SEPARATION AXIOMS IN X-TOP LATTICES AND PSI DECOMPOSITIONS

BY

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To my mother, family and to those who supported me.

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THESIS ABSTRACT

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TITLE OF STUDY: Separation Axioms in X-top Lattices and PSI decompositions

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The Zariski topology plays an important role in Commutative Algebra and Algebraic Geometry. The category of commutative rings has a non-natural correspondence with the category of spectral spaces. We study the natural duality between the category of bounded distributive lattices and the category of spectral spaces. Due to its importance, a generalization of the Zariski topology has been carried out to the so called X-top lattices by Abuhlail and Lomp. We study several separation axioms, the regularity and the normality of X-top lattices. We introduce the so called PSI decompositions of elements in complete lattices which generalize the strongly irreducible decompositions. We study further these decompositions in the special case of the lattice of sub(semi)modules of a left (semi)module over a (semi)ring. Moreover, we investigate their relation with primary decompositions of modules over commutative rings.

ملخص الرسالة

الاسم: عبدالمحسن عبدالوهاب الفرج

عنوان الدراسة: مسلمات الفصل في طوبولوجيا الشبكات التامة والتحلل الى العناصر الشبه غير قابلة للإختزال

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طوبولوجيا زارسكي تلعب دورا هاما في الجبر التبادلي و الهندسة الجبرية. سندرس الثنائية بين فئة الأطياف الرئيسة للحلقات الإبدالية وفئة شبكات التوزيع المحدودة. سندرس بعض الخصائص الطوبولجية لتعميم طوبولوجيا زاريسكي الى إطار أعم و أوسع وهو إطار الشبكات التامة. نقدم ما يسمى بالتحلل الى عناصر شبه غير قابلة للاختزال في الشبكات الكاملة. سندرس هذا التحلل في حالة شبكة أشباه الحلقيات الجزئية لشبه حلقية على شبه حلقة. سندرس أيضا علاقة هذا التحلل مع التحلل البدائي للحلقيات على الحلقات الإبدالية.

NOTATION

$(L, \wedge, \vee)$	a lattice
$(L, \wedge, \vee, 0, 1)$	a bounded lattice
$Sub(M)$	the lattice of submodules of a module M
$\mathbb{N}_0$	$\{0, 1, 2, 3, ...\}$
$B(n, i)$	the Alarcon & Anderson's semiring
$\mathbb{B}$	$\{0, 1\}$
$\mathcal{C}$	category of commutative rings
$\mathcal{S}$	category of spectral spaces
$\mathcal{D}$	category of distributive lattices
$Spec(R)$	the prime spectrum of a ring R
$V(I)$	the set of all prime ideals containing I
$V(a)$	$\{x \in X \mid a \leq x\}$
$V(L)$	$\{V(a) \mid a \in L\}$
$D(a)$	$X \backslash V(a)$
$SI(L)$	the set of all strongly irreducible elements of L
$I(A)$	$\bigwedge_{a \in A} a$

$\sqrt{a}$	$I(V(a))$
$C^X(L)$	the set of all radical elements of L with respect to X
ACC	the ascending chain condition
DCC	the descending chain condition
$Ideal(R)$	the lattice of ideals of a ring R
C_2	a chain of length 2
$\mathcal{T}_n$	a tree of height 1 with n minimal elements
$\mathcal{V}_n$	$\mathcal{T}_n$ in the dual poset
$(N :_R M)$	$\{r \in R \mid rM \subseteq N\}$
PSI	pseudo strongly irreducible
PSH	pseudo strongly hollow

INTRODUCTION

The prime spectrum of a commutative ring has the so called *Zariski topology* [6]. This topology is spectral, i.e. it is T_0, sober, compact, and has a basis of open compact sets closed under finite unions. This topology showed to be extremely important in the Theory of Commutative Rings as well as in Algebraic Geometry.

Hochster [16] showed that the spectra of commutative rings with the Zariski topology characterize spectral spaces. However, from the point of view of category theory, this correspondence is *not* natural. Esteban [11] proved that there is a *natural duality* between the category of bounded distributive lattices and the category of spectral spaces.

Several authors attempted to generalize the Zariski topology to spectra of special classes of submodules of a given module over a ring. Several notions of a prime submodule were introduced (e.g. [20], [23]). However, these did not necessarily yield a Zariski-like topology, mainly because the closed varieties were not necessarily closed under finite unions. Different attempts to find conditions under which such varieties form a topology were made. Among the first were attempts by Smith and McCasland ([19], [20]), where they defined *top modules* over commutative rings as those whose closed varieties are closed under finite unions. Moreover, different authors studied similar topologies on different types of submodules such as coprime, first, and second submodules (e.g. [1]).

Abuhlail and Lomp [3] [4], worked on a more general framework that recovered the Zariski-like topologies on (co)prime submodules as a special case. This led them to introduce the notion of *X-top lattices* where X is a proper subset of a complete lattice L. The generalization to lattices was motivated by the fact that the collection of sub(semi)modules of a (semi)module over a (semi)ring is a complete lattice.

Abuhlail and Hroub [2] studied further the properties of X-top lattices. They generalized many results on different Zariski-like topologies on modules to this general setting. Moreover, they studied the interplay between the algebraic properties of the lattice and the topological properties of these Zariski-like topology on X. Moreover, they studied conditions under which this topology would be spectral. In this thesis we continue the work of Abuhlail and Hroub and study in particular some separation axioms as well as the regularity and normality of such topologies. Moreover, we introduce a decomposition which has the flavor of the primary decomposition of a submodule of a Noetherian module over a commutative ring, which we call a *PSI decomposition*.

This thesis consists of 3 chapters. The first chapter is introductory and contains mainly some basic definitions and preliminaries, as well as results from the literature. Our main contributions are in chapters 2 and 3.

In Chapter 1, we explain why the correspondence between commutative rings and spectral spaces is not natural. Moreover, we provide a *detailed proof* of the natural duality between the category of bounded distributive lattices and the category of spectral spaces. The proof was given by Esteban [11] in a very compact form. In addition, we emphasize the fact that the spectrum of a commutative semiring is spectral, as shown in [22]. Combining several results from the literature, we obtain Theorem 1.32 which provides a complete characterization of spectral spaces that extends Hochster's characterization of such spaces.

In Chapter 2, we study the separation axioms of X-top-lattices. We focus on the quarter separation axioms, namely $T_{1/4}$, $T_{1/2}$, and $T_{3/4}$. The reason is that when the Zariski-like topology is T_1, the space is forced to be zero-dimensional. Moreover, we investigate the regularity and the normality of X-top-lattices. We illustrate our results with several examples and counter examples.

In Chapter 3, we study the so called *pseudo-strongly irreducible elements* (PSI for short) of a complete lattice with an action from a poset with a largest and smallest element. These were introduced in [15] along with the so called *pseudo-strongly hollow elements*, which, in some sense, serve as duals to PSI elements. The interest in studying PSIs stems from the fact that they build a new class larger than that of strongly irreducible elements and different from that of irreducible elements. We investigate the relation between them and other types of elements such as prime, coprime, and second elements. Moreover, we introduce and study *PSI decompositions* of elements, which generalize strongly irreducible decompositions. Lastly, we study PSI submodules and their relations with *primary submodules*. We study also the relation between PSI decompositions and *primary decompositions* of modules over commutative rings.

CHAPTER 1

LATTICE THEORY AND SPECTRAL SPACES

In this chapter, we introduce some basic definitions and results form Lattice Theory [14] and the theory of Semirings and Semimodules [13]. Moreover, we demonstrate the natural duality between the category of bounded distributive lattices and the category of spectral spaces as shown by Esteban [11]. We recover [22, Theorem 3.1], which states that the prime spectrum of a commutative semiring is spectral, and combine it with Esteban's results to obtain a complete characterization of spectral spaces in Theorem 1.32. Note that all rings and semirings in this thesis are with a 1.

1.1 Lattice Theory

In this section we recall some of the basic definitions and results from Lattice Theory that will be needed in this thesis. Any notions or results not stated here can be found in [14].

Definition 1.1. A **lattice** L is a partially ordered set such that the supremum and infimum of any two elements exist. We define two operations $\wedge$ and $\vee$ on L, such that $\forall\ a, b \in L$:

$$a \vee b := \sup\{a, b\} \qquad \text{and} \qquad a \wedge b := \inf\{a, b\}.$$

There are several types of lattices:

Definitions 1.2. A lattice $(L, \wedge, \vee)$ is called

1. **bounded**, if L has a *least element* 0 and a *largest element* 1.

2. **complete**, if *arbitrary* meets and *arbitrary* joins of elements exist in L (in this case, $0 = \bigwedge_{x \in L} x$ and $1 = \bigvee_{x \in L} x$).

3. **distributive**, if $\forall\ x, y, z \in L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{equivalently} \qquad x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z)$$

4. **modular**, if $\forall\ x, y, z \in L$:

$$x \leq z \implies x \vee (y \wedge z) = (x \vee y) \wedge z$$

Remark 1.3. One can easily see that every complete lattice is bounded, and every distributive lattice is modular.

We define now ideals and some special types of elements of lattices and their duals:

1.4. Let $(L, \wedge, \vee)$ be a lattice.

1. A subset $A \subseteq L$ is a **sublattice**, if A is closed under $\wedge$ and $\vee$.

2. A subset $I \subseteq L$ is said to be an **ideal**, if I is a sublattice and for all $x \in I$, $a \in L$, we have $x \wedge a \in I$, equivalently, for all $a, b \in I$, we have

$$a \vee b \in I \text{ and } \{a\}\downarrow := \{x \in L \mid x \leq a\} \subseteq I.$$

Moreover, the ideal generated by any $A \subseteq L$, denoted by $id(A)$, is the smallest ideal containing A. Clearly, for all $a, b \in L$ we have $id(a) = \{a\}\downarrow$, whence

$$id(a) \wedge id(b) = id(a \wedge b) \text{ and } id(a) \vee id(b) = id(a \vee b).$$

3. A proper ideal $I \subsetneq L$ is said to be a **prime ideal**, if for all $a, b \in L$ with $a \wedge b \in I$ we have $a \in I$ or $b \in I$.

We consider next some special types of elements of a lattice, which will be used in the sequel.

Definition 1.5. [4] Let $(L, \wedge, \vee)$ be a lattice. We call $x \in L$:

1. **irreducible**, if for all $a, b \in L$ with $a \wedge b = x$, we have $a = x$ or $b = x$. These are referred to as *meet-irreducible* elements in [14].

2. **strongly irreducible**, if for all $a, b \in L$ with $a \wedge b \leq x$, we have $a \leq x$ or $b \leq x$.

3. **hollow**, if for all $a, b \in L$ with $a \vee b = x$, we have that $a = x$ or $b = x$. These are referred to as *join-irreducible* elements in [14].

4. **strongly hollow**, if for all $a, b \in L$ with $a \vee b \geq x$, we have that $a \geq x$ or $b \geq x$.

Examples 1.6. 1. Let X be any non-empty set and 2^X be its power set. Then $L = (2^X, \cap, \cup, \emptyset, X)$ is a complete distributive lattice. In this case, the irreducible elements are $X \backslash \{a\}$, where $a \in X$, and the hollow elements are the singletons.

2. The lattice of two sided ideals of a (semi)ring R, $L = (ideal(R), \cap, +, 0, R)$, is a bounded lattice.

3. Let R be a ring, M a left (right) R-module, and $Sub(M)$ the collection of R-submodules of M. Then $L = (Sub(M), \cap, +, 0, M)$, is a complete *modular* lattice. Note that this example covers the lattice of left (right) ideals of R.

4. Let S be a semiring, M a left (right) S-semimodule, and $Sub(M)$ the collection of S-subsemimodules of M. Then $L = (Sub(M), \cap, +, 0, M)$ is a complete lattice that is *not* necessarily modular as we will see in this section. Note that this covers the lattice of left (right) ideals of S.

The following result provides a *graphical characterizations* of modular and distributive lattices.

Theorem 1.7. [14, Theorems 101, 102] *A lattice L is*

1. *modular if and only if L does not contain a pentagon (N_5) as a sublattice.*

2. *distributive if and only if L does not contain a pentagon (N_5) or a diamond (M_3) as sublattices.*

Examples 1.8. 1. The **Pentagon** N_5 in Figure 1.1 is the smallest *non-modular lattice*. Observe that $\{0, c\}$ is a prime ideal and $\{0, b\}$ is an ideal which is not prime.

2. The **Diamond** M_3 in Figure 1.1 is the smallest *non-distributive lattice.* Notice that M_3 is modular.

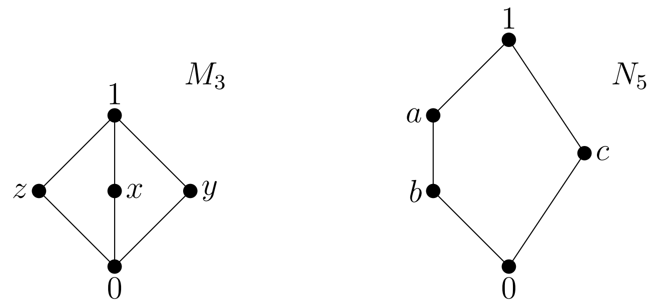


Figure 1.1: The lattices M_3 and N_5

We end the section with a lemma that will be needed in the third section of this chapter.

Lemma 1.9. [14, Lemma 5 and Corollary 116] *Let* $(L, \wedge, \vee)$ *be a lattice,* $H \subseteq L$ *a subset, and* $I \subseteq L$ *an ideal.*

1. $I = id(H)$ *if and only if*

$$I = \{x \mid x \leq h_1 \vee ... \vee h_n \textit{ for some } n \geq 1 \textit{ and } h_1, ..., h_n \in H\}.$$

2. *If* L *is distributive and* $a \in L \backslash I$*, then there exists a* prime *ideal* P *such that* $I \subseteq P \subseteq L \backslash \{a\}$.

1.2 Semirings and semimodules

In this section, we include some of the basic definitions and results from the theory or Semirings and Semimodules. We explain why the lattice of left (right) ideals of a semiring is not *necessarily modular*. Any notions or results not mentioned here can be found in [13].

Definition 1.10. [13] A **semiring** R is a set equipped with two operations, addition "+" and multiplication "·", and satisfies the following conditions:

1. $(R, +)$ is a commutative monoid with identity element 0;
2. $(R, \cdot)$ is a monoid with identity element 1;
3. Multiplication distributes over addition from either side;
4. $0 \cdot r = 0 = r \cdot 0$ for all $r \in R$;
5. $1 \neq 0$.

Examples 1.11. 1. Every ring is a semiring.

2. Every bounded distributive lattice is a commutative semiring.
3. $(\mathbb{N}_0, +, 0, \cdot, 1)$, where $\mathbb{N}_0 := \{0, 1, 2, \cdots\}$, is a commutative semiring.
4. The *Boolean algebra* $\mathbb{B} = \{0, 1\}$, where $1 + 1 = 1$, is a semiring. We call $\mathbb{B}$ the **Boolean semiring**.

Example 1.12. [13, Example 1.8] Consider $B(n,i) := (B(n,i), \oplus, 0, \odot, 1)$, where $B(n,i) = \{0, 1, 2, ..., n-1\}$ and:

- $x \oplus y = x + y$ if $x + y < n$; otherwise, $x \oplus y$ is the unique element c of $B(n,i)$ satisfying $c \equiv x + y \ mod \ (n-i)$;

- $x \odot y = xy$ if $xy < n$; otherwise, $x \odot y$ is the unique element c of $B(n,i)$ satisfying $c \equiv xy \ mod \ (n-i)$.

  Then $B(n,i)$ is a semiring. Observe that $B(2,1) = \mathbb{B}$ and $B(n,0) = \mathbb{Z}_n$.

The lattice of ideals of a semiring is *not* necessarily modular.

Example 1.13. Consider the semiring $\mathbb{N}_0$ and the ideals

$$I_1 = 2\mathbb{N}_0 \backslash \{2\},\ I_2 = 2\mathbb{N}_0,\ I_3 = \mathbb{N}_0 \backslash \{1,2,3\},\ I_4 = \mathbb{N}_0 \backslash \{1,2\} \text{ and } I_5 = \mathbb{N}_0 \backslash \{1\}.$$

Observe that $I_1 \subseteq I_2 \subseteq I_5$, $I_1 \subseteq I_3 \subseteq I_4 \subseteq I_5$, $I_3 \cap I_2 = I_1 = I_4 \cap I_2$, and $I_3 + I_2 = I_5 = I_4 + I_2$. Hence, $\{I_1, I_2, I_3, I_4, I_5\}$ forms a pentagon. By Theorem 1.7 (1), the lattice of ideals of $\mathbb{N}_0$ is not modular.

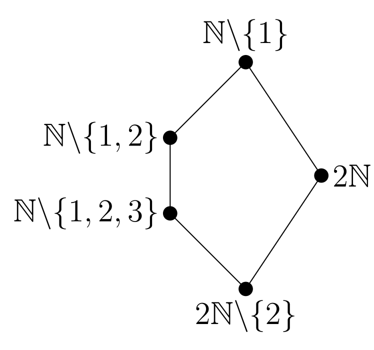


Figure 1.2: A pentagon in the lattice of ideals of $\mathbb{N}$

We define now semimodules over semirings and provide some examples on them.

Definition 1.14. [13] Let S be semiring. A **left S-semimodule** is a commutative monoid $(M, +, 0_M)$ with a map (scalar multiplication)

$$S \times M \to M \ , \ (s, m) \mapsto sm \ ,$$

which satisfies the following conditions for all $m, m_1, m_2 \in M$ and $s, s_1, s_2 \in S$:

1. $(s_1 s_2)m = s_1(s_2 m)$;

2. $(s_1 + s_2)m = s_1 m + s_2 m$;

3. $s(m_1 + m_2) = sm_1 + sm_2$;

4. $1_s m = m$;

5. $s0_M = 0_M = 0_S m$.

If M is a left S-semimodule, and $(N, +, 0_M) \leq (M, +, 0_M)$ is a submonoid such that $sn \in N$ for all $s \in S$ and $n \in N$, then we say that N is an **S-subsemimodule** of M and write $N \leq_S M$.

Example 1.15. The category of commutative monoids is nothing but the category of $\mathbb{N}_0$-semimodules.

Example 1.16. Consider the semiring $M_2(\mathbb{R}^+)$, where $M_2(\mathbb{R}^+)$ is the set of 2×2 matrices over $\mathbb{R}^+ := [0, \infty)$. Then

$$N_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\} \text{ and } N_2 = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}$$

are left $M_2(\mathbb{R}^+)$-semimodules and $\mathbb{R}^+$-semimodules.

Example 1.17. If M is a left R-semimodule and A is a non-empty set, then M^A is a left R-semimodule where $(f+g)(a) = f(a) + g(a)$ and $(rf)(a) = r(f(a))$, for all $f, g \in M^A$, $a \in A$ and $r \in R$.

