# ANALYSIS OF THIN PLATES SUPPORTED BY IRREGULAR LAYOUTS AND SHAPES OF INTERNAL SUPPORTS BY BOUNDARY POINT METHOD

BY

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I dedicate this work to my family members, the people of the Sudan, and all educational institutions I have passed through to reach this stage

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# LIST OF ABBREVIATIONS

| | | |
|---|---|---|
| **BEM** | : | Boundary Element Method |
| **BPM** | : | Boundary Point Method |
| **DBEM** | : | Direct Boundary Element Method |
| **FDM** | : | Finite Difference Method |
| **FEM** | : | Finite Element Method |
| **KLPT** | : | Kirchhoff–Love plate theory |
| **RBF** | : | Radial Basis Function |

# ABSTRACT

Full Name : Abubakr Elfatih Siddeeg Musa

Thesis Title : Analysis of Thin Plates Supported By irregular Layouts and Shapes of Internal Supports By Boundary Point Method

Major Field : Civil Engineering

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Analytical solutions for the bending of plates are limited to plates with simple geometries and supporting conditions. The analysis of plates with internal supports is much more complicated and therefore, researchers resort to numerical solutions such as the finite difference method (FDM), the finite element method (FEM) or the boundary element method (BEM), to mention a few.

In this work, to avoid having mesh inside or on the boundary of the plate, the boundary point method (BPM) is proposed to solve the problem. Owing to the complicated formulation and computations involved in the proposed BPM, a computer code is developed using the Wolfarm Mathematica software. This work investigates the effect of several variables such as: the shape of the plate, the shapes of the supporting columns or walls, and the different layouts of supports. Moreover, attention is paid, in this study, to cover the variation of the shear and bending moments around the internal supports which are the most stressed parts in the plate domain. The effectiveness and accuracy of the proposed method are assessed through several numerical examples and applications. The

results of the proposed method are verified by FEM software package COMSOL and other available results in the literature.

# ملخص الرسالة

**الاسم الكامل: أبوبكر الفاتح صديق موسى**

**عنوان الرسالة: تحليل البلاطات الرفيعة المستندة على مساند مختلفة الشكل و التوزيع باستخدام طريقة نقاط الجدار**

**التخصص: الهندسة المدنية**

**تاريخ الدرجة العلمية: يناير 2017**


تحليل البلاطات بالطرق التحليلية محدود للبلاطات ذات الاشكال البسيطة و ظروف تثبيت محدودة. تحليل البلاطات المستندة على مساند داخلية كالاعمده و الحوائط أكثر تعقيدا لذلك يلجأ الباحثون إلى استخدام الطرق العددية مثل طريقة الفرق المتناهي (FDM), و طريقة العنصر المتناهي (FEM), و طريقة العنصر الجداري (BEM).

في هذا العمل لتجنب تقسيم نطاق البلاطة أو الجدار الطرفي للبلاطة إلى عناصر تم اختيار طريقة النقاط الجدارية (BPM) لحل هذه المسألة. نتيجة للحسابات المعقدة التي تحتاجها هذه الطريقة تم عمل برمجة بالكمبيوتر باستخدام برنامج الماثماتيكا (Wolfram Mathematica). هذا العمل أمتد ليشمل أشكال مختلفة من البلاطات و أشكال مختلفة من المساند الداخلية كالحوائط و الأعمدة و توزيعات مختلفة و غير منتظمة لهذه المساند الداخلية. إضافة إلى ذلك فان جهدا مقدرا قد ألقى على تغيير مخطط القص و عزوم الإنحناء حول المساند الداخلية و التي تمثل أكثر المناطق إجهادا داخل مجال البلاطة. العديد من الأمثلة الرقمية قد حلت للتحقق من نتائج هذه الطريقة و قورنت هذه النتائج بإستخدام أحد برامج طريقة العنصر المتناهي الكومسول (COMSOL) و بالاضافة الى النتائج المتاحة لبعض الامثلة.

# CHAPTER 1

# INTRODUCTION

## 1.1 General

The plate bending problem is a vital for the design of plated structures and therefore, it has attracted many researchers in the scope of structural engineering. The analytical solution for the thin plates supported along the boundary has been carried a long time ago as it can be seen in the literature survey. The problem of thin plates supported by internal supports arises in new architectural designs and the trend for having different layout of the internal supports is followed by so many engineers worldwide. The flat plate floors and floor slabs reinforced by beams can be considered as plates with supports inside the plate domain.

The analysis of internally supported thin plates with an irregular layout of the internal supports is a major concern in the structural design of the plated structures which are commonly used in engineering practice. For analysis of such plates, it is difficult to obtain an analytical-based solution. The difficulties are increased when the layout of the internal supports is irregular; with different column shapes such as square, circular, and triangular; and different kinds of the applied loads. The numerical-based solutions are the best alternatives to the analytical solutions to deal with these kinds of complexities. The most common element-based numerical methods are the finite difference method (FDM), the finite element method (FEM) and the boundary element method (BEM). The proposed method is based on the boundary point method (BPM) which is a special type of BEM that does not require meshing. The plate and supports are modeled by properly distributed nodes inside the plate domain and all boundaries including those of the internal supports.

## 1.2 Problem Statement

In the design of reinforced concrete structures, the bending analysis of the flat plate floors supported by internal supports with regular column layout are carried by the direct-design method as recommended by the American Concrete Institute ACI 318 under the following conditions [1]:

1- The minimum of three spans in the direction considered.
2- The difference of the shorter to longer span is less than or equal to one third of the longer span.
3- The aspect ratio the rectangular panel must not be greater than 2.
4- The column offset should not be more than 0.10 times the span parallel to the offset.
5- The loads have to be gravity loads and uniformly distributed.
6- The ratio of the live load to the dead load should not be more than 2.

The free bending moment or so called the statical moment in the direct-method for the shaded part of the flat plate as in Figure 1 is calculated using Equation 1.1, this moment is to be divided into positive and negative moment as shown in Fig. (2) for the internal panel as it proposed by the shaded parts in Figure 2. For the external panels, However, Table 1 can be used for distribution of the statical moment to positive moment at the span and negative moment near the supports.

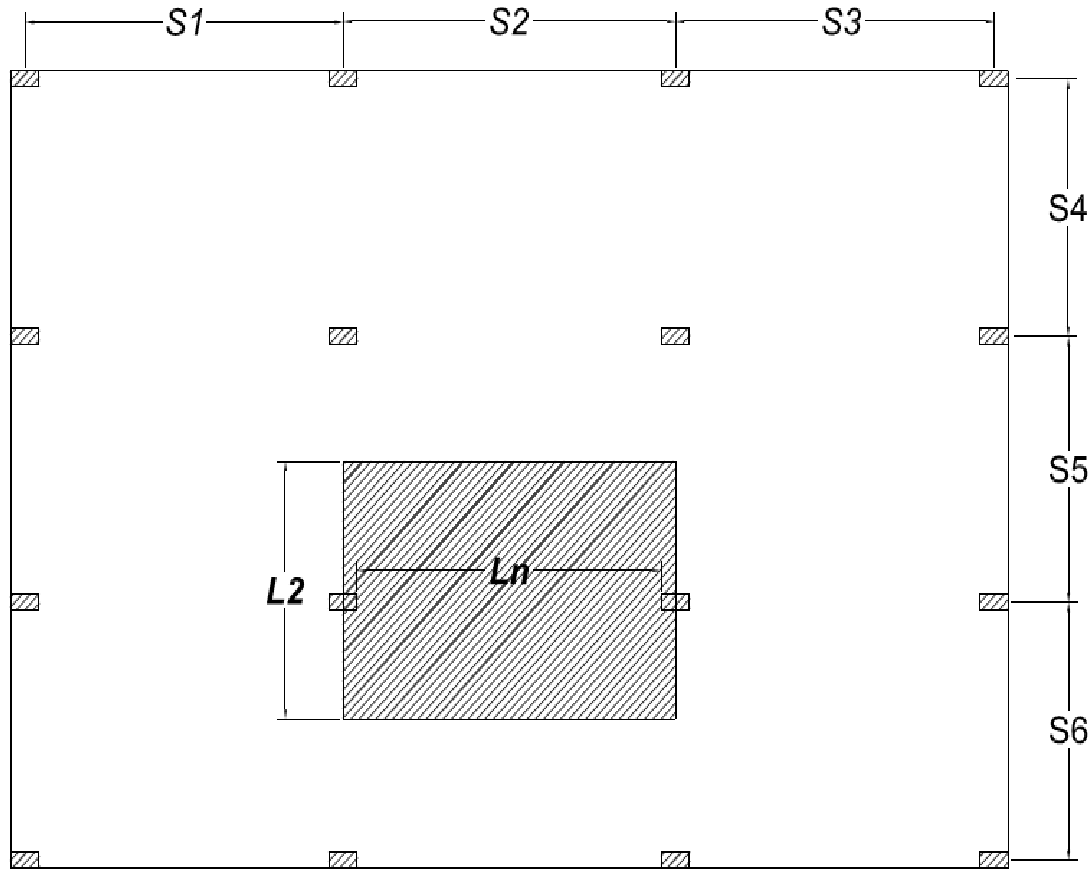



**Figure 1: Flat plate with regular column layout.**

The statical moment is represented by

$$M_o = \frac{q\, l^2}{8} \tag{1.1}$$

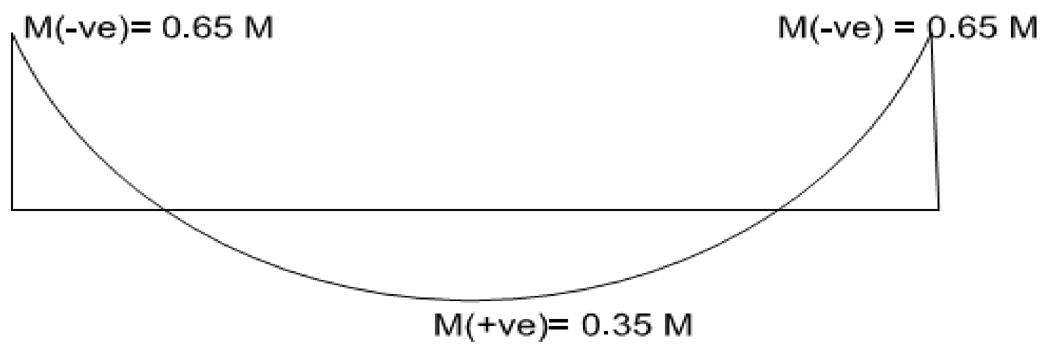



**Figure 2: Distribution of the statical moment to span and support moment for an internal panel.**

**Table 1: Distribution of the statical moment for the external panels[2].**

| | (1) | (2) | (3) | (4) | (5) |
|---|---|---|---|---|---|
| | | | Slab without beams between interior supports | | |
| | Exterior edge unrestrained | Slab with beams between all supports | Without edge beam | With edge beam | Exterior edge fully restrained |
| Interior negative factored moment | 0.75 | 0.70 | 0.70 | 0.70 | 0.65 |
| Positive factored moment | 0.63 | 0.57 | 0.52 | 0.50 | 0.35 |
| Exterior negative factored moment | 0 | 0.16 | 0.26 | 0.30 | 0.65 |

The positive and negative moments are to be distributed between the column strip and middle strip for flat plate without beams is as shown in Figure 3 below:

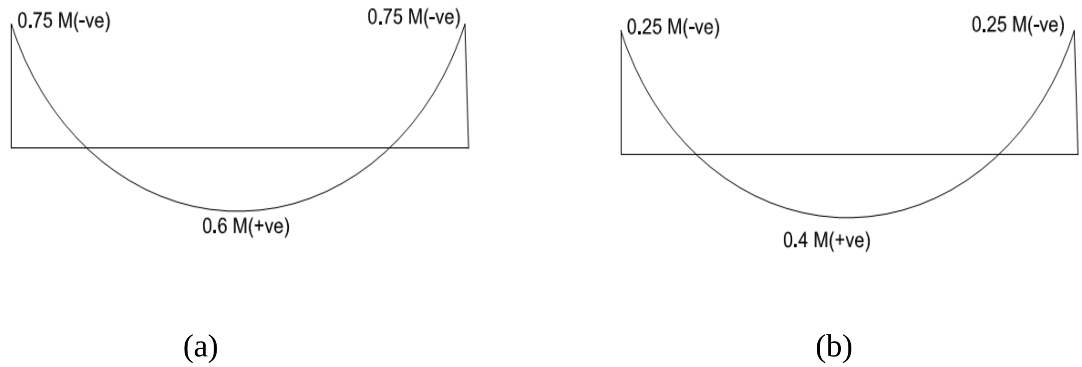



**Figure 3: The distribution of the positive and negative moment between (a) the column strip, and (b) the middle strip.**

On the other hand, for regular structures with large variety in spans and non uniform loading conditions the equivalent frame method can be used which divides the three dimensional problems to a series of two dimensional frame as shown in Figure 4. Each

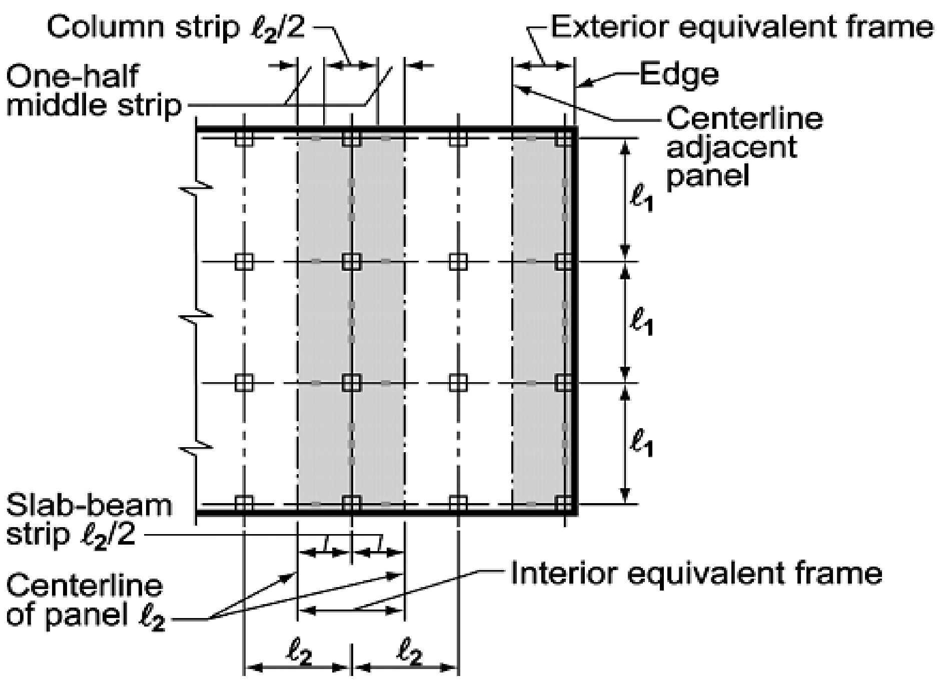



**Figure 4: Equivalent frame selection [1].**

frame can be analyzed independently under the applied loads and equivalent frames have to be taken in the transverse direction to model the whole slab.

In contrast, for irregular column layout as in Figure 5 and Figure 6 the direct method is not applicable and the equivalent frame method is difficult to be applied.

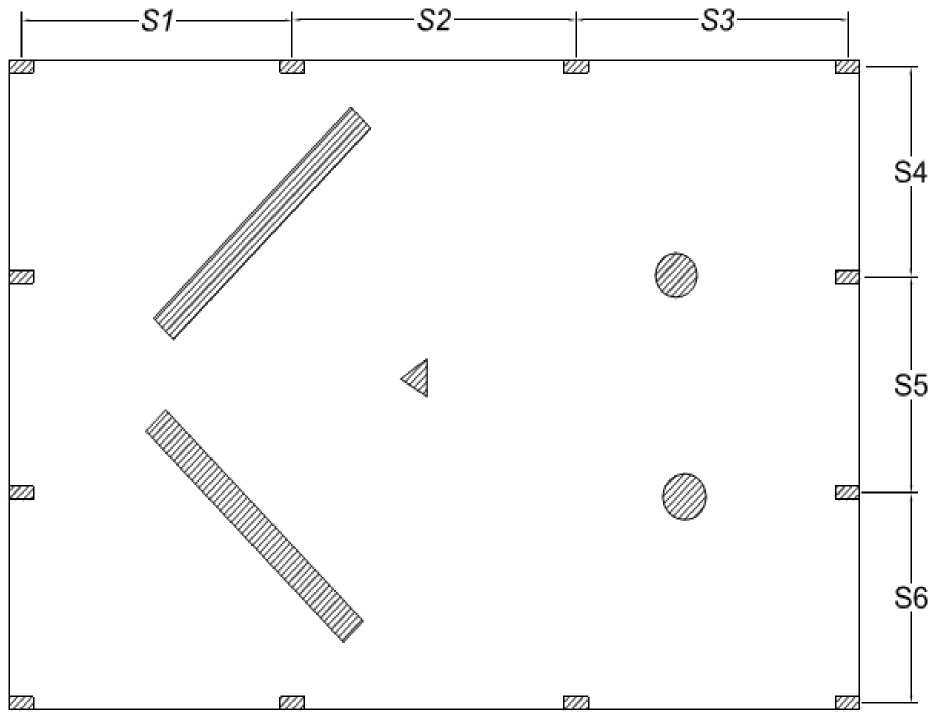



Figure 5: Flat plate with irregular layout of the internal supports and walls.

In addition, the zone of the plate around the column is highly stressed compared to other parts of the plate and, therefore, needs special attention for the bending and shear stresses in this zone. The distribution of the shear stress varies from great value at the corners and reducing at the zone near the middle of the column as shown in Figure 7.

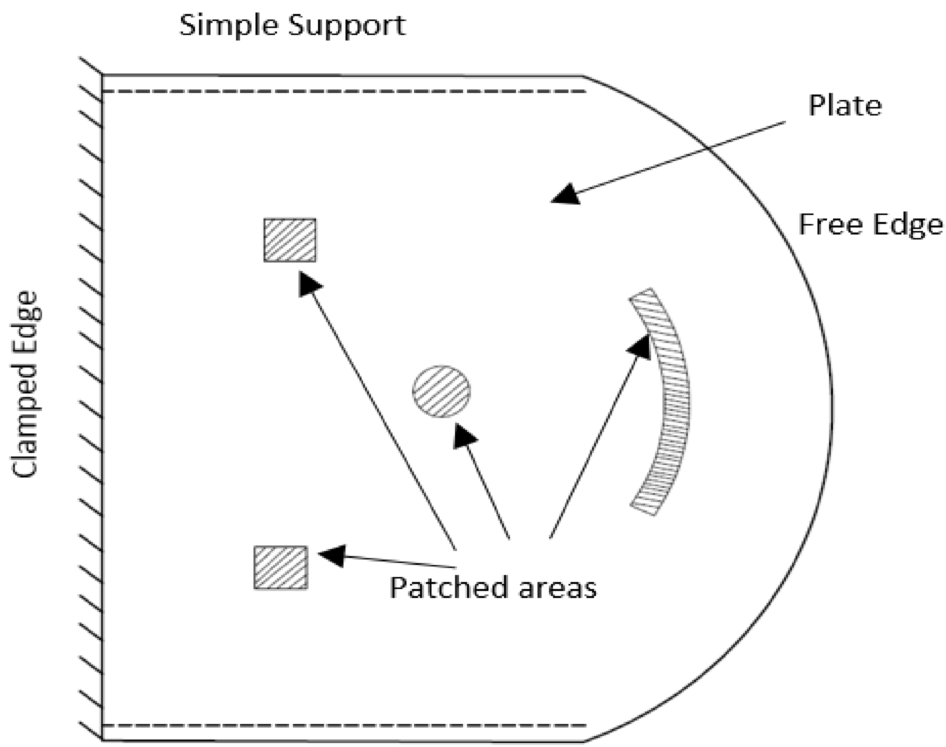



**Figure 6: Irregular flat plate with irregular columns and walls layout.**

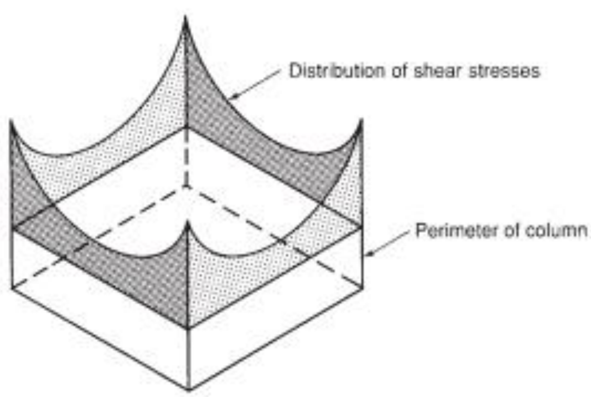



**Figure 7 The distribution of the shear stress around the rectangular column** [1]

The analytical solutions for the thin plate with irregular column layout have some difficulties and most of the recent researchers oriented their research to have approximate analytical techniques that can provide adequate methods to be applied in the design procedure of these types of plates. This study is focusing on the analysis of thin plates with irregular layouts and shapes of internal supports with different loading conditions. This kind of plates is commonly used in building floors and the irregular shape of internal supports can be found in the area of shear walls as well as different column cross-sections such as square, circular and triangular shape. The analysis, in this study, is based on BPM and the results are compared with the available analytical/numerical solutions in the literature and with FEM results obtained using COMSOL.

## 1.3 Objectives

It can be clearly concluded from the available literature that there is no clearly established analysis of thin plates with irregular layout of the internal supports especially for different shapes of the internal supports.

The primary objective of this study is to establish an analysis of thin plates with an irregular layout of the internal supports with different shapes using BPM. In addition, the study will investigate the behavior of the stresses at the high stressed zone around the internal supports and verify the results with the available finite element software programs. The specific objectives are to:

1- Develop a BPM model for the analysis of plates resting on irregular layout of internal supports.

2- Implement the BPM model in a Mathematica computer code capable of performing the analysis of plates with different layouts and shapes of internal supports.
3- Use the developed computer code to investigate the critical zones at the plate-support joint.
4- Verify the accuracy of the obtained BPM results with the available analytical solutions for cases involving simple geometries.
5- Compare the BPM results with those obtained by FEM for cases involving more complicated geometrical and supporting conditions.

## 1.4 Outlines of the Thesis

This thesis is organized as follows. Chapter 1 introduce the considered problem, and list the work objectives. The second chapter presents a brief literature review about the previous work in this problem and other related works. A step by step procedure to reach the general differential equation and boundary conditions of thin plate bending problem is explained in Chapter 3.

Chapter 4 is devoted to the mathematical formulation of the proposed BPM for plate with edge supports and solving some numerical examples. Through these examples, different problems are selected to boost the confidence of the reader, on one hand, and to explain the singularity associated with the considered problems and their possible solutions. In Chapter 5, the developed BPM Formulation is extended to tackle problems of plate containing rigid internal supports with different numerical examples, the selection of which is based on comparison the method with other available numerical solutions and test it for more complicated plate problems. The BPM formulation is extended, in Chapter 6, to deal with plate problems having internal flexible supports and two numerical example are given for the purpose of evaluating the accuracy of the proposed BPM method.

Finally, the summary of the completed works, conclusions, and recommendations for future works are presented in Chapter 7.

# CHAPTER 2

# LITERATURE REVIEW

## 2.1 Analysis of Single Plates

### 2.1.1 Analytical Techniques

The mile stone in the analysis of plates is the introduction of the mathematics in the analysis of plates and coming up with the general differential equation of plate bending problem subjected to lateral loads which is due to the French engineer L. Navier as described in [2] in 1823.To solve for some cases of rectangular plates, Navier came up with a method of transforming the general differential equation into a set of algebraic equations using the Fourier trigonometric series. However, this solution is limited to simply supported plates. The convergence of these solutions is found to be very fast in case of continuous and uniformly distributed loads but in case of the concentrated loads becomes very slow. The results of this method of analysis can be summarized as shown in the Table 2 and Table 3 below.

As far as closed-formed solutions are concerned, they are available for limited cases as shown in Figure 8.

M. Levy in 1900 [2] came up with another series-based solution of rectangular plates involving a single sine series instead of two sine series as done by L. Navier. Application of this method requires presence of opposite simply supported edges and the load can not vary along the direction parallel to opposite simply supported edges. The advantage of the

**Table 2 A series solutions for rectangular plates with different loads [3]**

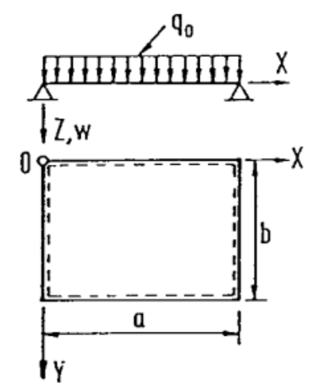
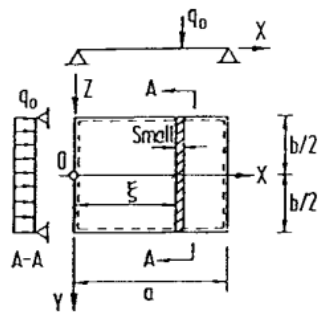
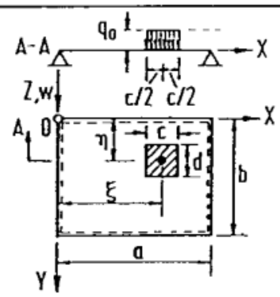
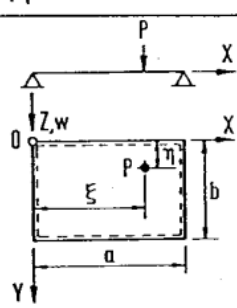
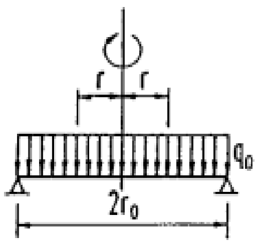
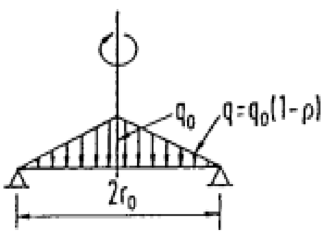
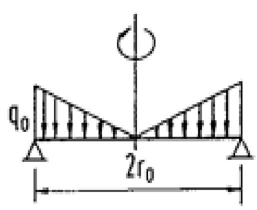
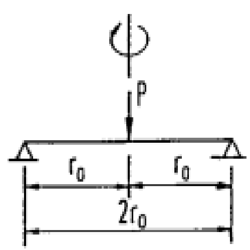
| Case No. | Structural System and Static Loading | Deflection and Internal Forces |
|---|---|---|
| 1 |  | $w = \frac{16q_0}{\pi^6 D}\sum_m\sum_n \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$<br>$m_x = \frac{16q_0a^2}{\pi^4}\sum_m\sum_n \frac{\left(m^2+\nu\frac{n^2}{\epsilon^2}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(m^2+\frac{n^2}{\epsilon^2}\right)^2}$<br>$m_y = \frac{16q_0a^2}{\pi^4}\sum_m\sum_n \frac{\left(\frac{n^2}{\epsilon^2}+\nu m^2\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left(m^2+\frac{n^2}{\epsilon^2}\right)^2}$<br>$\epsilon = \frac{b}{a}$, $m = 1, 3, 5, \ldots, \infty$; $n = 1, 3, 5, \ldots, \infty$ |
| 2 |  | $w = \frac{a^4}{D\pi^4}\sum_{m=1}^{\infty}\frac{P_m}{m^4}\left(1-\frac{2+\alpha_m\tanh\alpha_m}{2\cosh\alpha_m}\cos\lambda_m y + \frac{\lambda_m y\sinh\lambda_m y}{2\cosh\alpha_m}\right)\sin\lambda_m x$<br>where<br>$P_m = \frac{2q_0}{a}\sin\frac{m\pi\xi}{a}$ $\lambda_m = \frac{m\pi}{a}$<br>$m = 1, 2, 3, \ldots$ $\alpha_m = \frac{m\pi b}{2a}$ |
| 3 |  | $w = \frac{16q_0}{D\pi^6}\sum_m\sum_n \frac{\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\sin\frac{m\pi c}{b}\sin\frac{n\pi d}{2b}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$<br>$\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$<br>$m = 1, 2, 3, \ldots$<br>$n = 1, 2, 3, \ldots$ |
| 4 |  | $w = \frac{4P}{D\pi^4 ab}\sum_m\sum_n \frac{\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}$<br>$m = 1, 2, 3, \ldots$<br>$n = 1, 2, 3, \ldots$ |

**Table 3 A series solutions for circular plates with different loads [3]**

| Case No. | Structural System and Static Loading | Deflection and Internal Forces | |
|---|---|---|---|
| 1 |  | $w = \frac{q_0 r_0^4}{64D(1+\nu)}[2(3+\nu)C_1-(1+\nu)C_0]$<br>$m_r = \frac{q_0 r_0^2}{16}(3+\nu)C_1$<br>$m_\theta = \frac{q_0 r_0^2}{16}[2(1-\nu)-(1+3\nu)C_1]$<br>$q_r = \frac{q_0 r_0}{2}\rho$ | $\rho = \frac{r}{r_0}$<br>$C_0 = 1-\rho^4$<br>$C_1 = 1-\rho^2$ |
| 2 |  | $w = \frac{q_0 r_0^4}{14400D}\left[\frac{3(183+43\nu)}{1+\nu}-\frac{10(71+29\nu)}{1+\nu}\rho^2+225\rho^4-64\rho^5\right]$<br>$(m_r)_{\rho=0} = (m_\varphi)_{\rho=0} = \frac{q_0 r_0^4}{720}(71+29\nu);$<br>$(q_r)_{\rho=1} = -\frac{q_0 r_0}{6}$ | $\rho = \frac{r}{r_0}$ |
| 3 |  | $w = \frac{q_0 r_0^4}{450D}\left[\frac{3(6+\nu)}{1+\nu}-\frac{5(4+\nu)}{1+\nu}\rho^2+2\rho^5\right]$<br>$(m_r)_{\rho=0} = (m_\varphi)_{\rho=0} = \frac{q_0 r_0^2}{45}(4+\nu);$<br>$(q_r)_{\rho=1} = -\frac{q_0 r_0}{3}$ | $\rho = \frac{r}{r_0}$ |
| 4 |  | $w = \frac{P r_0^2}{16\pi D}\left[\frac{3+\nu}{1+\nu}C_1+2C_2\right]$<br>$m_r = \frac{P}{4\pi}(1+\nu)C_3$<br>$m_\varphi = \frac{P}{4\pi}[(1-\nu)-(1+\nu)C_3]$<br>$q_r = \frac{P}{2\pi r_0 \rho}$ | $C_1 = 1-\rho^2$<br>$C_2 = \rho^2 \ln\rho$<br>$C_3 = \ln\rho$<br>$\rho = \frac{r}{r_0}$ |

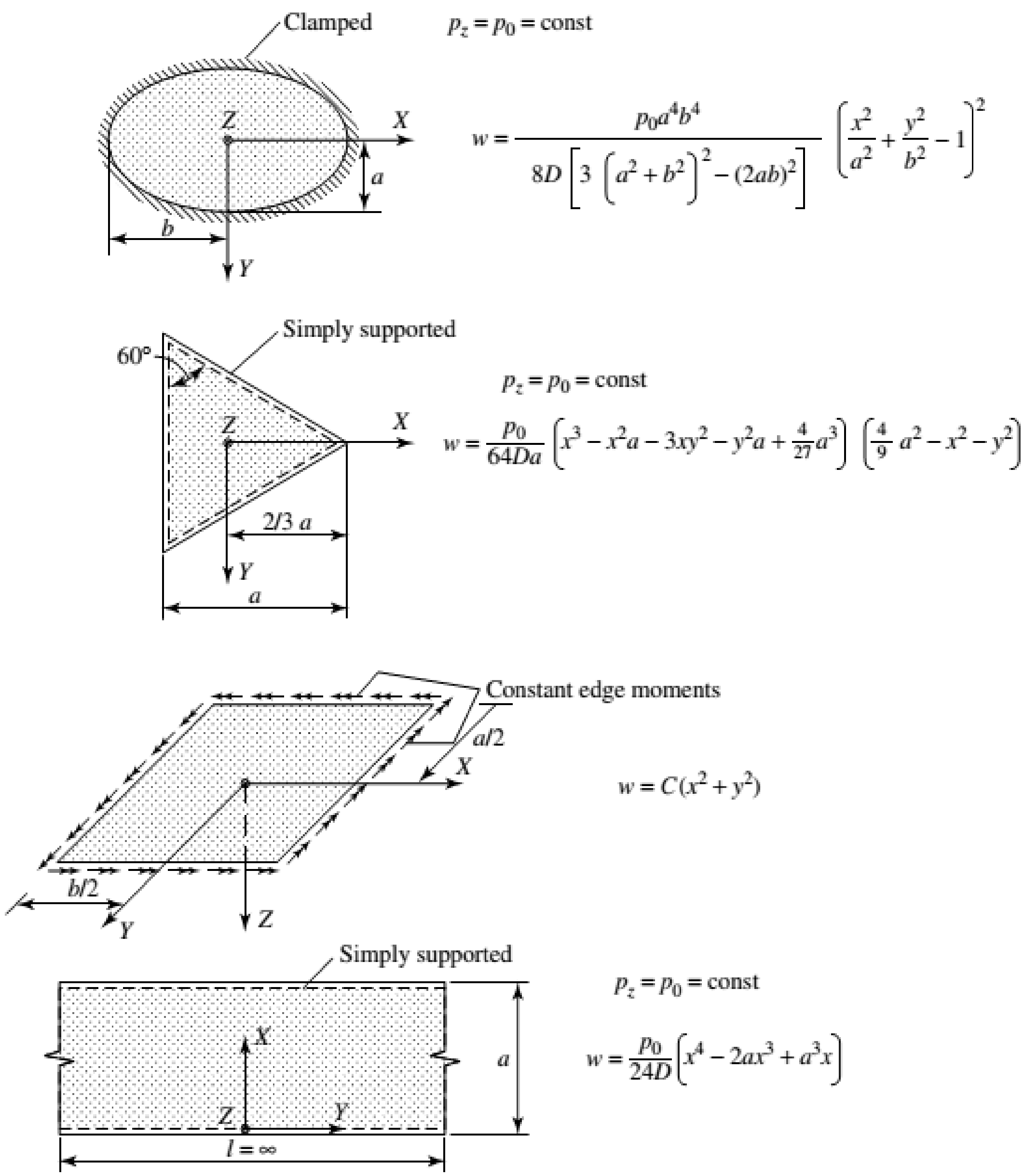



**Figure 8 Closed-form, exact solutions for some plate problems [2]**

Levy's method is simplicity of calculations but has the disadvantage of being limited to a certain loading and boundary conditions.

