ON SOME PROBLEMS ARISING FROM A SUSPENSION BRIDGE

BY

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A Dissertation Presented to the
DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICS

MAY 2016

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by **SOH EDWIN MUKIAWA** under the direction of his thesis adviser and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY IN MATHEMATICS** .

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To the Almighty God and to the Soh's family

ACKNOWLEDGMENTS

First, I would like to thank the Almighty God for the opportunity, strength and good health He gave me and my team throughout this research project. My sincere gratitude and appreciation goes to King Fahd University of Petroleum and Minerals as a whole and in particular to the Department of Mathematics and Statistics and all it faculty members for giving me this opportunity and putting in place research and teaching resources to make my Ph.D. program a success. Its an honour and a privilege to be supervised by Distinguished Professor Salim A. Messaoudi. My deepest gratitude and appreciations goes to him for his inspiration, invaluable academic support, motivation and guidance all through this work. May God reward him abundantly.

I would like to express my deepest gratitude and appreciation to my Co-advisor Doctor Ahmed Bonfoh for his invaluable support and advices throughout my years in KFUPM. Many thanks for your diligent support, motivation and for sharing with me your vast research experience.

My sincere gratitude to my committee members Professor Bilal Chanane, Professor Fiazud Zaman and Doctor Islam Mustafa for their plenty advices, encouragements, time and effort to see that I complete my Ph.D. program in KFUPM.

I would to thank my friends and brothers Dr. Ubong Monday, Cyril Enyi, John-

son Audu and Hashim Jamilu for their wonderful friendship and support in one way or the other. I enjoyed the every day struggle we have been through and the great stay I have with you all. Special thanks goes to my friends Tangwan Mary Amboh and Epie Godfred Essongolle for their unequivocal encouragement and support throughout this program, may the Lord bless you all abundantly. My deepest gratitude and appreciation to my parents Soh Ignatius and Soh Regina for their love and the prayers they offer on my behalf to God throughout this journey. Many thanks to my brothers and sisters Soh Kennedy, Rogers Tibah, Mah Juliette and Soh Erica for their support and encouragement in all my activities. I would also like to give a special thank you to my uncle Soh Michael Tibah of blessed memory and wife Soh Seraphia for playing a fundamental part in my journey to this level, may his gentle soul rest in perfect Peace. My appreciations and affectionate thanks to my Cousins Dr. Soh Maureen, Titi Wendy and Soh Blasius for their invaluable support and encouragement throughout my studies. Special thank to every other person whose name has not been mentioned who helped may in one way or the other during my studies.

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THESIS ABSTRACT

NAME: Soh Edwin Mukiawa

TITLE OF STUDY: On some problems arising from a suspension bridge

MAJOR FIELD: Mathematics

DATE OF DEGREE: May 2016

This dissertation aims at studying some fourth-order partial differential equations with non-classical boundary conditions, serving as models for describing the displacement of a suspension bridge in the downward direction. We shall establish well-posedness, stability and general decay results for weak solutions of some problems. We use the Faedo-Galerkin approximation method and/or the semigroup theory to establish the well-posedness and use the Multiplier method and/or existing techniques such as concavity and perturbation methods to establish the stability and general decay results.

الملخص

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عنوان الدراسة: على بعض المشاكل الناجمة عن جسر معلق

مجال التخصص : الرياضيات

تاريخ درجة : مايو ٢٠١٦

تهدف هذه الأطروحة إلى دراسة بعض المعادلات التفاضلية الجزئية من الرتبة الرابعة مزودة بشروط حدية غيرتقليدية، والتي عادة ما تنمذج لحركة الجسور المعلقة؛ حيث نقوم باثبات بعض نتائج الصياغة الجيدة (الوجود والوحدانية)، والاستقرار، والاضمحلال العام للحلول الضعيفة لهذه المسائل. وللحصول على نتائجنا سنستخدم طريقة التقريبات لفايدوغلاركين أو نظرية أنصاف الزمر لاثبات الصياغة الجيدة، كما نستخدم بعض التقنيات الأخرى مثل تقنية المضروبات، طريقة التقعر، والاضطراب للبرهان على الاستقرار ونتائج الاضمحلال العام.

CHAPTER 1

GENERAL INTRODUCTION

1.1 Suspension bridge

A suspension bridge is a type of bridge in which the load bearing region (the deck) is hung below suspension cables on vertical suspenders. In this type of bridge, the hangers are suspended between towers and with vertical suspended cables which carry the weight of deck below it as in figure 1.1.

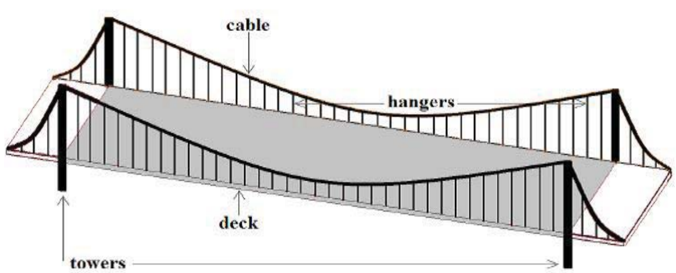


Figure 1.1: The structure of suspension Bridges

Since any external load added or applied to the suspension bridge is transformed into a tensional force, the cables of a suspension bridge must be anchored at each end of the Bridge. The main hangers (cables) of the bridge pass beyond the main towers (pillars) to the load bearing region (deck-level) supports, and are finally anchored into the ground.

Even though suspension bridges are in limited number compared to other type of bridges, a high attention has been directed towards the structural behaviour and the instability of these bridges. This is due to the large number of suspension bridge failures occurring after construction. In 1940, the Tacoma suspension

Figure 1.2: The Tacoma suspension bridge collapse, 1940

bridge (figure 1.2) categorized by many authors as one of the most historical suspension bridge disaster (because of available videos) occurred, see [3, 45, 88, 89] for web videos and update of other suspension bridge disasters. Also, in a PhD thesis at the University of Cambridge in 2004, Imhof [44] reported that a huge number of around 400 suspension bridges have collapsed after construction and about 70 of those failures occur after 2000. Thus, there is need for precise mathematical models to describe the structural behaviour and the instability of suspension bridges. However, not enough work has been done in this area.

In this dissertation, we consider a simple rectangular plate model by Ferrero and Gazzola [30] and investigates this model in the presence of damping due to internal friction and the viscosity, as well as the viscoelastic (viscous and elastic) behaviour of materials used in construction.

1.2 Ferrero and Gazzola's plate model

In this section, we describe the first attempt to model a suspension bridge as a simple rectangular plate by Ferrero and Gazzola [30] in 2015. They considered a simple thin rectangular plate representing the roadway of a suspension bridge of length L and width 2ℓ (assuming that $2\ell \cong \frac{L}{100}$) hinged in the vertical edges (y-axis) but free on the horizontal edges ($x-$ axis), and been anchored to the ground. In what follows, the thin rectangular plate is represented by $\Omega = (0, L) \times (-\ell, \ell)$. The thin plate Ω is assumed to have surface principal curvatures K_1 and K_2 of the graph of a function u that describes the downward vertical motion of the suspension bridge. For a deformed plate Ω, the bending energy can be modelled by the following energy functional

$$E_B(u) = \frac{\nu d_0^3}{6(1-\sigma)} \int_\Omega \left(\frac{K_1^2 + K_2^2}{2} + \sigma K_1 K_2 \right) dxdy, \tag{1.1}$$

where d_0 is the thickness of plate, $\sigma = \frac{\lambda}{2(\lambda+\nu)}$ is called the Poisson ratio and ν, λ are the Lamé moduli. Due to physical reasons, $\nu > 0$ and $\lambda > 0$; therefore

$$0 < \sigma < \frac{1}{2}. \tag{1.2}$$

When the deformation u of the rectangular plate Ω is small, the following approximations hold

$$(K_1 + K_2)^2 \approx (\Delta u)^2, \quad K_1 K_2 \approx det(D^2 u) = u_{xx}u_{yy} - u_{xy}^2.$$

Therefore,

$$\left(\frac{K_1^2+K_2^2}{2}+\sigma K_1K_2\right)\approx\frac{1}{2}(\Delta u)^2+(\sigma-1)det(D^2u).$$

Suppose that f is an external force including both live and dead load as well as gravity acting on the plate Ω, then it follows that the total bending energy for small deformation u of the plate takes the form

$$\begin{aligned}E_T(u) &= E_B(u)-\int_\Omega fudxdy\\ &= \frac{\nu d_0^3}{6(1-\sigma)}\int_\Omega\left(\frac{1}{2}(\Delta u)^2+(\sigma-1)(u_{xx}u_{yy}-u_{xy}^2)\right)dxdy-\int_\Omega fudxdy.\end{aligned}$$

If the external force f is replaced by $\frac{\nu d_0^3}{6(1-\sigma)}f$, then we can rewrite the total bending energy E_T up to a constant of multiplier as

$$E_T(u)=\int_\Omega\left[\left(\frac{1}{2}(\Delta u)^2+(\sigma-1)(u_{xx}u_{yy}-u_{xy}^2)\right)-fu\right]dxdy. \tag{1.3}$$

When the plate Ω is at the equilibrium position, the function u is a minimizer of the functional E_T and it is the unique solution of the Euler-Lagrange equation

$$\Delta^2u(x,y)=f(x,y),\ \text{ in }\Omega. \tag{1.4}$$

Now, since the bridge is usually anchored to the ground on the vertical edges ($x = 0, x = L$, that is the $y-$ axis) only, the boundary conditions in this case are

$$u(0, y) = u_{xx}(0, y) = u(L, y) = u_{xx}(L, y) = 0.$$

Figure 1.3: Thin rectangular plate Ω

Figure 1.3 shows the thin rectangular plate anchored at $x = 0$ and $x = L$ with the blue region indicating where hangers act.