1.3 Spectral Spaces

Hochster [16] defined the so called *spectral topological spaces*. He showed that the prime spectra of commutative rings characterize spectral spaces in the sense that every spectral space is the prime spectrum of some commutative ring. However, this correspondence had two problems: it is *not* natural in the sense of category theory, and the commutative ring R corresponding to a given spectral space τ is not given explicitly, in general.

Esteban [11] showed that there is a *natural duality* between the category $\mathcal{D}$ of bounded distributive lattices and the category $\mathcal{S}$ of spectral spaces; moreover, the bounded distributive lattice L corresponding to a given spectral space τ was given explicitly.

In this section, we explain why the correspondence between the category $\mathcal{C}$ of commutative rings and $\mathcal{S}$ is not natural. Moreover, we demonstrate a detailed proof of the natural duality between $\mathcal{D}$ and $\mathcal{S}$. We end this section by combining [22, Theorem 3.1] with the main theorems of [16] and [11] to provide a complete characterization of spectral spaces using spectra of commutative (semi)rings. For any notions or results from Category Theory not explained here, we refer to [17].

Definition 1.18. [2] Let X be a topological space.

1. A non-empty subset $A \subseteq X$ is said to be **irreducible** if for any two closed subsets C, D of X, such that $A \subseteq C \cup D$, we have $A \subseteq C$ or $A \subseteq D$.

2. A closed subset $C \subseteq X$ is said to have a **generic point** $x \in X$ if $\overline{\{x\}} = C$.

3. X is called **sober** if every closed irreducible subset of X has a unique generic point.

Definition 1.19. A topological space X is called a spectral space if

1. X is T_0.

2. X is *compact*.

3. X has a *compact open* sets closed under finite intersections.

4. X is *sober*.

We now define what we mean by an anti-equivalence of categories.

Definition 1.20. [17] Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories and $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A **natural isomorphism** $F \overset{\alpha}{\simeq} G$ is a family of morphisms $\alpha = \{\alpha_X : F(X) \longrightarrow G(X)\}_{X \in |C|}$ in $\mathcal{D}$ such that:

1. $F(X) \overset{\alpha_X}{\simeq} G(X)$ for every object X in $\mathcal{C}$.

2. For every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ we have

$$\alpha_Y \circ F(f) = G(f) \circ \alpha_X,$$

i.e. the following diagram commutes

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"'] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

Definition 1.21. [17] Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be **equivalent**, if there exist two *covariant* functors, $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$ along with two natural isomorphisms $\alpha : I_{\mathcal{D}} \longrightarrow FG$ and $\beta : I_{\mathcal{C}} \longrightarrow GF$. If F and G yield an equivalence and are *contravariant* functors, then we have a **dual equivalence** or a **natural duality** of categories. In this case, we say that the categories $\mathcal{C}$ and $\mathcal{D}$ are **dual** (to each other).

1.22. With $\mathcal{C}$ we denote the category whose objects are the commutative rings and whose arrows are maps which respect addition, multiplication, and the identity. With $\mathcal{D}$ we denote the category whose objects are the bounded distributive lattices and whose arrows are maps that respect meets and joins. With $\mathcal{S}$ we denote the category whose objects are the spectral spaces and whose arrows are the continuous maps such that the inverse image of a compact open set is compact (and open).

Remark 1.23. The map $Spec : \mathcal{C} \longrightarrow \mathcal{S}$, that assigns to any commutative ring R the prime spectrum $Spec(R)$ equipped with the Zariski topology, is a contravariant functor. Hochster [16] was able to show that the image of $Spec$ is actually the whole of $\mathcal{S}$. In other words, given any spectral space X, there exists some commutative ring R such that $Spec(R) = X$. However, he showed also that $Spec$ cannot be inverted on a number of subcategories on $\mathcal{S}$ concluding that it cannot be inverted on the whole of $\mathcal{S}$.

In fact, $\mathcal{C}$ and $\mathcal{S}$ are *not* dually equivalent through the functor $Spec$. Suppose there exists a contravariant functor $F : \mathcal{S} \longrightarrow \mathcal{C}$ such that for any commutative ring

R, $(F \circ Spec)(R) \cong R$. Noting that the spectra of all fields are homeomorphic, we would have

$$\mathbb{Z}_3 \cong (F \circ Spec)(\mathbb{Z}_3) \cong (F \circ Spec)(\mathbb{Z}_2) \cong \mathbb{Z}_2,$$

a contradiction.■

Next, we provide a detailed proof of Esteban's result [11] which shows that the category $\mathcal{S}$ of spectral spaces and the category $\mathcal{D}$ of bounded distributive lattices are dual.

We start by proving four lemmas, and consequently the main duality theorem.

Remark 1.24. Let $(L, \wedge, \vee)$ be a bounded distributive lattice. For any ideal $I \leq L$ and any $a \in L$ we define:

$$V(I) = \{P \in Spec(L) \mid I \subseteq P\} \text{ and } V(a) = \{P \in Spec(L) \mid a \in P\}.$$

Clearly, $V(0) = Spec(L)$, $V(L) = \emptyset$, and for any collection of ideals $\{I_j\}_{j \in J}$:

$$V(\bigvee_{j \in J} I_j) = V(id(\bigcup_{j \in J} I_j)) = \bigcap_{j \in J} V(I_j)$$

Also,

$$V(I) \cup V(J) = V(I \cap J) = V(I \wedge J),$$

since $a \wedge b \in P \iff a \in P$ or $b \in P$, for any P prime ideal of L. It follows that $\{V(I)\}_{I \in Ideal(L)}$ forms a basis of closed sets for a topology τ on $Spec(L)$, where $\{D(I) = Spec(L) \backslash V(I)\}_{I \in Ideal(L)}$ are the open sets. Moreover, for any $a \in L$, one can easily see that $V(a) = V(id(a))$.

Lemma 1.25. *Let $L \in \mathcal{D}$ be a bounded distributive lattice. Then,*

$$\{D(a) = Spec(L) \backslash V(a)\}_{a \in L}$$

are the compact open sets of $Spec(L)$.

Proof. Let $\{D(I_k)\}_{k \in K}$, be an open cover for $D(a)$, where $a \in L$.

Then, $D(a) \subseteq \bigcup_{k \in K} D(I_k)$, whence $V(a) \supseteq \bigcap_{k \in K} V(I_k) = V(\bigvee_{k \in K} I_k)$. By Lemma 1.9 (2), if $a \notin \bigvee_{k \in K} I_k$, then $V(a) \not\supseteq \bigcap_{k \in K} V(I_k) = V(\bigvee_{k \in K} I_k)$, a contradiction. By Lemma 1.9 (1), $a \in \bigvee_{c \in C} I_c$ for some finite subset $C \subseteq K$. Hence, $D(a) \subseteq \bigcup_{c \in C} D(I_c)$, i.e. $D(a)$ is compact and open.

Conversely, suppose there exists a non-principal ideal I of L with $D(I)$ compact. Since I is not principal, $\bigvee_{a \in A \subseteq I} id(a) \neq I$ for any finite subset A (recall that $id(a) \vee id(b) = id(a \vee b)$). Now, $I = \bigvee_{a \in I} id(a)$, i.e. $V(I) = V(\bigvee_{a \in I} id(a))$. Therefore, $D(I) = \bigcup_{a \in I} D(id(a))$ and $\{D(id(a))\}_{a \in L}$ has no finite subcover for $D(I)$, contradicting that $D(I)$ is compact.■

Lemma 1.26. *Let $L \in \mathcal{D}$ be a bounded distributive lattice. Then $Spec(L)$ is a spectral space.*

Proof. We showed in Lemma 1.25 that $Spec(L)$ has a basis of compact open sets and $D(1) = V(0) = Spec(L)$ is compact. It remains to show that $Spec(L)$ is T_0 and that every irreducible closed set has a unique generic point, i.e. the space is *sober*.

Claim I: $Spec(L)$ is T_0.

If $P \neq Q \in Spec(L)$, then $\overline{\{P\}} \neq \overline{\{Q\}}$.

Observe that:

$$\begin{aligned}\overline{\{P\}} &= \{K \in Spec(L) \mid \forall a \in L,\ K \in D(a) \Rightarrow P \in D(a)\} \\ &= \{K \in Spec(L) \mid K \supseteq P\} = V(P).\end{aligned}$$

Since $P \neq Q$ we have $\overline{\{P\}} = V(P) \neq V(Q) = \overline{\{Q\}}$. Hence, $Spec(L)$ is T_0.

Claim II: $Spec(L)$ is sober.

Let $V(I)$ be any closed irreducible set for some ideal $I \leq L$. Let $a \wedge b \in I$. Then $V(I) \subseteq V(a \wedge b) = V(a) \cup V(b)$. Thus, $V(I) \subseteq V(a)$ or $V(I) \subseteq V(b)$ as $V(I)$ is irreducible. So, $a \in I$ or $b \in I$, i.e. I is prime. Thus, $V(I) = \overline{\{I\}}$, i.e. $Spec(L)$ is sober.■

Lemma 1.27. *We have a contravariant functor* $Spec : \mathcal{D} \longrightarrow \mathcal{S}$ *is a contravariant functor, where for every arrow* $f : L_1 \longrightarrow L_2$ *in* $\mathcal{D}$*, we have*

$$Spec(f) : Spec(L_2) \longrightarrow Spec(L_1),\ P \mapsto f^{-1}(P).$$

Proof. Lemma 1.26 shows that $Spec(L) \in \mathcal{S}$ for every $L \in \mathcal{D}$. Let $f : L_1 \longrightarrow L_2$ be a morphism of distributive lattices. We prove first that $f^* := Spec(f)$ is a morphism of spectral spaces.

Claim I: f^* respect prime ideals.

Suppose that $a \wedge b \in f^{-1}(P)$ for some $a, b \in L_2$ and $P \in Spec(L_2)$. It follows that $f(a \wedge b) = f(a) \wedge f(b) \in P$. Since P is prime, $f(a) \in P$, whence $a \in f^{-1}(P)$, or $f(b) \in P$ whence $b \in f^{-1}(P)$.

Claim II: f^* is continuous.

Let W be a basic open set in $Spec(L_1)$. By Lemma 1.25, $W = D(a)$ for some

$a \in L$. It follows that

$$\begin{aligned} f^{*-1}(D(a)) &= \{P \in Spec(L_2) \mid f^{-1}(P) \in D(a)\} \\ &= \{P \in Spec(L_2) \mid a \notin f^{-1}(P)\} \\ &= \{P \in Spec(L_2) \mid f(a) \notin P\} \\ &= D(f(a)). \end{aligned}$$

It follows again by Lemma 1.25 that $D(f(a))$ is open (and compact).

Claim III: f^{*-1} of a compact open set is compact open (i.e. f^* is a spectral map).

By Lemma 1.25, the compact open sets are precisely $D(a)$ for some $a \in L$. From claim II we can see that f^{*-1} respects compact open sets.

One can check easily now that $Spec(id_L) = id_{Spec(L)}$ and $Spec(g \circ f) = Spec(f) \circ Spec(g)$ for all $L_1 \xrightarrow{f} L_2 \xrightarrow{g} L_3$ in $\mathcal{D}$. Consequently, $Spec$ is a contravariant functor.$\blacksquare$

Lemma 1.28. *We have a contravariant functors $F : \mathcal{S} \longrightarrow \mathcal{D}$ where*

$$X \mapsto \{C \subseteq X \mid X \backslash C \text{ is open and compact}\}$$

and for every morphism of spectral spaces $f : X \longrightarrow Y$, we have

$$F(f) : F(Y) \longrightarrow F(X),\ C \mapsto f^{-1}(C).$$

Proof. Let X be a spectral space. It is clear that $(F(X), \cup, \cap, X, \emptyset)$ is a bounded distributive lattice with $\cup$ as a meet, $\cap$ as a join, $0 = X$ and $1 = \emptyset$.

Let $f : X \longrightarrow Y$ be a morphism of spectral spaces.

Claim: $F(f) : F(X) \longrightarrow F(Y)$ is a morphism of distributive lattices.

Let $C \in F(Y)$ be such that $U := X \backslash C$ is open compact. Then $f^{-1}(C) = f^{-1}(X \backslash U) = X \backslash f^{-1}(U) \in F(X)$ as f is a spectral map.

Moreover, for all $C, D \subset F(Y)$, we have

$$f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D),\ \ f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D),$$
$$f^{-1}(\emptyset) = \emptyset,\ \text{and}\ f^{-1}(Y) = X, \tag{1.1}$$

i.e. $F(f)$ is morphism in $\mathcal{D}$.

It can be easily checked that $F(id_X) = id_{F(X)}$ and $F(g \circ f) = F(f) \circ F(g)$ for all $X \xrightarrow{f} Y \xrightarrow{g} Z$ in is $\mathcal{S}$. Consequently, F is a contravariant functor.■

Theorem 1.29. *The category $\mathcal{D}$ of bounded distributive lattices is dually equivalent to the category $\mathcal{S}$ of spectral spaces.*

Proof. By Lemma 1.27 and 1.28, $Spec : \mathcal{D} \longrightarrow \mathcal{S}$ and $F : \mathcal{S} \longrightarrow \mathcal{D}$ are contravariant functors. It remains to show that

$$\alpha : I_{\mathcal{D}} \longrightarrow F \circ Spec \text{ and } \beta : I_{\mathcal{S}} \longrightarrow Spec \circ F$$

are natural isomorphisms.

Step I: Given any $L \in \mathcal{D}$, we define

$$\alpha_L : L \longrightarrow (F \circ Spec)(L), \quad a \mapsto V(a).$$

The map is well defined, by Lemma 1.25.

We show that α_L is an isomorphism in $\mathcal{D}$. For all $a, b \in L$, we have:

1. $\alpha_L(0) = V(0) = Spec(L)$ and $\alpha_L(1) = V(1) = \emptyset$

2. $\alpha_L(a \vee b) = V(a \vee b) = V(a) \cap V(b) = \alpha_L(a) \vee \alpha_L(b)$.

3. $\alpha_L(a \wedge b) = V(a \wedge b) = V(a) \cup V(b) = \alpha_L(a) \wedge \alpha_L(b)$.

4. Suppose that $\alpha_L(a) = \alpha_L(b)$, i.e. $V(a) = V(b)$. If $a \neq b$, then either $a \notin id(b)$

or $b \notin id(a)$ and, by Lemma 1.9 (2), we reach a contradiction to $V(a) = V(b)$.

So, α_L is injective.

5. α_L is surjective by Lemma 1.25.

Step II: Let $f : L_1 \longrightarrow L_2$ be a morphism in $\mathcal{D}$. For all $a \in L$, we have

$$\begin{aligned}((F \circ Spec)(f) \circ \alpha_{L_1})(a) &= (F(f^*) \circ \alpha_{L_1})(a) &= {f^*}^{-1}(V(a)) \\ &= V(f(a)) &= \alpha_{L_2}(f(a)).\end{aligned}$$

It follows that

$$(F \circ Spec)(f) \circ \alpha_{L_1} = \alpha_{L_2} \circ f.$$

Step III: For every $X \in \mathcal{S}$, define

$$\beta_X : X \longrightarrow (Spec \circ F)(X), \quad x \mapsto \{C \in F(X) \mid x \in C\}.$$

We show that β_X is a homeomorphism of spectral spaces.

Claim 1: β_X is well defined.

For all $C, D \in \beta_X(x)$ we have $x \in C \cap D$ and $x \in C \cup D$, whence $C \vee D := C \cap D \in \beta_X(x)$ and $C \wedge D := C \cup D \in \beta_X(x)$, i.e. $\beta_X(x)$ is a sublattice. Now, if $C \in \beta_X(x)$ and $D \in F(X)$, then $x \in C \cup D = C \wedge D$, i.e. $C \wedge D \in \beta_X(x)$. Consequently, $\beta_X(x)$ is an ideal. Lastly, if $C, D \in F(X)$ such that $C \cup D = C \wedge D \in \beta_X(x)$, then $x \in C \cup D$ which implies that $x \in C$ or $x \in D$, i.e. $C \in \beta_X(x)$ or $D \in \beta_X(x)$. Thus, $\beta_X(x)$ is a prime ideal of $(F(X), \cup, \cap, X, \emptyset)$.

Claim 2: β_X is injective.

Suppose that $x \neq y$ in X. Then there exists $C \in F(X)$ such that $x \in C$ and $y \notin C$ since X is T_0. Thus, $\beta_X(x) \neq \beta_X(y)$.

Claim 3: β_X is surjective.

Suppose that $P \in Spec(F(X))$ and $\{C_i\}_{i=1}^n \subseteq P$. Then $\bigcap_{i=1}^n C_i \neq \emptyset$; if not, $1 = \emptyset \in P$ implies that $P = F(X)$ contradicting that P is prime. Thus, P is a family of closed sets in X with the finite intersection property and X compact implies that $\bigcap_{C \in P} C = C_0 \neq \emptyset$, for some $C_0 \in X$ (A closed subset of a compact space is compact). Note that $C_0 \in P$ since $C_0 = \bigcap_{C \in P} C = \bigvee_{C \in P} C \in P$. Now P is a prime and is a down set in $(F(X), \cup, \cap, X, \emptyset)$ implies that $P = \{C \in F(X) \mid C_0 \subset C\}$ and C_0 is irreducible, i.e. C_0 has a unique generic point $x \in X$. Hence, $P = \beta_X(x)$.

Claim 4: β_X is continuous.

For any ideal $I \leq F(X)$, we have that

$$\beta_X^{-1}(V(I)) = \{x \mid \beta_X(x) \in V(I)\} = \bigcap_{C \in I} C$$

which is closed.

Claim 5: β_X is closed.

Suppose that $C \subseteq X$ is closed in X, i.e. $C = \bigcap_{k \in K} C_k$ where $C_k \in F(X)$. Let I be the ideal of $F(X)$ generated by $\{C_k\}_{k \in K}$. Then

$$\beta_X(C) = \{C' \in F(X) \mid x \in C', \forall x \in C\} = \{C' \in F(X) \mid C \subseteq C'\} = V(I).$$

Step IV: We want to show that given a morphism $f : X \longrightarrow Y$, then $(Spec \circ F)(f) \circ \beta_X = \beta_Y \circ f$. For all $x \in X$, we have that

$$((Spec \circ F) \circ \beta_X)(x) = ((f^{-1})^* \circ \beta_X)(x) = \{C \in F(Y) \mid f^{-1}(C) \in \beta_X(x)\}$$

$$= \{C \in F(Y) \mid x \in f^{-1}(C)\} = \{C \in F(Y) \mid f(x) \in C\} = \beta_Y(f(x))$$

as desired.

Therefore, $\mathcal{D}$ and $\mathcal{S}$ are dually equivalent.$\blacksquare$

The following is the well known characterization of spectral spaces by Hochster [16].

Theorem 1.30. *A topological space is spectral if and only if its homeomorphic to the spectrum of a commutative ring.*

The following is a theorem proven in [22].

Theorem 1.31. [22, Theorem 3.1] *Let R be a commutative semiring. Then $Spec(R)$ is a spectral space.*

As a consequence of Theorem 1.31 and the fact that every ring is a semiring, one can deduce that a topological space is spectral if and only if it is homeomorphic to the spectrum of a commutative semiring. Thus, we reach the following characterization of spectral spaces.

