DISCONTINUOUS GALERKIN METHODS FOR FRACTIONAL DIFFUSION PROBLEMS

BY

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A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

APRIL, 2016

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN- 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

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I dedicate my Dissertation work to my family. A special feeling of gratitude to my loving mother, my father, my wife, my sons, my daughter, my brothers, my sisters

Acknowledgments

I am most grateful to Almighty ALLAH, the Beneficent, the Merciful, for enabling me to complete this work. Peace and blessings of ALLAH be upon his Last messenger Mohammed (Sallallah-Alaihe-Wasallam) and his family, who guided us to the right path.

First and the foremost acknowledgments are due to the King Fahd University of Petroleum and Minerals and to the Department of Mathematical Sciences for supporting my research work.

I wish to express My deep appreciation and heartfelt gratitude to Prof. Kassem Mustapha, my thesis advisor for his unfailing encouragement, advice and suggestions through this work and Co. advisor Prof. Bernardo Cockburn who guided me with his dedicated attention, expertise and knowledge throughout this research. I also wish to thank my Committee Members, Dr. Faisal Fairag, Prof. Khaled Furati, and Prof. Mohamed El-Gebeily, for their constructive guidance and support.

And last, but not the least, my cordial thanks and appreciation are due to my parents, my wife and children, brothers, friends and all members of my family who always support me with their love, patience, encouragement and constant prayers.

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DISSERTATION ABSTRACT

Name: Maher Mohammad Jaber Nour

Title: **Discontinuous Galerkin Methods for Fractional Diffusion Problems**

Major Field: MATHEMATICS

Date of Degree: April, 2016.

In this thesis, we propose and analyze various numerical schemes including finite elements, finite difference(Crank-Nicolson) and we use discontinuous Galerkin (DG) method for solving time-fractional diffusion problems. We prove the existence, uniqueness and stability of the approximate solutions, and analysis the order of convergence. To compensate the singular behavior of the continuous solution near t = 0, we employ a nonuniform graded mesh in time. In space, a class of regular, quasi uniform meshes will be used. However, to show the super-convergence properties of the optimal HDG method, we use a quasi uniform spacial meshes. We present a series of concrete numerical experiments to demonstrate our theoretical and also to illustrate the applicability of the proposed method.

ملخص الرسالة

الاسم: ماهر محمد جابر نور

عنوان الرسالة: طرق جالركن المنفصلة لحل المسائل التفاضلية الجزئية الكسرية عدديا.

التخصص: الرياضيات التطبيقية

تاريخ التخرج: 2016

في هذة الرسالة، إننا نقترح و نحلل المخططات العددية المختلفة بما في ذلك طريقة العناصر المحدودة، طريقه الفرق المحدود و جالركن المنفصلة لحل المسائل التفاضلية الجزئية الكسرية عدديا. سوف نثبت ان الحل التقريبي وحيد و مستقر، سنشتق الخطأ في القياس بين الحل التقريبي و الحل الدقيق. لتعويض السلوك الشاذ في الحل المتصل بالقرب من نقطة البداية في الزمن سوف نوظف تجزئة غير منتظمة تقوم على تركيز الاجزاء بالقرب من نقطة السلوك الشاذ للحل. بالنسبة للفضاء سوف نستخدم تجزئه شبة منتظمة لنبين خصائص فائقة التقارب ل(HDG). سوف نقدم سلسلة من التجارب لتوضيح النظريات و مدى التوافق بين الحل الدقيق والتقريبي.

Chapter 1

INTRODUCTION

1.1 Computational solutions for fractional diffusion models

This thesis considers numerical solutions for time-fractional (slow) diffusion problems by using discontinuous Galerkin (DG) and hybridizable DG (HDG) computational methods. Fractional diffusion problems under consideration arise in a variety of physical, biological and chemical applications [20, 24, 43, 46]. Over the past few decades, researchers have observed numerous porous media systems in which some key underlying random motion conforms to a model in which the diffusion is anomalously slow (fractional sub-diffusion) rather than to the classical model of diffusion.

The fractional diffusion model problem considered here captures the dynamics of some anomalous diffusion processes in which the growth of the mean square variance is slower than a Gaussian process, see [41]. In this regard, the diffusion process in the matrix phase can be modeled using fractional derivatives that account for some small-scale fractures. This approach has applications such as diffusion in a fractal geometry [39], highly heterogeneous aquifers [2] and underground environmental problems [15]. There are two distinct approaches to modeling fractional sub-diffusion; one is based on fractional Brownian motion and Langevin equations [23, 47], and the other is based on continuous time random walks (CTRW) and master equations with power law waiting time densities [30]. The fractional Brownian motion approach leads to a diffusion equation with a varying diffusion coefficient exhibiting a fractional power law scaling over time [47]. The CTRW approach leads to a diffusion equation with fractional order temporal derivatives operating on the spatial Laplacian [17, 30].

Also,we investigate regourisely the time-space DG method for solving numerically the following fractional diffusion problem:

$$ {}^{\mathrm{c}}\mathrm{D}^{\alpha}u(x,t) - \nabla\cdot\Big((k(x,t)\nabla u(x,t)\Big) = f(x,t), \quad \text{in } \Omega\times(0,T], \tag{1.1.1a}$$

$$ u(x,t) = g(x), \quad \text{on } \partial\Omega\times(0,T], \tag{1.1.1b}$$

$$ u(x,0) = u_0(x), \quad \text{in } \Omega. \tag{1.1.1c}$$

where $\Omega \subset R^n$ (with $n = 1, 2, 3$) is a convex polyhedral domain, $T > 0$ is fixed, ${}^{\mathrm{c}}\mathrm{D}^{\alpha}v$ is the time fractional Caputo derivative of order $0 < \alpha < 1$ defined by:

$$ {}^{\mathrm{c}}\mathrm{D}^{\alpha}v(t) = \mathrm{I}^{\alpha}\, v'(t), $$

$\mathrm{I}^{\alpha}\, v(t)$ is the Riemann-Liouville fractional integral of order α given by

$$ \mathrm{I}^{\alpha}\, v(t) := \int_0^t \omega_\alpha(t-s)\, v(t)\, ds \quad \text{for } 0 < \alpha < 1. \tag{1.1.2}$$

Here $\omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and Γ denotes the Euler's gamma function. In the above model problem, the source term f, the initial data g, and the generalized diffusivity k are assumed to be sufficiently regular on their respective domains. Also, we assume that $k(x,t) \geq k_0 > 0$ in $\Omega\times(0,T]$.

The nonlocal nature of the Caputo derivative operator ${}^{\mathrm{c}}\mathrm{D}^{\alpha}$ means that at each time level, it is required to efficiently evaluate a sum of integrals over all preceding time levels. This will increase substantially the number of iterations as well as the number of active operations (storage). Thus, reducing the number of time-steps and at the same time maintaining high accuracy is wanted.

For the spatial discretization of the model problem (1.1.1) , we focus on developing and analyzing a hybridizable discontinuous Galerkin (HDG) method. The HDG methods share with the classical (hybridized version of the) mixed finite element methods their remarkable convergence properties, as well as the way in which they can be efficiently implemented. They provide approximations that are more accurate than the ones given by any other DG method for second-order elliptic problems [9].

The time discretizatrion, low-order and high-order DG methods will be studied very extensively. Since their inception in the early 1970s, time-stepping DG methods were applied for various practical time-dependent model problems including fractional and classical parabolic problems and first order hyperbolic problems (conservation law). Their advantages include excellent stability properties even for highly non-uniform meshes and suitability for adaptive refinement based on a posteriori error estimates to handle problems with low regularity.

The stability and the convergence of the proposed numerical schemes will be studied in details. Variable time steps will be used to compensate for the singular behaviour of the exact solution near $t = 0$. Various numerical tests will be delivered to illustrate numerically theoretical convergence results.

1.2 Thesis outline

In the next chapter, some notations that will be used throughout the thesis will be introduced. We also state various technical results including some essential properties of the fractional integral and fractional derivative operators.

In Chapter 3, for exact time marching, we develop the HDG scheme in space for the fractional diffusion problem 1.1.1 that allow us to approximate the solution and the flux. For the convergence analysis, a specific discontinuous projection introduced by Bernardo and others [8] will be used as the comparison function. A convergence of order $r+1$ will be proved for approximate solutions in $L_2(\Omega)$ with respect to the space and $L_\infty(0,T)$ with respect to time. Here, r is the degrees of the HDG approximate solutions.

To demonstrate the theoretical convergence results of the semi-discrete HDG scheme, we discretize in time using Crank-Nicolson (CN) [see [31, 33]] in Chapter 4. This will define a fully discrete CN-HDG scheme. The existence and uniqueness of the CN-HDG solution will be proved. The CN scheme is second-order accurate provided that the continuous solution is sufficiently regular. However, due to the singular behavior of the exact solution near $t = 0$, and so, second order accuracy in time is not feasible. Nonuniform time graded meshes will be employed to compensate for the singularity of the continuous solution near $t = 0$.

In Chapter 5, we will seek a better approximation to u by means of an element-by-element HDG postprocessing. We show that the postproccessed HDG solution converges with a super rate of order h^{k+2} for $k \leq 1$ assuming that the mesh is quasi-uniform. As in Chapter 4, we discretize in time using the generalized CN scheme. The achieved convergence rate will be illustrated numerically on a sample of test problems.

In Chapter 6, we propose and analyze a piecewise-linear, time-stepping DG method to solve numerically the fractional diffusion equation (1.1.1). For completeness, a continuous standard finite element method for spatial discretization will be investigated which will then define a fully discrete scheme. Existence, uniqueness and stability of the numerical solution will be shown. Also, the convergence of the numerical scheme will be studied in details. Optimal convergence results from the spatial discretization is proved. For the error from the time-stepping DG scheme, a sub-optimal convergence rate of order $k^{2-\frac{(1-\alpha)}{2}}$ is proved, where nonuniform time graded meshes based on concentrating the time-steps near $t = 0$ will be employed to compensate for the singular behavior of the continuous solution. However, our numerical experiments suggest optimal convergence rates in both time and space variables.

In Chapter 7, we extend piecewise linear time-stepping DG method in Chapter 6 to high order. That is, we consider a DG scheme of order $m \geq 1$ for the time discretization. The error analysis will be carried out. Convergence rates of order $k^{s+1-\frac{1-\alpha}{2}}$ will be proved, where k is the maximum time step size. The achieved convergence rate is short by order $\frac{1-\alpha}{2}$ from being optimal. In the last chapter, we will combine the time-stepping DG with the spatial HDG method. The existence and uniqueness of the fully discrete DG scheme will be studied. The implementation of the fully DG scheme will be discussed. Some numerical results will be given

1.3 Literature review

Numerical methods for (1.1.1) were proposed and analyzed by several authors. For *one dimensional* case, Zhao et.al [51] considered the initial-boundary value problem for the time fractional differential equation (1.1.1) with Neumann boundary condition. He transformed the problem into an equivalent system of lower order differential equations, a novel technique was used to proof both stability and convergence. He obtained a global convergence of order in maximum norm is $(k^{2-\alpha+h^2})$. Li and Xu [22], they proposed a spectral method in both time and spatial discretizations for the time fractional diffusion equation (1.1.1). The convergence of the method was proven and numerical experiments were applied to confirm the theoretical results. With $\Omega = (0, L)$, Alikhanov [3] constructed a new difference analog of the time fractional Caputo derivative(called $L_2 - 1_\sigma$) with the order of approximation $O(k^{3-\mu})$. Zhao and Xu [52] proposed a compact difference scheme for (1.1.1), Stability and convergence properties of the scheme were proved. For time fractional convection-diffusion problems, Cui [13] studied a compact exponential scheme. The stability and the convergence analysis were showed assuming that the coefficients of the model problem are constants. For time independent coefficients, Saadatmandi et al. [42] investigated the Sinc-Legendre collocation method. Zhang et al. [50] studied two classes of finite difference (FD) methods. Stability properties were provided. Recently, a similar convergence rate was shown by Zeng et al. [49] where the fractional linear multistep method was used for the time discretization.

For two or three dimensional cases, A global uniform convergence of order $O(t^{2-\alpha}+h^2)$ was proved. Mustapha et.al [33], proposed and analyzed a time stepping discontinuous petrov-Galerkin(DPG) method combined with the standard continuous finite element method (FEM) in space. Jin et.al [19] investigated the numerial solution of (1.1.1) by using the seme-discrete Galerkin FEM and lumped mass Galerkin FEM for the spatial discretization, by using piecewise linear functions. They established optimal error with respect to the regularity error estimates, including the cases of smooth and nonsmooth initial data. In my paper [37], a hybridizable DG method in space was extensively studied by Mustapha et al. Mustapha [31] studied a semidiscrete in time and fully discrete schemes, Crank-Nicolson in time and finite elements in space, and derived error bounds for smooth initial data. For *three-dimensional* spatial domains, a fractional ADI scheme was proposed and analyzed by Chen et al. [7]. [40], the authors combined the order reduction approach and L_1 discretization of the fractional derivative that considered by Oldham and Spanier and constructed a box-type scheme.

Various numerical methods have been applied to the following fractional subdiffusion model:

$$u' + D^{1-\alpha}Au(t) = f(t) \qquad \text{in } \Omega \times (0, T]. \tag{1.3.1}$$

Equations (1.1.1) and (1.3.1) are equivalent under certain conditions on their exact solution(if f and the initial condition are given such that u is sufficiently regular, then they have the same solution). However, the numerical approach for these problems are formally different.

Many authors solved problem (1.3.1) for example, Mclean and Thomee [29] developed a numerical method based on spatial finite element discretization and Laplace transformation with quadrature in time for (1.3.1) with a homogeneous Dirichlet boundary data. They proved that maximum-norm error estimate of order $O(t^{-1-\alpha}h^2l_h^2)$, $l_h = \| \ln h\|$ for initial data $v \in L_\infty(\Omega)$.

Mustapha and Mclean [28] employed a piecewise-constant, discontinuous Galerkin method for the time discretization of a sub-diffusion equation. They proved an a priori error bound of order k under realistic assumptions on the regularity of the solution. They also showed that a spatial discretization using continuous, piecewise-linear finite elements leads to an additional error term of order $h_2 \max(1, \log k^{-1})$.

Mustapha and Mclean [35] used a piecewise-linear, discontinuous Galerkin method for the time discretization of a fractional diffusion equation involving $0 < \alpha < 1$. Their analysis showed that, for a time interval $(0, T)$ and a spatial domain Ω, the error in $L_\infty\Big((0,T); L_2(\Omega)\Big)$ is of order $k^{(2+\alpha)}$. They employed a non-uniform mesh based on concentrating the cells near the singularity. In the limiting case, $\alpha = 0$, we recover the known $O(k^2)$ convergence for the classical diffusion (heat) equation. They also considered a fully-discrete scheme that employs standard (continuous) piecewise-linear finite elements in space, and showed that the additional error is of order $h^2 \log(1/k)$.

Finite difference method was proposed by Yuste and Acedo [48] to solve fractional diffusion equation (1.3.1) at $n = 1$. They showed, a truncation error was $O(k + h^2)$ when u is sufficient smooth at $t = 0$. Chen *et al.* [6] solved the problem (1.3.1) at $n = 2$ by using the finite difference method for space and Grunwald–Letnikov expansion for time. The order of convergence was proved equals to $O(k + h^2)$.

Cui [11] proposed high-order compact finite difference scheme (After approximating the second-order derivative with respect to space by the compact finite difference, they used the Grunwald-Letnikov discretization of the Riemann-Liouville derivative to obtain a fully discrete implicit scheme) and he proved the method has accuracy of four in the spatial grid size and one in the fractional time step, provided u is sufficiently smooth.

Mustapha [31] studied an implicit finite-difference Crank-Nicolson method in time combined with spatial piecewise-linear finite elements(FEs) scheme for solving fractional diffusion equation (1.3.1) . Convergence of order $O(h^2 + k^{2+\alpha})$ was proven. A time-space FD scheme was studied recently in [34] where convergence of order $O(h^2 + k^{2+\alpha})$ was achieved.

A compact ADI scheme was studied recently in [12]. This method is used to split the original problem into two separate one-dimensional problems. The local truncation error was analyzed and the stability was discussed by the Fourier method. Recently, high order *hp*-DG methods with exponential rates of convergence for fractional diffusion (1.3.1) and also for fractional wave equations were studied in [32].

Chapter 2

PRELIMINARIES

In this chapter we introduce some basic notations, definitions and theorems that will be used throughout the dissertation.

2.1 Spaces

Definition 1 ([20]) We denote by $L_p(\mathbf{D})$ the space of all Lebesgue real-valued measurable functions v defined on any bounded domain $\mathbf{D} \subseteq \mathbb{R}^n$ for which $\|v\|_p < \infty$, where

$$\|v\|_p := \left(\int_{\mathbf{D}} v^p(x)\, dx \right)^{1/p}.$$

The space $L_2(\mathbf{D})$ is equipped with the inner product

$$(v, w) := \int_{\mathbf{D}} v(x)\, w(x)\, dx.$$

We use $\langle u, v \rangle_{\partial \mathbf{D}}$ for the L_2-inner product on $\partial \mathbf{D}$ (boundary of $\mathbf{D}$)

Definition 2 ([21, 45]) Given an integer $k \geq 1$, the distributional derivative of order k of $f \in L_{1,loc}$ is the linear functional,

$$D^m v(\phi) = (-1)^m \int_{\mathbf{D}} v(x) D^m \phi(x)\, dx, \qquad \forall \phi \in C_c^\infty(\mathbf{D}),\ |m| \leq k.$$

If there exists a locally integrable function g such that $D^m v(\phi) = g(\phi)$, namely

$$\int_{\mathbf{D}} g(x)\phi(x)\, dx = (-1)^m \int_{\mathbf{D}} v(x) D^m \phi(x)\, dx, \qquad \forall \phi \in C_c^\infty(\mathbf{D}),$$

then we say that g is the weak derivative of order k of f, where C_c^∞ denotes the space of continuous functions with compact support, having continuous derivatives of every order and $m = (m_1, \cdots, m_n)$ is a n-vector where the m_i are non-negative integers,

$|m| = \sum_{i=1}^{n} m_i$ and $D^m\phi = \frac{\partial^{|m|}\phi}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}$.

Definition 3 ([21]) We define $W^{r,p}(\mathbf{D})\,, r \geq 0,\; 1 \leq p \leq +\infty$, to be the space of all functions whose weak derivatives of order $\leq r$ belong to $L_p(\mathbf{D})$, i.e.,

$$W^{r,p}(\mathbf{D}) = \{v \in L_p(\mathbf{D}) : D^m v \in L_p(\mathbf{D}) \;\text{ for }\; |m| \leq r\}.$$

The norm of $W^{r,p}(\mathbf{D})$ is define as

$$\|v\|_{W^{r,p}}(\mathbf{D}) := \left(\sum_{|m|=r} \|D^m v\|^p_{L^p(\mathbf{D})}\right)^{1/p}$$

The space $H^r(\mathbf{D})$ denotes $W^{r,2}(\mathbf{D})$. The norm and the seminorm on the space $H^r(\mathbf{D})$ are denoted respectively:

$$\|v\|_{r,\mathbf{D}} := \|v\|_{H^r}(\mathbf{D}) = \left(\sum_{|m|\leq r} \|D^m v\|^2\right)^{1/2},$$

$$|v|_{r,\mathbf{D}} := |v|_{H^r} = \left(\sum_{|m|=r} \|D^m v\|^2\right)^{1/2}$$

We also denote $\|\cdot\|_{X(0,T;Y(\mathbf{D}))}$ by $\|\cdot\|_{X(Y)}$

where $D^m v$ is the weak derivative of order k of v.

Also, we introduce the space,

$$H_0^1 := \{v \in H^1 : \text{ with zero trace}\},$$

Remark 1

If $v \in C^k(\mathbf{D})$, then its weak derivative $D^m v$ of order $|m| \leq k$ coincides with the corresponding partial derivative (in the classical pointwise sense); $D^m v := \frac{\partial^{|m|}v}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}$.

Definition 4 ([16]) A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \epsilon$ for any finite collection $\{[a_i, b_i] : 1 \le i \le N\}$ of non-overlapping subintervals $[a_i, b_i]$ of $[a, b]$ with $\sum_{i=1}^{N} |b_i - a_i| < \delta$.

Definition 5 For $\ell \in \{0, 1\}$, we let $\mathcal{C}^\ell(J_n, L^2(\Omega))$ denote the space of functions $v : J_n \to L^2(\Omega)$ such that the restriction $v|_{I_j}$ extends to an ℓ-times continuously differentiable function on the closed interval $\overline{I}_j$ for $1 \le j \le n$.

Definition 6 $W^{m,p}\big((0, t_n), L^2(\mathbf{D})\big)$ is the space of functions $v : (0, t_n) \to L^2(\mathbf{D})$ such that $\|v\|_{L^2(\mathbf{D})} \in W^{m,p}(0, t_n)$

Definition 7 ([20]) *(i)* The Riemann–Liouville fractional integral operator I^α is defined by

$$\mathrm{I}^\alpha v(t) := \begin{cases} \int_0^t \omega_\alpha(t-s) v(s)\, ds, & \alpha > 0, \\ v(t), & \alpha = 0, \end{cases}$$

where

$$\omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} \quad \text{and} \Gamma(t) := \int_0^\infty x^{t-1}\, e^{-x}\, dx$$

is the standard gamma function.

(ii) The Riemann–Liouville fractional derivative ${}^R\mathrm{D}^\alpha$, $\alpha > 0$, is defined by

$${}^R\mathrm{D}^\alpha v(t) := \mathrm{D}^n \mathrm{I}^{\mathrm{n}-\alpha} v(t) = \frac{d^n}{dt^n} \int_0^t \omega_{n-\alpha}(t-s) v(s)\, ds \quad \text{where} \quad n = \lfloor \alpha \rfloor + 1,$$

if it exists, where $\lfloor \alpha \rfloor$ is the greatest integer, less than or equals to α.

(iii) The Caputo fractional derivative ${}^{c}\!D^{\alpha}$, $\alpha > 0$, is defined by

$$ {}^{c}\!D^{\alpha}v(t) := \mathrm{I}^{n-\alpha}\mathrm{D}^{n}v(t) = \int_0^t \omega_{n-\alpha}(t-s)\,\frac{d^n v(s)}{ds^n}\,ds, \quad \text{where } \ n = -\lfloor -\alpha \rfloor, $$

if it exists.

Properties 8 ([18, 20], Some properties of ${}^{R}\mathrm{D}^{\alpha}$ and I^{α})

(i) Let $0 < \alpha < 1$, $n = 1$. Assume that v is such that both ${}^{R}\mathrm{D}^{\alpha}v$ and ${}^{c}\mathrm{D}^{\alpha}v$ exist. Then

$$ {}^{R}\mathrm{D}^{\alpha}v(t) := \frac{v(0)\,t^{-\alpha}}{\Gamma(1-\alpha)} + {}^{c}\!D^{\alpha}v(t). $$

(ii) (**Semi Group Inequality)** If $\alpha > 0$ and $\beta > 0$, then

$$ I^{\alpha+\beta}v = I^{\alpha}I^{\beta}v, $$

is satisfied at almost every point in $[0, T]$ for $v \in L_p(0, T)$, $1 \le p \le \infty$.

2.2 Classical inequalities

In this section, we display some inequalities that we will use in the next chapters.

Inequalities 9

(i) ([45], **Cauchy-Schwarz Inequality)** If $v, w \in L_2(0, T)$, then $v\,w \in L_1(0, T)$ and

$$ |\langle v, w\rangle| \le \|v\|\|w\|. $$

(ii) ([21], Poincare's Inequality) If **D** is a bounded domain in $\mathbb{R}^n$, then there exists a constant C such that

$$\|v\|_1 \leq C\|\nabla v\|, \quad \forall v \in H_0^1.$$

(iii) (Geometric Arithmetic Mean Inequality) If $a, b \in \mathbb{R}$, then for any $\epsilon > 0$,

$$ab \leq \frac{\epsilon\, a^2}{2} + \frac{b^2}{2\,\epsilon}.$$

(iv) ([10], Lemma 4, integral inequality) Suppose that,

$$E^2(t) \leq A(t) + 2\int_0^t B(s)\, E(s)\, ds, \quad \text{for any t} \geq 0$$

for some nonnegative functions A and B.

Then,

$$E(T) \leq \max_{t\in(0,T)} A^{1/2}(t) + \int_0^T B(s)\, ds \;\; \text{for any} \;\; T > 0.$$

2.3 Additional results

Definition 10 ([4, 44]) A set $\{\varphi_j\}_{j=1}^{\infty}$ in a Hilbert space V is orthonormal if $(\varphi_i, \varphi_j) = \delta_{i,j}$, where $\delta_{i,j}$ defined by

$$\delta_{i,j} = \begin{cases} 1, & \text{for} \;\; i = j, \\ 0, & \text{for} \;\; i \neq j. \end{cases}$$

Definition 11 ([4, 44]) A set $\{\varphi_j\}_{j=1}^{\infty}$ is said to be a basis for a Hilbert space V if every g in V can be expressed uniquely in the form $g = \sum_{i=1}^{\infty} c_i\, \varphi_i$.

where c_i are the scalars. The numerical coefficient c_i are the coordinate of u in the basis.

Remark 2 ([4, 44])

A set $\{\varphi_j\}_{j=1}^{\infty}$ is said to be a complete orthonormal set for a Hilbert space V or orthonormal basis for a Hilbert space V if it is orthonormal in V and is a basis for V.

Equalities 12 For $0 < \alpha,\ \beta < 1$ and $t > 0$ we have

(i)

$$\int_0^t (t-s)^{\alpha-1}\, s^{\beta-1}\, ds := \frac{t^{\alpha+\beta-1}\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

(ii)

$$\int_0^t \omega_\alpha(t-s)\,\omega_\beta(s)\, ds := \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)},$$

(iii)

$$\int_s^\tau \omega_{1-\alpha}(t-s)\,\omega_\alpha(\tau-t)\, dt := 1,$$

Theorem 13 ([16],[20]:page 2, Fundamental Theorem of Calculus) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f is differentiable almost everywhere with integrable derivative such that $f(t) = \int_a^t f'(x)\, dx + f(a)$ holds if and only if f is absolutely continuous.

Theorem 14 ([21], Green's Formula) Let $u \in H^2(\mathbf{D})$ and $v \in H^1(\mathbf{D})$, then

$$\int_{\mathbf{D}} \nabla u\, \nabla v dx = \int_{\partial\mathbf{D}} \frac{\partial u}{\partial n} v ds - \int_{\mathbf{D}} \Delta u\, v dx,$$

where $\frac{\partial u}{\partial n} = n.\nabla u$ is the outward normal derivative of u on $\partial\mathbf{D}$.

Theorem 15 Let f be a function such that f and f_t are continuous in x and t in some region of the (t,x)-plane, including $g(t) \le x \le h(t),\ t_0 \le t \le t_1$. Also assume that $g(t)$ and $h(t)$ are continuous and their derivatives are also continuous for $t_0 \le t \le t_1$. Then for $t_0 \le t \le t_1$

$$\frac{d}{dt}\int_{g(t)}^{h(t)} f(t,x)\,dx = f(t,h(t)).h'(t) - f(t,g(t)).g'(t) + \int_{g(t)}^{h(t)} \frac{\partial}{\partial t} f(t,x)dx \qquad (2.3.1)$$

2.4 Fractional inequalities

In this section, we state some important fractional inequalities that we will use through the dissertation.

Lemma 16 For $1 \le n \le N$ and for $0 < \alpha < 1$, we have

(i) For v in $\mathcal{C}^1(J_n, L^2(\Omega))$(a function with first continuous derivative) or in $W^{1,1}((0,t_n), L^2(\Omega$ the fractional derivative ${}^R\mathrm{D}^\alpha$ satisfies:

$$\int_0^{t_n} \langle {}^R\mathrm{D}^\alpha v, v\rangle\,dt \ge c'_\alpha t_n^{-\alpha} \int_0^{t_n} \|v(t)\|^2\,dt \quad \text{with} \quad c'_\alpha = \pi^\alpha \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}} \cos\left(\frac{\alpha\pi}{2}\right).$$

(ii) The integral operator I^α satisfies: for $v,\ w \in \mathcal{C}(J_n, L^2(\Omega))$

$$\left|\int_0^{t_n} \langle I^\alpha v, w\rangle\,dt\right|^2 \le \frac{1}{c_\alpha^2}\int_0^{t_n} \langle I^\alpha v, v\rangle\,dt \int_0^{t_n} \langle I^\alpha w, w\rangle\,dt \quad \text{with} \quad c_\alpha := \cos(\frac{\alpha\pi}{2})$$

(iii) $2\left|\int_0^T (v, \mathcal{I}^\alpha w)\,dt\right| \le \int_0^T \left(\frac{1}{c_\alpha^2}(v,\mathcal{I}^\alpha v) + (w, \mathcal{I}^\alpha w)\right)dt$

(iv) $\int_0^T (\mathcal{I}^\alpha v(t), v(t))\,dt \ge c_\alpha \int_0^T \|\mathcal{I}^{\frac{\alpha}{2}} v(t)\|^2\,dt \quad$ for $v \in \mathcal{C}(0,T; L_2(\mathbf{D}))$

(v) $\int_0^T v(t)w(t)\,dt \le \frac{1}{c_\alpha}|v|_{-\alpha}\,|w|_\alpha$

(vi) $\int_0^T {}^R\mathrm{D}^\alpha\, v(t)w(t)\,dt \le \frac{1}{c_\alpha}|v|_{-\alpha}\,|w|_{-\alpha}$

Proof 1 The property (i) was proven in [26, Theorem A.1] by using the Laplace transform and Plancherel Theorem. For the proof of the property (ii), see [38, Lemma 3.1].

The inequality (iii) follows from (ii) and geometric arithmetic mean inequality. □

Lemma 17 If $0 < \alpha < 1$, $u \in AC[0,T]$ and ${}^{c}\mathcal{D}^{1-\frac{\alpha}{2}}u \in L_2(0,T)$, then we have

$$|u(t)|^2 \leq 2|u(0)|^2 + 2\frac{T^{1-\alpha}}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})} \int_0^T |{}^{c}\mathcal{D}^{1-\frac{\alpha}{2}}u(s)|^2\, ds. \tag{2.4.1}$$

for $0 < t \leq T$.

proof: see [1]

2.5 Notations

In this section, we state different notations that will use in the thesis.

(i) $|v|_\alpha^2 = \int_0^T {}^{R}\mathrm{D}^{\alpha} v\, v\, dt$,

(ii) $\|v\|_\alpha^2 = \int_0^T ({}^{R}\mathrm{D}^{\alpha} v, v)\, dt$ for $v \in [0,T] \times \mathbf{D}$

(iii) we introduce a partition of the time interval [0, T] given by $0 < t_1 < t_2 < t_3 < \cdots < t_N = T$, and $I_k = (t_{k-1}, t_k]$, then, we denote J_n as $J_n := \cup_{j=1}^{n} I_j$ where, N is a positive integer.

(iv) $\|v\|_{I_n} := \sup_{t \in I_n} \|v(t)\|$ and $\|v\|_J := \max_{n=1}^{N} \|v\|_{I_n}$.

Chapter 3

CONVERGENCE OF HDG METHODS FOR SPATIAL DISCRETIZATION

We use the hybridizable discontinuous Galerkin (HDG) method for the spatial discretization to solve the time fractional diffusion model, where this work was not in the literature review:

$$
\begin{aligned}
&{}^{\mathrm{c}}\mathrm{D}^{\alpha}u(x,t) - \Delta u(x,t) = f(x,t) && \text{in } \Omega \times (0,T], && (3.0.1a)\\
&q(x,t) = -\nabla u(x,t), && && (3.0.1b)\\
&u(x,t) = g(x) && \text{on } \partial\Omega \times (0,T], && (3.0.1c)\\
&u(x,0) = u_0(x) && \text{on } \Omega. && (3.0.1d)
\end{aligned}
$$

For each time $t \in [0,T]$, the HDG approximations$(u_h,\ \boldsymbol{q}_h)$ of (3.0.1) are taken to be piecewise polynomials of degree $k \geq 0$ on the spatial domain Ω. they are approximations to the exact solution u in the $L_\infty\big(0,T;L_2(\Omega)\big)$-norm and to ∇u in the $L_\infty\big(0,T;\mathbf{L}_2(\Omega)\big)$-norm, the order of convergence is proven with rate h^{k+1} provided that u is sufficiently regular, where h is the maximum diameter of the elements of the mesh.

3.1 The HDG method

This section is devoted to defining a scalar approximation $u_h(t)$ to $u(t)$, a vector approximation $\boldsymbol{q}_h(t)$ to the flux $\boldsymbol{q}(t)$, and a scalar approximation $\widehat{u}_h(t)$ to the trace of $u(t)$ on element boundaries for each time $t \in [0,T]$, using a spatial HDG method. We begin by discretizing the domain $\Omega \subseteq \mathbb{R}^n$ by a conforming triangulation (the commonality between triangles is an edge or vertex or nothing) $\mathcal{T}_h$ made of elements K; we denote by $\partial\mathcal{T}_h$ the set of all the boundaries ∂K of the elements K of $\mathcal{T}_h$. We denote by $\mathcal{E}_h$ the union of faces F of the simplexes K of the triangulation $\mathcal{T}_h$.

Next, we introduce the discontinuous finite element spaces:

$$W_h = \{w \in L^2(\Omega)\colon w|_K \in \mathcal{P}_k(K), \qquad \forall\, K \in \mathcal{T}_h\}, \tag{3.1.1a}$$

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in [L_2(\Omega)]^d\colon\ \boldsymbol{v}|_K \in [\mathcal{P}_k(K)]^d, \ \ \forall\, K \in \mathcal{T}_h\}, \tag{3.1.1b}$$

$$M_h = \{\mu \in L^2(\mathcal{E}_h) :\ \mu|_F \in \mathcal{P}_k(F), \qquad \forall\, F \in \mathcal{E}_h\}, \tag{3.1.1c}$$

where $\mathcal{P}_k(K)$ is the space of polynomials of degree at most k in the spatial variable, $[L_2(\Omega)]^d$ is a space of vectors such that each component is function in $L_2(\Omega)$ To describe HDG scheme, we choose the first two equation of the system (3.0.1):

$$\boldsymbol{q} + \nabla u = 0, \tag{3.1.2a}$$

$${}^{c}\mathrm{D}^{1-\alpha} u - \nabla \cdot \boldsymbol{q} = f, \tag{3.1.2b}$$

taking the inner product (3.1.2a) and (3.1.2b) with $\boldsymbol{\phi} \in H(\mathrm{div}, \Omega)$, and $\chi \in H^1(\Omega)$, respectively. We obtain

$$(\boldsymbol{q}, \boldsymbol{\phi}) + (\nabla u, \boldsymbol{\phi}) = 0 \qquad \forall\, \boldsymbol{\phi} \in H(\mathrm{div}, \Omega),$$

$$({}^{c}\mathrm{D}^{1-\alpha} u, \chi) + (\nabla \cdot \boldsymbol{q}, \chi) = (f, \chi) \qquad \forall\, \chi \in H^1(\Omega)$$

Using the green's formula, we get the weak formulation of our problem:

$$(\boldsymbol{q}, \boldsymbol{\phi}) - (u, \nabla \cdot \boldsymbol{\phi}) + \langle u, \boldsymbol{\phi} \cdot \boldsymbol{n} \rangle = 0 \qquad \forall\, \boldsymbol{\phi} \in H(\mathrm{div}, \Omega), \tag{3.1.3a}$$

$$({}^{c}\mathrm{D}^{1-\alpha} u, \chi) - (\boldsymbol{q}, \nabla \chi) + \langle \boldsymbol{q} \cdot \boldsymbol{n}, \chi \rangle = (f, \chi) \quad \forall\, \chi \in H^1(\Omega)\,. \tag{3.1.3b}$$

For each $t > 0$, the HDG method provides approximations $u_h(t) \in W_h$, $\boldsymbol{q}_h(t) \in$

$\boldsymbol{V}_h$, and $\widehat{u}_h(t) \in M_h$, the trace of $u(t)$. These are determined by requiring that

$$(\boldsymbol{q}_h, \boldsymbol{r}) - (u_h, \nabla\cdot\boldsymbol{r}) + \langle \widehat{u}_h, \boldsymbol{r}\cdot\boldsymbol{n}\rangle = 0, \qquad \forall\, \boldsymbol{r} \in \boldsymbol{V}_h, \tag{3.1.4a}$$

$$({}^{c}\mathrm{D}^{1-\alpha} u_h, w) - (\boldsymbol{q}_h, \nabla w) + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, w\rangle = (f, w), \qquad \forall\, w \in W_h, \tag{3.1.4b}$$

$$\langle \widehat{u}_h, \mu\rangle_{\partial\Omega} = \langle g, \mu\rangle_{\partial\Omega}, \quad \forall\, \mu \in M_h, \tag{3.1.4c}$$

$$\langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle - \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega} = 0, \qquad \forall\, \mu \in M_h, \tag{3.1.4d}$$

and take the numerical trace for the flux as

$$\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h + \tau\left(u_h - \widehat{u}_h\right)\boldsymbol{n} \quad \text{ on } \partial\mathcal{T}_h, \tag{3.1.4e}$$

for some nonnegative stabilization function τ defined on $\partial\mathcal{T}_h$; we assume that, for each element $K \in \mathcal{T}_h$, $\tau|_{\partial K}$ is constant on each of its faces. At $t = 0$, $u_h(0) \in W_h$ approximates the initial solution u_0.

The first two equations are inspired by the weak form of the fractional differential equations satisfied by the exact solution, (8.1.1). The form of the numerical trace given by (3.1.4d) allows us to express $(u_h, \boldsymbol{q}_h, \widehat{\boldsymbol{q}}_h)$ elementwise in terms of $\widehat{u}_h$ and f by using equations (3.1.4a), (3.1.4b) and (3.1.4e). Then, the numerical trace $\widehat{u}_h$ is determined by as the solution of the transmission condition (3.1.4d), which enforces the single-valuedness of the normal component of the numerical trace $\widehat{\boldsymbol{q}}_h$, and the boundary condition (3.1.4c). Thus, the only globally-coupled degrees of freedom are those of $\widehat{u}_h$.

3.2 Error estimates

In this section, we carry our a priori error analysis of the HDG method. We also use the integral inequality [4]

Next, we define the projections which play the comparison function role in the error analysis. For each $t \in (0, T]$, we assume that $\boldsymbol{q}(t) \in [H^1(\mathcal{T}_h)]^d$ and $u(t) \in H^1(\mathcal{T}_h)$, where $H^1(\mathcal{T}_h) = \prod_{K\in\mathcal{T}_h} H^1(K)$, the projections $\boldsymbol{\mathit{\Pi}}_V \boldsymbol{q}(t) \in \boldsymbol{V}_h$ and $\mathit{\Pi}_W u(t) \in W_h$ are defined by: on each simplex $K \in \mathcal{T}_h$ and for all faces F of K,

$$(\boldsymbol{\mathit{\Pi}}_V \boldsymbol{q}(t), \boldsymbol{v})_K = (\boldsymbol{q}(t), \boldsymbol{v})_K, \tag{3.2.1a}$$

$$(\mathit{\Pi}_W u(t), w)_K = (u(t), w)_K, \tag{3.2.1b}$$

$$\langle \boldsymbol{\mathit{\Pi}}_V \boldsymbol{q}(t) \cdot \boldsymbol{n} + \tau \mathit{\Pi}_W u(t), \mu \rangle_F = \langle \boldsymbol{q}(t) \cdot \boldsymbol{n} + \tau u(t), \mu \rangle_F, \tag{3.2.1c}$$

for al $\boldsymbol{v} \in [\mathcal{P}_{k-1}(K)]^d$, $w \in \mathcal{P}_{k-1}(K)$ and $\mu \in \mathcal{P}_k(F)$. This projection introduced in [8] to study HDG methods for the steady-state diffusion problem and also used in the error analyses of HDG methods for classical diffusion [5] as well as for fractional subdiffusion [10] problems. As mentioned in [8], The projection $\boldsymbol{\mathit{\Pi}}_V$ depends not only on q, but rather on both q and u. Similarly for the projection $\mathit{\Pi}_W$. Hence the notations $\boldsymbol{\mathit{\Pi}}_V$ and $\mathit{\Pi}_W$ are somewhat misleading but convenient.By the definition 3.2.1 Its approximation properties are described in the following result.