The case of fully anchored thin rectangular plate Ω (that is $u = \Delta u = 0$, on $\partial\Omega$) has been studied by Navier [82] in 1823. Since the horizontal edges ($y = -\ell, y = \ell$, that is the $x-$ axis) of the plate Ω are free, different types of boundary conditions should be considered. Rectangular plate problems in which the vertical edges are simply supported, has been discussed by several authors, see [57, 63, 81].

Realistically, a rectangular plate Ω in which the horizontal edges are free should necessarily be considered. In this case, (see Ventsel and Krauthammer [90]) the remaining boundary conditions are

$$\begin{cases} u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, \ \ for \ x, \in \ (0,L), \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0, \ \ for \ x \in \ (0,L), \end{cases} \tag{1.5}$$

To recapitulate, (see Ferrero and Gazzola [30] for more details), a rectangular thin plate Ω as a model for a stationary suspension bridge is represented by the fourth-order boundary-value problem

$$\begin{cases} \Delta^2 u(x,y) = f(x,y), \ \ in \ \ \Omega, \\ u(0,y) = u_{xx}(0,y) = u(L,y) = u_{xx}(L,y) = 0, \quad y \in (-\ell,\ell), \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, \quad x \in \ (0,L), \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0, \quad x \in \ (0,L). \end{cases} \tag{1.6}$$

Since the "suspension" bridge is suspended by cables (hangers), Ferrero and Gazzola [30] considered the action of the cables (hangers) by introducing a nonlinear function $h(x,y,u)$ that has a potential energy $\int_\Omega H(x,y,u)dxdy$. Therefore, the energy functional (1.3) takes the form

$$E_T(u) = \int_\Omega \left[\left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)(u_{xx}u_{yy} - u_{xy}^2) \right) + H(x,y,u) - fu \right] dxdy. \tag{1.7}$$

By minimizing the convex functional (1.7), the following stationary problem is obtained

$$\begin{cases} \Delta^2 u(x,y) + h(x,y,u(x,y)) = f(x,y), \;\; in \;\; \Omega, \\ u(0,y) = u_{xx}(0,y) = u(L,y) = u_{xx}(L,y) = 0, \quad y \in (-\ell, \ell), \\ u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = 0, \quad x \in \; (0,L), \\ u_{yyy}(x, \pm\ell) + (2-\sigma) u_{xxy}(x, \pm\ell) = 0, \quad x \in \; (0,L). \end{cases} \tag{1.8}$$

Supposing that the external load f is time dependent; that is $f = f(x,y,t)$, they added a kinetic energy term

$$\frac{1}{2}\int_\Omega u_t^2 dxdy$$

to the total static energy. Therefore, the total energy (1.7) takes the form

$$\begin{aligned} E_T(u) &= \frac{1}{2}\int_\Omega u_t^2 dxdy \\ &+ \int_\Omega \left[\left(\frac{1}{2}(\Delta u)^2 + (\sigma - 1)(u_{xx}u_{yy} - u_{xy}^2) \right) + H(x,y,u) - fu \right] dxdy, \end{aligned}$$

and the equation of motion of the bridge is then given by

$$u_{tt}(x,y,t) + \Delta^2 u(x,y,t) + h(x,y,u(x,y,t)) = f(x,y,t). \tag{1.9}$$

Due to internal friction or viscosity, they added a damping term δu_t ($\delta > 0$ is frictional constant). Thus, the equation of motion of the bridge (1.9) becomes

$$u_{tt}(x,y,t) + \delta u_t(x,y,t) + \Delta^2 u(x,y,t) + h(x,y,u(x.y,t)) = f(x,y,t). \tag{1.10}$$

Ferrero and Gazzola in [30] have studied equation (1.10) together with initial data and the boundary conditions of problem (1.8) and established the existence of unique weak global solution.

The model (1.10) is much more reliable and gives some realistic responses compared to beam models with less degree of freedom, however there is still room to study several forms of it in order to obtain further assured responses and suggestions for future designs. The above model (1.10) does not take into account the possible nonlinear behaviour of damping, as elastic materials are expected to exhibit large velocities when friction increases superlinearly. The damping may also be as a result of the viscoelastic (viscous and elastic) nature of the materials used, resulting to weak damping mechanism (e.g see [27]) as proposed in [48].

In this dissertation, we will consider several variants of equation (1.10), supplement them with the boundary conditions of (1.8) and initial data, and investigate the existence and long-time behaviour of weak solutions. Our main objectives are to treat these two situations; that is: when the linear damping is replaced by a nonlinear damping and in the presence of viscoelastic damping. Our future works will examine the effect of delay on the stability of the model.

1.3 Results Description

In this dissertation, we are mainly concerned with the investigation of the well-posedness, stability and general decay of weak solutions for fourth-order plate equations as models for the downward displacement of a suspension bridge. Our results take advantage of this first attempt to model suspension bridges as a thin rectangular plate by improving on several results and well-established techniques by Messaoudi in [70, 71, 72, 80] for second order hyperbolic equations. For other models that could be improved, see [25, 35].

This dissertation is divided into four chapters. Our contributions to literature begins in chapter two. For convenience, we set $L = \pi$ throughout this thesis.

Chapter Two: In this chapter, we consider the following fourth-order nonlinear wave equation

$$\begin{cases} u_{tt} + \delta|u_t|^{p-2}u_t + \Delta^2 u + h(x,y,u) = f, & \text{in} \quad \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \;\; u_t(x,y,0) = u_1(x,y), & \text{for} \quad (x,y) \in \Omega, \end{cases} \tag{1.11}$$

where $\Omega = (0,\pi) \times (-\ell,\ell) \subset \mathbb{R}^2$, $2 < p < \infty$, $\delta > 0$ is a constant, $0 < \sigma < \frac{1}{2}$ is the Poisson ratio. In this model, the function $u = u(x,y,t)$ represents the vertical displacement in the downward direction of the suspension bridge, $\delta|u_t|^{p-2}u_t$ is the nonlinear frictional damping, $h = h(x,y,u)$ is the restoring force of hangers, $f = f(x,y,t)$ is an external load. Under suitable assumptions on h and f, we will establish a well-posedness as well as a stability result for the energy functional of the problem (1.11).

Chapter Three: In chapter, we consider a model with a general frictional damping and a time dependent frictional damping coefficient

$$\begin{cases} u_{tt} + \Delta^2 u + \beta(t)g(u_t) + h(u) = 0, & \text{in} \quad \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), & \text{for} \quad (x,y) \in \Omega, \end{cases} \tag{1.12}$$

where β and g are known functions to be specified later and h is the hangers restoring force and other parameters remains the same as in chapter two. Our task here is to establish well-posedness and an explicit and general decay estimates for (1.12) with suitable hypothesis on β, g and h.

Chapter Four: In chapter four, we consider a model in the presence of a viscoelastic term, namely

$$\begin{cases} u_{tt} + \Delta^2 u + \int_0^t g(t-s)\Delta^2 u(x,y,s)ds = 0, & \text{in} \quad \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), & \text{for} \quad (x,y) \in \Omega, \end{cases} \tag{1.13}$$

where g is a positive, decreasing relaxation function to be specified later. Our main objective in this chapter is to establish a well-posedness and a general decay result for problem (1.13) with suitable conditions on the relaxation function g.