Theorem 1.32. *Let X be a topological space. The following statements are equivalent:*

1. *X is spectral.*

2. *X is homeomorphic to the spectrum of a bounded distributive lattice.*

3. *X is homeomorphic to the spectrum of a commutative semiring.*

4. *X is homeomorphic to the spectrum of a commutative ring.*

CHAPTER 2

SEPARATION AXIOMS IN X-TOP LATTICES

The prime spectrum of a commutative ring induces the so called Zariski topology [6]. This topology is *spectral*, i.e. it is T_0, sober, compact, and has a basis of open compact sets closed under finite unions. It has shown to be a very important tool in the study of commutative rings as it provides an important bridge between Commutative Algebra and Algebraic Geometry.

Motivated by this, many authors attempted to generalize the Zariski topology to different spectra of modules over rings. The first step to such an attempt was a generalization of the notion of primeness from ideals to submodules. This resulted in different notions of *prime submodules* (e.g. [20], [23]), all of which coincide with prime ideals when the base ring is considered as a module over itself. However, a significant difference was that these spectra do not automatically attain Zariski-like topologies (in general). The problem was mainly that the closed varieties are *not necessarily* closed under finite unions. Different attempts were carried out by several authors to find conditions under which one obtains a Zariski-like topology. Among those were Smith and McCasland (yi[19], [20]) who introduced the notion of *top modules* (modules whose prime spectra induce Zariski-like topologies). Other spectra, e.g. coprime submodules and second submodules were studied, among others, by Abuhlail (e.g. [1]).

Due to similarities in the results on these different spectra, Abuhlail and Lomp [3], [4] provided a more general framework. They introduced the notion of X*-top lattices* (complete lattice L for which a chosen spectrum from 2^X of some proper subset $X \subsetneq L$ induce a Zariski-like topology). Among the many advantages of this approach is the freedom to choose X and the possibility to make use of the duality (as every complete lattice attains a dual complete lattice with the reversed order). Moreover, they gave a complete characterization of such spaces [3, Theorem 2.2].

Abuhlail and Hroub [2] studied further the properties of X-top lattices. They succeeded in proving many results in this general framework that recovered results on special spectra of submodules of a given module over a ring as special cases. Moreover, they studied the interplay between the algebraic properties of the complete lattice and topological properties of the corresponding Zariski-like topology. Furthermore, they studied some of the conditions under which this topology is spectral.

The main goal of this chapter is to provide a focused study on the separation axioms of X-top lattices. From [2], we know that if X is a T_1 space, then X is of dimension 0. Hence, we focus on the quarter separation axioms, i.e. $T_{1/4}$, $T_{1/2}$, $T_{3/4}$ [22]. We provide some diagrammatic characterizations of these separation axioms as well. Moreover, we study the regularity, the complete regularity, the normality, the complete normality, and the perfect normality of X-top lattices which are not T_1.

2.1 Preliminaries

We start by introducing the required notation that will be used throughout this chapter. All lattices in this chapter are assumed to be complete.

2.1. ([3], [2]) Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a complete lattice and $X \subseteq L\backslash\{1\}$. For $a \in L$, we define $V(a)$, the **variety of** a, by $V(a) := \{x \in X \mid a \leq x\}$, and set $D(a) := X\backslash V(a)$. Moreover, we set

$$V(L) := \{V(a) \mid a \in L\}.$$

Note that $V(0) = X$, $V(1) = \emptyset$, and $V(L)$ is closed under arbitrary intersections as $\bigcap_{a\in A}(V(a)) = V(\bigvee_{a\in A} a)$ for any $A \subseteq L$. We say that L is an X**-top lattice** if and only if $V(L)$ is closed under finite unions. Moreover, if $X \subseteq SI(L)$ (i.e. X is a subset of the strongly irreducible elements of L), then L is called a **strongly** X**-top lattice**.

2.2. Let L be a complete lattice and $X \subseteq L\backslash\{1\}$. For any $Y \subseteq X$ and $a \in L$, we set

$$I(Y) := \bigwedge_{y\in Y} y \text{ and } \sqrt{a} := I(V(a)).$$

We say that a is an X**-radical element** if and only if $\sqrt{a} = a$, and define the set of X**-radical elements** of L to be

$$C^X(L) := \{a \in L \mid \sqrt{a} = a\}.$$

We drop X if it is clear from the context. Defining,

$$\tilde{\bigvee} Y := I(V(\bigvee Y)) = \sqrt{\bigvee Y}$$

it turns out that $(C(L), \wedge, \tilde{\vee}, \sqrt{0}, 1)$ is a complete lattice [3, p.3].

Now we state the main characterization of X-top lattices by Abuhlail and Lomp [3].

Theorem 2.3. [3, Theorem 2.2] *Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a complete lattice and $X \subseteq L \backslash \{1\}$. The following are equivalent:*

1. *$\mathcal{L}$ is an X-top lattice.*

2. *$V : (C(L), \wedge, \tilde{\vee}) \longrightarrow (\mathcal{P}(X), \cap, \cup)$ is an anti-homomorphism of lattices.*

3. *every element $x \in X$ is strongly irreducible in $(C(L), \wedge)$*

4. *$(C(L), \wedge, \tilde{\vee})$ is a distributive lattice and every element $x \in X$ is irreducible in $(C(L), \wedge)$.*

We prove now an important consequence of Theorem 2.3, which shows that for any X-top lattice L, and any non empty $Y \subseteq X$, L is a Y-top lattice as well. The importance of this result is that it provides access to a large class of examples.

Corollary 2.4. *Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be an X-top lattice for some $X \subseteq L \backslash \{1\}$. Then L is a Y-top lattice, for any non-empty $Y \subseteq X$. In fact, the corresponding topology on Y is the subspace topology inherited from X.*

Proof. We first show that $C^Y(L) \subseteq C^X(L)$. Let $a \in C^Y(L)$. Then $a = \bigwedge V_Y(a)$. Notice that $V_Y(a) \subseteq V_X(a)$ as $Y \subseteq X$, hence, $\bigwedge V_X(a) \leq \bigwedge V_Y(a) = a$. Since,

$a \leq \bigwedge V_X(a)$, we conclude that $a = \bigwedge V_X(a)$. By Theorem 2.3 (3), every $y \in Y$ is strongly irreducible in $(C^X(L), \wedge)$. As $C^Y(L) \subseteq C^X(L)$ and both share the same meet, we conclude that every $y \in Y$ is strongly irreducible in $(C^Y(L), \wedge)$. Once more, by Theorem 2.3 (3), L is a Y-top lattice. Notice that $V_Y(a) = V_X(a) \cap Y$ for all $a \in L$, i.e. the resulting topology on Y is nothing but the subspace topology inherited from X.■

Examples 2.5. 1. Let R be a (semi)ring and $\mathcal{I} = (Ideal(R), \cap, +, 0, R)$, the lattice of (two-sided) ideals of R. Clearly, $\mathcal{I}$ is $Spec(R)$-top and the topology on $Spec(R)$ is nothing but the usual Zariski topology on the prime spectrum of R. Moreover, for every $Y \subseteq Spec(R)$, it follows by Corollary 2.4 that $\mathcal{I}$ is Y-top. Of special importance is the topology on $Y = Max(R)$, the spectrum of *maximal* ideals of R.

2. Let L be a complete lattice and consider $SI(L)$ the set of strongly irreducible elements of L. For any $Y \subseteq SI(L)$, the lattice L is a Y-top lattice. This is a direct consequence of Theorem 2.3 (3).

3. Let M be a left (semi)module over a (semi)ring R and consider the spectrum $SI(M)$ of strongly irreducible sub(semi)modules of M. Then M is a Y-top (semi)module for every $Y \subseteq SI(M)$.

The following lemma, which combines results from [3] and [2], recalls some properties of the maps I and V defined in 2.1.

Lemma 2.6. *Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a complete lattice and $X \subseteq L\backslash\{1\}$. For all $x, y \in L$ and $A, B \subseteq L$, we have:*

1. *$A \subseteq B \implies I(B) \leq I(A)$.*

2. *$V(x) \subseteq V(y) \iff \sqrt{y} \leq \sqrt{x}$ (whence $V(x) = V(y) \iff \sqrt{y} = \sqrt{x}$).*

3. *$V(x) = V(\sqrt{x})$.*

4. *$\bigcap_{x \in A} V(x) = V(\bigvee_{x \in A} x)$.*

5. *$I \circ V \circ I = I$.*

6. *$V \circ I \circ V = V$.*

7. *$\mathcal{L}$ is an X-top $\iff V(x) \cup V(y) = V(x \wedge y)$ for any $x, y \in C(X)$.*

The following is a combination of [2, Lemma 1.11, Proposition 1.15, and Proposition 1.17].

Lemma 2.7. *Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be an X-top lattice for some $X \subseteq L\backslash\{1\}$.*

1. *X is a T_0 topological space.*

2. *The closure of any $Y \subseteq X$ is given by $\overline{Y} = V(I(Y))$.*

3. *For all $x \in X$, we have $\overline{\{x\}} = V(x)$.*

4. *$V(x)$ is irreducible for all $x \in X$.*

5. *For any closed subset $Y \subseteq X$, we have*

$$Y = \bigcup_{x \in Y} V(x) = V(\bigwedge_{x \in Y} x).$$

6. *Every finite closed irreducible subset of X has a unique generic point (whence sober, if X is finite).*

2.8. Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a complete lattice and $X \subsetneq L$ a proper subset. The set of maximal (resp. minimal) elements of X is denoted by $Max(X)$ (resp. $Min(X)$). Moreover, the set of maximal (resp. minimal) elements of L is $Max(L) := Max(L\backslash\{1\})$ (resp. $Min(L) := Min(L\backslash\{0\})$).

1. We say that X is **atomic**, if for every $x \in X$ there exists $y \in Min(X)$ such that $y \leq x$.

2. We say that X is **coatomic** iff for every $x \in X$ there exists $y \in Max(X)$ such that $x \leq y$.

3. We say that L is an **atomic lattice**, if for every $L\backslash\{0\}$ is atomic (i.e. for every $a \in L\backslash\{0\}$ there exists $m \in Min(L)$ such that $m \leq a$).

4. We say that L is a **coatomic lattice**, if $L\backslash\{1\}$ is coatomic (i.e. for every $a \in L\backslash\{1\}$ there exists $m \in Max(L)$ such that $a \leq m$).

5. L is said to have the **complete max property**, if

$$\bigwedge_{x \in Max(L)\backslash\{y\}} x \nleq y \text{ for every } y \in Max(L).$$

Remarks 2.9. Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$.

1. If X satisfies the descending chain condition (DCC), then X is atomic.

2. If X is atomic and $Min(X)$ is finite, then X is irreducible if and only if $Min(X)$ is a singleton.

3. If $F \subseteq X$ is closed and $0 \in F$, then $F = X$ (observe that $X = V(0) = \overline{\{0\}} \subseteq F$).

4. Suppose that $0 \in X$. For all $a \in L$, if $D(a)$ is non-empty, then $0 \in D(a)$. This is clear as the only $V(a)$ that contains 0 is $V(0)$, in which case $D(0)$ is empty. Moreover, X is irreducible as $Min(X) = \{0\}$, a singleton.

Before we proceed, we introduce some graphical definitions of posets that will be used throughout the rest of the chapter.

2.10. Let $(P, \leq)$ be a partially ordered set (a poset). For *incomparable* $x, y \in S$ (i.e. $x \nleq y$ and $y \nleq x$), we write $x \mid\mid y$.

1. If $(P, \leq)$ is a chain of n elements (abbreviated C_n), then we say that P is a chain of length $n-1$. Figure 2.1 shows what C_2 looks like.

2. A **tree** in P, is a subset $T \subseteq P$ satisfying the following conditions $\forall x, y, z \in T$:

    (a) if $x \mid\mid y \ \in T$, i.e. x and y are incomparable, then $\exists\ z \in T$ such that $z > x$ and $z > y$;

    (b) if $x < y$ and $x < z$, then y and z are comparable.

    If $Min(T)$ is finite, then we say that T is a **tree of finite base**. If T is of height 1 (i.e. the length of longest chain in T is 1) and $|Min(T)| = n$, then we explicitly denote it by $\mathcal{T}_n$. Note that in $\mathcal{T}_n$ there is a unique maximal element and n minimal elements. Observe that C_2 is just $\mathcal{T}_1$. Clearly, if T is a tree of a finite base, then T should be finite. A **forest** is a collection of disjoint trees. Figures 2.1 and 2.2 shows three examples of trees.

3. If $T \subseteq P^o$ is a tree in the dual poset $P^o = (P, \geq^o)$, then we say that T is a V. If $Max(V)$ is finite, then we say that V is of a **finite cover**. If V is of length 1 and $|Max(V)| = n$, then we explicitly denote it by $\mathcal{V}_n$. Note that in $\mathcal{V}_n$ there is a unique minimal element and n maximal elements. Figure 2.2 shows what $\mathcal{V}_4$ looks like.

The graphs in the following figures are not necessarily lattices. The nodes represent the elements, and the lines represent the order. A line connecting a higher node x to a lower node y means that $y \lneqq x$. If two incomparable elements are connected to a smaller or larger element, that does not necessarily mean it is the meet or the join of the two elements (opposed to the graphs of lattices).

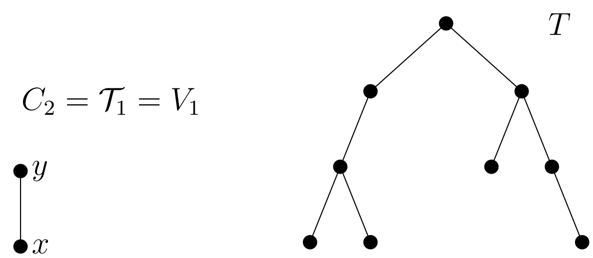


Figure 2.1: C_2 and a tree T with a finite base ($Min(T) = 4$).

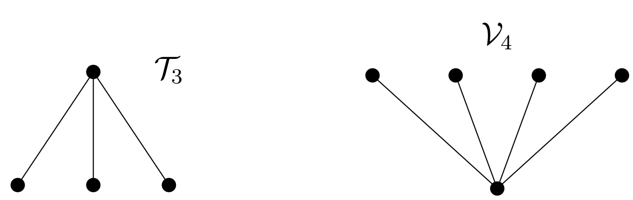


Figure 2.2: $\mathcal{T}_3$ and $\mathcal{V}_4$.

2.2 Separation Axioms

In this section, we study some separation axioms for the X-top lattice. We start by stating a couple of results from [2] that provide conditions equivalent to X being T_1 or discrete.

For the convenience of the reader, we recall some definitions. Note that all lattices in this section are complete.

Definitions 2.11.
1. We say that a topological space X is a T_1 **space**, if whenever $x \neq y$ in X there exist two open neighborhoods U_x of x and U_y of y, such that $x \notin U_y$ and $y \notin U_x$.
2. We say that a topological space X is a T_2 **space** (**Hausdorff**) if whenever $x \neq y$ in X there exist two open neighborhoods U_x of x and U_y of y, such that $U_x \cap U_y = \emptyset$.

Definition 2.12. Let L be an X-top lattice for some $X \subsetneq L$. Then X is a poset and we define the dimension of X, denoted by $dim(X)$, as the supremum of the lengths of all chains in X. This is nothing but the *Krull dimension* of the (semi)ring R if R is a commutative (semi)ring R, L is the lattice of ideals of R and $X = Spec(R)$, the prime spectrum of R.

Proposition 2.13. [2, Proposition 1.23] *Let L be an X-top lattice for some $X \subseteq L \backslash \{1\}$. The following are equivalent:*

1. X is T_1;

2. *$Max(X) = X = Min(X)$;*

3. *$dim(X) = 0$.*

Proposition 2.14. [2, Theorem 1.24] *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. Then $Max(X) = Max(C(L))$. Moreover, the following are equivalent:*

1. *X is T_1 and $C(L)$ satisfies the complete max property.*

2. *X is discrete.*

Proposition 2.15. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. If $C(L)$ satisfies the complete max property, then the following are equivalent:*

1. *$dim(X) = 0$;*

2. *$Max(X) = X = Min(X)$;*

3. *X is T_1;*

4. *X is T_2;*

5. *X is discrete.*

In the following example, we show that the T_1 and T_2 separation axioms do not coincide in arbitrary X-top lattices.

Example 2.16. Let $L := Ideal(\mathbb{Z})$, $X = Spec(\mathbb{Z})$ and $Y = Spec(\mathbb{Z})\backslash\{0\}$. By Corollary 2.4, L is Y-top. Since, $dim(Y) = 0$, we know Y is a T_1 by Proposition 2.13.

Claim: There exist *no* two disjoint open sets of Y.

Suppose not. Then there exist positive integers $m \neq n$, with $D(n\mathbb{Z}) \cap D(m\mathbb{Z}) = \emptyset$. It follows that

$$\begin{aligned} X \backslash V(nm\mathbb{Z}) &= X \backslash (V(n\mathbb{Z} \cap m\mathbb{Z})) = X \backslash (V(n\mathbb{Z}) \cup V(m\mathbb{Z})) \\ &= X \backslash V(n\mathbb{Z}) \cap X \backslash V(m\mathbb{Z}) = \emptyset. \end{aligned}$$

Hence, $V(nm\mathbb{Z}) = X$, i.e. every prime number divides nm which is absurd. Therefore, X is not T_2, as $|X| \geq 2$ and no two points of X can be separated.■

As we have seen, if L is an X-top lattice satisfying T_1 (or any higher separation axiom), then $dim(X) = 0$ making the study of such spaces less interesting. This reason led us to study separation axioms lower than T_1, rather than the higher ones. The findings are interesting and have some convenient graphical characterizations. The lower separation axioms we investigate are $T_{1/4}$, $T_{1/2}$, and $T_{3/4}$. One can easily show that

$$T_1 \implies T_{3/4} \implies T_{1/2} \implies T_{1/4} \implies T_0.$$

The interested reader might find more on these in [22].

Before defining these quarter separation axioms, we prepare the stage by providing some definitions to be used in the sequel.

Definitions 2.17. ([12], [10], and [24]) Let X be a topological space.

1. An element $x \in X$ is called **isolated**, if $\{x\}$ is an open set. The set of isolated points of X is denoted by $Isol(X)$.

2. An element $x \in X$ is called **kerneled**, if the intersection of all neighborhoods of x is equal to $\{x\}$. The set of kerneled points of X is denoted by $Ker(X)$.

We define now what we mean by a $T_{1/4}$ topological space.

Definition 2.18. [22] Let X be a topological space. We say that X is $T_{1/4}$ if each point of X is closed or kerneled.

To characterize $T_{1/4}$ X-top lattice, we start by characterizing the kerneled points of an X-top lattice.

