ON RISK AND UNCERTAINTY IN INVENTORY PROBLEMS WITH STOCHASTIC NATURE

BY

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To my parents for supporting me in every step of my life

To my wife for her love and patience

To my kids, Somayah, Abdulmalek, Halah, Omar and Sarah

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All praise and glory are due to Almighty ALLAH alone; we seek His help and ask for His forgiveness and guidance. May peace and blessings be upon the greatest human being that ever walked on the earth, the last and the final messenger, Prophet Muhammad, upon his household, his companions, and all those who follow their footsteps till the day of judgment.

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I	The set of potential markets, $\|I\| = m$.
i	Market index, $i = 1,\ 2,\ \ldots,\ m$.
J	The set of products, $\|J\| = p$.
j	Product index, $i = 1,\ 2,\ \ldots,\ p$.
r_{ij}	The selling price per unit of product j in market i.
e_j	The expediting (or equivalently, shortage or penalty) cost per unit of product j from a local supplier.
c_j	The purchasing cost per unit of product j from an external supplier.
v_j	The salvage value per unit of product j.
S_{ij}	The fixed cost per period of introducing product j in market i in the flexible market entry case.
S_i	The fixed cost per period of entering market i in the full market entry case.
$\mathfrak{s}_{ij}$	The fixed cost per period of introducing product j in market i in the partial market entry case.

s_i	The fixed cost per period of entering market i in the partial market entry case.
x_{ij}	A random variable representes the total demand per period of product j in market i.
μ_{ij}	The mean demand of product j in market i.
σ_{ij}^2	The variance of the demand of product j in market i.
y_{ij}	A binary selection variable of entering market i with product j, $y_{ij} = 1$ if market i is selected and 0 otherwise.
$\mathcal{Y}_i$	A binary selection variable of entering market i, $y_i = 1$ if market i is selected and 0 otherwise.
$\widehat{\boldsymbol{y}}$	A binary markets selection vector, $\widehat{\boldsymbol{y}} = (y_1,\ y_2, \ldots, y_m)$.
Q_j	The order quantity of product j from the external supplier.
$Q_{\widehat{\boldsymbol{y}}j}$	The order quantity of product j from the external supplier for the market selection vector $\widehat{\boldsymbol{y}}$.
$\xi_{\widehat{\boldsymbol{y}}j}$	$=\sum_{i=1}^{m} \xi_{ij} y_{ij}$, is the total value of parameter ξ for product j associated with the market selection vector $\widehat{\boldsymbol{y}}$.
$f\left(x_{ij}\right)$	The probability density function (p.d.f) of the random demand; x_{ij}.
$f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right)$	The p.d.f of the total random demand; $x_{\widehat{\boldsymbol{y}}j}$, for the market selection vector $\widehat{\boldsymbol{y}}$.
$F\left(x_{ij}\right)$	The cumulative distribution function (c.d.f) of the random demand; x_{ij}.

$F_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right)$	The c.d.f of the total random demand; $x_{\widehat{\boldsymbol{y}}j}$, for the market selection vector $\widehat{\boldsymbol{y}}$.
ρ_j	$= \frac{e_j - c_j}{e_j - v_j}$, is the well-known newsvendor quantile of product j.
α_j	A target service level for product j, $\alpha_j \in (0,\ 1)$.
η_j	The degree of risk-aversion for the decision maker for product j, $\eta_j \in [0, 1)$.
θ_j	The $(1 - \eta_j)$ quantile of the profit distribution for product j.
$L\left(t\right)$	The standard normal loss function; $L\left(t\right) = \int_t^{\infty} (u - t)\phi\left(u\right) du$.
Ψ	The adjustable parameter that controls the size of the box uncertainty set.
Ω	The adjustable parameter that controls the size of the ellipsoidal uncertainty set.
Γ	The adjustable parameter that controls the size of the polyhedral uncertainty set.
$\phi\left(u\right)$	The p.d.f. of the standard normal distribution.
$\Phi\left(u\right)$	The c.d.f. of the standard normal distribution.
$U(a,\ b)$	The uniform distribution with domain a to b inclusive.

THESIS ABSTRACT

NAME: Mohammad A. M. Abdel-Aal

TITLE OF STUDY: On Risk and Uncertainty in Inventory Problems with Stochastic Nature

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Traditional approaches of supply chain planning consider planning problems under the assumption that all demand sources should be satisfied. However, in the real world, firms have limited resources, therefore, planning approaches must take the capacity of these resources into consideration and select the optimal set of demands to satisfy. This dissertation investigates the impact of the risk preferences of the decision maker and the lack of demand information on the performance of inventory and demand selection problems. We focus on the multi-product multi-market newsvendor problem, termed as Multi-Product Selective Newsvendor Problem (MPSNVP). The dissertation studies variety of versions of the MPSNVP under different risk preferences of the decision makers and deficiency of the demand information.

In the first part of the dissertation, we study the risk-neutral MPSNVP for flexible market entry, full market entry and partial market entry cases of the MPSNVP. We analyze and formulate the mathematical models for each case of the risk-neutral MPSNVP, in order to maximize the expected profit. In addition, we incorporate service level constraints on the probability of satisfying the demand of each product. Under the assumption of independent normally distributed demands, the mathematical models for the above cases, with and without service level criteria, result in Integer Nonlinear Programs (NIP). We propose polynomial optimal solution algorithms as well as efficient heuristics for solving the obtained models.

In the second part of the dissertation, we consider the CVaR risk-averse MPSNVP for the above cases, i.e. flexible market entry, full market entry and partial market entry cases. Similar to the risk-neutral MPSNVP, the obtained mathematical models are NIP. We propose the optimal solution algorithms to solve these models in polynomial time. We deduce that the risk-averse decision maker orders less quantity of each product than the risk-neutral decision maker does. Then, we reformulate the NIP as Conic Quadratic Mixed Integer Programs (CQMIP), and compare the performance of the proposed solution algorithms with the performance of the state-of-the-art commercial solvers. The comparison demonstrates that the proposed solution algorithms outperform the commercial solvers in terms of the computational time and the solution quality.

In the third part of the dissertation, we study the MPSNVP when the demand

distribution at some markets is only partially specified. The demand uncertainty is characterized by an uncertainty set. We study the MPSNVP cases under different kinds of uncertainty set including; box, ellipsoidal, polyhedral uncertainty set and combinations of these sets. The robust counterpart models under these uncertainty sets are obtained and solution algorithms are proposed. The computational results and discussion are provided.

ملخص الرســالة

الاســم : محمـد عبدالعـال محمـد عبدالعـال

عنوان الرسـالة : عن المخاطرة وعدم التأكد في مسائل المخزون ذات الطبيعة العشوائية

التخصص : الهندســة الصنــاعيــة

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يتم في هذه الرسالة استقصاء تأثير أفضلية المخاطرة لدى صناع القرار ونقص المعلومات المتاحة عن الطلب في الاسواق على أداء المخزون واختيار الأسواق. يتم في هذه الأطروحة التركيز على مسائل بائع الصحف متعددة المنتجات والأسواق حيث يتم دراسة أشكال متعددة من تلك المسائل مع اعتبار أفضلية المخاطرة لدى صناع القرار ونقص المعلومات المتاحة عن الطلب في الأسواق.

يهتم الجزء الأول من هذه الرسالة بدراسة النزعة المتعادلة للمخاطرة لمسائل بائع الصحف متعددة المنتجات والأسواق. يتم دراسة تلك المسائل مع اعتبار حالات الدخول المرن للأسواق و الدخول الكلى للأسواق وأخيرًا الدخول الجزئي للأسواق. يتم تحليل وصياغة النماذج الرياضية لكلٍ من تلك الحالات بما يتيح تعظيم الربح المتوقع. وبالإضافة إلى ذلك فقد تم دمج شروط مستوى الخدمة لاحتمالية الوفاء بالطلب على كل منتج. يتم صياغة النماذج الرياضية للحالات سالفة الذكر مع الفرض بأن الطلب يتوزع توزيعًا طبيعيًا مستقلًا ونتيجة لذلك فإن النماذج المتحصلة تكون نماذج صحيحة لاخطية. يتم اقتراح طرق للحصول على الحلول المثلى وكذلك يتم اقتراح طرق للحصول على حلول استرشادية.

يتم في الجزء الثاني من هذه الرسالة تطبيق دراسة نزعة تجنب المخاطرة مع الحالات سابقة الذكر من مسائل بائع الصحف متعددة المنتجات والأسواق. يتم اعتبار القيمة المشروطة عند المخاطرة كمقياس لتجنب المخاطرة. وعلى غرار النتائج لدراسة النزعة المتعادلة للمخاطرة فإن النماذج الرياضية المتحصلة للحالات مع نزعة تجنب المخاطرة هي أيضاً نماذج صحيحة لاخطية. يتم اقتراح طرق للحصول على الحلول المثلى.

يتم أيضًا إعادة صياغة النماذج الصحيحة اللاخطية كنماذج صحية مختلطة مخروطية تربيعية ويتم مقارنة أداء طرق الحلول المقترحة مع أداء أفضل البرامج التجارية المتاحة لحل تلك النماذج. تبين النتائج أن صانع القرار ذو نزعة تجنب المخاطرة عليه أن يقوم بطلب كميات من كل منتج أقل مما لو كانت نزعته متعادلة بالنسبة للمخاطرة.

يهتم الجزء الثالث من هذه الرسالة بدراسة مسائل بائع الصحف متعددة المنتجات والأسواق عندما تكون المعلومات المتاح عن الطلب في الأسواق غير وافية أو متكاملة. عدم التأكد حيال الطلب السوقي يتم توصيفه عن طريق فئة عدم التأكد. يقوم صانع القرار بتحديد شكل وحجم فئة عدم التأكد للطلب السوقي. يتم دراسة مسائل بائع الصحف متعددة المنتجات والأسواق مع اعتبار فئة عدم التأكد مستطيلة الشكل ثم بيضاوية الشكل ثم متعددة السطوح وأخيرًا اعتبار فئة عدم التأكد مزيج من تلك الفئات. يتم صياغة النظائر المتينة للنماذج مع اعتبار فئات عدم التأكد المختلفة ويتم طرح طرق الحلول لها. يتم عرض ومناقشة النتائج الحسابية.

CHAPTER 1

INTRODUCTION

1.1 Overview and Motivation

Today's competitive world emphasizes the development of powerful paradigms and tools to manage the entire supply chain. Inventory management aspects are crucial issues in supply chain management. Several literature reviews were conducted to discuss the coordination of supply chain activities and flexibility in responding to changing market conditions [1, 2, 3].

Traditional models of supply chain planning used to treat planning decisions separately. Those models do not optimize the entire supply chain, because of the conflict between planning decisions, and the drastic changes in the dynamic business environment. One can notice a proliferation of recent research trend on integrating different planning decision. A crucial topic in supply chain planning is inventory planning.

Based on the available information about the demand nature; the inventory systems are classified into three main types:

- Deterministic systems: when there is precise information about the demand size before the demand size is realized.
- Stochastic systems: when the demand is uncertain and random in nature, but there is enough data and information to characterize a suitable demand probability distribution.
- Uncertain systems: when the demand is uncertain and random in nature, but the available demand data and information is not enough to characterize a suitable demand probability distribution; this is due to imprecision, vagueness, or ambiguity of the available information.

The inventory decision making process differs based on the inventory system:

- For deterministic systems "Under Certainty Decision Making Process": the decision maker has a complete knowledge of the demand states and he knows exactly the demand behavior; so that, the decision problem is simple. The most common example of this situation is the economic order quantity (EOQ) model for deterministic demand.
- For stochastic systems "Under Risk Decision Making Process": the decision maker has partial information about the demand states. The available information is expressed in terms of demand probability distributions, which enable decision maker to determine; for instance, the maximum expectation of profit or incorporating other decision maker's preferences; such as

risk-averse and risk-seeking preferences.

- For uncertainty systems "Under Uncertainty Decision Making Process": the decision maker has very little information about the demand states; such that, the demand probability distributions are not known. The known information in such situation might be demand mean and/or variance and/or the maximum and/or minimum demand amount. The best inventory control policy for these cases can be determined based on some suitable criterion such as minimax or maximax, or based on an appropriate approach like robust optimization techniques.

One of the most rigorously studied problems in the inventory management area is the well-known *Newsvendor Problem* (NVP). The decision maker of the classical NVP has to decide on the order quantity to be procured, in order to maximize (minimize) the total expected profit (cost). The procurement decision has to be taken prior to the realization of the actual demand. Upon demand realization, either leftover inventory or stock-out will occur at the end of the selling period. The decision maker should consider both of these possibilities during the decision-making process.

Another stream of literature that is closely related to inventory planning problems is the demand or market selection problems. This type of problems considers demands characteristics, and allows the supplier to select the markets to serve. Related papers on deterministic demand selection models and their variants include [4, 5, 6, 7]. For stochastic demand selection, see [8, 9, 10, 11, 12].

Taaffe et al. [8] introduced an integrated problem of the classical newsvendor problem and the demand selection decision in a single problem termed as the Selective Newsvendor Problem (SNVP). The SNVP is concerned with a firm sells a single product in a set of potential markets. The decision maker of the SNVP has to decide on the markets to cover and the order quantity to procure from an overseas supplier.

Consequently, a stream of studies discussing the SNVP were conducted including [13, 14, 9, 12, 10, 15, 16]. To the best of our knowledge, Strinka et al. [16] is the only published work that studied the MPSNVP. The authors presented several versions of the MPSNVP where most of them are solved employing the same solution procedure presented in [8]. They studied the direct extension to the work in [8] as well as the case they called 'General Case'. However, for the general case, the authors proposed a solution algorithm to obtain the optimal solution to the general case. The proposed solution algorithm runs in exponential time in the number of products. They also proposed a set of heuristics to solve the general case model.

However, Strinka et al. [16] studied some versions of the multi-product SNVP and proposed solution algorithms, still no polynomial optimal solution is available for this very practical problem. In addition to that, more general cases of the MPSNVP should be discussed. Another important issue to address is the market demand information availability and quality and its effect on market selection decisions. The study in this dissertation is really motivated by these clear gaps in the SNVP literature. Table 1.1 shows a comparison of the focus of the dissertation and the SNVP literature. Next section presents the problem statement and the MPSNVP to be studied in this dissertation.

Table 1.1: A comparison of the focus of the dissertation with SNVP literature.

Reference	Single product SNVP	MPSNVP			Risk-neutral SNVP	Risk-averse SNVP	Robust SNVP	Service level constraints
		Flexible	Full	Partial				
[8]	⋆				⋆			
[9]	⋆				⋆			
[14]	⋆				⋆			
[12]	⋆					⋆		
[13]	⋆					⋆		
[15]	⋆					⋆		⋆
[10]	⋆						⋆	
[16]		⋆	⋆		⋆			
This Dissertation		⋆	⋆	⋆	⋆	⋆	⋆	⋆

1.2 Problem Statement

This dissertation presents the SNVP with several products, termed as the Multi-Product Selective Newsvendor Problem (MPSNVP). The MPSNVP is concerned with a firm who sells several products in several markets or to several groups of customers. The decision maker of the firm selects the most profitable markets to serve, and determines the optimal order quantities of each product to be purchased from an external supplier. It is assumed that the realized demand of each product is always satisfied: either from the procured quantity or expedited from a local supplier at a higher purchasing cost. We discuss three cases of the MPSNVP:

- Flexible market entry: the firm has the ability to sell one or more products in the selected market. The firm pays a single cost per period for introducing a particular product into the selected market, this cost may consists of transportation, taxes, advertising and inventory maintaining costs.

- Full market entry: the firm sells the entire set of products in the selected market. The firm pays a single cost per period for introducing the entire set of products into the selected market, the cost might include transportation, taxes, advertising and inventory maintaining costs.

- Partial market entry: the firm pays an initial fixed cost, such as taxes costs, to enter the market and then it has the ability to sell any number of products in the selected market. The firm has to pay an additional cost for introducing a particular product into the selected market; the additional cost may include transportation, advertising and inventory maintaining costs.

For each of the above mentioned MPSNVP we will develop the mathematical modeling and propose the solution algorithms. We will study the risk-neutral, service level constrained, risk-averse MPSNVP as well as the robust MPSNVP.

1.3 Dissertation Outline

The reminder of this dissertation is organized as follows. In Chapter 2, we develop mathematical modeling for risk-neutral flexible, full and partial market entry MPSNVP. In addition, we incorporate service level constraints with each case. The mathematical manipulation for these cases results in binary nonlinear programs. We propose solution approaches that takes advantage of the special structure of the mathematical models. The proposed solution algorithms guarantee optimal solutions. In addition, we propose efficient heuristic algorithms.

Chapter 3 considers the risk-averse MPSNVP. The mathematical models for the risk-averse flexible, full and partial market entry cases are developed. The objective is to maximize profit under Conditional Value-at-Risk (CVaR) criterion. The obtained mathematical models have the same structures of the models studied in Chapter 2. Therefore, the solution approaches for the CVaR risk-averse case are similar to those for the risk-neutral case. In addition, we reformulate the mathematical models into Conic Quadratic Mixed Integer Programs (CQMIPs), and compare the performance of the proposed algorithms with that of the commercial solvers of NIP and CQMIP.

The demand uncertainty is the focus of Chapters 4, 5 and 6. In these three chapters, we discuss robust optimization approaches under box, ellipsoidal, polyhedral uncertainty sets and combinations of these uncertainty sets. The flexible, full and partial market entry MPSNVP under demand uncertainty are studied in Chapter 4, 5 and 6, respectively. The robust counterpart approach in most of the discussed cases results in Integer Linear Programs (ILP), which can be solved easily and efficiently with commercial solvers. For the cases where the robust counterpart approach results in NIP, we propose solution approaches that are either similar to the solution approaches presented in Chapter 1 or based on the reformulation the NIP into CQMIP. In addition, we provide discussion of computational results.

Finally, Chapter 7 provides some concluding remarks and some directions of potential future extensions.

CHAPTER 2

MULTI-PRODUCT SELECTIVE NEWSVENDOR PROBLEM WITH SERVICE LEVEL CONSTRAINTS

2.1 Introduction

As indicated in Chapter 1, Taaffe et al. [8] is the first study that integrated the classical newsvendor problem and the problem of market selection in a single problem known as the Selective Newsvendor Problem (SNVP). Consequently, the SNVP considers a firm that aims at maximizing its expected profit from selling a single product in a set of potential markets. The decision maker of the SNVP has to identify the optimal quantity to be manufactured or purchased from a supplier, as well as select the set markets to serve.

Practically, companies with newsvendor- structures such as sports, fashion, dairy, bakery, etc., sell more than one type of product. Lau and Lau [17] called these types of companies the Newsstand Problem (NSP). In this dissertation we present the SNVP with several products, termed as the Multi-Product Selective Newsvendor Problem (MPSNVP). The MPSNVP is concerned with a firm who sells several products in several markets or to several groups of customers. The decision maker of the firm selects the most profitable markets to serve, and determines the optimal order quantities of each product to be purchased from an external supplier. It is assumed that the realized demand of each product is always satisfied: either from the procured quantity or expedited from a local supplier at a higher purchasing cost. We discuss the three cases of the MPSNVP presented in Chapter 1:

1. Flexible market entry.

2. Full market entry.

3. Partial market entry.

In this chapter, we develop the mathematical models of the above three cases under profit risk-neutral preferences; i.e. the expected value of the profit, and propose polynomial optimal solution algorithms.

2.2 Literature Review

The NVP has received extensive interest over the past 50 years. Interested readers may refer to [18, 19] for a comprehensive review. The popularity of the NVP is due to its applicability in manufacturing and retailing industries. Service industries such as air transportation and hotels, and manufacturing industries such as dairy, sports, fashion, and electronic devices are typical newsvendor model examples. Typically, products with short life cycles aptly fit the NVP assumptions. The following literature review is organized into two parts: the SNVP literature, and the Multi-Products Newsvendor Problem (MPNVP) literature.

Taaffe et al. [8] introduced a new version of the newsvendor problem, known as the Selective Newsvendor Problem (SNVP). The SNVP considers a firm that sells a single product in a set of possible markets. The decision maker (newsvendor) is responsible for making decisions in order to cover the demand of some selected markets from a set of potential markets. The demand of the markets is assumed to be satisfied without backordering. In the case of inventory shortage, the firm subcontracts the shortage quantity from a local supplier. The modeling of this problem relies on the benefit of risk pooling effect by gathering the demand of multiple markets and ordering a single order quantity. Eppen [20] and Chen and Lin [21] studied and displayed the risk pooling effect and its benefits in inventory management. The obtained mathematical model of the problem boils down to an Integer Nonlinear Program (INLP). The authors presented an optimal algorithm to solve the problem.

Taaffe et al. [14] extended the SNVP model, where the demand follows a discrete probability distribution with all-or-nothing orders in single and multiple periods. A two-stage Stochastic Integer Program (SIP) model was developed, and then the authors proposed a tailored cutting plane algorithm based on the L-shaped method for solving the SIP.

Bakal et al. [9] provided a study of the market selection decisions and the accompanying implications on pricing policies of a firm offering a single product. The authors studied different pricing strategies for both Economic Order Quantity (EOQ) model and newsvendor model. For the newsvendor model they utilized the same solution procedure presented in [8].

Several studies were performed to incorporate risk aversion concepts into the SNVP models [13, 12, 15]. Furthermore, the application of robust optimization techniques in the case of limited information on the probability distribution of the markets' demand in the SNVP was studied in [10].

The second part of the literature review is concerned with the MPNVP. To the best of our knowledge, the first study of the MPNVP with constraints was conducted by Hadley and Whitin [22]. The authors presented a dynamic programming and Lagrange multipliers-based method to solve the model.

Lau and Lau [17] considered the *Newsstand Problem*, which is an MPNVP with multiple capacity constraints such as budget constraints, storage constraints, production constraints, etc. The authors presented the model formulation and developed a heuristic solution procedure for the problem. They are credited for noting that the optimal order quantity may assume a negative value if the non-negativity constraints are ignored. This is likely to happen when the number of products is large, with a tight budget constraint.

Abdel-Malek et al. [23] investigated the MPNVP with a budget constraint. They provided an exact solution to the MPNVP when the demand is uniformly distributed. The authors provided an approximate solution with a known level of error when the demand is exponentially distributed. For any general demand distribution, they came up with an iterative algorithm for solving the MPNVP. The iterative algorithm is called the Generic Iterative Method (GIM). The beauty of the GIM is that it gives the absolute gap within iterations.

Areeratchakul and Abdel-Malek [24] presented a simple approximate solution to the MPNVP with constraints. The proposed solution is based on quadratic programming and triangular representation of the area under the cumulative demand distribution curve. The authors obtained an exact solution to the case of uniformly distributed demand. Moreover, they provided an approximate solution to any other demand distribution.

Taleizadeh et al. [25, 26] provided genetic algorithms for solving the MPNVP with multi-constraints. They also considered total and incremental quantity discounts.

Zhang et al. [27] studied the MPNVP with budget constraints. They presented a bi-section search to obtain the optimal marginal budget benefit value of the products. The proposed solution produces an optimal or a near-optimal solution to the case of continuous demand distributions. However, it produces an approximate solution to the case of discrete demand distributions.

Recently, other papers also studied variations of the multi-product newsvendor problem with capacity and/or budget constraints and demand uncertainties [28, 29, 30, 31].

Strinka et al. [16] studied some versions of the multi-product SNVP and proposed solution algorithms that are exponential in the number of products.

In this chapter, we discuss the mathematical modeling of different cases of the MPSNVP. In addition, we propose a polynomial optimal solution to these models.

2.3 MPSNVP Mathematical Modeling

The following two questions are of critical importance for the decision maker of the SNVP:

1. What are the markets that the firm should select in order to maximize its profit?

2. What is the optimal total quantity to be procured from the external supplier?

The above two questions are generalized in this chapter by considering the case of marketing several products instead of a single product. This generalization is the main theme of the MPSNVP. We will consider the following three cases of MPSNVP: flexible, full and partial market entry. In this section, we develop the mathematical models of the above three cases with and without service level constraints.

Throughout this dissertation, we assume that $r_{ij} > e_j > c_j > v_j$, to avoid trivial solutions.

2.3.1 Case 1: MPSNVP with Flexible Market Entry

A product j is allowed to individually enter market i with an entry cost S_{ij} (paid once during the period). The profit $P(Q_j,\ y_{ij})$ depends on the realized demand and the ending inventory level, either a surplus or a shortage. The function $P(Q_j, y_{ij})$ is expressed as follows:

$$P(Q_j,\ y_{ij}) = \begin{cases} \sum\limits_{j\in J}\sum\limits_{i\in I}[r_{ij}x_{ij} - S_{ij}]\,y_{ij} + \sum\limits_{j\in J}\left[v_j\left(Q_j - \sum\limits_{i\in I}x_{ij}y_{ij}\right) - c_jQ_j\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } Q_j \geq \sum\limits_{i\in I}x_{ij}y_{ij}, \\ \sum\limits_{j\in J}\sum\limits_{i\in I}[r_{ij}x_{ij} - S_{ij}]\,y_{ij} - \sum\limits_{j\in J}\left[e_j\left(\sum\limits_{i\in I}x_{ij}y_{ij} - Q_j\right) + c_jQ_j\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } Q_j < \sum\limits_{i\in I}x_{ij}y_{ij}. \end{cases} \tag{2.1}$$

For any product j, the expected total demand per period for a given set of markets is:

$$E\left[\sum_{i\in I}x_{ij}y_{ij}\right] = \sum_{i\in I}y_{ij}E[x_{ij}] = \sum_{i\in I}\mu_{ij}y_{ij}.$$

Market demands during a period are assumed to be statistically independent, and since $y_{ij}^2 = y_{ij}$ the variance of the total demand is represented as:

$$Var\left[\sum_{i\in I} x_{ij}y_{ij}\right] = \sum_{i\in I} \sigma_{ij}^2 y_{ij}.$$

Then, the expected profit becomes:

$$\begin{aligned} E\left[P\left(Q_j,\ y_{ij}\right)\right] = &\sum_{j\in J}\sum_{i\in I}\left[r_{ij}\mu_{ij} - S_{ij}\right]y_{ij} + \sum_{j\in J} Q_j\left(v_j - c_j\right) - \sum_{j\in J}\sum_{i\in I} v_j\mu_{ij}y_{ij} \\ &- \sum_{j\in J}\left(e_j - v_j\right)\int_{\sum_{i\in I} x_{ij}y_{ij}=Q_j}^{\infty}\left(\sum_{i\in I} x_{ij}y_{ij} - Q_j\right) dF\left(x_{1j}, \cdots,\ x_{mj}\right). \end{aligned} \tag{2.2}$$

Maximizing the above profit function results in a stochastic mixed integer non-linear program, that is, involving stochastic variables with continuous probability distributions. Indeed, exact evaluation of this type of model is, in general, extremely difficult or even impossible [32, 33, 34]. The tractability of the model cannot be retrieved unless the integral term is further simplified. Given the vector $\widehat{\boldsymbol{y}}$, which specifies the selected markets, it is straightforward to show that the expected profit function $E\left[P\left(Q_j,\ y_{ij}\right)\right]$ for any product j, is concave over $Q_j \geq 0$. In this case, $E\left[P\left(Q_j,\ y_{ij}\right)\right]$ becomes a sum of separable NVP, where each NVP is formulated as

$$Q_{\widehat{\boldsymbol{y}}j}\left(v_j - c_j\right) - \left(e_j - v_j\right)\int_{x_{\widehat{\boldsymbol{y}}j}=Q_{\widehat{\boldsymbol{y}}j}}^{\infty}\left(x_{\widehat{\boldsymbol{y}}j} - Q_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right) dx_{\widehat{\boldsymbol{y}}j}\ \forall\ j \in J.$$

The optimal order quantity $Q^*_{\widehat{\boldsymbol{y}}j}$ that maximizes the above function (reduced problem) is well known, and is given by:

$$Q^*_{\widehat{\boldsymbol{y}}j} = F^{-1}_{\widehat{\boldsymbol{y}}} \left(\frac{e_j - c_j}{e_j - v_j} \right), \tag{2.3}$$

where $F^{-1}_{\widehat{\boldsymbol{y}}}$ is the inverse of $F_{\widehat{\boldsymbol{y}}}$. The function $E\left[P\left(Q_j,\ y_{ij}\right)\right]$ can be expressed as:

$$E\left[P\left(Q_j,\ y_{ij}\right)\right] = \sum_{j\in J}\sum_{i\in I}\left[\left(r_{ij} - v_j\right)\mu_{ij} - S_{ij}\right]y_{ij} - \sum_{j\in J} Q^*_{\widehat{\boldsymbol{y}}j}\left(c_j - v_j\right)$$
$$- \sum_{j\in J}\left(e_j - v_j\right)\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q^*_{\widehat{\boldsymbol{y}}j}\right),$$

where $\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q^*_{\widehat{\boldsymbol{y}}j}\right)$, is the loss function for the order quantity $Q^*_{\widehat{\boldsymbol{y}}j}$ and market selection vector $\widehat{\boldsymbol{y}}$:

$$\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j}\right) = \int_{x_{\widehat{\boldsymbol{y}}j}=Q_{\widehat{\boldsymbol{y}}j}}^{\infty} \left(x_{\widehat{\boldsymbol{y}}j} - Q_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right) dx_{\widehat{\boldsymbol{y}}j}. \tag{2.4}$$

For any product j, if the demand of each market is normally distributed with a negligible probability of negative demands, the loss function can be expressed as:

$$\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q^*_{\widehat{\boldsymbol{y}}j}\right) = \sqrt{\sum_{i\in I}\sigma^2_{ij}y_{ij}}\, L\left(z\left(\rho_j\right)\right), \tag{2.5}$$

where, $\rho_j = \frac{e_j - c_j}{e_j - v_j}$, $L\left(z\left(\rho_j\right)\right)$ is the standard normal loss function, and

$$z\left(\rho_j\right) = \frac{Q^*_{\widehat{\boldsymbol{y}}j} - \sum_{i\in I}\mu_{ij}y_{ij}}{\sqrt{\sum_{i\in I}\sigma^2_{ij}y_{ij}}} = \Phi^{-1}\left(\rho_j\right).$$

Now, the optimal order quantity, $Q^*_{\widehat{\boldsymbol{y}}j}$, for product j can be written as:

$$Q^*_{\widehat{\boldsymbol{y}}j} = \sum_{i \in I} \mu_{ij} y_{ij} + z(\rho_j) \sqrt{\sum_{i \in I} \sigma^2_{ij} y_{ij}}, \tag{2.6}$$

and the expected profit function, $E[P(Q_j,\ y_{ij})]$, can be expressed as:

$$\begin{aligned} E\left[P\left(\Phi^{-1}(\rho_j),\ y_{ij}\right)\right] = & \sum_{j \in J} \sum_{i \in I} [(r_{ij} - c_j)\mu_{ij} - S_{ij}] y_{ij} \\ & - \sum_{j \in J} (c_j - v_j)\Phi^{-1}(\rho_j) \sqrt{\sum_{i \in I} \sigma^2_{ij} y_{ij}} \\ & - \sum_{j \in J} \left(e_j - v_j\right) L\left(\Phi^{-1}(\rho_j)\right) \sqrt{\sum_{i \in I} \sigma^2_{ij} y_{ij}}. \end{aligned}$$

Let π_{ij} and $K(\rho_j)$ be defined as:

$$\pi_{ij} = (r_{ij} - c_j)\mu_{ij} - S_{ij},$$

$$K(\rho_j) = (c_j - v_j)\Phi^{-1}(\rho_j) + \left(e_j - v_j\right) L\left(\Phi^{-1}(\rho_j)\right).$$

Using the above definitions, the expected profit function, $E[P(Q_j,\ y_{ij})]$, becomes:

$$E\left[P\left(\Phi^{-1}(\rho_j),\ y_{ij}\right)\right] = \sum_{j \in J} \left[\sum_{i \in I} \pi_{ij} y_{ij} - K(\rho_j) \sqrt{\sum_{i \in I} \sigma^2_{ij} y_{ij}}\right]$$

In addition to that, for any product j, a target service level, $\alpha_j \in (0,\ 1)$, can be introduced to penalize shortages. The target service level in our case can be considered as the condition that the probability of satisfying the demand with current levels of inventory is at least α. This will minimize the probability of expediting for excess demand. We already know that the optimal order quantity for the reduced unconstrained problem is $Q_{\widehat{\boldsymbol{y}}j} = F_{\widehat{\boldsymbol{y}}}^{-1}(\rho_j)$. If the service level constraint is satisfied for product j; i.e., $F_{\widehat{\boldsymbol{y}}}^{-1}(\rho_j) \geq F_{\widehat{\boldsymbol{y}}}^{-1}(\alpha_j)$, or equivalently, $\rho_j \geq \alpha_j$, for all $j = 1, \ldots, p$, then the optimal solution of the reduced problem remains optimal, with respect to the service level constraint. On the other hand, for any product j, if $\rho_j < \alpha_j$, then the service level constraint is violated. For such a scenario, the optimal order quantity becomes $Q_{\widehat{\boldsymbol{y}}j} = F_{\widehat{\boldsymbol{y}}}^{-1}(\alpha_j)$. The service level constraints can be implemented as:

$$F_{\widehat{\boldsymbol{y}}}(Q_{\widehat{\boldsymbol{y}}j}) \geq \gamma_j$$

where $\gamma_j = \max\{\rho_j, \alpha_j\}$. To maximize the expected profit of the firm the following model of Case 1 should be solved:

Problem I:

$$\begin{aligned} \text{Max} \quad & \sum_{j \in J} \left[\sum_{i \in I} \pi_{ij} y_{ij} - K(\gamma_j) \sqrt{\sum_{i \in I} \sigma_{ij}^2 y_{ij}} \right], \\ \text{s.t.} \quad & \\ & y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I. \end{aligned} \tag{2.7}$$

The final model is a binary integer nonlinear program involving only the market selection variables. The above model can be considered as a summation of p independent SNVP.

2.3.2 Solution Algorithm to Case 1

The solution approach is based on the following theorem:

Theorem 1: *Let*

$$F_1 := \max_{w} \left\{ \sum_{i \in I} a_i w_i - \sqrt{\sum_{i \in I} b_i w_i} \;\middle|\; w_i \in \{0,\ 1\},\ a_i \in \mathbb{R},\, b_i > 0\ \forall\ i \in I \right\}.$$

Assume, without loss of generality, that a_i and b_i are indexed, such that:

$$\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \ldots \geq \frac{a_n}{b_n}.$$

Let

$$G_1(r) = \sum_{i=1}^{r} a_i - \sqrt{\sum_{i=1}^{r} b_i}.$$

If $b_i \geq 0$, then $z^* = \max_{1 \leq r \leq n} \{G_1(r)\}$ is the optimal solution value of F_1, and the optimal solution vector of F_1 is given as:

$$w_i^* = \begin{cases} 1\ \forall\ i \leq r, \\ 0\ \forall\ i > r. \end{cases}$$

Proof: See [35]; Theorem 4.2. □

In Problem I given by (2.7), the coefficient of the nonlinear term in the objective function is a squared value, and hence it is always positive. Therefore, we can directly apply Theorem 1 to solve the model; the only thing that we need is to sort the variables in a non-increasing order.

This approach was used for solving the SNVP in Taaffe et al. [8]. It is similar to the approach used in Shen et al. [35]. For each product j, the markets are sorted in a non-increasing order of the ratio of the expected Revenue to the Demand Uncertainty (RDU). The RDU is the ratio of the linear coefficient of y_{ij} to the nonlinear coefficient of y_{ij} in the objective function.

The sorting rule for each product is as follows:

$$R1: \ \frac{\pi_{[1]j}}{\sigma^2_{[1]j}} \geq \frac{\pi_{[2]j}}{\sigma^2_{[2]j}} \geq \cdots \geq \frac{\pi_{[m]j}}{\sigma^2_{[m]j}}, \quad j \in J. \tag{2.8}$$

The objective function of each product should be calculated for each set of the markets following the sequence in rule $R1$. The set of the markets that has the maximum expected total profit is the optimal selected markets set. The necessary condition of optimal solution to the above problem states that if $y_k = 1$, then $y_i = 1,\ i = 1,\ 2,\ \ldots,\ k{-}1$ (see Theorem 1). The solution procedure is summarized in the following algorithm:

Algorithm I:

Step 1: for every product j, sort the markets in the non-increasing order of the RDU ratio $R1$ presented in (2.8).

Step 2: evaluate $G_j(k)$ for $k = 1, \ldots,\ m$, defined as follows:

$$G_j\left(k\right) = \sum_{i=1}^{k} \pi_{[i]j} - K\left(\gamma_j\right)\sqrt{\sum_{i=1}^{k} \sigma^2_{[i]j}},\ k = 1, \ldots,\ m,\ \forall\ j \in J.$$

Step 3: Identify $\overline{k}$ for each $j \in J$, such that $G_j\left(\overline{k}\right) \geq G_j\left(k\right),\ k = 1,\ \ldots,\ m.$ Fix $y_{ij}^* = 1$ for the optimal set of markets K^*, where $K^* = \left\{[1],\ [2], \ldots,\ \left[\overline{k}\right]\right\}$ corresponds to the maximum (global optimal) $G_j\left(\overline{k}\right)$ for each product j, $y_{ij}^* = 0$ otherwise.

Using *Algorithm I*, we obtain p sequences; each one of them can be considered as a separate SNVP problem. The sorting sequences reduce the number of possible market selection combinations or candidate solutions for each product from 2^m to only $m+1$ possible sequences, and the computational time complexity becomes $\mathcal{O}\left(p(m \log m)\right)$.

