GENERAL ENERGY DECAY RATES FOR SOME VISCOELASTIC PROBLEMS

BY

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To my dear family

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THESIS ABSTRACT

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TITLE OF STUDY:	General Energy Decay Rates for Some Viscoelastic Problems
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The work in this dissertation is concerned with the longtime behavior of some viscoelastic problems. Precisely, we will consider four problems, where the relaxation function satisfies a new relation of the form

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0,$$

where $1 \leq p < \frac{3}{2}$ *and* $\xi : \mathbb{R}_+ \longrightarrow (0, +\infty)$ *is a nonincreasing differentiable function.*

Using the multiplier method, we could establish some new explicit decay results depending on p*,* ξ*, and other parameters in the problem such as the behavior of the feedback function and/or the degree of the nonlinearity of the frictional damping*

when it is present in the equation.

Our work generalized many results in the literature such as [27, 31, 36] and improved others. Particularly, it gaves a better rate of decay than that of [29]. In addition, our results answered partially the question in [2], namely, looking for decay rates induced by relaxation functions satisfying

$$g'(t) \leq -\xi(t)H(g(t)), \quad \forall t \geq 0,$$

for functions ξ *and* H.

ملخص الرسالة

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في هذا البحث ندرس السلوك التقاربي لبعض معادلات المرونة اللزجة، وبالتحديد نهتم بدراسة أربعة مسائل واعتبار دوال إسترخاء تحقق علاقة جديدة من الشكل

$$g'(t) \leq -\xi(t) g^{p}(t), \quad \forall t \geq 0,$$

حيث $1 \leq p < \frac{3}{2}$ والدالة $\xi: \mathbb{R}_{+} \longrightarrow (0,+\infty)$ غير متزايدة وقابلة للإشتقاق.

بإستخدام طريقة المضروبات استطعنا أن نبرهن بعض نتائج الإضمحلال والتي تعتمد كل منها على p، ξ، ومعاملات أخرى قد تظهر في المسألة كسلوك دالة التغذية التقاربية أو درجة أس التخميد الناتج من الاحتكاك.

هذا العمل يعمم ويحسن الكثير من النتائج في البحوث السابقة على سبيل المثال [27 ,31 ,36] وبالأخص تعطي معدل إضمحلال أفضل من ذلك المثبت في البحث [29]. بالإضافة إلى ذلك، فهو يجيب جزئياً عن التساؤل المطروح من قبل المؤلفين في [2] والذي مفاده ايجاد معدلات إضمحلال ناتجة عن دوال إسترخاء تحقق

$$g'(t) \leq -\xi(t)\, H\big(g(t)\big), \quad \forall t \geq 0$$

لدوال H و ξ.

CHAPTER 1

INTRODUCTION

The word “viscoelasticity” is derived from the words “elastic” and “viscous”. Elasticity is the property of a material (mostly solids) to recover its initial shape and size quickly after the deforming force (or load) has been removed. That is, all the energy stored in a “purely” elastic material during the loading is returned when the load is removed. On the other hand, viscosity is a characteristic of a material (mostly fluids) to resist the force of flow. This type of “purely’ viscous material deform under even smallest load and all the energy is lost as “pure damping” once the load is removed. However, there are materials which exhibit both the elastic and viscous properties in the deformation process. In fact, some of the energy stored in a viscoelastic system is recovered upon removal of the load, and the remainder is dissipated in the form of heat. This kind of materials can be seen in the nature, for example, human tissue, the disks in the human spine and wood.

Many early contributions have been devoted in modeling this phenomena. Boltzmann (1844 – 1906) proposed, in his model, that the stress at the current time depends not only on the current strain, but on the past strains as well. It was assumed that a strain at a distant past contributes less to the stress than a more recent strain. This is recognized as the familiar concept of fading memory. Furthermore, Boltzmann suggested the superposition principle which states that the response of a material to a given load is independent of the response of the material to any load, which is already on the material, which means that stress σ and strain ϵ are linearly dependant. As a result, the model is given by

$$\sigma(t) = \sigma_0\epsilon(t) - \int_0^t g(t-s)\epsilon(s)ds,$$

where g is a function that characterises the mechanical properties of the material and is referred to as "relaxation function". This function brings about damping effect of the solution to the problem. We will mainly be concerned with this phenomenon in our study.

1.1 Literature Review

Consider the following viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + a(x)|u_t|^{m-2}u_t = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, g is a positive nonincreasing function, a is a nonnegative function defined on Ω and $m \geq 2$. This problem models the motion of a viscoelastic body with the appearance of a "nonlinear" frictional damping acting on a part or on the whole body, with reference configuration Ω. This type of problems has attracted the attention of many scientists and several results of existence, nonexistence, stability and blow up have appeared. We begin with the pioneer work of Dafermos [6, 7] in 1970, where he proved some existence results for certain viscoelastic problems in one dimension. He also showed that when t goes to infinity, the solutions approach zero. However, he did not give any rate of decay. In 1985, Hrusa [18] considered the nonlinear one-dimensional viscoelastic equation

$$u_{tt} - cu_{xx} + \int_0^t m(t-s)(\psi(u_x(x,s)))_x ds = f(x,t)$$

and established several global existence results for large data and, also, proved an exponential decay result for strong solutions when $m(s) = e^{-s}$ and ψ satisfies certain conditions. For the higher-dimension problems, we mention the work of Dassios and Zafiropoulos [15], where they studied a viscoelastic model in $\mathbb{R}^3$ and proved that when the kernels are decaying exponentially, the solutions decay polynomially. Rivera [37] investigated a linear homogeneous viscoelastic equation in a bounded domain and in the whole space $\mathbb{R}^n$. In the bounded-domain case and for kernels decaying exponentially and regular solutions, he showed that the sum of the first and the second energy decays exponentially. He also imposed an extra relation between the kernel and its second derivative. For the whole-space case and for exponentially decaying memory kernels, he obtained an algebraic decay rate. This result was later improved by Cabanillas and Rivera [38], in the sense that they obtained an algebraic decay rate for the energy even when kernels are decaying algebraically. Cavalcanti *et al.* [13] established an exponential rate of decay for (1.1) with relaxation functions obeying the inequality

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \ t \geq 0,$$

for some positive constants ξ_1 and ξ_2, and $a(x) \geq a_0 > 0$ in a subdomain $\omega \subset \Omega$, with a positive measure, that satisfies some geometric restriction. Berrimi and Messaoudi [3] showed that the exponential decay can be obtained even for the function a vanishing on the whole domain Ω. So, no need for the geometric condition imposed by Cavalcanti *et al.* [13]. Moreover, Berrimi and Messaoudi [4] improved and extended Cavalcanti's result for a problem with a nonlinear source term. Messaoudi [27] investigated a semilinear viscoelastic problem of the type

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + |u|^\gamma u = 0, & \text{in } \Omega \times (0,+\infty) \\ u = 0, & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases}$$

for relaxation functions satisfying

$$g'(t) \leq -\xi g^p(t), \quad t \geq 0, \tag{1.2}$$

for positive constants ξ and $1 \leq p < \frac{3}{2}$. He obtained an exponential decay rate for $p = 1$ and a polynomial decay rate for $p > 1$.

In [14], Cavalcanti and Oquendo considered the viscoelastic problem

$$\begin{cases} u_{tt} - k_0 \Delta u + \displaystyle\int_0^t div[a(x) g(t-s) \nabla u(s)] ds + b(x) h(u_t) + f(u) = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases}$$

where a and b are two C^1-functions satisfying

$$a(x) + b(x) \geq \delta > 0, \;\; \forall x \in \Omega.$$

They obtained, for h linear (resp. nonlinear) exponential (resp. polynomial) stability for g decaying exponentially (resp. polynomially).

In 2008, Messaoudi [28, 29] considered the viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s) \Delta u(x,s) ds = b|u|^{\gamma} u, & \text{in } \Omega \times (0, +\infty) \\ u = 0, & \text{on } \partial\Omega \times (0, +\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.3}$$

with $b = 0$ and $b = 1$ and relaxation functions that satisfy

$$g'(t) \leq -\xi(t) g(t), \quad t \geq 0, \tag{1.4}$$

where $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nonincreasing differentiable function. For the first time, he obtained "general" decay rates that cover the exponential and algebraic decay rates as special cases. In fact, he showed that the energy E and the relaxation function g have the same rate of decay and it is given by

$$E(t) \leq Ce^{-\lambda \int_0^t \xi(s)ds}, \ \ \forall t \geq 0.$$

where C and λ are positive constants. This decay rate is of polynomial type if $\xi(t) = 1/(t+1)$ and of exponential type if $\xi \equiv 1$. After that a lot of papers, using the idea of [28, 29], have appeared. See for instance Liu [21, 22], Park and Park [40], Wu [44] and others. **Let us note here that, though the relation** (1.4) **allows an expanded class of relaxation functions, the decay results obtained in [28, 29] do not give the optimal result for polynomially decaying functions.** Alabau-Boussouira and Cannarsa [2] considered (1.3), with $b = 0$ and g, a positive function that satisfies

$$g'(t) \leq -H\left(g(t)\right), \quad t \geq 0, \tag{1.5}$$

where $H : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing and strictly convex on $(0, k_0]$, for some $k_0 > 0$, with $H(0) = H'(0) = 0$. In addition, they required that

$$\int_0^{k_0} \frac{dx}{H(x)} = +\infty, \qquad \int_0^{k_0} \frac{xdx}{H(x)} < 1, \qquad \liminf_{s \to 0^+} \frac{H(s)/s}{H'(s)} > \frac{1}{2}$$

and claimed an energy decay result. Moreover, if H satisfies

$$\limsup_{s\to 0^+} \frac{H(s)/s}{H'(s)} < 1 \qquad \text{and} \qquad g'(t) = -H\left(g(t)\right)$$

then an explicit decay rate is given. Mustafa and Messaoudi [39] weakened by a great deal the conditions made by Alabau-Boussouira and Cannarsa. They only required that the relaxation function satisfies (1.5) and

$$\int_0^{+\infty} \frac{g(s)}{D^{-1}(H^{-1}(-g'(s)))} ds < +\infty,$$

where D is a positive C^1 function, with $D(0) = 0$, for which $H(D)$ is strictly increasing and strictly convex C^2 function on $(0, k_0]$ for some $k_0 > 0$.

Cavalcanti *et al.* [9] looked into a quasilinear equation of the form

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau) d\tau - \gamma \Delta u_t = 0, \tag{1.6}$$

with $\rho > 0$ if $n = 1, 2$ and $0 < \rho \leq \frac{2}{n-2}$ if $n \geq 3$. They proved a global existence result for $\gamma \geq 0$ and an exponential stability for $\gamma > 0$. Messaoudi and Tatar [34] obtained the same result of [9] in the presence of a nonlinear source term. Furthermore, Messaoudi and Tatar [35, 36] established, for $\gamma = 0$, exponential (resp. polynomial) decay results if g decays exponentially (resp. polynomially) in the absence, as well as in the presence, of a source term. Han and Wang [17] considered (1.6) for $\gamma = 0$ and with a relaxation function satisfying (1.4) and proved a general decay result for which the previous results are only special cases.

Messaoudi and Mustafa [32] studied (1.6) for relaxation functions obeying (1.5) and obtained a uniform decay rate. In [10], Cavalcanti *et al.* considered (1.6), with $\gamma = 0$, and a relaxation function satisfying (1.5). In addition, they required

$$\liminf_{x\to 0^+}\{x^2 H''(x) - xH'(x) + H(x)\} \geq 0$$

and $y^{1-\alpha_0} \in L^1(1,+\infty)$, for some $\alpha_0 \in [0,1)$, where $y(t)$ is the solution of the problem

$$y'(t) + H(y(t)) = 0, \;\; y(0) = g(0) > 0.$$

They characterized the decay of the energy by the solution of a corresponding ODE, as in [19].

For stabilization by mean of boundary feedback, Cavalcanti *et al.* [12] studied the following problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds = 0, & \text{in } \Omega \times (0,+\infty) \\ u = 0, & \text{on } \Gamma_0 \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu} - \displaystyle\int_0^t g(t-s)\dfrac{\partial u}{\partial \nu}(s)ds + h(u_t) = 0, & \text{on } \Gamma_1 \times (0,+\infty) \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.7}$$

and proved a global existence result and gave some uniform decay rate results under some restrictive assumptions on both the memory kernel g and the damping function h. These restrictions were relaxed by Cavalcanti *et al.* [11] and further they established a uniform stability result depending on the behavior of h near the origin and on the behavior of g at infinity. Messaoudi and Mustafa [31, 33] used some properties of convex functions [1] to extend these results, by considering relaxation functions g satisfying (1.4), and proved an explicit and general stability result. In [43, 44], Wu proved existence results for (1.7) with nonlinear boundary/interior sources and obtained the same uniform decay rates as in [31, 33]. Further, he proved some blow up results.

1.2 Result Description

In this dissertation, we study the stabilization of certain viscoelastic problems when the relaxation function g satisfies

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0,\ 1 \leq p < \frac{3}{2},$$

and establish general decay rates that will cover the results in [28, 29, 30, 31, 32, 33] as well as the exponential decay and the polynomial decay results. Moreover, the optimal polynomial decay is easily and directly obtained without restrictive conditions as in [10].

Now we give a summary of the content of the dissertation. In Section 1.3, we present the technique used to prove the decay results. In Section 1.4, we prepare some material which is needed throughout the dissertation. We investigate the asymptotic behavior of the linear viscoelastic equation

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds = 0$$

with Dirichlet conidtions in Chapter 2 and with boundary feedback in Chapter 4. The stability of the quasilinear viscoelastic equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds = 0$$

is investigated in Chapter 3. Finally, we study the asymptotic behavior of the viscoelastic equation with a nonlinear frictional damping

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2}u_t = 0, \;\; m > 1.$$

For all the above problems, we established decay rates which generalized and improved some of the existing results in the literature and gave optimal rates for certain polynomial decay cases.

1.3 Methodology

We use the multiplier method to establish the desired stability results of the problems. The multiplier method relies mostly on defining an appropriate Lyapunov functional $\mathcal{L}$ which it is equivalent to the energy E. By the equivalence $\mathcal{L} \sim E$, we mean

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \quad \forall t \geq 0, \tag{1.8}$$

for two positive constants α_1 and α_2. In the case of a general decay result, the obtained decay rate depends on the relaxation function g which is assumed to satisfy the following two conditions

$(H1)$ $g : \mathbb{R}_+ \longrightarrow (0, +\infty)$ is a nonincreasing differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

$(H2)$ There exist a nonincreasing differentiable function $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\xi(0) > 0$, and a constant $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0.$$

We split the study of the stability into two cases.

Case of $p = 1$. We show that

$$(\xi(t)\mathcal{L}(t) + \beta_1 E(t))' \leq -c_1\xi(t)E(t), \ \forall t \geq 0.$$

After that, we exploit (1.8) to prove that

$$W(t) = \xi(t)\mathcal{L}(t) + \beta_1 E(t) \sim E(t). \tag{1.9}$$

Direct integration on $(0,t)$ gives

$$W(t) \leq K e^{-\lambda \int_0^t \xi(s)ds}, \quad \forall t \geq 0, \tag{1.10}$$

where K and λ are positive constants.

Case of $p > 1$. We prove that

$$\left(\xi^{\alpha+1}(t)E^{\alpha}(t)\mathcal{L}(t) + \beta_1 E(t)\right)' \leq -c_1 \xi^{\alpha+1}(t)E^{\alpha+1}(t), \ \forall t \geq 0,$$

where $\alpha = 2p - 2$. Then, from (1.8) we have

$$W(t) = \xi^{\alpha+1}(t)E^{\alpha}(t)\mathcal{L}(t) + \beta_1 E(t) \sim E(t). \tag{1.11}$$

Integrating over $(0,t)$ yields, for some positive constant C,

$$W(t) \leq C \left[\frac{1}{\int_{t_0}^t \xi^{2p-1}(s)ds + 1}\right]^{\frac{1}{2p-2}}, \quad \forall t \geq 0.$$

Under the condition

$$\int_0^{+\infty} \left[\frac{1}{\int_{t_0}^t \xi^{2p-1}(s)ds + 1}\right]^{\frac{1}{2p-2}} dt < +\infty,$$

we obtain the improved decay estimate

$$W(t) \leq C \left[\frac{1}{\int_{t_0}^t \xi^{p}(s)ds + 1}\right]^{\frac{1}{p-1}} \quad \forall t \geq 0.$$

This leads to the "optimality" of the polynomial decay for certain problems (see Examples 2.1, 4.1 and 5.1).

1.4 Preliminaries

Throughout this dissertation, we use the following notations

- $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + ... + \partial_{x_n}^2$
- $\nabla = (\partial_{x_1},\ \partial_{x_2}, ..., \partial_{x_n})$
- $u_t = \dfrac{\partial u}{\partial t}, \quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2},$
- $\mathcal{C}^1(\Omega)$ denotes the space of all continuously differentiable functions on Ω,
- $\mathcal{C}_0^1(\Omega)$ denotes the space of all continuously differentiable functions with compact support in Ω. The support of a continuous function f defined on Ω is the closure of the set of point where $f(x)$ is nonzero. That is

$$\mathrm{supp}(f) :\, = \overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

- $\mathcal{C}_0^\infty(\Omega)$ denotes the space of all continuously functions with compact support in Ω, having continuous derivatives of every order.
- $L^p(\Omega) = \{f : \Omega \to \mathbb{R};\ f \text{ is measurable function and } \int_\Omega |f|^p dx < +\infty\}$, where $1 \leq p < \infty$.