Lemma 2.19. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. Then $x \in X$ is kerneled if and only if x is minimal in X (i.e. $Ker(X) = Min(X)$).*

Proof. Observe that

$$\begin{aligned} \bigcap\{U \mid U \text{ is an open neighborhood of } x\} &= \bigcap\{X\backslash V(y) \mid y > x \text{ or } y \mid\mid x\} \\ &= \{z \in X \mid z \leq x\}. \end{aligned}$$

The last equality follows from the fact that any open set containing x should contain any $z \in X$ such that $z \leq x$. Therefore, x is kerneled if and only if $\{x\} = \{z \in X \mid z \leq x\}$ if and only if x is minimal in X.■

Lemma 2.20. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$ and $x \in X$. Then $\{x\}$ is closed if and only if x is maximal in X.*

Proof. We have by Lemma 2.7,

$$\overline{\{x\}} = V(x) = \{z \in X \mid z \geq x\}.$$

It follows that $\{x\}$ is closed if and only if $\{x\} = \{z \in X \mid z \geq x\}$ if and only if x is maximal in X.■

As a combination of Lemmas 2.19 and 2.20, we obtain the following characterization:

Theorem 2.21. *Let L be an X-top lattice for some $X \subseteq L \backslash \{1\}$. The following are equivalent:*

1. *X is $T_{1/4}$;*
2. *$X = Min(X) \cup Max(X)$;*
3. *$dim(X) \leq 1$.*

We now provide some examples of X-top lattice spaces that are $T_{1/4}$. Moreover, we will show that the $T_{1/4}$ and T_0 do not coincide in X-top lattice.

Example 2.22. As $dim(\mathbb{Z}) = 1$, any $Y \subseteq Spec(\mathbb{Z})$ is $T_{1/4}$. Figure 2.3 shows $Spec(\mathbb{Z})$.

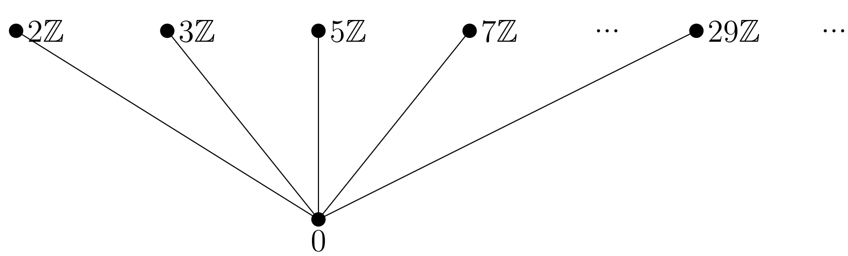


Figure 2.3: $Spec(\mathbb{Z})$, the prime spectrum of $\mathbb{Z}$.

Example 2.23. As $dim(Spec(\mathbb{N}_0)\backslash\{\mathbb{N}_0\backslash\{1\}\}) = 1$, any $Y \subseteq Spec(\mathbb{N}_0)\backslash\{\mathbb{N}_0\backslash\{1\}\}$ is $T_{1/4}$. Figure 2.4 describes $Spec(\mathbb{N}_0)$, the prime spectrum of $\mathbb{N}_0$.

Example 2.24. Any X-top lattice with $dim(X) > 1$ is T_0 but not $T_{1/4}$. In particular, $Spec(\mathbb{N}_0)$ is T_0, but not $T_{1/4}$ as $dim(\mathbb{N}_0) = 2$.

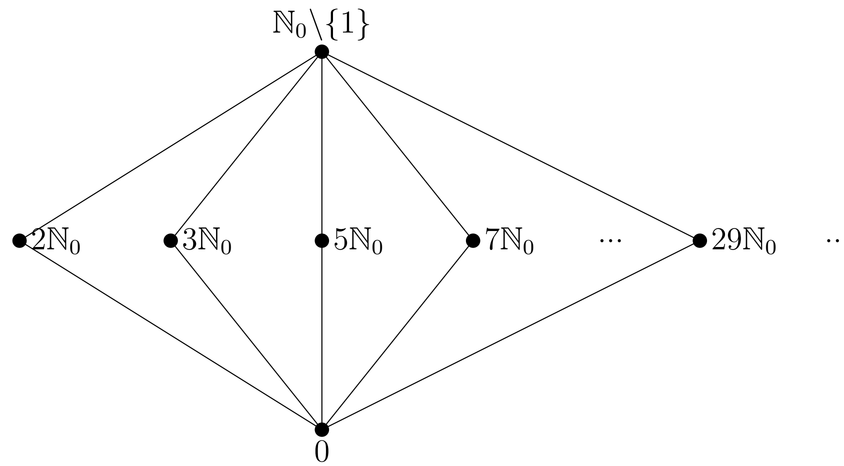


Figure 2.4: The prime spectrum of the semiring $\mathbb{N}_0$ [5].

A special class of elements in a lattice play an important role in the sequel.

Definition 2.25. Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. We call $x \in X$ **absolutely irreducible** in X, if for any $A \subseteq X$ and $\bigwedge_{y\in A} y \leq x$, we have $y \leq x$ for some $y \in A$. The set of absolutely irreducible elements of X is denoted by $AI(X)$.

Proposition 2.26. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. Then $x \in X$ is isolated if and only if x is absolutely irreducible and minimal in X, i.e.*

$$Isol(X) = AI(X) \cap Min(X). \tag{2.1}$$

Proof. $(\subseteq)$ Let $x \in X$ be isolated. Clearly, x is kerneled, whence minimal by Lemma 2.19. Let $\bigwedge_{y\in A} y \leq x$, for some $A \subseteq X$. Suppose that $y \nleq x$ for all $y \in A$. Since x is isolated, $\{x\} = X\backslash V(z)$ for some $z \in L$, whence $y \in V(z)$ for all $y \in A$. It follows that $z \leq \bigwedge_{y \in A} y \leq x$, whence $x \in V(z)$, a contradiction. Therefore, x is absolutely irreducible in X.

($\supseteq$) Assume that x is absolutely irreducible and minimal in X. Setting $z := \bigwedge_{y \in X \setminus \{x\}} y$, we have $z \nleq x$, i.e. $x \in D(z)$. Observe that $y \notin D(z)$ for all $y \in X \setminus \{x\}$ as $y \in V(z)$ by construction. Hence, $\{x\} = D(z)$, i.e. x is isolated.■

Corollary 2.27. *Let L be an X-top lattice for some $X \subseteq L \setminus \{1\}$. Then X is a discrete space if and only if every $x \in X$ is absolutely irreducible and minimal.*

Remark 2.28. Let L be an X-top lattice for some *finite* $X \subseteq L \setminus \{1\}$. Then $x \in X$ is absolutely irreducible in X if and only if x is strongly irreducible in $C(L)$. By Theorem 2.3, every $x \in X$ is absolutely irreducible in X.

We define now the $T_{1/2}$ topological spaces. Thereafter, we characterize the $T_{1/2}$ separation axiom in X-top lattices.

Definition 2.29. [10] We say that a topological space X is $T_{1/2}$, if each point of X is closed or isolated.

Theorem 2.30. *Let L be an X-top lattice for some $X \subseteq \{1\}$. Then X is $T_{1/2}$ if and only if every $x \in X$ is either maximal or both absolutely irreducible and minimal (i.e. $X = Max(X) \cup (AI(X) \cap Min(X))$).*

Proof. The result follows directly from the definitions, Lemma 2.20 and Proposition 2.26.■

Proposition 2.31. *Let L be an X-top lattice for some $X \subseteq L \backslash \{1\}$, and assume that $Min(X) \subseteq AI(X)$. Then X is $T_{1/2}$ if and only if X is $T_{1/4}$. In particular, if X is finite, then X is $T_{1/2}$ if and only if X is $T_{1/4}$.*

Proof. The right implication always holds. Let X be $T_{1/4}$. Then

$$\begin{aligned} X \text{ is } T_{1/4} &\iff X = Max(X) \cup Min(X) && \text{(Theorem 2.21)} \\ &\iff X = Max(X) \cup (AI(X) \cap Min(X)) && \text{(by the assumption)} \\ &\iff X \text{ is } T_{1/2} && \text{(Theorem 2.30)} \end{aligned}$$

If X is finite, then $X = AI(X)$ by Remark 2.28 and the result follows by an argument similar to the one above.■

We now provide graphical classes of examples of X-top lattices for which X is $T_{1/2}$.

Corollary 2.32. *Let L be an X-top lattice for some $X \subseteq L \backslash \{1\}$. If X is a finite combination of disjoint V_ks and $\mathcal{T}_n$s for all $n, k \in \mathbb{N}$, then X is $T_{1/2}$.*

Proof. Note that if X is a finite combination of $\mathcal{T}_n$s and V_ks, then X is finite. As every $\mathcal{T}_n$ and every V_k is of height 1, we have $dim(X) = 1$. Hence X is $T_{1/2}$ by Theorem 2.21 and Proposition 2.30.■

We provide now some examples of $T_{1/2}$ X-top lattices.

Examples 2.33.

1. Any X-top lattice with $X \subseteq L \backslash \{1\}$ *finite* and $dim(X) \leq 1$ is $T_{1/2}$. This follows from Proposition 2.31. In particular if X is C_2, then X is $T_{1/2}$.

2. Let R be a DVR (a discrete valuation domain). Since R is nothing but a principal ideal domain with a unique non-zero prime ideal [6], we have $|Spec(R)| = 2$ and

$dim(R) = 1$. It follows by Corollary 2.32 that $Spec(R)$ is $T_{1/2}$. Examples of DVRs include rings of power series $k[[x]]$ (where k is a field) and the ring of p-*adic* integers (where p is any prime).

In what follows we provide examples of X-top lattices which are $T_{1/4}$ but not $T_{1/2}$.

Examples 2.34. Denote with $\mathbb{P}$ the set of prime (positive) integers.

1. $Spec(\mathbb{Z})$ is $T_{1/4}$ but not $T_{1/2}$. Since $dim(\mathbb{Z}) = 1$, $Spec(\mathbb{Z})$ is $T_{1/4}$ by Theorem 2.21. Although $0 = \bigcap\limits_{p\ \in \mathbb{P}} p\mathbb{Z}$ is minimal in $Spec(\mathbb{Z})$, it is neither absolutely nor maximal. Whence, $Spec(\mathbb{Z})$ is not $T_{1/2}$ by Theorem 2.30.

2. $X = Spec(\mathbb{N}_0)\backslash(\mathbb{N}_0\backslash\{1\})$ is $T_{1/4}$ but not $T_{1/2}$. Since $dim(Spec(\mathbb{N}_0)) = 2$, we have $dim(X) = 1$ and it follows that X is $T_{1/4}$ by Theorem 2.21. However, $0 = \bigcap\limits_{p\ \in \mathbb{P}} p\mathbb{N}_0$ is minimal in X, but neither absolutely irreducible nor maximal. Whence X is not $T_{1/2}$ by Theorem 2.30.$\blacksquare$

We move now to $T_{3/4}$ X-top lattices. We start by defining $T_{3/4}$ topological spaces.

Definition 2.35. An open subset $A \subseteq X$ is called **regular open**, if $int(\overline{A}) = A$, i.e. A is equal to the interior of its closure.

Definition 2.36. [22] We say that a topological space X is $T_{3/4}$, if each point $x \in X$ is closed or $\{x\}$ is regular open.

Definition 2.37. Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. Then $x \in X$ is called **completely isolated** if $\sqrt{\bigwedge D(x)} = \sqrt{\bigwedge_{y \neq x} y}$. The set of completely isolated elements of X will be denoted by $CI(X)$.

The following theorem is a characterization of $T_{3/4}$ X-top lattices.

Theorem 2.38. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. Then*

$$X \text{ is } T_{3/4} \iff X = Max(X) \cup (Min(x) \cap AI(X) \cap CI(X)).$$

Proof. ($\Longrightarrow$) Assume that $x \in Min(x) \cap AI(X)$ is open regular. Suppose that $a = \sqrt{\bigwedge D(x)} \neq \sqrt{\bigwedge_{y \neq x} y} = b$, i.e. x is not completely isolated. Then, $x \in Min(x) \cap AI(X)$ implies that $a \nleq x$, i.e. $x \in D(a)$. Now $D(b) = \{x\}$ (x is open regular) and $a \neq b$ implies that $\exists\ z > x$ such that $z \notin V(a)$ and $z \in V(b)$. Thus, $\{x\} \subsetneq D(a) \subseteq V(x)$, a contradiction to the fact that x is open regular.

($\Longleftarrow$) Clearly X is $T_{1/2}$. Let $x \in Min(x) \cap AI(X) \cap CI(X)$ and suppose that x is not open regular. Then, $U := int(V(x)) \supsetneq \{x\}$ for some open set $U = D(z)$ where $z \in L$. Therefore,

$$D(x) \subseteq U^c = V(z) \subseteq \{y\ |y \neq x\} \implies \bigwedge D(x) \geq \bigwedge V(z) \geq \bigwedge_{y \neq x} x.$$

Thus, $a = \sqrt{\bigwedge V(z)} = \sqrt{\bigwedge_{y \neq x} y} = b$ as $x \in CI(X)$. Hence, $D(a) = D(b) = \{x\}$ and $a = \sqrt{\bigwedge V(z)} = \sqrt{z}$, i.e. $\{x\} = D(a) = D(\sqrt{z}) = D(z) = U$, a contradiction.

Example 2.39. Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$ with $|X| = 2$, i.e. $X = \{x, y\}$ for some $x \neq y$. If x and y are comparable, say $x < y$, then X is C_2 and X is $T_{1/2}$ but not $T_{3/4}$ since $int(V(x)) = int(X) = X \neq \{x\}$.

If $x\ ||\ y$, then X is clearly $T_{3/4}$ (In fact, X is discrete).

We provide now some graphical results about $T_{3/4}$ X-top lattices when X is finite.

Theorem 2.40. *Let L be an X-top lattice for some* finite $X \subseteq L\backslash\{1\}$.

1. *If X is $T_{3/4}$, then $dim(X) \leq 1$ and X does not contain a separate $\mathcal{V}_n$ or a separate C_2.*

2. *If X is forest of a finite number of disjoint $\mathcal{T}_{n_i}$s, where $n_i \geq 2$ for each $i \in \{1, 2, ..., m\}$, then X is $T_{3/4}$.*

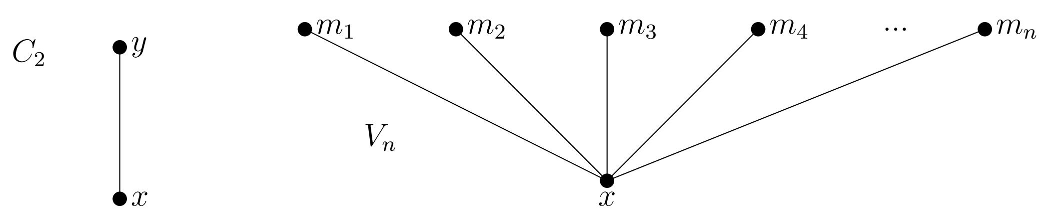
Proof. Let X be finite.

1. Let X be $T_{3/4}$. Then X is $T_{1/4}$, whence $dim(X) \leq 1$ by Theorem 2.21.

    - Assume that X contains a separate chain $C_2 : x < y$. Set $a := \bigwedge_{z \in X\backslash\{x,y\}} z \nleq y$ (and so $a \nleq x$) as X is finite and y is strongly irreducible in $C(L)$ by Theorem 2.3. It follows that $V(x) = \{x, y\}$ is open and

    $$int(V(x)) = D(a) = V(x) \neq \{x\},$$

    i.e. x is minimal but $\{x\}$ is not regular open, contradicting the fact that X is $T_{3/4}$.



    - Suppose that X contains a separate $\mathcal{V}_n$ with minimal element x.

    Set $a := \bigwedge_{w \in X\backslash\mathcal{V}_n}$ and notice that $a \nleq m$ for all $m \in Max(\mathcal{V}_n)$ (and so $a \nleq x$ the minimal element of $\mathcal{V}_n$ as each m is strongly irreducible in

$C(L)$). Therefore, $V(x) = \mathcal{V}_n$ is open and $int(V(x)) = V(x) \neq \{x\}$, i.e. x is minimal but $\{x\}$ is not regular open, contradicting the fact that X is $T_{3/4}$.

2. Suppose that $X = \bigcup_{i=1}^{m} \mathcal{T}_{n_i}$, where each $\mathcal{T}_{n_i}$ has the unique maximal element x_i. For all $i \in \{1, 2, ..., m\}$ and for all $y \in \mathcal{T}_{n_i} \backslash \{x_i\}$, notice that y is minimal, $V(y) = \{y, x_i\}$, and any open set $D(a)$ for some $a \in L$ containing $V(y)$ will contain every element less than or equal to x_i, i.e. $\mathcal{T}_{n_i} \subseteq D(a)$. Thus, $V(y)$ is not open as every $n_i \geq 2$ ($V(y) \subsetneq \mathcal{T}_{n_i}$). Moreover, $\{y\} = \bigcap_{z \in X \backslash \{y\}} X \backslash V(z)$ (X is finite, and y strongly irreducible in $C(L) \supseteq X$), which implies that $\{y\}$ is open. Thus, y is minimal and $\{y\}$ is regular open. Moreover, every x_i is maximal, i.e. $\{x_i\}$ is closed. Therefore, X is $T_{3/4}$.■

We provide now some concrete examples of $T_{3/4}$ X-top lattices. Moreover, we show that in X-top lattices the $T_{3/4}$ axiom does not coincide with either $T_{1/2}$ or T_1.

Example 2.41. Let R be a local (semi)ring with $|Spec(R)| = n \geq 3$ and $dim(R) \leq 1$, whence $Spec(R)$ is a $\mathcal{T}_{n-1}$. By Theorem 2.40 (2), $Spec(R)$ is $T_{3/4}$. For example, $R = K[[x, y]]/(xy)$, where $K[[x, y]]$ is the formal power series in x and y. If $(xy) \subseteq P$ for any prime P, then we have $x \in P$ or $y \in P$, i.e. $(x) \subset P$ or $(y) \subseteq P$. Hence, the only prime ideals of R are $\overline{(x)}$, $\overline{(y)}$, and $\overline{(x, y)}$, i.e. $|Spec(R)| = 3$ and $dim(R) = 1$. Therefore, R is local with $\overline{(x, y)}$ maximal and $Spec(R)$ is $T_{3/4}$. Observe that $Spec(R)$ is not T_1 as $dim(R) = 1$. Figure 2.5 shows the spectrum of R.

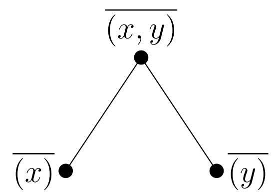


Figure 2.5: $Spec(K[[x,y]]/(xy))$.

Example 2.42. Consider the prime spectrum $Spec(\mathbb{N}_0)$ of the semiring $\mathbb{N}_0$. If $Y = \{\mathbb{N}_0 \backslash \{1\}, p_1\mathbb{N}_0, p_2\mathbb{N}_0, ..., p_n\mathbb{N}_0\}$ where $\{p_1, \cdots, p_n\}$ are prime numbers, then Y is $\mathcal{T}_n$ whence Y is $T_{3/4}$ by Theorem 2.40 (2) (figure 2.6 describes Y). If $W = \{0, p_1\mathbb{N}_0, p_2\mathbb{N}_0, ..., p_n\mathbb{N}_0\}$, then W is a $\mathcal{V}_n$ whence W is not $T_{3/4}$ by Theorem 2.40 (1). Notice that W is $T_{1/2}$ as $dim(Z) = 1$ and Z is finite (see Proposition 2.31).