1.4 Literature Review

1.4.1 History and results on suspension bridge models

As mentioned in section 1.1, the history of suspension bridge failures after construction is quiet large. The available video [89] of the Tacoma suspension bridge failure in 1940 has inspired a large number of studies in this direction, see [86] and reference therein. Thus, Mathematicians and Engineers have shown more interest in the last three decades to develop mathematical models that can be used in describing the instability and structural behaviour of such type of bridge. In this light, various nonlinear models have been developed to describe the dynamics of suspension bridges. However, the best nonlinearities responsible for the phenomena (torsional oscillation) exhibited during the Tacoma bridge failure is still obscure. The investigation of the dynamics of suspension bridges goes back to the pioneer work of Glover et al. [38] in which they considered the following damped coupled beam system

$$\begin{cases} u_{tt} + u_t + u_{xxxx} + \gamma_1 u_t + k(u-v)^+ = f, \\ \epsilon v_{tt} - v_{xx} + \gamma_2 v_t - k(u-v)^+ = g, \end{cases} \tag{1.14}$$

where $\gamma_1, \gamma_2 > 0$ are damping constants,

$$u, v : [0, L] \times \mathbb{R}^+ \longrightarrow \mathbb{R}$$

stand for the downward oscillation of a one-dimensional beam as suspension bridge (single-rod representing the roadway) and the vertical displacement of string respectively, f and g are external forcing terms, $k(u-v)^+$ is the string restoring force and $k > 0$ is a constant. They established the existence and stability of nonlinear oscillations for (1.14). The existence of global attractor for the coupled system (1.14) has also been established by Ma and Zhong in [62]. When the cables holding the suspended beam are replaced by rigid rod, the couple damped system (1.14) was reduced by Lazer and Mckenna [56] to the fourth-order equation

$$u_{tt} + u_{xxxx} + u_t + k^2 u^+ = f, \quad x \in (0, 1), \quad t > 0, \tag{1.15}$$

where $u = u(x,t)$ represents the downward displacement of the bridge in the vertical plane, $u^+ = max\{0, u\}$ the positive part of u, f is the forcing term and k is a positive constant. They proved the existence of periodic solutions by assuming the suspension bridge to be bending beam. Mckenna and collaborators in [66, 67, 68, 69] proved the existence of travelling wave solutions (by considering the suspension bridge as a vibrating beam) and torsional oscillations for a suspension bridge beam model. The key assumption used by Mckenna to establish travelling wave solutions was slackening the hangers as a nonlinear phenomenon.

It should be noted that, this (slackening phenomenon) is a well known assumption for Engineers, see also [17, 48]. Other authors, [28, 74], have as well treated the slackening phenomenon in complex beam models. Bochicchio et al. [16] and [2, 22] have carried out different analysis on the system (1.14) or its modified version.

Recently, in 2012, Bochicchio et al. [15] established explicit stationary solutions to the following beam system

$$\begin{cases} m_c v_{tt} - H v_{xx} + \delta_c v_t - k(u-v)^+ = q_c + f_c(x,t), \;\; x \in (0,\pi), t > 0, \\ m_b u_{tt} + EI u_{xxxx} + \delta_b u_t + [\gamma - M\|u_x\|_{L^2(0,\pi)}]u_{xx} + k(u-v)^+ = q_b + f_b(x,t), \end{cases} \tag{1.16}$$

where v and u represent the vertical displacements of the cable and the beam, m_c and m_b are the masses of cable and beam, q_c and q_b are forces due to dead loads of the cable and the beam, f_c and f_b are external forces of cable and the beam while others are constants that depend on the elasticity and physical materials. The nonlinear term $(u-v)^+$ was suggested by McKenna while the nonlocal term $[\gamma - M\|u_x\|_{L^2(0,\pi)}]u_{xx}$ was first suggested by Woinowsky-Krieger in [92]. In 1941, Ammann et al. [7] considered the appearance of torsional oscillations in their report on the Tacoma bridge disaster. Thus, it appears the beam model have less degree of freedom to describe the actual phenomenon of the Tacoma bridge before it collapses. In this regard, Mckenna et al. [67] also studied general models that account for torsional oscillations, and established numerically the sudden transition from vertical to torsional oscillation. The well-posedness of another interesting model have been established in [29].

Recently, Gazzola [31] revisited the nonlinear structural behaviour of suspension bridges, motivated by the results in [32, 33] in which the following non-linear beam equation was studied

$$w_{xxxx}(x) - Tw_{xx}(x) + f(w(x)) = 0, \tag{1.17}$$

where w stands for the vertical displacement of the beam, T the tension and f a non-linear restoring force. The existence of solutions with wide oscillations which blows up in finite space was established for this equation. Also, the structural instability of suspension bridges have been investigated recently in a number of models, see [6, 9, 14]. In 2001, Scott [86] emphasized in his book that the correct way to model a suspension bridge is through a thin rectangular plate. But, suspension bridge models as a plate appear complicated and not much work has been done in this area.

A first attempt to model suspension bridges through a thin rectangular plate was by Ferrero and Gazzola in [30] where their fundamental work established the fourth-order hyperbolic equation (1.10). Recent result by Al-Gwaiz et al.[6] have proved the the existence of bending and stretching energies of the rectangular plate model suggested in [30]. For more details on suspension bridge plate models by Gazzola and collaborators, we refer the reader to [10, 13, 36, 37].

1.4.2 Nonlinear plate problems

In 2000, Messaoudi in [73] considered the so-called Petrovsky system

$$\begin{cases} v_{tt} + \Delta^2 v + a|v_t|^{m-2}v_t = b|v|^{p-2}v, & \text{in} \quad \Omega \times (0,T), \\ v = \frac{\partial v}{\partial \eta} = 0, & \text{on} \quad \partial\Omega \times [0,T), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.18}$$

where $a, b > 0$ are constants and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial\Omega$. For $p > m$, he proved the existence of a weak local solution. In addition, he showed that, for negative initial energy (that is $E(0) < 0$), the local solution blows up in finite time. He also established the existence of global solution when $m \geq p$. The result in [73] has been improved by Chen and Zhou [21], where global nonexistence and a finite time blow for varnishing initial energy (that is $E(0) = 0$) have been established. Yang [93] studied the following plate problem

$$\begin{cases} v_{tt} + \Delta^2 v + \delta v_t = \sum_{i=1}^{n} \alpha_i \left(\frac{\partial v}{\partial x_i}\right), & \text{in} \quad \Omega \times (0,T), \\ v = \frac{\partial v}{\partial \eta} = 0, & \text{on} \quad \partial\Omega \times [0,T), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.19}$$

$\delta \geq 0$ is a constant, $\alpha_i = \alpha_i(s)$, $i = 1, 2, ..., N$, are known nonlinear functions and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. He established the existence of global weak solution and exponential decay result under suitable conditions on the nonlinear terms and initial data. Guesmia in [39] considered

$$\begin{cases} v_{tt}(x,t) + \Delta^2 v(x,t) + q(x)v(x,t) + g(v_t(x,t)) = 0, & \text{in} \quad \Omega \times (0, +\infty) \\ v = \frac{\partial v}{\partial \eta} = 0, & \text{on} \quad \partial\Omega \times [0, \infty) \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.20}$$

where g is a continuous, increasing function with $g(0) = 0$, $q : \Omega \to \mathbb{R}^+$ is a bounded function and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with a smooth boundary $\partial\Omega$. He proved global existence and a decay (exponential and polynomial) result with some suitable assumptions on g. Similar result have been obtained in [40] when the system (1.20) is coupled with a similinear wave equation. The system (1.20) has been studied by Aassila and Guesmia in [1] when the term $q(x)v(x,t) + g(v_t(x,t))$ is replaced by $\Delta^2 v_t(x,t) + \Delta(g(\Delta v(x,t)))$.

In 2014, Wang [91] considered the following fourth-order plate equation

$$v_{tt} + \delta v_t + \Delta^2 v + av = |v|^{p-2}v, \tag{1.21}$$

where $a = a(x,y,t)$ is a bounded and measurable sign-changing function and $2 < p < \infty$. He then supplement (1.21) with the boundary conditions of (1.8) and initial data, and proved the existence and uniqueness of a weak local solution when $2 < p < \infty$ and a finite time blow up result. For the analysis of other interesting plate problems, we refer the reader to [26, 49, 50, 53, 55, 87].

1.4.3 Viscoelastic plate problems

For a review of plate problems with memory term, we begin with the work of Cavalcanti et al. [20] in which the following Von Karman system was considered

$$\begin{cases} u_{tt} + \Delta^2 u - [u,v] - g * \Delta^2 u = 0, & \text{in} \quad \Omega \times (0,\infty), \\ \Delta^2 v + [u,u] = 0, & \text{in} \quad \Omega \times (0,\infty), \\ u = v = 0, & \text{on} \quad \partial\Omega \times [0,\infty), \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0, & \text{on} \quad \partial\Omega \times [0,\infty), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.22}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$,

$$(g * w)(t) = \int_0^t g(t-s)w(s)ds,$$

and

$$[u,v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}.$$

They established the well-posedness and sharp decay estimates when the relaxation function g satisfies the assumption

$$g'(s) \leq H(g(s)), \forall s > 0, \tag{1.23}$$

where H is a known continuous, convex, increasing and positive function with $H(0) = 0$. We note here that the condition (1.23) was first introduced by

Alabau-Boussouira and Cannarsa [5] in 2009. Other interesting analysis of the von Karman plate equation with memory or boundary memory can be found in [12, 23, 24, 41, 52, 65, 76, 77, 84].

