of X . Let $T_i : C \rightarrow C$ be nonexpansive mappings. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be as defined in (2.3.2). Suppose that $F \neq \emptyset$ and the following conditions are satisfied:*

$$(i) \quad 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1.$$

$$(ii) \quad b_n, a_n \in [a, b] \text{ where } 0 < a \leq b < 1.$$

If one of the T_i is completely continuous, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a common fixed point of T_i .

Proof. By Lemma 2.3.3, we have

$$\lim_{n \rightarrow \infty} d(T_1x_n, x_n) = \lim_{n \rightarrow \infty} d(T_2x_n, x_n) = \lim_{n \rightarrow \infty} d(T_3x_n, x_n) = 0. \quad (2.3.21)$$

Without loss of generality, let T_1 be completely continuous. By Lemma 2.3.1, $\{x_n\} \subset C$ is bounded and thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $T_1 x_{n_k}$ converges. Since $\lim_{n \rightarrow \infty} d(T_1 x_n, x_n) = 0$, $\{x_{n_k}\}$ converges to say q . By Lemma 2.3.1, we have that $\{x_n\}$ converges strongly to fixed point q of T_1 . It follows from (2.3.21) and the fact that T_2 , and T_3 are continuous that q is a fixed point of T_2 and T_3 . Therefore, $\{x_n\}$ converges strongly to common fixed point of T_i . From (2.3.16) and (2.3.20), we have that

$$\lim_{n \rightarrow \infty} y_n = q = \lim_{n \rightarrow \infty} z_n.$$

□

Next, we establish a Δ -convergence result.

Theorem 2.3.5. *Let (X, d, W) be a uniformly convex hyperbolic space and C a nonempty closed and convex subset of X . Let $T_i : C \rightarrow C$ be nonexpansive mappings. For a given $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be as defined in (2.3.2). Suppose that $F \neq \emptyset$ and the following conditions are satisfied:*

$$(i) \quad 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1,$$

$$(ii) \quad b_n, a_n \in [a, b] \text{ where } 0 < a \leq b < 1.$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$, Δ -converge to a common fixed point of T_i .

Proof. Since $\{x_n\}$ is bounded by Lemma 2.3.1, it has a unique centre say x . By Lemma 1.5.5 and (2.3.21), x is a fixed point of T_i . Let $\{u_n\}$ be any subsequence of $\{x_n\}$ with asymptotic centre u . It follows from Lemma 1.5.5 and (2.3.21) that u is a fixed point of T_i . Thus, $\lim_{n \rightarrow \infty} d(x_n, u)$ exists by Lemma 2.3.1. Suppose $x \neq u$. Then by the uniqueness of asymptotic centres, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(u_n, u) &< \limsup_{n \rightarrow \infty} d(u_n, x) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, x) \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, u) \\ &= \limsup_{n \rightarrow \infty} d(u_n, u). \end{aligned}$$

This is a contradiction. Thus $x = u$, that is, x is the unique asymptotic centre of every subsequence $\{u_n\}$ of $\{x_n\}$. This shows that $\{x_n\}$, Δ -converges to a common fixed point of T_i . The conclusion for $\{y_n\}$ and $\{z_n\}$ follows from (2.3.16) and (2.3.20), respectively. $\square$

2.3.3 Application to Convex Minimization Problem

Definition 2.3.6. Let (X, d) be a $CAT(0)$ space, E a convex subset of X and $f : E \rightarrow (-\infty, \infty]$ a mapping.

(i) f is convex if for any $\gamma : [a, b] \rightarrow E$, we have

$$f(\gamma(t)) \leq tf(\gamma(a)) + (1-t)f(\gamma(b)) \text{ for all } t \in [a, b].$$

(ii) f is lower semi-continuous at a point $x \in E$ if $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$ whenever $x_n \rightarrow x$. Moreover, if f is lower semi-continuous at any point of E , we say f is lower semi-continuous on E .

Definition 2.3.7. Let X be an arbitrary set, Y a totally ordered set and $f : X \rightarrow Y$ a function. The minimizers of f is the set

$$\operatorname{argmin}_{x \in X} f(x) = \{x \in X \mid f(x) \leq f(y) \ \forall y \in X\}.$$

Definition 2.3.8 ([63]). Let (X, d) be a $CAT(0)$ space and $\lambda > 0$, the Moreau-Yosida resolvent of $f : X \rightarrow (-\infty, \infty]$ is defined by

$$J_\lambda(x) = \operatorname{argmin}_{y \in X} \left[f(y) + \frac{1}{2\lambda} d^2(y, x) \right] \text{ for all } x \in X.$$

Lemma 2.3.9 ([63]). Let (X, d) be a complete $CAT(0)$ space and $f : X \rightarrow (-\infty, \infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent J_λ of f is a nonexpansive mapping.

Lemma 2.3.10 ([11]). Let (X, d) be a $CAT(0)$ space. For any $\lambda > 0$, the set of fixed points of the resolvent associated with f , $F(J_\lambda)$, equals the set $\operatorname{argmin}_{x \in X} f(x)$.

We now approximate common minimizers of three proper convex and lower semi-continuous mappings as follows.

Theorem 2.3.11. *Let (X, d) be a complete $CAT(0)$ space. Let $f_i : X \rightarrow (-\infty, \infty]$ ($i \in I$) be proper convex and lower semi-continuous functions. Let T_i ($i \in I$) be resolvent associated with f_i , respectively. Suppose that $\Omega = \cap_{i=1}^3 \operatorname{argmin} f_i(x) \neq \emptyset$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be as defined in (2.3.2). If*

$$(i) \quad 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1,$$

$$(ii) \quad b_n, a_n \in [a, b] \text{ where } 0 < a \leq b < 1,$$

then $\{x_n\}, \{y_n\}$ and $\{z_n\}$, Δ -converge to a common minimizer of f_i .

Proof. By Lemmas 2.3.9 and 2.3.10, T_i are nonexpansive and $F(T_i) = \operatorname{argmin}_{y \in X} f(y)$. Thus $\Omega \neq \emptyset$. So, by Theorem 2.3.5, $\{x_n\}, \{y_n\}$ and $\{z_n\}$, Δ -converge to a common fixed point of T_i . Since minimizer of f_i coincides with $F(T_i)$, the result follows. $\square$

As Hilbert spaces are $CAT(0)$ spaces, so we have:

Corollary 2.3.12. *Let H be a Hilbert space and f_i, T_i ($i \in I$) be as above. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be as defined in (2.3.2). If*

$$(i) \quad 0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} (\alpha_n + \beta_n) < 1,$$

$$(ii) \quad b_n, a_n \in [a, b] \text{ where } 0 < a \leq b < 1,$$

then $\{x_n\}, \{y_n\}$ and $\{z_n\}$, Δ -converge to a common minimizer of f_i .

2.3.4 Numerical Example

We illustrate approximation of a common minimizer of three proper convex semi-continuous mappings by means of the algorithm (2.3.2).

For this, we solve the following minimization problem in the Euclidean space $X = \mathbb{R}^4$.

Example 4. Find

$$\min_{x \in C} \left\{ \|x\|_1 + \frac{1}{2} \|x\|_2 + (2, 2, -2, 1)^T x + 6, \|A_1 x\|_2^2, \|A_2 x - b_2\|_2^2 \right\}$$

where $C = \{x = (w_1, w_2, w_3, w_4) : -10 \leq w_1, w_2, w_3, w_4 \leq 10\}$,

$$A_1 = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 0 & 1 \\ -1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 1 & 0 & 2 \\ 1 & 0 & 1 & -2 \\ 1 & 3 & -1 & 0 \\ -5 & 2 & 0 & 4 \end{pmatrix} \quad b_2 = \begin{pmatrix} 1 \\ 0 \\ -5 \\ 3 \end{pmatrix}$$

and

$$\|x\|_1 = \sum_{i=1}^4 |w_i|, \quad \|x\|_2 = \sqrt{\sum_{i=1}^4 w_i^2}.$$

For all $x \in C$, we define $f_i : C \rightarrow (-\infty, \infty]$, $i \in I$, by $f_1(x) = \|x\|_1 + \frac{1}{2}\|x\|_2 + (2, 2, -2, 1)x + 6$, $f_2(x) = \|A_1 x\|_2^2$, $f_3(x) = \|A_2 x - b_2\|_2^2$.

It is easy to verify that f_1, f_2 and f_3 are proper convex and lower semi-continous functions. Using the ideas of proximity operator [27] and soft thresholding operator [56], the resolvents of f_i , respectively, can be computed as follows:

$$\begin{aligned} J_1(x) &= \operatorname{argmin}_{y \in X} \left[f_1(y) + \frac{1}{2}d^2(y, x) \right] \\ &= \operatorname{prox}_{\frac{\|\cdot\|_2}{2}} \left(\frac{x - (2, 2, -2, 1)^T}{2} \right) \\ &= \left(\max \left\{ \frac{|w_1 - 2| - 1}{2}, 0 \right\} \operatorname{sgn}(w_1 - 2), \max \left\{ \frac{|w_2 - 2| - 1}{2}, 0 \right\} \operatorname{sgn}(w_2 - 2), \right. \\ &\quad \left. \max \left\{ \frac{|w_3 + 2| - 1}{2}, 0 \right\} \operatorname{sgn}(w_3 + 2), \max \left\{ \frac{|w_4 - 1| - 1}{2}, 0 \right\} \operatorname{sgn}(w_4 - 1) \right), \\ J_2(x) &= \operatorname{argmin}_{y \in X} \left[f_2(y) + \frac{1}{2}d^2(y, x) \right] = (I + A_1^T A_1)^{-1}x, \\ J_3(x) &= \operatorname{argmin}_{y \in X} \left[f_3(y) + \frac{1}{2}d^2(y, x) \right] = (I + A_2^T A_2)^{-1}(x + A_2^T b_2). \end{aligned}$$

Choose $T_i x = J_i(x)$, $i = 1, 2, 3$ and $a_n = \frac{n}{10n+1} = b_n = c_n, \alpha_n = \beta_n = \gamma_n = \frac{1}{4}$ in Theorem 2.3.11. Now, the algorithm (2.3.2) becomes:

$$\begin{aligned} z_n &= \frac{n}{10n+1} T_1 x_n + \frac{9n+1}{10n+1} x_n \\ y_n &= \frac{n}{10n+1} T_2 z_n + \frac{n}{10n+1} T_1 x_n + \frac{8n+1}{10n+1} x_n \\ x_{n+1} &= \frac{1}{4} T_3 y_n + \frac{1}{4} T_1 z_n + \frac{1}{4} T_1 x_n + \frac{1}{4} x_n. \end{aligned} \tag{2.3.22}$$

For the initial point $x_1 = (0, 0, 0, 0)^T$ in this algorithm, we get the numerical results in

Table 2.1 and Fig. 2.1.

Table 2.1: Numerical results for the algorithm (2.3.22)

n	$x_n = (w_1^{(n)}, w_2^{(n)}, w_3^{(n)}, w_4^{(n)})^T$	$\ x_n - x_{n-1}\ _2$
1	(0, 0, 0, 0)	
2	(-4.13223140e-03, -4.13223140e-03, 4.13223140e-03, 3.14418630e-18)	0.6110108045
3	(-0.37401024, -0.38833395, 0.29749375, -0.01284134)	0.3927601385
4	(-0.60587885, -0.6273907, 0.50250756, -0.01104301)	0.2535988776
5	(-0.7517022, -0.77512559, 0.64609222, -0.00653723)	0.1647547863
$\vdots$	$\vdots$	$\vdots$
26	(-0.999996873, -1.00012599, 0.999475999, $4.46560317 \times 10^{-5}$)	0.0001397104
$\vdots$	$\vdots$	$\vdots$
46	(-1.00000003, -1.00000029, 0.999998971, $9.33856363 \times 10^{-8}$)	$2.76898627435 \times 10^{-7}$
47	(-1.00000002, -1.00000021, 0.999999235, $6.94593460 \times 10^{-8}$)	$2.0590057241 \times 10^{-7}$
48	(-1.00000002, -1.00000016, 0.999999431, $5.16611441 \times 10^{-8}$)	$1.53106623952 \times 10^{-7}$
49	(-1.00000001, -1.00000012, 0.999999577, $3.84222330 \times 10^{-8}$)	$1.13849223161 \times 10^{-7}$
50	(-1.00000001, -1.00000009, 0.999999685, $2.85751546 \times 10^{-8}$)	$8.46575758251 \times 10^{-8}$

We note from Table 2.1 that $\{x_n\}$ converges to $(-1, -1, 1, 0)^T$ which is a common minimizer of the three proper convex and semi-lower continuous mappings f_i . Hence, $(-1, -1, 1, 0)^T$ is a solution of our minimization problem. For the three proper convex and semi-lower continuous mappings f_i defined above, choose $u = (1, 2, -1, 1)^T$, $\alpha_n = \frac{2n-1}{30n}$ and the initial point $x_1 = (0, 0, 0, 0)$. Then, Halpern-type algorithm studied in [60] takes the form:

$$\begin{aligned}
z_n &= T_1 x_n \\
y_n &= T_2 z_n \\
x_{n+1} &= \frac{2n-1}{30n} u + \frac{29n+1}{30n} T_3 y_n.
\end{aligned} \tag{2.3.23}$$

The corresponding numerical results of algorithm (2.3.23) are shown in Table 2.2.

From Table 2.2 and Fig 2.1, we note that the algorithm (2.3.22) approaches quickly to the common minimizer $(-1, -1, 1, 0)^T$ than algorithm (2.3.23).

Remark 2.3.13.

1. Our results in this section generalize and provide analogue of the corresponding known results in uniformly convex Banach space([116], [47]) and uniformly convex hyperbolic space([38], [70]).

Table 2.2: Numerical results for the algorithm (2.3.23)

n	$x_n = (w_1^{(n)}, w_2^{(n)}, w_3^{(n)}, w_4^{(n)})^T$	$\ x_n - x_{n-1}\ _2$
1	(0, 0, 0, 0)	
2	(-0.90465845, -0.89855279, 0.75361047, 0.0032449)	1.4811290652
3	(-0.96438928, -0.95911053, 0.91342019, 0.00946986)	0.1811433365
4	(-0.97689728, -0.96971717, 0.95889315, 0.00856403)	0.0483483274
5	(-0.98265731, -0.97588115, 0.97479045, 0.00713104)	0.0180540995
$\vdots$	$\vdots$	$\vdots$
26	(-0.99723692, -0.99598772, 0.99667877, 0.00123305)	0.0002518265
$\vdots$	$\vdots$	$\vdots$
46	(-0.998499255, -0.997818935, 0.998203695, $6.71404388 \times 10^{-4}$)	$7.27680342981 \times 10^{-5}$
47	(-0.998531208, -0.997865330, 0.998242123, $6.57149706 \times 10^{-4}$)	$6.9666376265 \times 10^{-5}$
48	(-0.998561829, -0.997909792, 0.998278941, $6.4348768210 \times 10^{-4}$)	$6.67588975247 \times 10^{-5}$
49	(-0.998591200, -0.997952440, 0.9.98314248, $6.30382110e10 \times 10^{-4}$)	$6.40297197854x10 \times 10^{-5}$
50	(-0.9.98619394, -0.997993382, 0.9.98348135, $6.17799674 \times 10^{-4}$)	$6.14645553081 \times 10^{-5}$

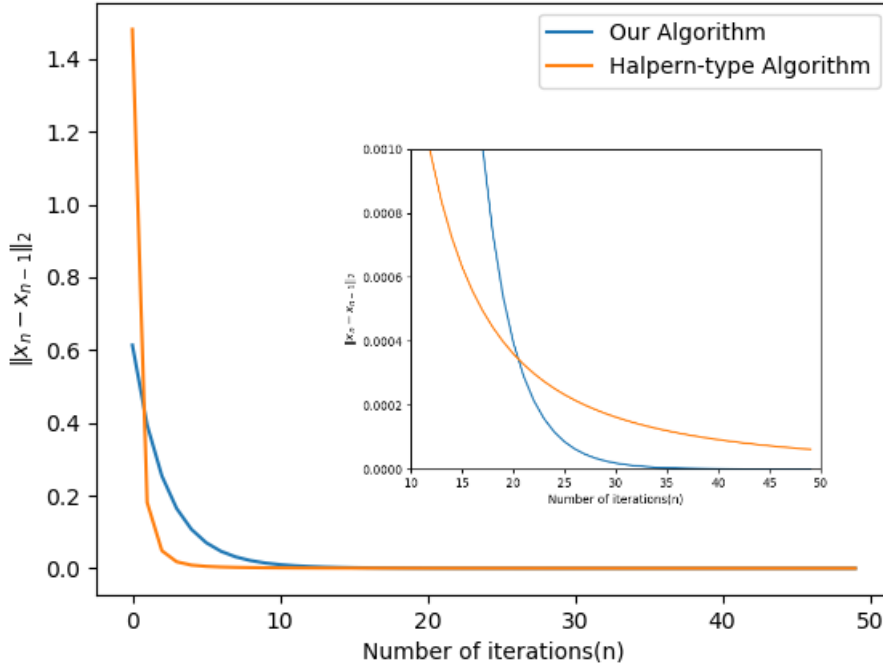


Figure 2.1: Comparison of the convergence of various algorithms

2. By means of a numerical example, we showed that our algorithm (2.3.22) converges quicker to a common minimizer than algorithm (2.3.23) known in literature.