2.3.3 Case 2: MPSNVP with Full Markets Entry

For this case, if any market i is selected, then a single fixed market entry cost, S_i, is incurred. The profit, $P\left(Q_j,\ \mathcal{Y}_i\right)$, can be expressed as:

$$P\left(Q_j,\ \mathcal{Y}_i\right) = \begin{cases} \sum\limits_{i \in I}\left(\sum\limits_{j \in J}\left(r_{ij}x_{ij}\right) - S_i\right)\mathcal{Y}_i + \sum\limits_{j \in J}\left[v_j\left(Q_j - \sum\limits_{i \in I} x_{ij}\mathcal{Y}_i\right) - c_jQ_j\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } Q_j \geq \sum\limits_{i \in I} x_{ij}\mathcal{Y}_i, \\ \sum\limits_{i \in I}\left(\sum\limits_{j \in J}\left(r_{ij}x_{ij}\right) - S_i\right)\mathcal{Y}_i - \sum\limits_{j \in J}\left[e_j\left(\sum\limits_{i \in I} x_{ij}\mathcal{Y}_i - Q_j\right) + c_jQ_j\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } Q_j < \sum\limits_{i \in I} x_{ij}\mathcal{Y}_i. \end{cases} \tag{2.9}$$

Following the same modeling procedure in Case 1, the total expected profit, $E\left[P\left(Q_j,\ \mathcal{Y}_i\right)\right]$, can be expressed as:

$$E\left[P\left(Q_j,\ \mathcal{Y}_i\right)\right] = \sum_{i\in I}\left(\sum_{j\in J}\left(r_{ij}\mu_{ij}\right) - S_i\right)\mathcal{Y}_i - \sum_{j\in J}\left(Q_j\left(c_j - v_j\right) + v_j\sum_{i\in I}\mu_{ij}\mathcal{Y}_i\right) - \sum_{j\in J}\left(e_j - v_j\right)\int_{x_{\widehat{\boldsymbol{y}}j}=Q_j}^{\infty}\left(x_{\widehat{\boldsymbol{y}}j} - Q_j\right) f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right) dx_{\widehat{\boldsymbol{y}}j}. \tag{2.10}$$

Substituting the loss function $\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q_j\right)$ from (2.4), and the optimal order quantity $Q^*_{\widehat{\boldsymbol{y}}j}$ from (2.3), in the above function, yields:

$$E\left[P\left(Q_j,\ \mathcal{Y}_i\right)\right] = \sum_{i\in I}\left(\sum_{j\in J}\left(r_{ij}\mu_{ij}\right) - S_i\right)\mathcal{Y}_i - \sum_{j\in J}\left(Q^*_{\widehat{\boldsymbol{y}}j}\left(c_j - v_j\right) + v_j\sum_{i\in I}\mu_{ij}\mathcal{Y}_i\right) - \sum_{j\in J}\left(e_j - v_j\right)\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q^*_{\widehat{\boldsymbol{y}}j}\right).$$

If the demand is normally distributed, then the loss function $\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q_j\right)$ is given by Equation (2.5), and the optimal order quantity, $Q^*_{\widehat{\boldsymbol{y}}j}$ is given by Equation (2.6). The expected profit for the normally distributed demand can be written as:

$$E\left[P\left(\Phi^{-1}\left(\rho_j\right),\ \mathcal{Y}_i\right)\right] = \sum_{i\in I}\left(\sum_{j\in J}\left(\left(r_{ij} - c_j\right)\mu_{ij}\right) - S_i\right)\mathcal{Y}_i - \sum_{j\in J}\left(c_j - v_j\right)\Phi^{-1}\left(\rho_j\right)\sqrt{\sum_{i\in I}\sigma_{ij}^2\mathcal{Y}_i} - \sum_{j\in J}\left(e_j - v_j\right) L\left(\Phi^{-1}\left(\rho_j\right)\right)\sqrt{\sum_{i\in I}\sigma_{ij}^2\mathcal{Y}_i}.$$

Defining π_i and $K(\rho_j)$as:

$$\pi_i = \sum_{j \in J} \left((r_{ij} - c_j)\, \mu_{ij} \right) - S_i,$$

$$K(\rho_j) = (c_j - v_j)\, \Phi^{-1}(\rho_j) + \left(e_j - v_j \right) L\left(\Phi^{-1}(\rho_j) \right).$$

Using the above definitions, the expected profit function, $E[P(Q_j,\ \mathcal{Y}_i)]$, becomes:

$$E\left[P\left(\Phi^{-1}(\rho_j),\ \mathcal{Y}_i\right)\right] = \sum_{i \in I} \pi_i \mathcal{Y}_i - \sum_{j \in J} K(\rho_j) \sqrt{\sum_{i \in I} \sigma_{ij}^2 \mathcal{Y}_i}.$$

If there is a target service level, $\alpha_j \in (0,\ 1)$, for product j, following a similar argument as in Case 1, this service level constraint can be implemented as:

$$F_{\hat{\boldsymbol{y}}}(Q_{\hat{\boldsymbol{y}}j}) \geq \gamma_j$$

where $\gamma_j = \max\{\rho_j, \alpha_j\}$. To maximize the expected profit of the firm the following model of Case 2 should be solved:

Problem II:

$$\begin{aligned} \text{Max} \quad & \sum_{i \in I} \pi_i \mathcal{Y}_i - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i \in I} \sigma_{ij}^2 \mathcal{Y}_i}, \\ \text{s.t.} \quad & \\ & \mathcal{Y}_i \in \{0,1\},\ \forall\ i \in I. \end{aligned} \tag{2.11}$$

Similar to Problem I, Problem II is a binary integer nonlinear program involving the market selection variables.

2.3.4 Solution Algorithm to Case 2

The solution approach is based on the following theoritical analysis:

Lemma 1: Let $f(x) : X \to \mathbb{R}$, $X \subset \mathbb{R}$ be a strictly concave function. For any scalars a, b, c where b, $c > 0$ and a, $a{+}b$, $a{+}c$, $a{+}b{+}c \in X$, we have;

$$f(a+b) - f(a) > f(a+b+c) - f(a+c). \tag{2.12}$$

Proof: Without loss of generality, assume that $b > c$, hence

$$f(a+c) = f\left(a\frac{b-c}{b} + (a+b)\frac{c}{b}\right) > \frac{b-c}{b}f(a) + \frac{c}{b}f(a+b).$$

Re-arranging terms we get:

$$\frac{f(a+b) - f(a)}{b} > \frac{f(a+b) - f(a+c)}{b-c}. \tag{2.13}$$

Similarly,

$$f(a+b) = f\left((a+c)\frac{c}{b} + (a+b+c)\frac{b-c}{b}\right) > \frac{c}{b}f(a+c) + \frac{b-c}{b}f(a+b+c).$$

Re-arranging terms we get

$$\frac{f(a+b) - f(a+c)}{b-c} > \frac{f(a+b+c) - f(a+c)}{b}. \tag{2.14}$$

Inequality (2.12) follows immediately from (2.13) and (2.14).

If $b = c$, then we have

$$f(a+b) = f\left(\frac{1}{2}a + \frac{1}{2}(a+b+c)\right) > \frac{1}{2}f(a) + \frac{1}{2}f(a+b+c),$$

$$2f(a+b) > f(a) + f(a+b+c).$$

Re-arranging the terms yields inequality (2.12). □

Theorem 2: *Let*

$$F_2 := \max_{w} \sum_{i\in I} a_i w_i - \sum_{j\in J} \sqrt{\sum_{i\in I} b_{ij} w_i}$$

s. t.

$$a_i \in \mathbb{R},\ i \in I,$$

$$b_{ij} > 0,\ \forall\ j \in J,\ \forall\ i \in I,$$

$$w_i \in \{0,\ 1\},\ i \in I.$$

Let $G_2(I) = \sum_{i\in I} a_i - \sum_{j\in J} \sqrt{\sum_{i\in I} b_{ij}}$, *and* $G_2(I\backslash r) = \sum_{i\in I\backslash r} a_i - \sum_{j\in J} \sqrt{\sum_{i\in I\backslash r} b_{ij}}$.

1. *If* $a_r \leq 0$ for some $r \in I$, *then* ${w_r}^* = 0$,

2. *If* $a_r > 0$ for some $r \in I$, *and* $G_2(I) \leq G_2(I\backslash r)$, *then* ${w_r}^* = 0$,

3. *If* $a_r > 0$ for some $r \in I$, *and* $G_2(r) = a_r - \sum_{j\in J} \sqrt{b_{rj}} > 0$, *then* ${w_r}^* = 1$.

Proof:

1. Property 1 follows immediately. Since $b_{ij} \geq 0\ \forall\ i \in I$ and $j \in J$, if $a_r \leq 0$, then for any solution $\overline{w}$ with $w_r = 1$, the objective function value is strictly

smaller than that of the solution obtained from $\overline{w}$ by setting ${w_r}^* = 0$.

2. For proving Property 2 we write the following: Let $\boldsymbol{w^*}$ be the optimal solution to F_2. Let $\Delta^* = \{i \mid {w_i}^* = 1 \text{ for some } i \in I\}$ be the optimal set of indices, then $G_2(\Delta^*) \geq G_2(\Delta)\ \forall\ \Delta \subseteq I$. If $G_2(I) \leq G_2(I \backslash r)$ for some $r \in R \subseteq I$, we need to prove that $\Delta^* \cap R = \emptyset$.

   a) By contradiction, assume that $\Delta^* = \{\Delta_1 \cup r_1\}\ \forall\ r_1 \in R$, then $G_2(\Delta_1 \cup r_1) \geq G_2(\Delta)\ \forall\ \Delta \subseteq I$, it must be true for $\Delta = \Delta_1$, then

$$\sum_{i\in\Delta_1} a_i + a_{r_1} - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j}} \geq \sum_{i\in\Delta_1} a_i - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij}},$$

   Hence,

$$a_{r_1} \geq \sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j}} - \sqrt{\sum_{i\in\Delta_1} b_{ij}}\right). \tag{2.15}$$

   given $G_2(I) \leq G_2(I \backslash r_1)$:

$$\begin{aligned}&\sum_{i\in\Delta_1} a_i + a_{r_1} + \sum_{n\in\Delta_2} a_n - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j} + \sum_{n\in\Delta_2} b_{nj}}\\ &\leq \sum_{i\in\Delta_1} a_i + \sum_{n\in\Delta_2} a_n - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}},\end{aligned}$$

   Then,

$$a_{r_1} \leq \sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j} + \sum_{n\in\Delta_2} b_{nj}} - \sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}}\right). \tag{2.16}$$

From (2.15) and (2.16):

$$
\begin{aligned}
&\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}}\right)\le a_{r_1}\\
&\le\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+\sum_{n\in\Delta_2} b_{nj}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+\sum_{n\in\Delta_2} b_{nj}}\right).
\end{aligned}
\tag{2.17}
$$

From Lemma 1, and since the square-root function is a strictly increasing concave function, then

$$
\begin{aligned}
&\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}}\\
&>\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+\sum_{n\in\Delta_2} b_{nj}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+\sum_{n\in\Delta_2} b_{nj}},
\end{aligned}
$$

therefore (2.17) doesn't hold.

b) Similarly, by contradiction, assume that $\Delta^*=\{\Delta_1\cup r_1\cup r_2\}$ for some $\{r_1,r_2\}\in R$, then $G_2\left(\Delta_1\cup r_1\cup r_2\right)\ge G_2\left(\Delta\right)\ \forall\ \Delta\subseteq I$, it must be true for $\Delta=\Delta_1$, then

$$
\sum_{i\in\Delta_1} a_i+a_{r_1}+a_{r_2}-\sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+b_{r_2j}}\ge\sum_{i\in\Delta_1} a_i-\sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij}},
$$

$$
a_{r_1}+a_{r_2}\ge\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+b_{r_2j}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}}\right).
\tag{2.18}
$$

given $G_2(I) \leq G_2(I \backslash r_1)$:

$$\sum_{i\in\Delta_1} a_i + a_{r_1} + \sum_{n\in\Delta_2} a_n - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j} + \sum_{n\in\Delta_2} b_{nj}}$$
$$\leq \sum_{i\in\Delta_1} a_i + \sum_{n\in\Delta_2} a_n - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}},$$

$$a_{r_1} \leq \sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j} + \sum_{n\in\Delta_2} b_{nj}} - \sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}}\right). \tag{2.19}$$

Also, given $G_2(I) \leq G_2(I \backslash r_2)$:

$$\sum_{i\in\Delta_1} a_i + a_{r_2} + \sum_{n\in\Delta_2} a_n - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_2 j} + \sum_{n\in\Delta_2} b_{nj}}$$
$$\leq \sum_{i\in\Delta_1} a_i + \sum_{n\in\Delta_2} a_n - \sum_{j\in J}\sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}},$$

$$a_{r_2} \leq \sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_2 j} + \sum_{n\in\Delta_2} b_{nj}} - \sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}}\right). \tag{2.20}$$

From (2.18), (2.19) and (2.20):

$$\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j} + b_{r_2 j}} - \sqrt{\sum_{i\in\Delta_1} b_{ij}}\right) \leq a_{r_1} + a_{r_2}$$
$$\leq \sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_1 j} + \sum_{n\in\Delta_2} b_{nj}} - \sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}}\right)$$
$$+ \sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij} + b_{r_2 j} + \sum_{n\in\Delta_2} b_{nj}} - \sqrt{\sum_{i\in\Delta_1} b_{ij} + \sum_{n\in\Delta_2} b_{nj}}\right),$$

Then,

$$
\begin{aligned}
&\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+b_{r_2j}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}}\right)\\
&+\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}}\right)\le a_{r_1}+a_{r_2}\\
&\le\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+\sum_{n\in\Delta_2} b_{nj}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+\sum_{n\in\Delta_2} b_{nj}}\right)\\
&+\sum_{j\in J}\left(\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_2j}+\sum_{n\in\Delta_2} b_{nj}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+\sum_{n\in\Delta_2} b_{nj}}\right).
\end{aligned}
\tag{2.21}
$$

From Lemma 1, and since the square-root function is a strictly increasing concave function,

$$
\begin{aligned}
&\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+b_{r_2j}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}}\\
&>\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_2j}+\sum_{n\in\Delta_2} b_{nj}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+\sum_{n\in\Delta_2} b_{nj}},\ \forall\, j\in J,
\end{aligned}
$$

$$
\begin{aligned}
&\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}}\\
&>\sqrt{\sum_{i\in\Delta_1} b_{ij}+b_{r_1j}+\sum_{n\in\Delta_2} b_{nj}}-\sqrt{\sum_{i\in\Delta_1} b_{ij}+\sum_{n\in\Delta_2} b_{nj}},\ \forall\, j\in J,
\end{aligned}
$$

therefore Equation (2.21) doesn't hold.

Following the same procedure in (a) and (b), we can show that $\Delta^*\cap R=\emptyset$, if $G_2(I)\le G_2(I\backslash r)$ for some $r\in R\subseteq I$.

3. Let Δ^* be the set optimal of indices, i.e., $G_2(\Delta^*)=F_2$. Let r be such that

$a_r > 0$ and $G_2(r) = a_r - \sum_{j\in J}\sqrt{b_{rj}} > 0$. If $r \in \Delta^*$, then the proof is complete. If $r \notin \Delta^*$, then consider the following:

$$G_2(\Delta^* \cup r) - G_2(\Delta^*) = a_r - \left(\sum_{j\in J}\sqrt{b_{rj} + \sum_{i\in\Delta^*} b_{ij}} - \sum_{j\in J}\sqrt{\sum_{i\in\Delta^*} b_{ij}}\right)$$

$$> \sum_{j\in J}\sqrt{b_{rj}} - \left(\sum_{j\in J}\sqrt{b_{rj} + \sum_{i\in\Delta^*} b_{ij}} - \sum_{j\in J}\sqrt{\sum_{i\in\Delta^*} b_{ij}}\right)$$

$$= \sum_{j\in J}\sqrt{b_{rj}} + \sum_{j\in J}\sqrt{\sum_{i\in\Delta^*} b_{ij}} - \sum_{j\in J}\sqrt{b_{rj} + \sum_{i\in\Delta^*} b_{ij}} > 0$$

The relation follows from the assumption that $G_2(r) = a_r - \sum\limits_{j\in J}\sqrt{b_{rj}} > 0$ and Lemma 1. The above inequality, i.e., $G_2(\Delta^* \cup r) - G_2(\Delta^*) > 0$ is a contradiction to the statement that Δ^* is the set of optimal indices. The contradiction occurred due to the false assumption that $r \notin \Delta^*$. Thus, $r \in \Delta^*$, where Δ^* is the set of optimal indices. □

The solution procedure for the MPSNVP with full market entry is based on the properties of the model structure and the properties from Theorem 2. To obtain the optimal solution to Problem II given by (2.11), we can apply the following solution procedure:

Algorithm II:

The following two algorithms are equivalently proposed to obtain the optimal solution to Problem II:

Forward Algorithm:

Step 1: calculate the marginal profit, P_i, for each market i, defined as:

$$P_i = \pi_i - \textstyle\sum_{j \in J} K(\gamma_j)\sigma_{ij}, \ \forall \ i \in I,$$

Step 2: set $k = 0$

- If $P_i \geq 0$ for some $i \in I$, then let $L_k = \{i \mid P_i > 0, \ i \in I\}$, if $L_k = I$, then go to Step 6, otherwise go to Step 3.
- If $P_i < 0 \ \forall \ i \in I$, then let $L_k = \left\{r \mid P_r = \max_{i \in I} \ P_i\right\}$, and go to Step 3.

Step 3: calculate:

$$G(L_k) = \sum_{i \in L_k} \pi_i - \sum_{j \in J} K\left(\gamma_j\right) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2}.$$

Step 4: for each market $r \in I \setminus L_k$ calculate:

$$G(L_k \cup r) = \sum_{i \in L_k} \pi_i + \pi_r - \sum_{j \in J} K\left(\gamma_j\right) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2 + \sigma_{rj}^2}, \ \forall \ r \in I \setminus L_k.$$

Step 5: for each market $r \in I \setminus L_k$ calculate

- If $G(L_k \cup r) \geq G(L_k)$ for some $r \in I \setminus L_k$, then let $U = \{r \mid G(L_k \cup r) \geq G(L_k), \ r \in I \setminus L_k\}$, and update $k = k + 1$ and $L_k = L_k \cup U$, if $L_k = I$, then go to Step 6, otherwise go to Step 4.
- If $G(L_k \cup r) < G(L_k) \ \forall r \in I \setminus L_k$, then let $\bar{r} = \left\{\bar{r} \mid G(L_k \cup \bar{r}) = \max_{r \in I \setminus L_k} \ G(L_k \cup r)\right\}$, and update $k = k+1$ and $L_k = L_k \cup \bar{r}$, if $L_k = I$, then go to Step 6, otherwise go to Step 4.

Step 6: calculate:

$$G(L_k) = \sum_{i \in L_k} \pi_i - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2}.$$

Step 7: The optimal solution is $\mathcal{Y}_i^* = 1 \ \forall \ i \in L^*$, where L^* is the set of optimal selected markets and it is defined as $L^* = \left\{ L \mid G(L) = \max_k \ G(L_k) \right\}$, $\mathcal{Y}_i^* = 0$ otherwise.

Backward Algorithm:

Step 1: set $k = 0$ and $L_k = I$.

Step 2: calculate

$$G(L_k) = \sum_{i \in L_k} \pi_i - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2}.$$

Step 3: for each market r, where $r \in L_k$, calculate:

$$G(L_k \setminus r) = \sum_{i \in L_k \setminus r} \pi_i - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i \in L_k \setminus r} \sigma_{ij}^2}.$$

- If $G(L_k \setminus r) \geq G(L_k)$ for some $r \in L_k$, then let $U = \{r \mid G(L_k \setminus r) \geq G(L_k),\ r \in L_k\}$, and update $k = k + 1$ and $L_k = L_k \setminus U$, if $L_k = \emptyset$ then set $G(L_k) = 0$ and go to Step 4, otherwise go to Step 2.

- If $G(L_k \setminus r) < G(L_k) \ \forall \ r \in L_k$, then let $\bar{r} = \left\{ \bar{r} \mid G(L_k \setminus \bar{r}) = \max_{r \in L_k} \ G(L_k \setminus r) \right\}$, and update $k = k + 1$ and $L_k = L_k \setminus \bar{r}$, if $L_k = \emptyset$ then set $G(L_k) = 0$ and go to Step 4, otherwise go to Step 2.

Step 4: The optimal solution is $\mathcal{Y}_i^* = 1 \ \forall \ i \in L^*$, where L^* is the set of optimal selected markets and it is defined as $L^* = \left\{L \mid G(L) = \max_k \ G(L_k)\right\}$, $\mathcal{Y}_i^* = 0$ otherwise.

The forward algorithm as well as the backward algorithm reduces the number of alternatives from 2^m to $m+1$, therefore the proposed algorithm obtains the optimal solution in polynomial time with computational complexity $\mathcal{O}((m+1)m\log m)$.

Heuristic I for solving Problem II:

We can further reduce the required computational time for solving the full market entry MPSNVP, Problem II shown in (2.11), by applying the following heuristic:

Step 1: sort the markets in a non-increasing order of the RDU ratio; R_{H_I}, $\frac{\sum_{j\in J}\pi_{ij}}{\sum_{j\in J}K^2(\gamma_j)\sigma_{ij}^2}$, $\forall\ i \in I$:

$$R_{H_I}:\ \frac{\sum_{j\in J}\pi_{[1]j}}{\sum_{j\in J}K^2\left(\gamma_j\right)\sigma_{[1]j}^2} \geq \frac{\sum_{j\in J}\pi_{[2]j}}{\sum_{j\in J}K^2\left(\gamma_j\right)\sigma_{[2]j}^2} \geq \cdots \geq \frac{\sum_{j\in J}\pi_{[m]j}}{\sum_{j\in J}K^2\left(\gamma_j\right)\sigma_{[m]j}^2}. \tag{2.22}$$

Step 2: following the sequence obtained from R_{H_I}, calculate the objective function values, $G\left(k\right)$, of Problem II as defined below

$$G\left(k\right) = \sum_{i=1}^{k}\pi_{[i]} - \sum_{j\in J}K\left(\gamma_j\right)\sqrt{\sum_{i=1}^{k}\sigma_{[i]j}^2},\ k = 1, \ldots,\ m,$$

Identify $\overline{k}$ such that $G\left(\overline{k}\right) \geq G\left(k\right),\ k = 1, \ldots,\ m.$ Fix $\mathcal{Y}_i^* = 1$ for the selected set of markets K^*, where $K^* = \left\{[1]\,,\ [2]\,, \ldots,\ \left[\overline{k}\right]\right\}$ corresponds to the maximum $G\left(\overline{k}\right)$, $\mathcal{Y}_i^* = 0$ otherwise.

The number of possible market selection combinations or candidate solutions based on the above heuristic is reduced 2^m to $m+1$ and the computational time complexity is $\mathcal{O}(m \log m)$.

2.3.5 Case 3: MPSNVP with Partial Markets Entry

This is the general case that generalized Case 1 and Case 2. For this case, if any market i is selected, then a fixed market entry cost per period, s_i, is incurred, and if a product j is selected to be sold in market i, then an additional cost per period, $\mathfrak{s}_{ij}$, is paid for introducing product j in market i. The profit, $P(Q_j,\, y_{ij}, \mathcal{Y}_i)$, for the partial market entry case can be expressed as:

$$P(Q_j,\; y_{ij}, \mathcal{Y}_i) = \begin{cases} \sum_{i\in I}\left(\sum_{j\in J}\left(\left(r_{ij}x_{ij} - \mathfrak{s}_{ij}\right)y_{ij}\right) - s_i\right)\mathcal{Y}_i \\ + \sum_{j\in J}\left(v_j\left(Q_j - \sum_{i\in I} x_{ij}y_{ij}\right) - c_jQ_j\right), \\ \qquad\qquad \text{for } Q_j \geq \sum_{i\in I} x_{ij}y_{ij}, \\ \sum_{i\in I}\left(\sum_{j\in J}\left(\left(r_{ij}x_{ij} - \mathfrak{s}_{ij}\right)y_{ij}\right) - s_i\right)\mathcal{Y}_i \\ - \sum_{j\in J}\left(e_j\left(\sum_{i\in I} x_{ij}y_{ij} - Q_j\right) + c_jQ_j\right), \\ \qquad\qquad \text{for } Q_j < \sum_{i\in I} x_{ij}y_{ij}. \end{cases} \tag{2.23}$$

Notice that, if the firm sells any product j in any market i, then both variables y_{ij} and $\mathcal{Y}_i$ are 1, however if the firm decides to enter market i this does not necessarily mean that the firm will sell product j in that market. We can now conclude that the following constraints control the relation between y_{ij} and $\mathcal{Y}_i$:

$$\mathcal{Y}_i \geq y_{ij},\; \forall j \in J,\; \forall\, i \in I.$$

Now, following the same modeling procedure in Cases 1 and 2, the total expected profit function, $E\left[P\left(Q_j, y_{ij},\ \mathcal{Y}_i\right)\right]$, can be expressed as:

$$\begin{aligned}E\left[P\left(Q_j,\ y_{ij},\mathcal{Y}_i\right)\right]=&\sum_{i\in I}\left(\sum_{j\in J}\left(\left(r_{ij}\mu_{ij}-\mathfrak{s}_{ij}\right)y_{ij}\right)-s_i\right)\mathcal{Y}_i\\&-\sum_{j\in J}Q_j\left(c_j-v_j\right)-\sum_{j\in J}\sum_{i\in I}v_j\mu_{ij}y_{ij}\\&-\sum_{j\in J}\left(e_j-v_j\right)\int_{x_{\widehat{\boldsymbol{y}}j}=Q_j}^{\infty}\left(x_{\widehat{\boldsymbol{y}}j}-Q_j\right)f\left(x_{\widehat{\boldsymbol{y}}j}\right)dx_{\widehat{\boldsymbol{y}}j}.\end{aligned}\tag{2.24}$$

Substituting the loss function, $\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q_j\right)$, from (2.4), and the optimal order quantity, $Q^*_{\widehat{\boldsymbol{y}}j}$, from (2.3), we get:

$$\begin{aligned}E\left[P\left(Q_j,\ y_{ij},\mathcal{Y}_i\right)\right]=&\sum_{i\in I}\left(\sum_{j\in J}\left(\left(r_{ij}\mu_{ij}-\mathfrak{s}_{ij}\right)y_{ij}\right)-s_i\right)\mathcal{Y}_i-\sum_{j\in J}Q^*_{\widehat{\boldsymbol{y}}j}\left(c_j-v_j\right)\\&-\sum_{j\in J}\sum_{i\in I}v_j\mu_{ij}y_{ij}-\sum_{j\in J}\left(e_j-v_j\right)\Lambda_{\widehat{\boldsymbol{y}}j}\left(Q^*_{\widehat{\boldsymbol{y}}j}\right).\end{aligned}$$

For normally distributed demand, similar to Case 1 and 2, the function $E\left[P\left(Q_j,\ y_{ij},\mathcal{Y}_i\right)\right]$ can be written as:

$$\begin{aligned}E\left[P\left(\Phi^{-1}\left(\rho_j\right),\ y_{ij},\mathcal{Y}_i\right)\right]=&\sum_{i\in I}\left(\sum_{j\in J}\left(\left(\left(r_{ij}-c_j\right)\mu_{ij}-\mathfrak{s}_{ij}\right)y_{ij}\right)-s_i\right)\mathcal{Y}_i\\&-\sum_{j\in J}\left(c_j-v_j\right)\Phi^{-1}\left(\rho_j\right)\sqrt{\sum_{i\in I}\sigma^2_{ij}y_{ij}}\\&-\sum_{j\in J}\left(e_j-v_j\right)L\left(\Phi^{-1}\left(\rho_j\right)\right)\sqrt{\sum_{i\in I}\sigma^2_{ij}y_{ij}}.\end{aligned}$$

Defining π_{ij} and $K(\rho_j)$ as:

$$\pi_{ij} = (r_{ij} - c_j)\,\mu_{ij} - \mathfrak{s}_{ij},$$

$$K(\rho_j) = (c_j - v_j)\,\Phi^{-1}(\rho_j) + \left(e_j - v_j\right) L\left(\Phi^{-1}(\rho_j)\right).$$

Using these definitions, and noting that $y_{ij}\mathcal{Y}_i = y_{ij}$ because both variables are binary and $\mathcal{Y}_i \geq y_{ij}$, then the expected profit function, $E\left[P\left(\Phi^{-1}(\rho_j),\ y_{ij}, \mathcal{Y}_i\right)\right]$, becomes:

$$E\left[P\left(\Phi^{-1}(\rho_j),\ y_{ij}, \mathcal{Y}_i\right)\right] = \sum_{i\in I}\sum_{j\in J}\pi_{ij}y_{ij} - \sum_{j\in J}K(\rho_j)\sqrt{\sum_{i\in I}\sigma_{ij}^2 y_{ij}} - \sum_{i\in I}s_i\mathcal{Y}_i.$$

If there is a target service level, $\alpha_j \in (0,\ 1)$, for product j, it can be implemented as:

$$F_{\hat{\boldsymbol{y}}}\left(Q_{\hat{\boldsymbol{y}}j}\right) \geq \gamma_j$$

where $\gamma_j = \max\{\rho_j, \alpha_j\}$. To maximize the expected profit of the firm we solve the following model of Case 3:

Problem III:

$$\begin{aligned}
&\text{Max} \quad \sum_{i\in I}\sum_{j\in J}\pi_{ij}y_{ij} - \sum_{j\in J}K(\gamma_j)\sqrt{\sum_{i\in I}\sigma_{ij}^2 y_{ij}} - \sum_{i\in I}s_i\mathcal{Y}_i,\\
&\text{s.t.}\\
&\quad \mathcal{Y}_i \geq y_{ij},\ \forall\, j \in J,\ \forall\, i \in I,\\
&\quad \mathcal{Y}_i, y_{ij} \in \{0, 1\},\ \forall\, j \in J,\ \forall\, i \in I.
\end{aligned} \tag{2.25}$$

Similar to Problems I and II, Problem III is a binary integer nonlinear program involving the market selection variables for each product.

2.3.6 Solution Algorithm to Case 3

The solution procedure for the MPSNVP with partial market entry is based on the properties of the model structure and the properties from Theorem 2. To obtain the optimal solution to Problem III given by (2.25), we can apply the following solution procedure which contains two stages:

Algorithm III:

Stage 1

We solve for the following part of Problem III shown in (2.25),

$$\text{Max} \quad \sum_{i\in I}\sum_{j\in J}\pi_{ij}y_{ij} - \sum_{j\in J}K(\gamma_j)\sqrt{\sum_{i\in I}\sigma_{ij}^2 y_{ij}},$$

s.t.

$$y_{ij} \in \{0,1\}, \ \forall\, j \in J, \ \forall\, i \in I.$$

The above reduced model is equivalent to Problem I presented in (2.7), therefore we apply *Algorithm I* to get the candidate markets for each product, i.e. $y_{ij} = 1 \ \forall\, i \in K^*$, where K^* is defined and obtained in *Algorithm I*.

Stage 2

In this stage we consider the candidate solutions for y_{ij} that are obtained from Stage 1. Stage 2 can be achieved by implementing one of the following algorithms:

Forward Algorithm:

Step 1: for any market i, if $y_{ij} = 0 \ \forall \ j \in J$, then fix $\mathcal{Y}_i^* = 0$. Let $T = \{i \mid y_{ij} = 0 \ \forall \ j \in J, \ i \in I\}$, update $I = I \setminus T$.

Step 2: calculate the marginal profit, P_i, for each market i, defined as:

$$P_i = \sum_{j \in J} \pi_{ij} - \sum_{j \in J} K(\gamma_j)\sigma_{ij} - s_i, \ \forall \ i \in I.$$

Step 3: set $k = 0$

- If $P_i \geq 0$ for some $i \in I$, then let $L_k = \{i \mid P_i > 0, \ i \in I\}$, if $L_k = I$, then go to Step 7, otherwise go to Step 4.
- If $P_i < 0 \ \forall \ i \in I$, then let $L_k = \left\{ r \mid P_r = \max_{i \in I} \ P_i \right\}$, and go to Step 4.

Step 4: calculate:

$$G(L_k) = \sum_{i \in L_k} \sum_{j \in J} \pi_{ij} - \sum_{j \in J} K\left(\gamma_j\right) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2} - \sum_{i \in L_k} s_i.$$

Step 5: for each market $r \in I \setminus L_k$ calculate:

$$G(L_k \cup r) = \sum_{i \in L_k} \pi_i + \pi_r - \sum_{j \in J} K\left(\gamma_j\right) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2 + \sigma_{rj}^2} - \sum_{i \in L_k} s_i - s_r, \ \forall \ r \in I \setminus L_k.$$

Step 6:

- If $G(L_k \cup r) \geq G(L_k)$ for some $r \in I \setminus L_k$, then let $U = \{r \mid G(L_k \cup r) \geq G(L_k), \ r \in I \setminus L_k\}$, and update $k = k + 1$ and $L_k = L_k \cup U$, if $L_k = I$, then go to Step 8, otherwise go to Step 5.
- If $G(L_k \cup r) < G(L_k) \ \forall \ r \in L_k \setminus I$, then let $\bar{r} =$

$\left\{\bar{r} \mid G(L_k \cup \bar{r}) = \max_{r \in I \setminus L_k} G(L_k \cup r)\right\}$, and update $k = k + 1$ and $L_k = L_k \cup \bar{r}$, if $L_k = I$, then go to Step 8, otherwise go to Step 5.

Step 7: calculate:

$$G(L_k) = \sum_{i \in L_k} \sum_{j \in J} \pi_{ij} - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2} - \sum_{i \in L_k} s_i.$$

Step 8: The optimal solution is $\mathcal{Y}_i^* = 1 \; \forall \; i \in L^*$, where L^* is the set of optimal selected markets and it is defined as $L^* = \left\{L \mid G(L) = \max_k G(L_k)\right\}$, $\mathcal{Y}_i^* = 0$ otherwise. While $y_{ij}^* = 1 \; \forall \; i \in L^* \cap K^*$, $y_{ij}^* = 0$ otherwise.

Backward Algorithm:

Step 1: for any market i, if $y_{ij} = 0 \; \forall \; j \in J$, then fix $\mathcal{Y}_i^* = 0$. Let $T = \{i \mid y_{ij} = 0 \; \forall \; j \in J, \; \forall \; i \in I\}$, update $I = I \setminus T$.

Step 2: set $k = 0$ and $L_k = I$,

Step 3: calculate

$$G(L_k) = \sum_{i \in L_k} \sum_{j \in J} \pi_{ij} - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i \in L_k} \sigma_{ij}^2} - \sum_{i \in L_k} s_i.$$

Step 4: for each market r, where $r \in L_k$, calculate:

$$G(L_k \setminus r) = \sum_{i \in L_k \setminus r} \pi_i - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i \in L_k \setminus r} \sigma_{ij}^2} - \sum_{i \in L_k} s_i.$$

- If $G(L_k \setminus r) \geq G(L_k)$ for some $r \in L_k$, then let $U = \{r \mid G(L_k \setminus r) \geq G(L_k), \; r \in L_k\}$, and update $k = k+1$ and $L_k = L_k \setminus U$, if

$L_k = \emptyset$ then set $G(L_k) = 0$ and go to Step 5, otherwise go to Step 3.

- If $G(L_k \setminus r) < G(L_k)\ \forall\ r \in L_k$, then let $\bar{r} = \left\{\bar{r} \mid G(L_k \setminus \bar{r}) = \max_{r \in L_k} G(L_k \setminus r)\right\}$, and update $k = k+1$ and $L_k = L_k \setminus \bar{r}$, if $L_k = \emptyset$ then set $G(L_k) = 0$ and go to Step 5, otherwise go to Step 3.

Step 5: The optimal solution is $\mathcal{Y}_i^* = 1\ \forall\ i \in L^*$, where L^* is the set of optimal selected markets and it is defined as $L^* = \left\{L \mid G(L) = \max_{k} G(L_k)\right\}$, $\mathcal{Y}_i^* = 0$ otherwise. While $y_{ij}^* = 1\ \forall\ i \in L^* \cap K^*$, $y_{ij}^* = 0$ otherwise.

Algorithm III reduces the number of possible market selection combinations or candidate solutions from 2^{pm} to $(p+1)(m+1)$, therefore the proposed algorithm obtains the optimal solution in polynomial time with computational complexity $\mathcal{O}(pm \log m + \sum_{n=1}^{m} n \log n)$.

Heuristic II as an Alternative to Stage 2:

We can further reduce the required computational time for solving the partial market entry MPSNVP, Problem III shown in (2.25), by applying Stage 1 in *Algorithm III* then replace Stage 2 by the following heuristic:

Step 1: for any market i, if $y_{ij} = 0\ \forall\ j \in J$, then fix $\mathcal{Y}_i^* = 0$. Let $T = \{i \mid y_{ij} = 0\ \forall\ j \in J,\ \forall\ i \in I\}$, update $I = I \setminus T$.

Step 2: sort the markets in a non-increasing order of the RDU ratio; $R_{H_{II}}$, $\frac{\sum_{j \in J} \pi_{ij} - s_i}{\sum_{j \in J} K^2(\gamma_j)\sigma_{ij}^2}$, $\forall\ i \in I$:

$$R_{H_{II}}: \frac{\sum_{j \in J} \pi_{[1]j} - s_{[1]}}{\sum_{j \in J} K^2(\gamma_j)\,\sigma_{[1]j}^2} \geq \frac{\sum_{j \in J} \pi_{[2]j} - s_{[2]}}{\sum_{j \in J} K^2(\gamma_j)\,\sigma_{[2]j}^2} \geq \cdots \geq \frac{\sum_{j \in J} \pi_{[m-l-t]j} - s_{[m]}}{\sum_{j \in J} K^2(\gamma_j)\,\sigma_{[m]j}^2}. \tag{2.26}$$

Step 3: following the sequence obtained from $R_{H_{II}}$, calculate the objective function values, $G(k)$, of Problem III as defined below

$$G(k) = \sum_{i=1}^{k} \sum_{j \in J} \pi_{[i]j} - \sum_{j \in J} K(\gamma_j) \sqrt{\sum_{i=1}^{k} \sigma_{[i]j}^2} - \sum_{i=1}^{k} s_{[i]},\ k = 1, \ldots,\ m.$$

Step 4: Identify $\underline{k}$ for each $j \in J$, such that $G_j(\underline{k}) \geq G_j(k),\ k = 1, \ldots,\ m.$ Fix $\mathcal{Y}_i^* = 1$ for the selected set of markets $\overline{K}^*$, where $\overline{K}^* = \{[1],\ [2], \ldots,\ [\underline{k}]\}$ corresponds to the maximum $G_j(\underline{k})$ for each product j, $\mathcal{Y}_i^* = 0$ otherwise. While $y_{ij}^* = 1\ \forall\ i \in \overline{K}^* \cap K^*$, $y_{ij}^* = 0$ otherwise.

Algorithm III based on Heuristic II reduces the number of possible market selection combinations or candidate solutions from 2^{pm} to $(p+1)(m+1)$, therefore the proposed algorithm obtains the solution in polynomial time with computational complexity $\mathcal{O}((p+1)m \log m)$.

