t-Reductions of Ideals in Integral Domains

BY

Abdulilah Kadri

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In loving memory of my father

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THESIS ABSTRACT

NAME: Abdulilah Kadri

TITLE OF STUDY: t-Reductions of Ideals in Integral Domains

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This Ph.D. thesis traverses two chapters which contribute to the study of multiplicative ideal theoretic properties of integral domains. Let R be a domain and I a nonzero ideal of R. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$ (See a brief summary on the t-operation in Section 1.1). An element $x \in R$ is t-integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. The set of all elements that are t-integral over I is called the t-integral closure of I. The first chapter investigates the t-reductions and t-integral closure of ideals. Our objective is to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. Namely, Section 1.2 identifies basic properties of t-reductions of ideals and features explicit examples discriminating between the notions of reduction and t-reduction. Section 1.3 investigates the concept of t-

integral closure of ideals, including its correlation with t-reductions. Section 1.4 studies the persistence and contraction of t-integral closure of ideals under ring homomorphisms. All along the chapter, the main results are illustrated with original examples.

An ideal I is t-basic if it has no t-reduction other than the trivial ones. The second chapter investigates t-reductions of ideals in pullback constructions. Section 2.2 examines the correlation between the notions of reduction and t-reduction in pseudo-valuation domains. Section 2.3 solves an open problem on whether the finite t-basic and v-basic ideal properties are distinct. We prove that these two notions coincide in any arbitrary domain. Section 2.4 features the main result, which establishes the transfer of the finite t-basic ideal property to pullbacks in line with Fontana-Gabelli's result on Prüfer v-Multiplication Domains (PvMDs) [16, Theorem 4.1] and Gabelli-Houston's result on v-domains [20, Theorem 4.15]. This allows us to enrich the literature with new families of examples, which put the class of domains subject to the finite t-basic ideal property strictly between the two classes of v-domains and integrally closed domains.

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ملخص الرسالة

الاسم الكامل: عبدالإله محمد قادري

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نقوم في هذه الرسالة بدراسة الاختزال-t للمثاليات في الحلقات الصحيحة والذي يُعدّ تعميما لمفهوم الاختزال. درسنا في الوحدة الأولى الخصائص الأساسية لهذا المفهوم، وأوضحنا بعض الأمثلة المميزة له عن مفهوم الاختزال. قمنا أيضا بتعريف الإغلاق-t للمثاليات، ودرسنا العلاقة بينه وبين الاختزال-t وخاصية تمديد وقصر هذا المفهوم من خلال التشاكل (الهومومورفزم) بين حلقتين صحيحتين. ترسخ النتائج التي حصلنا عليها نظائر-t لنتائج معروفة جيداً عن الاختزال والإغلاق.

نقول عن مثالي أنه أساسي-t إذا لم يقبل أي اختزال-t سوى نفسه. نقول عن حلقة أنها تمتلك الخاصية الأساسية-t (المنتهية) إذا كان كل مثالي (منته) هو مثالي أساسي-t . نقوم في الوحدة الثانية بدراسة هذه الخاصية للحلقات ونعطي شروطا لازمة وكافية لانتقال هذا الخاصية من خلال بُنى الانسحاب. في هذه الوحدة نتوصل لحل أحد المسائل المفتوحة عن العلاقة بين الخاصية الأساسية-t المنتهية والخاصية الأساسية-v المنتهية. احتوت الرسالة على العديد من التطبيقات، كما دُعِّمت النتائج بالعديد من الأمثلة.

Introduction

Throughout this thesis, all rings considered are commutative with identity. Let R be a ring and I a proper ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called a basic ideal [25, 44]. The notion of reduction was introduced by Northcott and Rees and its usefulness resides mainly in two facts: "First, it defines a relationship between two ideals which is preserved under homomorphisms and ring extensions; secondly, what we may term the reduction process gets rid of superfluous elements of an ideal without disturbing the algebraic multiplicities associated with it" [44]. The main purpose of this paper was to contribute to the analytic theory of ideals in Noetherian (local) rings via minimal reductions. Let $(R, \mathfrak{m})$ be a Noetherian local ring and I a proper ideal of R. One of the main results asserts that if I is non-basic, then it admits at least one minimal reduction [44, Section 2, Theorem 1]. Further, if the residue field is assumed to be infinite, then J is a minimal reduction of I if and only if $\mu(J) = l(I)$ [44, Section 4, Theorems 1&2], where $\mu(J)$ denotes the number of a minimal generating set of J and $l(I)$ denotes the analytic spread of I.

In [25, 26], Hays investigated reductions of ideals in more general settings of commutative rings (i.e., not necessarily local or Noetherian); particularly, Noetherian rings and Prüfer domains. He provided several sufficient conditions for an ideal to be basic. For instance, in Noetherian rings, an ideal is basic if and only if it is locally basic [25, Theorem 3.6]. He also introduced and studied the dual notion of a basic ideal; namely, an ideal is a C-ideal if it is not a reduction of any larger ideal. Several results about C-ideals are proved; including the fact that this notion is local for regular ideals in Noetherian rings [25, Corollary 5.10]. Moreover, domains in which every ideal is basic are examined. This class is shown to be strictly contained in the class of Prüfer domains; and a new characterization for Prüfer domains is provided; namely, a domain is Prüfer if and only if every finitely generated ideal is basic. The main result of these two papers states that in an integral domain, every ideal is basic if and only if it is a one-dimensional Prüfer domain [25, Theorem 6.1] combined with [26, Theorem 10]. Finally, he exhibited an explicit example in a valuation domain showing that most results on reductions in Noetherian rings do not extend beyond this class of rings, including the existence of minimal reductions.

Reductions happened to be a useful tool for the theory of integral dependence over ideals. Let I be an ideal in a ring R. An element $x \in R$ is integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in I^i$ for $i = 1, ..., n$. The set of all elements that are integral over I is called the integral closure of I, and is denoted by $\overline{I}$. If $I = \overline{I}$, then I is called integrally closed. It turned out that an element $x \in R$ is integral over I if and only if I is a reduction of $I + Rx$; and if I is finitely generated, then $I \subseteq \overline{J}$ if and only if J is a reduction of I. This correlation allowed to prove a number of crucial results in the theory including the facts that the integral closure of an ideal is an ideal. For a full treatment of this topic, we refer the reader to Huneke and Swanson's book [33].

This Ph.D. thesis traverses two chapters which contribute to the study of multiplicative ideal theoretic properties of integral domains. Let R be a domain and I a nonzero ideal of R. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$ (See a brief summary on the t-operation in Section 1.1). An element $x \in R$ is t-integral over I if there is an equation $x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. The set of all elements that are t-integral over I is called the t-integral closure of I. The first chapter investigates the t-reductions and t-integral closure of ideals. Our objective is to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. Namely, Section 1.2 identifies basic properties of t-reductions of ideals and features explicit examples discriminating between the notions of reduction and t-reduction. Section 1.3 investigates the concept of t-integral closure of ideals, including its correlation with t-reductions. Section 1.4 studies the persistence and contraction of t-integral closure of ideals under ring homomorphisms. All along the chapter, the main results are illustrated with original examples.

An ideal I is t-basic if it has no t-reduction other than the trivial ones. The second chapter investigates t-reductions of ideals in pullback constructions. Section 2.2 examines the correlation between the notions of reduction and t-reduction in pseudo-valuation domains. Section 2.3 solves an open problem on whether the finite t-basic and v-basic ideal properties are distinct. We prove that these two notions coincide in any arbitrary domain. Section 2.4 features the main result, which establishes the transfer of the finite t-basic ideal property to pullbacks in line with Fontana-Gabelli's result on Prüfer v-Multiplication Domains (PvMDs) [16, Theorem 4.1] and Gabelli-Houston's result on v-domains [20, Theorem 4.15]. This allows us to enrich the literature with new families of examples, which put the class of domains subject to the finite t-basic ideal property strictly between the two classes of v-domains and integrally closed domains.

CHAPTER 1

T-REDUCTIONS AND T-INTEGRAL CLOSURE OF IDEALS

This chapter investigates the t-reductions and t-integral closure of ideals. Our objective is to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. This work is accepted for publication in Rocky Mountain Journal of Mathematics under the title "t-Reductions and t-integral closure of ideals" (in collaboration with Dr. S. Kabbaj).

1.1 Introduction

Let R be a domain with quotient field K, I a nonzero fractional ideal of R, and let $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I. The ideal I is a v-ideal (or divisorial) if $I_v = I$ and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$, the set $F_t(R)$ of fractional t-ideals of R is a semigroup with unit R. An invertible element for this operation is called a t-invertible t-ideal of R. Recall that factorial domains, Krull domains, GCDs, and PvMDs can be regarded as t-analogues of the principal domains, Dedekind domains, Bézout domains, and Prüfer domains, respectively. For instance, a domain is Prüfer (resp., a Prüfer v-Multiplication Domain (PvMD)) if every nonzero finitely generated ideal is invertible (resp., t-invertible). For some relevant works on v- and t-operations, we refer the reader to [22, 32, 38, 39, 40, 45, 48, 49, 50].

This chapter investigates the t-reductions and t-integral closure of ideals with the aim to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. Namely, Section 1.2 identifies basic properties of t-reductions of ideals and features explicit examples discriminating between the notions of reduction and t-reduction. Section 1.3 investigates the concept of t-integral closure of ideals, including its correlation with t-reductions. Section 1.4 studies the persistence and contraction of t-integral closure of ideals under ring homomorphisms. All along the chapter, the main results are illustrated with original examples.

1.2 t-Reductions of ideals

This section identifies basic ideal-theoretic properties of the notion of t-reduction including its behavior under localizations. As a prelude to this, we provide explicit examples discriminating between the notions of reduction and t-reduction.

Recall that, in a ring R, a subideal J of an ideal I is called a reduction of I if $JI^n = I^{n+1}$ for some positive integer n [44]. An ideal which has no reduction other than itself is called a basic ideal [25, 26].

Definition 1.1 [cf. [30, Definition 1.1]] Let R be a domain and I a nonzero ideal of R. An ideal $J \subseteq I$ is a *t-reduction* of I if

$$(JI^n)_t = (I^{n+1})_t$$

for some integer $n \geq 0$ (and, a fortiori, the relation holds for $n \gg 0$). The ideal J is a *trivial t-reduction* of I if $J_t = I_t$. The ideal I is *t-basic* if it has no t-reduction other than the trivial t-reductions.

At this point, recall a basic property of the t-operation, which, in fact, holds for any star operation (see definition in Section 2.2) that will be used throughout the chapter. For any two nonzero ideals I and J of a domain, we have

$$(IJ)_t = (I_tJ)_t = (IJ_t)_t = (I_tJ_t)_t.$$

So, obviously, for nonzero ideals $J \subseteq I$, we always have:

$$J \text{ is a } t\text{-reduction of } I \Leftrightarrow J \text{ is a } t\text{-reduction of } I_t \Leftrightarrow J_t \text{ is a } t\text{-reduction of } I_t.$$

Notice also that a reduction is necessarily a t-reduction; and the converse is not true, in general, as shown by the next example which exhibits a domain R with two t-ideals $J \subsetneqq I$ such that J is a t-reduction but not a reduction of I.

Example 1.1 We use a construction from [34]. Let x be an indeterminate over $\mathbb{Z}$ and let

$$\begin{aligned} R &:= \mathbb{Z}[3x, x^2, x^3] \\ I &:= (3x, x^2, x^3) \\ J &:= (3x, 3x^2, x^3, x^4). \end{aligned}$$

Then $J \subsetneqq I$ are two finitely generated t-ideals of R such that:

$$JI^n \subsetneqq I^{n+1} \forall\ n \in \mathbb{N} \ \text{ and } \ (JI)_t = (I^2)_t.$$

Proof. I, being a height-one prime ideal [34], is a t-ideal of R. Next, we prove that J is a t-ideal. We first claim that $J^{-1} = \frac{1}{x}\mathbb{Z}[x]$. Indeed, notice that $\mathbb{Q}(x)$ is the quotient field of R and since $3x \in J$, then $J^{-1} \subseteq \frac{1}{3x}R$. So, let $f := \frac{g}{3x} \in J^{-1}$ where $g = \sum_{i=0}^{m} a_i x^i \in \mathbb{Z}[x]$ with $a_1 \in 3\mathbb{Z}$. Then the fact that $x^3 f \in R$ implies that $a_i \in 3\mathbb{Z}$ for $i = 0, 2, ..., m$; i.e., $g \in 3\mathbb{Z}[x]$. Hence $f \in \frac{1}{x}\mathbb{Z}[x]$, whence $J^{-1} \subseteq \frac{1}{x}\mathbb{Z}[x]$.