Alabau-Boussouira et al. [4] looked into the following problem

$$\begin{cases} v_{tt} + \Delta^2 v - \int_0^t g(t-s)\Delta^2 v(s)ds = f(v), & \text{in} \quad \Omega \times (0,T), \\ v = \frac{\partial v}{\partial \nu} = 0, & \text{on} \quad \partial\Omega \times [0,T), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.24}$$

and established exponential and polynomial decay results for sufficiently small initial data. Andrade et al. [8] and Liu et al. [60] considered the following system

$$\begin{cases} v_{tt} + \Delta^2 v - \Delta_p v - g * \Delta v - \Delta v_t + f(v) = 0, & \text{in} \quad \Omega \times (0,T), \\ v = \Delta v = 0, & \text{on} \quad \partial\Omega \times [0,T), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & \text{in} \quad \Omega, \end{cases} \tag{1.25}$$

where

$$\Delta_p v = div(|\nabla v|^{p-2}\nabla v)$$

is the usual $p-$Laplacian operator and

$$(g * \Delta v)(t) = \int_0^t g(t-s)\Delta v(s)ds.$$

By assuming that the relaxation kernel g decays exponentially, the authors proved the well-posedness and obtained an exponential decay result for the solution of problem (1.25). Rivera et al. in [75] studied the following fourth-order equation

$$v_{tt} + \gamma \Delta v_{tt} + \Delta^2 v - \int_0^t g(t-\tau)\Delta^2 v(\tau) d\tau = 0, \quad in \;\; \Omega \times (0,T) \tag{1.26}$$

together with initial and dynamical boundary conditions. They proved that the sum of the first and second energies decays exponentially (polynomially) if the kernel g decays exponentially (polynomially). Lin and Li [58] considered

$$v_{tt} + \gamma \Delta v_{tt} + \Delta^2 v - \int_0^t g(t-\tau)\Delta^2 v(\tau) d\tau = div\left(c(f(\nabla v)\nabla v)\right), \;\; in \;\; \Omega \times (0,T) \tag{1.27}$$

together with initial and dynamical boundary conditions similar to those imposed by Rivera et al. [75], and established a decay result, similar to that of Rivera et al.[75]. Mustafa and Ghassan [79] considered the Kirchoff plate equation

$$v_{tt} + \Delta^2 v = 0, \quad in \;\; \Omega \times (0, +\infty) \tag{1.28}$$

with viscoelastic boundary localized damping conditions and established a general decay result. Santos and Junior in [85] also considered (1.28) with memory boundary conditions and established that the solution decays exponentially (polynomially) if the relaxation function decays exponentially (polynomially) at the same rate. For similar result on equation (1.28), see the result of Horn and Lasiecka in [42]. For more results on the Kirchoff plate equation with memory, we refer the reader to [43, 51, 78].

1.5 Important Notations and inequalities

In this section, we state some notations and important inequalities that will be used repeatedly throughout this dissertation. The standard $L^2(\Omega)$, $H^2(\Omega)$ and $H^4(\Omega)$ Sobolev spaces will be used throughout this work.

- Suppose $\Omega \subset \mathbb{R}^N$ is a domain and $p \in \mathbb{R}$ such that $1 \leq p < \infty$; we set

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ } is \text{ } measurable \text{ } and \int_{\Omega} |f|^p < \infty \right\},$$

  with norm

$$\|f\|_p = \|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f|^p \right)^{\frac{1}{p}}.$$

- For $p = \infty$, we set

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \text{ } is \text{ } measurable \text{ } and \text{ } \exists C : \ |f(x)| \leq C \text{ } a.e \text{ } on \text{ } \Omega \},$$

  with norm

$$\|f\|_\infty = \|f\|_{L^\infty(\Omega)} = \inf \{ C > 0 : \ \ |f(x)| \leq C \text{ } a.e \text{ } on \text{ } \Omega \}.$$

- Let $m \geq 1$ be an integer and $1 \leq p < \infty$. we define

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \ \forall \ |\alpha| \leq m \},$$

with norm

$$\|u\|_{m,p} = \|u\|_{W^{m,p}(\Omega)} = \|u\|_p + \sum_{1\leq|\alpha|\leq m} \|D^\alpha u\|_p.$$

By $D^\alpha u$, we mean the $\alpha - th$ weak partial derivative g of the function u satisfying

$$\int_\Omega uD^\alpha\phi = (-1)^{|\alpha|}\int_\Omega g\phi, \quad \forall\ \phi \in C_0^\infty(\Omega).$$

When $p = 2$, we denote $W^{m,2}(\Omega) = H^m(\Omega)$

- $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N})$.
- $\Delta f = \frac{\partial^2 f}{\partial x_1 x_1} + \frac{\partial^2 f}{\partial x_2 x_2} + ... + \frac{\partial^2 f}{\partial x_N x_N}$.
- $\Delta^2 f = \displaystyle\sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\partial^2}{\partial x_j x_j}\left(\frac{\partial^2 f}{\partial x_i x_i}\right)$.
- $v_t = \frac{\partial v}{\partial t},\ v_{tt} = \frac{\partial^2 v}{\partial tt}$.
- $C_0^\infty(\Omega)$ denotes the space of continuously differentiable functions with compact support in Ω.
- Let $f \in C^k(\Omega)$ (space of $k-$ times continuously differentiable functions) and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ is a multi-index of length $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_N$, $|\alpha| \leq k$, then

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_N^{\alpha_N}}$$

- The support of f(continuous functions) is defined by

$$supp(f) := \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

Throughout this work, c and/or C (sometimes c_i and C_i, $i = 0, 1, 2, ...$) are generic constants that may change from line to line.

The following inequalities will be use repeatedly in this dissertation.

1. **Young's inequality.** Let $1 < r, s < \infty$ and $A, B > 0$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Then for all $\epsilon > 0$, we have

$$AB \leq \epsilon A^r + C_\epsilon B^s,$$

where $C_\epsilon = \frac{1}{s(\epsilon r)^{\frac{s}{r}}}$. In particular, when $r = s = 2$, we obtain

$$AB \leq \epsilon A^2 + \frac{1}{4\epsilon} B^2.$$

2. **Hölder's inequality.** Let $u \in L^r(\Omega)$, $v \in L^s(\Omega)$ and $1 \leq r, s \leq \infty$ such that $\frac{1}{r} + \frac{1}{s} = 1$. Then $uv \in L^1(\Omega)$ and

$$\int_\Omega uv \leq \|u\|_{L^r(\Omega)} \|v\|_{L^s(\Omega)}. \tag{1.29}$$

In the inequality (1.29), for $r = s = 2$, we obtain the **Cauchy-Schwarz inequality.**

3. **Green's formula.** Suppose Ω is a bounded domain with a regular boundary and let $v \in H^1(\Omega)$ and $u \in H^2(\Omega)$. Then we have

$$\int_\Omega v\Delta u = \int_{\partial\Omega} v\frac{\partial u}{\partial \eta} - \int_\Omega \nabla v \nabla u,$$

where $\frac{\partial u}{\partial \eta}$ is the normal derivative of u on the boundary $\partial\Omega$ of Ω.

4. **Jensen's Inequality.** Let φ be a convex function on $\mathbb{R}$, f an integrable function over Ω and $\varphi o f$ integrable over Ω also. Then

$$\varphi\left(\int_\Omega f\right) \leq \int_\Omega \varphi(f)$$

5. We say that two functions or functionals F and L are equivalent and denote it by $F \sim L$, if there exist two positive numbers α_1 and α_2 such that

$$\alpha_1 F(t) \leq L(t) \leq \alpha_2 F(t), \ \forall t \geq 0.$$

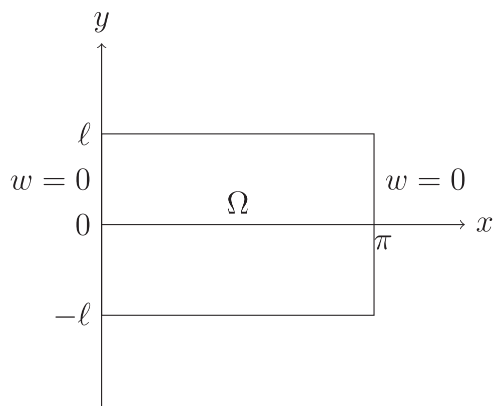
6. Let H be a Hilbert space equipped with the inner product $(,)$. A linear operator $A : D(A) \subset H \to H$ is said to be maximal monotone iff

   - For all $u \in D(A)$, we have $(Au, u)_H \geq 0$
   - For all $f \in H$ there exists a unique $u \in D(A)$ such that $(I + A)u = f$

1.6 Some useful materials

Throughout this work, $\Omega = (0,\pi)\times(-\ell,\ell)$. Let us introduce as in [30] the following Sobolev space

$$H^2_*(\Omega) = \left\{ w \in H^2(\Omega) : w(0,y) = w(\pi,y) = 0,\ \forall y \in (-\ell,\ell) \right\}, \tag{1.30}$$



together with the inner product

$$(u,v)_{H^2_*} = \int_\Omega [(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]dxdy, \tag{1.31}$$

and denote by

$$\mathcal{H}^*(\Omega) = dual\ \ of\ \ H^2_*(\Omega).$$

For the completeness of the space $H^2_*(\Omega)$, we recall some results by Ferrero and Gazzola in [30].