2.4 Computational Results

In this section, we evaluate the quality of the proposed heuristic solutions based on sorting rules similar to the optimal sorting rule in *Algorithm I.* In addition to the sorting rule in Heuristics I and II, for simplicity and consistency we call it here R_{1H}, two additional sorting rules, R_{2H} and R_{3H} are suggested for arranging the variables of the following general problem:

$$F := \max_w \sum_{i\in I} a_i w_i - \sum_{j\in J}\sqrt{\sum_{i\in I} b_{ij} w_i}$$

s. t.

$$a_i \in \mathbb{R},\ \forall\, i \in I,$$

$$b_{ij} > 0,\ \forall\, i \in I,\ \forall\, j \in J,$$

$$w_i \in \{0,\ 1\},\ \forall\, i \in I,$$

where $|I| = m$

The variables of the problem will be arranged in a non-increasing order of one of the following ratios:

$$R_{1H} = \frac{a_i}{\sum\limits_{j\in J} b_{ij}}, R_{2H} = \frac{a_i}{\sum\limits_{j\in J}\sqrt{b_{ij}}}, R_{3H} = \frac{a_i}{\left(\sum\limits_{j\in J}\sqrt{b_{ij}}\right)^2}.$$

Then, we calculate the objective function values, $G(k)$, as defined below

$$G(k) = \sum_{i=1}^{k} a_{[i]} - \sum_{j\in J}\sqrt{\sum_{i=1}^{k} b_{[i]j}},\ k = 1, \ldots,\ m,$$

where the sequence $[1],\ [2], \ldots,\ [m]$ follows a non-increasing order of R_{1H}, R_{2H} or R_{3H}.

We choose the set of variables $\{[1],\ [2], \ldots,\ [k]\}$ that corresponds to the maximum $G(k)$. We fix $w_i = 1$ for the variables $\{[1],\ [2], \ldots,\ [k]\}$.

We consider a problem with $|I| = 20$ variables and $|J| = 5$. The positive term coefficients i.e. a_i, are drawn from uniform distributions on [0, 2,000]. b_{ij} are also drawn from uniform distributions on [0, 1,500,000]. One thousand instances of the test problem are solved individually using the proposed sorting rules R_{1H}, R_{2H} and R_{3H}. The optimal solution is obtained via complete enumeration. The comparison between the optimal solution and the heuristic results shows the following:

1. The heuristic based on the proposed sorting rule R_{1H} always gives the optimal solution;

2. The heuristic based on R_{2H} fails in obtaining the optimal in 81 instances out of 1,000 runs;

3. The heuristic based on R_{3H} fails in obtaining the optimal solution in 27 instances out of 1,000 runs.

Now, we consider a larger problem with $|I| = 100$ variables and $|J| = 3$, 4 and 5. The positive term coefficients i.e. a_i, are drawn from uniform distributions on [0, 500]. b_{ij} are also drawn from uniform distributions on [0, 500,000]. One thousand instances of the test problem are solved individually using the proposed heuristic based on the sorting rules R_{1H}, R_{2H} and R_{3H}. In Table 1, we report the number of times a heuristic based on one rule beats the heuristic based on another rule. For instance, in the first row of the table, 626 indicates that out of 1000 runs, in 626 instances the heuristic based on R_{1H} gives a better objective function value than the heuristic based on R_{2H}. To improve the quality of the obtained solution

Table 2.1: A comparison between the heuristic solutions based on the proposed sorting ratios for MPSNVP.

	$\|J\| = 3$			$\|J\| = 4$			$\|J\| = 5$		
	R_{1H}	R_{2H}	R_{3H}	R_{1H}	R_{2H}	R_{3H}	R_{1H}	R_{2H}	R_{3H}
R_{1H}	-	626	334	-	718	467	-	771	557
R_{2H}	1	-	145	1	-	188	0	-	221
R_{3H}	2	558	-	1	619	-	0	630	-

using the proposed heuristic, we can apply a local search to obtain a local star. The local search provides a mechanism to compare the performance of the sorting rules.

In the following, we introduce the definition of a local star solution used in non-convex optimization [36, 37]. Loosely speaking, a solution is a local star if the objective function at this point is not less than its adjacent points. In the definition below, we define a local star point in the context of the problem on hand.

Definition:

Let $G(w) = \sum_{i \in I} a_i w_i - \sum_{j \in J} \sqrt{\sum_{i \in I} b_{ij} w_i}$, $a_i,\ b_{ij} \in \mathbb{R}\ \forall\ i \in I,\ \forall\ j \in J$; and the objective is to maximize $G(w)$.

A point $\boldsymbol{w}^* = (\ w_1^*,\ w_2^*,\ \ldots, w_m^*)$ is a local star solution if $G(\boldsymbol{w}^*) \geq G(\boldsymbol{w}^r)\,,\ r = 1,\ 2, \ldots,\ m$, where $\boldsymbol{w}^r = (\ w_1^r,\ w_2^r,\ \ldots, w_m^r)\ ,\ r = 1,\ 2,\ \ldots,\ m$ are the points adjacent to $\boldsymbol{w}^*$; i.e. $w_i^r = w_i^*$ for $i = 1,\ 2, \ldots,\ m,\ i \neq r,\ w_r^r = 1 - w_r^*$.

We evaluate the local search performance by considering the above large-scale problem. Applying the local search shows that:

1. In the case of using R_{1H} as the sorting rule, the objective function value is improved for seven instances out of 1,000 runs and the local search converged, at most, after three iterations of the local search.

2. In the case of using R_{2H} as the sorting rule, the objective function value is improved for 762 instances out of 1,000 runs and the local search converged, at most, after ten iterations of the local search.

3. In the case of using R_{3H} as the sorting rule, the objective function value is improved for 538 instances out of 1,000 runs and the local search converged, at most, after nine iterations of the local search.

The overall conclusion of these experiments implies that the sorting rule R_{1H} is a good choice to be used for constructing the proposed heuristic.

2.4.1 The relationship between the market selection decisions and the service level constraints

The relationship between the market selection decisions and the service level constraints In this section, we show how the service level constraint affects the market selection decision of the MPSNVP. Consider an MPSNVP problem that involves three products with $\rho_j = 0.2$ for each product and five possible markets to consider. The optimal solution to the problem without service level constraints is to select all markets. The heuristic solution based on the RDU ratio generates optimal solution. Figure 2.1 shows the optimal market selection vector for different service level values in the range $\alpha_j \in (0, 1)$. We examine the same service level value for all products. The markets are sorted in a non-increasing order of the RDU ratio. Figure 1 shows that optimal solution is given by $\widehat{\boldsymbol{y}} = [1,\ 1,\ 1,\ 1,\ 1]$ for $\alpha_j \leq 0.6$. On the other hand, as the service level increases the optimal solution vector changes to [1, 1, 1, 1, 0], [1, 1, 1, 0, 0], [1, 1, 0, 0, 0] and [1, 0, 0, 0, 0]. Increasing in the service level results in removing markets with less RDU ratio before the markets with large RDU ratio. This implies that exaggerating the service level value guarantees that shortage is reduced and the need for expediting is rare; however, it also results in less profit.

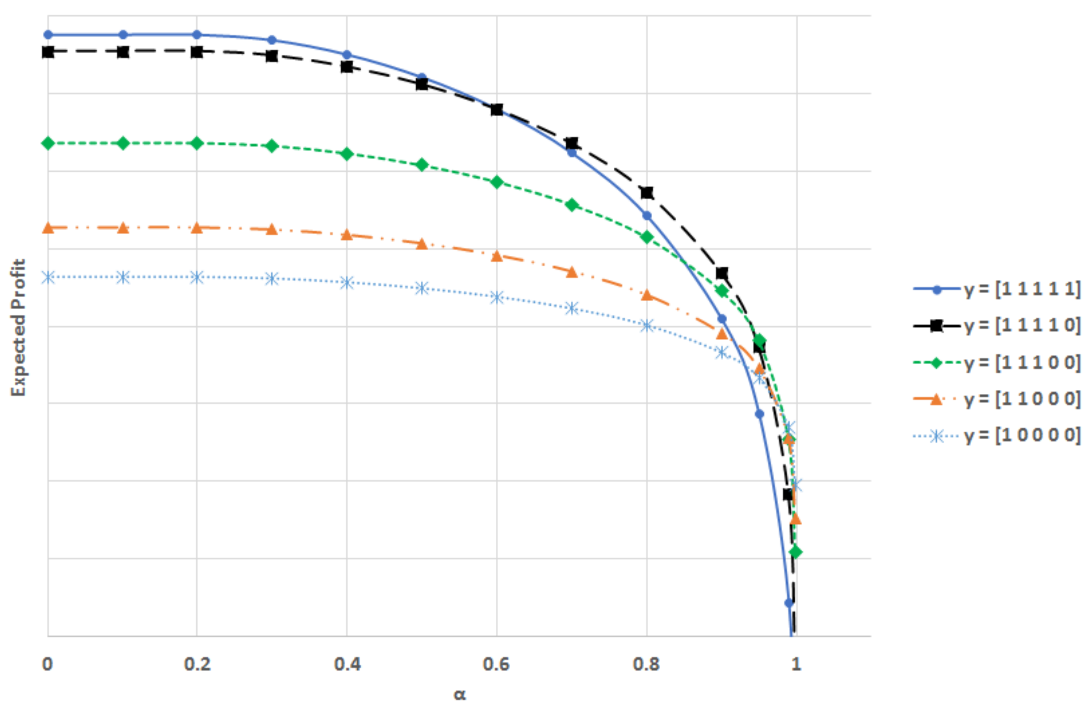


Figure 2.1: Expected profit as a function of the service level for different market selection decisions.

2.4.2 The relationship between the market selection decisions and the unit selling price

At contracting stage with markets, the firm should choose whether to allow the selling price per unit to be market dependent or fixed for all markets. The determination of the selling price per unit is related to other parameters such as the mean and variance of the market demand and the market entry cost. For the partial market entry MPSNVP, the attractive selling price for product j is

$$r_{ij} > \frac{\mathfrak{s}_{ij} + c_j \mu_{ij} + K(\gamma_j)\,\sigma_{ij}}{\mu_{ij}}$$

If the above expression of the selling price for product j is satisfied, then market i becomes a strong candidate for selection. The coefficient $\frac{\sigma_{ij}}{\mu_{ij}}$ is the coefficient of variation for market i demand. If the firm can reduce the coefficient of variation, then the selling price per unit can be reduced. This will also enable higher market share.

In addition, market i is selected if the following expression is satisfied,

$$\sum_{j=1}^{p} r_{ij}\mu_{ij} > \sum_{j=1}^{p} \left[K(\gamma_j)\,\sigma_{ij} + \mathfrak{s}_{ij} + c_j \mu_{ij}\right] + s_i$$

Controlling the coefficients of variation is possible by at least two ways. First, employing effective advertising to enhance the sales volume and reduce variability. Second, implementing strong forecasting and market study tools will reduce uncertainty and enable good estimation of the demand parameters.

Now, if the firm already contracted with a set of markets k and there is a potential market $k+1$ for the firm to add, the marginal profit due to this market should be checked. If market $k+1$ satisfies the following expression,

$$\sum_{j=1}^{p} r_{k+1,j}\mu_{k+1,j} > \sum_{j=1}^{p} K(\gamma_j) \left[\sqrt{\sum_{i=1}^{k+1} \sigma_{ij}} - \sqrt{\sum_{i=1}^{k} \sigma_{ij}}\right] + \sum_{j=1}^{p} \left[\mathfrak{s}_{k+1,j} + c_j \mu_{k+1,j}\right] + s_{k+1}$$

then, market $k+1$ should be added to the market list. Note that the risk-pooling concept plays an important role in this case. The more the contracted markets the less the demand variability [20, 38]. The risk pooling effect constructs a buffer in safety stock that accommodates the additional variability by the new market.

CHAPTER 3

RISK-AVERSE MULTI-PRODUCT SELECTIVE NEWSVENDOR PROBLEM UNDER CVAR CRITERION

3.1 Introduction

In the previous chapter, we study the risk-neutral MPSNVP. In this chapter, we study the risk-averse version of the MPSNVP. We consider the Conditional Value-at-Risk (CVaR) as the risk measure. Traditionally, the classical newsvendor problem and the SNVP are modeled to obtain the maximum expected profit or, equivalently, the minimum expected cost. This modeling approach is suitable for risk-neutral decision makers. However, there are decision makers with risk-taking preferences and others with risk-aversion preferences [19]. Empirical studies based on interviews with executives, and based on questionnaires from executives of international firms, showed that the behavior of decision-makers in the real world is always consistent with the loss aversion preferences [39]. Practically, not all companies have financial resources to support potential losses due to demand uncertainty; risk-aversion preference is suitable for this kind of companies [12]. In addition, in real world, companies might be concerned with achieving a predetermined target of profit or avoiding a certain level of losses [40]. These facts motivate the study of risk aversion preferences of decision makers and the consequences of these preferences.

3.2 Literature Review

In recent years, researchers have focused on the risk-averse newsvendor problem and have provided different approaches to incorporate risk-aversion to the newsvendor problem. Atkinson [41] studied the risk-aversion attitude of a manager, and showed that, such a manager will order a smaller quantity than a risk-neutral manager will. Some of the researchers exploited the utility function to model the risk-aversion in the newsvendor problem [42, 43, 44, 45]. Other studies maximized the probability of achieving a predetermined profit [44, 46]. Another approach to incorporate risk-aversion to the newsvendor problem is to optimize the mean-variance function of the newsvendor model [47, 48, 49]. The recent trend in the risk-averse newsvendor literature is focusing on the use of risk measures, such as Value-at-Risk (VaR) [50, 51, 52, 53], Conditional Value-at-Risk (CVaR) [40, 54, 12, 55, 56, 57], spectral measures of risk [58], and law invariant measures of risk [59, 60].

Several studies were performed to extend the SNVP by incorporating risk-aversion preferences to the SNVP.

Taaffe and Tirumalasetty [13] introduced the risk aversion concept into the SNVP. They provided two risk models; one of them is related to the critical predefined profit level and the other is related to minimizing the worst case profits of a given demand. The authors proposed heuristic procedures for solving each model of the two resulting stochastic integer programs.

Chahar and Taaffe [12] extended the all-or-nothing model of [14] to the risk aversion case. They applied the CVaR and the mean-CVaR approaches to control the demand risk.

Waring [15] studied the effect of the value-at-risk (VaR) as well as the CVaR and the mean-CVaR as risk measures on the optimal decisions and profit of the SNVP. She also evaluated the effect of the fluctuations of the risk preference levels on the SNVP performance.

The common approach to treating risk aversion is through utility function. However, several studies state that expected utility is not a dedicated risk measure and is difficult or even impossible to be implemented in practice [57, 60, 61]. Artzner et al. [62] introduced four coherency axioms, when a risk measure satisfies these axioms; it is known to be a coherent measure of risk.

Gotoh and Takano [40] and Choi et al. [59] justified the utilization of coherent risk measures; such as CVaR, as a strong alternative to utility function approach in expressing the risk aversion preferences of decision makers and demonstrate that optimizing the CVaR never conflicts with optimizing the expectation of any risk-averse utility function by stating that:

- Expected utility models as well as coherent risk measures are convex and consistent with stochastic dominance.
- Coherent risk measures satisfy the axioms of Translation Equivariance and Positive Homogeneity.
- For expected utility models, Translation Equivariance and Positive Homogeneity axioms typically do not hold.

Choi et al. [59] stated the following ”For a multi-product newsvendor, the Translation Equivariance axiom implies that adding a constant gain is equivalent to changing the vendors performance measure by the same amount; the Positive Homogeneity axiom guarantees that one obtains the same solution when considering the total profit or the profit rate (i.e., average profit per product), and when one changes the currency in which the profit is calculated.”

The above arguments demonstrate that implementing coherent risk measures to model risk aversion attitudes of the multi-product newsvendor problem can be more attractive than implementing the expected utility approaches due to the appealing properties of coherent risk measures.

Pflug [63] proved that CVaR is a coherent risk measure. The appealing property of CVaR; and in fact all coherent risk measures, is that it is consistent with the stochastic dominance conditions and this leads to convex optimization problems [57, 64].

In this chapter, we take risk preferences of decision makers into consideration. We study the CVaR risk-averse MPSNVP.

3.3 CVaR Risk-Averse MPSNVP Mathematical Modeling

In this section, we introduce three cases of the CVaR risk-averse mathematical modeling and optimization for the three cases of the MPSNVP discussed in Chapter 2, namely, flexible, full and partial market entry MPSNVP. For the sake of simplicity, throughout this chapter, we assume that the selling price for each product is the same in all markets, i.e. $r_{ij} = r_j$.

3.3.1 Case 1: CVaR Risk-Averse MPSNVP with Flexible Market Entry

The profit function for the flexible market entry MPSNVP; $P\left(Q_j,\ y_{ij}\right)$, is given by (2.1). The expected profit model or the risk-neutral version of the problem is presented in (2.2).

In this section, we present the flexible market entry risk-averse MPSNVP. We use the CVaR criterion as the risk measure. Specifically, CVaR at a certain level, say $1-\eta$, can be defined as the average profit in the left $(1-\eta)$ tail of the profit distribution. The literature of ordering decisions based on CVaR; e.g., [40, 55, 65, 66], assume that the ris-averse newsvendor aims at maximizing the expected profit that falls below some $(1-\eta)$ quantile of the profit distribution, we denote this quantile as θ. Following this definition of CVaR, we can write the risk-averse profit function based on the CVaR criterion as follows:

$$CVaR\left[P\left(Q_j,\ y_{ij}\right)\right]=\sum_{j\in J}\max_{\theta_j\in\mathbb{R}}\Pi_j\left(Q_j,\ y_{ij},\theta_j\right), \tag{3.1}$$

where $\Pi_j\left(Q_j,\ y_{ij},\theta_j\right)=\theta_j-\frac{1}{1-\eta_j}E\left[\left(\theta_j-P_j\left(Q_j,\ y_{ij}\right)\right)^+\right]$, $(z)^+=\max\{z,\ 0\}$ and

$$P_j\left(Q_j,\ y_{ij}\right)=\begin{cases}\sum_{i=1}^{m}\left(r_jx_{ij}-S_{ij}\right)y_{ij}+v_j\left(Q_j-\sum_{i=1}^{m}x_{ij}y_{ij}\right)-c_jQ_j, \\ \qquad\qquad\qquad \text{for } Q_j\geq\sum_{i=1}^{m}x_{ij}y_{ij}, \\ \sum_{i=1}^{m}\left(r_jx_{ij}-S_{ij}\right)y_{ij}-e_j\left(\sum_{i=1}^{m}x_{ij}y_{ij}-Q_j\right)-c_jQ_j, \\ \qquad\qquad\qquad \text{for } Q_j<\sum_{i=1}^{m}x_{ij}y_{ij}.\end{cases}$$

Proposition 1: *Given a vector* $\widehat{\boldsymbol{y}}$*; which specifies the selected markets, the optimal order quantity for any product* j*;* $Q^*_{\widehat{\boldsymbol{y}}j}$*, that maximizes the CVaR risk-averse profit function* (3.1)*, is given by*:

$$Q^*_{\widehat{\boldsymbol{y}}j}=F^{-1}_{\widehat{\boldsymbol{y}}}\left[\frac{\left(e_j-c_j\right)\left(1-\eta_j\right)}{e_j-v_j}\right],$$

where $F^{-1}_{\widehat{\boldsymbol{y}}}$ *is the inverse of* $F_{\widehat{\boldsymbol{y}}}$.

Proof:

We can write Equation (3.1), for a given vector of markets, i.e. $\widehat{\boldsymbol{y}}$, as follows:

$$\Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}},\theta_{\widehat{\boldsymbol{y}}j}\right)=\theta_{\widehat{\boldsymbol{y}}j}-\frac{1}{1-\eta_j}E\left[\theta_{\widehat{\boldsymbol{y}}j}-P_j\left(Q_j,\widehat{\boldsymbol{y}}\right)\right]^+,$$

where

$$P_j\left(Q_j,\widehat{\boldsymbol{y}}\right)=\left(r_jx_{\widehat{\boldsymbol{y}}j}-S_{\widehat{\boldsymbol{y}}j}\right)+v_j(Q_{\widehat{\boldsymbol{y}}j}-x_{\widehat{\boldsymbol{y}}j})^+-e_j(x_{\widehat{\boldsymbol{y}}j}-Q_{\widehat{\boldsymbol{y}}j})^+-c_jQ_{\widehat{\boldsymbol{y}}j}.$$

The above equation can be rewritten as:

$$\begin{aligned}
&\Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}},\theta_{\widehat{\boldsymbol{y}}j}\right)=\\
&\theta_{\widehat{\boldsymbol{y}}j}-\frac{1}{1-\eta_j}\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{Q_{\widehat{\boldsymbol{y}}j}}\left(\theta_{\widehat{\boldsymbol{y}}j}-\left(\left(r_j-v_j\right)x_{\widehat{\boldsymbol{y}}j}-S_{\widehat{\boldsymbol{y}}j}-\left(c_j-v_j\right)Q_{\widehat{\boldsymbol{y}}j}\right)\right)^+f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right)\\
&-\frac{1}{1-\eta_j}\int_{x_{\widehat{\boldsymbol{y}}j}=Q_{\widehat{\boldsymbol{y}}j}}^{\infty}\left(\theta_{\widehat{\boldsymbol{y}}j}-\left(\left(r_j-e_j\right)x_{\widehat{\boldsymbol{y}}j}-S_{\widehat{\boldsymbol{y}}j}+\left(e_j-c_j\right)Q_{\widehat{\boldsymbol{y}}j}\right)\right)^+f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right).
\end{aligned}$$

The optimal order quantity for each product of the above CVaR risk-averse newsvendor is determined by $Q^*_{\widehat{\boldsymbol{y}}j}=\arg\max_{Q_{\widehat{\boldsymbol{y}}j}\geq 0}\max_{\theta_{\widehat{\boldsymbol{y}}j}\in\mathbb{R}}\Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\widehat{\boldsymbol{y}},\theta_{\widehat{\boldsymbol{y}}j}\right)$. To find the optimal solution, we consider three ranges for $\theta_{\widehat{\boldsymbol{y}}j}$. In the first range, both integrand disappear, in the second range the first integrand disappears. We prove that the third range includes the optimal solution.

If $\theta_{\widehat{\boldsymbol{y}}j}\leq-\left(c_j-v_j\right)Q_{\widehat{\boldsymbol{y}}j}-S_{\widehat{\boldsymbol{y}}j}$, then $\Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\widehat{\boldsymbol{y}},\theta_{\widehat{\boldsymbol{y}}j}\right)=\theta_{\widehat{\boldsymbol{y}}j}<0$. Hence this range of $\theta_{\widehat{\boldsymbol{y}}j}$ does not contain the optimal solution.

If $-\left(c_j - v_j\right) Q_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j} < \theta_{\widehat{\boldsymbol{y}}j} \leq \left(r_j - c_j\right) Q_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j}$, then,

$$\begin{aligned} \Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}, \theta_{\widehat{\boldsymbol{y}}j}\right) = \ & \theta_{\widehat{\boldsymbol{y}}j} - \frac{1}{1-\eta_j} \int_{x_{\widehat{\boldsymbol{y}}j}=0}^{\frac{\theta_{\widehat{\boldsymbol{y}}j}+\left(c_j-v_j\right)Q_{\widehat{\boldsymbol{y}}j}+S_{\widehat{\boldsymbol{y}}j}}{r_j-v_j}} \\ & \left[\theta_{\widehat{\boldsymbol{y}}j} - \left(\left(r_j - v_j\right) x_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j} - \left(c_j - v_j\right) Q_{\widehat{\boldsymbol{y}}j}\right)\right] f_{\widehat{\boldsymbol{y}}}\ \left(x_{\widehat{\boldsymbol{y}}j}\right), \end{aligned} \tag{3.2}$$

Next, we investigate the conditions under which a maximum for (3.2) exists. Towards this end, we examine the partial derevative of $\Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}, \theta_{\widehat{\boldsymbol{y}}j}\right)$ with respect to $\theta_{\widehat{\boldsymbol{y}}j}$,

$$\frac{\partial \Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j}, \widehat{\boldsymbol{y}}, \theta_{\widehat{\boldsymbol{y}}j}\right)}{\partial \theta_{\widehat{\boldsymbol{y}}j}} = 1 - \frac{1}{1-\eta_j} F_{\widehat{\boldsymbol{y}}}\left(\frac{\theta_{\widehat{\boldsymbol{y}}j} + \left(c_j - v_j\right) Q_{\widehat{\boldsymbol{y}}j} + S_{\widehat{\boldsymbol{y}}j}}{r_j - v_j}\right), \tag{3.3}$$

Setting the right hand side of (3.3) equal to zero, we get:

$$\theta^*_{\widehat{\boldsymbol{y}}j} = \left(r_j - v_j\right) F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right) - \left(c_j - v_j\right) Q_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j}. \tag{3.4}$$

By substituting for $\theta^*_{\widehat{\boldsymbol{y}}j}$ from (3.4) into (3.2), we get:

$$\begin{aligned} \Pi^*_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}\right) = & \left(r_j - v_j\right) F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right) - \left(c_j - v_j\right) Q_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j} \\ & - \frac{1}{1-\eta_j} \int_{x_{\widehat{\boldsymbol{y}}j}=0}^{F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right)} \left(r_j - v_j\right) \left(F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right) - x_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right). \end{aligned}$$

which is a decreasing linear function of $Q_{\widehat{\boldsymbol{y}}j}$. Hence, the maximum of $\Pi^*_{\widehat{\boldsymbol{y}}j}$ in this range of $\theta_{\widehat{\boldsymbol{y}}j}$ is achieved at $Q^*_{\widehat{\boldsymbol{y}}j} = 0$. Again, this range of $\theta_{\widehat{\boldsymbol{y}}j}$ does not contain the optimal solution of the problem on hand.

Finally, we consider $\theta_{\widehat{\boldsymbol{y}}j} > (r_j - c_j)\, Q_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j}$, then,

$$\begin{aligned}
&\Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}, \theta_{\widehat{\boldsymbol{y}}j}\right) = \\
&\theta_{\widehat{\boldsymbol{y}}j} - \frac{1}{1-\eta_j}\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{Q_{\widehat{\boldsymbol{y}}j}} [\theta_{\widehat{\boldsymbol{y}}j} - ((r_j - v_j)\, x_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j} - (c_j - v_j)\, Q_{\widehat{\boldsymbol{y}}j})] f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right) \\
&- \frac{1}{1-\eta_j}\int_{\sum_{i\in I} x_{\widehat{\boldsymbol{y}}j}=Q_{\widehat{\boldsymbol{y}}j}}^{\frac{\theta_{\widehat{\boldsymbol{y}}j} - \left(e_j - c_j\right) Q_{\widehat{\boldsymbol{y}}j} + S_{\widehat{\boldsymbol{y}}j}}{r_j - e_j}} [\theta_{\widehat{\boldsymbol{y}}j} - ((r_j - e_j)\, x_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j} + (e_j - c_j)\, Q_{\widehat{\boldsymbol{y}}j})] f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right),
\end{aligned} \tag{3.5}$$

and,

$$\frac{\partial \Pi_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}, \theta_{\widehat{\boldsymbol{y}}j}\right)}{\partial \theta_{\widehat{\boldsymbol{y}}j}} = 1 - \frac{1}{1-\eta_j} F_{\widehat{\boldsymbol{y}}}\left(\frac{\theta_{\widehat{\boldsymbol{y}}j} - (e_j - c_j)\, Q_{\widehat{\boldsymbol{y}}j} + S_{\widehat{\boldsymbol{y}}j}}{r_j - e_j}\right), \tag{3.6}$$

Setting the right hand side of (3.6) equal to zero, we get:

$$\theta^*_{\widehat{\boldsymbol{y}}j} = (r_j - e_j)\, F_{\widehat{\boldsymbol{y}}}^{-1}(1-\eta_j) + (e_j - c_j)\, Q_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j}. \tag{3.7}$$

By substituting for $\theta^*_{\widehat{\boldsymbol{y}}j}$ from (3.7) into (3.5), we get:

$$\begin{aligned}
&\Pi^*{}_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}\right) = (r_j - e_j)\, F_{\widehat{\boldsymbol{y}}}^{-1}(1-\eta_j) + (e_j - c_j)\, Q_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j} \\
&- \frac{1}{1-\eta_j}\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{Q_{\widehat{\boldsymbol{y}}j}} \left[(r_j - e_j)\, F_{\widehat{\boldsymbol{y}}}^{-1}(1-\eta_j) + (e_j - v_j)\, Q_{\widehat{\boldsymbol{y}}j} - (r_j - v_j)\, x_{\widehat{\boldsymbol{y}}j}\right] f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right) \\
&- \frac{1}{1-\eta_j}\int_{x_{\widehat{\boldsymbol{y}}j}=Q_{\widehat{\boldsymbol{y}}j}}^{F_{\widehat{\boldsymbol{y}}}^{-1}(1-\eta_j)} (r_j - e_j)\left(F_{\widehat{\boldsymbol{y}}}^{-1}(1-\eta_j) - x_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right).
\end{aligned} \tag{3.8}$$

Then,

$$\frac{\partial \Pi^*_{\widehat{\boldsymbol{y}}j}\left(Q_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}\right)}{\partial Q_{\widehat{\boldsymbol{y}}j}} =$$
$$e_j - c_j - \frac{1}{1-\eta_j}\left[\left(e_j - v_j\right) F_{\widehat{\boldsymbol{y}}}\left(Q_{\widehat{\boldsymbol{y}}j}\right) + \left(r_j - e_j\right) F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right) - \left(r_j - e_j\right) Q_{\widehat{\boldsymbol{y}}j}\right]$$
$$- \frac{1}{1-\eta_j}\left(r_j - e_j\right)\left(F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right) - Q_{\widehat{\boldsymbol{y}}j}\right), \tag{3.9}$$

Setting the right hand side of (3.9) equal to zero, we get:

$$\left(e_j - v_j\right) F_{\widehat{\boldsymbol{y}}}\left(Q_{\widehat{\boldsymbol{y}}j}\right) = \left(e_j - c_j\right)\left(1-\eta_j\right).$$

Hence, for any product j, the optimal order quantity is:

$$Q^*_{\widehat{\boldsymbol{y}}j} = F_{\widehat{\boldsymbol{y}}}^{-1}\left[\frac{\left(e_j - c_j\right)\left(1-\eta_j\right)}{e_j - v_j}\right] = F_{\widehat{\boldsymbol{y}}}^{-1}\left(\beta_j\right), \tag{3.10}$$

where, $\beta_j = \frac{(e_j - c_j)(1-\eta_j)}{e_j - v_j}$. □

In the previous chapter we show that the order quantity for the risk-neutral MPSNVP is $Q^*_{\widehat{\boldsymbol{y}}j} = F_{\widehat{\boldsymbol{y}}}^{-1}\left(\frac{e_j - c_j}{e_j - v_j}\right)$. Proposition 1 demonstrates that the CVaR risk-averse MPSNVP orders smaller quantities than the risk-neutral MPSNVP do. This result is congruous with the results appeared in the literature (c.f. [12, 40, 55]). Also, we can notice that the optimal order quantity for any product j is independent of the selling price of the product, while it changes inversely with change in the purchasing cost. It is also worth noting that, the optimal order quantity for any product j increases/decreases as the expediting cost increases/decreases; the same can be said about the salvage value. The increase in the expediting cost will motivate the firm to order more in order to avoid the expediting losses. The increase in the salvage value will stimulate the firm to order more, because this leads to less loss at the end of the selling period, if there is excess inventory.

Proposition 2. *The CVaR risk-averse profit function* (3.1) *can be written as*:

$$CVaR\left[P\left(Q_j,\ y_{ij}\right)\right] = \sum_{j\in J}\left[\sum_{i\in I}\overline{\pi_{ij}}y_{ij} - K_{\eta_j}\left(\beta_j\right)\sqrt{\sum_{i\in I}\sigma_{ij}^2 y_{ij}}\right],$$

where,

$$\beta_j = \frac{\left(e_j - c_j\right)\left(1-\eta_j\right)}{e_j - v_j},$$

$$\overline{\pi_{ij}} = \left(r_j - c_j\right)\mu_{ij} - S_{ij},$$

$$K_{\eta_j}\left(\beta_j\right) = \left(\frac{1}{1-\eta_j} - 1\right)\left(r_j - e_j\right)\Phi^{-1}\left(1-\eta_j\right) - \left(e_j - c_j\right)\Phi^{-1}\left(\beta_j\right)$$
$$+\frac{1}{1-\eta_j}\left[\left(r_j - e_j\right)L\left(\Phi^{-1}\left(1-\eta_j\right)\right) + \left(e_j - v_j\right)\left(\Phi^{-1}\left(\beta_j\right) + L\left(\Phi^{-1}\left(\beta_j\right)\right)\right)\right].$$

Proof

The CVaR risk-averse profit function, from (3.8), can be expressed as:

$$\begin{aligned}\Pi^*{}_{\widehat{\boldsymbol{y}}j}\left(Q^*_{\widehat{\boldsymbol{y}}j},\ \widehat{\boldsymbol{y}}\right) &= \left(r_j - e_j\right)F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right) + \left(e_j - c_j\right)Q^*_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j} \\ &- \frac{1}{1-\eta_j}\left(e_j - v_j\right)\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{Q^*_{\widehat{\boldsymbol{y}}j}}\left(Q^*_{\widehat{\boldsymbol{y}}j} - x_{\widehat{\boldsymbol{y}}j}\right)f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right) \\ &- \frac{1}{1-\eta_j}\left(r_j - e_j\right)\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right)}\left(F_{\widehat{\boldsymbol{y}}}^{-1}\left(1-\eta_j\right) - x_{\widehat{\boldsymbol{y}}j}\right)f_{\widehat{\boldsymbol{y}}}\left(x_{\widehat{\boldsymbol{y}}j}\right).\end{aligned} \tag{3.11}$$

Now, assuming that market demands are independent and normally distributed with a negligible probability of negative demands, we can write the following,

$$\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{Q^*_{\widehat{\boldsymbol{y}}j}} (Q_{\widehat{\boldsymbol{y}}j} - x_{\widehat{\boldsymbol{y}}j}) f_{\widehat{\boldsymbol{y}}}(x_{\widehat{\boldsymbol{y}}j}) = \int_{x_{\widehat{\boldsymbol{y}}j}=0}^{F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j)} \left(F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j) - x_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}(x_{\widehat{\boldsymbol{y}}j}) =$$
$$\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{\infty} \left(F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j) - x_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}(x_{\widehat{\boldsymbol{y}}j}) + \int_{x_{\widehat{\boldsymbol{y}}j}=F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j)}^{\infty} \left(x_{\widehat{\boldsymbol{y}}j} - F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j)\right) f_{\widehat{\boldsymbol{y}}}(x_{\widehat{\boldsymbol{y}}j}).$$

Therefore,

$$\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{F_{\widehat{\boldsymbol{y}}}{}^{-1}(1-\eta_j)} \left(F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j) - x_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}(x_{\widehat{\boldsymbol{y}}j}) = F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j) - (\mu_{\widehat{\boldsymbol{y}}j}) + L\left(\Phi^{-1}(\beta_j)\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}},$$

and hence,

$$\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j)} \left(F_{\widehat{\boldsymbol{y}}}{}^{-1}(\beta_j) - x_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}(x_{\widehat{\boldsymbol{y}}j}) = \Phi^{-1}(\beta_j)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}} + L\left(\Phi^{-1}(\beta_j)\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}}. \tag{3.12}$$

Similarly,

$$\int_{x_{\widehat{\boldsymbol{y}}j}=0}^{F_{\widehat{\boldsymbol{y}}}{}^{-1}(1-\eta_j)} \left(F_{\widehat{\boldsymbol{y}}}{}^{-1}(1-\eta_j) - x_{\widehat{\boldsymbol{y}}j}\right) f_{\widehat{\boldsymbol{y}}}(x_{\widehat{\boldsymbol{y}}j}) =$$
$$\Phi^{-1}(1-\eta_j)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}} + L\left(\Phi^{-1}(1-\eta_j)\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}}. \tag{3.13}$$

The optimal order quantity in (3.10) can be written as:

$$Q^*_{\widehat{\boldsymbol{y}}j} = \mu_{\widehat{\boldsymbol{y}}j} + \Phi^{-1}(\beta_j)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}}. \tag{3.14}$$

By substituting the above results of (3.12), (3.13) and (3.14) into (3.11), we get:

$$\begin{aligned}\Pi^{**}{}_{\widehat{\boldsymbol{y}}j}\left(\widehat{\boldsymbol{y}}\right) = {} & (r_j - e_j)\left(\left(\mu_{\widehat{\boldsymbol{y}}j}\right) + \Phi^{-1}\left(1-\eta_j\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}}\right) \\ & + (e_j - c_j)\left(\left(\mu_{\widehat{\boldsymbol{y}}j}\right) + \Phi^{-1}\left(\beta_j\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}}\right) - S_{\widehat{\boldsymbol{y}}j} \\ & - \frac{1}{1-\eta_j}\left(r_j - e_j\right)\left(\Phi^{-1}\left(1-\eta_j\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}} + L\left(\Phi^{-1}\left(1-\eta_j\right)\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}}\right) \\ & - \frac{1}{1-\eta_j}\left(e_j - v_j\right)\left(\Phi^{-1}\left(\beta_j\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}} + L\left(\Phi^{-1}\left(\beta_j\right)\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}}\right).\end{aligned}$$

Now, we can write the above equation on the following form:

$$\Pi^{**}{}_{\widehat{\boldsymbol{y}}j}\left(\widehat{\boldsymbol{y}}\right) = \overline{\pi_{\widehat{\boldsymbol{y}}j}} - K_{\eta_j}\left(\beta_j\right)\sqrt{\sigma^2_{\widehat{\boldsymbol{y}}j}},$$

where,

$$\overline{\pi_{\widehat{\boldsymbol{y}}j}} = \left(r_j - c_j\right)\mu_{\widehat{\boldsymbol{y}}j} - S_{\widehat{\boldsymbol{y}}j},$$

$$\begin{aligned}K_{\eta_j}\left(\beta_j\right) = {} & \left(\frac{1}{1-\eta_j} - 1\right)\left(r_j - e_j\right)\Phi^{-1}\left(1-\eta_j\right) - \left(e_j - c_j\right)\Phi^{-1}\left(\beta_j\right) \\ & + \frac{1}{1-\eta_j}\left[\left(r_j - e_j\right)L\left(\Phi^{-1}\left(1-\eta_j\right)\right) + \left(e_j - v_j\right)\left(\Phi^{-1}\left(\beta_j\right) + L\left(\Phi^{-1}\left(\beta_j\right)\right)\right)\right]. \square\end{aligned}$$

To maximize the CVaR risk-averse profit of the MPSNVP with flexible market entry, we have to solve the following model:

Problem I-RA:

$$\begin{aligned}& \text{Max} \quad \sum_{j\in J}\left[\sum_{i\in I}\overline{\pi_{ij}}y_{ij} - K_{\eta_j}\left(\beta_j\right)\sqrt{\sum_{i\in I}\sigma^2_{ij}y_{ij}}\right], \\ & \text{s.t.} \\ & \quad y_{ij} \in \{0,1\},\ \forall\, j \in J,\ \forall\, i \in I.\end{aligned} \tag{3.15}$$

The final model is a binary integer nonlinear program involving only the market selection variables.

