# BOUNDARY CONTROL OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

BY

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**I dedicate my Dissertation work to my family.**

**A special feeling of gratitude to my loving parents, my wife, my son, my daughters, my brothers and sisters.**

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**THESIS ABSTRACT**

Name: Faez Ali Nasser AL-Qarni

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We propose to stabilize time and/or space fractional heat-wave-like equations using boundary control. We derive and investigate the controllers based on the backstepping method (transmutation) and show that these controllers stabilized the unstable fractional partial differential equations. Stability of the overall system-controller is demonstrated. We use the Caputo fractional derivative. It is well-know that the fractional diffusion and wave equations are derived from the classical diffusion and wave equations by changing the integer order derivative by appropriate orders of fractional derivatives. Numerical simulations are given to illustrate the effectiveness of the approaches.


DOCTOR OF PHILOSOPHY DEGREE


KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

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# ملخص بحث
# درجه الدكتوراه في الفلسفه

**الاسم: فائز علي ناصر القارني**

**عنوان البحث: السيطره الحديه للمعادلات التفاضليه الجزئيه الكسريه**

**التخصص: الرياضيات**

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درسنا في هذه الاطروحه السيطره الحديه للمعادلات التفاضليه الجزئيه الكسريه (معادله انتشار الحراره ومعادله الموجه) الغير مستقره . واستخدمنا طريقه معامل التحويل الخطي القابل للانعكاس ( باك ستبنق) لازاله الجزء الغير مستقر في النظام الاصلي وتحويل النظام عبر نظام اخر مستقر وبسيط لايجاد التحكم الحدي الذي بدوره يتحكم في النظام الاصلي وتحويله الي نظام مستقرر. ايضا في هذا البحث سوف نستخدم التفاضل الكسري المعرف باسم كابوتو.

نحصل علي معادله انتشار الحراره الكسريه ومعادله الموجه الكسريه بتحويل رتبه الاشتقاق بالنسبه للزمن في معادله الانتشار العاديه او معادله الموجه العاديه من عدد صحيح الي عدد كسري او اي عدد اخر، اذا كان العدد محصور بين 0 و 1 سوف نحصل علي معادله الانتشار الكسريه واذا كان العدد بين 1 و 2 سوف نحصل علي معادله الموجه الكسريه.

في الاخير سوف نوضح طريق الحل بجموعه من الامثله العدديه وبقيم مختلفه لمرتبه الاشتقاق وقيم مختلفه للحل الابتدائي.

# Chapter 1

# INTRODUCTION

## 1.1 Introduction

It has been observed in the last three decades that fractional calculus and differential equations provide adequate modelling tools for many physical devices and processes. The central idea is the concept of fractional derivative, which is more than three centuries old. It generalizes the notions of integration and differentiation to arbitrary orders (including complex orders). The idea of fractional derivative is simple and dates back to 1695 when L'Hospital wrote to Leibniz asking him about the simple fractional derivative $\partial^{\frac{1}{2}}u/\partial x^{\frac{1}{2}}$ [29]. If $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ exist, then $\partial^{\frac{1}{2}}u/\partial x^{\frac{1}{2}}$ may exist too.

The fractional calculus also finds applications in different fields of science and engineering [17], not limited to the theory of fractals, numerical analysis, physics, engineering, biology, economics and finance. For instance, some problems of viscoelasticity was formulated and solved by M. Caputo [4] with his own definition of fractional differentiation.

Also, fractional diffusion and wave equations are obtained from the classical diffusion and wave equations by replacing derivative of integer orders by fractional derivatives of appropriate orders. The most simple examples are fractional heat and wave equations obtained by replacing the order $\alpha$ of the time derivative of the classical heat and wave equations by a non-integer order satisfying $\alpha \in (0, 1)$ to obtain fractional heat equation and $\alpha \in (1, 2)$ to obtain fractional wave equation. Fractional integrals and derivatives also appear in the theory of control systems, where for the description of the controlled system and the controller, fractional differential equations are used. Below are the simplest well-know fractional-order partial differential equations (FOPDE) considered in the literature,

- Time fractional-order diffusion equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2}, \ \alpha \in (0, 1] \tag{1.1}$$

- Time fractional-order wave equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2}, \ \alpha \in (1, 2] \tag{1.2}$$

- Time fractional-order diffusion-wave equation

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}}, \ \alpha \in (0,2], \ \beta \in (1,2]. \tag{1.3}$$

The equation (1.3) represents a hyperbolic wave equation for $\beta = 2$, $\alpha = 2$, and parabolic diffusion equation for $\beta = 2$, $\alpha = 1$, such that the equation (1.3) can be interpreted as the interpolation between a hyperbolic and a parabolic equation. The existence and uniqueness of solution was studied by Fujita [25]. Diffusion-wave equation involving Riemann-Liouville derivative and Caputo derivative, have also been discussed by various researchers .

Mittag-Leffer functions; a generalization of exponential functions to fractional order are often used to express the solutions of the above equations. Various methods have been used to solve these equations for example Green's function method [49], finite sine transform method [15], method of images and Fourier transform [51], separation of variables method [36], finite difference method [38], Adomian decomposition method (ADM) [35]. However, controlling such a fractional differential equations is a very recent topic and so much remains to be done. Boundary controller of fractional differential equations is still new, and we expect that better controller topics will be explored in the coming years.

A system can be modeled in a classical way using integer order derivative or derivative of fractional-order. The controller can also be operated as a classical one or a fractional-order one. Several scenarios have been considered

1. integer-order controller for integer-order system
2. integer-order controller for fractional-order system
3. fractional-order controller for integer-order system
4. fractional-order controller for fractional-order system

with increasing difficulty. Publications dealing with the last problems are very scarce as compared to the previous cases. Even in this last case most of the research consider systems with constant coefficients. As an important application field of fractional calculus, the topic about fractional-order control and system recently has been the focus of many researchers [1], [2], [3], [4], [5].

For unstable systems, we may have an infinite output irrespective of whether the input is finite or not. This in itself produces some visible problems. For instance, an unstable arm of a robot may make the robot move in a dangerous manner. Sometimes, very costly damages are incurred by unstable systems. In any case, there are quiet a number of systems which are naturally unstable, e.g., a rocket lift off or a fighter jet to mention a few. However, it is possible to design controllers which stabilizes such systems. To do that, one must understand what stability is, how it is determined and why it is needed. The stability of control systems is an important property. Considering any bounded input signal of a system, and if the output signal of the system to such a signal is also bounded, then the system is called bounded-input-bounded-output stable. If the output signal does not show this property, the system is unstable. Stability is the base requirement for the design of a control system.

Control is also applied whenever one would like to change the operation of a system to a desired form over a period of time. For instance, trying to make a car maintain the speed of say 50km/h on a road, irrespective of the topography of the road or presence of pot holes. Another example is the process of trying to keep at a specified level, the temperature and pressure level in a reactor vessel in a chemical process plant. Or one may want an aircraft to fly at a specific altitude, direction and velocity, independent of wind gusts.

These above examples, are types of task which are being accomplished today by various control methods. Moreover, these examples employ the use of automatic control system, which needs no human intervention.

## 1.2 Literature overview

Fractional calculus and differential equations have been studied by several authors. Several recent books on the fractional calculus and control theory have been written, illustrating the usefulness of the theory in applications [1], [2], [3], [4], [5], [6], [7], [8], [9]. In 1974, fractional diffusion equation with $\frac{1}{2}$ order time derivative and first-order derivative in space was considered by Oldham and Spanier [10] and they discussed the relation between diffusion equation with integer order and fractional one. Om.P. Agrawal [15] defined a fractional diffusion-wave equation in a bounded space domain with Caputo fractional time derivative and used a finite sine and Laplace transform technique to find a general solution in terms of Mittag-Leffler functions. In 2015 Mophou and Tao [54] studied the existence and uniqueness of solutions for fractional diffusion equation and composite fractional equation using change of variables and eigenfunctions expansions.

Boundary control of diffusion equation with integer-order derivative was considered by many researcher when $\lambda = q(x)$ is function of $x$ or $\lambda = \lambda_0$ constant and also with Dirichlet, Neumann, and Robin boundary control [30], [31], [32], [33], [37], [39] where the boundary control is defined by using the backstepping method.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda(x)u(x,t), \ (x,t) \in \ [0,1] \times [0,\infty)$$

It was shown that without controller this system is unstable for large value of $\lambda$. The system can be stabilized using appropriate controller where the controller is the function on the boundary. Recently, many advances in control theory have been given, for example [21], [22] for stability properties of linear fractional differential systems and [26] for controllability and observability properties of linear fractional differential systems [27], [28]. In 2004, Liang, Chen and Fullmer [42] considered the simple form of one-dimensional boundary control of fractional wave equation,

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \ 1 < \alpha \le 2, \ 0 \le x \le 1, \ t > 0 \\ u(0,t) = 0, \ u_x(1,t) = f(t), \ t > 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ 0 \le x \le 1 \end{cases} \quad (1.1)$$

where $f(t)$ is the boundary control at the free end of the cable and is defined by $f(t) = -k_d u_t(1,t)$ where $k_d$ is the controller gain and subscript $d$ means that $k_d$ is a gain for derivative feedback.

Also they added disturbance force at the same point where the boundary control and defined the boundary force by

$$\widehat{f}(s) = (k_d + \frac{ks}{s^2 + \omega^2})\widehat{u_t}(1, s) + \widehat{n}(s) \tag{1.2}$$

where $\widehat{f}(s)$ is the Laplace transform of the combination of boundary control force and disturbance force $n(t)$. The authors observed and concluded from simulation that the boundary controller can stabilize the fractional order wave equation. In 2005, Liang, Chen, Vinagre and Podlubny [43] considered the same system (1.1) and defined the controller with fractional order by

$$f(t) = -k\frac{d^{\mu}u(1, t)}{dt^{\mu}}, 0 < \mu < 1 \tag{1.3}$$

where $k$ is the controller gain, $\mu$ is the order of fractional derivative of the displacement at the free end of the cable. They have also shown that the fractional order boundary controller is better than the integer order boundary control. In 2005, Liang, Zhang and Podlubny [44], used the Smith predictor method (the most famous method for control of systems with pure time delays) to solve the instability problem of the fractional wave equation (1.1) with a fractional-order boundary controller when the time delay in the boundary measurement is small or large. They concluded that for large delays which makes the system unstable, the fractional-order controller combined with the Smith predictor is able to compensate the time delay and is robust against a small difference between the assumed delay and the actual delay. In 2010, Ahmed [45] studied the boundary controllability of nonlinear fractional integrodifferential systems under sufficient conditions in Banach space. An elementary method of steering a fractional linear control system from a given initial state to a given final state was presented by Dzielinski and Malesza [41] in 2011. In 2011 Yanzhu Zhang, Wang Xiaoyan and Yanmei [46], considered the boundary control of the two models of the anti-stable fractional-order and integer-order vibration systems. The string vibration system can be modelled in the integer and fractional wave models.

(i) Integer-order

$$\begin{cases} u_{tt}(x,t) = u_{xx}(x,t) \\ u_x(0,t) = -a_1 u_t(0,t) \\ u_x(1,t) = f_1(t) \end{cases} \tag{1.4}$$

where $f_1(t)$ is the boundary controller and is defined by A. Smyshlyaev [34]

$$f_1(t) = \frac{ka_1(a_1+b_1)}{1+a_1b_1}u(0,t) - ku(1,t) - \frac{a_1+b_1}{1+a_1b_1}u_t(1,t) - \frac{k(a_1+b_{1)}}{1+a_1b_1}\int_0^1 u_t(y,t)dy \tag{1.5}$$

(ii) Fractional-order

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \ 1 < \alpha < 2 \\ u_x(0,t) = -a_2 u_t(0,t) \\ u_x(1,t) = f_2(t) \end{cases} \tag{1.6}$$

where $f_2(t)$ is the boundary controller defined by

$$f_2(t) = \frac{k_d a_2(a_2+b_2)}{1+a_2b_2}u(0,t) - k_d u(1,t) - \frac{a_2+b_2}{1+a_2b_2}u_t(1,t) - \frac{k_d(a_2+b_2)}{1+a_2b_2}\int_0^1 u_t(y,t)dy \tag{1.7}$$

where $b_1$and $b_2$ are appropriate constants.

In this thesis, we are interested in controlled systems described by fractional partial differential equations. The input can be a function in a boundary condition, and the output is the solution of the fractional partial differential equation. The input is called the control variable, or the control, and the output is called the state of the system.

And also, we shall consider the boundary control of fractional partial differential equations and present the backstepping method to design boundary controllers stabilizing the fractional PDE systems and show that these controllers work "irrespective" of the initial condition. The main feature of backstepping is to eliminate the destabilizing effects terms that appear throughout the domain while the control is acting only from the boundary. We pursue a continuum equivalent of this approach and build a change of variables, which involves a volterra integral operator that absorbs the destabilizing terms acting in the domain and allows the boundary control to completely eliminate their effect. One can pursue several different objectives in a control design for PDEs and fractional PDEs systems. If the system is already stable, a typical objective for feedback control would be to improve performance. Optimality methods are natural in such situations. Another control objective is stabilization. The objective of stabilization is to annihilate of the effect of perturbation of the system state in order to steer the system state to a given desired trajectory. For this purpose feedback laws are introduced, that allow to react to deviations of the system state from the desired trajectory. Since the deviations are a priority unknown, the feedback laws must be well defined for all possible system states.

This thesis is organized as follows: In chapter two, we present some basic definitions, lemmas, properties and notation needed later in this work. In chapter three, first we present our contribution, specifically the boundary control of time fractional diffusion equation in two cases, namely constant coefficient and space dependent coefficient. Thereafter, we show that the boundary control stabilizes the unstable fractional diffusion equation and finally give some examples to illustrate our contribution. In chapter 4, we show that the boundary control can improve the stability of time fractional wave equation. After that, in chapter five we present the diffusion-wave time and space fractional equation, which is represented by a hyperbolic wave equation or a parabolic diffusion equation, and show that the control is stabilizing. Finally, we consider the optimal control of fractional diffusion equation with space dependent coefficient and give some examples to illustrate our contributions to the problems considered.

# Chapter 2

# PRELIMINARIES

In this chapter, we give some notations and basic definitions of special functions such as Gamma and Mittag-Leffler function [8], [4]) used in this work. These notations and functions are most frequently used in the fractional calculus and especially in solving fractional differential equations,

**Definition 1** *[4] Let* $\Omega = [a, b]$, $(0 \leq a < b \leq \infty)$ *be a finite interval. We denote by* $L_p(a, b)$ $(1 \leqq p \leqq \infty)$ *the set of those Lebesgue real-valued measurable functions* $f$ *on* $\Omega$ *for which* $\|f\|_p < \infty$, *where*

$$\|f\|_p = \left( \int_a^b |f(x)|^p \, dt \right)^{\frac{1}{p}} \quad (1 \leqq p < \infty) \tag{2.1}$$

*and*

$$\|f\|_p = ess \sup_{a \leqq x \leqq b} |f(x)|, \; if \; p = +\infty \tag{2.2}$$

*Here* $ess \sup |f(x)|$ *is the essential supremum of* $|f(x)|$.

**Definition 2** *[4] Let* $\Omega = [a,b]$, $(-\infty < a < b < \infty)$ *be a finite interval and let* $AC[a,b]$ *be the space of functions* $f$ *which are absolutely continuous on* $[a,b]$. *It is known that* $AC[a,b]$ *coincides with the space of primitives of Lebesgue summable functions:*

$$f(.) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \phi(t)dt, \ (\phi(.) \in L(a,b)), \tag{2.3}$$

*and therefore an absolutely continuous function* $f(.)$ *has a summable derivative* $f'(x) = \phi(x)$ *almost everywhere on* $[a,b]$. *Thus (2.3) yields*

$$\phi(x) = f'(x) \quad and \quad c = f(a). \tag{2.4}$$

**Lemma 3** *[4] The space* $AC^n[a,b]$ *consists of those and only those functions which can be represented in the form*

$$f(x) = (I_a^n \phi)(x) + \sum_{k=0}^{n-1} c_k (x-a)^k, \tag{2.5}$$

*where* $\phi \in L(a,b)$, $c_k \, (k = 0,1,...,n-1)$ *are arbitrary constants, and*

$$(I_a^n \phi)(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \phi(t)dt. \tag{2.6}$$

*It follows from (2.5) that*

$$\phi(t) = f^{(n)}(t), \quad c_k = \frac{f^{(k)}(a)}{k!}, \ k = 0,1,...,n-1. \tag{2.7}$$

Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ and $m \in \mathbb{N}_0 = \{0, 1, ...\}$. We denote by $C^m(\Omega)$ the space of functions $f$ which are $m$ times continuously differentiable on $\Omega$ with the norm

$$\|f\|_{C^m} = \sum_{k=0}^{m} \left\|f^{(k)}\right\|_C = \sum_{k=0}^{m} \max_{x\in\Omega} \left|f^{(k)}(x)\right|, \ m \in \mathbb{N}_0. \tag{2.8}$$

In particular, for $m = 0$, $C^0(\Omega) \equiv C(\Omega)$ is the space of continuous functions $f$ on $\Omega$ with the norm

$$\|f\|_C = \max_{x\in\Omega} |f(x)|. \tag{2.9}$$

## 2.1 Derivative of Integrals Depending on a Parameter

The following formula holds when the integrated function $g(t, \tau)$ is integrable with respect to the second variable, its derivative $\frac{\partial}{\partial t} g(t, \tau)$ is continuous and $g(t, \tau)$ is defined in $a < \tau < t < \infty$

$$\frac{d}{dt} \int_a^t g(t, \tau) d\tau = g(t, t) + \int_a^t \frac{\partial}{\partial t} g(t, \tau) d\tau \tag{2.10}$$

## 2.2 The Gamma Function

In integer-order calculus, the factorial function plays an important role because it is one of the most fundamental combinatorial tools. The Gamma function has the same importance in the fractional-order calculus and denoted by $\Gamma(x)$. We will recall in this section some results on the gamma function which are important for other parts in this work.

**Definition 4** *[4] The gamma function, the Euler integral of the second kind* $\Gamma(.)$ *is defined by the integral*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \mathrm{Re}\, z > 0. \tag{2.11}$$

One of the most important properties is a generalization of the factorial function $n!$, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ and satisfies the following functional equation,

$$\Gamma(x+1) = x\Gamma(x),$$

which can be easily proved using integration by parts. The Gamma function has been extensively studied by many researchers [16].

## 2.3 The Mittag-Leffler Function

The one parameter Mittag-Leffler function is defined by

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

It generalizes the exponential function. The two parameter function of the Mittag-Leffler type, introduced by R. P. Agarwal, plays a very important role also in fractional calculus. It is defined by [4]

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \quad (\alpha > 0,\ \beta > 0). \tag{2.12}$$

For special choices of the values of the parameters $\alpha$, $\beta$ we obtain well-known classical functions,

$$\begin{aligned}
E_{1,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \\
E_{1,2}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z-1}{z} \\
E_{2,1}(z^2) &= \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z) \\
E_{2,2}(z^2) &= \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}
\end{aligned} \tag{2.13}$$

As we will see later, classical derivatives of the Mittag-Leffler function appear in solutions of fractional DEs. Since the series (2.12) is uniformly convergent we may differentiate term by term and obtain

$$E_{\alpha,\beta}^{(m)}(z) := \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{z^k}{\Gamma(\alpha k+\alpha m+\beta)} \tag{2.14}$$

## 2.4 The Laplace Transform

The Laplace transform is a very useful tool for solving linear ODEs with constant coefficients since it converts linear differential equations to linear algebraic equations which can be solved easily. The inverse transform of the result, is usually the most complicated part of this approach.

**Definition 5** *The Laplace transform of a function $\varphi(.)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is defined by*

$$(\mathcal{L}\varphi)(s) = \mathcal{L}[\varphi(.)](s) = \widetilde{\varphi}(s) := \int_0^\infty e^{-st}\varphi(t)dt, \ (s \in \mathbb{C}). \tag{2.15}$$

*The inverse Laplace transform is given for $x \in \mathbb{R}^+$ by the formula*

$$\begin{aligned}(\mathcal{L}^{-1}g)(x) &= \mathcal{L}^{-1}[g(.)](x) : \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}g(s)ds, \ (c = \Re(s) > c_0),\end{aligned} \tag{2.16}$$

*where $c_0$ lies in the right half plane of the absolute convergence of the Laplace integral (2.15). The direct and inverse Laplace transforms are inverse to each other for "sufficiently good" functions $\varphi$ and $g$ : $\mathcal{L}^{-1}\mathcal{L}\varphi = \varphi$ and $\mathcal{L}\mathcal{L}^{-1}g = g$.*

## 2.5 *The Bessel and the Modified Bessel Functions*

Bessel functions are named after Friedrich Wilhelm Bessel (1784 - 1846).

**Definition 6** *[13], [14] The second order differential equation*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \ for \ \ \nu \in \mathbb{C}, \tag{2.17}$$

*is called the Bessel equation and its solution is given by*

$$\mathcal{J}_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k+\nu}}{k!(\nu+k)!}, \tag{2.18}$$

*whereas the second order differential equation*

$$x^2 y'' + xy' - (x^2+\nu^2)y = 0, \;\; for\ \nu \in \mathbb{C}, \tag{2.19}$$

*is called the modified Bessel equation with solution given by*

$$\mathcal{I}_\nu(x) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k+\nu}}{k!(\nu+k)!}. \tag{2.20}$$

*Solutions to these differential equations are called Bessel and modified Bessel functions respectively.*

## 2.6 Fractional Integration and differentiation

If $f$ is a continuous function on the real line, then we can form the definite integral from $a$ to $t$

$$I^1 f(t) := \int_a^t f(\tau) d\tau_1.$$

Repeating this process gives

$$I^2 f(t) := \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau_2) d\tau_2,$$

and this can be extended arbitrarily to

$$I^n f(t) := \int_a^t \int_a^{\tau_1} ... \int_a^{\tau_{n-1}} f(\tau) d\tau_{n-1}...d\tau_2 d\tau_1$$

The Cauchy's formula for repeated integration [10], [4] is

$$\begin{aligned} I^n f(t) \ &: \ = \int_a^t \int_a^{\tau_1} ... \int_a^{\tau_{n-1}} f(\tau) d\tau ... d\tau_2 d\tau_1 \\ &= \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(x) d\tau, \end{aligned} \tag{2.21}$$

where $n \in \mathbb{N}$, $a, t \in \mathbb{R}$, $t > a$. If $n$ is substituted with a positive real number ($\alpha > 0$) and $(n-1)!$ by its generalization $\Gamma(\alpha)$, a formula for fractional integration is obtained as follows

**Definition 7** *[4], [5] Let $\Omega = [a, b]$ $(-\infty < a < b < \infty)$ be a finite real interval. The Riemann-Liouville fractional integrals $I^{\alpha}_{a+} f$ and $I^{\alpha}_{b-} f$ of order $\alpha > 0)$ is defined by*

$$\left(I^{\alpha}_{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds, \ \ x > a, \tag{2.22}$$

*and*

$$\left(I^{\alpha}_{b-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds, \ \ x < b, \tag{2.23}$$

*respectively. Here $\Gamma$ is the Gamma function. These integrals are called the right-sided and the left-sided fractional integrals. ([17], [4]).*

There are several types of fractional derivative, the most popular ones are the Riemann-Liouville and the Caputo derivatives ([17], [4]).

### 2.6.1 The Riemann-Liouville Fractional Differential Operator

**Definition 8** *[4], [5] The Riemann-Liouville fractional derivatives* $\left(D_{a+}^{\alpha}y\right)$ *and* $\left(D_{b-}^{\alpha}y\right)$ *of order* $\alpha \geq 0$, $n-1 \leq \alpha < n$, $n = [\mathcal{R}(\alpha)] + 1$ *are defined by*

$$
\begin{aligned}
\left(D_{a+}^{\alpha}y\right)(x) &= \left(\frac{d}{dx}\right)^{n} (I_{a+}^{n-\alpha}y)(x) \\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}(x-s)^{n-\alpha-1}y(s)ds,\ x>a
\end{aligned}
\tag{2.24}
$$

*and*

$$
\begin{aligned}
\left(D_{b-}^{\alpha}y\right)(x) &= \left(-\frac{d}{dx}\right)^{n} (I_{b-}^{n-\alpha}y)(x) \\
&= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{b}(s-x)^{n-\alpha-1}y(s)ds,\ x<b
\end{aligned}
\tag{2.25}
$$

*respectively, where* $[\mathcal{R}(\alpha)]$ *means the integer parts of* $\mathcal{R}(\alpha)$. *In particular, when* $\alpha = n \in \mathbb{N}_0 = \{0, 1, ...\}$, *then*

$$\left(D_{a+}^{0}y\right)(x) = \left(D_{b-}^{0}y\right)(x) = y(x); \tag{2.26}$$

$$\left(D_{a+}^{\alpha}y\right)(x) = y^{(n)}(x) \ \text{and} \ \left(D_{b-}^{\alpha}y\right)(x) = (-1)^{n}y^{(n)}(x) \quad (n \in \mathbb{N}) \tag{2.27}$$

*where* $y^{(n)}(x)$ *is the usual derivative of* $y(x)$ *of order* $n$.