Lemma 1.1 [*30*] *Assume* $0 < \sigma < \frac{1}{2}$. *Then, the norm* $\|.\|_{H^2_*(\Omega)}$ *given by* $\|.\|^2_{H^2_*(\Omega)} = (.,.)_{H^2_*}$ *is equivalent to the usual* $H^2(\Omega)$*-norm. Moreover,* $H^2_*(\Omega)$ *endowed with the scalar product* $(.,.)_{H^2_*}$ *is a Hilbert space.*

Theorem 1.1 *[30] Assume* $0 < \sigma < \frac{1}{2}$ *and let* $f \in L^2(\Omega)$*. Then, there exists a unique* $u \in H^2_*(\Omega)$ *such that*

$$\int_\Omega [\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]dxdy = \int_\Omega fv, \quad \forall v \in H^2_*(\Omega). \tag{1.32}$$

Furthermore, the weak solution $u \in H^2_*(\Omega)$*, of* (1.32)*, is in* $H^4(\Omega)$ *and there exists* $C = C(\ell, \sigma) > 0$ *such that*

$$\|u\|_{H^4(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{1.33}$$

Remark: The function $u \in H^2_*(\Omega)$ satisfying (1.32) is called the weak solution of the stationary problem

$$\begin{cases} \Delta^2 u = f, \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0. \end{cases} \tag{1.34}$$

We also recall the following important Lemmas that will be used in this work.

Lemma 1.2 *Suppose* $u \in H^4(\Omega)$ *satisfying the boundary conditions of* (1.34) *and* $v \in H^2_*(\Omega)$*. Then,*

$$\int_\Omega v\Delta^2 u = (u,v)_{H^2_*(\Omega)}. \tag{1.35}$$

Proof. Using Green's formula we obtain

$$\int_\Omega v\Delta^2 u = \int_\Omega \Delta u \Delta v + \int_{\partial\Omega} [v\frac{\partial \Delta u}{\partial \eta} - \Delta u \frac{\partial v}{\partial \eta}]. \tag{1.36}$$

Integrating the left hand side of (1.35) leads to

$$\begin{aligned}
\int_\Omega v\Delta^2 u \;\; = \;\; & \int_\Omega \Delta u \Delta v - \int_0^\pi v(x,-\ell)[u_{xxy}(x,-\ell) + u_{yyy}(x,-\ell)]dx \\
+ \;\; & \int_0^\pi v(x,\ell)[u_{xxy}(x,\ell) + u_{yyy}(x,\ell)]dx \\
+ \;\; & \int_0^\pi v_y(x,-\ell)[u_{xx}(x,-\ell) + u_{yy}(x,-\ell)]dx \\
- \;\; & \int_{-\ell}^\ell v_x(\pi,y)[\cancelto{0}{u_{xx}(\pi,y)} + \cancelto{0}{u_{yy}(\pi,y)}]dy \\
- \;\; & \int_0^\pi v_y(x,\ell)[u_{xx}(x,\ell) + u_{yy}(x,\ell)]dx \\
+ \;\; & \int_{-\ell}^\ell v_x(0,y)[\cancelto{0}{u_{xx}(0,y)} + \cancelto{0}{u_{yy}(0,y)}]dy].
\end{aligned}$$

This gives

$$\begin{aligned}
\int_\Omega v\Delta^2 u \;\; = \;\; & \int_\Omega \Delta u \Delta v - \int_0^\pi v(x,-\ell)[u_{xxy}(x,-\ell) + u_{yyy}(x,-\ell)]dx \\
+ \;\; & \int_0^\pi v(x,\ell)[u_{xxy}(x,\ell) + u_{yyy}(x,\ell)]dx \\
+ \;\; & \int_0^\pi v_y(x,-\ell)[u_{xx}(x,-\ell) + u_{yy}(x,-\ell)]dx \\
- \;\; & \int_0^\pi v_y(x,\ell)[u_{xx}(x,\ell) + u_{yy}(x,\ell)]dx. \qquad (1.37)
\end{aligned}$$

By using the boundary conditions of (1.34), we obtain

$$\begin{aligned}\int_\Omega v\Delta^2 u &= \int_\Omega \Delta u \Delta v + (1-\sigma)\int_0^\pi [v(x,-\ell)u_{xxy}(x,-\ell) - v(x,\ell)u_{xxy}(x,\ell)]dx \\ &+ (1-\sigma)\int_0^\pi [v_y(x,-\ell)u_{xx}(x,-\ell) - v_y(x,\ell)u_{xx}(x,\ell)]dx. \qquad (1.38)\end{aligned}$$

By performing similar integration by part on the right hand side of (1.31), we obtain (1.38). Hence the result. ∎

Lemma 1.3 *[91] Let* $u \in H^2_*(\Omega)$ *and assume* $1 \le p < +\infty$*. Then, there exists a positive constant* $C_e = C_e(\Omega, p) > 0$ *such that*

$$\|u\|_p^p \le C_e \|u\|^p_{H^2_*(\Omega)}. \qquad (1.39)$$

Lemma 1.4 (Compacity Lemma [59]) *Let* V, X *and* W *be Banach spaces such that* $V \subset X \subset W$, V *and* W *are reflexive and the embbeding* $V \subset W$ *is compact. If* (u_m) *is bounded in* $L^p((0,T);V)$ *and* (u_t^m) *is bounded in* $L^q((0,T);W)$, $1 < p, q < +\infty$*. Then there exists a subsequence* (u^l) *such that*

$$(u^l) \longrightarrow u \quad \textit{strongly} \quad \textit{in} \quad L^p((0,T);X)$$

Lemma 1.5 (Lions [59]) *Let* $V \subset H \subset V^*$*, with* V *a reflexive Banach space,* H *a Hilbert space and* V^* *the dual of* V *such that the embeddings are continuous.*

If $u \in L^p\left((0,T),V\right)$ *and* $u_t \in L^q\left((0,T),V^*\right)$, *where* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$u \in C([0,T],H).$$

Lemma 1.6 *Suppose* X *and* Y *are two Banach spaces with* X *continuously embedded in* Y. *If* $u \in L^\infty((0,T),X)$ *and* $u \in C_w([0,T],Y)$, *then* $u \in C_w([0,T],X)$.

Lemma 1.7 *For any* $a,b \in \mathbb{R}$ *and* $p \geq 2$, *we have*

$$(|a|^{p-2}a - |b|^{p-2}b)(a-b) \geq 2^{2-p}|a-b|^p. \tag{1.40}$$

Lemma 1.8 (Komornik [46]) *Let* $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *be non-increasing function. Assume that there exist* $\gamma > 0$ *and* $\lambda > 0$ *such that*

$$\int_s^\infty E^{1+\gamma}(t)dt \leq \lambda E(s), 0 \leq s < \infty.$$

Then, there exists $C,\omega > 0$ *such that*

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{\gamma}}},\ \forall t \geq 0,\ if\ \ \gamma > 0,$$

$$E(t) \leq Ce^{-\omega t}, \forall t \geq 0,\ if\ \ \gamma = 0.$$

Publications

In this dissertation, we published/submitted the following results

1. Salim A. Messaoudi and Soh Edwin Mukiawa, Existence and decay of solutions to a viscoelastic plate equation, Electronic Journal of Differential Equation, Vol. 2016 (2016), No. 22, pp. 1-14.

2. Salim A. Messaoudi and Soh E. Mukiawa, A suspension bridge problem: A semi-linear model, Springer's Proceedings of Mathematics and Statistics, Accepted.

3. Salim A. Messaoudi, Soh E. Mukiawa and Enyi D. Cyril, Finite dimensional global attractor for a suspension bridge problem with delay, Comptes rendus Mathematique, Accepted.

4. Salim A. Messaoudi and Soh Edwin Mukiawa , Existence and General decay rates for a fourth-order damped plate problem, Applicable Analysis, Submitted.

5. Salim A. Messaoudi , Ahmed Bonfoh, Soh E. Mukiawa and Cyril D. Enyi, The global attractor for a suspension bridge with memory and partially hinged boundary conditions, Journal of Nonlinear Analysis, Submitted.

CHAPTER 2

EXISTENCE AND STABILITY RESULTS FOR A FOURTH-ORDER NONLINEAR WAVE PROBLEM

This chapter is devoted to the global existence and stability of the following fourth-order nonlinear wave problem

$$\begin{cases} u_{tt} + \delta|u_t|^{p-2}u_t + \Delta^2 u + h(x,y,u) = f, & \text{in } \Omega \times (0,T) \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for } (y,t) \in (-\ell,\ell) \times (0,T) \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for } (y,t) \in (-\ell,\ell) \times (0,T) \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for } (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0, & \text{for} (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $u = u(x, y, t)$ stands for the vertical displacement in the downward direction of the suspension bridge, $\Omega = (0, \pi) \times (-l, l) \subset \mathbb{R}^2$, $2 < p < \infty$, $0 < \sigma < \frac{1}{2}$ is the Poission ratio, h is the hanger restoring force, $f = f(x, y, t)$ is the forcing term and $\delta |u_t|^{p-2} u_t$ is the nonlinear frictional damping with a damping constant $\delta > 0$. This chapter is divided into the following sections. In section 2.1, we present some assumptions to be used in this chapter. In section 2.2, we establish the well-posedness of problem (2.1). Finally, in section 2.3, we state and prove a stability result for problem (2.1) in the absence of the external load, that is when $f \equiv 0$.

2.1 Main Assumptions

In this section, we present some useful assumptions needed in this chapter. For this, we assume $f \in L^2(\Omega \times (0, T))$ and the function h admits the following hypotheses:

(A_1) $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is nondecreasing and locally Lipschitz with respect to s such that $h(.,.,0) = 0$.