Chapter 3

Halpern's Type Algorithm for α -nonexpansive Semigroup

3.1 Introduction

We start with the definition of α -nonexpansive semigroup.

Definition 3.1.1. A one-parameter α -nonexpansive semigroup is a family $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings on a closed and convex subset C of a metric space X satisfying :

- (i) $T(0)x = x$ for all $x \in C$.
- (ii) $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$.
- (iii) for each $t > 0$, $T(t)$ is α -nonexpansive i.e. for some $0 \leq \alpha < 1$ and $x, y \in C$,

$$d^2(T(t)x, T(t)y) \leq \alpha d^2(T(t)x, y) + \alpha d^2(T(t)y, x) + (1 - 2\alpha)d^2(x, y).$$

Finding common fixed point for classical nonexpansive semigroup has been extensively studied by several authors in different spaces under different conditions. Xu [121] studied the following implicit iterations for the nonexpansive semigroup in a Hilbert space:

$$x_n = \alpha_n u + (1 - \alpha_n)\sigma_{t_n}(x_n)$$

for each $n \geq 1$, where $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t} \int_0^t T(s)x ds \quad \text{for any } t > 0.$$

Later, Benavides et al. [13] and Aleyner and Censor [6] studied strong convergence of the Halpern's iterations

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n$$

under the assumption that the semigroup $\mathcal{T}$ satisfies uniform asymptotic regularity condition. Over the years, many researchers have extended the classical semigroup of nonexpansive mappings for a wider class of nonlinear mappings and spaces using various iterative schemes. For example, Cho and Kang [25] studied pseudo-contraction semigroups in Banach spaces; Cho et al. [24] studied nonexpansive semigroup in $CAT(0)$ space and Zegeye et al. [123] studied asymptotically nonexpansive semigroup in Banach spaces.

Inspired by work of Naraghirad [93] and Song et al. [113], we consider Halpern's type iterations for an α -nonexpansive semigroup $\mathcal{T}$ and establish strong convergence of this iteration in Banach spaces as well as $CAT(0)$ spaces. Throughout this chapter, we denote the set of fixed point of $T(t)$, $t \geq 0$ by $F(T(t))$ and set $F = \cap_{t \geq 0} F(T(t))$.

3.2 Preliminaries in Banach and $CAT(0)$ spaces

The following definitions and lemmas would be instrumental for the development of our results.

Definition 3.2.1. Let X be a real Banach space X with $C, D \subset X$ and $D \subset C$. A mapping $Q_D : C \rightarrow D$ is said to be :

- (a) sunny if $Q_D(Q_D x + t(x - Q_D x)) = Q_D x$ for each $x \in X$ and $t \geq 0$.
- (b) a retraction if $Q_D x = x$ for each $x \in C$.

If $X = H$ (real Hilbert space), then $Q_D = P_D$, the metric projection of C onto D .

Lemma 3.2.2 ([103]). *Let X be real Banach space with normalized duality mapping J . Let $C, D \subset X$ with $D \subset C$ and $Q_D : C \rightarrow D$ as retraction. Then Q_D is sunny nonexpansive if and only if*

$$\langle z - Q_D(z), J(w - Q_D(z)) \rangle \leq 0$$

for all $z \in C$ and $w \in D$.

Lemma 3.2.3 ([103]). *Let J be the normalized duality mapping of a real Banach space X . Then,*

$$\|z + w\|^2 \leq \|z\|^2 + 2\langle w, J(z + w) \rangle$$

for all $z, w \in X$.

Lemma 3.2.4 ([92]). *Let C be a nonempty subset of a uniformly convex Banach space X with the Opial conditon. Let $T : C \rightarrow C$ be an α -nonexpansive mapping for some $0 \leq \alpha < 1$.*

(i) If $F(T) \neq \emptyset$. then T is quasi-nonexpansive and $F(T)$ is closed and convex.

(ii) If $\{x_n\}$ converges weakly to y and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$, then $Ty = y$.

Definition 3.2.5. A one parameter semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ is said to be uniformly asymptotically regular (u.a.r.) if for any $s > 0$ and any bounded subset K of C , we have

$$\lim_{t \rightarrow \infty} \sup_{x \in K} d(T(s)T(t)x, T(t)x) = 0.$$

Here are examples of uniformly asymptotically regular semigroups.

Example 5 ([13]). Let H be a Hilbert space and C a nonempty bounded, closed and convex subset of H . If $T : C \rightarrow C$ is a contraction, then $\mathcal{T} = \{T^n : n \in \mathbb{N}\}$ is a u.a.r. contraction semigroup.

Example 6 ([5]). Let $X = l^2(\mathbb{N})$ be Hilbert space consisting of all functions x from $\mathbb{N}$ into $\mathbb{R}$ satisfying $\sum_{k \in \mathbb{N}} |x(k)|^2 < \infty$ with inner product $\langle x, y \rangle = \sum_{k \in \mathbb{N}} x(k)y(k)$. Define a bounded, closed and convex subset C of X by

$$C = \{x \in X : 0 \leq x(k) \leq p_k\}$$

where $p_k = 2^{\frac{-k}{2}}$. Then $\{T(t) : t \geq 0\}$ defined by

$$(T(t)x)(k) = \max \{x(k) - tp_k^2, 0\}$$

is a u.a.r. nonexpansive semigroup.

The concept of uniform asymptotic regularity extends to a sequence of mappings as follows:

Definition 3.2.6. A family $\{T_n\}$ of self-mappings on a nonempty set C is said to be uniformly asymptotically regular (u.a.r.) if, for each positive integer m and any bounded subset $K \subset C$, we have

$$\lim_{n \rightarrow \infty} \sup_{x \in K} d(T_m(T_n x), T_n x) = 0$$

Lemma 3.2.7 ([122]). *If $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n \quad \forall n \geq 1$$

where $0 \leq b_n \leq 1$ and $c_n \in \mathbb{R}$ satisfy the following conditions:

$$1. \sum_{n=1}^{\infty} b_n = \infty \quad 2. \limsup_{n \rightarrow \infty} c_n \leq 0 \text{ or } \sum_{n=1}^{\infty} |b_n c_n| < \infty,$$

then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 3.2.8 ([86]). *Let $\{a_n\}$ be a real sequence that has a subsequence $\{a_{n_k}\}$ which satisfies $a_{n_k} < a_{n_k+1}$ for all k . Then there exists an increasing sequence of integers $\{m_k\} \subset \mathbb{N}$ such that*

$$1. \lim_{k \rightarrow \infty} m_k = \infty \quad 2. a_{m_k} \leq a_{m_k+1} \quad 3. a_k \leq a_{m_k+1}$$

for all (sufficiently large) numbers $k \in \mathbb{N}$.

Lemma 3.2.9 ([64]). *Let X be a complete CAT(0) space, $\{x_n\}$ a sequence in X and $x \in X$. Then $\{x_n\}$ $\triangle$ -converges to x if and only if $\limsup_{n \rightarrow \infty} \langle x x_n, xy \rangle \leq 0$ for all $y \in X$.*

Lemma 3.2.10 ([102]). *Let X be a complete CAT(0) space, $\{x_n\}$ a sequence in X and $x \in X$. Suppose there exists a nonempty subset K of X satisfying :*

(i) For every $z \in K$, $\lim_{n \rightarrow \infty} d(x_n, z)$ exists

(ii) If a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is Δ -convergent to $x \in X$, then $x \in K$.

Then $\{x_n\}$ Δ -converges to $x \in K$.

Lemma 3.2.11 ([18], [32]). For any x, y, z, w in a $CAT(0)$ space X and $t \in [0, 1]$ with $u_t = tz \oplus (1 - t)w$, we have the following:

$$(i) \quad \langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t \langle \overrightarrow{z x}, \overrightarrow{u_t y} \rangle + (1 - t) \langle \overrightarrow{w x}, \overrightarrow{u_t y} \rangle.$$

$$(ii) \quad \langle \overrightarrow{u_t x}, \overrightarrow{z y} \rangle \leq t \langle \overrightarrow{z x}, \overrightarrow{z y} \rangle + (1 - t) \langle \overrightarrow{w x}, \overrightarrow{z y} \rangle.$$

$$(iii) \quad \langle \overrightarrow{u_t x}, \overrightarrow{w y} \rangle \leq t \langle \overrightarrow{z x}, \overrightarrow{w y} \rangle + (1 - t) \langle \overrightarrow{w x}, \overrightarrow{w y} \rangle.$$

Lemma 3.2.12 ([30]). Let X be a $CAT(0)$ space and $x, y, z \in X$. Then for each $\lambda \in [0, 1]$, we have

$$d^2(\lambda x \oplus (1 - \lambda)y, z) \leq \lambda^2 d^2(x, z) + (1 - \lambda)^2 d^2(y, z) + 2\lambda(1 - \lambda) \langle \overrightarrow{x z}, \overrightarrow{y z} \rangle.$$

Lemma 3.2.13 ([92]). Let E be a nonempty subset of a $CAT(0)$ space X . Let $K : E \rightarrow X$ be an α -nonexpansive mapping for some $0 \leq \alpha < 1$.

(i) If $F(K) \neq \emptyset$, then K is quasi-nonexpansive. Moreover, $F(K)$ is closed and convex.

(ii) If $\{x_n\}$ is a sequence in E such that $\lim_{n \rightarrow \infty} d(x_n, Kx_n) = 0$ and $\Delta - \lim_{n \rightarrow \infty} x_n = z$ for some $z \in X$, then $z \in E$ and $Kz = z$.

3.3 Convergence Results in Banach space

Throughout this section, X is considered to be a real uniformly convex Banach space which admits weakly sequentially continuous duality mapping J .

Theorem 3.3.1. Let C a nonempty closed and convex subset of X . Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be the u.a.r. semigroup of α -nonexpansive mappings from C into itself with $F \neq \emptyset$.

Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} u \in C, & x_1 \in C \quad (\text{chosen arbitrarily}) \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)[\lambda_n x_n + (1 - \lambda_n)T(t_n)x_n]. \end{cases} \quad (3.3.1)$$

Let $\{\alpha_n\}, \{\lambda_n\}$ be sequences in $[0, 1]$ and $t_n > 0$ satisfy the following assumptions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$
- (iii) $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1$
- (iv) $\lim_{n \rightarrow \infty} t_n = \infty$.

Then, the sequence $\{x_n\}$ defined in (3.3.1) converges to $Q_F u$, where Q_F is a sunny nonexpansive retraction from X onto F .

Proof. By Lemma 3.2.4(i), it follows that $T(t_n)$ is a quasi nonexpansive mapping and the sunny nonexpansive retraction from X onto F is well defined. Define $z = Q_F u$.

Step 1: $\{x_n\}$ and $\{T(t_n)x_n\}$ are bounded.

Let $y_n = \lambda_n x_n + (1 - \lambda_n)T(t_n)x_n$. Then $x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n$. Let $p \in F$. By Lemma 1.3.1, there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} \|y_n - p\|^2 &= \|\lambda_n(x_n - p) + (1 - \lambda_n)(T(t_n)x_n - p)\|^2 \\ &\leq \lambda_n \|x_n - p\|^2 + (1 - \lambda_n) \|T(t_n)x_n - p\|^2 \\ &\quad - \lambda_n(1 - \lambda_n)g(\|x_n - T(t_n)x_n\|) \\ &\leq \lambda_n \|x_n - p\|^2 + (1 - \lambda_n) \|x_n - p\|^2 \\ &\quad - \lambda_n(1 - \lambda_n)g(\|x_n - T(t_n)x_n\|) \\ &= \|x_n - p\|^2 - \lambda_n(1 - \lambda_n)g(\|x_n - T(t_n)x_n\|). \end{aligned}$$

Hence,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \lambda_n(1 - \lambda_n)g(\|x_n - T(t_n)x_n\|) \quad (3.3.2)$$

and

$$\|y_n - p\| \leq \|x_n - p\|. \quad (3.3.3)$$

Now, (3.3.3) implies that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n u + (1 - \alpha_n)y_n - p\| \\ &= \|\alpha_n(u - p) + (1 - \alpha_n)(y_n - p)\| \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|y_n - p\| \\ &\leq \alpha_n\|u - p\| + (1 - \alpha_n)\|x_n - p\|. \end{aligned}$$

By induction, we have

$$\|x_{n+1} - p\| \leq \max\{\|u - p\|, \|x_1 - p\|\}$$

for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. Since $\|T(t_n)x_n - p\| \leq \|x_n - p\|$, we also have that $\{T(t_n)x_n\}$ is bounded.

Step 2: For any $n \in \mathbb{N}$, we prove that

$$\|x_{n+1} - z\|^2 \leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z) \rangle. \quad (3.3.4)$$

By (3.3.1), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n u + (1 - \alpha_n)y_n - z\|^2 \\ &\leq \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|y_n - z\|^2. \end{aligned}$$

Using (3.3.2), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \alpha_n\|u - z\|^2 \\ &\quad + (1 - \alpha_n)[\|x_n - p\|^2 - \lambda_n(1 - \lambda_n)g(\|x_n - T(t_n)x_n\|)]. \end{aligned} \quad (3.3.5)$$

Setting $M = \sup\{\|u - z\|^2 - \|x_n - z\|^2 + \lambda_n(1 - \lambda_n)g(\|x_n - T(t_n)x_n\|)\}$, we have that

$$\lambda_n(1 - \lambda_n)g(\|x_n - T(t_n)x_n\|) \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M. \quad (3.3.6)$$

By Lemma 3.2.3 and (3.3.1), we can deduce that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n u + (1 - \alpha_n)y_n - z\|^2 \\
&\leq \|\alpha_n u + (1 - \alpha_n)y_n - z - \alpha_n(u - z)\|^2 \\
&\quad + 2\langle \alpha_n(u - z), J(x_{n+1} - z) \rangle \\
&= (1 - \alpha_n)\|y_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z) \rangle \\
&\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z) \rangle.
\end{aligned}$$

Step 3: If $\lim_{n \rightarrow \infty} \|x_n - T(t_n)x_n\| = 0$, then $\lim_{n \rightarrow \infty} \|x_n - T(s)x_n\| = 0$ for any $s > 0$.

By (3.3.1), we have that

$$y_n - x_n = (1 - \lambda_n)(x_n - T(t_n)x_n) \quad \text{and} \quad x_{n+1} - y_n = \alpha_n(u - y_n).$$

With the aid of the assumption $\lim_{n \rightarrow \infty} \|x_n - T(t_n)x_n\| = 0$ and condition (i), we can deduce that as $n \rightarrow \infty$,

$$\|y_n - x_n\| \rightarrow 0 \quad \text{and} \quad \|x_{n+1} - y_n\| \rightarrow 0. \quad (3.3.7)$$

Since $\|x_{n+1} - x_n\| \leq \|x_{n+1} - y_n\| + \|y_n - x_n\|$, we have that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.3.8)$$

Using $\mathcal{T}$ is u.a.r. and condition (iv), it follows that for any $s > 0$

$$\lim_{n \rightarrow \infty} \|T(s)T(t_n)x_n - T(t_n)x_n\| \leq \lim_{n \rightarrow \infty} \sup_{x \in K} \|T(s)T(t_n)x - T(t_n)x\| = 0 \quad (3.3.9)$$

where K is any bounded subset of C containing $\{x_n\}$ and $\{T(t_n)x_n\}$.