Theorem 18 ([8]) Suppose $\tau|_{\partial K}$ is nonnegative and $\tau_K^{\max} := \max \tau|_{\partial K} > 0$. Then the system (3.2.1) is uniquely solvable for $\boldsymbol{\mathit{\Pi}}_V \boldsymbol{q}$ and $\mathit{\Pi}_W u$. Furthermore, there is a constant C independent of K and τ such that for each $t \in (0,T]$,

$$\|e_{\boldsymbol{q}}(t)\|_K \leq C\, h_K^{k+1} \left(|\boldsymbol{q}(t)|_{\boldsymbol{H}^{k+1}(K)} + \tau_K^* \, |u(t)|_{H^{k+1}(K)} \right),$$
$$\|e_u(t)\|_K \leq C\, h_K^{k+1} \left(|u(t)|_{H^{k+1}(K)} + |\nabla \cdot \boldsymbol{q}(t)|_{H^k(K)} / \tau_K^{\max} \right)$$

where $e_{\boldsymbol{q}} := \boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{q}$, $e_u := \Pi_W u - u$, and h_K is the diameter of the spatial mesh element K. Here $\tau_K^* := \max \tau|_{\partial K \setminus F^*}$, where F^* is a face of K at which $\tau|_{\partial K}$ is maximum.

From theorem 18, we note that the approximation error of the projection is of order $k+1$ provided that the stabilization function is such that both τ_K^* and $1/\tau_K^{\max}$ are uniformly bounded and the exact solution is sufficiently regular.

Thus, the main task now is to estimate the terms $\varepsilon_u := \Pi_W u - u_h$ and $\boldsymbol{\varepsilon}_{\mathbf{q}} := \boldsymbol{\Pi}_V \boldsymbol{q} - \boldsymbol{q}_h$. For convenience, we further introduce the following notations: $\varepsilon_{\widehat{u}} := P_M u - \widehat{u}_h$ and $\boldsymbol{\varepsilon}_{\widehat{q}} := \boldsymbol{P}_M \boldsymbol{q} - \widehat{\boldsymbol{q}}_h$ where P_M denotes the L^2-orthogonal projection onto M_h, and $\boldsymbol{P}_M$ denotes the vector-valued projection each of whose components are equal to P_M. For later use, for each $t \in (0,T]$, (3.2.1c) is equivalent to

$$\langle \boldsymbol{\Pi}_V \boldsymbol{q}(t) \cdot \boldsymbol{n} + \tau \Pi_W u(t) - P_M(\boldsymbol{q}(t) \cdot \boldsymbol{n}) - \tau P_M u(t), \mu \rangle_F = 0 \quad \forall\, \mu \in \mathcal{P}_k(F)\,.$$

Since $\langle P_M(\boldsymbol{q}(t) \cdot \boldsymbol{n}), \mu \rangle_F = \langle \boldsymbol{q}(t) \cdot \boldsymbol{n}, \mu \rangle_F$ and $\langle P_M u(t), \mu \rangle_F = \langle u(t), \mu \rangle_F$.
Where $\mu \in P_k(F)$ so, it is in $L_2(F)$ Since $\boldsymbol{\Pi}_V \boldsymbol{q}(t) \cdot \boldsymbol{n} + \tau \Pi_W u(t) - P_M(\boldsymbol{q}(t) \cdot \boldsymbol{n}) - \tau P_M u(t) \in \mathcal{P}_k(F)$,

$$\boldsymbol{\Pi}_V \boldsymbol{q}(t) \cdot \boldsymbol{n} + \tau \Pi_W u(t) - P_M(\boldsymbol{q}(t) \cdot \boldsymbol{n}) - \tau P_M u(t) = 0 \quad \text{for each} \;\; t \in (0,T]\,. \quad (3.2.3)$$

The projection of the errors satisfy the equations stated in the next lemma.

Lemma 19 For each $t > 0$, we have

$$(\boldsymbol{\varepsilon}_{\mathbf{q}}, \boldsymbol{r}) - (\varepsilon_u, \nabla \cdot \boldsymbol{r}) + \langle \varepsilon_{\widehat{u}}, \boldsymbol{r} \cdot \boldsymbol{n} \rangle = (e_{\boldsymbol{q}}, \boldsymbol{r}), \qquad \forall\, \boldsymbol{r} \in \boldsymbol{V}_h \tag{3.2.4a}$$

$$(\mathcal{I}^{\alpha} \varepsilon_u', w) - (\boldsymbol{\varepsilon}_{\mathbf{q}}, \nabla w) + \langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, w \rangle = (\mathcal{I}^{\alpha} e_u', w), \quad \forall\, w \in W_h \tag{3.2.4b}$$

$$\langle \varepsilon_{\widehat{u}}, \mu \rangle_{\partial\Omega} = 0, \qquad \forall\, \mu \in M_h \tag{3.2.4c}$$

$$\langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \mu \rangle - \langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \mu \rangle_{\partial\Omega} = 0, \qquad \forall\, \mu \in M_h \tag{3.2.4d}$$

and we also have

$$\boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n} := \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \boldsymbol{n} + \tau(\varepsilon_u - \varepsilon_{\widehat{u}}) \quad \text{ on } \partial\mathcal{T}_h. \tag{3.2.4e}$$

Proof From (8.1.1), we recall that $\boldsymbol{q}$ and u satisfy the equations

$$\begin{aligned}
(\boldsymbol{q}, \boldsymbol{r}) - (u, \nabla \cdot \boldsymbol{r}) + \langle u, \boldsymbol{r} \cdot \boldsymbol{n} \rangle &= 0 \qquad \forall\, \boldsymbol{r} \in \boldsymbol{V}_h, \\
(\mathcal{I}^{\alpha} u', w) - (\boldsymbol{q}, \nabla w) + \langle \boldsymbol{q} \cdot \boldsymbol{n}, w \rangle &= (f, w) \quad \forall\, w \in W_h\,.
\end{aligned}$$

By the equalities $\boldsymbol{q} = \boldsymbol{\Pi}_V \boldsymbol{q} - e_{\boldsymbol{q}}$ and $u := \Pi_W u - e_u$, the fact that P_M is then L^2-projection into M_h and (3.2.1c), we get

$$\begin{aligned}
&(\boldsymbol{\Pi}_V \boldsymbol{q}, \boldsymbol{r}) - (u, \nabla \cdot \boldsymbol{r}) + \langle P_M u, \boldsymbol{r} \cdot \boldsymbol{n} \rangle = (e_{\boldsymbol{q}}, \boldsymbol{r}), \\
&(\mathcal{I}^{\alpha} (\Pi_W u)', w) - (\boldsymbol{q}, \nabla w) + \langle \boldsymbol{\Pi}_V \boldsymbol{q} \cdot \boldsymbol{n} + \tau(\Pi_W u - P_M u), w \rangle = (f + \mathcal{I}^{\alpha} e_u', w),
\end{aligned}$$

$\forall\, \boldsymbol{r} \in \boldsymbol{V}_h$ and $\forall\, w \in W_h$, given that, for each element $K \in \mathcal{T}_h$, τ is constant on each face F of K. Hence, by (3.2.1a) and (3.2.1b), we observe that

$$(\boldsymbol{\Pi}_V \boldsymbol{q}, \boldsymbol{r}) - (\Pi_W u, \nabla\cdot\boldsymbol{r}) + \langle P_M u, \boldsymbol{r}\cdot\boldsymbol{n}\rangle = (e_{\boldsymbol{q}}, \boldsymbol{r}), \quad \forall\, \boldsymbol{r} \in \boldsymbol{V}_h \tag{3.2.6}$$

$$(\mathcal{I}^\alpha(\Pi_W u)', w) - (\boldsymbol{\Pi}_V \boldsymbol{q}, \nabla w) + \langle \boldsymbol{\Pi}_V \boldsymbol{q}\cdot\boldsymbol{n} + \tau(\Pi_W u - P_M u), w\rangle$$
$$= (f + \mathcal{I}^\alpha e_u', w), \quad \forall\, w \in W_h\,. \tag{3.2.7}$$

We used the following properties:

$$(u, \nabla\cdot\boldsymbol{r})_k = (\Pi_W u, \nabla\cdot\boldsymbol{r}) \ \text{ since } \ \nabla\cdot\boldsymbol{r} \in P_{k-1}(K)$$

$$(\boldsymbol{q}, \nabla\omega)_k = (\boldsymbol{\Pi}_V \boldsymbol{q}, \nabla\omega) \ \text{ since } \ \nabla\omega \in \boldsymbol{P}_{k-1}(K)$$

and

$$P_M(\boldsymbol{q}\cdot\boldsymbol{n}) = \boldsymbol{\Pi}_V \boldsymbol{q}\cdot\boldsymbol{n} + \tau(\Pi_W u - P_M u)$$

Subtracting the equations (3.1.4a) and (3.1.4b) from (3.2.6) and (3.2.7), respectively, we obtain equations (3.2.4a) and (3.2.4b), respectively. The equation (3.2.4c) follows directly from the equation (3.1.4c) and (1.1.1b)

$$\langle \widehat{u}, \mu\rangle_{\partial\Omega} = \langle g, \mu\rangle_{\partial\Omega},$$
$$\langle \widehat{u}, \mu\rangle_{\partial\Omega} = \langle u, \mu\rangle_{\partial\Omega}, \quad \text{on the boundary}$$
$$\langle u, \mu\rangle_{\partial\Omega} = \langle P_M u, \mu\rangle_{\partial\Omega}, \quad \mathrm{P_M} \text{ is } L_2 \text{ projection}$$

So

$$\langle P_M u - u, \mu\rangle = 0 \quad \text{and} \quad \langle \varepsilon_{\widehat{u}}, \mu\rangle = 0.$$

By the definition of $\boldsymbol{\varepsilon}_{\widehat{q}}$ and since P_M is the L^2-projection into M_h, we have

$$\begin{aligned}
\langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \mu\rangle - \langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \mu\rangle_{\partial\Omega} &= \langle (\boldsymbol{P}_M \boldsymbol{q} - \widehat{\boldsymbol{q}}_h) \cdot \boldsymbol{n}, \mu\rangle - \langle (\boldsymbol{P}_M \boldsymbol{q} - \widehat{\boldsymbol{q}}_h) \cdot \boldsymbol{n}, \mu\rangle_{\partial\Omega}, \\
&= \langle (\boldsymbol{q} - \widehat{\boldsymbol{q}}_h) \cdot \boldsymbol{n}, \mu\rangle - \langle (\boldsymbol{q} - \widehat{\boldsymbol{q}}_h) \cdot \boldsymbol{n}, \mu\rangle_{\partial\Omega} \\
&= [\langle \boldsymbol{q} \cdot \boldsymbol{n}, \mu\rangle - \langle \boldsymbol{q} \cdot \boldsymbol{n}, \mu\rangle_{\partial\Omega}] - [\langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu\rangle - \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu\rangle_{\partial\Omega}] = 0,
\end{aligned}$$

where in the last equality we used that $\boldsymbol{q}$ is in $\mathbf{H}(\mathrm{div}, \Omega)$ and equation (3.1.4d). Thus, the identity (3.2.4d) holds. For the proof of (3.2.4e),

$$\begin{aligned}
\boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n} &= P_M(\boldsymbol{q} \cdot \boldsymbol{n}) - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \\
&= P_M(\boldsymbol{q} \cdot \boldsymbol{n}) - (\boldsymbol{q}_h \cdot \boldsymbol{n} + \tau\,(u_h - \widehat{u}_h)), && \text{by (3.1.4e)}, \\
&= (\boldsymbol{\mathit{\Pi}}_V \boldsymbol{q} \cdot \boldsymbol{n} + \tau\,(\mathit{\Pi}_W u - P_M u)) - (\boldsymbol{q}_h \cdot \boldsymbol{n} + \tau\,(u_h - \widehat{u}_h)), \\
&= (\boldsymbol{\mathit{\Pi}}_V \boldsymbol{q} \cdot \boldsymbol{n} - \boldsymbol{q}_h \cdot \boldsymbol{n}) + \tau\,(\mathit{\Pi}_W u - u_h) - \tau\,(P_M u - \widehat{u}_h), && \text{by (3.2.3)}, \\
&= \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \boldsymbol{n} + \tau(\varepsilon_u - \varepsilon_{\widehat{u}})\,. \quad \square
\end{aligned}$$

Lemma 20 Let $S_h := \|\sqrt{\tau}(\varepsilon_u - \varepsilon_{\widehat{u}})\|_{\partial\mathcal{T}_h}$. For $T > 0$,

$$\begin{aligned}
\int_0^T (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u')\,dt + \|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\|^2 + S_h^2(T) \le \|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) \\
+ \frac{1}{c_\alpha^2}\int_0^T (\mathcal{I}^\alpha e_u', e_u')\,dt + 2\int_0^T (e_{\boldsymbol{q}}', \boldsymbol{\varepsilon}_{\mathbf{q}})\,dt\,.
\end{aligned}$$

Proof 2 Since $(\varepsilon_u, \nabla\cdot\boldsymbol{r}) = -(\nabla\varepsilon_u, \boldsymbol{r}) + \langle \varepsilon_u, \boldsymbol{r}.\boldsymbol{n}\rangle$, (3.2.4a) can be rewritten as:

$$(\boldsymbol{\varepsilon}_{\mathbf{q}}, \boldsymbol{r}) + (\nabla\varepsilon_u, \boldsymbol{r}) + \langle \varepsilon_{\widehat{u}} - \varepsilon_u, \boldsymbol{r}\cdot\boldsymbol{n}\rangle = (e_{\boldsymbol{q}}, \boldsymbol{r})\,.$$

A time differentiation of both sides yields,

$$(\boldsymbol{\varepsilon}'_{\mathbf{q}}, \boldsymbol{r}) + (\nabla \varepsilon'_u, \boldsymbol{r}) + \langle \varepsilon'_{\widehat{u}} - \varepsilon'_u, \boldsymbol{r} \cdot \boldsymbol{n} \rangle = (e'_{\boldsymbol{q}}, \boldsymbol{r})\,.$$

Setting $\boldsymbol{r} = \boldsymbol{\varepsilon}_{\mathbf{q}}$ and choosing $w = \varepsilon'_u$ in equation (3.2.4b), we observe that

$$(\boldsymbol{\varepsilon}'_{\mathbf{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}}) + (\nabla \varepsilon'_u, \boldsymbol{\varepsilon}_{\mathbf{q}}) + \langle \varepsilon'_{\widehat{u}} - \varepsilon'_u, \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \boldsymbol{n} \rangle = (e'_{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}})\,,$$

$$(\mathcal{I}^\alpha \varepsilon'_u, \varepsilon'_u) - (\boldsymbol{\varepsilon}_{\mathbf{q}}, \nabla \varepsilon'_u) + \langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \varepsilon'_u \rangle = (\mathcal{I}^\alpha e'_u, \varepsilon'_u)\,.$$

Combining the above two equations and using

$$(\boldsymbol{\varepsilon}'_{\mathbf{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}}) = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2,$$

we obtain

$$(\mathcal{I}^\alpha \varepsilon'_u, \varepsilon'_u) + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + \psi_h = (\mathcal{I}^\alpha e'_u, \varepsilon'_u) + (e'_{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}})\,, \tag{3.2.8}$$

where

$$\psi_h = \langle \varepsilon'_{\widehat{u}} - \varepsilon'_u, \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \boldsymbol{n} \rangle + \langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \varepsilon'_u \rangle\,.$$

A time differentiation of (3.2.4c) followed by choosing $\mu = \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}$ and then using (3.2.4d) yields

$$\langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \varepsilon'_{\widehat{u}} \rangle_{\partial\Omega} = \langle \boldsymbol{\varepsilon}_{\widehat{q}} \cdot \boldsymbol{n}, \varepsilon'_{\widehat{u}} \rangle = 0.$$

Thus, by (3.2.4e),

$$\begin{aligned}\psi_h &= \langle \varepsilon'_{\widehat{u}} - \varepsilon'_u, (\boldsymbol{\varepsilon}_{\mathbf{q}} - \boldsymbol{\varepsilon}_{\widehat{q}}) \cdot \boldsymbol{n} \rangle,\\ &= \langle \tau(\varepsilon'_u - \varepsilon'_{\widehat{u}}), (\varepsilon_u - \varepsilon_{\widehat{u}}) \rangle,\\ &= \frac{1}{2}\frac{d}{dt} S_h^2(t)\,.\end{aligned}$$

Now, integrating (3.2.8) over the time interval $[0,T]$ and substituting the value of ψ_h, we get

$$\int_0^T (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u')\,dt + \frac{1}{2}\int_0^T \frac{d}{dt}[\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + S_h^2]\,dt = \int_0^T (\mathcal{I}^\alpha e_u', \varepsilon_u')\,dt + \int_0^T (e_{\boldsymbol{q}}', \boldsymbol{\varepsilon}_{\mathbf{q}})\,dt\,.$$

Therefore,

$$2\int_0^T (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u')\,dt + \|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\|^2 + S_h^2(T)$$
$$= \|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + 2\int_0^T (\mathcal{I}^\alpha e_u', \varepsilon_u')\,dt + 2\int_0^T (e_{\boldsymbol{q}}', \boldsymbol{\varepsilon}_{\mathbf{q}})\,dt\,. \quad (3.2.9)$$

An application of the continuity property of the fractional derivative operator $\mathcal{I}^\alpha$, (3), yields

$$2\Big|\int_0^T (\mathcal{I}^\alpha e_u', \varepsilon_u')\,dt\Big| \le \frac{1}{c_\alpha^2}\int_0^T (\mathcal{I}^\alpha e_u', e_u')\,dt + \int_0^T (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u')\,dt\,.$$

Finally, inserting this in (3.2.9) and simplifying completes the proof.

The task now is to estimate $\|\varepsilon_u(T)\|$ and $\|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\|$ in the following two lemmas.

Lemma 21 for $t \ge 0$, we have

$$\|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\|^2 + \int_0^T (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u')\,ds \le C\Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + \int_0^T \big(\frac{1}{4c_\alpha^2}\|\mathcal{I}^\alpha e_u'\|^2 + \|e_u'\|^2 + T\|e_{\boldsymbol{q}}'\|^2\big)ds\Big)$$

The constant C is independent of K and τ

Proof 3 From Lemma 20, for $t \geq 0$, we have $E^2(t) \leq A(t) + 2\int_0^t B(s)\,E(s)\,ds$ where

$$A(t) := \|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + \frac{1}{c_\alpha^2}\int_0^t (\mathcal{I}^\alpha e_u', e_u')\,ds, \quad B(t) := \|e_{\boldsymbol{q}}'(t)\|,$$
$$E(t) := \Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}(t)\|^2 + \int_0^t (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u')\,ds\Big)^{\frac{1}{2}}$$

Thus, an application of the integral inequality in Lemma 4, yields

$$E(T) \leq \max_{t\in(0,T)} A^{\frac{1}{2}}(t) + \int_0^T B(s)\,ds \quad \text{for any} \quad T > 0.$$

Hence,

Since $E(T)$, $\max_{t\in(0,T)} A^{\frac{1}{2}}(t)$ and $\int_0^T B(t)\,dt$ are nonnegative real values. Then

$$\begin{aligned}
E^2(T) &\leq \Big(\max_{t\in(0,T)} A^{\frac{1}{2}}(t) + \int_0^T B(t)\,dt\Big)^2 \\
&\leq \max_{t\in(0,T)} A(t) + \Big(\int_0^T B(t)\,dt\Big)^2 + 2\max_{t\in(0,T)} A^{\frac{1}{2}}(t)\int_0^T B(t)\,dt \\
&\leq 2\Big(\max_{t\in(0,T)} A(t) + (\int_0^T B(t)\,dt)^2\Big) \\
&\leq C\Big(\max_{t\in(0,T)} A(t) + \Big((\int_0^T 1^2\,dt)^{\frac{1}{2}}(\int_0^T B^2(t)\,dt)^{\frac{1}{2}}\Big)^2\Big) \\
&\leq C\Big(\max_{t\in(0,T)} A(t) + T\int_0^T B^2(t)\,dt\Big)
\end{aligned}$$

$$\begin{aligned}
&\|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\|^2 + \int_0^T (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u')\,ds \\
&\quad\leq C\Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + \int_0^T \big(\frac{1}{c_\alpha^2}(\mathcal{I}^\alpha e_u', e_u') + T\|e_{\boldsymbol{q}}'\|^2\big)ds\Big) \\
&\quad\leq C\Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + \int_0^T \big(\frac{1}{c_\alpha^2}\|\mathcal{I}^\alpha e_u'\|\,\|e_u'\| + T\|e_{\boldsymbol{q}}'\|^2\big)ds\Big) \\
&\quad\leq C\Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + \int_0^T \big(\frac{1}{4c_\alpha^2}\|\mathcal{I}^\alpha e_u'\|^2 + \|e_u'\|^2 + T\|e_{\boldsymbol{q}}'\|^2\big)ds\Big) \quad (3.2.10)
\end{aligned}$$

Lemma 22 For $t > 0$,

$$\|\boldsymbol{\varepsilon}_{\mathbf{q}}(t)\|^2 + \|\varepsilon_u(t)\|^2 \leq C_1^2(1+T)^2\Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + h^{2k+2}\Big).$$

The constant C_1 only depends on C, α, $\|u\|_{\mathcal{C}^1(H^{k+1})}$, and on $\|\boldsymbol{q}\|_{\mathcal{C}^1(H^{k+1})}$.

Proof 4 Firstly, we prove that $\varepsilon_u(t) = I^{1-\frac{\alpha}{2}}(\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u')(t)$:

$$\begin{aligned}
\varepsilon_u(t) &= \int_0^t \varepsilon_u'(s)\,ds + \varepsilon_u(0) \\
&= \int_0^t \varepsilon_u'(s)\,ds + \Pi_W u(0) - u_h(0) \\
&= \int_0^t \varepsilon_u'(s)\,ds + \Pi_W u(0) - \Pi_W u_0 \\
&= \int_0^t \varepsilon_u'(s)\,ds
\end{aligned}$$

On the other hand,

$$\begin{aligned}
I^{1-\frac{\alpha}{2}}(\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u')(t) &= \int_0^t w_{1-\frac{\alpha}{2}}(t-s)\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\,ds \\
&= \int_0^t w_{1-\frac{\alpha}{2}}(t-s)\int_0^s w_{\frac{\alpha}{2}}(s-q)\varepsilon_u'(q)\,dq\,ds
\end{aligned}$$

changing the order of the integrals, we get

$$I^{1-\frac{\alpha}{2}}(\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u')(t) = \int_0^t \int_q^t w_{1-\frac{\alpha}{2}}(t-s)w_{\frac{\alpha}{2}}(s-q)\,ds\,\varepsilon_u'(q)\,dq$$

changing of variables by assuming $\tau = s - q$. We get

$$\begin{aligned}
I^{1-\frac{\alpha}{2}}(\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u')(t) &= \int_0^t \int_0^{t-q} w_{1-\frac{\alpha}{2}}(t-q-\tau)w_{\frac{\alpha}{2}}(\tau)\,d\tau\,\varepsilon_u'(q)\,dq \\
&= \int_0^t (1)\varepsilon_u'(q)\,dq
\end{aligned}$$

So, $\varepsilon_u(t) = I^{1-\frac{\alpha}{2}}(\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u')(t)$.

By the Cauchy-Schwarz inequality and the coercivity property of the operator $\mathcal{I}^{\alpha}$, (16), we obtain,

$$
\begin{aligned}
\|\varepsilon_u(t)\|^2 &= \Big(\varepsilon_u(t), \varepsilon_u(t)\Big)\\
&= \Big(I^{1-\frac{\alpha}{2}}(\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u')(t), I^{1-\frac{\alpha}{2}}(\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u')(t)\Big)\\
&= \int_\Omega \Big[\int_0^t \omega_{1-\frac{\alpha}{2}}(t-q)\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(q)\,dq \int_0^t \omega_{1-\frac{\alpha}{2}}(t-s)\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\,ds\Big]\,dx\\
&= \int_\Omega \Big[\int_0^t \int_0^t \omega_{1-\frac{\alpha}{2}}(t-q)\omega_{1-\frac{\alpha}{2}}(t-s)\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(q)\,\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\,dq\,ds\Big]\,dx\\
&= \int_0^t \int_0^t \omega_{1-\frac{\alpha}{2}}(t-q)\omega_{1-\frac{\alpha}{2}}(t-s)\Big[\int_\Omega \mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(q)\,\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\,dx\Big]\,dq\,ds\\
&\le \int_0^t \int_0^t \omega_{1-\frac{\alpha}{2}}(t-q)\omega_{1-\frac{\alpha}{2}}(t-s)\|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(q)\|\,\|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\|\,dq\,ds
\end{aligned}
$$

Then separating the integrals and using the Cauchy-Schwarz inequality, we observe

$$
\begin{aligned}
\|\varepsilon_u(t)\|^2 &\le \int_0^t \omega_{1-\frac{\alpha}{2}}(t-s)\|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\|\,ds \int_0^t \omega_{1-\frac{\alpha}{2}}(t-q)\|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(q)\|\,dq\\
&\le \Big(\int_0^t \omega_{1-\frac{\alpha}{2}}(t-s)\,\|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\|\,ds\Big)^2\\
&\le \Big(\big(\int_0^t \omega_{1-\frac{\alpha}{2}}^2(t-s)\,ds\big)^{\frac{1}{2}}\big(\int_0^t \|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\|^2\,ds\big)^{\frac{1}{2}}\Big)^2\\
&\le \int_0^t \omega_{1-\frac{\alpha}{2}}^2(s)\,ds \int_0^t \|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\|^2\,ds\\
&= \frac{t^{1-\alpha}}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\int_0^t \|\mathcal{I}^{\frac{\alpha}{2}}\varepsilon_u'(s)\|^2\,ds\\
&\le \frac{t^{1-\alpha}}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})\,c_\alpha}\int_0^t (\mathcal{I}^\alpha\varepsilon_u', \varepsilon_u')\,ds\,.
\end{aligned}
$$

Therefore, combining (3.2.10) with the above bound, and apply Theorem 18 for the time derivative error projections e_u' and $e_{\boldsymbol{q}}'$

We know that $(\varPi_W u(t))' = \varPi_W u'$ and $(\boldsymbol{\varPi}_V \boldsymbol{q}(t))' = \boldsymbol{\varPi}_V \boldsymbol{q}'$

So,

$$e_u' = \mathit{\Pi}_W u' - u' \quad \text{and} \quad e_{\boldsymbol{q}}' = \boldsymbol{\Pi}_V \boldsymbol{q}' - \boldsymbol{q}'$$

$$\|e_u'\| \leq C_1 h^{k+1} \quad \text{and} \quad \|e_{\boldsymbol{q}}'\| \leq C_1 h^{k+1}$$

And

$$\begin{aligned} \|\mathcal{I}^{\alpha} e_u'(t)\| &\leq \int_0^t (\omega_\alpha(t-s))\|e_u'(s)\|\, ds \\ &\leq C_1 h^{k+1} \end{aligned}$$

Then

$$\begin{aligned} \|\boldsymbol{\varepsilon}_{\mathbf{q}}(t)\|^2 &\leq C\Big[\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + \int_0^T \frac{1}{4c_\alpha} C_1^2 h^{2k+2} + C_1^2 h^{2k+2} + T C_1^2 h^{2k+2}\, dt\Big] \\ &\leq C_1^2\Big[\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) + \frac{T}{4c_\alpha} h^{2k+2} + T h^{2k+2} + T^2 h^{2k+2}\Big] \end{aligned}$$

Doing the same procedure for the upper bound for$\|\varepsilon_u(T)\|$, we obtain the result

$$\begin{aligned} \|\varepsilon_u(T)\|^2 &\leq \frac{C_1^2\, T^{1-\alpha}}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})}\Big[\|\varepsilon_u(0)\|^2 + S_h^2(0) + T h^{2k+2} + T^2 h^{2k+2}\Big] \\ &\leq C_1^2\, T^{1-\alpha}\Big[\|\varepsilon_u(0)\|^2 + S_h^2(0) + (T + T^2) h^{2k+2}\Big] \end{aligned}$$

Now complete

Next, we show the main error bounds of the HDG method. We choose $u_h(0) = \mathit{\Pi}_W u_0$ and so, $\varepsilon_u(0) = 0$.

Theorem 23 Assume that $u \in \mathcal{C}^1(0, T; H^{k+1}(\Omega))$ and $\boldsymbol{q} \in \mathcal{C}^1(0, T; \boldsymbol{H}^{k+1}(\Omega))$. Then we have that

$$\|(u-u_h)(T)\| + \|(\boldsymbol{q}-\boldsymbol{q}_h)(T)\| \leq \ C_1(1+T)\,h^{k+1}.$$

Proof 5 From the decompositions: $u-u_h=\varepsilon_u-e_u$ and $\boldsymbol{q}-\boldsymbol{q}_h=\boldsymbol{\varepsilon}_{\mathbf{q}}-e_{\boldsymbol{q}}$, and the error projection in Theorem 18, we have

$$\|(u-u_h)(T)\| + \|(\boldsymbol{q}-\boldsymbol{q}_h)(T)\| \leq C_1\,h^{k+1} + \|\varepsilon_u(T)\| + \|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\|\,.$$

Since

$$\begin{aligned}
\|(u-u_h)(T)\| &\leq \|(\Pi_W u-u)(T)\| + \|(\Pi_W u-u_h)(T)\| \\
&\leq \|\varepsilon_u(T)\| + \|e_u(T)\| \\
&\leq \|\varepsilon_u(T)\| + \sum_{K\in\tau_h} \|e_u(T)\|_K \\
&\leq \|\varepsilon_u(T)\| + Ch^{k+1} \sum_{K\in\tau_h} \Big(|\boldsymbol{q}(T)|_{H^{k+1}(K)} + \tau^*|u(T)|_{H^{k+1}(K)}\Big) \\
&\leq \|\varepsilon_u(T)\| + C'\,h^{k+1}\Big(\|\boldsymbol{q}(T)\|_{H^{k+1}(\Omega)} + \tau^*\|u(T)\|_{H^{k+1}(\Omega)}\Big) \\
&\leq \|\varepsilon_u(T)\| + C_1 h^{k+1}
\end{aligned}$$

and

$$
\begin{aligned}
\|(\boldsymbol{q}-\boldsymbol{q}_h)(T)\| &\leq \|(\boldsymbol{\Pi}_V\boldsymbol{q}-\boldsymbol{q})(T)\| + \|(\boldsymbol{\Pi}_V\boldsymbol{q}-\boldsymbol{q}_h)(T)\| \\
&\leq \|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\| + \|e_{\boldsymbol{q}}(T)\| \\
&\leq \|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\| + \sum_{K\in\tau_h} \|e_{\boldsymbol{q}}(T)\|_K \\
&\leq |\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\| + Ch^{k+1} \sum_{K\in\tau_h} \left(|u(T)|_{H^{k+1}(K)} + \frac{1}{\tau^*} |\,\nabla\cdot\boldsymbol{q}(T)|_{H^k(K)} \right) \\
&\leq \|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\| + C^{''}h^{k+1} \left(\|u(T)\|_{H^{k+1}(\Omega)} + \frac{1}{\tau^*}\|\boldsymbol{q}(T)\|_{H^{k+1}(\Omega)} \right) \\
&\leq \|\boldsymbol{\varepsilon}_{\mathbf{q}}(T)\| + C_1 h^{k+1}.
\end{aligned}
$$

Now, we need to bound $\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0)$. Since

$$
(\varepsilon_u, \nabla\cdot\boldsymbol{r}) = -(\nabla\varepsilon_u, \boldsymbol{r}) + \langle \varepsilon_u, \boldsymbol{r}.\boldsymbol{n}\rangle,
$$

setting $\boldsymbol{r} = \boldsymbol{\varepsilon}_{\mathbf{q}}$ in (3.2.4a) and $w = \varepsilon_u$ in (3.2.4b) yield

$$
\begin{aligned}
\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + (\nabla\varepsilon_u, \boldsymbol{\varepsilon}_{\mathbf{q}}) + \langle \varepsilon_{\widehat{u}} - \varepsilon_u, \boldsymbol{\varepsilon}_{\mathbf{q}}\cdot\boldsymbol{n}\rangle &= (e_{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}}), \\
(\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u) - (\boldsymbol{\varepsilon}_{\mathbf{q}}, \nabla\varepsilon_u) + \langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, \varepsilon_u\rangle &= (\mathcal{I}^\alpha e_u', \varepsilon_u)\,.
\end{aligned}
$$

Adding the above equations, and using $\langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, \varepsilon_{\widehat{u}}\rangle = 0$ (this follows by choosing $\mu = \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}$ in (3.2.4c) and $\mu = \varepsilon_{\widehat{u}}$ in (3.2.4d)) and (3.2.4e), we obtain

$$
\begin{aligned}
(\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u) + \|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + \langle \varepsilon_{\widehat{u}} - \varepsilon_u, \boldsymbol{\varepsilon}_{\mathbf{q}}\cdot\boldsymbol{n}\rangle + \langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, \varepsilon_u\rangle - \langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, \varepsilon_{\widehat{u}}\rangle& \\
= (\mathcal{I}^\alpha e_u', \varepsilon_u) + (e_{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}})&
\end{aligned}
$$

Then, The right side becomes as:

$$
\begin{aligned}
&(\mathcal{I}^\alpha e_u', \varepsilon_u) + (e_{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}}) \\
&\qquad = (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u) + \|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + \langle \varepsilon_{\widehat{u}} - \varepsilon_u, (\boldsymbol{\varepsilon}_{\mathbf{q}} - \boldsymbol{\varepsilon}_{\widehat{q}}) \cdot \boldsymbol{n} \rangle \\
&\qquad = (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u) + \|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + \langle \varepsilon_{\widehat{u}} - \varepsilon_u, \tau(\varepsilon_{\widehat{u}} - \varepsilon_u) \rangle \\
&\qquad = (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u) + \|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + S_h^2.
\end{aligned}
$$

We rewrite the first term in the left side $\mathcal{I}^\alpha \varepsilon_u'$ in terms of Riemann-liouville fractional derivative.

$$
{}^R D^{1-\alpha} \varepsilon_u = \omega_{1-\alpha}(t) \varepsilon_u(0) + \mathcal{I}^\alpha \varepsilon_u' = \mathcal{I}^\alpha \varepsilon_u'
$$

Now, integrating over the time interval $[0, t]$, observing that

$$
\int_0^t (\mathcal{I}^\alpha \varepsilon_u', \varepsilon_u) \, ds = \int_0^t (\mathrm{D}^{1-\alpha} \varepsilon_u, \varepsilon_u) \, ds \geq 0,
$$

(in the first equality we used $\varepsilon_u(0) = 0$ and the last inequality follows from the non-negativity property of the Riemann–Liouville fractional derivative operator ${}^{\mathrm{R}}\mathrm{D}^{1-\alpha}$, see [28, Section 2]) and using the inequality $(e_{\boldsymbol{q}}, \boldsymbol{\varepsilon}_{\mathbf{q}}) \leq \frac{1}{2}\|e_{\boldsymbol{q}}\|^2 + \frac{1}{2}\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2$, we get

$$
\int_0^t [\frac{1}{2}\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|^2 + S_h^2] \, ds \leq \int_0^t (\|\mathcal{I}^\alpha e_u'\| \, \|\varepsilon_u\| + \frac{1}{2}\|e_{\boldsymbol{q}}\|^2) \, ds \, .
$$

Therefore, by the mean value theorem for integrals, there exist $t^*, \tilde{t} \in (0, t)$ such that

$$
t\Big(\frac{1}{2}\|\boldsymbol{\varepsilon}_{\mathbf{q}}(t^*)\|^2 + S_h^2(t^*)\Big) \leq t\Big(\|\varepsilon_u(\tilde{t})\| \max_{s \in (0,t)} \|\mathcal{I}^\alpha e_u'(s)\| + \frac{1}{2} \max_{s \in (0,t)} \|e_{\boldsymbol{q}}(s)\|^2\Big) \, .
$$

Finally, dividing by t, taking the limit when t goes to zero, and using again the fact that $\varepsilon_u(0) = 0$, we observe that $\|\boldsymbol{\varepsilon}_{\mathbf{q}}(0)\|^2 + S_h^2(0) \leq \|e_{\boldsymbol{q}}(0)\|^2 \leq C_1 h^{2k+2}$ and by lemma

22 the proof is now complete.

Chapter 4

CRANK NICOLSON-HDG METHOD

In this chapter, we apply the CN in time and HDG in space for sake of testing, the accuracy of the semi-discrete HDG method which proposed in the previous chapter.

4.1 CN-HDG scheme

To define our scheme, we introduce a uniform partition of the time interval $[0, T]$ given by the points: $t_i = i\delta$ for $i = 0, \cdots, N$, with $\delta = T/N$ being the time-step size. We take δ to be sufficiently small so that the spatial discretizations errors are dominant.

The time-stepping CN combined with the HDG method provides approximations $u_h^j \in W_h$, $\boldsymbol{q}_h^j \in \boldsymbol{V}_h^j$, and $\widehat{u}_h^j \in M_h$ of $u(t_j)$, $\boldsymbol{q}(t_j)$, and the trace of $u(t_j)$, respectively, for $j = 1, \cdots, N$. Starting from $u_h^0 = u_h(0) \approx u_0$, and with appropriate choices of $\boldsymbol{q}_h^0$ and $\widehat{u}_h^0$, our fully discrete scheme is defined by:

$$\begin{aligned}
(\boldsymbol{q}_h^{j-\frac{1}{2}}, \boldsymbol{r}) - (u_h^{j-\frac{1}{2}}, \nabla \cdot \boldsymbol{r}) + \langle \widehat{u}_h^{j-\frac{1}{2}}, \boldsymbol{r} \cdot \boldsymbol{n} \rangle &= 0, && \forall\, \boldsymbol{r} \in \boldsymbol{V}_h, \\
(\mathcal{J}_\alpha \overline{u}_h(t_j), w) - (\boldsymbol{q}_h^{j-\frac{1}{2}}, \nabla w) + \langle \widehat{\boldsymbol{q}}_h^{j-\frac{1}{2}} \cdot \boldsymbol{n}, w \rangle &= (f^{j-\frac{1}{2}}, w), && \forall\, w \in W_h, \\
\langle \widehat{u}_h^j, \mu \rangle_{\partial\Omega} &= \langle g, \mu \rangle_{\partial\Omega}, && \forall\, \mu \in M_h, \\
\langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, \mu_1 \rangle - \langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial\Omega} &= 0, && \forall\, \mu_1 \in M_h,
\end{aligned} \tag{4.1.1}$$

where $f^{j-\frac{1}{2}} := \frac{1}{2}(f(t_{j-1}) + f(t_j))$, $\widehat{\boldsymbol{q}}_h^j = \boldsymbol{q}_h^j + \tau \left(u_h^j - \widehat{u}_h^j\right)\boldsymbol{n}$ on $\partial\mathcal{T}_h$,

$$\mathcal{J}_\alpha \overline{u}_h(t_j) = \int_{t_{j-1}}^{t_j} \int_0^t \omega_\alpha(t-s)\overline{u}_h(s)\, ds\, dt,$$

with $\overline{u}_h(s) := \delta^{-1}(u_h^i - u_h^{i-1})$ for $s \in (t_{i-1}, t_i)$, $\boldsymbol{q}_h^{j-\frac{1}{2}} := \frac{1}{2}(\boldsymbol{q}_h^j + \boldsymbol{q}_h^{j-1})$, and the functions $u_h^{j-\frac{1}{2}}$, $\widehat{u}_h^{j-\frac{1}{2}}$, and $\widehat{\boldsymbol{q}}_h^{j-\frac{1}{2}}$ are similarly defined.