- $L^\infty(\Omega) = \{f : \Omega \to \mathbb{R};\ f$ is measurable function and there is a constant $C \geq 0$ such that $|f(x)| \leq C$ a.e. on $\Omega\}$.

- $L^p_{loc}(\Omega) = \{f : \Omega \to \mathbb{R};\ f$ is measurable function and $f \in L^p(K)$, $\forall\, K \subset \Omega, K$ compact$\}$.

We use the standard $W^{1,p}(\Omega)$ space that is defined as

$$W^{1,p}(\Omega) = \Big\{u \in L^p(\Omega); \exists g_1, g_2, \cdots, g_n \in L^p(\Omega) : \int_\Omega u\frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i\varphi, \quad \forall \varphi \in C_0^\infty(\Omega)\Big\}.$$

We set $g_i := \dfrac{\partial u}{\partial x_i}$ and $H^1(\Omega) = W^{1,2}(\Omega)$.
The space $H^1(\Omega)$ is equipped with the norm

$$\|u\|^2_{H^1(\Omega)} = \|u\|_2^2 + \|\nabla u\|_2^2,$$

where $\|u\|_2 = \|u\|_{L^2(\Omega)}$. We define the space

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega)\colon \exists \{u_m\}_0^\infty \subset C_0^1(\Omega), \text{ such that } u_m \to u \text{ in } W^{1,p}(\Omega)\}.$$

and set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. If Ω is bounded, the norm of H_0^1 can be taken

$$\|u\|^2_{H_0^1(\Omega)} = \|\nabla u\|_2^2.$$

Let X be a real Banach space with a norm $\|\quad\|$. We have the following definitions

- The space $L^p(0,T;X)$ consists of all measurable functions $u : [0,T] \to X$ with

  1. $$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p dt\right)^{1/p} < +\infty,$$

     for $1 \leq p < \infty$, and

  2. $$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(t)\| < +\infty,$$

     for $p = \infty$.

- The space $L^p_{loc}(0,T;X)$ consists of all measurable functions $u : (0,T) \to X$ with $u \in L^p([a,b];X)$ for every closed interval $[a,b] \subset (0,T)$.

- The space $C([0,T],X)$ consists of all continuous functions $u : [0,T] \to X$ with

  $$\|u\|_{\mathcal{C}([0,T],X)} := \max_{0 \leq t \leq T} \|u\| < +\infty$$

  .

- The space $C^1([0,T],X)$ consists of all continuously differentiable functions $u : [0,T] \to X$ with

  $$\|u\|_{\mathcal{C}^1([0,T],X)} := \max_{0 \leq t \leq T} \|u\| + \max_{0 \leq t \leq T} \|\frac{du}{dt}\| < +\infty$$

  .

The following inequalities are repeatedly used in the dissertation

1. **Hölder's inequality**. Let $1 \le p,\, q \le \infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |uv| \le \|u\|_p \|v\|_q.$$

   By taking $p = q = 2$, we have the **Cauchy-Schwarz inequality**.

2. **Young's inequality**. Let $1 < p,\, q < +\infty$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then for any $\varepsilon > 0$, we have

$$ab \le \varepsilon a^p + C_\varepsilon b^q, \quad \forall a,\, b \ge 0,$$

   where $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{\frac{q}{p}}}$. For $p = q = 2$, we have

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

3. **Green's formula (integration by parts)**. Let Ω be a bounded domain of $\mathbb{R}^n$ with a smooth boundary. Then

$$\int_\Omega u \Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} u \nabla v \cdot \nu ds, \qquad \forall u \in H^1(\Omega) \text{ and } v \in H^2(\Omega).$$

where ν is the outer unit normal to $\partial\Omega$. If $u \in H_0^1$, Green's formula becomes

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \cdot \nabla v dx, \ \ \forall u \in H_0^1(\Omega) \text{ and } v \in H^2(\Omega).$$

4. **Poincaré's inequality**. Let $1 \leq p < +\infty$ and Ω be a bounded domain of $\mathbb{R}^n$. Then there exists a constant C (depending on Ω and p only) such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \quad \forall u \in W_0^{1,p}(\Omega).$$

5. **Jensen's inequality**. Let G be a concave function on $[a,b]$ ($-G$ is convex), $f : \Omega \to [a,b]$ and h are integrable functions on Ω, with $h(x) \geq 0$, and $\int\limits_\Omega h(x)dx = k > 0$. Then

$$\frac{1}{k}\int\limits_\Omega G[f(x)]h(x)dx \leq G\left[\frac{1}{k}\int\limits_\Omega f(x)h(x)dx\right].$$

For the special case $G(y) = y^{\frac{1}{p}}$, $y \geq 0$, $p > 1$, we have

$$\frac{1}{k}\int\limits_\Omega [f(x)]^{\frac{1}{p}}h(x)dx \leq \left[\frac{1}{k}\int\limits_\Omega f(x)h(x)dx\right]^{\frac{1}{p}}. \tag{1.12}$$

Furthermore, we mostly use c to denote a generic positive constant.

To make our writing simpler, we introduce the following notation

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2 ds,$$

for any $v \in L^1_{loc}([0,+\infty); L^2(\Omega))$.

The following lemmas and corollary are crucial in establishing our stability results.

Lemma 1.1 *Assume that g satisfies $(H1)$ and $(H2)$ then*

$$\int_0^{+\infty} \xi(t) g^{1-\sigma}(t)dt < +\infty, \quad \forall \sigma < 2-p.$$

Proof. Recalling $(H2)$, we find that

$$\xi(t)g^{1-\sigma}(t) = \xi(t)g^{1-\sigma}(t)g^p(t)g^{-p}(t) \leq -g'(t)g^{1-\sigma-p}(t).$$

Integration then gives

$$\int_0^{+\infty} \xi(t)g^{1-\sigma}(t)dt \leq -\int_0^{+\infty} g'(t)g^{1-\sigma-p}(t)dt = -\frac{g^{2-p-\sigma}(t)}{2-p-\sigma}\Bigg]_0^{+\infty} < +\infty,$$

since $\sigma < 2-p$. ∎

Lemma 1.2 ([14]) *Suppose that $g \in C([0,+\infty))$, $w \in L^1_{loc}(0,+\infty)$, and $0 \leq \sigma \leq 1$; then, for any $\theta > 0$, we have, for all $t \geq 0$,*

$$\int_0^t |g(s)w(s)|ds \leq \left[\int_0^t |g(s)|^{1-\sigma}|w(s)|ds\right]^{\frac{1}{\theta+1}} \left[\int_0^t |g(s)|^{1+\frac{\sigma}{\theta}}|w(s)|ds\right]^{\frac{\theta}{\theta+1}}. \quad (1.13)$$

Proof. For any fixed $t \geq 0$ we see that

$$\int_0^t |g(s)w(s)|ds = \int_0^t \underbrace{|g(s)|^{\frac{1-\sigma}{\theta+1}}|w(s)|^{\frac{1}{\theta+1}}}_{:=w_1} \underbrace{|g(s)|^{1-\frac{1-\sigma}{\theta+1}}|w(s)|^{\frac{\theta}{\theta+1}}}_{:=w_2} ds, \ \forall t \geq 0.$$

Note that $w_1 \in L^s_{loc}(0,+\infty)$, $w_2 \in L^{s'}_{loc}(0,+\infty)$, where $s = \theta+1$ and $s' = \frac{\theta+1}{\theta}$. Using Hölder's inequality, we get (1.13). ∎

Lemma 1.3 ([14]) *Suppose that $u \in L^\infty(0,T;H^1_0(\Omega))$, for every $T>0$, and g is a continuous function. Then, there exists $C>0$ such that, for $0<\sigma<1$ and $\mu>0$,*

$$(g\circ\nabla u)(t) \leq C\left[\left(\int_0^t g^{1-\sigma}ds\right)\|u\|^2_{L^\infty(0,T,H^1_0(\Omega))}\right]^{\frac{1}{\sigma\mu+1}}\left[(g^{1+\frac{1}{\mu}}\circ\nabla u)(t)\right]^{\frac{\sigma\mu}{\sigma\mu+1}}, \tag{1.14}$$

for all $t\geq 0$.

Proof. For $u \in L^\infty(0,T;H^1_0(\Omega))$ and for every $t\geq 0$, the function

$$w(.,t) := \int_\Omega |\nabla u(t)-\nabla u(.)|^2dx \in L^1_{loc}(0,+\infty).$$

Hence Lemma 1.2, with $\theta=\sigma\mu$, gives

$$\begin{aligned}(g\circ\nabla u)(t) &= \int_0^t g(t-s)\underbrace{\int_\Omega |\nabla u(t)-\nabla u(s)|^2}_{:=w(t)}dsdx\\ &\leq \left[\int_0^t g^{1-\sigma}(t-s)w(s,t)ds\right]^{\frac{1}{\theta\mu+1}}\left[\int_0^t g^{1+\frac{1}{\mu}}(t-s)w(s,t)ds\right]^{\frac{\sigma\mu}{\sigma\mu+1}}\\ &\leq \left[(g^{1-\sigma}\circ\nabla u)(t)\right]^{\frac{1}{\sigma\mu+1}}\left[(g^{1+\frac{1}{\mu}}\circ\nabla u)(t)\right]^{\frac{\sigma\mu}{\sigma\mu+1}},\ \forall t\geq 0,\end{aligned}$$

and

$$
\begin{aligned}
(g^{1-\sigma} \circ \nabla u)(t) &= \int_0^t g^{1-\sigma}(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 dx ds \\
&\leq C \left(\int_0^t g^{1-\sigma}(s) ds \right) \|u\|^2_{L^\infty(0,T,H_0^1(\Omega))}, \quad \forall t \geq 0,
\end{aligned}
$$

from which (1.14) follows. ∎

Now by substituting $\mu = \frac{1}{p-1}$, where $p > 1$, we get the following lemma.

Lemma 1.4 ([27]) *Assume that g is a continuous function and $u \in \mathcal{C}(\mathbb{R}_+; H_0^1(\Omega))$ then there exists a positive constant C such that, for $0 < \sigma < 1$,*

$$
(g \circ \nabla u)(t) \leq C \left[\left(\int_0^t g^{1-\sigma}(s) ds \right) \right]^{\frac{p-1}{p-1+\sigma}} (g^p \circ \nabla u)^{\frac{\sigma}{p-1+\sigma}}(t), \quad \forall t \geq 0.
$$

By taking $\sigma = \frac{1}{2}$, we get

$$
(g \circ \nabla u)(t) \leq C \left[\int_0^t g^{\frac{1}{2}}(s) ds \right]^{\frac{2p-2}{2p-1}} (g^p \circ \nabla u)^{\frac{1}{2p-1}}(t), \quad \forall t \geq 0. \tag{1.15}
$$

Corollary 1.1 *Assume that g satisfies (H1) and (H2) and $u \in \mathcal{C}(\mathbb{R}_+; H_0^1(\Omega))$ then there exists a constant $C > 0$ such that*

$$
\xi(t)(g \circ \nabla u)(t) \leq C \left[(-g' \circ \nabla u)(t) \right]^{\frac{1}{2p-1}}, \quad \forall t \geq 0.
$$

Proof. Multiply both sides of (1.15) by $\xi(t)$ and recall Lemma 1.1 to get

$$\begin{aligned}
\xi(t)(g \circ \nabla u)(t) &\leq C\xi^{\frac{2p-2}{2p-1}}(t)\left[\int_0^t g^{\frac{1}{2}}(s)ds\right]^{\frac{2p-2}{2p-1}} \xi^{\frac{1}{2p-1}}(t)(g^p \circ \nabla u)^{\frac{1}{2p-1}}(t) \\
&\leq C\left[\int_0^t \xi(s)g^{\frac{1}{2}}(s)ds\right]^{\frac{2p-2}{2p-1}} (\xi g^p \circ \nabla u)^{\frac{1}{2p-1}}(t) \\
&\leq C\left[\int_0^{+\infty} \xi(s)g^{\frac{1}{2}}(s)ds\right]^{\frac{2p-2}{2p-1}} (-g' \circ \nabla u)^{\frac{1}{2p-1}}(t) \\
&\leq C\left[(-g' \circ \nabla u)(t)\right]^{\frac{1}{2p-1}}, \ \ \forall t \geq 0.
\end{aligned}$$

∎

CHAPTER 2

GENERAL AND OPTIMAL DECAY IN A LINEAR VISCOELASTIC EQUATION

This chapter is devoted for the study of the following viscoelastic equation

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds = 0, & \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. In Section 2.1, we introduce some assumptions needed throughout the chapter. In Section 2.2, we give some technical lemmas. The main result and its proof are given in Section 2.3. Finally, two illustrative examples are given in Section 2.4.

2.1 Assumptions

In this section, we introduce our assumptions and the "modified" energy functional. Throughout this chapter, all the functionals and inequalities are defined and valid for $t \geq 0$ unless it stated otherwise.

We impose the following hypotheses on the memory kernel g

$(H1)$ $g : \mathbb{R}_+ \longrightarrow (0, +\infty)$ is a nonincreasing differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

$(H2)$ There exist a nonincreasing differentiable function $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\xi(0) > 0$, and a constant $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0.$$

The "modified" energy functional associated to the problem (2.1) is given by

$$E(t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t),$$

where, for any $v \in L^1_{loc}([0, +\infty); L^2(\Omega))$, we set

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2 ds.$$

2.2 Technical Lemmas

In this section, we state, without proof, the global existence result of [12] and then set up certain lemmas that are required for the proof of our main theorem.

Definition 2.1 *A function $u \in \mathcal{C}([0,T]; H_0^1(\Omega))$ with $u_t \in \mathcal{C}([0,T]; L^2(\Omega))$ is said to be a "weak" solution of (2.1) if it satisfies*

1. $$\int_\Omega u_{tt}\phi dx + \int_\Omega \nabla u \cdot \nabla \phi dx - \int_0^t g(t-s)\int_\Omega \nabla u(s)\cdot \nabla\phi dx ds = 0,$$

   for all $\phi \in H_0^1(\Omega)$ and for almost all $t \in [0,T]$, and

2. $u(x,0) = u_0(x)$ *in* $H_0^1(\Omega)$ *and* $u_t(x,0) = u_1(x)$ *in* $L^2(\Omega)$.

Theorem 2.1 ([12]) *Assume that $(u_0, u_1) \in H_0^1(\Omega)\times L^2(\Omega)$ and g satisfies (H1). Then there exists a unique global "weak" solution u of (2.1) satisfying*

$$u \in \mathcal{C}(\mathbb{R}_+; H_0^1(\Omega)), \quad u_t \in \mathcal{C}(\mathbb{R}_+; L^2(\Omega)).$$

Moreover, if $(u_0, u_1) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$; then the weak solution becomes "strong" solution in the sense that

$$u \in \mathcal{C}(\mathbb{R}_+; H^2(\Omega)\cap H_0^1(\Omega)), \quad u_t \in \mathcal{C}(\mathbb{R}_+; H_0^1(\Omega)), \quad u_{tt} \in \mathcal{C}(\mathbb{R}_+; L^2(\Omega)).$$

Lemma 2.1 *The modified energy functional satisfies, along the solution,*

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \le \frac{1}{2}(g'\circ\nabla u)(t) \le 0.$$

Proof. Multiplying the first equation in the problem (2.1) by u_t and integrating over Ω gives

$$\frac{1}{2}\frac{d}{dt}\left(\|u_t\|_2^2 + \|\nabla u(t)\|_2^2\right) - \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\nabla u(s)ds \ dx = 0. \tag{2.2}$$

Noting that

$$\begin{aligned}
& -\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\nabla u(s)ds \ dx = \\
& -\int_0^t g(t-s)\int_\Omega \nabla u_t(t)\left[\nabla u(s)-\nabla u(t)\right]ds \ dx - \int_0^t g(t-s)\int_\Omega \nabla u_t(t)\cdot\nabla u(t)ds \ dx \\
= & \int_0^t g(t-s)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)-\nabla u(s)\|_2^2\right)ds - \int_0^t g(t-s)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_2^2\right)ds \\
= & \frac{1}{2}\frac{d}{dt}(g\circ\nabla u)(t) - (g'\circ\nabla u)(t) - \frac{1}{2}\frac{d}{dt}\left[\|\nabla u(t)\|_2^2\int_0^t g(s)ds\right] + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2
\end{aligned}$$

and combining with (2.2) yield the result. This result is established for strong solutions. However, it also holds for weak solutions by a simple density argument.