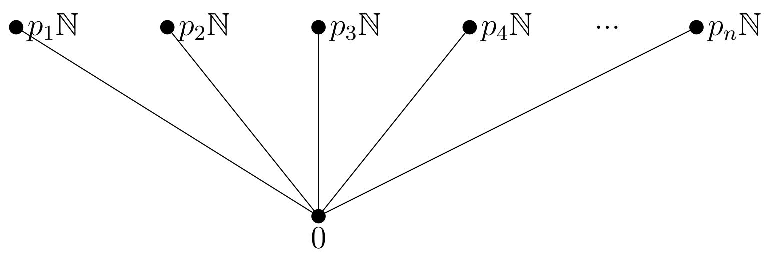


Figure 2.6: The poset W

Example 2.43. Any X-top lattice such that X is a C_2 is $T_{1/2}$ but not $T_{3/4}$. For example, any DVR (discrete valuation ring) R is a $T_{1/2}$ $Spec(R)$-top ring that is not $T_{3/4}$. This is clear by Theorem 2.40 (1).

We recall now a result describing the prime spectrum of the semiring $B(n,i)$ defined in Example 1.12. We use this result to prove several examples and counter examples.

Theorem 2.44. [5, Theorem 24] *Consider the semiring $B(n,i)$.*

1. *$dim(B(n,i)) = 0$ if $n = 2$ and either $i = 0$ or $i = 1$.*

2. *$dim(B(n,i)) = 1$ if $i = 1$ and $n > 2$. In this case the prime ideals of $B(n,i)$ are 0 and $pB(n,i)$ where p is a prime with $p|n-1$.*

3. *$dim(B(n,i)) = 1$ if $n > 2$ and $n = i+1$. In this case the prime ideals of $B(n,i)$ are 0 and $\{0, 2, 3, \cdots, n-1\}$.*

4. *$dim(B(n,i)) = 2$ if $n-1 > i > 2$. In this case the prime ideals are 0, $M = \{0, 2, 3, ..., n-1\}$ and $pB(n,i)$ where p is prime and $p|n-i$.*

Example 2.45. Consider the semiring $B(n,1)$, where $n > 2$, and let $\omega = \omega(n-1)$ denote the number of distinct prime divisors of $n-1$.

1. If $n-1$ is prime, then $Spec(B(n,i))$ is a C_2 by Theorem 2.44 (2), whence $T_{1/2}$ but not $T_{3/4}$ by Example 2.39.

2. If $n-1$ is not prime, then $Spec(B(n,i))$ is a $\mathcal{V}_\omega$ by Theorem 2.44 (2), whence $Spec(B(n,i))$ is $T_{1/2}$ by Corollary 2.32 but not $T_{3/4}$ by Theorem 2.40.■

Example 2.46. Consider the semiring $B(n,i)$, where $i > 1$, $n = i+1$. Then $Spec(B(n,i))$ is C_2, whence $T_{1/2}$ but not $T_{3/4}$ (see Example 2.39).

Example 2.47. Consider the semiring $B(n,i)$, where $n-1 > i > 2$, and let $\omega = \omega(n-i)$ denote the number of distinct prime divisors of $n-i$. By Theorem 2.44 (4),

$dim(B(n,i)) = 2$ whence $Spec(B(n,i))$ is T_0 but not $T_{1/4}$. Moreover, if $n-1$ is not prime, and $Y := Spec(B(n,i))\backslash\{0\}$, then $dim(Y) = 1$ and Y is a $\mathcal{T}_\omega$ whence $T_{3/4}$ by Theorem 2.40 but not T_1 by Proposition 2.13.

2.3 Regularity and Normality

We shift our focus now to studying regularity and normality of X-top lattices. We show that regularity (whence complete regularity) in X-top lattices imply that X is zero-dimensional.

We draw the attention of the reader that when studying the different regularity and normality notions of topological spaces, we consider the versions which drop the T_1 separation axiom (some authors assume that regular and normal spaces are T_1, e.g. Munkres [21]). The reason is that T_1 (whence any higher separation axiom) forces our topological spaces to be zero-dimensional (see Theorem 2.13).

We start with the definitions of these properties.

Definitions 2.48. Let X be a topological space.

1. We say that X is **regular**, if for any closed set $C \subseteq X$ and a point $p \notin C$, we can separate C and p by disjoint open neighborhoods. A T_1 regular space is called T_3.

2. We say that X is **completely regular**, if for any closed set $C \subseteq X$ and a point $p \notin C$, we can find a continuous map

$$f : X \longrightarrow \mathbb{R} \text{ such that } f(p) = 1 \text{ and } f(C) = 0.$$

A T_0 completely regular space is called $T_{3\frac{1}{2}}$ (or a **Tychonoff space**).

3. We say that X is **normal**, if for any two disjoint closed sets $C, D \subsetneq X$, we can separate C and D by two disjoint open sets. A T_1 normal space is called T_4.

4. We say that X is **completely normal**, if every subspace of X with the subspace topology is normal. A T_1 completely normal space is called T_5.

5. We say that X is **perfectly normal**, if every two disjoint closed sets C and D can be precisely separated by a a continuous map

$$f : X \longrightarrow \mathbb{R} \text{ such that } f^{-1}(0) = C \text{ and } f^{-1}(1) = D.$$

   Equivalently, X is perfectly normal if and only if X is normal and every closed set is a G_δ-set (a countable intersection of open sets). A T_1 perfectly normal space is called T_6.

Remark 2.49. [24, p.95, p.99] For any topological space X we have

$$\text{completely regular} \implies \text{regular}, \qquad \text{and}$$

$$\text{perfectly normal} \implies \text{completely normal} \implies \text{normal}.$$

Proposition 2.50. *Let L be a an X-top lattice for some $X \subseteq L \backslash \{1\}$.*

1. *X is completely regular if and only if X is $T_{3_{1/2}}$.*

2. *X is regular if and only if X is T_3.*

Proof. 1. This follows from the fact the X is T_0 (Lemma 2.7 (1)).

2. Every T_3 space is regular. We prove the converse.

   Claim: If X is regular, then $dim(X) = 0$.

   Assume that X is regular. Suppose $dim(X) > 0$, i.e. there exists $x < y$, for some $x, y \in X$. Notice that $x \notin V(y)$ and that x and $V(y)$ cannot be separated by disjoint open sets: every open set in X is of form $D(a)$, for some $a \in L$, and any $D(a)$ that contains $V(y)$ would contain x as well, a contradiction to the regularity of X.

   Since $dim(X) = 0$, it follows by Proposition 2.13 that X is T_1.■

De Marco and Orsatti [9] provided a characterization of commutative rings whose prime spectrum is normal. We restate this result in the context of X-top lattices, where X is spectral, using the fact that a topological space is spectral if and only if it is homeomorphic to the spectrum of a commutative ring. We define first what we mean by X is pm in the context of X-top lattices.

A ring is said to be a **pm-ring**, if every prime ideal of R is contained in a unique maximal ideal [9]. One can generalize this notion easily to **pm-semirings**.

Definition 2.51. Let L is an X-top lattice for some $X \subseteq L\backslash\{1\}$. Then, X is **pm** if and only if every element $x \in X$ is less than or equal to a unique maximal element of X, i.e. $|V(x) \cap Max(X)| = 1$.

We state the interpretation of [9, Theorem 2.1] in the context of X-top lattices.

Theorem 2.52. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$. If X is spectral, then the following are equivalent:*

1. *X is pm.*

2. *$Max(X)$ is a retract of X.*

3. *X is a normal space.*

Corollary 2.53. *Let R be a commutative semiring (e.g. a bounded distributive lattice). The following are equivalent:*

1. *R is pm.*

2. *$Max(R)$ is a retract of $Spec(R)$.*

3. *$Spec(R)$ is a normal space.*

Proof. By Theorem 1.32 $Spec(R)$ is spectral and so the result is a direct consequence of Theorem 2.52.■

Examples 2.54.

1. $\mathbb{N}_0$ is a pm-semiring (note that $\mathbb{N}_0$ is local). By Corollary 2.53, $Spec(\mathbb{N}_0)$ is a normal space.

2. The ring $C(X)$ of continuous real valued functions on a completely regular topological space is a pm-ring [8]. By Corollary 2.53, $Spec(C(X))$ is a normal space.

We prove next a partial generalization of Theorem 2.52 to X-top lattices, where X is not necessarily spectral.

Theorem 2.55. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$.*

1. *If X is coatomic and $Max(X)$ is a retract of X, then X is pm.*

2. *If X is finite, then*

$$Max(X) \text{ is a retract of } X \iff X \text{ is pm.}$$

Proof. Assume X is coatomic.

1. Let $Max(X)$ be a retract of X, where $r : X \longrightarrow Max(X)$ is the retraction map. For any $x \in X$, we have $r(x) = m$ for some $m \in Max(X)$. Hence, $x \in r^{-1}(V(m))$, which is closed as r is continuous. It follows that $\overline{\{x\}} = V(x) \subseteq r^{-1}(V(m))$. Then, for any $m' \in Max(X)$ such that $x \leq m'$, we have $m' \in r^{-1}(V(m))$. Thus, $r(m') \in V(m) = \{m\}$, i.e. $m' = m$. As X is coatomic, every x is less than or equal to at least one maximal element of X. Hence, every x is less than or equal to a unique maximal element of X.

2. Assume that X is finite and pm. The map

$$r : X \longrightarrow Max(X),\, x \mapsto m,$$

where m is the unique maximal element such that $m \geq x$, is well defined. For any $m \in Max(X)$, we have

$$r^{-1}(V(m)) = r^{-1}(m) = \{z \in X \mid z \leq m\} = V(\bigwedge_{z \leq m} z).$$

The last equality holds since X is finite and any $x \in X$ is strongly irreducible in $C(L)$ (Note that $y \in V(\bigwedge_{z \leq m} z)$ if and only if $\bigwedge_{z \leq m} z \leq y$ if and only if $y \leq z \leq m$, for some $z \leq m$). Since any closed set in $Max(X)$ is a finite union of $V(m)$s, r is a continuous map. Clearly, $r(m) = m$ for any $m \in Max(X)$, which implies that $Max(X)$ is a retract of X. The converse holds by (1) (Observe that if X is finite then X is coatomic).■

Theorem 2.56. *Let L be an X-top lattice for some $X \subseteq L \backslash \{1\}$.*

1. *If X is coatomic and normal, then X is pm.*

2. *Let X be atomic and assume $Min(x)$ is finite. Then*

$$X \text{ is coatomic and normal} \iff X \text{ is pm.}$$

Proof. Assume that X is coatomic.

1. Let X be normal. Let $x \in X$, $m \neq m' \in Max(X)$, $x \leq m$, and $x \leq m'$. Then $V(m)$ and $V(m')$ are closed, but cannot be separated by open disjoint sets. The reason is that any open set is of the form $D(a)$, for some $a \in L$, and if it contains $V(m)$ or $V(m')$ then it would contain x as well. So, one cannot find disjoint open sets separating $V(m)$ and $V(m')$; a contradiction to the normality of X. Consequently, $m = m'$ and X is pm.

2. ($\Longleftarrow$) Assume that X is pm, atomic, and $|Min(x)| = n$ for some $n \in \mathbb{N}$. By our assumption, one can see that

$$|Max(X)| \ \leq n = \ |Min(X)|.$$

For each $a_i \in Min(X)$, let $m_i \in Max(X)$ be such that $a_i \leq m_i$. Consider two closed disjoint sets $V(c)$ and $V(d)$. We can find two index sets $K, J \subseteq \{1, 2, ..., n\} = I$ such that $K \cap J = \emptyset$,

$$\{m_k | k \in K\} \subseteq V(c) \text{ and } \{m_j | j \in J\} \subseteq V(d).$$

Set $U := \bigcap_{i \in I \setminus K} D(a_i)$ and $V := \bigcap_{i \in I \setminus J} D(a_i)$. By construction, $V(c) \subseteq U$, $V(d) \subseteq V$, $U \cap V = \emptyset$ and U, V are open sets. Hence, X is normal.■

The following example shows that the converse of (1) in Theorem 2.56 is not true in general: if L is an X-top lattice and X is coatomic, then being pm does *not necessarily* imply that X is normal. Moreover, it shows that a subspace of a spectral space is not necessarily spectral.

Example 2.57. Let $X = Spec(\mathbb{Z}) \setminus \{0\}$. We saw in Example 2.16 that there exist no two disjoint open sets of X. Hence, we cannot separate the closed sets $V(2\mathbb{Z}) = \{2\mathbb{Z}\}$ and $V(3\mathbb{Z}) = \{3\mathbb{Z}\}$, i.e. X is *not* normal. However, X is clearly pm as $dim(X) = 0$ (every element is maximal). By Theorem 2.52, X is not spectral (showing that the subspace of a spectral space is *not necessarily* spectral).

The following example shows that the atomic condition in (2) of Theorem 2.56 is required.

Example 2.58. Consider the ring $\mathbb{Z}$ and let $\mathbb{P}$ be the set of prime (positive) numbers. Then, the set of strongly irreducible ideals of $\mathbb{Z}$ is

$$SI(\mathbb{Z}) = \{p^n\mathbb{Z} \mid p \in \mathbb{P} \text{ and } n \geq 1\} \cup \{0\}.$$

By Example 2.5, $\mathbb{Z}$ is a Y-top lattice for any $Y \subseteq SI(\mathbb{Z})$. In particular, we choose $Y = \{2^n\mathbb{Z} \mid n \geq 1\} \cup \{3\mathbb{Z}\}$. Then, Y is pm, $|Min(Y)| = 1$, and Y is not atomic (since $2\mathbb{Z}$ does not contain any minimal element of Y). Moreover, Y is not normal since $V(2\mathbb{Z}) = \{2\mathbb{Z}\}$ and $V(3\mathbb{Z}) = \{3\mathbb{Z}\}$ cannot be separated by disjoint open sets (any open set different from Y contains $2^m\mathbb{Z}$, for some positive integer m).

The following is a characterization of normal X-top lattices in a special case.

Proposition 2.59. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$ and assume that X is coatomic. If $\bigwedge_{x\in X} x \in X$, then*

$$X \text{ is normal} \iff |Max(x)| = 1.$$

Proof. ($\implies$) Assume that X is normal and $\bigwedge_{x\in X} x \in X$. By Theorem 2.56 (1), X is pm, which implies that $\bigwedge_{x\in X} x \leq m$ for some unique $m \in Max(X)$. Therefore, $|Max(x)| = 1$ since $\bigwedge_{x\in X} x$ is the smallest element in X.

($\impliedby$) Let $Max(X) = \{m\}$. Then $m \in V(a)$, for all $a \in L$, since X is coatomic. Thus, X is normal as no two closed sets of X are disjoint.■

Example 2.60. Let R be an entire commutative semiring (i.e. an integral domain). Then $0 \in Spec(R)$. By Proposition 2.59, $Spec(R)$ is normal if and only if R is local.

We provide now graphical conditions, one sufficient and another necessary, for X to be completely normal when L is an X-top lattice.

Theorem 2.61. *Let L be an X-top lattice for some $X \subseteq L\backslash\{1\}$.*

1. *If X is forest of a finite number of disjoint trees with a finite base, then X is completely normal.*

2. *If X is completely normal, then X does not contain any V with finite cover and $Max(V) \geq 2$.*

Proof. 1. Let X be a forest with the trees T_{n_i} of base n_i for $i \in \{1, ..., l\} = I$ and let m_i to be the *unique* maximal element of T_{n_i}.

Step 1: X is normal.

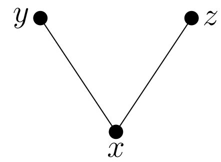
Let $V(a)$ and $V(b)$ be two disjoint closed sets of X. We can find two non-empty index sets $K, J \subseteq I$ such that $K \cap J = \emptyset$, $V(a) \subseteq \bigcup_{k\in K} T_{n_k}$, and $V(b) \subseteq \bigcup_{j\in J} T_{n_j}$. If not, then there exists some $i \in I$ such that T_{n_i} intersects both $V(a)$ and $V(b)$. Then $m_i \in V(a) \cap V(b)$, contradicting the fact that $V(a)$ and $V(b)$ are disjoint. Now, for all $i \in I$, $T_{n_i} = V(\bigwedge T_{n_i})$ as every x is strongly irreducible in $X \subseteq C(L)$ (Theorem 2.3 (3)). Thus, $U = \bigcap_{i\in I\setminus K} D(\bigwedge T_{n_i})$ and $V = \bigcap_{i\in I\setminus J} D(\bigwedge T_{n_i})$ are two disjoint open sets. It follows, by construction, that $V(a) \subseteq U$ and $V(b) \subseteq V$. Consequently, X is normal.

Step 2: Every subspace of X is normal.

Removing any non-maximal element of any T_{n_i}, one can easily see that the conditions of the tree are still preserved, which means that we produced a new tree of a finite base. On the other hand, removing the maximal element m_i from T_{n_i}, produces two new trees, each of which is of finite base.

Therefore, if we remove any number of elements from X, we will still have a forest of finite trees each of which has a finite base. By the first step, one can deduce that any subset of X is normal as desired.

2. Let X be completely normal. Suppose that X contains a V of finite cover and $Max(V) \geq 2$. Let x be the unique minimal element of V. Then, there exists $y \neq z$ in $Max(V) \subseteq X$ such that $x < y$ and $x < z$.



Then, $Y := \{x, y, z\}$ is atomic, not pm and $|Min(Y)| = 1$. By, Proposition 2.56 (2), Y is not normal, a contradiction to the fact that X is completely normal.■

The following results gives a necessary condition for X to be perfectly normal when L is an X-top lattice.

Proposition 2.62. *Let L be an X-top lattice for some $X \subseteq L \backslash \{1\}$. Then X is perfectly normal if and only if X is T_6.*

Proof. Claim: X does not contain C_2 (whence $dim(X) = 0$).

Let X be perfectly normal and suppose that there exists $C_2 \subseteq X$, i.e. we can find $x < y$ in X. Any open set in X is of the form $D(a)$, where $a \in L$, and so contains x whenever it contains $V(y)$. Now, $x \notin V(y)$ implies that $V(y)$ cannot be expressed as an intersection of open sets, i.e. not a G_δ-set, contradicting the fact that X is perfectly normal.■

The following example shows that, in X-top lattices, not every completely normal space is perfectly normal or completely regular.

Example 2.63. Let L be the lattice in Figure 2.7.

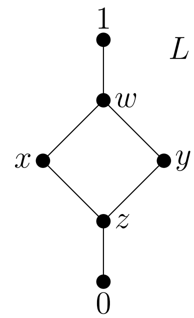
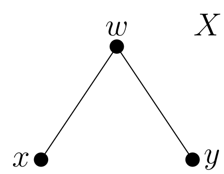


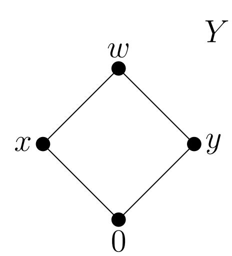
Figure 2.7: The Lattice L

Let $X = \{x, y, w\}$. Then L is an X-top lattice by Theorem 2.3. In this case, X is a $\mathcal{T}_2$ and $dim(X) = 1$. By Theorem 2.61, X is completely normal but not perfectly normal, by Proposition 2.62. Notice that, by Proposition 2.50, X is *not* regular as $dim(X) = 1$.