Ghosh and Som in 1978 [3] presented an analytical technique for the analysis of rectangular plates supported by flexible beams. The formulation was prepared in a way so that the deflection expression of the plate is equal to sum of three major terms: the first one is taking into account the effect of the stiffness of the edge beams, and the remaining two terms are representing the solution of the homogenous part of the governing differential equation in both x and y coordinates. In this work, the effect of the vertical deflection and the torsional rigidity of the edge beam have been taken into account and the compatibility conditions were satisfied leading to the vertical deflection is the same for both the spandrel beam and the plate. This compatibility conditions lead also to the edge bending moment in the plate is equal to the twisting moment in the spandrel beam at a specified conjunction point. The prepared formulation was then presented in a computer code using FORTRAN IV programming language. Experimental models have also been performed and both the analytical and experimental results shown a good agreement with an underestimation of the deflection in the proposed formulation compared to the experimental, but as an overall for the whole compared models the different in the vertical deflection is found to be less than 8%.

Boborykin in 2012 [4], proposed a method for the construction of Green's function for irregular shaped plates with different types of the boundary conditions. The constructed function was written in terms of the kernel vector function, the biharmonic equation, fundamental solution, and kernel functions of the inverse regular integral. The accuracy of this method was verified using some numerical examples of irregular shaped plates and it was found to be in a high accuracy level.

Tan and Zhang in 2013 [5], proposed a computational approach for the analysis of thin plate with different regular shape subjected to a point load using the regular hybrid boundary node method (RHBNM). The solution of this method was divided into two main parts, the first is the particular solution for the concentrated load which obtained by utilizing the fundamental solution. The second part is the complementary solution which was obtained by applying the RHBNM. The boundary variables where approximated using the moving least square (MLS) method whereas the domain variable were interpolated linearly from the solution of the biharmonic equation and Laplace's equation. The validity and efficiency of this method have been verified by some numerical examples.

Brezzi and Marini in 2013 [6], proposed an analysis of thin plate by using the virtual elements method. In this approach only unisolvence is needed for space of a given space functions with polynomials subset. The construction of the stiffness matrix is very simple and therefore can be used for more complex geometries because no more details are required for the non-polynomial part of the plate local spaces. This method is very simple compared to the traditional finite elements method because in the later the space of polynomials is needed for every element as well as every degree of freedom.

Li et al. in 2014 [7], presented an analytical benchmark bending solutions of thin plates of rectangular shape with point supports at some of plate corners and the other adjacent edges are simply supported, or clamped, or one is simply supported and the other is clamped. In this paper the analytical solution for plate bending was derived and the superposition method was used. The accuracy of this method was checked through several numerical examples and the results were compared with the finite elements available solutions and were found to be a good accuracy.

### 2.1.2 Numerical Techniques

Maiti and Chakrabarty in 1974 [8], developed an integral equation approach to study the bending response of uniformly loaded simply supported polygonal thin plates. The plate boundary solutions were utilized to establish the biharmonic function which led to computation of the deflection formula and the bending moments inside the plate domain. The presented solution was applied to a plate shape of rectangle, rharmonic, and hexagon and the results were found to be in an excellent accuracy as compared with the available solutions in the literature and the results were also shown that this method can be applied to other plate domains.

Bezine in 1978 [9], presented a solution for the plate bending with different boundary conditions using boundary elements method. In the prepared formulation, a numerical technique has been developed to solve the integral equations by discretization the plate boundary to some segments. The presented solutions have been done for a square plate with different boundary conditions and the results were found to be matched with the available analytical and finite elements method solutions and the accuracy of this formulation is not highly affected by the selected number of segments along the plate boundary.

Stern in 1978 [10], presented a direct method to obtain a general formulation for bending analysis of thin plates to get the bending moment, slope, deflection and the shear forces along the boundary. In this formulation, singular integral equations were coupled with the boundary conditions. One of the advantages of this method is that it can be used for the plates with curved boundaries because the boundary geometry in this formulation is not explicitly involved and no need to approximate the shape boundary of the plate. The accuracy of this approach has been verified by two numerical examples.

Burgess and Mahajerin in 1985 [11], presented a numerical techniques for the analysis of thin plates subjected to lateral loads. In this method the plate was treated as an embedded in a finite plane and the sources of point loads outside the plate boundary were used. In this approach, the solution for the arbitrary loaded polygon within the infinite domain was derived first and coupled with the lateral loading effects in the described collocation points. This method can be used for different shapes of the plates, boundary conditions, and distributions of the lateral loads. The accuracy of this method was found to be very high compared with the previous numerical solutions as had been verified using some numerical examples.

Johnson in 1987 [12], utilized the boundary point method to prepare a numerical solution for the plate bending problem. In this method, the source points were selected to be outside the plate domain and each one was associated with one of the boundary point of the plate, which have four edge conditions depending on the type of support clamped or simply supported. The application of the boundary conditions led to solve for the displacement and the stresses in the plate. The accuracy of this method was verified numerically and it can be getting with less number of boundary points.

Paris and Leon in 1987 [13], presented an analysis of thin plate using boundary element method (BEM). In this work, the biharrnonic field equation was coupled into two harmonic equations and there are two integrals appear with in the plate domain. The first domain integral for the problem and the second for the decomposition of the biharmonic equation. Both of them were evaluated to form the boundary integrations. Cells are not required for dealing with different kinds of support conditions and loading patterns. The work presented here is more flexible to be applied to the external effects such as temperature changes and elastic supports.

Hu and Hartely in 1994 [14] proposed an analysis of thin plates supported by flexible beams using the direct boundary element method. To deal with the elastic boundary conditions considered in this formulation the author adopted two different models: the first one deal with a continuous flexible edge beam boundary conditions in which the edge beam and the plate displaces in a compatible manner, this is called here as 'boundary condition' type. The second adopted model considers the edge beam is connected to the plate in a finite number of discrete points which is so called 'attached beam element' typ. This approach can be extended to solve for large variety of problems in this area, for example, building floor slabs in which both adopted models are applicable but in case of there is a discontinuity as it found in a discontinuous edge beam, the second model is preferable to avoid the singularity at the point of discontinuity by increasing the number of discrete points. Several numerical examples of rectangular plates have been solved for the sake of verification and a good accuracy was observed.

V. Sladek and J. Sladek in 1995 [15], proposed a an implementation of the boundary element formulation for the analysis of plate bending for planner plates with different shapes and boundary conditions. In this method, a direct and nonsingular formulation were utilized to work for large variety of plate shapes, boundary conditions, and loads. The discontinuity in plate geometry has been represented by 'corner' elements while the smooth parts of the plate boundary have been presented by employing the Overhauser elements. The proposed method has been verified with several numerical examples and the results were compared with the available solutions and were found to be in an excellent accuracy level.

Paiva in 1996 [16], presented an analysis of slab using boundary element formulation. In this work, the beams and columns are coupled using the three nodal displacements for the nodes along the plate boundary. The cross-sections of the columns are assumed to be flat after the plate deflection and the stress in the contact surface between the plate and column is assumed to be varied linearly and the plate is coupled to the beams and columns with their nodes and these supports are supposed to transmit the vertical forces only (i.e. the bending moments are not considered to be transmitted from the slab to the beams).

Tanaka et al. in 2000 [17] presented an elastic bending analysis of thin plates stiffened by beams using the boundary element method. In this work, the stiffener beams were treated as lines and the forces and moments in theses stiffeners as line loads and they were the unknowns of the problem. The boundary element for plate is then used for the numerical implementation of the prepared formulation and computer code was developed. The discretization in this problem was made for the boundary of the plate and the stiffener beams. It was found that no need for going to higher-order elements but constant elements was found to be adequate for the stiffener beams and quadratic elements for the elements along the plate boundary as well as the quantities within the plate domain to ensure more accurate results. The obtained results in this method were compared with the available Timoshenko solutions and a good agreement was observed for the slope and deflection results but for the shear results an excellent agreement was observed only when using quadratic elements for discretizing the plate boundary.

## 2.2 Analysis of Continuous Plates with Regular Layout of the Internal Supports

### 2.2.1 Analytical Techniques

Analytical solutions of continuous plate in one and two directions have been well explained in [2]. The solution in this method is based on the superposition principle and the main two assumptions are the internal supporting beams are rigid therefore no deflection and secondly the supporting beams allow rotation of the plate around the support. This kind of plate is statically indeterminate structure and the analysis is based on the equilibrium conditions as well as compatibility conditions involving the displacement and the rotations at the adjacent edges.

To apply the force method for this problem the force method is used and the redundant moments are released from the internal supports and the slope continuity has been taken into account therefore the slope at panel i and panel i+1 are the same. Applying the principle of superposition the total deflection at panel i can be expressed as a sum of the deflection due to simply supported plate, the deflection due to the moment from panel i-1 at support of panel i, and the deflection due to moment at panel i at the support of panel i as shown in Figure 9 above. The same method can be extended to work for plate continues in two directions.

Analysis of plate supported by equidistant column supports has been given in [18]. This analysis is based on the assumptions that the load is uniformly distributed over the whole plate, the distance between the columns are equals, and the columns dimensions are very small compared to the plate dimensions. These assumptions led to approximately similar

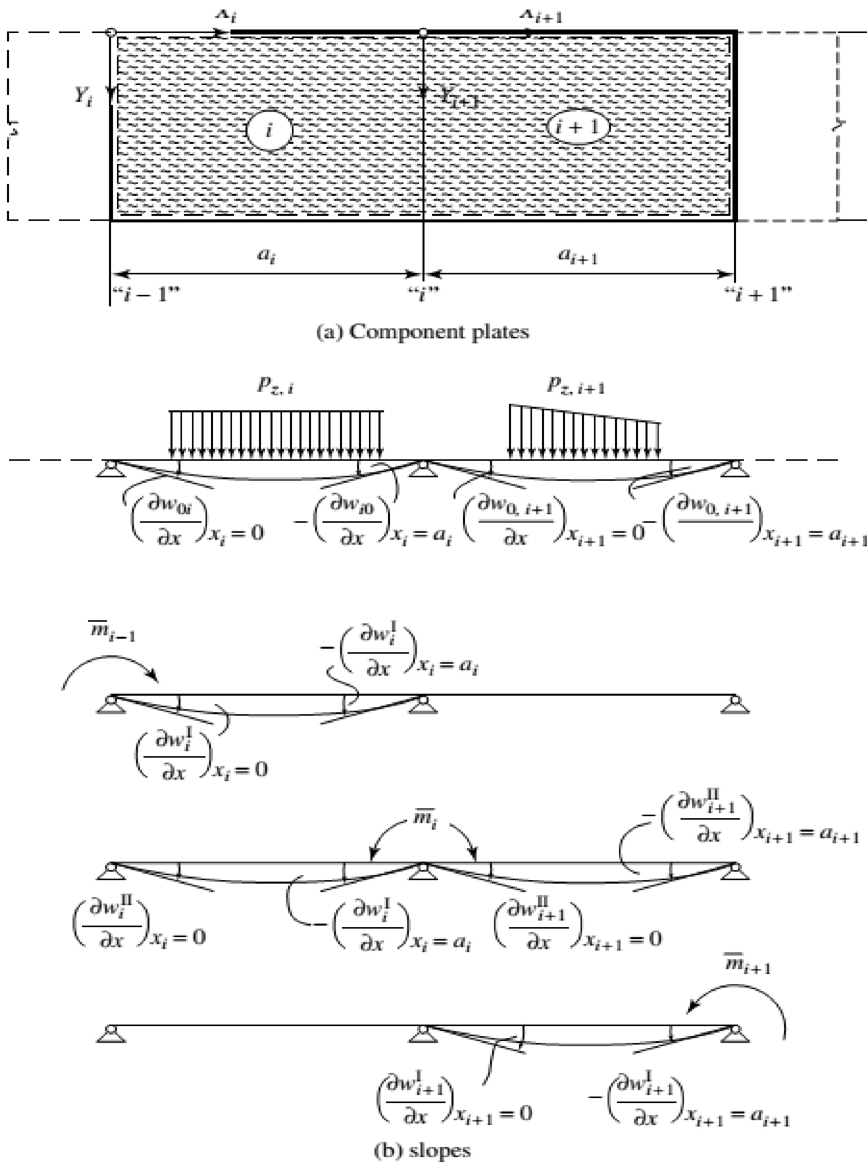



**Figure 9 Analysis of continuous plate in one direction [2]**

bending moments in the panels not adjacent to the plate boundary and therefore one typical internal panel as shown in Figure 10 below has been selected to represent the whole plate. This deflection of this point-supported panel is then expressed in terms of series form.

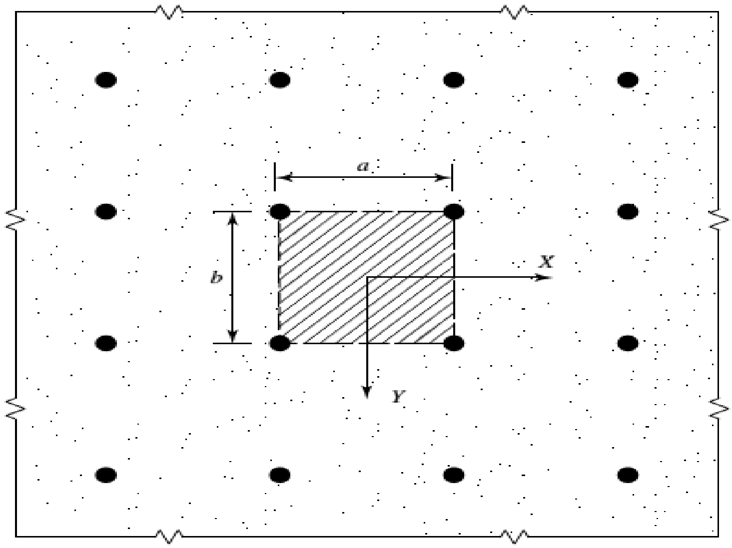



Figure 10 Interior panel of plate supported by internal point supports [2]

The maximum moment found as shown an the above paragraph needs to be modified as described in the in [2]. The modifications is based in calculation of the moments at the edge of the supporting column as shown in Figure 11.

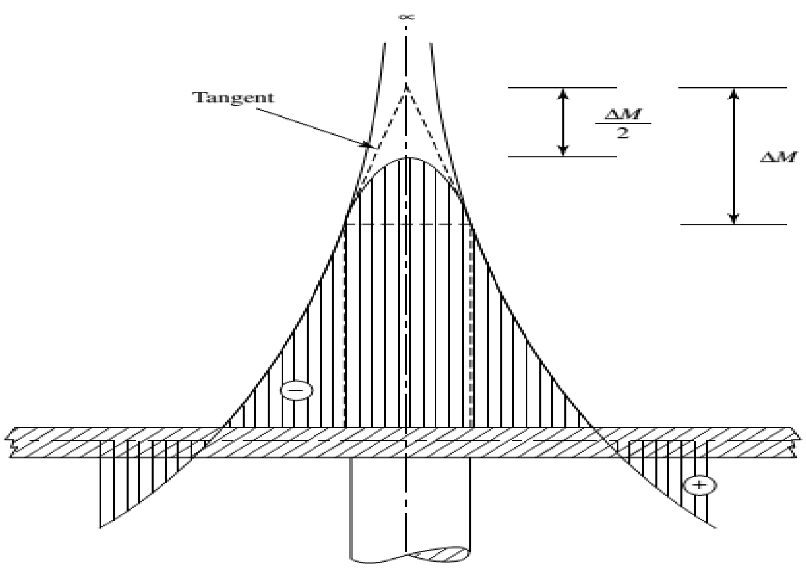



Figure 11 Modification of the maximum negative moments at the support [2]

Altekin and Altay in 2008 [19], prepared an analysis of super-elliptical isotropic plate resting on point supports and subjected to a uniformly distributed lateral load. The plate perimeter was varied from an elliptical to a rectangular shape. This approach was based on the Kirchhoff–Love plate theory (KLPT) and the Ritz method was utilized for the computations. The boundary conditions have been satisfied using Lagrange multipliers. The results were compared with those of a corner-supported rectangular plate and a good match was obtained for the bending moment and the deflection but the values of the twisting moment and the shear force did not match due to neglecting the transverse shear force in the KLPT.

### 2.2.2 Numerical Techniques

Modeling of the plate stiffened by beams has been studied by so many researchers with different method to model the stiffeners-plate interaction. There are three common procedures as they were briefly summarized in [20] in each method a certain limitations have to be met before applying it. The first method is modeling the plate as an orthotropic plate [21],[22]; therefore a modified plate stiffness to be calculated to account for the introduction of the stiffeners leading to simulation of the plate with an orthotropic properties but the results of this method is acceptable for small center to center spacing of the stiffeners. The second approach is so called the grillage approximation [23],[24], in this approximation the plates' effects are to be embedded into the stiffeners leading to higher second beam stiffness. This approximation has a problem in the accuracy of selection of the effective beam width or in other words the effective part of the plate that should be taken to increase the section of the beam. The third approximation and the most general one is analyzing the plate and the beam stiffeners separately [25],[26],[27] and then applying the compatibility and boundary conditions along the interfaces lines or points to have an overall solution of the system.

Paris and Leon in 1996 [28] presented the analysis of thin plate bending problem by using a couple of two Poison's equations. The classical way of Green's second theorem was used to represent the proposed formulation in an integral form. A series of a simple domain functions are used to approximate the domain integrals, these simple domain functions have coefficients evaluated at the collocation points in the boundary and the domain of the plate. The resulting approximate domain integrals are then transferred to boundary integrals to solve for the extra unknown coefficients. One of the advantages of this formulation is the overcoming of the singularity problems associated with solving the biharmonic equation using the fundamental solution. This formulation works for different boundary conditions, thermal effects, effect of an elastic foundation support, and rigid internal supports. Different numerical examples have been used to evaluate the accuracy of this approach and the results were compared with the available analytical solution and they found to have a good accuracy with small number of boundary points.

The direct boundary element method has been used by Tanaka and Bercin in 1998 [20] for an elastic bending analysis of thin plates stiffened by beams of different cross-sections which they are commonly used in the aircraft parts, vehicles, ships, and different buildings parts. The beams sections used here are varied in a wide range including open and closed sections and the structural effects of these sections such as the bending, eccentricity, warping rigidity, and the torsional rigidity have been taken into account. In this formulation, regularized boundary integrals were used so that it can be calculated by the use of Gaussian quadrature formulation instead of evaluation the Cauchy principal integral directly. To model the stiffeners adopted in this study, it was assumed that they are contacted with the plate along a line at the same plane of the shear center. The compatibility conditions states that the displacement of the stiffener is equal to the displacement of the plate at the same interface point, and the angle of the twist of the stiffener is equal to the slope of the plate at the same interface point. In addition, the equilibrium conditions were also satisfied. Several numerical examples of plate stiffened by singular stiffener and cross stiffeners were solved using the proposed formulation to verify the accuracy and a good agreement was observed.

Sapountzakis and Katsikadelis in 2000 [29] presented an analysis of ribbed plate systems under both transverse and in plane loading conditions using the boundary element method. The resulting deformations and the in plane forces of the plate and the axial deformations and the axial forces of the beam have been taken into account in the adopted formulation. The beams were isolated from the plate by creating sections parallel to the bottom part of the plate. The interface forces were found by applying the continuity conditions leading to a nonlinear couple of beam and plate problems which has been solved by analog equation method (AEM) as it is available in the literature. In this formulation, the shear forces at the interface between the beam and plate can be evaluated which is very important especially in composite construction where there are steel beams connected to the reinforced concrete plate through shear connecters. Some numerical examples have been solved using the prepared formulation and it was found that the in plane forces along the interface which is coming from the bending of the considered system are greater than the applied inplane load.

Fernandes and Venturini in 2005 [30], presented an analysis of slab with beam using the boundary elements method. This formulation was developed by assuming a zoned sub-regions for the beam elements and taking the rigidity of the beam into account. The global integral equation for the whole zoned plate was developed. Because of that two structural elements are working together, the effect of both bending and stretching is taken into account. Then applying the equilibrium and compatibility conditions led to solve for the unknowns. The accuracy of the presented formulation has been verified by comparing the results with analytical solutions and the numerical solutions prepared by finite elements software programs.

Bezine in 1981 [31], presented a mixed of boundary elements and finite elements method for flexure of thin plate with conditions within the domain as has been described by Kirchhoff's theory. A system of boundary integral equations for calculating the values of deflection, slope, bending moment and transverse shear force all around the edge by utilizing a matrix formulation. Some numerical examples have been used to check the accuracy of the proposed method.

Guminiak and Sygulski in 2007 [32], presented a static analysis of internally supported thin plates using the boundary elements method. Bettie theorem has been used to derive the integral equation over the boundary. In this paper, a modified formulation for bending of thin plates supported by internal supports has been included. In this modified formulation, the introduction of the equivalent shear forces over the plate boundary and the point forces at the corners of the plate were not required. The accuracy of this approach has been verified using some numerical examples and the results were found to be in agreement with the FEM results as well as the analytical results from the available literature.

Guminiak and Jankowiak in 2007 [33], presented a initial stability analysis of internally supported thin plates using the boundary elements method. Bettie theorem has been used to derive the integral equation over the boundary. In this paper, a modified boundary integral equation was used and the introduction of the equivalent shear forces over the plate boundary and the point forces at the corners of the plate are not required. The accuracy of this approach has been verified using numerical examples and the results were found to be matched with the finite elements solutions.

Sapountzakis and Mokos in 2008 [34] presented an analysis of stiffened plate by parallel doubly symmetric beamssystem with deformable connection under arbitrary transverse loading conditions using the boundary element method. The resulting deformations and the inplane forces of the plate and the axial deformations and the axial forces of the beam have been taken into account in the adopted formulation. Due to deformable connection adopted in this formulation the beams were isolated from the plate by creating sections parallel to the bottom part of the plate making a hypothesis that the there are no constrains for the slipping of the beam and plate and they can slip in all direction without separation (i.e. no differential vertical movement). The interface forces were found by applying the continuity conditions leading to a nonlinear couple of beam and plate problems which has been solved by analog equation method (AEM) as it is available in the literature. In this formulation, the shear forces at the interface between the beam and plate can be evaluated which is very important especially in composite construction where there are steel beams connected to the reinforced concrete plate through shear connecters. Some numerical examples have been solved using the prepared formulation and noteworthy accuracy was observed.

Guminiak in 2009 [35], presented an initial stability of thin plates with internal line supports. The modified form for the boundary integral equations were established and the boundary element method was utilized to get the final solution of  the problem. In this proposed formulation, the introduction of Kirochhoff forces at the corners of the plate and the equivalent shear forces along the edges of the plates were not required. The source points were selected to be outside the plate domain. The accuracy of this approach was checked using some numerical examples and the results were found to be agreed with the finite elements available solution.

Oliveira et al. in 2015 [36] prepared a formulation for the analysis of plate-beam interaction using boundary element method. Theplate-beam interaction considered in this formulation is commonly found in the floor slabs and the waffle slabs in which secondary beams or ribs are there in both directions. In this formulation, the plate was modeled by a boundary element formulation with three parameters per node as it was presented in [37]&[38], and the elements of the beams were replaced by their equivalent actions on the plate,a linearly distributed load and forces at the end of each beam element. The solution of the governing differential equation of the beam with linearly varying distributed load was utilized to express the tractions along the plate-beam interaction in terms of the beam nodal displacements. Thus, a final system of equations was prepared in terms of the nodal displacement of the beam and the boundary of the plate. Some numerical examples of plate with internal parallel and inclined beams as well as waffle slabs were solved to verify the accuracy of this formulation and excellent agreement with the finite element results were observed.

## 2.3 Analysis of Continuous Plates with Irregular Layout of the Internal Supports

### 2.3.1 Analytical Techniques

In the textbook of Timoshenko and Woinowsky-Krieger [18] an explanation to the effect of the rigid slab column connection to the moments in the slab is given. The main concentration is about the moments at the point of the column. The moments at the for the case of rigid connections was expressed assuming the moment is equal to zero at the point of inflection in the slab supported by internal columns which can be expressed as 22 percent of the span measured from the center of the supports as shown in Figure 12 below:

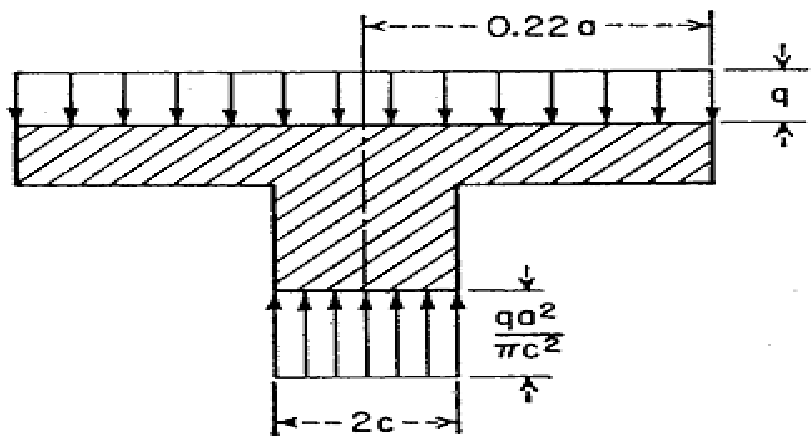



Figure 12 The rigid slab column connection [19]

These assumptions led to say that the zone around the column can be analyzed as an annular plate with radius equal to the distance from the center of the support to the inflection point and inner radius equal to the radius of the support, the boundary condition is simple supported plate at the outer radius and clamped at the inner radius. Thus the deflection expression can be found by summing the results of the two cases as shown in figure 13 below.

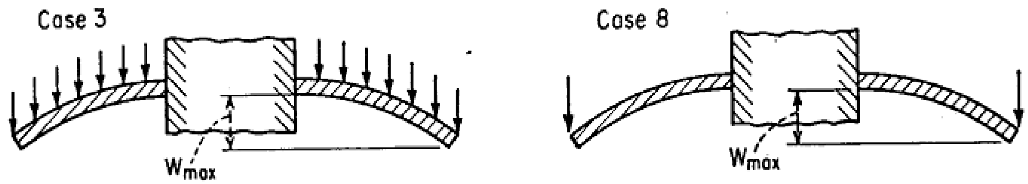



Figure 13 Annular plate clamped from inside and simply supported from outside [19]

Altekin in 2010 [39], presented a bending analysis of super-elliptical plates resting on an internal supports. An optimization process has been done for investigation the supports positions those leads to a minimum central deflection and bending moment at the support locations. The Ritz method and Lagrange multipliers were utilized in this work to improve the accuracy of the proposed method. The bending moment and the plate deflection are determined at different locations for plate shapes varying rectangular to an elliptical shape and the structural behavior was found to be highly sensitive to the positions of the supports. The results were compared with the available solutions for the rectangular and the elliptical plate shapes and were found to be accurate.

Dang and Truong in 2013 [40], presented an iterative approach for the analysis of thin plates with partial internal line supports PILSs. Two models were considered in this work the first with one PILS and the second with two PILSs. The solution of the problems was reduced to a sequence of boundary value problems for the gained Poison's equation and the available software was used to solve for the obtained second-order equation. The fast convergence of the proposed method was shown using several numerical examples and the accuracy of this method was also verified by comparing the obtained results with those from the literature.

### 2.3.2 Numerical Techniques

Katsikadelis et al. in 1990 [41], prepared a boundary element approach for the analysis of Kirchhoff's plates with arbitrary layout of internal supports (columns , walls, beams) in different shapes in addition to the boundary supports. The Green’s function for static problem and Gauss integration were utilized for establishing this approach. From the advantages of this approach is that all kinds of the boundary conditions can be treated with the same set of equations and this method can be used to solve for the nonlinear internal supports as well as the linear problems. The accuracy of this method had been evaluated using some numerical examples.

Hartley and Abdel-Akher prepared in 1992 [42] a couple of direct boundary element method (DBEM) and stiffness method for analysis of structural frames contains thin plates supported by rigid path supports. The DBEM was used to find the stiffness matrix of the plate and the patch reaction are treated to be rigid. The rigid patch was modeled as a set of points some of them at the corners of the patch and the rest inside the patch area. The reaction force in the patch is simply the sum of the reaction forces at theses points and the moment is the sum of the moments around the center of the patch. The compatibility condition was then applied at these patch points and it was found that a good accuracy can be found with a few patch points for rigid reaction as it was discussed in [43] by Abdel-Akher and [44] by Chen. The accuracy of the proposed method has been verified through several simple numerical examples and more comprehensive and complex examples have been reported in another research paper.

Hartley and Abdel-Akher published a comprehensive work [45] in 1993 about analysis of building frames consists of flat-plate with the irregular layout and shapes of structural walls and columns. The material considered in this work was selected to be linear elastic material due to the complexity associated with constitutive relations of reinforced concrete and making the hypothesis that the linear elastic material can provide a good judgment to the capability of structures to sustain under the applied load. In this work, a three dimensional analysis using the stiffness method was used by utilizing one dimensional structural elements such as beam element for the columns and walls. Regarding the slabs, however, the direct boundary element method (DBEM) for plate bending with the help of the fundamental solution was used for generating the stiffness of the slab. One of the major advantages of this method is that it can be used for the analysis of multistory building frames with no additional work required. The same work can also be extended to account for the analysis of building frames where there are column heads by including nonprismatic structural elements for such columns. The accuracy of the proposed method has been verified through several numerical examples and the results were found to be in an excellent agreement with the finite element results.