∎

Now combining Lemma 2.1 with Corollary 1.1, we obtain

$$\xi(t)(g\circ\nabla u)(t) \le C\left[-E'(t)\right]^{\frac{1}{2p-1}}. \tag{2.3}$$

To make our writing simpler, we introduce the following notation

$$(g\odot v)(t) = \int_\Omega \left|\int_0^t g(t-s)(v(t)-v(s))ds\right|^2 dx.$$

Lemma 2.2 *There exists a constant $c > 0$ such that the following inequalities hold, along the solution,*

$$(g \odot u)(t) \leq c(g \circ \nabla u)(t)$$

and

$$(g \odot \nabla u)(t) \leq c(g \circ \nabla u)(t).$$

Proof. Using Cauchy-Schwartz and Poincaré's inequalities and $(H1)$, we get

$$\begin{aligned}
(g \odot u)(t) &= \int_\Omega \left| \int_0^t \underbrace{\sqrt{g(t-s)}}\, \underbrace{\sqrt{g(t-s)}(u(t)-u(s))}\, ds \right|^2 dx \\
&\leq \int_\Omega \left(\int_0^t g(t-s)ds \right) \left(\int_0^t g(t-s)|u(t)-u(s)|^2 ds \right) dx \\
&\leq \left(\int_0^t g(s)ds \right) \int_\Omega \int_0^t g(t-s)|u(t)-u(s)|^2 dsdx \\
&\leq \int_\Omega \int_0^t g(t-s)|u(t)-u(s)|^2 dsdx \\
&\leq C \int_\Omega \int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2 dsdx \\
&= c(g \circ \nabla u)(t).
\end{aligned}$$

It is also obvious, from the above steps, that

$$(g \odot \nabla u)(t) \leq c(g \circ \nabla u)(t).$$

∎

We define the Lyapunov functional F by

$$F(t) := E(t) + \varepsilon_1 G_1(t) + \varepsilon_2 G_2(t), \tag{2.4}$$

where ε_1 and ε_2 are positive constants and

$$G_1(t) := \int_\Omega u u_t dx, \qquad G_2(t) := -\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx.$$

Lemma 2.3 ([27]) *The functional F is equivalent to E for ε_1 and ε_2 sufficiently small. That is, there exist two positive constants α_1 and α_2 such that*

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t). \tag{2.5}$$

Proof. Use of Young's inequality, with $p = q = 2$ and $\varepsilon = \frac{1}{2}$, and Poincaré's inequality followed by Lemma 2.2 gives

$$\begin{aligned} F(t) &\leq E(t) + \frac{\varepsilon_1}{2}\int_\Omega |u_t|^2 dx + \frac{\varepsilon_1}{2}\int_\Omega |u|^2 dx \\ &\quad + \frac{\varepsilon_2}{2}\int_\Omega |u_t|^2 dx + \frac{\varepsilon_2}{2}(g \odot u)(t) \\ &\leq E(t) + \left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)||u_t||_2^2 + \left(\frac{\varepsilon_1}{2}\right) C||\nabla u||_2^2 + C(g \circ \nabla u)(t) \\ &\leq \alpha_2 E(t). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
F(t) &= E(t) - \varepsilon_1 \int_\Omega (-uu_t)dx - \varepsilon_2 \int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\geq E(t) - \frac{\varepsilon_1}{2}\int_\Omega |u_t|^2 dx - \frac{\varepsilon_1}{2}\int_\Omega |u|^2 dx \\
&\quad - \frac{\varepsilon_2}{2}\int_\Omega |u_t|^2 dx - \frac{\varepsilon_2}{2}(g \odot u)(t) \\
&\geq \frac{l}{2}\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \left(\frac{\varepsilon_1 + \varepsilon_2}{2}\right)||u_t||_2^2 \\
&\quad - \left(\frac{\varepsilon_1}{2}\right) C||\nabla u||_2^2 - C(g \circ \nabla u)(t) \\
&\geq \alpha_1 E(t),
\end{aligned}
$$

for ε_1 and ε_2 small enough. ∎

Lemma 2.4 ([39]) *When conditions (H1) and (H2) hold, the functional*

$$
G_1(t) := \int_\Omega uu_t dx
$$

satisfies, along the solution,

$$
G_1'(t) \leq -\frac{l}{2}\|\nabla u\|_2^2 + \|u_t\|_2^2 + C(g \circ \nabla u)(t).
$$

Proof. Differentiation and use of (2.1) followed by Green's formula lead to

$$
\begin{aligned}
G_1'(t) &= \int_\Omega u_t^2 dx + \int_\Omega u\Delta u dx - \int_\Omega u \int_0^t g(s)\Delta u(t-s)dsdx \\
&\leq \int_\Omega u_t^2 dx - l\int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u(t) \cdot \int_0^t g(s)(\nabla u(t-s) - \nabla u(t))dsdx
\end{aligned}
$$

employing Young's inequality, with $p = q = 2$ and Lemma 2.2, we obtain, for $\delta > 0$,

$$\begin{aligned} & \int_\Omega \nabla u(t) \cdot \int_0^t g(s)(\nabla u(t-s) - \nabla u(t))dsdx \\ \leq \; & \delta \int_\Omega |\nabla u|^2 dx + \frac{1}{4\delta} \int_\Omega \left(\int_0^t g(s)|\nabla u(t-s) - \nabla u(t)|ds \right)^2 dx \\ \leq \; & \delta \int_\Omega |\nabla u|^2 dx + \frac{c}{\delta}(g \circ \nabla u)(t). \end{aligned}$$

Therefore

$$G_1'(t) \leq -(l-\delta)\|\nabla u\|_2^2 + \|u_t\|_2^2 + \frac{c}{\delta}(g \circ \nabla u)(t).$$

Choosing $\delta = \frac{l}{2}$ completes the proof. ∎

Lemma 2.5 ([39]) *When conditions (H1) and (H2) hold, the functional*

$$G_2(t) := -\int_\Omega u_t \int_0^t g(t-s)(u(t) - u(s))dsdx$$

satisfies, along the solution and for any $\delta > 0$,

$$G_2'(t) \leq -(\int_0^t g(s)ds - \delta)\|u_t\|_2^2 + \delta\|\nabla u\|_2^2 + \frac{C}{\delta}(g \circ \nabla u)(t) - \frac{C}{\delta}(g' \circ \nabla u)(t).$$

Proof. Differentiation and use of (2.1) and Green's formula, we have

$$
\begin{aligned}
G_2'(t) &= -\int_\Omega u_{tt}(t)\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\quad -\int_\Omega u_t(t)\left[\int_0^t g'(t-s)(u(t)-u(s))ds+\int_0^t g(t-s)(u_t(t))ds\right]dx \\
&= \int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&\quad -\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right)\cdot\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\
&\quad -\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2dx \\
&= \left(1-\int_0^t g(s)ds\right)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&\quad +\int_\Omega \left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2dx \\
&\quad -\int_\Omega u_t(t)\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2dx.
\end{aligned}
$$

Young's and Poincaré's inequalities and Lemma 2.2 lead to

$$
\begin{aligned}
&\left(1-\int_0^t g(s)ds\right)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
\le\ & \delta\int_\Omega |\nabla u|^2dx+\frac{c}{\delta}(g\circ\nabla u)(t)
\end{aligned}
$$

and

$$
-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx \le \delta\int_\Omega u_t^2dx-\frac{c}{\delta}(g'\mathrm{o}\nabla u)(t).
$$

Combination all the above estimates and taking in account Lemma 2.2 yield the assertion of the lemma. ∎

2.3 The Main Result

Theorem 2.2 *Assume that* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and* g *satisfies* $(H1)$ *and* $(H2)$*. Then for each* $t_0 > 0$*, there exist strictly positive constants* C *and* λ *such that the solution of* (2.1) *satisfies, for all* $t \geq t_0$,

$$E(t) \leq Ce^{-\lambda \int_{t_0}^{t} \xi(s)ds}, \qquad p = 1, \tag{2.6}$$

$$E(t) \leq C\left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(s)ds}\right]^{\frac{1}{2p-2}}, \qquad p > 1. \tag{2.7}$$

Moreover, if

$$\int_{t_0}^{+\infty} \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(s)ds}\right]^{\frac{1}{2p-2}} dt < +\infty, \quad 1 < p < \frac{3}{2}, \tag{2.8}$$

then

$$E(t) \leq K\left[\frac{1}{1 + \int_{t_0}^{t} \xi^{p}(s)ds}\right]^{\frac{1}{p-1}}, \qquad p > 1. \tag{2.9}$$

Remark 2.1 *Obviously,* (2.7) *and* (2.8) *yield*

$$\int_{t_0}^{+\infty} E(t)dt < +\infty.$$

Remark 2.2 *Estimate* (2.6) *was obtained first in [28].*

Proof. Differentiating (2.4) and using Lemmas 2.1, 2.4, 2.5, we get

$$
\begin{aligned}
F'(t) \;&\leq\; (\frac{1}{2} - \frac{\varepsilon_2 C}{\delta})(g' \circ \nabla u)(t) - \left(\varepsilon_2 \left(\int_0^t g(s)ds - \delta\right) - \varepsilon_1\right) \|u_t\|_2^2 \\
&\quad - \left[\frac{\varepsilon_1 l}{2} - \varepsilon_2 \delta\right] \|\nabla u(t)\|_2^2 + \left[\varepsilon_1 C + \varepsilon_2 \frac{C}{\delta}\right] (g \circ \nabla u)(t). \qquad (2.10)
\end{aligned}
$$

From the continuity and positiveness of g, we have, for any $t_0 > 0$,

$$
\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds =: g_0 > 0 \quad \forall t \geq t_0.
$$

Choose δ in (2.10) small enough so that

$$
g_0 - \delta > \frac{1}{2} g_0 \quad \text{and} \quad \frac{2}{l}\delta < \frac{1}{4} g_0.
$$

Whence δ is fixed, any choice of ε_1 and ε_2, satisfying

$$
\frac{1}{4} g_0 \varepsilon_2 < \varepsilon_1 < \frac{1}{2} g_0 \varepsilon_2, \qquad (2.11)
$$

makes

$$
\begin{aligned}
k_1 \;&:=\; \varepsilon_2 (g_0 - \delta) - \varepsilon_1 > \frac{\varepsilon_2 g_0}{2} - \varepsilon_1 > 0 \\
k_2 \;&:=\; \frac{\varepsilon_1 l}{2} - \varepsilon_2 \delta > \frac{\varepsilon_1 l}{2} - \varepsilon_2 \frac{l g_0}{8} = \frac{l}{2}(\varepsilon_1 - \frac{\varepsilon_2 g_0}{4}) > 0.
\end{aligned}
$$

We pick ε_2 even smaller so that (2.5) and (2.11) are satisfied and, further,

$$k_3 := \frac{1}{2} - \frac{\varepsilon_2 C}{\delta} > 0.$$

So, (2.10) becomes

$$\begin{aligned} F'(t) &\leq -k_1\|u_t\|_2^2 - k_2\|\nabla u(t)\|_2^2 + k_3(g' \circ \nabla u)(t) + C(g \circ \nabla u)(t) \\ &\leq -k_1\|u_t\|_2^2 - k_2\|\nabla u(t)\|_2^2 + C(g \circ \nabla u)(t), \;\; \forall t \geq t_0; \end{aligned}$$

which implies that, for some $m > 0$,

$$F'(t) \leq -mE(t) + C(g \circ \nabla u)(t), \;\; \forall t \geq t_0. \tag{2.12}$$

Multiplication by $\xi(t)$ yields

$$\xi(t)F'(t) \leq -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t), \;\; \forall t \geq t_0. \tag{2.13}$$

Case of $p = 1$

Recalling $(H2)$ and Lemma 2.1, we obtain, from (2.13),

$$\begin{aligned} \xi(t)F'(t) &\leq -m\xi(t)E(t) + C(\xi g \circ \nabla u)(t) \\ &\leq -m\xi(t)E(t) - C(g' \circ \nabla u)(t) \\ &\leq -m\xi(t)E(t) - CE'(t), \;\; \forall t \geq t_0; \end{aligned}$$

which leads to

$$\left(\xi F + CE\right)'(t) \leq -m\xi(t)E(t), \ \ \forall t \geq t_0. \tag{2.14}$$

Let $L(t) := \xi(t)F(t) + CE(t)$ then clearly $L \sim E$ and we have, for some $m_1 > 0$,

$$L'(t) \leq -m_1\xi(t)L(t), \ \ \forall t \geq t_0.$$

By a simple integration, we arrive at

$$L(t) \leq Ce^{-m_1\int_{t_0}^{t}\xi(s)ds}, \ \ \forall t \geq t_0,$$

and hence (2.6) by virtue of $L \sim E$.

Case of $p > 1$

To establish (2.7), we again consider (2.13) and use Corollary 1.1 to get

$$\xi(t)F'(t) \ \ \leq \ \ -m\xi(t)E(t) + C\left[-E'(t)\right]^{\frac{1}{2p-1}}, \ \ \forall t \geq t_0.$$

Multiplication of the last inequality by $\xi^{\alpha}E^{\alpha}(t)$, for $\alpha = 2p - 2$, leads to

$$\xi^{\alpha+1}E^{\alpha}(t)F'(t) \leq -m\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C\left(\xi E\right)^{\alpha}(t)\left[-E'(t)\right]^{\frac{1}{\alpha+1}}, \ \ \forall t \geq t_0.$$

Use of Young's inequality, with $q=\alpha+1$ and $q^*=\frac{\alpha+1}{\alpha}$, yields

$$\begin{aligned}\xi^{\alpha+1}E^{\alpha}(t)F'(t) &\leq -m\xi^{\alpha+1}(t)E^{\alpha+1}(t)+C\left[\varepsilon\xi^{\alpha+1}(t)E^{\alpha+1}(t)-C_{\varepsilon}E'(t)\right]\\ &= -(m-\varepsilon C)\xi^{\alpha+1}(t)E^{\alpha+1}(t)-CE'(t), \quad \forall\varepsilon>0,\ \forall t\geq t_0.\end{aligned}$$

We then choose $\varepsilon<\frac{m}{C}$ and recall that $\xi'\leq 0$ and $E'\leq 0$, to get

$$\left(\xi^{\alpha+1}E^{\alpha}F\right)'(t)\leq \xi^{\alpha+1}(t)E^{\alpha}(t)F'(t)\leq -c_1\xi^{\alpha+1}(t)E^{\alpha+1}(t)-CE'(t),\ \forall t\geq t_0;$$

which implies

$$\left(\xi^{\alpha+1}E^{\alpha}F+CE\right)'(t)\leq -c_1\xi^{\alpha+1}(t)E^{\alpha+1}(t),\ \forall t\geq t_0.$$

Let $W=\xi^{\alpha+1}E^{\alpha}F+CE\sim E$. Then

$$W'(t)\leq -C\xi^{\alpha+1}(t)W^{\alpha+1}(t)=-C\xi^{2p-1}(t)W^{2p-1}(t),\ \forall t\geq t_0.$$

Integrating over (t_0,t) and using the fact that $W\sim E$, we reach

$$E(t)\leq C\left[\frac{1}{\int_{t_0}^{t}\xi^{2p-1}(s)ds+1}\right]^{\frac{1}{2p-2}} \quad \forall t\geq t_0. \tag{2.15}$$

To establish (2.9), we consider (2.13) and recall Remark 2.1. So, we see that, for all $t \geq t_0$,

$$\begin{aligned}
\xi(t)F'(t) &\leq -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) \\
&= -m\xi(t)E(t) + C\frac{\eta(t)}{\eta(t)} \int_0^t [\xi^p(s)g^p(s)]^{\frac{1}{p}} \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds,
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
\eta(t) = \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds &\leq C \int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2 ds \\
&\leq C \int_0^t [E(t) + E(t-s)]\, ds \\
&\leq 2C \int_0^t E(t-s) ds \\
&= 2C \int_0^t E(s) ds < 2C \int_0^{+\infty} E(s) ds < +\infty.
\end{aligned}$$

Applying Jensen's inequality (1.12) for the second term of the right hand side of (2.16), with $G(y) = y^{\frac{1}{p}}$, $y > 0$, $f(s) = \xi^p(s)g^p(s)$ and $h(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2$, to get, for all $t \geq t_0$,

$$\xi(t)F'(t) \leq -m\xi(t)E(t) + C\eta(t) \left[\frac{1}{\eta(t)} \int_0^t \xi^p(s)g^p(s) \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \right]^{\frac{1}{p}},$$

where we assume that $\eta(t) > 0$, otherwise we get $\|\nabla u(t) - \nabla u(t-s)\| = 0$ and hence from (2.12) we have

$$E(t) \leq Ce^{-mt}, \ \forall t \geq t_0.$$

Therefore, we obtain

$$\begin{aligned}\xi(t)F'(t) &\leq -m\xi(t)E(t) + C\eta^{\frac{p-1}{p}}(t)\left[\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\nabla u(t)-\nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}} \\ &\leq -m\xi(t)E(t) + C(-g'\circ\nabla u)^{\frac{1}{p}}(t) \leq -m\xi(t)E(t) + C(-E'(t))^{\frac{1}{p}}, \ \forall t \geq t_0.\end{aligned}$$

Multiplying by $\xi^{\alpha}(t)E^{\alpha}(t)$, for $\alpha = p-1$, and repeating the same computations as in above, we arrive at

$$E(t) \leq C\left[\frac{1}{\int_{t_0}^t \xi^p(s)ds + 1}\right]^{\frac{1}{p-1}}, \quad \forall t \geq t_0.$$

This completes the proof of our main result. ∎

Remark 2.3 *Estimates* (2.6), (2.7) *and* (2.9) *are also true for* $t \in [0, t_0]$ *by virtue of continuity and boundedness of* $E(t)$ *and* $\xi(t)$.

Remark 2.4 *When* $\xi \equiv a$ *then* (2.9) *gives*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{p-1}}}, \ \forall t \geq t_0;$$

which is the optimal decay obtained by Cavalcanti et al. [13] and Messaoudi [28].

Remark 2.5 *Our theorem improves the result of Messaoudi [27, 28] which gives a general decay but without obtaining the optimal rate in the case of polynomial decay. Example 2.1 elaborates this point.*

2.4 Examples

The following examples illustrate our result and show the optimal decay rate in the polynomial case.