The reverse inclusion holds since

$$\frac{1}{x}J\mathbb{Z}[x] = (3, 3x, x^2, x^3)\mathbb{Z}[x] \subseteq R$$

proving the claim. Next, let $h \in (R : \mathbb{Z}[x]) \subseteq R$. Then $xh \in R$ forcing $h(0) \in 3\mathbb{Z}$ and thus $h \in (3, 3x, x^2, x^3)$. So, $(R : \mathbb{Z}[x]) \subseteq (3, 3x, x^2, x^3)$, hence

$$(R : \mathbb{Z}[x]) = \frac{1}{x}J.$$

It follows that

$$\begin{aligned} J_t &= J_v \\ &= \left(R : \tfrac{1}{x}\mathbb{Z}[x]\right) \\ &= x(R : \mathbb{Z}[x]) \\ &= J \end{aligned}$$

as desired.

Next, let $n \in \mathbb{N}$. It is easy to see that $x^3 x^{2n} = x^{2n+3}$ is the monic monomial with the smallest degree in JI^n. Therefore

$$x^{2(n+1)} = x^{2n+2} \in I^{n+1} \setminus JI^n.$$

That is, J is not a reduction of I. It remains to prove $(JI)_t = (I^2)_t$. We first claim that

$$(JI)^{-1} = \frac{1}{x^2}\mathbb{Z}[x].$$

Indeed, we have

$$(JI)^{-1} \subseteq (\mathbb{Z}[x] : JI\mathbb{Z}[x]) = (\mathbb{Z}[x] : x^2(9, 3x, x^3)\mathbb{Z}[x]) = x^{-2}(\mathbb{Z}[x] : (9, 3x, x^3)\mathbb{Z}[x]) = \frac{1}{x^2}\mathbb{Z}[x]$$

and the reverse inclusion holds since

$$\frac{1}{x^2}JI\mathbb{Z}[x] = (3, 3x, x^2, x^3)(3, x, x^2)\mathbb{Z}[x] \subseteq R$$

proving the claim. Now, observe that $I^2 = (9x^2, 3x^3, x^4, x^5)$. It follows that

$$\begin{aligned} (IJ)_t &= (IJ)_v \\ &= \left(R : \tfrac{1}{x^2}\mathbb{Z}[x]\right) \\ &= x^2(R : \mathbb{Z}[x]) \\ &= xJ \\ &\supseteq I^2. \end{aligned}$$

Thus $(IJ)_t \supseteq (I^2)_t$, as desired. □

Observe that the domain R in the above example is not integrally closed. Next, we provide a class of integrally closed domains where the notions of reduction and t-reduction are always distinct. For this purpose, recall that a domain R is said to be completely integrally closed if every nonzero ideal of R is v-invertible (with respect to the ideal v-multiplication $(I, J) \mapsto (IJ)_v$). The domain R is said to be a Mori domain if it satisfies the ascending chain condition on divisorial ideals; and R is a Krull domain if every nonzero ideal of R is t-invertible (with respect to the ideal t-multiplication $(I, J) \mapsto (IJ)_t$).

Example 1.2 Let R be any integrally closed Mori domain that is not completely integrally closed (i.e., not Krull). Then there always exist nonzero ideals $J \subsetneqq I$ in R such that J is a t-reduction but not a reduction of I.

Proof. These domains do exist; for instance, let $k \subsetneqq K$ be a field extension with k algebraically closed and let x be an indeterminate over K. Then, $R := k + xK[x]$ is an integrally closed Mori domain [20, Theorem 4.18] that is not completely integrally closed [24, Lemma 26.5] (see [19, p. 161]).

Now, by [30, Proposition 1.4], there exists a t-ideal A in R that is not t-basic; say, $B \subseteq A$ is a t-reduction of A with $B_t \subsetneqq A_t$. By [6, Theorem 2.1], there exist finitely generated ideals $F \subseteq A$ and $J \subseteq B$ such that $A^{-1} = F^{-1}$ and $B^{-1} = J^{-1}$; yielding $A_t = F_t$ and $B_t = J_t$. Let $I := F + J$. Then, one can easily see that J is a non-trivial t-reduction of I. Finally, we claim that J is not a reduction of I. Deny. Since I is finitely generated, $I \subseteq \overline{J}$ by [33, Corollary 1.2.5]. But, $\overline{J} \subseteq J_t$ by [43, Proposition 2.2]. It follows that $J_t = I_t$, the desired contradiction. $\square$

Another crucial fact concerns reductions of t-ideals. Indeed, if J is a reduction of a t-ideal, then so is J_t; and the converse is not true, in general, as shown by the following example which features a domain R with a t-ideal I and an ideal $J \subseteq I$ such that J_t is a reduction but J is not a reduction of I.

Example 1.3 Let k be a field and let x, y, z be indeterminates over k. Let

$$\begin{aligned} R &:= k[x] + (y,z)k(x)[[y,z]] \\ M &:= (y,z)k(x)[[y,z]] \\ J &:= M^2. \end{aligned}$$

Note that R is a classical pullback issued from the local Noetherian and integrally closed domain $T := k(x)[[y,z]]$. Then M is a divisorial ideal of R by [29, Corollary 5] and clearly, we have

$$\forall\, n \in \mathbb{N},\ M^{n+2} \subsetneqq M^{n+1}.$$

That is, J is not a reduction of M in R. On the other hand, notice that $(M : M) = T$ (since T is integrally closed) and M is not principal in T. Therefore, by [29, Theorem 13], we have

$$\begin{aligned} (R : (R : M^2)) &= (R : (M^{-1} : M)) \\ &= (R : ((M : M) : M)) \\ &= (R : (T : M)) \\ &= (R : M^{-1}) \\ &= M. \end{aligned}$$

So that

$$J_t = J_v = M.$$

Hence, J_t is trivially a reduction of M in R.

In the sequel, R will denote a domain. For convenience, recall that, for any nonzero ideals I, J, H of R, the equality $(IJ+H)_t = (I_tJ+H)_t$ always holds since $I_tJ \subseteq (I_tJ)_t = (IJ)_t \subseteq (IJ+H)_t$. This property will be used in the proof of the next basic result which examines the t-reduction of the sum and product of ideals.

Lemma 1.1 *Let $J \subseteq I$ and $J' \subseteq I'$ be nonzero ideals of R. If J and J' are t-reductions of I and I', respectively, then $J+J'$ is a t-reduction of $I+I'$ and JJ' is a t-reduction of II'.*

Proof. Let n be a positive integer. Then the following implication always holds

$$(JI^n)_t = (I^{n+1})_t \;\Rightarrow\; (JI^m)_t = (I^{m+1})_t \; \forall m \geq n. \tag{1.1}$$

Indeed, multiply the first equation through by I^{m-n} and apply the t-closure to both sides. By (1.1), let m be a positive integer such that

$$(JI^m)_t = (I^{m+1})_t \;\text{ and }\; (J'I'^m)_t = (I'^{m+1})_t. \tag{1.2}$$

By (1.2), we get

$$\begin{aligned}
\left((I+I')^{2m+1}\right)_t &\subseteq \left(I^{m+1}(I+I')^m + I'^{m+1}(I+I')^m\right)_t \\
&= \left((I^{m+1})_t(I+I')^m + (I'^{m+1})_t(I+I')^m\right)_t \\
&= \left((JI^m)_t(I+I')^m + (J'I'^m)_t(I+I')^m\right)_t \\
&= \left(JI^m(I+I')^m + J'I'^m(I+I')^m\right)_t \\
&\subseteq \left((J+J')(I+I')^{2m}\right)_t \\
&\subseteq \left((I+I')^{2m+1}\right)_t
\end{aligned}$$

and then equality holds throughout, proving the first statement. The proof of the second statement is straightforward via (1.2). □

The next basic result examines the transitivity for t-reduction.

Lemma 1.2 *Let $K \subseteq J \subseteq I$ be nonzero ideals of R. Then:*

(1) *If K is a t-reduction of J and J is a t-reduction of I, then K is a t-reduction of I.*

(2) *If K is a t-reduction of I, then J is a t-reduction of I.*

Proof. For any positive integer m, we always have

$$(JI^m)_t = (I^{m+1})_t \;\Rightarrow\; (J^nI^m)_t = (I^{m+n})_t \;\forall n \geq 1. \tag{1.3}$$

Indeed, multiply the first equation through by J^{n-1}, apply the t-closure to both sides, and conclude by induction on n. Let

$$(KJ^n)_t = (J^{n+1})_t \text{ and } (JI^m)_t = (I^{m+1})_t$$

for some positive integers n and m. By (1.3), we get

$$\begin{aligned}
(I^{m+n+1})_t &= (J^{n+1}I^m)_t \\
&= \left((J^{n+1})_t I^m\right)_t \\
&= \left((KJ^n)_t I^m\right)_t \\
&= (KI^{m+n})_t
\end{aligned}$$

proving (a). The proof of (b) is straightforward. □

The next basic result examines the t-reduction of the power of an ideal.

Lemma 1.3 *Let $J \subseteq I$ be nonzero ideals of R and let n be a positive integer. Then:*

(1) *J is a t-reduction of I if and only if J^n is a t-reduction of I^n.*

(2) *If $J = (a_1, ..., a_k)$, then: J is a t-reduction of I if and only if $(a_1^n, ..., a_k^n)$ is a t-reduction of I^n.*

Proof. (a) The "only if" implication holds by Lemma 1.1. For the converse, suppose $(J^nI^{nm})_t = (I^{nm+n})_t$ for some positive integer m. Then

$$(I^{nm+n})_t = (JJ^{n-1}I^{nm})_t \subseteq (JI^{nm+n-1})_t \subseteq (I^{nm+n})_t$$

and so equality holds throughout, as desired.

(b) Assume that J is a t-reduction of I. From [33, (8.1.6)], we always have the following equality

$$\left(a_1^n, ..., a_k^n\right)\left(a_1, ..., a_k\right)^{(k-1)(n-1)} = \left(a_1, ..., a_k\right)^{(n-1)k+1} \tag{1.4}$$

and, multiplying (1.4) through by J^{k-1}, we get

$$\left(a_1^n, ..., a_k^n\right) J^{nk-n} = J^{nk}.$$

Therefore $(a_1^n, ..., a_k^n)$ is a t-reduction of J^n and a fortiori of I^n by (a) and Proposition 1.2. The converse holds by (a) and Proposition 1.2. □

The next basic result examines the t-reduction of localizations.

Lemma 1.4 *Let $J \subseteq I$ be nonzero ideals of R and let S be a multiplicatively closed subset of R. If J is a t-reduction of I, then $S^{-1}J$ is a t-reduction of $S^{-1}I$.*

Proof. Assume that $(JI^n)_t = (I^{n+1})_t$ for some positive integer n. Let t_1 denote the t-operation with respect to $S^{-1}R$. By [40, Lemma 3.4], we have:

$$\begin{aligned}
((S^{-1}I)^{n+1})_{t_1} &= (S^{-1}(I^{n+1}))_{t_1} \\
&= (S^{-1}((I^{n+1})_t))_{t_1} \\
&= (S^{-1}((JI^n)_t))_{t_1} \\
&= (S^{-1}(JI^n))_{t_1} \\
&= ((S^{-1}J)(S^{-1}I)^n)_{t_1}.
\end{aligned}$$

proving the lemma. $\square$

It is worthwhile noting here that, in a PvMD, J is a t-reduction of I if and only if J is t-locally a reduction of I; i.e., JR_M is a reduction of IR_M for every maximal t-ideal M of R [30, Lemma 2.2].