Rashed in 2002 [46], presented a powerful method coupling the boundary elements method and the flexibility method to perform the analysis of thin plates with internal supports. In this approach, the solution is divided to two main steps; the first is the analysis of the plates by ignoring the presence of the internal supports or the plates to be considered as it is supported only at the boundary. The second step is to perform the analysis by putting a unit loads at the point of the supports and the compatibility equations at the points of the internal supports are utilized to solve for the redundants. The accuracy of this method is very high as has been verified with some numerical examples but the main disadvantages of this method is that it cannot be applied to a plate where the edges are free or supported internally only.

Oliveira Neto and Paiva in 2003 [37], presented a boundary elements formulation for the analysis of floor slabs resting on internal columns. In this paper, the three nodal displacements method is used. Therefore, there are three equations for each node in the plate boundary and the points of the internal supports. The columns is assumed to behave elastically and the cross-sections of the internal columns are assumed to be remained flat after loading leading to the linear stress variation over the contact surface between the plate and the column. The accuracy of this approach was verified using numerical examples and the results were compared with the available literature.

Rashed in 2005 [47], presented a modeling of flat plate floor subjected to gravity loading and supported by internal supports using boundary elements method. The effect of shear deformation in the plate bending is considered to have more accurate modeling results and the internal supports are treated using their actual cross-sectional dimensions which lead to get more accurate results for the transferred moment from the slab to the column. . In this model, the values of the shear forces and bending moments can be found everywhere in the slab even over the columns with no need for moment redistribution as required by the FEM. Some practical numerical examples have been done and the results are matched with the FEM solution.

Paiva and Mendanca in 2010 [48], prepared a new approach for the analysis of plate-beam interaction which is commonly found in the building floor slab supported by beams. In this paper, the three nodal values method was used and the beams elements are replaced by their effects or actions in on the plates a distributed loads as well as the end forces. The differential equation of the beam has been coupled with boundary elements method through the beam-plate interface. Several numerical examples have been solved to verify the accuracy of this method and the results are found to be in an excellent agreement with the finite element available solution.

Guminiak and Szajek in 2014 [49], prepared a static analysis of elliptic and circular thin plate supported internally by flexible supports using the boundary element method. The elastic supports have been considered here have the same characters of the elastic Winkler foundation taking into account the effect of bilateral and unilateral constraints. The Bèzine technique and static fundamental solution for the normal thin plate were utilized to introduce the supports inside the domain of the plate and finding the reactions and the deflection at the points of these supports. The accuracy of this approach has been checked using some numerical examples for both circular and elliptic plate.

Sasikala et al. in 2015 [50], prepared a nonsingular formulation for the analysis of thin plates supported by internal and external beams using a couple of boundary element method and the finite element method. In this method, the plate was treated using the BEM with deflection and the rotations in two normal directions. The beams were modeled using the FEM by utilizing the Timoshenko beam theory by considering the transverse shear forces, bending and twisting moments. The compatibility and equilibrium conditions at the interfaces were applied to couple the plate equations and the beams equations and solution for the unknowns degrees of freedom was obtained easily. The accuracy of this formulation was verified numerically and a C++ oriented program was developed.

To sum up, most of the previous numerical studies involve plates with regular shapes and layouts of columns and supports. Moreover, they do not pay enough attention to the zone of the plate column interaction. Furthermore, the available analytical solution is based on so many approximations related to the layout of the column and the supporting conditions. The proposed method addresses all the above issues and capable of investigating accurately the behavior at the plate-column zone.

# CHAPTER 3

# THE GOVERNING EQUATIONS OF PLATE

The bending of thin plate subjected to lateral loading is governed by a fourth order partial differential equation in terms of displacement. Once the solution for of the governing differential equation is achieved, one can simply solve for other secondary variables such as bending moments, stresses, strains and so on. The governing differential equation or the thin plate classical theory is based on the following assumptions as known as Kirckkoff's hypotheses:

1- The deflection is small compared to the thickness of the plate.
2- The midplane of the plate remains unrestrained.
3- Plane sections before bending remain plane after bending.
4- The stress in the z-direction is small compared to other plane stresses.

Based on the above mentioned assumptions all plate unknowns are expressed in terms of displacement.

## 3.1 Strain-Displacement Relations

The strains can be expressed in terms of displacement as shown below:

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{3.1}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{3.2}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \tag{3.3}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{3.4}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0 \tag{3.5}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \tag{3.6}$$

From Figure 14, the deflection does not vary along the thickness and it is, therefore, function of $x$ and $y$ only.

$$w = w(x, y)$$

For very small slope

$$\tan\left(\frac{\partial w}{\partial x}\right) \approx \frac{\partial w}{\partial x} = \frac{-u}{z}$$

Leading to

$$u = -z\frac{\partial w}{\partial x} \tag{3.7}$$

Similarly

$$v = -z\frac{\partial w}{\partial y} \tag{3.8}$$

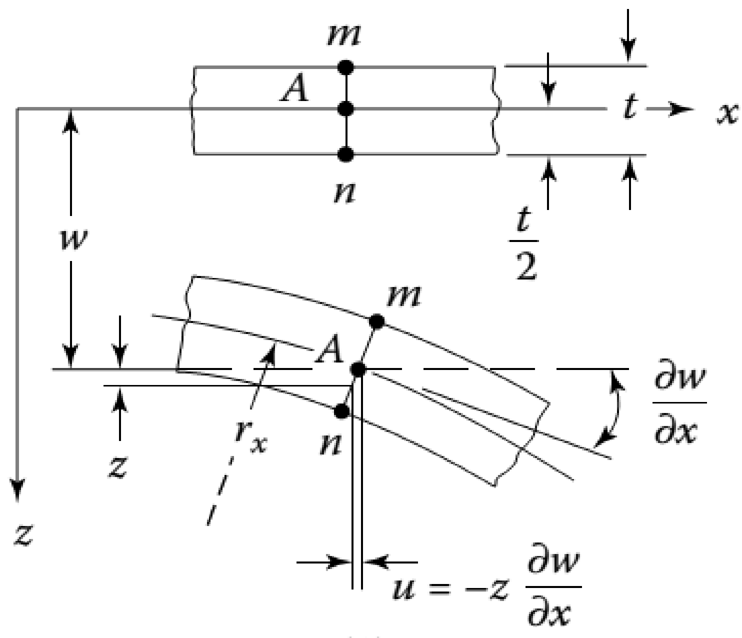



Figure 14 Part of the plate before and after deflection [1]

Substitution of Equations 3.7 and 3.8 into Equations 3.1 to 3.6, one can simply get

$$\varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \tag{3.9}$$

$$\varepsilon_y = -z\frac{\partial^2 w}{\partial y^2} \tag{3.10}$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{3.11}$$

## 3.2 Stress and Stress Resultants-Displacement Relations

The generalized Hooke's law for three dimensional state of stress for a homogeneous material is expressed as shown below:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] \tag{3.12}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \tag{3.13}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \tag{3.14}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{3.15}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} \tag{3.16}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{3.17}$$

Where the double subscript represents the shear stresses, the first subscript stands for the normal direction to the plane and the second subscript stands for the direction of the stress.

The shear modulus of elasticity can be expressed in terms of Poisson's ratio and the elastic modulus

$$G = \frac{E}{2(1 + \nu)} \tag{3.18}$$

Substitution of zero stains and Equation 3.18 into Equations 3.12 to 3.17 and expressing the non-zero stresses in terms of the non-zero strains yields

$$\sigma_x = \frac{E}{1 - \nu^2}(\varepsilon_x + \nu\varepsilon_y) \tag{3.19}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \tag{3.20}$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \tag{3.21}$$

Substitution of Equations 3.9 to 3.11 into Equations 3.19 to 3.21 yields

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{3.22}$$

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{3.23}$$

$$\tau_{xy} = -\frac{Ez}{(1+\nu)}\frac{\partial^2 w}{\partial x \partial y} \tag{3.24}$$

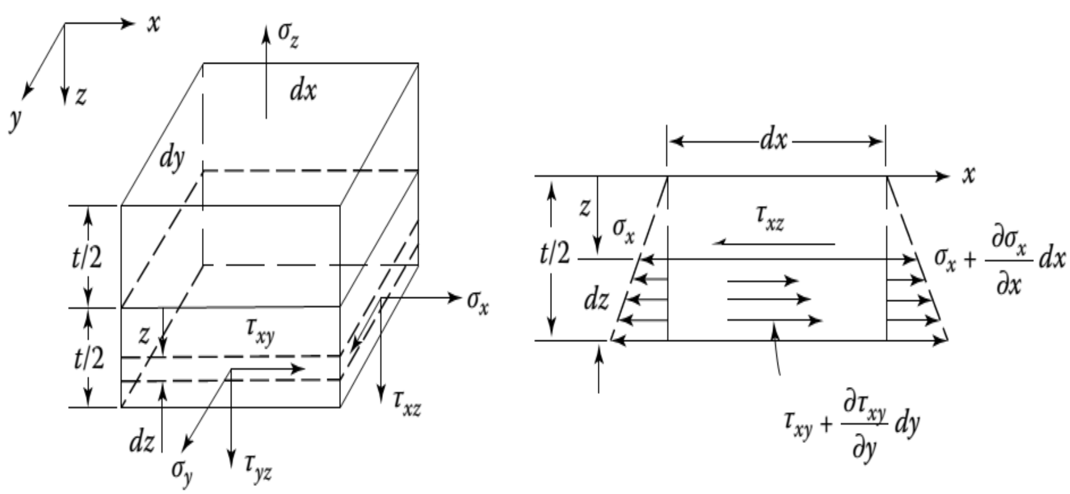



**Figure 15 Stresses at the bottom part of the plate element [51]**

As it can be shown in Figure 15, the normal stresses are varied linearly along the plate thickness and the resultant moments and shear can simply be expressed as

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} zdz$$

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} dz$$

Substitution of Equations 3.22 to 3.24 into above mentioned expressions and performing the integration the one can get

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) \tag{3.25}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \tag{3.26}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \tag{3.27}$$

Where D is the stiffness of the plate which is given by

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{3.28}$$

The stresses can also be expressed in terms of the moments by substituting Equations 3.25 to 3.27 into Equations 3.22 to 3.24 and simplifying. This leads to

$$\sigma_x = \frac{12M_x z}{t^3} \tag{3.29}$$

$$\sigma_y = \frac{12M_y z}{t^3} \tag{3.30}$$

$$\tau_{xy} = \frac{12M_{xy} z}{t^3} \tag{3.31}$$

## 3.3 The Governing Differential Equation for Plate Bending

As long as all the stresses, strains, moments, and shear forces are expressed in terms of the vertical displacements $w$, it would be simply finding the governing differential equations in terms of $w$.

The variation of shear and moment expressions with position can simply be expressed by the first two terms of the Taylor's expansion. The shear force $Q_x$, for example, changes after a distance $dx$ to be

$$Q_x + \frac{\partial Q_x}{\partial x} dx$$

As shown in Figure 16, the force equilibrium about the $z$ axis is given by

$$\frac{\partial Q_x}{\partial x} dxdy + \frac{\partial Q_y}{\partial y} dxdy + pdxdy = 0$$

Leading to

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \tag{3.32}$$

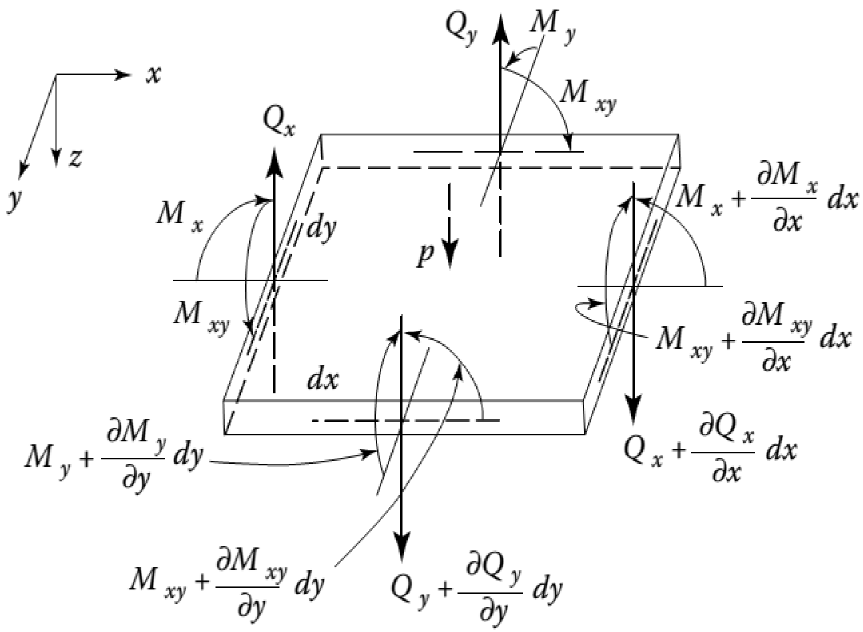



**Figure 16 Positives stresses resultants and load in a plate element**

The moment equilibrium about the $x$ axis is given by

$$\frac{\partial M_{xy}}{\partial x} dxdy + \frac{\partial M_y}{\partial y} dxdy - Q_y dxdy = 0$$

Or simply

$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \tag{3.33}$$

Similarly, the moment equilibrium about the $y$ axis gives

$$Q_x = \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} \tag{3.34}$$

Then, the introduction of the of $Q_x$ and $Q_y$ from Equations 3.33 and 3.34 into the equilibrium equation about $z$ axis yields to

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_x}{\partial x \partial y} + \frac{\partial^2 M_x}{\partial y^2} = -p \tag{3.35}$$

The shear forces $Q_x$ and $Q_y$ can be expressed in terms of displacement by using Equations 3.33 and 3.34 along with the moment expressions in Equations 3.25 to 3.27 leading to

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial x}(\nabla^2 w) \tag{3.36}$$

$$Q_x = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -D\frac{\partial}{\partial y}(\nabla^2 w) \tag{3.37}$$

Where

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Finally, substitution of the moment expression from Equations 3.25 to 3.27 into Equation 3.35 yields

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \tag{3.38}$$

Which is known as the general differential equation for plate bending of a plate with constant flexural rigidity. It also can be written in a more compact form by

$$\nabla^4 w = \frac{p}{D} \tag{3.39}$$

Where

$$\nabla^4 = (\nabla^2)^2 = \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)$$

## 3.4 Boundary Conditions

In general, to find all the stresses and stress resultants you need to have an expression for the deflection $w$. This $w$ has to satisfy the governing differential equation (3.39) and the boundary conditions.

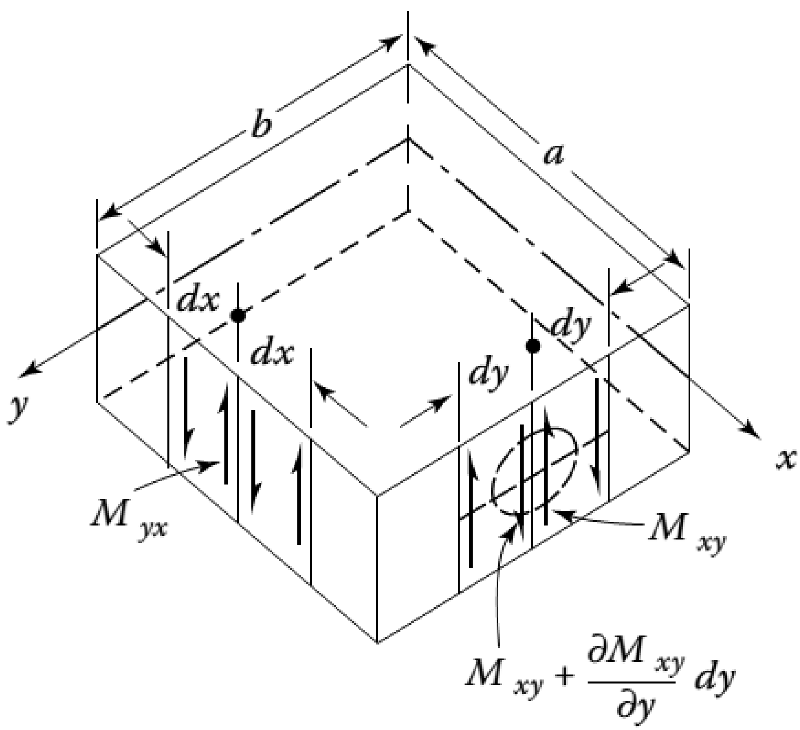



Figure 17 Effect of the twisting moment on the shear force

For a rectangular plate of length $a$ along the $x$ axis and width $b$ along the $y$ axis as shown in Figure 17 the shear force along the boundary parallel to $y$ axis can be modified by adding the terms coming from the twisting moment. The modified shear force is known as effective transverse shear force, $V_x$. This can be written as

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -D\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right) \tag{3.40}$$

Similarly, for a plate edge parallel to the $x$ axis

$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x} = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial y \partial x^2}\right) \tag{3.41}$$

### 3.4.1 Boundary Conditions of Straight Boundaries

The practical boundary conditions commonly used for rectangular plates are summarized in Figure 18 and Table 4. The edge $x = a$ is used to represent the boundary condition for each case.

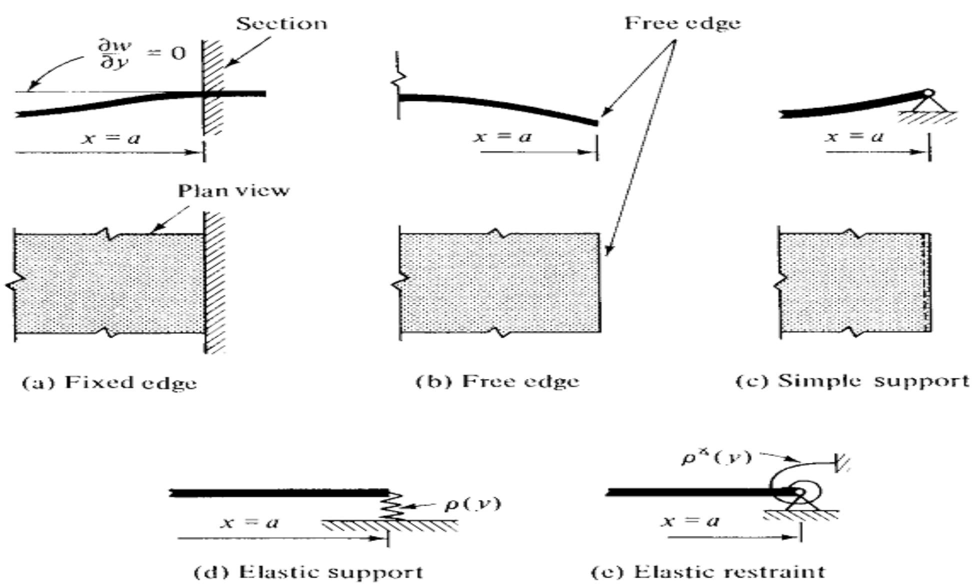



**Figure 18 The boundary conditions for commonly used cases [51]**

Table 4 Summary of the boundary conditions for straight boundaries

| **Type of Support Along the Edge** $x = a$ | **Mathematical Expression** |
|---|---|
| Simple support | $w = 0;\ M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0$ |
| Clamped support | $w = 0;\ \frac{\partial w}{\partial x} = 0$ |
| Free edge | $M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0;$ <br> $V_x = -D\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right) = 0$ |
| Partially clamped support | $(w) = 0;$ <br> $\frac{\partial w}{\partial x} = -\frac{D}{\rho^0}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$ |
| Elastic support | $M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) = 0;$ <br> $w = -\frac{D}{\rho}\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right)$ |

Note: $\rho^0$ =the rotational stiffness of the support; $\rho$ =the stiffness of the support.

### 3.4.2 Boundary Conditions of Curved Boundaries

For the case of curved boundary as shown in Figure 19 the expressions of the moment, shear, and slope are taken in the form of derivatives with respect to the normal direction.

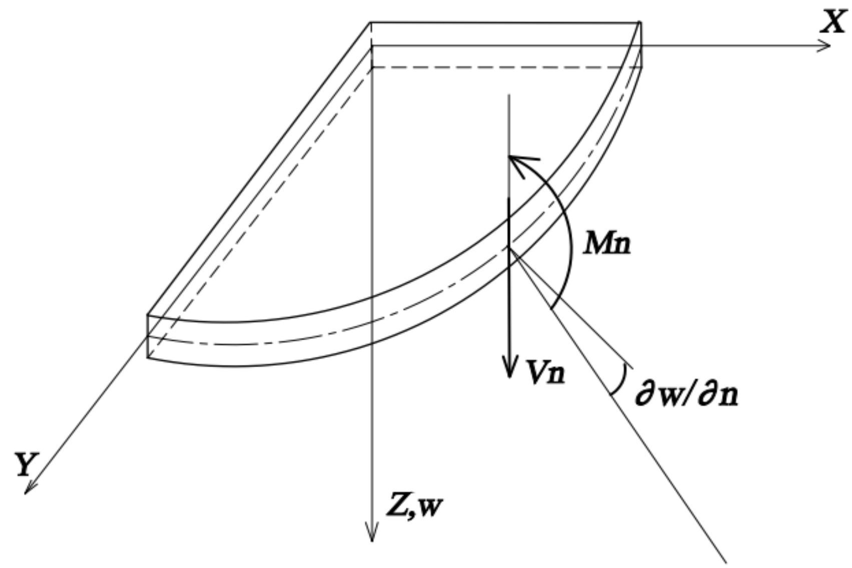



**Figure 19 Curved boundary of a plate**

The slope for the curved boundary is therefore given by

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} nx + \frac{\partial w}{\partial y} ny \tag{3.42}$$

Similarly, the moment

$$M_n = -D\left[\nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + (1-\nu)\left(\frac{\partial^2 w}{\partial x^2} nx^2 + \frac{\partial^2 w}{\partial x \partial y} nxny + \frac{\partial^2 w}{\partial y^2} ny^2\right)\right] \tag{3.43}$$

And the shear

$$V_n = D\left[\begin{array}{c} nx\left(\frac{\partial^3 w}{\partial x \partial y^2}\left(-1-2ny^2(\nu-1)\right)+\frac{\partial^3 w}{\partial x^3}\left(-1-2ny^2(\nu-1)\right)\right) \\ +ny\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2 \partial y}\left(-1+ny^2(\nu-1)\right)\right) \\ +nx^3\frac{\partial^3 w}{\partial x \partial y^2}(\nu-1)-nx^2 ny\left(2\frac{\partial^3 w}{\partial x^2 \partial y}-\frac{\partial^3 w}{\partial x^3}\right)(\nu-1) \end{array}\right] \tag{3.44}$$

The boundary conditions for some common cases can then be expressed as shown below:

For simply supported curved edge:

$$w = 0;\ M_n = 0$$

For clamped support:

$$w = 0;\ \frac{\partial w}{\partial n} = 0$$

And for free edge:

$$V_n = 0;\ M_n = 0$$

# CHAPTER 4

# BPM FOR PLATE WITHOUT INTERNAL SUPPORTS

In this chapter, the BPM formulation for plate with edge supports is prepared in details. A simple computer code is recommended for applying the prepared BPM formulation. In addition, numerical examples are solved and compared with analytical solutions.

## 4.1 BPM Formulation

### 4.1.1 The homogenous Solution

In the boundary point method (BPM), also known as the method of fundamental solution (MFS), the solution to the problem is approximated as a linear superposition of the fundamental solution of the problem with a set of discrete points $\underline{x}_s^j, (j = 1, \ldots, 2N_b)$, that are located on a "virtual boundary" outside the domain of the problem, called source points, and another set of discrete points $\underline{x}_b^j, (j = 1, \ldots, N_b)$ that are located on the actual boundary, called boundary or collocation points as shown in Figure 20. For the plate problem, we define the fundamental solution as the solution of the differential equation

$$\nabla^4 w^* = \Delta(\underline{x} - \underline{x}_s) \tag{4.1}$$

where $\underline{x}$ contains the coordinates of an arbitrary point inside the domain or on the boundary and $\Delta\left(\underline{x} - \underline{x_s}\right)$ denotes the Dirac delta function. It can be shown that the solution of (4.1) is given by

$$w^*(r) = \frac{r^2 \log(r)}{8\pi D} \tag{4.2}$$

where

$$r = \left| \underline{x} - \underline{x_s} \right|$$

Thus, the BPM homogenous solution of the plate problem can then be given by

$$w_h(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) \tag{4.3}$$

Where $c^j =$ are constants to be determined later from the system of equations.

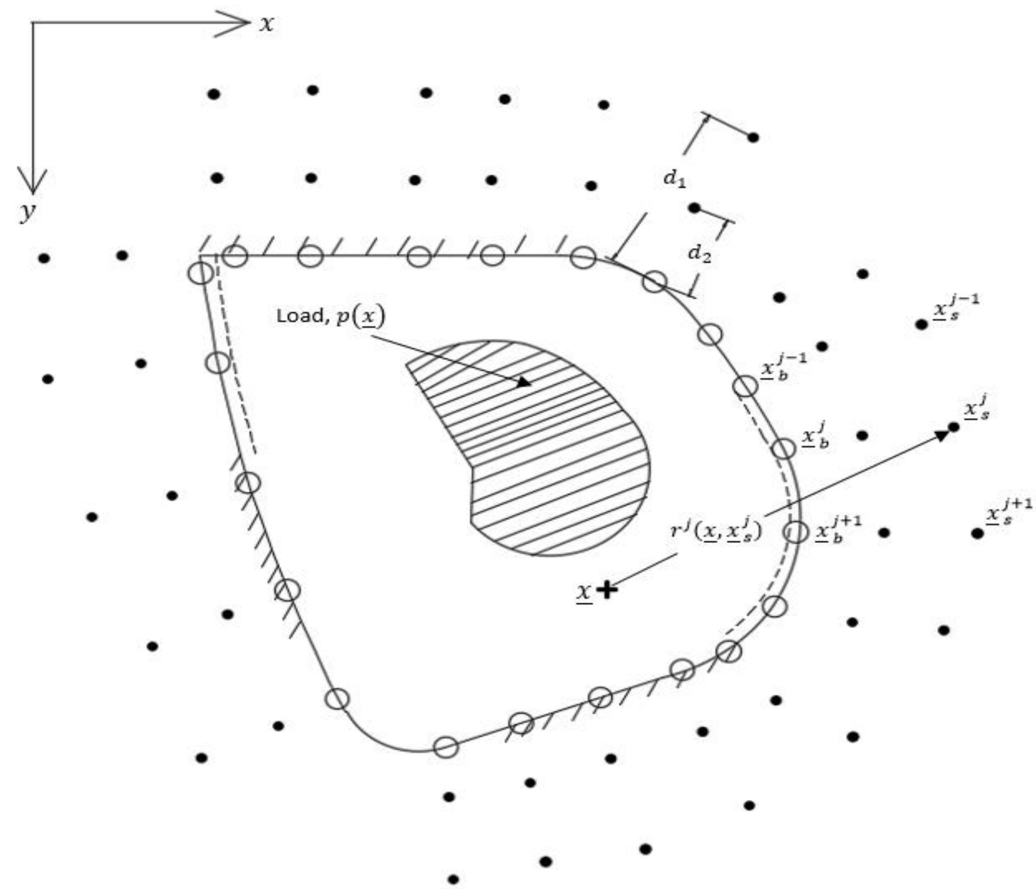



Figure 20 Plate subjected to domain load and different boundary conditions

The complete solution $w$ of the governing plate equation is can simply be found by superimposing the homogenous and particular solution to be

$$w(\underline{x}) = w_h(\underline{x}) + w_p(\underline{x})$$

Such that:

$$\nabla^4 w_h(\underline{x}) = \Delta(\underline{x} - \underline{x}_s)$$

And

$$\nabla^4 w_p(\underline{x}) = \frac{q(\underline{x})}{D}$$

### 4.1.2 The Particular Solution for Uniformly Distributed Load

The particular solution $w_p$ depends on the type of the applied load. For the case of uniformly distributed load the particular solution is given by

$$w_p(\underline{x}) = \frac{p_0}{48D}(x^4 + y^4) \qquad (4.2)$$

The complete solution becomes

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + \frac{p_0}{48D}(x^4 + y^4)$$

### 4.1.3 The Particular Solutions for Point Loads

If the applied loads are point loads $P^i$ on $(m)$ discrete domain points $\left(\underline{x}_m^i\right)$, the particular solution is simply the fundamental solution times the corresponding point loads. The complete solution becomes:

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + \sum_{j=1}^{m} P^i w^*(\underline{x}, \underline{x}_m^i) \tag{4.3}$$

### 4.1.4 The Particular Solution for Patched Loads

In most practical cases there are no point loads and the loads are distributed over a patched areas. To treat this kind of patched loads let us start by expanding the solution of one point load.

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + P w^*(\underline{x}, \underline{x}_m) \tag{4.4}$$

For the case one a pathed load $q\left(\underline{\xi}\right)$ distributed over a domain $\Omega_q$

$$q = \frac{P}{\Omega_q}$$

Leading to:

$$P = \int_{\Omega_q} q\left(\underline{\xi}\right) d\Omega_q$$

Therefore

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + \int_{\Omega_q} q\left(\underline{\xi}\right) w^*\left(\underline{x}, \underline{\xi}\right) d\Omega_q \tag{4.5}$$

Let

$$\nabla^2 v^* = w^* \tag{4.6}$$

Where:

$$v^* = \frac{r^4(3 - 2\log(r))}{256\pi D} \tag{4.7}$$

By utilizing Equation 4.6, the second part of the right hand side of Equation 4.5 can be expressed by

$$\int_{\Omega_q} q\left(\underline{\xi}\right) w^*\left(\underline{x}, \underline{\xi}\right) d\Omega_q = \int_{\Omega_q} q\left(\underline{\xi}\right) \nabla^2 v^*\left(\underline{x}, \underline{\xi}\right) d\Omega_q$$

Or simply

$$\int_{\Omega_q} q\left(\underline{\xi}\right) w^*\left(\underline{x}, \underline{\xi}\right) d\Omega_q = \int_{\Omega_q} \left(\frac{\partial^2 v^*}{\partial x^2} + \frac{\partial^2 v^2}{\partial y^2}\right) q\, d\Omega_q \tag{4.8}$$

Let us deal with the part in *x* direction of the integration from the right hand side of Equation 4.8

$$\int_{\Omega_q} \left(\frac{\partial^2 v^*}{\partial x^2}\right) q\, d\Omega_q = \int_{\Omega_q} \frac{\partial}{\partial x}\left(\frac{\partial v^*}{\partial x} q\right) d\Omega_q - \int_{\Omega_q} \left(\frac{\partial v^*}{\partial x}\frac{\partial q}{\partial x}\right) d\Omega_q$$

Use of the Divergence Theorem to transfer the domain integral to a boundary integral leads to

$$\int_{\Omega_q} \frac{\partial}{\partial x}\left(\frac{\partial v^*}{\partial x} q\right) d\Omega_q = \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial x} q\right) nx \, d\Gamma_q$$

Therefore

$$\int_{\Omega_q} \left(\frac{\partial^2 v^*}{\partial x^2}\right) q \, d\Omega_q = \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial x} q\right) nx \, d\Gamma_q - \int_{\Omega_q} \left(\frac{\partial v^*}{\partial x}\frac{\partial q}{\partial x}\right) d\Omega_q$$

Again

$$\int_{\Omega_q} \left(\frac{\partial v^*}{\partial x}\frac{\partial q}{\partial x}\right) d\Omega_q = \int_{\Omega_q} \frac{\partial}{\partial x}\left(v^* \frac{\partial q}{\partial x}\right) d\Omega_q - \int_{\Omega_q} v^* \left(\frac{\partial^2 q}{\partial x^2}\right) d\Omega_q$$

Or

$$\int_{\Omega_q} \left(\frac{\partial v^*}{\partial x}\frac{\partial q}{\partial x}\right) d\Omega_q = \int_{\Gamma_q} \left(v^* \frac{\partial q}{\partial x}\right) nx \, d\Gamma_q - \int_{\Omega_q} v^* \left(\frac{\partial^2 q}{\partial x^2}\right) d\Omega_q$$

Leading finally to

$$\int_{\Omega_q} \left(\frac{\partial^2 v^*}{\partial x^2}\right) q \, d\Omega_q = \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial x} q\right) nx \, d\Gamma_q - \int_{\Gamma_q} \left(v^* \frac{\partial q}{\partial x}\right) nx \, d\Gamma_q + \int_{\Omega_q} v^* \left(\frac{\partial^2 q}{\partial x^2}\right) d\Omega_q \qquad (4.9)$$

Similarly, in $y$ direction

$$\int_{\Omega_q} \left(\frac{\partial^2 v^*}{\partial y^2}\right) q \, d\Omega_q = \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial y} q\right) ny \, d\Gamma_q - \int_{\Gamma_q} \left(v^* \frac{\partial q}{\partial y}\right) ny \, d\Gamma_q$$

$$+ \int_{\Omega_q} v^* \left(\frac{\partial^2 q}{\partial y^2}\right) d\Omega_q \tag{4.10}$$

Substitution of Equations 4.9 and 4.10 into Equation 4.8 and simplifying leads to

$$\int_{\Omega_q} q\left(\underline{\xi}\right) w^*\left(\underline{x}, \underline{\xi}\right) d\Omega_q = \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial n} q\right) d\Gamma_q - \int_{\Gamma_q} \left(v^* \frac{\partial q}{\partial n}\right) d\Gamma_q$$

$$+ \int_{\Omega_q} v^* \nabla^2 q d\Omega_q \tag{4.11}$$

Eventually, the final expression for a plate subjected to patch load is then can simply be written as

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial n} q\right) d\Gamma_q - \int_{\Gamma_q} \left(v^* \frac{\partial q}{\partial n}\right) d\Gamma_q$$

$$+ \int_{\Omega_q} v^* \nabla^2 q d\Omega_q \tag{4.12}$$

If the load q is uniform over $\Omega_q$, the solution reduces to

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial n} q\right) d\Gamma_q \tag{4.13}$$

## 4.2 System of Equations of BPM for plate with Edge Supports

The BPM equations for a plate without internal supports can then be formed by applying the boundary conditions to solve for the constants $(c^j, j = 1, \dots, 2N_b)$. The system of equations can be expressed by

$$BC_1\left(w\left(\underline{x}_b^j\right)\right) = 0, \qquad j = 1, N_b \tag{4.15}$$

$$BC_2\left(w\left(\underline{x}_b^j\right)\right) = 0, \qquad j = 1, N_b \tag{4.16}$$

Where the $BC_1$, $BC_2$ are the differential operators of the first and second boundary conditions at each boundary point $(\underline{x}_b^j)$.