### 2.6.2 The Caputo Fractional Differential Operator

**Definition 9** *[4], [5] Let $[a, b]$ be a finite interval of the real line $\mathbb{R}$, and let*

$$D^{\alpha}_{a+}\left[y(s)\right](x) \equiv \left(D^{\alpha}_{a+}y\right)(x) \text{ and } D^{\alpha}_{b-}\left[y(s)\right](x) \equiv \left(D^{\alpha}_{b-}y\right)(x)$$

*be the Riemann-Liouville fractional derivatives of order $\alpha \in \mathbb{C}$ $(\mathcal{R}(\alpha) \geqq 0)$ defined by (2.22) and (2.23) respectively. The fractional derivatives*

$$\left({}^{C}D^{\alpha}_{a+}y\right)(x) \text{ and } \left({}^{C}D^{\alpha}_{b-}y\right)(x)$$

*of order $\alpha \in \mathbb{C}$ $(\mathcal{R}(\alpha) \geqq 0)$ on $[a, b]$ are defined via the above Riemann-Liouville fractional derivatives by*

$$\left({}^{C}D^{\alpha}_{a+}y\right)(x) = \left(D^{\alpha}_{a+}\left[y(s) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(s-a)^{k}\right]\right)(x) \tag{2.28}$$

*and*

$$\left({}^{C}D^{\alpha}_{b-}y\right)(x) = \left(D^{\alpha}_{b-}\left[y(s) - \sum_{k=0}^{n-1}\frac{y^{(k)}(b)}{k!}(b-s)^{k}\right]\right)(x) \tag{2.29}$$

*respectively, where*

$$n = [\mathcal{R}(\alpha)] + 1 \text{ for } \alpha \notin \mathbb{N}_0; \; n = \alpha \text{ for } \alpha \in \mathbb{N}_0. \tag{2.30}$$

*These derivatives are called right-sided and left-sided Caputo fractional derivatives of order $\alpha$*

**Lemma 10** *[4], [5] Let* $\mathcal{R}(\alpha) > 0$ *and let* $n$ *be given by (2.30) . If* $y(.) \in AC^n[a,b]$ *or* $y(.) \in C^n[a,b]$, *then*

$$\left(I_{a+}^{\alpha}\, {}^{C}D_{a+}^{\alpha} y\right)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k \tag{2.31}$$

*and*

$$\left(I_{b-}^{\alpha}\, {}^{C}D_{b-}^{\alpha} y\right)(x) = y(x) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!}(b-x)^k \tag{2.32}$$

*In particular, if* $0 < \mathcal{R}(\alpha) \leqq 1$ *and* $y(.) \in AC[a,b]$ *or* $y(.) \in C[a,b]$, *then*

$$\left(I_{a+}^{\alpha}\, {}^{C}D_{a+}^{\alpha} y\right)(x) = y(x) - y(a), \text{ and } \left(I_{b-}^{\alpha}\, {}^{C}D_{b-}^{\alpha} y\right)(x) = y(x) - y(b) \tag{2.33}$$

**Definition 11** *[4] Let* $\mathrm{Re}(\alpha) \in (m-1, m)$. *The right-sided and left-sided Riemann-Liouville derivatives of order* $\alpha$ *are defined as*

$$\begin{aligned} \left(D_{a+}^{\alpha} f\right)(x) &= D^m \left(I_{a+}^{m-\alpha} f\right)(x).\ x > a, \\ \left(D_{b-}^{\alpha} f\right)(x) &= (-D)^m \left(I_{b-}^{m-\alpha} f\right)(x).\ x < b. \end{aligned} \tag{2.34}$$

*Analogous formulas yield the left- and right-sided Caputo derivatives of order* $\alpha$ :

$$\begin{aligned} \left({}^{C}D_{a+}^{\alpha} f\right)(x) &= (I_{a+}^{m-\alpha} D^m f)(x).\ x > a, \\ \left({}^{C}D_{b-}^{\alpha} f\right)(x) &= I_{b-}^{m-\alpha}(-D^m f)(x).\ x < b. \end{aligned} \tag{2.35}$$

This operator is introduced in Caputo [12].

**Corollary 12** *[4] The relation between Riemann-Liouville fractional derivative and Caputo fractional derivative are given by (see Gorenflo and Mainardi [17])*

$$\begin{aligned} {}^{C}D^{\alpha}f(x) &= D^{\alpha}f(x) - \sum_{k=0}^{n-1}\frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0) \\ {}^{C}D^{\alpha}f(x) &= D^{\alpha}\left(f(x) - \sum_{k=0}^{n-1}\frac{x^{k}}{k!}f^{(k)}(0)\right) \end{aligned} \tag{2.36}$$

**Lemma 13** *Let $n-1<\alpha<n$, $n\in\mathbb{N}$, $\alpha$, $\lambda\in\mathbb{C}$ and the functions $f(.)$ and $g(.)$ be such that both ${}^{C}D^{\alpha}f(.)$ and ${}^{C}D^{\alpha}g(.)$ exist, the Caputo fractional derivative is a linear operator, i.e.,*

$${}^{C}D^{\alpha}(\lambda f(t)+g(t)) = \lambda{}^{C}D^{\alpha}f(t) + {}^{C}D^{\alpha}g(t). \tag{2.37}$$

*Similarly, the Riemann-Liouville operator satisfies*

$$D^{\alpha}(\lambda f(t)+g(t)) = \lambda D^{\alpha}f(t) + D^{\alpha}g(t) \tag{2.38}$$

### 2.6.3 Examples of Fractional Derivatives

In this subsection we shall find derivatives of some elementary functions, e.g., the constant, the power and the exponential function, as well as the sine and cosine function. The Caputo fractional derivatives of these functions are studied and compared with the Riemann-Liouvill fractional derivative [4], [18], [5].

### The Constant Function

Let $f(.)$ be a constant function i.e $f(x) = c$, then the Riemann-Liouville fractional derivative of $f(.)$ is given by

$$D^{\alpha}[f(x)] = D^{\alpha}[c] = \frac{c}{\Gamma(1-\alpha)} x^{-\alpha} \neq 0.$$

whereas Caputo fractional derivative is

$$^{C}D^{\alpha}[f(x)] = ^{C}D^{\alpha}[c] = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} (x-s)^{n-\alpha-1} c^{(n)} ds = 0 \tag{2.39}$$

### The Power Function

The Riemann-Liouville fractional and Caputo fractional derivatives of the power function satisfy

$$D^{\alpha} t^{p} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \quad n-1 < \alpha < n,\ p > -1,\ p \in \mathbb{R}. \tag{2.40}$$

$$^{C}D^{\alpha} t^{p} = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} & , n-1 < \alpha < n,\ p > n-1,\ p \in \mathbb{R}, \\ 0 & , n-1 < \alpha < n,\ p \leq n-1,\ p \in \mathbb{N}, \end{cases} \tag{2.41}$$

respectively [4].

**The Exponential Function**

We shall consider the exponential function, let $\alpha \in \mathbb{R}$, $n-1 < \alpha < n, n \in \mathbb{N}$ , $\lambda \in \mathbb{C}$. Then the Caputo fractional derivative of the exponential function has the form

$$^{C}De^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n} t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha} E_{1,n-\alpha+1}(\lambda t), \tag{2.42}$$

[4] where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-Leffer type.

**The Laplace Transform of Basic Fractional Operator**

Suppose that $p > 0$ and $F(s)$ is the Laplace transform of $f(t)$, then the following statements holds [4]:

- The Laplace transform of the fractional integral of order $\alpha$, $n-1 \leq \alpha < n$ is given by

$$L\{J^{\alpha} f(t); s\} = s^{-\alpha} F(s). \tag{2.43}$$

- The Laplace transform of the Riemann-Liouville fractional differential operator of order $\alpha$ is given by

$$\begin{aligned} L\{D^{\alpha} f(t); s\} &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0} \\ &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{n-k-1} [D^k J^{n-\alpha} f(t)]_{t=0}, \end{aligned} \tag{2.44}$$

- The Laplace transform of the Caputo fractional differential operator of order $\alpha$

is given by

$$L\{^C D^\alpha f(t); s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),\ n-1 \le \alpha < n. \tag{2.45}$$

- Let $\alpha,\ \beta,\ \lambda \in \mathbb{R}$ ,$\alpha,\ \beta > 0$, $p \in \mathbb{N}$, then the Laplace transform of the two-parameter function of Mittag-Leffler type is given by

$$L\{t^{\alpha p+\beta-1} E_{\alpha,\beta}^{(p)}(\pm\lambda t^\alpha); s\} = \frac{p! s^{\alpha-\beta}}{(s^\alpha \pm \lambda)^{p+1}},\ \ \mathrm{Re}(s) > |\lambda|^{1/\alpha}. \tag{2.46}$$

## 2.7 Fractional Ordinary Differential Equations

In this section fractional initial value problems are presented. These are fractional ordinary differential equations with classical initial conditions [4], [5]. Using the Laplace transform we shall obtain the general solution for a linear fractional ordinary differential equation with constant coefficients.

**Theorem 14** *Consider the linear initial value problem*

$$\begin{cases} {}^C D_t^{(\alpha)} y(t) - \lambda y(t) = 0,\ t > 0,\ n-1 < \alpha < n, \\ y^{(k)}(0) = b_k,\ b_k \in \mathbb{R},\quad k = 0, ..., n-1. \end{cases} \tag{2.47}$$

*Then the solution of problem (2.47) is given by*

$$y(t) = \sum_{k=0}^{n-1} b_k t^k \mathrm{E}_{\alpha,k+1}(\lambda t^\alpha), \tag{2.48}$$

*where $E_{\alpha,\beta}(z)$ is the two-parameter function of Mittag-Leffler type.*

**Proof.** [4], Applying Laplace transform to the fractional differential equation in (2.47), it becomes

$$s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}y^{(k)}(0) - \lambda Y(s) = 0, \tag{2.49}$$

where $Y(s)$ is the Laplace transform of $y(t)$ and $L\{-\lambda y(t); s\} = -\lambda Y(s)$. We solve equation (2.49) with respect to $Y(s)$ as follows

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda} y^{(k)}(0). \tag{2.50}$$

Substituting the initial conditions gives

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda} b_k.$$

Using Laplace transform of the two-parameter function of Mittag-Leffler types (2.46), it follows

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^{\alpha}-\lambda} b_k = \sum_{k=0}^{n-1} L\left\{t^k E_{\alpha,k+1}(\lambda t^{\alpha}); s\right\} b_k = L\left\{\sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha}); s\right\}. \tag{2.51}$$

That is the inverse Laplace transform, $y(t)$ is

$$y(t) = y(t,\alpha) = \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^{\alpha}). \tag{2.52}$$

■

It is clear that the form of the solution is given by the properties of the Mittag–Leffler function [20]. Figure 2-1 and Figure 2-2 show the graphs of the function for different values of $\alpha$ and $\lambda < 0$. As we can see, the behavior corresponds to a standard first-order decay for $\alpha \in (0, 1]$, is exponential for $\alpha = 1$, becomes a damped oscillation for $\alpha \in (1, 2]$, and oscillates for $\alpha = 2$. Figure 2-3 shows a growth toward infinity for different values of $\alpha$ and $\lambda > 0$,

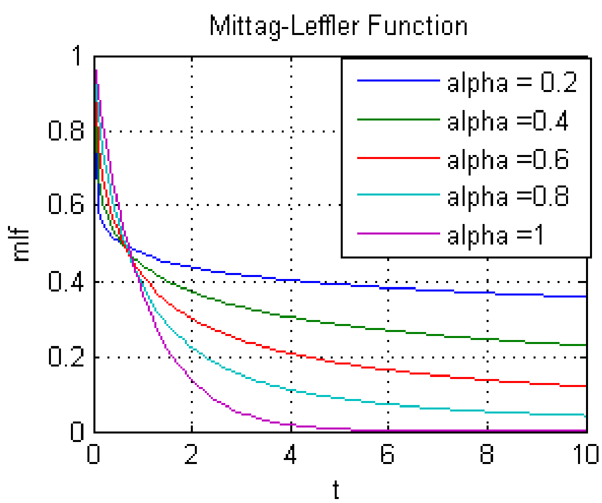


Figure 2-1: Mittag-Leffler Function, $0 < \alpha \leq 1$, $\lambda < 0$

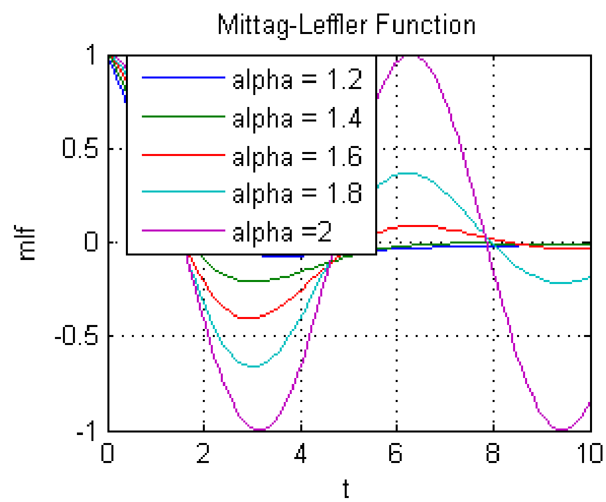


Figure 2-2: Mittag-Leffler Function, $1 < \alpha \leq 2$, $\lambda < 0$

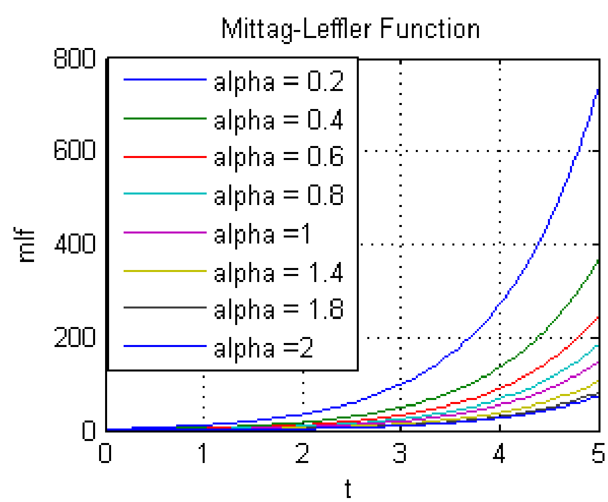


Figure 2-3: Mittag-Leffler Function, $0 < \alpha \leq 2$, $\lambda > 0$

## 2.8 Regular Sturm-Liouville Problems

### 2.8.1 Integer Order Case

Consider the following Regular Sturm-Liouville problem [67],

$$\begin{cases} L[y] = \frac{1}{w(x)} \left\{ \frac{d}{dx} \left[ p(x) \frac{d}{dx} y(x) \right] - q(x)y(x) \right\} = -\lambda y(x) \\ a_1 y(a) + a_2 p(a) y' = 0, \ b_1 y(b) + b_2 p(b) y' = 0, \end{cases}$$

where $p > 0$, $w > 0$, and $p$, $q$, $w$ are continuous functions on interval $[a, b]$ and $a_1^2 + a_2^2 \neq 0$ and $b_1^2 + b_2^2 \neq 0$. $\lambda$ is called an eigenvalue and the corresponding non-trivial solutions $y$ are called eigenfunctions. Below are some properties of the a regular SL problems

1. The eigenvalues are real

2. The eigenfunctions of corresponding to distinct eigenvalues are orthogonal with respect to weight function $w$ on $[a, b]$

3. The eigenvalues of a regular SL problems are simple. Thus an eigenfunction corresponding to an eigenvalue is unique up to a constant multiple.

### 2.8.2 Fractional Order Case

**Theorem 15** *Assume that (* $\frac{1}{2} < \alpha < 1$ *and* $p, q, w_\alpha$ *are given functions such that:* $p$ *is of class* $C^1$ *and* $p(x) > 0$; $q, w_\alpha$ *are continuous), the fractional Sturm–Liouville*

*Problem (FSLP)*

$$\begin{aligned}[\,{}^{C}D_{b-}^{(\alpha)}p(x)\,{}^{C}D_{a+}^{(\alpha)} + q(x)]y(x) &= \lambda w_{\alpha}(x)y(x) \\ y(a) &= y(b) = 0\end{aligned}$$

*has an infinite increasing sequence of eigenvalues* $\lambda^{(1)}$,$\lambda^{(2)}$, ...,$\lambda^{(n)}$,... *and to each eigenvalue* $\lambda^{(n)}$ *there corresponds an eigenfunction* $y^{(n)}$ *which is unique up to a constant factor. Furthermore, eigenfunctions* $y^{(n)}$ *form an orthogonal set of solutions [62]*

## 2.9 Stability of Fractional Differential Equation

Stability analysis is the most important problem when studying fractional differential equations. Recently, many stability results for fractional-order systems were derived, see, for instance, [21], [61]. These stability results are mainly concerned with the linear fractional differential system, a sufficient and necessary condition on asymptotic stability of linear fractional differential system with order $0 < \alpha < 1$ was first given. Then some other research on the stability of fractional-order systems appeared. In 1996, Matignon [21], studied stability of the following fractional differential system involving Caputo derivative

$$^{C}D_0^{\alpha}y(t) = Ay(t),\ 0 < \alpha \leq 1,\ y \in \mathbb{R} \tag{2.53}$$

with initial value $y(0) = y_0$ and $A \in \mathbb{R}^{n\times n}$. The stability of the equilibrium of system (2.53) was first defined and established by Matignon as follows.

**Definition 16** *The autonomous system (2.53) is said to be: (a) stable if and only if for any $y_0$, there exists $\epsilon > 0$ such that $\|y(t)\| \leq \epsilon$ for $t \geq 0$. (b) asymptotically stable if and only if $lim_{t\to\infty}\|y(t)\| = 0$.*

**Theorem 17** *[21], The autonomous system (2.53) is (a) asymptotically stable iff $|\arg(spec(A))| > \frac{\alpha\pi}{2}$. In this case, the components of the state decay towards 0 like $t^{-\alpha}$. (b) stable iff either its asymptotically stable, or those critical eigenvalues which satisfy $|\arg(spec(A))| = \frac{\alpha\pi}{2}$ have geometric multiplicity one. (Here $\arg(spec(A))$ denotes the arguments of the eigenvalues of the square matrix $A$).*

## 2.10 Eigenfunction Expansions and Exact Solution

In this subsection we shall use separation of variables to find the exact solution of a simple fractional PDE system.

Consider the fractional diffusion equation which includes a reaction term with boundary and initial conditions,

$$\begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda_0 u(x,t),\ 0 < x < 1,\ t > 0 \\ &u(x,0) = u_0(x),\ 0 \leq x \leq 1 \\ &u(0,t) = 0,\ u(1,t) = 0,\ t > 0. \end{aligned} \tag{2.54}$$

where $0 < \alpha \leq 1$, $\lambda_0 > 0$ and $u_0$ is a continuous function over $[0, 1]$, Let us find the solution to this fractional initial boundary value problem and determine for which values of the parameter $\lambda_0$ this system is unstable. We assume that the solution $u(x,t)$ can be written as a product of a function of the space variable and a function of the time variable.

$$u(x,t) = X(x)T(t). \tag{2.55}$$

Substitution into fractional PDE, gives,

$$X(x)T^{(\alpha)}(t) = X''(x)T(t) + \lambda_0 X(x)T(t) \tag{2.56}$$

Division by $X(x)T(t)$ gives,

$$\frac{T^{(\alpha)}(t)}{T(t)} = \frac{X''(x) + \lambda_0 X(x)}{X(x)} \tag{2.57}$$

In the above equation the left hand side depends only on time and the right hand side depends on the spatial variable, thus, the equality can hold only if both sides are constant, that is

$$\frac{T^{(\alpha)}(t)}{T(t)} = \frac{X''(x) + \lambda_0 X(x)}{X(x)} = \sigma \quad \text{(constant)}$$

Hence,

$$T^{(\alpha)}(t) = \sigma T(t), \; 0 < \alpha \leq 1, \; t > 0 \tag{2.58}$$

and

$$\begin{cases} X''(x) + (\lambda_0 - \sigma)X(x) = 0, \ 0 < x < 1 \\ X(0) = X(1) = 0 \end{cases} \tag{2.59}$$

That is, we are dealing with a regular Sturm-Liouville problem (2.59) which has the simple eigenvalues $\lambda_0 - \sigma_n = (n\pi)^2$, $n \geq 1$ with corresponding eigenfunctions

$$X_n(x) = A_n \sin(n\pi x), \ n \geq 1, \ A_n \neq 0. \tag{2.60}$$

Now,

$$T_n(t) = a_n E_{\alpha,1}(\sigma_n t^\alpha)$$

where $a_n$ are constant,

Superposition of the product solutions gives,

$$u(x,t) = \sum_{n\geq 1} C_n E_{\alpha,1}(\sigma_n t^\alpha) \sin(n\pi x)$$

that is,

$$u(x,t) = \sum_{n\geq 1} C_n E_{\alpha,1}((\lambda_0 - (n\pi)^2)t^\alpha) \sin(n\pi x) \tag{2.61}$$

where $C_n$ are the Fourier coefficients of $u(x,0) = u_0(x)$ that is

$$C_n = \frac{1}{2}\int_0^1 u_0(x) \sin(n\pi x) dx \tag{2.62}$$

Thus,

$$u(x,t) = 2\sum_{n=1}^{\infty} C_n E_{\alpha,1}((\lambda_0 - (n\pi)^2)t^\alpha)\sin(n\pi x),\ 0 < x < 1 \tag{2.63}$$

Let us look at the structure of the solution. It consists of the following elements:

· eigenvalues: $\lambda_0 - \pi^2 n^2$

· eigenfunctions: $\sin(\pi n x)$

· effect of initial conditions: $\int_0^1 u_0(x)\sin(n\pi x)dx$

The largest eigenvalue, $\lambda_0 - \pi^2(n = 1)$, dictates the rate of growth or decay of the solution if $u_0$ is not orthogonal to $\sin \pi x$ ! If $u_0$ is orthogonal to $\sin \pi x$ then it is $\lambda_0 - (2\pi)^2$, $(n = 2)$ which will dictate the rate of growth or decay. Therefore in the first case we impose $\lambda_0 < \pi^2$ for stability whereas in the second case we need $\lambda_0 < (2\pi)^2$ for stability and in general if $u_0$ is orthogonal to $\{\sin k\pi x,\ k = 1, 2, 3, ...N - 1\}$ and not orthogonal to $\sin(N\pi x)$ then the system is stable if $\lambda_0 < (N\pi)^2$.

## 2.11 Terminology

In this work we shall be using the following function spaces

| Name | Description | Norm |
| --- | --- | --- |
| $C^{(n)}[a,b]$ | $f, f', ..f^{(n)}$ continuous functions on $(a,b)$ | $\|f\|_\infty = \max_x |f(x)|$ |
| $L^1(a,b)$ | Integrable function: $\int |f(x)|\, dx < +\infty$ | $\|f\|_{L^1} = \int |f(x)|\, dx$ |
| $L^2(a,b)$ | Square integrable function: $\int |f(x)|^2\, dx < +\infty$ | $\|f\|_{L^2} = [\int |f(x)|^2\, dx]^{\frac{1}{2}}$ |
| $H^1(a,b)$ | Sobolev space: $f \in L^2$ and $f' \in L^2$ | $\|f\|_{H^1}^2 = \|f(x)\|_{L^2}^2 + \left\|f'(x)\right\|_{L^2}^2$ |

and in general

$H^m(a,b) = \{f \ | f, f', ...., f^{(m)} \in L^2(a,b)\}$ with the norm

$\|f\|^2_{H^m} = \sum_{j=0}^{m} \left\|f^{(j)}(x)\right\|^2_{L^2} = \|f\|^2_{H^{m-1}} + \left\|f^{(m)}\right\|^2_{L^2} = \left\|f'\right\|^2_{H^{m-1}} + \|f\|^2_{L^2}$ .

The inner product in $L^2(a,b)$ and $H^1(a,b)$ are defined as

$$\langle f, g\rangle = \int_a^b f(x)\bar{g}(x)dx$$

and

$$\langle f, g\rangle = \int_a^b [f(x)\bar{g}(x) + f'(x)\bar{g}'(x)]dx$$

respectively.

## 2.12 Existence and Uniqueness

The existence and uniqueness of solutions to initial and boundary-value problems for fractional differential equations has been studied by many authors; see for example [55], [57], [58], [56]. Some of the existence and uniqueness results have been obtained using the well-known Lax-Milgram theorem, and fixed point theorems [60], [59]. Many important results on the existence of solutions of various classes of fractional differential equations were given by Oldham and Spainer [10], Kilbas and Marichev [5], Miller and Ross [9], Podlubny [4] etc.

There is a need to improve or to adapt the existing methods and techniques from the classical case to the fractional one. At the first sight this process seems simple and direct but in fact this is a complicated process and it requires much attention mainly because the fractional calculus requires some additional conditions in order to be well defined.