1.3 t-Integral closure of ideals

This section investigates the concept of t-integral closure of ideals and its correlation with t-reductions. Our objective is to establish satisfactory t-analogues of (and in some cases generalize) well-known results, in the literature, on the integral closure of ideals and its correlation with reductions.

Definition 1.2 Let R be a domain and I a nonzero ideal of R. An element $x \in R$ is t-integral over I if there is an equation

$$x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0 \text{ with } a_i \in (I^i)_t \ \forall i = 1, ..., n.$$

The set of all elements that are t-integral over I is called the t-integral closure of I, and is denoted by $\widetilde{I}$. If $I = \widetilde{I}$, then I is called t-integrally closed.

Recall that a domain R is called a v-domain if all its nonzero finitely generated ideals are v-invertible (with respect to the ideal v-multiplication $(I, J) \mapsto (IJ)_v$); equivalently, if the t-integral closure of the ring R (also called pseudo-integral closure) is equal to R. Notice, at this point, that the t-integral closure of the ideal

R is always R, whereas the t-integral closure of the ring R may be larger than R; e.g., consider any non v-domain [3, 19]. Also, we have

$$J \subseteq I \Rightarrow \widetilde{J} \subseteq \widetilde{I}.$$

More ideal-theoretic properties are provided in Remark 1.7.

It is well-known that the integral closure of an ideal is an ideal which is integrally closed [33, Corollary 1.3.1]. Next, we establish a t-analogue for this result.

Theorem 1.4 *The t-integral closure of an ideal is an integrally closed ideal. In general, it is not t-closed and, a fortiori, not t-integrally closed.*

The proof of this theorem relies on the following lemma which sets a t-analogue for the notion of Rees algebra of an ideal [33, Chapter 5]. Recall, for convenience, that the Rees algebra of an ideal I (in a ring R) is the graded subring of $R[x]$ given by

$$R[Ix] := \bigoplus_{n \geq 0} I^n \, x^n$$

[33, Definition 5.1.1] and whose integral closure in $R[x]$ is the graded ring

$$\overline{R[Ix]} = \bigoplus_{n \geq 0} \overline{I^n} \, x^n$$

[33, Proposition 5.2.1].

Lemma 1.5 *Let R be a domain, I a t-ideal of R, and x an indeterminate over*

R. Let

$$R_t[Ix] := \bigoplus_{n\geq 0} (I^n)_t x^n.$$

Then $R_t[Ix]$ is a graded subring of $R[x]$ and its integral closure in $R[x]$ is the graded ring

$$\overline{R_t[Ix]} = \bigoplus_{n\geq 0} \widetilde{I^n} x^n.$$

Proof. That $R_t[Ix]$ is $\mathbb{N}$-graded follows from the fact that

$$(I^i)_t \cdot (I^j)_t \subseteq (I^{i+j})_t, \forall i, j \in \mathbb{N}.$$

By [33, Theorem 2.3.2], $\overline{R_t[Ix]}$ is an $\mathbb{N}$-graded ring. Let $k \in \mathbb{N}$ and let S_k denote the homogeneous component of $\overline{R_t[Ix]}$ of degree k. We shall prove that $S_k = \widetilde{I^k} x^k$. Let $s := s_k x^k \in S_k$, for some $s_k \in R$. Then

$$s^n + a_1 s^{n-1} + \cdots + a_n = 0$$

for some positive integer n and $a_i \in R_t[Ix]$, $i = 1, \ldots, n$. Expanding each $a_i = \sum_{j=0}^{k_i} a_{i,j} x^j$ with $a_{i,j} \in (I^j)_t$, the coefficient of the monomial of degree kn in the above equation is

$$s_k^n + \sum_{i=1}^{n} a_{i,ik} s_k^{n-i} = 0$$

with $a_{i,ik} \in (I^{ik})_t$. It follows that $s_k \in \widetilde{I^k}$ and thus $S_k \subseteq \widetilde{I^k}x^k$. For the reverse inclusion, let $z_k := y_k x^k \in \widetilde{I^k}x^k$, for some $y_k \in \widetilde{I^k}$. Then

$$y_k^n + a_1 y_k^{n-1} + \cdots + a_n = 0$$

for some positive integer n and $a_j \in (I^{kj})_t$, $j = 1, \ldots, n$. Multiplying through by x^{kn} yields the equation

$$z_k^n + a_1 x^k z_k^{n-1} + \cdots + a_n x^{kn} = 0$$

with

$$a_j x^{kj} \in (I^{kj})_t x^{kj} \subseteq R_t[Ix]$$

for $j = 1, \ldots, n$. That is, $z_k \in \overline{R_t[Ix]}$. But z_k is homogeneous of degree k in $\overline{R_t[Ix]}$. Therefore $z_k \in S_k$ and hence $\widetilde{I^k}x^k \subseteq S_k$, completing the proof of the lemma. □

Definition 1.3 The t-Rees algebra of an ideal I (in a domain R) is the graded subring of $R[x]$ given by

$$R_t[Ix] := \bigoplus_{n \geq 0} (I^n)_t x^n.$$

Proof of Theorem 1.4 Let R be a domain and I a nonzero ideal of R. Since $\widetilde{I} = \widetilde{I_t}$, we may assume I to be a t-ideal. We first prove that $\widetilde{I}$ is an ideal. Clearly, $\widetilde{I}$ is closed under multiplication. Next, we show that $\widetilde{I}$ is closed under addition.

Let $a, b \in \widetilde{I}$. Then, by Lemma 1.5, ax and $bx \in \overline{R_t[Ix]}$. Hence

$$ax + bx = (a+b)x \in \overline{R_t[Ix]}.$$

Again, by Lemma 1.5, $a + b \in \widetilde{I}$, as desired. Next, we prove that $\widetilde{I}$ is integrally closed. For this purpose, observe that, $\forall n \in \mathbb{N}$, $(S_1)^n \subseteq S_n$, forcing

$$\left(\widetilde{I}\right)^n \subseteq \widetilde{I^n} \ \forall n \in \mathbb{N}. \tag{1.5}$$

Consider the Rees algebra of the ideal $\widetilde{I}$

$$R[\widetilde{I}x] = \bigoplus_{n \geq 0} \left(\widetilde{I}\right)^n x^n.$$

Therefore $R[\widetilde{I}x] \subseteq \overline{R_t[Ix]}$ and hence

$$\overline{R[\widetilde{I}x]} \subseteq \overline{R_t[Ix]}.$$

Now, a combination of Lemma 1.5 and [33, Proposition 5.2.1] yields

$$\bigoplus_{n \geq 0} \overline{\left(\widetilde{I}\right)^n} x^n \subseteq \bigoplus_{n \geq 0} \widetilde{I^n} x^n.$$

In particular, $\overline{\widetilde{I}} \subseteq \widetilde{I}$; that is, $\widetilde{I}$ is integrally closed. The proof of the last statement of the theorem is handled by Example 1.9(b), where we provide a domain with an ideal I such that $\widetilde{I} \subsetneqq (\widetilde{I})_t$. That is, $\widetilde{I}$ is not a t-ideal and, hence, not t-integrally closed since $(\widetilde{I}\,)_t \subseteq \widetilde{\widetilde{I}}$ always holds. $\square$

The next result shows that the t-integral closure collapses to the t-closure in the class of integrally closed domains. It also completes two existing results in the literature on the integral closure of ideals (Gilmer [24] and Mimouni [43]).

Theorem 1.5 *Let R be a domain. The following assertions are equivalent:*

(1) *R is integrally closed;*

(2) *Every principal ideal of R is integrally closed;*

(3) *Every t-ideal of R is integrally closed;*

(4) *$\overline{I} \subseteq I_t$ for each nonzero ideal I of R;*

(5) *Every principal ideal of R is t-integrally closed;*

(6) *Every t-ideal of R is t-integrally closed;*

(7) *$\widetilde{I} = I_t$ for each nonzero ideal I of R.*

Proof. (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) are handled by [24, Lemma 24.6] and [43, Proposition 2.2], respectively. Also, (g) $\Leftrightarrow$ (f) $\Rightarrow$ (e) $\Rightarrow$ (b) are straightforward. So, it remains to prove (a) $\Rightarrow$ (g). Assume R is integrally closed and let I be a nonzero ideal of R. The inclusion $I_t \subseteq \widetilde{I}$ holds in any domain. Next, let $\alpha \in \widetilde{I}$.

Claim 1.1 *There exists a finitely generated ideal $J \subseteq I$ such that $\alpha \in \widetilde{J}$.*

Indeed, α satisfies an equation of the form

$$\alpha^n + a_1\alpha^{n-1} + ... + a_n = 0$$

with $a_i \in (I^i)_t$ $\forall i = 1, ..., n$. Now, let $i \in \{1, ..., n\}$. Hence, there exists a finitely generated ideal $F_i \subseteq I^i$ such that $a_i \in F_{iv}$. Further, each generator of F_i is a finite combination of elements of the form

$$\prod_{1 \leq j \leq i} c_j \in I^i.$$

Let J denote the subideal of I generated by all c_j's emanating from all F_i's. Clearly, $a_i \in (J^i)_t$ $\forall i = 1, ..., n$. That is, $\alpha \in \widetilde{J}$, proving the claim.

Claim 1.2 $\widetilde{J} \subseteq J_t$.

Indeed, we first prove that $J^{-1} = (\ \widetilde{J}\ )^{-1}$. Clearly, $(\ \widetilde{J}\ )^{-1} \subseteq J^{-1}$. For the reverse inclusion, let $x \in J^{-1}$ and $y \in \widetilde{J}$. Then y satisfies an equation of the form

$$y^n + a_1 y^{n-1} + ... + a_n = 0$$

with $a_i \in (J^i)_t$ $\forall i = 1, ..., n$. It follows that

$$(yx)^n + a_1 x(yx)^{n-1} + \cdots + a_n x^n = 0$$

such that

$$\begin{aligned} a_i x^i &\in (J^i)_t (J^{-1})^i \\ &\subseteq (J^i)_t (J^i)^{-1} \\ &= (J^i)_t ((J^i)_t)^{-1} \\ &\subseteq R. \end{aligned}$$

Hence $yx \in R$. Thus, $x \in (\widetilde{J})^{-1}$, as desired. Therefore

$$\widetilde{J} \subseteq (\widetilde{J})_v = J_v = J_t$$

proving the claim. Now, by the above claims, we have

$$\alpha \in \widetilde{J} \subseteq J_t \subseteq I_t.$$

Consequently, $\widetilde{I} = I_t$, completing the proof of the theorem. □

In case all ideals of a domain are t-integrally closed, then it must be Prüfer. This is a well-known result in the literature:

Corollary 1.6 ([24, Theorem 24.7]) *A domain R is Prüfer if and only if every ideal of R is (t-)integrally closed.*

Now, we examine the correlation between the t-integral closure and t-reductions of ideals. Recall that, for the trivial operation, two crucial results assert that

$$x \in \overline{I} \Leftrightarrow I \text{ is a reduction of } I + Rx \text{ [33, Corollary 1.2.2]}$$

and if I is finitely generated and $J \subseteq I$, then:

$$I \subseteq \overline{J} \Leftrightarrow J \text{ is a reduction of } I \text{ [33, Corollary 1.2.5].}$$

Next, we establish t-analogues of these two results.

Proposition 1.1 *Let R be a domain and let $J \subseteq I$ be nonzero ideals of R.*

(1) $x \in \widetilde{I} \Rightarrow I$ *is a t-reduction of $I + Rx$.*

(2) *Assume I is finitely generated. Then: $I \subseteq \widetilde{J} \Rightarrow J$ is a t-reduction of I.*

Moreover, both implications are irreversible in general.

Proof. (a) Let $x \in \widetilde{I}$. Then, $x^n + a_1x^{n-1} + \cdots + a_n = 0$ for some $a_i \in (I^i)_t$ for each $i \in \{1, \ldots, n\}$. Hence

$$\begin{aligned} x^n &\in I_tx^{n-1} + \cdots + (I^n)_t \\ &\subseteq (I_tx^{n-1} + \cdots + (I^n)_t)_t \\ &\subseteq \left(I(I + Rx)^{n-1}\right)_t. \end{aligned}$$

It follows that

$$(I + Rx)^n \subseteq \left(I(I + Rx)^{n-1}\right)_t.$$

Hence

$$((I + Rx)^n)_t = \left(I(I + Rx)^{n-1}\right)_t.$$

Thus, I is a t-reduction of $I + Rx$.