## 4.3 Computer Code

The proposed system of equations prepared in the previous section is generated and solved using Wolfram Mathematica software. The symbolic capability of the software enables the determination of the solution and its derivatives in functional forms which makes the post processing of the solution easy.

## 4.4 Numerical Examples

The verification of the proposed BPM model formulation for plates without internal supports is achieved by comparing the obtained solution with the available analytical solution using the following examples.

### 4.4.1 Elliptical Plate with Clamped Edges

Consider an elliptical plate with clamped edges as shown in Figure 21. The plate is subjected to a uniformly distributed load $p$, with Poison's ratio taken to be $\nu = 0.3$ .

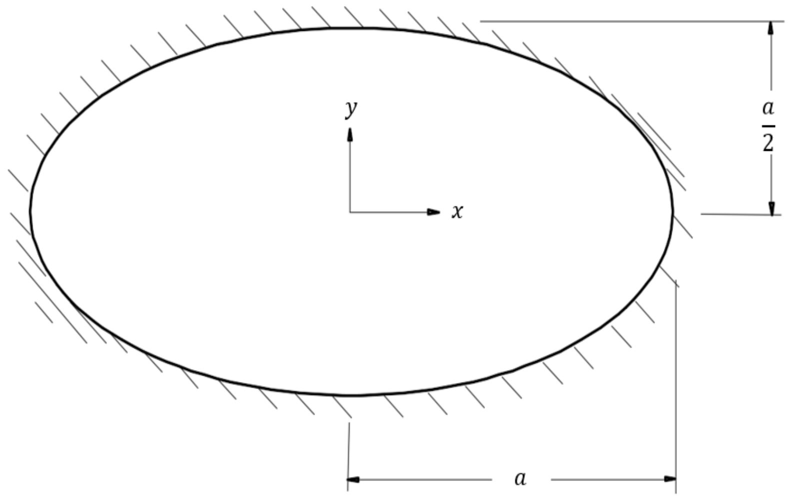



**Figure 21 Clamped elliptical plate subjected to uniformly distributed load**

The analytical (exact) solution of this problem is available in [2] and therefore no need to involve other numerical solutions in the comparison. The BPM model is based on 50 boundary points and two contours each containing 50 source points at distances of $0.075\ a$ and $0.15\ a$ away and normal to the plate boundary as shown in Figure 22.

The deformed shape of the plate is shown in Figure 23 below.

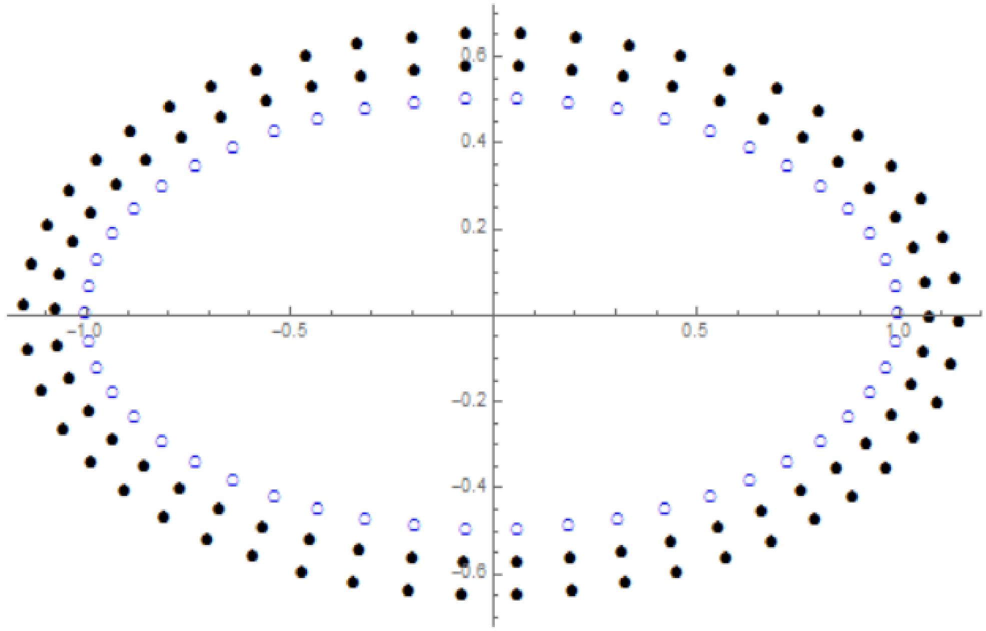



**Figure 22 Boundary points and the source points of the elliptical plate**

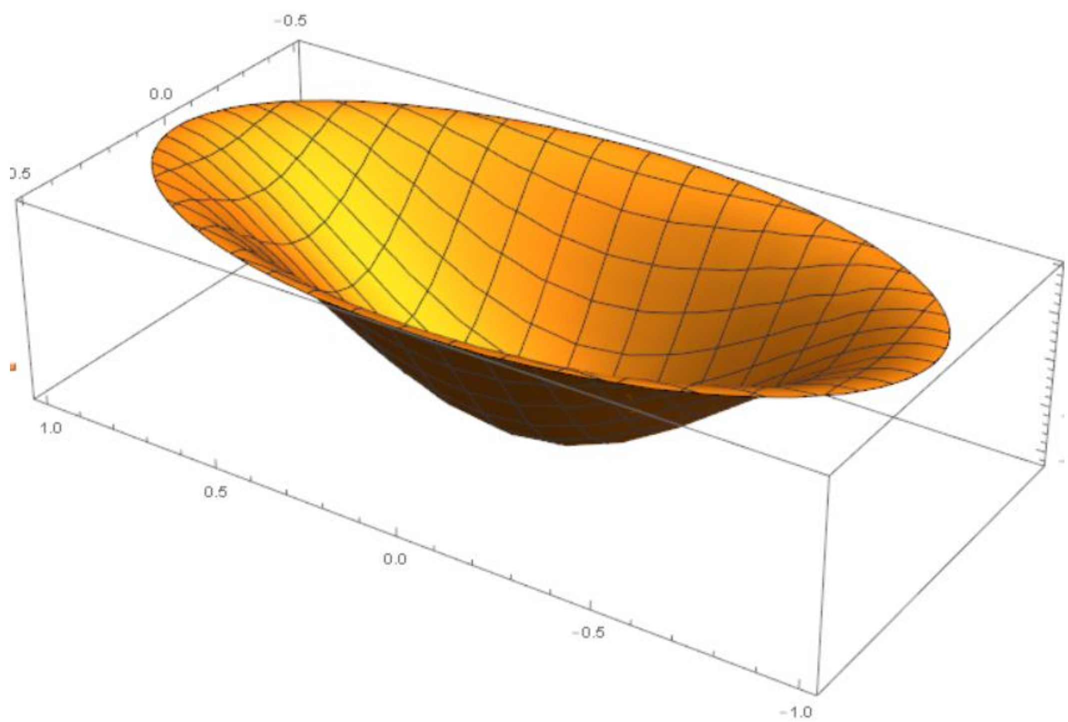



**Figure 23 A 3D-Model of the deformed elliptical plate**

The results of the deflection, bending moment $M_x$ , and bending moment $M_y$ along the line dividing the plate into two vertical parts are shown in Figures 24 through 26. An almost exact agreement with less than 0.2 % difference between the exact solution and the proposed BPM method is observed all through the presented line except some deviation very closed to the plate boundary ($x = 0.95\ a$) which increase the errors to be about 1.2 %, 0.7 %, and 5.5 % for deflection $w$, bending moment $M_x$, and bending moment $M_y$, respectively.

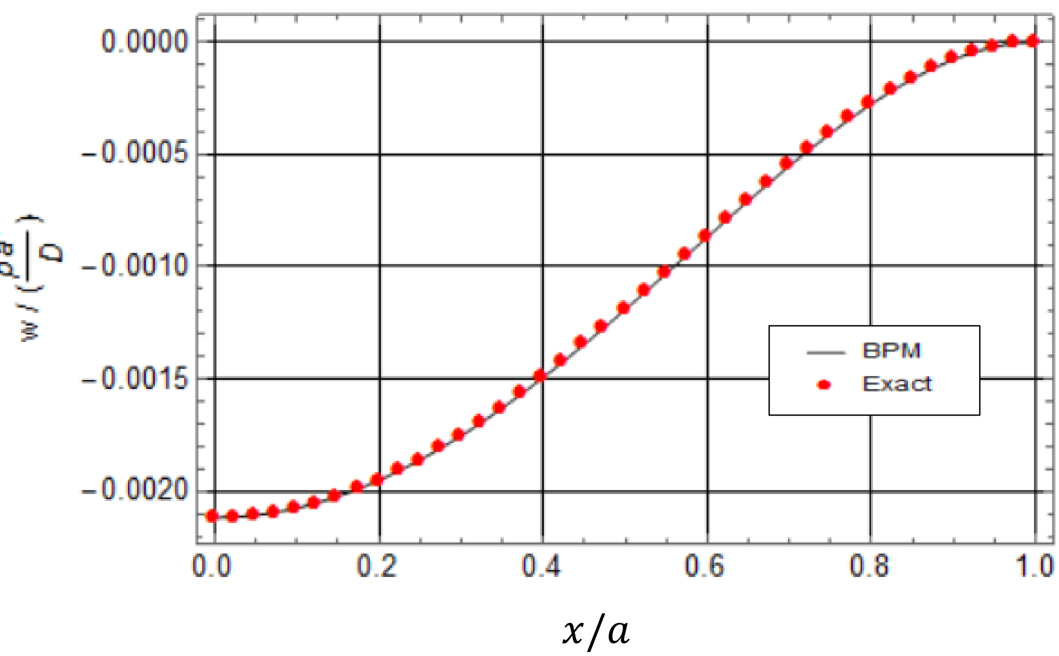



**Figure 24 Deflection of the plate along line at $y = 0$**

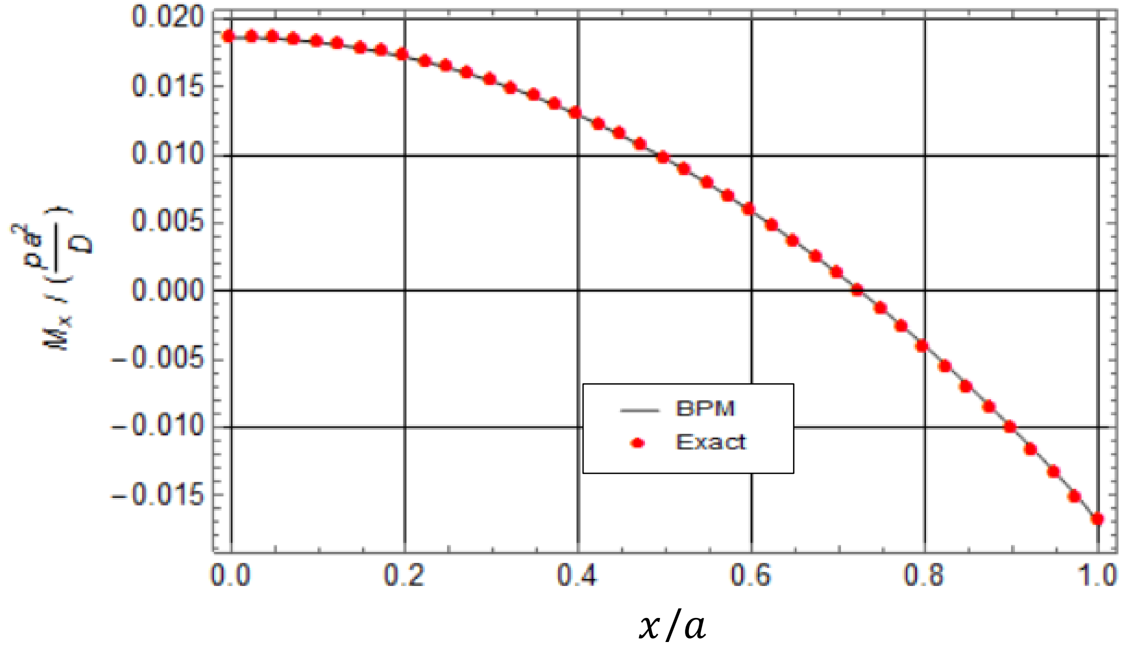



**Figure 25 Bending moment $M_x$ of the plate along line at $y = 0$**

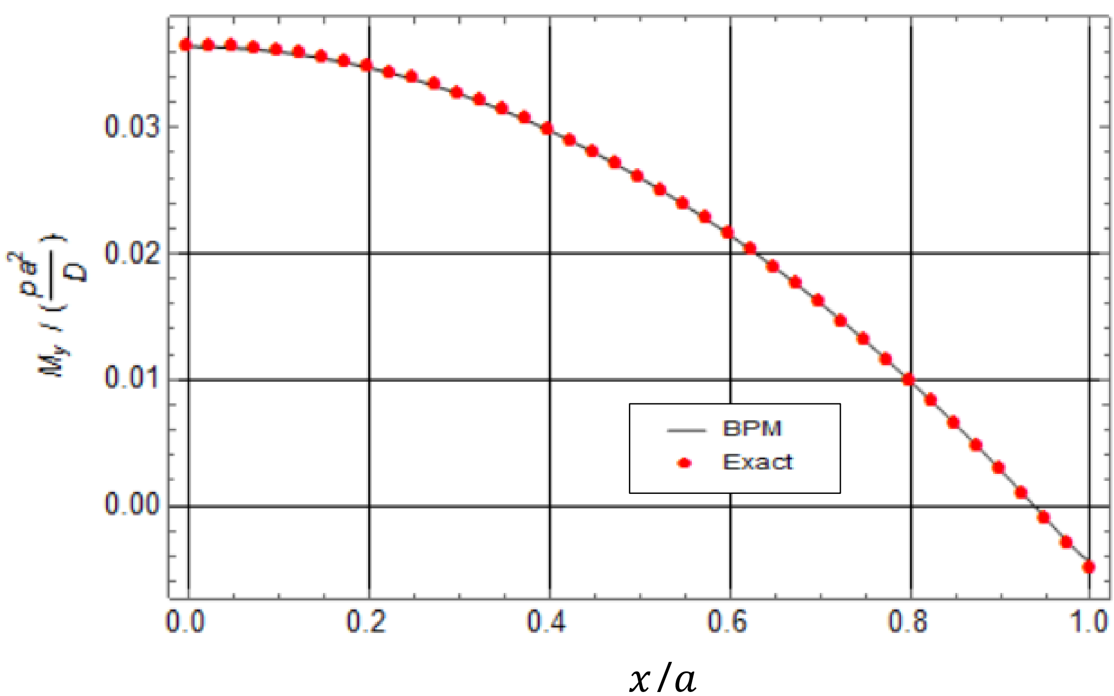



**Figure 26 Bending moment $M_y$ of the plate along line at $y = 0$**

The deviations of the results near the plate boundaries are mainly due to the following two reasons: (a) selection of the proper distance from the plate boundary, and (b) number of the boundary points.

### 4.4.2 A Simply Supported Square Plate Subjected to Central Point Load

The main idea of addressing this kind of problem is to test the capability of the proposed BPM model in capturing the singularity of the bending moment under the point load. The square plate ($a$ by $a$) is simply supported and subjected to a point load $P$ at its center as shown in Figure 27. The simple supporting conditions led to the use the available analytical (series-based) solution in [3] for the purpose of comparing with the BPM results.

Regarding the BPM Solution, application of Equation 4.3 to only one point load lead to

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + P\, w^*(\underline{x}, \underline{x}_m)$$

At the point where the load is applied $\underline{x} = \underline{x}_m$ , the radial distance $r(\underline{x}, \underline{x}_m) = 0$, therefore the term $P\, w^*(r = 0) = 0$ or, in short, at the point where the load is applied the concentrated load does not contribute to the deflection equation. On the other hand, the value of the bending moment can be found from the second derivative of the deflection. The term of the point load in the moment would be a multiple of $\log(r)$ which results in an infinite bending moment at the point of the applied load ($r = 0$) which agrees with the singularity expected for the moment under the point load.

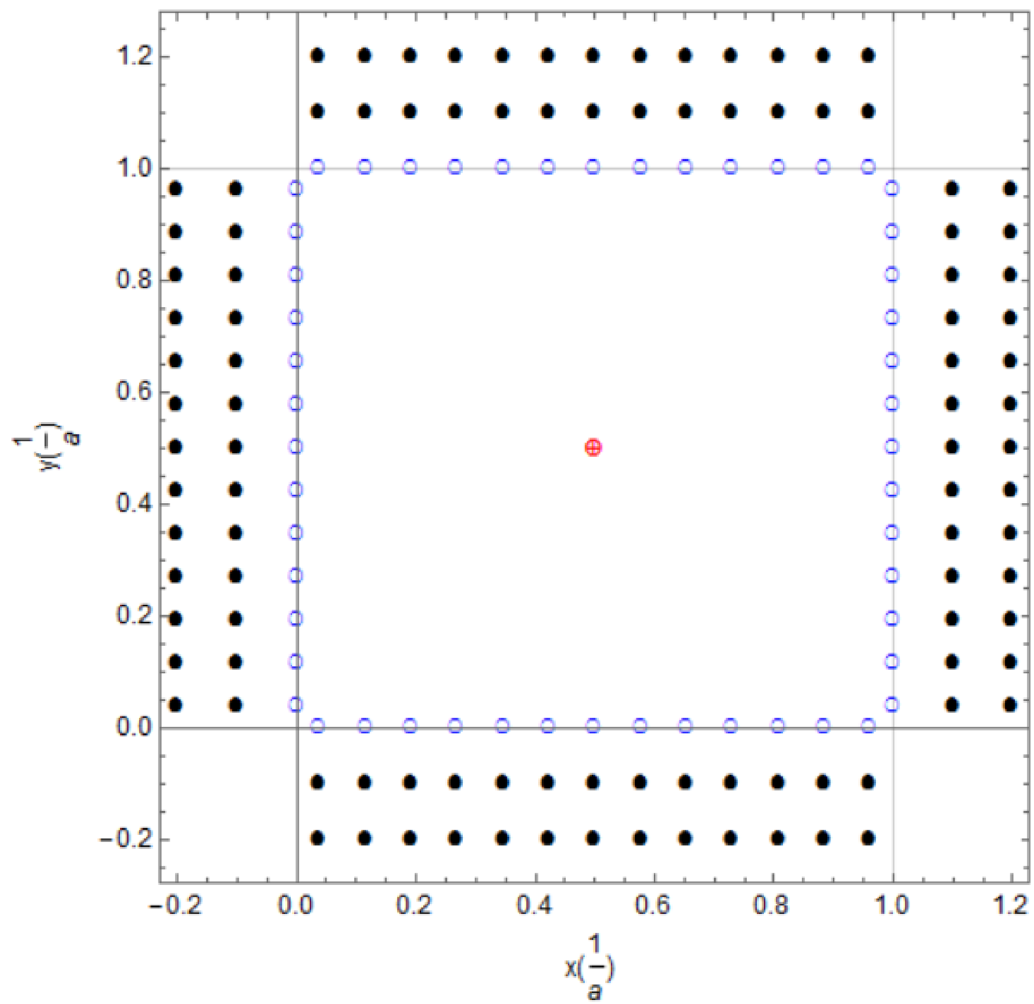



**Figure 27 Boundary points, source points, and the point of the point of the applied load**

The results of both the analytical and the BPM results along a line that divides the plate into two equal halves $y = a/2$ are presented in Figures 28 and 29 for both the deflection $w$ and bending moment $M_x$, respectively.

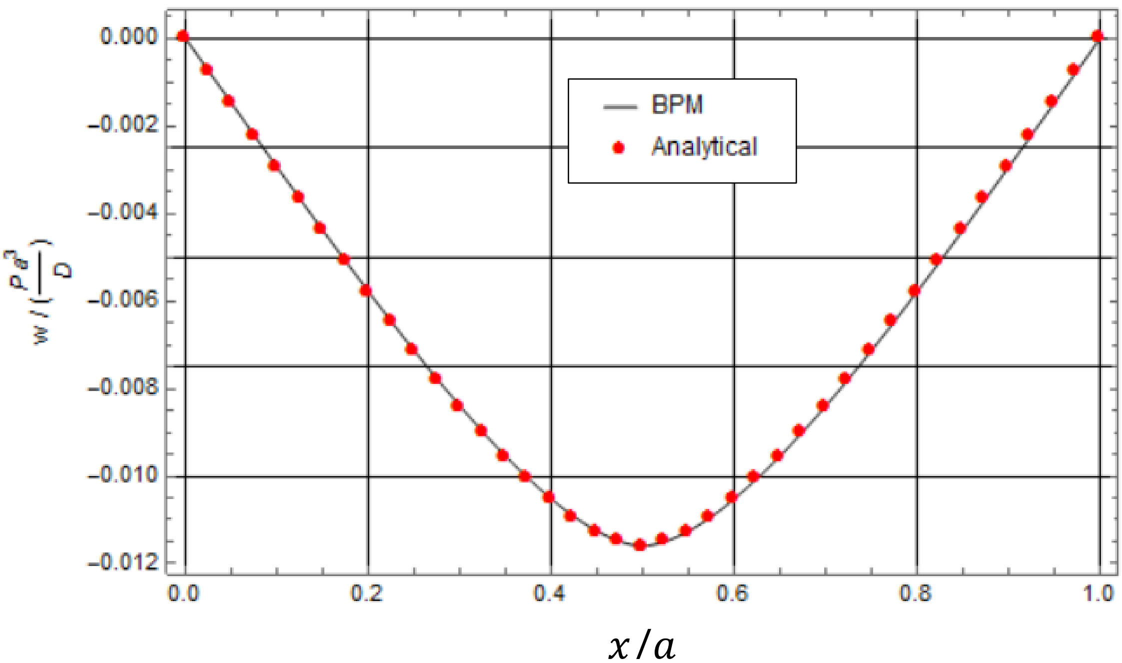



**Figure 28 The deflection of the plate $w$ along line at line at $y = a/2$**

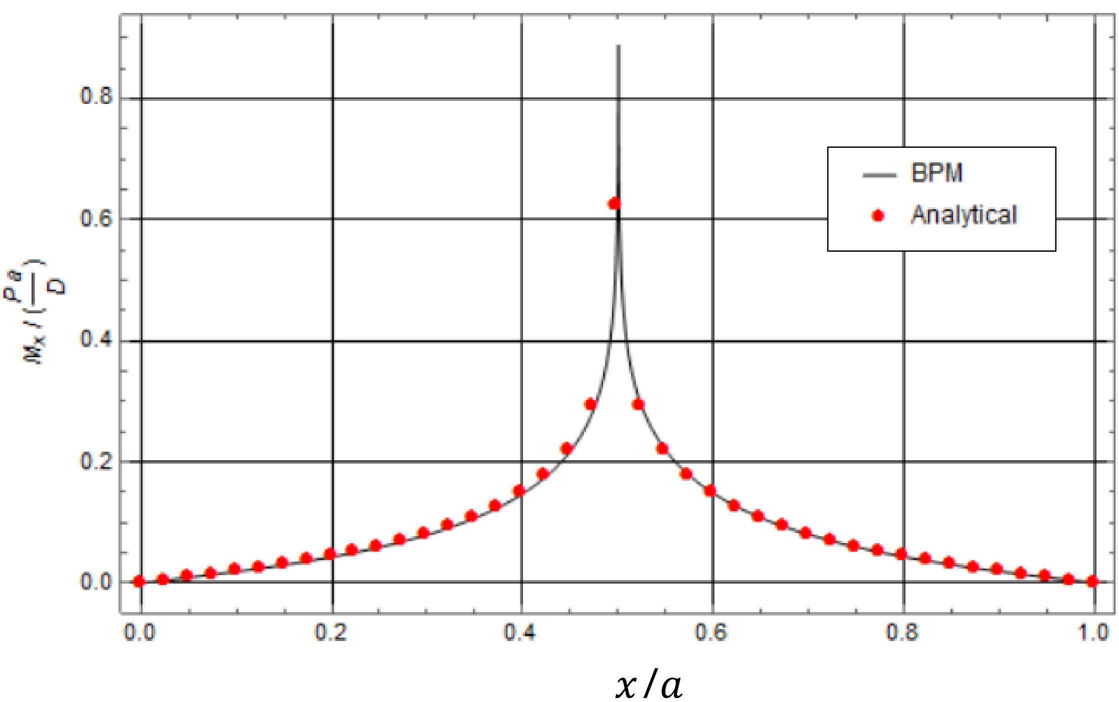



**Figure 29 Bending moment $M_x$ of the plate along line at $y = a/2$**

An almost exact agreement between the BPM and was observed with an error of less than 0.05 % in the deflection and 0.04 in the bending moment $M_x$ away from the point of the applied load. At the point of the applied load, we know that the bending moment is infinite and therefore it is not meaningful to compare the two solutions. However, the same figure clearly shows that the two solutions are in good agreement at the zone close to the point load.

### 4.4.3 A Simply Supported Square Plate Subjected to Central Patched Load

One of the possible remedies for resolving the singularity problems associated with the point load is to be more practical and replace the point load by a patched load which is more realistic in representing the load introduced in the plate due to a column. Consideration of this argument leads to deal with so called ‘patched load’ instead of point loads throughout the rest of this thesis.

An example for a patched load is shown in Figure 30, where a uniformly distributed load $q$ is applied over the shaded area as shown in the center of the plate. The plate is square in shape of side $a$ and simply supported all around. The Poison’s ratio $\nu = 0.3$ .

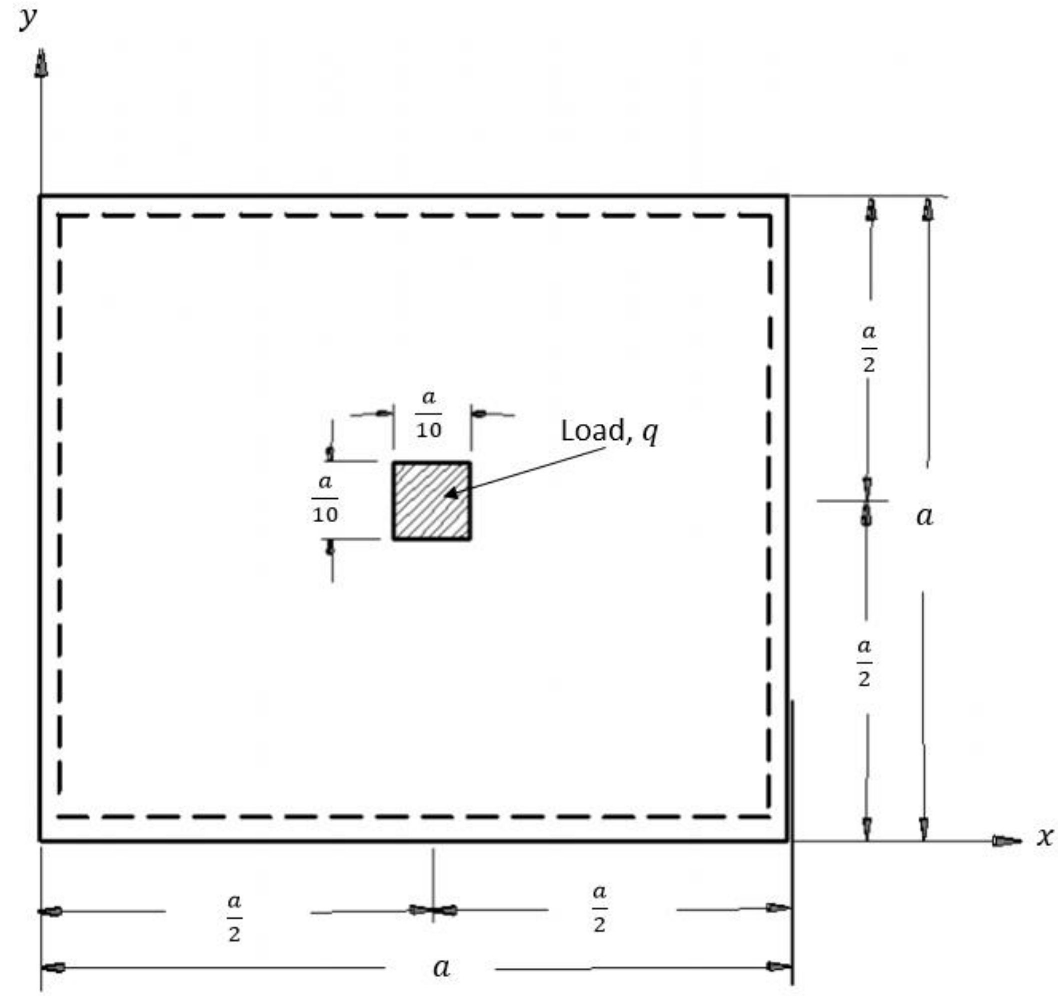



**Figure 30 A simply supported square plate subjected to central patched load**

Since the applied load in this problem is uniform, the solution can be obtained using Equation 4.13. Performance of the boundary integral was done numerically using Gaussian Quadrature. The boundary of the load was divided to 80 nodes as shown in Figure 31 and the numerical integration was performed between each two sequential points. BPM model is based on 80 boundary nodes and two contours each containing 80 nodes normal and away from the boundary.