## 2.13 Volterra Integral Equations

A linear Volterra integral equation (VIE) of the second kind is a functional equation of the form

$$u(x) = g(x) + \int_0^x k(x,y)u(y)dy, \;\; x \in I$$

Where $I := [0,b]$, $D = \{(x,y) : 0 \le y \le x \le b\}$. Here, $g(.)$ and $k(.,.)$ are given functions, and $u(.)$ is an unknown function. The function $k(.,.)$ is called the kernel of the VIE. A linear VIE of the first kind is given by

$$g(x) = \int_0^x k(x,y)u(y)dy, \;\; x \in I$$

Here, the unknown function occurs only under the integral sign. In 1896 Vito Volterra published the first of his fundamental papers on integral equations. It contains the following fundamental result [65], [66]

**Theorem 18** *Assume that kernel $k(.,.)$ of the linear Volterra integral equation*

$$u(x) = g(x) + \int_0^x k(x,y)u(y)dy, \;\; x \in [0,b]$$

*is continuous on $D := \{(x,y) : 0 \le y \le x \le b\}$. Then for any function $g(.)$ that is continuous on $[0,b]$ (that is, $g \in C([0,b])$), the VIE possesses a unique solution*

$u \in C([0,b])$. *This solution can be written in the form*

$$u(x) = g(x) + \int_0^x l(x,y)g(y)dy, \;\; x \in I,$$

*for some* $l \in C(D)$. *The function* $l = l(x,y)$ *is called the resolvent kernel of the given kernel* $k(x,y)$

If we define the integral operator $\mathbb{L} : C[0,b] \to C[0,b]$ by

$$(\mathbb{L}g)(x) := \int_0^x l(x,y)g(y)dy, x \in [0,b]$$

and if we write the VIE in operator form,

$$(I - \mathbb{K}) = g \;\Rightarrow\; u = (I + \mathbb{L})g$$

By last Theorem, the inverse operator $(I - \mathbb{K})^{-1}$ always exists, and hence (by uniqueness of $l(x,y)$)

$$(I - \mathbb{K})^{-1} = I + \mathbb{L}$$

# Chapter 3

# The Time-Fractional Diffusion Equation

## 3.1 Introduction

In this chapter we present the backstepping method to design Dirichlet boundary controllers stabilizing the fractional PDE systems and show that these controllers work "irrespective" of the initial condition. We shall also deal only with boundary control of fractional differential equations and the backstepping approach is particularly well suited for boundary control. The main feature of backstepping is to eliminate the destabilizing effects terms that appear throughout the domain while the control is acting only from the boundary and that is at first a highly surprising result. We pursue a continuum equivalent of this approach and build a change of variables, which involves a volterra integral operator that absorbs the destabilizing terms acting in the domain and allows the boundary control to completely eliminate their effect. One can pursue several different objectives in a control design for fractional PDE systems. If the system is already stable, a typical objective for feedback control would be to improve performance. Optimality methods are natural in such situations. Another control objective is stabilization.

## 3.2 Boundary Control of Time-Fractional Diffusion Equation with Constant Coefficient

In this section we shall apply the Dirichlet boundary control on one end and insulating the other. We also discuss two cases, the uncontrolled system and the boundary control applied to the one-dimensional fractional diffusion equations with Caputo fractional derivatives with respect to time. We shall consider the following fractional reaction–diffusion system with a destabilizing linear term on the right-hand side and Dirichlet boundary conditions

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda_0 u(x,t),\ 0 < \alpha \leq 1,\ 0 < x < 1,\ t > 0 \\ u(x,0) = u_0(x),\ 0 \leq x \leq 1 \\ u(0,t) = 0,\ t > 0 \\ u(1,t) = U(t),\ t > 0 \end{cases} \tag{3.1}$$

where $\alpha$ is the parameter describing the order of the time fractional derivative and $\lambda_0 > 0$. $U(t)$ is the boundary control at the free end of the boundary, $u_0(x)$ is the initial conditions of the displacement. When $\alpha = 1$, the problem (3.1) is reduced to the classical integer order unstable heat equation for positive and large values of $\lambda_0$.

### 3.2.1 The Free Time-Fractional Diffusion Equation (Uncontrolled System)

In this subsection we shall consider the uncontrolled system, where the boundary control is identically zero, $U(t) \equiv 0$,

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda_0 u(x,t), \; 0 < \alpha \leq 1, \lambda > 0, \; 0 \leq x \leq 1, \; t > 0 \\ u(x,0) = u_0(x), \; 0 \leq x \leq 1 \\ u(0,t) = 0, \; u(1,t) = 0, \; t > 0. \end{cases} \tag{3.2}$$

The general solution of (3.2) can be obtained by separation of variables as follows: let $u(x,t) = X(x)T(t)$ then we have

$$X(x)T^{(\alpha)}(t) = X''(x)T(t) + \lambda_0 X(x)T(t) \tag{3.3}$$

from which we get

$$\frac{T^{(\alpha)}(t)}{T(t)} = \frac{X''(x) + \lambda_0 X(x)}{X(x)} = -\mu \text{ (constant)}. \tag{3.4}$$

Hence, $T$ must satisfies the fractional ordinary differential equation

$$T^{(\alpha)}(t) = -\mu T(t), \; t > 0 \tag{3.5}$$

and $X$ satisfies the regular Sturm-Liouville problem

$$\begin{cases} -X''(x) - \lambda_0 X(x) = \mu X(x), \ 0 < x < 1 \\ X(0) = X(1) = 0 \end{cases} \tag{3.6}$$

with simple eigenvalues are $\mu_k = (k\pi)^2 - \lambda_0, \ k \geq 1$ and corresponding normalized eigenfunctions $X_k(x), k \geq 1, \left( \|X_k\|^2 = \int_0^1 |X_k(x)|^2 \, dx = 1 \right)$

$$X_k(x) = \sqrt{2} \sin(k\pi x), \ k \geq 1. \tag{3.7}$$

For each $k \geq 1$ we have

$$T_k(t) = d_k E_\alpha(-\mu_k t^\alpha), \ k \geq 1 \tag{3.8}$$

where $E_\alpha(z)$ is the Mittag-Leffler function, and $d_k$ are constants. Superposition of the product solutions gives,

$$u(x,t) = \sum_{k \geq 1} c_k E_\alpha(-\mu_k t^\alpha) X_k(x) \tag{3.9}$$

where $c_k$ are the Fourier coefficients of $u(x,0) = u_0(x)$ that is

$$c_k = 2 \int_0^1 u_0(x) X_k(x) dx. \tag{3.10}$$

Therefore, we have shown that the solution to the uncontrolled system is

$$u(x,t) = 2 \int_0^1 g(x, \xi, t) u_0(\xi) d\xi \tag{3.11}$$

where

$$g(x,\xi,t) = \sum_{k\geq 1} E_\alpha(-\mu_k t^\alpha) X_k(x) X_k(\xi). \tag{3.12}$$

We can see that a negative eigenvalues $\mu_k$ will make the solution of the uncontrolled system increase without bounds if the initial data $u_0$ is not orthogonal to the corresponding eigenfunction $X_{k_0}$. Hence, it follows that,

**Theorem 19** *Let $\{\mu_k, X_k\}$ be the sequence of eigenvalues and associated (normalized) eigenfunctions of the Sturm-Liouville problem( 3.6 ) if $\lambda_0 > (k_0\pi)^2$ gives $\mu_k < 0$, for some integer $k_0$then the uncontrolled system ( 3.2) is unstable if the initial data $u_0$ is not orthogonal to $X_{k0}$*

### 3.2.2 Boundary Control of Time-Fractional Diffusion Equation (Controlled System)

In controlled system, one-end of the rod is insulated where as the other end is regulated according to the measurement of averaged temperature over the whole rod. Physically, if the destabilizing heat is generated inside the rod, then we cool that controlled end to avoid over heating. In this subsection we shall apply the method of transmutation or transformation operator or back-stepping method. The objective of this method is to eliminate all unwanted terms from the equation. In other words, we want the closed-loop system to have the form of the target system. We shall transform the original system (3.1) into a target system which has some desired properties.

**Definition 20** *Let $L_0$ and $L_1$ be operators. The operator $T$ is a transmutation for* *the pair of operators* $(L_0, L_1)$ *if*

$$TL_0 = L_1 T$$

*and* $T$ *is a bounded, invertible with bounded inverse*

**Theorem 21** *Let* $L_0 = \frac{\partial^2}{\partial x^2}$ *and* $L_1 = \frac{\partial^2}{\partial x^2} + \lambda_0$ *where* $\lambda_0$ *is a given constant. Let* $\mathbb{K} : H^2(0,1) \to H^2(0,1)$ *be the operator defined by,*

$$(\mathbb{K})u(x,t) = \int_0^x k(x,y)u(y,t)dy \tag{3.13}$$

*where* $k$ *solves the Goursat problem,*

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = \lambda_0 k(x,y), \ 0 \le y \le x \le 1 \\ k(x,x) = -\frac{\lambda_0}{2}x, \ 0 \le x \le 1 \\ k(x,0) = 0, \ 0 \le x \le 1 \\ k(0,0) = 0 \end{cases} \tag{3.14}$$

*Then the operator* $I - \mathbb{K}$ *is a transmutation operator for the pair of operators* $\{L_0, L_1\}$*. In fact,*

$$(I - \mathbb{K})^{-1} = I + \mathbb{L} \text{ where } \mathbb{L} : H^2(0,1) \to H^2(0,1),$$

*is defined by*

$$(\mathbb{L}w)(x,t) = \int_0^x l(x,y)w(y,t)dy \tag{3.15}$$

*where $l$ solves the Goursat problem,*

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -\lambda_0 l(x,y), \ 0 \le y \le x \le 1 \\ l(x,x) = -\frac{\lambda_0}{2}x, \ 0 \le x \le 1 \\ l(x,0) = 0, \ 0 \le x \le 1 \\ l(0,0) = 0 \end{cases} \tag{3.16}$$

Proof: it has been proved for integer derivative by A.Smyshlyaev and M. Krstic [57], [40]

**Lemma 22** *[57]: The problems (3.14), (3.16) have unique solutions which are twice continuously differentiable in $0 \le y \le x \le 1$ and defined respectively by*

$$k(x,y) = -\lambda_0 y \frac{\mathcal{I}_1\sqrt{\lambda_0(x^2-y^2)}}{\sqrt{\lambda_0(x^2-y^2)}} \text{ and } l(x,y) = -\lambda_0 y \frac{\mathcal{J}_1\sqrt{\lambda_0(x^2-y^2)}}{\sqrt{\lambda_0(x^2-y^2)}}$$

*where $\mathcal{I}_1(z)$ is a first-order modified Bessel function of the first kind (2.20) and $\mathcal{J}_1(iz) = -i\mathcal{I}_1(z)$*

We shall transform the system (3.1), by applying $I - \mathbb{K}$ to both sides of the fractional PDE in (3.1). we get,

$$(I-\mathbb{K})\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = (I-\mathbb{K})[(\frac{\partial^2}{\partial x^2} + \lambda_0)u(x,t)], 0 < \alpha \le 1, \ 0 < x < 1, \ t > 0$$

interchanging $I - \mathbb{K}$ and $\frac{\partial^\alpha}{\partial t^\alpha}$ we obtain,

$$\frac{\partial^\alpha}{\partial t^\alpha}(I - \mathbb{K})u = (I - \mathbb{K})(\frac{\partial^2}{\partial x^2} + \lambda)u$$

$$\frac{\partial^\alpha}{\partial t^\alpha}(I - \mathbb{K})u = (I - \mathbb{K})\mathbb{L}_1 u$$

$$\frac{\partial^\alpha}{\partial t^\alpha}(I - \mathbb{K})u = \mathbb{L}_0(I - \mathbb{K})u$$

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2}$$

As for the boundary and initial conditions, we have

$$(I - \mathbb{K})u(0,t) = w(0,t) = 0 \text{ and } (I - \mathbb{K})u(x,0) = (I - \mathbb{K})u_0(x) = w_0(x)$$

Taking the other boundary condition to be $w(1,t) = 0$, leads to the target system for $w = (I - \mathbb{K})u$,

$$\begin{cases} \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial t^2},\ 0 < \alpha \leq 1, 0 < x < 1,\ t > 0 \\ w(0,t) = 0,\ w(1,t) = 0,\ t > 0 \\ w(x,0) = w_0(x),\ 0 < x < 1 \end{cases} \tag{3.17}$$

The boundary control in this case is obtained as follows. Since $u(x,t) = (I + \mathbb{L})w(x,t)$. It follows, $w(1,t) = 0$ and $u(1,t) = U(t)$ that is,

$$U(t) = \int_0^1 k(1,y)u(y,t)dy$$

Returning to the target system (3.17) separation of variables $w(x,t) = X(x)T(t)$ leads to the solution

$$w(x,t) = \sum_{k\geq 1} c_k E_\alpha(-(k\pi)^2)t^\alpha) X_k(x) \tag{3.18}$$

where $c_k$ are the Fourier coefficients of $w(x,0) = w_0(x)$ given by

$$c_k = \int_0^1 w_0(x)\sin(k\pi x)dx.$$

Replacing $c_k$ by their values in (3.18), we obtain the solution of the target system as

$$w(x,t) = \int_0^1 g(x,\xi,t)w_0(\xi)d\xi \tag{3.19}$$

where

$$g(x,\xi,t) = \sum_{k\geq 1} E_\alpha(-(k\pi)^2 t^\alpha)\sin(k\pi\xi)\sin(k\pi x). \tag{3.20}$$

The decay rate of the solution to zero is dictated by the first eigenvalues $\mu_{k_0} = (k_0\pi)^2$. Provided the initial data $w_0$ is not orthogonal to the corresponding eigenfunction $X_{k_0}$. Hence, we have the following,

**Theorem 23** *If $\mu_{k_0} = (k_0\pi)^2$, a boundary control which stabilizes (3.1) is given by*

$$U(t) = -\int_0^1 \lambda_0 y \frac{\mathcal{I}_1\sqrt{\lambda_0(1^2 - y^2)}}{\sqrt{\lambda_0(1^2 - y^2)}} u(y,t)dy$$

*where*

$$u(x,t) = w(x,t) - \lambda_0 \int_0^x y \frac{\mathcal{J}_1\sqrt{\lambda_0(x^2 - y^2)}}{\sqrt{\lambda_0(x^2 - y^2)}} w(x,t)dy.$$

*and $w(x,t)$ is the solution of (3.17)*

**Remark 1** *Note that the above choice for target system is good for several reasons: first, it's simple, well studied equation, which allows us to avoid any issues with well-posedness of the closed-loop system. Second, this equation is explicitly solvable, so that the exact closed loop eigenvalues are known and explicit closed-loop solutions can be obtained.*

## 3.3 Boundary Control of Time-Fractional Diffusion Equation with Space Dependent Coefficient

In this section we shall consider the following one-dimensional fractional diffusion equations with Caputo fractional derivative with respect to time.

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x)u(x,t),\ 0 < \alpha \leq 1; \lambda > 0, 0 \leq x \leq 1, t > 0 \\ u(x,0) = u_0(x), 0 \leq x \leq 1 \\ u(0,t) = 0, t > 0 \\ u(1,t) = U(t), t > 0 \end{cases} \tag{3.21}$$

where $\alpha$ is the parameter describing the order of the time fractional derivative, $u(x,t)$ is the displacement of the $(x,t) \in [0,1] \times [0,\infty)$, $U(t)$ is the boundary control at the free end of the boundary, $u_0(.) \in C[0,1]$ is the initial condition of the displacement and $q \in C^1[0,1]$.

### 3.3.1 The Free Time-Fractional Diffusion Equation (Uncontrolled System)

In this subsection we shall consider the uncontrolled system, $U(t) \equiv 0$,

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x)u(x,t),\ 0 < \alpha \leq 1,\ 0 < x < 1,\ t > 0 \\ u(x,0) = u_0(x),\ 0 \leq x \leq 1 \\ u(0,t) = 0,\ u(1,t) = 0,\ t > 0 \end{cases} \tag{3.22}$$

Separation of variables leads to the solution of the problem (3.22) as

$$u(x,t) = 2\int_0^1 g(x,\xi,t)u_0(\xi)d\xi \tag{3.23}$$

where

$$g(x,\xi,t) = \sum_{k\geq 1} E_\alpha(-\mu_k t^\alpha)X_k(x)X_k(\xi). \tag{3.24}$$

We can see that a negative eigenvalues $\mu_k$ will make the solution of the uncontrolled system increase without bounds if the initial data $u_0$ is not orthogonal to the corresponding eigenfunction $X_{k_0}$.

**Theorem 24** *Let $\{\mu_k, X_k\}$ be the sequence of eigenvalues and  and associated (normalized) eigenfunctions of the Sturm-Liouville problem if $\mu_k < 0$, for some integer $k_0$ then the uncontrolled system ( 3.22) is unstable if the initial data $u_0$ is not orthogonal to $X_{k_0}$*

### 3.3.2 Boundary Control of Time-Fractional Diffusion Equation (Controlled System)

In this subsection we shall again apply the method of transmutation operator. One of the main key of the transmutation (back-stepping) method is to transform a difficult system with nonhomogeneous boundary conditions to a simpler one with homogeneous boundary condition to which seperation of variables can be applied. We shall transform the original system (3.21) into a target system (3.30) which has some desired properties and then summarize the results in the following theorem

**Theorem 25** *Let* $L_0 = \frac{\partial^2}{\partial x^2} + q(x)$ *and* $L_1 = \frac{\partial^2}{\partial x^2} + \lambda_0$, *where* $q \in C^1([0,1])$ *and* $\lambda_0$ *is a given constant.*

*Let*

$$\mathbb{K} : H^2(0,1) \to H^2(0,1) \tag{3.25}$$

*be the operator defined by,*

$$(\mathbb{K})u(x,t) = \int_0^x k(x,y)u(y,t)dy \tag{3.26}$$

*where* $k$ *solves the Goursat problem,*

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = [q(x) - \lambda_0]k(x,y), \ 0 \le y \le x \le 2 \\ \frac{d}{dx}k(x,x) = -\frac{1}{2}[q(x) - \lambda_0], \ 0 \le x \le 1 \\ k(x,0) = 0, \ 0 \le x \le 1 \\ k(0,0) = 0 \end{cases} \tag{3.27}$$

*Then the operator $I-\mathbb{K}$ is a transmutation operator for the pair of operators $\{L_0, L_1\}$. In fact*

$$(I-\mathbb{K})^{-1} = I + \mathbb{L} \text{ where } \mathbb{L} : H^2(0,1) \to H^2(0,1),$$

*is defined by*

$$(\mathbb{L}w)(x,t) = \int_0^x l(x,y)w(y,t)dy \tag{3.28}$$

*where $l$ solves the Goursat problem,*

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) = -[q(x)-\lambda_0]l(x,y), \ 0 \le y \le x \le 1 \\ \frac{d}{dx}l(x,x) = -\frac{1}{2}\left[q(x)-\lambda_0\right], \ 0 \le x \le 1 \\ l(x,0) = 0, \ 0 \le x \le 1 \\ l(0,0) = 0 \end{cases} \tag{3.29}$$

The Goursat problems (3.27) and (3.29), have been shown to have unique solution that are twice continuously differentiable in $0 \le y \le x \le 1$ provided $q \in C^1([0,1])$. The theorem has been proved in the case of the boundary control of integer order diffusion equation in [37]

**Remark 2** *[37] The kernels PDEs (3.27), (3.29) can be shown to be well posed but, unlike the gain kernel equations for plants we considered before, it cannot be solved in closed form. However, one can solve it either symbolically, using the recursive procedure similar to the one given in (3.13) ,(3.15) or numerically with finite difference schemes.*

We shall transform the system (3.21), by applying $I-\mathbb{K}$ to both sides of the PDE in (3.21),

$$(I-\mathbb{K})\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = (I-\mathbb{K})\left[\left(\frac{\partial^2}{\partial x^2}+q\right)u(x,t)\right],$$

interchanging $I-\mathbb{K}$ and $\partial^\alpha/\partial t^\alpha$ we obtain,

$$\frac{\partial^\alpha}{\partial t^\alpha}(I-\mathbb{K})u = (I-\mathbb{K})\mathbb{L}_0 u$$
$$\frac{\partial^\alpha}{\partial t^\alpha}(I-\mathbb{K})u = \mathbb{L}_1(I-\mathbb{K})u$$
$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2} + \lambda_0 w(x,t)$$

As for the boundary conditions, we proceed as follows. Since, $u = (I+\mathbb{L})w$ and $w = (I-\mathbb{K})u$ we have

$$(I-\mathbb{K})u(x,0) = w_0(x) \text{ and } (I-\mathbb{K})u(0,t) = w(0,t) = 0$$

Taking the other boundary condition to be $w(1,t)=0$, we obtain the target system $w = (I-\mathbb{K})u$,

$$\begin{cases} \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial t^2} + \lambda_0 w(x,t),\ 0<x<1,\ \ t>0 \\ w(0,t)=0,\ \ w(1,t)=0,\ \ t>0 \\ w(x,0)=w_0(x),\ 0<x<1 \end{cases} \tag{3.30}$$

where $w_0(x)$ is a continuous function. The boundary control in this case is

$$U(t) = \int_0^1 l(1,y)w(y,t)dy \tag{3.31}$$

Returning to the target system (3.30). Separation of variables $w(x,t) = X(x)T(t)$, leads to the solution

$$w(x,t) = \sum_{k\geq 1} c_k E_\alpha(-((k\pi)^2 - \lambda_0)t^\alpha)X_k(x) \tag{3.32}$$

where $c_k$ are the Fourier coefficients of $w(x,0) = w_0(x)$ given by

$$c_k = \int_0^1 w_0(x)\sin(k\pi x)dx$$

Replacing $c_k$ in (3.32), we obtain

$$w(x,t) = \int_0^1 g(x,\xi,t)w_0(\xi)d\xi$$

where we have taken

$$g(x,\xi,t) = \sum_{k\geq 1} E_\alpha((\lambda_0 - (k\pi)^2)t^\alpha)\sin(k\pi\xi)\sin(k\pi x). \tag{3.33}$$

Thus, we have,

**Theorem 26** *If* $0 < \lambda_0 < (k_0\pi)^2$, *a boundary control which stabilizes (3.21) is given by*

$$U(t) = \int_0^1 k(1,y)u(y,t)dy$$

*where*

$$u(x,t) = w(x,t) + \int_0^x l(x,y)w(x,t)dy. \tag{3.34}$$

*$w(x,t)$ is the solution of (3.30)*

## 3.4 Examples and Simulation for Fractional Diffusion Equations

In this section we shall give several examples to illustrate the effectiveness of the method. In each examples, we shall plot the boundary control $U$ and the solution $u$ of the given system, the solution $u$ of the free system (i.e., uncontrolled $U(t) = 0$) and finally, the solution $w$ of the target system.