(b) Assume $I = (a_1, \ldots, a_n)$, for some integer $n \geq 1$ and $a_i \in R\ \forall i = 1, \ldots, n$. Suppose that $I \subseteq \widetilde{J}$. By (a), J is a t-reduction of $J + Ra_i$, for each $i \in \{1, \ldots, n\}$. By Lemma 1.1, J is a t-reduction of $J + (a_1, \ldots, a_n) = I$, as desired.

The converse of (a) is not true, in general, as shown by Example 1.9(a). Also, (b) can be irreversible even with I and J both being finitely generated. For instance, consider the integrally closed domain R of Example 1.2 with two ideals $J \subsetneqq I$, where J is a non-trivial t-reduction of I (i.e., $J_t \subsetneqq I_t$). By Theorem 1.5, $\widetilde{J} = J_t \not\supseteq I$. $\square$

Next, we collect some ideal-theoretic properties of the integral closure of ideals.

Remark 1.7 Let R be a domain and let I, J be nonzero ideals of R. Then:

(1) $I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}$. Example 1.8(a) features a t-ideal for which these three containments are strict. However, note that radical (and, a fortiori, prime) t-deals are necessarily t-integrally closed.

(2) $\widetilde{I \cap J} \subseteq \widetilde{I} \cap \widetilde{J}$. The inclusion can be strict, for instance, in any integrally closed domain that is not a PvMD by [1, Theorem 6] and Theorem 1.5. Another example is provided in the non-integrally closed case by Example 1.8(c).

(3) $\widetilde{I} + \widetilde{J} \subseteq \widetilde{I + J}$. The inclusion can be strict. For instance, in $\mathbb{Z}[x]$, we have $\widetilde{(2)} + \widetilde{(x)} = (2, x)$ and $(2, x)^{-1} = \mathbb{Z}[x]$ so that $\widetilde{(2, x)} = (2, x)_t = \mathbb{Z}[x]$ (via Theorem 1.5).

(4) By (1.5), $\forall\ n \geq 1$, $(\ \widetilde{I}\ )^n \subseteq \widetilde{I^n}$. The inclusion can be strict, as shown by Example 1.8(b).

(5) $\forall\ x \in R$, $x\ \widetilde{I} \subseteq \widetilde{xI}$. Indeed, let $y \in x\ \widetilde{I}$. Then, there is an equation of the form $y^n + (xa_1)y^{n-1} + \cdots + x^n a_n = 0$ with $x^i a_i \in x^i(I^i)_t = ((xI)^i)_t$, $i =$

$1, \ldots, n$. Hence, $y \in \widetilde{xI}$. Note that $x\,\widetilde{I} = \widetilde{xI}$, $\forall\, x \in R$ and $\forall\, I$ ideal $\Leftrightarrow$ R is integrally closed (Theorem 1.5).

We close this section by the two announced examples.

Example 1.8 Let x be an indeterminate over $\mathbb{Z}$ and let

$$\begin{aligned} R &:= \mathbb{Z}[\sqrt{-3}][2x, x^2, x^3] \\ I &:= (2x^2, 2x^3, x^4, x^5) \\ J &:= (x^3). \end{aligned}$$

Then I is a t-ideal of R such that

(1) $I \subsetneqq \overline{I} \subsetneqq \widetilde{I} \subsetneqq \sqrt{I}$.

(2) $(\widetilde{I})^2 \subsetneqq \widetilde{I^2}$.

(3) $\widetilde{J \cap I} \subsetneqq \widetilde{J} \cap \widetilde{I}$.

Proof. We first show that I is a t-ideal. Clearly, $\frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x] \subseteq I^{-1}$. For the reverse inclusion, let $f \in I^{-1} \subseteq x^{-4}R$. Then

$$f = \frac{1}{x^4}(a_0 + a_1 x + \cdots + a_n x^n)$$

for some $n \in \mathbb{N}$, $a_0 \in \mathbb{Z}[\sqrt{-3}]$, $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$, and $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \geq 2$. Since $2x^2 f \in R$, then $a_0 = a_1 = 0$. It follows that $f \in \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Therefore $I^{-1} = \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Next, let $g \in (R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq R$. Then $xg \in R$, forcing

$g(0) \in 2\mathbb{Z}[\sqrt{-3}]$ and hence $g \in (2, 2x, x^2, x^3)$. So

$$(R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq (2, 2x, x^2, x^3).$$

The reverse inclusion is obvious. Thus, $(R : \mathbb{Z}[\sqrt{-3}][x]) = (2, 2x, x^2, x^3)$. Consequently, we obtain

$$\begin{aligned} I_t &= I_v \\ &= (R : \tfrac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]) \\ &= x^2(R : \mathbb{Z}[\sqrt{-3}][x]) \\ &= I. \end{aligned}$$

(a) Next, we prove the strict inclusions

$$I \subsetneqq \overline{I} \subsetneqq \widetilde{I} \subsetneqq \sqrt{I}.$$

For $I \subsetneqq \overline{I}$, notice that $(1 + \sqrt{-3})x^2 \in \overline{I} \setminus I$ as $\big((1 + \sqrt{-3})x^2\big)^3 = -8x^6 \in I^3$ and $1 + \sqrt{-3} \notin 2\mathbb{Z}[\sqrt{-3}]$.

For $\overline{I} \subsetneqq \widetilde{I}$, we claim that $x^3 \in \widetilde{I} \setminus \overline{I}$. Indeed, let $f \in (I^2)^{-1} \subseteq x^{-8}R$. Then there are $n \in \mathbb{N}$, $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \in \{0, 2, \dots, n\}$, and $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$ such that

$$f = \frac{1}{x^8}(a_0 + a_1 x + \cdots + a_n x^n).$$

Since $4x^4f \in R$, then $a_0 = a_1 = a_2 = a_3 = 0$. Therefore, $(I^2)^{-1} \subseteq \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. The reverse inclusion is obvious. Hence, $(I^2)^{-1} = \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. It follows that

$$\begin{aligned}
(I^2)_t &= (I^2)_v \\
&= (R : \tfrac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]) \\
&= x^4(R : \mathbb{Z}[\sqrt{-3}][x]) \\
&= x^2 I.
\end{aligned}$$

Hence $x^6 \in (I^2)_t$ and thus $x^3 \in \widetilde{I}$. It remains to show that $x^3 \notin \overline{I}$. By [33, Corollary 1.2.2], it suffices to show that I is not a reduction of $I + (x^3)$. Let $n \in \mathbb{N}$. It is easy to see that x^4x^{3n} is the monic monomial with the smallest degree in $I\big(I + (x^3)\big)^n$. Therefore

$$x^{3(n+1)} = x^{3n+3} \in \big(I + (x^3)\big)^{n+1} \setminus I\big(I + (x^3)\big)^n.$$

Hence, I is not a reduction of $I + (x^3)$, as desired.

For $\widetilde{I} \subsetneqq \sqrt{I}$, we claim that $x^2 \in \sqrt{I} \setminus \widetilde{I}$. Obviously, $x^2 \in \sqrt{I}$. In order to prove that $x^2 \notin \widetilde{I}$, it suffices by Proposition 1.1 to show that I is not a t-reduction of $I + (x^2)$. To this purpose, notice that $I + (x^2) = (x^2)$. Suppose by way of contradiction that

$$(I(I + (x^2))^n)_t = ((I + (x^2))^{n+1})_t$$

for some $n \in \mathbb{N}$. Then

$$(x^2)^{n+1} = x^{2n+2} \in (I(I + (x^2))^n)_t = x^{2n} I.$$

Consequently, $x^2 \in I$, absurd.

(b) We first prove that

$$\widetilde{I} = (2x^2, (1+\sqrt{-3})x^2, x^3, x^4).$$

In view of (a) and its proof, we have

$$(2x^2, (1+\sqrt{-3})x^2, x^3, x^4) \subseteq \widetilde{I}.$$

Next, let $\alpha := (a + b\sqrt{-3})x^2 \in \widetilde{I}$ where $a, b \in \mathbb{Z}$. If $b = 0$, then $a \neq 1$ as $x^2 \notin \widetilde{I}$. Moreover, since $2x^2 \in \widetilde{I}$, a must be even; that is, $\alpha \in (2x^2)$. Now assume $b \neq 0$. If $a = 0$, then $b \neq 1$ as $\sqrt{-3}x^2 \notin \widetilde{I}$. Moreover, since $2\sqrt{-3}x^2 \in \widetilde{I}$, b must be even; that is, $\alpha \in (2x^2)$. So suppose $a \neq 0$. Then similar arguments force a and b to be of the same parity. Further, if a and b are even, then $\alpha \in (2x^2)$; and if a and b are odd, then $\alpha \in (2x^2, (1+\sqrt{-3})x^2)$. Finally, we claim that $\widetilde{I}$ contains no monomials of degree 1. Deny and let $ax \in \widetilde{I}$, for some nonzero $a \in 2\mathbb{Z}[\sqrt{-3}]$.

Then, by [33, Remark 1.1.3(7)], we obtain

$$\begin{aligned} ax &\in \widetilde{I} \\ &\subseteq \widetilde{(x^2)} \\ &= \overline{(x^2)} \\ &\subseteq \overline{x^2\mathbb{Z}[\sqrt{-3}][x]}. \end{aligned}$$

By [33, Corollary 1.2.2], (x^2) is a reduction of (ax, x^2) in $\mathbb{Z}[\sqrt{-3}][x]$, absurd. Consequently, $\widetilde{I} = (2x^2, (1+\sqrt{-3})x^2, x^3, x^4)$. Now, we are ready to check that

$$(\widetilde{I}\,)^2 \subsetneqq \widetilde{I^2}.$$

For this purpose, recall that $(I^2)_t = x^2 I$. So, $2x^4 \in \widetilde{I^2}$. We claim that $2x^4 \notin (\widetilde{I}\,)^2$. Deny. Then, $2x^4 \in (4x^4, 2(1+\sqrt{-3})x^4)$, which yields $x^2 \in (2x^2, (1+\sqrt{-3})x^2) \subseteq \widetilde{I}$, absurd.

(c) We claim that

$$x^3 \in \widetilde{I} \cap \widetilde{J} \setminus \widetilde{I \cap J}.$$

We proved in (a) that $x^3 \in \widetilde{I}$. So, $x^3 \in \widetilde{I} \cap \widetilde{J}$. Now, observe that $I \cap J = xI$ and assume, by way of contradiction, that $x^3 \in \widetilde{I \cap J} = \widetilde{xI}$. Then x^3 satisfies an equation of the form

$$(x^3)^n + a_1(x^3)^{n-1} + \cdots + a_n = 0$$

with

$$a_i \in ((xI)^i)_t = x^i(I^i)_t, \ i = 1, \dots, n.$$

For each i, let $a_i = x^i b_i$, for some $b_i \in (I^i)_t$. Therefore

$$(x^2)^n + b_1(x^2)^{n-1} + \cdots + b_n = 0.$$

It follows that $x^2 \in \widetilde{I}$, the desired contradiction. $\square$

Example 1.9 Let x be an indeterminate over $\mathbb{Q}$ and let

$$\begin{aligned} R &:= \mathbb{Z} + x\mathbb{Q}(\sqrt{2})[x] \\ I &:= (\tfrac{x}{\sqrt{2}}) \\ a &:= \tfrac{x}{2}. \end{aligned}$$

Then:

(1) I is a t-reduction of $I + aR$ and $a \notin \widetilde{I}$.

(2) $\widetilde{I} \subsetneqq (\,\widetilde{I}\,)_t$ and hence $\widetilde{I} \subsetneqq \widetilde{\widetilde{I}}$.