Example 2.1 *Let*

$$g(t) = \frac{a}{(1+t)^{\nu}}, \quad \nu > 2,$$

where $a > 0$ *is a constant so that* $\int_0^{+\infty} g(t)dt < 1$. *We have*

$$g'(t) \quad = \quad -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^{\nu}}\right)^{\frac{\nu+1}{\nu}} = -bg^p(t), \tag{2.17}$$

where $p = \frac{\nu+1}{\nu} < \frac{3}{2}$ *and* $b > 0$. *Therefore* (2.8), *with* $\xi(t) = b$, *yields*

$$\int_0^{+\infty} \left(\frac{1}{b^{2p-1}t+1}\right)^{\frac{1}{2p-2}} dt < \infty.$$

and hence by (2.9) *we get*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{p-1}}} = \frac{C}{(1+t)^{\nu}}, \quad \forall t \geq t_0,$$

which is the optimal decay rate. But, if we write (2.17) *as*

$$g'(t) = -\frac{\nu}{1+t} g(t)$$

and consider $\xi_2(t) = \frac{\nu}{1+t}$*, then according to Theorem 3.6 in [28] we have*

$$E(t) \leq \frac{C}{(1+t)^{\lambda\nu}}, \;\; \forall t \geq t_0,$$

which is not necessarily the optimal decay.

Example 2.2 *Let* $g(t) = ae^{-(1+t)^{\nu}}$, $0 < \nu \leq 1$*. where* $0 < a < 1$ *is chosen so that* $\int_0^{+\infty} g(t)dt < 1$*. Then*

$$g'(t) = -a\nu(1+t)^{\nu-1}e^{-(1+t)^{\nu}} = -\xi(t)g(t)$$

where $\xi(t) = \nu(1+t)^{\nu-1}$ *which is a decreasing function and* $\xi(0) > 0$*. Therefore we can use* (2.6) *to deduce*

$$E(t) \leq Ce^{-\lambda(1+t)^{\nu}}, \;\; \forall t \geq t_0.$$

CHAPTER 3

GENERAL AND OPTIMAL DECAY FOR A QUASILINEAR VISCOELASTIC EQUATION

This chapter is devoted for the study of the quasilinear viscoelastic problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = 0, & \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. In Section 3.1, we introduce some assumptions needed throughout the chapter. In Section 3.2, we give some technical lemmas. The main result is given in Section Section 3.3.

3.1 Assumptions

In this section, we present some material needed in the proof of our result. Throughout this chapter, all the functionals and inequalities are defined and valid for $t \geq 0$ unless it stated otherwise.

We impose the following assumptions on ρ and g

$(H1)$ ρ is a constant that satisfies

$$0 < \rho \leq \frac{2}{n-2}, \text{ if } n \geq 3$$
$$0 < \rho, \text{ if } n = 1, 2.$$

$(H2)$ $g : \mathbb{R}_+ \longrightarrow (0, +\infty)$ is a nonincreasing differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

$(H3)$ There exist a nonincreasing differentiable function $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\xi(0) > 0$, and a constant $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi(t) g^p(t), \quad \forall t \geq 0.$$

We introduce the "modified" energy functional

$$E(t) := \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t),$$

where, for any $v \in L^1_{loc}([0,+\infty);L^2(\Omega))$, we set

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t)-v(s)\|_2^2 ds.$$

A direct differentiation, using (3.1) and calculations as in the proof of Lemma 2.1, leads to

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \leq \frac{1}{2}(g' \circ \nabla u)(t) \leq 0. \tag{3.2}$$

3.2 Technical Lemmas

In this section, we state, without proof, the global existence result of [12] and then set up certain lemmas that are required for the proof of our main theorem.

Definition 3.1 *A function $u \in \mathcal{C}^1([0,T];H_0^1(\Omega))$ is said to be a "weak" solution of* (3.1) *if it satisfies*

1. $$\int_\Omega |u_t|^\rho u_{tt}\phi dx + \int_\Omega \nabla u \cdot \nabla \phi dx + \int_\Omega \nabla u_{tt} \cdot \nabla \phi dx - \int_0^t g(t-s)\int_\Omega \nabla u(s) \cdot \nabla \phi dx ds = 0,$$

   for all $\phi \in H_0^1(\Omega)$ and for almost all $t \in [0,T]$, and

2. $u(x,0) = u_0(x)$ *and* $u_t(x,0) = u_1(x)$ *in* $H_0^1(\Omega)$.

Theorem 3.1 *Assume that* $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *and the conditions (H1)-(H3) hold. Then Problem* (3.1) *has a unique global "weak" solution*

$$u \in L^\infty(\mathbb{R}_+; H_0^1(\Omega)), \quad u_t \in L^\infty(\mathbb{R}_+; H_0^1(\Omega)), \quad u_{tt} \in L^2(\mathbb{R}_+; H_0^1(\Omega)). \quad (3.3)$$

Remark 3.1 (3.3) *implies that* $u \in \mathcal{C}^1(\mathbb{R}_+; H_0^1(\Omega))$.

Lemma 3.1 ([32]) *Under the assumptions* $(H1)-(H3)$, *the functional*

$$G_1(t) := \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \int_\Omega \nabla u . \nabla u_t dx$$

satisfies, along the solution of (3.1), *the inequality*

$$G_1'(t) \leq -\frac{l}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx + \frac{1-l}{2l}(g \circ \nabla u)(t). \quad (3.4)$$

Proof. Direct differentiation of G_1, using (3.1), yields

$$\begin{aligned} G_1'(t) = &-\int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u(t). \int_0^t g(t-s)\nabla u(s) ds dx \\ &+ \int_\Omega |\nabla u_t|^2 dx + \frac{1}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx. \end{aligned} \quad (3.5)$$

Applying Young's inequality for the second term on the right side of (3.5) then adding and subtracting $\nabla u(t)$ yields

$$\begin{aligned} &\int_\Omega \nabla u(t). \int_0^t g(t-s)\nabla u(s) ds dx \\ &\quad \leq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega \left(\int_0^t g(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|) ds\right)^2 dx. \end{aligned}$$

By exploiting

$$\int_0^t g(s)ds < \int_0^{+\infty} g(s)ds = 1-l, \tag{3.6}$$

Lemma 2.2 and

$$(a+b)^2 \leq (1+\eta)a^2 + \left(1+\frac{1}{\eta}\right)b^2, \quad \forall \eta > 0,$$

we arrive at

$$\begin{aligned}
&\int_\Omega \nabla u(t). \int_0^t g(t-s)\nabla u(s)dsdx \\
&\quad \leq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2 dx \\
&\qquad\qquad + \frac{1}{2}(1+\eta)(1-l)^2 \int_\Omega |\nabla u|^2 dx \\
&\quad \leq \frac{1}{2}\left[1+(1+\eta)(1-l)^2\right]\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t).
\end{aligned}$$

By taking $\eta = \frac{l}{1-l}$, we find

$$\int_\Omega \nabla u(t). \int_0^t g(t-s)\nabla u(s)dsdx \leq \frac{2-l}{2}\int_\Omega |\nabla u|^2 dx + \frac{1-l}{2l}(g\circ\nabla u)(t). \tag{3.7}$$

Inserting (3.7) in (3.5), estimate (3.4) is established. This result is proved for regular solutions. However, it also holds for weak solutions by a simple density argument. ∎

Lemma 3.2 ([32]) *Under the assumptions* $(H1)-(H3)$, *the functional*

$$G_2(t) := \int_\Omega \left(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\right)\int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, along the solution of (3.1) *and for any* $\delta_1, \delta_2 > 0$, *the estimate*

$$\begin{aligned} G_2'(t) \le (1+2(1-l)^2)\delta_1 \int_\Omega |\nabla u|^2 dx - \frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\int_\Omega |u_t|^{\rho+2}dx \\ + (1-l)\left(2\delta_1 + \frac{1}{2\delta_1}\right)(g\circ\nabla u)(t) - \frac{g(0)}{4\delta_2}\left(1+\frac{C_p}{\rho+1}\right)(g'\circ\nabla u)(t) \\ + \left[\delta_2 + c\frac{\delta_2}{\rho+1}(2E(0))^\rho - \left(\int_0^t g(s)ds\right)\right]\int_\Omega |\nabla u_t|^2 dx, \end{aligned} \tag{3.8}$$

where c *is a positive constant and* C_p *is the Poincaré constant.*

Proof. Differentiating G_2 and making use of (3.1), we find

$$\begin{aligned} G_2'(t) = &\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ &- \int_\Omega \left(\int_0^t g(t-s)\nabla u(t)ds\right).\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx \\ &- \left(\int_0^t g(s)ds\right)\int_\Omega |\nabla u_t|^2 dx - \int_\Omega \nabla u_t(t).\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ &- \frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx \\ &- \frac{1}{\rho+1}\left(\int_0^t g(s)ds\right)\int_\Omega |u_t|^{\rho+2}dx. \end{aligned} \tag{3.9}$$

We estimate every term in the right hand side of (3.9), using repeatedly Cauchy-Schwarz' inequality, Young's inequality, (3.6) and Lemma 2.2. The first term may be estimated as follows

$$\begin{aligned} &\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ &\qquad \le \delta_1 \int_\Omega |\nabla u|^2 dx + \frac{1-l}{4\delta_1}(g\circ\nabla u)(t),\ \forall \delta_1 > 0. \end{aligned} \tag{3.10}$$

For the second term, we recall 2.2 and the fact that $(a+b)^2 \leq 2(a^2+b^2)$ to get

$$
\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right) . \left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right) dx \\
&\leq \delta_1 \int_\Omega \left| \int_0^t g(t-s)\nabla u(s)ds \right|^2 dx + \frac{1}{4\delta_1}\int_\Omega \left| \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds \right|^2 dx \\
&\leq \delta_1 \int_\Omega \left(\int_0^t g(t-s)\left(\left| \nabla u(t)-\nabla u(s) \right| + |\nabla u(t)|\right) ds\right)^2 dx \\
&\quad + \frac{1}{4\delta_1}\left(\int_0^t g(t-s)ds\right)(g\circ \nabla u)(t) \\
&\leq 2\delta_1 \int_\Omega \left(\int_0^t g(t-s)\left| \nabla u(t)-\nabla u(s) \right| ds\right)^2 dx + 2\delta_1 \left(\int_0^t g(s)ds\right)^2 \int_\Omega |\nabla u|^2 dx \\
&\quad + \frac{1}{4\delta_1}\left(\int_0^t g(t-s)ds\right)(g\circ \nabla u)(t) \\
&\leq \left(2\delta_1 + \frac{1}{4\delta_1}\right)(1-l)(g\circ\nabla u)(t) + 2\delta_1(1-l)^2\int_\Omega |\nabla u|^2 dx.
\end{aligned}
\tag{3.11}
$$

Concerning the fourth term, we use Young's inequality to get, $\forall \delta_2 > 0$,

$$
\begin{aligned}
&\int_\Omega \nabla u_t(t)\cdot \int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&\qquad \leq \delta_2 \int_\Omega |\nabla u_t|^2 dx + \frac{g(0)}{4\delta_2}\int_\Omega\int_0^t -g'(t-s)|\nabla u(t)-\nabla u(s)|^2 dsdx.
\end{aligned}
\tag{3.12}
$$

The fifth term may be handled as follows:

$$
\begin{aligned}
&\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&\quad \leq \frac{1}{\rho+1}\left[\delta_2\int_\Omega |u_t|^{2(\rho+1)}dx + \frac{g(0)}{4\delta_2}C_p\int_\Omega\int_0^t -g'(t-s)|\nabla u(t)-\nabla u(s)|^2 dsdx\right],
\end{aligned}
\tag{3.13}
$$

where C_p is the Poincaré constant. By exploiting the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega) \text{ for } 0<\rho\leq \frac{2}{n-2} \text{ if } n\geq 3 \text{ and } \rho>0 \text{ if } n=1,\,2$$

and the fact that $E(t) \leq E(0)$, $\forall t \geq 0$, we obtain

$$\int_\Omega |u_t|^{2(\rho+1)} dx \leq c(2E(0))^\rho \int_\Omega |\nabla u_t|^2 dx. \tag{3.14}$$

Therefore (3.13) takes the form

$$\begin{aligned} &\frac{1}{\rho+1} \int_\Omega |u_t(t)|^\rho u_t(t) \int_0^t g'(t-s)(u(t)-u(s)) ds dx \\ &\quad \leq c\delta_2 (2E(0))^\rho \int_\Omega |\nabla u_t|^2 dx - \frac{g(0)C_p}{4\delta_2(\rho+1)} (g' \circ \nabla u)(t). \end{aligned} \tag{3.15}$$

Combining (3.9)-(3.12) and (3.15), estimate (3.8) is established. This result is proved for regular solutions. However, it also holds for weak solutions by a simple density argument. ∎

3.3 The Main Result

In this section we state our main decay theorem. But first, we adopt the following lemma

Lemma 3.3 ([32]) *Assume that $(H1)-(H3)$ hold. Then, for any $t_0 > 0$, there exist positive constants M, ε, m, such that the functional*

$$F = ME + \varepsilon G_1 + G_2$$

satisfies, for all $t \geq t_0$

$$F \sim E \tag{3.16}$$

and

$$F'(t) \leq -mE(t) + C(g \circ \nabla u)(t), \tag{3.17}$$

Proof. To prove (3.16), we use Young's inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\rho+2}(\Omega)$ to obtain

$$\begin{aligned}
|F(t) - ME(t)| \leq & \frac{\varepsilon}{\rho+2}||u_t||_{\rho+2}^{\rho+2} + \frac{\varepsilon}{(\rho+1)(\rho+2)}||u||_{\rho+2}^{\rho+2} + \frac{\varepsilon}{2}||\nabla u_t||_2^2 \\
& + \frac{\varepsilon}{2}||\nabla u||_2^2 + \frac{1}{2(\rho+1)}||u_t||_{2(\rho+1)}^{2(\rho+1)} + \frac{1-l}{2(\rho+1)}C_p(g \circ \nabla u)(t) \\
& + \frac{1}{2}||\nabla u_t||_2^2 + \frac{1-l}{2}(g \circ \nabla u)(t) \\
\leq & \varepsilon E(t) + \varepsilon \frac{C^{\rho+2}}{(\rho+1)(\rho+2)} 2^{\frac{\rho+2}{2}} l^{-\frac{\rho+2}{2}} (E(0))^{\frac{\rho}{2}} E(t) + \varepsilon(1 + l^{-1})E(t) \\
& + \frac{C_p^{2(\rho+1)}}{\rho+1} 2^{\rho} (E(0))^{\rho} E(t) + \frac{(1-l)}{2(\rho+1)} C_p E(t) + E(t) + (1-l)E(t).
\end{aligned}$$

By fixing M large enough, we obtain $|F(t) - ME(t)| \leq C_1 E(t)$ which gives the desired result.

For the proof of (3.17), we notice that, for $t \geq t_0 > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0 > 0 \quad \forall t \geq t_0,$$

which is true since g is continuous, positive and $g(0) > 0$. Now using (3.2), (3.4) and (3.8), we have

$$
\begin{aligned}
L'(t) \leq & \left[\frac{M}{2} - \frac{g(0)}{4\delta_2}\left(1 + \frac{C_p}{\rho+1}\right)\right](g' \circ \nabla u)(t) + \frac{\varepsilon - g_0}{\rho+1}\int_\Omega |u_t|^{\rho+2} dx \\
& - \left[\varepsilon\frac{l}{2} - (1 + 2(1-l)^2)\delta_1\right]\int_\Omega |\nabla u|^2 dx \\
& - \left[g_0 - \varepsilon - \delta_2 - c\delta_2(2E(0))^\rho\right]\int_\Omega |\nabla u_t|^2 dx \\
& + (1-l)\left(\frac{\varepsilon}{2l} + 2\delta_1 + \frac{1}{2\delta_1}\right)(g \circ \nabla u)(t), \ \ \forall t \geq t_0.
\end{aligned} \tag{3.18}
$$

At this point, we choose our constant carefully. First, we pick $\varepsilon < g_0$, and then δ_1 and δ_2 small enough that

$$
\varepsilon\frac{l}{2} - (1 + 2(1-l)^2)\delta_1 > 0, \qquad g_0 - \varepsilon - \delta_2 - c\delta_2(2E(0))^\rho > 0.
$$

Then we take M sufficiently large so that (3.16) remains valid and

$$
\frac{M}{2} - \frac{g(0)}{4\delta_2}\left(1 + \frac{C_p}{\rho+1}\right) \geq 0.
$$

Therefore, (3.18) reduces to (3.17) for two positive constants m and C. ∎

Theorem 3.2 *Assume that $(u_0, u_1) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and the assumptions (H1)-(H3) hold. Then for each $t_0 > 0$, there exist strictly positive constants C and λ such that the solution of (3.1) satisfies, for all $t \geq t_0$,*

$$
E(t) \leq Ce^{-\lambda \int_{t_0}^t \xi(s)ds}, \qquad p = 1, \tag{3.19}
$$

$$
E(t) \leq C\left[\frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s)ds}\right]^{\frac{1}{2p-2}}, \qquad p > 1. \tag{3.20}
$$

Moreover, if

$$\int_0^{+\infty} \left[\frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s)ds} \right]^{\frac{1}{2p-2}} dt < +\infty, \qquad 1 < p < \frac{3}{2}, \tag{3.21}$$

then

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^t \xi^{p}(s)ds} \right]^{\frac{1}{p-1}}, \qquad p > 1. \tag{3.22}$$

Proof. The proof goes exactly like that of Theorem 2.2 by repeating the steps from (2.13) until the end of the proof. ∎

Remark 3.2 *Note that our result and that of [10] agree in giving the optimal decay for the polynomial case in a certain range* $(1 < p < \frac{3}{2})$*. However, we obtain our result directly, without solving any extra ODE. In addition, we do not see how their result can be applied in a direct way to Example 2.2.*

CHAPTER 4

GENERAL AND OPTIMAL DECAY FOR A VISCOELASTIC EQUATION WITH BOUNDARY FEEDBACK

In this chapter, we investigate the following viscoelastic wave equation with boundary feedback:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds = 0, & \text{in } \Omega \times (0,+\infty) \\ u = 0, & \text{on } \Gamma_0 \times (0,+\infty) \\ \dfrac{\partial u}{\partial \nu} - \displaystyle\int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds + h(u_t) = 0, & \text{on } \Gamma_1 \times (0,+\infty) \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint, with $meas(\Gamma_0) > 0$, ν is the unit outward normal to $\partial\Omega$, and g, h are specific functions. In Section 4.1, we present some assumptions needed throughout the chapter. In Section 4.2, we adopt some technical lemmas. The main result and its proof are given in Section 4.3. Finally, two illustrative examples are given in Section 4.4.