4.2 Existence and uniqueness of the CN-HDG solution

For each $1 \le j \le N$, (4.1.1) amounts to a square linear system. Thus the existence of the CN HDG solution follows from its uniqueness. We prove the uniqueness by induction hypothesis on j. We let $f^{i-\frac{1}{2}}$ (for $1 \le i \le j$) and g be identically zero in (4.1.1), we assume that $(u_h^i, \boldsymbol{q}_h^i, \widehat{u}_h^i) \equiv (0, \mathbf{0}, 0)$ for $1 \le i \le j-1$ and the task is to show that this holds true for $i = j$. To do so, choose $\boldsymbol{r} = \boldsymbol{q}_h^j$, $w = u_h^j$, $\mu = \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}$ and $\mu_1 = \widehat{u}_h^j$ in (4.1.1) and then simplify, yield

$$
\begin{aligned}
\|\boldsymbol{q}_h^j\|^2 - (u_h^j, \nabla \cdot \boldsymbol{q}_h^j) + \langle \widehat{u}_h^j, \boldsymbol{q}_h^j \cdot \boldsymbol{n} \rangle &= 0, \\
2(\mathcal{J}_\alpha \overline{u}_h(t_j), u_h^j) - (\boldsymbol{q}_h^j, \nabla u_h^j) + \langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, u_h^j \rangle &= 0, \\
\langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, \widehat{u}_h^j \rangle_{\partial\Omega} &= 0, \\
\langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, \widehat{u}_h^j \rangle - \langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, \widehat{u}_h^j \rangle_{\partial\Omega} &= 0
\end{aligned}
$$

Adding the last two equation, we observe:

$$
\begin{aligned}
\|\boldsymbol{q}_h^j\|^2 - (u_h^j, \nabla \cdot \boldsymbol{q}_h^j) + \langle \widehat{u}_h^j, \boldsymbol{q}_h^j \cdot \boldsymbol{n} \rangle &= 0, \\
2(\mathcal{J}_\alpha \overline{u}_h(t_j), u_h^j) - (\boldsymbol{q}_h^j, \nabla u_h^j) + \langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, u_h^j \rangle &= 0, \\
\langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, \widehat{u}_h^j \rangle &= 0.
\end{aligned}
$$

Since $(u_h^j, \nabla \cdot \boldsymbol{q}_h^j) = \langle u_h^j, \boldsymbol{q}_h^j \cdot \boldsymbol{n} \rangle - (\boldsymbol{q}_h^j, \nabla u_h^j)$, adding the above equations give

$$
\begin{aligned}
2(\mathcal{J}^\alpha \overline{u}_h(t_j), u_h^j) + \|\boldsymbol{q}_h^j\|^2 - \langle u_h^j, \boldsymbol{q}_h^j \cdot \boldsymbol{n} \rangle + (\boldsymbol{q}_h^j, \nabla u_h^j) \\
+ \langle \widehat{u}_h^j, \boldsymbol{q}_h^j \cdot \boldsymbol{n} \rangle - \langle \boldsymbol{q}_h^j, \nabla u_h^j \rangle + \langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, u_h^j \rangle - \langle \widehat{\boldsymbol{q}}_h^j \cdot \boldsymbol{n}, \widehat{u}_h^j \rangle = 0\,.
\end{aligned}
$$

Simplifying the above equation, gives

$$2(\mathcal{J}^\alpha \overline{u}_h(t_j), u_h^j) + \|\boldsymbol{q}_h^j\|^2 + \langle \widehat{u}_h^j - u_h^j, (\boldsymbol{q}_h^j - \widehat{\boldsymbol{q}}_h^j) \cdot \boldsymbol{n} \rangle = 0\,.$$

Hence, by the induction hypothesis and the identity $(\boldsymbol{q}_h^j - \widehat{\boldsymbol{q}}_h^j) \cdot \boldsymbol{n} = \tau \left(u_h^j - \widehat{u}_h^j \right)$ on $\partial \mathcal{T}_h$, we notes that

$$2(\mathcal{J}^\alpha \overline{u}_h(t_j), u_h^j) + \|\boldsymbol{q}_h^j\|^2 + \|\sqrt{\tau}(\widehat{u}_h^j - u_h^j)\|_{\partial \mathcal{T}_h}^2 = 0\,.$$

By the induction hypothesis $u_h^i = 0$ for all $i = 0, 1, 2, \cdots j-1$, so the above equation becomes to:

$$2\delta(\mathcal{J}^\alpha \overline{u}_h(t_j), \frac{u_h^j - u_h^{j-1}}{\delta}) + \|\boldsymbol{q}_h^j\|^2 + \|\sqrt{\tau}(\widehat{u}_h^j - u_h^j)\|_{\partial \mathcal{T}_h}^2 = 0\,.$$

Hence,

$$2\delta(\mathcal{J}^\alpha \overline{u}_h(t_j), \overline{u}_h(t)) + \|\boldsymbol{q}_h^j\|^2 + \|\sqrt{\tau}(\widehat{u}_h^j - u_h^j)\|_{\partial \mathcal{T}_h}^2 = 0 \quad \text{for} \quad t \in (t_{j-1}, t_j)$$

$$2\delta(\int_{t_{j-1}}^{t_j} \mathcal{I}^\alpha \overline{u}_h(t)\ \mathrm{dt}, \overline{u}_h(t)) + \|\boldsymbol{q}_h^j\|^2 + \|\sqrt{\tau}(\widehat{u}_h^j - u_h^j)\|_{\partial \mathcal{T}_h}^2 = 0\,.$$

Since $u_h^i = 0$, $\boldsymbol{q}_h^i = 0$ and $\widehat{u}_h^i = 0$ for all $i = 0, 1, 2, \cdots, j-1$, summing from $i = 1$ to $i = j$, we get:

$$2\delta \int_0^{t_j} (\mathcal{I}^\alpha \overline{u}_h(t), \overline{u}_h(t))\, dt + \|\boldsymbol{q}_h^j\|^2 + \|\sqrt{\tau}(\widehat{u}_h^j - u_h^j)\|_{\partial \mathcal{T}_h}^2 = 0.$$

Therefore, the use of the coercivity property of $\mathcal{I}^\alpha$, (iv) (16), yields the result, $\boldsymbol{q}_h^j = 0$, $u_h^j = 0$ and $\widehat{u}_h^j = 0$

4.3 Implementation of the CN-HDG scheme

In this section, we implement the CN-HDG scheme in one dimension. In this case, W_h and $\boldsymbol{V}_h$ are equal. We divide the spatial domain $\Omega = (0,1)$ and the time interval [0,T] into M and N subintervals, respectively. Assume that $I_j = (x_{j-1}, x_j)$, for $j = 1, 2, 3, ..., M$.

Recall that,

$$\boldsymbol{V}_h = W_h = \{\chi \in L^2(0,1),\ \chi|_{I_j} \in p_k(I_j), \quad 1 \le j \le M\}.$$

p_k is a space of all polynomials of degree less than or equal to k.

The one-dimensional CN-HDG scheme is then defined as follows:

$$(\boldsymbol{q}_h^{n-1/2}, \chi) - (u_h^{n-1/2}, \chi') + \hat{u}_h^{n-1/2}(x_m)\ \chi(x_m^-) - \hat{u}_h^{n-1/2}(x_{m-1})\ \chi(x_{m-1}^+) = 0 \quad \forall \chi \in W_h \tag{4.3.1a}$$

$$(I^\alpha \overline{u}_h(t), \chi) - (\boldsymbol{q}_h^{n-1/2}, \chi') + \hat{\boldsymbol{q}}_h^{n-1/2}(x_m)\ \chi(x_m^-) - \hat{\boldsymbol{q}}_h^{n-1/2}(x_{m-1})\ \chi(x_{m-1}^+) = (f^{n-\frac{1}{2}}, \chi) \tag{4.3.1b}$$

$$\begin{aligned}
\hat{u}_h^n(x_j)\big|_{j=0,M} &= u(x_j, t_n), \\
\hat{\boldsymbol{q}}_h^n(x_m)\big|_{m\neq 0,M} &= \frac{1}{2}\Big(-\big(u_h^n(x_m^+) - \boldsymbol{q}_h^n(x_m^+)\big) + u_h^n(x_m^-) + \boldsymbol{q}_h^n(x_m^-)\Big) \\
\hat{u}_h^n(x_m)\big|_{m\neq 0,M} &= \frac{1}{2}\Big(\big(u_h^n(x_m^+) - \boldsymbol{q}_h^n(x_m^+)\big) + u_h^n(x_m^-) + \boldsymbol{q}_h^n(x_m^-)\Big);
\end{aligned}$$

The numerical trace for the flux at x_0 and x_M is given by

$$-\hat{\boldsymbol{q}}_h^n(x_0) = -\boldsymbol{q}_h^n(x_0^+) + u_h^n(x_0^+) - \hat{u}_h^n(x_0), \quad \hat{\boldsymbol{q}}_h^n(x_M) = \boldsymbol{q}_h^n(x_M^-) + u_h^n(x_M^-) - \hat{u}_h^n(x_M)$$

Recall that, we used the following notation:

$$\hat{g}^{n-1/2}(x_m) = \frac{\hat{g}^n(x_m) + \hat{g}^{n-1}(x_m)}{2}$$

Now, we write u_h^n and $\boldsymbol{q}_h^n$ in terms of the legendre basis functions $\Big\{\varphi_i\Big\}_{i=1}^{M(k+1)}$ of the finite dimensional space W_h

$$u_h^n = \sum\nolimits_{i=1}^{M(k+1)} \alpha_i^n \varphi_{i(x)} \text{ and } \boldsymbol{q}_h^n = \sum\nolimits_{i=1}^{M(k+1)} \beta_i^n \varphi_{i(x)}\,.$$

To this end,

$$\varphi_{i(x)} = \begin{cases} p_{j(\frac{2x}{h}+1-2m)}, & x_{m-1} < x < x_m; \\ 0, & otherwise. \end{cases}$$

where, for $i = (k+1)(m-1)+j+1$, $1 \le m \le M$ is the index of the subinterval, $0 \le j \le k$, $P_j(x)$ is the legendre polynomial of dgree j; p_0, p_1, p_2, p_3 over $[-1,1]$ are shown in the above figure.

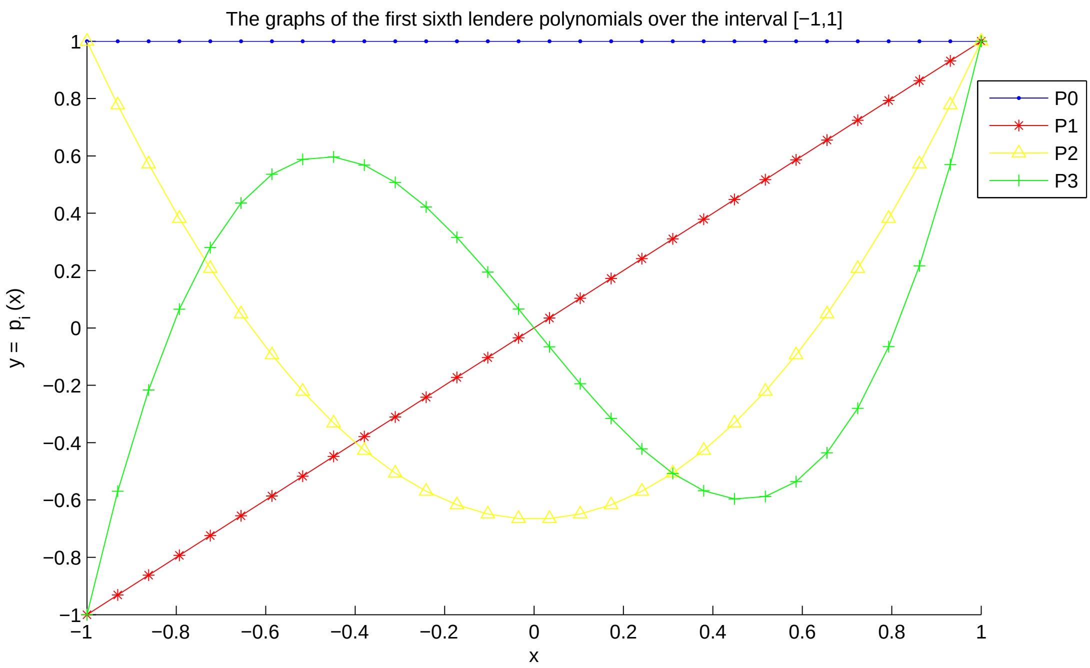
For example, let $M = 2$ and $k = 2$; then $1 \le i \le 6$ and the basis of W_h are graphed below:

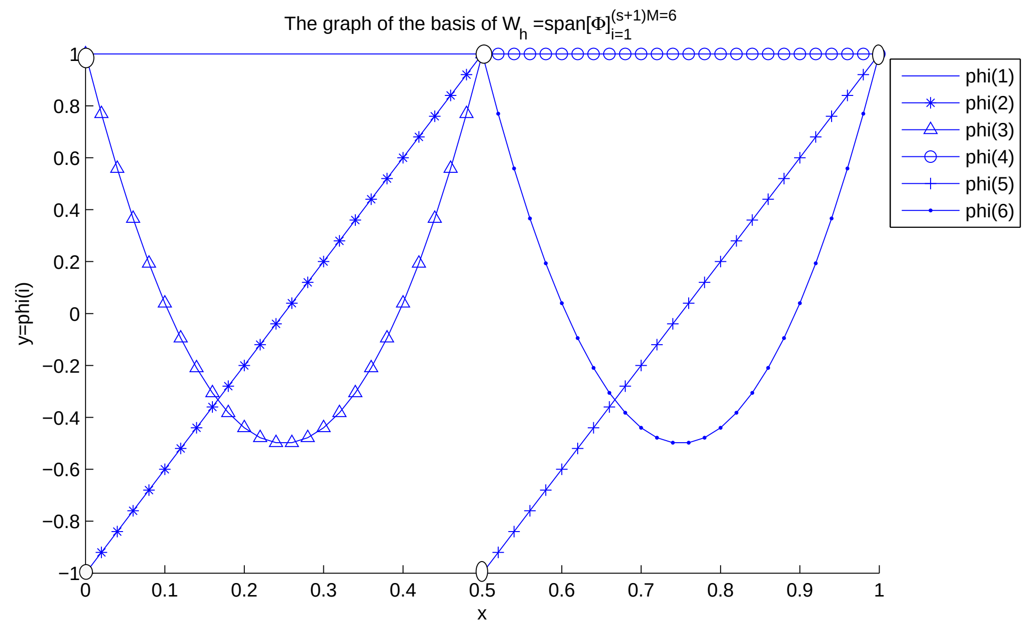
Since the legendre polynomials are orthogonal over the domain $[-1,1]$; $(p_j, p_i) = \frac{2}{2j+1}\delta_{i,j}$

Noting that the basis function $\{\varphi_i\}$ have the following properties:

- $(\varphi_j, \varphi_i) = \frac{h}{2\deg(\varphi_j)+1}\delta_{i,j}$ (Orthogonality)
- For $(m-1)(k+1)+1 \le i, j \le m(k+1)$

$$(\varphi_j, \varphi_i^{'})_{I_m} = \begin{cases} 2, & \deg(\varphi_i) > \deg(\varphi_j); \quad \deg(\varphi_i) + \deg(\varphi_j) := \text{odd number}\ ; \\ 0, & \text{otherwise.} \end{cases}$$





where, $\deg(\varphi_\ell)$ denotes the degree of the polynomial of function φ_ℓ

- In the limiting case,

$$\varphi_i(x_m^-) = \begin{cases} 1, & (k+1)(m-1)+1 \leq i \leq (k+1)m \ ; \\ 0, & \text{otherwise.} \end{cases}$$

$$\varphi_i(x_{m-1}^+) = \begin{cases} (-1)^{\deg(\varphi_i)}, & (k+1)(m-1)+1 \leq i \leq (k+1)m \ ; \\ 0, & \text{otherwise.} \end{cases}$$

therefore, From the forth equation of the scheme (4.1.1), the numerical trace $\hat{u}_h^n$ and the normal component of the numerical flux $\hat{\boldsymbol{q}}_h^n$ at the interior nods are given by:

$$\begin{aligned} \hat{u}_h^n(x_m) &= \frac{1}{2}\Big(u_h^n(x_m^+) + u_h^n(x_m^-) - \boldsymbol{q}_h^n(x_m^+) + \boldsymbol{q}_h^n(x_m^-)\Big) \\ &= \frac{1}{2}\sum_{j=1}^{M(k+1)} \Big(\alpha_j^n\varphi_j(x_m^+) + \alpha_j^n\varphi_j(x_m^-) - \beta_j^n\varphi_j(x_m^+) + \beta_j^n\varphi_j(x_m^-)\Big) \\ &= \frac{1}{2}\sum_{j=1}^{M(k+1)} \Big(\alpha_j^n\big(\varphi_j(x_m^+) + \varphi_j(x_m^-)\big) + \beta_j^n\big(-\varphi_j(x_m^+) + \varphi_j(x_m^-)\big)\Big) \end{aligned} \tag{4.3.3}$$

$$\begin{aligned} \hat{\boldsymbol{q}}_h^n(x_m) &= \frac{1}{2}\Big(-u_h^n(x_m^+) + u_h^n(x_m^-) + \boldsymbol{q}_h^n(x_m^+) + \boldsymbol{q}_h^n(x_m^-)\Big) \\ &= \frac{1}{2}\sum_{j=1}^{M(k+1)} \Big(\alpha_j^n\big(-\varphi_j(x_m^+) + \varphi_j(x_m^-)\big) + \beta_j^n\big(\varphi_j(x_m^+) + \varphi_j(x_m^-)\big)\Big) \end{aligned} \tag{4.3.4}$$

From (4.3.3), the expression $\hat{u}_h^n(x_m)\varphi_i(x_m^-) - \hat{u}_h^n(x_{m-1})\varphi_i(x_{m-1}^+)$ equals

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{M(k+1)} \Big(\alpha_j^n\big(\varphi_j(x_m^+) + \varphi_j(x_m^-)\big) + \beta_j^n\big(-\varphi_j(x_m^+) + \varphi_j(x_m^-)\big)\Big)\varphi_i(x_m^-) \\
&-\frac{1}{2}\sum_{j=1}^{M(k+1)} \Big(\alpha_j^n\big(\varphi_j(x_{m-1}^+) + \varphi_j(x_{m-1}^-)\big) + \beta_j^n\big(-\varphi_j(x_{m-1}^+) + \varphi_j(x_{m-1}^-)\big)\Big)\varphi_i(x_{m-1}^+) \\
=&\frac{1}{2}\sum_{j=d_1}^{(m+1)(k+1)} \alpha_j^n\Big(\big(\varphi_j(x_m^+) + \varphi_j(x_m^-)\big)\varphi_i(x_m^-) - \big(\varphi_j(x_{m-1}^+) + \varphi_j(x_{m-1}^-)\big)\varphi_i(x_{m-1}^+)\Big) \\
&+\frac{1}{2}\sum_{j=d_1}^{(m+1)(k+1)} \beta_j^n\Big(\big(-\varphi_j(x_m^+) + \varphi_j(x_m^-)\big)\varphi_i(x_m^-) - \big(-\varphi_j(x_{m-1}^+) + \varphi_j(x_{m-1}^-)\big)\varphi_i(x_{m-1}^+)\Big)
\end{aligned}
$$

where, d_1 is an index such that one of the following does not equal zero:

$$\varphi_{d_1}(x_m^-),\ \varphi_{d_1}(x_m^+),\ \varphi_{d_1}(x_{m-1}^-),\ \varphi_{d_1}(x_{m-1}^+); \quad \text{for all } i = 1,2,3,\ \cdots,\ M(k+1)$$

The above expression can be written in a matrix form as follows:

$$\hat{u}_h^n(x_m)\varphi_i(x_m^-) - \hat{u}_h^n(x_{m-1})\varphi_i(x_{m-1}^+) = D_m\,\alpha_m^n + E_m\,\beta_m^n$$

where, the matrices $D_m = \left[D_m^{i,j}\right]_{\ell_1\times\ell_2}$ and $E_m = \left[E_m^{i,j}\right]_{\ell_1\times\ell_2}$, with $\ell_1 = k+1,\ \ell_2 = 3k+3$ and $\alpha_m^n = \left[\alpha_n^i\right]_{\ell_1\times 1}$, $\beta_m^n = \left[\beta_n^i\right]_{\ell_1\times 1}$ are two column vector matrices.

Here,

$$D_m^{i,j} = \frac{1}{2}\Big(\big(\varphi_j(x_m^+) + \varphi_j(x_m^-)\big)\varphi_i(x_m^-) - \big(\varphi_j(x_{m-1}^+) + \varphi_j(x_{m-1}^-)\big)\varphi_i(x_{m-1}^+)\Big)$$

$$E_m^{i,j} = \frac{1}{2}\Big(\big(-\varphi_j(x_m^+) + \varphi_j(x_m^-)\big)\varphi_i(x_m^-) + \big(\varphi_j(x_{m-1}^+) - \varphi_j(x_{m-1}^-)\big)\varphi_i(x_{m-1}^+)\Big).$$

In a similar fashion, $\hat{\boldsymbol{q}}_h^n(x_m)\varphi_i(x_m^-) - \hat{\boldsymbol{q}}_h^n(x_{m-1})\varphi_i(x_{m-1}^+)$, equals

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{M(k+1)}\Big(\alpha_j^n\big(\varphi_j(x_m^-)-\varphi_j(x_m^+)\big)+\beta_j^n\big(\varphi_j(x_m^+)+\varphi_j(x_m^-)\big)\Big)\varphi_i(x_m^-)\\
&-\frac{1}{2}\sum_{j=1}^{M(k+1)}\Big(\alpha_j^n\big(\varphi_j(x_{m-1}^-)-\varphi_j(x_{m-1}^+)\big)+\beta_j^n\big(\varphi_j(x_{m-1}^+)+\varphi_j(x_{m-1}^-)\big)\Big)\varphi_i(x_{m-1}^+)\\
=&\frac{1}{2}\sum_{j=d_1}^{(m+1)(k+1)}\alpha_j^n\Big(\big(\varphi_j(x_m^-)-\varphi_j(x_m^+)\big)\varphi_i(x_m^-)-\big(\varphi_j(x_{m-1}^-)-\varphi_j(x_{m-1}^+)\big)\varphi_i(x_{m-1}^+)\Big)\\
&+\frac{1}{2}\sum_{j=d_1}^{(m+1)(k+1)}\beta_j^n\Big(\big(\varphi_j(x_m^+)+\varphi_j(x_m^-)\big)\varphi_i(x_m^-)-\big(\varphi_j(x_{m-1}^+)+\varphi_j(x_{m-1}^-)\big)\varphi_i(x_{m-1}^+)\Big)
\end{aligned}
$$

Again, for a matrix form,

$$
\hat{\boldsymbol{q}}_h^n(x_m)\varphi_i(x_m^-) - \hat{\boldsymbol{q}}_h^n(x_{m-1})\varphi_i(x_{m-1}^+) = G_m\,\alpha_m^n + H_m\,\beta_m^n
$$

with, $G_m = \Big[G_m^{i,j}\Big]_{\ell_1\times\ell_2}$, $H_m = \Big[H_m^{i,j}\Big]_{\ell_1\times\ell_2}$. Where,

$$
G_m^{i,j} = \frac{1}{2}\Big(\big(\varphi_j(x_m^-)-\varphi_j(x_m^+)\big)\varphi_i(x_m^-)-\big(\varphi_j(x_{m-1}^-)-\varphi_j(x_{m-1}^+)\big)\varphi_i(x_{m-1}^+)\Big)
$$

$$
H_m^{i,j} = \frac{1}{2}\Big(\big(\varphi_j(x_m^+)+\varphi_j(x_m^-)\big)\varphi_i(x_m^-)-\big(\varphi_j(x_{m-1}^+)+\varphi_j(x_{m-1}^-)\big)\varphi_i(x_{m-1}^+)\Big)
$$

Recall that, the above two matrices form expression are valid at the interior nods. Now, for the boundary nodes, x_0, x_M, we have

$$
\hat{u}_h^n(x_0) = u(x_0,t_n)\,; \quad -\hat{\boldsymbol{q}}_h^n(x_0) = -\boldsymbol{q}_h^n(x_0^+) + u_h^n(x_0^+) - \hat{u}_h^n(x_0)
$$

$$
\hat{u}_h^n(x_M) = u(x_M,t_n)\,; \quad \hat{\boldsymbol{q}}_h^n(x_M) = \boldsymbol{q}_h^n(x_M^-) + u_h^n(x_M^-) - \hat{u}_h^n(x_M)
$$

$$\hat{u}_h^n(x_1)\varphi_i(x_1^-) - u(x_0,t_n)\varphi_i(x_0^+) =$$
$$\frac{1}{2}\sum_{j=1}^{2(k+1)}\Big(\alpha_j^n\big(\varphi_j(x_1^+)+\varphi_j(x_1^-)\big)+\beta_j^n\big(-\varphi_j(x_1^+)+\varphi_j(x_1^-)\big)\Big)\varphi_i(x_1^-) - u(x_0,t_n)\varphi_i(x_0^+)$$

$$\begin{aligned}
&\hat{\boldsymbol{q}}_h^n(x_1)\varphi_i(x_1^-) - \hat{\boldsymbol{q}}_h^n(x_0)\varphi_i(x_0^+) = \\
&\frac{1}{2}\sum_{j=1}^{2(k+1)}\Big(\alpha_j^n\big(\varphi_j(x_1^-)-\varphi_j(x_1^+)\big)+\beta_j^n\big(\varphi_j(x_1^+)+\varphi_j(x_1^-)\big)\Big)\varphi_i(x_1^-)+ \\
&\sum_{j=1}^{2(k+1)}\Big(\alpha_j^n\varphi_j(x_0^+)-\beta_j^n\varphi_j(x_0^+)\Big)\varphi_i(x_0^+) - u(x_0,tn)\varphi_i(x_0^+) \\
=&\sum_{j=1}^{2(k+1)}\alpha_j^n\Big(\frac{1}{2}\big(\varphi_j(x_1^-)-\varphi_j(x_1^+)\big)\varphi_i(x_1^-)+\varphi_j(x_0^+)\varphi_i(x_0^+)\Big)+ \\
&\sum_{j=1}^{2(k+1)}\beta_j^n\Big(\frac{1}{2}\big(\varphi_j(x_1^+)+\varphi_j(x_1^-)\big)\varphi_i(x_1^-)-\varphi_j(x_0^+)\varphi_i(x_0^+)\Big) - u(x_0,t_n)\varphi_i(x_0^+)
\end{aligned}$$

For all $i = 1,\, 2,\, 3, \cdots (k+1)$

So, in the matrix form on the first subinterval (m=1),we write D_1, E_1, H_1 and G_1 as:

$$D_1^{i,j} = \Big[\frac{1}{2}\big(\varphi_j(x_1^+)+\varphi_j(x_1^-)\big)\varphi_i(x_1^-)\Big]; \quad E_1^{i,j} = \Big[\frac{1}{2}\big(-\varphi_j(x_1^+)+\varphi_j(x_1^-)\big)\varphi_i(x_1^-)\Big]$$

$$\begin{aligned}
G_1^{i,j} &= \Big[\frac{1}{2}\big(\varphi_j(x_1^-)-\varphi_j(x_1^+)\big)\varphi_i(x_1^-)+\varphi_j(x_0^+)\varphi_i(x_0^+)\Big] \\
H_1^{i,j} &= \Big[\frac{1}{2}\big(\varphi_j(x_1^+)+\varphi_j(x_1^-)\big)\varphi_i(x_1^-)-\varphi_j(x_0^+)\varphi_i(x_0^+)\Big]
\end{aligned}$$

For convenience, we introduce the following notations:

- For, $n = 1, 2, 3, \cdots N$, $u_0^n = \Big[u(x_0, t_n)\varphi_i(x_0^+)\Big]_{\ell_1 \times 1}$ is column vector of size ℓ_1
- $1 \le n \le N$, $U_0^n = \begin{pmatrix} u_0^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{M(k+1)\,\mathrm{x}\,1}$

For the second boundary point x_M,

$$u(x_M, t_n)\varphi_i(x_M^-) - \hat{u}_h^n(x_{M-1})\varphi_i(x_{M-1}^+) =$$

$$u(x_M, t_n)\varphi_i(x_M^-) - \frac{1}{2}\sum_{j=d_1}^{M(k+1)} \Big(\alpha_j^n\big(\varphi_j(x_{M-1}^+) + \varphi_j(x_{M-1}^-)\big) + \beta_j^n\big(-\varphi_j(x_{M-1}^+) + \varphi_j(x_{M-1}^-)\big)\Big)\varphi_i(x_{M-1}^+)$$

$$\begin{aligned}
&\hat{\boldsymbol{q}}_h^n(x_M)\varphi_i(x_M^-) - \hat{\boldsymbol{q}}_h^n(x_{M-1})\varphi_i(x_{M-1}^+) = \\
&\sum_{j=d_2}^{M(k+1)} \Big(\alpha_j^n\varphi_j(x_M^-) + \beta_j^n\varphi_j(x_M^-)\Big)\varphi_i(x_M^-) - u(x_M, tn)\varphi_i(x_M^-) \\
&- \frac{1}{2}\sum_{j=d_2}^{M(k+1)} \Big(\alpha_j^n\big(\varphi_j(x_{M-1}^-) - \varphi_j(x_{M-1}^+)\big) + \beta_j^n\big(\varphi_j(x_{M-1}^+) + \varphi_j(x_{M-1}^+)\big)\Big)\varphi_i(x_{M-1}^+) \\
= &\sum_{j=d_2}^{M(k+1)} \alpha_j^n\Big(\varphi_j(x_M^-)\varphi_i(x_M^-) - \frac{1}{2}\big(\varphi_j(x_{M-1}^-) - \varphi_j(x_{M-1}^+)\big)\varphi_i(x_{M-1}^+)\Big) \\
&+ \sum_{j=d_2}^{M(k+1)} \beta_j^n\Big(\varphi_j(x_M^-)\varphi_i(x_M^-) - \frac{1}{2}\big(\varphi_j(x_{M-1}^+) + \varphi_j(x_{M-1}^-)\big)\varphi_i(x_{M-1}^+)\Big) - \\
&u(x_M, t_n)\varphi_i(x_M^-)\,,
\end{aligned}$$

where $d_2 = (M-2)(k+1)+1$ For all $(M-1)(k+1)+1 \ \le i \ \le M(k+1)$

So, in the matrix form on the final subinterval (m=M),we write D_M, E_M, H_M and

G_M as the following:

$$
\begin{aligned}
D_M^{i,j} &= \Big[-\frac{1}{2}\big(\varphi_j(x_{M-1}^{+}) + \varphi_j(x_{M-1}^{-})\big)\varphi_i(x_{M-1}^{+})\Big] \\
E_M^{i,j} &= \Big[\frac{1}{2}\big(\varphi_j(x_{M-1}^{+}) - \varphi_j(x_{M-1}^{-})\big)\varphi_i(x_{M-1}^{+})\Big] \\
G_M^{i,j} &= \Big[\varphi_j(x_{M}^{-})\varphi_i(x_{M}^{-}) - \frac{1}{2}\big(\varphi_j(x_{M-1}^{-}) - \varphi_j(x_{M-1}^{+})\big)\varphi_i(x_{M-1}^{+})\Big] \\
H_M^{i,j} &= \Big[\varphi_j(x_{M}^{-})\varphi_i(x_{M}^{-}) - \frac{1}{2}\big(\varphi_j(x_{M-1}^{+}) + \varphi_j(x_{M-1}^{-})\big)\varphi_i(x_{M-1}^{+})\Big]
\end{aligned}
$$

We introduce next some additional notations:

- $u_M^n = \Big[u(x_M, t_n)\varphi_i(x_M^-)\Big]$ is column vector of length $k+1$, for $n = 1, 2, 3, \cdots, N$.
- $U_M^n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ u_M^n \end{pmatrix}_{M(k+1)\times 1}$ is column vector of size $M(k+1)$

The other terms in the scheme are written in matrix form as follows:

$\big(u_h^n, \varphi_i\big)_{I_m} = \sum_{j=(m-1)(k+1)+1}^{m(k+1)} \alpha_j^n \big(\varphi_j, \varphi_i\big)_{I_m}$, $\quad \big(\boldsymbol{q}_h^n, \varphi_i\big)_{I_m} = \sum_{j=(m-1)(k+1)+1}^{m(k+1)} \beta_j^n \big(\varphi_j, \varphi_i\big)_{I_m}$, and $\big(u_h^n, \varphi_i'\big)_{I_m} = \sum_{j=(m-1)(k+1)+1}^{m(k+1)} \alpha_j^n \big(\varphi_j, \varphi_i'\big)_{I_m}$

The matrices A_m and B_m are as:

$$
\Big[\big(\varphi_j, \varphi_i\big)\Big]_{(k+1)\,\mathrm{x}\,(k+1)} \quad ; \quad \Big[\big(\varphi_j, \varphi_i'\big)\Big]_{(k+1)\,\mathrm{x}\,(k+1)}
$$

respectively. With size equal to $(k+1)\,\mathrm{x}\,(k+1)$ and $(m-1)(k+1)+1 \le i, j \le m(k+1)$.

Let

$$F^n = \begin{pmatrix} (f_{(\frac{t_{n-1}+t_n}{2})}, \varphi_1) \\ (f_{(\frac{t_{n-1}+t_n}{2})}, \varphi_2) \\ \vdots \\ (f_{(\frac{t_{n-1}+t_n}{2})}, \varphi_{M(k+1)}) \end{pmatrix},$$

$$\alpha^n = \begin{pmatrix} \alpha^n_{\mathbf{1}} \\ \alpha^n_{\mathbf{2}} \\ \vdots \\ \alpha^n_{\mathbf{M}} \end{pmatrix},$$

$$\beta^n = \begin{pmatrix} \beta^n_{\mathbf{1}} \\ \beta^n_{\mathbf{2}} \\ \vdots \\ \beta^n_{\mathbf{M}} \end{pmatrix}$$

are vectors of dimension equals $M(k+1)$. Now, we are ready to write the equation in (4.3.1a) in a matrix form. To do so, we choose $\chi = \varphi_i$ in (4.3.1a) and get

$$\Big(\big(\boldsymbol{q}^n_h, \varphi_i\big)_{I_m} - \big(u^n_h, \varphi_i^{'}\big)_{I_m} + \hat{u}^n_{h(x_m)}\ \varphi_{i(x_m^-)} - \hat{u}^n_{h(x_{m-1})}\ \varphi_{i(x^+_{m-1})} \Big)$$
$$+ \Big(\big(\boldsymbol{q}^{n-1}_h, \varphi_i\big)_{I_m} - \big(u^{n-1}_h, \varphi_i^{'}\big)_{I_m} + \hat{u}^{n-1}_{h(x_m)}\ \varphi_{i(x_m^-)} - \hat{u}^{n-1}_{h(x_{m-1})}\ \varphi_{i(x^+_{m-1})} \Big) = 0\,;$$

where, $(m-1)(k+1)+1 \leq i \leq m(k+1)$

Therefore, (4.3.1a) can be written in the following matrix form:

$$\Big(A+E\Big)\big(\beta^n + \beta^{n-1}\big) + \Big(D-B\Big)\big(\alpha^n + \alpha^{n-1}\big) = U_u^{n+\frac{1}{2}}$$

where $A = \Big[A_m\Big]_{m=1}^{M}$, $B = \Big[B_m\Big]_{m=1}^{M}$, $D = \Big[D_m\Big]_{m=1}^{M}$, $E = \Big[E_m\Big]_{m=1}^{M}$ are block diagonal matrices.

We rearrange the above equation as follows:

$$\Big(A+E\Big)\beta^{n}+\Big(D-B\Big)\alpha^{n}=U_{u}^{n+\frac{1}{2}}-\Big(A+E\Big)\beta^{n-1}-\Big(D-B\Big)\alpha^{n-1} \tag{4.3.5}$$

where, $U_u^{n+\frac{1}{2}} = U_0^n + U_0^{n-1} - U_M^n - U_M^{n-1}$

Now, setting $\chi = \varphi_i$ in (4.3.1b) and rearranging the equation, we get

$$\begin{aligned}
&2\Big(\int_0^t w_{\alpha(t-s)}\overline{u}_h(t)\,\mathrm{ds},\varphi_i\Big)_{I_m}-\Big(\boldsymbol{q}_h^n,\varphi_i^{'}\Big)_{I_m}+\hat{\boldsymbol{q}}_h^n(x_m)\varphi_i(x_m^-)-\hat{\boldsymbol{q}}_h^n(x_{m-1})\varphi_i(x_{m-1}^+)\\
&\quad=2\big(f^{n+\frac{1}{2}},\varphi_i\big)_{I_m}+\Big(\big(\boldsymbol{q}_h^{n-1},\varphi_i^{'}\big)_{I_m}+\hat{\boldsymbol{q}}_h^{n-1}(x_m)\varphi_i(x_m^-)-\hat{\boldsymbol{q}}_h^{n-1}(x_{m-1})\varphi_i(x_{m-1}^+)\Big)
\end{aligned}$$

where $(m-1)(k+1)+1 \le i \le m(k+1) \quad \forall\ m = 1,2,3,\cdots M$ and $t \in [t_{n-1}, t_n]$.

Integrating the above equation with respect to t, the first term on the left side becomes

$$\begin{aligned}
&\Big(\int_{t_{n-1}}^{t_n}\int_0^t w_{\alpha(t-s)}\overline{u}_h(t)\,\mathrm{ds}\,\mathrm{dt},\varphi_i\Big)\\
&=\Big(\int_{t_{n-1}}^{t_n}\Big(\int_0^{t_{n-1}} w_{\alpha(t-s)}\overline{u}_h(t)\,\mathrm{ds}+\int_{t_{n-1}}^{t} w_{\alpha(t-s)}\overline{u}_h(t)\,\mathrm{ds}\Big)\,\mathrm{dt},\varphi_i\Big)\\
&=\Big(\frac{u_h^n-u_h^{n-1}}{k_n}\int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} w_{\alpha(t-s)}\,\mathrm{ds}\,\mathrm{dt}+\sum_{j=1}^{n-1}\Big(\frac{u_h^j-u_h^{j-1}}{k_j}\int_{t_{n-1}}^{t_n}\int_{t_{j-1}}^{t_j} w_{\alpha(t-s)}\,\mathrm{ds}\,\mathrm{dt}\Big),\varphi_i\Big)\\
&=\Big(\frac{u_h^n-u_h^{n-1}}{k_n}w_{\alpha+2(k_n)}+\sum_{j=1}^{n-1} g_j\Big(u_h^j-u_h^{j-1}\Big),\varphi_i\Big)\\
&=\frac{w_{\alpha+2(k_n)}}{k_n}\Big(\big(u_h^n,\varphi_i\big)-\big(u_h^{n-1},\varphi_i\big)\Big)+\sum_{j=1}^{n-1}\Big[g_j\Big(\big(u_h^j,\varphi_i\big)-\big(u_h^{j-1},\varphi_i\big)\Big)\Big]
\end{aligned}$$

where, $g_j = \frac{1}{k_j}\big(w_{\alpha+2(t_n-t_{j-1})} - w_{\alpha+2(t_{n-1}-t_{j-1})} - w_{\alpha+2(t_n-t_j)} + w_{\alpha+2(t_{n-1}-t_j)}\big)$

Therefore, (4.3.1b) can be written in the following matrix form:

$$\frac{2}{k_n} A\Big[\frac{w_{\alpha+2(k_n)}}{k_n}\big(\alpha^n - \alpha^{n-1}\big) + \sum_{j=1}^{n-1}\Big(g_j\big(\alpha^j - \alpha^{j-1}\big)\Big)\Big] - B\,\beta^n + G\,\alpha^n + H\,\beta^n - U_0^n - U_M^n$$
$$= 2F^n - \Big(- B\,\beta^{n-1} + G\,\alpha^{n-1} + H\,\beta^{n-1} - U_0^{n-1} - U_M^{n-1}\Big)$$

where $G = \Big[G_m\Big]_{m=1}^{M}$, $H = \Big[H_m\Big]_{m=1}^{M}$ are block diagonal matrices.