**Example 1:**

$$u_0(x) = \begin{cases} 2x, & x \leq 0.5 \\ 2(1-x), & 0.5 < x \leq 1 \end{cases}, \ \lambda_0 = 12, x \in [0,\ 1], t \in [0,\ 5],\ \alpha = 0.9$$

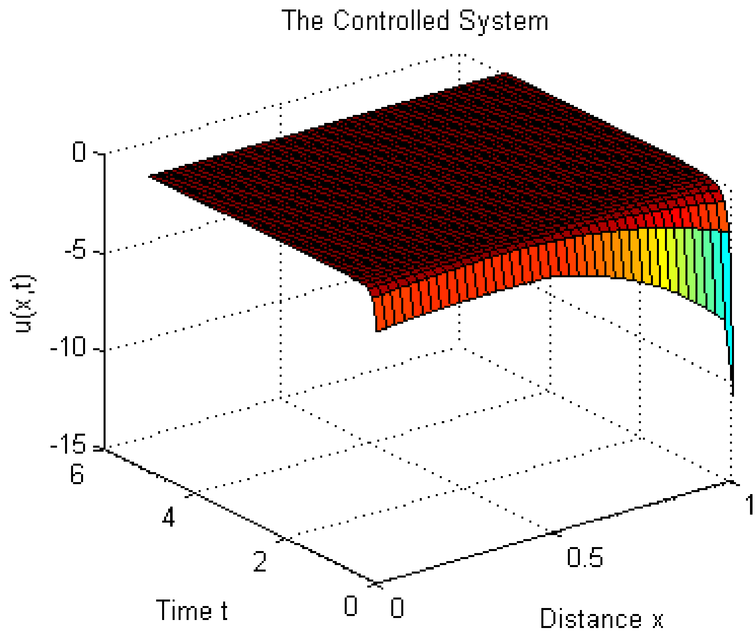


Figure 3-1: Controlled system for Example 1

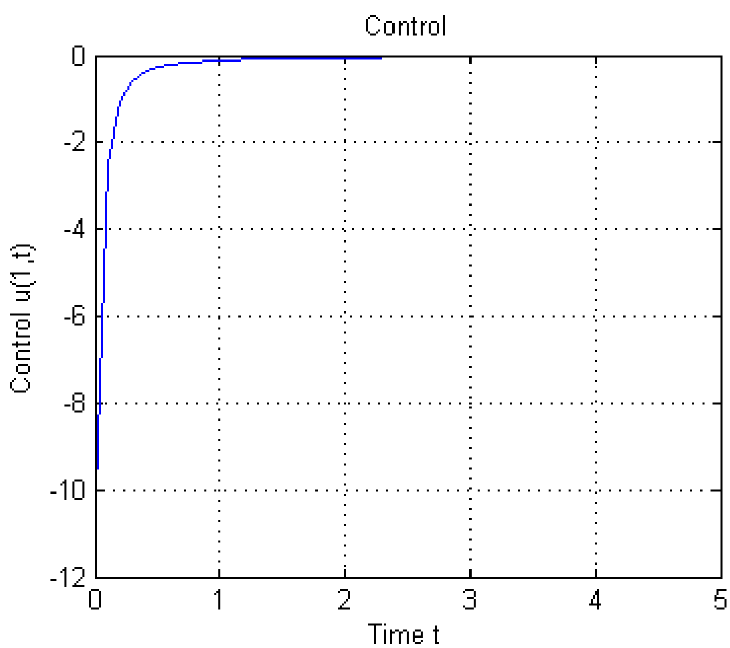


Figure 3-2: $U(t)$ Control for Example 1

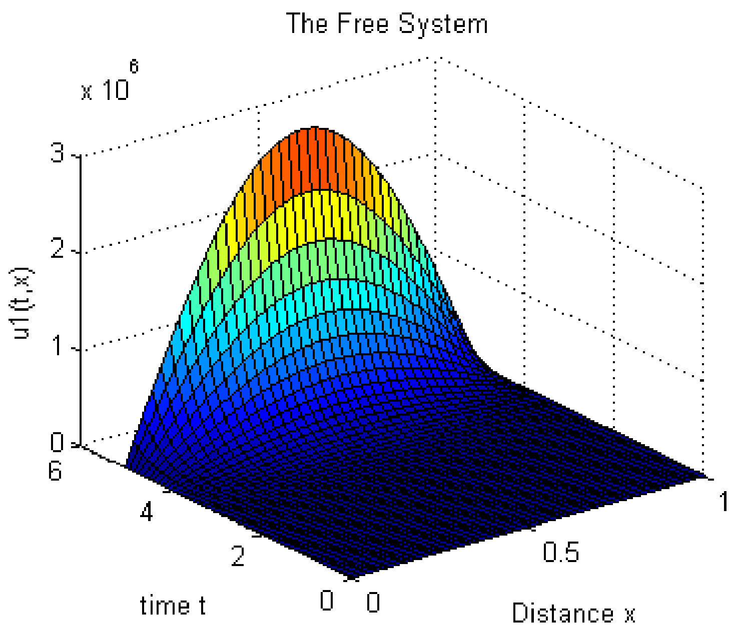


Figure 3-3: Free system for Example 1

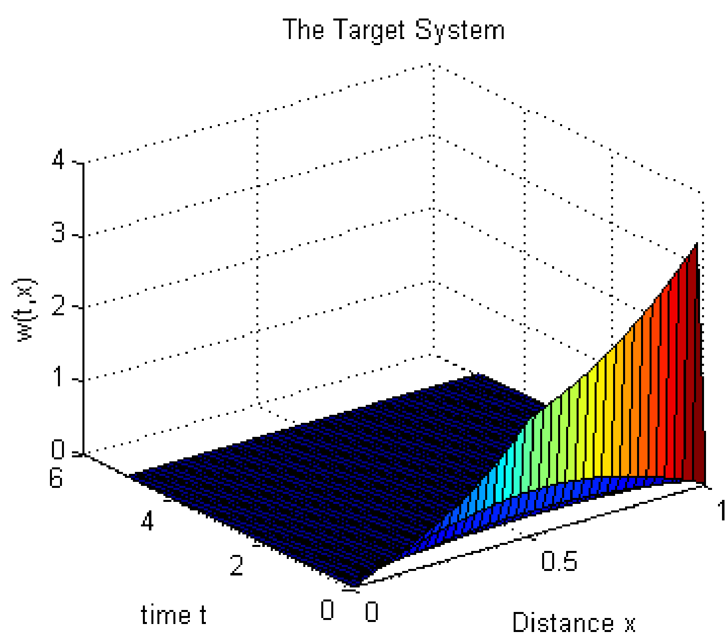


Figure 3-4: Target system for Example 1

**Example 2:**

$$u_0(x) = \sin(\pi x),\ \ \lambda_0 = 12,\ \ x \in [0,\ 1], t \in [0,\ 5],\ \ \alpha = 0.5$$

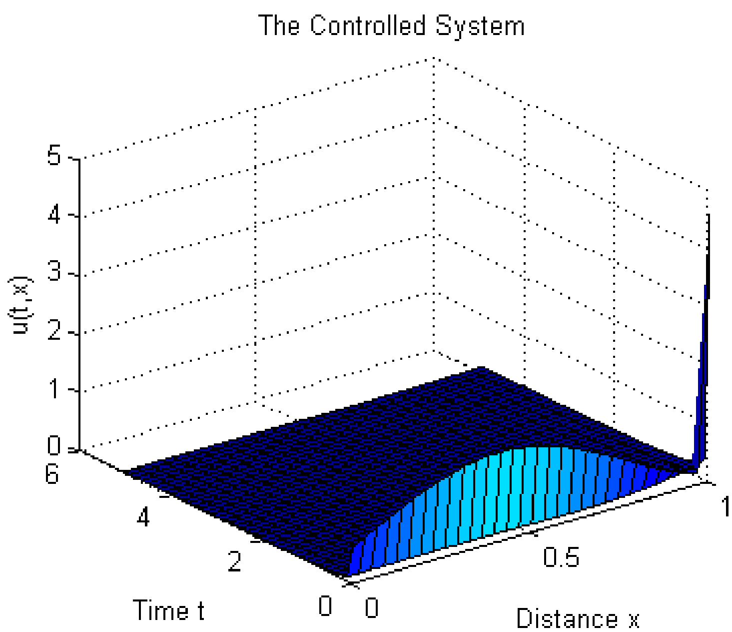


Figure 3-5: Controlled system for Example 2

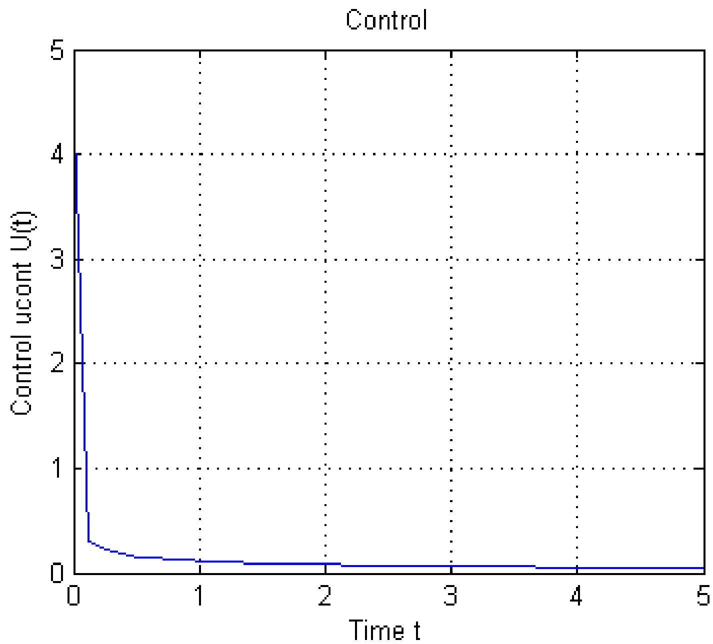


Figure 3-6: $U(t)$ Control for Example 2

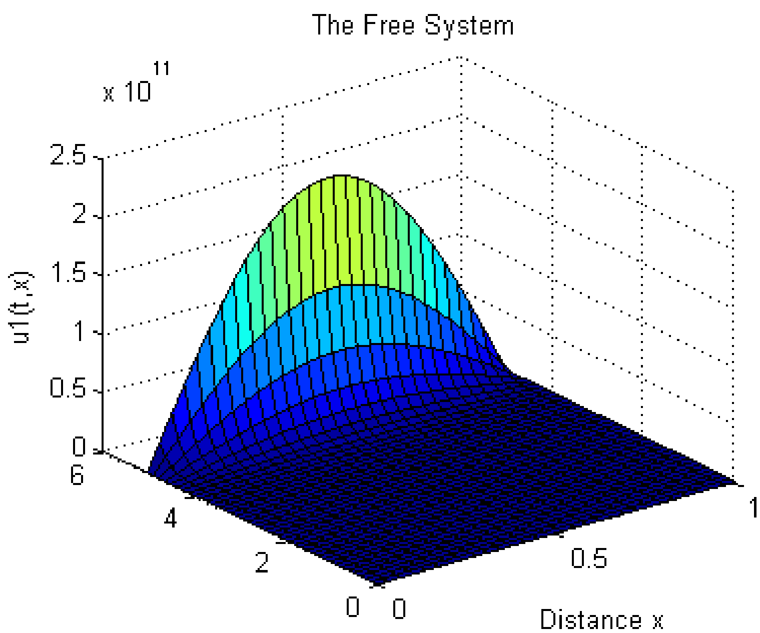


Figure 3-7: Free system for Example 2

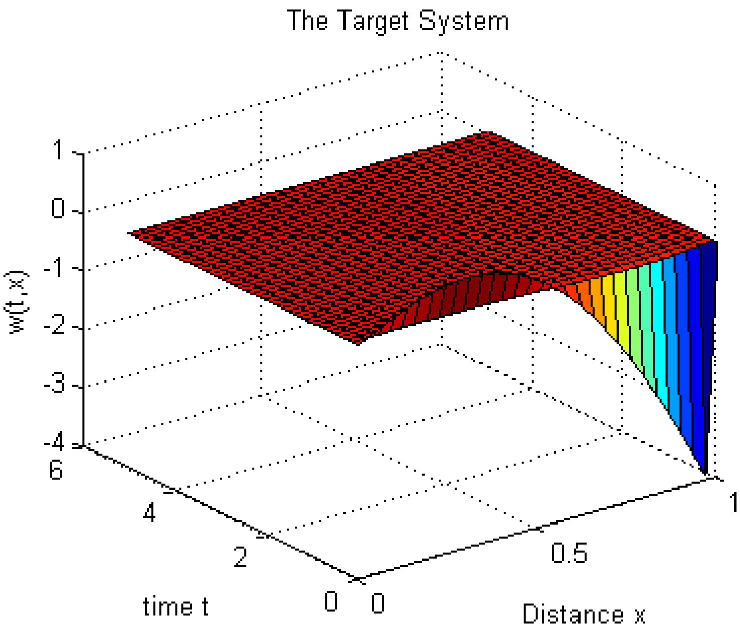


Figure 3-8: Target system for Example 2

**Example 3:**

$$u_0(x) = \sin(\pi x),\ \lambda_0 = 5,\ t \in [0,\ 10],\ \alpha = 0.75,\ q(x) = -11(x^2 + 1)$$

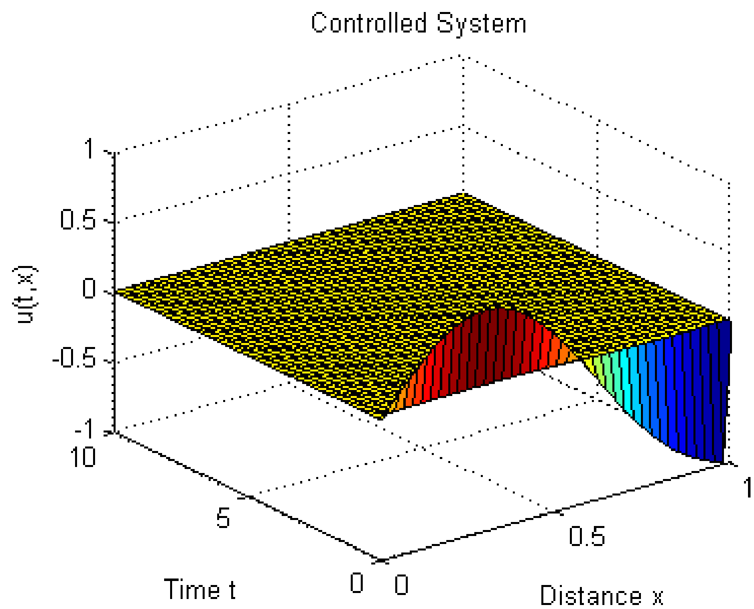


Figure 3-9: Controlled System for Example 3

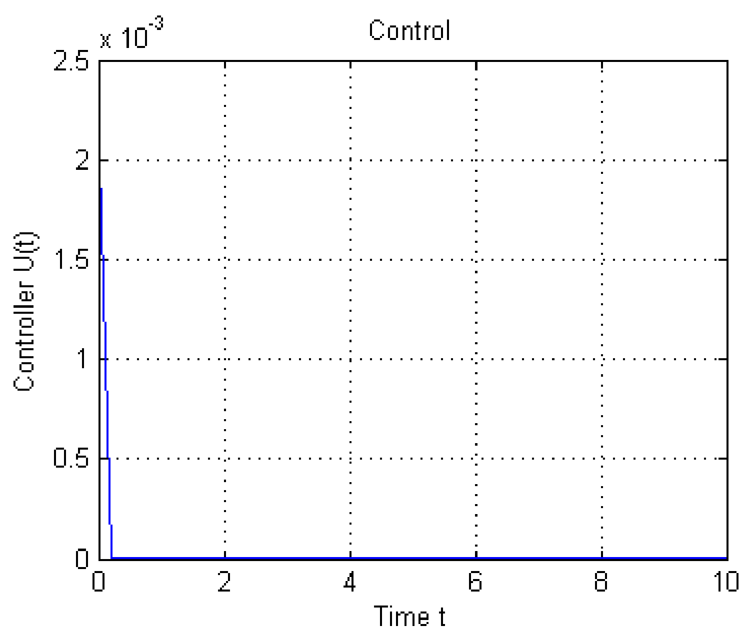


Figure 3-10: $U(t)$ Control for Example 3

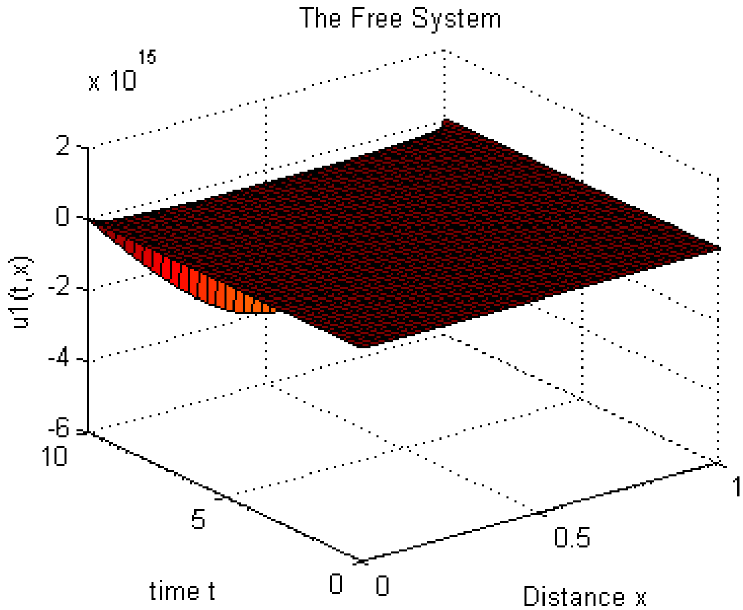


Figure 3-11: Free System for Example 3

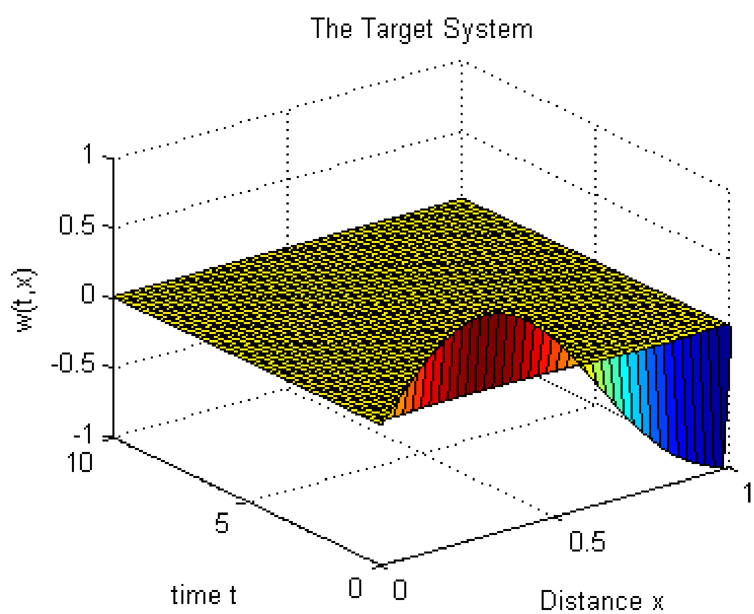


Figure 3-12: Target System for Example 3

**Example 4:**

$$u_0(x) = \sin(\pi x),\ \lambda_0 = 5,\ x \in [0,\ 1],\ t \in [0,\ 5],\ \alpha = 0.75,\ q(x) = -20\cos(x)$$

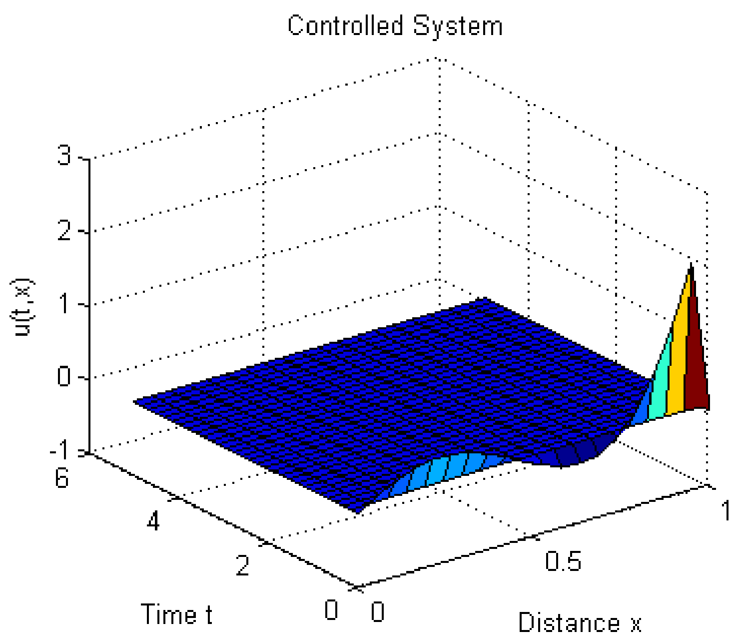


Figure 3-13: Controlled System for Example 4

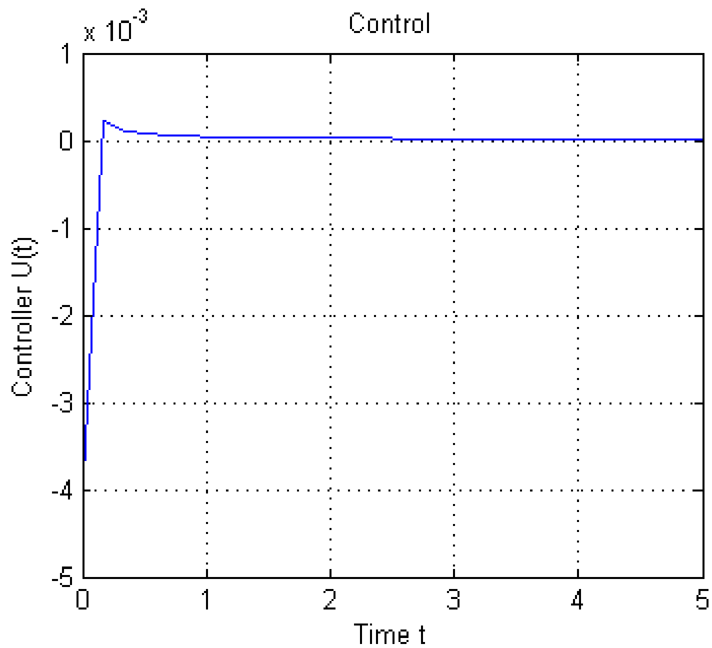


Figure 3-14: $U(t)$ Control for Example 4

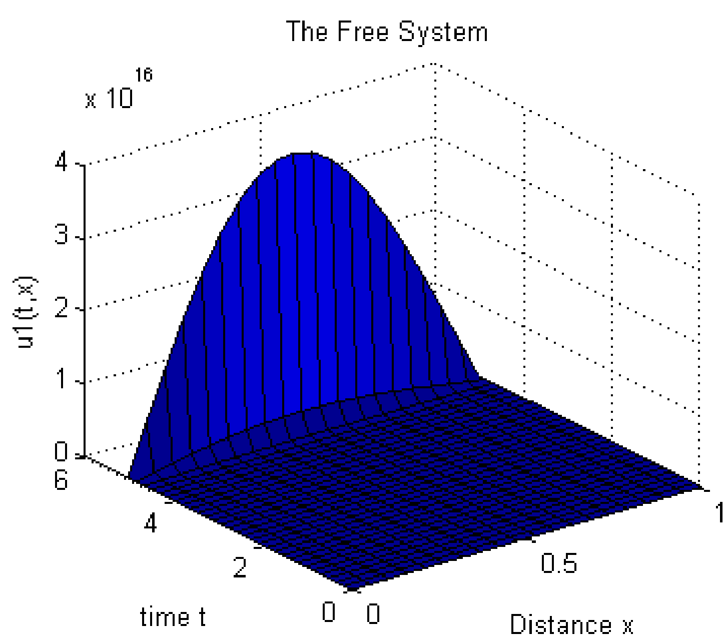


Figure 3-15: Free System for Example 4

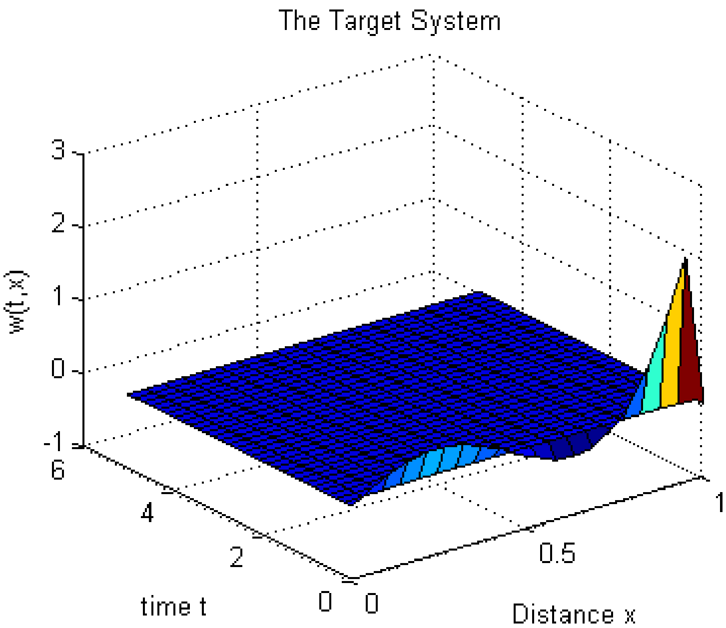


Figure 3-16: Target System for Example 4

**Example 5:**

$$
\begin{aligned}
u_0(x) &= \begin{cases} \frac{4}{3}x & x \leq \frac{3}{4} \\ 4(1-x), & \frac{3}{4} < x \leq 1 \end{cases} \\
\lambda_0 &= 5,\ x \in [0,\ 1],\ t \in [0,\ 2],\ \alpha = 0.6,\ q(x) = -10(x+1)
\end{aligned}
$$