Proof. (a) First, we prove that

$$(I(I + aR))_t = ((I + aR)^2)_t.$$

It suffices to show that $a^2 \in (I(I+aR))_t$. For this purpose, let

$$\begin{aligned} f &\in (I(I+aR))^{-1} \\ &= (\tfrac{x^2}{2}, \tfrac{x^2}{2\sqrt{2}})^{-1} \\ &\subseteq (\tfrac{x^2}{2})^{-1} \\ &= \tfrac{2}{x^2}R. \end{aligned}$$

Then, $f = \frac{2}{x^2}(a_0 + a_1x + \ldots + a_nx^n)$, for some $n \geq 0$, $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{Q}(\sqrt{2})$ for $i \geq 1$. Since $\frac{x^2}{2\sqrt{2}}f \in R$, $a_0 = 0$. It follows that

$$(I(I+aR))^{-1} \subseteq \frac{1}{x}\mathbb{Q}(\sqrt{2})[x].$$

On the other hand, we have

$$(I(I+aR))(\frac{1}{x}\mathbb{Q}(\sqrt{2})[x]) \subseteq R$$

so that

$$\left(I(I+aR)\right)^{-1} = \left(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}}\right)^{-1} = \frac{1}{x}\mathbb{Q}(\sqrt{2})[x] \tag{1.6}$$

Now, clearly, $a^2(I(I+aR))^{-1} \subseteq R$. Therefore

$$a^2 \in (I(I+aR))_v = (I(I+aR))_t$$

as desired.

Next, we prove that

$$a \notin \widetilde{I} = \overline{I}.$$

By [33, Corollary 1.2.2], it suffices to show that I is not a reduction of $I + aR$. Deny and suppose that $I(I + aR)^n = (I + aR)^{n+1}$, for some positive integer n. Then

$$a^{n+1} = (\frac{x}{2})^{n+1} \in I(I + aR)^n = \frac{x}{\sqrt{2}}(\frac{x}{\sqrt{2}}, \frac{x}{2})^n.$$

One can check that this yields

$$1 \in \sqrt{2}(\sqrt{2}, 1)^n \subseteq (\sqrt{2})$$

in $\mathbb{Z}[\sqrt{2}]$, the desired contradiction.

(b) We claim that $a \in (\ \widetilde{I}\ )_t$. Notice first that $x \in \widetilde{I}$ as $x^2 \in I^2 = (I^2)_t$. Therefore, $A := (x, \frac{x}{\sqrt{2}}) \subseteq \widetilde{I}$. Clearly, $A = \frac{2}{x}(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}})$. Hence, by (1.6), $A^{-1} = \mathbb{Q}(\sqrt{2})[x]$. However, $aA^{-1} \subseteq R$. Whence, $a \in A_v = A_t \subseteq (\ \widetilde{I}\ )_t$. Consequently, $a \in (\ \widetilde{I}\ )_t \setminus \widetilde{I}$. □

1.4 Persistence and contraction of t-integral closure

Recall that the persistence and contraction of integral closure describe, respectively, the facts that for any ring homomorphism $\varphi : R \to T$, we have:

- $\varphi(\overline{I}\,) \subseteq \overline{\varphi(I)T}$ for every ideal I of R

- $\overline{\varphi^{-1}(J)} = \varphi^{-1}(J)$ for every integrally closed ideal J of T.

This section studies the persistence and contraction of t-integral closure. To this purpose, we first introduce the concept of t-compatible homomorphism which extends the well-known notion of t-compatible extension [2]. Throughout, we denote by t (resp. t_1) and v (resp. v_1) the t- and v- closures in R (resp., T).

Lemma 1.6 *Let $\varphi : R \longrightarrow T$ be a homomorphism of domains. Then, the following statements are equivalent:*

(1) $\varphi(I_v)T \subseteq \left(\varphi(I)T\right)_{v_1}$*, for each nonzero finitely generated ideal I of R;*

(2) $\varphi(I_t)T \subseteq \left(\varphi(I)T\right)_{t_1}$*, for each nonzero ideal I of R;*

(3) $\varphi^{-1}(J)$ *is a t-ideal of R for each t_1-ideal J of T such that $\varphi^{-1}(J) \neq 0$.*

Proof. (a) $\Rightarrow$ (c) Let J be a t_1-ideal of T and let A be any finitely generated ideal of R contained in $\varphi^{-1}(J)$. Then, $\varphi(A)T \subseteq J = J_{t_1}$. Further, $\varphi(A)T$ is finitely generated. Hence, $\left(\varphi(A)T\right)_{v_1} \subseteq J$. It follows, via (a), that

$$\varphi(A_v)T \subseteq \left(\varphi(A)T\right)_{v_1} \subseteq J.$$

Therefore, $A_v \subseteq \varphi^{-1}(J)$ and thus $\varphi^{-1}(J)$ is a t-ideal.

(c) $\Rightarrow$ (b) Let I be a nonzero ideal of R. The ideal $J := \left(\varphi(I)T\right)_{t_1}$ is clearly a t_1-ideal of T with $\varphi^{-1}(J) \neq 0$. By (c), $\varphi^{-1}(J)$ is a t-ideal of R. Consequently, we obtain

$$\begin{aligned} I_t &\subseteq \Big(\varphi^{-1}(\varphi(I)T)\Big)_t \\ &\subseteq \Big(\varphi^{-1}\big(\varphi(I)T\big)_{t_1}\Big)_t \\ &= \big(\varphi^{-1}(J)\big)_t \\ &= \varphi^{-1}(J). \end{aligned}$$

So that $\varphi(I_t)T \subseteq J = \big(\varphi(I)T\big)_{t_1}$, as desired.

(b) $\Rightarrow$ (a) Trivial. $\square$

Definition 1.4 A homomorphism of domains $\varphi : R \longrightarrow T$ is called t-compatible if it satisfies the equivalent conditions of Lemma 1.6.

When φ denotes the natural embedding $R \subseteq T$, this definition matches the notion of t-compatible extension (i.e., $I_tT \subseteq (IT)_{t_1}$ for every ideal I of R) well studied in the literature [2, 7, 11, 16].

Next, we announce the main result of this section which establishes persistence and contraction of t-integral closure under t-compatible homomorphisms.

Proposition 1.2 *Let $\varphi : R \longrightarrow T$ be a t-compatible homomorphism of domains, I an ideal of R, and J an ideal of T. Then:*

(1) $\varphi(\ \widetilde{I}\ )T \subseteq \widetilde{\varphi(I)T}$.

(2) $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\ \widetilde{J}\ )$. *Moreover, if J is t-integrally closed, then*

$$\widetilde{\varphi^{-1}(J)} = \varphi^{-1}(J).$$

Proof. (a) Let $x \in \widetilde{I}$, $y := \varphi(x)$, and $z \in T$. We shall prove that $yz \in \widetilde{\varphi(I)T}$. Suppose that x satisfies the equation $x^n + a_1x^{n-1} + ... + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. Then, apply φ to this equation and multiply through by z^n to obtain

$$(yz)^n + b_1z(yz)^{n-1} + ... + b_{n-1}z^{n-1}(yz) + b_nz^n = 0$$

where, by t-compatibility, we get

$$\begin{aligned} b_i &:= \varphi(a_i) \\ &\in \varphi((I^i)_t)T \\ &\subseteq (\varphi(I^i)T)_{t_1} \\ &= \left((\varphi(I)T)^i\right)_{t_1}. \end{aligned}$$

Hence $b_iz^i \in \left((\varphi(I)T)^i\right)_{t_1}$, for $i = 1, ..., n$. Consequently, $yz \in \widetilde{\varphi(I)T}$.

(b) Let $H := \varphi(\varphi^{-1}(J))T$. Then, by (a), we have

$$\varphi(\widetilde{\varphi^{-1}(J)})T \subseteq \widetilde{H} \subseteq \widetilde{J}.$$

It follows that $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J})$, as desired. Now, if J is t-integrally closed, then

$$\begin{aligned} \widetilde{\varphi^{-1}(J)} &\subseteq \varphi^{-1}(\widetilde{J}) \\ &= \varphi^{-1}(J) \\ &\subseteq \widetilde{\varphi^{-1}(J)}. \end{aligned}$$

and hence the equality holds. $\square$

In the special case when both R and T are integrally closed, persistence of t-integral closure coincides with t-compatibility by Theorem 1.5. This shows that the t-compatibility assumption in Proposition 1.2 is imperative.

Corollary 1.10 *Let $R \subseteq T$ be a t-compatible extension of domains and I an ideal of R. Then:*

(1) $\widetilde{I}T \subseteq \widetilde{IT}$.

(2) $\widetilde{I} \subseteq \widetilde{\widetilde{I}T \cap R} \subseteq \widetilde{IT} \cap R$.

Moreover, the above inclusions are strict in general.

Proof. (a) and (b) are direct consequences of Proposition 1.2. The inclusion in (a) and second inclusion in (b) can be strict as shown by Example 1.12. The first inclusion in (b) can also be strict. For instance, let R be an integrally closed domain and let $P \subsetneqq Q$ be prime ideals of R with $x \in Q \setminus P$. Then $\widetilde{(x)} = (x)$ by Theorem 1.5. While

$$x\widetilde{R_P \cap R} = \widetilde{R_P \cap R} = R.$$

That is, $\widetilde{(x)} \subsetneqq (x)\widetilde{R_P \cap R}$. $\square$

Corollary 1.11 *Let R be a domain, I an ideal of R, and S a multiplicatively closed subset of R. Then $S^{-1}\widetilde{I} \subseteq \widetilde{S^{-1}I}$.*

Proof. It is well-known that flatness implies t-compatibility [16, Proposition 0.6]. Hence, Corollary 1.10 leads to the conclusion. $\square$

For the integral closure, we always have $S^{-1}\overline{I} = \overline{S^{-1}I}$ [33, Proposition 1.1.4]. But in the above corollary the inclusion can be strict, as shown by the following example.

Example 1.12 We use a construction due to Zafrullah [48]. Let E be the ring of entire functions and x an indeterminate over E. Let S denote the set generated by the principal primes of E. Then, we claim that $R := E + xS^{-1}E[x]$ contains a prime ideal P such that $S^{-1}\widetilde{P} \subsetneqq \widetilde{S^{-1}P}$. Indeed, R is a P-domain that is not a PvMD [48, Example 2.6]. By [49, Proposition 3.3], there exists a prime t-ideal P in R such that PR_P is not a t-ideal of R_P. By Theorem 1.5, we have

$$\begin{aligned} \widetilde{P}R_P &= PR_P \\ &\subsetneqq R_p \\ &= (PR_P)_t \\ &= \widetilde{PR_P}. \end{aligned}$$

since R is integrally closed. Also notice that

$$\begin{aligned} P &= \widetilde{PR_P \cap R} \\ &\subsetneqq \widetilde{PR_P} \cap R \\ &= R. \end{aligned}$$

Corollary 1.13 *Let R be a domain and I a t-ideal that is t-locally t-integrally closed (i.e., I_M is t-integrally closed in R_M for every maximal t-ideal M of R). Then I is t-integrally closed.*

Proof. Let $Max_t(R)$ denote the set of maximal t-ideals of R. By Corollary 1.11, we have

$$\begin{aligned}
\widetilde{I} &\subseteq \bigcap_{M_i \in Max_t(R)} (\widetilde{I})_{M_i} \\
&\subseteq \bigcap_{M_i \in Max_t(R)} \widetilde{I_{M_i}} \\
&= \bigcap_{M_i \in Max_t(R)} I_{M_i} \\
&= I.
\end{aligned}$$

Consequently, I is t-integrally closed. □

CHAPTER 2

ON T-REDUCTIONS OF IDEALS IN PULLBACKS

This chapter investigates t-reductions of ideals in pullback constructions of type $\square$ (See definition in Section 2.4). This work is submitted for publication under the title "On t-reductions of ideals in pullbacks" (in collaboration with Dr. S. Kabbaj and Dr. A. Mimouni).

2.1 Introduction

Let R be a ring and I a proper ideal of R. Recall that an ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. In [25, 26], Hays investigated reductions of ideals in Noetherian rings and Prüfer domains. He provided several conditions for an ideal to be basic. His two main results asserted that a domain R is Prüfer (resp., one-dimensional Prüfer) if and only if R has the finite basic ideal property (resp., basic ideal property).