For any $s > 0$, let $x = T(t_n)x_n$, $y = x_n$ and $T = T(s)$ in Lemma 1.5.4 . Then, we have

$$\begin{aligned}
\|T(t_n)x_n - T(s)x_n\|^2 &\leq \frac{1 + \alpha}{1 - \alpha} \|T(t_n)x_n - T(s)T(t_n)x_n\|^2 + \|T(t_n)x_n - x_n\|^2 \\
&\quad + \frac{2\alpha}{1 - \alpha} \|T(t_n)x_n - x_n\| \|T(t_n)x_n - T(s)T(t_n)x_n\| \\
&\quad + \frac{2}{1 - \alpha} \|T(s)x_n - T(s)T(t_n)x_n\| \|T(t_n)x_n - T(s)T(t_n)x_n\| \\
&\leq \frac{1 + \alpha}{1 - \alpha} \|T(t_n)x_n - T(s)T(t_n)x_n\|^2 + \|T(t_n)x_n - x_n\|^2 \\
&\quad + \frac{2}{1 - \alpha} V \|T(t_n)x_n - T(s)T(t_n)x_n\|
\end{aligned}$$

where $V = \sup\{\alpha\|T(t_n)x_n - x_n\| + \|T(s)T(t_n)x_n - T(s)T(t_n)x_n\|\}$.

Using the assumption $\lim_{n \rightarrow \infty} \|x_n - T(t_n)x_n\| = 0$ of Step 3 above and (3.3.9), we have

$$\lim_{n \rightarrow \infty} \|T(t_n)x_n - T(s)x_n\| = 0. \quad (3.3.10)$$

Also,

$$\begin{aligned} \|x_{n+1} - T(s)x_n\|^2 &= \|\alpha_n u + (1 - \alpha_n)y_n - T(s)x_n\|^2 \\ &= \|\alpha_n u + (1 - \alpha_n)[\lambda_n x_n + (1 - \lambda_n)T(t_n)x_n] - T(s)x_n\|^2 \\ &= \|\alpha_n(u - T(t_n)x_n) + (1 - \alpha_n)\lambda_n(x_n - T(t_n)x_n) \\ &\quad + T(t_n)x_n - T(s)x_n\|^2 \\ &\leq 2\alpha_n\|u - T(t_n)x_n\| + 4(1 - \alpha_n)\lambda_n\|x_n - T(t_n)x_n\|^2 \\ &\quad + 4\|T(t_n)x_n - T(s)x_n\|^2. \end{aligned}$$

By $\lim_{n \rightarrow \infty} \|x_n - T(t_n)x_n\| = 0$, condition (i) and (3.3.10), we get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - T(s)x_n\| = 0. \quad (3.3.11)$$

Hence for any $s > 0$,

$$\begin{aligned} \|x_{n+1} - T(s)x_n\| &= \|x_n - x_{n+1} + x_{n+1} - T(s)x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T(s)x_n\|. \end{aligned}$$

From (3.3.8) and (3.3.11), we conclude that

$$\lim_{n \rightarrow \infty} \|x_n - T(s)x_n\| = 0. \quad (3.3.12)$$

Step 4: We prove that $x_n \rightarrow z$ as $n \rightarrow \infty$.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - z\|\}_{n=n_0}^\infty$ is nonincreasing.

Then $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists. Using conditions (i),(iii) and (3.3.6), we deduce that

$$\lim_{n \rightarrow \infty} g(\|x_n - T(t_n)x_n\|) = 0.$$

By the properties of g , we have that

$$\lim_{n \rightarrow \infty} \|x_n - T(t_n)x_n\| = 0. \quad (3.3.13)$$

Thus by Step 3, (3.3.12) holds. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k+1}\}$ converges weakly to y . It follows from (3.3.12) and Lemma

3.2.4 that $y \in F$. We have that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z, J(x_{n+1} - z) \rangle &= \lim_{k \rightarrow \infty} \langle u - z, J(x_{n_k+1} - z) \rangle \\ &= \langle u - z, J(y - z) \rangle. \end{aligned}$$

Using Lemma 3.2.2, we conclude that

$$\limsup_{n \rightarrow \infty} \langle u - z, J(x_{n+1} - z) \rangle \leq 0. \quad (3.3.14)$$

Let $a_n = \|x_{n+1} - z\|^2$, $b_n = \alpha_n$ and $c_n = \langle u - z, J(x_{n+1} - z) \rangle$ in (3.3.4). Then, we deduce by conditions (i), (ii) and Lemma 3.2.7 that $x_n \rightarrow z$.

Case 2: Suppose $\{\|x_n - z\|\}_{n=n_o}^\infty$ is not nonincreasing, i.e. that there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\|x_{n_k} - z\| < \|x_{n_k+1} - z\| \quad \forall k \in \mathbb{N}.$$

Then by Lemma 3.2.8, there exists an increasing sequence of integers $\{m_k\} \subset \mathbb{N}$ such that

$$1. \lim_{k \rightarrow \infty} m_k = \infty \quad 2. \|x_{m_k} - z\| \leq \|x_{m_k+1} - z\| \quad 3. \|x_k - z\| \leq \|x_{m_k+1} - z\|$$

for all (sufficiently large) numbers $k \in \mathbb{N}$. This, together with (3.3.6) implies that

$$\begin{aligned} \lambda_{m_k}(1 - \lambda_{m_k})g(\|x_{m_k} - T(t_{m_k})x_{m_k}\|) &\leq \|x_{m_k} - z\|^2 - \|x_{m_k+1} - z\|^2 \\ &+ \alpha_{m_k}M \\ &\leq \alpha_{m_k}M. \end{aligned}$$

Conditions (i),(iii) and properties of g imply

$$\lim_{k \rightarrow \infty} \|x_{m_k} - T(t_{m_k})x_{m_k}\| = 0.$$

As shown before, we have

$$\limsup_{k \rightarrow \infty} \langle u - z, J(x_{m_k+1} - z) \rangle \leq 0. \quad (3.3.15)$$

From (3.3.4), we obtain

$$\|x_{m_k+1} - z\|^2 \leq (1 - \alpha_{m_k})\|x_{m_k} - z\|^2 + 2\alpha_{m_k}\langle x - z, J(x_{m_k+1} - z) \rangle. \quad (3.3.16)$$

This together with $\|x_{m_k} - z\| \leq \|x_{m_k+1} - z\|$ implies that

$$\alpha_{m_k} \|x_{m_k} - z\|^2 \leq 2\alpha_{m_k} \langle x - z, J(x_{m_k+1} - z) \rangle$$

and thus,

$$\|x_{m_k} - z\|^2 \leq 2 \langle x - z, J(x_{m_k+1} - z) \rangle.$$

It follows from (3.3.15) that

$$\lim_{k \rightarrow \infty} \|x_{m_k} - z\| = 0.$$

Consequently, (3.3.16) ensures that

$$\lim_{k \rightarrow \infty} \|x_{m_k+1} - z\| = 0.$$

Recalling that for sufficiently large $k \in \mathbb{N}$,

$$\|x_k - z\| \leq \|x_{m_k+1} - z\|,$$

and so we conclude that $\lim_{k \rightarrow \infty} \|x_k - z\| = 0$. Hence, $x_n \rightarrow z$ as $n \rightarrow \infty$. $\square$

Corollary 3.3.2. *Let C a nonempty closed and convex subset of X . Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be the u.a.r. semigroup of α -nonexpansive mappings from C into itself with $F \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by*

$$\begin{cases} u \in C, \quad x_1 \in C \quad (\text{chosen arbitrarily}) \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) S_\lambda(t_n) x_n. \end{cases} \quad (3.3.17)$$

where $S_\lambda x = \lambda x + (1 - \lambda)T(t)x$ for any $\lambda \in (0, 1)$ and $t > 0$. Let $\{\alpha_n\}$ in $[0, 1]$ and $t_n > 0$ satisfy the following assumptions:

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$
- (iii) $\lim_{n \rightarrow \infty} t_n = \infty$.

Then, the sequence $\{x_n\}$ defined in (3.3.17) converges to $Q_F u$, where Q_F is a sunny nonexpansive retraction from X onto F .

Proof. Put $\lambda_n = \lambda$ for all n in Theorem 3.3.1. $\square$

Corollary 3.3.3. *Let C a nonempty closed and convex subset of X . Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be the u.a.r. semigroup of any of the following self-mappings on C :*

1. *Mean nonexpansive mappings.*
2. *Nonspreading mappings.*
3. *Hybird mappings.*

Suppose that $F \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} u \in C, & x_1 \in C \text{ (chosen arbitrarily)} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)[\lambda_n x_n + (1 - \lambda_n)T(t_n)x_n]. \end{cases} \quad (3.3.18)$$

Let $\{\alpha_n\}, \{\lambda_n\}$ in $[0, 1]$ and $t_n > 0$ satisfy :

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$
- (iii) $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1$
- (iv) $\lim_{n \rightarrow \infty} t_n = \infty$.

Then, the sequence $\{x_n\}$ defined in (3.3.18) converges to $Q_F u$, where Q_F is a sunny nonexpansive retraction from X onto F .

Proof. (1). Put $\alpha = \frac{b}{2}$ in Theorem 3.3.1 .

(2). Put $\alpha = \frac{1}{2}$ in Theorem 3.3.1 .

(3). Put $\alpha = \frac{1}{3}$ in Theorem 3.3.1 . □

Replacing the terms $T(t_n)$ and $T(s)$ of Theorem 3.3.1 with the terms T_n and T_m , respectively, we have the following result which improves and extends ([113], Theorem 3.4) and ([96], Theorem 4.1).

Theorem 3.3.4. *Let C a nonempty closed and convex subset of X . Let $\{T_n\}$ be the sequence of u.a.r. α -nonexpansive mappings from C into itself with $F \neq \emptyset$. Let $\{x_n\}$ be*

the sequence defined by

$$\begin{cases} u \in C, & x_1 \in C \quad (\text{chosen arbitrarily}) \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)[\lambda_n x_n + (1 - \lambda_n)T_n x_n]. \end{cases} \quad (3.3.19)$$

Let $\{\alpha_n\}, \{\lambda_n\}$ be sequences in $[0, 1]$ and

$$1. \lim_{n \rightarrow \infty} \alpha_n = 0, \quad 2. \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 3. 0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1.$$

Then, the sequence $\{x_n\}$ defined in (3.3.19) converges to $Q_F u$, where Q_F is a sunny nonexpansive retraction from X onto F .

3.4 Convergence Results in $CAT(0)$ space

Theorem 3.4.1. Let X be a complete $CAT(0)$ space and C a nonempty closed and convex subset of X . Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be the u.a.r semigroup of α -nonexpansive mappings from C into itself with $F \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by

$$\begin{cases} u \in C, & x_1 \in C \quad (\text{chosen arbitrary}) \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)[\lambda_n x_n \oplus (1 - \lambda_n)T(t_n)x_n]. \end{cases} \quad (3.4.1)$$

Let $\{\alpha_n\}, \{\lambda_n\}$ in $[0, 1]$ and $t_n > 0$ satisfy the following conditions :

- (i) $\lim_{n \rightarrow \infty} \alpha_n = 0$
- (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$
- (iii) $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 1$
- (iv) $\lim_{n \rightarrow \infty} t_n = \infty$.

Then, the sequence $\{x_n\}$ defined in (3.4.1) converges to $\bar{x} = P_F u$, where P_F is the metric projection of X onto $\mathcal{F}$.

Proof. Let $z \in F$ and $y_n = \lambda_n x_n \oplus (1 - \lambda_n)T(t_n)x_n$.

Step 1: $\{x_n\}$ and $\{T(t_n)x_n\}$ are bounded.

By (CN) inequality and Lemma 3.2.13(i), we obtain

$$\begin{aligned}
d^2(y_n, z) &\leq \lambda_n d^2(x_n, z) + (1 - \lambda_n) d^2(T(t_n)x_n, z) \\
&\quad - \lambda_n(1 - \lambda_n) d^2(x_n, T(t_n)x_n) \\
&\leq \lambda_n d^2(x_n, z) + (1 - \lambda_n) d^2(x_n, z) - \lambda_n(1 - \lambda_n) d^2(x, T(t_n)x_n) \\
&= d^2(x_n, z) - \lambda_n(1 - \lambda_n) d^2(x, T(t_n)x_n) \\
&\leq d^2(x_n, z).
\end{aligned}$$

By replacing $\|\cdot - \cdot\|$ with $d(\cdot, \cdot)$ in Step 1 of the proof of Theorem 3.3.1, we have

$$d(x_{n+1}, z) \leq \max\{d(u, z), d(x_1, z)\}.$$

Hence, $\{x_n\}$ and $\{T(t_n)x_n\}$ are bounded.

Step 2: If $\lim_{n \rightarrow \infty} d(x_n, T(t_n)x_n) = 0$, then $\lim_{n \rightarrow \infty} d(x_n, T(s)x_n) = 0$ for any $s > 0$.

By replacing $\|\cdot - \cdot\|$ with $d(\cdot, \cdot)$ in Step 3 of the proof of Theorem 3.3.1 and using properties of d in a $CAT(0)$ space, it is easy to verify that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (3.4.2)$$

Moreover, using arguments similar to those in Step 3 of the proof of Theorem 3.3.1 and replacing $\|\cdot - \cdot\|$ with $d(\cdot, \cdot)$, we get

$$\lim_{n \rightarrow \infty} d(T(t_n)x_n, T(s)x_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(T(s)T(t_n)x_n, T(t_n)x_n) = 0. \quad (3.4.3)$$

Now,

$$\begin{aligned}
d(y_n, T(s)x_n) &\leq d(y_n, T(t_n)x_n) + d(T(t_n)x_n, T(s)x_n) \\
&= \lambda_n d(x_n, T(t_n)x_n) + d(T(t_n)x_n, T(s)x_n).
\end{aligned}$$

Taking limit as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} d(y_n, T(s)x_n) = 0. \quad (3.4.4)$$

By (CN) inequality, we have

$$\begin{aligned}
d^2(x_{n+1}, T(s)x_n) &= d^2(\alpha_n u \oplus (1 - \alpha_n)y_n, T(s)x_n) \\
&\leq \alpha_n d^2(u, T(s)x_n) + (1 - \alpha_n) d^2(y_n, T(s)x_n) \\
&\quad - \alpha_n(1 - \alpha_n) d^2(u, y_n) \\
&\leq \alpha_n d^2(u, T(s)x_n) + (1 - \alpha_n) d^2(y_n, T(s)x_n).
\end{aligned}$$

By condition (i) and (3.4.4), it follows that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, T(s)x_n) = 0. \quad (3.4.5)$$

Hence, for any $s > 0$,

$$d(x_n, T(s)x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, T(s)x_n)$$

and

$$\lim_{n \rightarrow \infty} d(x_n, T(s)x_n) = 0. \quad (3.4.6)$$

Step 3: $x_n \rightarrow \bar{x} \in F$ as $n \rightarrow \infty$.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{d(x_n, z)\}_{n=n_0}^\infty$ is nonincreasing.

Then $\lim_{n \rightarrow \infty} d(x_n, z)$ exists. With the aid of Lemma 1.3.2 and reasoning as in Step 2 of the proof of Theorem 3.3.1, we can show that

$$\lambda_n(1 - \lambda_n) d^2(x_n, T(t_n)x_n) \leq d^2(x_n - z) - d^2(x_{n+1}, z) + \alpha_n M.$$

Since $\lim_{n \rightarrow \infty} d(x_n, z)$ exists, it follows from conditions (i) and (iii) that

$$\lim_{n \rightarrow \infty} d(x_n, T(t_n)x_n) = 0. \quad (3.4.7)$$

Consequently, we deduce from Step 2 above that

$$\lim_{n \rightarrow \infty} d(x_n, T(s)x_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(x_n, y_n) = 0. \quad (3.4.8)$$

Since $\{x_n\}$ is bounded, it follows by Lemma 1.4.4 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that $\triangle$ -converges to $\bar{x} \in X$. Using (3.4.6) and Lemma 3.2.13, we have that $\bar{x} \in F$.

Now, Lemma 3.2.10 ensures that $\{x_n\}$ $\triangle$ -converges to $\bar{x} \in F$.