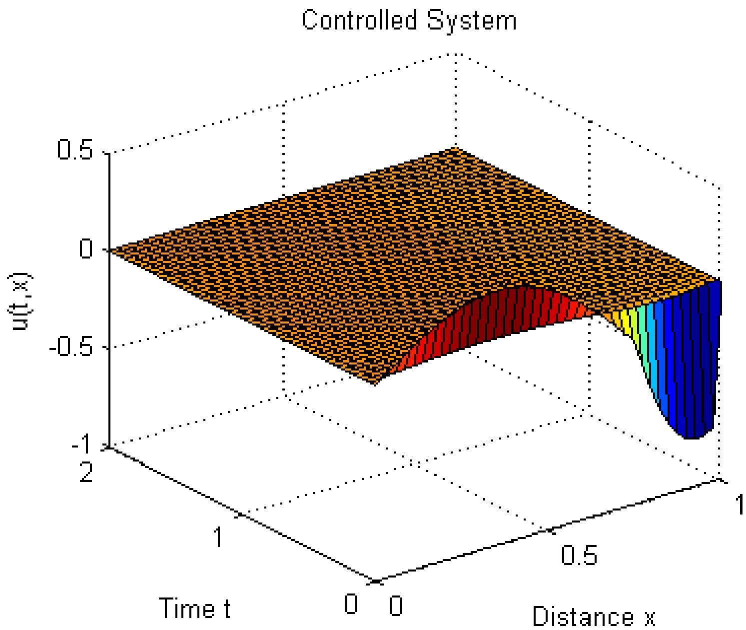


Figure 3-17: Controlled System for Example 5

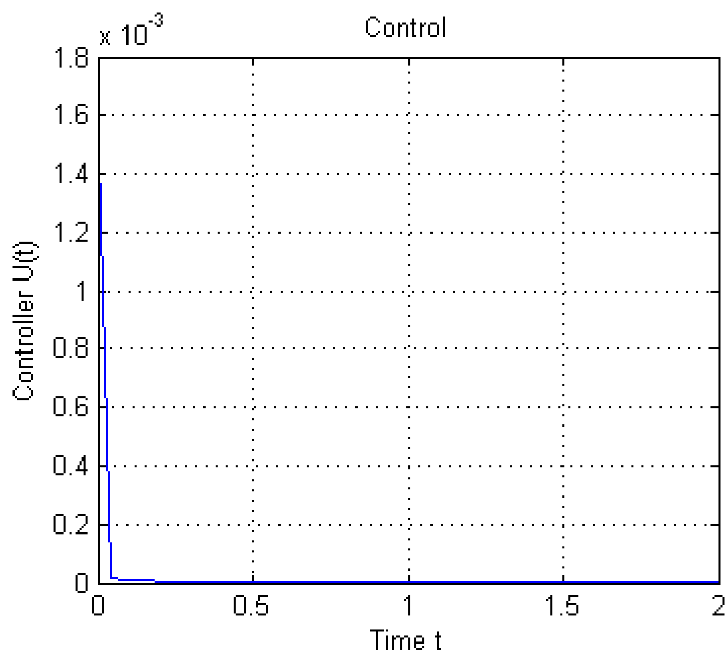


Figure 3-18: $U(t)$ Control for Example 5

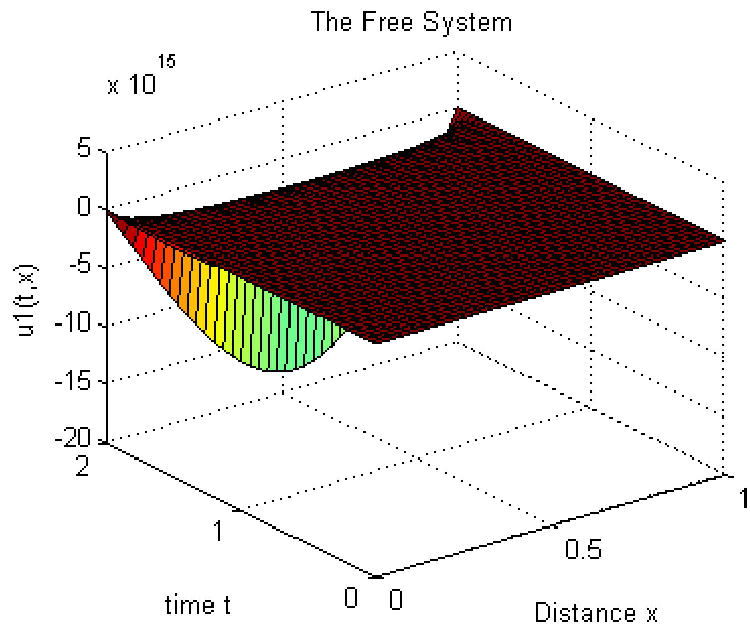


Figure 3-19: Free System for Example 5

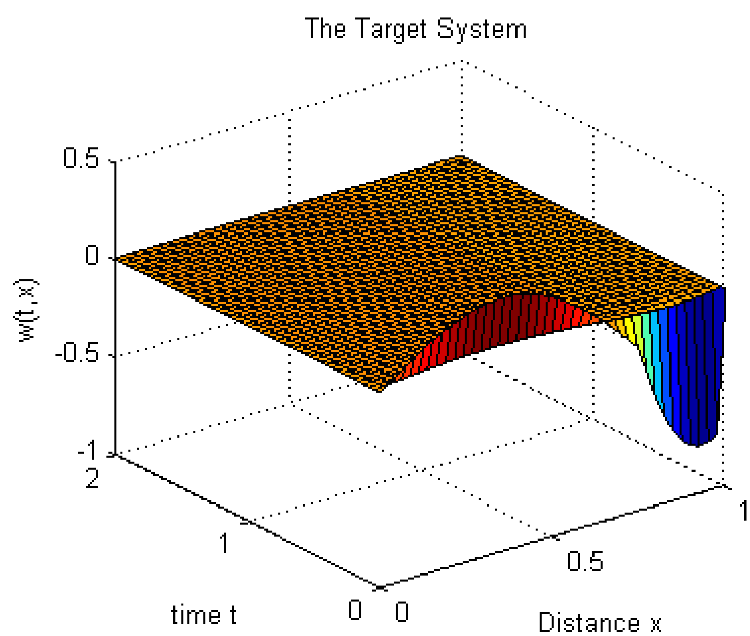


Figure 3-20: Target System for Example 5

# Chapter 4

# The Time-Fractional Wave Equation

## 4.1 Introduction

In this chapter, we shall consider the boundary control of fractional hyperbolic partial differential equations, more specifically the time fractional wave equation which model various oscillatory phenomena, i.e., string vibrations. We shall use again the back-stepping method. The main feature of a wave fractional equation is that the order derivative of time $\alpha$ is between $1 < \alpha \leq 2$ and the solution is oscillatory. The objective of stabilization is to annihilate the effect of perturbation of the system state in order to steer the system state to a given desired trajectory. For this purpose feedback laws are introduced, that allow to react to deviations of the system state from the desired trajectory. Since the deviations are a-priori unknown, the feedback laws must be well defined for all possible system states.

## 4.2 Boundary Control of Time-Fractional Wave Equation with Constant Coefficients

The control for wave fractional systems are typically developed to damp out their oscillations. We shall consider one-dimensional fractional wave equations with fractional derivatives with respect to time and constant coefficients, fixed at one end stabilized by boundary controller at the other end. The system can be represented by

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda_0 u(x,t), \ 0 \le x \le 1, \ t > 0 \\ u(x,0) = u_0(x), \ 0 \le x \le 1 \\ u_t(x,0) = u_1(x), \ 0 \le x \le 1 \\ u(0,t) = 0, \ t > 0 \\ u(1,t) = U(t), \ t > 0 \end{cases} \tag{4.1}$$

where $1 < \alpha \le 2$, is the parameter describing the order of the time fractional derivative. $U(.)$ is the boundary control at the free end of the boundary, $u_0(.)$ is the initial condition of displacement and $u_1(.)$ is velocity, $\lambda_0$ is an arbitrary positive constant and both $u_0$ and $u_1 \in C\left([0,1]\right)$

### 4.2.1 The Free Time-Fractional Wave Equation (Uncontrolled System)

In this subsection we shall consider the uncontrolled system, $U(t) \equiv 0$,

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \lambda_0 u(x,t),\ 1 < \alpha \leq 2,\ \lambda_0 > 0,\ 0 \leq x \leq 1,\ t > 0 \\ u(x,0) = u_0(x),\ 0 \leq x \leq 1 \\ u_t(x,0) = u_1(x),\ 0 \leq x \leq 1 \\ u(0,t) = 0,\ t > 0 \\ u(1,t) = 0,\ t > 0 \end{cases} \tag{4.2}$$

Separation of variables $u(x,t) = X(x)T(t)$ leads to the solution,

$$u(x,t) = \sum_{k \geq 1} \left[ b_{1k} E_{\alpha,1}(-((k\pi)^2 - \lambda_0)t^\alpha) + b_{2k} t E_{\alpha,2}(-((k\pi)^2 - \lambda_0)t^\alpha) \right] \sin(k\pi x) \tag{4.3}$$

where $b_{1n}$ and $b_{2k}$ are the Fourier coefficients of $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$ respectively given by

$$\begin{aligned} b_{1k} &= 2\int_0^1 u_0(x) \sin(k\pi x) dx \\ b_{2k} &= 2\int_0^1 u_1(x) \sin(k\pi x) dx. \end{aligned}$$

When $\lambda_0$ is large and positive the solution $u(x,t)$ grows to infinity without bound.

### 4.2.2 Boundary Control of Time-Fractional Wave Equations with Constant Coefficients (Controlled System)

In this subsection we introduce the main ideas for backstepping control of fractional wave equations. We shall use the same transmutations (3.13), (3.15) as for the fractional diffusion equation with constant coefficients defined in Theorem (21).

Now transform the system (4.1), by applying $I - \mathbb{K}$ to both sides of the PDE in (4.1).

$$(I - \mathbb{K})\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}u(x,t) = (I - \mathbb{K})[(\frac{\partial^2}{\partial x^2} + \lambda_0)u(x,t)],$$

interchanging $I - \mathbb{K}$ and $\partial^\alpha/\partial t^\alpha$ we obtain,

$$\frac{\partial^\alpha}{\partial t^\alpha}(I - \mathbb{K})u = (I - \mathbb{K})\mathbb{L}_1 u$$

$$\frac{\partial^\alpha}{\partial t^\alpha}(I - \mathbb{K})u = \mathbb{L}_0(I - \mathbb{K})u$$

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2}.$$

leading to the target system,

$$\begin{cases} \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial t^2},\ 1 < \alpha \leq 2,\ 0 < x < 1,\ t > 0 \\ w(0,t) = 0,\ w(1,t) = 0,\ t > 0 \\ w(x,0) = w_0(x) = (I - \mathbb{K})u_0(x),\ 0 < x < 1 \\ w_t(x,0) = w_1(x) = (I - \mathbb{K})u_1(x),\ 0 < x < 1 \end{cases} \quad (4.4)$$

where $w_0(.)$ and $w_1(.)$ are continuous functions. defined by,

$$\begin{aligned} w_0(x) &= u_0(x) - \int_0^x k(x,y)u_0(y)dy, \\ w_1(x) &= u_1(x) - \int_0^x k(x,y)u_1(y)dy. \end{aligned}$$

The boundary control in this case is

$$u(1,t) = \int_0^1 k(1,y)u(y,t)dy.$$

Returning to the target system (4.4) separation of variables $w(x,t) = X(x)T(t)$ leads to the solution given by,

$$w(x,t) = \sum_{k\geq 1} [c_{1k}E_{\alpha,1}(-(k\pi)t^{\alpha}) + c_{2k}tE_{\alpha,2}(-(k\pi)t^{\alpha})]\sin(k\pi x) \tag{4.5}$$

where $c_{1k}$ and $c_{1k}$ are the Fourier coefficients of $w(x,0) = w_0(x)$ and $w_t(x,0) = w_1(x)$ respectively that is

$$\begin{aligned} c_{k1} &= 2\int_0^1 w_0(x)\sin(k\pi x)dx, \\ c_{k2} &= 2\int_0^1 w_1(x)\sin(k\pi x)dx \end{aligned}$$

The boundary control is therefore,

$$U(t) = -\int_0^1 \lambda_0 y \frac{\mathcal{I}_1\sqrt{\lambda_0(1^2-y^2)}}{\sqrt{\lambda_0(1^2-y^2)}}u(y,t)dy \tag{4.6}$$

where the solution of (**??**) is given by

$$u(x,t) = w(x,t) - \lambda_0 \int_0^x y \frac{\mathcal{J}_1 \sqrt{\lambda_0 (x^2 - y^2)}}{\sqrt{\lambda_0 (x^2 - y^2)}} w(x,t) dy. \tag{4.7}$$

**Theorem 27** *If $\lambda_0$ is large and positive the solution of the uncontrolled system (4.2) grows to infinity without bound. The controlled system (4.1) is stabilized using the control given by (4.6)*

## 4.3 Boundary Control of Time-Fractional Wave Equation with Space Dependent Coefficients

In this section we shall consider one-dimensional linear fractional wave equations with fractional derivatives with respect to time and space dependent coefficients

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x) u(x,t), \ 0 < x < 1, \ t > 0 \\ u(x,0) = u_0(x), \ 0 \leq x \leq 1 \\ u_t(x,0) = u_1(x), \ 0 \leq x \leq 1 \\ u(0,t) = 0, \ t > 0 \\ u(1,t) = U(t), \ t > 0 \end{cases} \tag{4.8}$$

where $1 < \alpha \leq 2$, is the parameter describing the order of the time fractional derivative. $q \in C^1([0,1])$, $U(t)$ is the boundary control at the free end of the boundary, $u_0(.)$ and $u_1(.) \in C([0,1])$

### 4.3.1 The Free Time-Fractional Wave Equation (Uncontrolled System)

In this subsection we shall consider the uncontrolled system, $U(t) \equiv 0$

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x)u(x,t) \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \\ u(0,t) = 0, \ u(1,t) = 0, \end{cases} \tag{4.9}$$

Separation of variables $u(x,t) = X(x)T(t)$ leads to solution of (4.9) given by

$$u(x,t) = \sum_{k\geq 1} \left[b_{1k} E_{\alpha,1}(-\mu_k \ t^\alpha) + b_{2k} t E_{\alpha,2}(-\mu_k \ t^\alpha)\right] X_k(x) \tag{4.10}$$

where $b_{1n}$ and $b_{2k}$ are the Fourier coefficients of $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$ respectively that is

$$b_{1k} = 2\int_0^1 u_0(x)X_k(x)dx \text{ and } b_{2k} = 2\int_0^1 u_1(x)X_k(x)dx$$

If $q$ is a positive and large function then the solution $u(x,t)$ of (4.9) grows to infinity without bound as we will see later in the examples and simulation.

### 4.3.2 Boundary Control of Time-Fractional Wave Equations with Space Dependent Coefficients

In this section we shall consider one-dimensional fractional wave equations with fractional derivatives with respect to time and space dependent coefficients

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + q(x)u(x,t),\ 0 \le x \le 1,\ \ t > 0 \\ u(x,0) = u_0(x),\ \ 0 \le x \le 1 \\ u_t(x,0) = u_1(x),\ \ 0 \le x \le 1 \\ u(0,t) = 0,\ \ t > 0 \\ u(1,t) = U(t),\ \ t > 0 \end{cases} \tag{4.11}$$

where $1 < \alpha \le 2$, is the parameter describing the order of the time fractional derivative, $q \in C^1[0,1]$, $u_0(.)$ and $u_1(.) \in C\left([0,1]\right)$. $U(t)$ is the boundary control at the free end of the boundary. We shall again use the same transmutations (3.26), (3.28) as for the fractional diffusion equation with space dependent coefficients defined in Theorem (25).

Now, we shall transform the system (4.8), by applying $I - \mathbb{K}$ to both sides of the PDE in (4.8), then, interchanging with $\partial^\alpha/\partial t^\alpha$ and using the transmutation rule we get,

$$(I-\mathbb{K})\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = (I-\mathbb{K})[(\tfrac{\partial^2}{\partial x^2} + q(x))u(x,t)]$$

$$\frac{\partial^\alpha}{\partial t^\alpha}(I-\mathbb{K})u = (I-\mathbb{K})\mathbb{L}_0 u$$

$$\frac{\partial^\alpha}{\partial t^\alpha}(I-\mathbb{K})u = \mathbb{L}_1(I-\mathbb{K})u$$

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2} + \lambda_0 w(x,t)$$

leading to the target system,

$$\begin{cases} \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial t^2} + \lambda_0 w(x,t),\ 0 < x < 1,\ \ t > 0 \\ w(0,t) = 0,\ \ w(1,t) = 0,\ \ t > 0 \\ w(x,0) = (I - \mathbb{K})u_0(x) = w_0(x),\ 0 \le x \le 1 \\ w_t(x,0) = (I - \mathbb{K})u_1(x) = w_1(x),\ \ 0 \le x \le 1 \end{cases} \tag{4.12}$$

where $w_0(x)$ and $w_1(x)$ are continuous functions defined by

$$w_0(x) = (I - \mathbb{K})u_0(x)$$
$$w_1(x) = (I - \mathbb{K})u_1(x).$$

The boundary control in this case is

$$u(1,t) = \int_0^1 u(1,y)u(y,t)dy$$

Returning to the target system (4.12) separation of variables $w(x,t) = X(x)T(t)$ leads to the solution,

$$w(x,t) = \sum_{k \ge 1} \left[c_{1k} E_{\alpha,1}(-((k\pi)^2 - \lambda_0)t^\alpha) + c_{2k} t E_{\alpha,2}(-((k\pi)^2 - \lambda_0)t^\alpha)\right] \sin(k\pi x) \tag{4.13}$$

where $c_k$ are the Fourier coefficients of $w(x,0) = w_0(x)$ and $w_t(x,0) = w_1(x)$ respectively that is

$$c_{k1} = 2\int_0^1 w_0(x)\sin(k\pi x)dx \text{ and } c_{k2} = 2\int_0^1 w_1(x)\sin(k\pi x)dx$$

The boundary control is therefore,

$$u(1,t) = \int_0^1 k(1,y)u(y,t)dy \tag{4.14}$$

and the solution of the (4.11)

$$u(x,t) = w(x,t) + \int_0^1 l(1,y)w(y,t)dy. \tag{4.15}$$

where $k(x,y)$ and $l(x,y)$ are the solutions of the kernels PDE (3.27), (3.29)

**Theorem 28** *If $q$ is a positive and large function then the solution of the uncontrolled fractional system (4.9) grows to infinity without bound. The controlled system (4.11) is stabilized using the control given by (4.14)*

## 4.4 Examples and Simulations for Fractional Wave Equations

We present examples to demonstrate the efficiency and simplicity of the method and to show the behavior of the solution of the fractional wave equation

**Example 1:** Consider the following

$$\begin{aligned} u_0(x) &= \sin(\pi x),\ u_1(x) = \begin{cases} 2x & x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1 \end{cases}, \\ \alpha &= 1.9,\ \lambda_0 = 12,\ x \in [0,\ 1],\ xp = 50,\ t \in [0,\ 20],\ tp = 60 \end{aligned}$$

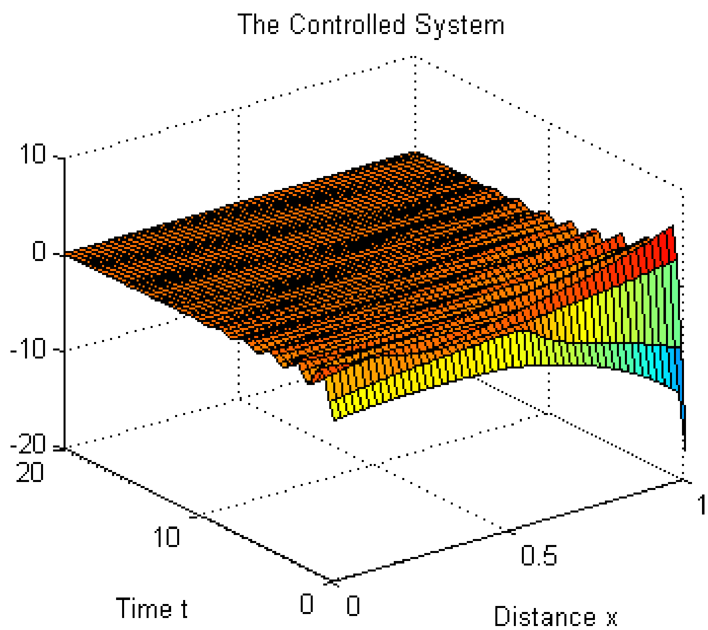


Figure 4-1: Controlled System for Example 1

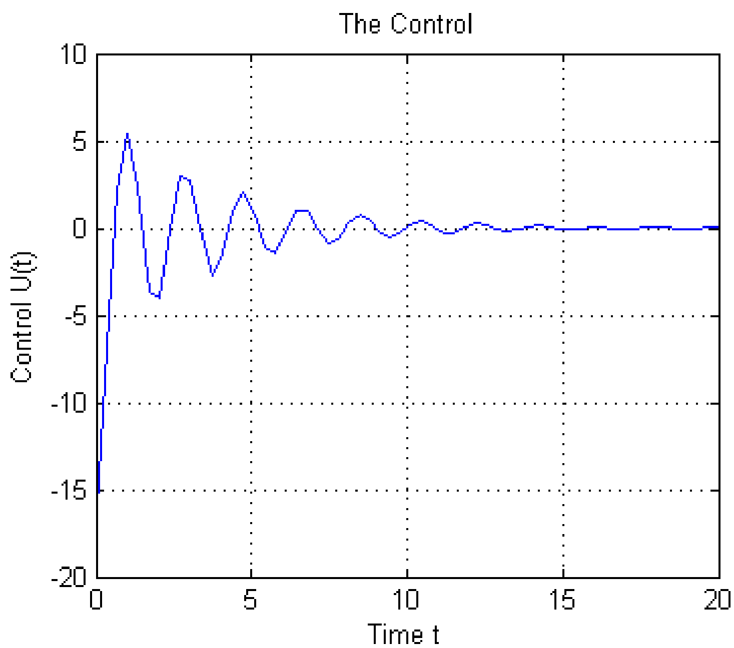


Figure 4-2: $U(t)$ Control for Example 1

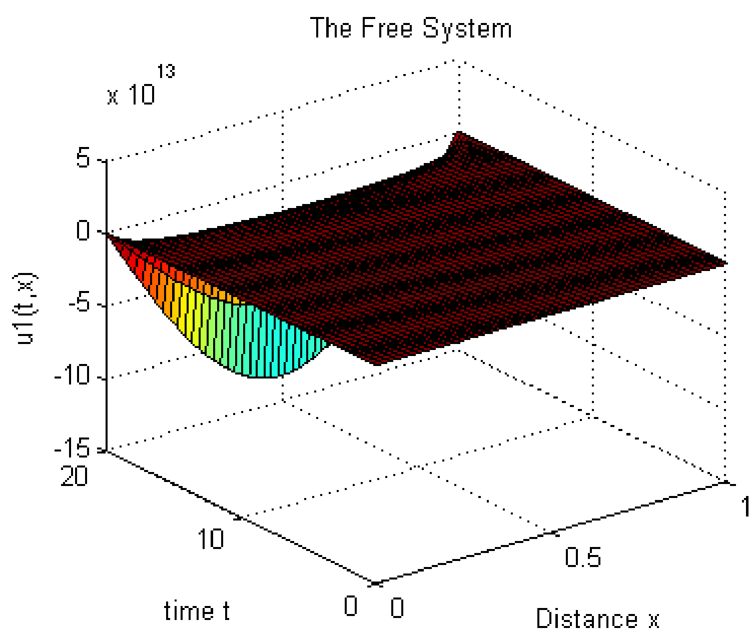


Figure 4-3: Free System for Example 1

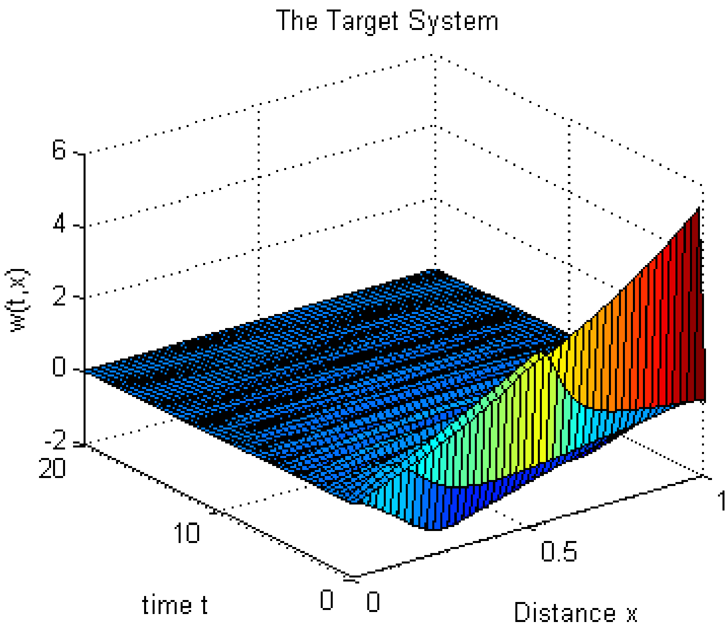


Figure 4-4: Target System for Example 1

**Example 2:** Consider the following

$$\begin{aligned} u_0(x) &= \sin(\pi x),\ u_1(x) = \begin{cases} \frac{4}{3}x & x \leq \frac{3}{4} \\ 4(1-x), & \frac{3}{4} < x \leq 1 \end{cases}, \\ \alpha &= 1.75,\ \lambda_0 = 5,\ x \in [0,\ 1],\ t \in [0,\ 10],\ q(x) = -15(1+x^2) \end{aligned}$$