Let R be a domain and I a nonzero fractional ideal of R. Recall, for convenience, that the v-, t-, and w-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$, $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I, and $I_w = \cap IR_M$ where M ranges over the set of maximal t-ideals of R. Recall that a domain is a Prüfer v-Multiplication Domain (PvMD)) if every nonzero finitely generated ideal is t-invertible (with respect to the ideal t-multiplication $(I, J) \mapsto (IJ)_t$).

In [30], the authors extended Hays' aforementioned results to PvMDs; namely, a domain has the finite w-basic ideal property (resp., w-basic ideal property) if and only if it is a PvMD (resp., PvMD of t-dimension one). They also investigated relations among the classes of domains subject to various $\star$-basic properties for a given $\star$-operation (See definition in Section 2.2). In this vein, the problem of whether the finite t- and v-basic ideal properties are distinct was left open. In the first chapter, we investigated the t-reductions and t-integral closure of ideals establishing satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. One of our main result (Theorem 1.3.5) asserts that the t-closure and t-integral closure of an ideal coincide in the class of integrally closed domains.

This chapter investigates t-reductions of ideals in pullback constructions of type $\square$ (See definition in Section 2.4). Section 2.2 examines the correlation between the notions of reduction and t-reduction in pseudo-valuation domains. Section 2.3 solves an open problem raised in [30] on whether the finite t-basic and

v-basic ideal properties are distinct. We prove that these two notions coincide in any arbitrary domain. Section 2.4 features the main result, which establishes the transfer of the finite t-basic (equiv., v-basic) ideal property to pullbacks in line with Fontana-Gabelli's result on PvMDs [16, Theorem 4.1] and Gabelli-Houston's result on v-domains [20, Theorem 4.15]. This allows us to enrich the literature with new families of examples, which put the class of domains subject to the finite t-basic ideal property strictly between the two classes of v-domains and integrally closed domains.

For a full treatment of the topic of reduction theory, we refer the reader to [33].

2.2 t-Reductions in pseudo-valuation domains

Let R be a domain with quotient field K, and let $F(R)$ denote the set of nonzero fractional ideals of R. A map

$$\begin{array}{cccc} \star & : & F(R) & \rightarrow & F(R) \\ & & I & \mapsto & I^{\star} \end{array}$$

is said to be a *star operation on* R if the following conditions hold for every nonzero $a \in K$ and $I, J \in F(R)$:

(1) $(aI)^{\star} = aI^{\star}$ and $R^{\star} = R$.

(2) $I \subseteq I^{\star}$ and $I \subseteq J$ implies $I^{\star} \subseteq J^{\star}$.

(3) $I^{\star\star} = I^{\star}$.

For more details about star operations, we refer to [17] and [24, Sections 32 and 34]. We first recall the definitions of $\star$-reduction and related concepts such as the trivial $\star$-reduction and (finite) $\star$-basic ideal property.

Definition 2.1 ([30, 36]) Let R be a domain and I a nonzero ideal of R.

(1) An ideal $J \subseteq I$ is a *$\star$-reduction* of I if $(JI^n)^{\star} = (I^{n+1})^{\star}$ for some integer $n \geq 0$. The ideal J is a *trivial $\star$-reduction* of I if $J^{\star} = I^{\star}$.

(2) I is *$\star$-basic* if it has no $\star$-reduction other than the trivial $\star$-reductions.

(3) R has the (finite) $\star$-basic ideal property if every nonzero (finitely generated) ideal of R is $\star$-basic.

This is not to be confused with Epstein's c-reduction [12, 13, 14], which generalizes the original notion of reduction in a different way and was studied in different settings. Namely, let c be a closure operation. An ideal $J \subseteq I$ is a c-reduction of I if $J^c = I^c$. Thus, for $c := \star$, Epstein's c-reduction coincides with the trivial $\star$-reduction.

In the sequel, we will be using the following obvious facts, for nonzero ideals $J \subseteq I$, without explicit mention:

$$J \text{ is a } t\text{-reduction of } I \Leftrightarrow J \text{ is a } t\text{-reduction of } I_t \Leftrightarrow J_t \text{ is a } t\text{-reduction of } I_t.$$

Recall that R is a pseudo-valuation domain if R is local and shares its maximal ideal with a valuation overring V or, equivalently, if R is a pullback issued from the following diagram

$$\begin{array}{ccc} R = \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & K := V/M \end{array}$$

where (V, M) is a valuation domain and k is a subfield of K (cf. [27, 28] and also [4, 5, 9, 10, 46]).

Note that a reduction is necessarily a t-reduction; and the converse is not true in general. The next result investigates necessary and sufficient conditions for the notions of reduction and t-reduction to coincide in pseudo-valuation domains. This result can be used readily to provide examples discriminating between the two notions of reduction and t-reduction.

Theorem 2.1 *Let R be a pseudo-valuation domain issued from (V, M, k) and set $K := V/M$. Then, the following statements are equivalent:*

(i) *For every nonzero ideals $J \subseteq I$, J is a t-reduction of I if and only if J is a reduction of I.*

(ii) *For each k-subspace W of K containing k, W^n is a field for some positive integer n.*

Proof. $(i) \Rightarrow (ii)$ Let W be a k-subspace of K with $k \subsetneqq W \subsetneqq K$. Let $0 \neq a \in M$ and consider the ideals of R

$$J := aR \subseteq I := a\varphi^{-1}(W).$$

Let $r \geq 1$. Then, the fact that $k \subsetneq W$ yields

$$(R : I^r) = a^{-r}\varphi^{-1}(k : W^r) = a^{-r}M$$

and then

$$(I^r)_v = a^r M^{-1} = a^r V.$$

By [31, Proposition 4.3], the t- and v- operations coincide in R. Hence, we have

$$\begin{aligned} (JI)_t &= (aI)_t \\ &= aI_t \\ &= aI_v \\ &= a^2 V \\ &= (I^2)_v \\ &= (I^2)_t \end{aligned}$$

and so J is a t-reduction of I. By (i), J must be a reduction of I and so

$$\begin{aligned} a^{n+1}\varphi^{-1}(W^n) &= JI^n \\ &= I^{n+1} \\ &= a^{n+1}\varphi^{-1}(W^{n+1}) \end{aligned}$$

for some positive integer n. It follows that $\varphi^{-1}(W^n) = \varphi^{-1}(W^{n+1})$; i.e., $W^n = W^{n+1}$. Therefore $W^n = (W^n)^2$ and thus W^n is a ring. In particular, let $0 \neq \lambda \in K$

and let $W_o := k + \lambda k$. Then, there is some positive integer m such that

$$\begin{aligned} k + \lambda k + \cdots + \lambda^m k &= W_o^m \\ &= W_o^{m+1} \\ &= k + \lambda k + \cdots + \lambda^{m+1} k. \end{aligned}$$

So, $\lambda^{m+1} \in k + \lambda k + \cdots + \lambda^m k$. Therefore λ is algebraic over k and thus K is algebraic over k. Consequently, W^n is a field, as desired.

$(ii) \Rightarrow (i)$ Let $J \subseteq I$ be a t-reduction of I; i.e., $(JI^n)_t = (I^{n+1})_t$ for some positive integer n. If I is an ideal of V, then both JI^n and I^{n+1} are ideals of V so that JI^n and I^{n+1} are divisorial ideals of R by [27, Theorem 2.13]. Therefore, we obtain

$$\begin{aligned} JI^n &= (JI^n)_v \\ &= (JI^n)_t \\ &= (I^{n+1})_t \\ &= (I^{n+1})_v \\ &= I^{n+1}. \end{aligned}$$

That is, J is a reduction of I. Next, assume that I is not an ideal of V. Then, by [8, Theorem 2.1(n)], we have

$$I = a\varphi^{-1}(W)$$

for some nonzero $a \in M$ and some k-vector space W with $k \subseteq W \subset K$. Assume that $k = W$; i.e., $I = aR$. Then $J_t = aR$. Now, if $J \subsetneqq aR$, then $a^{-1}J \subsetneqq R$, hence $a^{-1}J \subseteq M$, whence $J \subseteq aM$. Since M is a divisorial ideal of R [29, Corollary 5], we obtain

$$\begin{aligned} aR &= J_t \\ &\subseteq (aM)_t \\ &= aM_t \\ &= aM. \end{aligned}$$

which is a contradiction. So, necessarily, $J = I$. Next, assume $k \subsetneqq W$. Suppose J is an ideal of V. Then JI^n would be an ideal of V and hence a divisorial ideal of R yielding

$$\begin{aligned} a^n J &= JI^n \\ &= (JI^n)_v \\ &= (JI^n)_t \\ &= (I^{n+1})_t \\ &= (I^{n+1})_v \\ &= a^{n+1}V. \end{aligned}$$

where the last equality is already handled in (i) $\Rightarrow$ (ii). It follows that

$$J = aV = IV \supseteq I \supseteq J.$$

That is, $J = I$ is an ideal of V, absurd. Hence, J is not an ideal of V. So, since $J \subseteq I$, we may assume that $J = a\varphi^{-1}(F)$, where F is a k-subspace of W. Now by hypothesis, $W^s = W^{s+1}$ is a field for some $s \geq 1$. It follows that

$$FW^s = W^{s+1}$$

yielding

$$\begin{aligned} JI^s &= a^{s+1}\varphi^{-1}(FW^s) \\ &= a^{s+1}\varphi^{-1}(W^{s+1}) \\ &= I^{s+1}. \end{aligned}$$

Hence J is a reduction of I, completing the proof of the theorem. □

Note that the condition (ii) in the above result forces K to be algebraic over k. In this vein, this fact can be used readily to provide examples of domains where the two notions of reduction and t-reduction are distinct.

Example 2.2 Let R be a pseudo-valuation domain issued from (V, M, k) and set $K := V/M$.

(1) Assume that K is a transcendental extension of k. Then, the notions of reduction and t-reduction are distinct in R. For instance, pick a transcendental element $\lambda \in K$ over k and let $W := k + k\lambda$, $I := a\phi^{-1}(W)$ and $J =: aR$. Then, J is a proper t-reduction of I, but I is basic in R, as seen the proof of (i) $\Rightarrow$ (ii) of the above theorem.

(2) Assume that $[K : k]$ is finite. Then for every k-submodule W of K with $k \subseteq W \subseteq K$, some power of W is a field, and hence the notions of reduction and t-reduction coincide in R.

Given nonzero ideals $J \subseteq I$, if J_t is a reduction of I_t, then J is a t-reduction of I. The converse is not true in general as shown by Example 1.2.2 which consists of a domain containing two t-ideals $J \subsetneqq I$ such that J is a t-reduction but not a reduction of I. The next result provides a class of (integrally closed) pullbacks where the two assumptions are always equivalent.

Proposition 2.1 *Let R be a pseudo-valuation domain and let $J \subseteq I$ be nonzero ideals of R. Then, J is a t-reduction of I if and only if J_t is a reduction of I_t.*

Proof. Sufficiency is trivial. For the necessity, assume R is issued from (V, M, k) and, without loss of generality, $R \subsetneqq V$. Next, let J be a t-reduction of I. Then, J_t is a t-reduction of I_t and hence we may assume that J and I are both t-ideals. So $(JI^n)_t = (I^{n+1})_t$, for some integer $n \geq 1$. If I is an ideal of V, as in the proof of Theorem 2.1(ii)$\Rightarrow$(i), we get

$$\begin{aligned} JI^n &= (JI^n)_t \\ &= (I^{n+1})_t \\ &= I^{n+1}. \end{aligned}$$

That is, J is a reduction of I. Next, suppose that I is not an ideal of V. By [8, Theorem 2.1(n)], we have

$$I = a\varphi^{-1}(W)$$

for some nonzero $a \in M$ and some k-vector space W with

$$k \subseteq W \subset K := V/M.$$

We claim that $k = W$. Otherwise, we would get, via [31, Proposition 4.3], that

$$I = I_t = I_v = aV$$

where the last equality is already handled in the proof of Theorem 2.1(i)$\Rightarrow$(ii). It follows that I is an ideal of V, the desired contradiction. So, necessarily, $k = W$ and then $I = aR$. By [30, Lemma 1.2], I is t-basic; i.e., $J = I$, completing the proof. $\square$

The class of Prüfer domains is, so far, the only known class of domains where these two notions of reduction and t-reduction coincide. We close this section with the next result, which features necessary conditions for such a coincidence. For this purpose, recall that a domain where the trivial and w-operations are the same is called a DW-domain [23, 32, 42]. Common examples of DW-domains are pseudo-valuation domains, Prüfer domains, and quasi-Prüfer domains (i.e., domains with Prüfer integral closure) [18, Page 190].