We rewrite the above equation as follows:

$$\big(\frac{2\,w_{\alpha+2(k_n)}}{k_n^2} A + G\big)\alpha^n + \big(- B + H\big)\beta^n$$
$$= 2F^n + U_{\boldsymbol{q}}^{n+\frac{1}{2}} + \big(\frac{2\,w_{\alpha+2(k_n)}}{k_n^2}\,A - G\big)\,\alpha^{n-1} + \big(B - H\big)\,\beta^{n-1} - \frac{2}{k_n}\sum_{j=1}^{n-1} g_j\,A\,\big(\alpha^j - \alpha^{j-1}\big) \tag{4.3.6}$$

Where $U_{\boldsymbol{q}}^{n+\frac{1}{2}} = U_0^n + U_M^n + U_0^{n-1} + U_M^{n-1}$

4.4 Numerical experiments for CN-HDG

In this section, we present numerical experiments devise to validate our theoretical predictions from HDG spatial discretizations. By using the fully discrete CN HDG scheme (4.1.1). We take the (uniform) time steps δ to be sufficiently small so that the HDG spatial discretizations errors are dominant. This is achieved by fixing the ratio $\frac{\delta^2}{h^{k+2}}$ to a given number less than the unit because the time stepping CN scheme is second-order accurate provided that the exact solution is sufficiently regular.

We choose the spatial domain Ω to be the unit interval $(0, 1)$ and $T = 1$ in (1.1.1). We impose homogenous Dirichlet boundary conditions and choose the source term f and the initial data u_0 so that the exact solution is $u(x, t) = t^{3-\alpha} sin(\pi x)$. For different values of α, we obtain the history of convergence of the errors $\|(u-u_h)(T)\|$, $\|(\boldsymbol{q}-\boldsymbol{q}_h)(T)\|$ for different values of the polynomial degree, $k = 0, 1, 2$. To compute the spatial L_2-norm, we apply a composite Gauss quadrature rule with 4 points on each interval of the finest spatial mesh. The numerical results (errors and convergence rates) of the experiments are presented in Tables 4 .1 and 4 .2 and their figures. In full agreement with our theoretical results, we obtain optimal convergence rates for the HDG scheme.

N	$k=0$			
4	5.269e-01		7.899e-01	
8	3.027e-01	0.799	4.028e-01	0.972
16	1.616e-01	0.905	2.025e-01	0.992
32	8.342e-02	0.954	1.014e-01	0.997
64	4.237e-02	0.977	5.072e-02	0.999
128	2.135e-02	0.989	2.537e-02	0.999
	$k=1$			
4	6.031e-02		5.936e-02	
8	1.502e-02	2.005	1.321e-02	2.165
16	4.144e-03	1.858	3.487e-03	1.924
32	1.048e-03	1.983	8.649e-04	2.011
64	2.697e-04	1.958	2.199e-04	1.976
	$k=2$			
4	3.960e-03		4.596e-03	
8	5.059e-04	2.969	4.868e-04	3.239
16	6.352e-05	2.993	5.652e-05	3.107
32	7.957e-06	2.997	7.117e-06	2.989
32	7.957e-06	2.997	7.117e-06	2.989

Table 4 .1: The errors $\|(u_h-u)(T)\|$, $\|(\boldsymbol{q}_h-\boldsymbol{q})(T)\|$ and the corresponding rates of convergence for $\alpha = 0.5$ with HDG solutions of degree $k = 0, 1, 2$. We observe optimal convergence of order h^{k+1} for the errors in u_h and $\boldsymbol{q}_h$.

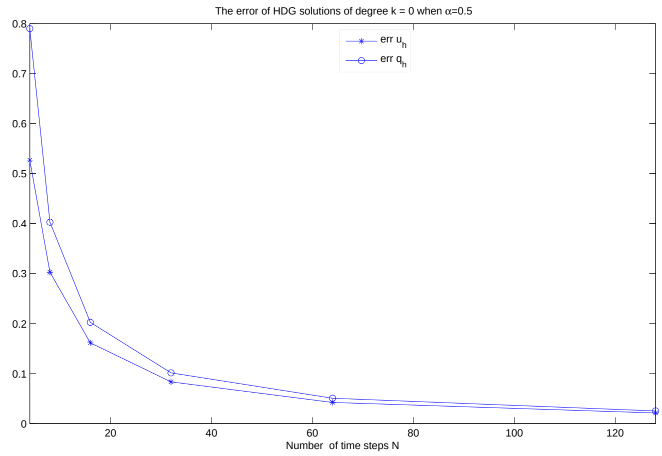


Figure 4 .1: HDG errors for piecewise constant solution($k = 0$) when $\alpha = 0.5$

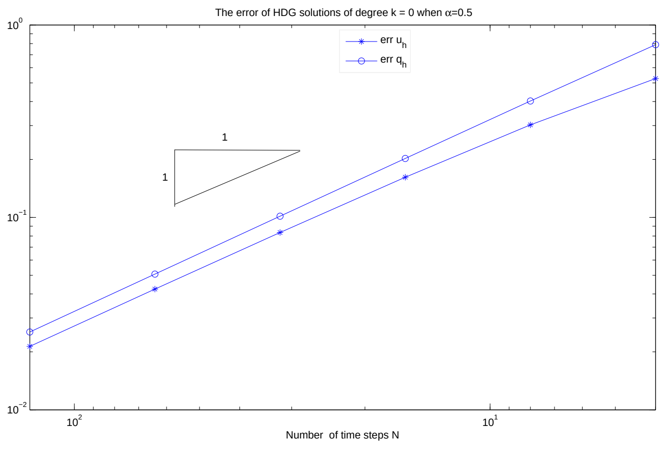


Figure 4 .2: HDG errors for piecewise constant solution($k = 0$) when $\alpha = 0.5$, log-log scaling

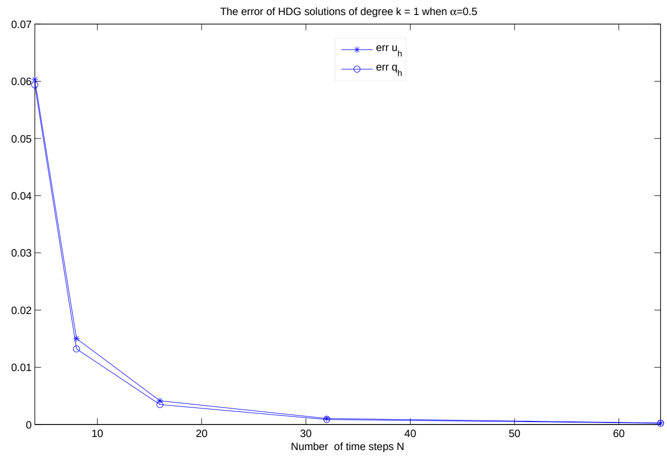


Figure 4 .3: HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.5$

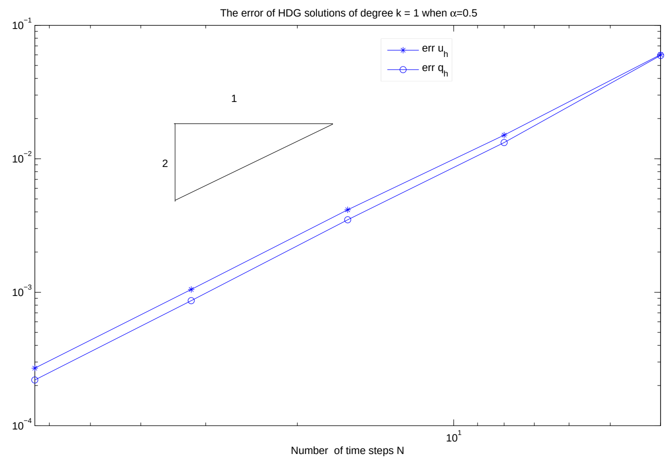


Figure 4 .4: HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.5$, log-log scaling

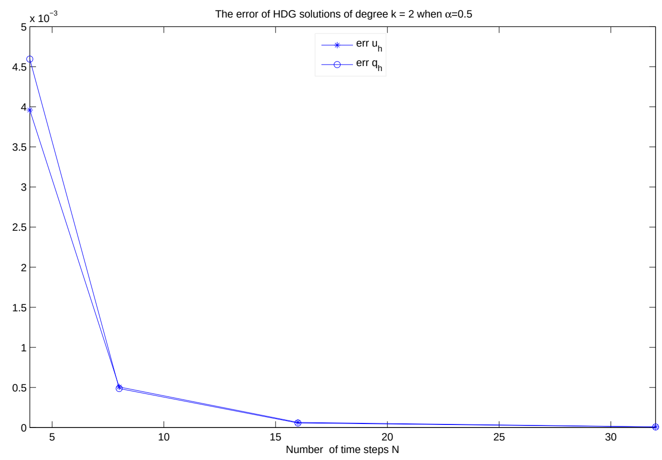


Figure 4 .5: HDG errors for piecewise quadratic solution($k = 2$) when $\alpha = 0.5$

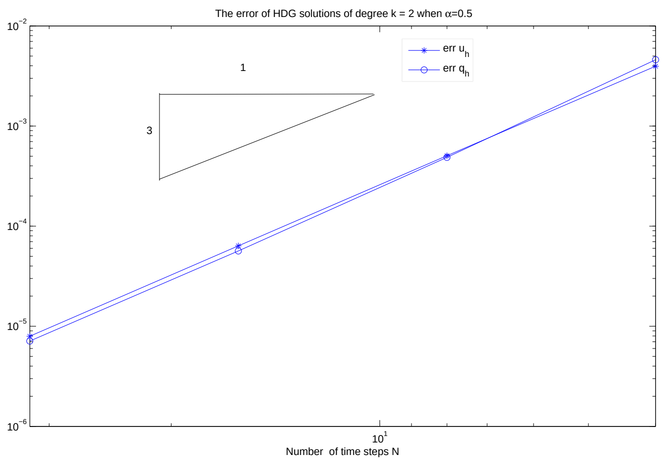


Figure 4 .6: HDG errors for piecewise quadratic solution($k = 2$) when $\alpha = 0.5$, log-log scaling

N	$k=0$			
4	5.455e-01		7.705e-01	
8	3.122e-01	0.805	3.088e-01	0.9884
16	1.661e-01	0.910	1.939e-01	1.002
32	8.558e-02	0.957	9.674e-02	1.003
64	4.342e-02	0.979	4.830e-02	1.002
128	2.187e-02	0.989	2.413e-02	1.001
	$k=1$			
4	6.081e-02		6.005e-02	
8	1.501e-02	2.018	1.321e-02	2.185
16	4.154e-03	1.854	3.485e-03	1.922
32	1.048e-03	1.987	8.434e-04	2.047
64	2.682e-04	1.966	2.143e-04	1.977
	$k=2$			
4	4.025e-03		4.978e-03	
8	5.088e-04	2.984	5.014e-04	3.312
16	6.367e-05	2.998	5.701e-05	3.137
32	7.892e-06	3.012	7.031e-06	3.019
32	7.892e-06	3.012	7.031e-06	3.019

Table 4 .2: The errors $\|(u_h - u)(T)\|$, $\|(\boldsymbol{q}_h - \boldsymbol{q})(T)\|$ and the corresponding rates of convergence for $\alpha = 0.7$ with HDG solutions of degree $k = 0, 1, 2$.

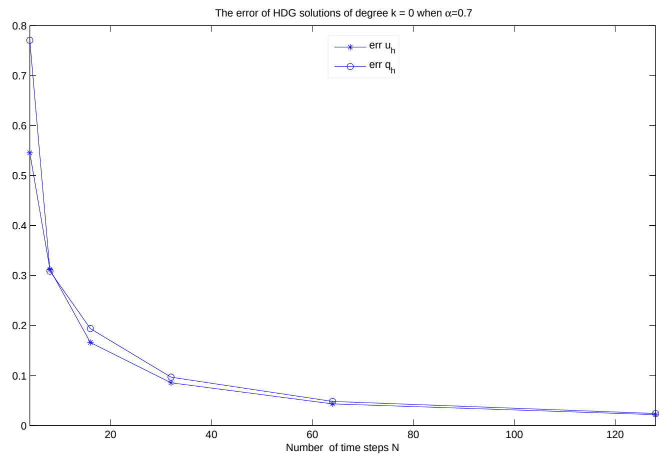


Figure 4 .7: HDG errors for piecewise constant solutions($k = 0$) with $\alpha = 0.7$

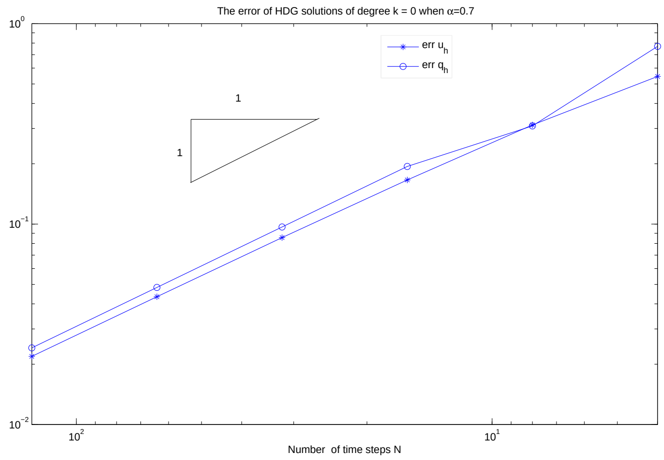


Figure 4 .8: HDG errors for piecewise constant solutions($k = 0$) with $\alpha = 0.7$, log-log scaling

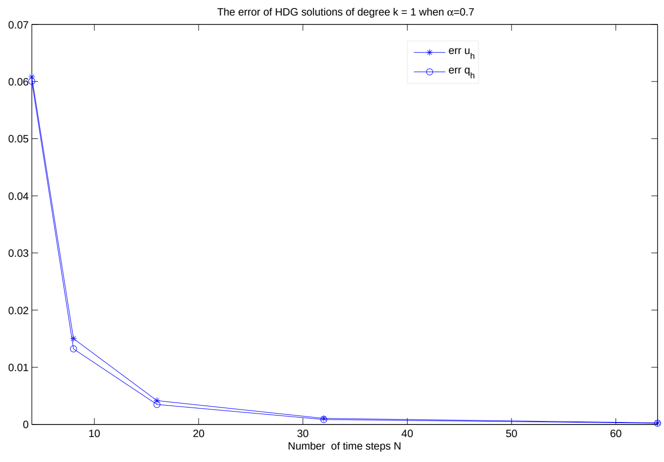


Figure 4 .9: HDG errors for piecewise linear solutions($k = 1$) with $\alpha = 0.7$

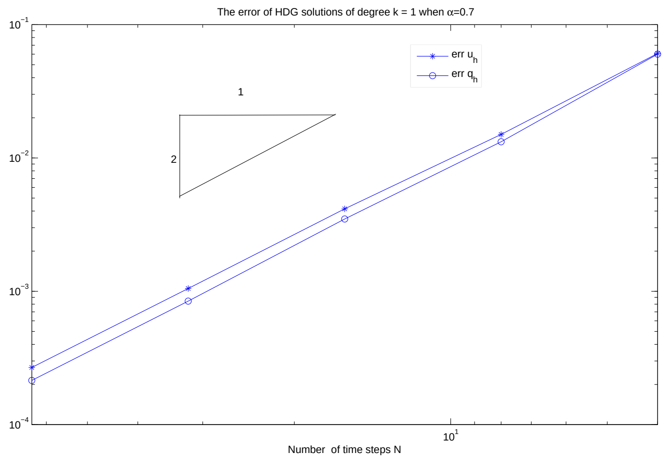


Figure 4 .10: HDG errors for piecewise linear solutions($k = 1$) with $\alpha = 0.7$, log-log scaling

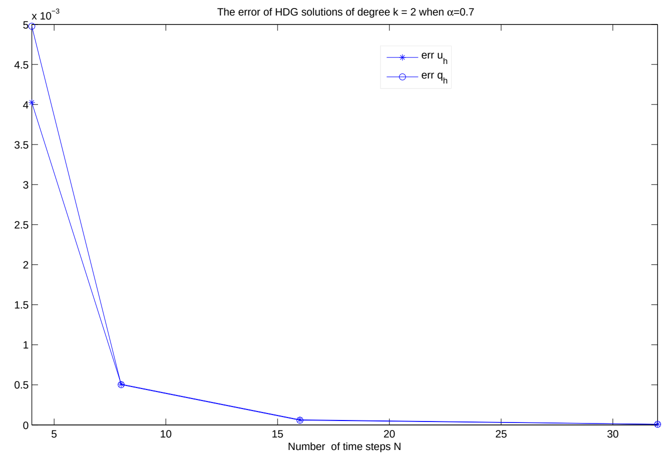


Figure 4 .11: HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$

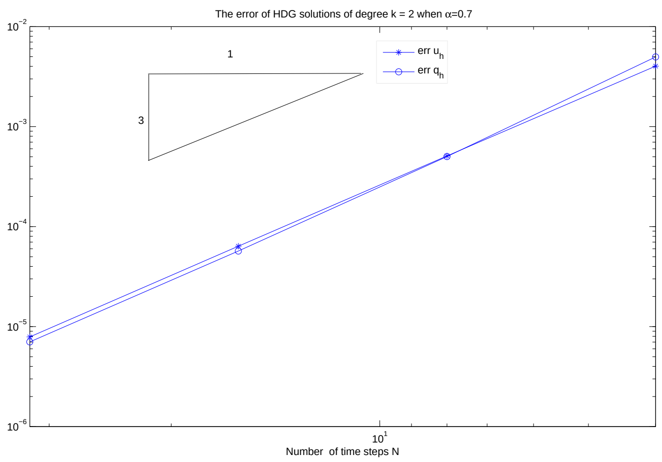


Figure 4 .12: HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, log-log scaling

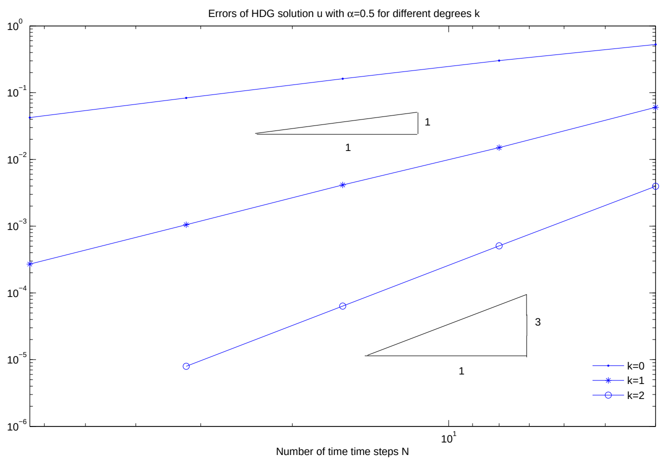


Figure 4 .13: HDG errors for various degrees of the solution $u_h(k = 0, 1, 2)$ with $\alpha = 0.5$, log-log scaling

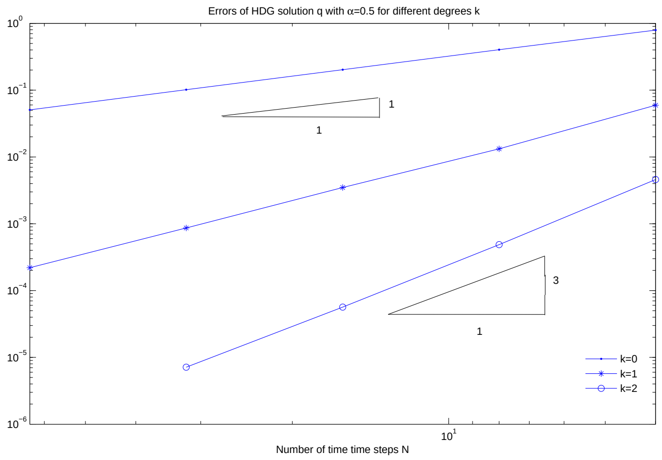


Figure 4 .14: HDG errors for various degrees of the solution $q_h(k = 0, 1, 2)$ with $\alpha = 0.5$, log-log scaling

Chapter 5

SUPERCONVERGENCE OF the HDG METHOD

In this chapter, we seek a better approximation to u by means of an element-by-element postprocessing HDG method. In this section, we describe such approximation, then we show how to get our superconvergence result by a duality argument.

5.1 Post-processing

Following the work in [5, 14], for each $t \in [0,T]$, we define the postprocessed HDG solution $u_h^\star(t) \in \mathcal{P}_{k+1}(K)$ to $u(t)$ for each simplex $K \in \mathcal{T}_h$, as follows:

$$(u_h^\star(t), 1)_K = (u_h(t), 1)_K \tag{5.1.1a}$$

$$(\nabla u_h^\star(t), \nabla w)_K = -\,(\boldsymbol{q}_h(t), \nabla w)_K \qquad \forall\; w \in \mathcal{P}_{k+1}(K). \tag{5.1.1b}$$

For each fixed $t \in (0,T]$.

Existence and uniqueness of $u_h^\star$

Since (5.1.1) amounts to a square linear system (for each fixed $t \in (0,T]$), the existence of the postprocessed HDG solution follows from its uniqueness. To this end, we let $u_h(t)$ and $\boldsymbol{q}_h(t)$ to be identically zero in the right hand side of (5.1.1). The task now is to show that $u_h^\star(t) \equiv 0$ for each $t \in (0,T]$. We choose $w = u_h^\star(t)$ in (5.1.1b), then $\|\nabla u_h^\star\|_K^2 = 0$. Therefore $u_h^\star(t)$ is equal to a constant c_0 on K. Hence, by (5.1.1a), $c_0\ measure(K) = 0$. This implies $c_0 = 0$.

5.2 Super convergence estimates

For showing the super convergence property of $u_h^\star$, splitting the postprocessed error as:

$$u - u_h^\star = (u - P_{k+1}u) + P_0\zeta + (\zeta - P_0\zeta)$$

where $\zeta = P_{k+1}u - u_h^\star$ and P_ℓ (for $\ell \geq 0$) be the $L^2(\Omega)$-projection into the space of functions which are polynomials of total degree $\leq \ell$ on each element $K \in \mathcal{T}_h$.

That is,

$$P_\ell|_K : L_2(K) \to P_\ell(K)$$

such that for $\phi \in L_2(K)$,

$$(P_\ell \; \phi, \chi)_K = (\phi, \chi)_K \quad \forall \chi \in P_\ell(K).$$

Hence, by the triangle inequality and the error properties of the projection P_ℓ,

$$\|u - u_h^\star\| \leq C\, h_K^{k+2}\, |u|_{H^{k+2}(K)} + \|P_0\zeta\|_K + Ch\|\nabla\zeta\|_K \tag{5.2.1}$$

By the property of the L_2 projection and the definition of $u_h^\star$, we have

$$\begin{aligned}
\|P_0\zeta\|_K^2 &= (P_0\zeta, P_0\zeta)_K \\
&= (\zeta, P_0\zeta)_K, && \text{Since } P_0 \text{ is } L_2 \text{ projection} \\
&= (P_{k+1}u - u_h^*, P_0\zeta)_K \\
&= (P_{k+1}u, P_0\zeta)_K - (u_h^*, P_0\zeta), \\
&= (P_{k+1}u, P_0\zeta)_K - (u_h, P_0\zeta)_K && \text{Since } P_0\zeta \text{ is constant on } K, \text{ we use (5.1.1a).}
\end{aligned}$$

By the properties of L_2 and $\varPi_W$ projections, we obtain

$$\begin{aligned}\|P_0\zeta\|_K^2 &= (P_{k+1}u, P_0\zeta)_K - (\varPi_W u - \varepsilon_u, P_0\zeta)_K\\ &= (P_{k+1}u, P_0\zeta)_K - (\varPi_W u, P_0\zeta)_K + (\varepsilon_u, P_0\zeta)_K\\ &= (P_{k+1}u, P_0\zeta)_K - (u, P_0\zeta)_K + (P_0\varepsilon_u, P_0\zeta)_K,\\ &= (P_{k+1}u - u, P_0\zeta)_K + (P_0\varepsilon_u, P_0\zeta)_K\end{aligned}$$

Therefore,

$$\begin{aligned}&\|P_0\zeta\|_K^2 \le \|P_{k+1}u - u\|_K\,\|P_0\zeta\|_K + \|P_0\varepsilon_u\|_K\,\|P_0\zeta\|_K\\ &\|P_0\zeta\|_K \le C\,h_K^{k+2}\,|u|_{H^{k+2}(K)} + \|P_0\varepsilon_u\|_K\end{aligned}$$

By the definition of $u_h^\star$, (5.1.1b) and (3.1.2a), we obtain

$$\begin{aligned}\|\nabla\zeta\|_K^2 &= (\nabla P_{k+1}u - \nabla u_h^*, \nabla\zeta)\\ &= (\nabla P_{k+1}u, \nabla\zeta) - (\nabla u_h^*, \nabla\zeta)\\ &= (\nabla P_{k+1}u + \boldsymbol{q}_h, \nabla\zeta)_K && \text{by (5.1.1b)}\\ &= (\nabla P_{k+1}u + \boldsymbol{q}_h, \nabla\zeta)_K - (\nabla u + \boldsymbol{q}, \nabla\zeta)_K && \text{since } \nabla u + \boldsymbol{q} = 0\\ &= (\nabla(P_{k+1}u - u), \nabla\zeta)_K - (\mathbf{q} - \boldsymbol{q}_h, \nabla\zeta)_K\,.\end{aligned}$$

Hence, applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\|\nabla\zeta\|_K^2 \le \|\nabla(P_{k+1}u - u)\|_K\,\|\nabla\zeta\|_K + \|\boldsymbol{q} - \boldsymbol{q}_h\|_K\,\|\nabla\zeta\|_K\\ &\|\nabla\zeta\|_K \le C\,h_K^{k+1}\,|u|_{H^{k+2}(K)} + \|\boldsymbol{q} - \boldsymbol{q}_h\|_K\end{aligned}$$

Substitute the above achieved inequalities in 5.2.1, we notice that

$$\|(u-u_h^\star)(T)\|_K \le C\, h_K^{k+2}\, |u(T)|_{H^{k+2}(K)} + \|P_0\varepsilon_u(T)\|_K + C\, h\, \|(\mathbf{q}-\boldsymbol{q}_h)(T)\|_K.$$

By theorem 23, we have $\|(\mathbf{q}-\boldsymbol{q}_h)(T)\| \le C_1(T+1)\; h^{k+1}$. Therefore

$$\|(u-u_h^\star)(T)\|_K \le C_1\, h^{k+2} + \|P_0\varepsilon_u(T)\|_K \tag{5.2.2}$$

It remains to show that the term $\|P_0\varepsilon_u(T)\|$ is of order $O(h^{k+2})$. Then the postprocessed approximation $u_h^\star$ would converge faster than the original approximation u_h. Noting that $\|P_0\varepsilon_u(T)\| = \sup_{\Theta\in C_0^\infty(\Omega)} \frac{(P_0\varepsilon_u(T),\Theta)}{\|\Theta\|}$. To estimate the expression $(P_0\varepsilon_u(T),\Theta)$, we use the traditional duality approach by using the solution of the dual problem

$$\boldsymbol{\Phi} + \nabla\Psi = 0 \quad \text{and} \quad (\mathcal{I}^{\alpha *}\Psi)' - \nabla\cdot\boldsymbol{\Phi} = 0 \quad \text{on } \Omega\times(0,T), \tag{5.2.3}$$

with $\Psi = 0$ on $\partial\Omega\times(0,T)$ and $\Psi(T) = \Theta$ on Ω, where $\mathcal{I}^{\alpha *}$ is the adjoint operator of $\mathcal{I}^{\alpha}$ defined by [36]:

$$\mathcal{I}^{\alpha *}\psi(t) = \int_t^T \omega_\alpha(s-t)\,\psi(s)\,ds\,.$$

Integrating $(\mathcal{I}^{\alpha *}\Psi)' - \nabla\cdot\boldsymbol{\Phi} = 0$ over the time interval (t,T), we obtain

$$-\int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}\Psi(s)\,ds - \int_t^T \nabla\cdot\boldsymbol{\Phi}(s)\,ds = 0$$
$$-\mathcal{I}^{\alpha *}\Psi - \int_t^T \nabla\cdot\boldsymbol{\Phi}(s)\,ds = 0$$

$$\mathcal{I}^{\alpha *}\Psi(t) + \int_t^T \nabla\cdot\boldsymbol{\Phi}(s)\,ds = 0\,. \tag{5.2.4}$$

We define now the adjoint ${}^R\mathrm{D}^{\alpha *}$ of the Riemann–Liouville fractional derivative operator ${}^{R}\mathrm{D}^{\alpha}$ as follows [36]: for $t \in (0,T)$,

$$ {}^{R}\mathrm{D}^{\alpha^*} v(t) = -\frac{\partial}{\partial t} \int_t^T \omega_{1-\alpha}(s-t)\, v(s)\, ds \quad \text{for any } v \in \mathcal{C}^1(0,T)\,. $$

Change the order of integrals and use the equality, $\int_t^q \omega_{1-\alpha}(s-t)\,\omega_\alpha(q-s)\,ds = 1$,

$$ \begin{aligned} {}^{R}\mathrm{D}^{\alpha^*}(\mathcal{I}^{\alpha *}\Psi)(t) &= -\frac{\partial}{\partial t} \int_t^T \omega_{1-\alpha}(s-t) \int_s^T \omega_\alpha(q-s)\Psi(q)\, dq\, ds \\ &= -\frac{\partial}{\partial t} \int_t^T \int_s^T \omega_{1-\alpha}(s-t)\,\omega_\alpha(q-s)\Psi(q)\, dq\, ds \\ &= -\frac{\partial}{\partial t} \int_t^T \int_t^q \omega_{1-\alpha}(s-t)\,\omega_\alpha(q-s)\, ds\,\Psi(q)\, dq \\ &= -\frac{\partial}{\partial t} \int_t^T (1)\,\Psi(q)\, dq = \Psi(t) \end{aligned} $$

Therefore, $\mathcal{I}^{\alpha *}$ is the *right-inverse* of ${}^{R}\mathrm{D}^{\alpha *}$.

Hence, using this after applying the operator ${}^{R}\mathrm{D}^{\alpha *}$ to both sides of (5.2.4), yields

$$ \Psi(t) + {}^{R}\mathrm{D}^{\alpha^*}\Big(\int_t^T \nabla \cdot \boldsymbol{\Phi}(q)\, dq \Big) = 0\,. \tag{5.2.5} $$

The second term on the left-hand side of (5.2.5) equals

$$ \begin{aligned} -\frac{\partial}{\partial t} \int_t^T \omega_{1-\alpha}(s-t) \int_s^T \nabla \cdot \boldsymbol{\Phi}(q)\, dq\, ds &= -\frac{\partial}{\partial t} \int_t^T \nabla \cdot \boldsymbol{\Phi}(q) \int_t^q \omega_{1-\alpha}(s-t)\, ds\, dq \\ &= -\frac{\partial}{\partial t} \int_t^T \omega_{2-\alpha}(q-t)\, \nabla \cdot \boldsymbol{\Phi}(q)\, dq \\ &= \int_t^T \omega_{1-\alpha}(s-t)\, \nabla \cdot \boldsymbol{\Phi}(s)\, ds, \end{aligned} $$

Using this after, differentiating both sides of (5.2.5) with respect to t, yield

$$ \Psi' - \nabla \cdot {}^{R}\mathrm{D}^{\alpha^*} \boldsymbol{\Phi} = 0. $$

Therefore, an alternative formulation of the dual problem (5.2.3) is given by:

$$\boldsymbol{\Phi} + \nabla \Psi = 0 \qquad \text{on } \Omega \times (0,T), \tag{5.2.6a}$$

$$\Psi' - \nabla \cdot {}^{R}\mathrm{D}^{\alpha^*} \boldsymbol{\Phi} = 0 \qquad \text{on } \Omega \times (0,T), \tag{5.2.6b}$$

$$\Psi = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{5.2.6c}$$

$$\Psi(T) = \Theta \qquad \text{on } \Omega. \tag{5.2.6d}$$

In the next lemma, an expression for the quantity $(P_0 \varepsilon_u(T), \Theta)$ in terms of the errors ε_u', $\boldsymbol{\varepsilon}_{\mathbf{q}}$, the projection errors $e_{\boldsymbol{q}}$ and e_u', and the solution of the dual problem will be given.

Lemma 24 Assume that the degree of the HDG solution $k \geq 1$. Then, for any $T > 0$, we have

$$\begin{aligned}(P_0 \varepsilon_u(T), \Theta) = \int_0^T &[(\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^{R}\mathrm{D}^{\alpha^*}(\nabla \mathrm{I}_h \Psi - \mathbf{\Pi}^{\mathrm{BDM}} \nabla \Psi)) \\ &+ (e_{\boldsymbol{q}}, {}^{R}\mathrm{D}^{\alpha^*}(\mathbf{\Pi}^{\mathrm{BDM}} \nabla \Psi - \nabla P_W \Psi)) + (\varepsilon_u' - e_u', P_0 \Psi - \mathrm{I}_h \Psi)]\, dt.\end{aligned}$$

where, In it, I_h is any interpolation operator from $L^2(\Omega)$ into $W_h \cap H_0^1(\Omega)$, P_W is the L^2-projection into W_h and $\mathbf{\Pi}^{\mathrm{BDM}}$ is the well-known projection associated to the lowest-order Brezzi-Douglas-Marini (BDM) space.

Proof 6 Since $\Psi(T) = \Theta$ by (5.2.6d) and $\varepsilon_u(0) = 0$, we have

$$\begin{aligned}(P_0 \varepsilon_u(T), \Theta) &= \int_0^T [((P_0 \varepsilon_u)', \Psi) + (P_0 \varepsilon_u, \Psi')]\, dt \\ &= \int_0^T [(\varepsilon_u', P_0 \Psi) + (\varepsilon_u, P_0 \nabla \cdot {}^{R}\mathrm{D}^{\alpha^*} \boldsymbol{\Phi})]\, dt\end{aligned}$$

by the definition of the L^2-projection P_0 and by (5.2.6b). By the commutativity property $P_0\nabla\cdot = \nabla\cdot\mathbf{\Pi}^{\text{BDM}}$ and the first error equation (3.2.4a) with $\boldsymbol{r} := {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi}$ (since $k \ge 1$), we get for each $t \in (0,T]$,

$$
\begin{aligned}
(\varepsilon_u, P_0\nabla\cdot {}^R\mathrm{D}^{\alpha^*}\boldsymbol{\Phi}) &= (\varepsilon_u, \nabla\cdot {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi}),\\
&=(\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi}) + \langle \varepsilon_{\widehat{u}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle - (e_{\boldsymbol{q}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi})\\
&=(\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi}) - (e_{\boldsymbol{q}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi})\\
&=(\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^R\mathrm{D}^{\alpha^*}(-\mathbf{\Pi}^{\text{BDM}}\nabla\Psi + \nabla\mathrm{I}_h\Psi)) - (\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^R\mathrm{D}^{\alpha^*}(\nabla\mathrm{I}_h\Psi)) - (e_{\boldsymbol{q}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi})\,.
\end{aligned}
$$

Noting that, in the second last equality we used

$$
\langle \varepsilon_{\widehat{u}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle = \langle \varepsilon_{\widehat{u}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle_{\partial\Omega} = 0
$$

which follows from (3.2.4d) (because ${}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi} \in \mathbf{H}(\text{div},\Omega)$) and the fact that $\varepsilon_{\widehat{u}} = 0$ on $\partial\Omega$ by (3.2.4c). But, by the error equation (3.2.4b) with $w := {}^R\mathrm{D}^{\alpha^*}(\mathrm{I}_h\Psi)$,

$$
(\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^R\mathrm{D}^{\alpha^*}(\nabla\mathrm{I}_h\Psi)) = (\mathcal{I}^\alpha(\varepsilon_u' - e_u'), {}^R\mathrm{D}^{\alpha^*}(\mathrm{I}_h\Psi)) + \langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, {}^R\mathrm{D}^{\alpha^*}(\mathrm{I}_h\Psi)\rangle\,.
$$

Now, putting together all the above intermediate steps,

$$
\begin{aligned}
(P_0\varepsilon_u(T),\Theta) = \int_0^T &[(\varepsilon_u', P_0\Psi) + (\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^R\mathrm{D}^{\alpha^*}(\nabla\mathrm{I}_h\Psi) - \mathbf{\Pi}^{\text{BDM}}\nabla\Psi)\\
&- ({}^R\mathrm{D}^{\alpha}\mathcal{I}^\alpha(\varepsilon_u' - e_u'), \mathrm{I}_h\Psi) - \langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, {}^R\mathrm{D}^{\alpha^*}(\mathrm{I}_h\Psi)\rangle - (e_{\boldsymbol{q}}, {}^R\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\text{BDM}}\boldsymbol{\Phi})]\,dt. \quad (5.2.8)
\end{aligned}
$$

But, $\langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, {}^R\mathrm{D}^{\alpha^*}(\mathrm{I}_h\Psi)\rangle = \langle \boldsymbol{\varepsilon}_{\widehat{q}}\cdot\boldsymbol{n}, {}^R\mathrm{D}^{\alpha^*}(\mathrm{I}_h\Psi)\rangle_{\partial\Omega} = 0$ by (3.2.4d) and the identity $\mathrm{I}_h\Psi = 0$ on $\partial\Omega$ by the boundary condition of the dual problem (5.2.6c). Using this and ${}^{R}\mathrm{D}^{\alpha}(\mathcal{I}^{\alpha}(\varepsilon_u' - e_u'))(t) = (\varepsilon_u - e_u)'(t)$ identity in (5.2.8) , Since

$$
\begin{aligned}
{}^{R}D^{\alpha}\,\mathcal{I}^{\alpha}\,v(t) &= \frac{\partial}{\partial t}\int_0^t \omega_{1-\alpha}(t-s)\int_0^s \omega_{\alpha}(s-q)v(q)\,dq\,ds \\
&= \frac{\partial}{\partial t}\int_0^t \Big(\int_q^t \omega_{1-\alpha}(t-s)\omega_{\alpha}(s-q)\,ds\Big)\,v(q)\,dq \\
&= \frac{\partial}{\partial t}\int_0^t v(q)\,dq = v(t)
\end{aligned}
$$

we observe that,

$$
\begin{aligned}
(P_0\varepsilon_u(T), \Theta) = \int_0^T &[(\boldsymbol{\varepsilon}_{\mathbf{q}}, {}^{R}\mathrm{D}^{\alpha^*}(\nabla \mathrm{I}_h\Psi - \mathbf{\Pi}^{\mathrm{BDM}}\nabla\Psi)) \\
&- (e_{\boldsymbol{q}}, {}^{R}\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\mathrm{BDM}}\boldsymbol{\Phi}) + (\varepsilon_u', P_0\Psi - \mathrm{I}_h\Psi) + (e_u', \mathrm{I}_h\Psi)]\,dt.
\end{aligned}
$$

Therefore, the desired result now follows after noting that

$$
-\int_0^T (e_{\boldsymbol{q}}, {}^{R}\mathrm{D}^{\alpha^*}\mathbf{\Pi}^{\mathrm{BDM}}\boldsymbol{\Phi})\,dt = \int_0^T (e_{\boldsymbol{q}}, {}^{R}\mathrm{D}^{\alpha^*}(\mathbf{\Pi}^{\mathrm{BDM}}\nabla\Psi - \nabla P_W\Psi))\,dt,
$$

(by (5.2.6a), the fact that P_W is the L^2-projection into W_h, and the orthogonality property of the projection $\boldsymbol{\Pi}_V$, (3.2.1a)) and that $(e_u', \mathrm{I}_h\Psi) = (e_u', \mathrm{I}_h\Psi - P_0\Psi)$ (by the fact that $P_0\Psi$ is constant on each element $K \in \mathcal{T}_h$, and the orthogonality property of the projection Π_W, (3.2.1b)). The proof is completed now.

In the next theorem we state the superconvergence estimate of the postprocessed HDG approximation. For the proof, we follow the derivation in [10, Section 5] step-by-step and use Lemma 24 instead of [10, Lemma 7], and we also use the achieved HDG error estimates in Theorem 23 and the following lemma.