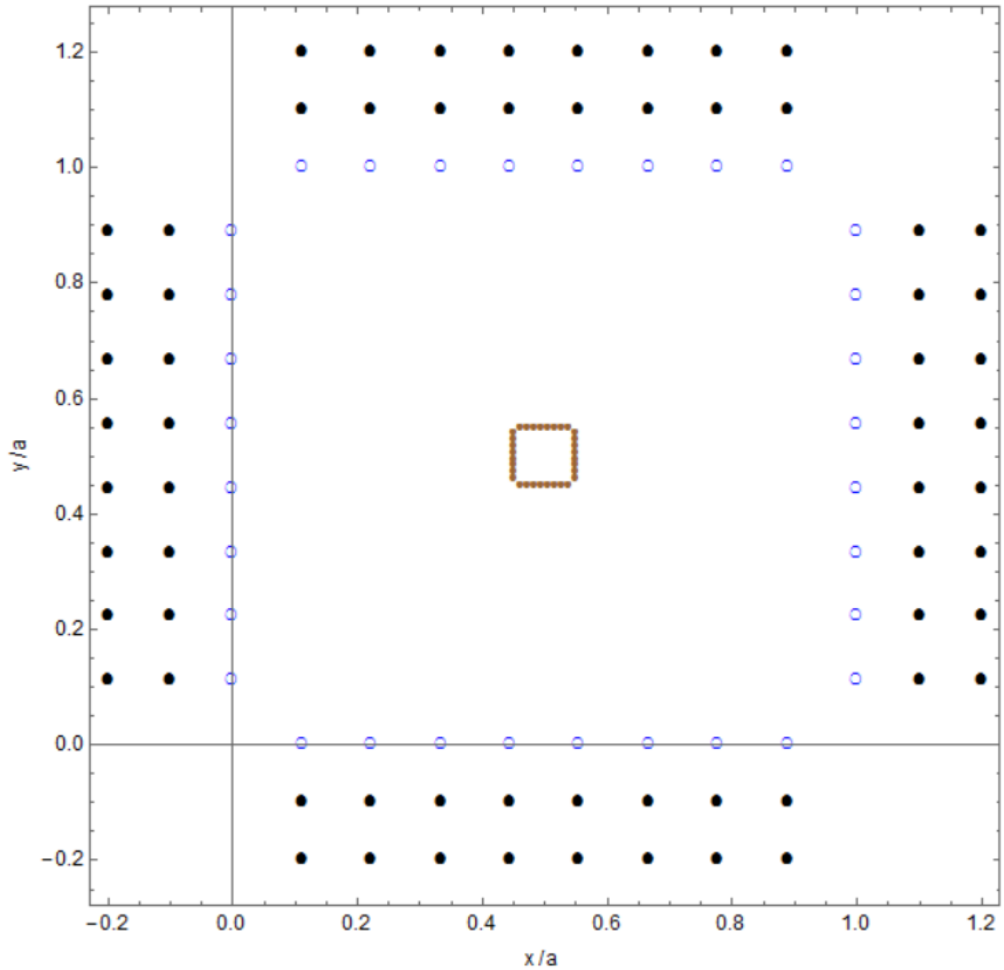



Figure 31 Boundary points, source points, and boundary points of the patched area

The results of the deflection $w$, bending moment $M_x$, and the shear force $V_x$ along a line at $y = a/\,2$ are shown in Figures 32 through 34, respectively. A comparison with the available analytical (series-based) solution in [3] with 100 terms in the series has been used and a very closed agreement was observed. The percentage of error is less than 0.4% in case of the deflection and bending moment and less than 1.0 % in case of the shear force.

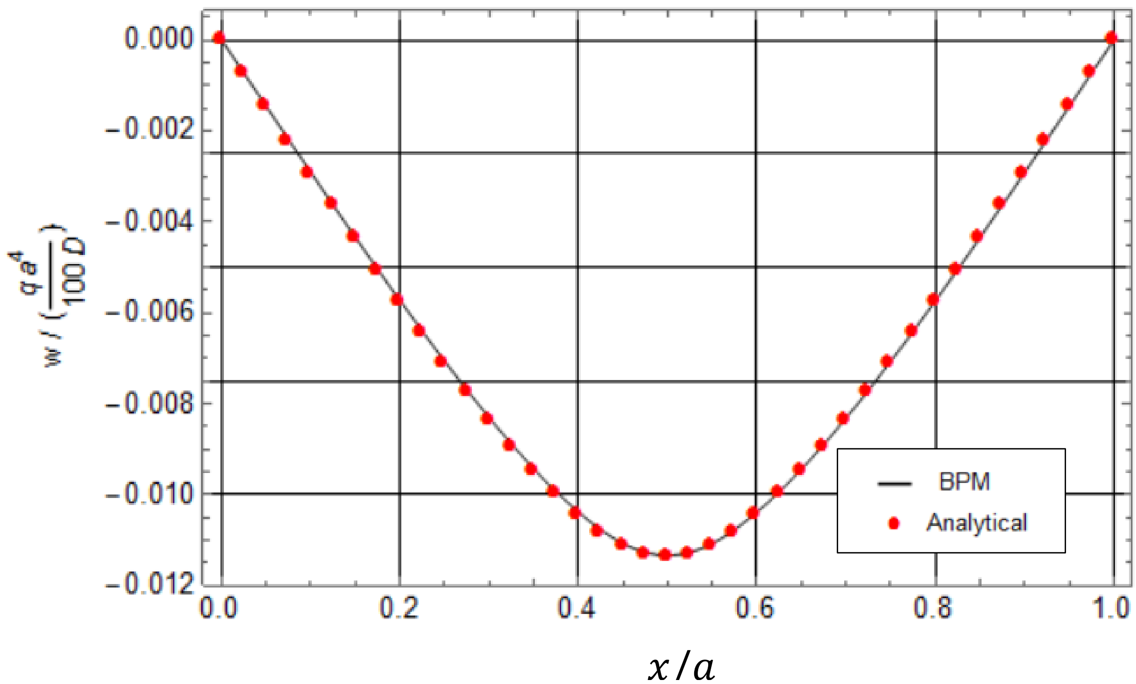



**Figure 32 Deflection $w$ of the plate along line at $y = a/2$**

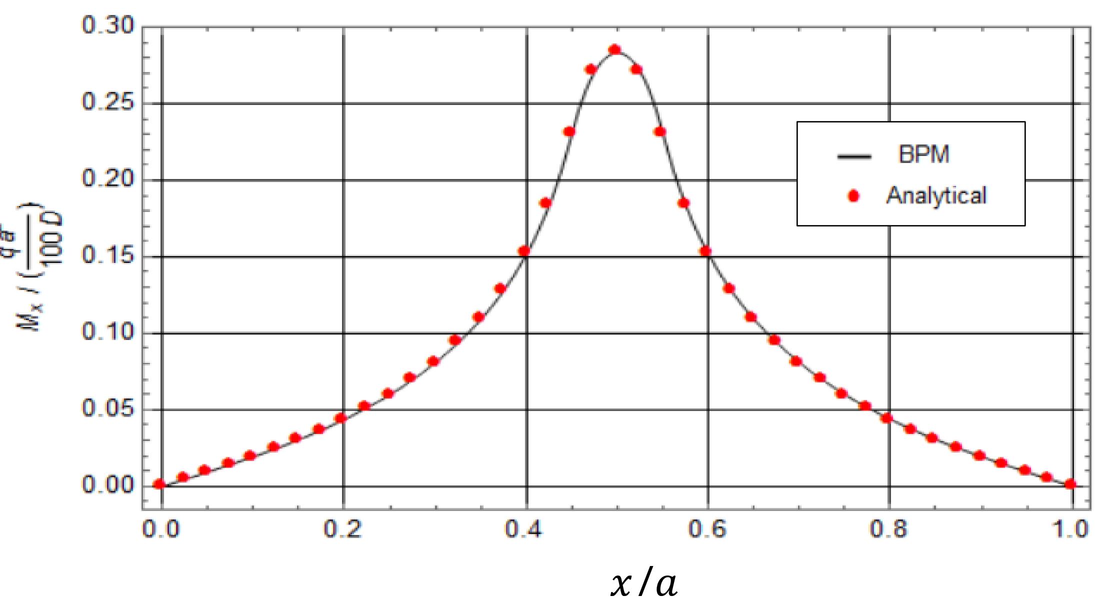



**Figure 33 Bending moment $M_x$ of the plate along line at $y = a/2$**

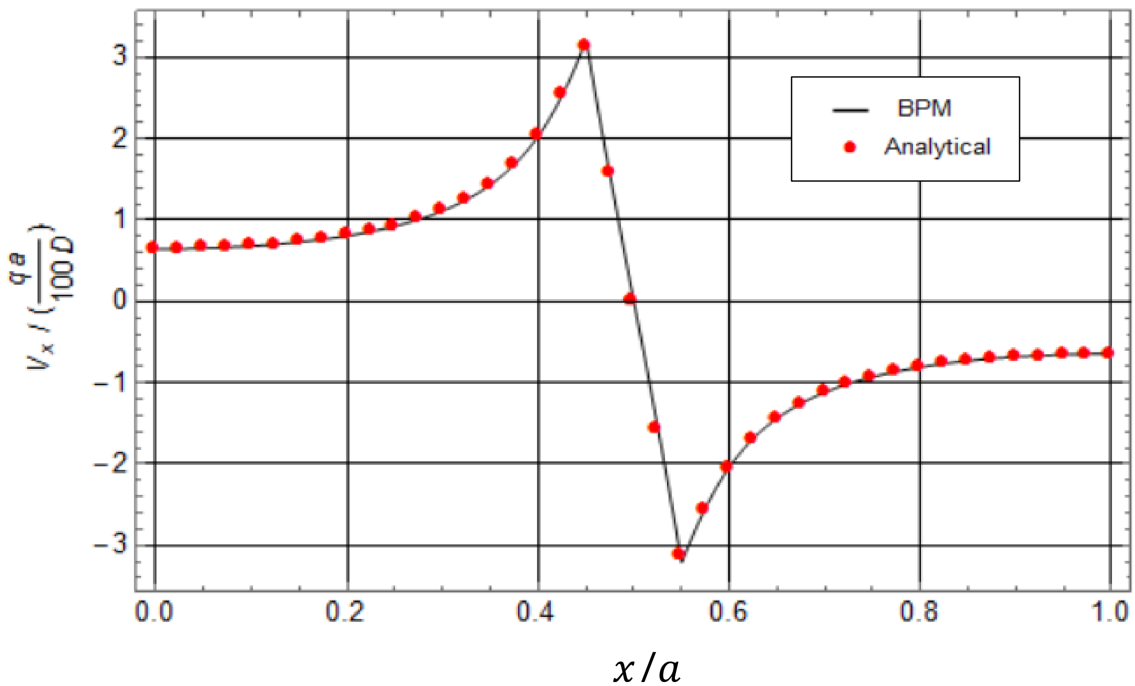



Figure 34 Shear force $V_x$ of the plate along line at $y = a/2$

The increase of the error for the case of shear force may results from the slow convergence associated with the shear due to the third derivatives and further reduction on the percentage of error can be obtained by increasing the terms in the series.

This chapter leads to the following important conclusion, which is the backbone of the next two chapters: the column load should be represented by a patch load rather than a point load. This conclusion is justified by two reasons. First, the patch load is more realistic and represent the real life application. Second, the singularity due to the application of point load is avoided.

# CHAPTER 5

# BPM FOR PLATE WITH INTERNAL RIGID SUPPORTS

A BPM solution for the analysis of thin Kirchhoff plates with internal rigid supports is presented in this chapter. The model is capable of handling supports of different shapes and layouts. The rigidity condition used extensively in this chapter refers to zero deflection over all the patched supported area. A typical application to the aforementioned is the analysis of floor slab systems supported with columns or walls, in which the stiffness of the columns and walls is considered infinitely large. To accurately model the plate-support interaction, each patched support area is divided to a group cells and the compatibility conditions between the plate and the supported patched areas are satisfied. Three numerical examples are presented and their results are compared with the finite element methods (FEM) as well as with the available results in the literature. The comparison verifies the accuracy of the proposed solution.

## 5.1 BPM Formulation for Plate with Internal Rigid Support

Since the material is assumed to be linearly elastic, the superposition principle can be used to obtain the solution of the plate due to the applied load $q(x, y)$ and the reactions from the internal supports.

The internal supports are modeled using the actual patched area of the support in order to accurately satisfy the compatibility of the deflection between the plate and the support. Each patched supported area is divided into a group of cells as shown in Figure 35, where the reaction at each cell $R_c^i$ is assumed to be uniformly distributed over the cell domain $\Omega_c^i$.

Therefore, the plate deflection due to superposition of the applied load, and the uniformly distributed load of $\frac{R_c^i}{\Omega_c^i}$ over the cells of the patched areas becomes

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^* + \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial n} q\right) d\Gamma_q + \sum_{i=1}^{N_c} \int_{\Gamma_c^i} \left(\frac{\partial v^*}{\partial n} \frac{R_c^i}{\Omega_c^i}\right) d\Gamma_c^i \qquad (5.1)$$

Where $N_c =$ the total number of cells in all the supports. The boundary integrals in the last term of the above equation can be perform numerically by dividing the boundary $\Gamma_c{}^i$ of the cell $(i)$ to boundary points called $\underline{x}_{bc^i}^n$, for $(n, 1, \dots, N_{pc}^i)$ as shown in Figure 36, where $N_{pc}^i$ is the number of boundary points in cell $(i)$. The computation of the boundary integrals of Equation (5.1) is to be performed using Gaussian quadrature rules.

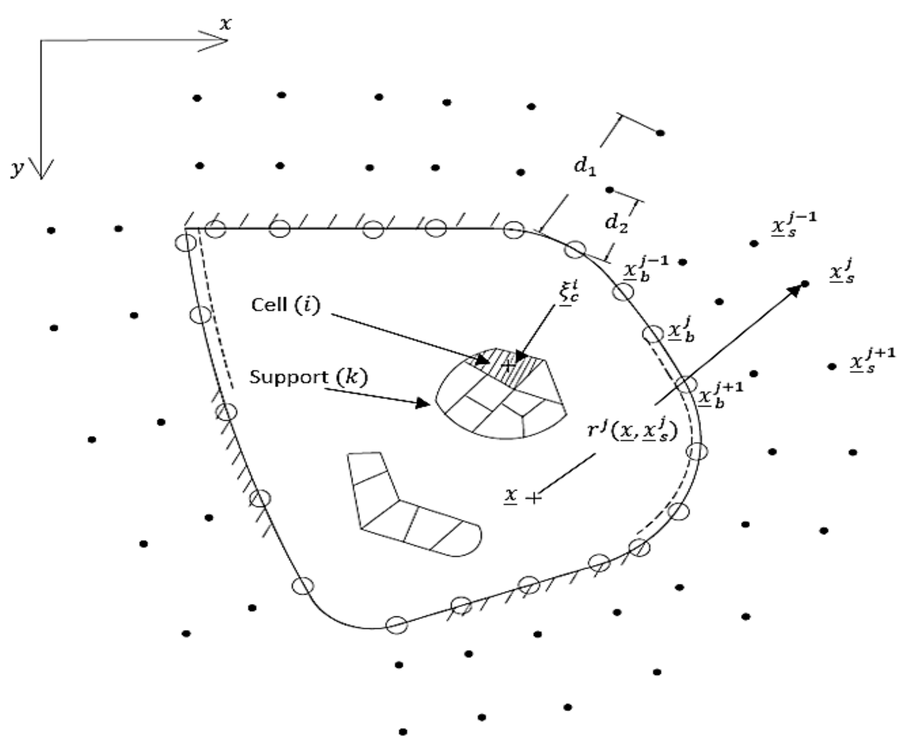



Figure 35 Plate with internal pathed areas divided to cells

Equation (5.1) involves $2N_b + N_c$ unknowns representing the homogenous solution coefficients ($c^j$, $j = 1, 2N_b$) and the support reactions ($R_c^i$ , $j = 1, N_c$), the solution of which can be obtained by applying the boundary conditions at the $N_b$boundary collocation points and forcing $w$ given by Equation (5.1) to be zero at the $N_c$ internal cells centers of the patched support area, i.e.

$$BC_1\left(w(\underline{x}_b^j)\right) = 0, \qquad j = 1, N_b \tag{5.2}$$

$$BC_2\left(w(\underline{x}_b^j)\right) = 0, \qquad j = 1, N_b \tag{5.3}$$

$$w\left(\underline{\xi}_c^j\right) = 0, \qquad j = 1, N_c \tag{5.4}$$

Where $\underline{\xi}_c^i = (\xi_c^i, \eta_c^i)$ the locations of the center of the cell $(i)$ in both $x$ and $y$ respectively.

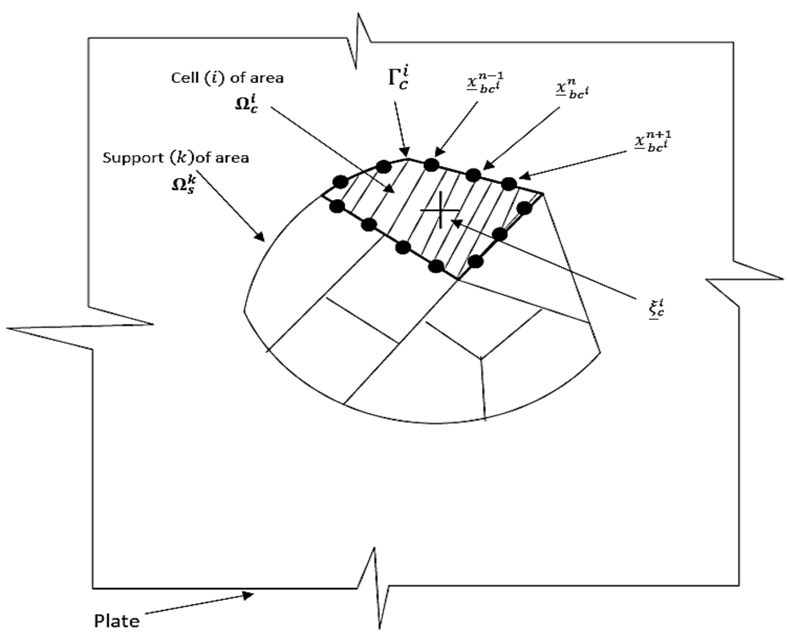



Figure 36 Plate with internal pathed areas divided to group of cells

Once the solution of Equations (5.2-5.4) is obtained, it can be used in Equation (5.1) to yield the deflection of the plate. Other plate secondary variables such as, the slope, the bending moments, and the shear forces can be found by direct analytical differentiation of the obtained deflection.

## 5.2 Applications of BPM formulation of Plate with Internal Rigid Supports

Application of the proposed method is illustrated through the next following examples. Since there are no available analytical solutions, all the presented examples are compared with the finite element method (FEM) and one of them with other numerical methods solutions such as Direct Boundary Element Method (DBEM) and Boundary Element Method (BEM) in addition to FEM. The main objective of presenting the first example is to study the convergence of the number of cells that is adequate to model the plate-support interaction. The second and third examples give the reader more confident about the proposed BPM method.

### 5.2.1 A Simply Supported Square Plate with Internal Rigid Column

A simply supported square plate with one square column as shown in Figure 37, is selected to study the convergence for number of cells required. The column is considered to be infinitely rigid and capable of preventing the plate deflection at the plate-column contact. The Poison's ratio is $\nu = 0.3$ ; the size of the column $= a/10$ .

An FEM model with 163220 domain elements and 1605 boundary elements have been used to solve the problem. Figure 38 and 39 compare the two solutions for the deflection $w$ and bending moment $M_x$, respectively. Figure 38 shows that at least 16 cells per support area has to be employed in order to satisfy the zero deflection condition over all points of the support area.

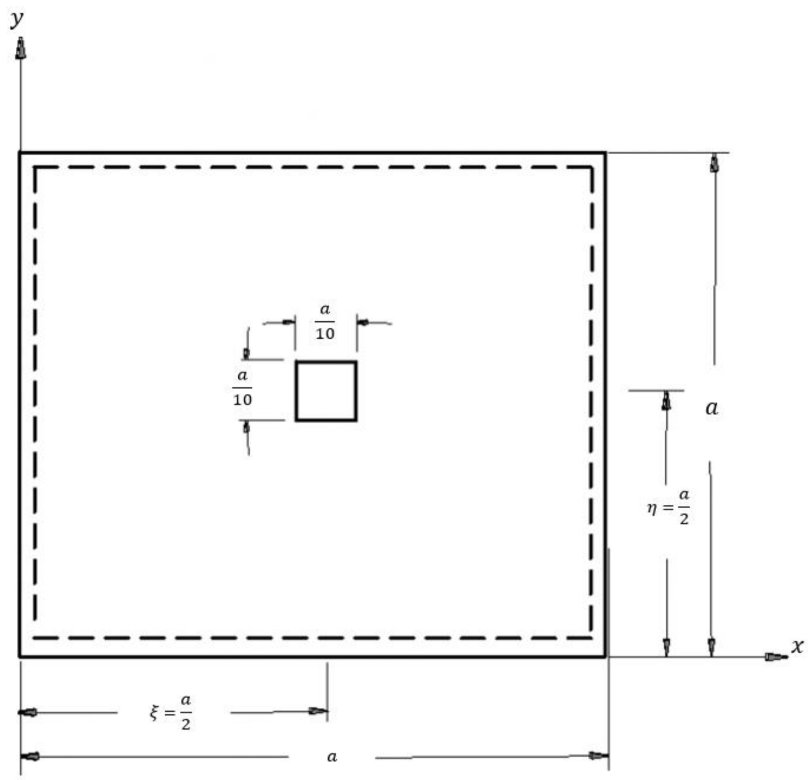



Figure 37 Simply supported square plate with internal rigid column

The comparison for the bending moment results is given in Figure 39, which shows that an accurate estimate of the bending moment at the support face can be obtained even with a very small number of cells. The same cannot be said of the results at the center of the support. This is due to the fact that the assumption of the support being rigid requires too many cells in order to satisfy the zero deflection (and therefore, zero bending moment) at enough number of points within the support. This difficulty has been overcome in the FEM solution by using a very fine mesh at the support. It should be noted, however, that in design practice, the critical bending moment, which is located at the face of the support, is more important and can be obtained very accurately by the proposed method with a 16-cell discretization.

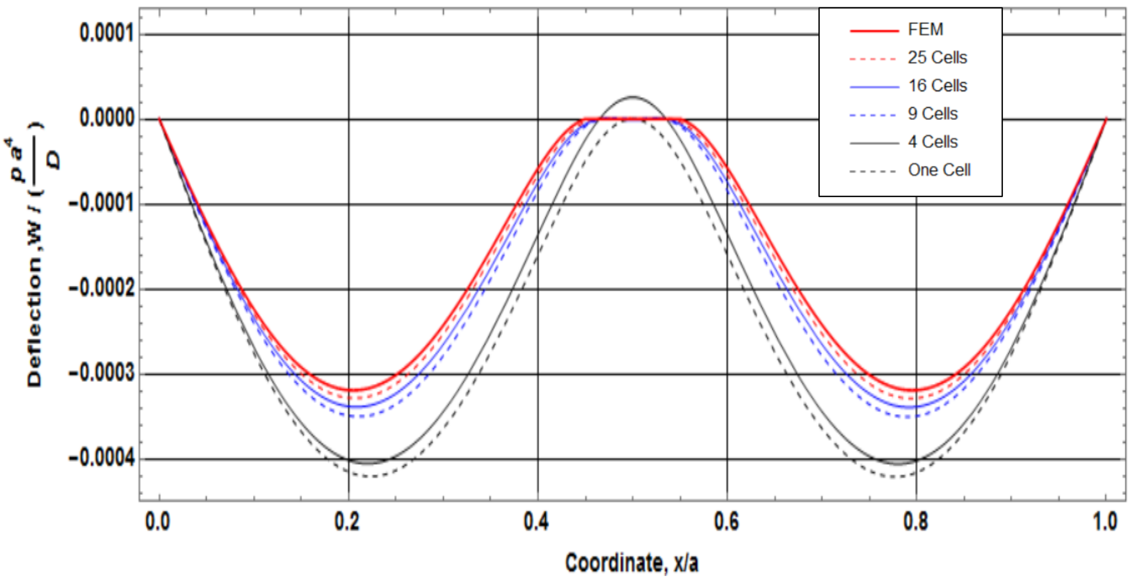



Figure 38 The deflection $w$ at $y = a/2$

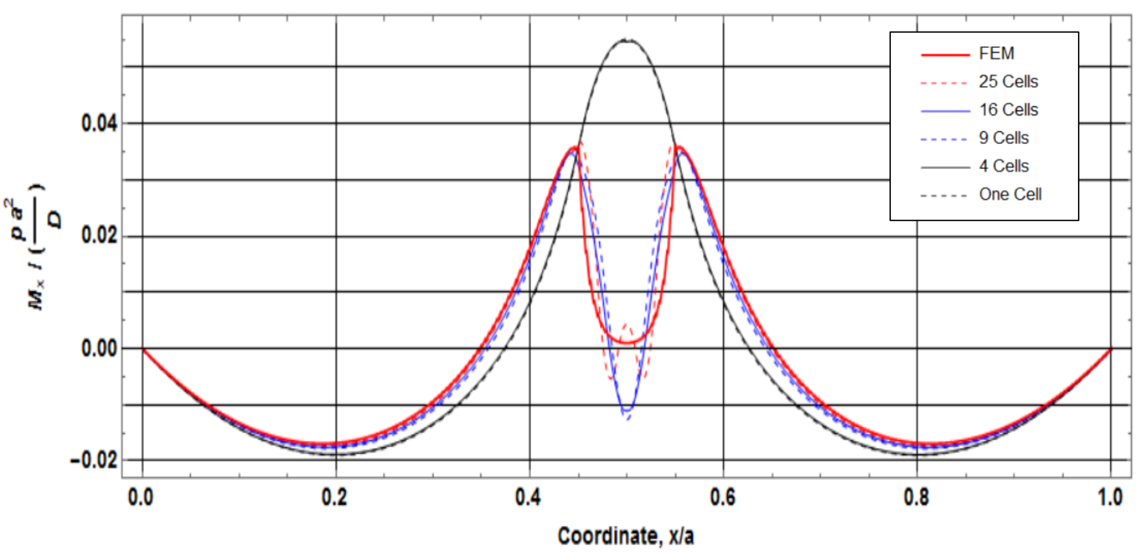



Figure 39 The deflection $M_x$ at $y = a/2$

### 5.2.2 A Free Edge Floor Slab with Four Internal Rigid Supports

Consider a free edges flat slab with four internal columns shown in Figure 40. For comparison purpose, the following unit-less data assumed in references [42] and [47] are used: $q = 0.0694, E = 480,000, \upsilon = 0.35$ and $L_x = L_y = 25$.

The main objective of this example is to evaluate the accuracy of the proposed BPM method by comparing the results with those obtained by DBEM in [42], BEM [47], and FEM. It should be noted that out of three DBEM models proposed in [42], two models, referred here as DBEM-mod-1 and DBEM-mod-2, have been selected for this comparison because they are based on the same assumption of rigid supports. In DBEM-mod-1, the rigid support is replaced by a set of five point reactions while in DBEM-mod-2, the rigid support is modeled as a hole whose boundary is clamped and modeled by four boundary elements.

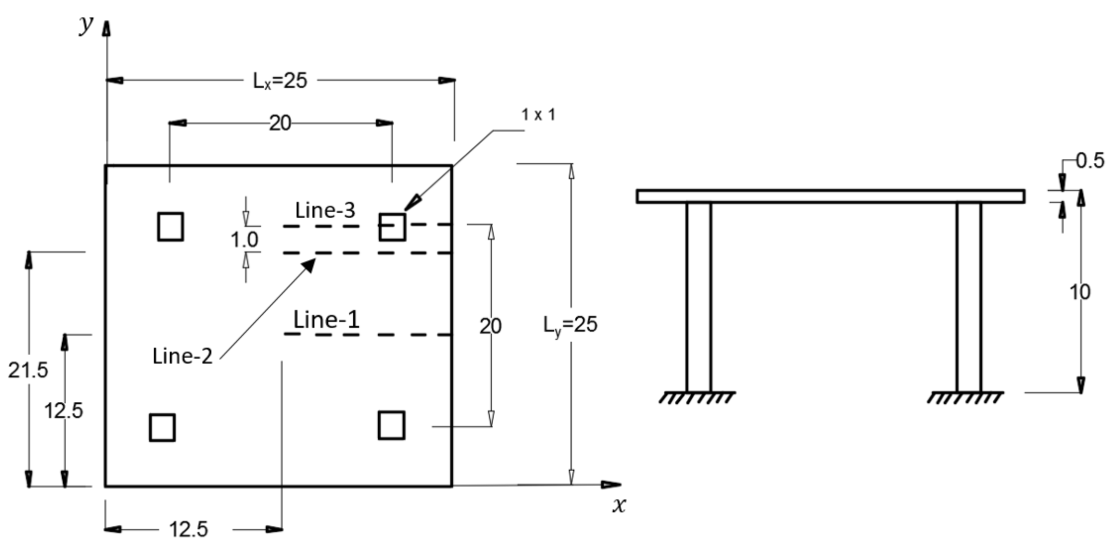



Figure 40 **Simple frame structure of free edges slab with four internal columns**

The discretized model used in the proposed BPM method is given in Figure 41, which shows the collocation points, the source points and the support cells. The BPM model has been represented by 80 uniformly distributed boundary points and 160 source points equally distributed at equal distances away from the plate edges. The deflection results computed by the five methods along Line 1 and 2, drawn in Figure 40, are given in

Figures 42 and 43, respectively. For both lines, the deflection curves of BPM and BEM [47] are closer to FEM curve than others. For Line-2, which is closer to the internal supports, all curves coincide at the plate free edge.

The results of the bending moment $M_x$ along Line 1 and 2 are given in Figures 44 and 45, respectively. For Line 1, all curves have small deviations from each other with BPM and BEM [47] curves being closer to FEM curve up to the $x$-location of the support. Over the segment between $x$-location of the support and the free edge, BPM and BEM [47] curves continue in full agreement with FEM curve and all three yield the expected zero value at the free edge. On the other hand, both DBEM curves [42] deviate from the correct solution with large values at the free edge. For Line 2, all curves are in agreement except over the region close to the support where all, excluding BPM curve, deviate from FEM curve. Similar trend is observed for the results of bending moment $M_y$ as shown in Figures 46, and 47.