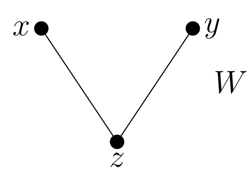
The following example shows that, in X-top lattices, not every normal space is regular or completely normal.

Example 2.64. Let L be the lattice in Figure 2.7 and $Y = \{0, x, y, w\}$. Then L is a Y-top lattice by Theorem 2.3. In this case, $0 \in Y$, $Max(Y) = \{w\}$, and $\{0, x, y\} \subseteq Y$ is a V_2.



Notice that Y is normal, by Proposition 2.59, but not completely normal, by Theorem 2.61. As $dim(Y) = 2$, Y is not regular and not perfectly normal, by Proposition 2.50 and Proposition 2.62.

Remark 2.65. Let L be the lattice in Figure 2.7 and $Z = \{z, x, y\}$. Observe that L is not a Z-top lattice. The reason is that $x \wedge y = z$, i.e. $z \in Z$ is not strongly irreducible in $C(L)$. Thus, for any $W \subseteq L \backslash \{1\}$, such that $Z \subseteq W$, L is not a W-top lattice.



CHAPTER 3

PSI DECOMPOSITIONS

In this chapter we study the so called *pseudo-strongly irreducible elements* (*PSI* for short) of a lattice with an action from a bounded lattice. The interest in studying PSIs stems from the fact that they build a class larger than the class of *strongly irreducible elements* and different from that of *irreducible elements*. This type of elements was introduced in [15] along with the *pseudo-strongly hollow elements* (PSH for short) which are, in some sense, dual to the PSIs. We investigate their relation with other types of elements such as prime, coprime, and second elements [15].

Our main interest is in decompositions of elements in a lattice with an S-action analogous to those of decompositions into (strongly) irreducible elements [4]. We provide *Existence* and *Uniqueness Theorems* for a special class *of* PSI-decompositions, which we call *faithfully PSI-decompositions*. These decomposition theorems generalize [3, Proposition 1.22], which provides a Uniqueness Theorem for strongly irreducible representations. In fact, our study is carried out in a very general setting that allows dualizing to PSH-representations. Moreover, we study applications of these decompositions and representation to the special lattices of sub(semi)modules of a given (semi)module over a (semi)ring.

Lastly, we investigate PSI-submodules of a module over a ring. In particular, we show that if M is a Noetherian module, then every PSI-submodule is *primary* (Theorem 3.53). Moreover, we study the relation between PSI-decompositions and primary decompositions of modules. We prove also a couple of results related to PSI-submodules of semisimple modules analogous to the corresponding ones for PSH-submodules in [15].

3.1 Preliminaries

In this section we state the basic definitions and results of a lattice L with an S-action, where S is a given poset with a smallest and a largest element. This generalizes the action of $Ideal(R)$, the lattice of ideals of a given (semi)ring R, on the lattice $Sub(M)$ of R-sub(semi)modules of a given left R-(semi)module M. We recall from [15] definitions for different types of elements of L which generalize their counterparts in $Sub(M)$.

Definition 3.1. [15, 3.1] Let $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ be a bounded lattice and $\mathcal{S} := (S, \leq, 0_S, 1_S)$ be a bounded poset. We say that $(L, \rightharpoonup)$ is a **lattice with an S-action** if it satisfies the following conditions $\forall s, s' \in S$ and $\forall x, y, \in \mathcal{L}$:

1. If $s \leq s'$, then $s \rightharpoonup x \leq s' \rightharpoonup x$;

2. If $x \leq y$, then $s \rightharpoonup x \leq s \rightharpoonup y$;

3. $s \rightharpoonup x \leq x$;

4. $0_S \rightharpoonup x = 0$ and $1_S \rightharpoonup 1 = 1$.

Example 3.2. Let R be a (semi)ring and M a left R-(semi-)module. The complete lattice $Sub(M)$ has an $Ideal(R)$-action defined by $I \rightharpoonup N := IN$.

Remark 3.3. The two conditions in (4) of Definition 3.1 above were not assumed in [15]. These two extra conditions shall be needed for some results in our work. Assuming them is not very restrictive since they are satisfied in our prototype (Example 3.2).

The following is a generalization of the notion of a multiplication module [15].

Definition 3.4. [15] A bounded lattice L with an S-action is a **multiplication lattice**, if for every element $x \in L$ there is some $s \in S$ such that $x = s \rightharpoonup 1$.

We recall now some definitions of special classes of elements of a lattice with an action.

Definitions 3.5. [15] Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a bounded lattice with an S-action from a bounded lattice $(S, \leq 0_S, 1_S)$.

1. $x \in L \backslash \{1\}$ is **strongly irreducible**, if for all $y, z \in L$:

$$y \wedge z \leq x \quad \Longrightarrow \quad y \leq x \quad or \quad z \leq x \tag{3.1}$$

2. $x \in L \backslash \{1\}$ is **pseudo strongly irreducible** (*PSI* for short), if for all $y \in L$ and $s \in S$:

$$s \rightharpoonup 1 \wedge y \leq x \quad \Longrightarrow \quad s \rightharpoonup 1 \leq x \quad or \quad y \leq x \tag{3.2}$$

3. $x \in L \backslash \{1\}$ is **prime**, if for all $y \in L$ and $s \in S$:

$$s \rightharpoonup y \leq x \quad \Longrightarrow \quad s \rightharpoonup 1 \leq x \quad or \quad y \leq x \tag{3.3}$$

4. $x \in L \backslash \{1\}$ is **coprime**, if for all $s \in S$:

$$s \rightharpoonup 1 \leq x \quad \text{or} \quad s \rightharpoonup 1 \vee x = 1 \tag{3.4}$$

5. $x \in L \backslash \{0\}$ is **strongly hollow**, if for all $y, z \in L$:

$$x \leq y \vee z \quad \Longrightarrow \quad x \leq y \quad or \quad x \leq z \tag{3.5}$$

6. $x \in L \backslash \{0\}$ is **pseudo strongly hollow** (*PSH* for short), if for all $y \in L$ and $s \in S$:

$$x \leq s \rightharpoonup 1 \vee y \quad \Longrightarrow \quad x \leq s \rightharpoonup 1 \quad or \quad x \leq y \tag{3.6}$$

7. $x \in L \backslash \{0\}$ is **second**, if for all $s \in S$:

$$s \rightharpoonup x = x \quad \text{or} \quad s \rightharpoonup x = 0 \tag{3.7}$$

3.2 Pseudo Strongly Irreducible Elements

In this section we focus on studying PSI elements of a *bounded* lattice $(L, \rightharpoonup)$ with an action. We investigate mainly the relation between PSIs and other types of elements under certain conditions. Moreover, we provide examples which illustrate the facts that the class of PSI-elements is larger than the class of strongly irreducible elements, and different from that of irreducible elements.

The class of *distributive lattices* [14] is an important class of lattices. However, a property weaker than that of distributivity is sufficient for some of our results.

Definition 3.6. Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action. We say that L is

1. $\wedge$-**pseudo distributive**, if $\forall s \in S$ and $\forall x, y \in L$:

$$x \wedge (s \rightharpoonup 1 \vee y) = (x \wedge s \rightharpoonup 1) \vee (x \wedge y)$$

2. $\vee$-**pseudo distributive** if $\forall s \in S$ and $\forall x, y \in L$:

$$x \vee (s \rightharpoonup 1 \wedge y) = (x \vee s \rightharpoonup 1) \wedge (x \vee y)$$

The following result clarifies the relationship between the class of PSI-elements and other classes of elements.

Proposition 3.7. *Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a bounded lattice with an S-action.*

1. *If 1 is second in L, then x is PSI for all $x \in L \backslash \{1\}$*

2. *If x is prime in L, then x is PSI.*

3. *If L is $\wedge$-pseudo distributive, then every coprime in L is PSI.*

4. *If L is $\vee$-pseudo distributive, then every irreducible in L is PSI.*

5. *Let L be a multiplication. Then $x \in L$ is PSI $\Longleftrightarrow$ x is strongly irreducible.*

Proof. 1. This follows immediately from the definitions.

2. Let x be prime in L. Suppose that $s \rightharpoonup 1 \wedge y \leq x$. As as $s \rightharpoonup y \leq s \rightharpoonup 1$ and $s \rightharpoonup y \leq y$, we have

$$s \rightharpoonup y \leq s \rightharpoonup 1 \wedge y \leq x$$

Since x is prime, it follows that $s \rightharpoonup 1 \leq x$ or $y \leq x$, i.e. x is PSI.

3. Assume that L is $\wedge$-pseudo distributive. Let $x \in L$ be coprime, and suppose that $s \rightharpoonup 1 \wedge y \leq x$. If $s \rightharpoonup 1 \leq x$, then we are done. If not, then $s \rightharpoonup 1 \vee x = 1$ (since x is coprime). Taking the meet of both sides with y, we get by the $\wedge$-pseudo distributive of L :

$$(y \wedge s \rightharpoonup 1) \vee (y \wedge x) = y,$$

which implies that $y \leq x \vee (y \wedge x) = x$.

4. Assume that L is $\vee$-pseudo distributive. Let $x \in L$ be irreducible, and suppose that $s \rightharpoonup 1 \wedge y \leq x$. Taking the join of both sides with x, we get

$$(x \vee s \rightharpoonup 1) \wedge (x \vee y) = x.$$

Since x is irreducible, $x \vee s \rightharpoonup 1 = x$ or $x \vee y = x$, i.e. $s \rightharpoonup 1 \leq x$ or $y \leq x$ and we are done.

5. This follows directly from the definitions.■

The following examples illustrate that the condition in (1) is necessary and that neither the converse of (2) nor that of (4) in Proposition 3.7 holds in general.

Example 3.8. Consider the ring of integers $\mathbb{Z}$ as a module over itself. Then, $4\mathbb{Z}$ is a PSI submodule that is not prime. Moreover, $6\mathbb{Z} = 3\mathbb{Z} \cap 2\mathbb{Z}$ is not PSI, whence $\mathbb{Z}$ is not second.

Example 3.9. Consider $M = \mathbb{B}[x]$ as an $\mathbb{N}_0$-Semimodule where $\mathbb{B} = \{0, 1\}$ is the Boolean semiring (with $1 + 1 = 1$). For any ideal $I \leq \mathbb{N}_0$, we have $IM = 0$ (if $I = 0$) or $IM = M$ (if $I \neq 0$). Therefore, M is second, whence every subsemimodule of M is PSI.

Consider $N := xM$ and $U := (x + 1)M$. Then

$$N \cap U = (x^2 + x)M$$

is PSI but not irreducible (and indeed *not* strongly irreducible). In addition, by (5) M is not a multiplication.■

We provide now an example of a complete lattice L which is $\wedge$-pseudo distributive and $\vee$-pseudo distributive but *not* distributive; moreover, we give an element in L which is PSI but not irreducible.

Example 3.10. Consider the $\mathbb{Z}$-module $M := \mathbb{Z}_2[x]$. Since M is second, $Sub(M)$ is $\vee$-pseudo distributive and $\wedge$-pseudo distributive. Setting $N := xM$, $U := (x + 1)M$ and $K := \mathbb{Z}_2$, we have

$$K \cap (N + U) = K \neq 0 = (K \cap U) + (K \cap N),$$

i.e. $Sub(M)$ is *not* distributive. Moreover, $N \cap U = (x^2 + x)M$ is PSI but not irreducible.■

3.3 PSI Decompositions and PSH Representations

In this section we study elements, in a *complete lattice* with an S-action, that can be decomposed as a meet of a finite number of PSI elements.

We introduce a special subclass of PSI elements which we call *faithfully PSI elements* (*FPSI* for short). We prove two Uniqueness Theorems for FPSI decompositions which generalize the uniqueness theorem of strongly irreducible decompositions [3, Proposition 1.22]. The proofs will be carried out in a more general setting, in which we consider the $FPSI_a$ elements for any $a \in L$.

We start be defining PSI_a elements, where $a \in L$ is arbitrary. Note that the PSI_1 elements are just the PSI elements. Although the definition works for arbitrary bounded lattices with an S-action, we assume the lattice L is *complete* as we need arbitrary meets and joins to exist in decomposition theorems. The poset $(S, \leq, 0, 1)$ is still assumed to be with a smallest and a largest element.

Definition 3.11. Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a bounded lattice with an S-action and $a \in L$. We say that $x \in L \backslash \{1\}$ is PSI_a, if for all $y \in L$ and $s \in S$:

$$(s \rightharpoonup a) \wedge y \leq x \quad \Longrightarrow \quad s \rightharpoonup a \leq x \quad \textit{or} \quad y \leq x. \tag{3.8}$$

The set of all PSI_a elements of L is denoted by $PSI_a(L)$.

3.12. Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a *complete* lattice with an S-action, $a \in L$ and assume that S is Noetherian (i.e. every ascending chain of elements in S stabilizes).

For $x \in L$ a PSI_a, set

$$A_x := \{s \in S \quad | \quad s \rightharpoonup a \leq x\}, \quad K_x := Max(A_x) \quad \text{and} \quad O(x) := \bigvee_{s \in K_x} (s \rightharpoonup a). \tag{3.9}$$

Notice that A_x is not empty since $0_S \rightharpoonup a = 0$, and that K_x exists since S is Noetherian. We drop x from our notation, if it is clear from the context. We say that x is K-PSI_a.

3.13. Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action and $a \in L$. We say that x has a PSI_a-**decomposition**, if x is a finite meet of PSI_a elements in L. We say that

$$x = \bigwedge_{i=1}^{n} x_i,$$

where each x_i is K_i-PSI_a, is a **minimal** PSI_a-**decomposition** for x if

1. $K_1, K_2, ..., K_n$ are distinct.
2. $\bigwedge\limits_{i=1, i \neq j}^{n} x_i \nleq x_j$ for all $j \in \{1, ..., n\}$.

Proposition 3.14. *Let S be Noetherian, $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action, $a \in L$ and $x, y \in PSI_a(L)$ be incomparable. Then $x \wedge y$ is K-PSI_a if and only if x and y are K-PSI_a.*

Proof. ($\Longrightarrow$) We first show that $K_x \subseteq K_{x \wedge y}$. Let $s \in K_x$. Since $s \rightharpoonup a \leq x$, it follows that $(s \rightharpoonup a) \wedge y \leq x \wedge y$. As x and y are incomparable and $x \wedge y$ is PSI, we conclude that $s \rightharpoonup a \leq x \wedge y$, i.e. $s \in A_{x \wedge y}$. Moreover, $s \in K_{x \wedge y}$ as $x \wedge y < x$. On the other hand, let $s \in K_{x \wedge y}$. Then, $s \rightharpoonup a \leq x \wedge y < x$ and so $s \in A_x$. Suppose $\exists t > s$

such that $t \rightharpoonup a \leq x$. Then, $(t \rightharpoonup a) \wedge y \leq x \wedge y$ and so $t \rightharpoonup a \leq x \wedge y$, i.e. $t \in K_{x\wedge y}$, contradiction. Therefore $s \in K_x$. Consequently, $K_x = K_{x\wedge y}$. One can show similarly that $K_y = K_{x\wedge y}$.

($\Longleftarrow$) We first show that $K_{x\wedge y} = K$ knowing that $K_x = K = K_y$. Let $s \in K$, so that $s \rightharpoonup a \leq x$ and $s \rightharpoonup a \leq y$. It follow that $s \rightharpoonup a \leq x \wedge y$, i.e. $s \in A_{x\wedge y}$. As $x \wedge y < x$ and $s \in K_{x\wedge y}$. On the other hand, if $s \in K_{x\wedge y}$, then $s \rightharpoonup a \leq x \wedge y < x$ and so $s \in K$ (otherwise, we reach the same contradiction above).
We show now that $x \wedge y$ is PSI_a. Assume that $(s \rightharpoonup a) \wedge z \leq x \wedge y$ for some $s \in S$ and $z \in L$. Then, $(s \rightharpoonup a) \wedge z \leq x$ and $(s \rightharpoonup a) \wedge z \leq y$, whence $s \rightharpoonup a \leq x$ or $z \leq x$, and $s \rightharpoonup a \leq y$ or $z \leq y$ as both x and y are PSI_a. Without loss of generality, suppose that $s \rightharpoonup a \leq x$. Since S is Noetherian, $\exists\ t \geq s$ in K such that $t \rightharpoonup a \leq x$ Therefore, $s \rightharpoonup a \leq t \rightharpoonup a \leq y$, i.e. $s \rightharpoonup a \leq x$ if and only if $s \rightharpoonup a \leq y$. Hence, either $s \rightharpoonup a \leq x \wedge y$ or $z \leq x \wedge y$.■

We provide now an Existence Theorem of a *minimal* PSI_a decomposition of a PSI_a-decomposable element in a complete lattice with an S-action.

Theorem 3.15. (Existence of a minimal PSI_a decomposition) *Let S be Noetherian, $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action and $a \in L$. Then every PSI_a decomposable $x \in L$ has a minimal PSI_a decomposition.*

Proof. Let $a = \bigwedge\limits_{i\in I} x_i$, where I is a finite index and each x_i is K_i-PSI_a. We start by removing all x_j such that $\bigwedge\limits_{i\in I, i\neq j} x_i \nleq x_j$ and $j \in I$, which is possible by the finiteness of I. Next, we gather the indices of all x_js that are K-PSI_a, i.e. those which share the same K, in an index set J. Now, $\bigwedge_{j\in J}$ is K-PSI_a by Proposition 3.14. Clearly, this process yields a *minimal* PSI_a decomposition of x.■

To prove the uniqueness theorems for minimal PSI_a decompositions, we will have to assume an extra condition on the PSI-elements. This new class of elements will be called *faithfully PSI-elements* (*FPSI* for short).

Definition 3.16. Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action and $a \in L$. An element $x \in L$ is called **faithfully** PSI_a, if x is a PSI_a and

$$O(x) \leq y \;\Rightarrow\; x \leq y. \tag{3.10}$$

Remark 3.17. If L is a *multiplication* lattice, then every PSI element is faithfully PSI (and strongly irreducible) since $O(x) = x$.

We construct examples of FPSI elements in lattices with an S-action.

Example 3.18. Consider the modular lattices L and S in Figure 3.1 below. We define an S-action on L as follows:

$$\begin{aligned} s &\rightharpoonup 1 = a,\ t \rightharpoonup 1 = b,\ u \rightharpoonup 1 = c,\ 1_S \rightharpoonup 1 = 1,\ 0_S \rightharpoonup 1 = 0, \\ v &\rightharpoonup z = 0 \text{ for all } v \in S \text{ and } z \in L\backslash\{1\}. \end{aligned}$$

Notice that x is faithfully $\{u, t, s\}$-PSI and

$$O(x) = (s \rightharpoonup 1) \vee (t \rightharpoonup 1) \vee (s \rightharpoonup 1).$$

Note that x is not strongly irreducible as $y \wedge w \leq x$ while $y \nleq x$ and $w \nleq x$.