Proposition 2.2 *Let R be a domain where the notions of reduction and t-reduction coincide for all ideals of R. Then:*

(1) *Every nonzero prime ideal of R is a t-ideal. In particular, R is a DW-domain.*

(2) *R is integrally closed if and only if R has the finite t-basic ideal property.*

(3) *R is a PvMD if and only if R is a Prüfer domain.*

Proof. (a) Let P be a nonzero prime ideal of R. Clearly, P is a t-reduction of P_t. By hypothesis, P is then a reduction of P_t. But every prime ideal is a C-ideal (i.e., it is not a proper reduction of any larger ideal) [25, Page 58]. It follows that $P = P_t$, as desired. In particular, every maximal ideal of R is a t-ideal and, hence, R is a DW-domain by [42, Proposition 2.2].

(b) Assume that R is integrally closed and let I be a finitely generated ideal of R and J a t-reduction of I. By hypothesis, J is a reduction of I. So, by a combination of [33, Corollary 1.2.5] and [43, Proposition 2.2(iii)], we get

$$I \subseteq \overline{J} \subseteq J_t$$

where $\overline{J}$ denotes the integral closure of J. It follows that $J_t = I_t$; i.e., I is t-basic, as desired. The converse is true for any arbitrary domain R by [30, Lemma 1.3].

(c) Assume R is a PvMD. By hypothesis, the notions of reduction and t-reduction coincide in R and, hence, R is a DW-domain by (1) above. By [23, Theorem 1.2], R is a Prüfer domain. The converse is trivial. □

2.3 Equivalence of the finite t- and v-basic ideal properties

For the reader's convenience, recall that a domain R is called a v-domain if all nonzero finitely generated ideals of R are v-invertible (with respect to the ideal v-multiplication $(I, J) \mapsto (IJ)_v$); an excellent reference for v-domains is Fontana

& Zafrullah's comprehensive survey paper [19]. Also, recall from [30] the following diagram of implications, which puts into perspective the finite basic ideal property for each of the t-, v-, and w-operations:

$$\begin{array}{c} \text{Krull domain} \\ \Downarrow \\ \text{P}v\text{MD} = \text{Finite } w\text{-basic ideal property} \\ \Downarrow \\ v\text{-domain} \\ \Downarrow \\ \text{Finite } v\text{-basic ideal property} \\ \Downarrow \\ \text{Finite } t\text{-basic ideal property} \\ \Downarrow \\ \text{Integrally closed domain} \end{array}$$

The problem of whether the fourth implication is reversible was left open in [30, Section 3]. The main result of this section (Theorem 2.4) solves this open problem. For this purpose, recall from the first chapter the following: Let R be a domain and I a nonzero ideal of R. An element $x \in R$ is t-integral over I if there is an equation

$$x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0 \ \text{ with } \ a_i \in (I^i)_t \ \forall i = 1, ..., n.$$

Consider the two sets:

$$\widetilde{I} := \{x \in R \mid x \text{ is } t\text{-integral over } I\}$$

$$\widehat{I} := \{x \in R \mid I \text{ is a } t\text{-reduction of } (I, x)\}.$$

$\widetilde{I}$ is called the t-integral closure of I and is an integrally closed ideal by Theorem 1.3.2. We always have

$$I_t \subseteq \widetilde{I} \subseteq \widehat{I}$$

where the first containment is trivial and the second is asserted by Proposition 1.3.7 and can be strict as shown by Example 1.3.10(a). However, for the trivial operation, it is well-known that the equality $\widetilde{I} = \widehat{I}$ always holds [33, Corollary 1.2.2]; this fact was used to show that the integral closure of an ideal is an ideal [33, Corollary 1.3.1]. Finally, in order to put Theorem 2.4 into perspective, recall the following important (partial) result from the first chapter.

Theorem 2.3 *For a domain R, the following two assertions are equivalent:*

(i) $I_t = \widetilde{I}$ *for each nonzero (finitely generated) ideal I of R;*

(ii) *R is integrally closed.*

Now, to the main result of this section.

Theorem 2.4 *For a domain R, the following assertions are equivalent:*

(i) $I_t = \widehat{I}$ *for each nonzero (finitely generated) ideal I of R;*

(ii) *R has the finite t-basic ideal property;*

(iii) *R has the finite v-basic ideal property.*

The proof of this result requires the following two lemmas.

Lemma 2.1 (**[30, Lemma 1.7]**) *Let R be a domain and let I be a finitely generated ideal of R. If $J \subseteq I$ is a t-reduction of I, then there exists a finitely generated ideal $K \subseteq J$ such that K is a t-reduction of I.*

Note that, for any given $\star$-operation, $\star$-reductions of (integral) ideals can be naturally extended to fractional ideals. The following lemma collects basic results on $\star$-reductions of (fractional) ideals.

Lemma 2.2 *For a domain R, let $K \subseteq J \subseteq I$ and $J' \subseteq I'$ be nonzero fractional ideals of R.*

(1) *If J and J' are $\star$-reductions of I and I', respectively, then $J + J'$ is a $\star$-reduction of $I + I'$ and JJ' is a $\star$-reduction of II'.*

(2) *If K is a $\star$-reduction of J and J is a $\star$-reduction of I, then K is a $\star$-reduction of I.*

(3) *If K is a $\star$-reduction of I, then J is a $\star$-reduction of I.*

(4) *J is a $\star$-reduction of I if and only if J^n is a $\star$-reduction of I^n.*

(5) *If $J = (a_1, ..., a_k)$, then: J is a $\star$-reduction of I if and only if $(a_1^n, ..., a_k^n)$ is a $\star$-reduction of I^n.*

Proof. Substitute "$\star$" for "t" and "fractional ideals" for "(integral) ideals" in the proofs of Lemmas 1.2.5, 1.2.6 and 1.2.7. $\square$

Proof of Theorem 2.4 In view of the aforementioned diagram, we only need to prove (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

First, let us prove that if the equality $\widehat{I} = I_t$ holds for all nonzero finitely generated ideals then it holds for all nonzero ideals. Indeed, let I be an ideal of R and $x \in R$ such that I is a t-reduction of (I, x). So,

$$(I(I,x)^n)_t = ((I,x)^{n+1})_t$$

for some positive integer n. Hence, $x^{n+1} \in (I(I,x)^n)_t$. Whence, $x^{n+1} \in A_v$ for some finitely generated ideal $A \subseteq I(I,x)^n$. Moreover, there exist finitely generated subideals $F_o, F_1 \ldots, F_n$ of I such that

$$A \subseteq F_o(F_1,x)(F_2,x)\cdots(F_n,x).$$

Set $F := \sum_{i=o}^{n} F_i \subseteq I$. Then, $A \subseteq F(F,x)^n$ and so

$$x^{n+1} \in (F(F,x)^n)_v = (F(F,x)^n)_t.$$

It follows that

$$((F,x)^{n+1})_t = (F(F,x)^n, x^{n+1})_t \subseteq (F(F,x)^n)_t.$$

Thus, F is a t-reduction of (F, x). Since F is finitely generated, then by hypothesis $x \in \widehat{F} = F_t \subseteq I_t$. Consequently, $\widehat{I} \subseteq I_t$ and, as mentioned above, the reverse inclusion always holds by Proposition 1.3.7.

Next, assume that R has the finite t-basic ideal property and let I be a finitely generated ideal of R and $x \in \widehat{I}$. Necessarily, we have

$$I_t = (I, x)_t$$

which forces $x \in I_t$. Consequently, $\widehat{I} = I_t$. Conversely, assume that (i) holds. Let $I := (a_1, \ldots, a_n)$ be a nonzero finitely generated ideal of R ($n \geq 1$) and let J be a t-reduction of I. By Lemma 2.1, we may assume that J is finitely generated. Clearly, we have

$$J \subseteq (J, a_1, \ldots, a_{n-1}) \subseteq I.$$

By Lemma 1.2.6, $(J, a_1, \ldots, a_{n-1})$ is a t-reduction of I which can be regarded as $\big((J, a_1, \ldots, a_{n-1}), a_n\big)$. Hence, by hypothesis

$$a_n \in (J, \widehat{a_1, \ldots, a}_{n-1}) = (J, a_1, \ldots, a_{n-1})_t.$$

It follows that

$$I_t = (J, a_1, \ldots, a_{n-1})_t.$$

But J, being a t-reduction of I_t, is also a t-reduction of $(J, a_1, \ldots, a_{n-1})$. Therefore, we re-iterate the above process by removing one generator at each step.

Eventually, we get $I_t = J_t$, as desired. This proves (i) $\Leftrightarrow$ (ii).

Assume that R has the finite t-basic ideal property and let I be a finitely generated ideal of R and J a v-reduction of I. So

$$J_v = \bigcap_{\lambda \in \Lambda} (a_\lambda)$$

where the (a_λ)'s are the nonzero principal fractional ideals of R containing J by [24, Theorem 34.1]. By Lemma 2.2, $(a_\lambda) = (J, a_\lambda)$ is a v-reduction of (I, a_λ) for each $\lambda \in \Lambda$. Hence (a_λ) is a t-reduction of (I, a_λ) as both ideals are finitely generated. Since R has the finite t-basic ideal property, one can easily verify that every nonzero *fractional* ideal of R is t-basic. Hence

$$(a_\lambda) = (I, a_\lambda)_t$$

for each $\lambda \in \Lambda$. Therefore

$$\begin{aligned} I_v &= I_t \\ &\subseteq \textstyle\bigcap_{\lambda \in \Lambda} (a_\lambda) \\ &= J_v. \end{aligned}$$

Hence, $I_v = J_v$; that is, I is v-basic. This proves (ii) $\Rightarrow$ (iii), completing the proof of the theorem. $\square$

New examples of domains subject to the finite t-basic (equiv., v-basic) ideal property will be provided in the next section. We close this section with the following open question:

Question 2.5 Is $\widehat{I}$ always an ideal?

2.4 Transfer of the finite t-basic ideal property to pullbacks

Let us fix notation for this section. Let T be a domain, M a maximal ideal of T, K its residue field, $\varphi : T \longrightarrow K$ the canonical surjection, and D a proper subring of K with quotient field k. Let R be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{cccc} & R & \longrightarrow & D \\ (\square) & \downarrow & & \downarrow \\ & T & \xrightarrow{\varphi} & K = T/M. \end{array}$$

So, we have

$$R := \varphi^{-1}(D) \subsetneq T.$$

This section establishes necessary and sufficient conditions for a pullback of type $\square$ issued from local domains to inherit the finite t-basic (equiv., v-basic) ideal property. Recall, at this point, that a domain with the t-basic ideal property is completely integrally closed [30, Proposition 1.4]. Therefore, by [24, Lemma 26.5], a pullback of type $\square$ never has the t-basic ideal property.

It is worthwhile recalling that the finite t-basic ideal property lies between the two notions of v-domain and integrally closed domain [30]; and that the finite w-

basic ideal property coincides with the PvMD property [30, Theorem 2.1]. Also, the transfer of the notions of PvMD and v-domain to pullbacks was established, respectively, by Fontana & Gabelli in [16] and by Gabelli & Houston in [20], which summarizes as follows:

Theorem 2.6 ([16, Theorem 4.1] & [20, Theorem 4.15]) *Let R be a pullback of type $\square$. Then, the following assertions are equivalent:*

(i) *R is PvMD (resp., v-domain);*

(ii) *T and D are PvMDs (resp., v-domains), T_M is a valuation domain, and $k = K$.*

Finally, recall that if T is integrally closed, then the integral closure of R is $\varphi^{-1}(\overline{D})$, where $\overline{D}$ denotes the integral closure of D in K. This follows easily from the fact that R and T have the same quotient field. Next, we announce the main result of this section which allows us to enrich the literature with new families of examples, putting the new class of domains subject to the finite t-basic ideal property strictly between the two classes of v-domains and integrally closed domains.