Lemma 25 ([10]) Let $(\boldsymbol{\Phi}, \Psi)$ be the solution of the dual problem with $\Theta := P_h\,\chi$

where $\chi \in W_h$. Then

$$\int_0^T \|\Psi_t\|\,dt \le \frac{C}{1-\alpha}\sqrt{\log(\kappa)}\|\chi\|$$

And

$$\int_0^T \|\nabla\Psi\|\,\|{}^R\mathrm{D}^{\alpha^*}\nabla\Psi\|\,dt \le \frac{C}{1-\alpha}\log(\kappa)\|\chi\|^2$$

Where, $\kappa > 1$ is a solution of $\kappa^{1-\alpha}\log(\kappa) = C_{k,d}^2 T^{1-\alpha}/\rho^2$. $\rho = min_{K\in\tau_h}\rho_K$ and ρ_K denotes the radius of the largest ball included in the simplex K, and $C_{k,d}$ depends on the dimension of Ω and k

Let $W_{h'}$ be the space of continuous functions which are polynomials of degree k on each element of $\tau_{h'}$. Where, $\tau_{h'}$ is a triangulation of Ω obtained by refining each of $K \in \tau_h$,then P_h is defined as the L^2 projection from W_h into $W_{h'}$

Theorem 26 Let Ω be convex. Assume that $u \in \mathcal{C}^1(0,T;H^{k+2}(\Omega))$ and $\boldsymbol{q} \in \mathcal{C}^1(0,T;\boldsymbol{H}^{k+1}(\Omega))$. Assume also that τ_K^* and $1/\tau_K^{\max}$ are bounded by C. Then, we have

$$\|(u-u_h^*)(T)\| \le C_2\,\max\{1,\sqrt{\log(Th^{-2/(1-\alpha)})}\}\,h^{k+2} \quad \text{for} \quad k \ge 1,$$

where the constant C_2, only depends on C, α, T, $\|u\|_{\mathcal{C}^1(H^{k+2})}$, and on $\|\boldsymbol{q}\|_{\mathcal{C}^1(H^{k+1})}$.

Proof 7 Using Lemma 24, Cauchy-Schwarz inequality, lemma 25 and 16, we have

$$\begin{aligned}\|(P_0\varepsilon_u(T),\Theta)\| \le \|\boldsymbol{\varepsilon}_{\mathbf{q}}\|_{L^\infty(L^2)}\,\|{}^R\mathrm{D}^{\alpha^*}(\nabla \mathrm{I}_h\Psi - \boldsymbol{\Pi}^{\text{BDM}}\nabla\Psi)\|_{L^1(L^2)}+ \\ \|e_{\boldsymbol{q}}\|_{L^\infty(L^2)}\,\|\|{}^R\mathrm{D}^{\alpha^*}(\boldsymbol{\Pi}^{\text{BDM}}\nabla\Psi - \nabla P_W\Psi)\|_{L^1(L^2(\tau_h))}+ \\ \Big(\|\varepsilon_u'\|_\alpha + \|e_u'\|_\alpha\Big)\,\|P_0\Psi - \mathrm{I}_h\Psi\|_{-\alpha} \quad (5.2.10)\end{aligned}$$

$$H_1(\Theta) := \mathrm{Sup}_{\Theta\in C_0^\infty(\Omega)} \max\Big\{\frac{\|{}^R\mathrm{D}^{\alpha^*}(\nabla \mathrm{I}_h\Psi - \boldsymbol{\Pi}^{\mathrm{BDM}}\nabla\Psi)\|_{L^1(L^2)}}{\|\Theta\|}, \frac{\|{}^R\mathrm{D}^{\alpha^*}(\boldsymbol{\Pi}^{\mathrm{BDM}}\nabla\Psi - \nabla P_W\Psi)\|_{L^1(L^2(\tau_h))}}{\|\Theta\|}\Big\}$$

$$H_2(\Theta) := \mathrm{Sup}_{\Theta\in C_0^\infty(\Omega)} \frac{\|P_0\Psi - \mathrm{I}_h\Psi\|_{-\alpha}}{\|\Theta\|}$$

Following the work of cockburn and mustapha,

$$\begin{aligned} H_1(\Theta) &\le C\,h\,\mathrm{Sup}_{\Theta\in C_0^\infty(\Omega)} \frac{\|\Psi_t\|_{L^1(L^2)}}{\|\Theta\|} \\ H_2(\Theta) &\le C\,h\,\mathrm{Sup}_{\Theta\in C_0^\infty(\Omega)} \frac{1}{\|\Theta\|}\Big(\int_0^T \|\nabla\Psi\|\,\|{}^R\mathrm{D}^{\alpha^*}\nabla\Psi\|\,dt\Big)^{\frac{1}{2}} \end{aligned}$$

By using the lemma 25 and the mesh is quasi uniform,

$$\kappa^{1-\alpha} < \kappa^(1-\alpha)\log(\kappa) = C_{k,d}^2 T^{1-\alpha}/\rho^2 \le C C_{k,d}^2 T^{1-\alpha}/h^2, \quad \text{for } \kappa > e$$

So $\sqrt{\log(\kappa)} \le C'_{k,d}\sqrt{\log(T\,h^{-2/(1-\alpha)})}$.

Substitute these bounds in equation (5.2.10),

$$\begin{aligned} \|P_0\varepsilon_u(T)\| &\le \Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|_{L^\infty(L^2)} + \|e_{\boldsymbol{q}}\|_{L^\infty(L^2)}\Big) H_1(\Theta) + \Big(\|\varepsilon_u'\|_\alpha + \|e_u'\|_\alpha\Big) H_2(\Theta) \\ &\le C\,h\,\frac{\sqrt{\log(T\,h^{-2/(1-\alpha)})}}{1-\alpha}\Big(\|\boldsymbol{\varepsilon}_{\mathbf{q}}\|_{L^\infty(L^2)} + \|e_{\boldsymbol{q}}\|_{L^\infty(L^2)} + \|\varepsilon_u'\|_\alpha + \|e_u'\|_\alpha\Big)\|\chi\| \end{aligned}$$

Insert this in equation (5.2.2) will complete the proof. $\square$

5.3 Implementation of post-processing in one dimension

Let

$$u_h^n = \sum_{j=0}^{s} \alpha_j^n \varphi_j(x), \quad \boldsymbol{q}_h^n = \sum_{j=0}^{s} \beta_j^n \varphi_j(x), \quad u_h^*(x) = \sum_{j=0}^{s+1} \gamma_j^n \varphi_j(x), \quad x \in I_m = (x_{m-1}, x_m]$$

where, $\{\varphi_j\}_{j=0}^k$ are defined by

$$\varphi_j(x) = p_j(\frac{2x}{h} + 1 - 2m), \quad x \in I_m.$$

Recall that, k is the degree of the approximate solution and $\{p_j\}_{j=0}^k$ are the legendre polynomial functions of degree j.

Finding the values of the unknowns γ_j^n, we substitute u_h^n, $\boldsymbol{q}_h^n$ and u_h^* in the system (5.1.1) and get

$$\begin{aligned}(u_h^n(x), 1)_{I_m} &= (u_h^*(x), 1)_{I_m} \\ &= \sum_{j=0}^{k+1} \gamma_j^n \, (\varphi_j(x), 1)_{I_m} \\ &= h\, \gamma_0^n\end{aligned} \tag{5.3.1}$$

$$\begin{aligned}(\boldsymbol{q}_h^n(x), {\varphi'}_i(x))_{I_m} &= -({u^*}'_h(x), {\varphi'}_i(x))_{I_m} \\ &= \sum_{j=0}^{k+1} \gamma_j^n \left({\varphi'}_j(x), {\varphi'}_i(x)\right)_{I_m}\end{aligned} \tag{5.3.2}$$

For $i = 1, 2, 3, \cdots k$, noting that ${\varphi'}_0(x) = 0$.

Also, we know that

$${p'}_{j+1} = (2j+1)p_j + (2(j-2)+1)p_{j-2} + (2(j-4)+1)p_{j-4} + \cdots$$

So,

$$\varphi'_{j+1}(x) = \frac{2}{h}\Big((2j+1)\varphi_j(x) + (2(j-2)+1)\varphi_{j-2}(x) + (2(j-4)+1)\varphi_{j-4}(x) + \cdots\Big)$$

since, $\varphi'_{j+1}(x) = \frac{2}{h}\,p'_{j+1}(\frac{2x}{h}+1-2m)$. For example, $\varphi'_1(x) = \frac{2}{h}\,\varphi_0(x)$, $\varphi'_2(x) = \frac{2}{h}\,(3\varphi_1(x))$, $\varphi'_3(x) = \frac{2}{h}\,(5\varphi_2(x)+\varphi_0(x))$, and $\varphi'_8(x) = \frac{2}{h}\,(15\varphi_7(x)+11\varphi_5(x)+7\varphi_3(x)+3\varphi_1(x))$,

By using $\Big(\varphi_j, \varphi_i\Big)_{I_m} = \frac{h}{2j+1}\delta_{ji}$, for $j = 0, 1, \cdots, k+1$, we conclude that

- $$\Big(\varphi'_{j+1}, \varphi'_k\Big)_{I_m} = \begin{cases} \frac{4}{h}\big((2j+1) + (2(j-2)+1) + (2(j-4)+1) + \cdots\big), \\ \text{If both (j and k) are even or both odd natural numbers;} \\ 0, \qquad \text{otherwise.} \end{cases}$$

- Let $j+1 < k$,

$$\Big(\varphi_j, \varphi'_k\Big)_{I_m} = \frac{2}{h}\Big(\varphi_j(x), (2(k-1)+1)\varphi_{k-1}(x) + (2(k-3)+1)\varphi_{k-3}(x) + (2(k-5)+1)\varphi_{k-5}(x) + \cdots\Big)_{I_m}$$
$$= \frac{2}{h}\Big(\varphi_j, (2j+1)\varphi_j\Big)_{I_m} = 2\delta_{j,k-l}$$

we rewrite now (5.1.1) in the matrix form as follows:

Assume that A and $\mathbf{G}^n$ are defined:

$$A = \Big[(\varphi'_j, \varphi'_i)_{I_m}\Big], \quad \text{and,} \quad \mathbf{G}^n = \begin{pmatrix} (\boldsymbol{q}_h^n(x), \varphi'_0(x))_{I_m} \\ (\boldsymbol{q}_h^n(x), \varphi'_1(x))_{I_m} \\ \vdots \\ (\boldsymbol{q}_h^n(x), \varphi'_k(x))_{I_m} \end{pmatrix}$$

Then, we have to solve the following system,

$$A\,\gamma^n = \mathbf{G}^n \quad \text{where,} \;\; \gamma^n = \begin{bmatrix} \gamma_1^n \\ \gamma_2^n \\ \gamma_3^n \\ \vdots \\ \gamma_{k+1}^n \end{bmatrix} \tag{5.3.3}$$

5.4 Numerical experiments for HDG post-processing

In this section, we present some numerical experiments to validate our theoretical results from the postprocessed HDG spatial discretizations. We impose homogenous Dirichlet boundary conditions and choose the source term f and u_0 so that the exact solution is $u(x,t) = t^{3-\alpha} sin(\pi x)$. So we substitute that coefficients of the Solutions u_h and $\boldsymbol{q}_h$ to find $u_h^{\star}$.We obtain the history of supperconvergence of the errors $\|(u - u_h^{\star})(T)\|$ for different values of the polynomial degree, $k = 0, 1, 2$. The numerical results (errors and convergence rates $= \frac{\log(\frac{\|error\|_{\frac{h}{2}}}{\|error\|_h})}{\log(2)}$) of the experiments are presented in Tables 5 .1 and 5 .2 and their figures. In full agreement with our theoretical results, we obtain $O(h^{k+2})$ superconvergence rates for the postprocessed HDG scheme.

N	$k=0$		$k=1$		$k=2$	
4	5.048e-01		7.401e-03		8.902e-04	
8	2.922e-01	0.788	8.835e-04	3.066	5.497e-05	4.017
16	1.566e-01	0.899	1.142e-04	2.951	3.416e-06	4.008
32	8.098e-02	0.951	1.420e-05	3.008	2.153e-07	3.988
64	4.117e-02	0.976	1.812e-06	2.970	2.153e-07	3.988
128	2.076e-02	0.988				

Table 5 .1: The errors $\|(u_h^\star - u)(T)\|$, and the corresponding rates of convergence for $\alpha = 0.5$ with HDG-postprocessed solutions of degree $k = 0, 1, 2$. We observe superconvergence rates of order h^{k+2} (when $k \geq 1$) for the error from the postprocessed HDG solution $u_h^\star$.

N	$k=0$		$k=1$		$k=2$	
4	5.240e-01		7.898e-03		1.079e-03	
8	3.020e-01	0.795	9.403e-04	3.070	6.698e-05	4.010
16	1.612e-01	0.905	1.218e-04	2.949	4.167e-06	4.007
32	8.320e-02	0.954	1.506e-05	3.015	2.641e-07	3.980
64	4.225e-02	0.978	1.953e-06	2.947		
128	2.128e-02	0.989				

Table 5 .2: The errors $\|(u_h^\star - u)(T)\|$, and the corresponding rates of convergence for $\alpha = 0.7$ with HDG-postprocessed solutions of degree $k = 0, 1, 2$.

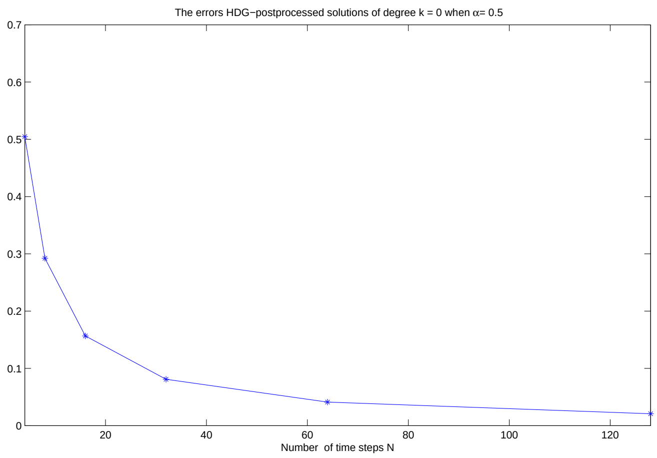


Figure 5 .1: HDG-postprocessing errors for piecewise constant solution($k = 0$) with $\alpha = 0.5$

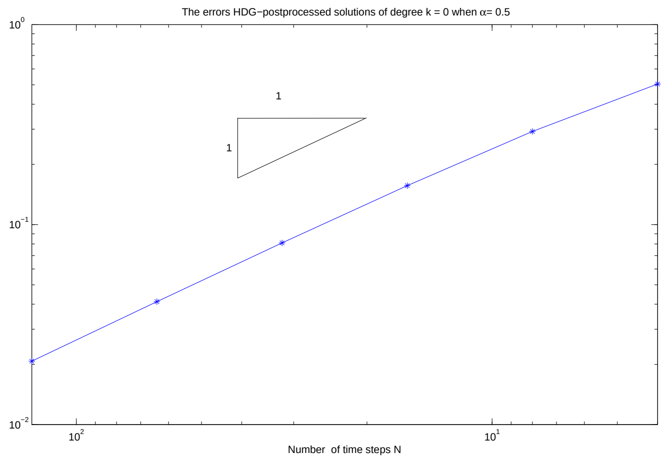


Figure 5 .2: HDG-postprocessing errors for piecewise constant solution($k = 0$) with $\alpha = 0.5$, log-log scaling

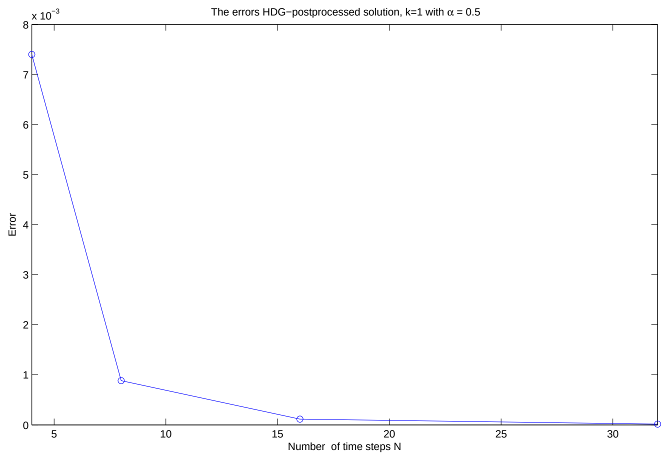


Figure 5 .3: HDG-postprocessing errors for piecewise linear solution($k = 1$) with $\alpha = 0.5$

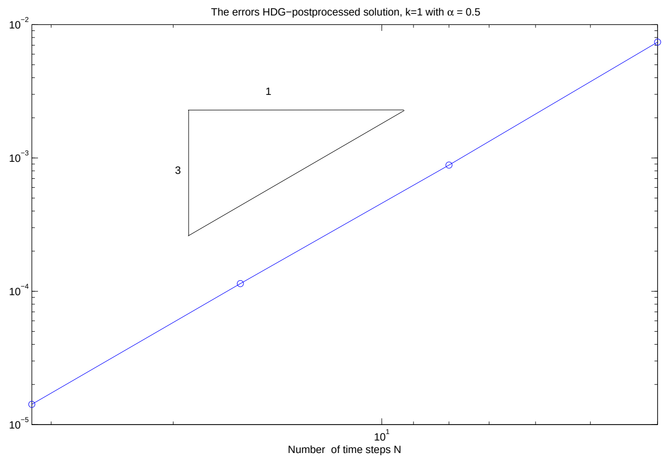


Figure 5 .4: HDG-postprocessing errors for piecewise linear solutions($k = 1$) with $\alpha = 0.5$, log-log scaling

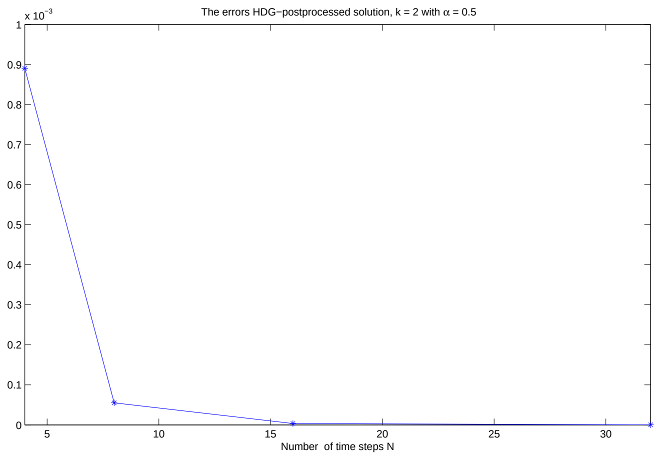


Figure 5 .5: HDG-postprocessing errors for piecewise linear solution($k = 2$) with $\alpha = 0.5$

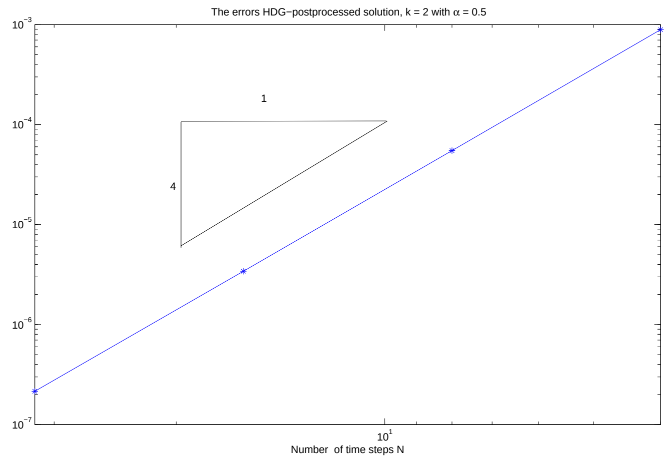


Figure 5 .6: HDG-postprocessing errors for piecewise constant solutions($k = 2$) with $\alpha = 0.5$, log-log scaling

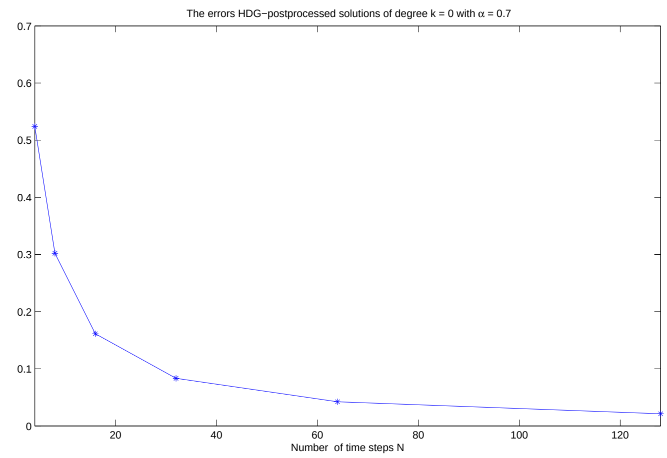


Figure 5 .7: HDG-postprocessing errors for piecewise linear solution($k = 0$) with $\alpha = 0.7$

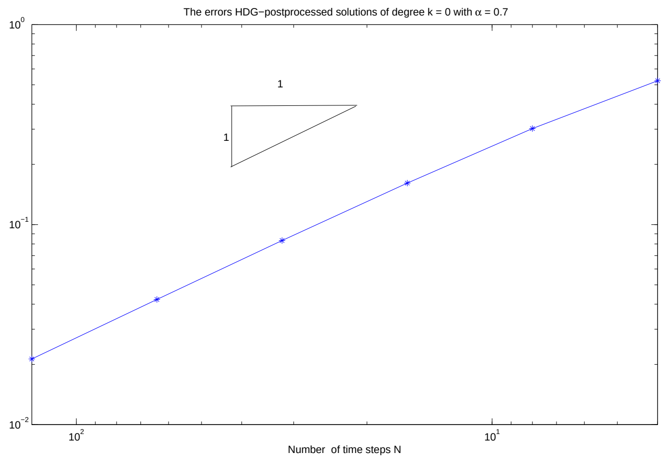


Figure 5 .8: HDG-postprocessing errors for piecewise constant solutions($k = 0$) with $\alpha = 0.7$, log-log scaling

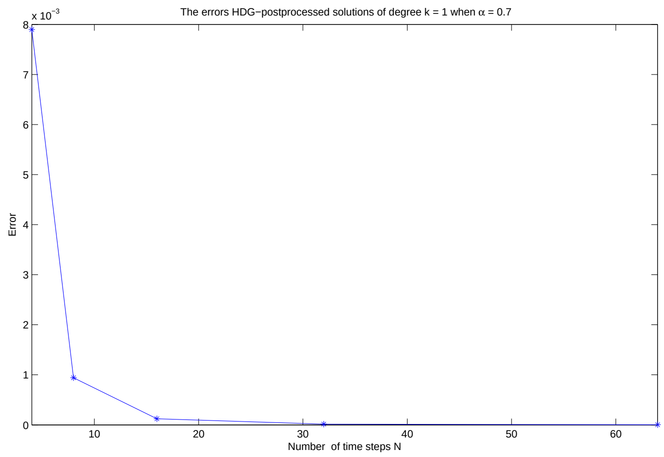


Figure 5 .9: HDG-postprocessing errors for piecewise linear solution($k = 1$) with $\alpha = 0.7$

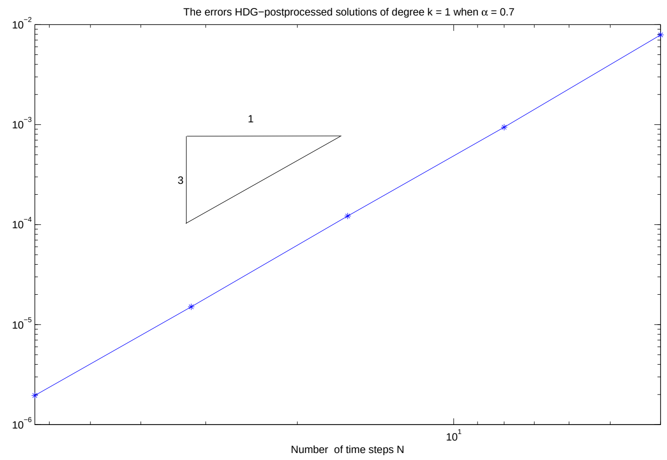


Figure 5 .10: HDG-postprocessing errors for piecewise constant solution($k = 1$) with $\alpha = 0.7$, log-log scaling

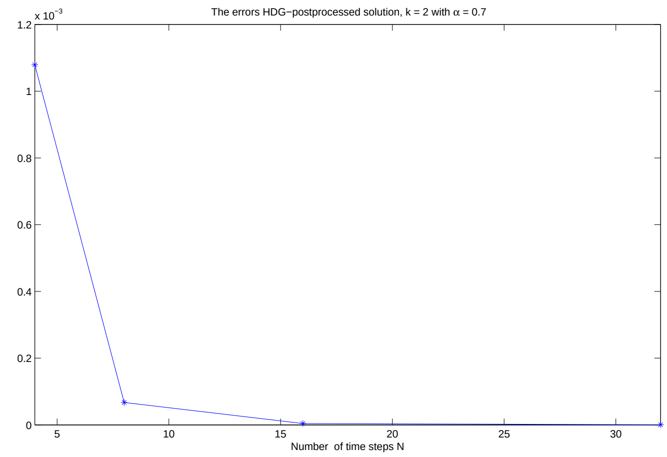


Figure 5 .11: HDG-postprocessing errors for piecewise linear solution($k = 2$) with $\alpha = 0.7$

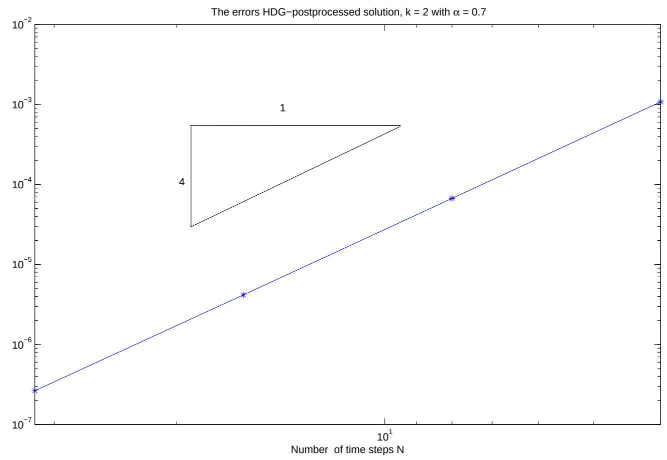


Figure 5 .12: HDG-postprocessing errors for piecewise constant solution($k = 2$) with $\alpha = 0.7$, log-log scaling

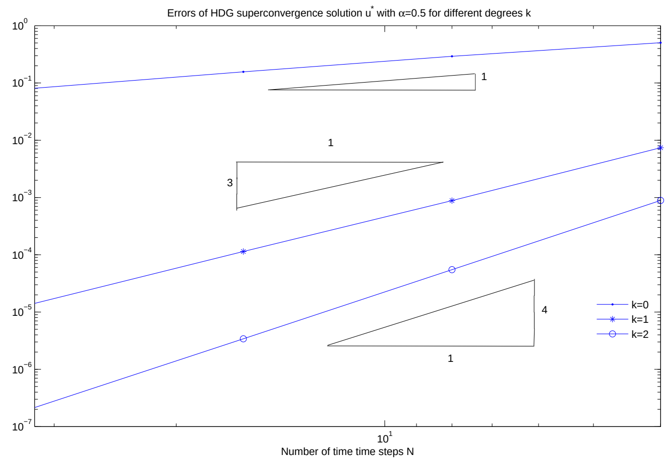


Figure 5 .13: HDG-postprocessing errors for various degrees of the solution $u_h^{\star}(k = 0, 1, 2)$ with $\alpha = 0.5$, log-log scaling

Chapter 6

DISCONTINUOUS GALERKIN FINITE ELEMENT SCHEME

In this chapter, we propose a time-stepping discontinuous Galerkin method with finite elements in space to solve numerically a time fractional diffusion equation involving Caputo derivative of order $1-\alpha \in (0,1)$ with $0 < \alpha < 1$ and variable coefficients. We solve the following problem:

$$\begin{aligned} {}^{c}\mathcal{D}^{1-\alpha}u(x,t) - \nabla\cdot(\mathcal{A}(x,t)\nabla u(x,t)) &= f(x,t) && \text{on } \Omega\times(0,T], \\ u(x,0) &= u_0(x) && \text{on } \Omega, \\ u(x,t) &= 0 && \text{on } \partial\Omega\times(0,T], \end{aligned} \tag{6.0.1}$$

where we assume that $\mathcal{A} \in \mathcal{C}^1([0,T], L^\infty(\Omega))$ and satisfies

$$0 < a_{\min} < \mathcal{A}(x,t) < a_{\max} < \infty \quad \text{on } \overline{\Omega}\times[0,T]. \tag{6.0.2}$$

Well-posedness of the fully discrete scheme and error analysis will be shown. For a time interval $(0,T)$ and a spatial domain Ω, our analysis suggest that the error in $L^2\big((0,T), L^2(\Omega)\big)$-norm is $O(k^{2-\frac{(1-\alpha)}{2}} + h^2)$ (that is, short by order $\frac{1-\alpha}{2}$ from being optimal in time) . However, our numerical experiments indicate optimal $O(k^2+h^2)$ error bound in the stronger $L^\infty\big((0,T), L^2(\Omega)\big)$-norm. Variable time steps are used to compensate the singularity of the continuous solution near $t = 0$.

6.1 The numerical method

To describe our fully discrete DG FE method, we introduce a time partition of the interval $[0,T]$ given by the points: $0 = t_0 < t_1 < \cdots < t_N = T$. We set $I_n = (t_{n-1}, t_n]$ and $k_n = t_n - t_{n-1}$ for $1 \le n \le N$ with $k := \max_{1\le n\le N} k_n$. Let

$$S_h = \Big\{v \in C(\Omega) : v|_K \in p_k(K), v|_{\partial\Omega} = 0\Big\},$$

where, p_k is the space of polynomials of degree less than or equals k with respect to a quasi-uniform partition of Ω into conforming triangular finite elements, with maximum diameter h. Next, we introduce our time-space finite dimensional DG FE space:

$$\mathcal{W} = \{v \in L^2((0,T), S_h) : \quad v|_{I_n} \in p_1(S_h) \text{ for } 1 \leq n \leq N\}$$

where $p_1(S_h)$ denotes the space of linear polynomials in the time variable t, with coefficients in S_h. We denote the left-hand limit, right-hand limit and jump at t_n by

$$w^n := w(t_n) = w(t_n^-), \quad w_+^n := w(t_n^+), \quad [w]^n := w_+^n - w^n,$$

respectively.

The weak form of the fractional diffusion equation in (6.0.1) is

$$\int_{I_n} \left[\langle {}^cD^{1-\alpha} u, v \rangle + a\big(t, u, v\big) \right] dt = \int_{I_n} \langle f, v \rangle \, dt, \quad \forall \; v \in L^2\big(I_n, H^1(\Omega)\big) \,. \tag{6.1.1}$$

For each fixed $t \in (0,T]$, $a(t,\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the bilinear form

$$a(t, v, w) = \langle \mathcal{A}(\cdot, t) \nabla v, \nabla w \rangle = \int_\Omega \mathcal{A}(x,t) \nabla v(x) \cdot \nabla w(x) \, dx$$

associated with the operator $\nabla \cdot (\mathcal{A}(\cdot, t) \nabla)$ which is symmetric and positive definite (by (6.0.2)), that is, there exist positive constants c_0 and c_1 such that

$$c_0 \|v(t)\|_1^2 \leq |v(t)|_1^2 := a(t, v, v) \leq c_1 \|v(t)\|_1^2 \quad \forall \;\; v(t) \in H_0^1(\Omega) \,. \tag{6.1.2}$$

The DG FE approximation $U \in \mathcal{W}$ is defined as follows: Given $U(t)$ for $0 \leq t \leq t_{n-1}$, the solution $U \in p_1(S_h)$ on I_n is determined by requesting that for $1 \le n \le N$,

$$\int_{I_n} \big[\langle {}^{c}D^{1-\alpha}U + \sum_{j=0}^{n-1} \omega_\alpha(t-t_j)\,[U]^j, X\rangle + a\big(t, U, X\big)\big]\, dt = \int_{I_n} \langle f, X\rangle\, dt,\ \forall\, X \in p_1(S_h), \tag{6.1.3}$$

with $U_-^0 = U^0 \in S_h$ is a suitable approximation of the initial data u_0. Since

$$ {}^{\mathrm{R}}\mathrm{D}^{1-\alpha}U(t) = {}^{c}D^{1-\alpha}U(t) + \omega_\alpha(t)U^0 + \sum_{i=1}^{n-1} \omega_\alpha(t-t_i)\,[U]^i \quad \text{for} \quad t \in I_n, \tag{6.1.4}$$

We write ${}^{\mathrm{R}}\mathrm{D}^{1-\alpha}U(t)$ as the following:

$$\begin{aligned}
{}^{\mathrm{R}}\mathrm{D}^{1-\alpha}U(t) =& \frac{d}{dt}\Big(\sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \omega_\alpha(t-s)U(s)\,ds + \int_{t_{n-1}}^{t} \omega_\alpha(t-s)U(s)\,ds\Big) \quad \text{for } t \in \mathrm{I}_n \\
=& \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \omega_{\alpha-1}(t-s)U(s)\,ds + \frac{d}{dt}\int_{t_{n-1}}^{t} \omega_\alpha(t-s)U(s)\,ds \\
=& \sum_{j=1}^{n-1} \Big(\omega_\alpha(t-t_{j-1})U(t_{j-1}^+) - \omega_\alpha(t-t_j)U(t_j^-)\Big) \\
&+ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \omega_\alpha(t-s)U'(s)\,ds + \frac{d}{dt}\int_{t_{n-1}}^{t} \omega_\alpha(t-s)U(s)\,ds
\end{aligned}$$

The final term is written as follows:

$$
\begin{aligned}
\frac{d}{dt}\int_{t_{n-1}}^{t} \omega_\alpha(t-s)U(s)\,ds =& \frac{d}{dt}\Big(\omega_{\alpha+1}(t-t_{n-1})U(t_{n-1}^{+}) + \int_{t_{n-1}}^{t} \omega_{\alpha+1}(t-s)U'(s)\,ds\Big)\\
=& \omega_\alpha(t-t_{n-1})U(t_{n-1}^{+}) + \frac{d}{dt}\int_{t_{n-1}}^{t} \omega_{\alpha+1}(t-s)U'(s)\,ds\\
=& \omega_\alpha(t-t_{n-1})U(t_{n-1}^{+}) + \int_{t_{n-1}}^{t} \omega_\alpha(t-s)U'(t-t_{n-1})\,ds\\
&+ \omega_{\alpha+1}(0)U'(t-t_{n-1}).
\end{aligned}
$$

The numerical scheme (6.1.3) can be rewritten in a compact form as follows: for $1 \le n \le N$,

$$
\int_{I_n} \big[\big\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, X\big\rangle + a\big(t, U, X\big)\big]\,dt = \int_{I_n} \langle f + \omega_\alpha(t)U^0, X\rangle\,dt \quad \forall\, X \in p_1(S_h). \quad (6.1.5)
$$

6.2 Stability of the numerical solution

To show the stability of the DG FE scheme (6.1.5), the identity: $v(t) = I^{1-\alpha}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} v)(t)$ for any $v \in \mathcal{W}$, will be proved in the next lemma.

Lemma 27 *If $v \in \mathcal{W}$ and $0 < \alpha < 1$, then*

$$
v(t) = I^{1-\alpha}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} v)(t) \quad \text{for} \quad t \in I_n \quad \text{with} \quad 1 \le n \le N.
$$

Proof 8 Since v has possible discontinuities at the time nodes $t_0, t_1, \cdots, t_{j-1}$, from (6.1.4),

$$
{}^{\mathrm{R}}\mathrm{D}^{1-\alpha} v(s) = \omega_\alpha(s)v_+^0 + \sum_{i=1}^{j-1} \omega_\alpha(s-t_i)\,[v]^i + {}^{c}D^{1-\alpha} v(s) \;\; \text{for} \;\; s \in I_j\,. \quad (6.2.1)
$$

Applying the operator $I^{1-\alpha}$ to both sides and using $I^{1-\alpha}({}^{c}D^{1-\alpha}v)(t) = \int_0^t v'(s)\,ds$, since

$$\begin{aligned} I^{\mu}({}^{c}D^{\mu}v)(t) &= \int_0^t \omega_{\mu}(t-s)\,({}^{c}D^{\mu}v)(s)\,ds \\ &= \int_0^t \omega_{\mu}(t-s)\Big(\int_0^s \omega_{1-\mu}(s-q)v'(q)\,dq\Big)\,ds \\ &= \int_0^t \int_q^t \omega_{\mu}(t-s)\,\omega_{1-\mu}(s-q)v'(q)\,ds\,dq \\ &= \int_0^t (1)v'(q)\,dq \end{aligned}$$

we observe

$$\begin{aligned} I^{1-\alpha}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}v)(t) = v_+^0 &+ \sum_{j=2}^{n-1}\int_{I_j}\omega_{1-\alpha}(t-s)\sum_{i=1}^{j-1}\omega_{\alpha}(s-t_i)\,[v]^i\,ds \\ &+ \int_{t_{n-1}}^t \omega_{1-\alpha}(t-s)\sum_{i=1}^{n-1}\omega_{\alpha}(s-t_i)\,[v]^i\,ds + \int_0^t v'(s)ds \quad \text{for} \quad t\in I_n\,. \end{aligned}$$

Now, changing the order of summations and rearranging the terms yield

$$\begin{aligned}
I^{1-\alpha}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}v)(t) =& v_+^0 + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\int_{I_j} \omega_{1-\alpha}(t-s)\omega_\alpha(s-t_i)\,[v]^i\,ds \\
&+ \sum_{i=1}^{n-1}\int_{t_{n-1}}^{t} \omega_{1-\alpha}(t-s)\omega_\alpha(s-t_i)\,[v]^i\,ds + \sum_{j=1}^{n}\int_{t_{j-1}}^{\min\{t,t_j\}} v'(s)ds \\
=& v_+^0 + \sum_{i=1}^{n-2}\int_{t_i}^{t_{n-1}} \omega_{1-\alpha}(t-s)\omega_\alpha(s-t_i)\,[v]^i\,ds \\
&+ \sum_{i=1}^{n-2}\int_{t_{n-1}}^{t} \omega_{1-\alpha}(t-s)\omega_\alpha(s-t_i)\,[v]^i\,ds \\
&+ \int_{t_{n-1}}^{t} \omega_{1-\alpha}(t-s)\omega_\alpha(s-t_{n-1})\,[v]^{n-1}\,ds + \sum_{j=1}^{n}\int_{t_{j-1}}^{\min\{t,t_j\}} v'(s)ds \\
=& v_+^0 + \sum_{i=1}^{n-2}\int_{t_i}^{t} \omega_{1-\alpha}(t-s)\omega_\alpha(s-t_i)\,[v]^i\,ds \\
&+ \int_{t_{n-1}}^{t} \omega_{1-\alpha}(t-s)\omega_\alpha(s-t_{n-1})\,[v]^{n-1}\,ds + \sum_{j=1}^{n}\int_{t_{j-1}}^{\min\{t,t_j\}} v'(s)ds\,.
\end{aligned}$$

Integrating and simplifying, then we have

$$I^{1-\alpha}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}v)(t) = v_+^0 + \sum_{i=1}^{n-1}[v]^i + \sum_{j=1}^{n-1}(v^j - v_+^{j-1}) + v(t) - v_+^{n-1} = v(t) \quad \text{for} \quad t \in I_n\,.$$

The proof is completed now. □

The next theorem shows the stability of the DG FE scheme.