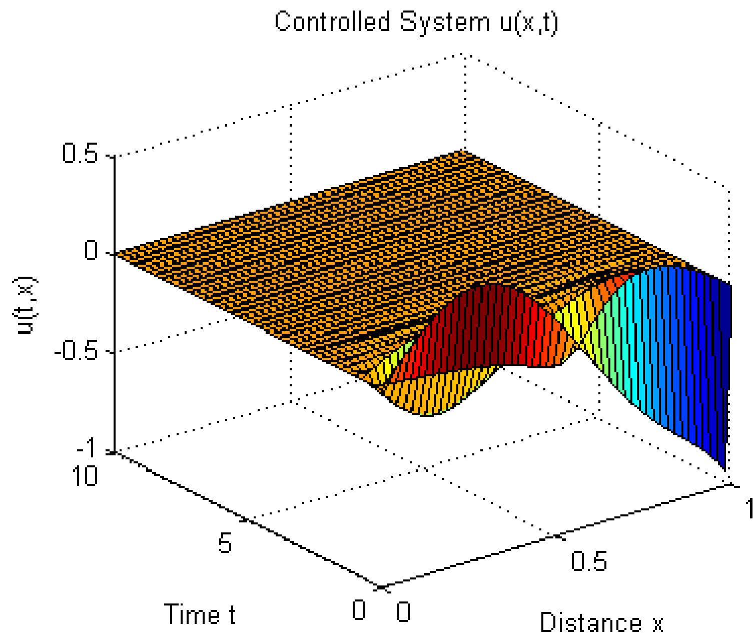


Figure 4-5: Controlled System for Example 2

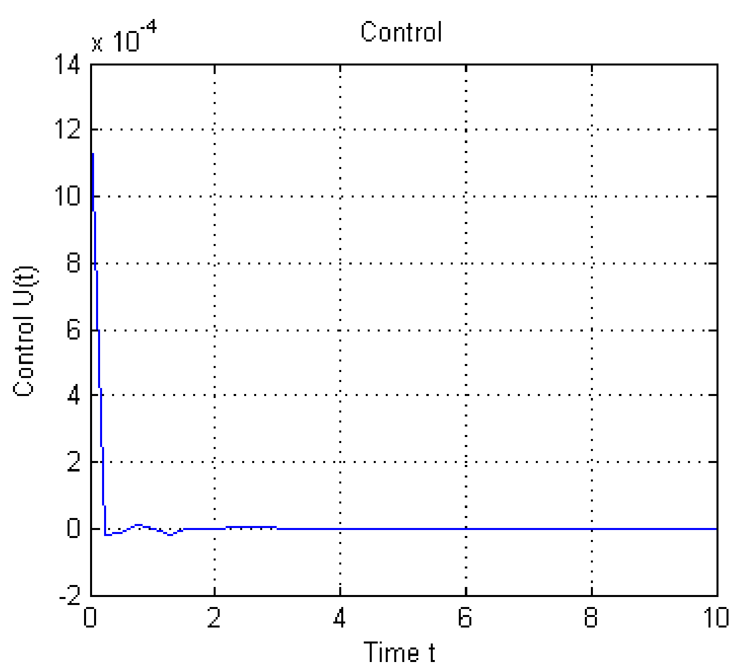


Figure 4-6: $U(t)$ Control for Example 2

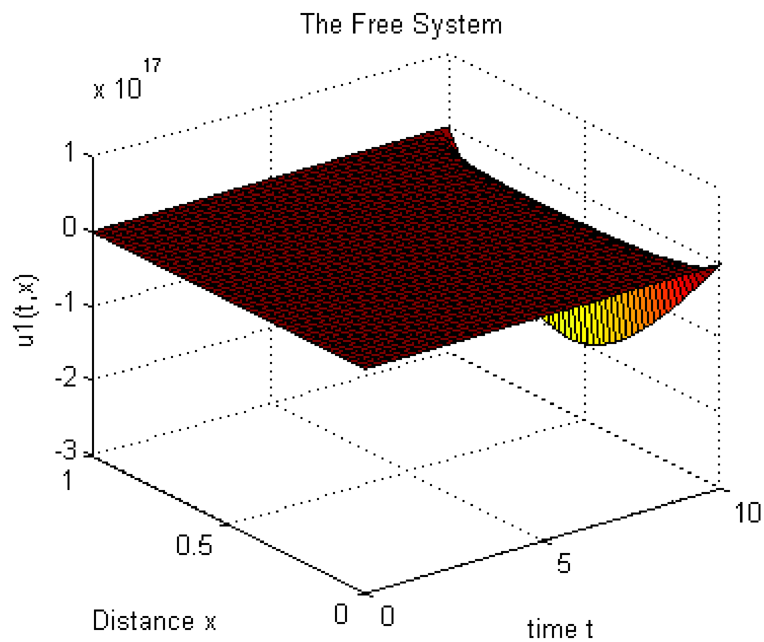


Figure 4-7: Free System for Example 2

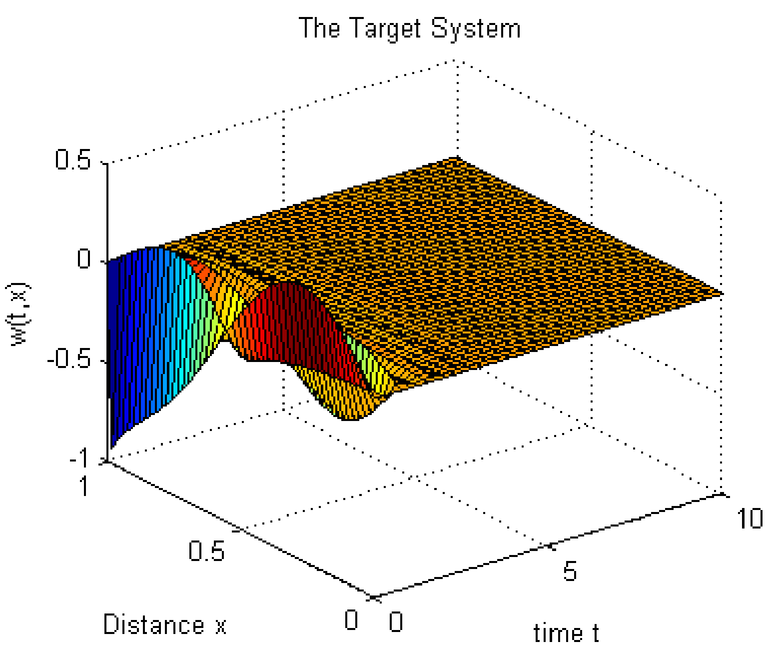


Figure 4-8: Target System for Example 2

# Chapter 5

# The Fractional Diffusion-Wave Equations

## 5.1 Introduction

A fractional diffusion-wave equation is a linear integro partial differential equation obtained from the classical diffusion or wave equation by replacing the first or second-order time derivative term by a fractional derivative of order $\alpha$, $0 < \alpha \leq 2$, and the second space derivative by a fractional derivative of order, $\beta$, $1 < \beta \leq 2$. The simplest fractional diffusion-wave equation is

$$^{C}D_t^{(\alpha)}u(x,t) = \ ^{C}D_x^{(\beta)}u(x,t),\ 0 < \alpha \leq 2,\ 1 < \beta \leq 2.\ 0 < x < 1,\ t \geq 0 \tag{5.1}$$

where the time and space fractional differential operators $^{C}D_t^{(\alpha)}$ and $^{C}D_x^{(\beta)}$ are defined in Caputo sense. (5.1) represents a hyperbolic wave equation for $\beta = 2$, $\alpha = 2$, and parabolic diffusion equation for $\beta = 2$, $\alpha = 1$. So equation (5.1) can be interpreted as the interpolation between a hyperbolic and parabolic equation. The method of the solution [23] is based on applying the operator $\boldsymbol{I}^{(\alpha)} = \boldsymbol{I}_0^{(\alpha)}$ , the inverse of the operator ${}^C D_t^{(\alpha)}$ to both sides of equation (5.1) to obtain

$$\begin{aligned}
u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} &= I^{(\alpha)}( {}^C D_x^{(\beta)} u) \\
u(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} + I^{(\alpha)}( {}^C D_x^{(\beta)} u) \\
u(x,t) &= \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} ( {}^C D_x^{(\beta)}) u(x,\tau) d\tau
\end{aligned}$$

and a series solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{5.2}$$

was given by the Adomian's decomposition method where the components $u_n(x,t)$ are determined recursively as follows

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} ( {}^C D_x^{(\beta)}) u(x,\tau) d\tau$$

where

$$u_0(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} \text{ and } u_n(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{( {}^C D_x^{(\beta)}) u_{n-1}(x,\tau) d\tau}{(t-\tau)^{1-\alpha}}$$

which particularizes to the cases

$$m = 1;\ u_0(x,t) = u(x,0) \text{ and } m = 2;\ u_0(x,t) = u(x,0) + \frac{\partial u(x,0)}{\partial t}$$

The convergence of the decomposition series has been investigated by several authors, see for instance [24].

## 5.2 The Fractional Diffusion-wave Equation with Constant Coefficients

In this section we shall consider the space and time fractional Diffusion-wave equation.

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} + \lambda_0 u(x,t) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ 0 < x < 1 \\ u(0,t) = 0,\ t \geq 0 \\ u(1,t) = U(t),\ t \geq 0 \end{cases} \tag{5.3}$$

where $\alpha, \beta$ are parameters describing the order of time and space fractional derivatives respectively, $U(.)$ is the boundary control at the free end of the boundary, $u_0(.)$ is the initial condition of displacement and $u_1(.)$ is velocity, $\lambda_0$ is an arbitrary positive constant and both $u_0$ and $u_1 \in C([0,1])$, $u(x,t)$ is defined in $[0,1] \times [0,\infty)$. We refer to the equation (5.3 ) to as the space and time fractional diffusion and to the space and time fractional wave equation in cases $\{0 < \alpha \leq 1,\ 1 \leq \beta \leq 2\}$ and

$\{1 \leq \alpha \leq 2,\ 1 < \beta \leq 2\}$ respectively.

### 5.2.1 The free Fractional Diffusion-wave Equation with Constant Coefficients (uncontrolled system)

We shall consider the system without control, $U(t) \equiv 0$,

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} + \lambda_0 u(x,t),\ 0 < \alpha \leq 1,\ 1 \leq \beta \leq 2 \\ u(x,0) = u_0(x),\ 0 < x < 1 \\ u(0,t) = 0,\ u(1,t) = 0,\ t \geq 0 \end{cases} \tag{5.4}$$

The general solution of (5.4) can be obtained by separation of variables as follows, let $u(x,t) = X(x)T(t)$ then we have

$$X(x)T^{(\alpha)}(t) = X^{^{(\beta)}}(x)T(t) + \lambda_0 X(x)T(t) \tag{5.5}$$

from which we get

$$\frac{T^{^{(\alpha)}}(t) - \lambda_0 T(t)}{T(t)} = \frac{X^{^{(\beta)}}(x)}{X(x)} = -\mu \quad \text{(constant)}.$$

Hence, $T$ satisfies the fractional ordinary differential equation

$$T^{(\alpha)}(t) = -(\mu - \lambda_0)T(t),\ t > 0 \tag{5.6}$$

and $X$ satisfies the regular fractional Sturm-Liouville problem

$$\begin{cases} X^{(\beta)}(x) = \mu X(x), \ 0 < x < 1 \\ X(0) = X(1) = 0. \end{cases} \tag{5.7}$$

The solution of the fractional differential equation in (5.7) is given by

$$X(x) = AE_{\beta,1}(\mu x^{\beta}) + BxE_{\beta,2}(\mu x^{\beta}). \tag{5.8}$$

Using the boundary condition $X(0) = 0$ gives $A = 0$ and $X(1) = 0$ gives $BE_{\beta,2}(\mu) = 0$. To get a nontrivial solution, $B$ must be different from zero, so,

$$E_{\beta,2}(\mu) = 0.$$

Solving the last equation to obtain the simple eigenvalues $\mu_k$, $k \geq 1$ and corresponding normalized eigenfunctions $X_k(x), k \geq 1$, $\left(\|X_k\|^2 = \int_0^1 |X_k(x)|^2 \, dx = 1\right)$,

$$X_k(x) = a_k x E_{\beta,2}(\mu_k x^{\beta}), \ k \geq 1. \tag{5.9}$$

where

$$a_k = \left(\int_0^1 x^2 E^2_{\beta,2}(\mu_k x^{\beta}) dx\right)^{-\frac{1}{2}}.$$

For each $k \geq 1$ we have

$$T_k(t) = d_k E_{\alpha,1}(-(\mu_k - \lambda_0)t^{\alpha}), \ k \geq 1$$

where $E_\alpha(z)$ is the Mittag-Leffler function, and $d_k$ are constants. Superposition of the product solutions gives,

$$u(x,t) = \sum_{k\geq 1} c_k E_{\alpha,1}(-(\mu_k - \lambda_0)t^\alpha) X_k(x)$$

where $c_k$ are the Fourier coefficients of $u(x,0) = \phi_0(x)$, that is

$$c_k = 2\int_0^1 u_0(x) X_k(x) dx$$

Therefore, we have shown that the solution to the uncontrolled system is

$$u(x,t) = 2\int_0^1 g(x,\xi,t) u_0(\xi) d\xi$$

where

$$g(x,\xi,t) = \sum_{k\geq 1} E_{\alpha,1}(-(\mu_k - \lambda_0)t^\alpha) X_k(x) X_k(\xi).$$

So that we conclude, if $\lambda_0$ is positive and large the solution of the uncontrolled system (5.4) grows to infinity without bound as we will see in the examples simulation

### 5.2.2 Boundary Control of Fractional Diffusion-wave Equation with Constant Coefficients (Controlled System)

In this part, we will also apply the transmutation method. However, we shall add and subtract from right hand side of the PDE in (5.3) $D_x^{(2)} u(x,t)$ to rewrite the system in the form,

$$\begin{cases} {}^{C}D_t^{(\alpha)}u(x,t) = D_x^{(2)}u(x,t) + \lambda_0 u(x,t) + \left[ {}^{C}D_x^{(\beta)} - D_x^{(2)} \right] u(x,t), \\ u(x,0) = u_0(x),\ 0 < x < 1 \\ u(0,t) = 0,\ t \geq 0 \\ u(1,t) = U(t),\ t \geq 0 \end{cases} \tag{5.10}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $0 \leq x \leq 1$, $t > 0$

Let $u = \sum_{j\geq 1} u_j$ and $U(t) = \sum_{j\geq 1} U_j(t)$ then substitute into differential equation (5.10) to get

$$\sum_{j\geq 1} {}^{C}D_t^{(\alpha)}u_j = \sum_{j\geq 1} \left(D_x^{(2)}u_j + \lambda_0 u_j\right) + \sum_{j\geq 1} \left[ {}^{C}D_x^{(\beta)} - D_x^{(2)} \right] u_j$$

Now, define $u_1$ by

$$\begin{cases} {}^{C}D_t^{(\alpha)}u_1(x,t) = D_x^{(2)}u_1(x,t) + \lambda_0 u_1(x,t) \\ u_1(x,0) = u_0(x) \\ u_1(0,t) = 0 \\ u_1(1,t) = U_1(t) \end{cases} \tag{5.11}$$

and $u_n$ by

$$\begin{cases} {}^{C}D_t^{(\alpha)}u_n(x,t) = D_x^{(2)}u_n(x,t) + \lambda_0 u_n(x,t) + F_n(x,t),\ n \geq 2 \\ u_n(x,0) = 0,\ u_n(0,t) = 0 \\ u_n(1,t) = U_n(t) \end{cases} \tag{5.12}$$

where $F_n$ are defined by

$$F_1(x,t) = 0 \text{ and } F_n(x,t) = \left( {}^C D_x^{(\beta)} - D_x^{(2)} \right) u_{n-1}(x,t) \tag{5.13}$$

We shall consider in the next section the boundary control of nonhomogeneous fractional diffusion equation

## 5.3 Boundary Control of Nonhomogeneous Fractional Diffusion Equations with constant coefficients

In this section, we shall consider the system of nonhomogeneous fractional diffusion equation with Dirichlet boundary control

$$\begin{cases} {}^C D_t^{(\alpha)} v(x,t) = D_x^{(2)} v(x,t) + \lambda_0 v(x,t) + F(x,t) \\ \quad v(x,0) = 0, \;\; v(0.t) = 0 \\ \quad v(1,t) = V(t). \end{cases} \tag{5.14}$$

where $\alpha \in (0,1)$ is parameter describing the time order derivative, $V(.)$ is the boundary control at the free end of the boundary, $v_0(x)$ is the initial condition, $\lambda_0$ is arbitrary positive constant and $v(x,t)$ is defined in $[0,1] \times [0,\infty)$. Consider the change of variables

$$w(x,t) = (I - \mathbb{K})v = v(x,t) - \int_0^x k(x,y)v(y,t)dy$$

by applying $I - \mathbb{K}$ to both sides of the PDE in (5.14), then, interchanging with $\partial^\alpha / \partial t^\alpha$ and using the transmutation rule we get,

$$\begin{aligned}
(I-\mathbb{K})\frac{\partial^\alpha v}{\partial t^\alpha} &= (I-\mathbb{K})\left(\frac{\partial^2}{\partial x^2}+\lambda_0\right)v + (I-\mathbb{K})F(x,t) \\
\frac{\partial^\alpha (I-\mathbb{K})v}{\partial t^\alpha} &= (I-\mathbb{K})\left(\frac{\partial^2}{\partial x^2}+\lambda_0\right)v + (I-\mathbb{K})F(x,t) \\
\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} &= \frac{\partial^2 w(x,t)}{\partial x^2} + f(x,t)
\end{aligned} \tag{5.15}$$

leading to the target system

$$\begin{cases} \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2} + f(x,t), \ x \in (0,1), \ t \geq 0 \\ w(x,0) = 0, \ x \in (0,1) \\ w(0,t) = 0, \ w(1,t) = 0 \end{cases} \tag{5.16}$$

where $w(x,0) = (I-\mathbb{K})v(x,0) = 0$ and $f(x,t) = (I-\mathbb{K})F(x,t)$. Finding a series solution of the form

$$w(x,t) = \sum_{n\geq 0} X_n(x)T_n(t)$$

where $X_n(x)$ are the eigenfunctions we find when solving the associated homogeneous problem

$$\begin{cases} \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2}, \ x \in (0,1), \ t \geq 0 \\ w(0,t) = w(1,t) = 0, \ t \geq 0 \end{cases} \tag{5.17}$$

and $T_n(t)$ are functions which can be found by solving a sequence of ODEs. Recall that the solution of (5.17) is of the form

$$w(x,t) = \sum_{n\geq 0} A_n E_{\alpha,1}(-\mu_n t^\alpha) X_n(x)$$

where $\mu_n$ and $X_n(x)$ are the eigenvalues and eigenfunctions of the problem (5.17). Returning to (5.16), we expand $f(x,t)$ in eigenfunctions expantion as

$$f(x,t) = f_1(t)X_1(x) + f_2(t)X_2(x) + ... + f_n(t)X_n(x) + ...$$

and find the response $w_n(x,t) = X_n(x)T_n(t)$ to each of these individual components. The solution to our problem will then be $w(x,t) = \sum_{n\geq 1} w_n(x,t)$ where $\mu_n = (n\pi)^2$, $n \geq 1$,

$$X_n(x) = \sin(n\pi x), \; n \geq 1. \tag{5.18}$$

and

$$f_n(t) = 2\int_0^1 w_0(x)\sin(n\pi x)dx.$$

Replacing $f(x,t)$ by its expansion, we shall seek a solution in the form

$$w(x,t) = \sum_{n\geq 1} T_n(t)\sin(n\pi x).$$

Thus,

$$\begin{cases} \sum_{n\geq 1} T_n^{(\alpha)}(t)\sin(n\pi x) = \sum_{n\geq 1} (n\pi)^2 T_n(t)\sin(n\pi x) + \sum_{n\geq 1} f_n(t)\sin(n\pi x) \\ \sum_{n\geq 1} T_n(t)\sin(0) = \sum_{n\geq 1} T_n(t)\sin(\pi) = 0 \\ \sum_{n\geq 1} T_n(0)\sin(n\pi x) = 0 \end{cases}$$

we are left with

$$\begin{cases} \sum_{n\geq 1} \left[T_n^{(\alpha)}(t) + (n\pi)^2 T_n(t) - f_n(t)\right]\sin(n\pi x) = 0 \\ \sum_{n\geq 1} T_n(0)\sin(n\pi x) = 0 \end{cases}$$

Thus $T_n$ must satisfy the initial value problem

$$\begin{cases} T_n^{(\alpha)}(t) + (n\pi)^2 T_n(t) - f_n(t) = 0 \\ T_n(0) = 0 \end{cases} \tag{5.19}$$

The solution to (5.19) is

$$T_n(t) = A_n E_{\alpha,1}(-(n\pi)^2 t^\alpha) + \int_0^t f_n(\tau) E_{\alpha,2}(-(n\pi)^2 (t-\tau)^\alpha) d\tau \tag{5.20}$$

thus the solution to (5.16) is

$$w(x,t) = \sum_{n\geq 1} \left[A_n E_{\alpha,1}(-(n\pi)^2 t^\alpha) + \int_0^t f_n(\tau) E_{\alpha,2}(-(n\pi)^2 (t-\tau)^\alpha) d\tau\right]\sin(n\pi x).$$

Since $w(x,0) = 0$, the Fourier coefficent $A_n = 0$ for all $n$, So that, the solution of the target system becomes

$$w(x,t) = \sum_{n\geq 1} \int_0^t f_n(\tau) E_{\alpha,2}(-(n\pi)^2 (t-\tau)^\alpha) d\tau) \sin(n\pi x) \tag{5.21}$$

leading to the solution of (5.14) as

$$v(x,t) = w(x,t) + \int_0^x l(x,y) w(y,t) dy \tag{5.22}$$

and the control as

$$v(1,t) = \int_0^1 k(x,y) v(y,t) dy \tag{5.23}$$

where $k$ and $l$ are solve the Goursat problem (3.14), (3.16) respectively.

**Theorem 29** *A boundary control for the nonhomogeneous system (5.14) is given by (5.23) and the state is given by (5.22). The boundary control stabilizes the overall system.*

Returning to (5.10), the solution is given by

$$u(x,t) = \sum_{n\geq 1} u_n(x,t) \tag{5.24}$$

under the boundary control

$$U(t) = \sum_{n\geq 1} U_n(t) \tag{5.25}$$

where $u_n$ and $U_n$ are the state solution and the boundary control of (5.12) for each $n$ respectively.

To summarize, we have the following theorem,

**Theorem 30** *The boundary control of fractional diffusion-wave equation (5.10) is given by* $U(t) = \sum_{n\geq 1} U_n(t)$ *and the solution is* $u(x,t) = \sum_{n\geq 1} u_n(x,t)$*. While the solution of the uncontrolled system (5.4) grows to infinity, the solution of the controlled system (5.10) stabilizes by the control given by (5.25)*

## 5.4 The Fractional Diffusion-wave Equation with Space Dependent Coefficients

In this section we shall consider time and space fractional derivatives diffusion-wave equation with space dependent coefficient

$$\begin{cases} {}^{C}D_t^{(\alpha)}(x,t) = {}^{C}D_x^{(\beta)}u(x,t) + q(x)u(x,t),\ 0<\alpha\leq 1,\ 1<\beta\leq 2,\ 0\leq x\leq 1,\ t>0 \\ u(x,0) = u_0(x),\ 0<x<1 \\ u(0,t) = 0,\ t\geq 0 \\ u(1,t) = U(t),\ t\geq 0 \end{cases} \tag{5.26}$$

where $\alpha, \beta$ are parameters describing the order of time and space fractional derivatives respectively, $u(x,t)$ is the field defined in the space domain $[0,1]$, and time $t \in [0,\infty)$, $U(.)$ is the boundary control at the free end of the boundary, $u_0(.)$ is the initial condition and $u_0 \in C([0,1])$

### 5.4.1 Boundary Control of Fractional Diffusion-wave with Space Dependent Coefficients (Controlled System)

As before, we shall add and subtract $D_x^{(2)}u(x,t)$ to the right hand side of the equation ( 5.26) above and rewrite the system as

$$\begin{cases} {}^C D_t^{(\alpha)}(x,t) = D_x^{(2)}u(x,t) + q(x)u(x,t) + \left[ {}^C D_x^{(\beta)} - D_x^{(2)} \right] u(x,t) \\ u(x,0) = u_0(x), \ 0 < x < 1 \\ u(0,t) = 0, \ t \geq 0 \\ u(1,t) = U(t), \ t \geq 0. \end{cases} \tag{5.27}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $0 \leq x \leq 1$, $t > 0$. Let $u = \sum_{j\geq 1} u_j$ and $U(t) = \sum_{j\geq 1} U_j(t)$ then substitute into differential equation (5.27) to get

$$\sum_{j\geq 1} {}^C D_t^{(\alpha)} u_j = \sum_{j\geq 1} \left( D_x^{(2)} u_j + q(x)u_j \right) + \sum_{j\geq 1} \left[ {}^C D_x^{(\beta)} - D_x^{(2)} \right] u_j$$

Now, define $u_1$ by

$$\begin{cases} {}^C D_t^{(\alpha)} u_1(x,t) = D_x^{(2)} u_1(x,t) + q(x)u_1(x,t) \\ u_1(x,0) = u_0(x) \\ u_1(0,t) = 0 \\ u_1(1,t) = U_1(t) \end{cases} \tag{5.28}$$

and $u_n$ by

$$\begin{cases} {}^C D_t^{(\alpha)} u_n(x,t) = D_x^{(2)} u_n(x,t) + q(x)u_n(x,t) + F_n(x,t), \ n \geq 2 \\ u_n(x,0) = 0, \ u_n(0,t) = 0 \\ u_n(1,t) = U_n(t) \end{cases} \tag{5.29}$$

where $F_n$ are defined by

$$F_1(x,t) = 0, \text{ and } F_n(x,t) = \left( {}^C D_x^{(\beta)} - D_x^{(2)} \right) u_{n-1}(x,t), \ n \geq 2 \tag{5.30}$$

We shall consider in the next section the boundary control of nonhomogeneous fractional diffusion equation with space dependent coefficients

## 5.5 Boundary Control of Nonhomogeneous Fractional Diffusion Equations with space dependent coefficients

we shall consider the boundary control of nonhomogeneous fractional diffusion equations with space dependent coefficients

$$\begin{cases} {}^C D_t^{(\alpha)} v(x,t) = D_x^{(2)} v(x,t) + q(x)v(x,t) + F(x,t) \\ v(x,0) = 0, \ v(0.t) = 0 \\ v(1,t) = V(t) \end{cases} \tag{5.31}$$

where $0 < \alpha \leq 1$, $(x,t) \in [0,1] \times [0,\infty)$. Consider the change of variables

$$w = (I - \mathbb{K})v = v(x,t) - \int_0^x k(x,y)v(y,t)dy$$

by applying $I - \mathbb{K}$ to both sides of the PDE in (5.31), then, interchanging with $\partial^\alpha/\partial t^\alpha$ and using the transmutation rule we get,

$$\begin{aligned} (I-\mathbb{K})\frac{\partial^\alpha v}{\partial t^\alpha} &= (I-\mathbb{K})\left(\frac{\partial^2}{\partial x^2}+q\right)v + (I-\mathbb{K})F(x,t) \\ \frac{\partial^\alpha (I-\mathbb{K})v}{\partial t^\alpha} &= (I-\mathbb{K})\left(\frac{\partial^2}{\partial x^2}+q\right)v + (I-\mathbb{K})F(x,t) \\ \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} &= \frac{\partial^2 w(x,t)}{\partial x^2} + \lambda_0 w(x,t) + f(x,t), \end{aligned} \tag{5.32}$$

leading to the target system

$$\begin{cases} \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \frac{\partial^2 w(x,t)}{\partial x^2} + \lambda_0 w(x,t) + f(x,t) \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(1,t) = 0 \end{cases} \tag{5.33}$$

where $w(x,0) = (I-\mathbb{K})v(x,0)$, $f(x,t) = (I-\mathbb{K})F(x,t)$. The solution for the target system (5.33) is given by

$$w(x,t) = \sum_{n\geq 1} \int_0^t f_n(\tau) E_{\alpha,2}(\lambda_0 - (n\pi)^2)(t-\tau)^\alpha) d\tau) \sin(n\pi x) \tag{5.34}$$

and the solution of (5.31) is given by

$$v(x,t) = (I - \mathbb{L})w(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy \tag{5.35}$$

and the boundary control is

$$v(1,t) = \int_0^1 k(x,y)v(y,t)dy \tag{5.36}$$

where $k$ and $l$ solve the Goursat problem (3.27), (3.29) respectively.