3.3.2 Case 2: CVaR Risk-Averse MPSNVP with Full Market Entry

In the full market entry MPSNVP case, if any market i is selected, then a single fixed market entry cost, S_i, is incurred. The profit function for the full market entry MPSNVP; $P(Q_j,\ \mathcal{Y}_i)$, is given by (2.9). The expected profit model or the risk-neutral version of the problem is presented in (2.10).

In order to find the CVaR risk-averse value of the profit function presented in (2.9), we can follow the same modeling procedure in Case 1 provided in the previous section, where CVaR is expressed as in equation (3.1). Thereafter, we follow the same derivation sequence as in Proposition 1, we find that the optimal order quantity $Q^*_{\widehat{\boldsymbol{y}}j}$ for a given vector of markets $\widehat{\boldsymbol{y}}$, for each product j, is obtained by equation (3.10). Finally, the CVaR risk-averse profit function for Case 2, can be expressed as:

$$CVaR\left[P\left(Q_j,\ \mathcal{Y}_i\right)\right]=\sum_{i\in I}\overline{\pi}_i\mathcal{Y}_i-\sum_{j\in J}K_{\eta_j}\left(\beta_j\right)\sqrt{\sum_{i\in I}\sigma_{ij}^2\mathcal{Y}_i},$$

where,

$$\beta_j=\frac{\left(e_j-c_j\right)\left(1-\eta_j\right)}{e_j-v_j},$$

$$\overline{\pi}_i=\sum_{j\in J}\left(r_j-c_j\right)\mu_{ij}-S_i,$$

$$K_{\eta_j}(\beta_j) = \left(\frac{1}{1-\eta_j} - 1\right)(r_j - e_j)\,\Phi^{-1}(1-\eta_j) - (e_j - c_j)\,\Phi^{-1}(\beta_j)$$
$$+ \frac{1}{1-\eta_j}\left[(r_j - e_j)\,L\left(\Phi^{-1}(1-\eta_j)\right) + (e_j - v_j)\left(\Phi^{-1}(\beta_j) + L\left(\Phi^{-1}(\beta_j)\right)\right)\right].$$

To maximize the CVaR risk-averse profit of the firm, the following model of Case 2 should be solved:

Problem II-RA:

$$\begin{aligned} &\text{Max} \quad \sum_{i\in I} \overline{\pi}_i \mathcal{Y}_i - \sum_{j\in J} K_{\eta_j}(\beta_j)\sqrt{\sum_{i\in I}\sigma_{ij}^2 \mathcal{Y}_i}, \\ &\text{s.t.} \\ &\quad \mathcal{Y}_i \in \{0,1\}, \ \forall\, i \in I. \end{aligned} \tag{3.16}$$

Similar to Problem I-RA, Problem II-RA is a binary integer nonlinear program involving the market selection variables.

3.3.3 Case 3: CVaR Risk-Averse MPSNVP with Partial Market Entry

For this case, if any market i is selected, then a fixed market entry cost per period, s_i, is incurred, and if a product j is selected to be sold in market i, then an additional cost per period, $\mathfrak{s}_{ij}$, is paid for introducing product j into market i. The profit function for the partial market entry MPSNVP; $P(Q_j,\ y_{ij}, \mathcal{Y}_i)$, is given by (2.23). The expected profit model or the risk-neutral version of the problem is presented in (2.24).

Following the same modeling procedure in Cases 1 and 2 discussed in the previous two sections, we can obtain the CVaR risk-averse solution of the above profit function. CVaR is expressed as in equation (3.1). Afterwards, we find that the optimal order quantity $Q^*_{\widehat{\boldsymbol{y}}j}$ for a given vector of markets $\widehat{\boldsymbol{y}}$, for each product j, is obtained by equation (3.10). Therefore, we follow the same sequence as in Section 3.3.1. Now, we can write the CVaR risk-averse profit function for Case 3 as follows:

$$CVaR\left[P\left(Q_j,\ y_{ij}, \mathcal{Y}_i\right)\right] = \sum_{i\in I}\sum_{j\in J}\overline{\pi_{ij}}y_{ij} - \sum_{j\in J}K_{\eta_j}\left(\beta_j\right)\sqrt{\sum_{i\in I}\sigma^2_{ij}y_{ij}} - \sum_{i\in I}s_i\mathcal{Y}_i,$$

where,

$$\beta_j = \frac{\left(e_j - c_j\right)\left(1-\eta_j\right)}{e_j - v_j},$$

$$\overline{\pi_{ij}} = \left(r_j - c_j\right)\mu_{ij} - \mathfrak{s}_{ij},$$

$$K_{\eta_j}\left(\beta_j\right) = \left(\frac{1}{1-\eta_j} - 1\right)\left(r_j - e_j\right)\Phi^{-1}\left(1-\eta_j\right) - \left(e_j - c_j\right)\Phi^{-1}\left(\beta_j\right)$$
$$+\frac{1}{1-\eta_j}\left[\left(r_j - e_j\right)L\left(\Phi^{-1}\left(1-\eta_j\right)\right) + \left(e_j - v_j\right)\left(\Phi^{-1}\left(\beta_j\right) + L\left(\Phi^{-1}\left(\beta_j\right)\right)\right)\right].$$

We solve the following model of Case 3 in order to maximize the CVaR risk-averse profit of the firm:

Problem III-RA:

$$\begin{aligned} &\text{Max} \quad \sum_{i\in I}\sum_{j\in J}\overline{\pi_{ij}}y_{ij} - \sum_{j\in J} K_{\eta_j}\left(\beta_j\right)\sqrt{\sum_{i\in I}\sigma_{ij}^2 y_{ij}} - \sum_{i\in I} s_i\mathcal{Y}_i, \\ &\text{s.t.} \\ &\quad \mathcal{Y}_i \geq y_{ij},\ \forall\ j\in J,\ \forall\ i\in I, \\ &\quad \mathcal{Y}_i,\ y_{ij}\in\{0,1\},\ \forall\ j\in J,\ \forall\ i\in I. \end{aligned} \tag{3.17}$$

Similar to Problems I-RA and II-RA, Problem III-RA is a binary integer nonlinear program involving the market selection variables for each product.

3.3.4 Solution Algorithms

Problems I-RA, II-RA and III-RA have the same structure as Problems I, II and III given by (2.7), (2.11) and (2.25), respectively. For Problems I-RA, II-RA and III-RA if $\eta_j = 0$, then we retrieve the risk neutral models that presented in Problems I, II and III, respectively.

Therefore, we obtain the optimal solution to Problem I-RA by applying *Algorithm I* discussed in Section 2.3.2. The solution to Problem II-RA is obtained by applying *Algorithm II* or Heuristic I discussed in Section 2.3.4. And lastly, The solution to Problem III-RA is obtained by applying *Algorithm III* or Heuristic II provided in Section 2.3.6.

3.3.5 Conic Quadratic Mixed Integer Reformulation of Cases 2 and 3

The obtained models for Cases 2 and 3 are more complex than the model obtained for Case 1. Therefore, the computational effort required for solving Problems II-RA and III-RA is higher than that required for solving Problem I-RA. In this section, we reformulate Problems II-RA and III-RA and put them in the form of Conic Quadratic Mixed-Integer Programs (CQMIP). In fact, this reformulation enables the use of standard optimization software packages, such as CPLEX, LINDO, XPRESS or MOSEK. This CQMIP transformation of the nonlinearities in the objective function has been used in the literature to reformulate the location-inventory models [67, 68, 69, 70], and the shortest path problem [71, 72]. We introduce auxiliary variables ω_j to represent the nonlinear terms in the objective function, then we transform Problem II-RA as follows:

Problem IV-RA:

$$\text{Max} \quad \sum_{i \in I} \overline{\pi_i} \mathcal{Y}_i - \sum_{j \in J} K_{\eta_j}(\beta_j) \omega_j,$$

s.t.

$$\sum_{i \in I} \sigma_{ij}^2 \mathcal{Y}_i \leq \omega_j^2, \ \forall\, j \in J, \tag{3.18}$$

$$\omega_j \geq 0, \ \forall\, j \in J,$$

$$\mathcal{Y}_i \in \{0, 1\}, \ \forall\, i \in I.$$

Similarly, Problem III-RA is reformulated as:

Problem V-RA:

$$\text{Max} \quad \sum_{i \in I} \sum_{j \in J} \overline{\pi_{ij}} y_{ij} - \sum_{j \in J} K_{\eta_j} (\beta_j) \omega_j - \sum_{i \in I} s_i \mathcal{Y}_i,$$

s.t.

$$\begin{aligned} & \sum_{i \in I} \sigma_{ij}^2 y_{ij} \leq \omega_j^2, \ \forall \ j \in J, \\ & \omega_j \geq 0, \ \forall \ j \in J, \\ & \mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I, \\ & \mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I. \end{aligned} \tag{3.19}$$

Now, we can use these transformations of the formulation of Cases 2 and 3 to compare the performance of the proposed solution strategies in the previous section, with state-of-the-art commercial solvers.

The following section presents computational tests to investigate the performance of the proposed solution procedures.

3.4 Computational Results

Through these computational experiments, we consider a CVaR risk-averse MP-SNVP for which the market entry costs and the product demand distributions are market dependent, while the selling prices, purchasing costs, expediting costs, and the salvage values are product dependent. The solution to Problem I-RA of the flexible market entry case is easier than Problems II-RA and III-RA; therefore, we will not discuss it here and we focus on sloving Problems II-RA and III-RA and their reformulations in Problems IV-RA and V-RA.

3.4.1 Computational Efficiency of the Solution Algorithms

The following computational tests are conducted to study the performance of the proposed solution algorithms and evaluate the quality of the achieved solution, and then, compare it with that of the state-of-the-art commercial solvers.

We consider MPSNVP with three, five and ten products. We consider six market pool sizes: 50, 100, 250, 500, 1,000 and 5,000. Table 1 shows the details of the products' selling prices, purchasing costs, expediting costs, salvage values and the degrees of risk-aversion; these values are fixed for each type of the products. Table 3.1 provides the detailed cost values for the products in the MPSNVP. The purchasing cost per unit of each product, the expediting cost per unit of each product, and the salvage value per unit of each product are shown in the table. Table 3.2 provides the nominal demand as well as the demand variance for each

Table 3.1: Costs of the products in the risk-averse MPSNVP.

Parameter	Product									
	1	2	3	4	5	6	7	8	9	10
r	15	150	20	1,500	300	25	100	10	250	1,000
e	10	120	15	1,200	250	24	90	7	220	900
c	7	100	10	1,000	200	15	85	6	200	800
v	5	50	5	600	50	5	30	5	100	200
η	0.7	0.4	0.8	0.2	0.4	0.8	0.4	0.6	0.2	0.7

market. In addition, the market entry cost for each type of the products are presented. These input data are drawn from uniform distributions as shown in Table 3.2.

Table 3.2: Input parameters for the risk-averse MPSNVP.

Product	Parameters		
	μ	σ	$\mathfrak{s}$
1	U(400, 600)	U(50, 100)	U(1,000, 2,000)
2	U(300, 400)	U(40, 75)	U(2,000, 3,000)
3	U(600, 800)	U(100, 150)	U(2,000, 4,000)
4	U(40, 60)	U(5, 10)	U(6,000, 10,000)
5	U(200, 500)	U(20, 50)	U(4,000, 8,000)
6	U(300, 400)	U(60, 70)	U(2,000, 3,000)
7	U(200, 220)	U(45, 50)	U(1,000, 3,000)
8	U(300, 500)	U(50, 60)	U(3,000, 5,000)
9	U(100, 120)	U(20, 25)	U(3,000, 5,000)
10	U(120, 150)	U(25, 30)	U(6,000, 10,000)

Example 1: We use the details of the CVaR risk-averse MPSNVP for the case of full market entry, i.e., Case 2, as presented in Table 1 and the first two columns of Table 3.2.

For each market pool size, we solve three problems, 3-product, 5-product and 10-product problems. For the 3-product problem, we use the input data of products 1 to 3, and the market entry cost is drawn from the uniform distribution U(20,000, 40,000). For the 5-product problem we use the input data of the products 1 to 5, and the market entry cost is drawn from the uniform distribution U(40,000, 80,000). Lastly, for the 10-product problem we use the input data of products 1 to 10, and the market entry cost is drawn from the uniform distribution U(80,000, 100,000).

Algorithm II is applied for solving the risk-averse full market entry problem, i.e., Problem II-RA. The obtained results are compared with the results of BARON and CPLEX. The performance of the proposed algorithm and the state-of-the-art commercial solvers are shown in Table 3.3.

Table 3.3 shows the obtained objective function values and the computational times for Baron, CPLEX and *Algorithm II* for 18 problem instances. The reported results provide a clear evidence of the efficiency of using *Algorithm II* for solving the optimization model of Case 2.

Algorithm II succeeds in obtaining at least as higher objective values as those

Table 3.3: Comparisons of NIP and CQMIP solvers with *Algorithm II* for risk-averse full market entry MPSNVP.

$\|J\|$	$\|I\|$	BARON		CPLEX		Heuristic I	
		Gap%	Time	Gap%	Time	Gap%	Time
3	50	0.0%	0.44	0.0%	0.27	0.0%	0.002
	100	0.0%	0.54	0.0%	0.31	0.0%	0.01
	250	0.2%	0.65	<0.1%	0.37	0.0%	0.02
	500	<0.1%	0.77	0.1%	0.61	0.0%	0.03
	1000	<0.1%	0.89	<0.1%	0.83	0.0%	0.08
	5000	<0.1%	4.15	<0.1%	4.33	0.0%	0.51
5	50	0.7%	0.72	0.0%	0.32	0.0%	0.003
	100	0.5%	0.63	0.0%	0.31	0.0%	0.004
	250	0.5%	0.64	<0.1%	0.49	0.0%	0.02
	500	0.3%	0.76	<0.1%	0.57	0.0%	0.05
	1000	0.2%	0.99	0.0%	0.63	0.0%	0.09
	5000	0.3%	4.27	<0.1%	3.74	0.0%	1.19
10	50	0.0%	0.75	0.0%	0.29	0.0%	0.005
	100	0.0%	0.73	0.0%	0.27	0.0%	0.01
	250	0.0%	0.75	0.0%	0.31	0.0%	0.04
	500	0.0%	0.78	0.0%	0.39	0.0%	0.06
	1000	0.0%	1.09	0.0%	0.45	0.0%	0.18
	5000	0.0%	6.28	0.0%	1.49	0.0%	0.98

obtained by the commercial solvers for all problem instances. *Algorithm II* outperforms the NIP solver in ten problem instances out of 18 instances. It also outperforms the CQMIP solver in seven problem instances out of 18 instances.

The *Gap*% of the obtained objective values are reported in Table 3.3. The *Gap*% is calculated as the percent relative gap from the best-obtained objective value, which is consistently obtained by *Algorithm II*. For some of the solved instances there is a positive relative gap from the optimal solution. The highest observed gap value is 0.7%.

The computational time required by *Algorithm II* is much smaller than the computational time required by the solvers. This reflects the efficiency of *Algorithm II* in terms of the computational effort.

Example 2: We utilize the details of the MPSNVP for the case of partial market entry, i.e., Case 3, as presented in Table 3.1 and Table 3.2.

We solve three problems, 3-product, 5-product and 10-product problems, for each market pool size. The 3-product problem uses the input data of products 1 to 3, the 5-product problem uses the input data of the products 1 to 5, and finally, the 10-product problem uses the input data of products 1 to 10. The market entry cost is drawn from the uniform distribution $U(15{,}000,\ 30{,}000)$.

We apply *Algorithm III* to solve the partial market entry problem, i.e., Problem III-RA. The obtained results are then compared with the results of BARON and CPLEX. The comparisons of the results are shown in Table 3.4.

Table 3.4 represents the performance of the commercial solvers; Baron and CPLEX, and *Algorithm III* for solving Case 3. The obtained objective function values and the computational times for 18 problem instances are reported in Table 3.4. The results clearly demonstrate the efficiency of using *Algorithm III* for solving the optimization model of Case 3.

Algorithm III succeeds in obtaining at least as higher objective values as those

Table 3.4: Comparisons of NIP and CQMIP solvers with *Algorithm III* for risk-averse partial market entry MPSNVP

$\|J\|$	$\|I\|$	BARON		CPLEX		Heuristic II	
		Gap%	Time	Gap%	Time	Gap%	Time
3	50	0.0%	0.56	0.0%	0.32	0.0%	0.003
	100	0.0%	0.78	0.0%	0.35	0.0%	0.005
	250	0.0%	1.62	0.0%	0.53	0.0%	0.009
	500	<0.1%	1.77	<0.1%	0.79	0.0%	0.034
	1000	<0.1%	2.41	<0.1%	1.15	0.0%	0.09
	5000	<0.1%	28.36	7.3%	8.38	0.0%	2.39
5	50	0.0%	0.74	0.0%	0.23	0.0%	0.007
	100	0.0%	0.90	0.0%	0.36	0.0%	0.01
	250	0.0%	1.13	0.0%	0.47	0.0%	0.03
	500	0.0%	2.08	0.0%	0.69	0.0%	0.07
	1000	0.0%	4.65	0.0%	0.93	0.0%	0.15
	5000	0.0%	58.52	0.0%	6.16	0.0%	3.05
10	50	<0.1%	0.91	<0.1%	0.35	0.0%	0.01
	100	<0.1%	1.01	0.0%	0.47	0.0%	0.02
	250	0.0%	2.24	<0.1%	0.83	0.0%	0.06
	500	0.0%	4.52	0.0%	1.19	0.0%	0.12
	1000	<0.1%	11.71	<0.1%	2.64	0.0%	0.28
	5000	0.0%	212.02	<0.1%	49.2	0.0%	6.42

obtained by the commercial solvers for all problem instances. *Algorithm III* outperforms the NIP solver in six problem instances out of 18 instances. It also outperforms the CQMIP solver in seven problem instances out of 18 instances.

The relative Gap% of the obtained objective values are reported in Table 3.4. For some of the solved instances there is a positive relative gap from the optimal solution. The highest observed gap value is 7.3%.

Algorithm III presents huge savings in computational effort, as it requires much smaller computational time than that required by the commercial solvers, especially for large-scale problems.

The computational results of *Examples* 1 and 2, which are presented in Tables 3.3

and 3.4, show the efficiency of the proposed algorithms in solving the models in polynomial time. One can say, in general, that the proposed algorithms succeed in obtaining a better solution and in less computational time than the commercial solvers. These findings show the practicality of the proposed solution algorithms in solving large-scale real-life supply chain and logistics problems.

3.4.2 Effect of the Risk Aversion Degree on Profit

In this section, computational tests are conducted to study the effect of the risk aversion degree on the profit. This effect is determined in terms of the absolute difference percentage (%AD) and the relative difference percentage (%RD) for specified values of the risk aversion degree; η_k, in the range from 0.1 to 0.9. The $\%\text{AD}_k$ and $\%\text{RD}_k$ at each risk aversion degree; η_k, are defined, respectively, as follows:

$$\%\text{AD}_k = \left[1 - \frac{\text{Risk Averse Profit at } \eta_k}{\text{Risk Neutral Profit}}\right] \times 100$$

$$\%\text{RD}_k = \left[1 - \frac{\text{Risk Averse Profit at } \eta_k}{\text{Risk Averse Profit at } \eta_{k-1}}\right] \times 100$$

Figure 3.1 shows %AD and %RD for the full market entry MPSNVP with market pool size 50 for different number of products. One can notice that, the %AD is increasing almost linearly as a function of the risk aversion degree in the range from 0.1 to 0.5, while the increase has much higher rate in the range of η from 0.5 to 0.9. On the other hand, for %RD, the curve seems to be flat in the range of η from 0.1 to 0.5, then it increases with much higher rate in the range of η from 0.5 to 0.9. It is also notable, from Figure 3.1, that both %AD and %RD are decreasing as the number of products increases.

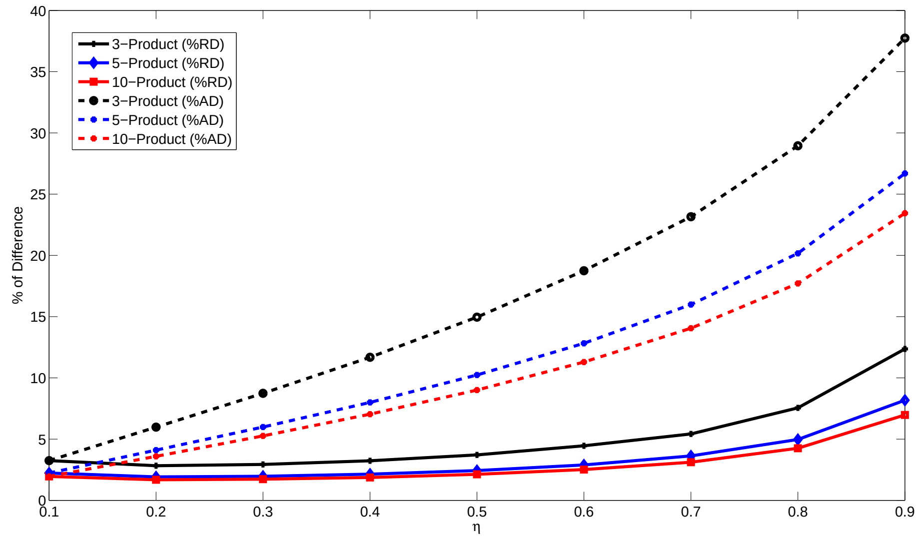


Figure 3.1: Percentage of relative and absolute differences for the CVaR profit values of the full market entry MPSNVP as a function of η for market pool size 50.

Figure 3.2 presents %AD and %RD for the full market entry MPSNVP with market pool size 5,000 for different number of products. The behavior of %AD and %RD in Figure 3.2 is similar to that in Figure 3.1. Comparing the values of %AD and %RD for market pool sizes 50 and 5,000, implies that, both of %AD and %RD are decreasing as the market pool size increases.

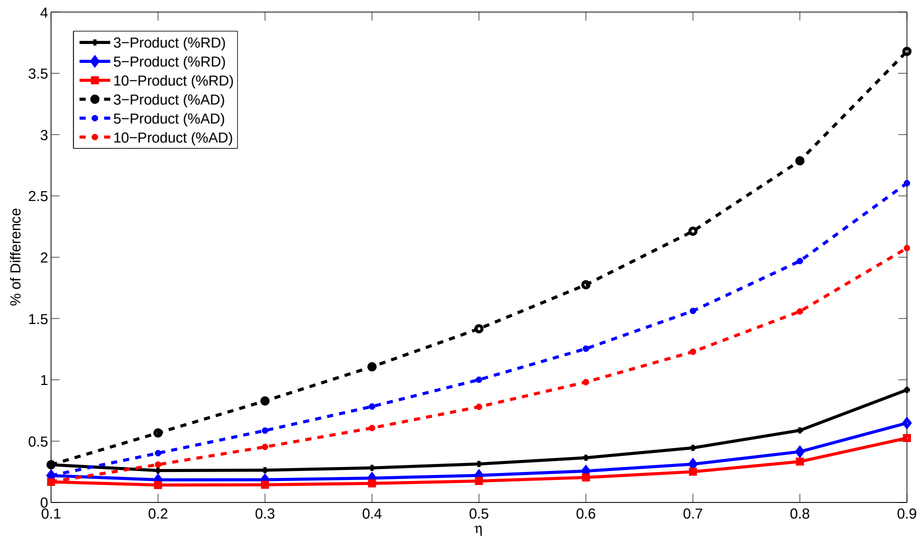


Figure 3.2: Percentage of relative and absolute differences for the CVaR profit values of the full market entry MPSNVP as a function of η for market pool size 5,000.

Figure 3.3 shows %AD and %RD for the partial market entry MPSNVP with market pool size 50 for different number of products. It is notable that, the %AD is increasing almost linearly as a function of the risk aversion degree in the range from 0.1 to 0.5, while it increases at a much higher rate in the range of η from 0.5 to 0.9. For %RD, the curve is almost flat in the range of η from 0.1 to 0.5, then it increases at a much higher rate in the range of η from 0.5 to 0.9. Both %AD and %RD are decreasing as the number of products increases.

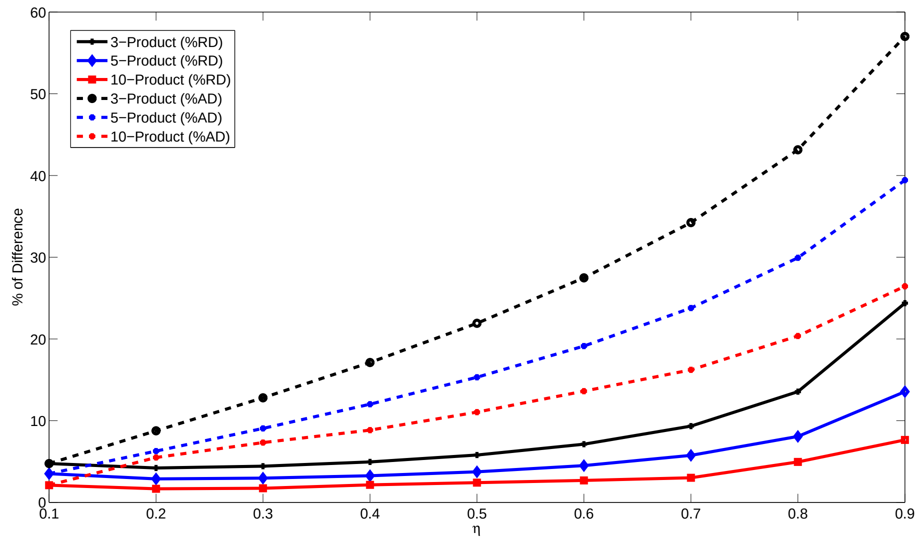


Figure 3.3: Percentage of relative and absolute differences for the CVaR profit values of the partial market entry MPSNVP as a function of η for market pool size 50.

Figure 3.4 provides %AD and %RD for the partial market entry MPSNVP with market pool size 5,000 for different number of products. The behavior of %AD and %RD in Figure 3.4 is similar to that in Figure 3.3. the values of %AD and %RD for market pool sizes 50 and 5,000, indicates that, both of %AD and %RD are decreasing as the market pool size increases.

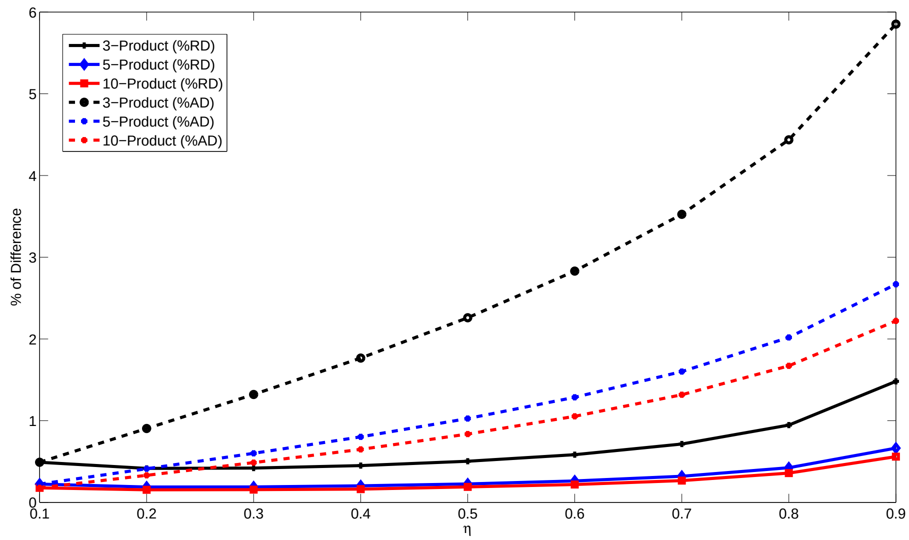


Figure 3.4: Percentage of relative and absolute differences for the CVaR profit values of the partial market entry MPSNVP as a function of η for market pool size 5,000.

Examination of the behavior of the %RD leads to the following conclusions. Obviously, the degree of risk aversion will affect the profit; the higher the value of η, the lesser is the profit. If the decision maker has low risk aversion preference, e.g. $\eta \leq 0.5$, then the change in η within this range will not result in large variation in the profit. The above figures show, for the case of 50 markets, that an increase in η of 0.1 will result in a relative drop in profit of at most 5.8%, this relative drop in the profit decreases with the increase in products number. However, the same change in η, if there are 5,000 markets, will cause a drop of at most 0.5%. Hence, a precise choice of η for this decision maker is not critical. On the other hand, if the decision maker is highly risk averse, then he/she has to choose η carefully, since the increase in its value within the range of $0.5 < \eta < 1$, causes a significant drop in the profit. For instance, the above figures show for the case of 50 markets that an increase in η of 0.1 will result in large relative drop in profit, this relative drop approaches 25%. The drop in the profit increases with the decrease in the market pool size and/or the number of products. A similar behavior is observable for the %AD; however the profit drop, as a result of the increase in η, is much more severe.

CHAPTER 4

ROBUST MULTI-PRODUCT SELECTIVE NEWSVENDOR

4.1 Introduction

The parameter values in optimization problems are usually assumed to be precisely known. However, this is not always the case in real world. Ignoring parameter uncertainties might have significant influence on the solution optimality, moreover, in most cases, it affects model feasibility. Therefore, parameter uncertainties should be taken into consideration in both modeling and analysis stages.

Uncertain model parameters may follow known probability distributions, while in many cases the available information about the probability distributions are limited. When the probability distribution of an uncertain parameter is known, the appropriate modeling approach is *Stochastic Programming*. However, when the probability distribution of the uncertain parameter is unknown, *Robust Opti-*

mization is the appropriate modeling approach.

In previous chapters, we presented the stochastic optimization of the MPSNVP. In this chapter, we are motivated to discuss the robust optimization of the MPSNVP with uncertain parameters. We will consider the cases where product's demand is uncertain with unknown probability distribution.

In the next section we give a review of the relevant robust optimization literature.

4.2 Literature Review on Robust Optimization

Making decisions in inventory problems under limited parameters information used to be done by developing distribution-free approach. The first appearance of the distribution-free approach with min-max objective for the classical newsvendor model was in the study by Scarf [73]. Scarf's model was extended in several studies, such as [74, 75, 76, 77, 78, 79, 80, 81, 82].

In another line of work, researchers quantified the regret of different decisions and made the optimal decision based on the minimax regret criterion. The minimax regret approach is less conservative than the min-max approach. The interested reader may refer to [83, 84] The recent trend of modeling lack of information is by using a predefined *Uncertainty Set* to describe the parameters' uncertainty[85, 86]. The uncertainty set is defined as the set of all possible realizations of the uncertain parameter that will be considered in the robust problem [85, 86]. The approach of defining an uncertainty set is known as *Robust Optimization* approach. The known uncertainty sets in the literature are, box, interval, ellipsoidal, polyhedral uncertainty set and combinations of theses sets. The shape of the selected uncertainty set will affect the tractability of the resulting robust optimization counterpart.

Robust optimization is a relatively recent approach in optimization. Researchers started developing robust optimization seriously and extensively 15 year ago [86, 87]. Today, robust optimization has a wide range of applications, including finance, energy systems, supply chain, facility location, inventory management, health-care, scheduling, marketing, queuing networks, machine learning; etc. [86, 87, 85].

Robust optimization is tailored to deal with the lack of information, while leading to tractable formulation. The uncertain parameters in robust optimization are taken at their worst case values, therefore the robust optimization approach results in a solution that is immunized against uncertainty [87].

The following paragraphs will discuss some of the studies in the literature of inventory robust optimization.

Ben-Tal et al. [88] developed an adjustable robust counterpart of a linear programming model under uncertain parameters with ellipsoidal and polyhedral uncertainty sets. They applied the counterpart reformulation on a multi-stage uncertain inventory system. Ben-Tal et al. [89] presented a multi-echelon supply chain with multi-period inventory control policy. An affinely adjustable robust counterpart reformulation of the original model is developed based on polyhedral uncertainty set.

Bienstock and Özbay [90] considered the optimization model of the base-stock level for a single buffer when demand is uncertain. The authors developed two robust counterparts, one of them is based on box uncertainty set, while the other is based on polyhedral uncertainty set.

See and Sim [91] proposed a robust counterpart reformulation of a multi-period inventory control problem under uncertain demand with limited information. The robust counterpart reformulation is based on the combination of interval-ellipsoidal and the combination of interval- polyhedral uncertainty sets.

Bertsimas and Thiele [92] determined the robust counterpart reformulation of a set of inventory problems based on the polyhedral uncertainty set.

Sözüer and Theiele [93] provide a recent survey and discussion of the most recent developments and applications of robust optimization.

To the best of our knowledge, the only available work discussing the robust optimization for an SNVP with a single product was showed in [10]. The authors discussed the minimax regret optimization of the SNVP where the demand is uncertain and the uncertainty set is an interval set.

In the next sections, we will describe the robust counterpart reformulations of the MPSNVP under different uncertainty sets. We will also propose the solution procedures for the obtained models.

4.3 Robust Counterpart Based on Uncertainty Sets

To ensure computational tractability of robust problems, the parameter uncertainty should be defined carefully. The uncertainty set should be specified by the decision maker [85, 86]. The size and shape of the uncertainty set reflect the degree of conservativeness and the preferences of the decision maker, respectively [94]. Typically applied uncertainty sets are box, ellipsoidal, polyhedral and combinations of them [95]. Generally speaking, consider the following Mixed-Integer Linear programming problem (MILP):

$$\mathcal{P}: \ \text{Max} \ \sum_j \tau_j X_j + \sum_k \lambda_k Y_k,$$

$$\text{s. t.}$$

$$\sum_j \tilde{a}_{ij} X_j + \sum_k \tilde{d}_{ik} Y_k \leq b_i, \ \forall \ i \in I,$$

$$X_j \in \mathbb{R}, \ \forall \ j \in J,$$

$$Y_k \in \mathbb{Z}, \ \forall \ k \in K.$$

Suppose, without loss of generality, that only the left-hand-side parameters in the constraints of model $\mathcal{P}$ have uncertain data. This assumption is valid because of the following:

- If the coefficients in the objective function have uncertain data, then the objective function can be written as a constraint.

- In any constraint i, if the right-hand-side parameter is subject to uncertainty, then it can be written as $\sum_j \tilde{a}_{ij} X_j + + \sum_k \tilde{d}_{ik} Y_k - \tilde{b}_i \leq 0$, therefore we end up with a constraint that has uncertain parameters on the left-hand-side only.

Assuming that only parameters $\tilde{a}_{ij}$ and $\tilde{d}_{ik}$ are subjected to uncertainty, then any constraint i in model $\mathcal{P}$ can be expressed as:

$$\sum_{j \notin J_i} a_{ij} X_j + \sum_{j \in J_i} \tilde{a}_{ij} X_j + \sum_{k \notin K_i} d_{ik} Y_k + \sum_{k \in K_i} \tilde{d}_{ik} Y_k \leq b_i, \tag{4.1}$$

where, J_i and K_i denote the sets of uncertain parameters in constraint i, and $\tilde{a}_{ij}$, $\tilde{d}_{ik}$ represent the true values of the uncertain parameters. In order to acquire control of the conservativeness degree of the robust formulation, the true value of the uncertain parameters $\tilde{a}_{ij}$ and $\tilde{d}_{ik}$ are represented as follows:

$$\begin{aligned} \tilde{a}_{ij} &= a_{ij} + \xi_{ij} \hat{a}_{ij}, \\ \tilde{d}_{ik} &= d_{ik} + \xi_{ik} \hat{d}_{ik}, \end{aligned} \tag{4.2}$$

where a_{ij} and d_{ik} are the nominal values and $\hat{a}_{ij}$ and $\hat{d}_{ik}$ represent the deviation magnitudes from the nominal values of the uncertain parameters $\tilde{a}_{ij}$ and $\tilde{d}_{ik}$ respectively. In addition, ξ_{ij} and ξ_{ik} are variables that take values in the interval $[-1, 1]$, indeed, these variables provide perturbations to the uncertain parameters. Next sections present the robust counterpart based on different uncertainty sets.

4.3.1 Robust Counterpart Based on Interval Uncertainty Set

To the best of our knowledge, the first work on robust optimization appeared in Soyster [96]. The author considered a simple perturbation in the data for uncertain parameters in a linear programming formulation, then he provided a reformulation of the original problem in order to obtain a solution that is feasible for all possible realizations of the uncertain parameters. Soyster's approach considers all uncertain parameters to take their worst case values, therefore this approach is considered to be the most conservative robust optimization approach. In terms of modeling complexity, robust counterpart approach based on interval uncertainty set preserves exactly the same complexity of the original model. This makes it a good candidate for complex problems such as robust discrete optimization problems including robust network problems, see [97] for a recent survey on the utilization of interval uncertainty set in discrete optimization problems.