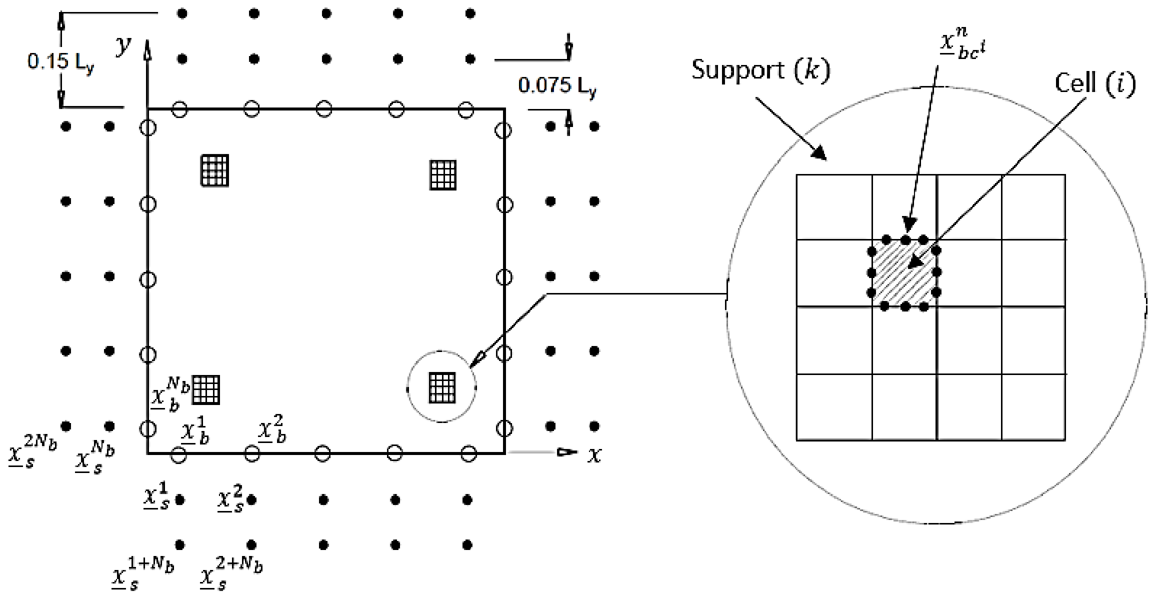



**Figure 41 The discretized model**

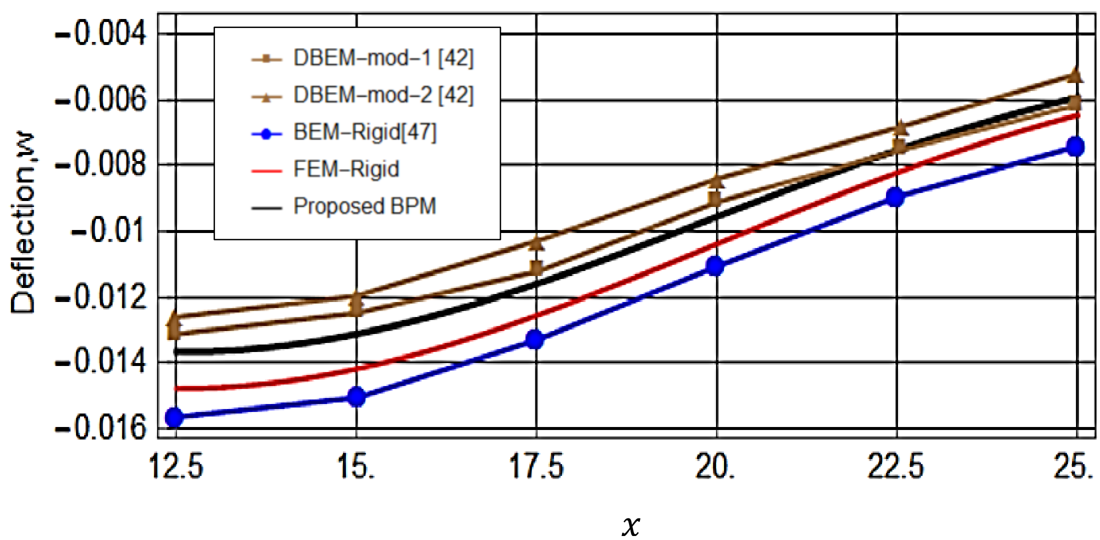



**Figure 42 Deflection *w* along Line-1**

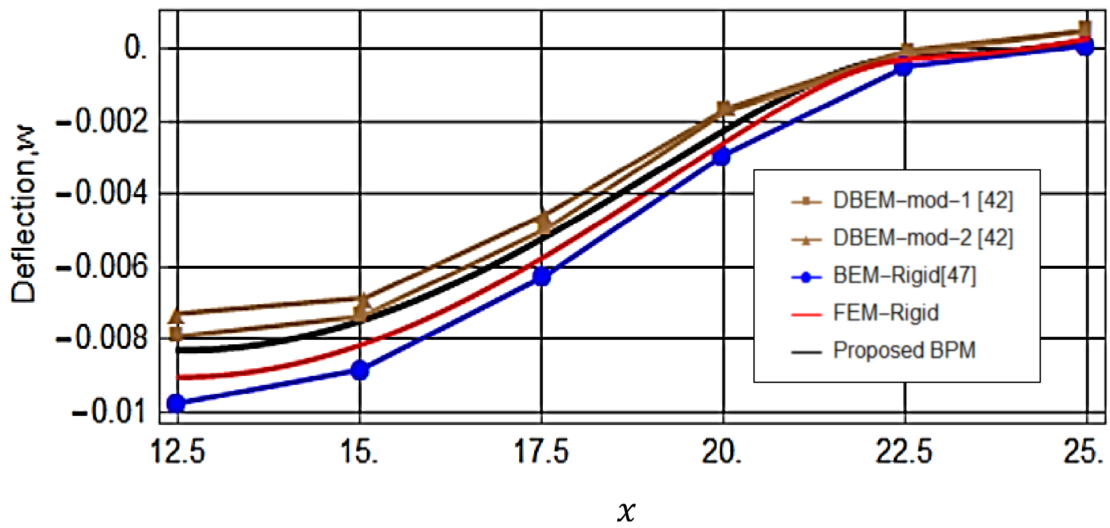



**Figure 43** **Deflection $w$ along Line-2**

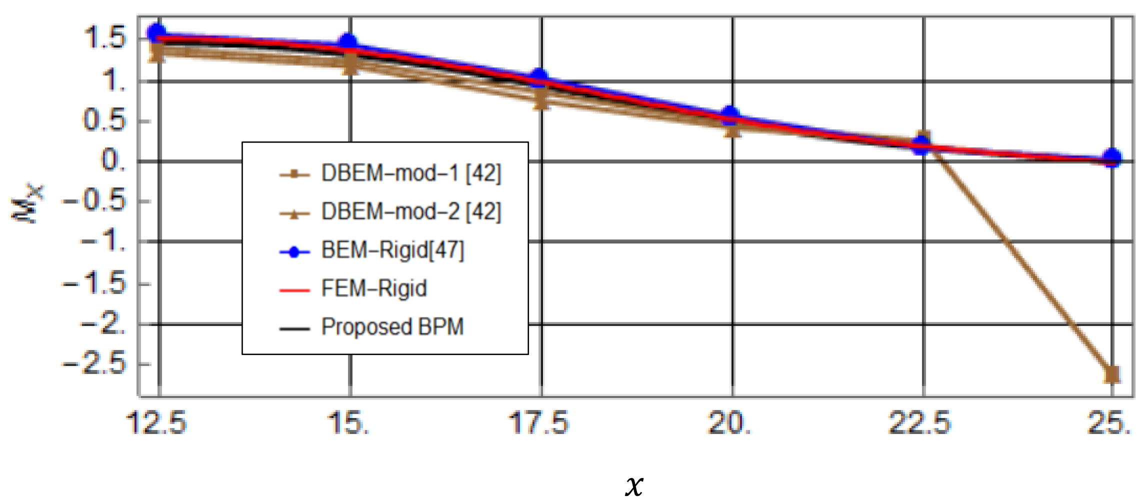



**Figure 44** **Bending moment $M_x$ along Line-1**

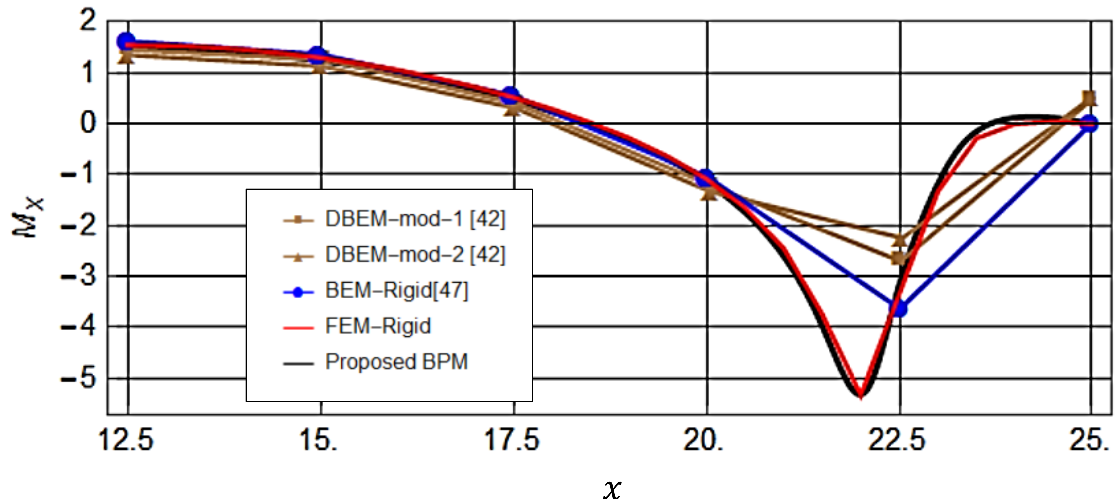



**Figure 45 Bending moment $M_x$ along Line-2**

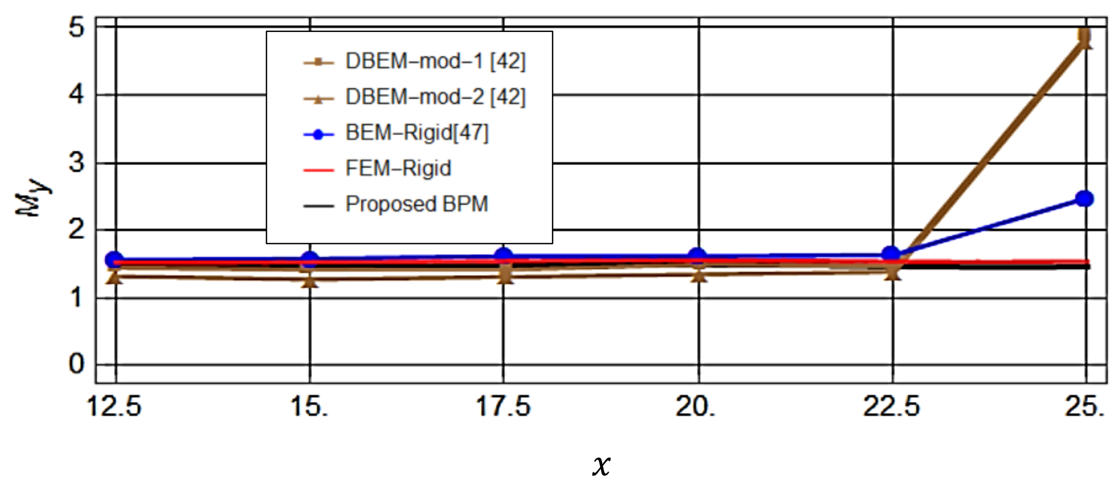



**Figure 46 Bending moment $M_y$ along Line-1**

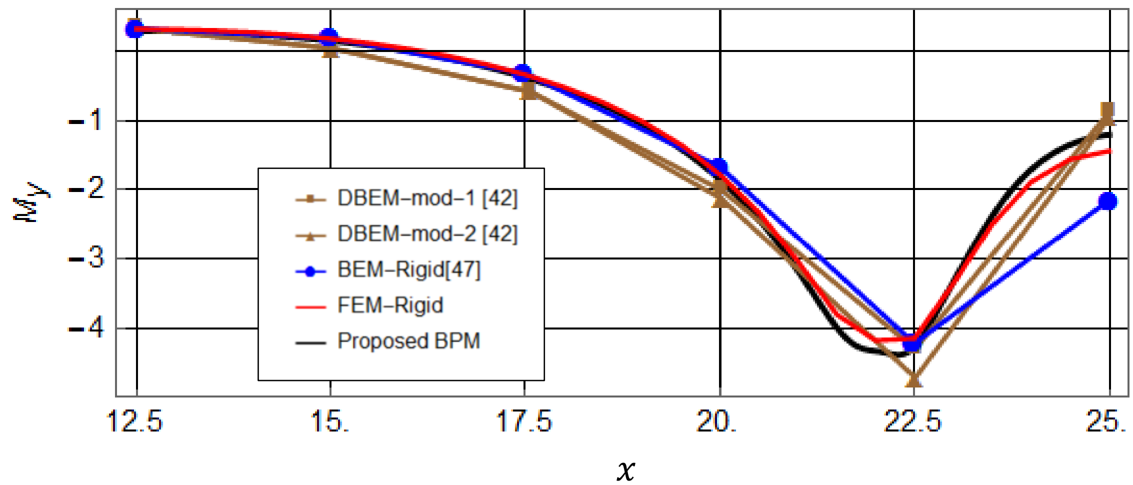



**Figure 47 Bending moment $M_y$ along Line-2**

A more critical verification of the method accuracy is to compute the bending moment $M_x$ along Line-3 passing through the center of the support. The results for that line have not been reported by BEM [47] and DBEM [42], and therefore the comparison is given here

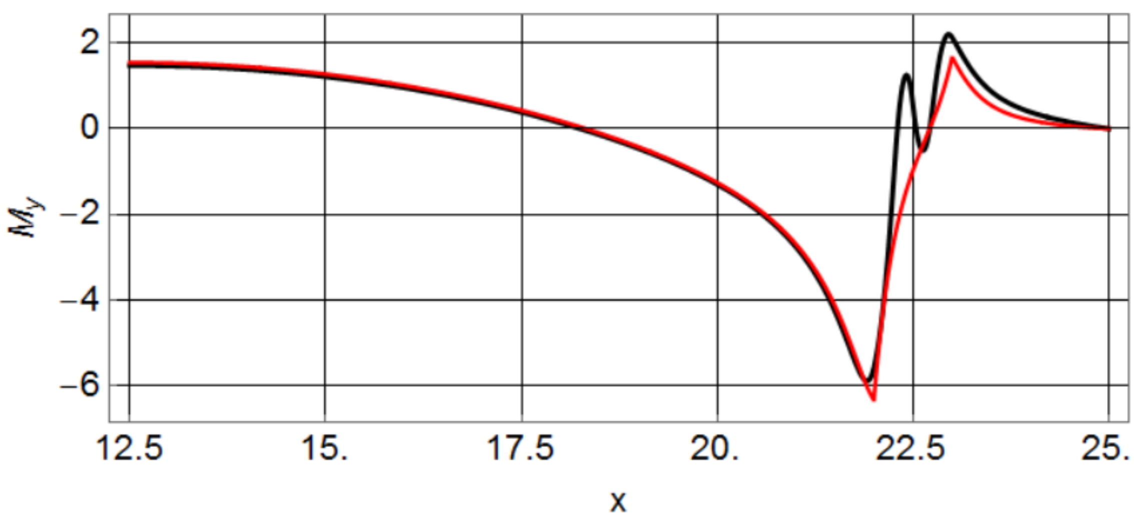



Figure 48 **Bending moment $M_x$ along Line-3**

with FEM only. Figure 48 shows a good agreement between the proposed BPM and FEM curves. Furthermore, the results of Figure 48 confirm the design practice of assuming the maximum moment to be at the face of the support.

To sum this example, the one can simply say that the results of the proposed method are very closed to the FEM results, and some difference observed with BEM and more with DBEM at some locations on the plate. This gives more confidence to evaluate the accuracy of the proposed method for moderately complex plate as shown in the following example.

### 5.2.3 A Curved Plate with Internal Rigid Patches of Different Layout and Shapes

Consider a plate that has the form of a quarter of a circle with an internal square column and an L-shape wall as shown in Figure 49. The plate is simply supported along the bottom edge, clamped along the curved edge and free along the left edge. The results are to be represented in terms of the plate stiffness $D$, the uniformly distributed load $q$, and the radius $a$ with $\upsilon = 0.30$. The solutions for deflection, bending moments and shear forces are sought at lines 1 and 2 as indicated in Figure 49.

To solve this example, the plate boundary has been represented by 60 boundary points, the square column has been divided to 16 equal square cells, and the L-shape wall has been divided to 21 equal square cells. The two contours of the source points are placed at uniform distances of $0.05\ a$ and $0.1\ a$ away from the plate edges (see Figure 50).

In the preparation of the FEM model the radius $a$ is taken to be the unity and the thickness of the plate is 0.02 to keep thin plate theory assumptions. The elastic modulus $E$ was then calculated with the help of Equation 3.28 to give unit stiffness $(D = 1)$. These simple calculations to give the results in a non-dimensional form. The model was meshed to 157839 domain elements and 2830 boundary elements.

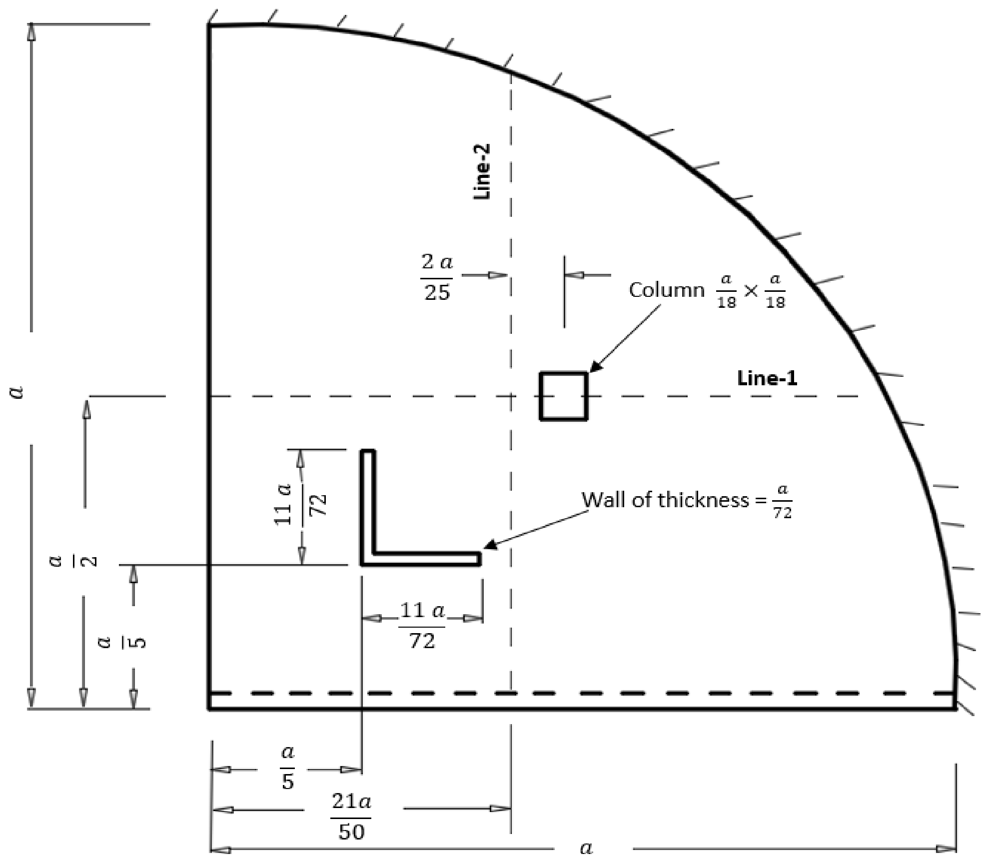



Figure 49 **Curved plate with an internal column and L-shape wall**

Figure 51 through 54, show the results of deflection, bending moment $M_x$, twisting moment $M_{xy}$, and the shear force $V_x$, respectively, along Line-1. Figure 51 shows an excellent agreement between the deflection curves of BPM and FEM. Furthermore, the two curves satisfy the zero deflection condition over the support. The results for bending moments, twisting moment, and shear forces (Figures 52-54) show a reasonable agreement between the two methods with some variations over the support. It should be noted, however, that the largest deviation takes place for the bending moment $M_x$ at the center of the support, which is not as critical as its value at the face of the support where the two methods pick up the maximum bending moment with a very close agreement.

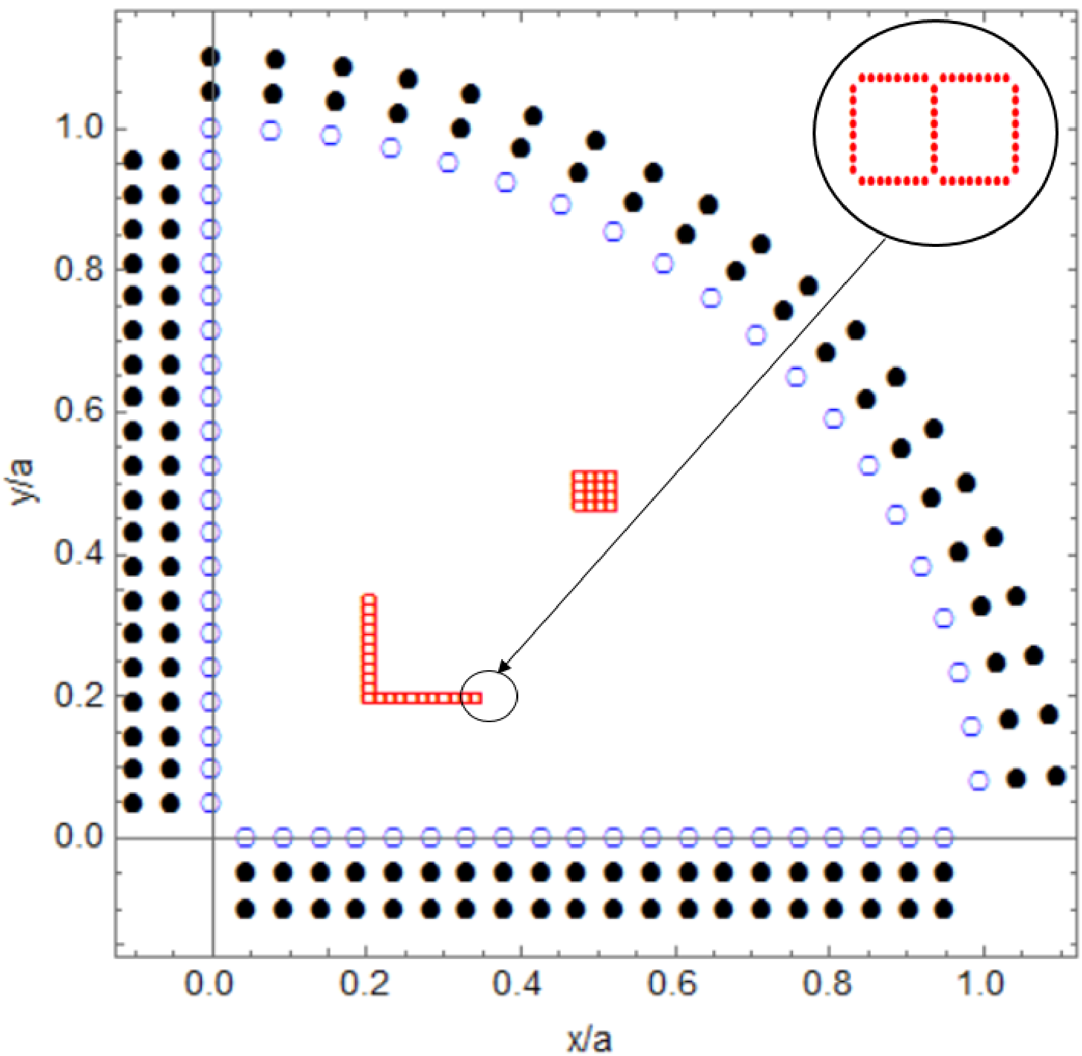



Figure 50 Boundary points, source points, and boundary points of the support cells

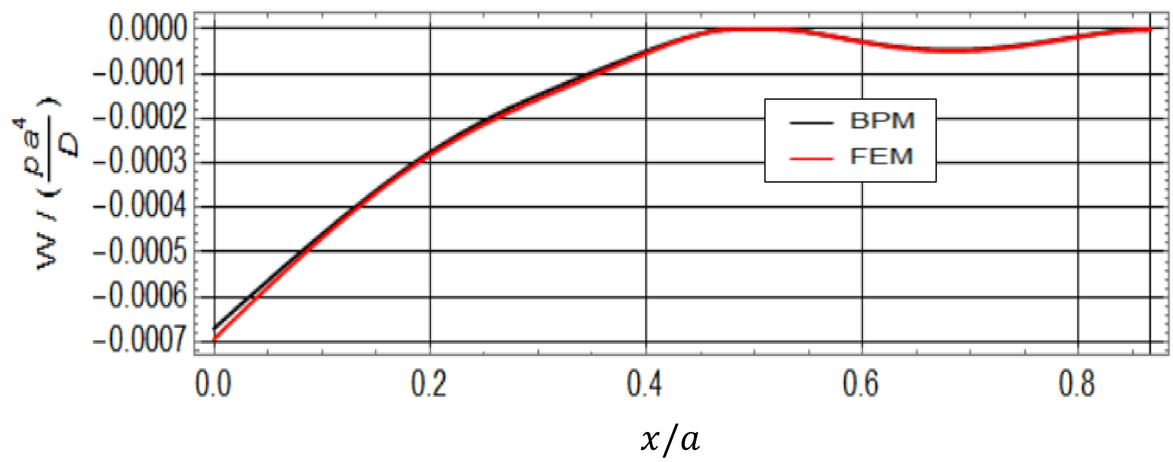



**Figure 51 Deflection $w$ along Line-1**

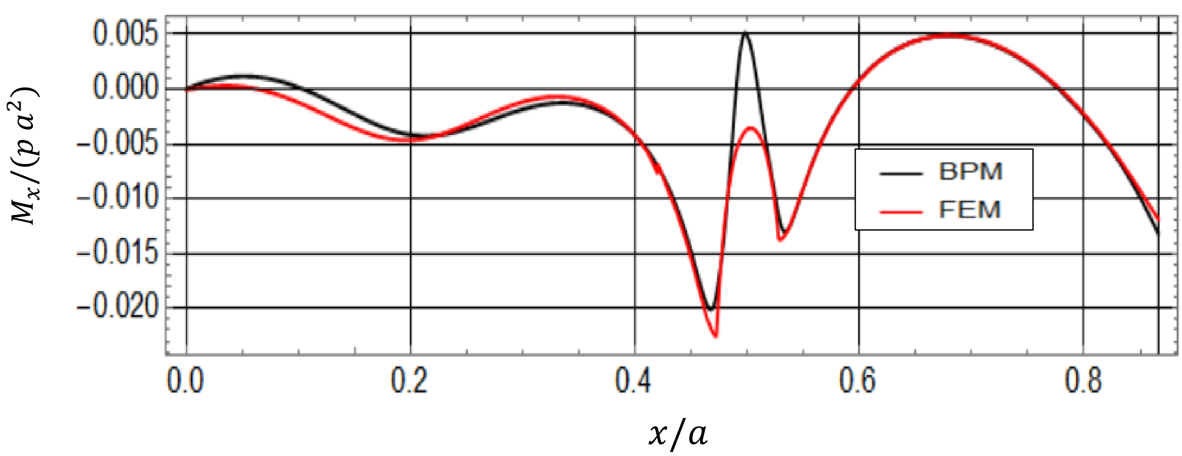



**Figure 52 Bending moment $M_x$ along Line-1**

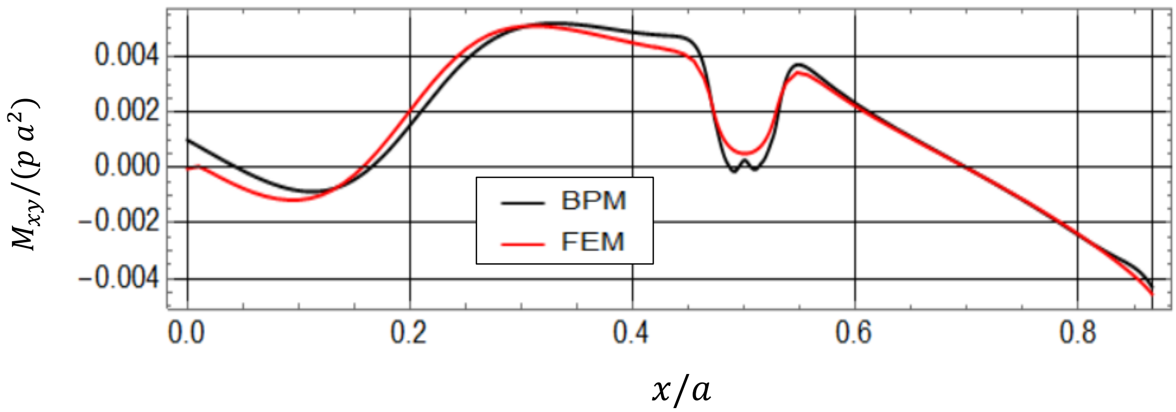



Figure 53 Twisting moment $M_{xy}$ along Line-1

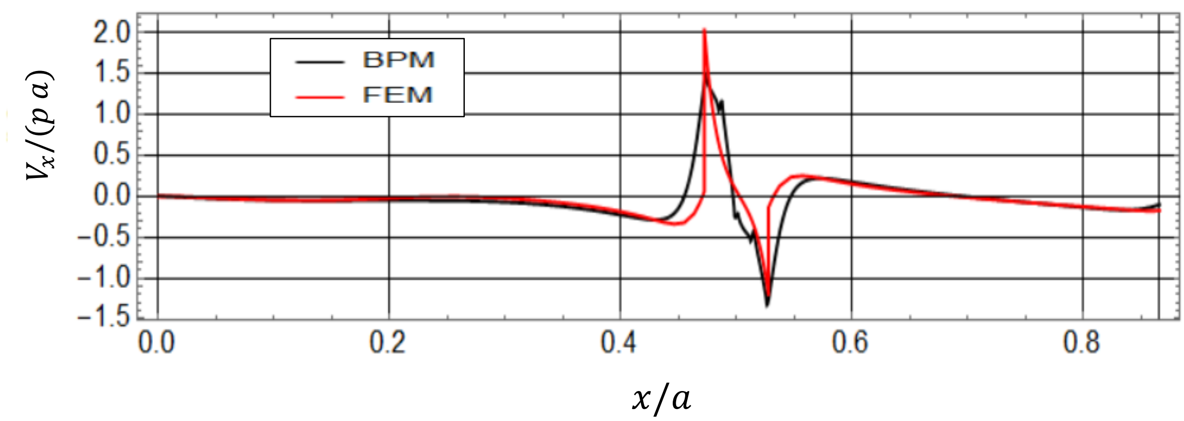



Figure 54 Shear force $V_x$ along Line-1

Figures 55 through 58, represent the results of deflection, bending moment $M_y$, twisting moment $M_{xy}$, and shear force $V_y$, respectively along Line-2. The results for all variables yield very close agreements between BPM and FEM.

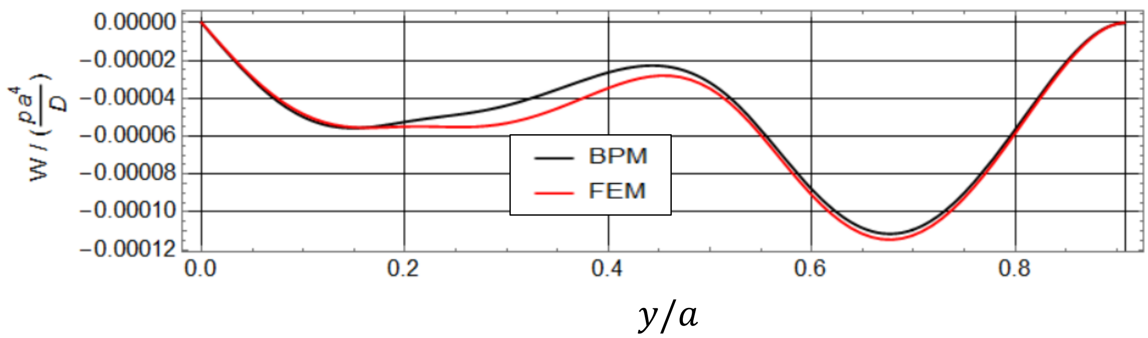



**Figure 55 Deflection *w* along Line-2**

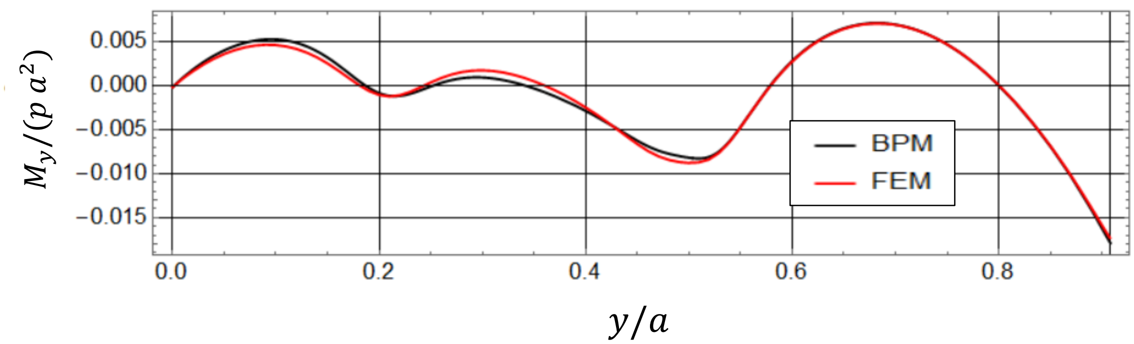



**Figure 56 Bending moment $M_y$ along Line-2**

Some deviations are shown very closed to the plate boundaries in case of the twisting moment and shear force as shown in Figure 57 and 58. This is mainly due to the following reasons: (a) selection of the proper distance from the plate boundary, and (b) number of the boundary points.