(A_2) $H(x, y, s) = \displaystyle\int_0^s h(x, y, \tau) d\tau$ is positive such that

$$sh(.,.,s) - H(.,.,s) \geq 0, \ \forall s \in \mathbb{R}.$$

Examples: An example of a function that satisfy assumptions (A_1) and (A_2) is

$$h(s) = |s|^{p-2} s, \ p \geq 2.$$

1. To prove our existence and uniqueness result, we exploit the method of Lions [59] with modifications to suit our problem.

2.2 Well posedness

In this section, we state and prove a well-posedness result for problem (2.1). First, we give the definition of a weak solution of problem (2.1).

Definition 2.1 *A function*

$$u \in C([0,T], H_*^2(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], \mathcal{H}^*(\Omega))$$

and

$$u_t \in L^p(\Omega \times (0,T))$$

is called a weak solution of (2.1), *if*

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u_t w + \delta \int_\Omega |u_t|^{p-2} u_t w + (u,w)_{H_*^2} + \int_\Omega h(x,y,u) w = \int_\Omega f w, \quad \forall w \in H_*^2(\Omega) \\ u(x,y,0) = u_0(x,y), \ u_t(x,y,0) = u_1(x,y), \\ for \ a.e \ t \in (0,T). \end{cases} \tag{2.2}$$

Now, we give an existence and uniqueness result for problem (2.1).

Theorem 2.1 *Suppose* $f \in L^2(\Omega \times (0,T))$ *and the assumptions* (A_1) *and* (A_2) *hold. Let* $(u_0, u_1) \in H_*^2(\Omega) \times L^2(\Omega)$ *be given. Then problem* (2.1) *has a unique global weak solution.*

Proof. We exploit the Galerkin approximation method. Note that $H_*^2(\Omega)$ is separable, so it has a countable basis. Let $\{w_j\}_{j=1}^{\infty}$ be a basis of $H_*^2(\Omega)$ and define the finite-dimensional subspace

$$W_m = span\{w_1, ..., w_m\}.$$

Let

$$u_0^m(x,y) = \sum_{j=1}^{m} \alpha_j w_j(x,y) \quad and \quad u_1^m(x,y) = \sum_{j=1}^{m} \beta_j w_j(x,y)$$

be sequences in $H_*^2(\Omega)$ and $L^2(\Omega)$ respectively, such that

$$u_0^m \longrightarrow u_0 \;\; in \;\; H_*^2(\Omega), \quad u_1^m \longrightarrow u_1 \;\; in \;\; L^2(\Omega). \tag{2.3}$$

We look for a solution of the form

$$u^m(x,y,t) = \sum_{j=1}^{m} a_j(t) w_j(x,y)$$

which satisfies the following approximation problem

$$\begin{cases} \displaystyle\int_\Omega u_{tt}^m w_j + \delta \int_\Omega |u_t^m|^{p-2} u_t^m w_j + (u^m, w_j)_{H_*^2} + \int_\Omega h(x,y,u^m) w_j = \int_\Omega f w_j, \\ \qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2.., m \\ u^m(x,y,0) = u_0^m(x,y), \;\; u_t^m(x,y,0) = u_1^m(x,y), \\ for \;\; a.e \;\; t \in (0,T). \end{cases} \tag{2.4}$$

We note that (2.4) is equivalent to a finite-dimensional system of ODE, thus standard ODE theory guarantees the existence of functions $a_j, \;\; j = 1, 2, ..., m$

such that

$$a_j : [0, t_m) \to \mathbb{R},\ 0 < t_m \leq T,$$

for which (2.4) is satisfied for a.e $0 < t < t_m$.

Next, we show that $t_m = T, \forall\, m \geq 1$. For this, we multiply (2.4) by a_j' and sum over $j = 1, 2, ..., m$ to get

$$\int_\Omega u_{tt}^m u_t^m + \delta \int_\Omega |u_t^m|^p + (u^m, u_t^m)_{H_*^2} + \int_\Omega h(x, y, u^m) u_t^m = \int_\Omega f u_t^m.$$

This gives

$$\frac{d}{dt}\left[\frac{1}{2}\int_\Omega |u_t^m|^2 + \frac{1}{2}\|u^m\|_{H_*^2}^2 + \int_\Omega H(x, y, u^m)\right] + \delta \int_\Omega |u_t^m|^p = \int_\Omega f u_t^m. \tag{2.5}$$

Integrating equation (2.5) over $(0, t),\ 0 < t < t_m$, we obtain

$$E^m(t) + \delta \int_0^t \int_\Omega |u_t^m|^p = E^m(0) + \int_0^t \int_\Omega f u_t^m, \tag{2.6}$$

where

$$E^m(t) = \frac{1}{2}\int_\Omega |u_t^m|^2 + \frac{1}{2}\|u^m\|_{H_*^2}^2 + \int_\Omega H(x, y, u^m). \tag{2.7}$$

Using Young's inequality, we get, $\forall\ t \in (0, t_m)$,

$$E^m(t) + \delta \int_0^t \int_\Omega |u_t^m|^p \leq E^m(0) + \epsilon \int_0^t \int_\Omega |u_t^m|^2 + C_\epsilon \int_0^T \int_\Omega |f|^2. \tag{2.8}$$

We know that

$$H(x,y,s)=\int_0^s h(x,y,\tau)d\tau.$$

Thus, on the account of assumption (A_2) and Lemma 1.3, we get

$$\int_\Omega |H(x,y,u_0^m)|dxdy \leqslant \int_\Omega |u_0^m(x,y)||h(x,y,u_0^m)|dxdy \le C\|u_0^m\|^2_{H^2_*(\Omega)}. \quad (2.9)$$

Now, combining (2.8) and (2.9), with the fact that (u_0^m) and (u_1^m) are bounded sequences (as convergent sequences) in $H^2_*(\Omega)$ and $L^2(\Omega)$ respectively , we obtain

$$\frac{1}{2}\int_\Omega |u_t^m|^2+\frac{1}{2}\|u^m\|^2_{H^2_*}+\int_\Omega H(x,y,u^m)+\delta\int_0^t\int_\Omega |u_t^m|^p \leqslant C_\epsilon+\epsilon T\sup_{(0,\ t_m)}\int_\Omega |u_t^m|^2. \quad (2.10)$$

Observe that each term in (2.10) is positive , so we can take supremum over $t\in(0,t_m)$ to get

$$\frac{1}{2}\sup_{(0,t_m)}\int_\Omega |u_t^m|^2+\frac{1}{2}\sup_{(0,t_m)}\|u^m\|^2_{H^2_*}+\sup_{(0,t_m)}\int_\Omega H(x,y,u^m) + \delta\int_0^T\int_\Omega |u_t^m|^p \le 4C_\epsilon+4\epsilon T\sup_{(0,t_m)}\int_\Omega |u_t^m|^2.$$

We choose $\epsilon=\frac{1}{16T}$ and obtain

$$\frac{1}{4}\sup_{(0,\ t_m)}\int_\Omega |u_t^m|^2+\frac{1}{2}\sup_{(0,\ t_m)}\|u^m\|^2_{H^2_*} + \sup_{(0,\ t_m)}\int_\Omega H(x,y,u^m)+\delta\int_0^T\int_\Omega |u_t^m|^p \leqslant C, \quad (2.11)$$

where C is a constant independent of m. So, we can extend the interval of solution of (2.4) to $[0,T)$. Moreover, we deduce from (2.11) that

$$\begin{cases} (u^m) \text{ is bounded in } L^\infty\left((0,T), H_*^2(\Omega)\right), \\ (u_t^m) \text{ is bounded in } L^\infty\left((0,T), L^2(\Omega)\right) \cap L^p\left(\Omega \times (0,T)\right). \end{cases} \tag{2.12}$$

Therefore, we can extract a subsequence (u^k) of (u^m) such that

$$\begin{cases} u^k \rightharpoonup u \text{ weakly star in } L^\infty\left((0,T), H_*^2(\Omega)\right) \text{ and weakly in } L^2\left((0,T), H_*^2(\Omega)\right), \\ u_t^k \rightharpoonup u_t \text{ weakly star in } L^\infty\left((0,T), L^2(\Omega)\right) \text{ and weakly in } L^2\left((0,T), L^2(\Omega)\right), \\ u_t^k \rightharpoonup u_t \text{ weakly in } L^p\left((0,T), L^p(\Omega)\right). \end{cases} \tag{2.13}$$

Since u_t^k is bounded in $L^p\left((\Omega \times (0,T))\right)$, we have that

$$\int_0^T \int_\Omega ||u_t^k|^{p-2}u_t^k|^{\frac{p}{p-1}} = \int_0^T \int_\Omega ||u_t^k|^{p-1}|^{\frac{p}{p-1}} = \int_0^T \int_\Omega |u_t^k|^p \leqslant C.$$

So,

$$(|u_t^k|^{p-2}u_t^k) \text{ is a bounded sequence in } L^{\frac{p}{p-1}}(\Omega \times (0,T)).$$