To immunize against uncertainty, we apply the robust counterpart approach to the original constraint (4.1) under the uncertainty set (4.2), this yields,

$$\sum_{j \notin J_i} a_{ij} X_j + \max_{\xi_{ij}} \left[\sum_{j \in J_i} \left(a_{ij} + \xi_{ij} \hat{a}_{ij} \right) X_j \right] + \sum_{k \notin K_i} d_{ik} Y_k + \max_{\xi_{ik}} \left[\sum_{k \in K_i} \left(d_{ik} + \xi_{ik} \hat{d}_{ik} \right) Y_k \right] \leq b_i .$$

The above constraint reduces to,

$$\sum_{j \in J} a_{ij} X_j + \max_{\xi_{ij}} \left[\sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} X_j \right] + \sum_{k \in K} d_{ik} Y_k + \max_{\xi_{ij}} \left[\sum_{k \in K_i} \xi_{ik} \hat{d}_{ik} Y_k \right] \leq b_i . \quad (4.3)$$

The above constraint indicates that each uncertain parameter will take its boundary value and yields,

$$\sum_{j\in J} a_{ij}X_j + \sum_{j\in J_i} \hat{a}_{ij}\,|X_j| + \sum_{k\in K} d_{ik}Y_k + \sum_{k\in K_i} \hat{d}_{ik}\,|Y_k| \le b_i. \tag{4.4}$$

The absolute value operator in (4.4) can be removed directly if the variable $X_j,\ j \in J_i$, is positive, and if the variable $Y_k,\ k \in K_i$, is positive or binary, hence the robust formulation for model $\mathcal{P}$ based on Soyster's approach becomes:

$$\text{Max} \quad \sum_j \tau_j X_j + \sum_k \lambda_k Y_k,$$

$$\text{s. t.}$$

$$\sum_{j\in J} a_{ij}X_j + \sum_{j\in J_i} \hat{a}_{ij}X_j + \sum_{k\in K} d_{ik}Y_k + \sum_{k\in K_i} \hat{d}_{ik}Y_k \le b_i,\ \forall\ i \in I,$$

$$X_j \in \mathbb{R}^+,\ \forall\ j \in J_i,$$

$$Y_k \in \mathbb{Z}^+ \text{or} \in \{0,1\},\ \forall\ k \in K_i,$$

$$X_j \in \mathbb{R},\ \forall\ j \in J \setminus J_i,$$

$$Y_k \in \mathbb{Z},\ \forall\ k \in K \setminus K_i.$$

However if the variable $X_j,\ j \in J_i$, is not positive and the variable $Y_k,\ k \in K_i$, is neither positive nor binary, the robust formulation for model $\mathcal{P}$ based on Soyster's approach becomes:

$$\mathcal{P} - RC_I: \ \text{Max} \quad \sum_j \tau_j X_j + \sum_k \lambda_k Y_k,$$

s. t.

$$\sum_{j\in J} a_{ij}X_j + \sum_{j\in J_i} \hat{a}_{ij}w_j + \sum_{k\in K} d_{ik}Y_k + \sum_{k\in K_i} \hat{d}_{ik}u_k \leq b_i,\ \forall\ i \in I,$$

$$-w_j \leq X_j \leq w_j,\ \forall\ j \in J_i,$$

$$-u_k \leq Y_k \leq u_k,\ \forall\ k \in K_i,$$

$$w_j,\ u_k \geq 0,\ \forall\ j \in J_i,\ \forall\ k \in K_i,$$

$$X_j \in \mathbb{R},\ \forall\ j \in J,$$

$$Y_k \in \mathbb{Z},\ \forall\ k \in K,$$

where, w_j and u_k are auxiliary variables.

4.3.2 Robust Counterpart Based on Box Uncertainty Set

Recently, Li et al. [95] provided a comprehensive study on the robust counterpart formulation for linear and MILP. They gave the mathematical proof of the robust counterpart to linear and MILP using different uncertainty sets. The proposed uncertainty sets are formulated based on different norms of the perturbation variables.

The box uncertainty set is formulated based on the Chebyshev norm of the perturbation variables, it is presented as follows:

$$U_\infty = \left\{\xi_j \mid \|\xi_j\|_\infty \leq \Psi\right\} = \{\xi_j \mid \xi_j \leq \Psi\}, \tag{4.5}$$

where Ψ is the adjustable parameter that control the uncertainty set size, and hence controlling the degree of conservatism. If $\Psi = 1$, then the resulting uncertainty set is the interval uncertainty set which is a special case of the box uncertainty set.

The robust counterpart of model $\mathcal{P}$ under box uncertainty set (4.5) is given as follows:

$$\mathcal{P} - RC_B : \text{Max} \quad \sum_j \tau_j X_j + \sum_k \lambda_k Y_k,$$

$$\text{s. t.}$$

$$\sum_{j \in J} a_{ij} X_j + \sum_{k \in K} d_{ik} Y_k + \Psi_i \left[\sum_{j \in J_i} \hat{a}_{ij} w_j + \sum_{k \in K_i} \hat{d}_{ik} u_k \right] \leq b_i, \ \forall \ i \in I,$$

$$- w_j \leq X_j \leq w_j, \ \forall \ j \in J_i,$$

$$- u_k \leq Y_k \leq u_k, \ \forall \ k \in K_i,$$

$$w_j, \ u_k \geq 0, \ \forall \ j \in J_i, \ \forall \ k \in K_i,$$

$$X_j \in \mathbb{R}, \ \forall \ j \in J,$$

$$Y_k \in \mathbb{Z}, \ \forall \ k \in K,$$

where w_j and u_k are auxiliary variables.

The proof of model $\mathcal{P} - RC_B$ is available in [95].

If $X_j, \ j \in J_i$, is positive, then we can remove the constraint $-w_j \leq X_j \leq w_j$ and replace w_j by X_j in model $\mathcal{P} - RC_B$. If $Y_k, \ k \in K_i$, is binary or positive, then we can remove the constraint $-u_k \leq Y_k \leq u_k$ and replace u_k by Y_k in model

$\mathcal{P} - RC_B$.

4.3.3 Robust Counterpart Based on Ellipsoidal Uncertainty Set

Li et al. [95] studied the robust counterpart formulation under the pure ellipsoidal uncertainty set. The ellipsoidal uncertainty set is defined as follows:

$$U_2 = \{\xi_j \mid \|\xi_j\|_2 \le \Omega\} = \left\{\xi_j \mid \sqrt{\sum_{j \in J_i} \xi_j^2} \le \Omega\right\}, \tag{4.6}$$

where Ω is the radius of the uncertainty set; it also represent the degree of conservatism. The ellipsoidal uncertainty set is formulated based on the 2-norm of the perturbation variables.

The robust counterpart of model $\mathcal{P}$ as follows:

$$\mathcal{P} - RC_E : \text{Max} \quad \sum_j \tau_j X_j + \sum_k \lambda_k Y_k,$$

s. t.

$$\sum_{j \in J} a_{ij} X_j + \sum_{k \in K} d_{ik} Y_k + \Omega_i \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 X_j^2 + \sum_{k \in K_i} \hat{d}_{ik}^2 Y_k^2} \le b_i, \ \forall\, i \in I,$$

$$X_j \in \mathbb{R}, \ \forall\, j \in J,$$

$$Y_k \in \mathbb{Z}, \ \forall\, k \in K.$$

The proof of model $\mathcal{P} - RC_E$ is available in [95].

4.3.4 Robust Counterpart Based on Polyhedral Uncertainty Set

Li et al. [95] also discussed the robust counterpart formulation under the pure polyhedral uncertainty set. The polyhedral uncertainty set is defined below:

$$U_1 = \left\{ \xi_j \mid \|\xi_j\|_1 \leq \Gamma \right\} = \left\{ \xi_j \mid \sum_{j \in J_i} |\xi_j| \leq \Gamma \right\}, \tag{4.7}$$

where Γ is the parameter that controls of the uncertainty set size; it also known as the *budget of robustness* or *price of robustness*. The polyhedral uncertainty set is formulated based on the 1-norm of the perturbation random variables.

The robust counterpart of model $\mathcal{P}$ is given as follows:

$$\mathcal{P} - RC_P : \text{Max} \quad \sum_j \tau_j X_j + \sum_k \lambda_k Y_k,$$

s. t.

$$\sum_{j \in J} a_{ij} X_j + \sum_{k \in K} d_{ik} Y_k + z_i \Gamma_i \leq b_i, \ \forall \ i \in I,$$

$$z_i \geq \hat{a}_{ij} w_j, \ \forall \ j \in J_i, \ \forall \ i \in I,$$

$$z_i \geq \hat{b}_{ik} u_k, \ \forall \ k \in K_i, \ \forall \ i \in I,$$

$$- w_j \leq X_j \leq w_j, \ \forall \ j \in J_i,$$

$$- u_k \leq Y_k \leq u_k, \ \forall \ k \in K_i,$$

$$w_j, \ u_k, \ z_i \geq 0, \ \forall \ j \in J_i, \ \forall \ k \in K_i, \ \forall \ i \in I,$$

$$X_j \in \mathbb{R}, \ \forall \ j \in J,$$

$$Y_k \in \mathbb{Z},\ \forall\ k \in K,$$

where w_j and u_k are auxiliary variables.

The proof of model $\mathcal{P} - RC_P$ is available in [95].

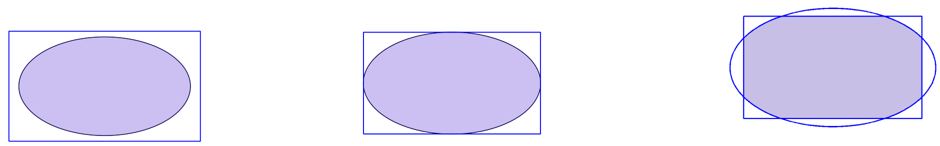
If $X_j,\ j \in J_i$, is positive, then we can remove the constraint $-w_j \leq X_j \leq w_j$ and replace w_j by X_j in model $\mathcal{P} - RC_P$. If $Y_k,\ k \in K_i$, is binary or positive, then we can remove the constraint $-u_k \leq Y_k \leq u_k$ and replace u_k by Y_k in model $\mathcal{P} - RC_P$.

4.3.5 Robust Counterpart Based on Interval-Ellipsoidal Uncertainty Set

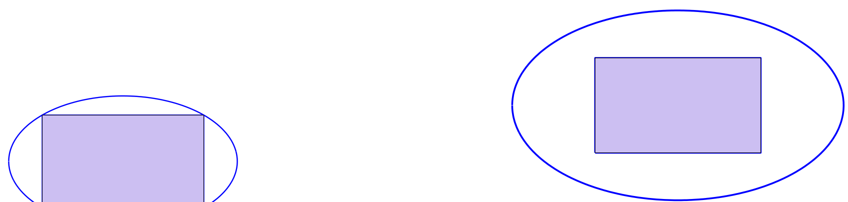
The rigorous development of robust optimization traced back to the early 2000's. Among the pioneering studies in the field are Ben-Tal and Nemirovski [98] and Bertsimas and Sim [99, 100].

Ben-Tal and Nemirovski [98] employed the ellipsoidal uncertainty set (4.6), this yields a robust optimization counterpart that is less conservative than the one obtained in Soyster's approach. The proposed robust counterpart formulation allows for a trade-off between the performance and the robustness or equivalently between the objective value and the conservativeness degree.

Figure 4.1 illustrates different geometry representations of the combined interval-ellipsoidal uncertainty set based on the value of the adjustable parameter Ω.



(a) $0 < \Omega < 1$. (b) $\Omega = 1$. (c) $1 < \Omega < \sqrt{|J_i| + |K_i|}$.



(d) $\Omega = \sqrt{|J_i| + |K_i|}$. (e) $\sqrt{|J_i| + |K_i|} < \Omega$.

Figure 4.1: Illustration of the combined interval-ellipsoidal uncertainty set.

The robust counterpart formulation of model $\mathcal{P}$ based on the combination of the interval uncertainty set (4.5) with $\Psi = 1$ and the ellipsoidal uncertainty set (4.6) is given by,

$$\mathcal{P} - RC_{I-E} : \text{Max} \quad \sum_{j} \tau_j X_j + \sum_{k} \lambda_k Y_k,$$

s. t.

$$\sum_{j \in J} a_{ij} X_j + \sum_{k \in K} d_{ik} Y_k + \sum_{j \in J_i} \hat{a}_{ij} w_{ij} + \sum_{k \in K_i} \hat{d}_{ik} u_{ik}$$

$$+ \Omega_i \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 t_{ij}^2 + \sum_{k \in K_i} \hat{d}_{ik}^2 z_{ik}^2} \leq b_i, \ \forall\, i \in I,$$

$$- w_{ij} \leq X_j - t_{ij} \leq w_{ij}, \ \forall\, j \in J_i, \ \forall\, i \in I,$$

$$- u_{ik} \leq Y_k - z_{ik} \leq u_{ik}, \ \forall\, k \in K_i, \ \forall\, i \in I,$$

$$w_{ij},\ t_{ij} \geq 0,\ \forall\ j \in J_i,\ \forall\ i \in I,$$

$$u_{ik},\ z_{ik} \geq 0,\ \forall\ k \in K_i,\ \forall\ i \in I,$$

$$X_j \in \mathbb{R},\ \forall\ j \in J,$$

$$Y_k \in \mathbb{Z},\ \forall\ k \in K,$$

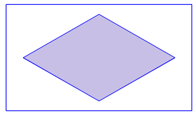
where t_{ij} and z_{ik} are positive dual variables, and w_{ij} and u_{ik} are auxiliary variables. The proof of model $\mathcal{P} - RC_{I-E}$ is available in [87, 95, 98].

4.3.6 Robust Counterpart Based on Interval-Polyhedral Uncertainty Set

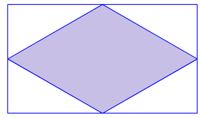
Although the approach proposed by Ben-Tal and Nemirovski [98]; which was discussed in Section 4.3.5, solved the problem of over conservatism in Soyster's approach [96]; which was discussed in Section 4.3.1, the resulting formulation $\mathcal{P} - RC_{I-E}$ is nonlinear, this results in computational complexities in solving mixed-integer nonlinear optimization problems.

Bertsimas and Sim [99, 100] provided the robust counterpart formulation that overcomes the drawbacks in both Soyster's approach and Ben-Tal and Nemirovski's approach. They presented the combination of the interval uncertainty set (4.5) with $\Psi = 1$ and the polyhedral uncertainty set (4.7).

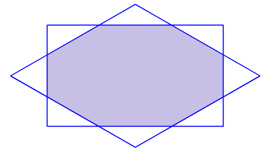
Figure 4.2 illustrates different geometry representations of the combined interval-polyhedral uncertainty set based on the value of the adjustable parameter Γ.



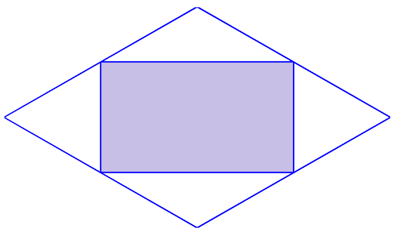
(a) $0 < \Gamma < 1$.



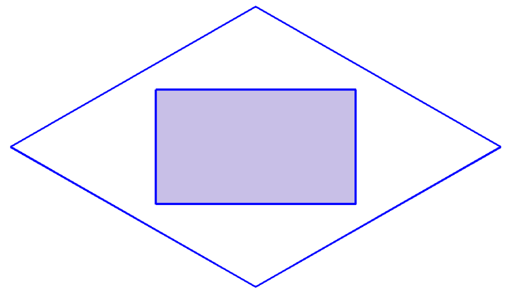
(b) $\Gamma = 1$.



(c) $1 < \Gamma < |J_i| + |K_i|$.



(d) $\Gamma = |J_i| + |K_i|$.



(e) $|J_i| + |K_i| < \Gamma$.

Figure 4.2: Illustration of the combined interval-polyhedral uncertainty set.

The robust counterpart formulation of model $\mathcal{P}$ based on the interval-polyhedral uncertainty combination is expressed as,

$$\mathcal{P} - RC_{I-P} : \text{Max} \quad \sum_j \tau_j X_j + \sum_k \lambda_k Y_k,$$

s. t.

$$\sum_{j \in J} a_{ij} X_j + \sum_{k \in K} d_{ik} Y_k + \sum_{j \in J_i} t_{ij} + \sum_{k \in K_i} p_{ik} + z_i \Gamma_i \le b_i, \ \forall \ i \in I,$$

$$z_i + t_{ij} \ge \hat{a}_{ij} w_j, \ \forall \ j \in J_i, \ \forall \ i \in I,$$

$$z_i + p_{ik} \ge \hat{b}_{ik} u_k, \ \forall \ k \in K_i, \ \forall \ i \in I,$$

$$- w_j \le X_j \le w_j, \ \forall \ j \in J_i,$$

$$- u_k \le Y_k \le u_k, \ \forall \ k \in K_i,$$

$$w_j,\ u_k,\ z_i,\ t_{ij},\ p_{ik} \geq 0,\ \forall\ j \in J_i,\ \forall\ k \in K_i,\ \forall\ i \in I,$$

$$X_j \in \mathbb{R},\ \forall\ j \in J, ,\ \forall\ k \in K_i,$$

$$Y_k \in \mathbb{Z},\ \forall\ k \in K,$$

where t_{ij}, p_{ik} and z_i are positive dual variables, and w_j and u_k are auxiliary variables.

The proof of model $\mathcal{P} - RC_{I-P}$ is available in [95, 99, 100].

If $X_j,\ j \in J_i$, is positive, then we can remove the constraint $-w_j \leq X_j \leq w_j$ and replace w_j by X_j in model $\mathcal{P} - RC_{I-P}$. If $Y_k,\ k \in K_i$, is binary or positive, then we can remove the constraint $-u_k \leq Y_k \leq u_k$ and replace u_k by Y_k in model $\mathcal{P} - RC_{I-P}$.

4.4 Robust Multi-Product Selective Newsvendor Problem with Flexible Market Entry

The profit function for the deterministic flexible market entry case of the multi-product selective newsvendor problem is given by (2.1) which equivalently can be expressed as:

$$P\left(Q_j,\ y_{ij}\right) = \sum_{i \in I} \left(\sum_{j \in J} \left(r_{ij} - v_j\right) x_{ij} - S_{ij} \right) y_{ij} - \sum_{j \in J} \left(e_j - v_j\right) \left(\sum_{i \in I} x_{ij} y_{ij} - Q_j \right)^+ - \sum_{j \in J} \left(c_j - v_j\right) Q_j. \tag{4.8}$$

where $(\zeta)^+ = \zeta$ if $\zeta \geq 0$, and 0 otherwise. It is also notable that the above model is separable in products; j. The resulting deterministic flexible market entry selective newsvendor problem for each product is given as:

$$\begin{aligned} P\left(Q_j,\ y_{ij}\right) = & \sum_{i \in I} \left(\left(r_{ij} - v_j\right) x_{ij} - S_{ij}\right) y_{ij} - \left(e_j - v_j\right) \left(\sum_{i \in I} x_{ij} y_{ij} - Q_j\right)^+ \\ & - \left(c_j - v_j\right) Q_j. \end{aligned} \tag{4.9}$$

In that function, we can replace $\left(\sum_{i \in I} x_{ij} y_{ij} - Q_j\right)^+$ by z_j and add the constraints $\sum_{i \in I} x_{ij} y_{ij} - Q_j \leq z_j$ and $z_j \geq 0$. Thus, the optimization model becomes:

$$\mathcal{SNVP}_{Flex}: \ \text{Max} \ \sum_{i \in I} \left(\left(r_{ij} - v_j\right) x_{ij} - S_{ij}\right) y_{ij} - \left(e_j - v_j\right) z_j - \left(c_j - v_j\right) Q_j,$$

$$\text{s. t.}$$

$$\sum_{i \in I} x_{ij} y_{ij} - Q_j \leq z_j,$$

$$Q_j, z_j \geq 0,$$

$$y_{ij} \in \{0, 1\}, \ \forall \ i \in I.$$

Note that, the optimal order quantity for each product j; Q_j, should satisfy the demand of that product from the selected markets, i.e.,

$$Q_j = \sum_{i \in I} x_{ij} y_{ij}, \tag{4.10}$$

and this causes $z_j = 0$ at optimality.

Hence, the deterministic flexible market entry selective newsvendor problem for product j can be expressed as:

$$\mathcal{SNVP}_{Flex-D}: \text{ Max } \sum_{i \in I} \left(\left(r_{ij} - c_j \right) x_{ij} - S_{ij} \right) y_{ij},$$

$$\text{s. t.}$$

$$y_{ij} \in \{0, 1\}, \ \forall \ i \in I.$$

The optimal solution to the deterministic flexible market entry selective newsvendor problem $\mathcal{SNVP}_{Flex-D}$, is:

$$\begin{aligned} y_{ij}^{*} &= 1 \text{ if } \left(r_{ij} - c_j \right) x_{ij} - S_{ij} > 0, \\ y_{ij}^{*} &= 0 \text{ otherwise.} \end{aligned} \tag{4.11}$$

The optimal order quantity is given by (4.10).

Now, suppose that the demand of the products; x_{ij}, is uncertain in the market set $I_k \subseteq I$, therefore, the demand value is considered as,

$$\tilde{x}_{ij} = x_{ij} + \xi_{ij} \hat{x}_{ij}, \ \forall \ i \in I_k, \tag{4.12}$$

where x_{ij} is the nominal value of the uncertain demand of product j in market i, $\hat{x}_{ij}$ represents the deviation magnitude from the nominal value and ξ_{ij} is the variable that controls demand perturbation and takes a value in the interval $[-1, 1]$. The model $\mathcal{SNVP}_{Flex}$ can be rewritten as:

$$\begin{aligned}
&\text{Max} \quad \sum_{i\in I\setminus I_k} \left(\left(r_{ij}-v_j\right)x_{ij}-S_{ij}\right)y_{ij}+\sum_{i\in I_k}\left(\left(r_{ij}-v_j\right)\tilde{x}_{ij}-S_{ij}\right)y_{ij} \\
&\qquad -\left(e_j-v_j\right)z_j-\left(c_j-v_j\right)Q_j, \\
&\text{s. t.} \\
&\qquad \sum_{i\in I\setminus I_k} x_{ij}y_{ij}+\sum_{i\in I_k}\tilde{x}_{ij}y_{ij}-Q_j\leq z_j, \\
&\qquad Q_j,\ z_j\geq 0, \\
&\qquad y_{ij}\in\{0,1\},\ \forall\ i\in I.
\end{aligned}$$

where x_{ij} and $\tilde{x}_{ij}$ represent the deterministic and uncertain values of the demand of product j in market i, respectively.

The above model can be rewritten as:

$$\begin{aligned}
&\mathcal{SNVP}_{Flex-U}:\text{Max}\ \ \delta, \\
&\qquad \text{s. t.} \\
&\qquad\quad \delta-\sum_{i\in I\setminus I_k}\left(\left(r_{ij}-v_j\right)x_{ij}-S_{ij}\right)y_{ij}-\sum_{i\in I_k}\left(\left(r_{ij}-v_j\right)\tilde{x}_{ij}-S_{ij}\right)y_{ij} \\
&\qquad\quad +\left(e_j-v_j\right)z_j+\left(c_j-v_j\right)Q_j\leq 0, \\
&\qquad\quad \sum_{i\in I\setminus I_k} x_{ij}y_{ij}+\sum_{i\in I_k}\tilde{x}_{ij}y_{ij}-Q_j\leq z_j, \\
&\qquad\quad Q_j,\ z_j\geq 0, \\
&\qquad\quad y_{ij}\in\{0,1\},\ \forall\ i\in I.
\end{aligned}$$

Next, we present the robust counterpart reformulation of the MPSNVP with flexible market entry; $\mathcal{SNVP}_{Flex-U}$, under different uncertainty sets.

4.4.1 Robust MPSNVP with Flexible Market Entry Based on Box Uncertainty Set

We apply the approach presented in Section 4.3.2 to obtain the robust counterpart of the uncertain model of the MPSNVP with flexible market entry, $\mathcal{SNVP}_{Flex-U}$, based on the box uncertainty set (4.5). Note that the uncertain demands are coefficients of the decision variables y_{ij}, which are binary variables, therefore the robust counterpart approach yields:

$$
\begin{aligned}
\text{Max} \quad & \delta, \\
\text{s.t.} \quad & \\
& \delta - \sum_{i \in I} \left(\left(r_{ij} - v_j \right) x_{ij} - S_{ij} \right) y_{ij} + \Psi \sum_{i \in I_k} \left(\left(r_{ij} - v_j \right) \hat{x}_{ij} \right) y_{ij} \\
& + \left(e_j - v_j \right) z_j + \left(c_j - v_j \right) Q_j \leq 0, \\
& \sum_{i \in I} x_{ij} y_{ij} + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} y_{ij} - Q_j \leq z_j, \\
& Q_j, \; z_j \geq 0, \\
& y_{ij} \in \left\{ 0, 1 \right\}, \; \forall \; i \in I.
\end{aligned}
$$

This robust counterpart results in an MILP formulation. This preserves the tractability of original model with the same number of variables and constraints. Notice that, we can get Soyster's robust counterpart to the above model by taking

$\Psi = 1$ and $\Psi_j = 1$.

The optimal solution to the above model yields:

$$
\begin{aligned}
Q_j^* &= \sum_{i \in I} x_{ij} y_{ij}^* + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} y_{ij}^*, \\
z_j^* &= 0.
\end{aligned}
\tag{4.13}
$$

Hence, the above model reduces to:

$$
\begin{aligned}
\text{Max} \quad & \sum_{i \in I} \left(\left(r_{ij} - v_j \right) x_{ij} - S_{ij} \right) y_{ij} - \Psi \sum_{i \in I_k} \left(\left(r_{ij} - v_j \right) \hat{x}_{ij} \right) y_{ij} \\
& - \left(c_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} y_{ij} \right), \\
\text{s.t.} \quad & \\
& y_{ij} \in \{0, 1\}, \ \forall \ i \in I.
\end{aligned}
$$

This model can be rewritten as:

$\mathcal{SNVP}_{Flex-U} - RC_B$:

$$
\begin{aligned}
\text{Max} \quad & \sum_{i \in I \setminus I_k} \left(\left(r_{ij} - c_j \right) x_{ij} - S_{ij} \right) y_{ij} \\
& + \sum_{i \in I_k} \left(\left(r_{ij} - c_j \right) x_{ij} - \Psi \left(r_{ij} - v_j \right) \hat{x}_{ij} - \Psi_j \left(c_j - v_j \right) \hat{x}_{ij} - S_{ij} \right) y_{ij}, \\
\text{s.t.} \quad & \\
& y_{ij} \in \{0, 1\}, \ \forall \ i \in I.
\end{aligned}
$$

The solution to model $\mathcal{SNVP}_{Flex-U} - RC_B$ is as follows:

$$
\begin{aligned}
&\text{For all } i \in I \setminus I_k: \\
&\qquad y_{ij}^{*} = 1 \text{ if } \left(r_{ij} - c_j\right) x_{ij} - S_{ij} > 0, \\
&\qquad y_{ij}^{*} = 0 \text{ otherwise}, \\
&\text{For all } i \in I_k: \\
&\qquad y_{ij}^{*} = 1 \text{ if } \left(r_{ij} - c_j\right) x_{ij} - \Psi \left(r_{ij} - v_j\right) \hat{x}_{ij} - \Psi_j \left(c_j - v_j\right) \hat{x}_{ij} - S_{ij} > 0, \\
&\qquad y_{ij}^{*} = 0 \text{ otherwise},
\end{aligned}
\tag{4.14}
$$

while the optimal order quantity for each product j is given by (4.13).

4.4.2 Robust MPSNVP with Flexible Market Entry Based on Ellipsoidal Uncertainty Set

We apply the approach provided in Section 4.3.3 to obtain the robust counterpart of the uncertain model of the MPSNVP with flexible market entry, $\mathcal{SNVP}_{Flex-U}$, this can be expressed as:

$$
\begin{aligned}
&\text{Max} \quad \delta, \\
&\text{s.t.} \\
&\quad \delta - \sum_{i \in I} \left(\left(r_{ij} - v_j\right) x_{ij} - S_{ij}\right) y_{ij} + \Omega \sqrt{\sum_{i \in I_k} \left(r_{ij} - v_j\right)^2 \hat{x}_{ij}^2 y_{ij}^2} \\
&\quad + \left(e_j - v_j\right) z_j + \left(c_j - v_j\right) Q_j \leq 0,
\end{aligned}
$$

$$\sum_{i \in I} x_{ij} y_{ij} + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^2} - Q_j \leq z_j,$$

$$Q_j,\ z_j \geq 0,$$

$$y_{ij} \in \{0,1\},\ \forall\ i \in I.$$

This robust counterpart preserves the same number of variables and constraints, however it results in an MINLP formulation, which causes computational complexities.

The optimal solution to the above model yields,

$$\begin{aligned} Q_j^* &= \sum_{i \in I} x_{ij} y_{ij}^* + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^{*2}}, \\ z_j^* &= 0. \end{aligned} \tag{4.15}$$

Hence, the above model reduces to:

$$\begin{aligned} \text{Max} \quad & \sum_{i \in I} \left(\left(r_{ij} - v_j \right) x_{ij} - S_{ij} \right) y_{ij} - \Omega \sqrt{\sum_{i \in I_k} \left(r_{ij} - v_j \right)^2 \hat{x}_{ij}^2 y_{ij}^2} \\ & - \left(c_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^2} \right), \\ \text{s.t.} & \\ & y_{ij} \in \{0,1\},\ \forall\ i \in I. \end{aligned}$$

The above model can be rewritten as:

$$\text{Max} \quad \sum_{i\in I\setminus I_k} \left(\left(r_{ij}-c_j\right)x_{ij}-S_{ij}\right)y_{ij}+\sum_{i\in I_k}\left(\left(r_{ij}-c_j\right)x_{ij}-S_{ij}\right)y_{ij}$$
$$-\,\Omega\sqrt{\sum_{i\in I_k}\left(r_{ij}-v_j\right)^2\hat{x}_{ij}^2y_{ij}^2}-\left(c_j-v_j\right)\Omega_j\sqrt{\sum_{i\in I_k}\hat{x}_{ij}^2y_{ij}^2},$$

s.t.

$$y_{ij}\in\{0,1\}\,,\ \forall\ i\in I.$$

The above model can be separated into the following two sub-models:

$$\mathcal{SNVP}_{Flex-U}-D:\ \text{Max}\sum_{i\in I\setminus I_k}\left(\left(r_{ij}-c_j\right)x_{ij}-S_{ij}\right)y_{ij},$$

s.t.

$$y_{ij}\in\{0,1\}\,,\ \forall\ i\in I\setminus I_k.$$

$$\mathcal{SNVP}_{Flex-U}-RC_E:\ \text{Max}\sum_{i\in I_k}\left(\left(r_{ij}-c_j\right)x_{ij}-S_{ij}\right)y_{ij}-\Omega\sqrt{\sum_{i\in I_k}\left(r_{ij}-v_j\right)^2\hat{x}_{ij}^2y_{ij}^2}$$
$$-\left(c_j-v_j\right)\Omega_j\sqrt{\sum_{i\in I_k}\hat{x}_{ij}^2y_{ij}^2},$$

s.t.

$$y_{ij}\in\{0,1\}\,,\ \forall\ i\in I_k.$$

The optimal solution to sub-model $\mathcal{SNVP}_{Flex-U}-D$ is as follows:

For all $i\in I\setminus I_k$:

$$y_{ij}^*=1 \text{ if } \left(r_{ij}-c_j\right)x_{ij}-S_{ij}>0,$$
$$y_{ij}^*=0 \text{ otherwise.} \tag{4.16}$$

The optimal solution to sub-model $\mathcal{SNVP}_{Flex-U} - RC_E$ can be determined by applying the same solution procedure that was applied in Section 2.3.2 for solving the risk-neutral MPSNVP with flexible market entry. The RDU ratio for sub-model $\mathcal{SNVP}_{Flex-U} - RC_E$ is:

$$RDU_{Flex-RC_E} = \frac{(r_{ij} - c_j)\, x_{ij} - S_{ij}}{\Omega^2 \left(r_{ij} - v_j\right)^2 \hat{x}_{ij}^2 + \Omega_j^2 \left(c_j - v_j\right)^2 \hat{x}_{ij}^2}. \tag{4.17}$$

The optimal order quantity for each product j is given be (4.15).

4.4.3 Robust MPSNVP with Flexible Market Entry Based on Polyhedral Uncertainty Set

We apply the approach provided in Section 4.3.4 to obtain the robust counterpart of the uncertain model of the MPSNVP with flexible market entry, $\mathcal{SNVP}_{Flex-U}$. Note that the uncertain demands are coefficients of the decision variables y_{ij}, which are binary variables, therefore applying the robust counterpart based on the polyhedral uncertainty set yields:

$$\text{Max} \;\; \delta,$$

s.t.

$$\delta - \sum_{i \in I} \left((r_{ij} - v_j)\, x_{ij} - S_{ij} \right) y_{ij} + u\Gamma + (e_j - v_j)\, z_j$$

$$+ (c_j - v_j)\, Q_j \leq 0,$$

$$u \geq (r_{ij} - v_j)\, \hat{x}_{ij} y_{ij}, \;\; \forall \; i \in I_k,$$

$$\sum_{i \in I} x_{ij} y_{ij} + u_j \Gamma_j - Q_j \leq z_j,$$

$$u_j \geq \hat{x}_{ij} y_{ij}, \ \forall \ i \in I_k,$$

$$u, \ u_j, \ Q_j, \ z_j \geq 0,$$

$$y_{ij} \in \{0, 1\}, \ \forall \ i \in I.$$

This robust counterpart results in MILP. This preserves the tractability of original model, however, the number of variables increases as well as the number of constraints. The number of binary variables remains the same as in the original formulation $\mathcal{SNVP}_{Flex-U}$.

We can further reduce the size of the above model. Note that the optimal solution to the above model; given the values of y_{ij}^* and u_j^*, results in:

$$\begin{aligned} Q_j^* &= \sum_{i \in I} x_{ij} y_{ij}^* + u_j^* \Gamma_j, \\ z_j^* &= 0. \end{aligned} \tag{4.18}$$

Substituting from (4.18) into the above model:

$$\text{Max} \quad \sum_{i \in I} \left(\left(r_{ij} - v_j \right) x_{ij} - S_{ij} \right) y_{ij} - u\Gamma - \left(c_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} + u_j \Gamma_j \right),$$

s.t.

$$u \geq \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij}, \ \forall \ i \in I_k,$$

$$u_j \geq \hat{x}_{ij} y_{ij}, \ \forall \ i \in I_k,$$

$$u, \ u_j \geq 0,$$

$$y_{ij} \in \{0,1\}, \; \forall\, i \in I.$$

The above model can be rewritten as:

$$\mathcal{SNVP}_{Flex-U} - RC_P : \text{Max} \;\; \sum_{i \in I} \left(\left(r_{ij} - c_j \right) x_{ij} - S_{ij} \right) y_{ij} - u\Gamma - \left(c_j - v_j \right) u_j \Gamma_j,$$

s.t.

$$u \geq \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij}, \; \forall\, i \in I_k,$$

$$u_j \geq \hat{x}_{ij} y_{ij}, \; \forall\, i \in I_k,$$

$$u, \; u_j \geq 0,$$

$$y_{ij} \in \{0,1\}, \; \forall\, i \in I.$$

The reduced model $\mathcal{SNVP}_{Flex-U} - RC_P$ is an MILP problem, it can be solved using commercial solvers such as CPLEX. Then, the results are used to get the optimal order quantities Q_j from (4.18).

4.4.4 Robust MPSNVP with Flexible Market Entry Based on Interval-Ellipsoidal Uncertainty Set

We apply the approach provided in Section 4.3.5 to obtain the robust counterpart of the uncertain model of the MPSNVP with flexible market entry, $\mathcal{SNVP}_{Flex-U}$, this can be expressed as:

$$\text{Max} \;\; \delta,$$

s.t.

$$
\begin{aligned}
&\delta - \sum_{i \in I} \left(\left(r_{ij} - v_j \right) x_{ij} - S_{ij} \right) y_{ij} + \sum_{i \in I_k} \left(r_{ij} - v_j \right) \hat{x}_{ij} p_{ij} \\
&+ \Omega \sqrt{\sum_{i \in I_k} \left(r_{ij} - v_j \right)^2 \hat{x}_{ij}^2 u_{ij}^2} + \left(e_j - v_j \right) z_j + \left(c_j - v_j \right) Q_j \le 0, \\
&- p_{ij} \le y_{ij} - u_{ij} \le p_{ij}, \ \forall \ i \in I_k, \\
&\sum_{i \in I} x_{ij} y_{ij} + \sum_{i \in I_k} \hat{x}_{ij} t_{ij} + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^2} - Q_j \le z_j, \\
&- t_{ij} \le y_{ij} - w_{ij} \le t_{ij}, \ \forall \ i \in I_k, \\
&u_{ij}, \ w_{ij}, \ p_{ij}, \ t_{ij} \ge 0, \ \forall \ i \in I_k, \\
&Q_j, \ z_j \ge 0, \\
&y_{ij} \in \{0, 1\}, \ \forall \ i \in I,
\end{aligned}
$$

where, p_{ij} and t_{ij} are auxiliary variables, while u_{ij} and w_{ij} are positive dual variables.

This robust counterpart results in an MINLP formulation with number of variables and constraints greater than that in the original formulation $\mathcal{SNVP}_{Flex-U}$. The size of the model can be reduced by performing preprocessing of some variables. In addition, the tractability of the above model can be retrieved by linearizing the nonlinear terms.

When the optimal values of y_{ij}^*, t_{ij}^* and w_{ij}^* are known, then the optimal solution to the above model is:

$$Q_j^* = \sum_{i \in I} x_{ij} y_{ij}^* + \sum_{i \in I_k} \hat{x}_{ij} t_{ij}^* + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^{*2}},$$
$$z_j^* = 0. \tag{4.19}$$

We also substitute for the nonlinear terms as follows:

$$\sum_{i \in I_k} (r_{ij} - v_j)^2 \hat{x}_{ij}^2 u_{ij}^2 \le q_j^2,$$
$$\sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^2 \le h_j^2.$$

Now, the above model can be represented as:

$$\text{Max} \quad \sum_{i \in I} \left((r_{ij} - v_j) x_{ij} - S_{ij} \right) y_{ij} - \sum_{i \in I_k} (r_{ij} - v_j) \hat{x}_{ij} p_{ij}$$
$$- \Omega q_j - (c_j - v_j) \left(\sum_{i \in I} x_{ij} y_{ij} + \sum_{i \in I_k} \hat{x}_{ij} t_{ij} + \Omega_j h_j \right),$$

s.t.

$$- p_{ij} \le y_{ij} - u_{ij} \le p_{ij}, \ \forall \ i \in I_k,$$

$$- t_{ij} \le y_{ij} - w_{ij} \le t_{ij}, \ \forall \ i \in I_k,$$

$$\sum_{i \in I_k} (r_{ij} - v_j)^2 \hat{x}_{ij}^2 u_{ij}^2 \le q_j^2,$$

$$\sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^2 \le h_j^2,$$

$$u_{ij}, \ w_{ij}, \ p_{ij}, \ t_{ij} \ge 0, \ \forall \ i \in I_k,$$

$$h_j, \ q_j \ge 0,$$

$$y_{ij} \in \{0,1\}, \ \forall \ i \in I,$$

The above model can be expressed as:

$$\mathcal{SNVP}_{Flex-U} - RC_{I-E} : \text{Max} \ \sum_{i \in I} \left(\left(r_{ij} - c_j \right) x_{ij} - S_{ij} \right) y_{ij} - \sum_{i \in I_k} \left(r_{ij} - v_j \right) \hat{x}_{ij} p_{ij}$$

$$- \Omega q_j - \left(c_j - v_j \right) \sum_{i \in I_k} \hat{x}_{ij} t_{ij} - \left(c_j - v_j \right) \Omega_j h_j,$$

s.t.

$$- p_{ij} \leq y_{ij} - u_{ij} \leq p_{ij}, \ \forall \ i \in I_k,$$

$$- t_{ij} \leq y_{ij} - w_{ij} \leq t_{ij}, \ \forall \ i \in I_k,$$

$$\sum_{i \in I_k} \left(r_{ij} - v_j \right)^2 \hat{x}_{ij}^2 u_{ij}^2 \leq q_j^2,$$

$$\sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^2 \leq h_j^2,$$

$$u_{ij}, \ w_{ij}, \ p_{ij}, \ t_{ij} \geq 0, \ \forall \ i \in I_k,$$

$$h_j, \ q_j \geq 0,$$

$$y_{ij} \in \{0,1\}, \ \forall \ i \in I.$$

Model $\mathcal{SNVP}_{Flex-U} - RC_{I-E}$ is an CQMIP problem, which can be solved efficiently and in reasonable computational time using of-the-shelf MILP commercial solvers such as CPLEX.