Exploiting properties of quasilinearization in (1.3.1), Cauchy-Schwartz inequality and

Lemma 3.2.12, we have

$$\begin{aligned}
d^2(x_{n+1}, \bar{x}) &= d^2(\alpha_n u \oplus (1 - \alpha_n)y_n, \bar{x}) \\
&\leq \alpha_n^2 d^2(u, \bar{x}) + (1 - \alpha_n)^2 d^2(y_n, \bar{x}) + 2(\alpha_n(1 - \alpha_n) \langle \overrightarrow{u\bar{x}}, \overrightarrow{y_n\bar{x}} \rangle) \\
&\leq \alpha_n^2 d^2(u, \bar{x}) + (1 - \alpha_n)^2 d^2(x_n, \bar{x}) \\
&+ 2\alpha_n(1 - \alpha_n) [\langle \overrightarrow{u\bar{x}}, \overrightarrow{y_n x_n} \rangle + \langle \overrightarrow{u\bar{x}}, \overrightarrow{x_n \bar{x}} \rangle] \\
&\leq (1 - \alpha_n) d^2(x_n, \bar{x}) \\
&+ \alpha_n \left[\alpha_n d^2(u, \bar{x}) + 2(1 - \alpha_n) [d(u, \bar{x})d(y_n, x_n) + \langle \overrightarrow{u\bar{x}}, \overrightarrow{x_n \bar{x}} \rangle] \right].
\end{aligned}$$

Since $\{x_n\}$ Δ -converges to $\bar{x}$, it follows by Lemma 3.2.9 that $\limsup_{n \rightarrow \infty} \langle \overrightarrow{u\bar{x}}, \overrightarrow{x_n \bar{x}} \rangle \leq 0$. This inequality, (3.4.8) and condition(i) imply

$$\limsup_{n \rightarrow \infty} \left[\alpha_n d^2(u, x) + 2(1 - \alpha_n) [d(u, \bar{x})d(y_n, x_n) + \langle \overrightarrow{u\bar{x}}, \overrightarrow{x_n \bar{x}} \rangle] \right] = 0.$$

By Lemma 3.2.7, we have that $\lim_{n \rightarrow \infty} d(x_n, \bar{x}) = 0$. Hence $x_n \rightarrow \bar{x} \in F$ as $n \rightarrow \infty$.

Case 2: Suppose $\{d(x_n, z)\}_{n=n_0}^\infty$ is not nonincreasing. So, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$d(x_{n_k}, z) < d(x_{n_k+1}, z) \quad \forall k \in \mathbb{N}.$$

Then by Lemma 3.2.8, there exists an increasing sequence of integers $\{m_k\} \subset \mathbb{N}$ such that

$$1. \lim_{k \rightarrow \infty} m_k = \infty \quad 2. d(x_{m_k}, z) \leq d(x_{m_k+1}, z) \quad 3. d(x_k, z) \leq d(x_{m_k+1}, z)$$

for all (sufficiently large) numbers $k \in \mathbb{N}$. As shown before, we have

$$\lambda_{m_k}(1 - \lambda_{m_k})d^2(x_{m_k}, T(t_{m_k})x_{m_k}) \leq d^2(x_{m_k}, z) - d^2(x_{m_k+1}, z) + \alpha_n M.$$

This inequality, condition (i) and $d(x_{m_k}, z) \leq d(x_{m_k+1}, z)$ imply

$$\lim_{k \rightarrow \infty} d(x_{m_k}, T(t_{m_k})x_{m_k}) = 0.$$

Moreover, we have from Step 2 that for any $s > 0$,

$$\lim_{k \rightarrow \infty} d(x_{m_k}, y_{m_k}) = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} d(x_{m_k}, T(s)x_{m_k}) = 0.$$

As in Case 1 above, $\{x_{m_k}\}$ Δ -converges to $\bar{x} \in F$ and

$$\begin{aligned} d^2(x_{m_k+1}, \bar{x}) &\leq (1 - \alpha_{m_k})d^2(x_{m_k}, \bar{x}) \\ &+ \alpha_{m_k} \left[\alpha_{m_k}d^2(u, \bar{x}) + 2(1 - \alpha_{m_k})[d(u, \bar{x})d(y_{m_k}, x_{m_k}) + \langle \overrightarrow{u\bar{x}}, \overrightarrow{x_{m_k}\bar{x}} \rangle] \right]. \end{aligned}$$

Since $d(x_{m_k}, z) \leq d(x_{m_k+1}, z)$, we have that

$$d^2(x_{m_k}, \bar{x}) \leq \alpha_{m_k}d^2(u, \bar{x}) + 2(1 - \alpha_{m_k})[d(u, \bar{x})d(y_{m_k}, x_{m_k}) + \langle \overrightarrow{u\bar{x}}, \overrightarrow{x_{m_k}\bar{x}} \rangle].$$

Condition (i) and the fact that $\limsup_{n \rightarrow \infty} \langle u\bar{x}, x_{m_k}\bar{x} \rangle \leq 0$ imply that $\lim_{k \rightarrow \infty} d(x_{m_k}, \bar{x}) = 0$ and consequently, $\lim_{k \rightarrow \infty} d(x_{m_k+1}, \bar{x}) = 0$. Finally,

$$d(x_k, \bar{x}) \leq d(x_{m_k+1}, \bar{x}) \text{ implies that } \lim_{k \rightarrow \infty} d(x_k, \bar{x}) = 0.$$

Hence the desired result follows.

Step 4: Now we show that $\bar{x} \in F$ is the nearest common fixed point of $\mathcal{T}$ to u i.e. $\bar{x} = P_F u$.

For any $z \in F$, it follows from (1.3.1) that

$$\begin{aligned} d^2(x_n, z) &= d^2(\alpha_n u \oplus (1 - \alpha_n)y_n, z) \\ &\leq \alpha_n d^2(u, z) + (1 - \alpha_n)d^2(y_n, z) - \alpha_n(1 - \alpha_n)d^2(u, y_n) \\ &\leq d^2(u, z) + d^2(x_n, z) - \alpha_n(1 - \alpha_n)d^2(u, y_n). \end{aligned} \tag{3.4.9}$$

Since $d(y_n, \bar{x}) \leq d(x_n, \bar{x})$, we have that $y_n \rightarrow \bar{x}$ as $n \rightarrow \infty$. Taking limit as $n \rightarrow \infty$ in (3.4.9), we have

$$d^2(\bar{x}, z) \leq d^2(u, z) + d^2(x_n, z) - \alpha_n(1 - \alpha_n)d^2(u, y_n).$$

This implies $d(u, \bar{x}) \leq d(u, z) \ \forall z \in F$. Hence $\bar{x} = P_F u$. □

Remark 3.4.2.

1. An analogue of Theorem 3.3.1 for a sequence of u.a.r. α -nonexpansive mappings on a $CAT(0)$ space can be easily established on the lines of its proof.
2. Theorem 3.3.1 and Corollaries 3.3.2 - 3.3.3 extend and improve Theorem 3.3 of Song et al. [113], Theorem 3.2 of Benavides et al. [13] and Theorem 20 of Aleyner and Censor [6]

Chapter 4

Common Endpoints of Multivalued Mappings in Hyperbolic Space

In this chapter, we introduce a new modified iteration process for approximating common endpoint of a multivalued α -nonexpansive mapping and a multivalued mapping satisfying condition (E') in uniformly convex hyperbolic spaces. As a by-product, we improve the main results of Panyanak [100] with a new and faster algorithm for two multivalued mappings. Moreover, we give a numerical example to substantiate our results. Our work is new and holds simultaneously in uniformly convex Banach spaces as well as $CAT(0)$ spaces.

4.1 Introduction and Preliminaries

Let U be a nonempty subset of metric space (X, d) . Recently, Pant and Shukla [98] introduced a mapping $R : U \rightarrow U$ as generalized α -nonexpansive mapping if for $0 \leq \alpha < 1$ and any $m, w \in U$, $\frac{1}{2}d(m, Rw) \leq d(m, w)$ implies

$$d(Rm, Rw) \leq \alpha d(Rm, w) + \alpha d(Rw, w) + (1 - 2\alpha)d(m, w).$$

They showed by means of an illustrative example that it is independent of α -nonexpansive mapping.

Extending valid single-valued fixed point results to the multivalued case is natural

in view of many useful applications of the latter mappings in applied sciences.(See [83], [75]).

In what follows, the notations $K(U)$ and $CB(U)$ will stand for the family of compact and closed and bounded subsets of U , respectively.

We shall employ the following terminology in the sequel:

The distance from $m \in X$ to $U = d(m, U) := \inf\{d(m, w) : w \in U\}$,

The radius of U relative to $m = \eta(m, U) := \sup\{d(m, w) : w \in U\}$,

Diameter of $U = Diam(U) := \sup\{d(m, w) : m, w \in U\}$,

The Hausdorff metric H on $CB(X)$; $H(U, Q) := \max\{\sup_{m \in U} d(m, Q), \sup_{q \in Q} d(q, U)\}$.

Definition 4.1.1. For a multivalued mapping $G : X \rightarrow 2^X$, we say $m \in X$ is (i) a fixed point of G if $m \in Gm$. (ii) an endpoint of G if $Gm = \{m\}$.

Endpoint is also referred to as stationary point [61] or strict fixed point [72]. Moreover, by $End(G)$, we mean the set of endpoints.

Definition 4.1.2 ([99]). A multivalued $G : U \rightarrow CB(U)$ is said to have an approximate endpoint property if $\inf_{w \in U} \eta(w, Gw) = 0$. Equivalently, G has an approximate endpoint property if it has an approximate endpoint sequence $\{w_n\}$ (i.e. $\lim_{n \rightarrow \infty} \eta(w_n, Gw_n) = 0$).

Remark 4.1.3. Note that $End(G) \subseteq F(G)$. Moreover, the existence of fixed point of G does not imply the existence of its endpoint(See [99]). For a single-valued mapping, these two notions coincide.

A multivalued mapping $R : X \rightarrow 2^X$ is:

(1) nonexpansive if

$$H(Rm, Rw) \leq d(m, w) \quad \forall m, w \in X.$$

(2) quasi-nonexpansive if $F(R) \neq \emptyset$ and

$$H(Rm, R\bar{w}) \leq d(m, w) \quad \forall \bar{w} \in F(R), m \in X.$$

Motivated by the work of Garcia-Falset et al. [44], several authors have studied multivalued mappings satisfying condition (E) (See [76], [4]).

Definition 4.1.4 ([4]). We say a multivalued mapping $G : U \rightarrow CB(U)$ satisfies condition (E) if for some $\mu \geq 1$, we have

$$d(m, Gw) \leq \mu d(m, Gm) + d(m, w) \text{ for all } m, w \in U.$$

Contrary to Proposition 4 in [76], a multivalued mapping satisfying condition (E) need not be quasi-nonexpansive.

Example 7. Let $U = [-1, 1]$ with the usual metric. Define $G : U \rightarrow CB(U)$ by

$$Gw = \begin{cases} [-w, 1], & w \in [-1, 0) \\ [-1, 1 - w], & w \in [0, 1] \end{cases}.$$

We note that $F(G) = [0, \frac{1}{2}]$ and G satisfies condition (E). However, for $m = -1$ and $w = 0$, we get $H(Gm, G0) = 2$ and $d(m, w) = 1$. Therefore, G can not be quasi-nonexpansive.

Inspired by the contributions in [3], we consider a rich subclass of mappings satisfying condition (E).

Definition 4.1.5. If for some $\mu \geq 1$, $G : U \rightarrow CB(U)$ satisfies

$$H(Gm, Gw) \leq \mu d(m, Gm) + d(m, w) \text{ for all } m, w \in U,$$

then, we say G satisfy condition (E').

Example 8 ([59], [108]). A multivalued generalized α -nonexpansive mapping $G : U \rightarrow CB(U)$ satisfies condition (E'). In fact, for any $m, w \in U$, we have

$$H(Gm, Gw) \leq 2 \frac{1 + \alpha}{1 - \alpha} d(m, Gm) + d(m, w).$$

Definition 4.1.6 ([55]). A mapping $R : U \rightarrow CB(U)$ is α -nonexpansive if for $\alpha \in [0, 1)$ and any $m, w \in C$, we have

$$H^2(Rm, Rw) \leq \alpha d^2(Rm, w) + \alpha d^2(Rw, m) + (1 - 2\alpha) d^2(m, w).$$

Hajisharifi [55] showed that the class of α -nonexpansive mapping contains the class of multivalued mean nonexpansive mapping introduced in [23].

We present an example to show that the set of mappings satisfying condition (E') and the class of α -nonexpansive mappings are different.

Example 9. Let $U = \{(4, 0), (4, 5), (5, 4), (0, 0), (2, 0), (0, 4)\} \subset \mathbb{R}^2$. Define a metric $d((w_1, m_1), (w_2, m_2)) = |w_1 - w_2| + |m_1 - m_2|$ and the mapping G by

$$Gm = \begin{cases} \{(0, 0), (2, 0)\}, & m \in \{(2, 0), (0, 4), (4, 0)\} \\ \{(4, 0)\}, & m = (4, 5) \\ \{(0, 4)\}, & m = (5, 4) \\ \{(0, 0)\}, & m = (0, 0) \end{cases}.$$

Then G satisfies condition (E') , $F(G) = \{(0, 0), (2, 0)\}$ and $End(G) = \{(0, 0)\}$. For $x = (5, 4)$ and $y = (4, 5)$, we have

$$\begin{aligned} H^2(Gm, Gw) &= 64 > 4 + 42\alpha \\ &= 4(1 - 2\alpha) + 25\alpha + 25\alpha \\ &= (1 - 2\alpha)d^2(m, w) + \alpha d^2(m, Gw) + \alpha d^2(Gm, w). \end{aligned}$$

Hence, G is not α -nonexpansive for any $\alpha < 1$.

Many researchers have shown that fixed point and endpoint exist for different multivalued contractive-type mappings. (See [61], [9]). Some applications of endpoint results to optimization problems can be found in the works of Lin and Du [85] and Corley [28]. Corley, for example, showed that maximization in relation to a cone is comparable to the case of finding endpoints of multivalued mappings. Several authors ([99], [23], [109], [80], [43], [57]) have studied existence and approximation results for endpoint of nonexpansive mappings.

In 2018, Panayak [100] introduced a modified Ishikawa iteration for approximating endpoints of multivalued nonexpansive mapping $S : U \rightarrow K(U)$ in a uniformly convex Banach space. Particularly, he showed that under some given conditions, the following

algorithm converges to an endpoint of a multivalued nonexpansive mapping:

$$\begin{aligned} z_n &= \beta_n y_n + (1 - \beta_n) w_n \\ w_{n+1} &= \alpha_n v_n + (1 - \alpha_n) w_n, \end{aligned}$$

where $w_1 \in U$, $y_n \in Sw_n$ with $\|w_n - y_n\| = \eta(w_n, Sw_n)$ and $v_n \in Sz_n$ with $\|z_n - v_n\| = \eta(z_n, Sz_n)$.

This result naturally gives rise to the following question:

Question: Can common endpoints of two multivalued nonexpansive mappings be approximated with a faster iteration process ?.

For a possible solution to the above question, we follow new results of Piri et al. [101] and Iqbal et al. [59] and introduce a new algorithm for approximating common endpoint in uniformly convex hyperbolic spaces. Consequently, we find an affirmative answer. In Section 4.2, we show that endpoint of multivalued α -nonexpansive mappings and mappings satisfying condition (E') exist. In section 4.3, we establish convergence theorems for the proposed algorithm. In section 4.4, we provide an example to substantiate our results. Moreover, we show the efficiency of our algorithm in approximating common endpoints of two nonexpansive mappings by comparing its results with common fixed point results of Fukhar-ud-din et al. [41] and Chang et al. [22].

The following known lemmas are needed to prove our main results.