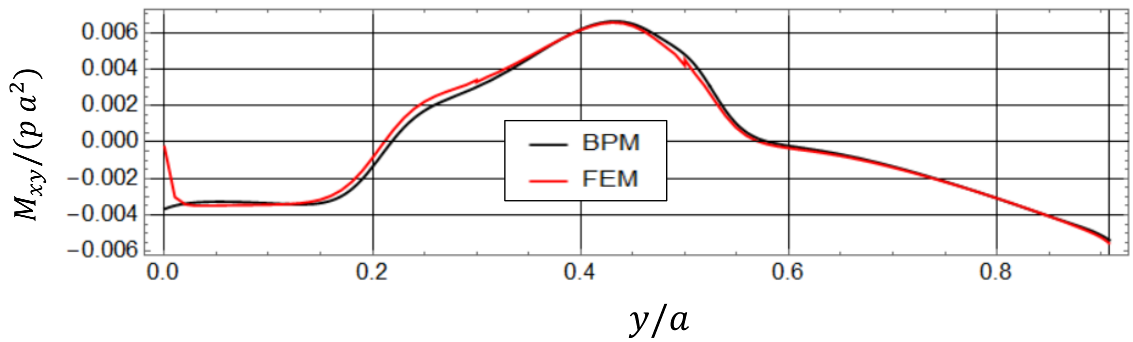



**Figure 57 Twisting moment $M_{xy}$ along Line-2**

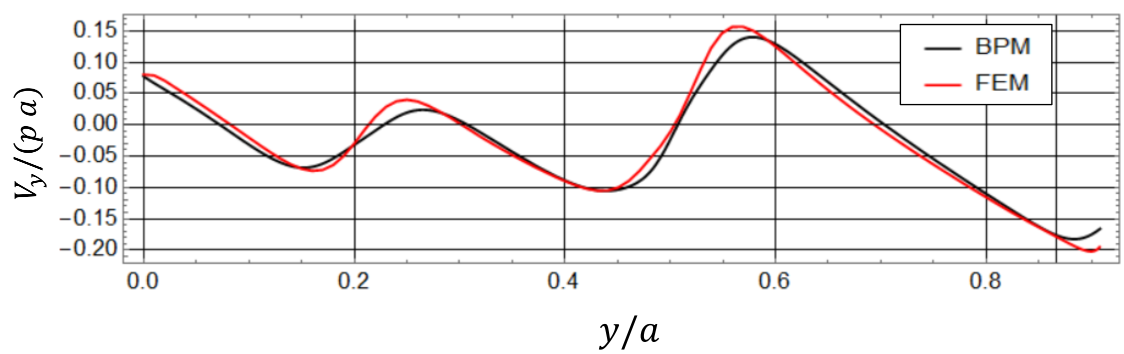



**Figure 58 Shear force $V_y$ along Line-2**

This example gives the reader more confident about the reliability of the proposed method especially when the shear force is under consideration, different boundary conditions, and different shapes.

# CHAPTER 6

# BPM FOR PLATE WITH INTERNAL FLEXIBLE SUPPORTS

In this chapter, the prepared BPM formulation for plate with internal rigid supports is extended to tackle the presence of the internal flexible supports. Two numerical examples have been solved and compared with other numerical solutions.

## 6.1 BPM Formulation for Plate with Internal Flexible Support

This section is given to explain, in more details, modeling of the flexible supports at the plate column interaction zone. Availability of flexible supports is very common and most of supporting members have a finite stiffness leading to have some deformations under the sustain load. A practical example to floor system having plate and columns can be shown in Figure 59.

In the present study, a BPM-based solution is developed for the solution of thin plates with mixed boundary conditions and arbitrary layouts and shapes of internal flexible supports (columns). In order to accurately model the plate-internal support interaction, the supports are modeled using their real patched areas and the equilibrium of the plate-support interaction involves three generalized internal forces: a force normal to the patched area and two bending moments about the two principal axes of the patched area. These three forces are introduced to the plate in terms of stresses over the patched area by dividing each support into a group of cells. The compatibility conditions of the deflection are applied at the centers of the cells while the slopes conditions are applied at the centers of the supports. The proposed method is simple to formulate and program compared to FEM and BEM, yet yields a reasonably accurate solution.

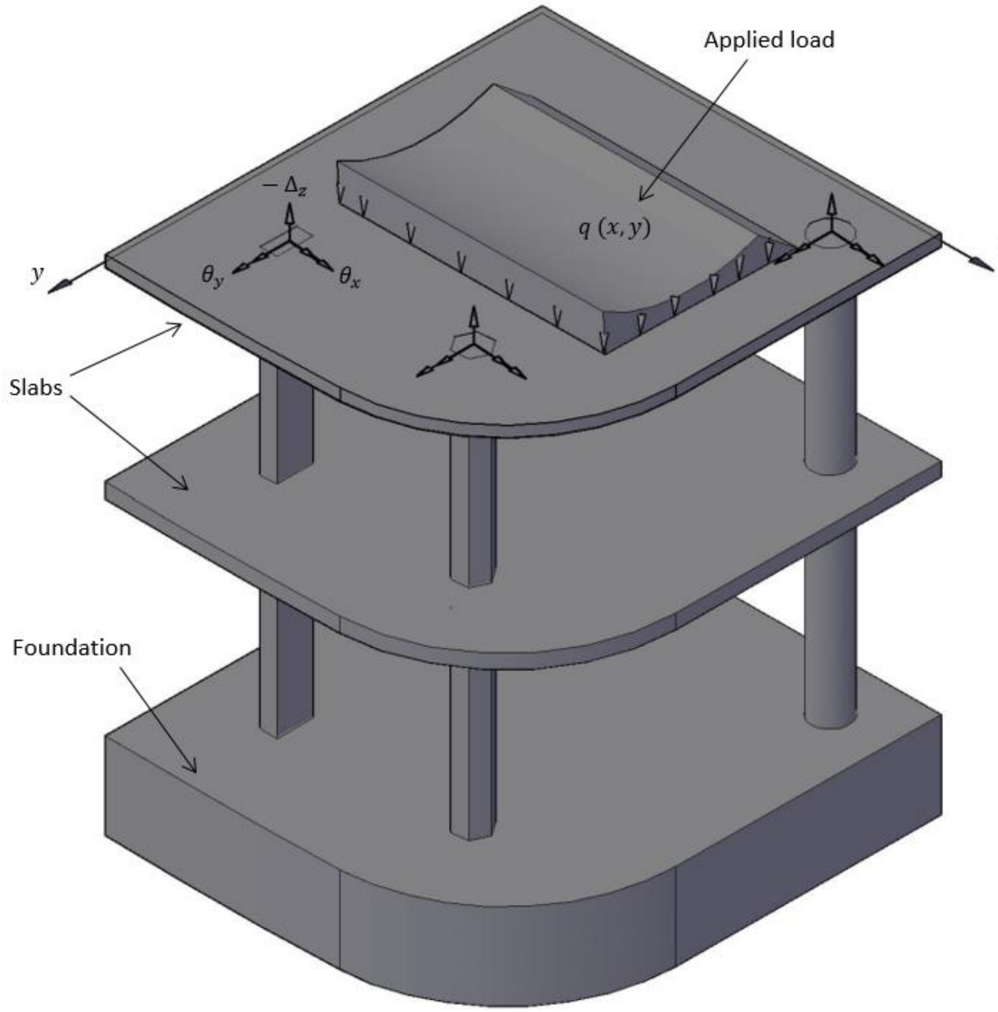



Figure 59 A multistory building involving slabs with internal flexible supports

In the presence of internal supports, Equation 4.13 should be modified to include the patch loads due to the internal supports, i.e.

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + \int_{\Gamma_q} \left(\frac{\partial v^*}{\partial n} q\right) d\Gamma_q + \sum_{k=1}^{N_s} \int_{\Omega_s^k} w^* \, q_s^k \, d\Omega_s \tag{6.1}$$

where $q_s^k$ and $\Omega_s^k$ are the patch load and area for support (k), respectively, and $N_s$ is the number of supports. Unlike the case of rigid supports where $q_s^k$ can be assumed uniform equal to the reaction of the support divided by the contact area, $q_s^k$ for flexible supports is composed of a non-uniform stress caused by the normal support reaction in addition to the flexural stresses caused by the bending moments about the principal axes of the patch area. The conversion of the domain integrals in Equation (6.1) into boundary integrals can be obtained through the following procedure. Dividing the support $(k)$ to a group of cells $(n_c^k)$ produces a group of cell reaction forces $(R_c)_i^k$ $(i = 1, n_c^k; k = 1, N_s)$ as shown in Figure 60. The moment resistance of the support (column) can also be introduced without dividing the support to cells as shown in Figure 61. The stress $\sigma_s^k$ over support $(k)$ can be expressed in terms of the local coordinates of the internal support leading to:

$$\sigma_s^k = \frac{M_{xs}^k}{I_{xs}^k}\, y_s^k - \frac{M_{ys}^k}{I_{ys}^k}\, x_s^k + \sum_{i}^{n_c^k} \frac{(R_c)_i^k}{(\Omega_c)_i^k} \tag{6.2}$$

Where $M_{xs}^k$ , $M_{ys}^k$ are the moments over the $x_s^k$ and $y_s^k$ axes of the supporting column $(k)$ respectively; $I_{xs}^k$ , $I_{ys}^k$ are the second moments of area of the supporting column $(k)$ about $x_s^k$ and $y_s^k$ axes.

The stress on the plate at its interaction with support *(k)*, $q_s^k$, is equal and opposite to $\sigma_s^k$. Then transformation to the global $xy$ system by replacing $x_s^k$ with $(x - \xi_s^k)$ and $y_s^k$ with $(y - \eta_s^k)$ yields the following expression for $q_s^k$:

$$q_s^k = -\frac{M_{xs}^k}{I_{xs}^k}\,(y-\eta_s^k) + \frac{M_{ys}^k}{I_{ys}^k}\,(x-\xi_s^k) - \sum_i^{n_c^k} \frac{(R_c)_i^k}{(\Omega_c)_i^k} \qquad (6.3)$$

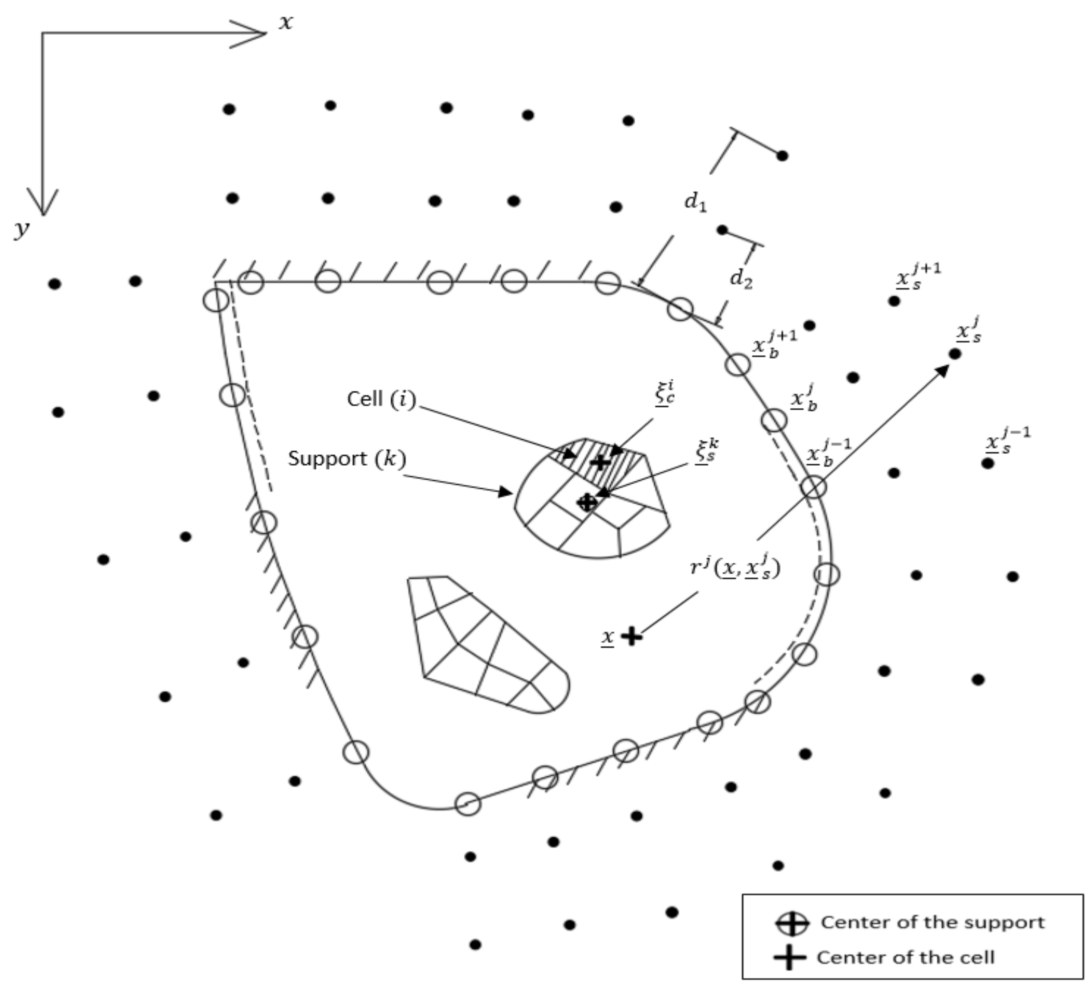



Figure 60 Plate with internal flexible supports divided to cells

Using Equation (6.3) in (6.1) and applying the divergence theorem, we get the total deflection due to the applied load $q$ and the patch loads $q_s^k$ due the internal supports, i.e.

$$w(\underline{x}) = \sum_{j=1}^{2N_b} c^j w^*(\underline{x}, \underline{x}_s^j) + \int_{\Gamma_q} \left(\frac{\partial \mathrm{v}^*}{\partial \mathrm{n}} \mathrm{q}\right) \mathrm{d}\Gamma_\mathrm{q}$$

$$- \sum_{k=1}^{N_s} \sum_{i=1}^{n_c^k} \int_{(\Gamma_c)_i^k} \left(\frac{\partial v^*}{\partial n} \frac{(R_c)_i^k}{(\Omega_c)_i^k}\right) (d\Gamma_c)_i^k$$

$$+ \sum_{k=1}^{N_s} \int_{\Gamma_s^k} \left(\frac{\partial v^*}{\partial n} \left(-\frac{M_{xs}^k}{I_{xs}^k} (y - \eta_s^k) + \frac{M_{ys}^k}{I_{ys}^k} (x - \xi_s^k)\right)\right) d\Gamma_s^k \tag{6.4}$$

$$- \sum_{k=1}^{N_s} \int_{\Gamma_s^k} \left(v^* \left( -\frac{M_{xs}^k}{I_{xs}^k} ny + \frac{M_{ys}^k}{I_{ys}^k} nx\right)\right) d\Gamma_s^k$$

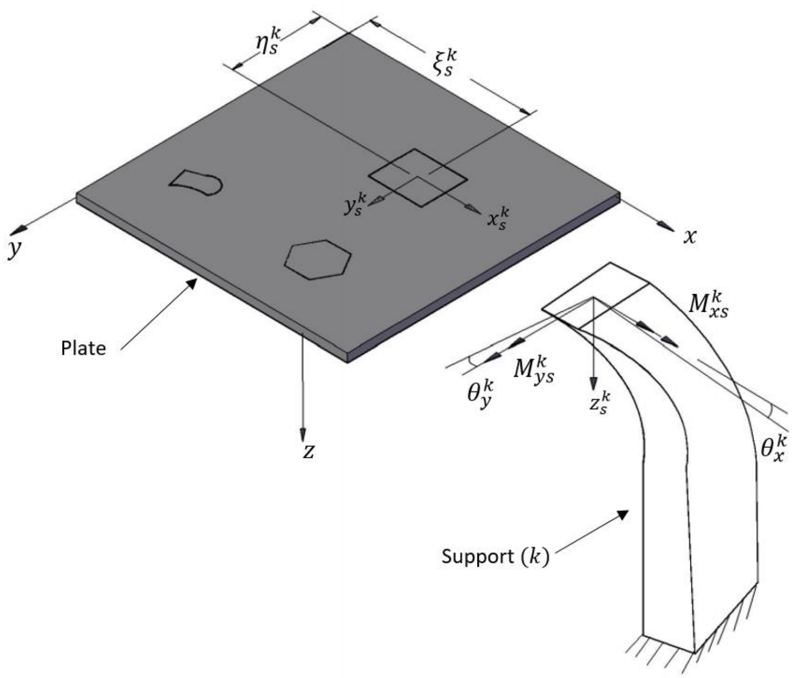



**Figure 61 3D model of Plate with deformed internal flexible supports**

In the above equation, there are three types of boundary integrals: the first one is over the boundary of the area over which the load is applied ($\Gamma_q$), the second is over i$^{th}$ cell boundary of k^th support, $(\Gamma_c)_i^k$, and the third is over the boundary of k^th support, $\Gamma_s{}^k$ (Figure 62). All boundary integrals can be performed numerically using one-dimensional Gaussian Quadrature.

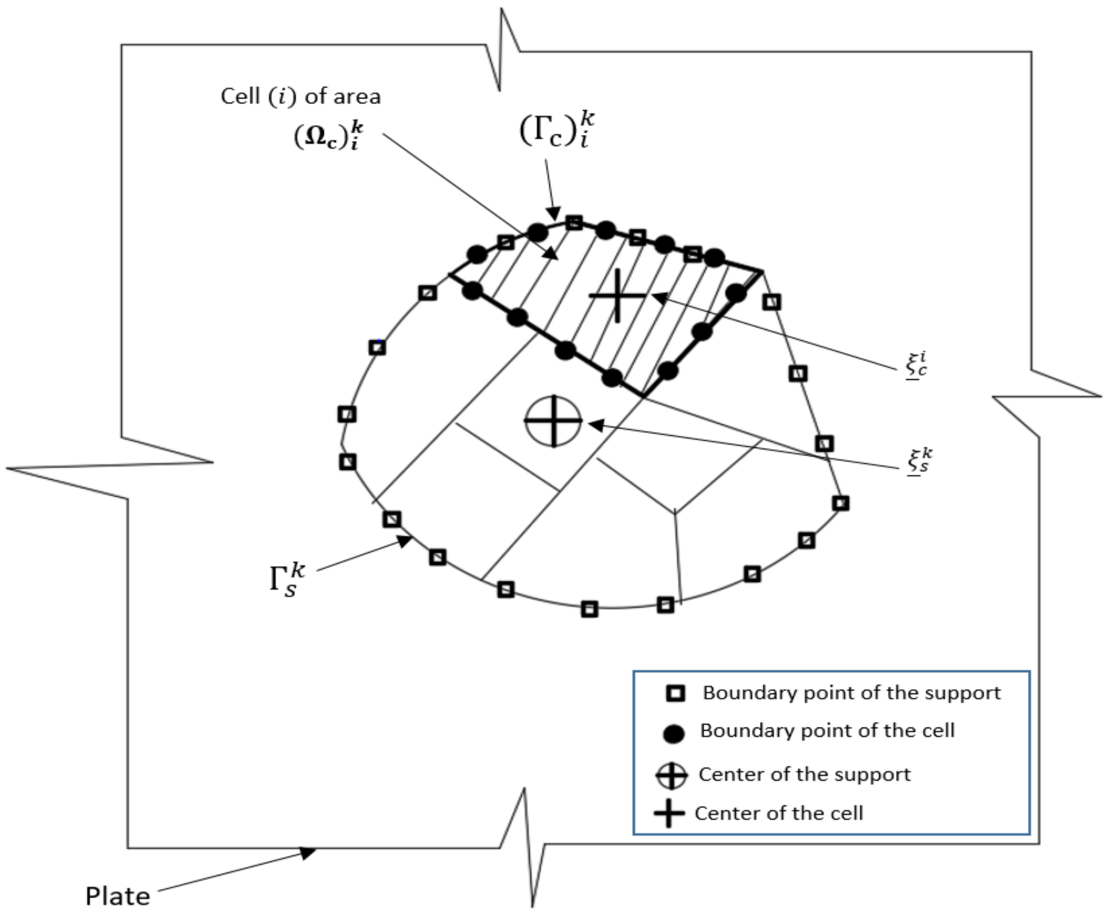



Figure 62 An enlarged support patched area divided to group of cells

Equation (6.4) has $2N_b$ unknown coefficients ($c^j$, $j, 1, \ldots, 2N_b$), $N_s$ unknown moments ($M_{xs}^k, k, 1, \ldots, N_s$), $N_s$ unknown moments ($M_{ys}^k, k, 1, \ldots, N_s$), and $N_c$ total number of cell reactions ($(R_c)_i^k, i = 1, n_c^k; k = 1, N_s$), summing up to a total of $2N_b + 2N_s + N_c$

unknowns. The Application of the two boundary conditions over each boundary point $\underline{x}_b^j$ yields the following $2N_b$ equations:

$$BC_1\left(w\left(\underline{x}_b^j\right)\right) = 0, \qquad j = 1, N_b \tag{6.5}$$

$$BC_2\left(w\left(\underline{x}_b^j\right)\right) = 0, \qquad j = 1, N_b \tag{6.6}$$

Where the $BC_1$, $BC_2$ are the differential operators of the first and second boundary conditions. The remaining required equations are obtained from the compatibility equations for deflection and slopes at the plate-support interaction. For the support (column), we will assume that the deflection varies linearly over its contact area with the plate according to the following formula:

$$w(x,y)_s^k = \sum_i^{n_c^k} \frac{(R_c)_i^k\, L}{(\Omega_c)_i^k\, E} + \left.\frac{\partial w}{\partial x}\right|_{(\xi_s^k,\eta_s^k)} (x - \xi_s^k) + \left.\frac{\partial w}{\partial y}\right|_{(\xi_s^k,\eta_s^k)} (y - \eta_s^k) \tag{6.7}$$

Where: $\underline{\xi}_s^k = (\xi_s^k, \eta_s^k)$ is the locations of the center of the support $(k)$ in both $x$ and $y$ respectively. Then, we can equate the support deflection given by Equation (6.7) to the plate deflection given by Equation (6.4) to generate $N_c$ equations applied at the $N_c$ centers of the cells. The last required $2N_s$equations can be obtained by equating the $x$ and $y$ slopes of the plate to those of the support at the their centers, i.e.

$$\frac{\partial w}{\partial x}(\xi_s^k, \eta_s^k) = \frac{M_{xs}^k\, L}{k_{xs}\, E\, I_{xs}^k}, \qquad k = 1, N_s \tag{6.8}$$

$$\frac{\partial w}{\partial y}(\xi_s^k, \eta_s^k) = -\frac{M_{ys}^k\, L}{k_{ys}\, E\, I_{ys}^k}, \qquad k = 1, N_s \tag{6.9}$$

Where: $L =$ the height of the supporting column, $E =$ the elastic modulus of the support material, $k_{xs} = k_{ys}$ = 4 if the far end of the supporting column is clamped and equal to 3 if the far end is hinged.

Once the solution of the above $2N_b + 2N_s + N_c$ equations is obtained, the deflection of the plate is completely determined by Equation 6.4.

## 6.2 Applications of Plate with Internal Flexible Supports

Application of the proposed BPM method is illustrated through the following two examples. The results are compared with those obtained by the available numerical solutions.

### 6.2.1 A Free Edge Floor Slab with Four Internal Flexible Columns

A free edges flat slab with four internal columns as solved in Section 5.2.2 was again analyzed using the proposed BPM method and the main difference here is that in the previous solution the supports were treated as a rigid support and here, in this section, the supports were treated as flexible supports.

The main objective of this example is to evaluate the accuracy of the proposed BPM method for flexible support with respect to results of DBEM-model-3 [42], BEM in [47] where all solutions are based on the same flexibility assumptions.

Due to the symmetry of the problem, only the top right quarter of the plate has been modeled. Lines 1, along the horizontal centerline of the plate, and Line 2, at a distance of one unit away from the center of the column, have been selected to compare the four methods (Figure 40).

The BPM model is based on 80 uniform boundary points, 80 uniform source points at a distance of 0.5 unit away from and normal to the boundary, and the remaining 80 source points are at a distance of one unit away and normal to the plate boundary too. The contact area of the column has been divided to 16 equal square cells. Each column has been assumed clamped at the bottom with a height of 9.75 units to the centerline of the plate. The deformed shape of the top right quarter of the plate is shown in Figure 63, which confirms the symmetry of the solution about the diagonal of the plate. The deformed shape of the part of the plate in contact with the column is given in Figure 64, which confirms the linearity assumed in Equation 6.7.

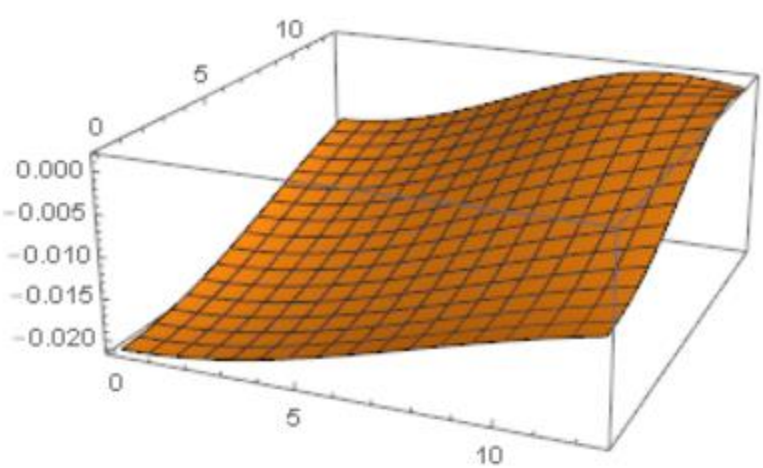



**Figure 63 A 3D-model of the deformed shape of quarter of the plate**

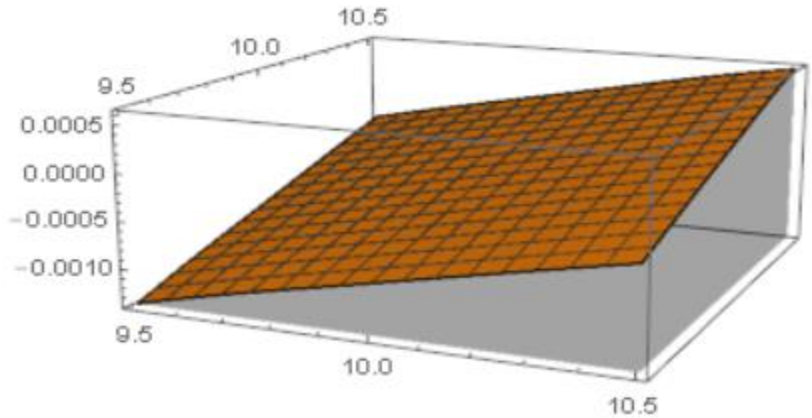



**Figure 64 A 3D-model of the deformed shape of the plate over the plate-column interaction zone**

The results for the plate deflection as computed by the four methods along Line 1 and 2 are given in Figures 65 and 66, respectively. For Line 1, the BPM curve perfectly coincides with FEM curve while the curves of BEM [47] and DBEM [42] deviate downward and upward of BPM-FEM curves, respectively. For Line-2, the same is true with more deviation of the curve of DBEM [42], which largely underestimates the deflection at the center of the plate.

The results for the bending moment $M_x$ along Line 1 and 2 are given in Figures 67 and 68, respectively. For Line 1, the three methods are reasonably in agreement with FEM except at the free edge where all fail to yield the expected value of zero with less deviation by the present method. For Line 2, all curves are in agreement except over the region close to the column where all excluding the proposed curve deviate from FEM curve.