4.1 Assumptions

In this section, we introduce our assumptions and the "modified" energy functional. Throughout this chapter, all the functionals and inequalities are defined and valid for $t \geq 0$ unless it stated otherwise.

We impose the following assumptions

$(H1)$ $g : \mathbb{R}_+ \longrightarrow (0, +\infty)$ is a nonincreasing differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

$(H2)$ There exist a nonincreasing differentiable function $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\xi(0) > 0$, and a constant $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi(t) g^p(t), \quad \forall t \geq 0.$$

$(H3)$ $h : \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing continuous function such that there exist a strictly increasing function $h_0 \in C^1([0, +\infty))$, with $h_0(0) = 0$, and positive constants c_1, c_2, ε such that

$$h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|), \;\; \forall\, |s| \leq \varepsilon$$

$$c_1|s| \leq |h(s)| \leq c_2|s|, \;\; \forall\, |s| \geq \varepsilon.$$

In addition, we assume that the function H, defined by $H(s) = \sqrt{s}h_0(\sqrt{s})$, is a strictly convex C^2 function on $(0, r^2]$, for some $r > 0$, when h_0 is nonlinear.

Remark 4.1 *Hypothesis (H3) implies that $sh(s) > 0$, $\forall s \neq 0$.*

The "modified" energy functional corresponding to the problem (4.1) is

$$E(t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) \|\nabla u\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t),$$

where, for any $v \in L^2_{loc}(\mathbb{R}_+; L^2(\Omega))$, we set

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2 ds.$$

A direct differentiation, using (4.1) and some manipulation as in the proof of Lemma 2.1, leads to

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \int_{\Gamma_1} u_t h(u_t) d\Gamma \leq 0. \tag{4.2}$$

4.2 Technical Lemmas

In this section, we state the global existence result of [12] and then we set up certain lemmas that are required for the proof of our main theorem.

Let $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$.

Definition 4.1 *A function $u \in \mathcal{C}([0,T]; V)$ with $u_t \in \mathcal{C}([0,T]; L^2(\Omega))$ is said to be a "weak" solution of (4.1) if it satisfies*

1.

$$\int_\Omega u_{tt}\phi dx + \int_\Omega \nabla u \cdot \nabla \phi dx - \int_0^t g(t-s) \int_\Omega \nabla u(s) \cdot \nabla \phi dx ds + \int_{\Gamma_1} h(u_t)\phi d\Gamma = 0,$$

for all $\phi \in V$ and for almost all $t \in [0,T]$, and

2. *$u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$ for all $x \in \Omega$.*

Theorem 4.1 *Let $(u_0, u_1) \in V \times L^2(\Omega)$ be given. Assume that (H1)-(H3) are satisfied. Then problem (4.1) has a unique global "weak" solution*

$$u \in \mathcal{C}(\mathbb{R}_+; V) \cap \mathcal{C}^1(\mathbb{R}_+; L^2(\Omega)).$$

Moreover, if

$$(u_0, u_1) \in (H^2(\Omega) \cap V) \times V$$

and satisfies the compatibility condition

$$\frac{\partial u_0}{\partial \nu} + h(u_1) = 0 \quad on\, \Gamma_1,$$

then the "strong" solution satisfies

$$u \in L^{\infty}(\mathbb{R}_+; H^2(\Omega) \cap V) \cap W^{1,\infty}(\mathbb{R}_+; V) \cap W^{2,\infty}(\mathbb{R}_+; L^2(\Omega)).$$

Lemma 4.1 ([31]) *Let u be the solution of* (4.1). *Assume that (H1)-(H3) hold. Then the functional*

$$G_1(t) := \int_{\Omega} u u_t dx$$

satisfies

$$G_1'(t) \leq -\frac{l}{2}\|\nabla u\|_2^2 + \|u_t\|_2^2 + C(g \circ \nabla u)(t) + C\int_{\Gamma_1} h^2(u_t) d\Gamma.$$

Proof. Differentiating G_1 and making use of (4.1) yield

$$\begin{aligned}
G_1'(t) &= \int_{\Omega} u_t^2 dx + \int_{\Omega} u\Delta u dx - \int_{\Omega} u \int_0^t g(s)\Delta u(t-s) ds dx \\
&\leq \int_{\Omega} u_t^2 dx - l\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \nabla u. \int_0^t g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\
&\quad - \int_{\Gamma_1} u h(u_t) d\Gamma.
\end{aligned}$$

Using Young's inequality, followed by Lemma 2.2, gives

$$\begin{aligned}
&\int_{\Omega} \nabla u. \int_0^t g(s)(\nabla u(t-s) - \nabla u(t)) ds dx \\
\leq\ & \delta\int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\delta}\int_{\Omega}\left(\int_0^t g(s)|\nabla u(t-s) - \nabla u(t)| ds\right)^2 dx \\
\leq\ & \delta\int_{\Omega} |\nabla u|^2 dx + \frac{c}{\delta}(g \circ \nabla u)(t).
\end{aligned}$$

Also, using Young's and Poincaré's inequalities and the trace theorem, we get

$$\begin{aligned} -\int_{\Gamma_1} uh(u_t)d\Gamma &\leq \delta \int_{\Gamma_1} u^2 d\Gamma + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma \\ &\leq C\delta \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma. \end{aligned}$$

Therefore

$$G_1'(t) \leq -(l - C\delta)\|\nabla u\|_2^2 + \|u_t\|_2^2 + \frac{c}{\delta}(g \circ \nabla u)(t) + \frac{1}{4\delta}\int_{\Gamma_1} h^2(u_t)d\Gamma.$$

Choosing δ small enough completes the proof. This result is established for strong solutions. However, it also holds for weak solutions by a simple density argument.

▮

Lemma 4.2 ([31]) *Under the assumptions (H1)-(H3), the functional*

$$G_2(t) := -\int_{\Omega} u_t(t)\int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, along the solution and for any $\delta > 0$,

$$G_2'(t) \leq -(\int_0^t g(s)ds - \delta)\|u_t\|_2^2 + \delta\|\nabla u\|_2^2 + \frac{C}{\delta}(g \circ \nabla u)(t) - \frac{C}{\delta}(g' \circ \nabla u)(t) + C\int_{\Gamma_1} h^2(u_t)d\Gamma.$$

Proof. By exploiting (4.1) and integrating by parts, we have

$$\begin{aligned} G_2'(t) &= \int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ &\quad -\int_\Omega (\int_0^t g(t-s)\nabla u(s)ds)\cdot(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds)dx \\ &\quad +\int_{\Gamma_1}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)h(u_t)d\Gamma \\ &\quad -\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx-(\int_0^t g(s)ds)\int_\Omega u_t^2dx \\ &= (1-\int_0^t g(s)ds)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\ &\quad +\int_\Omega \left|\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\right|^2 dx \\ &\quad +\int_{\Gamma_1}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)h(u_t)d\Gamma \\ &\quad -\int_\Omega u_t(t)\int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx-(\int_0^t g(s)ds)\int_\Omega u_t^2dx. \end{aligned}$$

Employing Young's and Poincaré's inequalities and Lemma 2.2, we get

$$(1-\int_0^t g(s)ds)\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \le \delta\int_\Omega |\nabla u|^2dx+\frac{c}{\delta}(g\circ\nabla u)(t),$$

$$\int_{\Gamma_1}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)h(u_t)d\Gamma \le C(g\circ\nabla u)(t)+C\int_{\Gamma_1}h^2(u_t)d\Gamma,$$

and

$$-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx \le \delta\int_\Omega u_t^2dx-\frac{c}{\delta}(g'\circ\nabla u)(t).$$

Combination all the above inequalities yield the assertion of the lemma. This result is established for strong solutions. However, it also holds for weak solutions by a simple density argument. ∎

Lemma 4.3 ([31]) *Under the assumptions (H1)-(H3), the solution satisfies the estimate*

$$\int_{\Gamma_1} h^2(u_t)d\Gamma \leq -CE'(t), \ \ \forall t \geq t_0, \tag{4.3}$$

if h_0 is linear; and

$$\int_{\Gamma_1} h^2(u_t)d\Gamma \leq CH^{-1}(\lambda(t)) - CE'(t), \ \ \forall t \geq t_0, \tag{4.4}$$

if h_0 is nonlinear, where

$$\lambda(t) := \frac{1}{|\Gamma_{12}|}\int_{\Gamma_{12}} u_t h(u_t)d\Gamma$$

and

$$\Gamma_{12} = \{x \in \Gamma_1 : |u_t| \leq \varepsilon_1\}.$$

Proof. If h_0 is linear then hypothesis $(H3)$ yields

$$c_1'|s| \leq |h(s)| \leq c_2'|s|, \ \ \forall s,$$

and so, in the view of Remark 4.1, we get

$$h^2(s) \leq c_2' s h(s), \ \ \forall s \in \mathbb{R}.$$

Integrating over Γ_1 and using (4.2) gives (4.3).

In the case of h_0 is nonlinear, we first assume that $\max\{r, h_0(r)\} < \varepsilon$; otherwise we take r smaller. Let $\varepsilon_1 = \min\{r, h_0(r)\}$; then for $\varepsilon_1 \leq |s| \leq \varepsilon$, using $(H3)$, we have

$$|h(s)| \leq \frac{h_0^{-1}(|s|)}{|s|}|s| \leq \frac{h_0^{-1}(\varepsilon)}{\varepsilon_1}|s| \quad \text{and} \quad |h(s)| \geq \frac{h_0(|s|)}{|s|}|s| \geq \frac{h_0(\varepsilon_1)}{\varepsilon}|s|.$$

So, we deduce that

$$\begin{cases} h_0(|s|) \leq |h(s)| \leq h_0^{-1}(|s|), & \text{for all } |s| \leq \varepsilon_1 \\ c_1'|s| \leq |h(s)| \leq c_2'|s|, & \text{for all} |s| \geq \varepsilon_1. \end{cases} \tag{4.5}$$

Since $H(s^2) = |s|h_0(|s|)$, then, using (4.5), we obtain

$$H(h^2(s)) \leq sh(s) \quad \forall\, |s| \leq \varepsilon_1,$$

which gives

$$h^2(s) \leq H^{-1}(sh(s)) \quad \forall\, |s| \leq \varepsilon_1. \tag{4.6}$$

We consider the following partition of Γ_1:

$$\Gamma_{11} = \{x \in \Gamma_1 : |u_t| > \varepsilon_1\}, \quad \Gamma_{12} = \{x \in \Gamma_1 : |u_t| \leq \varepsilon_1\}$$

Recalling the definition of ε_1 and using (4.5), we obtain on Γ_{12}

$$u_t h(u_t) \leq \varepsilon_1 h_0^{-1}(\varepsilon_1) \leq h_0(r)r = H(r^2) \tag{4.7}$$

and

$$u_t h(u_t) \leq \varepsilon_1 h_0^{-1}(\varepsilon_1) \leq r h_0^{-1}(h_0(r)) = r^2. \tag{4.8}$$

Applying Jensen's inequality on $\lambda(t)$ leads to

$$H^{-1}(\lambda(t)) \leq C \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) d\Gamma. \tag{4.9}$$

Thus, using (4.5)-(4.9), we get

$$\begin{aligned}
\int_{\Gamma_1} h^2(u_t) d\Gamma &= \int_{\Gamma_{12}} h^2(u_t) d\Gamma + \int_{\Gamma_{11}} h^2(u_t) d\Gamma \\
&\leq \int_{\Gamma_{12}} H^{-1}(u_t h(u_t)) d\Gamma + C \int_{\Gamma_{11}} u_t h(u_t) d\Gamma \\
&\leq C H^{-1}(\lambda(t)) - C E'(t),
\end{aligned}$$

which proves (4.4). ∎

4.3 The Main Result

In this section we state our main decay theorem. But, first, we need to adopt the following lemma

Lemma 4.4 ([31], Inequality (3.7)) *For any $t_0 > 0$, there exist positive constants N_1, N_2, m, such that the functional F, given by*

$$F(t) := N_1 E(t) + G_1 + N_2 G_2,$$

is equivalent to E and satisfies

$$F'(t) \le -mE(t) + C(g \circ \nabla u)(t) + C \int_{\Gamma_1} h^2(u_t) d\Gamma, \qquad \forall t \ge t_0. \tag{4.10}$$

Proof. Since g is continuous, positive and $g(0) > 0$, then, for $t_0 > 0$, we have

$$\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds =: g_0 > 0 \quad \forall t \ge t_0.$$

By using (4.2), (4.1) and (4.2), we easily see that

$$\begin{aligned} F'(t) \le & -\frac{l}{4}|| \nabla u||_2^2 - \left(N_2 g_0 - \frac{l}{4} - 1\right) ||u_t||_2^2 + \left(\frac{4C}{l} N_2^2 + C\right)(g \circ \nabla u)(t) \\ & + \left(\frac{1}{2}N_1 - \frac{4C}{l}N_2^2\right)(g' \circ \nabla u)(t) + (CN_2 + C)\int_{\Gamma_1} h^2(u_t)d\Gamma, \ \forall t \ge t_0. \end{aligned}$$

First we choose N_2 large enough in order that

$$\alpha := N_2 g_0 - \frac{l}{4} - 1 > 0$$

and then N_1 large enough so that

$$\frac{1}{2}N_1 - \frac{4C}{l}N_2^2 > 0.$$

So, we arrive at

$$F'(t) \le -\frac{l}{4}|| \nabla u||_2^2 - \alpha||u_t||_2^2 + C(g \circ \nabla u)(t) + C\int_{\Gamma_1} h^2(u_t)d\Gamma, \ \forall t \ge t_0. \tag{4.11}$$

Therefore, (4.11) reduces to (4.10), for two positive constants m and C. On the other hand (see Lemma 2.3), we can choose N_1 even larger (if needed) so that

$$F \sim E. \tag{4.12}$$

∎

We are ready now to state and prove the main result of this chapter.

Theorem 4.2 *Let $(u_0, u_1) \in V \times L^2(\Omega)$ be given. Assume that (H1)-(H3) are satisfied. Then there exist strictly positive constants k_1, k_2, k_3 and k_4 such that the solution of (4.1) satisfies, for all $t \geq t_0$,*

$$E(t) \leq k_1 H_1^{-1}\left(k_2 \int_{t_0}^{t} \xi(s)ds\right), \qquad p = 1, \tag{4.13}$$

and

$$E(t) \leq k_3 H_1^{-1}\left(k_4 \int_{t_0}^{t} \xi^{2p-1}(s)ds\right), \qquad 1 < p < \frac{3}{2}. \tag{4.14}$$

Moreover, if

$$\int_0^{+\infty} H_1^{-1}\left(k_4 \int_{t_0}^{t} \xi^{2p-1}(s)ds\right) dt < +\infty, \qquad 1 < p < \frac{3}{2}, \tag{4.15}$$

then

$$E(t) \leq k_3 (\hat{H}_1)^{-1}\left(k_4 \int_{t_0}^{t} \xi^{p}(s)ds\right), \qquad p > 1. \tag{4.16}$$

where

$$H_1(t) = \int_t^1 \frac{1}{s^{2p-1}H'(\varepsilon_0 s)}ds \quad and \quad \hat{H}_1(t) = \int_t^1 \frac{1}{s^p H'(\varepsilon_0 s)}ds.$$

Remark 4.2 *Simple calculations show that* (4.14), (4.15) *and the fact that* H_1 *is strictly decreasing on* $(0,1]$ *yield*

$$\int_{t_0}^{+\infty} E(t)dt < +\infty.$$

Proof. Multiplying (4.10) by $\xi(t)$ gives

$$\xi(t)F'(t) \leq -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) + C\xi(t)\int_{\Gamma_1} h^2(u_t)d\Gamma, \ \ \forall t \geq t_0. \quad (4.17)$$

Case of $p = 1$. Recalling $(H2)$ and (4.2), we obtain, from (4.17) and for all $t \geq t_0$

$$\begin{aligned} \xi(t)F'(t) &\leq -m\xi(t)E(t) + C(\xi g \circ \nabla u)(t) + C\xi(t)\int_{\Gamma_1} h^2(u_t)d\Gamma \\ &\leq -m\xi(t)E(t) - C(g' \circ \nabla u)(t) + C\xi(t)\int_{\Gamma_1} h^2(u_t)d\Gamma \\ &\leq -m\xi(t)E(t) - CE'(t) + C\xi(t)\int_{\Gamma_1} h^2(u_t)d\Gamma. \quad (4.18) \end{aligned}$$

Now, when h_0 is linear then in light of (4.3), estimate (4.18) gives

$$\xi(t)F'(t) \leq -m\xi(t)E(t) - CE'(t), \ \ \forall t \geq t_0.$$

which leads to

$$\xi(t)F'(t) + CE'(t) \leq -m\xi(t)E(t), \ \ \forall t \geq t_0.$$

Let $L_1(t) := \xi(t)F(t) + CE(t)$ then clearly $L_1 \sim E$ and we have, for some $m_1 > 0$,

$$L_1'(t) \leq -m_1\xi(t)L_1(t), \ \ \forall t \geq t_0.$$

By a simple integration, we arrive at

$$L_1(t) \leq Ce^{-m_1 \int_{t_0}^{t} \xi(s)ds}, \ \ \forall t \geq t_0.$$

and hence (4.13) by virtue of $L_1 \sim E$.