Lemma 4.1.7 ([58]). *Let U and V be nonempty subsets of a metric space X , and let $m \in X$. Then*

$$d(m, U) \leq d(m, V) + H(U, V).$$

Lemma 4.1.8 ([99]). *Let U be as in Lemma 4.1.7, $G : U \rightarrow CB(X)$ a multivalued mapping and $w_n \in U$. Then $\lim_{n \rightarrow \infty} \eta(w_n, Gw_n) = 0$ if and only if $\lim_{n \rightarrow \infty} d(w_n, Gw_n) = 0$ and $\lim_{n \rightarrow \infty} \text{Diam}(Gw_n) = 0$. Moreover, $\eta(m, Gm) = 0$ if and only if $m \in \text{End}(G)$.*

Lemma 4.1.9 ([109]). *Let U be as in Lemma 4.1.7 and $G : U \rightarrow CB(X)$ a multivalued mapping. Let $m, w \in U$. If $w' \in Gm$, then it follows that*

$$d(m, w') \leq \eta(m, Gm) + H(Gm, Gw).$$

4.2 Existence of Endpoint

Lemma 4.2.1. *Let U be a nonempty subset of a metric space (X, d) . For a multivalued α -nonexpansive mapping $G : U \rightarrow CB(U)$, we have*

$$\begin{aligned} d^2(m, Gw) &\leq d^2(m, w) \\ &+ \left[\frac{(1 + \alpha)d(m, Gm) + 2\alpha d(m, w) + 2H(Gm, Gw)}{1 - \alpha} \right] d(m, Gm). \end{aligned} \quad (4.2.1)$$

Proof. For any $m, w \in X$, Lemma 4.1.7 implies that

$$\begin{aligned} d^2(m, Gw) &\leq [d(m, Gm) + H(Gm, Gw)]^2 \\ &= d^2(m, Gm) + 2d(m, Gm)H(Gm, Gw) + H^2(Gm, Gw). \end{aligned}$$

By the definition of the mapping G , we have

$$\begin{aligned} d^2(m, Gw) &\leq d^2(m, Gm) + 2d(m, Gm)H(Gm, Gw) + \alpha d^2(m, Gw) \\ &+ \alpha d^2(w, Gm) + (1 - \alpha)d^2(m, w). \end{aligned} \quad (4.2.2)$$

Using

$$\begin{aligned} d^2(w, Gm) &\leq [d(w, m) + d(m, Gm)]^2 \\ &= d(m, w)^2 + 2d(m, w)d(m, Gm) + d^2(m, Gm) \end{aligned} \quad (4.2.3)$$

in (4.2.2), we have

$$\begin{aligned} (1 - \alpha)d^2(m, Gw) &\leq (1 - \alpha)d^2(m, w) \\ &+ [(1 + \alpha)d(m, Gm) + 2\alpha d(m, w)]d(m, Gm) \\ &+ 2H(Gm, Gw)d(m, Gm). \end{aligned}$$

Hence the result. □

Proposition 4.2.2. *Let U be as in Lemma 4.2.1. Then, a multivalued α -nonexpansive mapping $G : U \rightarrow CB(U)$ is a quasi-nonexpansive mapping.*

Proof. It follows by (4.2.1) and (4.2.3). □

Theorem 4.2.3. *Let U be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X . Then, a multivalued α -nonexpansive mapping, $G : U \rightarrow K(U)$, has a bounded approximate endpoint sequence if and only if G has an endpoint.*

Proof. Assume that $\{w_n\}$ is a bounded approximate endpoint sequence of G . Then,
 $\lim_{n \rightarrow \infty} \eta(w_n, Gw_n) = 0$ i.e.

$$\lim_{n \rightarrow \infty} d(w_n, Gw_n) = 0 \text{ and } \lim_{n \rightarrow \infty} \text{Diam}(Gx_n) = 0. \quad (4.2.4)$$

We may assume $\{w_n\}$ is regular relative to U by passing through a subsequence. It follows by Lemma 1.4.4 that $\{w_n\}$ has a unique asymptotic center w . By Lemma 4.2.1, we have

$$\begin{aligned} d^2(w_n, Gw) &\leq d^2(w_n, w) \\ &+ \left[\frac{(1 + \alpha)d(w_n, Gw_n) + 2\alpha d(w_n, w) + 2H(Gw_n, Gw)}{1 - \alpha} \right] d(w_n, Gw_n). \end{aligned}$$

Taking limit superior as $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} d(w_n, Gw) \leq \limsup_{n \rightarrow \infty} d(w_n, w). \quad (4.2.5)$$

Compactness of Gw implies there exists $m_n \in Gw$, such that

$$d(w_n, Gw) = d(w_n, m_n),$$

for each $n \in \mathbb{N}$. Moreover, and we have a subsequence $\{m_{n_k}\}$ of $\{m_n\}$, such that $m_{n_k} \rightarrow m \in Gw$. Hence,

$$\begin{aligned} \limsup_{k \rightarrow \infty} d(w_{n_k}, m_{n_k}) &= \limsup_{k \rightarrow \infty} d(w_{n_k}, Gw) \\ &\leq \limsup_{n \rightarrow \infty} d(w_n, Gw) \\ &\leq \limsup_{n \rightarrow \infty} d(w_n, w). \end{aligned}$$

Consequently, we get $\limsup_{k \rightarrow \infty} d(w_{n_k}, m) \leq \limsup_{n \rightarrow \infty} d(w_n, w)$. This inequality and regularity of $\{w_n\}$ imply that

$$r_U(w_{n_k}) \leq \limsup_{k \rightarrow \infty} d(w_{n_k}, m) \leq \limsup_{n \rightarrow \infty} d(w_n, w) = r_U(w_n) = r_U(w_{n_k}).$$

Hence, $m, w \in A_U(w_{n_k})$ and $w = m \in Gw$.

Next we show that w is an endpoint of G . Using (4.2.3), (4.2.5) and the definition of G , it follows that

$$\limsup_{n \rightarrow \infty} H(Gw_n, Gw) \leq \limsup_{n \rightarrow \infty} d(w_n, w). \quad (4.2.6)$$

Let h be an arbitrary element in Gw . Then there exists c_n in Gw_n such that $d(h, c_n) =$

$d(h, Gw_n)$. Then,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} d(w_n, h) &\leq \limsup_{n \rightarrow \infty} d(w_n, c_n) + \limsup_{n \rightarrow \infty} d(c_n, h) \\
&\leq \limsup_{n \rightarrow \infty} \eta(w_n, Gw_n) + \limsup_{n \rightarrow \infty} H(Gw_n, Gw) \\
&= \limsup_{n \rightarrow \infty} H(Gw_n, Gw).
\end{aligned}$$

Using (4.2.6), we have $\limsup_{n \rightarrow \infty} d(w_n, h) \leq \limsup_{n \rightarrow \infty} d(w_n, w)$. Then, uniqueness of the asymptotic center of $\{w_n\}$ implies that $w = h$. Therefore, $Gw = \{w\}$

Conversely, If G has an endpoint, then obviously G has a bounded approximate endpoint sequence. $\square$

Theorem 4.2.4. *Let U be as in Theorem 4.2.3. Then, a multivalued mapping satisfying condition (E') , $G : U \rightarrow K(U)$, has a bounded approximate endpoint sequence if and only if G has an endpoint.*

Proof. Let w_n, m_n, m and w be as in the proof of Theorem 4.2.3. Then, by condition (E') and Lemma 4.1.7, we get

$$\begin{aligned}
d(w_{n_k}, m) &\leq d(w_{n_k}, m_{n_k}) + d(m_{n_k}, m) \\
&= d(w_{n_k}, Gw) + d(m_{n_k}, m) \\
&\leq d(w_{n_k}, Gw_{n_k}) + H(Gw_{n_k}, Gw) + d(m_{n_k}, m) \\
&\leq (\mu + 1)d(w_{n_k}, Gw_{n_k}) + d(w_{n_k}, w) + d(m_{n_k}, m),
\end{aligned} \tag{4.2.7}$$

and $\limsup_{k \rightarrow \infty} d(w_{n_k}, m) \leq \limsup_{n \rightarrow \infty} d(w_n, w)$. Moreover, it follows from condition (E') that

$$\limsup_{n \rightarrow \infty} H(Gw_n, Gw) \leq \limsup_{n \rightarrow \infty} d(w_n, w). \tag{4.2.8}$$

By (4.2.7), (4.2.8) and reasons similar to those in the proof of Theorem 4.2.3, we get that w is an endpoint of G . $\square$

Theorem 4.2.5. *Let U be as in Theorem 4.2.3, $G : U \rightarrow K(U)$ be a multivalued mapping satisfying condition (E') and $R : U \rightarrow U$ be a α -nonexpansive mapping. If G has a bounded approximate endpoint sequence in $\text{End}(R) = F(R)$, then R and G have a common endpoint in U .*

Proof. Let $\{w_n\}$ be an approximate endpoint sequence for G in $\text{End}(R)$. We may assume $\{w_n\}$ is regular relative to U with a unique asymptotic center w . By Theorem 4.2.3, $w \in \text{End}(G)$. From (4.2.5), we have

$$\limsup_{n \rightarrow \infty} d(w_n, R(w)) \leq \limsup_{n \rightarrow \infty} d(w_n, w).$$

Since asymptotic center of $\{w_n\}$ is unique, we have that $R(w) = w$. Hence the result. $\square$

In the example below, we observe that commutativity of R and G does not guarantee existence of a common endpoint.

Example 10 ([80]). Let $U = [0, 1]$ with the usual metric. Define $G : U \rightarrow K(U)$ and $R : U \rightarrow U$ as follows:

$$R(w) = \frac{w}{2} \quad \text{and} \quad Gw = [w, 1] \quad \text{for all } w \in U.$$

It is easy to check that R is α -nonexpansive and G satisfies condition (E') with $\mu = 2$. Moreover, R and G are commuting mappings. However, $\text{End}(R) = \{0\}$ and $\text{End}(G) = \{1\}$.

4.3 Approximation of Common Endpoints

By Theorem 4.2.3 and Theorem 4.2.4, we can easily deduce the following technical result.

Lemma 4.3.1. *Let U be a nonempty closed and convex subset of a uniformly convex hyperbolic space X and $G : U \rightarrow K(U)$ be one of the following:*

1. *Multivalued α -nonexpansive mapping*
2. *Multivalued mapping satisfying condition (E') .*

If a sequence $\{m_n\}$ in X Δ -converges to m and $\lim_{n \rightarrow \infty} \eta(m_n, Gm_n) = 0$, then $m \in \text{End}(G)$.

Inspired by the works of Iqbal et al. [59] and Panyanak [100], we come up with a new modified algorithm to approximate common endpoint of two multivalued mappings $G : U \rightarrow K(U)$ and $R : U \rightarrow K(U)$ as follows:

For any $x_1 = x \in U$, let $\{x_n\}$ be the sequence given by

$$\begin{cases} x_{n+1} = W(w_n, v_n, \alpha_n) \\ z_n \in Rt_n \text{ and } d(t_n, z_n) = \eta(t_n, Rt_n) \\ t_n = W(x_n, u_n, \beta_n) \end{cases} \quad (4.3.1)$$

where $0 < \alpha_n, \beta_n < 1$, $w_n \in Rz_n$ with $d(z_n, w_n) = \eta(z_n, Rz_n)$, $u_n \in Rx_n$ with $d(x_n, u_n) = \eta(x_n, Rx_n)$ and $v_n \in Gw_n$ with $d(v_n, w_n) = \eta(w_n, Gw_n)$.

Next, we consider an important class of multivalued mappings called (L') -type.

Definition 4.3.2. Let U be a nonempty subset of a metric space (X, d) and $R : U \rightarrow CB(U)$. We say R is (L') -type if for any $w \in U$ and $\lim_{n \rightarrow \infty} d(w_n, Rw_n) = 0$, then we have

$$\limsup_{n \rightarrow \infty} H(Rw_n, Rw) \leq \limsup_{n \rightarrow \infty} d(w_n, w). \quad (4.3.2)$$

Example 11. The α -nonexpansive mapping in Theorem 4.2.3 and mapping satisfying condition (E') in Theorem 4.2.4 are (L') -type.

Lemma 4.3.3. Let U be a nonempty closed and convex subset of a uniformly convex hyperbolic space X . Let $G : U \rightarrow K(U)$ be a multivalued mapping satisfying condition (E') and $R : U \rightarrow K(U)$ be a multivalued α -nonexpansive mapping. Suppose that $\mathcal{F} = \text{End}(R) \cap \text{End}(G) \neq \emptyset$. Let $\{x_n\}$ be the sequence given by (4.3.1). Then $\lim_{n \rightarrow \infty} d(x_n, m)$ exists for all $m \in \mathcal{F}$. Moreover, $\lim_{n \rightarrow \infty} \eta(x_n, Rx_n) = \lim_{n \rightarrow \infty} \eta(x_n, Gx_n) = 0$.

Proof. By Proposition 4.2.2 and condition (E') , R and G are quasi-nonexpansive. Let $m \in \mathcal{F}$,

Step 1: We show that $\lim_{n \rightarrow \infty} d(x_n, m)$ exists and $\{x_n\}$ is bounded.

$$\begin{aligned} d(t_n, m) &= d(W(x_n, u_n, \beta_n), m) \\ &\leq \beta_n d(x_n, m) + (1 - \beta_n) d(u_n, m) \\ &= \beta_n d(x_n, m) + (1 - \beta_n) d(u_n, Rm) \\ &\leq \beta_n d(x_n, m) + (1 - \beta_n) H(Rx_n, Rm) \\ &\leq \beta_n d(x_n, m) + (1 - \beta_n) d(m, x_n) \\ &= d(x_n, m). \end{aligned} \quad (4.3.3)$$

Also,

$$\begin{aligned}
d(z_n, m) &= d(z_n, Rm) \\
&\leq H(Rt_n, Rm) \\
&\leq d(t_n, m) \\
&\leq d(x_n, m).
\end{aligned} \tag{4.3.4}$$

Similarly, we have

$$d(u_n, m) \leq d(x_n, m), \quad d(w_n, m) \leq d(z_n, m) \leq d(x_n, m) \tag{4.3.5}$$

$$\text{and } d(v_n, m) \leq d(w_n, m) \leq d(x_n, m).$$

Now, using (4.3.2) and (4.3.3), we have

$$\begin{aligned}
d(x_{n+1}, m) &= d(W(w_n, v_n, \alpha_n), m) \\
&\leq \alpha_n d(w_n, m) + (1 - \alpha_n) d(v_n, m) \\
&= \alpha_n d(w_n, Rm) + (1 - \alpha_n) d(v_n, Gm) \\
&\leq \alpha_n H(Rz_n, Rm) + (1 - \alpha_n) H(Gw_n, Gm) \\
&\leq \alpha_n d(z_n, m) + (1 - \alpha_n) d(w_n, m) \\
&\leq \alpha_n d(x_n, m) + (1 - \alpha_n) d(x_n, m) \\
&= d(x_n, m).
\end{aligned} \tag{4.3.6}$$

This implies that for any $m \in \mathcal{F}$, $\{d(x_n, m)\}$ is nonincreasing and thus convergent.

Step 2: We prove that $\lim_{n \rightarrow \infty} \eta(x_{n+1}, Gx_{n+1}) = 0$.