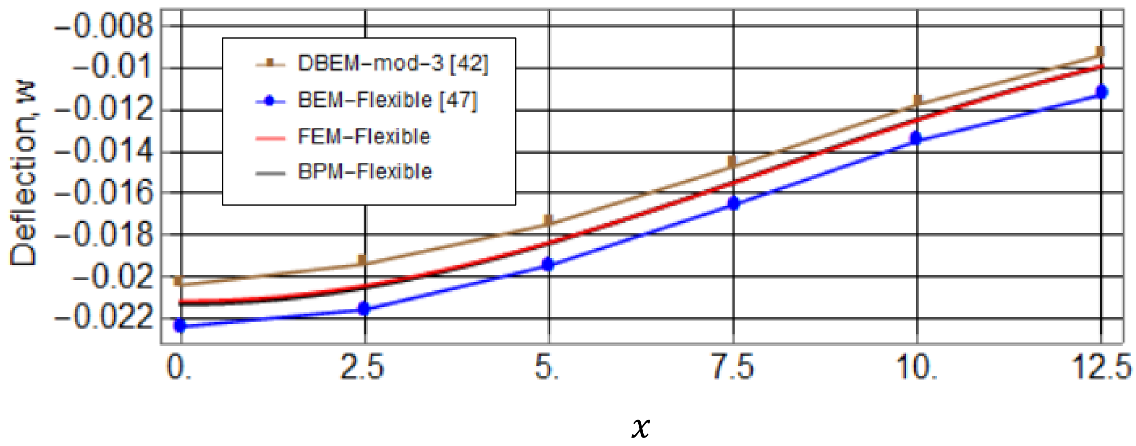



**Figure 65 Deflection *w* along Line-1**

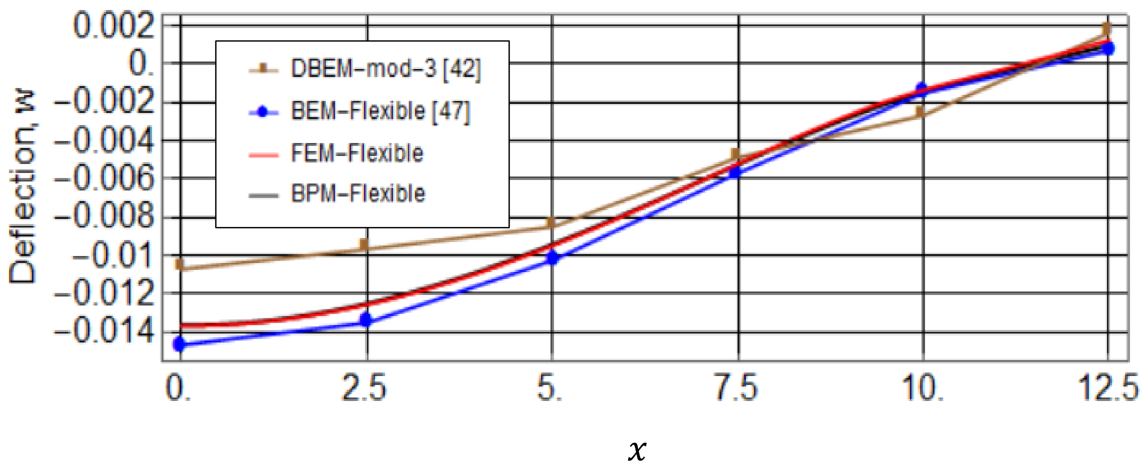



**Figure 66 Deflection *w* along Line-2**

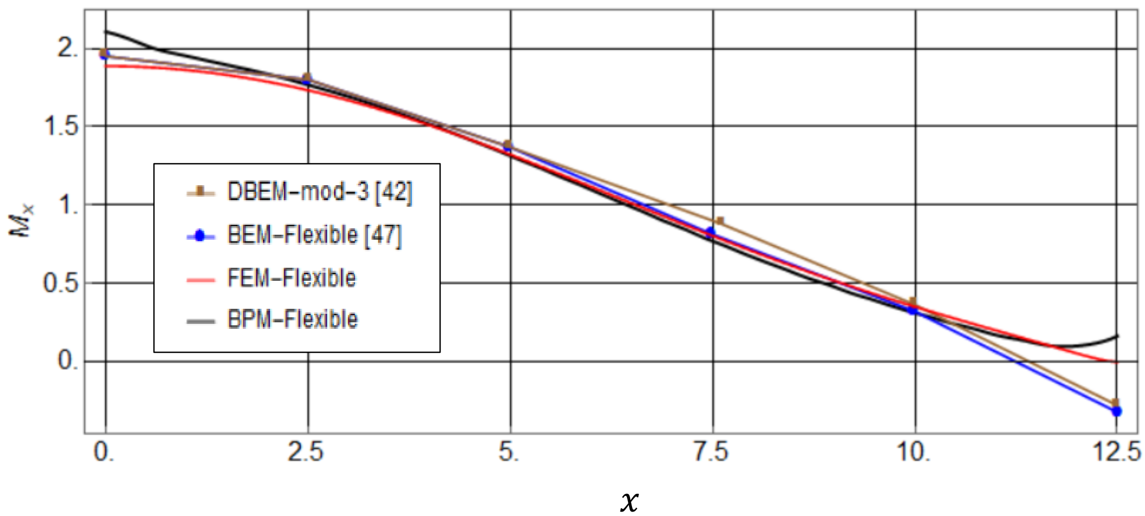



Figure 67 **Bending moment $M_x$ along Line-1**

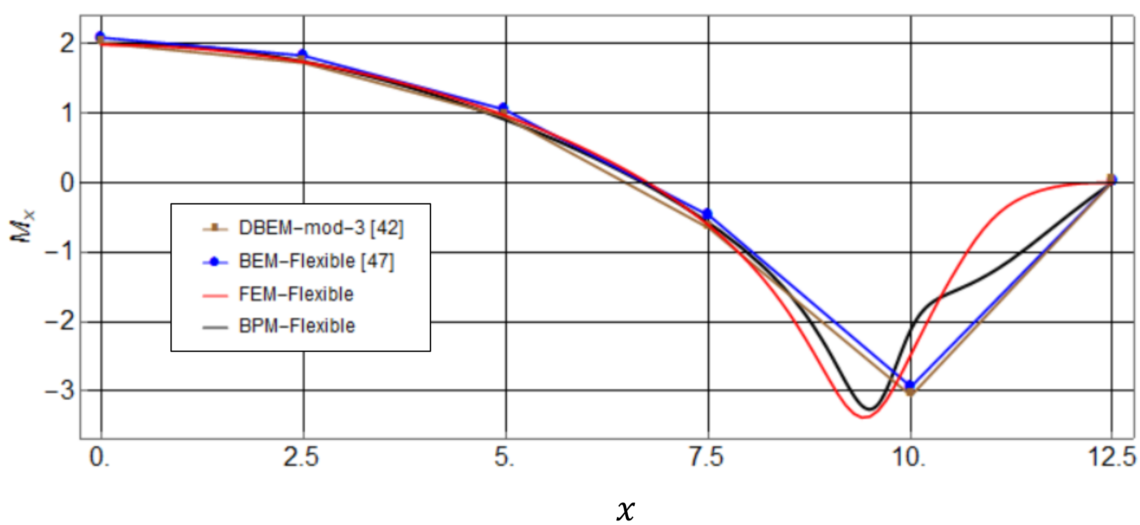



Figure 68 **Bending moment $M_x$ along Line-2**

The results for the bending moment $M_y$ along Line 1 and 2 are given in Figures 69 and 70, respectively. For Line 1, all curves stay constant up to the location of the column and then all deviate from FEM curve with lesser amounts by BEM [47] and the proposed BPM. For Line 2, all methods are reasonably in agreement with FEM except over the zone between the column and the free edge with less deviation by BEM [47].

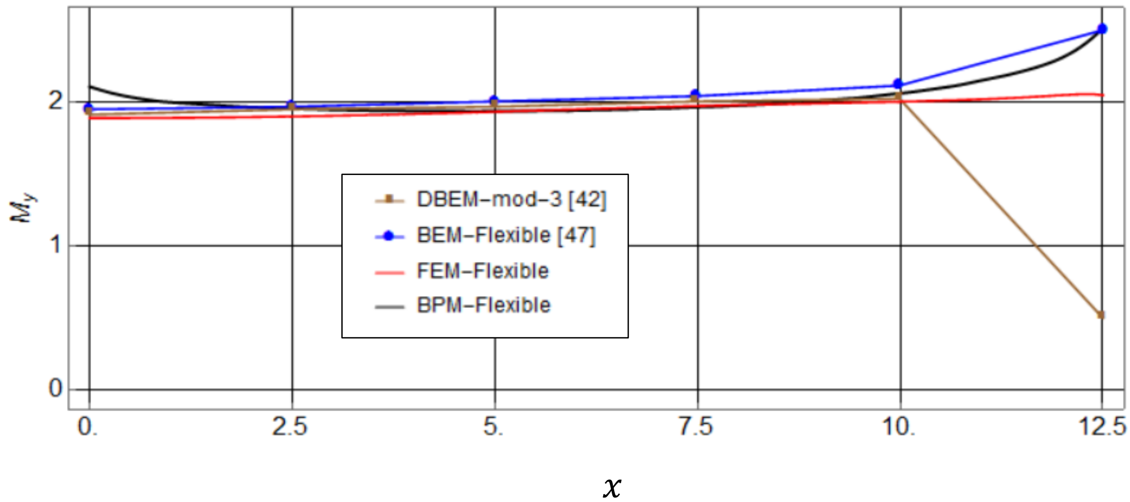



Figure 69 **Bending moment $M_y$ along Line-1**

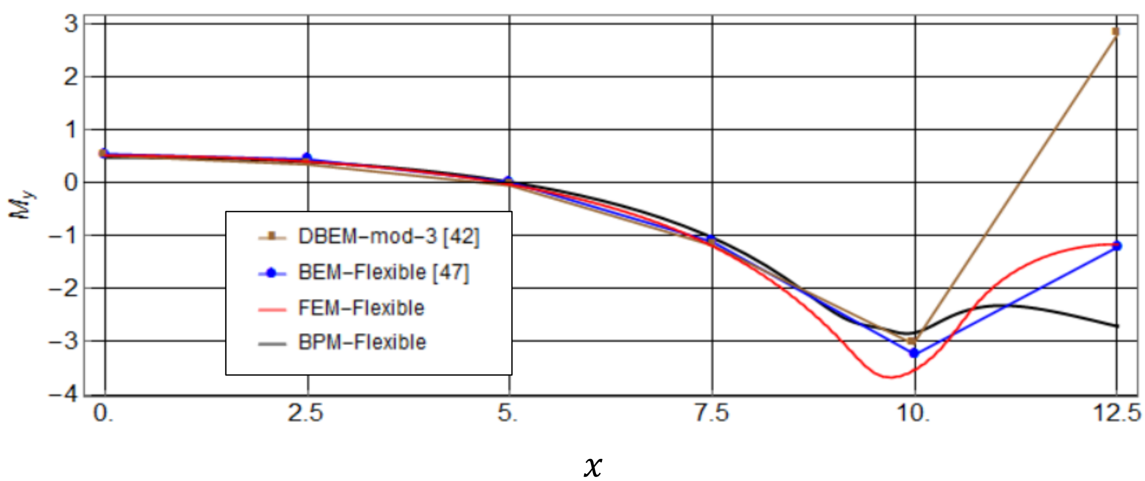



Figure 70 **Bending moment $M_y$ along Line-2**

To summarize this example, the one can simply say that the results of the proposed method are in a good agreement with the FEM results.  This gives more confidence to evaluate the accuracy of the proposed method for an irregular plate as shown in the following example.

### 6.2.2 An Irregular Floor Slab with Three Internal Flexible Columns

Consider an irregular floor slab simply supported at its outer edge and internally supported by three internal columns as shown in Figure 71. This example serves two purposes. First, to assess the accuracy of BPM against the robust FEM for a more complicated problem. Second, to compare the flexible support-based solution with the rigid support-based solution and study the effect of either assumption on the response of the plate in terms of maximum deflection and bending moments.

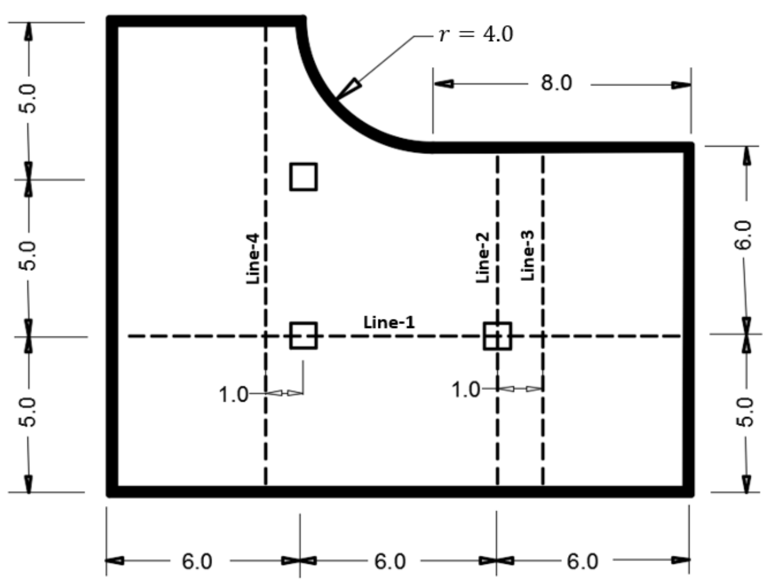



Figure 71 **Building floor slab with three internal columns and edge beam**

The material and geometric data are assumed to be: $q = 20\ KN/m^2$ , $E = 30\ GPa$, $\upsilon = 0.30$, $t = 0.25\ m$. Each column has been assumed clamped at the bottom with a size is $0.40\ m \times 0.40\ m$ and a height of 3.0 m to the centerline of the plate. The BPM is based on 160 boundary points and two contours each containing 160 source at a distance of $0.5\ m$ and $1\ m$, respectively, away and normal to the plate boundary. Each contact area of the column has been modeled by16 equal square cells. In the FEM model, however, the problem was modeled, as a complete frame analyzed using a shell-beam element interaction with a 2-D contact area.

The four lines indicated in Figure 71 have been targeted for comparing BPM and FEM methods. The deflection results along the four lines are shown in Figure 72 through 75. In general, the two methods are in good agreement for both rigid-based and flexible-based solutions along the four lines.

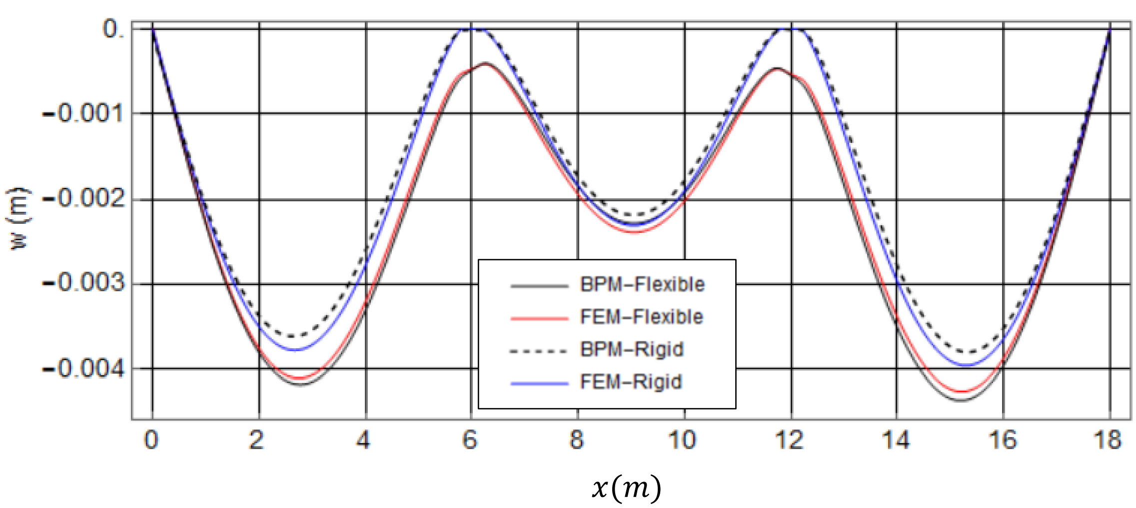



Figure 72 **Deflection $w$ along Line-1**

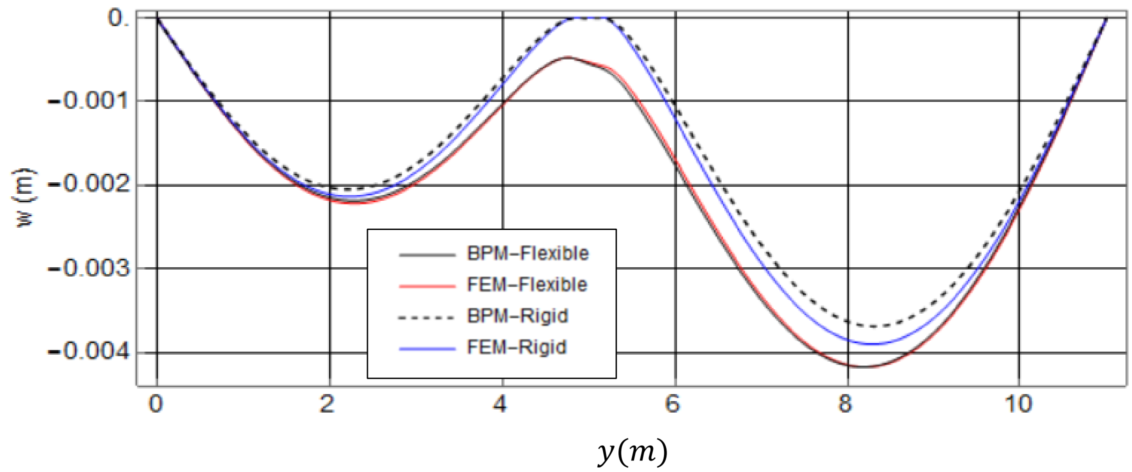



Figure 73 **Deflection *w* along Line-2**

The satisfaction of the rigidity assumption by both methods is confirmed by the uniform zero deflection over the patched areas of the columns as clearly shown in Figure 72 and 73. The results also show, as expected, that the rigid-based solutions by both methods underestimate the plate deflection as compared to the flexible-based solutions.

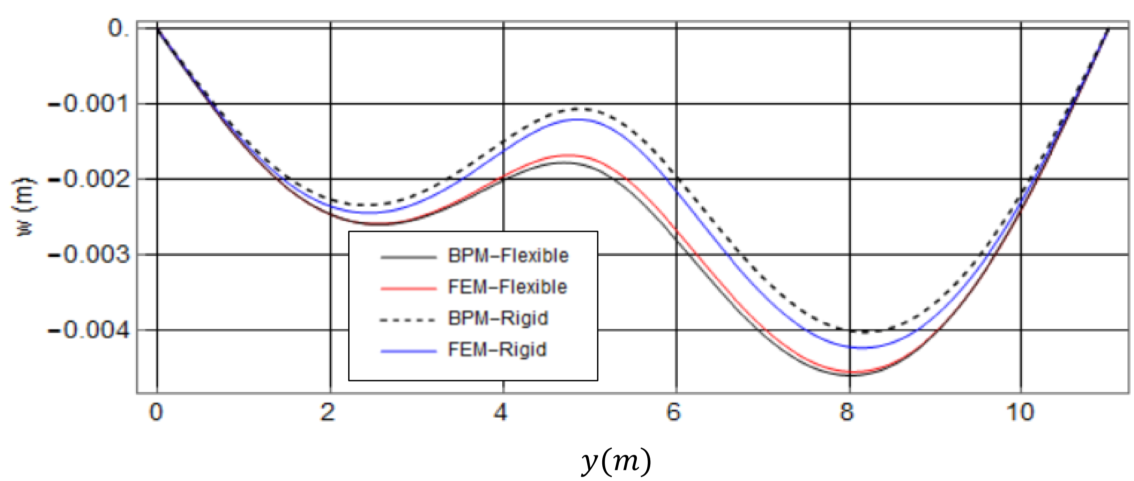



Figure 74 **Deflection *w* along Line-3**

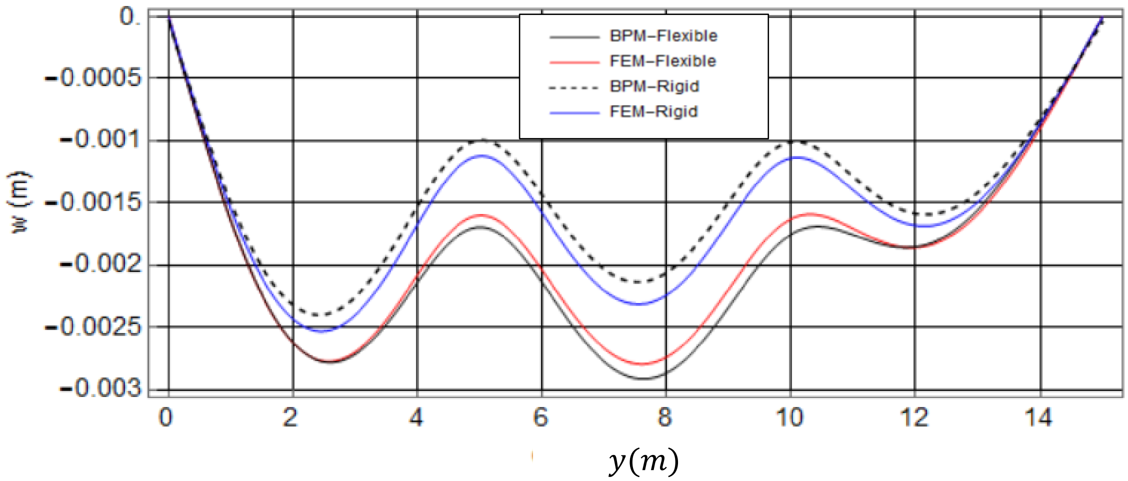



Figure 75 **Deflection *w* along Line-4**

The results for bending moments $M_x$ along Line 1 and $M_y$ along Lines 2, 3 and 4 are shown in Figures 76 through 79. All four curves are in perfect agreement except at the column zone, which indicates that the effect of rigidity/flexibility assumption is localized at the column zone and has no effect on the plate behavior away from the column. As far as the comparison between the BPM and FEM solutions is concerned, they are in close agreement with small variations at the patched areas of the columns.

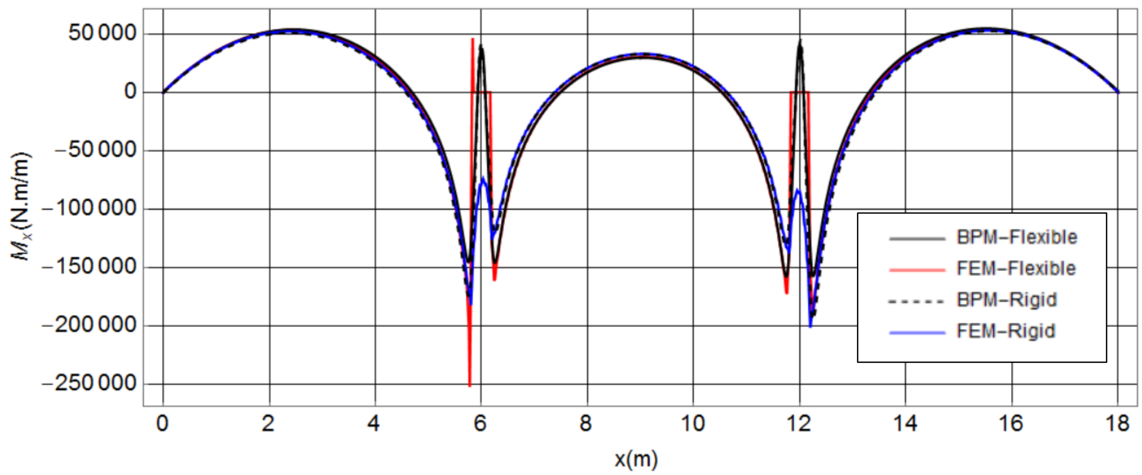



Figure 76 **Bending moment $M_x$ along Line-1**

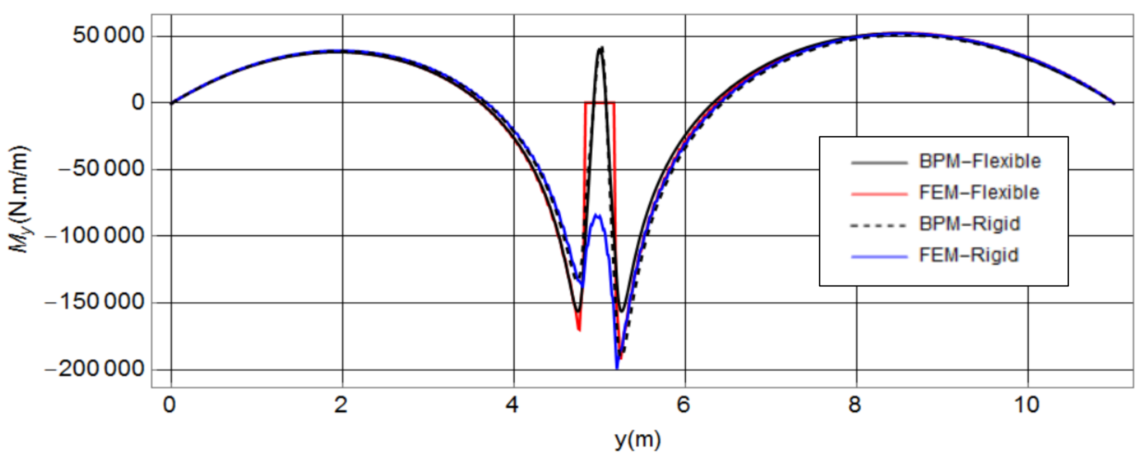



Figure 77 **Bending moment $M_y$ along Line-2**

The difference between the results of rigid and flexible cases for bending moment becomes clearer and easily distinguished as the moments are plotted away from the column (Lines 3 and 4). The results also show that, in contrast to the deflection results, the rigid supports assumption overestimates the maximum bending moments. In this example the overestimation by the rigid support assumption is about 15 % in the negative moments.

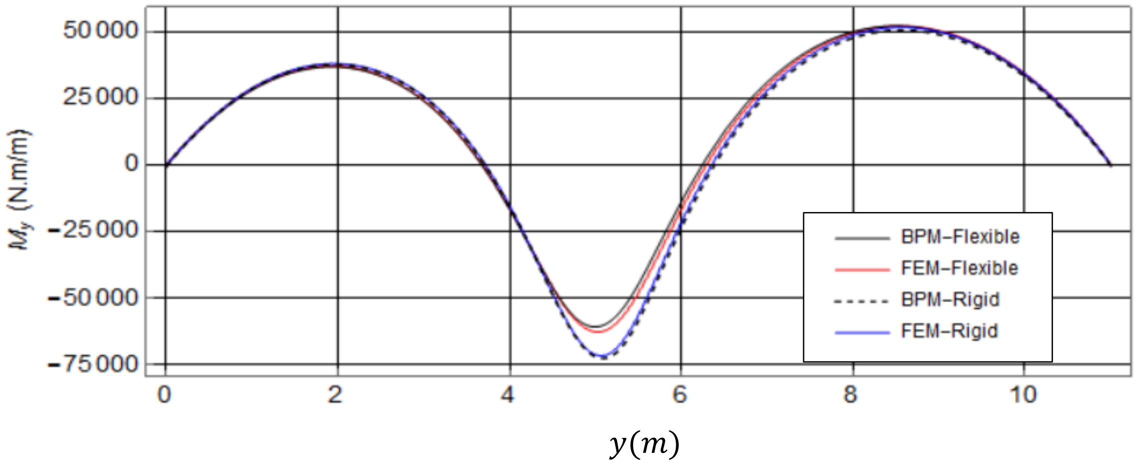



Figure 78 **Bending moment $M_y$ along Line-3**

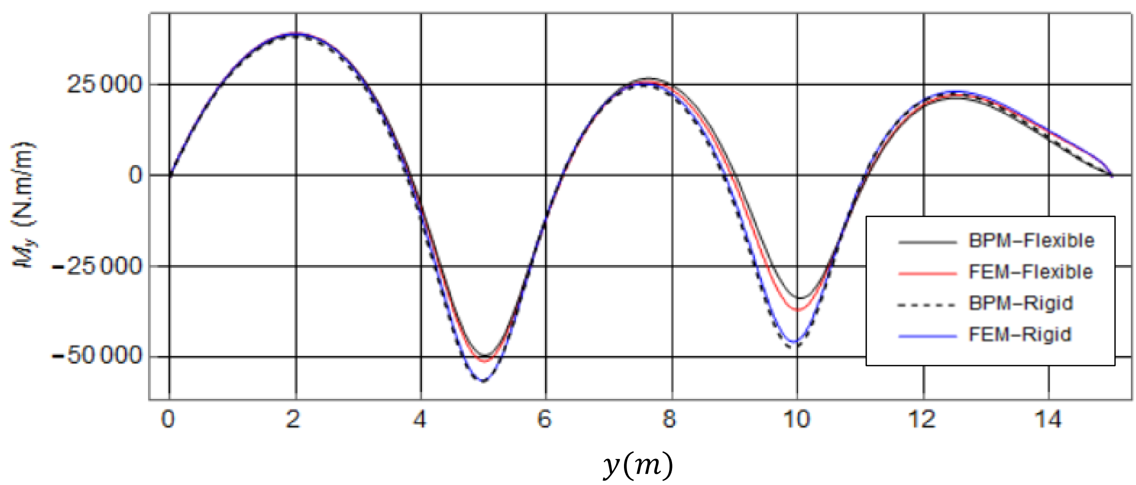



Figure 79 **Bending moment $M_y$ along Line-4**

To sum up this example, the general comparison between rigid and flexible support models reveals that the rigid model underestimates the deflection and overestimates the negative bending moment. This is meaningful for structural engineers in design offices in two aspects: (a) in the stage of strength design, which is concerned more about bending moments, the rigid support-based model overestimates the bending moment and hence leads to a safe design with less computational effort, and (b) in the stage of serviceability design, which is concerns more about deflections and vibration, the flexible support-based model becomes more conservative and therefore safer for this stage of design. Thus, the rigid support-based model offers a safe and quick solution for the strength-based design. The flexible support model, on the other hand, is more accurate as it represents the actual behavior of the structure but requires more modeling and computation efforts.

# CHAPTER 7

# SUMMARY, CONCLUSIONS, AND RECOMMONDATIONS

The main objective of this thesis is the development of an BPM method capable of modeling the problem of thin plates with internal rigid and flexible supports of different shapes and layouts. This chapter summarizes the main accomplishments of the research work undertaken, then it outlines the conclusions drawn from the results obtained and finally it recommends some directions for future research.

## 7.1 Summary and Conclusions

A BPM model for the analysis of thin plates with internal rigid and flexible supports of different shapes and layout has been developed. This procedure involved three main steps. The first one, is the development of a BPM model for the problem of plates supported on its edges only (without internal supports), to basically establish its reliability in preparation for the more complicated case of plates supported both internally and at the edges. The model was then implemented into a computer code using the MATHEMATICA software. Different numerical examples were used to check the validity of this code against the analytical solutions and an excellent agreement has been observed.

The second step is to extend the developed BPM model for plates with internal rigid supports. To model the plate-support interaction, each support has been divided to a series of cells where each cell was considered to be a patched area of a uniform pressure coming from the reaction. The compatibility condition was applied at the center of each cell to generate a number of equations equal in number to the number of reaction at the centers of the cells. Several numerical examples have been solved using the modified model to tackle practical cases involving different shapes of plates, different types of boundary conditions and different shapes and layouts of internal supports. The obtained results have been verified by comparing BPM solution with the available numerical solutions in the literature (DBEM and BEM) as well as with FEM solution.

The last step is the modification of the previously developed BPM model for rigid supports to account for flexible supports. As in the previous step, each support was divided to cells but additional terms are introduced to account for the effect of the bending moment at plate-support joint. The compatibility conditions for this case involves the deflection and the two normal slopes at the center of each support. Two numerical examples have been solved and the results were found to be more accurate than those obtained by the numerical methods available in the literature as confirmed by comparison with the reliable FEM solutions.

Based on the developed BPM models and the results presented in the last three chapters, the following conclusions can be drawn:

1- The BPM offers an excellent alternative to other available numerical methods in tackling -problems of internally supported thin plates. This is mainly due to its simplicity, ease of dealing with the complicated plate geometry and boundary/supporting conditions. One more advantage is that the solution can be obtained in a functional form which makes it possible to compute the secondary variables such as bending moments and stresses by direct differentiation of the deflection function, a unique feature not available in other numerical methods such as FEM and FDM.

2- The proposed BPM model is based on the collocation of the fundamental solution, which has the capability of capturing the singularity of the solution at the face of the internal support. Furthermore, the model does not require any domain integration as compared with the other numerical methods.

3- In general, BPM solutions for all the studied cases of different plate geometries and boundary conditions are closer to FEM compared to other numerical solutions.

4- The proper selection of the distance “d” locating the off-boundary source points of BPM plays a very important role in getting accurate results and stable solutions.

5- An accurate estimate of the bending moment at the support (column) face can be obtained even with a small number of cells. The same cannot be said of the results at the center of the support. This is due to the fact that the assumption of the support being rigid requires too many cells in order to satisfy the zero deflection (and therefore, zero bending moment) at enough number of points within the support. This difficulty has been overcome in the FEM solution by using a very fine mesh at the support zone. It should be noted, however, that in design practice, the critical bending moment, which is located at the face of the support, is more important and can be obtained very accurately by the proposed BPM model with a 16-cell discretization.

6- As a general comparison between rigid and flexible support models, the rigid model underestimates the deflection and overestimates the negative bending moment. Thus, the rigid support model offers a safe and quick solution for the strength-based design, which is based on the bending moments but care should be taken if the serviceability is of concern. The flexible support model, on the other hand, is more accurate as it represents the actual behavior of the structure, but requires more modeling and computation effort.

## 7.2 Recommendations for Future Works

Finally, as is the case with any considered numerical method, there is a scope for further application and enhancement of the proposed BPM method. Below are some of the recommendations that can be made:

1- A comprehensive study should be done to investigate the proper edge distance and the proper number of boundary points for different boundary conditions. An attempt also should be done to see if there is any relation between the two.

2- The employed BPM models were based on uniform node spacing. Non-uniform spacing should be tried for possible improvement in the solution accuracy.

3- In the more general case of semi-rigid plate column connection where the two slopes are not equal, the factors $k_{xs}$ and $k_{ys}$ should be modified. The modified factors should take in to account the relative stiffness of both the plate and the column.

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# Vitae

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**Work Experience**

- 5 Sep 2013–20 Aug 2014, Civil Engineer at University of Khartoum Consultancy Corporation (UKCC). Help to design of structures including houses, hospital, office blocks, and work in close partnership with architects. In addition to examine existing buildings and other structures to test if they are structurally sound and still fit for their intended purpose.

- 5 Sep. 2013–20 Aug 2014, part time teaching assistant at University of Khartoum, Faculty of Engineering, Civil Engineering department. Helping under-graduate students to deal with design of steel and concrete structures.

**Puplications**

- Abubakr E. S. Musa and Husain J. Al-Gahtani "Series-Based Solution for Analysis of Simply Supported Rectangular Thin Plate with Internal Rigid Supports." *Advances in Civil Engineering* 2017 (2017).