4.4.5 Robust MPSNVP with Flexible Market Entry Based on Interval-Polyhedral Uncertainty Set

We apply the approach provided in Section 4.3.6 to obtain the robust counterpart of the uncertain model of the MPSNVP with flexible market entry, $\mathcal{SNVP}_{Flex-U}$. Note that the uncertain demands are coefficients of the decision variables y_{ij}, which are binary variables, therefore applying the robust counterpart presented in Section 4.3.6 yields:

$$
\begin{aligned}
&\text{Max} \;\; \delta, \\
&\text{s.t.} \\
&\quad \delta - \sum_{i \in I} \left(\left(r_{ij} - v_j \right) x_{ij} - S_{ij} \right) y_{ij} + \sum_{i \in I_k} p_{ij} + u\Gamma + \left(e_j - v_j \right) z_j \\
&\quad + \left(c_j - v_j \right) Q_j \leq 0, \\
&\quad u + p_{ij} \geq \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij}, \;\; \forall \; i \in I_k, \\
&\quad \sum_{i \in I} x_{ij} y_{ij} + \sum_{i \in I_k} t_{ij} + u_j \Gamma_j - Q_j \leq z_j, \\
&\quad u_j + t_{ij} \geq \hat{x}_{ij} y_{ij}, \;\; \forall \; i \in I_k, \\
&\quad p_{ij}, \;\; t_{ij} \geq 0, \;\; \forall \; i \in I_k, \\
&\quad u, \;\; u_j, \;\; Q_j, \;\; z_j \geq 0, \\
&\quad p_{ij} \geq 0, \;\; \forall \; i \in I_k, \\
&\quad y_{ij} \in \left\{ 0, 1 \right\}, \;\; \forall \; i \in I.
\end{aligned}
$$

This robust counterpart results in MILP. This preserves the tractability of original model, however, the number of variables increases as well as the number of constraints. The number of binary variables remains the same as that in the original formulation $\mathcal{SNVP}_{Flex-U}$.

The size of the above model can be reduced by noting that the optimal values of Q_j and z_j are given as follows:

$$
\begin{aligned}
&Q_j^* = \sum_{i \in I} x_{ij} y_{ij}^* + \sum_{i \in I_k} t_{ij}^* + u_j^* \Gamma_j, \\
&z_j^* = 0.
\end{aligned} \tag{4.20}
$$

We substitute for Q_j^* and z_j^* in the above model, this yields the following:

$$
\begin{aligned}
\text{Max } & \sum_{i \in I} \left(\left(r_{ij} - v_j \right) x_{ij} - S_{ij} \right) y_{ij} - \sum_{i \in I_k} p_{ij} - u\Gamma \\
& - \left(c_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} + \sum_{i \in I_k} t_{ij} + u_j \Gamma_j \right), \\
\text{s.t.} & \\
& u + p_{ij} \geq \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij}, \ \forall \ i \in I_k, \\
& u_j + t_{ij} \geq \hat{x}_{ij} y_{ij}, \ \forall \ i \in I_k, \\
& p_{ij}, \ t_{ij} \geq 0, \ \forall \ i \in I_k, \\
& u, \ u_j \geq 0, \\
& y_{ij} \in \{0, 1\}, \ \forall \ i \in I.
\end{aligned}
$$

The above model can be rewritten as follows:

$$\mathcal{SNVP}_{Flex-U} - RC_{I-P} : \text{Max} \sum_{i \in I} \left(\left(r_{ij} - c_j \right) x_{ij} - S_{ij} \right) y_{ij} - \sum_{i \in I_k} p_{ij} - u\Gamma$$

$$- \left(c_j - v_j \right) \sum_{i \in I_k} t_{ij} - \left(c_j - v_j \right) u_j \Gamma_j,$$

s.t.

$$u + p_{ij} \geq \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij}, \; \forall \; i \in I_k,$$

$$u_j + t_{ij} \geq \hat{x}_{ij} y_{ij}, \; \forall \; i \in I_k,$$

$$p_{ij}, \; t_{ij} \geq 0, \; \forall \; i \in I_k,$$

$$u, \; u_j \geq 0,$$

$$y_{ij} \in \{0, 1\}, \; \forall \; i \in I.$$

The reduced model $\mathcal{SNVP}_{Flex-U} - RC_{I-P}$ is an MILP problem, it can be solved using commercial solvers such as CPLEX. Then, the results are used to get the optimal order quantities Q_j^* from (4.20).

4.5 Computational Results

In this section, we study and compare the implementation and solution of robust MPSNVP with flexible market entry. The study includes the robust counterpart formulations under different uncertainty sets discussed in the previous sections, namely, box, ellipsoidal, polyhedral, interval-ellipsoidal and interval-polyhedral based robust counterparts. The study compares the objective values under different uncertainty sets. In addition, we study the effect of changing the adjustable parameters on the performance of the robust counterparts.

It has been shown in the previous sections that the robust counterpart formulation of the MPSNVP with flexible market entry is a separable problem and it reduces to a single product problem for each product. Therefore, we apply the robust counterpart analysis on one product, and then similarly, the analysis procedure can be applied to other products.

As an example, we take the data of product 2 from Tables 4.1 and 4.2. Table 4.1 provides the detailed cost values for the products in the MPSNVP. The purchasing cost per unit of each product, the expediting cost per unit of each product, and the salvage value per unit of each product are shown in the table. Table 4.2 provides the selling prices as well as the nominal demand for each market. In addition, the market entry cost for each type of the products are presented. These input data are drawn from uniform distributions as shown in Table 4.2.

Table 4.1: Costs of the products in an MPSNVP with uncertain demand.

Parameter	Product									
	1	2	3	4	5	6	7	8	9	10
e	10	120	15	1,200	250	24	90	7	220	900
c	7	100	10	1,000	200	15	85	6	200	800
v	5	50	5	600	50	5	30	5	100	200

We consider three market pool sizes: 10, 100 and 1000. For each market pool size we apply different robust counterpart formulations to product 2. In addition, we assume that 50% of the markets have uncertain demand data. The demand of the product is assumed to be 10% perturbed around its nominal value.

Table 4.2: Input parameters for an MPSNVP with uncertain demand.

Product	Parameters		
	r	x	S
1	U(12, 20)	U(400, 600)	U(1,000, 2,000)
2	U(150, 200)	U(200, 300)	U(5,000, 10,000)
3	U(15, 25)	U(600, 800)	U(3,000, 5,000)
4	U(1200, 1800)	U(40, 60)	U(15,000, 20,000)
5	U(200, 300)	U(200, 500)	U(10,000, 15,000)
6	U(20, 30)	U(300, 400)	U(3,000, 5,000)
7	U(80, 120)	U(200, 220)	U(4,000, 8,000)
8	U(8, 12)	U(300, 500)	U(1,000, 2,000)
9	U(200, 300)	U(100, 120)	U(5,000, 10,000)
10	U(900, 1,100)	U(120, 150)	U(15,000, 20,000)

We set the values of Ω and Ω_j at the same level, they are equal to $\Psi\sqrt{|I_k|}$, where $|I_k|$ is the cardinality of the set of markets with uncertain demand; e.g. for market pool size 10 with 50% of the markets have uncertain demand, $|I_5|= 5$. Moreover, we set the values of Γ and Γ_j at the same level, they are equal to $\Psi|I_k|$.

Figures 4.3, 4.4 and 4.5 show the optimization results for 10, 100 and 1000 markets pool sizes respectively. We notice the following:

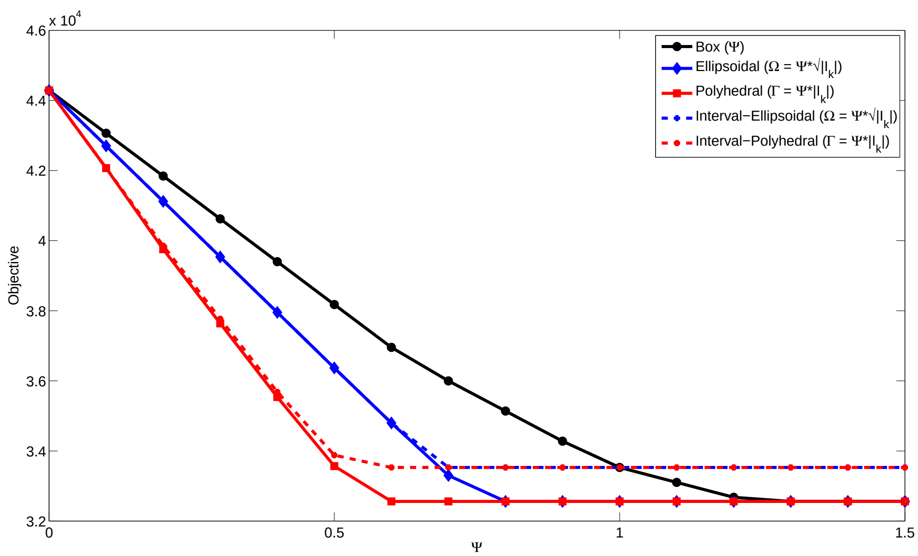


Figure 4.3: Robust MPSNVP with flexible market entry for market pool size 10.

- For $\Psi = 0$, all robust counterpart formulations yield the same solution, which is the solution of the deterministic problem.

- For $0 < \Psi < 1$, the solution of the robust counterpart based on box uncertainty set is better than the solution of any other uncertainty set, this happens because the box uncertainty set is smaller than any other uncertainty set in that range of the values of Ψ.

- For $\Omega \leq 1$, the solution of the robust counterpart based on the ellipsoidal uncertainty set has the same value of that solution based on the combination of interval and ellipsoidal uncertainty sets. The reason behind this is that the corresponding uncertainty sets for both robust counterparts are the same; see Figures 4.1a and 4.1b.

- For $1 < \Omega < \sqrt{|I_k|}$, the solution of the robust counterpart based on the combination of interval and ellipsoidal uncertainty sets is better than that solution based on the ellipsoidal uncertainty set because the corresponding uncertainty set for the former is smaller than that of the latter; see Figure 4.1c.

- For $\Omega \geq \sqrt{|I_k|}$, the solution of the robust counterpart based on the combination of interval and ellipsoidal uncertainty sets reaches its worst value and does not decrease any more. This is due to the fact that the corresponding uncertainty set becomes exactly the interval uncertainty set and it does not change; see Figures 4.1d and 4.1e.

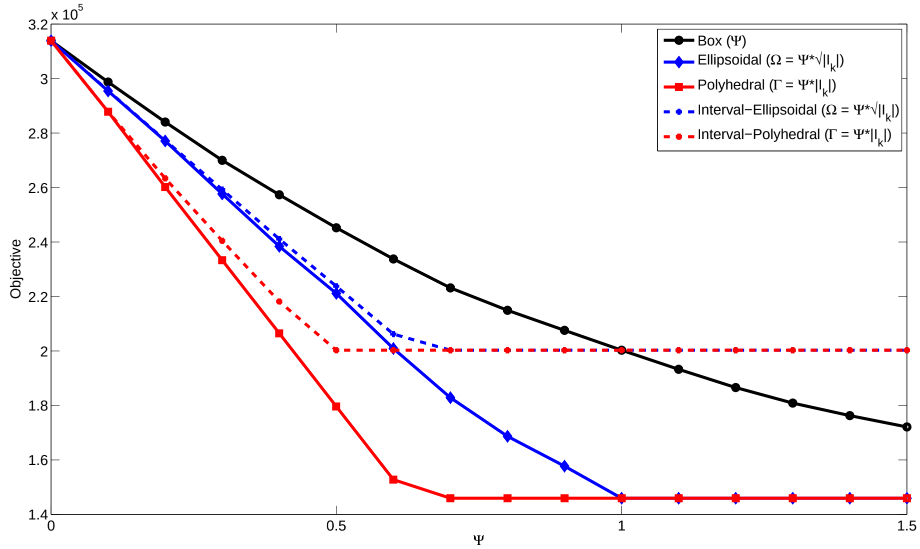


Figure 4.4: Robust MPSNVP with flexible market entry for market pool size 100.

- For $\Gamma \leq 1$, the solution of the robust counterpart based on the polyhedral uncertainty set has the same value of that solution based on the combination of interval and polyhedral uncertainty sets. The reason is that the corresponding uncertainty sets for both robust counterparts are the same; see Figures 4.2a and 4.2b.

- For $1 < \Gamma < |I_k|$, the solution of the robust counterpart based on the combination of interval and polyhedral uncertainty sets is better than that solution based on the polyhedral uncertainty set because the corresponding uncertainty set for the former is smaller than that of the polyhedral uncertainty set; see Figure 4.2c.

- For $\Gamma \geq |I_k|$, the solution of the robust counterpart based on the combination of interval and polyhedral uncertainty sets reaches its worst value and does not decrease any more. This happens because the corresponding uncertainty

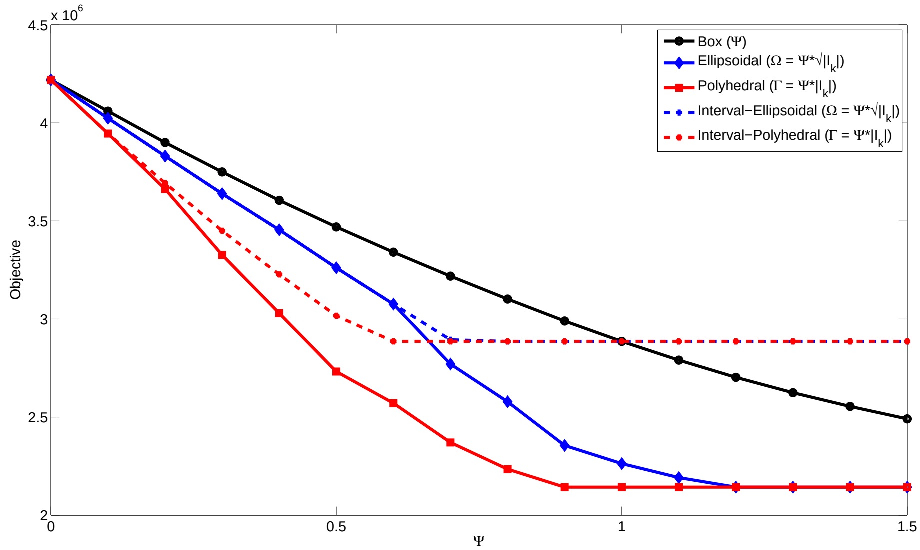
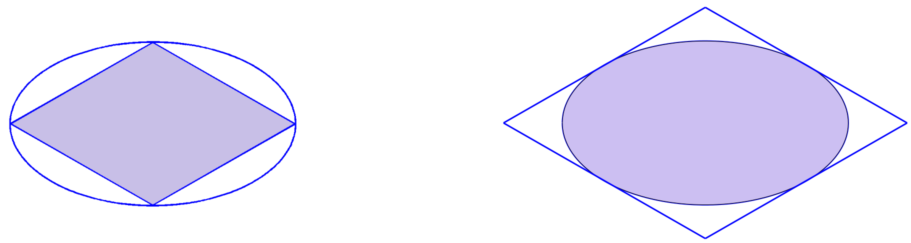


Figure 4.5: Robust MPSNVP with flexible market entry for market pool size 1000.

set becomes exactly the interval uncertainty set and it does not change; see Figures 4.2d and 4.2e.

- For $\Omega = \Gamma$, the solution of the robust counterpart based on the combination of interval and polyhedral uncertainty sets is better than that of the robust counterpart based on the combination of interval and ellipsoidal uncertainty sets, because the combination of interval and polyhedral uncertainty sets in this case is smaller than the combination of interval and ellipsoidal uncertainty sets. In this case, the corresponding uncertainty set of the former is completely covered by the corresponding uncertainty set of the latter; see Figure 4.6a.

- For $\Gamma = \Omega\sqrt{|I_k|}$, the solution of the robust counterpart based on the combination of interval and ellipsoidal uncertainty sets is better than that of the robust counterpart based on the combination of interval and polyhedral uncertainty sets, because the corresponding uncertainty set of the former is completely circumvented by the corresponding uncertainty set of the latter; see Figure 4.6b.



(a) $\Omega = \Gamma$.　　(b) $\Gamma = \Omega\sqrt{|I_k|}$.

Figure 4.6: Illustration of the relationship between ellipsoidal and polyhedral uncertainty set.

- For the robust counterparts based on box, ellipsoidal and polyhedral uncertainty set, there is a certain value of Ψ at which none of the markets with uncertain demand is selected, because the adjustable uncertainty set parameters take large values, these values indicate that the markets with uncertain demands are completely unreliable.

CHAPTER 5

ROBUST MULTI-PRODUCT SELECTIVE NEWSVENDOR WITH FULL MARKET ENTRY

5.1 Introduction

In this chapter, we discuss robust counterpart reformulations of MPSNVP with full market entry under different uncertainty sets of uncertain demand. Some of the obtained models can be solved in closed form solution, some of these models are MILP and the rest are CQMILP which can be solved efficiently using commercial solvers such as CPLEX.

5.2 Robust Multi-Product Selective Newsvendor Problem with Full Market Entry

The profit function for the deterministic full market entry case of the multi-product selective newsvendor problem is given by (2.9), or equivalently,

$$P\left(Q_j,\ \mathcal{Y}_i\right)=\sum_{i\in I}\left(\sum_{j\in J}\left(r_{ij}-v_j\right)x_{ij}-S_i\right)\mathcal{Y}_i-\sum_{j\in J}\left(e_j-v_j\right)\left(\sum_{i\in I}x_{ij}\mathcal{Y}_i-Q_j\right)^+$$
$$-\sum_{j\in J}\left(c_j-v_j\right)Q_j. \tag{5.1}$$

In that function, we can replace $\left(\sum_{i\in I}x_{ij}\mathcal{Y}_i-Q_j\right)^+$ by z_j and add the constraints $\sum_{i\in I}x_{ij}\mathcal{Y}_i-Q_j\leq z_j$ and $z_j\geq 0$. Thus, the optimization model becomes:

$$\mathcal{SNVP}_{Full}:\text{Max }\sum_{i\in I}\left(\sum_{j\in J}\left(r_{ij}-v_j\right)x_{ij}-S_i\right)\mathcal{Y}_i-\sum_{j\in J}\left(e_j-v_j\right)z_j$$
$$-\sum_{j\in J}\left(c_j-v_j\right)Q_j,$$

s. t.

$$\sum_{i\in I}x_{ij}\mathcal{Y}_i-Q_j\leq z_j,\ \forall\ j\in J,$$

$$Q_j,\ z_j\geq 0,\ \forall\ j\in J,$$

$$\mathcal{Y}_i\in\{0,1\},\ \forall\ i\in I.$$

Note that, the optimal order quantity for each product j; Q_j, should satisfy the demand of that product from the selected markets, i.e.,

$$Q_j = \sum_{i \in I} x_{ij} \mathcal{Y}_i, \ \forall \ j \in J, \tag{5.2}$$

and this causes $z_j = 0$ at optimality.

Hence, the deterministic full market entry selective newsvendor problem can be expressed as:

$$\mathcal{SNVP}_{Full-D} : \text{Max} \ \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - c_j) \, x_{ij} - S_i \right) \mathcal{Y}_i,$$

$$\text{s. t.}$$

$$\mathcal{Y}_i \in \{0, 1\} \, , \ \forall \ i \in I.$$

The optimal solution to the deterministic full market entry MPSNVP $\mathcal{SNVP}_{Full-D}$ is:

$$\begin{aligned} &\mathcal{Y}_i^* = 1 \text{ if } \sum_{j \in J} (r_{ij} - v_j) \, x_{ij} - S_i > 0, \\ &\mathcal{Y}_i^* = 0 \text{ otherwise.} \end{aligned} \tag{5.3}$$

The optimal order quantity Q_j^* is given by (5.2).

Suppose that the demand of the products; x_{ij}, is uncertain in the market set $I_k \subseteq I$, therefore, the demand value is considered as presented in (4.12). Hence, model $\mathcal{SNVP}_{Full}$ can be rewritten as:

$$\text{Max} \sum_{i\in I\setminus I_k}\left(\sum_{j\in J}(r_{ij}-v_j)\,x_{ij}-S_i\right)\mathcal{Y}_i+\sum_{i\in I_k}\left(\sum_{j\in J}(r_{ij}-v_j)\,\tilde{x}_{ij}-S_i\right)\mathcal{Y}_i$$
$$-\sum_{j\in J}(e_j-v_j)\,z_j-\sum_{j\in J}(c_j-v_j)\,Q_j,$$

s. t.

$$\sum_{i\in I\setminus I_k}x_{ij}\mathcal{Y}_i+\sum_{i\in I_k}\tilde{x}_{ij}\mathcal{Y}_i-Q_j\le z_j,\ \forall\ j\in J,$$

$$Q_j,\ z_j\ge 0,\ \forall\ j\in J,$$

$$\mathcal{Y}_i\in\{0,1\}\,,\ \forall\ i\in I.$$

where x_{ij} and $\tilde{x}_{ij}$ represent the deterministic and uncertain values of the demand of product j in market i, respectively.

The above model can be rewritten as:

$\mathcal{SNVP}_{Full-U}$:

$$\text{Max}\ \ \delta,$$

s. t.

$$\delta-\sum_{i\in I\setminus I_k}\left(\sum_{j\in J}(r_{ij}-v_j)\,x_{ij}-S_i\right)\mathcal{Y}_i-\sum_{i\in I_k}\left(\sum_{j\in J}(r_{ij}-v_j)\,\tilde{x}_{ij}-S_i\right)\mathcal{Y}_i$$
$$+\sum_{j\in J}(e_j-v_j)\,z_j+\sum_{j\in J}(c_j-v_j)\,Q_j\le 0,$$

$$\sum_{i\in I\setminus I_k}x_{ij}\mathcal{Y}_i+\sum_{i\in I_k}\tilde{x}_{ij}\mathcal{Y}_i-Q_j\le z_j,\ \forall\ j\in J,$$

$$Q_j,\ z_j\ge 0,\ \forall\ j\in J,$$

$\mathcal{Y}_i \in \{0,1\}\,, \; \forall \; i \in I.$

Next, we present the robust counterpart reformulation of the MPSNVP with full market entry; $\mathcal{SNVP}_{Full-U}$, under different uncertainty sets.

5.2.1 Robust MPSNVP with Full Market Entry Based on Box Uncertainty Set

We apply the approach presented in Section 4.3.2 to obtain the robust counterpart of the uncertain model of the MPSNVP with full market entry, $\mathcal{SNVP}_{Full-U}$, based on the box uncertainty set. Note that the uncertain demands are coefficients of the decision variables $\mathcal{Y}_i$, which are binary variables, therefore applying the robust counterpart approach in Section 4.3.2 yields:

$$\text{Max} \;\; \delta,$$

s.t.

$$\delta - \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - v_j)\, x_{ij} - S_i \right) \mathcal{Y}_i + \Psi \sum_{i \in I_k} \left(\sum_{j \in J} (r_{ij} - v_j)\, \hat{x}_{ij} \right) \mathcal{Y}_i$$

$$+ \sum_{j \in J} (e_j - v_j)\, z_j + \sum_{j \in J} (c_j - v_j)\, Q_j \leq 0,$$

$$\sum_{i \in I} x_{ij} \mathcal{Y}_i + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} \mathcal{Y}_i - Q_j \leq z_j, \;\; \forall \; j \in J,$$

$$Q_j, \;\, z_j \geq 0, \; \forall \; j \in J,$$

$$\mathcal{Y}_i \in \{0,1\}\,, \; \forall \; i \in I.$$

This robust counterpart results in an MILP formulation. This preserves the tractability of original model with the same number of variables and constraints. Note that, we can get Soyster's robust counterpart to the model given in Section 4.3.1 by taking $\Psi = 1$ and $\Psi_j = 1$ in the above model.

The optimal solution to the above model yields,

$$
\begin{aligned}
& Q_j^* = \sum_{i \in I} x_{ij} \mathcal{Y}_i^* + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} \mathcal{Y}_i^*, \ \forall \ j \in J, \\
& z_j^* = 0, \ \forall \ j \in J.
\end{aligned}
\tag{5.4}
$$

Hence, the above model reduces to:

$$
\begin{aligned}
\text{Max} \ \ & \delta, \\
\text{s.t.} \ \ & \\
& \delta - \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - v_j) \, x_{ij} - S_i \right) \mathcal{Y}_i + \Psi \sum_{i \in I_k} \left(\sum_{j \in J} (r_{ij} - v_j) \, \hat{x}_{ij} \right) \mathcal{Y}_i \\
& + \sum_{j \in J} (c_j - v_j) \left(\sum_{i \in I} x_{ij} \mathcal{Y}_i + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} \mathcal{Y}_i \right) \leq 0, \\
& \mathcal{Y}_i \in \{0, 1\}, \ \forall \ i \in I.
\end{aligned}
$$

This model can be represented as:

$$
\begin{aligned}
\mathcal{SNVP}_{Full-U} - RC_B : \ \text{Max} \ & \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - c_j) \, x_{ij} - S_i \right) \mathcal{Y}_i \\
& - \Psi \sum_{i \in I_k} \left(\sum_{j \in J} (r_{ij} - v_j) \, \hat{x}_{ij} \right) \mathcal{Y}_i \\
& - \sum_{i \in I_k} \left(\sum_{j \in J} \Psi_j \, (c_j - v_j) \, \hat{x}_{ij} \right) \mathcal{Y}_i,
\end{aligned}
$$

s.t.

$$\mathcal{Y}_i \in \{0,1\}\,,\ \forall\, i \in I.$$

The optimal solution to model $\mathcal{SNVP}_{Full-U} - RC_B$ is as follows:

$$\begin{aligned}
&\text{For all } i \in I \setminus I_k :\\
&\quad \mathcal{Y}_i^* = 1 \text{ if } \sum_{j\in J} (r_{ij} - c_j)\, x_{ij} - S_i > 0,\\
&\quad \mathcal{Y}_i^* = 0 \text{ otherwise,}\\
&\text{For all } i \in I_k :\\
&\quad \mathcal{Y}_i^* = 1 \text{ if } \sum_{j\in J} (r_{ij} - c_j)\, x_{ij} - \Psi \sum_{j\in J} (r_{ij} - v_j)\, \hat{x}_{ij}\\
&\qquad\qquad - \sum_{j\in J} \Psi_j\, (c_j - v_j)\, \hat{x}_{ij} - S_i > 0,\\
&\quad \mathcal{Y}_i^* = 0 \text{ otherwise.}
\end{aligned} \tag{5.5}$$

The optimal order quantity Q_j^* for each product j is given by (5.4).

5.2.2 Robust MPSNVP with Full Market Entry Based on Ellipsoidal Uncertainty Set

We apply the approach provided in Section 4.3.3 to obtain the robust counterpart of the uncertain model of the MPSNVP with full market entry, $\mathcal{SNVP}_{Full-U}$, based on the ellipsoidal uncertainty set, this can be expressed as:

$$
\begin{aligned}
&\text{Max} \;\; \delta, \\
&\text{s.t.} \\
&\delta - \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - v_j)\, x_{ij} - S_i \right) \mathcal{Y}_i + \Omega \sqrt{\sum_{i \in I_k} \sum_{j \in J} (r_{ij} - v_j)^2 \, \hat{x}_{ij}^2 \mathcal{Y}_i^2} \\
&+ \sum_{j \in J} (e_j - v_j)\, z_j + \sum_{j \in J} (c_j - v_j)\, Q_j \leq 0, \\
&\sum_{i \in I} x_{ij} \mathcal{Y}_i + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 \mathcal{Y}_i^2} - Q_j \leq z_j, \;\; \forall \, j \in J, \\
&Q_j, \; z_j \geq 0, \; \forall \, j \in J, \\
&\mathcal{Y}_i \in \{0, 1\}, \; \forall \, i \in I.
\end{aligned}
$$

This robust counterpart preserves the same number of variables and constraints, however it results in an MINLP formulation, which causes computational complexities.

Noting that, the optimal solution to the above model results in,

$$
\begin{aligned}
&Q_j^* = \sum_{i \in I} x_{ij} \mathcal{Y}_i^* + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 \mathcal{Y}_i^{*2}}, \; \forall \, j \in J, \\
&z_j^* = 0, \; \forall \, j \in J.
\end{aligned}
\tag{5.6}
$$

Therefore, the above model becomes:

$$
\text{Max} \; \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - v_j)\, x_{ij} - S_i \right) \mathcal{Y}_i - \Omega \sqrt{\sum_{i \in I_k} \sum_{j \in J} (r_{ij} - v_j)^2 \, \hat{x}_{ij}^2 \mathcal{Y}_i^2}
$$

$$- \sum_{j\in J} (c_j - v_j) \left(\sum_{i\in I} x_{ij}\mathcal{Y}_i + \Omega_j \sqrt{\sum_{i\in I_k} \hat{x}_{ij}^2 \mathcal{Y}_i^2} \right),$$

s.t.

$$\mathcal{Y}_i \in \{0,1\}\,,\ \forall\ i \in I.$$

This model is rewritten as:

$$\text{Max} \sum_{i\in I\setminus I_k} \left(\sum_{j\in J} (r_{ij} - c_j)\, x_{ij} - S_i \right) \mathcal{Y}_i + \sum_{i\in I_k} \left(\sum_{j\in J} (r_{ij} - c_j)\, x_{ij} - S_i \right) \mathcal{Y}_i$$
$$- \Omega \sqrt{\sum_{i\in I_k} \sum_{j\in J} (r_{ij} - v_j)^2\, \hat{x}_{ij}^2 \mathcal{Y}_i^2} - \sum_{j\in J} \Omega_j\, (c_j - v_j) \sqrt{\sum_{i\in I_k} \hat{x}_{ij}^2 \mathcal{Y}_i^2},$$

s.t.

$$\mathcal{Y}_i \in \{0,1\}\,,\ \forall\ i \in I.$$

The last model can be separated into the following two sub-models:

$$\mathcal{SNVP}_{Full-U} - D:\ \text{Max} \sum_{i\in I\setminus I_k} \left(\sum_{j\in J} (r_{ij} - c_j)\, x_{ij} - S_i \right) \mathcal{Y}_i,$$

s.t.

$$\mathcal{Y}_i \in \{0,1\}\,,\ \forall\ i \in I \setminus I_k.$$

$$\mathcal{SNVP}_{Full-U} - RC_E : \text{Max} \sum_{i\in I_k} \left(\sum_{j\in J} (r_{ij} - c_j)\, x_{ij} - S_i \right) \mathcal{Y}_i$$
$$- \Omega \sqrt{\sum_{i\in I_k} \sum_{j\in J} (r_{ij} - v_j)^2\, \hat{x}_{ij}^2 \mathcal{Y}_i^2}$$

$$- \sum_{j \in J} \Omega_j \left(c_j - v_j\right) \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 \mathcal{Y}_i^2},$$

s.t.

$$\mathcal{Y}_i \in \{0, 1\}, \ \forall \ i \in I_k.$$

The optimal solution to sub-model $\mathcal{SNVP}_{Full-U} - D$ is as follows:

For all $i \in I \setminus I_k$:

$$\begin{aligned} &\mathcal{Y}_i^* = 1 \text{ if } \sum_{j \in J} \left(r_{ij} - c_j\right) x_{ij} - S_i > 0, \\ &\mathcal{Y}_i^* = 0 \text{ otherwise.} \end{aligned} \tag{5.7}$$

The optimal solution to sub-model $\mathcal{SNVP}_{Full-U} - RC_E$ can be determined by appliyng the same solution procedure that was applied in Section 2.3.4 for solving the risk-neutral MPSNVP with flexible market entry. The RDU ratio for sub-model $\mathcal{SNVP}_{Full-U} - RC_E$ is:

$$RDU_{Full-RC_E} = \frac{\sum_{j \in J} \left(r_{ij} - c_j\right) x_{ij} - S_i}{\sum_{j \in J} \Omega^2 \left(r_{ij} - v_j\right)^2 \hat{x}_{ij}^2 + \sum_{j \in J} \Omega_j^2 \left(c_j - v_j\right)^2 \hat{x}_{ij}^2}. \tag{5.8}$$

The optimal order quantity Q_j^* for each product j is given by (5.6).

5.2.3 Robust MPSNVP with Full Market Entry Based on Polyhedral Uncertainty Set

We apply the approach shown in Section 4.3.4 to obtain the robust counterpart of the uncertain model of the MPSNVP with full market entry, $\mathcal{SNVP}_{Full-U}$, based on the polyhedral uncertainty set. Note that the uncertain demands are coefficients of the decision variables $\mathcal{Y}_i$, which are binary variables, therefore applying the robust counterpart given in Section 4.3.4 yields:

$$
\begin{aligned}
&\text{Max} \;\; \delta, \\
&\text{s.t.} \\
&\delta - \sum_{i\in I}\left(\sum_{j\in J}(r_{ij}-v_j)\,x_{ij} - S_i\right)\mathcal{Y}_i + u\Gamma + \sum_{j\in J}(e_j - v_j)\,z_j \\
&+\sum_{j\in J}(c_j - v_j)\,Q_j \le 0, \\
&u \ge (r_{ij}-v_j)\,\hat{x}_{ij}\mathcal{Y}_i, \;\forall\, j\in J, \;\forall\, i \in I_k, \\
&\sum_{i\in I} x_{ij}\mathcal{Y}_i + u_j\Gamma_j - Q_j \le z_j, \;\forall\, j \in J, \\
&u_j \ge \hat{x}_{ij}\mathcal{Y}_i, \;\forall\, j\in J, \;\forall\, i\in I_k, \\
&u \ge 0, \\
&u_j, \; Q_j, \; z_j \ge 0, \;\forall\, j \in J, \\
&\mathcal{Y}_i \in \{0,1\}, \;\forall\, i \in I.
\end{aligned}
$$

This robust counterpart results in MILP. This preserves the tractability of original model, however, the number of variables increases as well as the number of constraints. The number of binary variables remains the same as in the original formulation in $\mathcal{SNVP}_{Full-U}$.

The size of model $\mathcal{SNVP}_{Full-U} - RC_P$ can be reduced by noting that; when the values of $\mathcal{Y}_i^*$ and u_j^* are given, then the optimal solution to model $\mathcal{SNVP}_{Full-U} -$

RC_P results in:

$$Q_j^* = \sum_{i \in I} x_{ij} \mathcal{Y}_i^* + u_j^* \Gamma_j, \ \forall \ j \in J, \qquad (5.9)$$

$$z_j^* = 0, \ \forall \ j \in J.$$

Substituting from (5.9) into model $\mathcal{SNVP}_{Full-U} - RC_P$:

$$\text{Max} \ \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - v_j)\, x_{ij} - S_i \right) \mathcal{Y}_i - u\Gamma - \sum_{j \in J} (c_j - v_j) \left(\sum_{i \in I} x_{ij} \mathcal{Y}_i + u_j \Gamma_j \right),$$

s.t.

$$u \geq (r_{ij} - v_j)\, \hat{x}_{ij} \mathcal{Y}_i, \ \forall \ j \in J, \ \forall \ i \in I_k,$$

$$u_j \geq \hat{x}_{ij} \mathcal{Y}_i, \ \forall \ j \in J, \ \forall \ i \in I_k,$$

$$u, \geq 0,$$

$$u_j \geq 0, \ \forall \ j \in J,$$

$$\mathcal{Y}_i \in \{0, 1\}, \ \forall \ i \in I.$$

The above model can be rewritten as:

$$\mathcal{SNVP}_{Full-U} - RC_P : \text{Max} \ \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - c_j)\, x_{ij} - S_i \right) \mathcal{Y}_i - u\Gamma$$

$$- \sum_{j \in J} (c_j - v_j)\, u_j \Gamma_j,$$

s.t.

$$u \geq (r_{ij} - v_j)\, \hat{x}_{ij} \mathcal{Y}_i, \ \forall \ j \in J, \ \forall \ i \in I_k,$$

$$u_j \geq \hat{x}_{ij} \mathcal{Y}_i, \ \forall \ j \in J, \ \forall \ i \in I_k,$$

$$u \geq 0,$$

$$u_j \geq 0, \ \forall \ j \in J,$$

$$\mathcal{Y}_i \in \{0, 1\}, \ \forall \ i \in I.$$

Model $\mathcal{SNVP}_{Full-U} - RC_P$ is an MILP problem, it can be solved using commercial solvers such as CPLEX. Then, the results are used to get the optimal order quantities Q_j from (5.9).