Assume that

$$\lim_{n \rightarrow \infty} d(x_n, m) = k. \tag{4.3.7}$$

It follows from (4.3.2), (4.3.3) and (4.3.4) that

$$\limsup_{n \rightarrow \infty} d(v_n, m) \leq k, \quad \limsup_{n \rightarrow \infty} d(u_n, m) \leq k, \quad \limsup_{n \rightarrow \infty} d(w_n, m) \leq k \tag{4.3.8}$$

and

$$\limsup_{n \rightarrow \infty} d(w_n, m) \leq k, \quad \limsup_{n \rightarrow \infty} d(t_n, m) \leq k. \tag{4.3.9}$$

By (4.3.7) and Lemma 1.4.3,

$$k = \lim_{n \rightarrow \infty} d(x_{n+1}, m) = \lim_{n \rightarrow \infty} d(W(w_n, v_n, \alpha_n), m)$$

implies that

$$\lim_{n \rightarrow \infty} d(w_n, v_n) = 0. \quad (4.3.10)$$

Hence,

$$\lim_{n \rightarrow \infty} \eta(w_n, Gw_n) = 0. \quad (4.3.11)$$

Notice that $\limsup_{n \rightarrow \infty} \eta(v_n, Gv_n) \leq \limsup_{n \rightarrow \infty} H(Gw_n, Gv_n)$. The fact that S is an (L') -type mapping gives

$$\limsup_{n \rightarrow \infty} \eta(v_n, Gv_n) \leq \limsup_{n \rightarrow \infty} d(w_n, v_n).$$

Therefore, (4.3.10) implies that

$$\lim_{n \rightarrow \infty} \eta(v_n, Gv_n) = 0. \quad (4.3.12)$$

For each n , let $x'_{n+1} \in Gx_{n+1}$ be such that

$$\eta(x_{n+1}, Gx_{n+1}) - \frac{1}{n} \leq d(x_{n+1}, x'_{n+1}) \leq \eta(x_{n+1}, Gx_{n+1}).$$

A combination of G is an (L') -type mapping, (4.12) and Lemma 4.1.7 gives

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(v_n, x'_{n+1}) &\leq \lim_{n \rightarrow \infty} \eta(v_n, Gv_n) + \limsup_{n \rightarrow \infty} H(Gv_n, Gx_{n+1}) \\ &\leq \limsup_{n \rightarrow \infty} d(v_n, x_{n+1}) \\ &= \limsup_{n \rightarrow \infty} d(v_n, W(w_n, v_n, \alpha_n)) \\ &= \limsup_{n \rightarrow \infty} \alpha_n d(v_n, w_n). \end{aligned}$$

From (4.3.10), we get

$$\lim_{n \rightarrow \infty} d(v_n, x'_{n+1}) = 0. \quad (4.3.13)$$

On the same lines as before, we have

$$\lim_{n \rightarrow \infty} d(w_n, x'_{n+1}) = 0. \quad (4.3.14)$$

Taking limit as $n \rightarrow \infty$ in the following expression

$$\begin{aligned} d(x_{n+1}, x'_{n+1}) &= d(W(w_n, v_n, \alpha_n), x'_{n+1}) \\ &\leq \alpha_n d(w_n, x'_{n+1}) + (1 - \alpha_n) d(v_n, x'_{n+1}), \end{aligned}$$

we obtain by (4.3.13) and (4.3.14) that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x'_{n+1}) = 0.$$

Hence,

$$\eta(x_{n+1}, Gx_{n+1}) \leq \frac{1}{n} + d(x_{n+1}, x'_{n+1})$$

implies

$$\lim_{n \rightarrow \infty} \eta(x_{n+1}, Gx_{n+1}) = 0.$$

Step 3: We show that $\lim_{n \rightarrow \infty} \eta(x_n, Rx_n) = 0$.

Using (4.3.5) in

$$d(x_{n+1}, m) \leq \alpha_n d(w_n, m) + (1 - \alpha_n) d(v_n, m),$$

we obtain $d(x_{n+1}, m) \leq d(w_n, m)$. Consequently, $k \leq \liminf_{n \rightarrow \infty} d(w_n, m)$. This inequality and (4.3.8) give

$$\lim_{n \rightarrow \infty} d(w_n, m) = k.$$

It follows easily by (4.3.2), (4.3.3) and (4.3.4) that $\lim_{n \rightarrow \infty} d(t_n, p) = k$. Moreover, we obtain from (4.3.6), (4.3.7) and Lemma 4.1.7 that

$$\lim_{n \rightarrow \infty} d(x_n, u_n) = 0.$$

Therefore,

$$\lim_{n \rightarrow \infty} \eta(x_n, Rx_n) = \lim_{n \rightarrow \infty} d(x_n, u_n) = 0. \quad (4.3.15)$$

Step 4: We prove that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$.

Combining (4.3.15), Lemma 4.1.9 and the fact that R is a (L') -type mappings, we obtain

$$\begin{aligned}
\limsup_{n \rightarrow \infty} d(u_n, w_n) &\leq \limsup_{n \rightarrow \infty} d(u_n, x_n) + \limsup_{n \rightarrow \infty} d(x_n, w_n) \\
&\leq \limsup_{n \rightarrow \infty} d(w_n, x_n) \\
&\leq \limsup_{n \rightarrow \infty} \eta(x_n, Rx_n) + \limsup_{n \rightarrow \infty} H(Rx_n, Rz_n) \\
&\leq \limsup_{n \rightarrow \infty} d(z_n, x_n) \\
&\leq \limsup_{n \rightarrow \infty} \eta(x_n, Rx_n) + \limsup_{n \rightarrow \infty} H(Rx_n, Rt_n) \\
&\leq \limsup_{n \rightarrow \infty} d(t_n, x_n) \\
&= \limsup_{n \rightarrow \infty} d(x_n, W(x_n, u_n, \beta_n)) \\
&\leq \limsup_{n \rightarrow \infty} \beta_n d(x_n, u_n).
\end{aligned}$$

Also, (4.3.15) implies that

$$\lim_{n \rightarrow \infty} d(u_n, w_n) = 0. \quad (4.3.16)$$

Moreover, (4.3.15), (4.3.16) and $d(w_n, x_n) \leq d(w_n, u_n) + d(u_n, x_n)$ give

$$\lim_{n \rightarrow \infty} d(w_n, x_n) = 0. \quad (4.3.17)$$

Similarly, using (4.3.10) and (4.3.17), we obtain $\lim_{n \rightarrow \infty} d(v_n, x_n) = 0$.

Now,

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(W(w_n, v_n, \alpha_n), x_n) \\
&\leq \alpha_n d(w_n, x_n) + (1 - \alpha_n) d(v_n, x_n),
\end{aligned}$$

gives that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0. \quad (4.3.18)$$

Step 5: We prove that $\lim_{n \rightarrow \infty} \eta(x_n, Gx_n) = 0$.

G is an (L') -type mapping and thus,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} d(x_n, Gx_n) &\leq \limsup_{n \rightarrow \infty} d(x_n, x_{n+1}) + \limsup_{n \rightarrow \infty} d(x_{n+1}, Gx_{n+1}) \\
&\quad + \limsup_{n \rightarrow \infty} H(Gx_{n+1}, Gx_n) \\
&\leq 2 \limsup_{n \rightarrow \infty} d(x_n, x_{n+1}).
\end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} d(x_n, Gx_n) = 0. \quad (4.3.19)$$

For each $n \in \mathbb{N}$, Gx_n is compact. Thus, there exist p_n and q_n in Gx_n such that

$$\text{Diam}(Gx_n) = d(p_n, q_n).$$

By Lemma 4.1.9, we have

$$\begin{aligned} \text{Diam}(Gx_n) = d(p_n, q_n) &\leq d(p_n, x_{n+1}) + d(x_{n+1}, q_n) \\ &\leq 2H(Gx_n, Gx_{n+1}) + 2\eta(x_{n+1}, Gx_{n+1}). \end{aligned}$$

It follows from (4.3.18) and Step 3 that $\lim_{n \rightarrow \infty} d(p_n, q_n) = 0$ and thus,

$$\lim_{n \rightarrow \infty} \text{Diam}(Gx_n) = 0. \quad (4.3.20)$$

Combining (4.3.19) and (4.3.20), we get $\lim_{n \rightarrow \infty} \eta(x_n, Gx_n) = 0$. $\square$

Theorem 4.3.4. *Let U be a nonempty compact and convex subset of a complete uniformly convex hyperbolic space X . Let $G : U \rightarrow K(U)$ be mapping satisfying condition (E') and $R : U \rightarrow K(U)$ be a multivalued α -nonexpansive mapping with $\mathcal{F} \neq \emptyset$. Then, $\{x_n\}$ given by (4.3.1) converges strongly to a common endpoint of R and G .*

Proof. By Lemma 4.3.3, we have that $\{x_n\}$ is bounded and

$$\lim_{n \rightarrow \infty} \eta(x_n, Rx_n) = \lim_{n \rightarrow \infty} \eta(x_n, Gx_n) = 0. \quad (4.3.21)$$

There exists a subsequence x_{n_k} of $\{x_n\}$ such that x_{n_k} converges to some $m \in U$ since U is compact. Then, it follows from (4.3.21) and Lemma 4.3.1 that $m \in \mathcal{F}$. Consequently, the existence of $\lim_{n \rightarrow \infty} d(x_n, x)$ (as in Lemma 4.3.3) gives $x_n \rightarrow m \in \mathcal{F}$ as $n \rightarrow \infty$. $\square$

Multivalued version of Condition (I) of Fukhar-ud-din and Khan [40] and Senter and Dotson [111] in relation to endpoint is defined as follows:

Definition 4.3.5. We say two multivalued mappings $R, G : U \rightarrow 2^U$ satisfy Condition (AV) if we can find a nondecreasing function $\Theta : [0, \infty) \rightarrow [0, \infty)$ with $\Theta(0) = 0$, $\Theta(\mu) > 0$ for any $\mu \geq 0$ and $\frac{1}{2}[\eta(w, Gw) + \eta(w, Rw)] \geq \Theta(d(w, \mathcal{F}))$ for all $w \in U$.

Theorem 4.3.6. *Let U be as in Lemma 4.3.3, $G : U \rightarrow K(U)$ be a mapping satisfying condition (E') and $R : U \rightarrow K(U)$ be a multivalued α -nonexpansive mapping with $\mathcal{F} \neq \emptyset$. If R and G satisfy condition (AV) , then $\{x_n\}$ generated by (4.3.1) converges strongly to a common endpoint of R and G .*

Proof. Using the condition (AV) , we have

$$\Theta(d(x_n, \mathcal{F})) \leq \frac{1}{2}[\eta(x_n, Gx_n) + \eta(x_n, Rx_n)].$$

So by (4.3.21), $\lim_{n \rightarrow \infty} \Theta(d(x_n, \mathcal{F})) = 0$. Since Θ is nondecreasing and $\Theta(0) = 0$, we get $\lim_{n \rightarrow \infty} d(x_n, \mathcal{F}) = 0$. Now, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $m_k \in \mathcal{F}$ such that $d(x_{n_k}, m_k) \leq \frac{1}{2^k}$ for all $k \in \mathbb{N}$. Using (4.3.18), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} d(x_{n_k+1}, x_{n_k}) &\leq \lim_{k \rightarrow \infty} d(x_{n_k+1}, m_k) + \lim_{k \rightarrow \infty} d(m_k, x_{n_k}) \\ &\leq \lim_{k \rightarrow \infty} d(m_k, x_{n_k}) + \lim_{k \rightarrow \infty} d(x_{n_k}, m_k) \\ &\leq \frac{1}{2^{k-1}}. \end{aligned}$$

Therefore, it follows that $\{x_{n_k}\}$ converges to a point $w \in U$. Since $\{x_n\}$ is bounded, it follows that $\lim_{n \rightarrow \infty} x_n = w$. By (4.3.21) and Lemma 4.3.1, we get $w \in \mathcal{F}$. $\square$

Definition 4.3.7. We say a multivalued mapping $G : U \rightarrow 2^U$ is semi-compact if for any approximate endpoint sequence $\{w_n\}$ in U , there is a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and $w \in U$ such that $\lim_{k \rightarrow \infty} w_{n_k} = w$.

Theorem 4.3.8. *Let U be as in Lemma 4.3.3, $R : U \rightarrow K(U)$ be a multivalued α -nonexpansive mapping and $G : U \rightarrow K(U)$ be a multivalued mapping satisfying condition (E') with $\mathcal{F} \neq \emptyset$. If R and G are semi-compact, then $\{x_n\}$ generated by (4.3.1) converges strongly to a common endpoint of R and G .*

Proof. By Lemma 4.3.3, we have

$$\lim_{n \rightarrow \infty} \eta(x_n, Rx_n) = \lim_{n \rightarrow \infty} \eta(x_n, Gx_n) = 0.$$

Then, semi-compactness of R and G implies that there is a subsequence x_{n_k} of $\{x_n\}$ which converges to w . Subsequently, it follows from (4.3.21) and Lemma 4.3.1 that $w \in \mathcal{F}$. Since $\lim_{k \rightarrow \infty} d(x_n, w)$ exists by Lemma 4.3.3, we have that $\{x_n\}$ converges strongly to $w \in \mathcal{F}$. $\square$

Theorem 4.3.9. *Let U be as in Lemma 4.3.3, $G : U \rightarrow K(U)$ be a multivalued mapping satisfying condition (E') and $R : U \rightarrow K(U)$ be a multivalued α -nonexpansive mapping with $\mathcal{F} \neq \emptyset$. Then, $\{x_n\}$ given by (4.3.1) Δ -converges to a common endpoint of R and G .*

Proof. Set $\mathcal{W}_\Delta(\{x_n\}) = \bigcup_{\{u_n\} \subset \{x_n\}} A_U(\{u_n\})$. Proving our result is equivalent to showing that $\mathcal{W}_\Delta(\{x_n\}) \subset \mathcal{F} = \text{End}(R) \cap \text{End}(G)$ and $\mathcal{W}_\Delta(\{x_n\})$ is singleton. Let $w \in \mathcal{W}_\Delta(\{x_n\})$. Then, we have a subsequence $\{c_n\}$ of $\{x_n\}$ with asymptotic center w . By Lemma 1.4.5, there exists a subsequence $\{r_n\}$ of $\{c_n\}$ with $\Delta - \lim_{n \rightarrow \infty} r_n = b \in U$. Using (4.3.21) and Lemma 4.3.1, we have that $b \in \mathcal{F}$. Since $\{d(x_n, b)\}$ converges, it follows from Lemma 1.4.6 that $b = w$. Hence $w \in \mathcal{F}$ and $\mathcal{W}_\Delta(\{x_n\}) \subset \mathcal{F}$. Let w be an arbitrary element of $\mathcal{W}_\Delta(\{x_n\})$ and z be the unique asymptotic center of $\{x_n\}$. By definition of $\mathcal{W}_\Delta(\{x_n\})$, there is a subsequence $\{c_n\}$ of $\{x_n\}$ with asymptotic center w . Since $w \in \mathcal{W}_\Delta(\{x_n\}) \subset \mathcal{F}$ and limit of $\{d(x_n, w)\}$ exists by Lemma 4.3.3, we conclude using Lemma 1.4.6 that $w = z$. Hence the result. $\square$

On the lines of proof of Theorems 4.3.6 and 4.3.8, we can established the following common endpoint result.

Theorem 4.3.10. *Let U be as in Lemma 4.3.3, $R : U \rightarrow K(U)$ and $G : U \rightarrow K(U)$ be either multivalued α -nonexpansive mappings or multivalued mappings satisfying condition (E') with $\mathcal{F} \neq \emptyset$. If R and G are semi-compact or satisfy condition (AV) , then $\{x_n\}$ given by (4.3.1) converges strongly to a common endpoint of R and G .*

As a consequence of the above theorems, we get an extension of ([100], Theorem 3.5) as follows:

Corollary 4.3.11. *Let U be as in Lemma 4.3.3 and $R, G : U \rightarrow K(U)$ be multivalued nonexpansive mappings with $\mathcal{F} \neq \emptyset$. If R and G are semi-compact or satisfy condition (AV) , then $\{x_n\}$ generated by (4.3.1) converges strongly to a common endpoint of R and G .*

4.4 Numerical Experiment

Example 12. Let $U = [0, 1]$ with the usual metric. Define $R, G : U \rightarrow K(U)$ as follows :

$$Rm = \begin{cases} [m, 1 - m], & m \in [0, \frac{1}{2}) \\ [1 - m, m], & m \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad Gm = \begin{cases} [m, \frac{2m+1}{4}], & m \in [0, \frac{1}{2}] \\ \{\frac{1}{2}\}, & m \in [\frac{1}{2}, 1]. \end{cases}$$

Note that $F(R) \cap F(G) = [0, \frac{1}{2}]$, $End(R) \cap End(G) = \{\frac{1}{2}\}$ and U is compact. Moreover, it is easy to show that R and G are multivalued nonexpansive mappings.