Thus, we can extract a subsequence denoted by $(|u_t^l|^{p-2}u_t^l)$ such that

$$|u_t^l|^{p-2}u_t^l \rightharpoonup \psi \text{ weakly in } L^{\frac{p}{p-1}}(\Omega \times (0,T)). \tag{2.14}$$

Now, we have that

$$\begin{cases} u^k \rightharpoonup u \text{ weakly in } L^2\left((0,T), H_*^2(\Omega)\right), \\ u_t^k \rightharpoonup u_t \text{ weakly in } L^2\left((0,T), L^2(\Omega)\right). \end{cases} \tag{2.15}$$

Hence due to the compact embedding of $H_*^2(\Omega)$ in $L^2(\Omega)$ and Lemma 1.4, we can extract a subsequence, still denoted by (u^l) such that

$$u^l \to u \text{ strongly in } L^2\left((0,T), L^2(\Omega)\right) \tag{2.16}$$

and

$$u^l \longrightarrow u \text{ a.e in } \Omega \times (0,T).$$

Thus, using the lipschitz continuity of h, we obtain

$$h(x,y,u^l(x,y,t)) \longrightarrow h(x,y,u(x,y,t)) \;\; a.e \; in \; \Omega \times (0,T).$$

Now, we replace (u^m) by (u^l) in (2.4) and integrate over $(0,t)$ to get

$$\begin{aligned} &\int_\Omega u_t^l w_j + \delta \int_0^t \int_\Omega |u_t^l|^{p-2} u_t^l w_j + \int_0^t (u^l, w_j)_{H_*^2} \\ &+ \int_0^t \int_\Omega h(x,y,u^l) w_j = \int_\Omega u_1^l w_j + \int_0^t \int_\Omega f w_j, \; \forall j \leqslant l. \end{aligned} \tag{2.17}$$

By using (2.12) and the embedding of $H_*^2(\Omega)$ in $L^\infty(\Omega)$ (since $\Omega \subset \mathbb{R}^2$), we get that (u^l) is bounded in $L^\infty(\Omega \times (0,T))$. Thus, there exists a constant $C > 0$ such that

$$\|u^l\|_{L^\infty(\Omega\times(0,T))} \leq C,\ \forall\ l \geq 1. \tag{2.18}$$

It follows from (A_1), Hölder's inequality and (2.16) that

$$\begin{aligned}
&\left|\int_0^t\int_\Omega h(x,y,u^l)w_j - \int_0^t\int_\Omega h(x,y,u)w_j\right| \\
&\leq \int_0^t\int_\Omega |h(x,y,u^l) - h(x,y,u)||w_j| \\
&\leq C\|w_j\|_{L^2(\Omega)}\int_0^t \|u^l - u\|_{L^2(\Omega)} \\
&\leq CT \sup_{t\in(0,T)} \|u^l - u\|_{L^2(\Omega)} \to 0,\ as\ l \to +\infty.
\end{aligned} \tag{2.19}$$

By letting $l \longrightarrow +\infty$ in equation (2.17), we obtain

$$\begin{aligned}
&\int_\Omega u_t w_j + \delta\int_0^t\int_\Omega \psi w_j + \int_0^t (u, w_j)_{H_*^2} \\
&+ \int_0^t\int_\Omega h(x,y,u)w_j = \int_\Omega u_1 w_j + \int_0^t\int_\Omega f w_j,\ \forall j \geqslant 1,
\end{aligned}$$

which gives

$$\begin{aligned}
\int_\Omega u_t w \;&=\; -\delta\int_0^t\int_\Omega \psi w - \int_0^t (u,w)_{H_*^2} - \int_0^t\int_\Omega h(x,y,u)w \\
&+\; \int_\Omega u_1 w + \int_0^t\int_\Omega f w,\ \forall w \in H_*^2(\Omega).
\end{aligned} \tag{2.20}$$

Observe that each term on the right- hand side of (2.20) is absolutely continuous as functions of t defined by integrals over $(0,t)$, hence differentiable almost everywhere. Thus, differentiating equation (2.20) for almost every $t \in (0,T)$, we obtain

$$\frac{d}{dt}\int_\Omega u_t w + \delta\int_\Omega \psi w + (u,w)_{H_*^2} + \int_\Omega h(x,y,u)w = \int_\Omega fw,\ \forall w \in H_*^2(\Omega). \quad (2.21)$$

Next, we show that

$$|u_t|^{p-2}u_t = \psi.$$

Let $A(w) = |w|^{p-2}w$ and define χ^l as follows

$$\chi^l = \int_0^T\int_\Omega (A(u_t^l) - A(w))(u_t^l - w),\ \ \forall w \in L^p((0,T), H_*^2(\Omega)). \quad (2.22)$$

Then $\chi^l \geqslant 0$ by Lemma 2.4. We have that

$$\begin{aligned} \chi^l &= \int_0^T\int_\Omega A(u_t^l)u_t^l - \int_0^T\int_\Omega A(u_t^l)w - \int_0^T\int_\Omega A(w)(u_t^l - w) \\ &= \int_0^T\int_\Omega |u_t^l|^p - \int_0^T\int_\Omega A(u_t^l)w - \int_0^T\int_\Omega A(w)(u_t^l - w) \quad (2.23) \end{aligned}$$

By replacing (u^m) by (u^l) in (2.5), integrating over $(0,T)$ and replacing in (2.23), we get

$$\begin{aligned} \chi^l &= -\frac{1}{2\delta}[\int_\Omega |u_t^l|^2 + \|u^l\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u^l)] \\ &+ \frac{1}{2\delta}[\int_\Omega |u_1^l|^2 + \|u_0^l\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u_0^l)] \\ &+ \frac{1}{\delta}\int_0^T\int_\Omega fu_t^l - \int_0^T\int_\Omega A(u_t^l)w - \int_0^T\int_\Omega A(w)(u_t^l - w) \quad (2.24) \end{aligned}$$

Now, we know that, if $x_n \rightharpoonup x$ in X (X a Banach space), then

$$\|x\|_X \leq \liminf \|x_n\|_X. \tag{2.25}$$

Also, using the same reasoning as in (2.18)-(2.19), Hölder's inequality and the fact that $u_0^l \to u_0$ strongly in $H_*^2(\Omega)$, we have (Mean Value Theorem)

$$\begin{aligned} &\left| \int_\Omega (H(.,.,u_0^l) - H(.,.,u_0)) \right| \leq \int_\Omega |h(x,y,\tilde{u_0})||u_0^l - u_0| \\ &\leq C \int_\Omega |\tilde{u_0}||u_0^l - u_0| \leq \tilde{C}\|u_0^l - u_0\|_{H_*^2(\Omega)} \to 0,\ as\ l \to \infty, \end{aligned} \tag{2.26}$$

where $\tilde{u_0} = \alpha u_0^l + (1-\alpha)u_0,\ \alpha \in (0,1)$. Similarly, using (2.16) we can show that

$$\int_\Omega H(.,.,u^l) \to \int_\Omega H(.,.,u) \quad as \quad l \to \infty. \tag{2.27}$$

Combining (2.25)-(2.27) and letting $l \to +\infty$, we obtain

$$
\begin{aligned}
0 &\leq \limsup \chi^l \\
&\leq \limsup -\frac{1}{2\delta}[\int_\Omega |u_t^l|^2 + \|u^l\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u^l)] \\
&+ \limsup \frac{1}{2\delta}[\int_\Omega |u_1^l|^2 + \|u_0^l\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u_0^l)] \\
&+ \limsup \frac{1}{\delta}\int_0^T\int_\Omega fu_t^l - \int_0^T\int_\Omega A(u_t^l)w - \int_0^T\int_\Omega A(w)(u_t^l - w) \\
&\leq -\liminf \frac{1}{2\delta}[\int_\Omega |u_t^l|^2 + \|u^l\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u^l)] \\
&+ \frac{1}{2\delta}[\int_\Omega |u_1|^2 + \|u_0\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u_0)] \\
&+ \frac{1}{\delta}\int_0^T\int_\Omega fu_t - \int_0^T\int_\Omega A(u_t)w - \int_0^T\int_\Omega A(w)(u_t - w) \\
&\leq -\frac{1}{2\delta}[\int_\Omega |u_t|^2 + \|u\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u)] \\
&+ \frac{1}{2\delta}[\int_\Omega |u_1|^2 + \|u_0\|_{H_*^2}^2 + 2\int_\Omega H(.,.,u_0)] \\
&+ \frac{1}{\delta}\int_0^T\int_\Omega fu_t - \int_0^T\int_\Omega \psi w - \int_0^T\int_\Omega A(w)(u_t - w) \qquad (2.28)
\end{aligned}
$$

Replacing w by u_t in (2.21) and integrating over $(0,T)$ leads to

$$
\begin{aligned}
\frac{1}{\delta}\int_0^T\int_\Omega fu_t &= \frac{1}{2\delta}[\int_\Omega |u_t(.,.,T)|^2 + \|u(.,.,T)\|_{H_*^2}^2] \\
&- \frac{1}{2\delta}[\int_\Omega |u_1(.,.,T)|^2 + \|u_0\|_{H_*^2}^2] \\
&+ \int_0^T\int_\Omega \psi u_t + \frac{1}{\delta}[\int_\Omega (H(.,.,u(.,.,T)) - H(.,.,u_0))] \qquad (2.29)
\end{aligned}
$$