5.2.4 Robust MPSNVP with Full Market Entry Based on Interval-Ellipsoidal Uncertainty Set

We apply the approach provided in Section 4.3.5 to obtain the robust counterpart of the uncertain model of the MPSNVP with full market entry, $\mathcal{SNVP}_{Full-U}$, this can be expressed as:

$$\text{Max} \quad \delta,$$

$$\text{s.t.}$$

$$\delta - \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - v_j) x_{ij} - S_i \right) \mathcal{Y}_i + \sum_{i \in I_k} \sum_{j \in J} (r_{ij} - v_j) \hat{x}_{ij} p_{ij}$$
$$+ \Omega \sqrt{\sum_{i \in I_k} \sum_{j \in J} (r_{ij} - v_j)^2 \hat{x}_{ij}^2 u_{ij}^2} + \sum_{j \in J} (e_j - v_j) z_j$$
$$+ \sum_{j \in J} (c_j - v_j) Q_j \leq 0,$$

$$- p_{ij} \leq \mathcal{Y}_i - u_{ij} \leq p_{ij}, \ \forall \ j \in J, \ \ \forall \ i \in I_k,$$

$$\sum_{i \in I} x_{ij} \mathcal{Y}_i + \sum_{i \in I_k} \hat{x}_{ij} t_{ij} + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^2} - Q_j \leq z_j, \ \forall \ j \in J,$$

$$-\,t_{ij} \leq \mathcal{Y}_i - w_{ij} \leq t_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$u_{ij},\ w_{ij},\ p_{ij},\ t_{ij} \geq 0,\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$Q_j,\ z_j \geq 0,\ \forall\ j \in J,$$

$$\mathcal{Y}_i \in \{0,1\}\,,\ \forall\ i \in I,$$

where, p_{ij} and t_{ij} are auxiliary variables, while u_{ij} and w_{ij} are positive dual variables.

This robust counterpart results in an MINLP formulation with number of variables and constraints greater than that in the original formulation $\mathcal{SNVP}_{Full-U}$.

The size of the above model can be reduced by performing preprocessing of some variables. In addition, the tractability of this model can be retrieved by linearizing the nonlinear terms.

Given the optimal values of $\mathcal{Y}_i^*$, t_{ij}^* and w_{ij}^*, the optimal solution to the above model will result in:

$$\begin{aligned} Q_j^* &= \sum_{i\in I} x_{ij}\mathcal{Y}_i^* + \sum_{i\in I_k} \hat{x}_{ij} t_{ij}^* + \Omega_j \sqrt{\sum_{i\in I_k} \hat{x}_{ij}^2 w_{ij}^{*2}},\ \forall\ j \in J, \\ z_j^* &= 0,\ \forall\ j \in J. \end{aligned} \tag{5.10}$$

We can also substitute for the nonlinear terms as follows:

$$\sum_{i\in I_k}\sum_{j\in J} \left(r_{ij} - v_j\right)^2 \hat{x}_{ij}^2 u_{ij}^2 \leq q^2,$$

$$\sum_{i\in I_k} \hat{x}_{ij}^2 w_{ij}^2 \leq h_j^2,\ \forall\ j \in J.$$

This leads to a reformulation of the above model as an CQMIP problem, which can be solved efficiently and in reasonable computational time using commercial solvers such as CPLEX.

Now, the above model can be represented in the following CQMIP formulation:

$$\begin{aligned}
\mathcal{SNVP}_{Full-U} - RC_{I-E} : \text{Max } & \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - c_j)\, x_{ij} - S_i \right) \mathcal{Y}_i \\
& - \sum_{i \in I_k} \sum_{j \in J} (r_{ij} - v_j)\, \hat{x}_{ij} p_{ij} - \Omega q \\
& - \sum_{j \in J} (c_j - v_j) \left(\sum_{i \in I_k} \hat{x}_{ij} t_{ij} + \Omega_j h_j \right), \\
\text{s.t.} & \\
& - p_{ij} \leq \mathcal{Y}_i - u_{ij} \leq p_{ij},\ \forall\, j \in J,\ \forall\, i \in I_k, \\
& - t_{ij} \leq \mathcal{Y}_i - w_{ij} \leq t_{ij},\ \forall\, j \in J,\ \forall\, i \in I_k, \\
& \sum_{i \in I_k} \sum_{j \in J} (r_{ij} - v_j)^2\, \hat{x}_{ij}^2 u_{ij}^2 \leq q^2, \\
& \sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^2 \leq h_j^2,\ \forall\, j \in J, \\
& u_{ij},\ w_{ij},\ p_{ij},\ t_{ij} \geq 0,\ \forall\, j \in J,\ \forall\, i \in I_k, \\
& q \geq 0, \\
& h_j \geq 0,\ \forall\, j \in J, \\
& \mathcal{Y}_i \in \{0, 1\},\ \forall\, i \in I.
\end{aligned}$$

5.2.5 Robust MPSNVP with Full Market Entry Based on Interval-Polyhedral Uncertainty Set

We apply the approach presented in Section 4.3.6 to obtain the robust counterpart of the uncertain model of the MPSNVP with full market entry, $\mathcal{SNVP}_{Full-U}$. Note that the uncertain demands are coefficients of the decision variables $\mathcal{Y}_i$, which are binary variables, therefore, applying the robust counterpart approach given Section 4.3.6 yields:

$$
\begin{aligned}
\text{Max} \;\; & \delta, \\
\text{s.t.} \;\; & \\
& \delta - \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - v_j)\, x_{ij} - S_i \right) \mathcal{Y}_i + \sum_{i \in I_k} \sum_{j \in J} p_{ij} + u\Gamma + \sum_{j \in J} (e_j - v_j)\, z_j \\
& + \sum_{j \in J} (c_j - v_j)\, Q_j \leq 0, \\
& u + p_{ij} \geq (r_{ij} - v_j)\, \hat{x}_{ij} \mathcal{Y}_i, \;\; \forall\, j \in J, \;\; \forall\, i \in I_k, \\
& \sum_{i \in I} x_{ij} \mathcal{Y}_i + \sum_{i \in I_k} t_{ij} + u_j \Gamma_j - Q_j \leq z_j, \;\; \forall\, j \in J, \\
& u_j + t_{ij} \geq \hat{x}_{ij} \mathcal{Y}_i, \;\; \forall\, j \in J, \;\; \forall\, i \in I_k, \\
& u \geq 0, \\
& u_j, \;\; Q_j, \;\; z_j \geq 0, \;\; \forall\, j \in J, \\
& p_{ij}, \;\; t_{ij} \geq 0, \;\; \forall\, j \in J, \;\; \forall\, i \in I_k, \\
& \mathcal{Y}_i \in \{0, 1\}, \;\; \forall\, i \in I.
\end{aligned}
$$

This robust counterpart results in MILP. This preserves the tractability of original model, however, the number of variables increases as well as the number of constraints. The number of binary variables remains the same as that in the original formulation $\mathcal{SNVP}_{Full-U}$.

We can reduce the size of the last model by noting that the optimal values of Q_j and z_j are given as follows:

$$\begin{aligned}
&Q_j^* = \sum_{i \in I} x_{ij} \mathcal{Y}_i^* + \sum_{i \in I_k} t_{ij}^* + u_j^* \Gamma_j, \ \forall \ j \in J, \\
&z_j^* = 0, \ \forall \ j \in J.
\end{aligned} \tag{5.11}$$

Substituting for Q_j^* and z_j^* in the above model yields the following:

$$\begin{aligned}
\mathcal{SNVP}_{Full-U} - RC_{I-P}: \ \text{Max} \ & \sum_{i \in I} \left(\sum_{j \in J} (r_{ij} - c_j) \, x_{ij} - S_i \right) \mathcal{Y}_i - \sum_{i \in I_k} \sum_{j \in J} p_{ij} \\
& - u\Gamma - \sum_{i \in I_k} \sum_{j \in J} (c_j - v_j) \, t_{ij} - \sum_{j \in J} (c_j - v_j) \, u_j \Gamma_j, \\
\text{s.t.} & \\
& u + p_{ij} \geq (r_{ij} - v_j) \, \hat{x}_{ij} \mathcal{Y}_i, \ \forall \ j \in J, \ \forall \ i \in I_k, \\
& u_j + t_{ij} \geq \hat{x}_{ij} \mathcal{Y}_i, \ \forall \ j \in J, \ \forall \ i \in I_k, \\
& u \geq 0, \\
& u_j \geq 0, \ \forall \ j \in J, \\
& p_{ij}, \ t_{ij} \geq 0, \ \forall \ j \in J, \ \forall \ i \in I_k, \\
& \mathcal{Y}_i \in \{0, 1\}, \ \forall \ i \in I.
\end{aligned}$$

The reduced model $\mathcal{SNVP}_{Full-U} - RC_{I-P}$ is an MILP problem, it can be solved using commercial solvers such as CPLEX. Then, the results are used to get the optimal order quantities Q_j^* from (5.11).

5.3 Computational Results

In this section, we implement the robust counterpart reformulations from previous sections on a numerical examples of full market entry MPSNVP subjected to demand uncertainty.

We consider an MPSNVP with three, five and ten products. In addition, we consider three market pool sizes: 10, 100 and 1000. The details of the input data are given in Tables 4.1 and 4.2.

In the following examples, we assume that 50% of the markets have uncertain demand data. We set the values of Ω and Ω_j at the same level, they are equal to $\Psi\sqrt{|I_k|}$, where $|I_k|$ is the cardinality of the set of markets with uncertain demand. In addition, we take the values of Γ and Γ_j at the same level, they are equal to $\Psi|I_k|$.

For the full market entry MPSNVP with 3 products, we consider the input data for the first three products in Tables 4.1 and 4.2, i.e. products 1, 2 and 3. The demand of these products is assumed to be 20%, 30% and 10% perturbed around their nominal values, respectively. The value of the market entry cost is considered to be uniformly distributed on U(15,000, 20,000).

For the full market entry MPSNVP with 5 products, we consider the input data for the first five products in Tables 4.1 and 4.2, i.e. products 1, 2, 3, 4 and 5. The demand of these products is assumed to be 20%, 30%, 10%, 10% and 25% perturbed around their nominal values, respectively. The value of the market entry cost is considered to be uniformly distributed on U(60,000, 120,000).

For the full market entry MPSNVP with 10 products, we consider the input data in Tables 4.1 and 4.2. The demand of products 1 to 10 is assumed to be 20%, 30%, 10%, 10%, 25%, 10%, 20%, 20%, 10% and 25%, perturbed around their nominal values, respectively. The value of the market entry cost is considered to be uniformly distributed on U(150,000, 200,000).

Figures 5.1, 5.2 and 5.3 show the optimization results of the full market entry MPSNVP with 3 products for 10, 100 and 1000 markets pool sizes respectively. Figures 5.4, 5.5 and 5.6 show the optimization results of the full market entry MPSNVP with 5 products for 10, 100 and 1000 markets pool sizes respectively. Figures 5.7, 5.8 and 5.9 show the optimization results of the full market entry MPSNVP with 10 products for 10, 100 and 1000 markets pool sizes respectively. For these figures, we notice a similar behavior to that behavior in Figures 4.3, 4.4 and 4.5. Therefore, the same discussion presented in Section 4.5 is applicable to the results in this section.

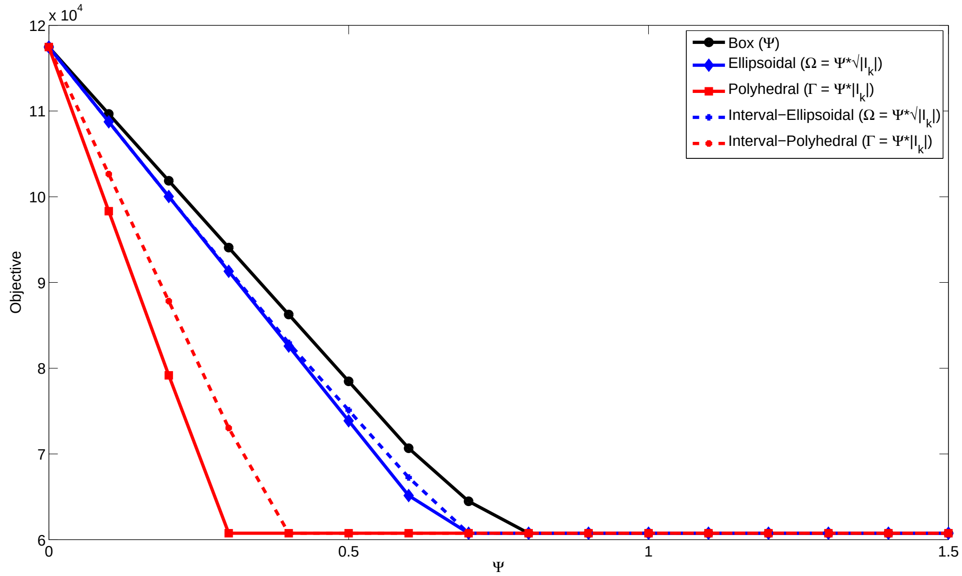


Figure 5.1: A 3-product robust MPSNVP with full market entry for market pool size 10.

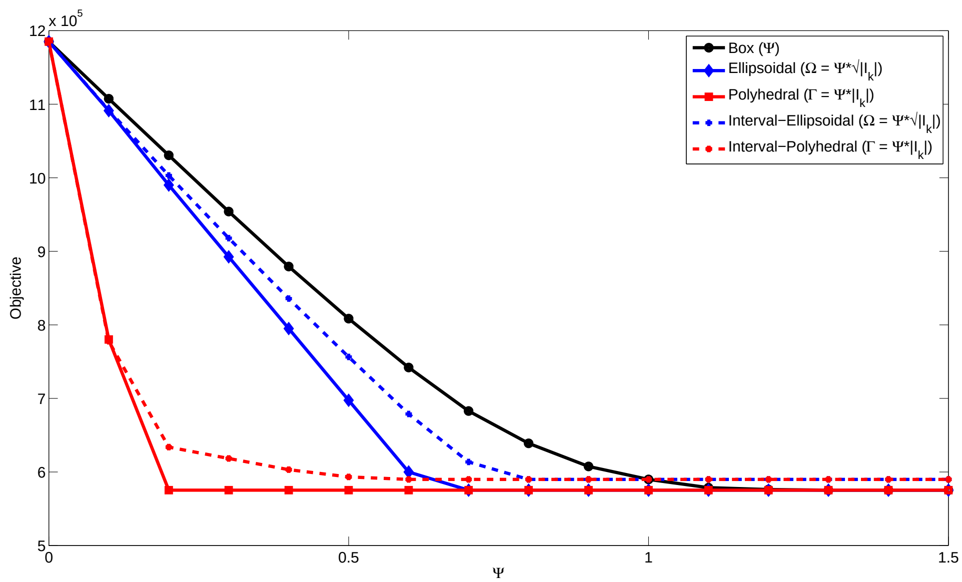


Figure 5.2: A 3-product robust MPSNVP with full market entry for market pool size 100.

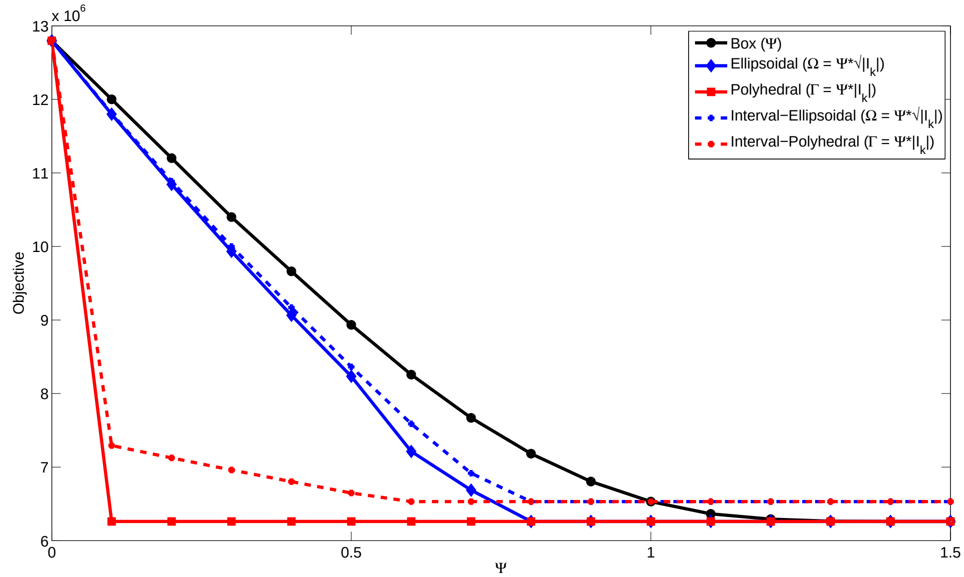


Figure 5.3: A 3-product robust MPSNVP with full market entry for market pool size 1000.

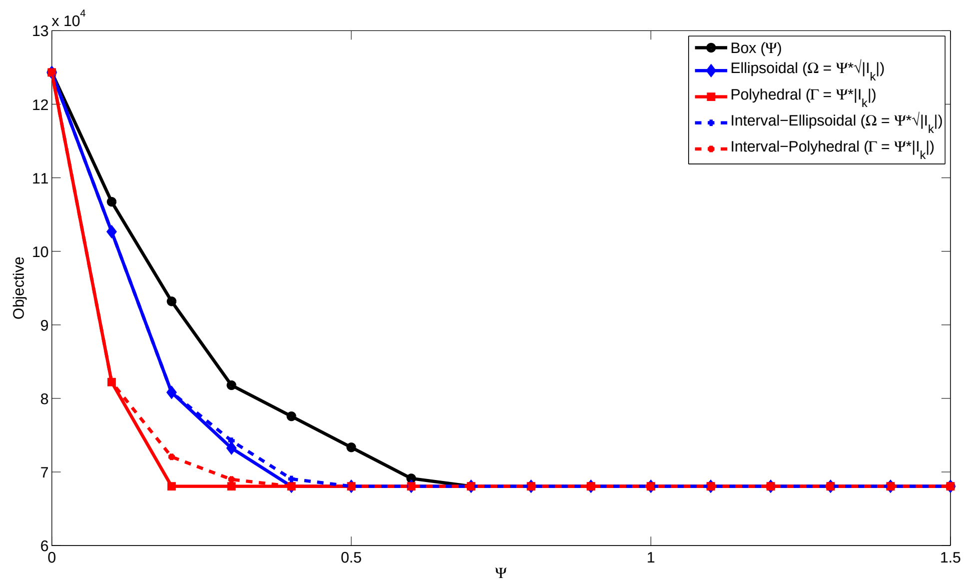


Figure 5.4: A 5-product robust MPSNVP with full market entry for market pool size 10.

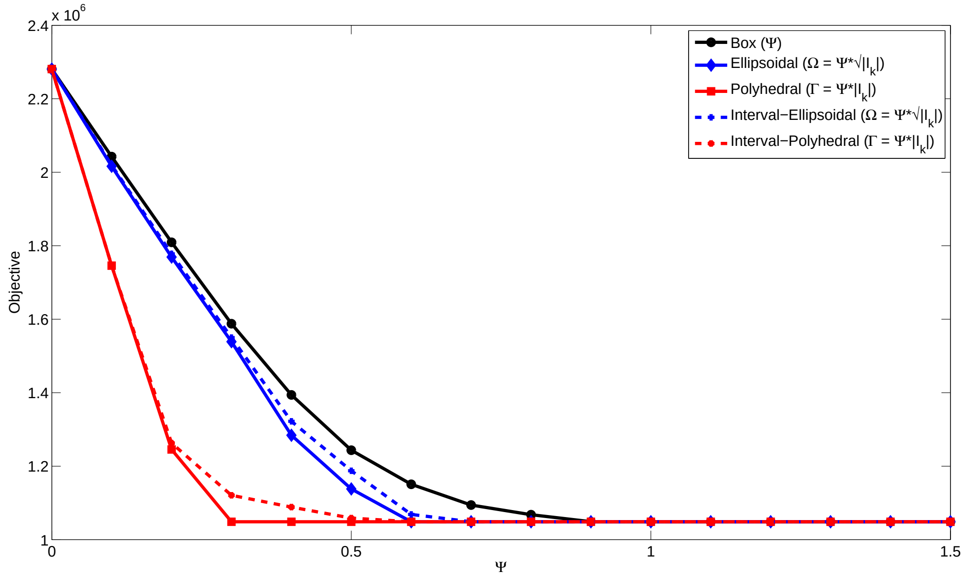


Figure 5.5: A 5-product robust MPSNVP with full market entry for market pool size 100.

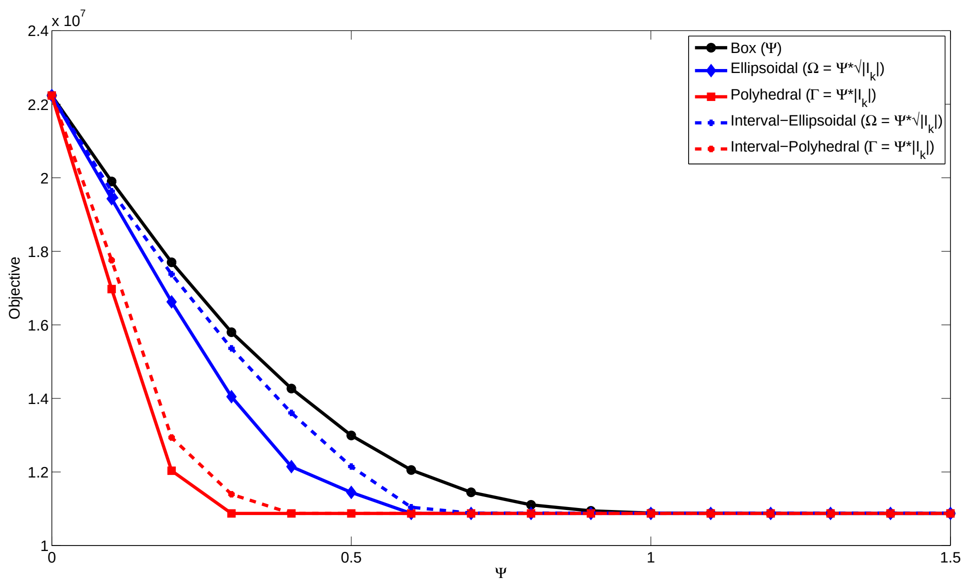


Figure 5.6: A 5-product robust MPSNVP with full market entry for market pool size 1000.

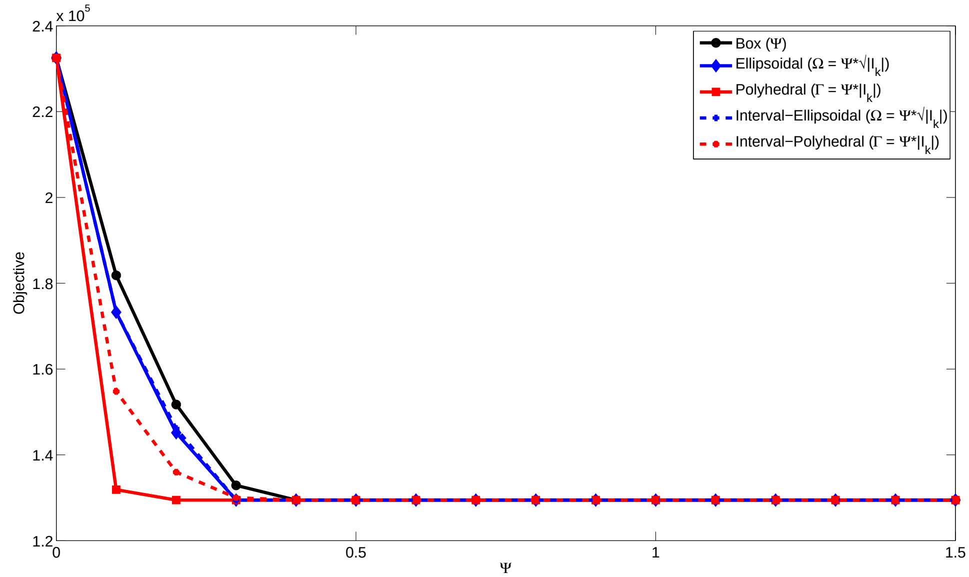


Figure 5.7: A 10-product robust MPSNVP with full market entry for market pool size 10.

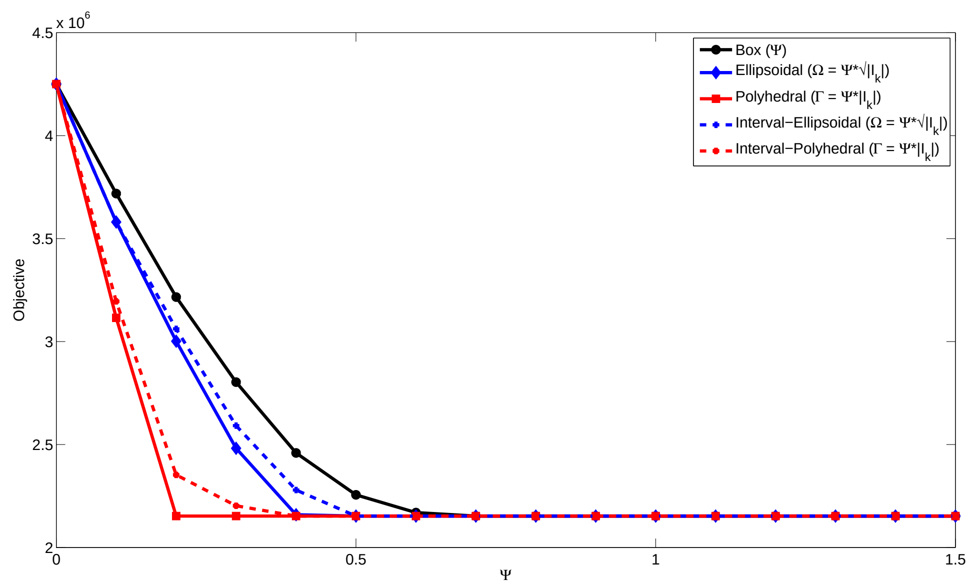


Figure 5.8: A 10-product robust MPSNVP with full market entry for market pool size 100.

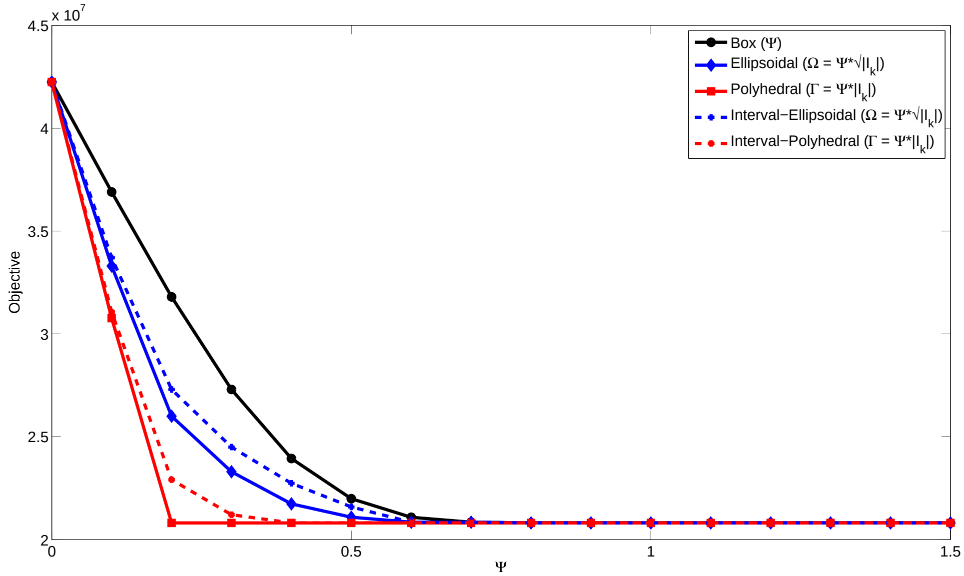


Figure 5.9: A 10-product robust MPSNVP with full market entry for market pool size 1000.

CHAPTER 6

ROBUST MULTI-PRODUCT SELECTIVE NEWSVENDOR WITH PARTIAL MARKET ENTRY

6.1 Introduction

In this chapter, we consider the MPSNVP with partial market entry when market demands are uncertain. We develop robust counterpart reformulations under different uncertainty sets. Numerical examples are presented and results are discussed.

6.2 Robust Multi-Product Selective Newsvendor Problem with Partial Market Entry

The profit function for the partial market entry case of the multi-product selective newsvendor problem is given by (2.23); equivalently, it can be expressed as:

$$\begin{aligned}
\text{Max} \ & \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \right) \mathcal{Y}_i \\
& - \sum_{j \in J} \left(e_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} - Q_j \right)^+ - \sum_{j \in J} \left(c_j - v_j \right) Q_j,
\end{aligned}$$

s. t.

$$\mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I,$$

$$\mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I.$$

Since $\mathcal{Y}_i$ and y_{ij} are binary variables and $\mathcal{Y}_i \geq y_{ij}$, then $\mathcal{Y}_i y_{ij} = y_{ij}$, and hence, the above model can be written as:

$$\begin{aligned}
\mathcal{SNVP}_{Part} : \text{Max} \ & \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - s_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) \\
& - \sum_{j \in J} \left(e_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} - Q_j \right)^+ - \sum_{j \in J} \left(c_j - v_j \right) Q_j,
\end{aligned}$$

s. t.

$$\mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I,$$

$$\mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I.$$

The profit for the deterministic partial market entry case of the multi-product selective newsvendor problem is given by the above model. In that model, we can replace $\left(\sum_{i\in I} x_{ij}y_{ij} - Q_j\right)^+$ by z_j and add the constraints $\sum_{i\in I} x_{ij}y_{ij} - Q_j \leq z_j$ and $z_j \geq 0$. Thus, the optimization model becomes:

$$\mathcal{SNVP}_{Part-D}: \ \text{Max} \ \sum_{i\in I}\left(\sum_{j\in J}\left(\left(r_{ij}-v_j\right)x_{ij}-\mathfrak{s}_{ij}\right)y_{ij}-s_i\mathcal{Y}_i\right)-\sum_{j\in J}\left(e_j-v_j\right)z_j$$
$$-\sum_{j\in J}\left(c_j-v_j\right)Q_j,$$

s. t.

$$\sum_{i\in I} x_{ij}y_{ij} - Q_j \leq z_j, \ \forall\, j\in J,$$

$$\mathcal{Y}_i \geq y_{ij}, \ \forall\, j\in J, \ \forall\, i\in I,$$

$$Q_j, \ z_j \geq 0, \ \forall\, j\in J,$$

$$\mathcal{Y}_i, \ y_{ij} \in \{0,1\}, \ \forall\, j\in J, \ \forall\, i\in I.$$

Suppose that the demand of the products; x_{ij}, is uncertain in the market set $I_k \subseteq I$, therefore, the demand value is considered as shown in (4.12). Subsequently, the above model can be rewritten as:

$$\text{Max} \ \sum_{i\in I\setminus I_k}\left(\sum_{j\in J}\left(\left(r_{ij}-v_j\right)x_{ij}-\mathfrak{s}_{ij}\right)y_{ij}-s_i\mathcal{Y}_i\right)$$
$$+\sum_{i\in I_k}\left(\sum_{j\in J}\left(\left(r_{ij}-v_j\right)\tilde{x}_{ij}-\mathfrak{s}_{ij}\right)y_{ij}-s_i\mathcal{Y}_i\right)$$
$$-\sum_{j\in J}\left(e_j-v_j\right)z_j-\sum_{j\in J}\left(c_j-v_j\right)Q_j,$$

s. t.

$$\sum_{i\in I\setminus I_k} x_{ij}y_{ij} + \sum_{i\in I_k} \tilde{x}_{ij}y_{ij} - Q_j \le z_j,\ \forall\ j \in J$$

$$\mathcal{Y}_i \ge y_{ij},\ \forall\ j \in J,\ \forall\ i \in I$$

$$Q_j,\ z_j \ge 0,\ \forall\ j \in J,$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0,1\},\ \forall\ j \in J,\ \forall\ i \in I,$$

where x_{ij} and $\tilde{x}_{ij}$ represent the deterministic and uncertain values of the demand of product j in market i, respectively.

The above model can be rewritten as:

$$\mathcal{SNVP}_{Part-U} : \text{Max}\ \ \delta,$$

s. t.

$$\delta - \sum_{i\in I\setminus I_k} \left(\sum_{j\in J} \left(\left(r_{ij} - v_j\right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right)$$

$$- \sum_{i\in I_k} \left(\sum_{j\in J} \left(\left(r_{ij} - v_j\right) \tilde{x}_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right)$$

$$+ \sum_{j\in J} \left(e_j - v_j\right) z_j + \sum_{j\in J} \left(c_j - v_j\right) Q_j \le 0,$$

$$\sum_{i\in I\setminus I_k} x_{ij}y_{ij} + \sum_{i\in I_k} \tilde{x}_{ij}y_{ij} - Q_j \le z_j,\ \forall\ j \in J,$$

$$Q_j,\ z_j \ge 0,\ \forall\ j \in J,$$

$$\mathcal{Y}_i \ge y_{ij},\ \forall\ j \in J,\ \forall\ i \in I.$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0,1\},\ \forall\ j \in J,\ \forall\ i \in I.$$

Next, we present the robust counterpart reformulation of the uncertain model of the MPSNVP with partial market entry; $\mathcal{SNVP}_{Part-U}$, under different uncertainty sets.

6.2.1 Robust MPSNVP with Partial Market Entry Based on Box Uncertainty Set

We apply the approach provided in Section 4.3.2 to obtain the robust counterpart of the uncertain model of the MPSNVP with partial market entry, $\mathcal{SNVP}_{Part-U}$, based on the box uncertainty set. Note that the uncertain demands are coefficients of the decision variables y_{ij}, which are binary variables, therefore applying the robust counterpart discussed in Section 4.3.2 yields:

$$
\begin{aligned}
\text{Max} \quad & \delta, \\
\text{s.t.} \quad & \\
& \delta - \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) + \Psi \sum_{i \in I_k} \sum_{j \in J} \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij} \\
& + \sum_{j \in J} \left(e_j - v_j \right) z_j + \sum_{j \in J} \left(c_j - v_j \right) Q_j \leq 0, \\
& \sum_{i \in I} x_{ij} y_{ij} + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} y_{ij} - Q_j \leq z_j, \ \forall \ j \in J, \\
& \mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I, \\
& Q_j, \ z_j \geq 0, \ \forall \ j \in J, \\
& \mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I.
\end{aligned}
$$

This robust counterpart results in an MILP formulation. This reformulation preserves the tractability of original model with the same number of variables and constraints. Notice that, we can get Soyster's robust counterpart to the above model gby taking $\Psi = 1$ and $\Psi_j = 1$.

The optimal solution to the above modelresults in:

$$\begin{aligned} Q_j^* &= \sum_{i \in I} x_{ij} y_{ij}^* + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} y_{ij}^*, \ \forall \ j \in J, \\ z_j^* &= 0, \ \forall \ j \in J. \end{aligned} \tag{6.1}$$

Hence, the above model reduces to:

$$\begin{aligned}
&\text{Max} \ \ \delta, \\
&\ \text{s.t.} \\
&\quad \delta - \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) + \Psi \sum_{i \in I_k} \sum_{j \in J} \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij} \\
&\quad + \sum_{j \in J} \left(c_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} + \Psi_j \sum_{i \in I_k} \hat{x}_{ij} y_{ij} \right) \leq 0, \\
&\quad \mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I, \\
&\quad \mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I.
\end{aligned}$$

The last model can be rewritten as:

$$\mathcal{SNVP}_{Part-U} - RC_B : \text{Max} \ \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - c_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right)$$

$$- \Psi \sum_{i \in I_k} \sum_{j \in J} (r_{ij} - v_j)\, \hat{x}_{ij} y_{ij} - \sum_{i \in I_k} \sum_{j \in J} \Psi_j (c_j - v_j)\, \hat{x}_{ij} y_{ij},$$

s.t.

$$\mathcal{Y}_i \geq y_{ij},\ \forall\ j \in J,\ \forall\ i \in I,$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0, 1\},\ \forall\ j \in J,\ \forall\ i \in I.$$

The optimal solution to model $\mathcal{SNVP}_{Part-U} - RC_B$ is as follows:

For all $i \in I \setminus I_k$:

$$y_{ij}^* = 1 \text{ if } (r_{ij} - c_j)\, x_{ij} - \mathfrak{s}_{ij} > 0 \text{ and } \mathcal{Y}_i^* = 1,$$

$$y_{ij}^* = 0 \text{ otherwise},$$

$$\mathcal{Y}_i^* = 1 \text{ if } \sum_{j \in J} ((r_{ij} - c_j)\, x_{ij} - \mathfrak{s}_{ij})\, y_{ij}^* - s_i > 0,$$

$$\mathcal{Y}_i^* = 0 \text{ otherwise},$$

For all $i \in I_k$:

$$y_{ij}^* = 1 \text{ if } (r_{ij} - c_j)\, x_{ij} - \Psi (r_{ij} - v_j)\, \hat{x}_{ij} - \Psi_j (c_j - v_j)\, \hat{x}_{ij} - \mathfrak{s}_{ij} > 0 \text{ and } \mathcal{Y}_i^* = 1,$$

$$y_{ij}^* = 0 \text{ otherwise},$$

$$\mathcal{Y}_i^* = 1 \text{ if } \sum_{j \in J} ((r_{ij} - c_j)\, x_{ij} - \mathfrak{s}_{ij})\, y_{ij}^* - \Psi \sum_{j \in J} (r_{ij} - v_j)\, \hat{x}_{ij} y_{ij}^*$$

$$- \sum_{j \in J} \Psi_j (c_j - v_j)\, \hat{x}_{ij} y_{ij}^* - s_i > 0,$$

$$\mathcal{Y}_i^* = 0 \text{ otherwise}.$$

y_{ij}^{*} given above are considered as candidate selected markets when the first part of the condition is satisfied, then we check for $\mathcal{Y}_i^{*}$ with those candidate selected markets; i.e. $y_{ij}^{*} = 1$. If the condition of $\mathcal{Y}_i^{*} = 1$ is satisfied, then y_{ij}^{*} candidates become actual selected markets. In addition, the optimal order quantity Q_j^{*} for each product j is given by (6.1).

6.2.2 Robust MPSNVP with Partial Market Entry Based on Ellipsoidal Uncertainty Set

We apply the approach provided in Section 4.3.3 to obtain the robust counterpart of the uncertain model of the MPSNVP with partial market entry, $\mathcal{SNVP}_{Part-U}$, this can be expressed as:

$$
\begin{aligned}
&\text{Max} \quad \delta, \\
&\quad \text{s.t.} \\
&\qquad \delta - \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) + \Omega \sqrt{\sum_{i \in I_k} \sum_{j \in J} \left(r_{ij} - v_j \right)^2 \hat{x}_{ij}^2 y_{ij}^2} \\
&\qquad + \sum_{j \in J} \left(e_j - v_j \right) z_j + \sum_{j \in J} \left(c_j - v_j \right) Q_j \leq 0, \\
&\qquad \sum_{i \in I} x_{ij} y_{ij} + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^2} - Q_j \leq z_j, \ \forall\, j \in J, \\
&\qquad \mathcal{Y}_i \geq y_{ij}, \ \forall\, j \in J, \ \forall\, i \in I, \\
&\qquad Q_j, \ z_j \geq 0, \ \forall\, j \in J, \\
&\qquad \mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall\, j \in J, \ \forall\, i \in I.
\end{aligned}
$$

This robust counterpart preserves the same number of variables and constraints, however it results in an MINLP formulation, which causes computational complexities.