Theorem 28 Assume that $u_0 \in L^2(\Omega)$ and $f \in L^2((0,T), L^2(\Omega))$. Then,

$$\int_0^T \|U\|_1^2\,dt \le CT^\alpha\|U^0\|^2 + C\int_0^T \|f\|^2\,dt\,.$$

Proof 9 Choosing $X = U$ in the DG FE scheme (6.1.5), and then summing over n, we obtain

$$\int_0^T \left[\left\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, U\right\rangle + a(t, U, U)\right] dt = \int_0^T \langle f + \omega_\alpha(t) U^0, U\rangle\, dt.$$

Since $a(\cdot, U, U) \geq c_0 \|U\|_1^2$ by (6.1.2) and $\langle f, U\rangle \leq \frac{1}{2c_0}\|f\|^2 + \frac{c_0}{2}\|U\|^2$, by Cauchy-Schwarz Inequality and Geometric Arithmetic Mean Inequality, we have

$$\int_0^T \left[\left\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, U\right\rangle + \frac{c_0}{2}\|U\|_1^2\right] dt \leq \int_0^T \left(\langle \omega_\alpha(t) U^0, U\rangle + \frac{1}{2c_0}\|f\|^2\right) dt\,.$$

Using the identity $U(t) = I^{1-\alpha}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U)(t)$ from Lemma 27, Lemma 16 (ii), the inequality $ab \leq \frac{a^2}{4} + b^2$, and the identity $I^{1-\alpha}\omega_\alpha(t) = 1$, yield

$$\begin{aligned}
\int_0^T \langle \omega_\alpha(t) U^0, U\rangle\, dt &= \int_0^T \langle \omega_\alpha(t) U^0, I^{1-\alpha}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U)\rangle\, dt \\
&\leq \frac{1}{4}\int_0^T \left\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, U\right\rangle dt + \sec^2(1-\alpha\pi/2)\int_0^T \omega_\alpha(t)(I^{1-\alpha}\omega_\alpha)(t)\, dt \|U^0\|^2 \\
&\leq \frac{1}{4}\int_0^T \left\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, U\right\rangle dt + C\, T^\alpha \|U^0\|^2\,. \qquad (6.2.2)
\end{aligned}$$

To complete the proof, we combine the above two equations and use the positivity property of the operator ${}^{\mathrm{R}}\mathrm{D}^{1-\alpha}$ given by Lemma 16 (ii). $\square$

Noting that, since the DG FE scheme (6.1.5) amounts to a square linear system, the existence of the numerical solution U follows from its uniqueness. The uniqueness follows immediately from the above stability theorem.

6.3 Projections and errors

In this section, we introduce time and space projections, and then derive some bounds and errors properties that will be used later in our convergence analysis.

6.3.1 Projection in space

For each $t \in [0,T]$, the elliptic projection operator $R_h : H_0^1(\Omega) \to S_h$ is defined by

$$a(t, R_h v - v, \chi) = 0 \quad \forall \;\; \chi \in S_h\,. \tag{6.3.1}$$

By the assumption $\mathcal{A} \in \mathcal{C}^1([0,T], L^\infty(\Omega))$, for each $t \in (0,T)$, the projection error $\xi := R_h u - u$ has the well-known approximation property:

$$\|\xi(t)\| + h\|\nabla\xi(t)\| \le C\, h^2 \|u(t)\|_2 \quad \text{for} \;\; u(t) \in H^2(\Omega) \cap H_0^1(\Omega)\,. \tag{6.3.2}$$

Moreover, By [[8](3.3)]

$$\|\xi'(t)\| \le C\, h^2(\|u(t)\|_2 + \|u'(t)\|_2) \quad \text{for} \;\; u \in W^{1,\infty}((0,T), H^2(\Omega) \cap H_0^1(\Omega))\,. \tag{6.3.3}$$

6.3.2 Projection in time

The local L^2-projection operator $\mathit{\Pi}_W : \mathcal{C}(\overline{I}_n, L^2(\Omega)) \to \mathcal{C}(\overline{I}_n, p_1(L^2(\Omega))$ defined by:

$$\int_{I_n} \langle \mathit{\Pi}_W v - v,\, w\rangle\, dt = 0 \;\; \forall \;\; w \in p_1(L^2(\Omega)) \quad \text{for} \;\; 1 \le n \le N,$$

where $p_1(L^2(\Omega))$ is the space of linear polynomials in the time variable t, with coefficients in $L^2(\Omega)$. Explicitly,

$$\mathit{\Pi}_W v(t) = \frac{12}{k_n^3}(t - t_{n-\frac{1}{2}}) \int_{I_n} (s - t_{n-\frac{1}{2}})\, v(s)\, ds + \frac{1}{k_n} \int_{I_n} v(s)\, ds \quad \text{for } t \in I_n\,,$$

where $t_{n-\frac{1}{2}} := (t_{n-1} + t_n)/2$. Hence,

$$(\mathit{\Pi}_W v)'(t) = \frac{12}{k_n^3} \int_{I_n} (t - t_{n-\frac{1}{2}})\, v(t)\, dt$$

Integrating by parts yields

$$\begin{aligned}
(\mathit{\Pi}_W v)'(t) &= \frac{6}{k_n^3}\Big[\frac{k_n^2}{4} v(t_n) - \frac{k_n^2}{4} v(t_{n-1}) - \int_{I_n} (t - t_{n-\frac{1}{2}})^2 v'\, dt\Big] \\
&= \frac{6}{k_n^3}\Big[\int \frac{k_n^2}{4} v'(t)\, dt - \int_{I_n} (t - t_{n-\frac{1}{2}})^2 v'\, dt\Big] \\
&= \frac{6}{k_n^3}\int \frac{k_n^2}{4} - (t - t_{n-\frac{1}{2}})^2 v'\, dt \\
&= \frac{3}{2k_n^3}\int \Big(k_n - 2(t - t_{n-\frac{1}{2}})\Big)\Big(k_n + 2(t - t_{n-\frac{1}{2}})\Big) v'\, dt \\
&= \frac{6}{k_n^3}\int_{I_n} (t_n - t)(t - t_{n-1})\, v'(t)\, dt\,.
\end{aligned}$$

Taking the norm for $\mathit{\Pi}_W v$ and $(\mathit{\Pi}_W v)'$, and using the inequality $(t_n - t)(t - t_{n-1}) \le \frac{k_n^2}{4}$ for $t \in I_n$, we get

$$\begin{aligned}
\|\mathit{\Pi}_W v(t)\| &\le \frac{12}{k_n^3}\frac{k_n}{2}\int_{I_n} \frac{k_n}{2}\, \|v(s)\|\, ds + \frac{1}{k_n}\int_{I_n} \|v(s)|\, ds, \\
\|(\mathit{\Pi}_W v)'(t)\| &\le \frac{6}{k_n^3}\int_{I_n} \frac{k_n^2}{4}\, \|v'(t)\|\, dt.
\end{aligned}$$

Therefore, for $1 \le n \le N$, we have

$$\|\mathit{\Pi}_W v(t)\| \le \frac{4}{k_n}\int_{I_n} \|v(t)\|\, dt \quad \text{and} \quad \|(\mathit{\Pi}_W v)'(t)\| \le \frac{3}{2k_n}\int_{I_n} \|v'(t)\|\, dt\,. \tag{6.3.4}$$

Setting $\eta_v = \mathit{\Pi}_W v - v$, we have the projection error bound, for $\ell = 1, 2$

$$\|\eta_v(t)\| + k_n \|\eta_v'(t)\| \le C_\ell k_n^{\ell-1} \int_{I_n} \|v^{(\ell)}(s)\|\, ds \quad \text{for} \quad t \in I_n. \tag{6.3.5}$$

Next, we show an error bound property of $\mathit{\Pi}_W$ that involves the fractional operator ${}^{\mathrm{R}}\mathrm{D}^{1-\alpha}$.

Lemma 29 Let $v^{(\ell)}|_{I_n} \in L^1\big((0, t_n), L^2(\Omega)\big)$ for $\ell \in \{1, 2\}$. Then, for $0 < \alpha < 1$, and $1 \le n \le N$, we have

$$\int_{I_n} \big\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} \eta_v, \eta_v \big\rangle \, dt \le C\, k_n^\alpha \max_{j=1}^{n} k_j^{2\ell-2} \Big(\int_{I_j} \|v^{(\ell)}\|\, dt \Big)^2 \quad \text{for} \quad 1 \le n \le N.$$

Proof 10 We integrate by parts and notice that

$$\begin{aligned}
\int_{I_n} \big\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} \eta_v, \eta_v \big\rangle \, dt &= \langle I^\alpha \eta_v(t), \eta_v(t) \rangle \Big|_{t_{n-1}^+}^{t_n} - \int_{I_n} \langle I^\alpha \eta_v, \eta_v' \rangle \, dt \\
&= \langle \mathcal{I}^n(t_n), \eta_v(t_n) \rangle + \langle I^\alpha \eta_v(t_{n-1}), \eta_v(t_n) \rangle - \langle I^\alpha \eta_v(t_{n-1}), \eta_v(t_{n-1}) \rangle \\
&\quad - \int_{I_n} \langle I^\alpha \eta_v, \eta_v' \rangle \, dt \\
&= \langle \mathcal{I}^n, \eta_v(t_n) \rangle + \int_{I_n} \langle I^\alpha \eta_v(t_{n-1}), \eta_v'(t) \rangle \, dt - \int_{I_n} \langle I^\alpha \eta_v, \eta_v' \rangle \, dt \\
&= \langle \mathcal{I}^n(t_n), \eta_v(t_n) \rangle - \int_{I_n} \langle \mathcal{I}^n(t), \eta_v'(t) \rangle \, dt,
\end{aligned} \tag{6.3.6}$$

where for $t \in I_n$,

$$\begin{aligned}
\mathcal{I}^n(t) &:= I^\alpha \eta_v(t) - I^\alpha \eta_v(t_{n-1}) \\
&= \int_0^{t_{n-1}} [\omega_\alpha(t-s) - \omega_\alpha(t_{n-1}-s)] \eta_v(s)\, ds + \int_{t_{n-1}}^t \omega_\alpha(t-s) \eta_v(s)\, ds\,.
\end{aligned}$$

Simplifying then integrating, we observe

$$\begin{aligned}\|\mathcal{I}^n(t)\| &\le \Big(\int_0^{t_{n-1}} [\omega_\alpha(t_{n-1}-s)-\omega_\alpha(t-s)]\,ds + \int_{t_{n-1}}^{t} \omega_\alpha(t-s)\,ds\Big)\|\eta_v\|_{J_n} \\ &\le [2\,\omega_{1+\alpha}(t-t_{n-1})+\omega_{1+\alpha}(t_{n-1})-\omega_{1+\alpha}(t)]\|\eta_v\|_{J_n} \\ &\le 2\,\omega_{1+\alpha}(t-t_{n-1})\|\eta_v\|_{J_n} \\ &\le 2\,\omega_{1+\alpha}(k_n)\max_{j=1}^{n}\|\eta_v\|_{I_j} \quad \text{for} \quad t\in I_n\,.\end{aligned}$$

Therefore, an application of the Cauchy-Schwarz inequality gives

$$\int_{I_n} |\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_v, \eta_v\rangle|\,dt \le 2\,\omega_{1+\alpha}(k_n)\max_{j=1}^{n}\|\eta_v\|_{I_j}\left(\|\eta_v(t_n)\| + \int_{I_n}\|\eta_v'\|\,dt\right),$$

and hence, using the error projection in (6.3.5), we obtain the desired bound. □

Now, we prove that ${}^{R}\mathrm{D}^{1-\alpha}v(t) = \omega_\alpha(t)v(0) + I^\alpha v'(t)$ and $\|\eta_v\|_{I_n} \le 5\|v\|_{I_n}$ as follows:

$$\begin{aligned}{}^{R}\mathrm{D}^{1-\alpha}v(t) =& \frac{d}{dt}\Big(\omega_{\alpha+1}(t)v(0) - \int_0^t \omega_{\alpha+1}(t-s)v'(s)\,ds\Big) \quad \text{integration by parts} \\ =& \omega_\alpha(t)v(0) + \int_0^t \omega_\alpha(t-s)v'(s)\,ds \quad \text{by theorem 15}\end{aligned}$$

$$\|\eta_v\|_{I_n} \le \|\varPi_W v\| + \|v\| \le \frac{4}{k_n}(k_n)\|v\|_{I_n} + \|v\| \le 5\|v\|_{I_n}$$

By the above equality and inequality , we have

$$\int_{I_n} |\langle {}^{R}\mathrm{D}^{1-\alpha}v, \eta_v\rangle|\,dt \le 5\|v\|_{I_n}\int_{I_n}(\omega_\alpha(t)\|v(0)\| + \|I^\alpha v'\|)\,dt.$$

Summing over n and using the inequality

$$\begin{aligned}\int_0^T \|I^\alpha v'\|)\,dt &\le \int_0^T \int_0^t \omega_\alpha(t-s)\|v'(s)\|\,ds\,dt\\ &= \int_0^T \int_s^T \omega_\alpha(t-s)\|v'(s)\|\,dt\,ds\\ &\le \int_0^T \omega_\alpha(T-s)\|v'(s)\|\,ds\end{aligned}$$

to reach:

$$\int_0^T |\langle {}^R\mathrm{D}^{1-\alpha} v, \eta_v\rangle|\,dt \le 5\,\|v\|_J(\omega_{1+\alpha}(T)\|v(0)\| + \int_0^T \omega_{1+\alpha}(T-s)\|v'(s)\|\,ds)\,. \tag{6.3.7}$$

On the other hand, by integration by parts with respect to t,

$$\int_0^T \langle {}^R\mathrm{D}^{1-\alpha}\mathit{\Pi}_W v, v\rangle\,dt = \langle I^\alpha \mathit{\Pi}_W v(T), v(T)\rangle - \int_0^T \langle I^\alpha \mathit{\Pi}_W v, v'\rangle\,dt.$$

Hence, an application of the Cauchy-Schwarz inequality and the first inequality in (6.3.4), tells us that

$$\begin{aligned}\Big|\int_0^T \langle {}^R\mathrm{D}^{1-\alpha}\mathit{\Pi}_W v, v\rangle\,dt\Big| &\le \|\mathit{\Pi}_W v\|_J \int_0^T \Big[\omega_\alpha(T-t)\|v(T)\| + \int_0^t \omega_\alpha(t-s)\,ds\|v'(t)\|\Big]dt\\ &\le 4\|v\|_J\Big(\|v(T)\|\omega_{1+\alpha}(T) + \int_0^T \omega_{1+\alpha}(t)\|v'(t)\|\,dt\Big)\,.\end{aligned} \tag{6.3.8}$$

We combine the above two inequalities, (6.3.7), (6.3.8) and use that

$$\|v\|_J \le \|v(0)\| + \int_0^T \|v'\|\,dt$$

and

$$\|v(0)\| \le \|v(T)\| + \int_0^T \|v'(t)\|\,dt$$

, we obtain the bound below.

$$\Big|\int_0^T \langle {}^R\mathrm{D}^{1-\alpha} v, \eta_v\rangle|\, dt\Big| + \Big|\int_0^T \langle {}^R\mathrm{D}^{1-\alpha} \mathit{\Pi}_W v, v\rangle\, dt\Big| \le C\, T^\alpha \Big(\|v(T)\| + \int_0^T \|v'\|\, dt\Big)^2 . \tag{6.3.9}$$

The above bound will be use later on to show the convergence of DG FE scheme (6.1.5)

6.4 Error estimates

This section is devoted to investigate the convergence of the DG FE scheme, (6.1.5). To do so, we decompose the error as follows:

$$U - u = \zeta + \mathit{\Pi}_W \xi + \eta_u \quad \text{with} \;\; \zeta = U - \mathit{\Pi}_W R_h u\, . \tag{6.4.1}$$

Recall that, $\xi = R_h u - u$ and $\eta_u = \mathit{\Pi}_W u - u$. The main task now is to estimate the term ζ:

Lemma 30

$$\int_0^T |\zeta|_1^2 dt \le C\, T^\alpha \|\xi(0)\|^2 + C \int_0^T \Big(\big\langle {}^R\mathrm{D}^{1-\alpha} \eta_u, \eta_u \big\rangle + \big\langle {}^R\mathrm{D}^{1-\alpha} \mathit{\Pi}_W \xi, \mathit{\Pi}_W \xi \big\rangle \Big)\, dt$$
$$+ \sum_{n=1}^N \Big| \int_{I_n} a(t, \mathit{\Pi}_W \xi + \eta_u, \zeta) dt \Big| . \tag{6.4.2}$$

Proof 11 We start our proof by taking the inner product of the model problem (6.0.1) with ζ, using the identity ${}^c D^{1-\alpha} u(t) =^R \mathrm{D}^{1-\alpha} u(t) - \omega_\alpha(t) u_0$, and then integrating over the time subinterval I_n, The above equation, the DG FE scheme (6.1.5) and the decomposition in (6.4.1) imply after simplifications:

$$\int_0^T \left(\left\langle {}^R\mathrm{D}^{1-\alpha}\zeta, \zeta\right\rangle + |\zeta|_1^2\right) dt = \int_0^T \langle \omega_\alpha(t)\, \xi(0), \zeta\rangle\, dt$$
$$- \int_0^T \left[\left\langle {}^R\mathrm{D}^{1-\alpha}(\mathit{\Pi}_W\xi + \eta_u), \zeta\right\rangle + a(t, \mathit{\Pi}_W\xi + \eta_u, \zeta)\right] dt\,. \quad (6.4.3)$$

Now, using the continuity property, Lemma 16 (ii) and the identity $I^{1-\alpha}({}^R\mathrm{D}^{1-\alpha}\, v(t)) = v(t)$ plus the inequality3, we notice that

$$\Big|\int_0^T \left\langle {}^R\mathrm{D}^{1-\alpha}\eta_u, \zeta\right\rangle dt\Big| \le C \int_0^T \left\langle {}^R\mathrm{D}^{1-\alpha}\eta_u, \eta_u\right\rangle dt + \frac{1}{4}\int_0^T \left\langle {}^R\mathrm{D}^{1-\alpha}\zeta, \zeta\right\rangle dt,$$
$$\Big|\int_0^T \left\langle {}^R\mathrm{D}^{1-\alpha}\mathit{\Pi}_W\xi, \zeta\right\rangle dt\Big| \le C \int_0^T \left\langle {}^R\mathrm{D}^{1-\alpha}\mathit{\Pi}_W\xi, \mathit{\Pi}_W\xi\right\rangle dt + \frac{1}{4}\int_0^T \left\langle {}^R\mathrm{D}^{1-\alpha}\zeta, \zeta\right\rangle dt\,.$$

In addition, following the steps in (6.2.2), we observe

$$\int_0^T \langle \omega_\alpha(t)\, \xi(0), \zeta\rangle\, dt \le \frac{1}{4}\int_0^T \left\langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\zeta, \zeta\right\rangle dt + C\, T^\alpha \|\xi(0)\|^2$$

Inserting the above three inequalities in (6.4.3), then simplifying, and using the positivity property of ${}^{\mathrm{R}}\mathrm{D}^{1-\alpha}$, Lemma 16 (i), yield the desired bound.

Lemma 31

$$\Big|\int_{I_n} a(t, \mathit{\Pi}_W\xi + \eta_u, \zeta)\, dt\Big| \le C'\, k_n^2 \int_{I_n} \|\nabla\xi\|^2 + \|\nabla\eta_u\|^2)\, dt + \frac{1}{2c_0}\int_{I_n} |\zeta|_1^2\, dt \quad (6.4.4)$$

Proof 12

$$
\begin{aligned}
&\int_{I_n} \langle \mathcal{A}(t_n)\nabla(\mathit{\Pi}_W\xi + \eta_u), \nabla\zeta\rangle\, dt \\
&= \int_{I_n} \langle \mathcal{A}(t_n)\nabla(\mathit{\Pi}_W R_h u - \mathit{\Pi}_W u + \mathit{\Pi}_W u - u), \nabla\zeta\rangle\, dt \\
&= \int_{I_n} \langle \mathcal{A}(t_n)\nabla(\mathit{\Pi}_W R_h u - R_h u + R_h u - u), \nabla\zeta\rangle\, dt \\
&= \int_{I_n} \langle \mathit{\Pi}_W(\mathcal{A}(t_n)\nabla R_h u) - (\mathcal{A}(t_n)\nabla R_h u), \nabla\zeta\rangle + \langle \mathcal{A}(t_n)\nabla(R_h u - u), \nabla\zeta\rangle\, dt \\
&= \int_{I_n} \langle \mathcal{A}(t_n)\nabla\xi, \nabla\zeta\rangle\, dt \\
&= \int_{I_n} \langle [\mathcal{A}(t_n) - \mathcal{A}(t)]\nabla\xi, \nabla\zeta\rangle\, dt
\end{aligned}
$$

and so,

$$
\begin{aligned}
\Big|\int_{I_n} a(t, \mathit{\Pi}_W\xi + \eta_u, \zeta) dt\Big| &= \Big|\int_{I_n} \langle \mathcal{A}(t_n)\nabla(\mathit{\Pi}_W\xi + \eta_u) + [\mathcal{A}(t) - \mathcal{A}(t_n)]\nabla(\mathit{\Pi}_W\xi + \eta_u), \nabla\zeta\rangle\, dt\Big| \\
&= \Big|\int_{I_n} \langle [\mathcal{A}(t) - \mathcal{A}(t_n)]\nabla(\eta_\xi + \eta_u), \nabla\zeta\rangle\, dt\Big| \quad \text{Since, } \eta_\xi = \mathit{\Pi}_W\xi - \xi \\
&\le C k_n \int_{I_n} \|\nabla(\eta_\xi + \eta_u)\|\, \|\nabla\zeta\|\, dt \quad \text{Since, } |\mathcal{A}(t) - \mathcal{A}(t_n)| \le C\, k_n\,.
\end{aligned}
$$

Thus, by the inequality $\|\nabla\eta_\xi(t)\| \le \|\nabla\xi(t)\| + 4k_n^{-1}\int_{I_n} \|\nabla\xi(s)\|\, ds$ (follows from the triangle inequality and the first property of Π_k in (6.3.4) with setting $(t - t_{n-\frac{1}{2}})$, $(s -$

$t_{n-\frac{1}{2}}) \le \frac{k_n}{2})$ for $s, t \in I_n$, and property (6.1.2),

$$\begin{aligned}
\Big|\int_{I_n} a(t, \varPi_W\xi + \eta_u, \zeta)\,dt\Big| &\le C\,k_n \int_{I_n} \Big(\|\nabla\eta_\xi\| + \|\nabla\eta_u\|\Big)\,\|\nabla\zeta\|\,dt \\
&\le C^2\,\frac{k_n^2}{2}\int_{I_n}\Big(\|\nabla\eta_\xi\| + \|\nabla\eta_u\|\Big)^2 dt + \frac{1}{2}\int_{I_n}\|\nabla\zeta\|^2\,dt \\
&\le C^2\,\frac{k_n^2}{2}\int_{I_n}\Big(\|\nabla\eta_\xi\|^2 + \|\nabla\eta_u\|^2 + 2\,\|\nabla\eta_\xi\|\,\|\nabla\eta_u\|\Big)\,dt \\
&\quad + \frac{1}{2c_0}\int_{I_n}|\nabla\zeta|_1^2\,dt \\
&\le C^2\,k_n^2\int_{I_n}\Big(\|\nabla\eta_\xi\|^2 + \|\nabla\eta_u\|^2\Big)\,dt + \frac{1}{2c_0}\int_{I_n}|\nabla\zeta|_1^2\,dt \\
&\le C'\,k_n^2\int_{I_n}\Big(\|\nabla\xi\|^2 + \frac{16}{k_n^2}\int_{I_n}1^2\,dt\int_{I_n}\|\nabla\xi\|^2\,dt + \|\nabla\eta_u\|^2\Big)\,dt \\
&\quad + \frac{1}{2c_0}\int_{I_n}|\nabla\zeta|_1^2\,dt \\
&\le C'\,k_n^2\int_{I_n}\|\nabla\xi\|^2 + \|\nabla\eta_u\|^2)\,dt + \frac{1}{2c_0}\int_{I_n}|\zeta|_1^2\,dt
\end{aligned}$$

Inserting this in (6.4.2) and using (6.3.2) for $t = 0$, we get

$$\begin{aligned}
&\int_0^T |\zeta|_1^2 dt \le C\,h^4\|u_0\|_2^2 \\
&\quad + C\sum_{n=1}^N\int_{I_n}\Big(\big\langle {}^R\mathrm{D}^{1-\alpha}\eta_u, \eta_u\big\rangle + \big\langle {}^R\mathrm{D}^{1-\alpha}\varPi_W\xi, \varPi_W\xi\big\rangle + k_n^2(\|\nabla\xi\|^2 + \|\nabla\eta_u\|^2)\Big)\,dt\,.
\end{aligned}$$

But, for $t \in I_n$ and for $\ell \in \{1, 2\}$,

$$\begin{aligned}
\int_{I_n}\big\langle {}^R\mathrm{D}^{1-\alpha}\eta_u, \eta_u\big\rangle\,dt &\le Ck_n \max_{j=1}^n k_j^{2\ell-3+\alpha}\Big(\int_{I_j}\|u^{(\ell)}\|\,dt\Big)^2 \quad \text{by Lemma 29,} \\
\|\nabla\xi(t)\| &\le C\,h\|u(t)\|_2 \quad \text{by the elliptic projection error (6.3.2),} \\
\|\nabla\eta_u(t)\| &\le C_\ell k_n^{\ell-1}\int_{I_n}\|\nabla u^{(\ell)}(s)\|\,ds \quad \text{by the time projection error (6.3.5),}
\end{aligned}$$

where in the first inequality we also used the non-increasing time step assumption. So,

$$\int_0^T |\zeta|_1^2 dt \le C\, h^4 \|u_0\|_2^2 + C \int_0^T \langle {}^R\mathrm{D}^{1-\alpha} \Pi_W \xi, \Pi_W \xi \rangle dt + C\, h^2 k^2 \int_0^T \|u\|_2^2 dt + C \max_{n=1}^{N} k_n^{2\ell-3+\alpha} \Big(\Big(\int_{I_n} \|u^{(\ell)}\|\, dt \Big)^2 + k_n^{1-\alpha} \Big(\int_{I_n} \|\nabla u^{(\ell)}\|\, dt \Big)^2 \Big). \tag{6.4.5}$$

It remains to estimate $\int_0^T \langle {}^R\mathrm{D}^{1-\alpha} \Pi_W \xi, \Pi_W \xi \rangle\, dt$. From the decomposition:

$$\int_{I_n} \langle {}^R\mathrm{D}^{1-\alpha} \Pi_W \xi, \Pi_W \xi \rangle\, dt = \int_{I_n} \Big[\langle {}^R\mathrm{D}^{1-\alpha} \eta_\xi, \eta_\xi \rangle + \langle {}^R\mathrm{D}^{1-\alpha} \xi, \eta_\xi \rangle + \langle {}^R\mathrm{D}^{1-\alpha} \Pi_W \xi, \xi \rangle \Big]\, dt\,. \tag{6.4.6}$$

By Lemma 29,

$$\int_{I_n} \langle {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} \eta_\xi, \eta_\xi \rangle\, dt \le C\, k_n^\alpha \max_{j=1}^{n} \Big(\int_{I_j} \|\xi'\|\, dt \Big)^2 \le C\, k_n \max_{j=1}^{n} \Big(k_j^{-\frac{1-\alpha}{2}} \int_{I_j} \|\xi'\|\, dt \Big)^2.$$

Inserting the above bound in (6.4.6), then summing over n and using the achieved bound in (6.3.9), we obtain

$$\int_0^T |\langle {}^R\mathrm{D}^{1-\alpha} \Pi_W \xi, \Pi_W \xi \rangle|\, dt \le C \max_{n=1}^{N} \Big(k_n^{-\frac{1-\alpha}{2}} \int_{I_n} \|\xi'\|\, dt \Big)^2 + C \Big(\|\xi(0)\| + \int_0^T \|\xi'\|\, dt \Big)^2. \tag{6.4.7}$$

Finally, to complete the proof, we combine (6.4.5) and (6.4.7). In the next theorem we show our main convergence results of the DG FE solution. Typically, the exact solution u of problem (6.0.1) satisfies the finite regularity assumptions:

$$\|u'(t)\|_2 + t\|u''(t)\|_1 \le \mathbf{M}\, t^{\sigma-1} \quad \text{for} \quad t > 0, \tag{6.4.8}$$

for some positive constants $\mathbf{M}$ and σ. Due to the singular behavior u near $t = 0$, we employ a family of non-uniform meshes, where the time-steps are graded towards

$t = 0$; see [27, 35]. More precisely, for a fixed parameter $\gamma \geq 1$, we assume that

$$t_n = (n/N)^\gamma T \quad \text{for } 0 \leq n \leq N. \tag{6.4.9}$$

We define k_j by using (6.4.9) as follows

$$\text{Noting that,} \quad k_j = t_j - t_{j-1} = [(\frac{j}{N})^\gamma - (\frac{j-1}{N})^\gamma]T = [(j)^\gamma - (j-1)^\gamma]\frac{T}{N^\gamma}$$

The function $q(x) = x^\gamma - (x-1)^\gamma$ is nondecreasing and so the sequence $\{k_j\}_{j=1}^N$ is also nondecreasing. That is, $k_i \leq k_j$ for $1 \leq i \leq j \leq N$. One can also show the following mesh property:

$$k_j \leq \gamma k t_j^{1-1/\gamma}\,. \tag{6.4.10}$$

as follows:

$$\begin{aligned} k_j = [(j)^\gamma - (j-1)^\gamma]\frac{T}{N^\gamma} &= \gamma \frac{T}{N^\gamma} \int_{j-1}^{j} x^{\gamma-1}\, dx \\ &\leq (j)^{(\gamma-1)}\,\frac{\gamma T}{N^\gamma} \\ &= \gamma \frac{T}{N}(\frac{j}{N})^{(\gamma-1)} \\ &\leq \gamma k (t_j^{\frac{1}{\gamma}})^{(\gamma-1)} \\ &= \gamma k t_j^{1-1/\gamma} \end{aligned}$$

Theorem 32 Assume that the solution u of (6.0.1) satisfies the regularity property (6.4.8) with $\sigma > (1-\alpha)/2$. Let U be the DG FE solution defined by (6.1.5). Then, we have the following error estimate:

$$\int_0^T \|U-u\|^2\, dt \leq C\,(h^4 + k^{\gamma(2\sigma-1+\alpha)}) \quad \text{for} \quad 1 \leq \gamma \leq \frac{3+\alpha}{2\sigma - 1 + \alpha},$$

where C is a constant that depends on T, α, γ, σ, and on $\mathbf{M}$, but independent of h and k.

Proof 13 From the decomposition of the error in (6.4.1), the triangle inequality, the bound in equation(6.4.5), the inequality $\|\mathit{\Pi}_W \xi\|_{L^2(L^2)} \leq \|\xi\|_{L^2(L^2)}$ (we will explain that next), the elliptic projection error (6.3.2), the error from the time projection (6.3.5), we have

$$\int_0^T \|U - u\|^2 \, dt \leq Ch^4 C_1(k, u) + C\, C_2(k, u) + C\, h^2(h^2 + k^2) \int_0^T \|u\|_{H^2}^2 \, dt.$$

From the orthogonality property of Π_k in (6.3.4), we have

$$\int_{I_n} \big\langle \mathit{\Pi}_W \xi - \xi, \mathit{\Pi}_W \xi \big\rangle \, dt = 0$$

Hence,

$$\begin{aligned}
\int_{I_n} \|\mathit{\Pi}_W \xi\|^2 \, dt &\leq \int_{I_n} \|\mathit{\Pi}_W \xi\| \, \|\xi\| \, dt \\
&\leq \frac{1}{2} \int_{I_n} (\|\mathit{\Pi}_W \xi\|^2 + \|\xi\|^2) \, dt \\
&\leq \int_{I_n} \|\xi\|^2 \, dt
\end{aligned}$$

By the regularity assumption (6.4.8), the inequality $h^2 k^2 \leq \frac{1}{2}(h^4 + k^4)$ and the inequality $\int_{I_n} t^{\sigma-1}\,dt = (t_{n-1}+k_n)^\sigma - t_{n-1}^\sigma \le C\,k_n^\sigma$ (by binomial theorem), we observe

$$\begin{aligned}\int_0^T \|U-u\|^2\,dt &\le Ch^4 \max_{n=1}^{N} \Big(k_n^{-\frac{1-\alpha}{2}} \int_{I_n} t^{\sigma-1}\,dt\Big)^2 + Ch^4\Big(1+\int_0^T t^{\sigma-1}\,dt\Big)^2 \\ &\quad + C\,k_1^{-1+\alpha}\left(\int_{I_1} t^{\sigma-1}\,dt\right)^2 + C\max_{n=2}^{N} k_n^{1+\alpha}\left(\int_{I_n} t^{\sigma-2}\,dt\right)^2 + C\,h^2k^2 \\ &\le C(h^4\max_{n=1}^{N} k_n^{2\sigma-1+\alpha} + h^4 + k_1^{2\sigma-1+\alpha} + \max_{n=2}^{N} k_n^{3+\alpha}t_n^{2\sigma-4} + k^4) \\ &\le C\,(h^4 + k^{\min\{\gamma(2\sigma-1+\alpha),3+\alpha\}})\end{aligned}$$

where in the last inequality we used the following:

- $k_n^{3+\alpha}{t_n}^{2(\sigma-2)} \le C\,k^{3+\alpha}t_n^{2(\sigma-2)+3+\alpha-(3+\alpha)/\gamma} \le C'\,k^{3+\alpha}$ by the mesh property (6.4.10),
- $k_1 \le \gamma k t_1^{1-\frac{1}{\gamma}} = \gamma k k_1^{1-\frac{1}{\gamma}}$,
- $k_1 \le c_\gamma\, k^\gamma$.

Therefore, the proof of this theorem is completed now. □

6.5 Numerical results

We present some numerical tests using a model problem in one space dimension, of the form (6.0.1) with $\Omega = (0,1)$, $[0,T] = [0,1]$, and $\mathcal{A}(x,t) = 1 + t^{3/2}$. We choose $u_0(x) = \sin(\pi x)$ for the initial data and choose the source term f so that

$$u(t) = (1+t^\alpha)\sin(\pi x)\,. \tag{6.5.1}$$

One can verify that the regularity condition (6.4.8) holds for $\sigma = \alpha$.

The numerical tests below reveal faster rates of convergence than suggested by Theorem 32, and that our regularity assumptions are more restrictive than is needed in practice. More precisely, the theoretical results in Theorem 32 show a suboptimal (in time) convergence of order $O(k^{2-\frac{1-\alpha}{2}}+h^2)$ for sufficiently graded time meshes in the time-space L^2-norm. However, we demonstrate numerically optimal (in both time and space) rates of convergence in the stronger $L^\infty(L^2)$-norm. To this end, We introduce a finer mesh

$$\mathcal{G}^m = \{\, t_{j-1} + \ell k_j/m : j = 1,\, 2,\, \ldots,\, N \text{ and } \ell = 0,\, 1,\, \ldots,\, m\,\}, \tag{6.5.2}$$

and define the discrete maximum norm $\|v\|_{\mathcal{G}^m} = \max_{t\in\mathcal{G}^m} \|v(t)\|$, so that, for sufficiently large values of m, $\|U_h - u\|_{\mathcal{G}^m}$ approximates the uniform error $\|U_h - u\|_{L^\infty(L^2)}$. In all tables, we choose $m = 10$.

For the numerical illustration of the convergence rates in time, we choose M (the number of uniform spatial subintervals) to be sufficiently large such that the spatial error is negligible compared to the error from the time discretization. We employ a time mesh of the form (6.4.9). Tables 6 .1, 6 .2 and 6 .3 and their figures show the error (in the stronger $L^\infty(L^2)$) and the rates of convergence when $\alpha = 0.7,\, 0.5,\, 1/3$ and 0.3 respectively, for various choices of N and γ. We observe optimal rates of order $O(k^{\gamma\sigma})$ for various choices of $1 \le \gamma \le \frac{2}{\sigma}$ which is faster than the rate $O(k^{\frac{\gamma}{2}(2\sigma-1+\alpha)})$ for $1 \le \gamma \le \frac{3+\alpha}{2\sigma-1+\alpha}$ predicted by our theory in Theorem 32. Noting that, in Table 6 .3, $\sigma \le 1-\alpha$ and thus the assumption $\sigma > (1-\alpha)/2$ in this theorem is not sharp.

Next, we test the performance of the spatial FEs discretizaton of the scheme (6.1.5). A uniform spatial mesh that consists of M subintervals where each is of width h will be used. We refine the time mesh such that the spatial error is dominating. By Theorem 32, a convergence of order $O(h^2)$ is expected. We illustrate these results in Table 6 .4.

Table 6 .1: Errors and time convergence rates with $\alpha = 0.7$ for various choices of γ.

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
10	5.8997e-03		1.1252e-03		9.9332e-04	
20	3.5981e-03	0.71339	4.1163e-04	1.4507	2.5524e-04	1.9604
40	2.1827e-03	0.72111	1.5008e-04	1.4556	6.4530e-05	1.9838
80	1.3208e-03	0.72468	5.4700e-05	1.4562	1.6137e-05	1.9996
160	7.9804e-04	0.72692	1.9995e-05	1.4519	4.0085e-06	2.0092
320	4.8168e-04	0.72840	7.3478e-06	1.4443	9.9164e-07	2.0152

Table 6 .2: Errors and time convergence rates with $\alpha = 0.5$ for various choices of γ.

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$		$\gamma = 4$	
10	1.149e-02		3.262e-03		1.560e-03		1.882e-03	
20	7.641e-03	0.589	1.619e-03	1.011	5.972e-04	1.385	4.869e-04	1.951
40	5.151e-03	0.569	8.037e-04	1.010	2.192e-04	1.446	1.209e-04	2.009
80	3.641e-03	0.500	3.997e-04	1.008	7.867e-05	1.478	2.933e-05	2.044
160	2.570e-03	0.503	1.992e-04	1.005	2.797e-05	1.492	7.011e-06	2.064
320	1.812e-03	0.504	9.940e-05	1.003	9.908e-06	1.497	1.774e-06	1.982

Table 6 .3: Errors and time convergence rates for various choices of γ.

$\alpha = 1/3$

N	$\gamma = 1$		$\gamma = 2$		$\gamma = 4$		$\gamma = 6$	
10	1.677e-02		7.579e-03		3.416e-03		3.261e-03	
20	1.327e-02	0.338	4.677e-03	0.696	1.393e-03	1.294	9.087e-04	1.843
40	1.044e-02	0.346	3.036e-03	0.623	5.553e-04	1.327	2.471e-04	1.879
80	8.191e-03	0.350	1.940e-03	0.646	2.205e-04	1.332	6.435e-05	1.941
160	6.427e-03	0.350	1.229e-03	0.658	8.753e-05	1.333	1.643e-05	1.970

$\alpha = 0.3$

N	$\gamma = 1$		$\gamma = 3$		$\gamma = 5$		$\gamma = 7$	
10	1.792e-02		5.149e-03		3.625e-03		3.991e-03	
20	1.446e-02	0.309	2.905e-03	8.258e-01	1.318e-03	1.459	1.121e-03	1.832
40	1.160e-02	0.318	1.577e-03	8.810e-01	4.673e-04	1.496	3.052e-04	1.877
80	9.290e-03	0.321	8.479e-04	8.955e-01	1.652e-04	1.499	7.981e-05	1.935
160	7.447e-03	0.319	4.547e-04	8.989e-01	5.843e-05	1.500		

Table 6 .4: Errors and convergence rates in space with $\alpha = 0.7, 0.5$ and 0.3.