**Theorem 31** *A boundary control for the nonhomogeneous system (5.31) is given by (5.36) and the state is given by (5.35). The boundary control stabilizes the overall system.*

Returning to (5.27), the general solution of (5.27) is given by

$$u(x,t) = \sum_{n\geq 1} u_n(x,t) \tag{5.37}$$

under the boundary control

$$U(t) = \sum_{n\geq 1} U_n(t) \tag{5.38}$$

where $u_n$ and $U_n$ are the state solution and the boundary control of (5.27) for each $n$ respectively.

To summarize, we have the following theorem,

**Theorem 32** *The boundary control of fractional diffusion-wave equation (5.26) is* *given by* $U(t) = \sum_{n\geq 1} U_n(t)$ *and the solution is* $u(x,t) = \sum_{n\geq 1} u_n(x,t)$. *While the solution of the uncontrolled system grows to infinity, the solution of the controlled system (5.26) stabilizes by the control given by (5.38)*

## 5.6 Examples and Simulations for Fractional Diffusion-Wave Equations

We present examples to show the efficiency and simplicity of the method and to demonstrate the behavior of the solution of the fractional diffusion-wave equation as the order of the time and space-fractional derivatives are changes.

**Example 1:** We shall consider

$$\begin{aligned} u_0(x) &= \sin(\pi x), q(x) = \lambda_0 = 15, \\ x &\in [0,\ 1],\ t \in [0,\ 10],\ \alpha = 0.9,\ \beta = 1.9 \end{aligned}$$

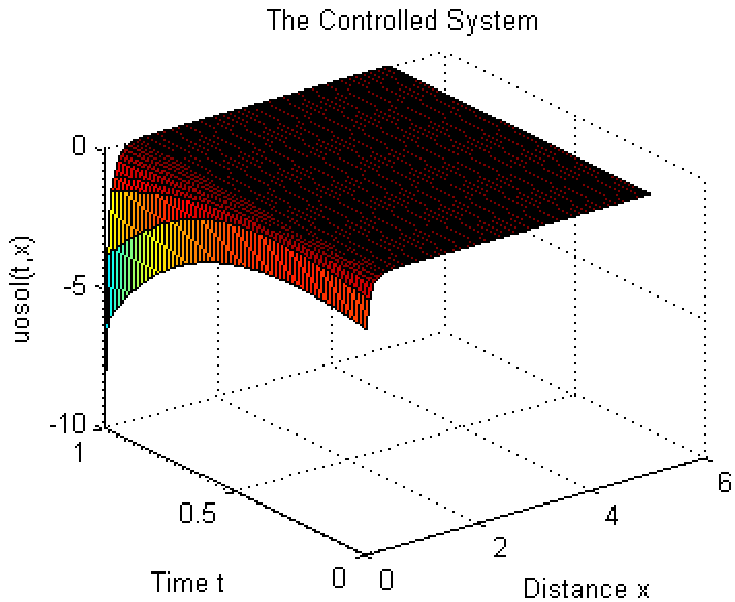


Figure 5-1: The Controlled System $u_1$ for Example 1

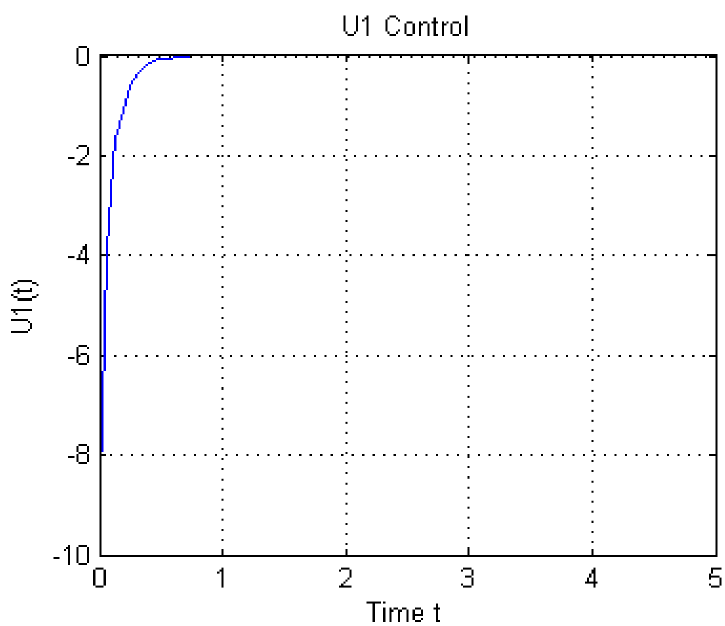


Figure 5-2: The Control $U_1$ for Example 1

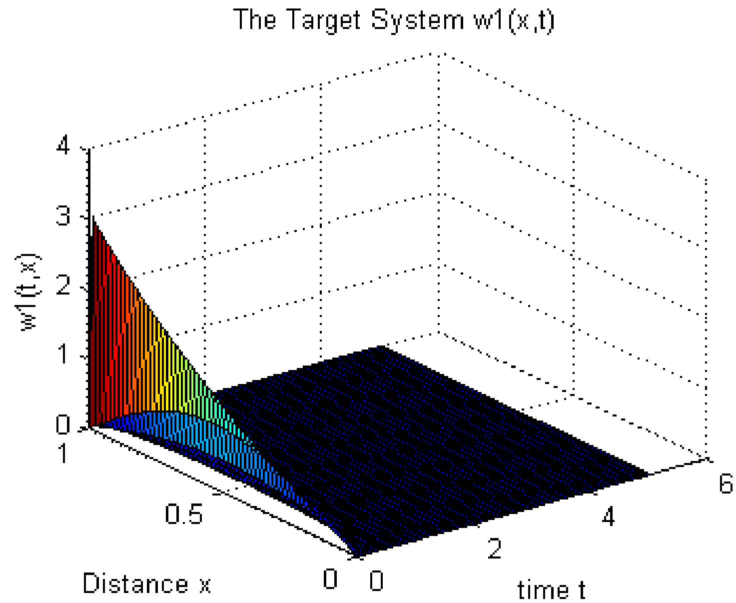


Figure 5-3: The Target System $w_1$ for Example 1

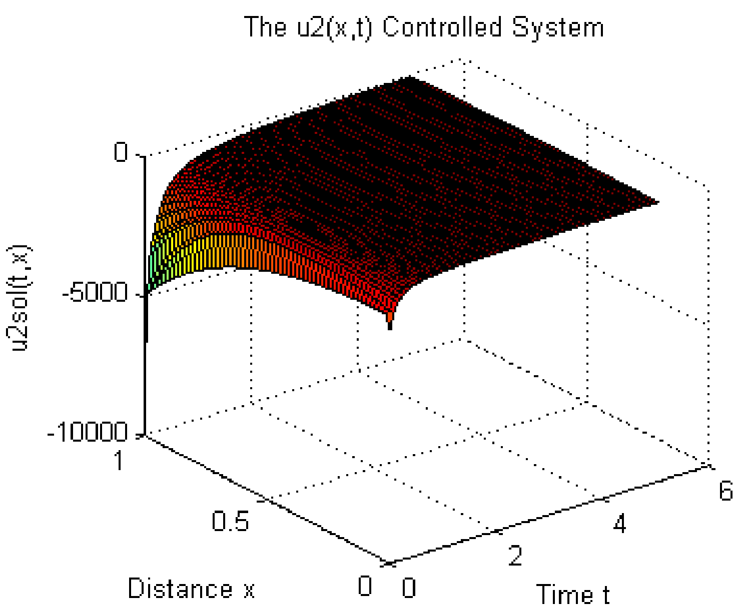


Figure 5-4: The Controlled System $u_2$ for Example 1

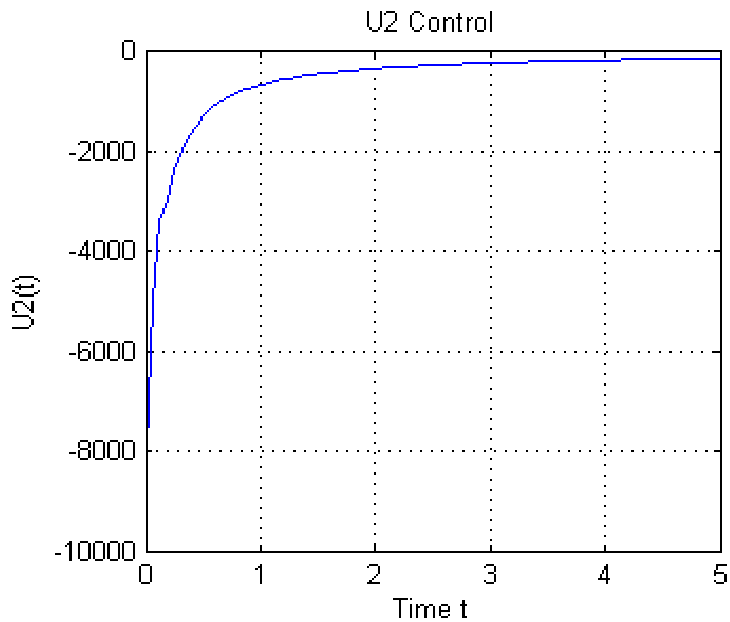


Figure 5-5: The Control $U_2$ for Example 1

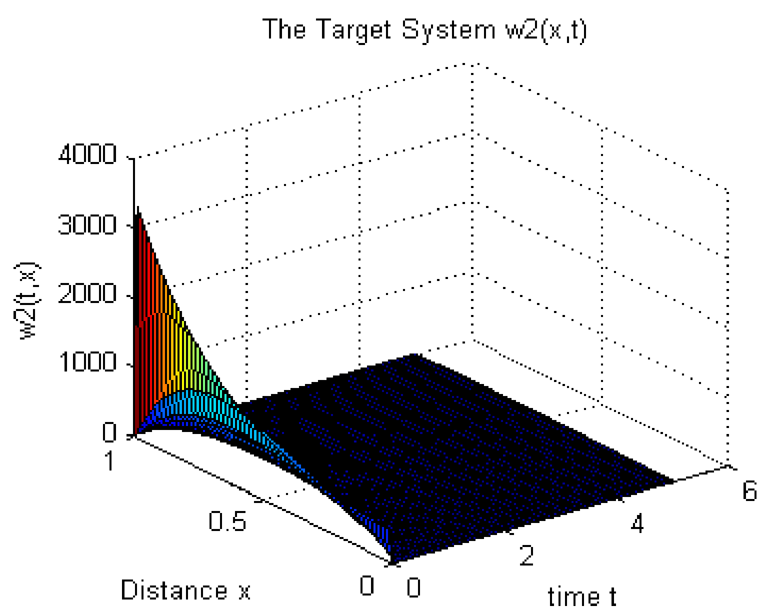


Figure 5-6: The Target System $w_2$ for Example 1

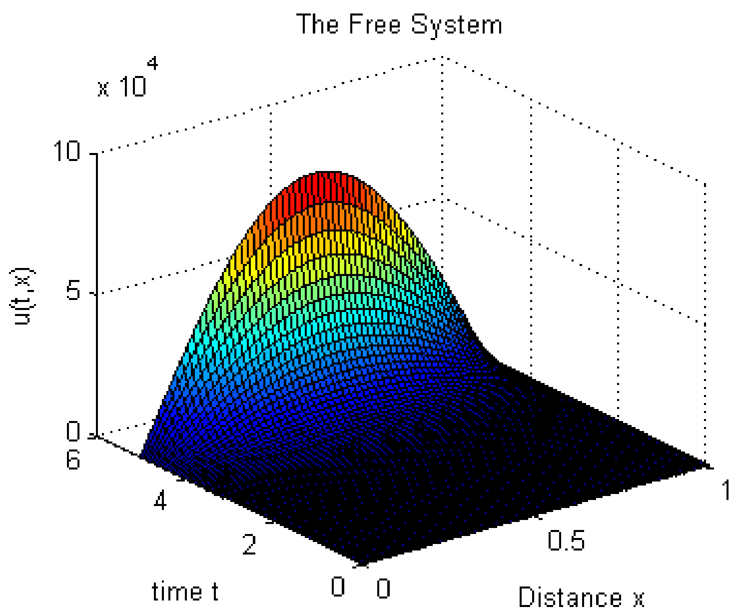


Figure 5-7: The Free System for example 1

# Chapter 6

# Optimal Control of Time-Fractional Diffusion Equation

## 6.1 Introduction

Optimal control theory is concerned with finding control functions that minimize cost functionals for systems described by differential equations. In the area of calculus of variations and optimal control of fractional differential equations little has been done compared to a differential equation with integer time derivative. The first record of the formulation of the fractional optimal control problem was given by O.P Agrawal [64] and he presented a general formulation and solution scheme for the fractional optimal control problem. In 2011, Mophou [63] considered the optimal control of the fractional diffusion equation in Rieman-Liouville sense. We are now in a position to formulate the optimal control problems. We shall consider the linear quadratic optimal control of the time-fractional PDE system given by,

$$\textbf{Min}_f \ J \triangleq \int_0^T \|u(.,t) - \widehat{u}(.,t)\|_w^2 \, dt \tag{6.1}$$

subject to,

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{w(x)} \left\{ \frac{\partial}{\partial x} \left[ p(x) \frac{\partial}{\partial x} u(x,t) \right] - q(x)u(x,t) \right\} + f(x,t), \ 0 < x < 1, \ 0 < t \leq T \\ u(x,0) = u_0(x), \ 0 \leq x \leq 1 \\ a_1 p \frac{\partial}{\partial x} u - a_2 u = 0, \ \text{at } x = 0, \ 0 < t \leq T \\ b_1 p \frac{\partial}{\partial x} u - b_2 u = 0, \ \text{at } x = 1, \ 0 < t \leq T \end{cases} \tag{6.2}$$

where $\widehat{u}(x,t)$ is a given function, $0 < \alpha \leq 1$ , with $p, w > 0$ and $f$ is the control

## 6.2 Method of solution

We shall associate to (6.2) the following Regular Sturm-Liouville problem,

$$\begin{cases} \frac{1}{w(x)} \left\{ \frac{d}{dx} \left[ p(x) \frac{d}{dx} v(x) \right] - q(x)v(x) \right\} = -\lambda v(x), \ a < x < b \\ a_1 p \frac{d}{dx} v - a_2 v = 0, \ \text{at } x = a \\ b_1 p \frac{d}{dx} v - b_2 v = 0, \ \text{at } x = b \end{cases} \tag{6.3}$$

It is well known that it has an infinite sequence of simple eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots \uparrow \infty$ with corresponding orthogonal eigenfunctions $v_0, v_1, v_2, \cdots$ with respect to the inner product $< g, h >_w = \int_a^b w(x)g(x)h(x)dx$. We shall assume that they are normalized using the induced norm $\|g\|_w = \left( \int_a^b w(x) \, |g(x)|^2 \, dx \right)^{1/2}$.Since the system of eigenfunctions is complete in $L^2_w([a,b])$, we expand $f(x,t)$ as

$$f(x,t) = \sum_{n=0}^{\infty} f_n(t) v_n(x)$$

and seek a solution of (6.2) in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) v_n(x)$$

Replacing into (6.2)

$$\begin{aligned}
\frac{\partial^{(\alpha)}}{\partial t^{(\alpha)}} \sum_{j=0}^{\infty} u_j v_j &= \frac{1}{w}\left[\frac{\partial}{\partial x} p \frac{\partial}{\partial x} \sum_{j=0}^{\infty} u_j v_j - q \sum_{j=0}^{\infty} u_j v_j\right] + \sum_{j=0}^{\infty} f_j v_j \\
\sum_{j=0}^{\infty} v_j \frac{\partial^{(\alpha)}}{\partial t^{(\alpha)}} u_j &= \sum_{j=0}^{\infty} u_j \left\{\frac{1}{w}\left[\frac{\partial}{\partial x}\left(p \frac{\partial}{\partial x} v_j\right) - q u_j v_j\right]\right\} + \sum_{j=0}^{\infty} f_j v_j \\
\sum_{j=0}^{\infty} v_j \frac{\partial^{(\alpha)}}{\partial t^{(\alpha)}} u_j &= \sum_{j=0}^{\infty} u_j \left(-\lambda_j v_j\right) + \sum_{j=0}^{\infty} f_j v_j
\end{aligned}$$

and taking the inner product with $v_j$ we obtain the decoupled infinite system of fractional ODEs

$$u_j^{(\alpha)}(t) = -\lambda_j u_j(t) + f_j(t),\ 0 < t \leq T,\ j = 0, 1, 2, \cdots \tag{6.4}$$

whose solution is

$$u_j(t) = E_\alpha(-\lambda_j t^\alpha) d_j + \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha) f_j(\tau) d\tau,\ j = 0, 1, 2, \cdots \tag{6.5}$$

where $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are the Mittag-Leffler functions, $d_j$ are arbitrary constants.

Thus, the solution to the PDE and boundary conditions in (6.2) is

$$\begin{aligned} u(x,t) &= \sum_{j=0}^{\infty} \left\{ E_\alpha(-\lambda_j t^\alpha) d_j + \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha) f_j(\tau) d\tau \right\} v_j(x) \\ &= \sum_{j=0}^{\infty} E_\alpha(-\lambda_j t^\alpha) v_j(x) d_j + \sum_{j=0}^{\infty} \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha) v_j(x) f_j(\tau) d\tau \end{aligned} \tag{6.6}$$

Letting $t = a$, we get, $d_j$ as the Fourier coefficients of $u(x, 0) = u_0(x)$

$$d_j = \int_a^b u_0(\xi) v_j(\xi) d\xi \tag{6.7}$$

Now, recall the Fourier coefficient $f_j(t)$ of $f(x,t)$,

$$f_j(t) = \int_a^b f(\xi, t) v_j(\xi) d\xi \tag{6.8}$$

Hence,

$$\begin{aligned} u(x,t) = &\sum_{j=0}^{\infty} E_\alpha(-\lambda_j t^\alpha) v_j(x) \int_a^b u_0(\xi) v_j(\xi) d\xi \\ &+ \sum_{j=0}^{\infty} \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha) v_j(x) \int_a^b f(\xi,\tau) v_j(\xi) d\xi d\tau \end{aligned}$$

$$\begin{aligned} u(x,t) = &\int_a^b \sum_{j=0}^{\infty} E_\alpha(-\lambda_j t^\alpha) v_j(x) v_j(\xi) u_0(\xi) d\xi \\ &+ \int_0^t \int_a^b \sum_{j=0}^{\infty} E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha) v_j(x) v_j(\xi) f(\xi,\tau) d\tau d\xi \end{aligned} \tag{6.9}$$

that is, the solution is

$$u(x,t) = \int_a^b g(x,\xi,t)u_0(\xi)d\xi + \int_a^b \int_0^t g(x,\xi,t-\tau)f(\xi,\tau)d\tau d\xi \tag{6.10}$$

where

$$g(x,\xi,t) = \sum_{j=0}^{\infty} E_\alpha(-\lambda_j t^\alpha)v_j(x)v_j(\xi) \tag{6.11}$$

Hence, we have proved the theorem below,

**Theorem 33** *Let $\{\lambda_j, v_j\}_{j\geq 0}$ be the sequence of eigenvalues and associated (normalized) eigenfunctions of the regular Sturm-Liouville problem (6.3), then the system (6.3) has the unique solution given by (6.10).*

## 6.3 Optimal Control

Replacing the control $f$, the state $u$ and the reference state $\widehat{u}$ by their eigenfunction expansions,

$$f(x,t) = \sum_{n=0}^{\infty} f_n(t)v_n(x),\ u(x,t) = \sum_{n=0}^{\infty} u_n(t)v_n(x),\ \widehat{u}(x,t) = \sum_{n=0}^{\infty} \widehat{u}_n(t)v_n(x) \tag{6.12}$$

in the integral in (6.1) we get,

$$\begin{aligned}
&\int_0^T \int_a^b w(x) \left\{ \sum_{n=0}^{\infty} [u_n(t) - \widehat{u}_n(t)]\, v_n(x) \right\}^2 dxdt \\
&\quad = \int_0^T \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [u_i(t) - \widehat{u}_i(t)]\, [u_j(t) - \widehat{u}_j(t)] \int_a^b w(x) v_i(x)\, v_j(x)\, dxdt \\
&\quad = \int_0^T \sum_{j=0}^{\infty} [u_j(t) - \widehat{u}_j(t)]^2\, dt = \sum_{j=0}^{\infty} \int_0^T [u_j(t) - \widehat{u}_j(t)]^2\, dt. \qquad (6.13)
\end{aligned}$$

So the problem (6.1), (6.2) becomes

$$\mathbf{Min}_f \sum_{j=0}^{\infty} \int_0^T [u_j(t) - \widehat{u}_j(t)]^2\, dt \qquad (6.14)$$

subject to (6.5) and the boundary conditions.