If h_0 is nonlinear, then we use (4.4) and the boundedness of ξ in (4.18) to get

$$L_2'(t) \leq -m\xi(t)E(t) + C\xi(t)H^{-1}(\lambda(t)), \ \ \forall t \geq t_0.$$

where $L_2 = \xi F + CE$ which is clearly equivalent to E.

When $p = 1$, we refer the reader to Messaoudi and Mustafa [31]. So we only consider the case $p > 1$.

Case of h_0 is linear. To establish (4.14), we consider (4.17) and use (4.3) and the fact that ξ is bounded to get

$$\xi(t)F'(t) \leq -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) - CE'(t), \ \ \forall t \geq t_0.$$

Let $L(t) := \xi(t)F(t) + CE(t)$ then clearly $L \sim E$ and, we have

$$L'(t) \leq -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t), \ \ \forall t \geq t_0. \tag{4.19}$$

Use of Corollary 1.1 in (4.19) gives

$$L'(t) \ \ \leq \ \ -m\xi(t)E(t) + C\left[-E'(t)\right]^{\frac{1}{2p-1}}, \ \ \forall t \geq t_0.$$

Multiplication of the last inequality by $\xi^{\alpha}E^{\alpha}(t)$, where $\alpha = 2p-2$, leads to

$$\xi^{\alpha}E^{\alpha}(t)L'(t) \leq -m\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C\left(\xi E\right)^{\alpha}(t)\left[-E'(t)\right]^{\frac{1}{\alpha+1}}, \ \ \forall t \geq t_0.$$

Use of Young's inequality, with $q = \alpha + 1$ and $q' = \frac{\alpha+1}{\alpha}$, yields

$$\begin{aligned} \xi^{\alpha}E^{\alpha}(t)L'(t) \ &\leq \ -m\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C\left[\varepsilon\xi^{\alpha+1}(t)E^{\alpha+1}(t) - C_{\varepsilon}E'(t)\right] \quad (4.20) \\ &= \ -(m - \varepsilon C)\xi^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t), \quad \forall \varepsilon > 0, \ \ \forall t \geq t_0. \end{aligned} \tag{4.21}$$

We then choose $\varepsilon < \frac{m}{C}$, and recall that $\xi' \leq 0$ and $E' \leq 0$, to get

$$\left(\xi^\alpha E^\alpha L\right)'(t) \leq \xi^\alpha(t)E^\alpha(t)L'(t) \leq -c_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t), \;\; \forall t \geq t_0; \quad (4.22)$$

which implies

$$\left(\xi^\alpha E^\alpha L + CE\right)'(t) \leq -c_1\xi^{\alpha+1}(t)E^{\alpha+1}(t), \;\; \forall t \geq t_0. \quad (4.23)$$

Let $W = \xi^\alpha E^\alpha L + CE \sim E$. Then

$$W'(t) \leq -C\xi^{\alpha+1}(t)W^{\alpha+1}(t) = -C\xi^{2p-1}(t)W^{2p-1}(t), \;\; \forall t \geq t_0. \quad (4.24)$$

Integrating over (t_0, t) and using the fact that $W \sim E$, we obtain

$$E(t) \leq C\left[\frac{1}{\int_{t_0}^t \xi^{2p-1}(s)ds + 1}\right]^{\frac{1}{2p-2}} \qquad \forall t \geq t_0. \quad (4.25)$$

Since, in this case, $H(s) = \sqrt{s}h_0(\sqrt{s}) = cs$ we have $H_1(t) = \frac{C}{2p-2}(t^{2p-2} - 1)$ and

$$E(t) \leq C\left[\frac{1}{\int_{t_0}^t \xi^{2p-1}(s)ds + 1}\right]^{\frac{1}{2p-2}} = C_1 H_1^{-1}\left(C_2\int_{t_0}^t \xi^{2p-1}(s)ds\right) \qquad \forall t \geq t_0.$$

To establish (4.16), we consider (4.19) and recall Remark 4.2. So, we have

$$\begin{aligned} L'(t) &\le -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) \\ &= -m\xi(t)E(t) + C\frac{\eta(t)}{\eta(t)}\int_0^t [\xi^p(s)g^p(s)]^{\frac{1}{p}}\|\nabla u(t) - \nabla u(t-s)\|_2^2, \ \ \forall t \ge t_0, \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} \eta(t) = \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds &\le C\int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2 ds \\ &\le C\int_0^t [E(t) + E(t-s)]\, ds \le 2C\int_0^t E(t-s)ds \\ &= 2C\int_0^t E(s)ds < 2C\int_0^{+\infty} E(s)ds < +\infty. \end{aligned} \tag{4.27}$$

Applying Jensen's inequality (1.12) for the second term of the right hand side of (4.26), with $G(y) = y^{\frac{1}{p}}$, $y > 0$, $f(s) = \xi^p(s)g^p(s)$ and $K(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2$, we get, for all $t \ge t_0$,

$$L'(t) \le -m\xi(t)E(t) + C\eta(t)\left[\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}}, \tag{4.28}$$

where we assume that $\eta(t) > 0$, otherwise we get $\|\nabla u(t) - \nabla u(t-s)\| = 0$ and hence we have, from (4.10) and (4.3),

$$E(t) \leq Ce^{-mt}, \ \forall t \geq t_0.$$

Therefore, we obtain

$$\begin{aligned} L'(t) &\leq -m\xi(t)E(t) + C\eta^{\frac{p-1}{p}}(t)\left[\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}} \\ &\leq -m\xi(t)E(t) + C(-g' \circ \nabla u)^{\frac{1}{p}}(t) \leq -m\xi(t)E(t) + C(-E'(t))^{\frac{1}{p}}, \ \forall t \geq t_0. \end{aligned} \tag{4.29}$$

Multiplying by $\xi^\alpha(t)E^\alpha(t)$, for $\alpha = p-1$, and repeating the same computations as in above, we arrive at

$$E(t) \leq C\left[\frac{1}{\int_{t_0}^t \xi^p(s)ds + 1}\right]^{\frac{1}{p-1}}, \quad \forall t \geq t_0. \tag{4.30}$$

Since, in this case, $H(s) = \sqrt{s}h_0(\sqrt{s}) = cs$ we have $\hat{H}_1(t) = \frac{C}{p-1}(t^{p-1} - 1)$. and

$$E(t) \leq C\left[\frac{1}{\int_{t_0}^t \xi^p(s)ds + 1}\right]^{\frac{1}{p-1}} = C_1\hat{H}_1^{-1}\left(C_2\int_{t_0}^t \xi^p(s)ds\right), \quad \forall t \geq t_0.$$

Case of h_0 is nonlinear. Again we consider (4.17) and use (4.4) to get

$$L_2'(t) \leq -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) + C\xi(t)H^{-1}(\lambda(t)), \ \forall t \geq t_0. \tag{4.31}$$

where $L_2 = \xi F + CE$ which is clearly equivalent to E.

From Corollary 1.1, we obtain

$$L_2'(t) \leq -m\xi(t)E(t) + C\left[-E'(t)\right]^{\frac{1}{2p-1}} + C\xi(t)H^{-1}(\lambda(t)), \;\; \forall t \geq t_0. \tag{4.32}$$

Multiplying (4.32) by $\xi^\alpha(t)E^\alpha(t)$, where $\alpha = 2p-2$ and repeating the calculations as in (4.20)-(4.24), we arrive at

$$W_1' \leq -m\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C\xi^{\alpha+1}(t)E^\alpha(t)H^{-1}(\lambda(t)), \;\; \forall t \geq t_0 \tag{4.33}$$

where $W_1 = \xi^\alpha E^\alpha L_2 + CE$ and is also equivalent to E. For $\varepsilon_0 < r^2$ and $c_0 > 0$, let

$$F_1(t) := H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) W_1(t) + c_0 E(t), \;\; \forall t \geq t_0.$$

Clearly F_1 satisfies, for some positive constants α_1, α_2,

$$\alpha_1 F_1(t) \leq E(t) \leq \alpha_2 F_1(t), \;\; \forall t \geq t_0. \tag{4.34}$$

and

$$\begin{aligned} F_1'(t) &= \varepsilon_0 \frac{E'(t)}{E(0)} H''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) W_1(t) + H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) W_1'(t) + c_0 E'(t) \\ &\leq -m\xi^{\alpha+1}(t)E^{\alpha+1}(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + C\xi^{\alpha+1}(t)E^\alpha(t)H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}(\lambda(t)) \\ &+ c_0 E'(t), \;\; \forall t \geq t_0 \end{aligned} \tag{4.35}$$

Let $H^*(s) := \sup_{\tau\in(0,r^2]}\{s\tau - H(\tau)\}$ for $s \in (0, H'(r^2)]$ be the dual function of H. From $(H3)$ we conclude that H' is increasing and defines a bijection from $(0, r^2]$ to $(0, H'(r^2)]$ and then for any $s \in (0, H'(r^2)]$, the function $\tau \mapsto s\tau - H(\tau)$ reaches its maximum on $(0, r^2]$ at the unique point $(H'(s))^{-1}$. Hence

$$H^*(s) = s(H')^{-1}(s) - H\left((H')^{-1}(s)\right) \quad \forall s \in (0, H'(r^2)]$$

and $H^*(s)$ satisfies the general Young inequality:

$$AB \le H^*(A) + H(B) \quad \forall A \in (0, H'(r^2)], B \in (0, r^2]. \tag{4.36}$$

We apply (4.36) on the second term on the right hand side of (4.35) with $A = H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = H^{-1}(\lambda(t))$ and use (4.2) and the fact that $H^*(s) \le s(H')^{-1}(s)$ to arrive at

$$\begin{aligned}
F_1'(t) \;&\le\; -m\xi^{\alpha+1}(t)E^{\alpha+1}(t)H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \varepsilon_0 C\xi^{\alpha+1}(t)\frac{E^{\alpha+1}(t)}{E(0)}H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\
&+\; C\xi^{\alpha+1}(t)E^{\alpha}(t)\lambda(t) + c_0E'(t)\\
&\le\; -(mE^{\alpha+1}(0) - \varepsilon_0 C)\xi^{\alpha+1}(t)\left(\frac{E(t)}{E(0)}\right)^{\alpha+1} H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\\
&+\; (c_0 - C\xi^{\alpha+1}(0)E^{\alpha}(0))E'(t), \;\; \forall t \ge t_0.
\end{aligned}$$

With a proper choice of ε_0 and c_0, we get

$$F_1'(t) \leq -C\xi^{\alpha+1}(t)\left(\frac{E(t)}{E(0)}\right)^{\alpha+1} H'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -C\xi^{2p-1}(t)H_2\left(\frac{E(t)}{E(0)}\right), \ \forall t \geq t_0 \tag{4.37}$$

where $H_2(\tau) := \tau^{\alpha+1}H'(\varepsilon_0\tau) = \tau^{2p-1}H'(\varepsilon_0\tau)$.

From the properties of H and keeping in mind that $p > 1$, we find that

$$H_2'(\tau) = (2p-1)\tau^{2p-2}H'(\varepsilon_0\tau) + \varepsilon_0\tau^{2p-1}H''(\varepsilon_0\tau) > 0, \ \forall \tau \in (0,1].$$

Therefore the functional Ψ defined by

$$\Psi(t) := \frac{\alpha_1 F_1(t)}{E(0)}$$

is equivalent to E and, in addition, taking in account (4.34) and (4.37), we obtain

$$\Psi'(t) \leq -C\xi^{2p-1}(t)H_2(\Psi(t)), \ \forall t \geq t_0.$$

Thus

$$-\int_{t_0}^{t}\frac{\Psi'(s)}{H_2(\Psi(s))}ds \geq C\int_{t_0}^{t}\xi^{2p-1}(s)ds, \ \forall t \geq t_0,$$

and with substitution $y = \Psi(t)$ on the left hand side, we get

$$\int_{\Psi(t)}^{1}\frac{1}{H_2(y)}dy \geq \int_{\Psi(t)}^{\Psi(t_0)}\frac{1}{H_2(y)}dy \geq C\int_{t_0}^{t}\xi^{2p-1}(s)ds, \ \forall t \geq t_0. \tag{4.38}$$

Since $H_2(\tau) > 0$ for all $\tau \in (0, 1]$, the function H_1 defined as

$$H_1(\tau) := \int_\tau^1 \frac{1}{H_2(s)} ds,$$

is strictly decreasing on $(0, 1]$, thus, using $\Psi \sim E$, (4.38) becomes

$$E(t) \leq C_1 H_1^{-1} \left(C_2 \int_{t_0}^t \xi^{2p-1}(s) ds \right), \quad \forall t \geq t_0. \tag{4.39}$$

To establish (4.16) we consider (4.31) and repeat all the steps of (4.26)-(4.29) to reach

$$L_2'(t) \leq -m\xi(t)E(t) + C(-E'(t))^{\frac{1}{p}} + C\xi(t)H^{-1}(\lambda(t)), \;\; \forall t \geq t_0.$$

Multiplication of the last inequality by $\xi^\alpha(t)E^\alpha(t)$ where $\alpha = p-1$ and repeating, again, the same procedure as in (4.33)-(4.38) we arrive at

$$E(t) \leq C_3 \hat{H}_1^{-1} \left(C_4 \int_{t_0}^t \xi^p(s) ds \right), \;\; \forall t \geq t_0.$$

where $\hat{H}_1(\tau) := \int_\tau^1 \frac{1}{\hat{H}_2(s)} ds$ and $\hat{H}_2(s) = s^p H'(\varepsilon_0 s)$. This completes the proof of our main result. ∎

4.4 Examples

The following examples illustrate our result and show the optimal decay rate in the polynomial case.

Example 4.1 *Let* $g(t) = \frac{a}{(1+t)^\nu}$, $\nu > 2$, *where* $a > 0$ *is a constant so that* $\int_0^{+\infty} g(t)dt < 1$ *and assume that* h_0 *is linear. We have*

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^\nu}\right)^{\frac{\nu+1}{\nu}} = -bg^p(t), \quad p = \frac{\nu+1}{\nu} < \frac{3}{2}, \quad b > 0.$$

Therefore (4.15), *with* $\xi(t) = b$ *and* $H_1^{-1}(t) = \left(\frac{1}{Ct+1}\right)^{\frac{1}{2p-2}}$, *yields* $\int_0^{+\infty} \left(\frac{1}{Cb^{2p-1}t+1}\right)^{\frac{1}{2p-2}} dt < +\infty$ *and hence by* (4.16) *we get*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{p-1}}} = \frac{C}{(1+t)^\nu},$$

which is the optimal decay rate.

Example 4.2 *If* $h_0(s) = s^q$ *where* $q > 1$ *then* $H(s) = s^{\frac{q+1}{2}}$ *is a strictly convex* C^2 *function on* $(0, \infty)$. *Therefore Therorem 4.2 is applicable and, with* $H_1^{-1}(t) = (Ct+1)^{-\frac{2}{q+4p-5}}$, *we obtain*

$$\begin{aligned} E(t) &\leq k_1\left(k_2\int_{t_0}^t \xi(s)ds\right)^{-\frac{2}{q-1}} \quad \text{if } p = 1, \\ E(t) &\leq k_3\left(k_4\int_{t_0}^t \xi^{2p-1}(s)ds\right)^{-\frac{2}{q+4p-5}} \quad \text{if } 1 < p < \frac{3}{2}. \end{aligned}$$

If (4.15) *is satisfied, i.e.* $\int_0^{+\infty} (Ct\xi^{2p-1}(t)+1)^{-\frac{2}{q+4p-5}} dt < +\infty$, *then we have the* *improved decay rate*

$$E(t) \leq k_3 \left(k_4 \int_{t_0}^{t} \xi^p(s)ds \right)^{-\frac{2}{q+4p-5}} \quad \textit{if } 1 < p < \frac{3}{2}.$$

CHAPTER 5

GENERAL AND OPTIMAL DECAY FOR A VISCOELASTIC EQUATION WITH A NONLINEAR TERM

In this chapter, we investigate the following viscoelastic problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + a|u_t|^{m-2}u_t = 0, & \text{in } \Omega \times (0,+\infty) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $m > 1$ and $a > 0$ are constants and g is a nonincreasing function. In Section 5.1, we introduce the assumptions needed in this chapter. Some technical lemmas are presented in Section 5.2. The main result and its proof are given in Section 5.3. Finally, some examples will given in Section 5.4.