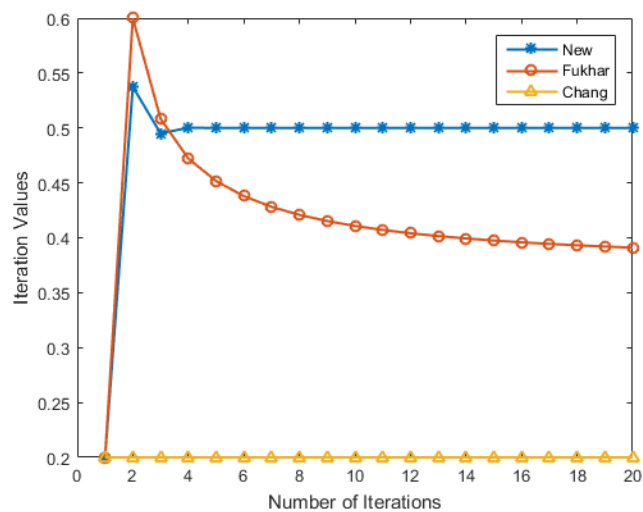


Fig 4.1a : Initial value = 0.2

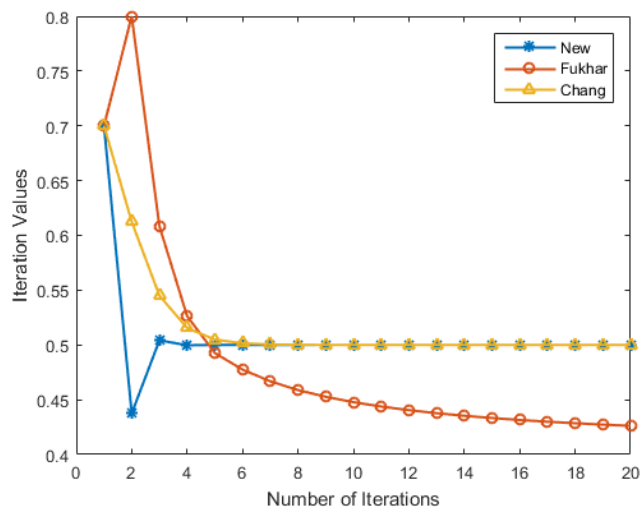


Fig 4.1b : Initial value = 0.7

First, we note that for any $m \in [0, 1]$ and $w \in Rm$, $d(w, m) = \eta(m, Rm)$ imply that

Table 4.1: Iterations for comparison

New Iteration	$z_n = \beta_n x_n + (1 - \beta_n)(1 - x_n)$ $y_n = (1 - z_n)$ $x_{n+1} = \alpha_n(1 - y_n) + (1 - \alpha_n)c_n$
Fukhar-ud-din et al. [41]	$y_n = \beta_n P_R x_n + \beta'_n x_n + (1 - \beta_n - \beta'_n)u_n$ $x_{n+1} = \alpha_n P_G y_n + \alpha'_n x_n + (1 - \alpha_n - \alpha'_n)v_n$
Chang et al. [22]	$y_n = \beta_n P_G x_n + (1 - \beta_n)x_n$ $x_{n+1} = \alpha_n P_G x_n + (1 - \alpha_n)P_R y_n$

$w = 1 - m$. Moreover, for any $m \in [0, 1]$, define $w \in Gm$ as follows :

$$w = \begin{cases} \frac{2m+1}{4}, & m \in [0, \frac{1}{2}] \\ \frac{1}{2}, & m \notin [\frac{1}{2}, 1] \end{cases}.$$

Then, $d(w, m) = \eta(m, Gm)$. Thus, our algorithm (4.3.1) becomes the new algorithm in Table 4.1 where

$$c_n = \begin{cases} \frac{1}{2}, & y_n \in [0, \frac{1}{2}] \\ \frac{3-2y_n}{4}, & y_n \in [\frac{1}{2}, 1] \end{cases}.$$

Moreover, some needed components of the other algorithms in Table 4.1 turn out to be

$P_R x = \{y \in Rx : d(y, x) = d(x, Rx)\} = \{x\}$ and

$$P_G x = \begin{cases} x, & x \in [0, \frac{1}{2}] \\ \frac{1}{2}, & x \in [\frac{1}{2}, 1] \end{cases},$$

where $P_R x$ is the best approximation operator of R (See [66] for more details). It is easy

to show that P_G and P_R are nonexpansive. Take $\alpha_n = \frac{n}{2(n+1)}$, $\alpha'_n = \frac{n^2+2n-2}{2n^2+2n}$, $\beta_n = \frac{1}{2(n^2+1)}$, $\beta'_n = \frac{n^2}{n^2+1}$, $u_n = \frac{1}{n}$ and $v_n = \frac{1}{n^2}$.

For initial values $x_1 = 0.2$ and $x_1 = 0.7$, respectively, we depict numerical results in Table 4.2.

From Figures 4.1a, 4.1b, Table 4.2 and Table 4.3, we make the following observations:

1. Unlike known algorithms that converge to common fixed point of two multivalued mappings, our algorithm approximates common endpoint of the mappings irrespective of initial values and without any restrictions on the parameter sequences.

Table 4.2: Numerical results for several iterations

$x_1 = 0.2$				$x_1 = 0.7$			
Iter.	Fukhar	Chang	New	Iter.	Fukhar	Chang	New
1	0.200000	0.20000	0.200000	1	0.700000	0.700000	0.700000
2	0.600000	0.20000	0.537500	2	0.800000	0.612500	0.437500
3	0.508333	0.20000	0.495000	3	0.608333	0.545000	0.504167
4	0.472106	0.20000	0.500188	4	0.526273	0.515469	0.499713
5	0.451627	0.20000	0.499992	5	0.492575	0.504914	0.500007
6	0.437905	0.20000	0.500000	6	0.477489	0.501488	0.500000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	0.410755	0.20000	0.5000000	13	0.437705	0.5000000	0.5000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	0.391996	0.20000	0.5000000	19	0.427287	0.5000000	0.5000000
20	0.390972	0.20000	0.5000000	20	0.426170	0.5000000	0.5000000

Table 4.3: Inferences from Table 4.2

	Initial Value	Iter. Value (6 decimal places)	Iter. No.
Chang et al. [22]	0.2	0.2	1
New iteration	0.2	0.5	6
Chang et al. [22]	0.7	0.5	13
New iteration	0.7	0.5	6

2. In Example 12, Chang et al. algorithm converges to the initial values x_1 for $x_1 \in [0, \frac{1}{2})$. However, the algorithm converges to the endpoint $\frac{1}{2}$ whenever $x \geq \frac{1}{2}$. Since this can not be said for multivalued nonexpansive mappings in general, the work in [22] is not very useful as it stands for approximating common endpoint.
3. Our algorithm converges faster than Chang et al. algorithm to common endpoint of two multivalued nonexpansive mappings. For example, in Table 4.3, we see that our algorithm approximates the endpoint $\frac{1}{2}$ in 6th iteration while the algorithm of Chang et al. does the same job in 13th iteration.
4. For a multivalued nonexpansive mapping, our algorithm converges faster to an endpoint than an Ishikawa-type iteration process. Hence, our work significantly improves recent results of Panyanak [100].

4.5 Suggestions For Future Work

This thesis opens the door for possible research in the following aspects.

1. Extension of Theorem 2.1.6 in the context of some important subclasses of a metric space as motivated by the work of Alfuraidan [7] and Alfuraidan et al. ([8], [1])
2. Finding applications of α -nonexpansive semigroups to differential equations and minimization problems.
3. Introduction and investigation of fast algorithms that would approximate fixed point of α -nonexpansive mappings as well as solve proximal split minimization problems using recent work of Abbas et al. [2] and Izuchukwu et al. [60] as a starting point.
4. Investigation of sufficient conditions for the existence of common fixed point of non-commutative mappings.
5. Investigation of the endpoint set of multivalued mappings studied in Theorem 4.2.3 and Theorem 4.2.4 by following the ideas of Khan [66].

Bibliography

- [1] M Abbas, M. R. Alfuraidan, T. Nazir, *Common fixed points of multivalued F -contractions on metric spaces with a directed graph*, Carpathian J. Math. , 32(2016), 1-12.
- [2] M. Abbas, M. Alshahrani, Q. H. Ansari, O. S. Iyiola, Y. Shehu, *Iterative methods for solving proximal split minimization problems*, Numer. Algor. , 78(2018), 193-215.
- [3] A. Abkar, M. Eslamain, *Common fixed point results in $CAT(0)$ spaces*, Nonlinear Anal. , 74(2011), 1835-1840.
- [4] A. Abkar, M. Eslamain, *Convergence theorems for a finite family of generalized nonexpansive multivalued mappings in $CAT(0)$ spaces*, Nonlinear Anal. , 75(2012), 1895-1903.
- [5] G. L. Acedo, T. Suzuki, *Browder's convergence for uniformly asymptotically regular nonexpansive semigroups in Hilbert Spaces*, Fixed Point Theory Appl. , 418030(2010), doi:10.1155/2010/418030 .
- [6] A. Aleyner, Y. Censor, *Best approximation to common fixed points of a semigroup of nonexpansive operators*, J. Nonlinear Convex Anal. , 6(2005), 137-151.
- [7] M. R. Alfuraidan, *The contraction principle for multivalued mappings on a modular metric space with a graph*, Can. Math. Bull. , 59(2016), 3-12.
- [8] M. R. Alfuraidan, M. Bachar, M.A. Khamsi , *A graphical version of Reich's fixed point theorem*, J. Nonlinear Sci. Appl. , 9(2016), 3931-3938.

- [9] A. Amini-Harandi, *Endpoints of set-valued contractions in metric spaces*, Nonlinear Anal. , 72(2010), 132-134.
- [10] K. Aoyama, F. Kohsaka, *Fixed point theorem for α -nonexpansive mappings in Banach spaces*, Nonlinear Anal. , 74 (2011), 4387-4391.
- [11] D. Ariza-Ruiz, L. Leustean, G. Lopez, *Firmly nonexpansive mappings in classes of geodesic spaces*, Trans. Am. Math. Soc. 366(2014), 4299-4322.
- [12] D. Ariza-Riuz , C. H. Linares , E. Llorens-Fuster, E. Moreno-G'alvez , *On α -nonexpansive mappings in Banach spaces*, Carpathian J. Math. , 32 (2016), 13-28.
- [13] T. D. Benavides, G. L. Acedo, H. K. Xu, *Construction of sunny nonexpansive retractions in Banach spaces*, Bull. Austral. Math. Soc. , 66(2002), 9-16.
- [14] I. D. Berg and I. G. Nikolaev, *Quasilinearization and curvature of Alexandrov spaces*, Geom. Dedic. , 133(2008), 195-218.
- [15] S. C. Bose, *Common fixed points of mappings in a uniformly convex Banach space*, J. London Math. Soc. , 18(1978), 151-156.
- [16] R. K. Bisht, *A note on the fixed point theorem of Górnicki*. J. Fixed Point Theory Appl. , 21(2019):54 .
- [17] R. K. Bisht, N. K. Singh , *On asymptotic regularity and common fixed points*. Journal of Analysis, <https://doi.org/10.1007/s41478-019-00213-0>.
- [18] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*. Springer Verlag, Berlin, Heidelberg, New York, 1999.
- [19] F. E. Browder, W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*. Bull. Am. Math. Soc. , 72(1966), 571-575.
- [20] D. W. Boyd, J. S. W. Wong, *On nonlinear contractions*. Proc. Am. Math. Soc. , 20(1969), 458-464.

- [21] F. Bruhat, J. Tits, *Groupes reductifs sur un corps local. I. Donnees radicielles values*, Inst. Hautes Etudes Sci. Publ. Math. , 41 (1972), 5-251.
- [22] S. Chang, G. Wang, L. Wang, Y. K. Tang, Z. L. Ma, *Δ -convergence theorems for multivalued nonexpansive mappings in hyperbolic spaces*, Appl. Math. Comput. , 249(2014), 535-540.
- [23] L. Chen, L. Gao, D. Chen, *Fixed point theorems of mean nonexpansive set-valued mappings in Banach spaces*, J. Fixed Point Theory Appl. ,19(2017), 2129-2143.
- [24] Y. J. Cho, L. Ćirić, Sheng-hua Wan , *Convergence theorems for nonexpansive semi-groups in $CAT(0)$ spaces*, Nonlinear Anal. , 74(2011), 6050-6059.
- [25] S. Y. Cho, S.M. Kang, *Approximation of fixed points of pseudo-contraction semi-groups based on a viscosity iterative process*, Appl. Math. Lett. , 24(2011), 224-228.
- [26] L. Ćirić , *On contraction type mappings*. Math. Balk. , 1(1971), 52-57.
- [27] P. L. Combettes, J. C Pesquet, *Proximal splitting methods in signal processing* In: Bauschke, H.H., Burachik, R., Combettes, P.L., Elser, V., Luke, D.R., Wolkowicz, H. (eds.) Fixed-Point Algorithms for Inverse Problems in Science and Engineering (2011), Springer, New York, 185-212 .
- [28] H. W. Corley, *Some hybrid fixed point theorems related to optimization*, J. Math. Anal. Appl. , 120(1986), 528-532.
- [29] G. Crombez, *Image recovery by convex combination of projections*, J. Math. Anal. Appl. , 155(1991), 413-419.
- [30] H. Dehghan, J. Rooin, *A characterization of metric projection in $CAT(0)$ spaces*, Proc. Inter. Con. Func. Equation, Geo. Func. Appl. (2012), 41-43.
- [31] A. Dehici, *Some remarks on generalized Kirk's process in Banach spaces and application*, Turk. J. Math. , 43(2019), 1395-1406, doi:10.3906/mat-1807-166.

- [32] S. Dhompongsa, B. Panyanak, *On $\triangle$ -convergence theorems in $CAT(0)$ spaces*, Comput Math. Appl. , 56(2008), 2572-2579.
- [33] S. Dhompongsa, W. Inhakon , W. Takahashi, *A weak convergence theorem for common fixed points of some generalized nonexpansive mappings and nonspreading mappings in a Hilbert space*, Optimization, 60(2011), 769-779.
- [34] A. Dixit, D. R. Sahu, A. K. Singh, T. Som, *Application of a new accelerated algorithm to regression problems*, Soft Computing, 24(2020), 1539-1552.
- [35] M. Eslamian. *Convergence theorems for nonspreading mappings and nonexpansive multivalued mappings and equilibrium problems*, Optim. Lett. , 7(2013), 547-557.
- [36] H. Fukhar-ud-din, *One step iterative scheme for a pair of nonexpansive mappings in a convex metric space*, Hacettepe Math. Stat. , 5(2015), 1023-1031.
- [37] H. Fukhar-ud-din, *Iterative process for an α -nonexpansive mapping and a mapping satisfying condition(C) in a convex metric space*, Iranian Journal of Mathematical Sciences and Informatic, 1(2019), 167-179.
- [38] H. Fukhar-ud-din, M. A. Khamsi, *Approximating common fixed points in hyperbolic spaces*, Fixed Point Theory Appl. , 113(2014).
- [39] H. Fukhar-ud-din, A. R. Khan, *Approximation of common fixed point of two quasi-nonexpansive mappings in convex metric spaces*, Mediterr. J. Math. , 15(2018):77.
- [40] H. Fukhar-ud-din, A. R. Khan, *Strong and $\triangle$ -convergence of Moudafi's iterative algorithm for generalized nonexpansive mappings in convex metric spaces*, Rendiconti del Circolo Matematico di Palermo Series, 2 <https://doi.org/10.1007/s12215-019-00418-z>.
- [41] H. Fukhar-ud-din, A. R. Khan, M. Ubaid-ur-rahmon, *Ishikawa type algorithm of two multi-valued quasi-nonexpansive maps on nonlinear domains*, Ann. Funct. Anal. , 2(2013), 97-109.

- [42] H. Fukhar-ud-din, K. Saleh, *One-Step iterations for a finite family of generalized nonexpansive mappings in $CAT(0)$ spaces*, Bull. Malays. Math. Sci. Soc. , 41(2018), 597-608.
- [43] J. Garcia-Falset, E. Llorens-Fuster, E. Moreno-Galvez, *Fixed point theory for multivalued generalized nonexpansive mappings*, Appl. Anal. Discrete Math. , 6(2012), 265-286.
- [44] J. Garcia-Falset, E. Llorens-Fuster, T. Suzuki, *Fixed point theory for a class of generalized nonexpansive mappings*, J. Math. Anal. Appl. , 375(2011), 185-195.
- [45] M. A. Geraghty, *On contractive mappings*. Proc. Am. Math. Soc. , 40(1973), 604-608.
- [46] S. H. Ghoncheh, A. Razani, *Fixed point theorems for some generalized nonexpansive mappings in Ptolemy spaces* , Fixed Point Theory Appl., 76(2014).
- [47] P. Glubudom, *Weak and strong convergence theorems for approximating common fixed points of three nonexpansive mappings*, Thai J. Math. , 8(2010), 447-457.
- [48] K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.
- [49] K. Goebel, S. Reich, *Uniform convexity, Hyperbolic geometry, and Nonexpansive mappings*, Series of Monographs and Textbooks in Pure and Applied Mathematics, 83(1984), Dekker, New York.
- [50] J. Górnicki , *Remarks on asymptotic regularity and fixed points*. J. Fixed Point Theory Appl. , 21(2019):29 .
- [51] J. Górnicki, *On some mappings with a unique fixed point*. J. Fixed Point Theory Appl. , 22(2020):8.
- [52] J. P. Gossez, E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. , 40(1972), 565-573.