M	$\alpha = 0.7$		$\alpha = 0.5$		$\alpha = 0.3$	
10	1.2156e-02		1.2780e-02		1.2563e-02	
20	3.1130e-03	1.9653	3.2743e-03	1.9646	3.1768e-03	1.9836
40	7.8803e-04	1.9820	8.2897e-04	1.9818	7.9873e-04	1.9918
80	1.9826e-04	1.9909	2.0864e-04	1.9903	2.0029e-04	1.9956
160	4.9724e-05	1.9954	5.2355e-05	1.9946	5.1065e-05	1.9717

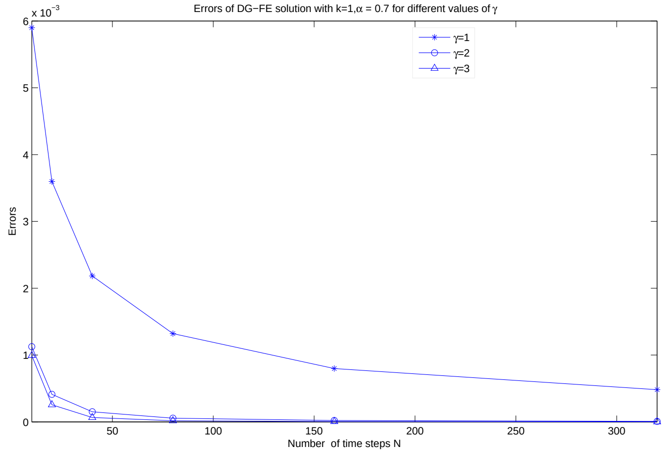


Figure 6 .1: Graphical errors for $\alpha = 0.7$ and different values of γ

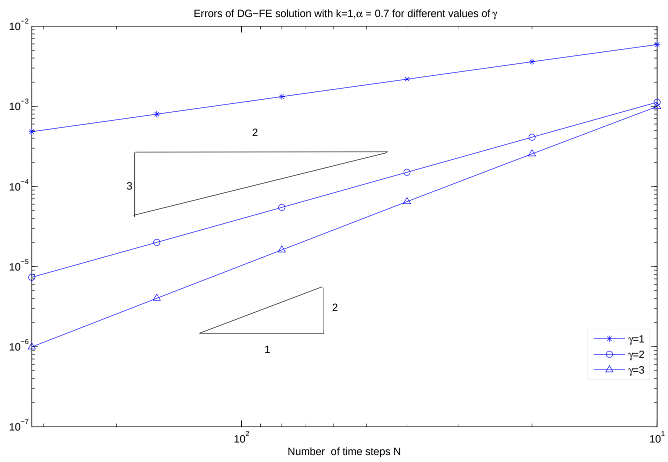


Figure 6 .2: Graphical errors for $\alpha = 0.7$ and different values of γ, log-log scaling

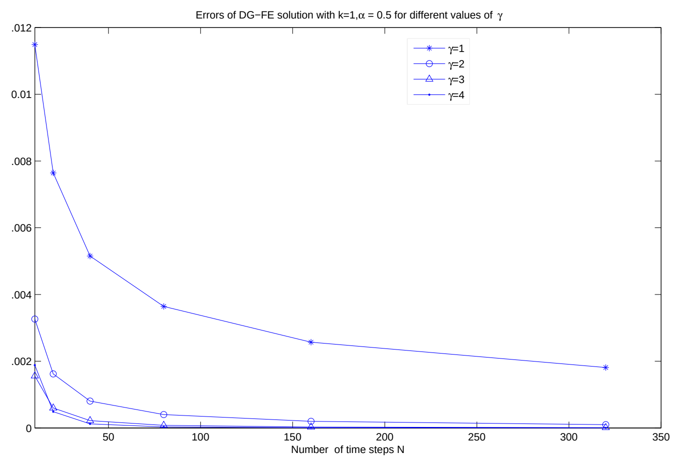


Figure 6 .3: Graphical errors for $\alpha = 0.5$ and different values of γ

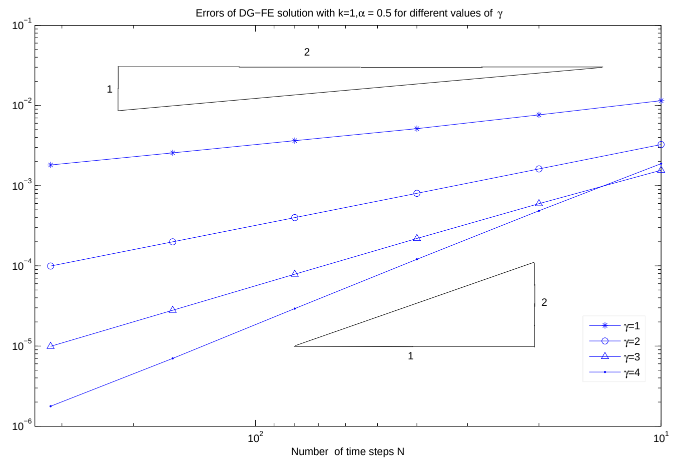


Figure 6 .4: Graphical errors for $\alpha = 0.5$ and different values of γ, log-log scaling

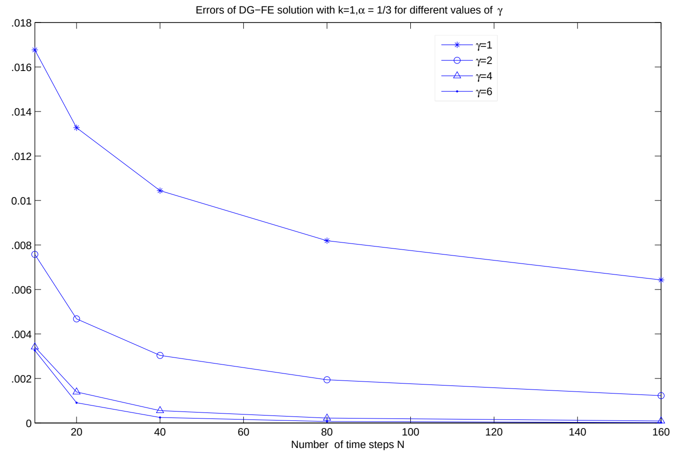


Figure 6 .5: Graphical errors for $\alpha = \frac{1}{3}$ and different values of γ

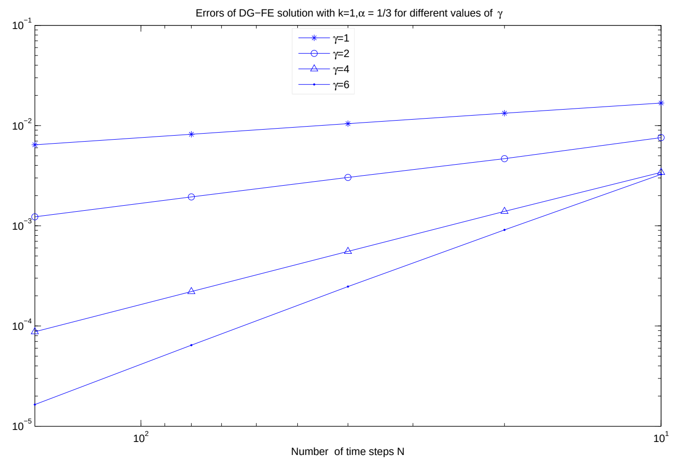


Figure 6 .6: Graphical errors for $\alpha = \frac{1}{3}$ and different values of γ, log-log scaling

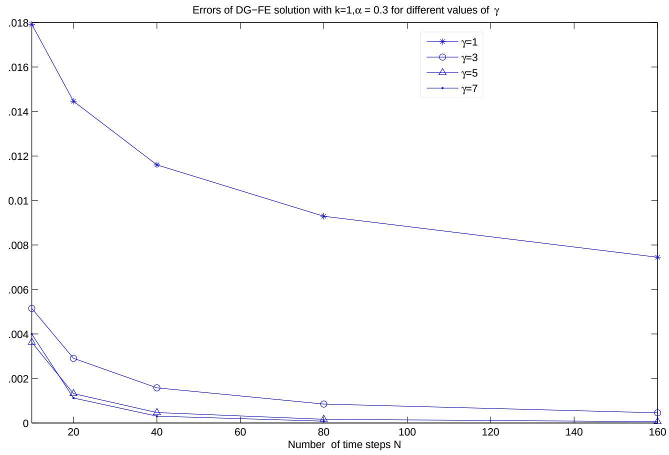


Figure 6 .7: Graphical errors for $\alpha = 0.3$ and different values of γ

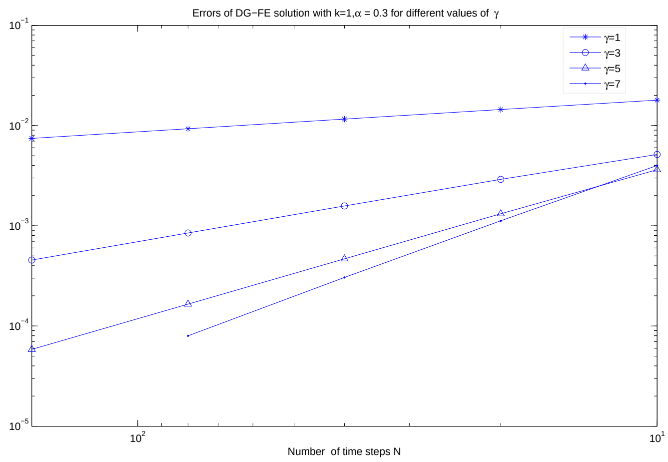


Figure 6 .8: Graphical errors for $\alpha = 0.3$ and different values of γ, log-log scaling

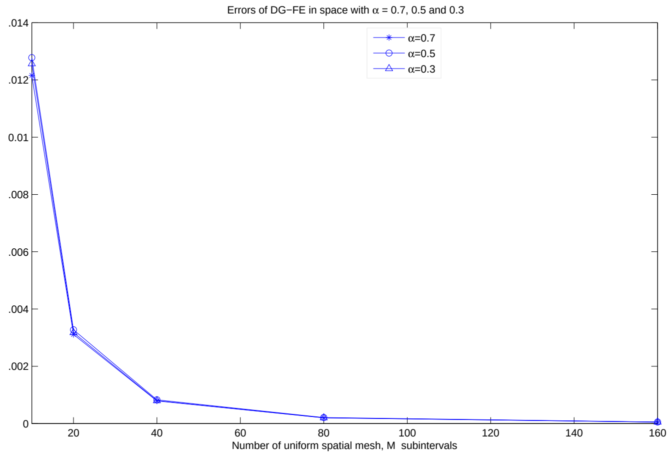


Figure 6 .9: Graphical errors different values of α

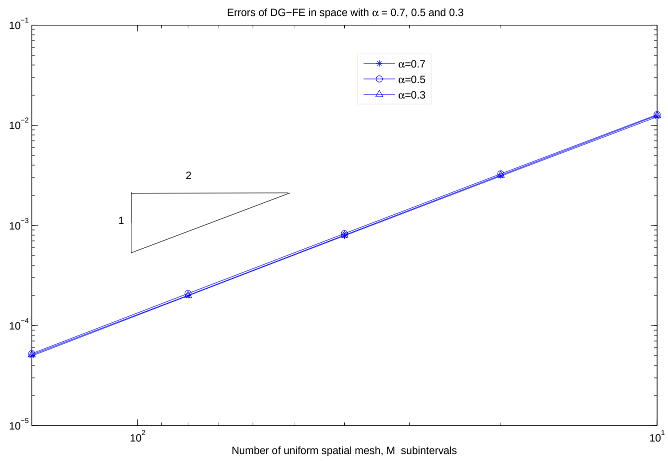


Figure 6 .10: Graphical errors for different values of α, log-log scaling

Chapter 7

HIGH ORDER, TIME STEPPING DISCONTINUOUS GALERKIN METHODS

In the previous chapter, a piecewise-linear time stepping DG method for (6.0.1) was studied. This chapter is devoted extend this numerical method to high order. Indeed the convergence analysis for this case is different from the preceding ones. Some ideas from chapter 6 will be used. For sake of simplicity, we choose the diffusivity coefficient A to be a constant to 1. So, (6.0.1) reduces to:

$$\begin{aligned} {}^{c}\!D^{1-\alpha}u(x,t) - \Delta u(x,t) &= f(x,t) && \text{on } \Omega \times (0,T], \\ u(x,0) &= u_0(x) && \text{on } \Omega, \\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T], \end{aligned} \tag{7.0.1}$$

We prove convergence of order $O(k^{m+1-\frac{1-\alpha}{2}})$ in the $L^2\big((0,T),L^2(\Omega)\big)$-norm, where m is the uniform degree of the DG solution. So the achieved convergence rate is short by order $\frac{1-\alpha}{2}$ from being optimal in time.

7.1 The numerical method

To describe the time-stepping DG method for (7.0.1) we define our time stepping DG finite dimensional space W as follows:

$$W := \{v \in L_2\big((0,T);H_0^1(\Omega)\big) : v|_{I_n} \in P_m(H^1(\Omega)), 1 \le n \le N\}$$

, where $P_m(H_0^1)$ is the space of polynomials in t of degree $\le m$ with coefficients in $H_0^1(\Omega)$.A trial function $U \in W$ is continuous from the left at each node t_n.We denote the left hand limit, right hand limit and the jump at t_n as follows:

$$v^n := v(t_n) = v(t_n^-), \quad v_+^n := v(t_n^+), \quad [v]^n = v_+^n - v^n, \quad \text{respectively.}$$

The weak formulation for the model problem (7.0.1) is

$$\int_{I_n} \Big[\big({}^{\mathrm{c}}\mathrm{D}^{1-\alpha} u(t), w(t)\big) + \big(\nabla u(t), \nabla w(t)\big) \Big] \,\mathrm{dt} = \int_{I_n} \big(f(t), w(t)\big) \,\mathrm{dt} \tag{7.1.1}$$

$$\text{However,} \quad {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} w(s) = \omega_\alpha(s) w_+^0 + \sum_{i=1}^{j-1} \omega_\alpha(s - t_i)\,[w]^i + {}^{c}D^{1-\alpha} w(s), \quad \text{so}$$

$$\int_{I_n} \Big[\big({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} u(t), w(t)\big) + \big(\nabla u(t), \nabla w(t)\big) \Big] \,\mathrm{dt} = \int_{I_n} \big(f(t) + \omega_\alpha(t) u_0, w(t)\big) \,\mathrm{dt}, \tag{7.1.2}$$

for all $w(t) \in L_2\big(I_n, H_0^1(\Omega)\big)$

The semi discrete DG solution $U \in W$ of problem (7.0.1) is defined as follows:

$$\int_{I_n} \Big[\big({}^{\mathrm{c}}\mathrm{D}^{1-\alpha} U(t) + \sum_{j=0}^{n-1} \omega_\alpha(t - t_j)[U]^j, \chi(t)\big) + \big(\nabla U(t), \nabla \chi(t)\big) \Big] \,\mathrm{dt} = \int_{I_n} \big(f(t), \chi(t)\big) \,\mathrm{dt}, \tag{7.1.3}$$

for all $\chi \in W$, $1 \le n \le N$ with $U(0) = u_0$.

Our scheme can be rewritten using Riemman Liouville fractional derivative operator ${}^{\mathrm{R}}\mathrm{D}^{1-\alpha}$ as follows:

$$\int_{I_n} \Big[\big({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U(t), \chi(t)\big) + \big(\nabla U(t), \nabla \chi(t)\big) \Big] \,\mathrm{dt} = \int_{I_n} \big(f(t) + \omega_\alpha(t) u_0, \chi(t)\big) \,\mathrm{dt} \quad \text{for all} \quad \chi \in W \tag{7.1.4}$$

7.2 Projection and errors

The local L_2-projection time operator $\Pi_m : C(\overline{I}_n; L^2(\Omega)) \to C(\overline{I}_n; p_m(I_n, L^2(\Omega)))$ is defined by:

$$\int_{I_n} (\Pi_m w - w, v) \,\mathrm{dt} = 0 \quad \text{for all} \quad v \in p_m(I_n, L^2(\Omega))$$

For all $1 \leq n \leq N$.

Where $p_m(I_n, L^2(\Omega)$ is the space of all polynomials of degree at most m in the time variable t, with coefficients in $L^2(\Omega)$.

Lemma 33 Let P_j be the legendre polynomial of degree j on the interval $[-1, 1]$. Then the time projection $\Pi_m w(t)$ can be written as:

$$\Pi_m w(t) = \sum_{j=0}^{m} \Big[\frac{2j+1}{k_n} \int_{I_n} P_j(\frac{2}{k_n}(t - t_{n-1}) - 1) w(t)\, \mathrm{dt}\ P_j(\frac{2}{k_n}(t - t_{n-1}) - 1)\Big].$$

Proof 14 Since $\Pi_m w$ is polynomial of degree m on I_n, it can be written in terms of the legendre polynomial basis functions $p_0, p_1, \cdots, p_m$ on each I_n as follows:

$$\Pi_m w = a_m p_m(\frac{2}{k_n}(t-t_{n-1})-1) + a_{m-1} P_{m-1}(\frac{2}{k_n}(t-t_{n-1})-1) + a_{m-2} P_{m-2}(\frac{2}{k_n}(t-t_{n-1})-1) + \cdots + a_1 P_1(\frac{2}{k_n}(t - t_{n-1}) - 1) + a_0 \quad (7.2.1)$$

By the definition of Π_m and the properties of the lagendre polynomials, we have

$$\int_{I_n} \big(\Pi_m w - w\big) P_j(\frac{2}{k_n}(t - t_{n-1}) - 1)\, \mathrm{dt} = 0$$

and

$$\int_{I_n} P_j(\frac{2}{k_n}(t - t_{n-1}) - 1) P_i(\frac{2}{k_n}(t - t_{n-1}) - 1)\, \mathrm{dt} = \begin{cases} \frac{k_n}{2j+1}, & \text{i=j;} \\ 0, & \text{otherwise.} \end{cases}$$

respectively.

The first equation is equivalent to:

$$\int_{I_n} \Pi_m w(t) P_j(\frac{2}{k_n}(t - t_{n-1}) - 1)\, \mathrm{dt} = \int_{I_n} w(t) P_j(\frac{2}{k_n}(t - t_{n-1}) - 1)\, \mathrm{dt}$$

Multiplying 7.2.1 by $P_j(\frac{2}{k_n}(t-t_{n-1})-1)$ and integrating with respect to t over I_n give us:

$$\int_{I_n} a_j\ P_j^2(\frac{2}{k_n}(t-t_{n-1})-1)\,\mathrm{dt} = \int_{I_n} w(t)P_j(\frac{2}{k_n}(t-t_{n-1})-1)\,\mathrm{dt}$$

Then

$$a_j = \frac{2j+1}{k_n}\int_{I_n} w(t)P_j(\frac{2}{k_n}(t-t_{n-1})-1)\,\mathrm{dt} \quad \text{for all} \quad j=0,1,\cdots,m$$

The proof is completed now. $\square$

Next, by using lemma 33, we show next upper bound for $\Pi_m w$ and its time partial derivative. Since,

$$\max_{t\in I_n}|P_j(\frac{2}{k_n}(t-t_{n-1})-1)| = 1 \quad \text{for all} \quad j=0,1,\cdots,m$$

Then

$$\|\Pi_m w(t)\| \leq \frac{1}{k_n}\sum_{j=0}^{m}(2j+1)\ \int_{I_n}\|w(t)\|\,\mathrm{dt} \quad \text{for all} \quad j=0,1,\cdots,m.$$

Noting that,

$$\begin{aligned}
P_{l+1}^{'}(t) &= (2l+1)P_l(t) + (2(l-2)+1)P_{l-2}(t) + (2(l-4)+1)P_{l-4}(t) + \quad \cdots \quad ,\\
P_l^{'}(t) &= (2(l-1)+1)P_{l-1}(t) + (2(l-3)+1)P_{l-3}(t) + (2(l-5)+1)P_{l-5}(t) + \quad \cdots\\
&= \sum_{i=0}^{\lfloor\frac{l-1}{2}\rfloor}(2(l-2i-1)+1)P_{l-2i-1}(\frac{2}{k_n}(t-t_{n-1})-1) \quad \text{and}\\
P_{l-1}^{'}(t) &= (2(l-2)+1)P_{l-2}(t) + (2(l-4)+1)P_{l-4}(t) + (2(l-6)+1)P_{l-6}(t) + \quad \cdots \quad .
\end{aligned}$$

Thus,

$$P'_{l+1}(t) - P'_{l-1}(t) = (2l+1)P_l(t). \tag{7.2.2}$$

Using the above equation, $(\Pi_m w(t))'$ can be written as:

$$\sum_{j=1}^{m} \Big[\frac{2(2j+1)}{k_n^2} \int_{I_n} P_j(\frac{2}{k_n}(t-t_{n-1})-1)w(t)\,\mathrm{dt} \\ \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} (2(j-2i-1)+1)P_{j-2i-1}(\frac{2}{k_n}(t-t_{n-1})-1)\Big]. \tag{7.2.3}$$

Integration by parts for the following integral and use the identity $P_{j+1}(z) = P_{j-1}(z)$ for $z = 1, -1$ give

$$\int_{I_n} P_j(\frac{2}{k_n}(t-t_{n-1})-1)w(t)\,\mathrm{dt} = \\ \frac{k_n}{2(2j+1)}\Big(P_{j+1}(\frac{2}{k_n}(t-t_{n-1})-1) - P_{j-1}(\frac{2}{k_n}(t-t_{n-1})-1)\Big)w(t)]_{t_{n-1}}^{t_n} \\ - \frac{k_n}{2(2j+1)} \int_{I_n} \Big(P_{j+1}(\frac{2}{k_n}(t-t_{n-1})-1) - P_{j-1}(\frac{2}{k_n}(t-t_{n-1})-1)\Big)w'(t)\,\mathrm{dt} \\ = -\frac{k_n}{2(2j+1)} \int_{I_n} (P_{j+1}(\frac{2}{k_n}(t-t_{n-1})-1) - P_{j-1}(\frac{2}{k_n}(t-t_{n-1})-1))w'(t)\,\mathrm{dt}.$$

Therefore,

$$\|(\Pi_m w(t))'\| \leq \frac{2}{k_n} \sum_{j=1}^{m} [\sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} (2j-4i-1)] \int_{I_n} \|w'(t)\|\,\mathrm{dt}. \tag{7.2.4}$$

As before, Let $\eta_w = \Pi_m w - w$. The following projection error bounds are well-known:

$$\|\eta_w(t)\| + k_n \left\|\eta'_w(t)\right\| \leq C_m k_n^{m+l-1} \int_{I_n} \left\|w^{(m+l)}(t)\right\| \;\mathrm{dt} \;\text{ for } l = 0, 1 \tag{7.2.5}$$

By using lemma33, we get:

$$\nabla \Pi_m w(t) = \sum_{j=0}^{m} \Big[\frac{2j+1}{k_n} \int_{I_n} P_j(\frac{2}{k_n}(t-t_{n-1})-1)\nabla w(t)\,\mathrm{dt}\ P_j(\frac{2}{k_n}(t-t_{n-1})-1)\Big]$$
$$= \Pi_m \nabla w(t)$$

Hence,

$$\nabla \eta_{w(t)} = \eta_{\nabla w(t)}.$$

Therefore, by the projection bound in (7.2.5),

$$\|\nabla \eta_w(t)\| \le C_l k_n^{m+l-1} \int_{I_n} \left\|\nabla w^{(m+l)}(t)\right\| \ \mathrm{dt} \tag{7.2.6}$$

Lemma 34 Let $w^{(\ell)}|_{I_n} \in L^1(I_n, L^2(\Omega))$ for $\ell \in \{0,\, 1\}$. Then, for $0 < \alpha < 1$, we have

$$\int_{I_j} ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_w, \eta_w)\,\mathrm{dt} \le C\ k_j^{\alpha} \max_{i=1}^{j} k_i^{2(m+l)-2} \Big(\int_{I_i} \left\|w^{(m+l)}\right\| \mathrm{dt}\Big)^2 \ \text{ for } l = 0, 1$$

proof Following the proof of lemma 29 step-by-step, we obtain desired bound.

7.3 Error estimates

In this section we study the convergence analysis of the DG scheme (7.1.4). To do so, we decompose the error as:

$$U - u = \xi + \eta_u \ \text{ with } \xi = U - \Pi_m u \text{ and } \eta_u = \Pi_m u - u \tag{7.3.1}$$

Then, by the triangle inequality, and (7.2.5)

$$\|U-u\| \leq \|\xi\| + \|\eta_u\| \leq \|\xi\| + C_m k_n^{m+l-1} \int_{I_n} \left\|u^{(m+l)}(t)\right\| \tag{7.3.2}$$

In next theorem we bound ξ.

Theorem 35 For $1 \leq n \leq N$ We have

$$\int_0^T \|\xi\|_{H^1}^2 \, \mathrm{dt} \leq C \max_{n=1}^{N} k_n^{2(m+l)-3+\alpha} \Big(\int_{I_n} \left\|\nabla u^{(m+l)}\right\| \, \mathrm{dt} \Big)^2 \quad \text{for} \quad l = 0, 1$$

Proof 15 Subtracting equations (7.1.4) and equation (7.1.2), gives

$$\int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\xi, \xi) + \|\nabla \xi\|^2 \, \mathrm{dt} = -\int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_u, \xi) + (\nabla \eta_u, \nabla \xi) \, \mathrm{dt} \tag{7.3.3}$$

By the continuity property lemma(16), we have

$$\left| \int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_u, \xi) \right| \mathrm{dt} \leq \frac{1}{C_\alpha} \int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_u, \eta_u) \, \mathrm{dt} + \frac{1}{4} \int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\xi, \xi) \, \mathrm{dt}$$

Substitute this in (7.3.3) and apply cauchy schwarz inequality yield

$$\begin{aligned}
\int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\xi, \xi) + \|\nabla \xi\|^2 \, \mathrm{dt} &\leq \left| \int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_u, \xi) \right| \mathrm{dt} + \int_0^T \|\nabla \eta_u\| \, \|\nabla \xi\| \, \mathrm{dt} \\
&\leq \frac{1}{C_\alpha} \int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_u, \eta_u) \, \mathrm{dt} + \frac{1}{4} \int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\xi, \xi) \, \mathrm{dt} \\
&\quad + \frac{1}{2} \int_0^T [\|\nabla \eta_u\|^2 + \|\nabla \xi\|^2] \, \mathrm{dt}
\end{aligned}$$

We set $n = N$

$$\int_0^T \frac{3}{4}({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\xi, \xi) + \frac{1}{2}\|\nabla\xi\|^2 \ \mathrm{dt} \le \frac{1}{C_\alpha^2}\int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_u, \eta_u)\ \mathrm{dt} + \frac{1}{2}\int_0^T \|\nabla\eta_u\|^2$$

Using lemma(16)(i), we notice that

$$\int_0^T \|\xi\|_{H^1}^2 \le \ C\int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\xi, \xi)\, dt \le C[\int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_u, \eta_u)\ \mathrm{dt} + \int_0^T \|\nabla\eta_u\|^2\ \mathrm{dt}]$$

Summing over N for the inequality of lemma(34) gives

$$\int_0^T ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}\eta_w, \eta_w)\,\mathrm{dt} \le C\ \max_{i=1}^{N} k_i^{2(m+l)-3+\alpha}\Big(\int_{I_i} \big\|w^{(m+l)}\big\|\,\mathrm{dt}\Big)^2 \ \text{ for } l = 0, 1$$

Taking the square of the second inequality of (7.2.5), then Integrating over $(0, T)$, we obtain,

$$\int_0^T \|\nabla\eta_u(t)\|^2\, dt \le C_l k_n^{2\,(m+l)-2}\Big(\int_{I_n} \big\|\nabla u^{(m+l)}(t)\big\|\ \mathrm{dt}\Big)^2$$

Hence,

$$\begin{aligned}
\int_0^T \|\xi\|_{H^1}^2 &\le C[\max_{j=1}^{N} k_j^{2(m+l)-3+\alpha}\Big(\int_{I_j} \big\|u^{(m+l)}\big\|\ \mathrm{dt}\Big)^2 + \int_0^T \|\nabla\eta_u\|^2\ \mathrm{dt}] \\
&\le C[\max_{n=1}^{N} k_n^{2(m+l)-3+\alpha}\Big(\int_{I_n} \big\|u^{(m+l)}\big\|\ \mathrm{dt}\Big)^2 + \max_{n=1}^{N} k_n^{2(m+l)-2}\Big(\int_{I_n} \big\|\nabla u^{(m+l)}\big\|\ \mathrm{dt}\Big)^2] \\
&\le C\max_{n=1}^{N} k_n^{2(m+l)-3+\alpha}\Big(\int_{I_n} \big\|\nabla u^{(m+l)}\big\|\ \mathrm{dt}\Big)^2
\end{aligned}$$

For $l = 0, 1$ $\quad\square$

We used the Poincar inequality $\|u\| \leq \|\nabla u\|$. In the next theorem we show the main convergence results of the DG solution. Typically which is of degree m, if we assume that, the exact solution u of problem (7.0.1) satisfies the finite regularity assumptions and other assumptions that in chapter 6:

Theorem 36 Let u be the solution of (7.0.1) satisfying the regularity property:

$$t^s \|u^{(s+1)}(t)\| \leq C t^{\sigma-1}, \quad 0 \leq s \leq m. \tag{7.3.4}$$

$$\int_0^T \|U - u\|^2 \, dt \leq C \, k^{\gamma(2\sigma+2\,m-3+\alpha)} \quad \text{for} \quad 1 \leq \gamma \leq \frac{2m+1+\alpha}{2\sigma+2\,m-3+\alpha}$$

where C is a constant that depends on T, α, γ, σ, and on $\mathbf{M}$, but independent of h and k, with $\sigma > 0$. where C is a constant that depends on T, α, but independent of k

Proof 16 we follow the derivation in the proof of theorem 32. We assume that u of the problem (7.0.1) satisfies the regularity property (7.3.4). For the proof of this inequality we can refer to the work by [25]. Indeed we need to impose some regularity assumption on the source term f and the initial data u_0 $\square$

Chapter 8

A FULLY DISCRETE SCHEME DG

In this chapter, we use the time-stepping DG method combined with the HDG method for the numerical solutions of fractional diffusion problem (1.1.1) where we assume $k(x) = 1$ for sake of simplicity. We will introduce the scheme of DG-HDG, show the existence-uniqueness of approximate DG-HDG solution. The implementation of the numerical scheme will be discussed. Some numerical results will be defined at the end of this chapter.

8.1 The time-DG, spacial-HDG method

Recall that $\boldsymbol{q} = -\nabla u$. So the weak formulation of the solution (u, q) of the model problem (1.1.1) is:

$$(\boldsymbol{q}, \phi)_{I_j} - (u, \nabla \cdot \phi)_{I_j} + \langle u, \phi \cdot \boldsymbol{n} \rangle_{I_j} = 0, \tag{8.1.1a}$$

$$({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}, \chi)_{I_j} - (\boldsymbol{q}, \nabla \chi)_{I_j} + \langle \boldsymbol{q} \cdot \boldsymbol{n}, \chi \rangle_{I_j} = (f + \omega_\alpha(t) u_0, w)_{I_j}, \tag{8.1.1b}$$

where we use the identity

$$\text{However,} \quad {}^{\mathrm{R}}\mathrm{D}^{1-\alpha} w(s) = \omega_\alpha(s) w_+^0 + \sum_{i=1}^{n-1} \omega_\alpha(s - t_i)\, [w]^i + {}^{c}D^{1-\alpha} w(s),$$

For all $\phi \in \mathbf{H}(div, \Omega)$ and $\chi \in H^1(\Omega)$. A scalar approximation $U \in W_{k,h}$ to u, a vector approximations $\boldsymbol{Q} \in \boldsymbol{V}_{k,h}$ to $\boldsymbol{q}$, and a scalar approximation $\widehat{U} \in M_{k,h}$ to the trace of u on element boundaries will be seeked, where $W_{k,h}$, $\boldsymbol{V}_{k,h}$ and $M_{k,h}$ are time-space finite dimensional spaces. To define these spaces, we discretize the spatial domain Ω as it is chapter 3. Following the convention in chapter 3.

$$
\begin{aligned}
W_{k,h} &= \{w : [0,T] \to L^2(\Omega) \colon w|_{I_j} \in \mathbb{P}_p(W_h), && \forall\, 1 \le j \le N\}, && \text{(8.1.2a)}\\
\boldsymbol{V}_{k,h} &= \{\boldsymbol{v} : [0,T] \to \mathbf{L}^2(\Omega) \colon\ \boldsymbol{v}|_{I_j} \in \mathbb{P}_p(\boldsymbol{V}_h), && \forall\, 1 \le j \le N\}, && \text{(8.1.2b)}\\
M_{k,h} &= \{\mu : [0,T] \to L^2(\mathcal{E}_h) \colon\ \mu|_{I_j} \in \mathbb{P}_p(M_h), 1 \le j \le N\}, && && \text{(8.1.2c)}
\end{aligned}
$$

where $\mathbb{P}_p(W_h)$ denotes the space of polynomials of degree $\le p$ in the time variable with coefficients in a finite dimensional space W_h. The spaces $\mathbb{P}_p(\boldsymbol{V}_h)$ and $\mathbb{P}_p(M_h)$ will be defined in similar fashion. The fully discrete DG scheme is now defined as follows: Given $(U, \boldsymbol{Q}, \widehat{U})$ for $t \in I_{j-1}$ with $U(0^+) \approx u_0$ (is suitably chosen), the numerical solution $(U, \boldsymbol{Q}, \widehat{U})$ on the next time-step I_j is determined by requesting that

$$
\begin{aligned}
(\boldsymbol{Q}, \boldsymbol{r})_{I_j,\mathcal{T}_h} - (U, \nabla \cdot \boldsymbol{r})_{I_j,\mathcal{T}_h} + \langle \widehat{U}, \boldsymbol{r} \cdot \boldsymbol{n} \rangle_{I_j,\partial\mathcal{T}_h} &= 0, && \text{(8.1.3a)}\\
({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, w)_{I_j,\mathcal{T}_h} - (\boldsymbol{Q}, \nabla w)_{I_j,\mathcal{T}_h} + \langle \widehat{\boldsymbol{Q}} \cdot \boldsymbol{n}, w \rangle_{I_j,\partial\mathcal{T}_h} &= (f + \omega_\alpha(t) U_+^0, w)_{I_j,\mathcal{T}_h} && \text{(8.1.3b)}\\
\langle \widehat{U}, \mu \rangle_{I_j,\partial\Omega} &= \langle g, \mu \rangle_{I_j,\partial\Omega}, && \text{(8.1.3c)}\\
\langle \widehat{\boldsymbol{Q}} \cdot \boldsymbol{n}, \mu \rangle_{I_j,\partial\mathcal{T}_h \setminus \partial\Omega} &= 0 && \text{(8.1.3d)}
\end{aligned}
$$

hold for all $\boldsymbol{r} \in \mathbb{P}_p(\boldsymbol{V}_h)$, $w \in \mathbb{P}_p(W_h)$, and $\mu \in \mathbb{P}_p(M_h)$,

$$
\widehat{\boldsymbol{Q}} = \boldsymbol{Q} + \tau \left(U - \widehat{U}\right)\boldsymbol{n} \quad \text{on } \partial\mathcal{T}_h, \qquad \text{(8.1.3e)}
$$

for some nonnegative stabilization function τ defined on $\partial\mathcal{T}_h$; we assume that, for each element $K \in \mathcal{T}_h$, $\tau|_{\partial K}$ is constant on each of its faces. Note that the first three equations are inspired in the weak form of the differential equations satisfied by the exact solution, equations (8.1.1).

8.2 Well-posedness of DG solutions

In this section, we show that the fully discrete DG scheme (8.1.3) is well defined. Since (8.1.3a)-(8.1.3e) amounts to a finite square linear system, the existence of the approximate solution follows from its uniqueness. To this end, we let f and g to be identically zero and set $U_-^0 = 0$ in (8.1.3a)-(8.1.3e). The task is to show that $(U, \boldsymbol{Q}, \widehat{U}) \equiv (0, \mathbf{0}, 0)$.