5.1 Assumptions

In this section we present some material needed in the proof of our result. Throughout this chapter, all the functionals and inequalities are defined and valid for $t \geq 0$ unless it stated otherwise.

We impose the following hypotheses on m and g

$(H1)$ $g : \mathbb{R}_+ \longrightarrow (0, +\infty)$ is a nonincreasing differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)ds = l > 0.$$

$(H2)$ There exist a nonincreasing differentiable function $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, with $\xi(0) > 0$, and a constant $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0.$$

$(H3)$ m is a constant such that

$$1 < m \leq \frac{2n}{n-2}, \text{ if } n > 2 \text{ and } m > 1, \text{ if } n = 1, 2.$$

We introduce the "modified" energy functional

$$E(t) := \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t),$$

where, for any $v \in L^1_{loc}([0, +\infty); L^2(\Omega))$, we set

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2 ds.$$

A direct differentiation, using (5.1), leads to

$$\begin{aligned} E'(t) &= -a\int_\Omega |u_t|^m dx + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ &\leq -a\int_\Omega |u_t|^m dx + \frac{1}{2}(g' \circ \nabla u)(t) \leq 0, \end{aligned} \tag{5.2}$$

This inequality is established for regular solutions. However, it also holds for weak solutions by a simple density argument.

We end this section with the following proposition, which will be used in the proof of our stability result.

Proposition 5.1 ([25]) *Let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be an increasing $\mathcal{C}^2$-function such that*

$$\phi(0) = 0 \ \textit{and} \ \lim_{t\to+\infty}\phi(t) = +\infty.$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_{\tau}^{+\infty} \psi^{q+1}(t)\phi'(t)dt \leq A\psi(\tau), \;\; 0 \leq \tau < +\infty,$$

then we have, for all $t \geq 0$,

$$\psi(t) \;\; \leq \;\; c\psi(0)(1+\phi(t))^{-\frac{1}{q}}, \;\; if\, q > 0,$$

$$\psi(t) \;\; \leq \;\; c\psi(0)e^{-\omega\phi(t)}, \;\; if\, q = 0,$$

where c and ω are positive constants independent of $\psi(0)$.

5.2 Technical Lemmas

In this section, we state, without proof, the global existence result of [30] and then establish certain lemmas required for the proof of our main theorem.

Definition 5.1 *A function $u \in \mathcal{C}([0,T]; H_0^1(\Omega))$ with $u_t \in \mathcal{C}([0,T]; L^2(\Omega)) \cap L^m(\Omega \times [0,T])$ is said to be a "weak" solution of (5.1) if it satisfies*

1. $$\int_{\Omega} u_{tt}\phi dx + \int_{\Omega} \nabla u \cdot \nabla \phi dx - \int_0^t g(t-s) \int_{\Omega} \nabla u(s) \cdot \nabla \phi dx ds + a \int_{\Omega} |u_t|^{m-2} u_t \phi dx = 0,$$

   for all $\phi \in H_0^1(\Omega)$ and for almost all $t \in [0,T]$, and

2. $u(x,0)=u_0(x)$ *in* $H_0^1(\Omega)$ *and* $u_t(x,0)=u_1(x)$ *in* $L^2(\Omega)$.

Theorem 5.1 ([30]) *Let* $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *be given. Assume that (H1) and (H3) are satisfied. Then Problem* (5.1) *has a unique global solution*

$$u\in\mathcal{C}(\mathbb{R}_+;H_0^1(\Omega))$$

$$u_t\in\mathcal{C}(\mathbb{R}_+;L^2(\Omega))\cap L^m(\Omega\times(0,\infty)).$$

Moreover, if $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$*; then the weak solution becomes "strong" solution in the sense that*

$$u\in\mathcal{C}(\mathbb{R}_+;H^2(\Omega)\cap H_0^1(\Omega)),\;\; u_t\in\mathcal{C}(\mathbb{R}_+;H_0^1(\Omega))\cap L^m(\Omega\times(0,\infty)),\;\; u_{tt}\in\mathcal{C}(\mathbb{R}_+;L^2(\Omega)).$$

Lemma 5.1 ([30]) *Under the assumptions* $(H1)-(H3)$, *the functional*

$$\Psi(t):=\int_\Omega uu_t dx$$

satisfies, along the solution, the estimate

$$\begin{aligned}\psi'(t)\le -\frac{l}{4}||\;\nabla u||_2^2+||u_t||_2^2+\frac{1-l}{2l}(g\circ\nabla u)(t)\\ +C\int_\Omega |u_t|^m dx,\qquad \text{if}\, m\ge 2\end{aligned}\tag{5.3}$$

and

$$\psi'(t) \le -\frac{l}{4}||\ \nabla u||_2^2 + ||u_t||_2^2 + \frac{1-l}{2l}(go\nabla u)(t) \\ + c(\delta, ||a||_\infty, \Omega)\left(\int_\Omega a(x)|u_t|^m dx\right)^{\frac{2m-2}{m}}, \qquad \textit{if } m < 2. \tag{5.4}$$

Proof. Case $m \ge 2$**.**

By using (5.1), we easily see that

$$\psi'(t) = ||u_t||_2^2 - ||\ \nabla u||_2^2 - a\int_\Omega |u_t|^{m-2}u_t u dx \\ + \int_\Omega \nabla u(t). \int_0^t g(s)\nabla u(t-s) ds dx. \tag{5.5}$$

We now estimate the third term of the right hand side of (5.5), using Young's inequality and $(H3)$,

$$\int_\Omega |u_t|^{m-2}u_t u dx \le \delta \int_\Omega |u|^m dx + C_\delta \int_\Omega |u_t|^m dx \\ \le \delta C||\nabla u||_2^{m-2}||\nabla u||_2^2 + C_\delta \int_\Omega |u_t|^m dx \\ \le \delta C E^{\frac{m-2}{2}}(0)||\nabla u||_2^2 + C_\delta \int_\Omega |u_t|^m dx. \tag{5.6}$$

The fourth term in the right hand side of (5.5) can be handled as follows

$$\int_\Omega \nabla u(t). \int_0^t g(t-s)\nabla u(s) ds dx \\ \le \frac{1}{2}||\nabla u||_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s)| ds\right)^2 dx \\ \le \frac{1}{2}||\nabla u||_2^2 + \frac{1}{2}\int_\Omega\left(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| + |\nabla u(t)| ds\right)^2 dx. \tag{5.7}$$

We then use Cauchy−Schwarz inequality, Young's inequality, Lemma 2.2 and the fact that

$$\int_0^t g(s)ds < \int_0^{+\infty} g(s)ds = 1-l,$$

to obtain, for any $\eta > 0$,

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)| + |\nabla u(t)|ds\right)^2 dx \\
&\quad \le \int_\Omega \left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2 dx + \int_\Omega \left(\int_0^t g(t-s)|\nabla u(t)|ds\right)^2 dx \\
&\qquad + 2\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)\left(\int_0^t g(t-s)|\nabla u(t)|ds\right)dx \\
&\quad \le \left(1+\frac{1}{\eta}\right)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|ds\right)^2 dx \\
&\qquad + (1+\eta)\int_\Omega \left(\int_0^t g(t-s)|\nabla u(t)|ds\right)^2 dx \\
&\quad \le \left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t) + (1+\eta)(1-l)^2||\nabla u||^2.
\end{aligned} \tag{5.8}$$

By combining (5.5)-(5.8), we arrive at

$$\begin{aligned}
\psi'(t) \le ||u_t||_2^2 - \frac{1}{2}\left[1-(1+\eta)(1-l)^2 - 2\delta C E^{\frac{m-2}{2}}(0)\right] ||\ \nabla u||_2^2 \\
+ \frac{1}{2}\left(1+\frac{1}{\eta}\right)(1-l)(g\circ\nabla u)(t) + C_\delta \int_\Omega |u_t|^m dx.
\end{aligned} \tag{5.9}$$

By choosing $\eta = \frac{l}{1-l}$ and $\delta = \frac{l}{4CE^{\frac{m-2}{2}}(0)}$, (5.3) is established.

For the case of $m < 2$, we re-estimate (5.6), as follows

$$\begin{aligned}
&\int_\Omega |u_t|^{m-2}u_t u dx \\
&\quad \le \delta||u||^2 + C_\delta \int_\Omega |u_t|^{2m-2}dx \\
&\quad \le \delta C||\nabla u||^2 + C_\delta \left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}.
\end{aligned} \tag{5.10}$$

By combining (5.5), (5.7), (5.8), (5.10) and choosing the same values of η and δ, (5.4) is obtained. ∎

Lemma 5.2 ([30]) *Under the assumptions* $(H1)-(H3)$*, the functional*

$$\chi(t) := -\int_\Omega u_t \int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, along the solution and for any $\delta > 0$*, the estimate*

$$\begin{aligned}\chi'(t) \leq \delta[1+2(1-l)^2]\|\nabla u\|_2^2 + \left(\delta - \int_0^t g(s)ds\right)\|u_t\|_2^2 + C_\delta(g \circ \nabla u)(t) \\ + \frac{g(0)}{4\delta}(-(g' \circ \nabla u))(t) + C_\delta \int_\Omega |u_t|^m dx, \qquad \text{if } m \geq 2\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\chi'(t) \leq \delta[1+2(1-l)^2]\|\nabla u\|_2^2 + \left(\delta - \int_0^t g(s)ds\right)\|u_t\|_2^2 + C_\delta(g \circ \nabla u)(t) \\ + \frac{g(0)}{4\delta}(-(g' \circ \nabla u))(t) + C_\delta \left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \quad \text{if } m < 2.\end{aligned} \tag{5.12}$$

Proof. **Case** $m \geq 2$**.**

By using (5.1), we easily see that

$$\begin{aligned}\chi'(t) &= \int_\Omega \nabla u(t). \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\right) dx \\ &- \int_\Omega \left(\int_0^t g(t-s)\nabla u(s)ds\right). \left(\int_0^t g(t-s)(u(t)-u(s))ds\right) dx \\ &+ a\int_\Omega |u_t|^{m-2}u_t \int_0^t g(t-s)(u(t)-u(s))dsdx \\ &- \int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx - \left(\int_0^t g(s)ds\right)\int_\Omega u_t^2 dx.\end{aligned} \tag{5.13}$$

Similarly to (5.5), we estimate the right hand side terms of (5.13). So for any $\delta > 0$, we have the estimate of the first term

$$\int_\Omega \nabla u(t). \left(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\right) dx \le \delta || \nabla u||_2^2 + \frac{1-l}{4\delta}(g \circ \nabla u)(t), \tag{5.14}$$

the second term

$$\begin{aligned}
&\int_{\Omega}\left(\int_0^t -g(t-s)\nabla u(s)ds\right).\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)dx \\
&\quad \leq \delta\int_{\Omega}\left(\int_0^t g(t-s)ds|\nabla u(s)|ds\right)^2 dx \\
&\quad + \frac{1}{4\delta}\int_{\Omega}\left(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\right)^2 dx \\
&\quad \leq \delta\int_{\Omega}\left(\int_0^t g(t-s)ds\left(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|ds\right)\right)^2 dx \\
&\quad + \frac{1}{4\delta}\left(\int_0^t g(t-s)ds\right)\int_{\Omega}\int_0^t g(t-s)\left|\left|\nabla u(t)-\nabla u(s)\right|\right|^2 dsdx \\
&\quad \leq 2\delta\int_{\Omega}\left(\int_0^t g(t-s)ds\left(|\nabla u(t)-\nabla u(s)|ds\right)\right)^2 dx \\
&\quad + 2\delta(1-l)^2\int_{\Omega}|\nabla u|^2 dx + \frac{1}{4\delta}(1-l)(g\circ\nabla u)(t) \\
&\quad \leq \left(2\delta+\frac{1}{4\delta}\right)(1-l)(g\circ\nabla u)(t) + 2\delta(1-l)^2||\ \nabla u||_2^2,
\end{aligned} \tag{5.15}$$

the third term

$$\begin{aligned}
&\int_{\Omega}|u_t|^{m-2}u_t\int_0^t g(t-s)(u(t)-u(s))dsdx \\
&\quad \leq \delta\int_{\Omega}\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^m dx + C_\delta\int_{\Omega}|u_t|^m dx \\
&\quad \leq \delta\left(\int_0^t g(t-s)ds\right)^{m-1}\int_{\Omega}\int_0^t g(t-s)|u(t)-u(s)|^m dsdx + C_\delta\int_{\Omega}|u_t|^m dx \\
&\quad \leq \delta(1-l)^{m-1}C\int_0^t g(t-s)||\nabla u(t)-\nabla u(s)||_2^m ds + C_\delta\int_{\Omega}|u_t|^m dx \\
&\quad \leq \delta(1-l)^{m-1}C_p\left(\frac{2E(0)}{l}\right)^{\frac{m-2}{2}}(g\circ\nabla u)(t) + C_\delta\int_{\Omega}|u_t|^m dx
\end{aligned} \tag{5.16}$$

and the forth term

$$-\int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))dsdx \le \delta||u_t||_2^2 + \frac{g(0)}{4\delta} C_p \int_\Omega (-g' \circ \nabla u)(t). \tag{5.17}$$

A combination of (5.13)-(5.17) then yields (5.11).

For the **Case** $m < 2$**,** we re-estimate (5.16), as follows

$$\begin{aligned} &\int_\Omega |u_t|^{m-2} u_t \int_0^t g(t-s)(u(t)-u(s))dsdx \\ &\le \delta C(g \circ \nabla u)(t) + C_\delta \left(\int_\Omega |u_t| dx \right)^{\frac{2m-2}{m}}. \end{aligned} \tag{5.18}$$

Hence, a combination of (5.13)-(5.15), (5.17) and (5.18) then gives (5.12). ∎

5.3 The Main Results

In this section we state and prove our main results. For this purpose we introduce the following lemma

Lemma 5.3 ([30]) *Assume that* $(H1)-(H3)$*. Then, for any* $t_0 > 0$*, there exist strictly positive constants* ε_1, ε_2, m_1, m_2, c_1, C *such that the functional*

$$F(t) = E(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t)$$

satisfies, $\forall t \ge t_0$,

$$F \sim E, \tag{5.19}$$

$$F'(t) \leq -m_1 E(t) + C(g \circ \nabla u)(t), \qquad if\, m \geq 2, \tag{5.20}$$

and

$$L'(t) \leq -m_2 E(t) + C(g \circ \nabla u)(t) + c_1 \left(\int_\Omega |u_t|^m dx \right)^{\frac{2m-2}{m}}, \qquad if\, 1 < m < 2. \tag{5.21}$$

Proof. The proof of (5.19) is the same proof of Lemma 2.5.

Now, we prove inequality (5.20). Since $g(0) > 0$, then there exists t_0 such that

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds := g_0 > 0, \;\; \forall t \geq t_0.$$

By using (5.2), (5.3) and (5.11), we obtain

$$\begin{aligned} F'(t) \leq & -\left[1 - (\varepsilon_1 + \varepsilon_2)C_\delta\right] a \int_\Omega |u_t|^m dx \\ & - \left[\varepsilon_2 \left(g_0 - \delta\right) - \varepsilon_1\right] ||u_t||_2^2 - \left[\frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta (1 + 2(1-l)^2)\right] ||\; \nabla u||_2^2 \\ & + \left[\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C\right] (g' \circ \nabla u)(t) + \left[\varepsilon_1 \frac{1-l}{2l} + \varepsilon_2 C_\delta\right] (g \circ \nabla u)(t). \end{aligned} \tag{5.22}$$

At this point we choose δ so small that

$$g_0 - \delta > \frac{1}{2} g_0$$

and

$$\frac{4}{l} \delta \left(1 + 2(1-l)^2\right) < \frac{1}{4} g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2}g_0\varepsilon_2 \tag{5.23}$$

will make

$$k_1 := \varepsilon_2\left(g_0 - \delta\right) - \varepsilon_1 > 0$$

and

$$k_2 := \frac{\varepsilon_1 l}{4} - \varepsilon_2\delta\left(1 + 2(1-l)^2\right) > 0.$$

We then pick ε_1 and ε_2 so small that (5.19) and (5.23) remain valid and, further,

$$1 - (\varepsilon_1 + \varepsilon_2)C_\delta > 0$$

and

$$\frac{1}{2} - \varepsilon_2\frac{g(0)}{4\delta}C > 0$$

Therefore (5.20) is established for two positive constants $m_1, C > 0$.

The same calculations, for $m < 2$, using (5.2), (5.4) and (5.12), give (5.21). ∎

Theorem 5.2 *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that (H1)-(H3) are satisfied and* $\underline{\mathbf{m \geq 2}}$*. Then for each* $t_0 > 0$*, there exist strictly positive con-*

stants K and λ such that the solution of (5.1) *satisfies, for all* $t \geq t_0$,

$$E(t) \leq K e^{-\lambda \int_{t_0}^{t} \xi(s) ds}, \qquad \text{if } p = 1, \tag{5.24}$$

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(s) ds} \right]^{\frac{1}{2p-2}}, \qquad \text{if } p > 1. \tag{5.25}$$

Moreover, if

$$\int_{0}^{+\infty} \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(s) ds} \right]^{\frac{1}{2p-2}} dt < +\infty, \qquad 1 < p < \frac{3}{2}, \tag{5.26}$$

then

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{p}(s) ds} \right]^{\frac{1}{p-1}}, \qquad p > 1. \tag{5.27}$$

Proof. The proof goes exactly like that of Theorem 2.2 by repeating the steps from (2.13) until the end of the proof. ∎

Remark 5.1 *The same results hold for* (5.1)*, with* $a = 0$.