- [53] Z. H. Gu, *Ishikawa iterative for mean nonexpansive mappings in uniformly convex Banach spaces*, J. Guangzhou Econ. Manag. Coll., 8(2006), 86-88.
- [54] Z. Gu, Y. Li, *Approximation methods for common fixed points of mean non-expansive mapping in Banach Spaces*, Fixed Point Theory Appl., 471532(2008), <https://doi.org/10.1155/2008/471532>.
- [55] H. R. Hajisharifi, *On some generalization of multivalued nonexpansive mappings*, Bull. Iran. Math. Soc. ,46(2020), 557-571 , <https://doi.org/10.1007/s41980-019-00275-7>.
- [56] E. T. Hale, W. Yin, Y. Zhang, *A fixed-point continuation method for l_1 regularized minimization with applications to compressed sensing*. Tech. Rep., CAAM TR0707 (2007).
- [57] M. Hosseini, K. Nourouzi, D. O'Regan, *Stationary points of set-valued contractive and nonexpansive mappings on ultrametric spaces*, Fixed Point Theory, 19(2018), 587-594.
- [58] S. Hu, N. S. Papageorgious, *Handbook of multivalued analysis vol. I: Theory*, Kluwer, Dordrecht, The Netherlands, 1997.
- [59] H. Iqbal, M. Abbas, S. M. Husnine, *Existence and approximation of fixed points of multivalued generalized α -nonexpansive mappings in Banach spaces*, Numer. Algor. <https://doi.org/10.1007/s11075-019-00854-z>.
- [60] C. Izuchukwu, G. C. Ugwumnadi, O. T. Mewomo, A. R. Khan, M. Abbas, *Proximal-type algorithms for split minimization problem in p -uniformly convex metric spaces*, Numer. Algor., 82(2019), 909-935.
- [61] J. Jachymski, *A stationary point theorem characterizing metric completeness*, Appl. Math. Lett. , 24(2011) , 169-171.

- [62] J. H. Jo, *Some generalizations of fixed point theorems and common fixed point theorems*, J. Fixed Point Theory Appl. , 20(2018):144.
- [63] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math Helv , 70(1995), 659-673.
- [64] A. Kakavandi, *Weak topologies in complete $CAT(0)$ metric spaces*, Proc. Am. Math. Soc. , 141(2013), 1029-1039.
- [65] R. Kannan, *Some results on fixed points*, Bull. Calc. Math. Soc. , 60(1968), 71-76.
- [66] A. R. Khan, *Properties of fixed point set of a multivalued map*, J. Appl. Math. Stoch. Anal. , 3(2005), 323-332.
- [67] A. R. Khan, A. A. Domlo, H. Fukhar-ud-din, *Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach space*, J. Math. Anal. Appl. 341(2008), 1-11.
- [68] A. R. Khan, M. A. Khamsi, H. Fukhar-ud-din, *Strong convergence of a general iteration scheme in $CAT(0)$ spaces*, Nonlinear Anal., 74(2011), 783-791.
- [69] A. R. Khan, H. Fukhar-ud-din, M. A. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces*, Fixed Point Theory Appl., 54(2012).
- [70] A. R. Khan, H. Fukhar-ud-din, A. Kalsoom, B. S Lee, *Convergence of a general algorithm of asymptotically nonexpansive maps in uniformly convex hyperbolic space*, Appl. Math. Comput. , 238(2014), 547-556.
- [71] A. R. Khan, S. A. Shukri, *Best proximity points in the Hilbert ball*, J. Nonlinear Convex Analysis, 17(2016), 1083-1094.
- [72] F. Khojasteh, V. Rakocević, *Some new common fixed point results for generalized contractive multi-valued non-self-mappings*, Appl. Math. Lett. , 25(2012), 287-293.

- [73] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. ,68(2008), 3689-3696.
- [74] W. A. Kirk, *On successive approximations for nonexpansive mappings*, Glasgow Math. J. , 2(1971), 6-9.
- [75] W. A. Kirk, S. Massa, *Remarks on asymptotic and Chebyshev centers*, Houst. J. Math. , 16(1990), 375-364.
- [76] J. K. Kim, R. P. Pathak, S. Dashputre, S. D. Diwan, R. Gupta, *Convergence theorems for generalized nonexpansive multivalued mappings in hyperbolic spaces*, Springerplus 5(2016) :912.
- [77] T. Kuczumow, *An almost convergence and its applications*, Ann. Univ. Mariae Curie-Sklodowska Sect. , 32(1978), 79-88.
- [78] P. K. F. Kuhfittig, *Common fixed points of nonexpansive mappings by iteration*, Pacific J. Math. , 97(1981), 137-139.
- [79] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Am. Math. Soc. , 357 (2005), 89-128.
- [80] A. Kudtha, B. Panyanak, *Common endpoints for Suzuki mappings in uniformly convex hyperbolic spaces*, Thai J. Math. , Special issue(2018), 159-168.
- [81] T. Laokul, B. Panyanak , *Common endpoints for non-commutative Suzuki mappings*, Thai J. Math. , 17(2019), 821-828.
- [82] L. Leustean, *Nonexpansive iteration in uniformly convex W-hyperbolic spaces*, In: A. Leizarowitz, B. S. Mordukhovich, I. Shafrir, A. Zaslarski, Nonlinear Analysis and Optimization I : Nonlinear Analysis, Contemporary Mathematics, Am. Math. Soc. , 513(2010), 193-209.
- [83] T. C. Lim, *A fixed point theorem for multivalued mappings in a uniformly convex Banach space*, Bull. Am. Math. Soc. , 80(1974), 1123-1126.

- [84] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Am. Math. Soc. , 60(1976), 179-182.
- [85] L. J. Lin, W. S. Du, *From an abstract maximal element principle to optimization problems, stationary point theorems and common fixed point theorems*, J. Glob. Optim. , 46(2010), 261-271.
- [86] P. E. Mainge, *Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization*, Set-Valued Anal. , 16(2008), 899-912.
- [87] C. Mongkolkeha, Y. J. Cho, P. Kumam, *Weak convergence theorems of iterative sequence in Hilbert space*, J. Nonlinear Convex Anal. , 15(2014), 1303-1317.
- [88] M. Moosaei, *Fixed points and common fixed points for fundamentally nonexpansive mappings on Banach Space*, Journal of Hyperstructures , 4(2015), 50-56.
- [89] M. Moosaei, *On fixed points of fundamentally nonexpansive mappings in Banach spaces*, Int. J. Nonlinear Anal. Appl. , 7(2016), 219-224
- [90] S. Moradi, F. Khojasteh, *Endpoints of multi-valued generalized weak contraction mappings*, Nonlinear Anal. , 74(2011), 2170-2174.
- [91] K. Muangchoo-in, P. Kumam, Y. J. Cho, *Approximating common fixed points of two α -nonexpansive mappings*, Thai J. Math. , 16(2018), 139-145.
- [92] E. Naraghirad, N. C. Wong, J. C. Yao, *Approximating fixed points of α -nonexpansive mappings in uniformly convex Banach spaces and $CAT(0)$ spaces*, Fixed Point Theory Appl. , 57(2013).
- [93] E. Naraghirad, *Approximating fixed points of 2-generalized hybrid mappings in Banach spaces and $CAT(0)$ spaces*, Fixed Point Theory, 15(2014), 501-528.
- [94] A. Nicolae, *Asymptotic behaviour of averaged and firmly nonexpansive mappings in geodesic space*, Nonlinear Anal. , 87(2013), 102-115.

- [95] M. Norouzian, A. Abkar, *Tripartite coincidence-best proximity points and convexity in generalized metric spaces*, Bull. Braz. Math. Soc. , New Series. <https://doi.org/10.1007/s00574-019-00137-0>
- [96] J. G. O'Hara , P. Pillay, H. K. Xu, *Iterative approaches to convex feasibility problems in Banach spaces*, Nonlinear Anal. , 64(2006), 2022-2042.
- [97] A. Pant, R. P. Pant, *Fixed points and continuity of contractive maps* , Filomat ,31(2017), 3501-3506.
- [98] R. Pant, R. Shukla, *Approximating fixed points of generalized α -nonexpansive mapping in Banach space*, Numer. Funct. Anal. Optim. , 38(2017), 248-266.
- [99] B. Panyanak, *Endpoints of multivalued nonexpansive mappings in geodesic spaces*, Fixed Point Theory Appl. , 147(2015), 1-11.
- [100] B. Panyanak, *Approximating endpoints of multivalued nonexpansive mappings in Banach spaces*, J. Fixed Point Theory Appl. , 20(2018):77.
- [101] H. Piri, B. Daraby, S. Rahrovi, M. Ghasemi, *Approximating fixed points of generalized α -nonexpansive mappings in Banach spaces by new faster iteration process*, Numer. Algor. , 81(2018), 1120-1148.
- [102] S. Ranjbar, H. Khatibzadeh, *Δ -convergence and w -convergence of the modified Mann iteration for a family of asymptotically nonexpansive type mappings in complete $CAT(0)$ spaces*, Fixed Point Theory, 17(2016), 151-158.
- [103] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. , 67(1979), 274-276.
- [104] S. Reich, *Some remarks concerning contraction mappings*. Can. Math. Bull. , 14(1971), 121-124.
- [105] S. Reich, I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal., 15(1990), 537-558.

- [106] B. E. Rhoades, *Finding common fixed points of nonexpansive mappings by iteration*, Bull. Austral. Math. Soc. , 43(1991), 153-159.
- [107] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. , 14(1976), 877-898.
- [108] R. Sadhu, C. Nahak, *Generalized α -nonexpansive multivalued mappings in $CAT(0)$ space*, J. Adv. Math. Stud. , 3(2019), 268-283.
- [109] S. Saejung, *Remarks on endpoints of multivalued mappings on geodesic spaces*, Fixed Point Theory Appl. , 52(2016).
- [110] H. Salahifard, S. M. Vaezpour, S. Dhompongsa, *Fixed point theorems for some generalized nonexpansive mappings in $CAT(0)$ spaces*, J. Nonlinear Anal. Optim. , 4 (2013), 241-248.
- [111] H. F. Senter, W. G. Dotson, *Approximating fixed points of nonexpansive mappings*, Proc. Am. Math. Soc. , 44(1974), 375-380.
- [112] T. Shimizu, W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*. Topol. Methods Nonlinear Anal. , 8 (1996), 197-203.
- [113] Y. Song, K. Muangchoo-in, P. Kunam, Y. J. Cho, *Successive approximations for common fixed points of a family of α -nonexpansive mappings*, J. Fixed Point Theory Appl. , 20(2018):10.
- [114] C. Suanoom, C. Klin-eam, *Remark on fundamentally nonexpansive mappings in hyperbolic spaces*, J. Nonlinear Sci. Appl. , 9(2016), 1952-1956.
- [115] W. Takahashi, *A convexity in metric spaces and nonexpansive mappings*, Kodai Math. Sem. Rep. , 22 (1970), 142-149.
- [116] W. Takahashi, T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Anal. , 5(1998), 45-56.

- [117] D. Van Dulst, *Equivalent norms and the fixed point theory for nonexpansive mappings*, Nonlinear Anal. , 65(2006), 762-772.
- [118] K. Włodarczyk, D. Klim, R. Plebaniak, *Existence and uniqueness of endpoints of closed set-valued asymptotic contractions in metric spaces*, J. Math. Anal. Appl. , 328(2007), 46-57.
- [119] C. Wu, L. Zhang, *Fixed points for mean nonexpansive mappings*, Acta Mathematicae Applicatae Sinica, English Series 23, 489(2007), <https://doi.org/10.1007/s10255-007-0388-x>.
- [120] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. , 16(1991), 1127-1138.
- [121] H. K. Xu, *Approximations to fixed points of contraction semigroups in Hilbert spaces*, Numer. Funct. Anal. Optim. , 19(1998), 157-163.
- [122] H. K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl. , 116(2003), 659-678.
- [123] H. Zegeye, N. Shahzad, O. A. Daman, *Strong convergence theorems for a semigroup of asymptotically nonexpansive mappings*, Math. Comput. Model. , 54(2011), 2077-2086.
- [124] S. S. Zhang, *About fixed point theory for mean nonexpansive mapping in Banach spaces*, Journal of Sichuan University, 2(1975), 67-78.
- [125] J. Zhou, Y. Cui, *Fixed point theorems for mean nonexpansive mappings in $CAT(0)$ spaces*, Numer. Funct. Anal. Optim. , 9(2015), 1224-1238.

OYETUNBI DOLAPO MUHAMMED

PERSONAL INFORMATION

Surname: Oyetunbi
Given name: Dolapo Muhammed
Nationality: Nigerian
Permanent Address: No 17, Lane 5 Temidire Oganke Ibadan, Nigeria
Telephone: 08166091107
Email: muhammed.oyetunbi@gmail.com

RESEARCH INTERESTS

My research interests lie in Functional Analysis, and Fixed point theory.

EDUCATION

January 2018 – May 2020	M. Sc in Mathematics King Fahd University of Petroleum and Minerals
January 2011- February 2015	B. Sc (Honours) First class in Mathematics University of Ibadan, Oyo state, Nigeria
September 2004- June 2010	Secondary School Certificate Examination SSCE Periscope College

PUBLICATIONS

- (1) A. R. Khan, H. Fukhar-ud-din, **D. M Oyetunbi**, Kirk type Iteration Process and Convex Feasibility Problem in $CAT(0)$ spaces, Submitted 2019.
- (2) A. R. Khan, H. Fukhar-ud-din, **D. M Oyetunbi**, An algorithm of three nonexpansive mappings with application to common minimization problem, Submitted 2019.
- (3) **D. M. Oyetunbi**, A. R. Khan, Strong convergence of Halpern's type iteration for alpha nonexpansive semigroup in Banach spaces and $CAT(0)$ spaces, Submitted 2020.
- (4) A. R. Khan, **D. M. Oyetunbi**, On some mappings with a unique common fixed point, J. Fixed Point Theory Appl. , Accepted.
- (5) **D. M. Oyetunbi**, A. R. Khan , Approximating common endpoints of multivalued generalized nonexpansive mappings in hyperbolic space, Submitted 2020.

WORK EXPERIENCE

September 2018- May 2020	Course Grader Department of Mathematics, King fahd University of Petroleum and Minerals, Saudi Arabia
	✓ Grading MAT 208: Introduction to Differential Equations and Linear Algebra ✓ Graded MATH 210: Introduction to Sets and Structures ✓ Graded MATH 225: Linear algebra ✓ Graded MATH 435: Ordinary differential Equation

October 2015- October 2016

Graduate Teaching Assistant (Under the National Youth Service Scheme NYSC)

Department of Mathematical Science, Taraba State University

- ✓ Explained lecture materials to freshmen during and after classes
- ✓ Helped students in understanding and solving homework problems

AWARDS AND DISTINCTION

January 2018- Till date	King Fahd University of Petroleum and Minerals Scholarship
March 2018	Participant, Mathematics in Five Minutes (M5M) competition, Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Saudi Arabia
2015	Second-Best Graduating Mathematics Student
2015	Secured third position in B.sc in the Faculty of Science, University of Ibadan
2012-2014	Three times winner of faculty of science award of excellence
2014	Abdul Kabir Undergraduate Scholarship
2007-2010	Periscope College Scholarship

UNIVERSITY SERVICE

2014-2015	Academic Coordinator National Association of Mathematical Science Students of Nigeria, University of Ibadan ✓ Organized and supervised tutorial classes ✓ Encouraged and assisted fellow students toward academic excellence
2014	Chairman of Orientation Program Committee Faculty of science, Muslim Students Society of Nigeria, University of Ibadan ✓ Sourced for funds used to organize the orientation program ✓ Organized the orientation program
2013-2015	Class Representative Mathematics department class of 2013/14 ✓ Encouraged fellow colleagues ✓ Fostered good relationship between lecturers and students