Theorem 37 Let f and g to be identically zero and set $U(0^+) = 0$ in (8.1.3a)-(8.1.3e). Then, for $1 \le n \le N$, the DG solution $(U, \boldsymbol{Q}, \widehat{U})$ of (8.1.3) satisfies

$$\|\boldsymbol{Q}\|^2_{J_n,\mathcal{T}_h} + \|U\|^2_{\alpha,J_n,\mathcal{T}_h} + \|\sqrt{\tau}(U - \widehat{U})\|^2_{J_n,\mathcal{T}_h} = 0$$

Proof 17 Taking $r = \boldsymbol{Q}$ in (8.1.3a), $w = U$ in (8.1.3b), $\mu = \widehat{\boldsymbol{Q}} \cdot \boldsymbol{n}$ in (8.1.3c) and $\mu = \widehat{U}$ in (8.1.3d), then

$$\begin{aligned}
(\boldsymbol{Q}, \boldsymbol{Q})_{I_j,\mathcal{T}_h} - (U, \nabla \cdot \boldsymbol{Q})_{I_j,\mathcal{T}_h} + \langle \widehat{U}, \boldsymbol{Q} \cdot \boldsymbol{n} \rangle_{I_j,\partial\mathcal{T}_h} &= 0 \\
({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, U)_{I_j,\mathcal{T}_h} - (\boldsymbol{Q}, \nabla U)_{I_j,\mathcal{T}_h} + \langle \widehat{\boldsymbol{Q}} \cdot \boldsymbol{n}, U \rangle_{I_j,\partial\mathcal{T}_h} &= 0 \\
\langle \widehat{U}, \widehat{\boldsymbol{Q}} \cdot \boldsymbol{n} \rangle_{I_j,\partial\Omega} &= 0 \\
\langle \widehat{\boldsymbol{Q}} \cdot \boldsymbol{n}, \widehat{U} \rangle_{I_j,\partial\mathcal{T}_h} - \langle \widehat{\boldsymbol{Q}} \cdot \boldsymbol{n}, \widehat{U} \rangle_{I_j,\partial\Omega} &= 0
\end{aligned}$$

For all $1 \le j \le n$.If we add the above equation, we obtain:

$$\|\boldsymbol{Q}\|^2_{I_j,\mathcal{T}_h} + ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha} U, U)_{I_j,\mathcal{T}_h} + \int_{I_j} \psi_h \, \mathrm{d}t = 0$$

Where,

$$
\begin{aligned}
\psi_h := &- (U, \nabla\cdot \boldsymbol{Q})_{I_j,\mathcal{T}_h} + \langle \widehat{U}, \boldsymbol{Q}\cdot\boldsymbol{n}\rangle_{I_j,\partial\mathcal{T}_h} - (\boldsymbol{Q}, \nabla U)_{I_j,\mathcal{T}_h} \\
&+ \langle \widehat{\boldsymbol{Q}}\cdot\boldsymbol{n}, U\rangle_{I_j,\partial\mathcal{T}_h} - \langle \widehat{U}, \widehat{\boldsymbol{Q}}\cdot\boldsymbol{n}\rangle_{I_j,\partial\mathcal{T}_h} \\
=& - \langle U, \boldsymbol{Q}\cdot\boldsymbol{n}\rangle_{I_j,\mathcal{T}_h} + \langle \widehat{U}, \boldsymbol{Q}\cdot\boldsymbol{n}\rangle_{I_j,\partial\mathcal{T}_h} + \langle \widehat{\boldsymbol{Q}}\cdot\boldsymbol{n}, U\rangle_{I_j,\partial\mathcal{T}_h} - \langle \widehat{U}, \widehat{\boldsymbol{Q}}\cdot\boldsymbol{n}\rangle_{I_j,\partial\mathcal{T}_h} \\
=& \quad \langle \widehat{U} - U, \boldsymbol{Q}\cdot\boldsymbol{n}\rangle_{I_j,\partial\mathcal{T}_h} - \langle \widehat{U} - U, \widehat{\boldsymbol{Q}}\cdot\boldsymbol{n}\rangle_{I_j,\partial\mathcal{T}_h} \quad \text{by green's formula} \\
=& \quad \langle \widehat{U} - U, \boldsymbol{Q}\cdot\boldsymbol{n} - \widehat{\boldsymbol{Q}}\cdot\boldsymbol{n}\rangle_{I_j,\partial\mathcal{T}_h} \\
=& \quad \langle \sqrt{\tau}(\widehat{U} - U), \sqrt{\tau}(\widehat{U} - U)\rangle_{I_j,\partial\mathcal{T}_h} \quad \text{by (8.1.3e)} \\
=& \quad \|\sqrt{\tau}(\widehat{U} - U)\|^2_{\partial\mathcal{T}_h}
\end{aligned}
$$

Taking the sum over j yields the following identity:

$$
\|\boldsymbol{Q}\|^2_{J_n,\mathcal{T}_h} + ({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}U, U)_{J_n,\mathcal{T}_h} + \|\sqrt{\tau}(\widehat{U} - U)\|^2_{J_n,\partial\mathcal{T}_h} = 0
$$

Now, by lemma s16(i) and 37, we have $({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}U, U)_{J_n,\mathcal{T}_h} = 0$, $\boldsymbol{Q} = \boldsymbol{0}$ and $U = \widehat{U}$. Then $U = \widehat{U} = 0$

8.3 Implementation of the numerical scheme

The main aim of this section is to implement DG-HDG scheme (8.1.3) in one dimension solutions U and $\boldsymbol{Q}$ have the following expansion:

$$
U(t,x) = \sum_{i=0}^{st} \psi_n^i(t)\, U_i^n(x) \quad \text{and} \quad \boldsymbol{Q}(t,x) = \sum_{i=0}^{st} \sum_{j=1}^{k+1} \psi_n^i(t)\boldsymbol{Q}_i^n(x) \quad \text{for } t \in I_n,\ x \in J_m = [x_{m-1}, x_
$$

where $\psi_n^k(t) = \left(\frac{t-t_{n-1}}{k_n}\right)^k$, $U_i^n(x) = \sum_{j=1}^{k+1} \alpha_j^{i,n}\, \varphi_j(x)$, $\boldsymbol{Q}_i^n(x) = \sum_{j=1}^{k+1} \beta_j^{i,n}\, \varphi_j(x)$, and st is the degree of U and $\boldsymbol{Q}$ with respect to t. Now, substituting U and $\boldsymbol{Q}$ in the first two equations of the DG-HDG scheme and integrating, give:

From the first equation, for all $j = 1, 2, 3, \cdots, M, M+1, \cdots, M\,k$, we obtain

$$\sum_{k=0}^{st}\Big[(\boldsymbol{Q}_k^n(x)\,\psi_n^k(t), \psi_n^l(t)\varphi_j(x))_{I_n,J_m} - (U_k^n(x)\,\psi_n^k(t), \psi_n^l(t)\varphi_j'(x))_{I_n,J_m}\Big] + \int_{I_n} \psi_n^l(t)\Big(\widehat{U}(x_m)\,\varphi_j(x_m^-) - \widehat{U}(x_{m-1})\,\varphi_j(x_{m-1}^+)\Big)\,dt = 0. \quad (8.3.1)$$

Now, $\widehat{U}(t, x_i)$ and $\widehat{\boldsymbol{Q}}(t, x_i)$ are defined as follows:

$$\widehat{U}(t,x_i) = \begin{cases} u(t,x_i), & i = 0 \text{ or } M; \\ \frac{1}{2}\Big(U(t,x_i^+) + U(t,x_i^-) - \boldsymbol{Q}(t,x_i^+) + \boldsymbol{Q}(t,x_i^-))\Big), & 0 < i < M. \end{cases}$$

$$\widehat{\boldsymbol{Q}}(t,x_i) = \begin{cases} \boldsymbol{Q}(t,x_0^+) - U(t,x_0^+) + u(t,x_0), & i = 0; \\ \frac{1}{2}\Big(-U(t,x_i^+) + U(t,x_i^-) + \boldsymbol{Q}(t,x_i^+) + \boldsymbol{Q}(t,x_i^-))\Big), & 0 < i < M; \\ \boldsymbol{Q}(t,x_M^-) + U(t,x_M^-) - u(t,x_M), & i = M. \end{cases}$$

Substituting $\widehat{U}^n$ as $\hat{u}_h^n$ is in section 4.3, we obtain

$$\sum_{k=0}^{st} \frac{k_n}{k+l+1}\Big[(\boldsymbol{Q}_k^n(x), \varphi_j(x))_{J_m} - (U_k^n(x), \varphi_j'(x))_{J_m}\Big] + U_{J_m} = 0$$

Where

$$U_{J_m} = \int_{I_n} \psi_n^l(t)\Big(\widehat{U}(x_m)\,\varphi_j(x_m^-) - \widehat{U}(x_{m-1})\,\varphi_j(x_{m-1}^+)\Big)\,dt$$

$$= \begin{cases} \sum_{k=0}^{st} \frac{k_n}{2(k+l+1)}\Big(U_k^n(x_m^+) - \boldsymbol{Q}_k^n(x_m^+) + U_k^n(x_m^-) + \boldsymbol{Q}_k^n(x_m^-))\Big)\varphi_j(x_m^-) & \\ -\int_{I_n} u(t,x_0)\psi_n^l(t)\,\varphi_j(x_{m-1}^+)\,dt, & m=1; \\ \sum_{k=0}^{st} \frac{k_n}{k+l+1}\Big[\frac{1}{2}\Big(U_k^n(x_m^+) - \boldsymbol{Q}_k^n(x_m^+) + U_k^n(x_m^-) + \boldsymbol{Q}_k^n(x_m^-))\Big)\varphi_j(x^-) & \\ -\frac{1}{2}\Big(U_k^n(x_{m-1}^+) - \boldsymbol{Q}_k^n(x_{m-1}^+) + U_k^n(x_{m-1}^-) + \boldsymbol{Q}_k^n(x_{m-1}^-))\Big)\varphi_j(x_m^+)\Big], & 1<m<M; \\ \int_{I_n} u(t,x_M),\psi_n^l(t)\,\varphi_j(x_m^-)\,dt & \\ -\sum_{k=0}^{st} \frac{k_n}{2(k+l+1)}\Big(U_k^n(x_{m-1}^+) - \boldsymbol{Q}_k^n(x_{m-1}^+) + U_k^n(x_{m-1}^-) + \boldsymbol{Q}_k^n(x_{m-1}^-))\Big)\varphi_j(x_m^+), & m=M. \end{cases}$$

The integral, $\int_{I_n} \psi_n^l(t)\psi_n^k(t)\,dt = \frac{k_n}{k+l+1}$, for all $k, l = 0, 1, 2, \cdots, st$. We are ready now to write the above equation in matrix form (globally) as we did in section 4.3 as the following:

$$\sum_{k=0}^{st} \frac{k_n}{k+l+1}\Big[(A+B)\beta^{k,n} + (D-B)\alpha^{k,n}\Big] = U_u^{l,n} \tag{8.3.2}$$

where, $U_u^{l,n} = U_0^{l,n} - U_M^{l,n}$

$$\beta^{k,n} = \begin{pmatrix} \beta_0^{k,n} \\ \beta_1^{k,n} \\ \vdots \\ \beta_{Ms}^{k,n} \end{pmatrix}, \quad U_0^{l,n} = \begin{pmatrix} u_0^{l,n} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad U_M^{l,n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ u_M^{l,n} \end{pmatrix},$$

And,

$$u_0^{l,n} = \begin{pmatrix} \int_{I_n} u(t,x_0), \psi_n^l(t)\,\varphi_1(x_0^+)\,dt \\ \int_{I_n} u(t,x_0), \psi_n^l(t)\,\varphi_2(x_0^+)\,dt \\ \vdots \\ \int_{I_n} u(t,x_0), \psi_n^l(t)\,\varphi_{s+1}(x_0^+)\,dt \end{pmatrix}, \quad u_M^{l,n} = \begin{pmatrix} \int_{I_n} u(t,x_M), \psi_n^l(t)\,\varphi_1(x_M^-)\,dt \\ \int_{I_n} u(t,x_M), \psi_n^l(t)\,\varphi_2(x_M^-)\,dt \\ \vdots \\ \int_{I_n} u(t,x_M), \psi_n^l(t)\,\varphi_{s+1}(x_M^-)\,dt \end{pmatrix}$$

The last task is to write the second equation of the numerical scheme in a matrix form. We rewrite the first term of the scheme as follows:

$$\begin{aligned}
\Big({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}U, \psi_n^l(t)\,\varphi_i\Big)_{I_n,J_m} &= \Big(\frac{\partial}{\partial t}\int_0^t \omega_\alpha(t-s)U(s,x)\,ds, \psi_n^l(t)\,\varphi_i(x)\Big)_{I_n,J_m} \\
&= \Big(\frac{\partial}{\partial t}\sum_{j=1}^{n-1}\int_{I_j} \omega_\alpha(t-s)\sum_{k=0}^{st}\psi_j^k(s)U_k^j(x)\,ds, \psi_n^l(t)\,\varphi_i(x)\Big)_{I_n,J_m} \\
&\quad + \Big(\frac{\partial}{\partial t}\int_{t_{n-1}}^t \omega_\alpha(t-s)\sum_{k=0}^{st}\psi_j^k(s)U_k^n(x)\,ds, \psi_n^l(t)\,\varphi_i(x)\Big)_{I_n,J_m} \\
&= \sum_{j=1}^{n-1}\sum_{k=0}^{st}\int_{I_n}\frac{\partial}{\partial t}\int_{I_j}\omega_\alpha(t-s)\,\psi_j^k(s)\,ds.\psi_n^l(t)\,dt\,\Big(U_k^j(x),\varphi_i(x)\Big)_{J_m} \\
&\quad + \sum_{k=0}^{st}\int_{I_n}\frac{\partial}{\partial t}\int_{t_{n-1}}^t\omega_\alpha(t-s)\,\psi_j^k(s)\,ds.\psi_n^l(t)\,dt\,\Big(U_k^n(x),\varphi_i(x)\Big)_{J_m} \\
&= \sum_{j=1}^{n-1}\sum_{k=0}^{st} h(k,j,l,n)\,\Big(U_k^j(x),\varphi_i(x)\Big)_{J_m} \\
&\quad + \sum_{k=0}^{st} h_t(k,n,l)\,\Big(U_k^n(x),\varphi_i(x)\Big)_{J_m}
\end{aligned}$$

where,

$$h(k,j,l,n) = \int_{I_n}\frac{\partial}{\partial t}\int_{I_j}\omega_\alpha(t-s)\,\psi_j^k(s)\,ds.\psi_n^l(t)\,dt,$$

$$h_t(k,n,l) = \int_{I_n}\frac{\partial}{\partial t}\int_{t_{n-1}}^t\omega_\alpha(t-s)\,\psi_j^k(s)\,ds.\psi_n^l(t)\,dt$$

Therefore,

$$\Big({}^{\mathrm{R}}\mathrm{D}^{1-\alpha}U, \psi_n^l(t)\,\varphi_j\Big)_{I_n,\tau_h} = \sum_{k=0}^{st}\Big(\sum_{j=1}^{n-1} h(k,j,l,n)\,A\,\alpha^{k,j} + h_t(k,n,l)\,A\,\alpha^{k,n}\Big)$$

The second equation of the scheme (8.1.3) can be written a the matrix form as follows:

$$\begin{aligned}&\sum_{k=0}^{st}\Big[\sum_{j=1}^{n-1} h(k,j,l,n)\Big(U_k^j(x),\varphi_i(x)\Big)_{J_m} + h_t(k,n,l)\Big(U_k^n(x),\varphi_i(x)\Big)_{J_m}\\ &\quad -\frac{k_n}{k+l+1}\Big(\boldsymbol{Q}_k^n(x),\varphi_i'(x)\Big)_{J_m}\Big] + \int_{I_n}\psi_n^l(t)\Big(\widehat{\boldsymbol{Q}}^n(x_m)\,\varphi_i(x_m^-) - \widehat{\boldsymbol{Q}}^n(x_{m-1})\,\varphi_i(x_{m-1}^+)\Big)\,dt\\ &= \int_{I_n}\Big(f(t,x),\psi_n^l(t)\varphi_i(x)\Big)_{J_m}\,dt + \omega_\alpha^l(I_n)\Big(U(0^+),\varphi_i(x)\Big)_{J_m}\end{aligned} \tag{8.3.3}$$

for $l = 0,1,2,\cdots,st$, $m = 1,2,\cdots,M$ and $i = 1,2,3,\cdots,k\,M$ (dimension of W_h).Hence,

$$\omega_\alpha^l(I_n) = \int_{I_n}\omega_\alpha(t)\psi_n^l(t)\,dt = \sum_{i=1}^{l}(-1)^{i+1}\frac{l!}{(l+i+1)!\,k_n^{i-1}}\,\omega_{\alpha+i}(t_n) - \frac{l!}{k_n^l}\,\omega_{\alpha+l}(t_{n-1})$$

substituting $\widehat{\boldsymbol{Q}}^n$ as $\hat{\boldsymbol{q}}_h^n$ is in section 4.3, we get

$$\begin{aligned}\boldsymbol{Q}_{J_m} &= \int_{I_n}\psi_n^l(t)\Big(\widehat{\boldsymbol{Q}}^n(x_m)\,\varphi_i(x_m^-) - \widehat{\boldsymbol{Q}}^n(x_{m-1})\,\varphi_i(x_{m-1}^+)\Big)\,dt\\ &= \begin{cases}\sum_{k=0}^{st}\frac{k_n}{k+l+1}\Big[\frac{1}{2}\Big(-U_k^n(x_m^+)+\boldsymbol{Q}_k^n(x_m^+)+U_k^n(x_m^-)+\boldsymbol{Q}_k^n(x_m^-)\Big)\varphi_j(x_m^-)\\ +\Big(U_k^n(x_{m-1}^+)-\boldsymbol{Q}_k^n(x_{m-1}^+)\Big)\varphi_j(x_{m-1}^+)\Big] - \int_{I_n}u(t,x_0)\psi_n^l(t)\,\varphi_j(x_{m-1}^+)\,dt, & m=1;\\ \sum_{k=0}^{st}\frac{k_n}{k+l+1}\Big[\frac{1}{2}\Big(-U_k^n(x_m^+)+\boldsymbol{Q}_k^n(x_m^+)+U_k^n(x_m^-)+\boldsymbol{Q}_k^n(x_m^-))\Big)\varphi_j(x_m^-)\\ -\frac{1}{2}\Big(-U_k^n(x_{m-1}^+)+\boldsymbol{Q}_k^n(x_{m-1}^+)+U_k^n(x_{m-1}^-)+\boldsymbol{Q}_k^n(x_{m-1}^-))\Big)\varphi_j(x_{m-1}^+)\Big], & 1<m<M;\\ \sum_{k=0}^{st}\frac{k_n}{k+l+1}\Big[\Big(U_k^n(x_m^-)+\boldsymbol{Q}_k^n(x_m^-)\Big)-\frac{1}{2}\Big(-U_k^n(x_{m-1}^+)+\boldsymbol{Q}_k^n(x_{m-1}^+)\\ +U_k^n(x_{m-1}^-)+\boldsymbol{Q}_k^n(x_{m-1}^-)\Big)\varphi_j(x_m^+)\Big] - \int_{I_n}u(t,x_M),\psi_n^l(t)\,\varphi_j(x_m^-)\,dt, & m=M.\end{cases}\end{aligned}$$

Hence, equation (8.3.3) can be written as:

$$\sum_{k=0}^{st}\Big[\sum_{j=1}^{n-1} h(k,j,l,n)\Big(U_k^j(x),\varphi_i(x)\Big)_{J_m} + h_t(k,n,l)\Big(U_k^n(x),\varphi_i(x)\Big)_{J_m} - \frac{k_n}{k+l+1}\Big(\boldsymbol{Q}_k^n(x),\varphi_i'(x)\Big)_{J_m}\Big] + \boldsymbol{Q}_{J_m} = \int_{I_n}\Big(f(t,x),\psi_n^l(t)\varphi_i(x)\Big)_{J_m}dt + \delta_m$$

where

$$\delta_m = \begin{cases} \omega_\alpha^l(I_n)\Big(U_0^1(x),\varphi_i(x)\Big)_{J_m}, & m=1;\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$F^{l,n} = \begin{pmatrix} \int_{I_n}\Big(f(t,x),\psi_n^l(t)\varphi_1(x)\Big)_{\tau_h}dt + \delta_1 \\ \int_{I_n}\Big(f(t,x),\psi_n^l(t)\varphi_2(x)\Big)_{\tau_h}dt + \delta_1 \\ \vdots \\ \int_{I_n}\Big(f(t,x),\psi_n^l(t)\varphi_{s+1}(x)\Big)_{\tau_h}dt + \delta_1 \\ \int_{I_n}\Big(f(t,x),\psi_n^l(t)\varphi_{s+2}(x)\Big)_{\tau_h}dt \\ \vdots \\ \int_{I_n}\Big(f(t,x),\psi_n^l(t)\varphi_{(s+1).M}(x)\Big)_{\tau_h}dt \end{pmatrix}$$

Finally second equation of the scheme (8.1.3) can be written in a matrix form as:

$$\sum_{k=0}^{st}\Big[\sum_{j=1}^{n-1} h(k,j,l,n)\Big(A\,\alpha^{k,j}\Big) + h_t(k,n,l)\Big(A\,\alpha^{k,n}\Big) - \frac{k_n}{k+l+1}\Big(B\,\beta^{k,n}\Big) + \frac{k_n}{k+l+1}\Big(G\,\alpha^{k,n} + H\,\beta^{k,n}\Big)\Big] = F^{l,n} + U_0^{l,n} + U_M^{l,n} \quad (8.3.4)$$

8.4 Numerical experiments for DG-HDG

We introduce some numerical experiments using a model problem in one space dimension, of the form (1.1.1) with $\Omega = (0,1)$, $[0,T] = [0,1]$, and $k(x) = 1$. We choose $u_0(x) = \sin(\pi x)$ for the initial data and choose the source term f so that

$$u(t) = (1 + t^{\alpha}) \sin(\pi x)\,. \tag{8.4.1}$$

One can verify that the regularity condition (7.3.4) holds.

The numerical tests below reveal faster rates of convergence than suggested by Theorem 36, and that our regularity assumptions are more restrictive than is needed in practice. More precisely, the theoretical results in Theorem 36 show a suboptimal (in time) convergence of order $O(k^{m+1-\frac{1-\alpha}{2}}$ for sufficiently graded time meshes in the time-space L^2-norm. However, we demonstrate numerically optimal time rate of convergence in the stronger $L^\infty(L^2)$-norm. To this end, We introduce a finer mesh

$$\mathcal{G}^m = \{\, t_{j-1} + \ell k_j/m : j = 1,\, 2,\, \ldots,\, N \text{ and } \ell = 0,\, 1,\, \ldots,\, m\,\}, \tag{8.4.2}$$

and define the discrete maximum norm $\|v\|_{\mathcal{G}^m} = \max_{t\in\mathcal{G}^m} \|v(t)\|$, so that, for sufficiently large values of m, $\|U - u\|_{\mathcal{G}^m}$ and $\|\boldsymbol{Q} - \boldsymbol{q}\|_{\mathcal{G}^m}$ approximate the uniform errors $\|U - u\|_{L^\infty(L^2)}$ and $\|\boldsymbol{Q} - \boldsymbol{q}\|_{L^\infty(L^2)}$. In all tables, we choose $m = 5$. For different values of the polynomial degree, $k = 1, 2$. The numerical results (errors and convergence rates) of the experiments are presented in Tables 8 .1 and 8 .2 and their figures

N	$\gamma = 1$			
4	9.9950e-03		2.6107e-02	
8	4.9655e-03	1.0093e+00	1.5528e-02	7.4959e-01
16	2.9905e-03	7.3157e-01	9.3931e-03	7.2516e-01
32	1.8000e-03	7.3237e-01	5.6548e-03	7.3211e-01
64	1.0887e-03	7.2541e-01	3.4202e-03	7.2540e-01
	$\gamma = 2$			
4	2.9948e-03		9.3953e-03	
8	1.0887e-03	1.4598e+00	3.4202e-03	1.4578e+00
16	4.1451e-04	1.3932e+00	1.3022e-03	1.3931e+00
32	1.5680e-04	1.4025e+00	4.9261e-04	1.4025e+00
64	5.9333e-05	1.4020e+00	1.8640e-04	1.4020e+00
	$\gamma = 3$			
4	3.7538e-03		1.1758e-02	
8	1.0382e-03	1.8542e+00	3.2611e-03	1.8503e+00
16	2.4802e-04	2.0656e+00	7.7918e-04	2.0653e+00
32	5.7696e-05	2.1039e+00	1.8126e-04	2.1039e+00
64	1.4812e-05	1.9617e+00	4.6535e-05	1.9617e+00

Table 8 .1: The errors $\|(U-u)(T)\|_{L^\infty(L^2)}$, $\|(\boldsymbol{Q}-\boldsymbol{q})(T)\|_{L^\infty(L^2)}$ and the corresponding rates of convergence for $\alpha = 0.7$ with DG-HDG solutions of degree $k = 1$.

N	$\gamma = 1$			
4	3.5706e-03		1.1203e-02	
8	2.1984e-03	6.9971e-01	6.9063e-03	6.9792e-01
16	1.3502e-03	7.0327e-01	4.2418e-03	7.0325e-01
32	8.2700e-04	7.0720e-01	2.5981e-03	7.0720e-01
64	5.0612e-04	7.0842e-01	1.5900e-03	7.0842e-01
	$\gamma = 2$			
4	1.3502e-03		4.2418e-03	
8	5.0612e-04	1.4156e+00	1.5900e-03	1.4156e+00
16	1.9000e-04	1.4135e+00	5.9691e-04	1.4135e+00
32	7.1649e-05	1.4070e+00	2.2509e-04	1.4070e+00
64	2.7094e-05	1.4030e+00	8.5117e-05	1.4030e+00
	$\gamma = 3$			
4	6.7116e-04		2.1084e-03	
8	1.5370e-04	2.1265e+00	4.8286e-04	2.1265e+00
16	3.5339e-05	2.1208e+00	1.1102e-04	2.1208e+00
32	8.2055e-06	2.1066e+00	2.5778e-05	2.1066e+00
64	1.9118e-06	2.1017e+00	6.0061e-06	2.1017e+00
	$\gamma = 4$			
4	8.0678e-04		2.5344e-03	
8	1.1323e-04	2.8329e+00	3.5572e-04	2.8328e+00
16	1.4873e-05	2.9285e+00	4.6725e-05	2.9285e+00
32	2.0665e-06	2.8475e+00	6.4920e-06	2.8475e+00
64	2.7436e-07	2.9130e+00	8.6193e-07	2.9130e+00

Table 8 .2: The errors $\|(U-u)(T)\|_{L^\infty(L^2)}$, $\|(\boldsymbol{Q}-\boldsymbol{q})(T)\|_{L^\infty(L^2)}$ and the corresponding rates of convergence for $\alpha = 0.7$ with DG-HDG solutions of degree $k = 2$.

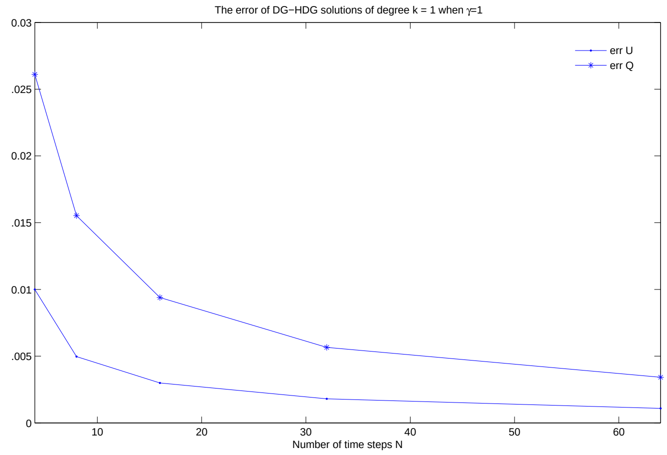


Figure 8 .1: DG-HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.7$
,$\gamma = 1$

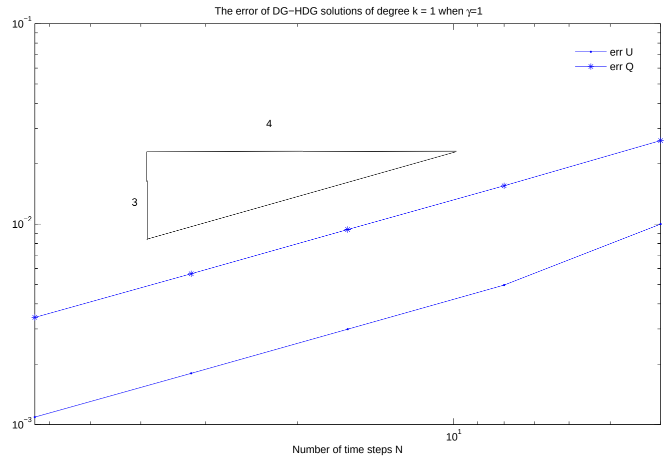


Figure 8 .2: DG-HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.7$,$\gamma = 1$, log-log scaling

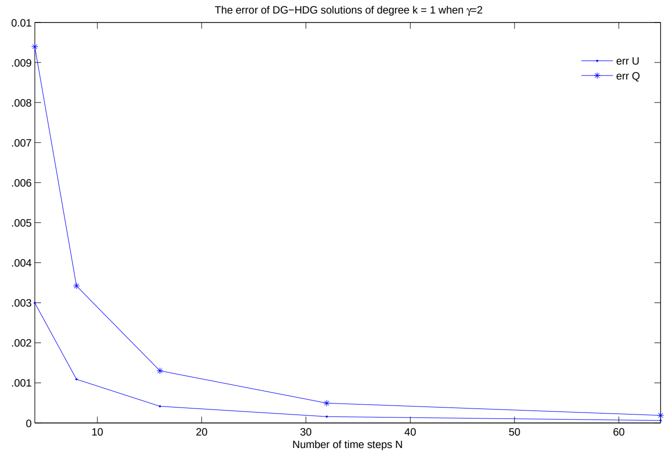


Figure 8 .3: DG-HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.7$
, $\gamma = 2$

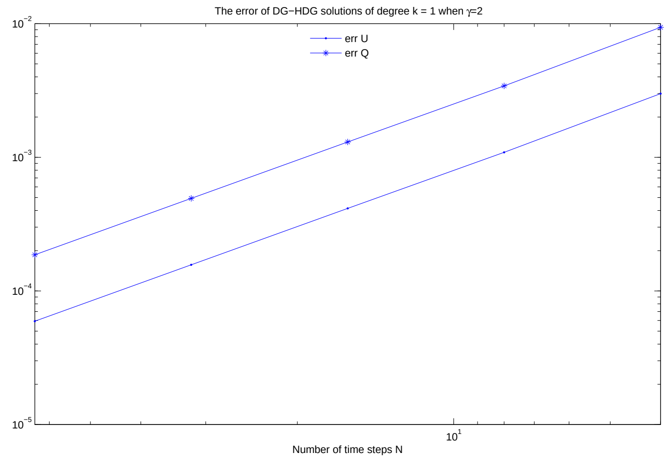


Figure 8 .4: DG-HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.7$, $\gamma = 2$ log-log scaling

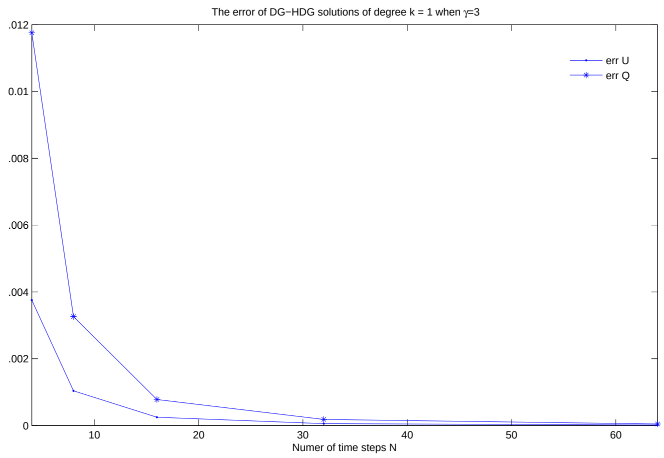


Figure 8 .5: DG-HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.7$, $\gamma = 3$

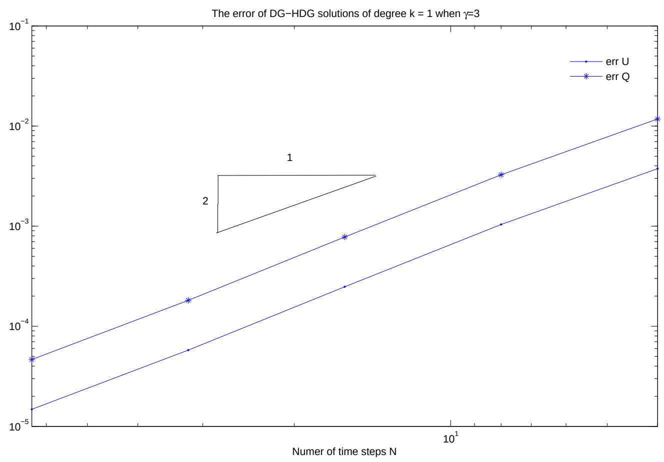


Figure 8 .6: DG-HDG errors for piecewise linear solution($k = 1$) when $\alpha = 0.7$,$\gamma = 3$ log-log scaling

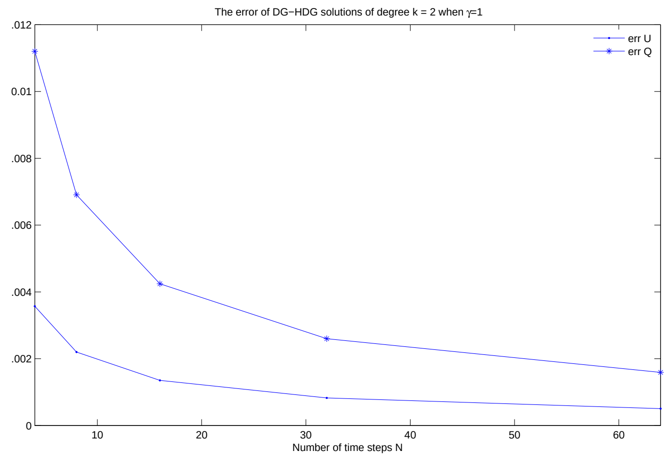


Figure 8 .7: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, $\gamma = 1$

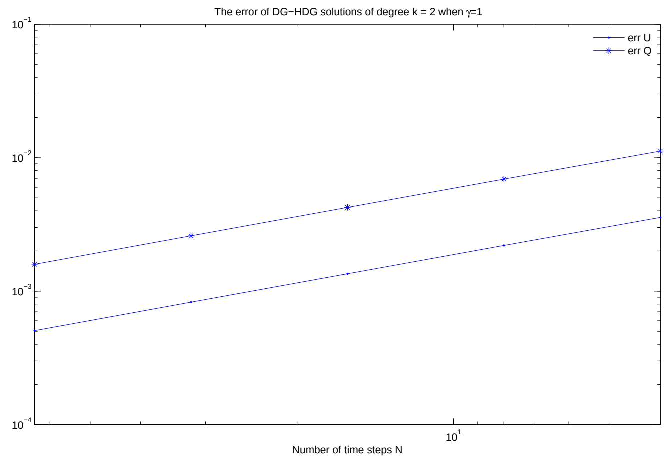


Figure 8 .8: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, $\gamma = 1$, log-log scaling

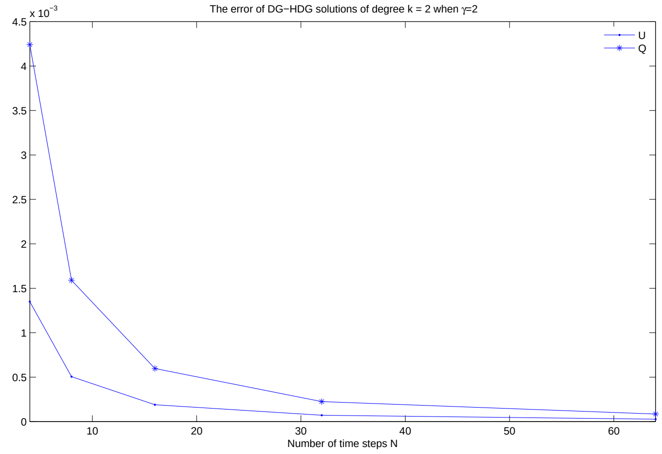


Figure 8 .9: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, $\gamma = 2$

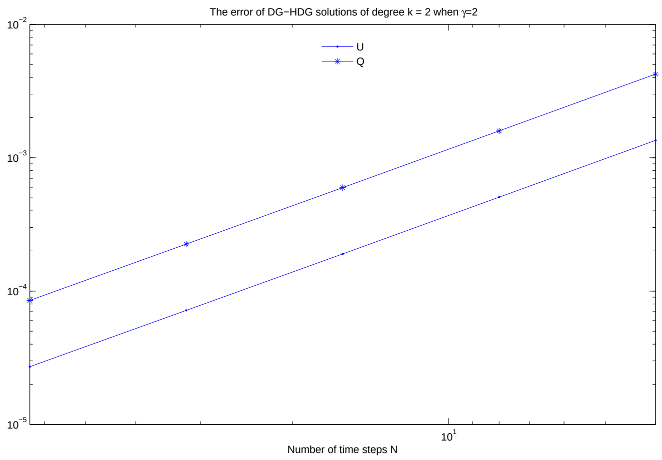


Figure 8 .10: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$,$\gamma = 2$, log-log scaling

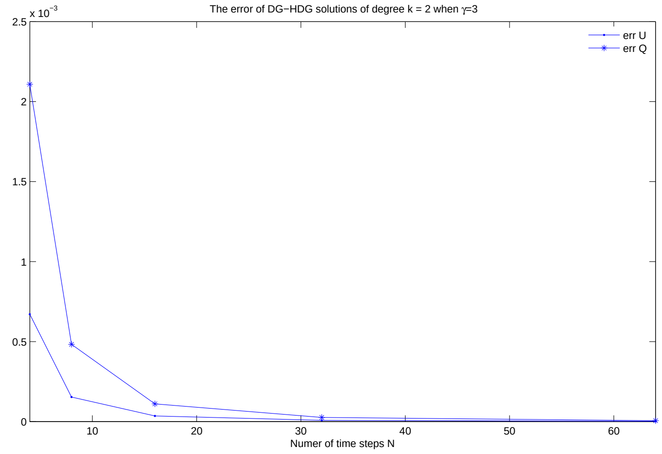


Figure 8 .11: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, $\gamma = 3$

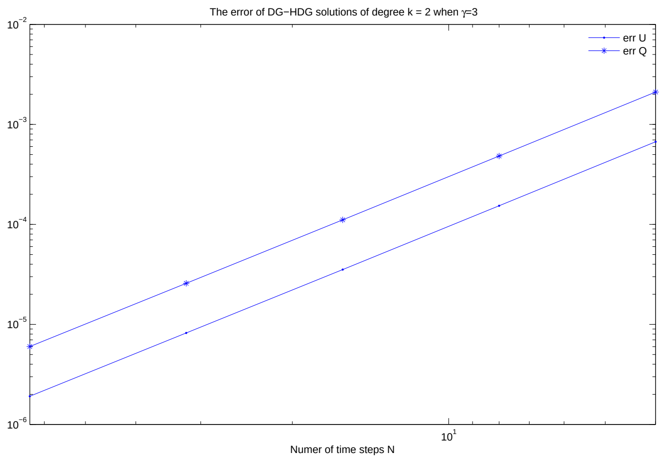


Figure 8 .12: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, $\gamma = 3$, log-log scaling

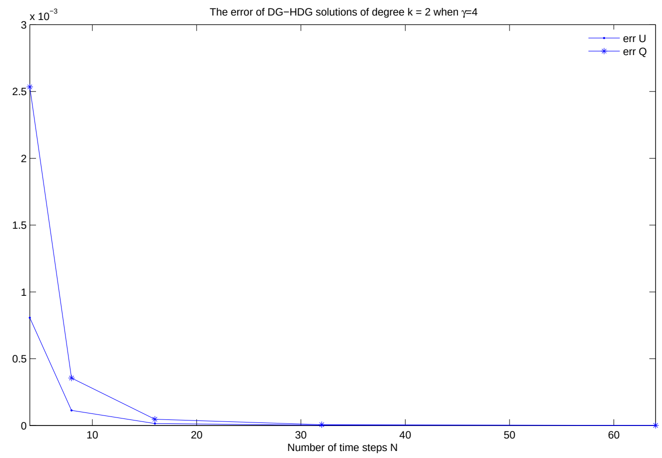


Figure 8 .13: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, $\gamma = 4$

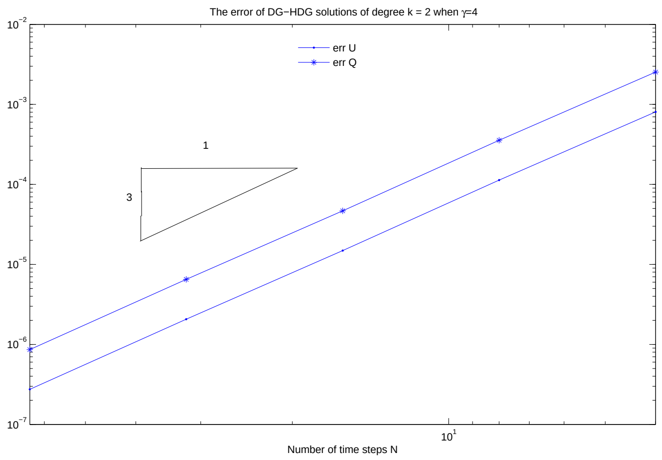


Figure 8 .14: DG-HDG errors for piecewise quadratic solutions($k = 2$) with $\alpha = 0.7$, $\gamma = 4$, log-log scaling

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RESUME

1. PERSONAL DETAILS

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Nationality: Jordanian

2. CONTACT DETALS

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King Fahd University of Petroleum and Minerals (KFUPM)

3. BACKGROUND

3.1. *Education*

University	Year	Degree Obtained
King Fahd University of Petroleum and Minerals	20011 – 2016 4 April 2016	PhD. In Mathematics (Numerical and Computational) Thesis defended
Birzeit University (Palestine)	2007 –2010	M Sc. in Mathematics
Birzeit University	2002 – 2006	M Sc. in Computational of Science
Birzeit University	1997 – 2001	B.Sc. Mathematics and Physics

3.2. *Awards and Scholarships*

Year	**Award or Scholarship**
2011 – present	Teaching and Research Assistantship (KFUPM)
2007–2008	Teaching and Research Assistantship (Birzeit University in Math Department)
2002–2004	Teaching and Research Assistantship (Birzeit University in Physics Dep.)
1999–2001	Teaching and Research Assistantship (Birzeit University in Physics Dep.)

4. AREA OF RESEARCH INTEREST

My current research focus is on proposing efficient computable high-order hybrid discontinuous Galerkin (HDG) methods for time-fractional diffusion models. I am working on both: the theory and the computer implementation of the HDG methods. I published two interesting articles in this field, in collaboration with my PhD advisor and others. I my previous research, I worked on the finite difference and finite element schemes for two-dimensional heat models. I wrote several computer codes to implement these numerical methods.

5. *TEACHING*

5.1 Teaching Experience

University	**Period of Service**	**Position**
KFUPM	2011 – 2016	Lecturer B
Birzeit University in Math Department	2008 – 2011	Lecturer A
Birzeit University in Math Department	2007 – 2008	Teaching and Research Assistant
Birzeit University in Physics Department	2002 – 2007	Teaching and Research Assistant

5.2. Course *Taught*

- **King Fahd University of Petroleum and Minerals**

  Calculus I: Lecturing and recitation

  Calculus II.
- **Birzeit University**

  Calculus I, Calculus II, Calculus III.

- **Birzeit University**
  Physics Labs (I, II, III)

6. Publications

1. M. Nour, Convergence and superconvergence analyses of HDG methods for time fractional diffusion problems, Adv. Comput. Math., To appear 2015 (with K., Mustapha and B. Cockburn)

2. M. Nour, a discontinuous Galerkin method for time fractional diffusion equations with variable coefficients, Numer. Algo 2015 (with K. Mustapha, B. Abdallah, K.M. Furati)

3. A full DG method for a 2- dimensional fractional diffusion model to be Submitted soon (with K., Mustapha and B. Cockburn) .

7. SEMINARS at KFUPM

Title	Location
Numerical solutions for a class of parabolic problems	Math. & Stat. departmental seminar
Using Discontinuous Galerkin Methods for Solving Fractional Diffusion	Math. & Stat. departmental seminar

8. COMPUTER SKILLS

8.1 ***Computer tools :*** Pascal, C++, Java, MATLAB, Mathematica, Maple

8.2 ***Typesetting Software:*** Latex, MS-Word, Excel, Power Point.

.

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