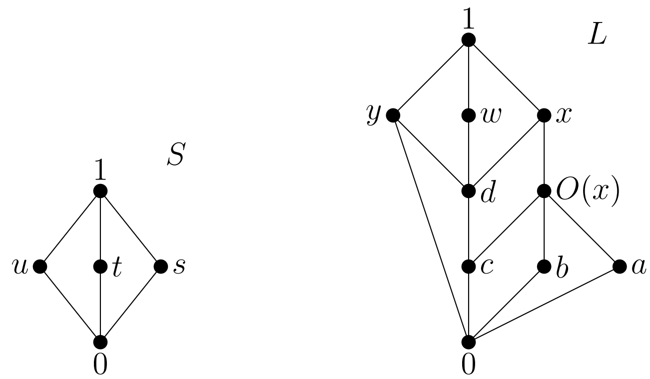


Figure 3.1: Lattice L with S-action

Example 3.19. Consider the lattice $L = \{0, a, b, 1\}$, where $0 < a < b < 1$ in Figure 3.2. Define a $\{0\}$-action (i.e. $S = \{0\}$) on L as follows: $0 \rightharpoonup x = 0$ for all $x \in L$. Then, $0, a$, and b are PSI where $O(b) = O(a) = O(0) = 0$. Hence, 0 and a are faithfully PSI, but b is not faithfully PSI as $O(b) = 0 < a < b$.

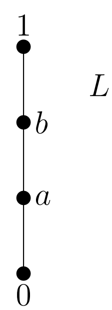


Figure 3.2: The Lattice L

We prove now our first uniqueness theorem for $FPSI$ decompositions.

Theorem 3.20. (First Uniqueness Theorem of FPSI Decompositions) *Let S be Noetherian, $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action and $a \in L$.*

If

$$\bigwedge_{i=1}^{n} x_i = x = \bigwedge_{j=1}^{m} y_j$$

are two minimal PSI_a decompositions for some $x \in L$ such that each x_i is K_i-$FPSI_a$ and each y_j is K'_j-$FPSI_a$. Then $n = m$, $\{K_1, ..., K_n\} = \{K'_1, ..., K'_n\}$, and $K_i = K'_j$ whenever $O(x_i) = O(y_j)$.

Proof. Notice that $K_x = Max(A_x)$ exists since S is Noetherian.

Step I: We prove that $\forall\ i \in \{1, ..., n\}$, $\exists\ j \in \{1, ..., m\}$ such that $x_i \geq O(y_j)$.

Suppose $\exists\ i \in \{1, ..., n\}$ such that $x_i \ngeq O(y_j)\ \forall j \in \{1, ..., m\}$. Then $\forall j \in \{1, ..., m\}$, $\exists\ t_j \in K'_j$ such that $x_i \ngeq t_j \rightharpoonup a$. But $x_i \geq x = \bigwedge_{j=1}^{m} y_j \geq \bigwedge_{j=1}^{m} (t_j \rightharpoonup a)$. Since x_i is PSI_a, we conclude that $t_j \rightharpoonup a \leq x_i$ for some j, a contradiction.

Step II: Fix $i \in \{1, ..., n\}$. We show that $\exists\ j \in \{1, ..., m\}$ such that $O(x_i) = O(y_j)$.

By Step I, we can find $j \in \{1, ..., m\}$ such that $x_i \geq O(y_j)$. It follows that $x_i \geq s \rightharpoonup a$ for all $s \in K'_j$, i.e. $s \in A_{x_i}$ for all $s \in K'_j$. Hence,

$$O(x_i) = \bigvee_{u \in K_i} (u \rightharpoonup a) \geq \bigvee_{s \in K'_j} (s \rightharpoonup a) = O(y_j).$$

Claim: $O(x_i) = O(y_j)$.

By similarity, we can find some $i' \in \{1, ..., n\}$ such that $O(y_j) \geq O(x_{i'})$. Since $x_{i'}$ is $FPSI$, we have

$$x_i \geq O(x_i) \geq O(y_j) \geq x_{i'}.$$

Thus $x_i = x_{i'}$, whence, $O(x_i) = O(y_j)$.

Step III: We show now that $O(x_i) = O(y_i)$ implies $K_i = K'_j$.

Let $s \in K_i$. It follows that $s \rightharpoonup a \leq x_i$ and so $s \rightharpoonup a \leq O(x_i) = O(y_j) \leq y_j$. Thus, there exists $t \in K'_j$ such that $t \geq s$ and

$$t \rightharpoonup a \leq O(y_j) = O(x_i) \leq x_i.$$

However, s is also maximal with this property, i.e. $s = t$ and $s \in K'_j$. Similarly, one can show that $K'_j \subseteq K_i$. Hence, $K'_j = K_i$.■

The following result recovers [4, Proposition 1.22] without the assumption that L is *modular*. This shows that our Uniqueness Theorem of $FPSI_a$ decompositions serves as a generalization to the Uniqueness Theorem of Strongly Irreducible Decompositions in the above mentioned paper.

Corollary 3.21. *Let L be a complete lattice and $\bigwedge_{i=1}^{n} x_i = x = \bigwedge_{j=1}^{m} y_j$ be two irredundant meets of strongly irreducible elements in L. Then, $n = m$, and* $\{x_1, ..., x_n\} = \{y_1, ..., y_n\}$.

Proof. We define an L-action on L such that for all $x \in L$,

$$x \rightharpoonup 1 = x \text{ and } x \rightharpoonup y = 0 \text{ for all } y \in L \backslash \{1\}.$$

Since every strongly irreducible element $x \in L$ is a PSI and $O(x) = x \rightharpoonup 1 = x$, it follows that x is $FPSI$ with $K_x = \{x\}$, by construction. Therefore, the two irredundant strongly irreducible decompositions are in fact *minimal PSI decompositions*. We proceed using the First Uniqueness Theorem 3.20. Notice that the assumption that L is Noetherian is not needed here as $K_x = \{x\}$, for all $x \in L$.■

We state now a Second Uniqueness Theorem for FPSI decompositions of 0 in a complete lattice with an S action.

Theorem 3.22. (Second uniqueness theorem of FPSI decompositions) *Let S be Noetherian, $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action and $a \in L$. Let*

$$\bigwedge_{i=1}^{n} x_i = 0 = \bigwedge_{j=1}^{n} y_j$$

be two minimal PSI_a decompositions of 0 such that each x_i is K_i-$FPSI_a$ and each y_j is K'_j-$FPSI_a$. If A_m is minimal in $\{A_1, ..., A_n\}$ and $O(x_m)$ is PSI_a, then $x_m = y_m$.

Proof. Let A_m be minimal in $\{A_1, ..., A_n\}$ such that $O(x_m)$ is PSI_a. For any $i \neq m$, $\exists\ s_i \in A_i \backslash A_m$, i.e. $s_i \rightharpoonup a \nleq x_m$. But $\bigwedge_{i \neq m} (s_i \rightharpoonup a) \wedge x_m \leq 0 \leq O(x_m)$. Hence, $x_m \leq O(x_m) \leq x_m$, i.e. $O(x_m) = x_m$, as $O(x_m)$ is PSI_a. Similarly $O(y_m) = y_m$. By the First Uniqueness Theorem 3.20, we obtain

$$x_m = O(x_m) = O(y_m) = y_m. \blacksquare$$

3.3.1 PSH Representations

In this subsection we cover what can be considered as the dual of PSI-decompositions. We state without proof the main Existence and Uniqueness Theorems for the so called faithfully pseudo strongly hollow elements (FPSH for short).

3.23. Let S be Artinian (i.e. every descending chain in S stabilizes) and $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action. For a PSH element $x \in L$. We set

$$B_x := \{s \in S \quad | \quad s \rightharpoonup 1 \geq x\}, \quad H_x = Min(B_x) \quad \text{and} \quad In(x) = \bigwedge_{s \in K_x} (s \rightharpoonup 1). \tag{3.11}$$

Notice that B_x is not empty since $1_S \rightharpoonup 1 = 1$, and that H_x exists since S is Artinian. We drop x when it is clear from the context and refer to x as H-PSH.

3.24. Let $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action and $a \in L$. We say that x is **PSH representable element**, if x can be written as a finite join of PSH elements in L. Moreover, we say that $x = \bigvee_{i=1}^{n} x_i$, where each x_i is H_i-PSH, is a **minimal PSH representation** for x, if

1. $H_1, H_2, ..., H_n$ are distinct.
2. $\bigvee_{i=1, i \neq j}^{n} x_i \ngeq x_j$ for all $j \in \{1, ..., n\}$.

Definition 3.25. Let S be Artinian and $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action. An element $x \in L$ is called **faithfully PSH** (FPSH for short), if it is PSH and $\forall y \in L$,

$$y \leq In(x) \implies y \leq x.$$

In the following, we demonstrate the duality between the PSI elements and the PSH elements in a bounded lattice with an S-action. We will be defining a suitable dual action on the dual lattice.

3.26. Let $(S, \leq, 0_S, 1_S)$ be a Noetherian poset and $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action. Notice that $(S^o, \geq, 1_S, 0_S)$ is Artinian and the dual lattice $(L^o, \vee, \wedge, 1, 0)$ is complete and has an S^o-action given by

$$s \rightharpoonup^o x = s \rightharpoonup x \text{ for all } s \in S \text{ and } x \in L.$$

It follows directly that

1. x is PSH in L if and only if x is PSI_0 in L^o. Moreover:

   (a) $B_x = \{s \in S \quad | \quad s \rightharpoonup 1 \geq x\} = \{s \in S \quad | \quad s \rightharpoonup^o 1 \leq^o x\} = A_x^o$;

   (b) $H_x = Min(B_x) = Max(A_x^o) = K_x^o$;

   (c) $In(x) = O^o(x)$.

2. x is faithfully H-PSH in L if and only if x is faithfully H-PSI_0 in L^o.

3. $x = \bigvee\limits_{i=1}^{n} x_i$ is a minimal PSH (FPSH) representation for $x \in L$ if and only if it is a minimal PSI_0 ($FPSI_0$) representation for $x \in L^o$.

4. S is Noetherian if and only if S^o is Artinian.

Taking into consideration the dualization process explained in 3.26, one can restate Proposition 3.15 and Theorems 3.20 and 3.22 to obtain corresponding Uniqueness Theorems for PSH representations.

Theorem 3.27. (Existence of a minimal PSH representation) *Let S be Artinian, $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ a complete lattice with an S-action and $a \in L$. Then every PSH representable $x \in L$ has a minimal PSH representation.*

Theorem 3.28. (First Uniqueness Theorem of Faithfully PSH Representations) *Let S be Artinian, $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action, and $\bigvee\limits_{i=1}^{n} x_i = x = \bigvee\limits_{j=1}^{m} y_j$ be two minimal PSH representations for $x \in L$ such that each x_i is*

H_i-$FPSH$ and each y_j is H'_j-$FPSH$. Then $n = m$, $\{H_1, ..., H_n\} = \{H'_1, ..., H'_n\}$ and $H_i = H'_j$ whenever $In(x_i) = In(y_j)$.

Theorem 3.29. *(*Second Uniqueness Theorem of Faithfully PSH Representations) *Let S be Artinian, $(L, \rightharpoonup) = (L, \wedge, \vee, 0, 1)$ be a complete lattice with an S-action, and $\bigvee_{i=1}^{n} x_i = 1 = \bigvee_{j=1}^{n} y_j$ be two minimal PSH representations for $1 \in L$ such that each x_i is H_i-FPSH and each y_j is H'_j-FPSH. If B_m is minimal in $\{B_1, ..., B_n\}$ and $In(x_m)$ is PSH, then $x_m = y_m$.*

3.4 PSI Subsemimodules

In this section, we work in the special context of semimodules over semirings. Throughout, R is an *associative* semiring, M is a left R-semimodule and the complete lattice $L := Sub(M)$ of R-subsemimodules of M is considered with the canonical S-action, where $S := Ideal(R)$ is the bounded lattice of ideals of R and $I \rightharpoonup K = IK$ for $I \in S$ and $K \in L$. Recall that every left (right) module over a ring is is a left (right) semimodule.

We apply some of the general results in Section 3.3 to this special case and provide several examples of PSI decompositions of subsemimodules of M. Moreover, we prove some results on PSI submodules in the context of semisimple modules. Lastly, we consider primary submodules and *primary decompositions* and investigate their relation with PSI decompositions.

We start by recalling the definition of *PSI subsemimodules* and *PSI decompositions.*

Definition 3.30. Let $N \lneqq_R M$ be a *proper* R-subsemimodule of M.

1. N is a **PSI subsemimodule** of M, if for all $K \leq_R M$ and $I \leq R$:

$$IM \cap K \subseteq N \quad \Longrightarrow \quad IM \subseteq N \text{ or } K \subseteq M.$$

2. N is a **faithfully PSI subsemimodule** of M, if N is PSI, and for any R-subsemimodule $L \leq_R M$ we have

$$(N :_R M)M \subseteq L \implies N \subseteq L.$$

3.31. Let $N \leq_R M$ an R-subsemimodule of M. Adapting the notation of the previous section to this special context, we have

$$\begin{array}{lclcl} A_N & = & \{I \leq R \mid IM \subseteq N\} & & \\ K_N & = & Max(A_N) & = & (N :_R M) \\ O(N) & = & \sum\limits_{I \in K_N} IM & = & (N :_R M)M. \end{array}$$

Notice that in this special case, K_N exists and $O(N)$ is well defined without assuming that the semiring R is Noetherian.

Even in the special context of a module (semimodule) M over a commutative ring (semiring), not every PSI submodule (subsemimodule) of M is faithfully PSI. Moreover, not every prime or strongly irreducible submodule (subsemimodule) is faithfully PSI.

Example 3.32. Consider $M = \mathbb{Z}_2[x]$ as a $\mathbb{Z}$-module. For a positive integer n, we have

$$n\mathbb{Z}M = \begin{cases} 0, & \text{if } n \text{ is even} \\ \\ M, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, every submodule of M is $\{2\mathbb{Z}\}$-PSI. Consider $N := (x)M$ and $K := (x^2)M$. Then $0 = O(N) \subsetneqq K \subsetneqq N$, i.e. N is not FPSI. Moreover, 0 is an FPSI submodule which is *not* strongly irreducible, since $(x)M \cap \mathbb{Z}_2 = 0$.

Example 3.33. Consider $M = \mathbb{Z}_2[x]$ as a $\mathbb{Z}$-module. Then $N = (x)$ is prime, strongly irreducible and $\{2\mathbb{Z}\}$-PSI. However, N is *not* FPSI as $O(N) = 0 \subseteq (x+1)$ and $(x+1)$ is not comparable with $N = (x)$.

We discuss now what we mean by a minimal PSI decomposition for a subsemimodule in the special case of the lattice of subsemimodules of a semimodule.

3.34. Let $N \lneqq_R M$ be a proper R-subsemimodule of M.

1. We say that N has a **PSI decomposition** (or **PSI decomposable**), if N can be written as a finite intersection of PSI subsemimodules of M.

2. We say that $N = \bigcap\limits_{i=1}^{n} N_i$, where each N_i is $(N_i :_R M)$-PSI, a **minimal PSI decomposition** of N if

   (a) $(N_1 :_R M), (N_2 :_R M), \cdots, (N_n :_R M)$ are distinct.

   (b) $\bigcap\limits_{i=1, i\neq j}^{n} N_i \nsubseteq N_j$ for all $j \in \{1, \cdots, n\}$.

We restate the main Existence and Uniqueness Theorems for PSI decompositions for semimodules over semirings. We draw the attention of the reader that given an R-semimodule M, and N, L PSI subsemimodules of M, we have

$$A_N \subseteq A_L \iff (N :_R M) \subseteq (L :_R M).$$

This permits us to state the Second Uniqueness Theorem of PSI decompositions in a more elegant form.

Theorem 3.35. (Existence of Minimal PSI decompositions) *Every PSI decomposable $N \leq_R M$ has a minimal PSI decomposition.*

Theorem 3.36. (First Uniqueness Theorem of FPSI Decompositions) *Let $\bigcap\limits_{i=1}^{n} N_i = N = \bigcap\limits_{i=1}^{m} L_j$ be two minimal PSI decompositions of $N \lneqq_R M$ such that each N_i is $(N_i :_R M)$-FPSI and each L_j is $(L_j :_R M)$-FPSI. Then $n = m$ and $\{(N_1 :_R M), \cdots, (N_n :_R M)\} = \{(L_1 :_R M), \cdots, (L_n :_R M)\}$.*

Theorem 3.37. (Second Uniqueness Theorem of FPSI Decompositions) *Let $\bigcap\limits_{i=1}^{n} N_i = 0 = \bigcap\limits_{i=1}^{n} L_j$ be two minimal FPSI decompositions of 0 such that each N_i is $(N_i :_R M)$-PSI and each L_j is $(L_j :_R M)$-PSI. If $(N_m :_R M)$ is minimal in $\{(N_1 :_R M), \cdots, (N_n :_R M)\}$ and $(N_m :_R M)M$ is PSI, then $N_m = L_m$.*

We provide now examples of PSI decompositions in the context of semimodules. Moreover, we show some relationships between PSI-elements and other special elements as a special application of the general case.

Examples 3.38. 1. Any subsemimodule of a $\cup$-pseudo distributive semimodule, that is a finite intersection of irreducible subsemimodules, is PSI-decomposable.

2. Let M be a second semimodule over a semiring R. Then every subsemimodule of M is PSI, whence has a *trivial* PSI-decomposition.

3. Let M be a $\cap$-pseudo distributive semimodule. Then every coprime subsemimodule is PSI. Hence, any subsemimodule that is a finite intersection of coprime subsemimodules is PSI-decomposable. One can easily see that every maximal subsemimodule is coprime. Therefore, if M is $\cap$-pseudo distributive, then any finite intersection of maximal subsemimodules of M is a PSI decomposition.

4. Every prime subsemimodule of a semimodule M is PSI. Hence, every *semiprime* subsemimodule N, which is the intersection of finitely many primes has a PSI-decomposition.

5. If ${}_RM$ is a finitely cogenerated module (i.e. if 0 is an intersection of a family of submodules, then 0 is an intersection of a finite number of these submodules) and 0 is the intersection of PSI submodules, then 0 is PSI-decomposable.