The optimal solution to the above model yields,

$$
\begin{aligned}
&Q_j^* = \sum_{i \in I} x_{ij} y_{ij}^* + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^{*2}}, \ \forall\, j \in J, \\
&z_j^* = 0, \ \forall\, j \in J.
\end{aligned}
\tag{6.2}
$$

Hence, the above model becomes:

$$
\begin{aligned}
\text{Max} \quad & \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) - \Omega \sqrt{\sum_{i \in I_k} \sum_{j \in J} \left(r_{ij} - v_j \right)^2 \hat{x}_{ij}^2 y_{ij}^2} \\
& - \sum_{j \in J} \left(c_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} + \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^2} \right), \\
\text{s.t.} \quad & \\
& \mathcal{Y}_i \geq y_{ij}, \ \forall\, j \in J, \ \forall\, i \in I, \\
& \mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall\, j \in J, \ \forall\, i \in I.
\end{aligned}
$$

This model can be rewritten as:

$$
\begin{aligned}
\text{Max} \quad & \sum_{i \in I \setminus I_k} \left(\sum_{j \in J} \left(\left(r_{ij} - c_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) \\
& + \sum_{i \in I_k} \left(\sum_{j \in J} \left(\left(r_{ij} - c_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) \\
& - \Omega \sqrt{\sum_{i \in I_k} \sum_{j \in J} \left(r_{ij} - v_j \right)^2 \hat{x}_{ij}^2 y_{ij}^2} - \sum_{j \in J} \left(c_j - v_j \right) \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^2},
\end{aligned}
$$

s.t.

$$\mathcal{Y}_i \geq y_{ij},\ \forall\ j \in J,\ \forall\ i \in I,$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0,1\},\ \forall\ j \in J,\ \forall\ i \in I.$$

The last model can be separated into the following two sub-models:

$$\mathcal{SNVP}_{Part-U} - D : \text{Max} \quad \sum_{i \in I \setminus I_k} \left(\sum_{j \in J} \left(\left(r_{ij} - c_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right),$$

s.t.

$$\mathcal{Y}_i \geq y_{ij},\ \forall\ j \in J,\ \forall\ i \in I \setminus I_k,$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0,1\},\ \forall\ j \in J,\ \forall\ i \in I \setminus I_k.$$

$$\begin{aligned}\mathcal{SNVP}_{Part-U} - RC_E : \text{Max} \ & \sum_{i \in I_k} \left(\sum_{j \in J} \left(\left(r_{ij} - c_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) \\ & - \Omega \sqrt{\sum_{i \in I_k} \sum_{j \in J} \left(r_{ij} - v_j \right)^2 \hat{x}_{ij}^2 y_{ij}^2} \\ & - \sum_{j \in J} \left(c_j - v_j \right) \Omega_j \sqrt{\sum_{i \in I_k} \hat{x}_{ij}^2 y_{ij}^2},\end{aligned}$$

s.t.

$$\mathcal{Y}_i \geq y_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0,1\},\ \forall\ j \in J,\ \forall\ i \in I_k.$$

The optimal solution to sub-model $\mathcal{SNVP}_{Part-U} - D$ is as follows:

For all $i \in I \setminus I_k$:

$$
\begin{aligned}
& y_{ij}^* = 1 \text{ if } \left(r_{ij} - c_j\right) x_{ij} - \mathfrak{s}_{ij} > 0 \text{ and } \mathcal{Y}_i^* = 1, \\
& y_{ij}^* = 0 \text{ otherwise}, \\
& \mathcal{Y}_i^* = 1 \text{ if } \sum_{j \in J} \left(\left(r_{ij} - c_j\right) x_{ij} - \mathfrak{s}_{ij}\right) y_{ij}^* - s_i > 0, \\
& \mathcal{Y}_i^* = 0 \text{ otherwise}
\end{aligned}
\tag{6.3}
$$

The optimal solution to sub-model $\mathcal{SNVP}_{Part-U} - RC_E$ can be determined by applying the same solution procedures that was applied in Section 2.3.6 for solving the risk-neutral MPSNVP with flexible market entry. For applying Heuristic II, The RDU ratio for sub-model $\mathcal{SNVP}_{Part-U} - RC_E$ is:

$$
RDU_{Part-RC_E} = \frac{\sum\limits_{j \in J} \left[\left(r_{ij} - c_j\right) x_{ij} - \mathfrak{s}_{ij}\right] y_{ij}^* - s_i}{\sum\limits_{j \in J} \Omega^2 \left(r_{ij} - v_j\right)^2 \hat{x}_{ij}^2 + \sum\limits_{j \in J} \Omega_j^2 \left(c_j - v_j\right)^2 \hat{x}_{ij}^2}. \tag{6.4}
$$

The optimal order quantity for each product j is given be (6.2).

6.2.3 Robust MPSNVP with Partial Market Entry Based on Polyhedral Uncertainty Set

We apply the approach presented in Section 4.3.4 to obtain the robust counterpart of the uncertain model of the MPSNVP with partial market entry, $\mathcal{SNVP}_{Part-U}$, based on the polyhedral uncertainty set. Note that the uncertain demands are coefficients of the decision variables y_{ij}, which are binary variables, therefore applying the robust counterpart approach yields:

$$
\begin{aligned}
&\text{Max} \quad \delta, \\
&\text{s.t.} \\
&\delta - \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) \\
&+ u\Gamma + \sum_{j \in J} \left(e_j - v_j \right) z_j + \sum_{j \in J} \left(c_j - v_j \right) Q_j \leq 0, \\
&u \geq \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I_k, \\
&\sum_{i \in I} x_{ij} y_{ij} + u_j \Gamma_j - Q_j \leq z_j, \ \forall \ j \in J, \\
&u_j \geq \hat{x}_{ij} y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I_k, \\
&\mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I, \\
&u \geq 0, \\
&u_j, \ Q_j, z_j \geq 0, \ \forall \ j \in J, \\
&\mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I.
\end{aligned}
$$

This robust counterpart results in MILP. This preserves the tractability of original model, however, the number of variables increases as well as the number of constraints. The number of binary variables remains the same as in the original formulation $\mathcal{SNVP}_{Part-U}$.

We can reduce the size of the last model by noting that, when the values of y^*_{ij}

and u_j^* are given, then the optimal solution to the above model results in:

$$\begin{aligned} Q_j^* &= \sum_{i \in I} x_{ij} y_{ij}^* + u_j^* \Gamma_j, \ \forall \ j \in J, \\ z_j^* &= 0, \ \forall \ j \in J. \end{aligned} \tag{6.5}$$

By substitution from (6.5) into the above model:

$$\begin{aligned} \text{Max} \ & \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - v_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) - u\Gamma \\ & - \sum_{j \in J} \left(c_j - v_j \right) \left(\sum_{i \in I} x_{ij} y_{ij} + u_j \Gamma_j \right), \\ \text{s.t.} & \\ & u \geq \left(r_{ij} - v_j \right) \hat{x}_{ij} y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I_k, \\ & u_j \geq \hat{x}_{ij} y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I_k, \\ & \mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I, \\ & u \geq 0, \\ & u_j \geq 0, \ \forall \ j \in J, \\ & \mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I. \end{aligned}$$

This model is expressed as:

$$\begin{aligned} \mathcal{SNVP}_{Part-U} - RC_P : \ \text{Max} \ & \sum_{i \in I} \left(\sum_{j \in J} \left(\left(r_{ij} - c_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) - u\Gamma \\ & - \sum_{j \in J} \left(c_j - v_j \right) u_j \Gamma_j, \end{aligned}$$

s.t.

$$u \geq (r_{ij} - v_j)\, \hat{x}_{ij} y_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$u_j \geq \hat{x}_{ij} y_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$\mathcal{Y}_i \geq y_{ij},\ \forall\ j \in J,\ \forall\ i \in I,$$

$$u \geq 0,$$

$$u_j \geq 0,\ \forall\ j \in J,$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0, 1\},\ \forall\ j \in J,\ \forall\ i \in I.$$

Model $\mathcal{SNVP}_{Part-U} - RC_P$ is an MILP problem, it can be solved using commercial solvers such as CPLEX. Then, the results are used to get the optimal order quantities Q_j from (6.5).

6.2.4 Robust MPSNVP with Partial Market Entry Based on Interval-Ellipsoidal Uncertainty Set

We apply the approach presented in Section 4.3.5 to obtain the robust counterpart of the uncertain model of the MPSNVP with partial market entry, $\mathcal{SNVP}_{Part-U}$, this can be expressed as:

$$\text{Max}\ \ \delta,$$

s.t.

$$\delta - \sum_{i \in I} \left(\sum_{j \in J} \left((r_{ij} - v_j)\, x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right)$$

$$+\sum_{i\in I_k}\sum_{j\in J}(r_{ij}-v_j)\,\hat{x}_{ij}p_{ij}+\Omega\sqrt{\sum_{i\in I_k}\sum_{j\in J}(r_{ij}-v_j)^2\,\hat{x}_{ij}^2u_{ij}^2}$$

$$+\sum_{j\in J}(e_j-v_j)\,z_j+\sum_{j\in J}(c_j-v_j)\,Q_j\leq 0,$$

$$-\,p_{ij}\leq y_{ij}-u_{ij}\leq p_{ij},\ \forall\ j\in J,\ \forall\ i\in I_k,$$

$$\sum_{i\in I}x_{ij}y_{ij}+\sum_{i\in I_k}\hat{x}_{ij}t_{ij}+\Omega_j\sqrt{\sum_{i\in I_k}\hat{x}_{ij}^2w_{ij}^2}-Q_j\leq z_j,\ \forall\ j\in J,$$

$$-\,t_{ij}\leq y_{ij}-w_{ij}\leq t_{ij},\ \forall\ j\in J,\ \forall\ i\in I_k,$$

$$\mathcal{Y}_i\geq y_{ij},\ \forall\ j\in J,\ \forall\ i\in I,$$

$$u_{ij},\ w_{ij},\ p_{ij},\ t_{ij}\geq 0,\ \forall\ j\in J,\ \ \forall\ i\in I_k,$$

$$Q_j,\ z_j\geq 0,\ \forall\ j\in J,$$

$$\mathcal{Y}_i,\ y_{ij}\in\{0,1\}\,,\ \forall\ j\in J,\ \forall\ i\in I,$$

where, p_{ij} and t_{ij} are auxiliary variables, while u_{ij} and w_{ij} are positive dual variables.

This robust counterpart results in an MINLP formulation with number of variables and constraints greater than that in the original formulation $\mathcal{SNVP}_{Part-U}$.

We can reduce the size of the above model by performing preprocessing of some variables, as it discusses below. In addition, we can retrieved the tractability of the above model by linearizing the nonlinear terms.

Given the optimal values of y_{ij}^*, t_{ij}^* and w_{ij}^*, the optimal solution to the above model will results in:

$$Q_j^* = \sum_{i\in I} x_{ij} y_{ij}^* + \sum_{i\in I_k} \hat{x}_{ij} t_{ij}^* + \Omega_j \sqrt{\sum_{i\in I_k} \hat{x}_{ij}^2 w_{ij}^{*2}},\ \forall\ j \in J,$$
$$z_j^* = 0,\ \forall\ j \in J. \tag{6.6}$$

In addition, we can substitute for the nonlinear terms as follows:

$$\sum_{i\in I_k} \sum_{j\in J} (r_{ij} - v_j)^2 \hat{x}_{ij}^2 u_{ij}^2 \leq q^2,$$
$$\sum_{i\in I_k} \hat{x}_{ij}^2 w_{ij}^2 \leq h_j^2,\ \forall\ j \in J.$$

This leads to a reformulation of the above model as an CQMIP problem, which can be solved efficiently and in reasonable computational time using commercial solvers such as CPLEX.

We express the above model in the following CQMIP formulation:

$$\mathcal{SNVP}_{Part-U} - RC_{I-E} : \text{Max} \ \sum_{i\in I} \left(\sum_{j\in J} \left(\left(r_{ij} - c_j \right) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right)$$
$$- \sum_{i\in I_k} \sum_{j\in J} \left(r_{ij} - v_j \right) \hat{x}_{ij} p_{ij} - \Omega q$$
$$- \sum_{j\in J} \left(c_j - v_j \right) \left(\sum_{i\in I_k} \hat{x}_{ij} t_{ij} + \Omega_j h_j \right),$$

s.t.

$$- p_{ij} \leq y_{ij} - u_{ij} \leq p_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$- t_{ij} \leq y_{ij} - w_{ij} \leq t_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$\sum_{i \in I_k} (r_{ij} - v_j)^2 \hat{x}_{ij}^2 u_{ij}^2 \leq q^2,$$

$$\sum_{i \in I_k} \hat{x}_{ij}^2 w_{ij}^2 \leq h_j^2, \ \forall \ j \in J,$$

$$\mathcal{Y}_i \geq y_{ij}, \ \forall \ j \in J, \ \forall \ i \in I,$$

$$u_{ij}, \ w_{ij}, \ p_{ij}, \ t_{ij} \geq 0, \ \forall \ j \in J, \ \forall \ i \in I_k,$$

$$h_j, \geq 0, \ \forall \ j \in J,$$

$$q \geq 0,$$

$$\mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall \ j \in J, \ \forall \ i \in I.$$

6.2.5 Robust MPSNVP with Partial Market Entry Based on Interval-Polyhedral Uncertainty Set

We apply the approach provided in Section 4.3.6 to obtain the robust counterpart of the uncertain model of the MPSNVP with partial market entry, $\mathcal{SNVP}_{Part-U}$. Note that the uncertain demands are coefficients of the decision variables y_{ij}, which are binary variables, therefore applying the robust counterpart presented in Section 4.3.6 yields:

$$\text{Max} \ \ \delta,$$

s.t.

$$\delta - \sum_{i \in I} \left(\sum_{j \in J} \left((r_{ij} - v_j) x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right) + \sum_{i \in I_k} \sum_{j \in J} p_{ij} + u\Gamma$$

$$+ \sum_{j \in J} (e_j - v_j) z_j + \sum_{j \in J} (c_j - v_j) Q_j \leq 0,$$

$$u + p_{ij} \geq (r_{ij} - v_j)\, \hat{x}_{ij} y_{ij}, \ \forall\, j \in J, \ \forall\, i \in I_k,$$

$$\sum_{i \in I} x_{ij} y_{ij} + \sum_{i \in I_k} t_{ij} + u_j \Gamma_j - Q_j \leq z_j, \ \forall\, j \in J,$$

$$u_j + t_{ij} \geq \hat{x}_{ij} y_{ij}, \ \forall\, j \in J, \ \forall\, i \in I_k,$$

$$\mathcal{Y}_i \geq y_{ij}, \ \forall\, j \in J, \ \forall\, i \in I,$$

$$u \geq 0,$$

$$u_j, \ Q_j, \ z_j \geq 0, \ \forall\, j \in J,$$

$$p_{ij}, \ t_{ij} \geq 0, \ \forall\, j \in J, \ \forall\, i \in I_k,$$

$$\mathcal{Y}_i, \ y_{ij} \in \{0, 1\}, \ \forall\, j \in J, \ \forall\, i \in I.$$

This robust counterpart results in MILP. This preserves the tractability of original model, however, the number of variables increases as well as the number of constraints. The number of binary variables remains the same as that in the original formulation $\mathcal{SNVP}_{Part-U}$.

The size of the above model can be reduced by noticing that the optimal values of Q_j and z_j are given as follows:

$$\begin{aligned} Q_j^* &= \sum_{i \in I} x_{ij} y_{ij}^* + \sum_{i \in I_k} t_{ij}^* + u_j^* \Gamma_j, \ \forall\, j \in J, \\ z_j^* &= 0, \ \forall\, j \in J. \end{aligned} \tag{6.7}$$

By substituting for Q_j^* and z_j^* from (6.7) into the above model, we get:

$$\mathcal{SNVP}_{Part-U} - RC_{I-P} : \text{Max} \ \sum_{i \in I} \left(\sum_{j \in J} \left((r_{ij} - c_j)\, x_{ij} - \mathfrak{s}_{ij} \right) y_{ij} - s_i \mathcal{Y}_i \right)$$

$$-\sum_{i\in I_k}\sum_{j\in J} p_{ij} - u\Gamma - \sum_{i\in I_k}\sum_{j\in J}(c_j - v_j)\, t_{ij}$$

$$-\sum_{j\in J}(c_j - v_j)\, u_j \Gamma_j,$$

s.t.

$$u + p_{ij} \geq (r_{ij} - v_j)\, \hat{x}_{ij} y_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$u_j + t_{ij} \geq \hat{x}_{ij} y_{ij},\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$\mathcal{Y}_i \geq y_{ij},\ \forall\ j \in J,\ \forall\ i \in I,$$

$$u \geq 0,$$

$$u_j \geq 0,\ \forall\ j \in J,$$

$$p_{ij},\ t_{ij} \geq 0,\ \forall\ j \in J,\ \forall\ i \in I_k,$$

$$\mathcal{Y}_i,\ y_{ij} \in \{0, 1\},\ \forall\ j \in J,\ \forall\ i \in I.$$

Model $\mathcal{SNVP}_{Part-U} - RC_{I-P}$ is an MILP problem, it can be solved using commercial solvers such as CPLEX. Then, the results are used to get the optimal order quantities Q_j^* from (6.7).

6.3 Computational Results

In this section, we implement the robust counterpart reformulations from previous sections on a numerical examples of partial market entry MPSNVP subjected to demand uncertainty.

We consider the same input data of the MPSNVP discussed in Section 5.3 with the following exceptions:

- For the partial market entry MPSNVP with 3 products, the value of the market entry cost is considered to be uniformly distributed on U(5,000, 10,000).
- For the partial market entry MPSNVP with 5 products, the value of the market entry cost is considered to be uniformly distributed on U(10,000, 20,000).
- For the partial market entry MPSNVP with 10 products, the value of the market entry cost is considered to be uniformly distributed on U(20,000, 30,000).

Figures 6.1, 6.2 and 6.3 show the optimization results of the partial market entry MPSNVP with 3 products for 10, 100 and 1000 markets pool sizes respectively. Figures 6.4, 6.5 and 6.6 show the optimization results of the partial market entry MPSNVP with 5 products for 10, 100 and 1000 markets pool sizes respectively. Figures 6.7, 6.8 and 6.9 show the optimization results of the partial market entry MPSNVP with 10 products for 10, 100 and 1000 markets pool sizes respectively. For these figures, we notice a similar behavior to that behavior in Figures 4.3, 4.4 and 4.5. Therefore, the same discussion presented in Section 4.5 is applicable to the results in this section.

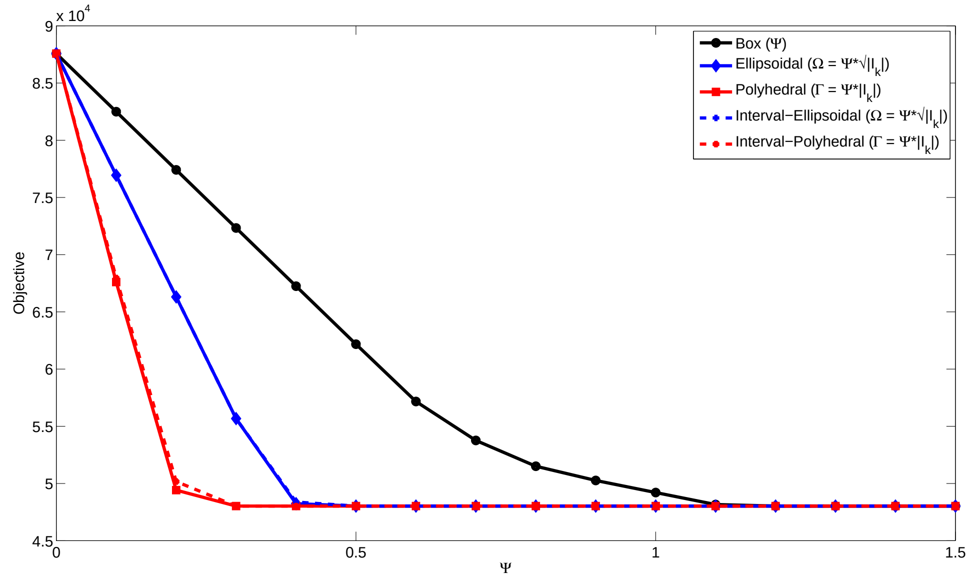


Figure 6.1: A 3 products MPSNVP with partial market entry for market pool size 10.

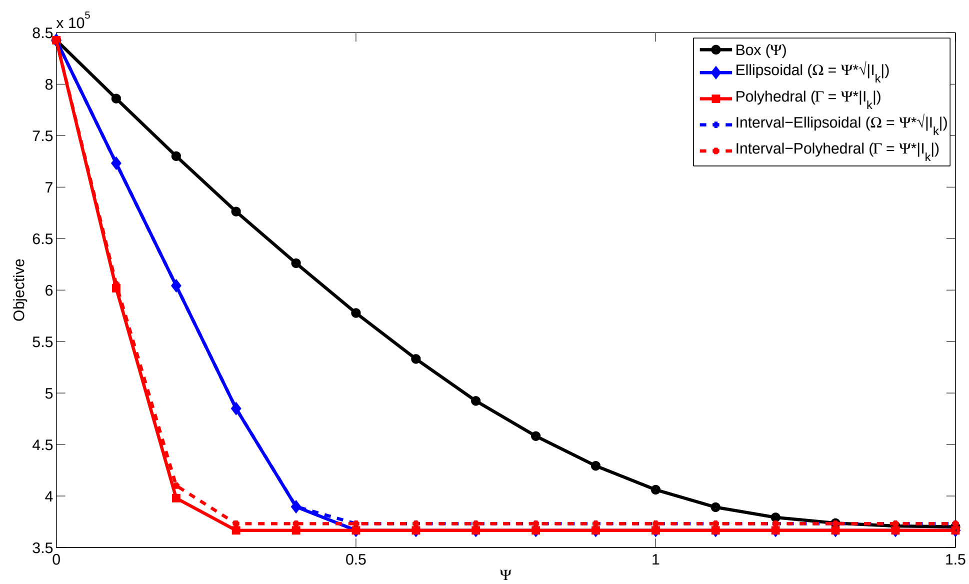


Figure 6.2: A 3 products MPSNVP with partial market entry for market pool size 100.

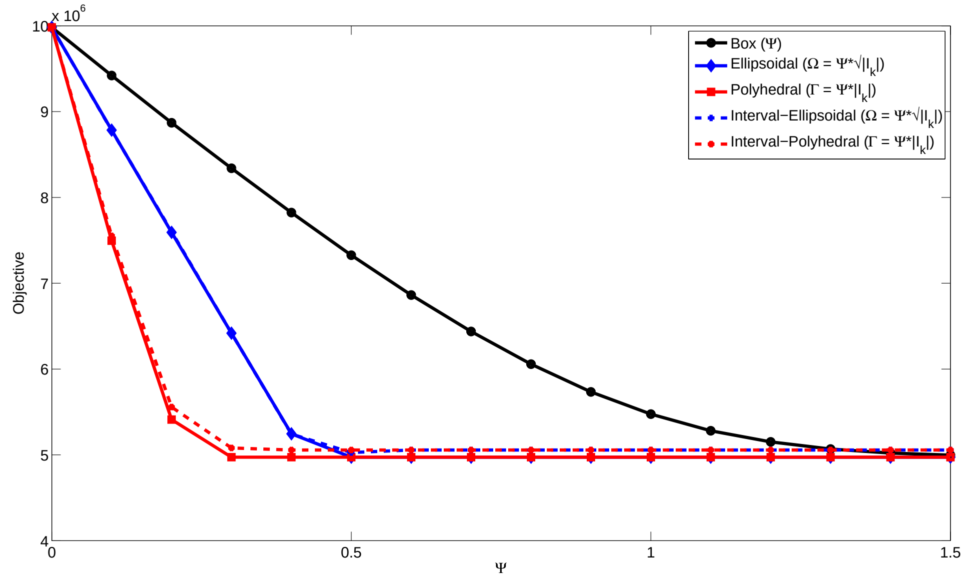


Figure 6.3: A 3 products MPSNVP with partial market entry for market pool size 1000.

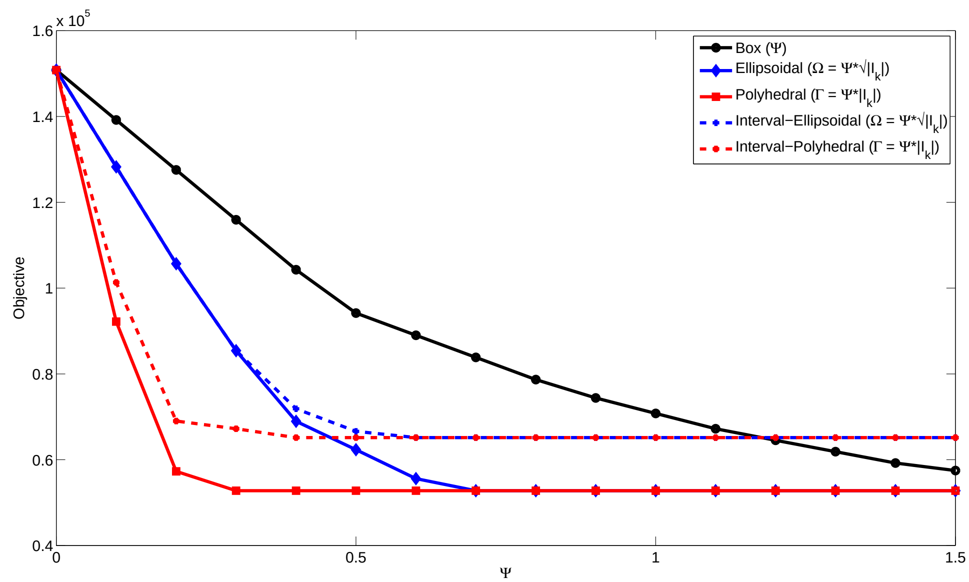


Figure 6.4: A 5 products MPSNVP with partial market entry for market pool size 10.

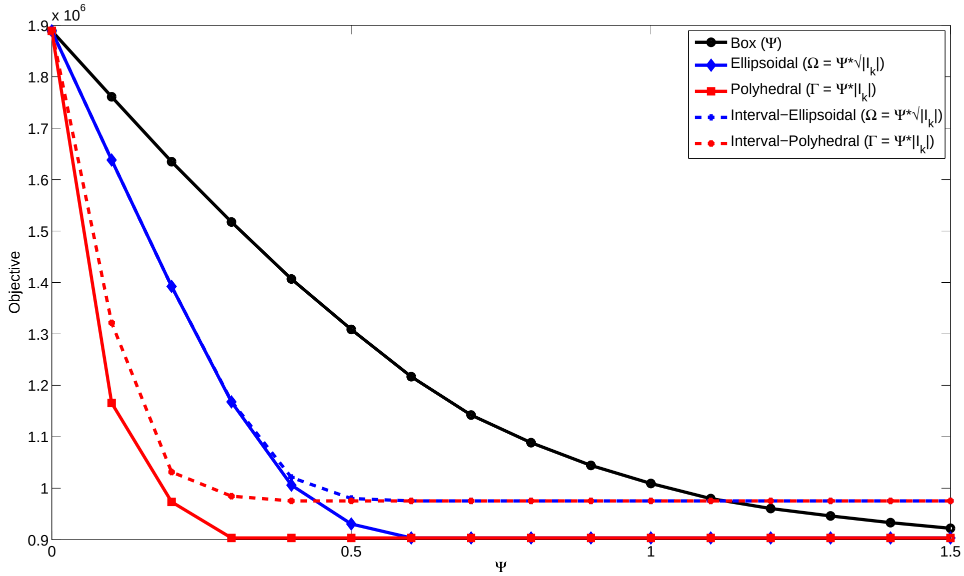


Figure 6.5: A 5 products MPSNVP with partial market entry for market pool size 100.

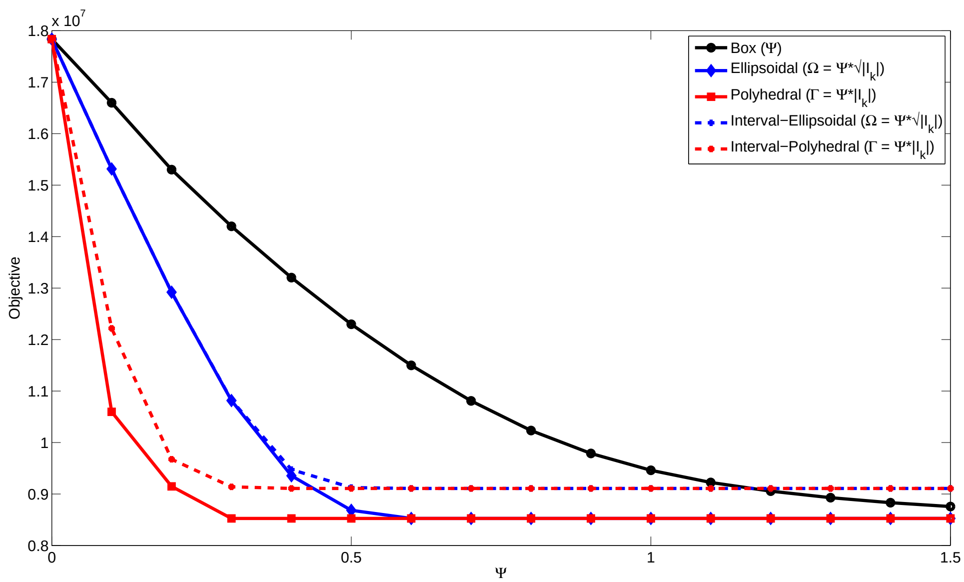


Figure 6.6: A 5 products MPSNVP with partial market entry for market pool size 1000.

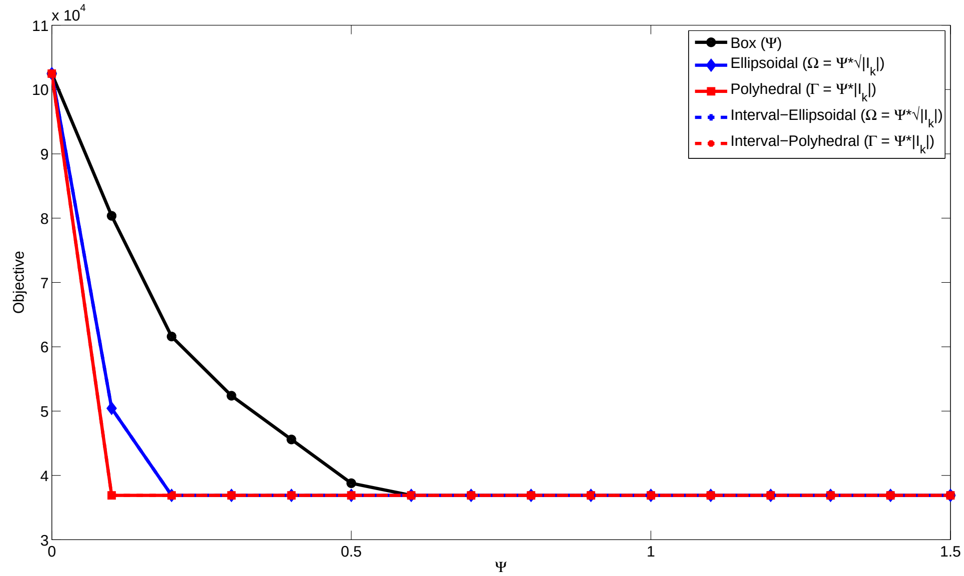


Figure 6.7: A 10 products MPSNVP with partial market entry for market pool size 10.

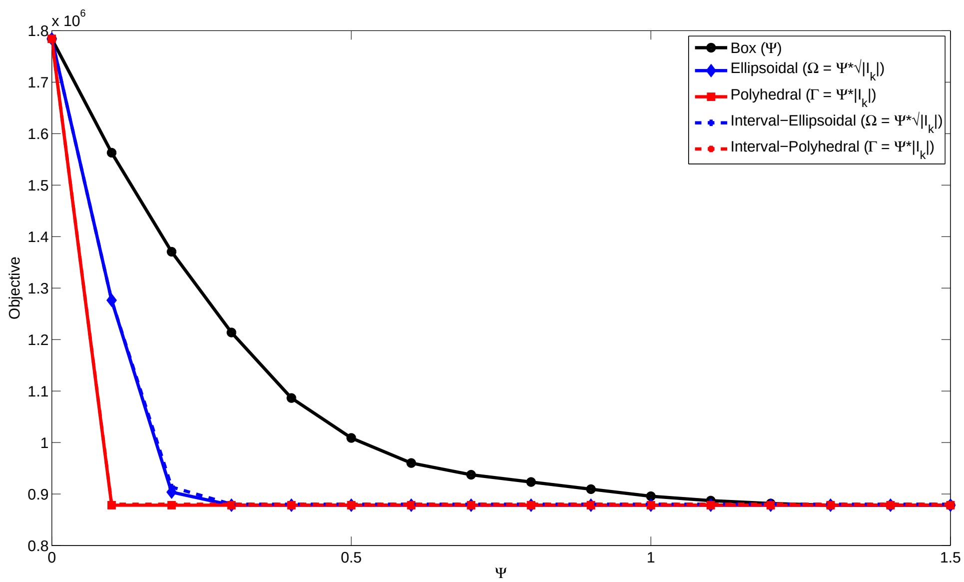


Figure 6.8: A 10 products MPSNVP with partial market entry for market pool size 100.

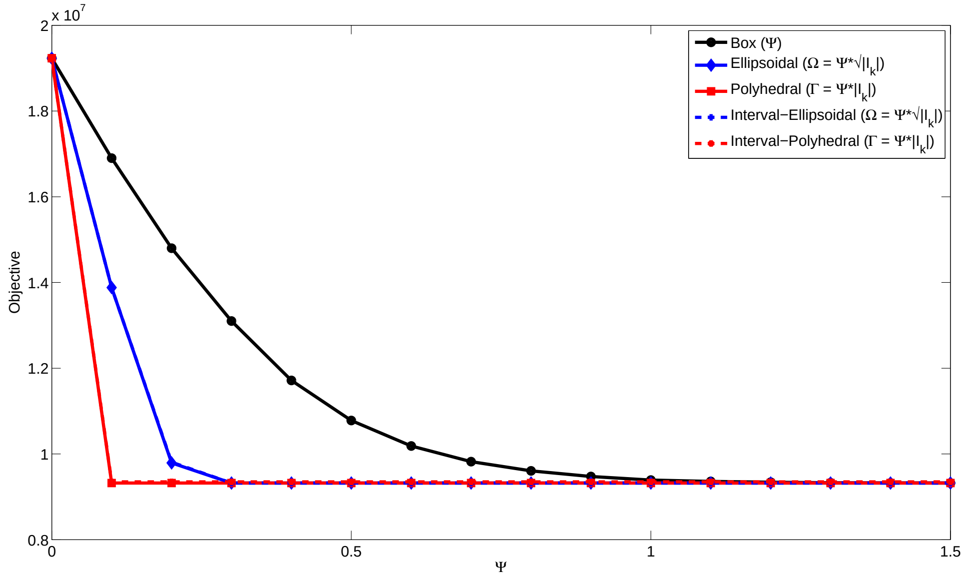


Figure 6.9: A 10 products MPSNVP with partial market entry for market pool size 1000.

CHAPTER 7

CONCLUSION

One of the very practical problems in supply chain planning is when the decision maker has freehand to select demands to satisfy, especially when resources are limited. In this dissertation, we studied the so-called *Multi-Product Selective Newsvendor Problem* (MPSNVP). The MPSNVP is multi-product multi-market newsvendor problem where the decision maker could select some markets to serve. A single study have been conducted in the literature to consider the MPSNVP. That study discussed the risk-neutral version of the problem and suggested a solution procedure that is exponential in the number of products. In this dissertation, we studied challenging general cases of the MPSNVP under risk-neutral as well as risk-averse preferences. In addition, we analyzed the MPSNVP with limited demand information.

For the MPSNVP with risk-neutral preferences, we discussed the flexible market entry, the full market entry and the partial market entry cases. The latter case is introduced by us to generalize the former two cases. For each case, we incorporate service level constraints. Then, we utilized the special structure of the developed models to provide polynomial optimal solution algorithms that obtain optimal markets to select and optimal order quantity to procure.

Conditional Value-at-Risk (CVaR) is a common coherent risk measure that has wide applications in finance industry, energy applications and supply chain planning. We examined the risk-averse preferences of the MPSNVP under CVaR risk criterion. We provided polynomial optimal solution algorithms that outperforms the stat-of-the-art commercial solvers in terms of the solution quality and computational time. In addition, we studied the effect of the risk-aversion degree on the objective value and gave some managerial insights.

The availability and quality of the demand information is always questionable. To overcome these limitations, we examined different cases of the MPSNVP under limited demand information. We analyzed mathematical models development of the robust counterparts under box, ellipsoidal, polyhedral uncertainty sets and combinations of these sets. We were able to propose solution algorithms to the developed robust counterparts. In addition, we were able to interpret the computational results and give some insights.

There are many direction for extending the work presented in this dissertation. One limitation of the presented work in Chapter 2 and Chapter 3 is the assumption that the market demands are independent normally distributed. It will be very interesting to relax this assumption, this will lead to different mathematical models and consequently, will require different solution approaches.

Another potential direction of extension for the risk-averse cases is to study the incorporation of other new risk measures such as the spectral risk measure.

It would also be very interesting to study the constrained versions of the MP-SNVP.

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  2. **Abdel-Aal, M.A.M.**, Selim, S.Z., under 2nd round of revision, Risk-Averse Multi-Product Selective Newsvendor Problem with Market Selection Flexibility under CVaR Criterion. Computers & Industrial Engineering (ISI Journal, Q1, IF: 2.086).

  3. Selim, S.Z., **Abdel-Aal, M.A.M.**, under 1st round of revision, A Generalized Targeting Model and an Application Involving Multiple Products. Computers & Industrial Engineering (ISI Journal, Q1, IF: 2.086).

  4. Ben-Daya, M., **Abdel-Aal, M.A.M.**, Nahas, N., *ready for submission*, A Production Remanufacturing Inventory System with Infinite Production Rates.