Thus, since the equations are decoupled, the problem becomes

$$\mathbf{Min}_{f_j}\ J_j \triangleq \int_0^T [u_j(t) - \widehat{u}_j(t)]^2\, dt \qquad (6.15)$$

subject to $u_j^{(\alpha)}(t) = -\lambda_j u_j(t) + f_j(t),\ 0 < t \le T, u_j(0) = d_j$

for $j = 0, 1, 2, \cdots$

$$\begin{aligned}
J_j &= \int_0^T [u_j(t) - \widehat{u}_j(t)]^2\, dt \\
&= \int_0^T \left[ E_\alpha(-\lambda_j t^\alpha) d_j + \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha) f_j(\tau) d\tau - \widehat{u}_j(t) \right]^2 dt
\end{aligned}$$

Let

$$f_j(t) = \sum_{i=0}^{\infty} f_{ji}\phi_i(t)$$

for each $j$, where $\{\phi_i\}_{i\geq 0}$ is some basis of $L^2([0,T])$

$$\mathbf{Min}_{f_{ji}} \int_0^T [u_j(t) - \widehat{u}_j(t)]^2 \, dt \tag{6.16}$$
$$\text{subject to } u_j^{(\alpha)}(t) = -\lambda_j u_j(t) + \sum_{i=0}^{n} f_{ji}\phi_i(t), \; 0 < t \leq T, u_j(0) = d_j$$

$$J_j[f_j] = \int_0^T [u_j(t) - \widehat{u}_j(t)]^2 \, dt =$$
$$\int_0^T \left[ E_\alpha(-\lambda_j t^\alpha)d_j + \sum_{i=0}^{n} f_{ji} \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha)\phi_i(\tau)d\tau - \widehat{u}_j(t) \right]^2 \, dt$$

Let $\frac{\partial J_j}{\partial f_{ji}} = 0$, for $i = 0, 1, 2, , , , , n$

$$\frac{\partial J_j}{\partial f_{ji}} = 2\int_0^T \left\{ \begin{array}{c} \left[ E_\alpha(-\lambda_j t^\alpha)d_j + \sum_{i=0}^{n} f_{ji} \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha)\phi_i(\tau)d\tau - \widehat{u}_j(t) \right] \\ \left[ \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha)\phi_i(\tau)d\tau \right] \end{array} \right\} dt$$

$$= 2\int_0^T [E_\alpha(-\lambda_j t^\alpha)d_j - \widehat{u}_j(t)] \left[ \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha)\phi_i(\tau)d\tau \right] dt +$$
$$2\int_0^T \sum_{i=0}^{n} \left[ f_{ji} \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha)\phi_i(\tau)d\tau \right] \left[ \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha)\phi_i(\tau)d\tau \right] dt$$
$$= 0.$$

Then we have,

$$\sum_{i=0}^{m} f_{ji} \int_0^T \left( \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^{\alpha}) \phi_i(\tau) d\tau \right) \left( \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^{\alpha}) \phi_i(\tau) d\tau \right) dt$$
$$= - \int_0^T (E_{\alpha}(-\lambda_j t^{\alpha}) d_j - \widehat{u}_j(t)) \left( \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^{\alpha}) \phi_i(\tau) d\tau \right) dt$$

implies,

$$\sum_{k=0}^{m} f_{jk} a_{kji} = b_{ji} \ , \ i = 0, 1, 2, , , , , n, \ \ j = 0, 1, 2, , , , , m \tag{6.17}$$

for a given $j$ we have $A_j f_j^* = B_j$ where

$$A_j = (n+1) * (m+1), \ B_j = (n+1) * 1, \quad f_j^* : (m+1)(1),$$
$$A_j = (a_{k,i,j})_{k,i}, \ B_j = (b_{ji})_j$$
$$a_{kji} = \int_0^T \left( \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^{\alpha}) \phi_k(\tau) d\tau \right) \left( \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^{\alpha}) \phi_i(\tau) d\tau \right) dt$$
$$b_{ji} = - \int_0^T (E_{\alpha}(-\lambda_j t^{\alpha}) d_j - \widehat{u}_j(t)) \left( \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^{\alpha}) \phi_i(\tau) d\tau \right) dt$$

Let

$$b(t, j, i) = \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^{\alpha}) \phi_i(\tau) d\tau, \ b(t_l, j, i) = \int_0^{t_l} E_{\alpha,2}(-\lambda_j (t_l-\tau)^{\alpha}) \phi_i(\tau) d\tau$$
$$b_{l,j,i} = \delta_t \sum_{k=1}^{l} E_{\alpha,2}(-\lambda_j (t_l - t_k)^{\alpha}) \phi_i(t_k)$$

where $t_l = l\delta_t$ , $l = 1, ..., n$ and $\delta_t = 1/n$. Solving the system of linear equations (6.17) for $f_j$ for $j = 0, 1, 2, \cdots m$ so that, we get

$$f_j^*(t) = \sum_{i=0}^{m} f_{ji}^* \varphi_i(t), \;\; u_j^* = E_\alpha(-\lambda_j t^\alpha) d_j + \sum_{i=0}^{n} f_{ji}^* \int_0^t E_{\alpha,2}(-\lambda_j (t-\tau)^\alpha) \phi_i(\tau) d\tau \quad (6.18)$$

and

$$f(x,t) = \sum_{n=0}^{\infty} f_n^*(t) v_n(x), \;\; u(x,t) = \sum_{n=0}^{\infty} u_n^*(t) v_n(x). \quad (6.19)$$

Thus we have

**Theorem 34** *The optimal control and optimal state of (6.1), (6.2) are given by (6.19) respectively.*

Applying the optimal control $f$ to the system will lead to the optimal state $u$ following $\widehat{u}(x,t)$. In the reconsidering expansion below. we shall draw the graphs of $u(\overline{x}, t)$ and $\widehat{u}(\overline{x}, t)$ for different values of $\overline{x}$.

## 6.4 Examples and Simulations for Optimal Control of Time-Fractional Diffusion Equation

In this chapter we shall present some examples to illustrate our method. For any practical purpose, we shall truncate the series say to $N$ and introduce

$$g_N(x, \xi, t) = \sum_{j=0}^{N} E_\alpha(-\lambda_j t^\alpha) v_j(x) v_j(\xi)$$

as well as the approximate solution,

$$u_N(x,t) = \int_a^b g_N(x,\xi,t)u_0(\xi)d\xi + \int_a^b \int_0^t g_N(x,\xi,t-\tau)f(\xi,\tau)d\tau d\xi$$

and the optimal control is

$$f_N(x,t) = \sum_{n=0}^{N} f_n^*(t)v_n(x)$$

We shall present in the sequel a few examples with different $p$, $q$, $w$, $u_0$ and $\widehat{u}$ and obtain the optimal $f$ in each case.

**Example 1:** We shall consider the following

$$w(x) = x^2+1,\ p(x) = x^2+1,\ q(x) = -20\cos(x),$$

$$\alpha = 0.7,\ x \in [0,1],\ xp = 60,\ t \in [0,0.9],\ tp = 50,\ N = 10$$

$$u_0(x) = \begin{cases} \frac{4}{3}x, & 0 \le x \le 0.75 \\ 4(1-x), & 0.75 < x \le 1 \end{cases},\ \widehat{u}(x,t) = e^{-t}\sin(\pi x)$$

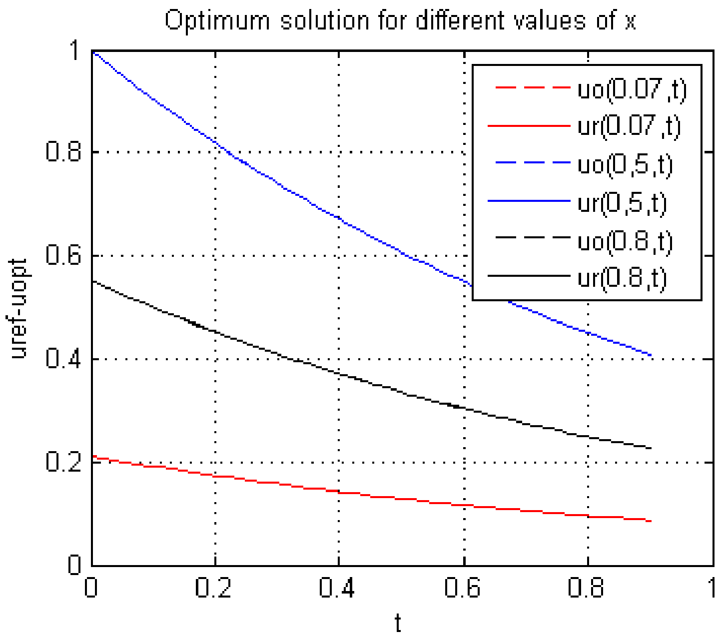


Figure 6-1: Optimum solution for different values of x for example 1

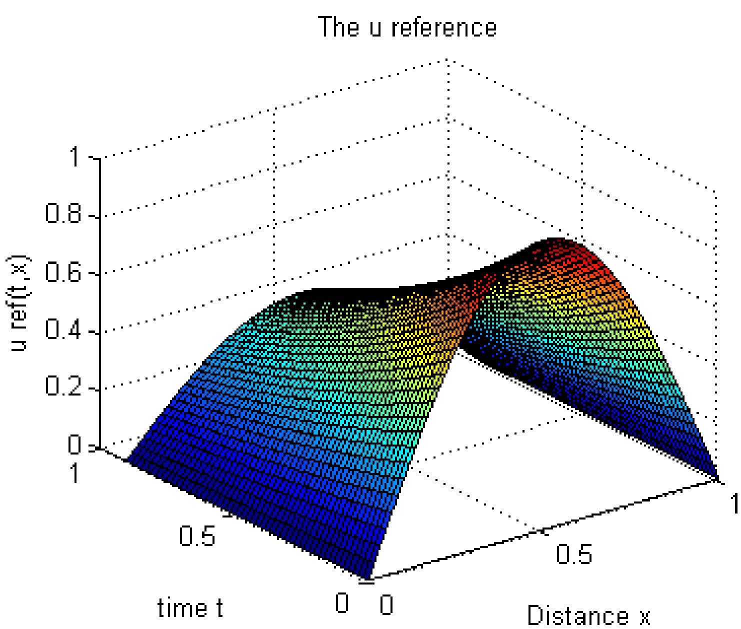


Figure 6-2: The $u$ References for example 1

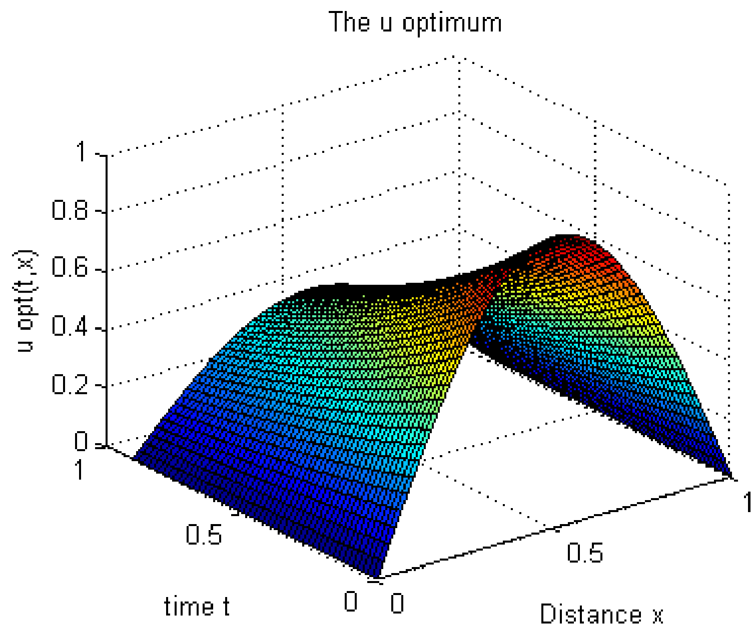


Figure 6-3: The $u$ Optimum Control for Example 1

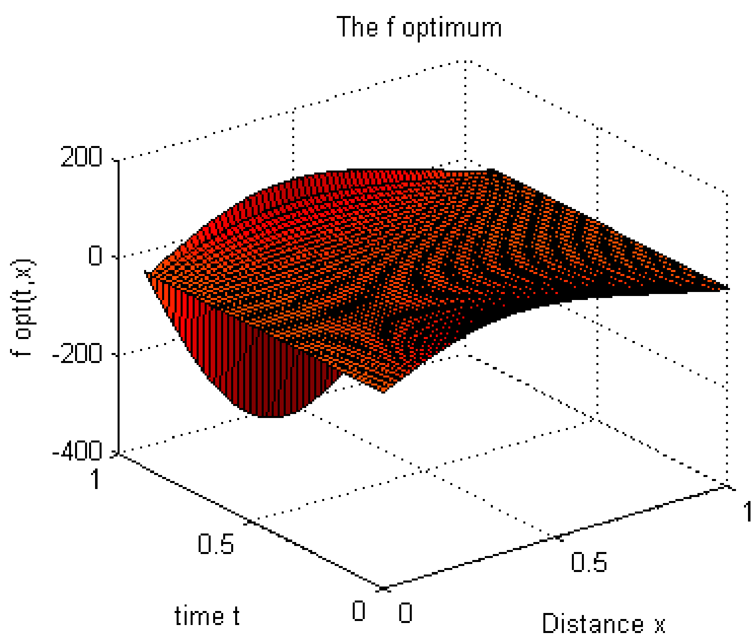


Figure 6-4: The $f$ Optimum Control for Example 1

**Example 2:** We shall consider the following

$$w(x) = x^2 + 1,\ p(x) = x^2 + 1\ ,\ x \in [0, 1],\ xp = 40,\ t \in [0, 3],\ tp = 50$$

$$u_0(x) = \sin(\pi x),\ q(x) = -20x,\ \alpha = 0.8,\ \widehat{u}(x, t) = e^{(xt)} \sin(\pi x)$$

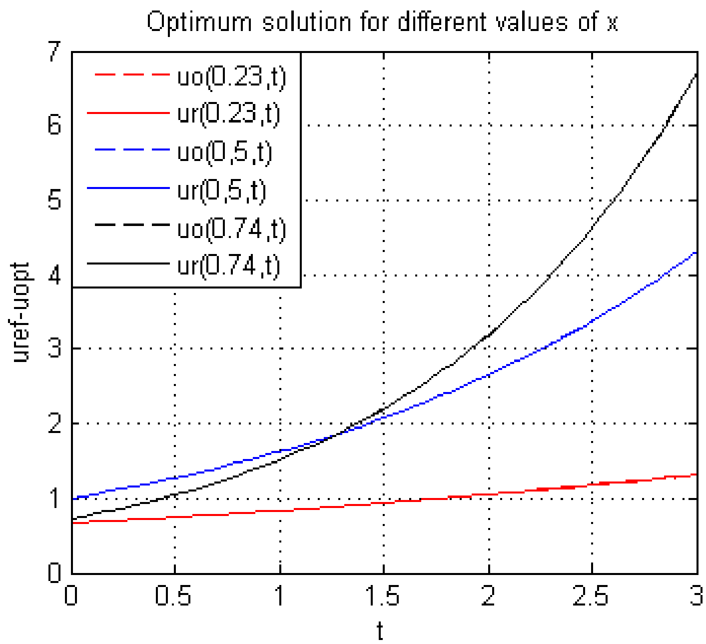


Figure 6-5: Optimum Solution for different values of $x$ for Example 2

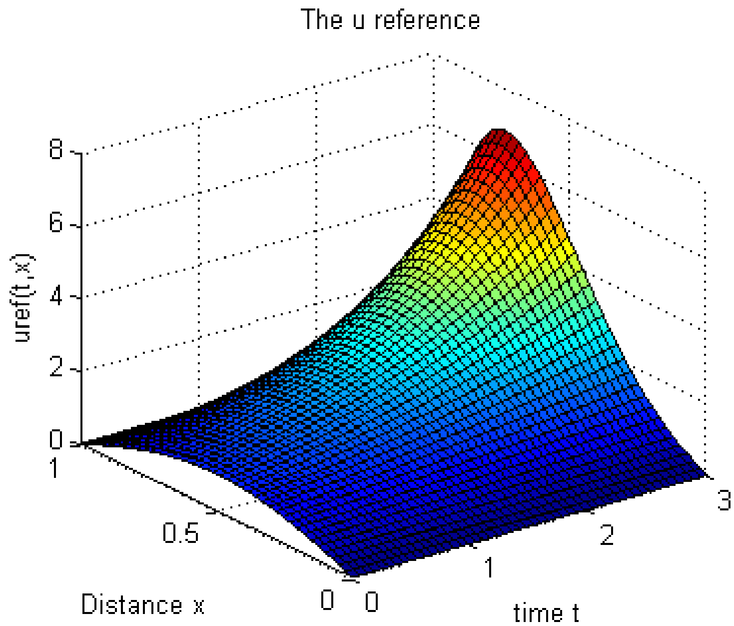


Figure 6-6: The $u$ Reference for Example 2

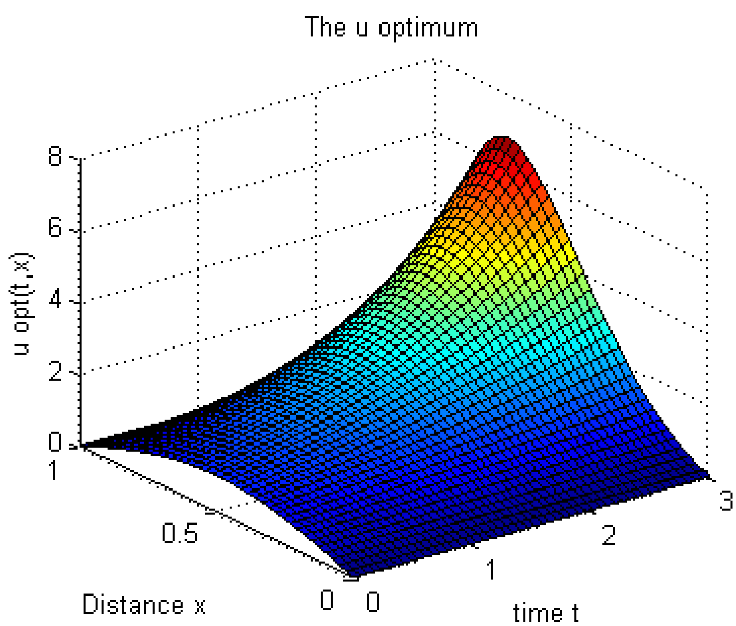


Figure 6-7: The $u$ Optimum for example 2

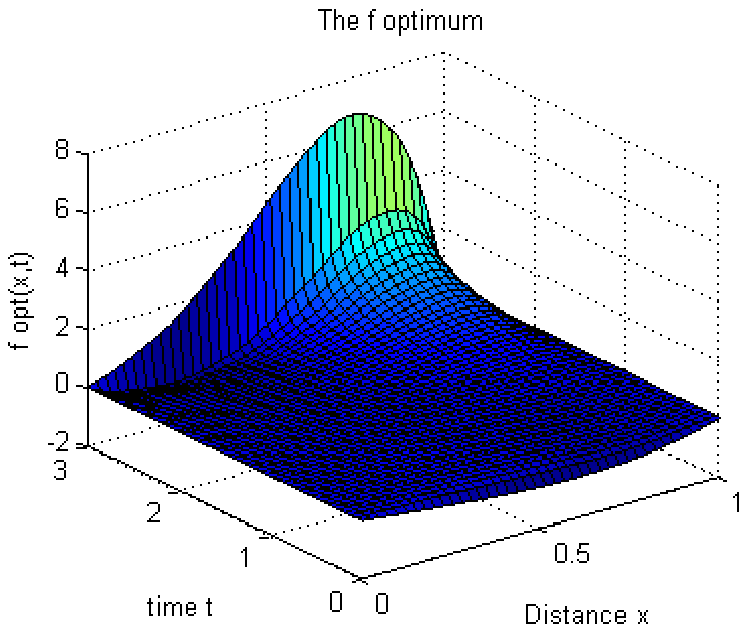


Figure 6-8: The $f$ Optimum Control for Example 2

**Example 3:** We shall consider the following

$$\begin{aligned} w(x) &= x^2+1,\ p(x)=x^2+1,\ q(x)=-50(x^2+1), \widehat{u}(x,t)=xte^{-t}, \alpha=0.5, \\ u_0(x) &= \begin{cases} 2x, & x \le 0.5 \\ 2(1-x), & 0.5 \le x \le 1 \end{cases},\ x\in[0,1], xp=60,\ t\in[0,10],\ tp=50 \end{aligned}$$

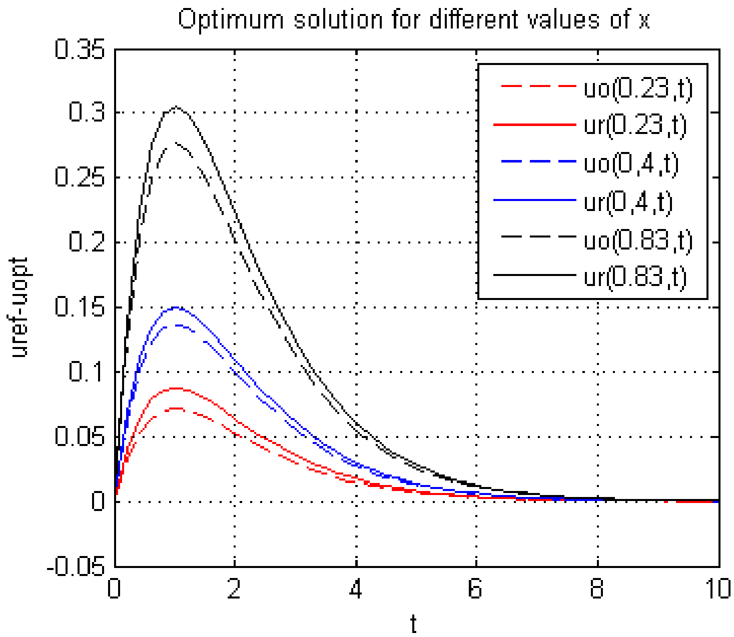


Figure 6-9: Optimum Solution for different values of $x$ for Example 3

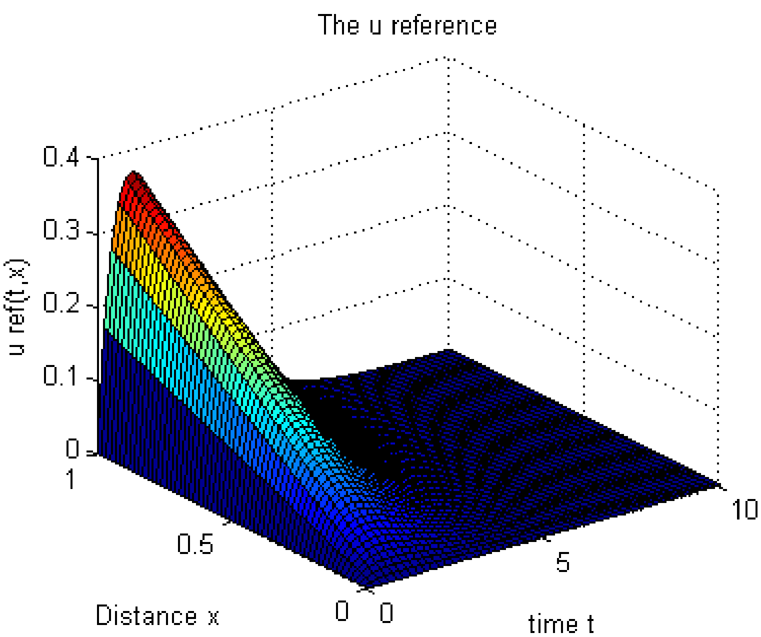


Figure 6-10: The $u$ Reference for Example 3

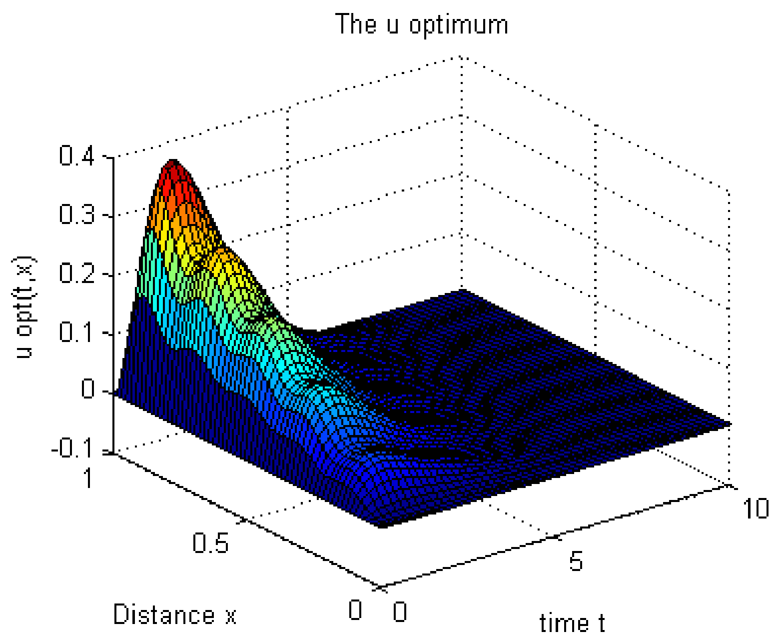


Figure 6-11: The $u$ Optimum for example 3

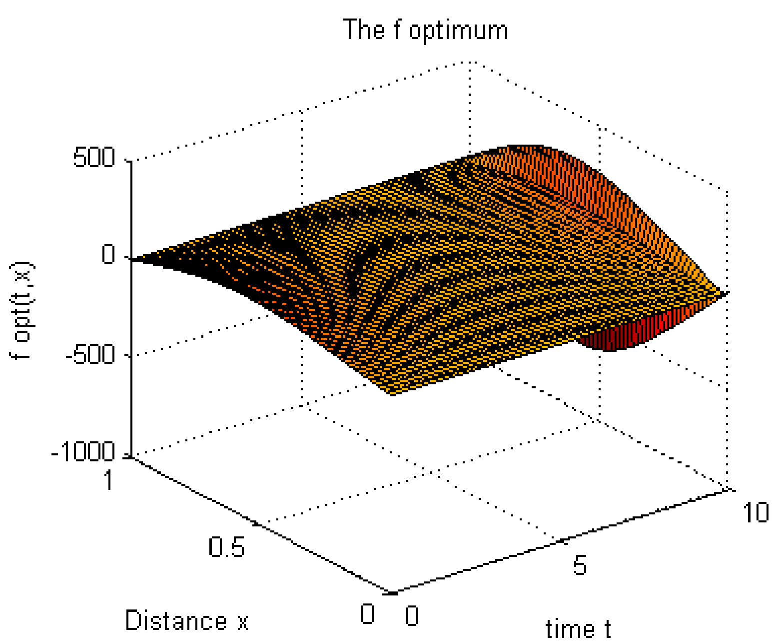


Figure 6-12: The $f$ Optimum Control for Example 3

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- **List of Publications**

  - Bilal Chanane, Faez Ali Alqarni, A. Boucherif, Boundary Control of Time-Fractional Diffusion Equations with Space Dependent Coefficient. (Submitted).
  - Bilal Chanane, Faez Ali Alqarni, A. Boucherif, Optimal Control of Time-Fractional Diffusion Equation with Space Dependent Coefficients. (Submitted).
  - Bilal Chanane, Faez Ali Alqarni, A. Boucherif, Boundary Control of Linear Parabolic PDEs using Transmutations. (Submitted).