Theorem 5.3 *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *be given. Assume that (H1) and (H2) are satisfied and* $\underline{\mathbf{1 < m < 2}}$. *Then for each* $t_0 > 0$*, there exist strictly positive constants* K *and* λ *such that the solution of* (5.1) *satisfies, for all* $t \geq t_0$,

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^{t} \xi^{2p-1}(s) ds} \right]^{(2m-2) \Big/ (2-m+4(p-1)(m-1))}, \qquad 1 \leq p < \frac{3}{2}. \tag{5.28}$$

Moreover, if

$$\int_0^{+\infty} \left[\frac{1}{1+\int_{t_0}^{t} \xi^{2p-1}(s)ds}\right]^{(2m-2)\Big/(2-m+4(p-1)(m-1))} dt < +\infty, \qquad 1 < p < \frac{3}{2}, \tag{5.29}$$

then

$$E(t) \le K \left[\frac{1}{1+\int_{t_0}^{t} \xi^{p}(s)ds}\right]^{(2m-2)\Big/(2-m+2(p-1)(m-1))}, \qquad 1 < p < \frac{3}{2}. \tag{5.30}$$

Remark 5.2 *The inequalities* (5.28) *and* (5.29) *yield*

$$\int_{t_0}^{+\infty} E(t)dt < +\infty.$$

Proof. Multiplying (5.21) by $\xi(t)$ gives

$$\xi(t)F'(t) \le -m_2\xi(t)E(t) + C\xi(t)(g\circ\nabla u)(t) + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \ \forall t \ge t_0. \tag{5.31}$$

Case of $p = 1$. Recalling $(H2)$ and (5.2), we obtain, from (5.31), for all

$t \geq t_0$,

$$\begin{aligned}
\xi(t)F'(t) &\leq -m_2\xi(t)E(t) + C(\xi g \circ \nabla u)(t) + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}} \\
&\leq -m\xi(t)E(t) - C(g' \circ \nabla u)(t) + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}} \\
&\leq -m\xi(t)E(t) - CE'(t) + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}
\end{aligned}$$

which leads to

$$(\xi F + CE)'(t) \leq -m_2\xi(t)E(t) + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \ \forall t \geq t_0. \tag{5.32}$$

Let $L(t) := \xi(t)F(t) + CE(t)$ then clearly $L \sim E$ and we have

$$m_2\xi(t)E(t) \leq -L'(t) + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \ \forall t \geq t_0.$$

We multiply by $E^k(t)$ where $k = \frac{2-m}{2m-2}$ and use (5.2) and the fact that $E \sim L$ to get

$$m_2\xi(t)E^{k+1}(t) \leq -CL^k(t)L'(t) + c_1\xi(t)E^k(t)\left(-E'(t)\right)^{\frac{1}{k+1}}, \ \forall t \geq t_0.$$

Applying Young's inequality with $q = k+1$ and $q^* = \frac{k+1}{k}$ yields, for every $\varepsilon > 0$,

$$m_2\xi(t)E^{k+1}(t) \leq -CL^k(t)L'(t) + \varepsilon c_1\xi(t)E^{k+1}(t) - c_1C_\varepsilon\xi(t)E'(t), \ \forall t \geq t_0.$$

We choose $\varepsilon < \frac{m_2}{c_1}$ to obtain

$$\xi(t)E^{k+1}(t) \leq -CL^k(t)L'(t) - C\xi(t)E'(t), \ \ \forall t \geq t_0. \tag{5.33}$$

By recalling that $\xi'(t) \leq 0$ and integrating (5.33) over (S,T), $S \geq t_0$, we get

$$\int_S^T \xi(t)E^{k+1}(t)dt \leq CL^{k+1}(S) + CE(S) \leq AE(S), \ \ \forall t \geq t_0.$$

for some positive constant A. Therefore, Proposition 5.1 gives, for all $t \geq t_0$,

$$E(t) \leq K\left[\frac{1}{1+\int_{t_0}^t \xi(s)ds}\right]^{\frac{1}{k}} = K\left[\frac{1}{1+\int_{t_0}^t \xi(s)ds}\right]^{\frac{2m-2}{2-m}}$$

which is the estimate (5.28) with $p = 1$.

Case of $p > 1$. We again consider (5.31) and use Corollary 1.1 to get

$$\xi(t)F'(t) \ \ \leq \ \ -m_2\xi(t)E(t) + C\left[-E'(t)\right]^{\frac{1}{2p-1}} + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \ \ \forall t \geq t_0.$$

Multiplication of the last inequality by $\xi^\alpha E^\alpha(t)$, where $\alpha = 2p-2$,

$$\begin{aligned}\xi^{\alpha+1}E^\alpha(t)F'(t) \ \ &\leq \ \ -m_2\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C(\xi E)^\alpha(t)\left[-E'(t)\right]^{\frac{1}{2p-1}} \\ &\quad + c_1\xi^{\alpha+1}(t)E^\alpha(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}, \ \ \forall t \geq t_0.\end{aligned}$$

Use of Young's inequality, with $q = \alpha + 1$ and $q^* = \frac{\alpha+1}{\alpha}$, yields

$$\begin{aligned}
\xi^{\alpha+1} E^{\alpha}(t) F'(t) \;&\leq\; -(m_2 - \varepsilon C)\xi^{\alpha+1}(t) E^{\alpha+1}(t) - C E'(t) \\
&\quad + c_1 \xi^{\alpha+1}(t) E^{\alpha}(t) \left(\int_{\Omega} |u_t|^m dx \right)^{\frac{2m-2}{m}}, \;\; \forall t \geq t_0.
\end{aligned}$$

We then choose $\varepsilon < \frac{m_2}{C}$, and recall that $\xi' \leq 0$ and $E' \leq 0$, to get

$$\begin{aligned}
\left(\xi^{\alpha+1} E^{\alpha} F\right)'(t) \;&\leq\; \xi^{\alpha+1}(t) E^{\alpha}(t) F'(t) \\
&\leq\; -m_4 \xi^{\alpha+1}(t) E^{\alpha+1}(t) - C E'(t) \\
&\quad + c_1 \xi^{\alpha+1}(t) E^{\alpha}(t) \left(\int_{\Omega} |u_t|^m dx \right)^{\frac{2m-2}{m}}, \;\; \forall t \geq t_0;
\end{aligned}$$

which implies

$$\left(\xi^{\alpha+1} E^{\alpha} F + C E\right)'(t) \leq -m_4 \xi^{\alpha+1}(t) E^{\alpha+1}(t) + c_1 \xi^{\alpha+1}(t) E^{\alpha}(t) \left(\int_{\Omega} |u_t|^m dx \right)^{\frac{2m-2}{m}}.$$

Let $W = \xi^{\alpha+1} E^{\alpha} F + C E \sim E$. Then

$$m_4 \xi^{\alpha+1}(t) E^{\alpha+1}(t) \leq -W'(t) + c_1 \xi^{\alpha+1}(t) E^{\alpha}(t) \left(\int_{\Omega} |u_t|^m dx \right)^{\frac{2m-2}{m}}, \;\; \forall t \geq t_0.$$

We Multiply the last inequality by $E^k(t)$ where $k = \frac{2-m}{2m-2}$ and use (5.2) and the fact that $E \sim W$ to get

$$m_4 \xi^{\alpha+1}(t) E^{k+\alpha+1}(t) \leq -C W^k(t) W'(t) + c_1 \xi^{\alpha+1}(t) E^{\alpha}(t) E^k(t) \left(\int_{\Omega} -E'(t) \right)^{\frac{1}{k+1}}, \;\; \forall t \geq t_0. \tag{5.34}$$

Again we use Young's inequality with $q = k+1$ and $q^* = \frac{k+1}{k}$ to obtain, for every $\varepsilon > 0$,

$$m_4 \xi^{\alpha+1}(t) E^{k+\alpha+1}(t) \leq -W^k(t) W'(t) + \varepsilon c_1 \xi^{\alpha+1}(t) E^{k+\alpha+1}(t) - c_1 C_\varepsilon \xi^{\alpha+1}(t) E^\alpha E'(t), \ \ \forall t \geq t_0. \tag{5.35}$$

By the choice $\varepsilon < \frac{m_4}{c_1}$ we have

$$\xi^{\alpha+1}(t) E^{k+\alpha+1}(t) \leq -CW^k(t) W'(t) - C\xi^{\alpha+1}(t) E^\alpha E'(t), \ \ \forall t \geq t_0. \tag{5.36}$$

By recalling that $\xi'(t) \leq 0$ and integrating over (S,T), $S \geq t_0$, we reach

$$\int_S^T \xi^{\alpha+1}(t) E^{k+\alpha+1}(t) dt \leq CW^{k+1}(S) + CE(S) \leq AE(S), \ \ \forall t \geq t_0. \tag{5.37}$$

for some positive constant A. Therefore, Proposition 5.1 gives, for all $t \geq t_0$,

$$E(t) \leq K \left[\frac{1}{1 + \int_{t_0}^t \xi^{\alpha+1}(s) ds} \right]^{\frac{1}{k+\alpha}} = K \left[\frac{1}{1 + \int_{t_0}^t \xi^{2p-1}(s) ds} \right]^{(2m-2) \Big/ (2-m+4(p-1)(m-1))},$$

which is the estimate (5.28).

To establish (5.30), we consider (5.31) and recall Remark 5.2. So, we have

$$\begin{aligned} \xi(t) F'(t) &\leq -m_2 \xi(t) E(t) + C\xi(t)(g \circ \nabla u)(t) + c_1 \xi(t) \left(\int_\Omega |u_t|^m dx \right)^{\frac{2m-2}{m}} \\ &= -m_2 \xi(t) E(t) + C \frac{\eta(t)}{\eta(t)} \int_0^t [\xi^p(s) g^p(s)]^{\frac{1}{p}} \|\nabla u(t) - \nabla u(t-s)\|_2^2 \\ &+ c_1 \xi(t) \left(\int_\Omega |u_t|^m dx \right)^{\frac{2m-2}{m}}, \end{aligned} \tag{5.38}$$

where

$$\eta(t) = \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2 ds \leq C\int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2 ds$$
$$\leq C\int_0^t [E(t) + E(t-s)]\, ds \leq 2C\int_0^t E(t-s)ds$$
$$= 2C\int_0^t E(s)ds < 2C\int_0^{+\infty} E(s)ds < +\infty.$$

Applying Jensens's inequality (1.12) for the second term of the right hand side of (5.38), with $G(y) = y^{\frac{1}{p}}$, $y > 0$, $f(s) = \xi^p(s)g^p(s)$ and $h(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2$, to get

$$\xi(t)F'(t) \leq -m_2\xi(t)E(t) + C\eta(t)\left[\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}}$$
$$+ c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}$$

where we assume that $\eta(t) > 0$. (For the case of $\eta(t) \equiv 0$, see Remark 5.3 below). Therefore, we obtain

$$\xi(t)F'(t) \leq -m_2\xi(t)E(t) + C\eta^{\frac{p-1}{p}}(t)\left[\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2 ds\right]^{\frac{1}{p}}$$
$$+ c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}$$
$$\leq -m_2\xi(t)E(t) + C(-g' \circ \nabla u)^{\frac{1}{p}}(t)$$
$$\leq -m_2\xi(t)E(t) + C(-E'(t))^{\frac{1}{p}} + c_1\xi(t)\left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}$$

Multiplying by $\xi^\alpha(t)E^\alpha(t)$, for $\alpha = p-1$, and repeating the same computations as in above, we arrive at

$$E(t) \leq C\left[\frac{1}{1+\int_{t_0}^{t} \xi^{\alpha+1}(s)ds}\right]^{\frac{1}{k+\alpha}} = C\left[\frac{1}{1+\int_{t_0}^{t} \xi^{p}(s)ds}\right]^{(2m-2)\Big/(2-m+2(p-1)(m-1))}, \quad \forall t \geq t_0.$$

This completes the proof of Theorem 5.2. ∎

Remark 5.3 *In* (5.38), *if* $\eta(t) \equiv 0$ *then* $\|\nabla u(t) - \nabla u(t-s)\| = 0$ *and hence from* (5.21) *we have*

$$F'(t) \leq -m_2 E(t) + c_1 \left(\int_\Omega |u_t|^m dx\right)^{\frac{2m-2}{m}}.$$

Multiplying by $E^k(t)$ *where* $k = \frac{2-m}{2m-2}$ *and using the same ideas in* (5.34)-(5.39), *we get*

$$\int_S^T E^{k+1}(t)dt \leq AE(S), \ \ \forall t \geq t_0, \tag{5.39}$$

for some positive constant A. Therefore, Proposition 5.1, with $\phi(t) = t$, *gives*

$$E(t) \leq K\left[\frac{1}{1+t}\right]^{\frac{2m-2}{2-m}}, \quad \forall t \geq t_0.$$

5.4 Examples

The following examples illustrate our results and show the optimal decay rate in the polynomial case.

Example 5.1 *Let* $g(t) = \frac{a}{(1+t)^\nu}$, $\nu > 2$, *where* $a > 0$ *is a constant so that*

$\int_0^{+\infty} g(t)dt < 1$. *We have*

$$g'(t) \;=\; -\frac{a\nu}{(1+t)^{\nu+1}} = -b\left(\frac{a}{(1+t)^{\nu}}\right)^{\frac{\nu+1}{\nu}} = -bg^p(t), \quad p = \frac{\nu+1}{\nu} < \frac{3}{2}, \quad b > 0.$$

If $m \geq 2$ *then* (5.26), *with* $\xi(t) = b$, *yields* $\int_0^{+\infty} \left(\frac{1}{b^{2p-1}t+1}\right)^{\frac{1}{2p-2}} dt < +\infty$ *and hence by* (5.27) *we get*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{p-1}}} = \frac{C}{(1+t)^{\nu}},$$

which is the optimal decay obtained in [36].

If $1 < m < 2$ *then, by* (5.28), *we have*

$$E(t) \leq C(1+t)^{-(2m-2)\Big/(2-m+4(p-1)(m-1))} = C(1+t)^{-\nu(2m-2)\Big/(2\nu-\nu m+4(m-1))}.$$

Example 5.2 *Let* $g(t) = e^{-at}$, *where* a *is chosen so that* $\int_0^{+\infty} g(t)dt < 1$. *Then* $g'(t) = -ag(t)$ *and hence* (5.24) *and* (5.28), *with* $\xi(t) = a$, *assert that,* $\forall t \geq t_0$,

$$E(t) \;\leq\; Ke^{-\lambda t}, \qquad \text{if } m \geq 2$$

$$E(t) \;\leq\; K\left[\frac{1}{1+t}\right]^{\frac{2m-2}{2-m}}, \qquad \text{if } 1 < m < 2.$$

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

In this work we considered four viscoelastic problems and obtained general decay rates for the energy associated to the each problem where the relaxation functions satisfy a relation of the form

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \geq 0,\ 1 \leq p < \frac{3}{2}. \tag{6.1}$$

Our decay results generalize many existing results in the literature as follow: When $p = 1$, the estimates (2.6), (3.19), (4.13), (5.24) and (5.28) cover the results in [28, 30, 31]. When $1 < p < \frac{3}{2}$ and $\xi \equiv 1$, the estimates (2.9), (3.22), (4.16), (5.27) and (5.30) generalize the polynomial decay rates in [12, 27, 35] among others.

To the best of our knowledge, the decay rates in Theorems 2.2, 3.2, 4.2, 5.2 and 5.3, when $1 < p < \frac{3}{2}$, are obtained for the first time in the literature. Moreover, these decay rates lead to the optimal rate of decay in the polynomial decay as shown in Examples 2.1, 4.1 and 5.1.

6.2 Future Work

Investigating other viscoelastic problems

There exist viscoelastic problems that can be studied for relaxation functions satisfying (6.1). Here are some examples:

1. Viscoelastic Kirchhoff Equations.
2. Moore-Gibson-Thompson Equation with Memory.
3. Coupled System of Nonlinear Viscoelastic Equations.

The multiplier method and the technical lemmas in our dissertation are strongly expected to pave the way for a change to general decay rates that will improve many results in the literature.

Cauchy viscoelastic problem

In [42], Said-Houari and Messaoudi investigated the following problem

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds = 0, & x \in \mathbb{R}^n,\ t \geq 0, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{6.2}$$

where g satisfies

$$g'(t) \leq -\xi(t)g(t), \quad t \geq 0,$$

where $\xi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nonincreasing differentiable function. The authors used the energy method in the Fourier space to establish general decay results. It is interesting to study this problem considering relaxation functions that satisfy (6.1).

Extending the range of optimality

In (6.1), if ξ is a constant then g could be decaying even for $p \in [1, 2)$ instead of what we assumed $p \in [1, \frac{3}{2})$. It would be interesting to extend the range of optimality in (6.1) using the ideas of [20].

Lebesgue and Sobolev spaces with variable exponents

Another interesting research direction is to study the well-posedness as well as the asymptotic behavior of Problem (5.1) with $m = m(x)$. This requires replacing the usual Lebesgue space $L^p(\Omega)$ and Sobolev space $W^{1,p}(\Omega)$ by the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ and the Sobolev space $W^{1,p(.)}(\Omega)$. The field of variable exponent function spaces has witnessed an explosive growth in recent years. The standard reference article for basic properties is only 25 years old.

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