Example 3.39. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_n$, where $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ is the prime factorization of n, and set

$$N_i = (p_i^{m_i}) = (p_i^{m_i}\mathbb{Z})M = (N_i :_{\mathbb{Z}} M)M.$$

Note that each submodule of $\mathbb{Z}_n$ is of the form $d\mathbb{Z}_n$ for some divisor d of n. Assume that $IM \cap K \subseteq N_i$ for some some ideal I of $\mathbb{Z}$ and some $\mathbb{Z}$-submodule of M. If

$IM \cap K = 0$ while both are not 0, then $K = (ap_i^{m_i})$ or $IM = (ap_i^{m_i})$ for some $a \in \mathbb{Z}$, and so $K \subseteq N_i$ or $IM \subseteq N_i$. If $IM \cap K \neq 0$, then we also have $K \subseteq N_i$ or $IM \subseteq N_i$. If not, $K = (ap_i^{l_1})$ and $IM = (bp_i^{l_2})$ where a,$b \in \mathbb{Z}$, $0 \leq l_1 \leq m_i - 1$, and $0 \leq l_2 \leq m_i - 1$, whence $IM \cap K \subseteq (p_i^{max(l_1,l_2)}) \nsubseteq N_i$, a contradiction (to $IM \cap K \subseteq N_i$). Therefore, each N_i is $(p_i^{m_i})$-FPSI. One can easily see that $0 = \bigcap_{i=1}^{k} N_i$ is a minimal PSI decomposition of 0 in M. By the First and Second Uniqueness Theorems 3.20 and 3.37 of FPSI decompositions, this decomposition is unique.■

The following example gives a class of PSI subsemimodules (submodules) that are *not necessarily* irreducible. At the same time, it provides a class of examples of PSI decompositions that are neither irreducible decompositions nor strongly irreducible decompositions.

Example 3.40. Let M be a semimodule (module) over a commutative local semiring (ring) $(R, \mathbf{m})$. Then, $\mathbf{m}M$ is an $\{\mathbf{m}\}$-FPSI subsemimodule (submodule). Moreover, every subsemimodule (submodule) $N \supseteq \mathbf{m}M$ is $\{\mathbf{m}\}$-PSI. Notice that, in general, $\mathbf{m}M$ is not necessarily irreducible. Indeed, consider the R-semimodule (R-module) $M = R \times R$. Then $(R \times \mathbf{m}) \cap (\mathbf{m} \times R) = (\mathbf{m} \times \mathbf{m}) = \mathbf{m}M$, whence, $\mathbf{m}M$ is not irreducible. Moreover $\mathbf{m}M = \mathbf{m} \times \mathbf{m}$ is a PSI decomposition of itself, however, it is not an irreducible decomposition. Observe also that $R \times \mathbf{m}$ and $\mathbf{m} \times R$ are both $\{\mathbf{m}\}$-PSI and irreducible, thus, $\mathbf{m}M = (R \times \mathbf{m}) \cap (\mathbf{m} \times R)$ is an irreducible decomposition and a PSI decomposition. In this case, however, the PSI decomposition is not minimal as both $(R \times \mathbf{m})$ and $(\mathbf{m} \times R)$ are $\{\mathbf{m}\}$-PSI.■

3.4.1 PSI submodules of a semisimple module

In this subsection we prove a couple of results related to PSI submodules of a semisimple module M. These results are analogous to those in [15, Theorem 3.37].

Lemma 3.41. *Let $M = \bigoplus_{S \in A} S$ be semisimple (where S is a simple left R-module for every $S \in A$) and set $P_S := \bigoplus_{S' \in A \setminus \{S\}} S'$. Then, every P_S is a maximal R-submodule of M and every maximal R-submodule of M is of the form P_S for some $S \in A$. Moreover, for all $S \in A$, P_S is prime, P_S is $Ann(S)$-PSI, and $\bigcap_{S \in A} P_S = 0$.*

Proof. It is clear that every P_S is maximal, as $M/P_S \cong S$ is simple. If K is a maximal R-submodule of M which is not of the form P_S, then $K = \bigoplus_{S \in A \setminus C} S$ where $C \subseteq A$ with $|C| \geq 2$. Pick $S \neq S'$ from C. Then $K \subsetneq P_S$ as $S' \subseteq P_S$ and $S' \nsubseteq K$, a contradiction as K is maximal.

Suppose now that $IN \subseteq P_S$, for some $I \leq R$ and $N \leq_R M$. If $N \nsubseteq P_S$, then $S \subseteq N$ and so $IS \subseteq IN \subseteq M$. Thus, $IM \subseteq P_S$ and so P_S is prime, whence PSI, and one can easily see that $(P_S : M) = Ann(S)$. Hence, P_S is prime and $Ann(S)$-PSI. Lastly, as every P_S misses one simple submodule $S \in A$, it follows that $\bigcap_{S \in A} P_S = 0$.■

Definition 3.42. We say the left R-module M is a **comultiplication**, if for every $K \leq_R M$ we have

$$K = (0 :_M (0 :_R K)).$$

Theorem 3.43. *Let ${}_RM$ be semisimple and B the set of minimal prime submodules* *of M. If*

$$Ann(M) \neq \bigcap_{P \in B \setminus \{Q\}} (P :_R M) \text{ for all } Q \in B, \tag{3.12}$$

then the following are equivalent:

1. *M is a multiplication;*

2. *Every PSI submodule of M is maximal;*

3. *Every prime submodule of M is maximal;*

4. *M is a comultiplication.*

Proof. Let $M = \bigoplus_{S \in A} S$ where S is simple for all $S \in A$.

(1) $\implies$ (2) If M is a multiplication, then any proper submodule of M is PSI if and only if it is strongly irreducible. Suppose that there exists a strongly irreducible $K \leq_R M$ that is not maximal, i.e. $K = \bigoplus_{S \in A \setminus C} S$ where $C \subseteq A$ with $|C| \geq 2$. Let $S \neq S'$ be in C. Then $S \cap S' = 0 \subseteq K$ and neither $S \subseteq K$ nor $S' \subseteq K$, a contradiction to the fact that K is strongly irreducible. Hence, every PSI is maximal.

(2) $\implies$ (3) This follows from the fact that every prime submodule of M is PSI.

(3) $\implies$ (1) From Lemma 3.41, every maximal is prime and of form P_S for some $S \in A$. By assumption, every submodule of M is prime if and only if it is maximal if and only if it is of the form P_S for some $S \in A$. In addition, the condition $Ann(M) \neq \bigcap_{P \in B \setminus \{Q\}} (P :_R M)$ for all $Q \in B$ can be rewritten as $Ann(M) \neq \bigcap_{S \in A \setminus \{S'\}} Ann(S)$ for all $S' \in A$. Let $K = \bigoplus_{S \in C \subsetneq A} S$ be any proper submodule of M, and set $I := \bigcap_{S \in A \setminus \{C\}} Ann(S)$. Then, clearly $IM \subseteq K$. Suppose now that $IM \subsetneq K$. Then

$IS' = 0$ for some $S' \in C$, and it follows that $\bigcap_{S \neq S'} Ann(S) \subseteq \bigcap_{S \in A} Ann(S) = Ann(M)$, contradicting our assumption. Therefore, $K = IM$, i.e. M is a multiplication as K is an arbitrary submodule of M.

(3) $\implies$ (4) : We use the assumptions of the previous argument. Let $K = \bigoplus_{S \in C \subsetneq A} S$ be any proper submodule of M, and set $I := (0 :_R K)$. Clearly, $K \subseteq (0 :_M I)$. Suppose now that $K \subsetneqq (0 :_M I)$. Then $IS' = 0$ for some $S' \in A \backslash \{C\}$, i.e. $I \subseteq Ann(S')$. Therefore, $\bigcap_{S \in C} Ann(S) \subseteq I \subseteq Ann(S')$, that is $\bigcap_{S \neq S'} Ann(S) = Ann(M)$, contradicting our assumption. Therefore, $K = (0 :_M I)$, i.e. M is a comultiplication as K is an arbitrary submodule of M.

(4) $\implies$ (3) Assume that M is a comultiplication and there exists a prime submodule $Q \lneq M$ that is not maximal. Then, $Q = \bigoplus_{S \in A \backslash C} S$ where $C \subseteq A$ with $|C| \geq 2$. Let $S' \neq S''$ in C. Since $_RM$ is comultiplication, we have

$$S' = (0 :_M (0 :_R S')) = (0 :_M Ann(S'))$$

whence $Ann(S')S'' \neq 0$. Hence, $Ann(S') \nsubseteq (Q :_R M)$ and $Ann(S')S' = 0 \subseteq Q$, contradicting that Q is prime (as $S' \nsubseteq Q$).■

The following result shows that, under a certain condition, the equivalent statements of Theorem 3.43 of this section and [15, Theorem 3.37] will actually follow.

Proposition 3.44. *Let $M = \bigoplus_{S \in A} S$ be a semisimple, where S is simple for all $S \in A$. If $Ann(M) \neq \bigcap_{S \in A \backslash \{S'\}} Ann(S)$ for all $S' \in A$, then:*

1. *M is multiplication.*

2. *Every PSI submodule is maximal.*

3. Every prime submodule is maximal.

4. Every PS-hollow submodule is simple.

5. Every second submodule is simple.

6. M is a comultiplication.

Proof. We show first that (3) holds. Suppose there exists a prime submodule $Q \leq M$ that is not maximal. Then, $Q = \bigoplus\limits_{S \in A \setminus C} S$ where $C \subseteq A$ with $|C| \geq 2$. Let $S' \neq S''$ in C. Then $Ann(M) \neq \bigcap\limits_{S \in A \setminus \{S''\}} Ann(S')$, whence $Ann(S') \nsubseteq Ann(S'')$, i.e. $Ann(S')S'' \neq 0$. Thus, $Ann(S') \nsubseteq (Q :_R M)$ and $Ann(S')S' = 0 \subseteq Q$, contradicting that Q is prime (as $S' \nsubseteq Q$).

We note now that to prove (1) $\implies$ (2), (2) $\implies$ (3), and (4) $\implies$ (3) in Theorem 3.43, we did not use the $Ann(M) \neq \bigcap\limits_{P \in B \setminus \{Q\}} (P : M)$ for all $Q \in B$. Moreover, in [15, Theorem 2.27], the assumption $Ann(M) \neq \bigcap\limits_{S \in B \setminus \{S'\}} Ann(S)$, where $S' \in B$ (and B is the set of all maximal second submodules of M), was not used in proving (1) $\implies$ (2), (2) $\implies$ (3), and (4) $\implies$ (3). Therefore, these implications hold here too.

In addition, assuming (3) in Theorem 3.43, $Ann(M) \neq \bigcap\limits_{P \in B \setminus \{Q\}} (P : M)$, for all $Q \in B$, is equivalent to $Ann(M) \neq \bigcap\limits_{S \in A \setminus \{S'\}} Ann(S)$, for all $S' \in A$. Thus, (3) $\implies$ (1) and (3) $\implies$ (4) in Theorem 3.43 follow here too. Moreover, when assuming condition (3) in [15, Theorem 2.27], $Ann(M) \neq \bigcap\limits_{S \in B \setminus \{S'\}} Ann(S)$ where $S' \in B$ and B is the set of all maximal second submodules of M, is equivalent to $Ann(M) \neq \bigcap\limits_{S \in A \setminus \{S'\}} Ann(S)$, for all $S' \in A$. Thus, (3) $\implies$ (1) and (3) $\implies$ (4) in [15, Theorem 2.27] follow here too.■

We provide now some examples illustrating the above mentioned results.

Examples 3.45. Consider the $\mathbb{Z}$ module $M = \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i}$, where p_i is prime for all i and $p_i \neq p_j$, for all $i \neq j$. Then M is semisimple and the condition of Proposition 3.44 holds. Therefore, M is a multiplication, every PSI and every prime submodule of M is maximal, every PS-hollow and every second submodule of M is simple, and M is a comultiplication. Moreover, $\bigcap_{i=1}^{n} P_{\mathbb{Z}_{p_i}} = 0$ is a unique FPSI decomposition of 0.■

The following is an example of a finitely generated module all submodules of which are PSI decomposable.

Example 3.46. Let M be a finitely generated (Artinian/Noetherian) semisimple module. Then, every proper submodule $K \leq M$ is PSI decomposable. To see this, let $M = \bigoplus_{S \in A} S$, where S is simple for all $S \in A$, and $K = \bigoplus_{S \in C \subsetneq A} S$. Then $K = \bigcap_{S \in A \setminus C} P_S$ where each P_S is maximal and prime by Lemma 3.41.■

The following example shows that Condition 3.12 in Theorem 3.43 cannot be dropped.

Example 3.47. Consider the semisimple $\mathbb{Z}$-module $M = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{p_i}$, where p_i is prime and $p_i \neq p_j$, for all $i \neq j$. Notice that Condition 3.12 of Theorem 3.43 does *not* hold. One can easily see that every PSI $\mathbb{Z}$-submodule *of* M is maximal. However, M is not a multiplication as $\mathbb{Z}_{p_1} \neq IM$, for any $I \leq \mathbb{Z}$.■

3.4.2 Primary Decompositions

In this subsection, we investigate the relation between the PSI decompositions and the primary decompositions. Throughout, R is a commutative ring and M is an R-module.

Definition 3.48. [7] Let $N \lneqq_R M$ be *a proper* R-submodule of M. For any $a \in R$, consider the R-linear map

$$f_a : M/N \longrightarrow M/N,\ \bar{x} \mapsto a\bar{x}.$$

We say that N is a **primary submodule** of M, if f_a is either injective or nilpotent for all $a \in R$. If $N \leq_R M$ is primary, then $P = \sqrt{Ann_R(M/N)} = \sqrt{(N :_R M)}$ is a prime ideal of R and we say that N is P-**primary**.

Definition 3.49. A primary decomposition $N = \bigcap\limits_{i=1}^{n} N_i$, where each N_i is P_i-primary, is said to be a **reduced primary decomposition** of N if

1. $P_1, P_2, \cdots, P_n$ are distinct.
2. $\bigcap\limits_{i=1, i\neq j}^{n} N_i \nsubseteq N_j$ for all $j \in \{1, ..., n\}$.

Theorem 3.50. [7, Theorem 2.1.6] (Existence Theorem of Primary Decompositions) *If $_RM$ is Noetherian module, then every proper R-submodule of M has a reduced primary decomposition.*

Theorem 3.51. [7] (First Uniqueness Theorem of Primary Decompositions) *Let* $\bigcap_{i=1}^{n} N_i = N = \bigcap_{i=1}^{m} L_j$ *be two minimal primary decompositions for* $N \leq_R M$ *such that each* N_i *is* P_i*-primary and each* L_j *is* Q_j*-primary. Then* $n = m$ *and* $\{P_1, ..., P_n\} = \{Q_1, ..., Q_n\}$.

Theorem 3.52. [7] (Second Uniqueness Theorem of Primary Decompositions) *Let* $\bigcap_{i=1}^{n} N_i = N = \bigcap_{i=1}^{n} L_j$ *be two minimal primary decompositions for* $N \leq_R M$ *such that each* N_i *is* P_i*-primary and each* L_j *is* Q_j*-primary. If* P_m *is minimal in* $\{P_1, ..., P_n\}$*, then* $N_m = L_m$.

The follows result suggests a relation between PSI decompositions and primary decompositions.

Theorem 3.53. *If* ${}_RM$ *is Noetherian module, then every PSI submodule of* M *is primary.*

Proof. Suppose that $N \lneqq_R M$ is PSI but not primary. Then there exists $a \in R$ such that $f_a : M/N \longrightarrow M/N$ is neither injective nor nilpotent. Consider the ascending chain

$$ker(f_a) \subseteq ker(f_a^2) \subseteq \cdots \subseteq ker(f_a^k) \subseteq ker(f_a^{k+1}) \subseteq \cdots$$

Since M is Noetherian, this chain stabilizes, and so there exists some positive integer n such that $ker(f_a^n) \subseteq ker(f_a^{n+k})$ for all $k \geq 1$. Setting $g = f_a^n$, we have $ker(g) = ker(g^2)$. Notice that f_a is not injective implies that $ker(g) \neq \bar{0}$ and f_a not nilpotent implies that $im(g) \neq \bar{0}$.

<u>Claim II:</u> $ker(g) \cap im(g) = \bar{0}$.

Let $\bar{z} \in ker(g) \cap im(g)$. Then, $\bar{z} = g(\bar{y})$ for some $\bar{y} \in M/N$. Hence, $g^2(\bar{y}) = g(\bar{z}) = \bar{0}$, i.e. $\bar{y} \in ker(g^2) = ker(g)$, which implies that $\bar{z} = g(\bar{y}) = 0$. Now $im(g) \neq 0$ means that there exists $\bar{x} \in M/N$ such that $g(\bar{x}) = a^n\bar{x} \neq 0$, i.e. $a^n x \notin N$. Hence, the submodule $(a^n)M \nsubseteq N$.

Assume $ker(g) = L_1/N$ and $im(g) = L_2/N$ for some $N \leq_R L_1 \leq_R M$ and $N \leq_R L_2 \leq_R M$. One can easily see that $(a^n)M \subseteq L_2$.

<u>Claim II:</u> $L_1 \cap L_2 \subseteq N$.

Suppose not, then there exists $x \in (L_1 \cap L_2)\backslash N$, i.e. $\bar{x} \neq 0$. But, $\bar{x} \in L_1/N \cap L_2/N = ker(g) \cap im(g) = 0$, a contradiction as $x \notin N$. Therefore, $L_1 \cap (a^n)M \subseteq L_1 \cap L_2 \subseteq N$, where $(a^n)M \nsubseteq N$ and $L_1 \nsubseteq N$ as $ker(g) \neq 0$, contradicting the fact that N is PSI. Thus, N is primary.■

Corollary 3.54. *Let ${}_RM$ be Noetherian and $N \lneqq_R M$ a proper R-submodule of M. If $\bigcap\limits_{i=1}^{n} N_i$ is a reduced primary decomposition of N, where each N_i is a PSI, then $\bigcap\limits_{i=1}^{n} N_i$ is a minimal PSI decomposition of N.*

Proof. Let $N = \bigcap\limits_{i=1}^{n} N_i$ be a minimal primary decomposition of N, where each N_i is P_i-primary. Clearly, all $N_i's$ are incomparable and the ideals $(N_i :_R M)$, for $i \in \{1, 2, \cdots, n\}$ are distinct. If not, then there exists some $i \neq j$ in $\{1, 2, \cdots, n\}$ such that $(N_i :_R M) = (N_j :_R M)$, whence

$$P_i = \sqrt{(N_i :_R M)} = \sqrt{(N_j :_R M)} = P_j,$$

contradicting the minimality of the primary decomposition. Therefore, $\bigcap\limits_{i=1}^{n} N_i$ is a minimal PSI decomposition of N.■

The following example shows that the converse of Theorem 3.53 in not true in general, even if ${}_RM$ is Noetherian.

Example 3.55. Let $R := k[x, y]$, where k is a field, considered as a module over itself and recall that an ideal is PSI if and only if it is strongly irreducible. Notice that $I = (x^2, xy, y^2)$ is (x, y)-primary and *not irreducible* since $I = (x, y^2) \cap (x^2, y)$, whence not PSI. Therefore, I is primary but not PSI in R.

Example 3.56. Let ${}_RM$ be second and Noetherian. Then every proper submodule of M is PSI, whence *primary* by Theorem 3.53. In this case, the minimal primary decomposition of any submodule coincides with the minimal PSI decomposition.

Example 3.57. If ${}_RM$ is Noetherian and multiplication, then any minimal PSI decomposition of a submodule of M is a reduced primary decomposition.

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