Theorem 2.7 *Let R be a pullback of type $\square$ such that T is local. Then, the following assertions are equivalent:*

(i) *R has the finite t-basic ideal property;*

(ii) *T and D have the finite t-basic ideal property and $k = K$.*

Proof. (i) $\Longrightarrow$ (ii) Assume that R has the finite t-basic ideal property. We first prove that $k = K$. Assume, by way of contradiction, that $k \subsetneqq K$. By [20, Proposition 2.4], there is an element $x \in T \setminus R$ with

$$(R : (1, x)) = M.$$

Hence

$$\begin{aligned} x^2(R : (1, x)) &= x^2 M \\ &\subseteq TM \\ &\subseteq R. \end{aligned}$$

That is, $x^2 \in (1, x)_v$. Therefore, for any nonzero $m \in M$, we have

$$x^2 m^2 \in (m^2, xm^2)_v = (m^2, xm^2)_t$$

and so

$$\begin{aligned} ((m, xm)^2)_t &= (m^2, xm^2)_t \\ &= (m(m, xm))_t. \end{aligned}$$

forcing (m) to be a t-reduction of (m, xm) in R. Whence

$$(m, xm)_t = (m).$$

It follows that $xm \in (m)$ and thus $x \in R$, the desired contradiction.

Next, we prove that T has the finite t-basic ideal property. Below, we denote by v_1 and t_1 the v- and t- operations with respect to T. Let I be a nonzero finitely generated proper ideal of T and J a t-reduction of I. So $(JI^n)_{t_1} = (I^{n+1})_{t_1}$ for some positive integer n. We may assume, by Lemma 2.1, that J is finitely generated. If $(I^{n+1})_{v_1}$ is principal; say

$$(I^{n+1})_{t_1} = (I^{n+1})_{v_1} = (a)$$

for some nonzero $a \in T$, then

$$\begin{aligned} aJ_{t_1} &= (JI^{n+1})_{t_1} \\ &= (I^{n+2})_{t_1} \\ &= aI_{t_1} \end{aligned}$$

yielding $J_{t_1} = I_{t_1}$. Next, suppose that $(JI^n)_{v_1} = (I^{n+1})_{v_1}$ is not principal. Since $k = K$, then T is a localization of R (cf. [15, 35]). So

$$J = BT \text{ and } I = AT$$

for some nonzero finitely generated ideals $B \subseteq A$ of R. By [20, Proposition 2.7(1)(b)], we obtain

$$\begin{aligned}(A^{n+1})_t &= (A^{n+1})_v \\ &= (I^{n+1})_{v_1} \\ &= (I^{n+1})_{t_1} \\ &= (JI^n)_{t_1} \\ &= (JI^n)_{v_1} \\ &= (BA^n)_v \\ &= (BA^n)_t.\end{aligned}$$

It follows that B is a t-reduction of A and thus $B_t = A_t$. By [40, Lemma 3.4], we get

$$\begin{aligned}J_{t_1} &= (B_tT)_{t_1} \\ &= (A_tT)_{t_1} \\ &= I_{t_1}.\end{aligned}$$

Therefore, in both cases, we showed that J is a trivial t-reduction of I, as desired.

Next, we show that D has the finite t-basic ideal property. Let A be a nonzero finitely generated ideal of D and let B be a t-reduction of A. Let t_D denote the t-operation with respect to D. So, $(BA^n)_{t_D} = (A^{n+1})_{t_D}$ for some positive integer n. We may assume, by Lemma 2.1, that B is finitely generated. By [16, Corollary 1.7], we have

$$I := \varphi^{-1}(A) \text{ and } J := \varphi^{-1}(B)$$

are two nonzero finitely generated ideals of R (containing M). Since $k = K$, by

[16, Proposition 1.6(a) & Proposition 1.8(a3)], we obtain

$$\begin{aligned}(JI^n)_t &= (\varphi^{-1}(BA^n))_t \\ &= \varphi^{-1}((BA^n)_{t_D}) \\ &= \varphi^{-1}((A^{n+1})_{t_D}) \\ &= (\varphi^{-1}(A^{n+1}))_t \\ &= (I^{n+1})_t.\end{aligned}$$

Hence J is a t-reduction of I and thus $J_t = I_t$. It follows that

$$\begin{aligned}B_{t_D} &= \varphi(\varphi^{-1}(B_{t_D})) \\ &= \varphi(J_t) \\ &= \varphi(I_t) \\ &= \varphi(\varphi^{-1}(A_{t_D})) \\ &= A_{t_D}.\end{aligned}$$

completing the proof of the "only if" assertion.

(ii) $\Longrightarrow$ (i) Assume that T and D have the finite t-basic ideal property and $k = K$. Notice that, in presence of the latter assumption, M cannot be finitely generated [20, Lemma 4.1]. Also, recall that we always have $M_v = M$ [29, Corollary 5]. Next, let I be a nonzero finitely generated ideal of R and let J be a finitely generated subideal of I with $(JI^n)_t = (I^{n+1})_t$ for some positive integer n. By [21, Proposition 1.6], any ideal of R is comparable to M. So, we envisage two cases:

Case 1: Suppose that $M \subsetneqq I$. We first claim that

$$M \subsetneqq I^{n+1}.$$

Otherwise, $I^{n+1} \subseteq M$ yields, by [16, Proposition 1.1], the following

$$\begin{aligned} T &= (IT)^{n+1} \\ &= I^{n+1}T \\ &\subseteq MT \\ &= M. \end{aligned}$$

which is absurd. Moreover, we have $M \subsetneqq J$; otherwise, we would obtain

$$\begin{aligned} J &\subseteq M \\ &\subsetneqq I^{n+1} \\ &\subseteq J_t \\ &= J_v. \end{aligned}$$

which is absurd. Further, we claim that

$$M \subsetneqq JI^n.$$

Otherwise, $JI^n \subseteq M$ yields via [16, Proposition 1.1]

$$\begin{aligned} T &= (JT)(IT)^n \\ &= (JI^n)T \\ &\subseteq MT \\ &= M. \end{aligned}$$

which is absurd. Now, let $A := \varphi(I)$ and $B := \varphi(J)$, two nonzero finitely generated ideals of D. Therefore, by [16, Proposition 1.6(b) & Proposition 1.8(b3)], we get

$$\begin{aligned} (BA^n)_{t_D} &= (\varphi(JI^n))_{t_D} \\ &= \varphi((JI^n)_t) \\ &= \varphi((I^{n+1})_t) \\ &= (\varphi(I^{n+1}))_{t_D} \\ &= (A^{n+1})_{t_D}. \end{aligned}$$

Hence B is a t-reduction of A and thus $B_{t_D} = A_{t_D}$. It follows that

$$\begin{aligned} J_t &= \varphi^{-1}(\varphi(J_t)) \\ &= \varphi^{-1}(B_{t_D}) \\ &= \varphi^{-1}(A_{t_D}) \\ &= \varphi^{-1}(\varphi(I_t)) \\ &= I_t. \end{aligned}$$

Case 2: Suppose that $I \subsetneqq M$. If $II^{-1} \nsubseteq M$, then there is a nonzero $x \in \mathrm{qf}(R)$

with $M \subsetneqq xI \subseteq R$, hence $xJ_t = xI_t$ by Case 1, whence $J_t = I_t$. So, we may assume $II^{-1} \subseteq M$. Now, note that $(JI^n)^{-1} = (I^{n+1})^{-1}$. So, by [21, Proposition 2.2(1)], we have

$$\begin{aligned}
(JI^nT)_{t_1} &= (JI^nT)_{v_1} \\
&= ((JI^nT)^{-1})^{-1} \\
&= ((JI^n)^{-1}T)^{-1} \\
&= ((I^{n+1})^{-1}T)^{-1} \\
&= ((I^{n+1}T)^{-1})^{-1} \\
&= (I^{n+1}T)_{v_1} \\
&= (I^{n+1}T)_{t_1}.
\end{aligned}$$

Hence JT is a t-reduction of IT. It follows, via [21, Proposition 2.2(1)], that

$$\begin{aligned}
J^{-1}T &= (JT)^{-1} \\
&= ((JT)_{v_1})^{-1} \\
&= ((JT)_{t_1})^{-1} \\
&= ((IT)_{t_1})^{-1} \\
&= ((IT)_{v_1})^{-1} \\
&= (IT)^{-1} \\
&= I^{-1}T.
\end{aligned}$$

On the other hand, the assumption $II^{-1} \subseteq M$ yields

$$\begin{aligned}(IT)(IT)^{-1} &= II^{-1}T \\ &\subseteq MT \\ &= M.\end{aligned}$$

Hence IT is not invertible and, a fortiori, not principal in T. Therefore, by [20, Proposition 2.7(a)], we get

$$\begin{aligned}J^{-1} &\subseteq J^{-1}T \\ &= I^{-1}T \\ &= (IT)^{-1} \\ &= (M : I) \\ &= I^{-1} \\ &\subseteq J^{-1}.\end{aligned}$$

Consequently,

$$I_t = I_v = J_v = J_t$$

completing the proof of the theorem. □

Theorem 2.7 allows us to enrich the literature with new families of examples, which put the class of domains subject to the finite t-basic ideal property strictly between the two classes of integrally closed domains and v-domains.

Example 2.8 Consider any non-trivial pseudo-valuation domain R issued from (V, M, k) with k algebraically closed in $K := V/M$. Then:

(1) R is an integrally closed domain by [8, Theorem 2.1].

(2) R does not have the finite t-basic ideal property by Theorem 2.7.

Moreover, the two notions of reduction and t-reduction are distinct in R by Proposition 2.2(b).

Example 2.9 Consider any pullback R of type $\square$ issued from (T, M, D) where $\mathrm{qf}(D) = T/M$, T is a non-valuation local v-domain, and D is a v-domain. Then:

(1) R has the finite t-basic ideal property by [30, Proposition 1.6] and Theorem 2.4 and Theorem 2.7.

(2) R is not a v-domain by [20, Theorem 4.15].

One can easily build non-valuation local v-domains via pullbacks through [20, Theorem 4.15].

Here is a specific example, where we ensure, moreover, that the two notions of reduction and t-reduction are distinct.

Example 2.10 Let X, Y, Z be indeterminates over $\mathbb{Q}$ and let

$$\begin{aligned} T &:= \mathbb{Q}(X)[[Y, Z]] \\ M &:= (Y, Z)\mathbb{Q}(X)[[Y, Z]] \\ R &:= \mathbb{Z}[X] + M. \end{aligned}$$

Clearly, T and $D := \mathbb{Z}[X]$ have the finite t-basic property (since both are Noetherian Krull domains). By Theorem 2.7, R has the finite t-basic property. Also R is not a v-domain since T is a non-valuation local domain. Next, let $0 \neq a \in \mathbb{Z}$ and consider the finitely generated ideal of R given by

$$I := (a, X)\mathbb{Z}[X] + M = aR + XR.$$

Clearly $I^{-1} = R$ and so $(I^s)^{-1} = R$, for every positive integer s. In particular, we have

$$\begin{aligned} (I^2 I)_t &= (I^3)_t \\ &= (I^3)_v \\ &= R \\ &= (I^2)_v \\ &= (I^2)_t. \end{aligned}$$

and hence I^2 is a t-reduction of I. However, I^2 is not a reduction of I; otherwise, if $I^{n+2} = I^2 I^n = I^{n+1}$, for some $n \geq 1$, this would contradict [41, Theorem 76]. It follows that the notions of reduction and t-reduction are distinct in R, as desired.

We close this section with the following two open questions.

Question 2.11 Is Theorem 2.7 valid for the classical pullbacks $R = D + M$ issued from $T := K + M$ not necessarily local? The idea here is that (since $k = K$, then) $T = S^{-1}R$ for $S := D \setminus \{0\}$. Clearly, the current proof of the "only if" assertion works for this context.

Question 2.12 Is Theorem 2.7 valid for the non-local case through an additional assumption on T_M? The idea here is that, "($k = K$ and hence) $R_M = T_M$" is a necessity for the finite t-basic property; and for the PvMD and v-domain notions, $R_M = T_M$ is a valuation domain. So, one needs to investigate this localization for the t-basic ideal property in this context.

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