Thus, substituting (2.29) into (2.28), we obtain

$$\begin{aligned} 0 &\le \limsup \chi^l \\ &\le \int_0^T \int_\Omega \psi u_t - \int_0^T \int_\Omega \psi w - \int_0^T \int_\Omega A(w)(u_t - w) \\ &= \int_0^T \int_\Omega (\psi - A(w))(u_t - w) \end{aligned}$$

So,

$$\int_0^T \int_\Omega (\psi - A(w))(u_t - w) \ge 0, \ \ \forall w \in L^p((0,T), H^2_*(\Omega)).$$

Now, let $w = \lambda v + u_t$ for $\lambda \neq 0$ and $v \in L^p(\Omega \times (0,T))$ (by density of $H^2_*(\Omega)$ in $L^p(\Omega)$). Then we get

$$-\lambda \int_0^T \int_\Omega (\psi - A(\lambda v + u_t))v \ge 0, \ \forall \lambda \neq 0, \ \forall v \in L^p(\Omega \times (0,T)).$$

For $\lambda > 0$, we get

$$\int_0^T \int_\Omega (\psi - A(\lambda v + u_t))v \le 0. \tag{2.30}$$

For $\lambda < 0$, we obtain

$$\int_0^T \int_\Omega (\psi - A(\lambda v + u_t))v \ge 0. \tag{2.31}$$

Claim: As $\lambda \searrow 0$, we obtain

$$\int_0^T \int_\Omega (\psi - A(u_t))v \le 0, \ \forall \ v \in L^p(\Omega \times (0,T)). \tag{2.32}$$

To see this, we let (λ_n) be any sequence such that $\lambda_n \searrow 0$ as $n \to \infty$ and consider the sequence of functions $J(\lambda_n) = A(\lambda_n w + u_t)$. By using the continuity of A, $J(\lambda_n) \to A(u_t)$ as $n \to \infty$ (pointwise). Also, since (λ_n) is bounded (as a convergent sequence), we have

$$\begin{aligned} |J(\lambda_n)| &= |\lambda_n w + u_t|^{p-1}|v| \leq 2^{p-2}(|\lambda_n|^{p-1}|w|^{p-1} + |u_t|^{p-1})|v| \\ &\leq C(|w|^{p-1} + |u_t|^{p-1})|v| \in L^1(\Omega \times (0,T)). \end{aligned}$$

By applying dominated convergent Theorem to (2.30), we obtain the claim.

Similarly, as $\lambda \nearrow 0$, we get

$$\int_0^T \int_\Omega (\psi - A(u_t))v \geq 0. \tag{2.33}$$

Combining (2.32) and (2.33), it follows that

$$\psi = A(u_t) = |u_t|^{p-2}u_t.$$

Therefore equation (2.21) becomes

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u_t w + \delta \int_\Omega |u_t|^{p-2}u_t w + (u,w)_{H^2_*} \\ &\qquad + \int_\Omega h(x,y,u)w = \int_\Omega fw, \ \forall w \in L^p((0,T), H^2_*(\Omega)). \end{aligned} \tag{2.34}$$

Now, from (2.13), we have that

$$\begin{cases} u^l \rightharpoonup u \ weakly \ in \ L^2\left((0,T), H^2_*(\Omega)\right), \\ u^l_t \rightharpoonup u_t \ weakly \ in \ L^2\left((0,T), L^2(\Omega)\right). \end{cases} \tag{2.35}$$

Thus by Lions' Lemma (i.e Lemma 1.5), we conclude that, up to a subsequence,

$$u^l \to u \ in \ C([0,T), L^2(\Omega)). \tag{2.36}$$

Now, as in [30] and [64], let $\phi \in C_0^\infty(0,T)$ and replace u^m by u^l in (2.4) to get for any $j \leq l$

$$\begin{aligned} -\int_0^T \left(u^l_t(t), w_j\right)_{L^2(\Omega)} \phi'(t)dt &= -\delta \int_0^T \left(|u^l_t(t)|^{p-2} u^l_t(t), w_j\right)_{L^2(\Omega)} \phi(t)dt \\ &- \int_0^T \left(u^l(t), w_j\right)_{H^2_*(\Omega)} \phi(t)dt \\ &- \int_0^T \left(h(.,.,u^l(t)), w_j\right)_{L^2(\Omega)} \phi(t)dt \\ &+ \int_0^T \left(f(t), w_j\right)_{L^2(\Omega)} \phi(t)dt. \end{aligned} \tag{2.37}$$

On the account of (2.19) and letting $l \longrightarrow +\infty$, we obtain that

$$\begin{aligned} -\int_0^T \left(u_t(t), w_j\right)_{L^2(\Omega)} \phi'(t)dt &= -\delta \int_0^T \left(|u_t(t)|^{p-2} u_t(t), w_j\right)_{L^2(\Omega)} \phi(t)dt \\ &- \int_0^T \left(u(t), w_j\right)_{H^2_*(\Omega)} \phi(t)dt \\ &- \int_0^T \left(h(.,.,u(t)), w_j\right)_{L^2(\Omega)} \phi(t)dt \\ &+ \int_0^T \left(f(t), w_j\right)_{L^2(\Omega)} \phi(t)dt, \ \forall j \geq 1, \end{aligned}$$

which gives

$$
\begin{aligned}
-\int_0^T (u_t(t), w)_{L^2(\Omega)}\, \phi'(t)dt \;&=\; -\delta \int_0^T \left(|u_t(t)|^{p-2} u_t(t), w\right)_{L^2(\Omega)} \phi(t)dt \\
&-\; \int_0^T (u(t), w)_{H^2_*(\Omega)}\, \phi(t)dt \\
&-\; \int_0^T (h(.,.,u(t)), w)_{L^2(\Omega)}\, \phi(t)dt \\
&+\; \int_0^T (f(t), w)_{L^2(\Omega)}\, \phi(t)dt,\ \forall w \in H^2_*(\Omega).
\end{aligned}
$$

So, $u_{tt} \in L^2\left([0,T], \mathcal{H}^*(\Omega)\right)$. Thus,

$$
u_t \in L^2\left((0,T), L^2(\Omega)\right),\ u_{tt} \in L^2\left([0,T], \mathcal{H}^*(\Omega)\right) \Longrightarrow u_t \in C([0,T), \mathcal{H}^*(\Omega)). \tag{2.38}
$$

Next, we show that our solution u satisfies the regularity in definition (2.1). It follows from (2.13) and (2.36) that

$$
u \in L^\infty((0,T), H^2_*(\Omega)) \cap C([0,T], L^2(\Omega)).
$$

Thus, we get from Lemma 1.6 that

$$
u \in C_w([0,T], H^2_*(\Omega)). \tag{2.39}
$$

Again, from (2.13) and (2.39), we obtain that

$$
u_t \in L^\infty((0,T), L^2(\Omega)) \cap C([0,T], \mathcal{H}^*(\Omega)).
$$

So, we obtain from Lemma 1.6 that

$$u \in C^1_w([0,T], L^2(\Omega)), \tag{2.40}$$

where C_w means weak continuity. It follows from (2.39) and (2.40) that

$$u \in C_w([0,T], H^2_*(\Omega)) \cap C^1_w([0,T], L^2(\Omega)). \tag{2.41}$$

To have the conclusion in definition 2.1 (strong continuity), we just show that the function

$$t \longmapsto g(t) = \|u_t\|^2_{L^2} + \|u\|^2_{H^2_*} \text{ is continuous in } [0, \mathrm{T}].$$

For this, we multiply (2.34) by u_t and integrate over Ω to obtain

$$\frac{d}{dt} g(t) = -2\delta \|u_t\|^p_{L^p} - 2(h(.,.,u), u_t)_{L^2} + 2(f, u_t)_{L^2}. \tag{2.42}$$

Integrating both sides between t_1 and t_2 leads to

$$\begin{aligned} |g(t_2) - g(t_1)| &= \left| \int_{t_1}^{t_2} (-2\delta \|u_t\|^p_{L^p} - 2(h(.,.,u), u_t)_{L^2} + 2(f, u_t)_{L^2}) \, dt \right| \\ &\leq \int_{t_1}^{t_2} \left(2\delta \|u_t\|^p_{L^p} + 2\|u_t\|^2_{L^2} + C_e \|u\|^2_{H^2_*} + \|f\|^2_{L^2} \right) dt. \end{aligned} \tag{2.43}$$

Observe that from (2.11), the right-hand side of (2.43) is $L^1([0,T], \mathbb{R})$, thus converges to 0 as $t_2 \to t_1$. Thus, we obtain that

$$|g(t_2) - g(t_1)| \to 0 \text{ as } t_2 \to t_1.$$

Hence,

$$u \in C([0,T], H^2_*(\Omega)) \cap C^1([0,T], L^2(\Omega)). \tag{2.44}$$

For the initial conditions, we let $\phi \in C^2([0,T], H^2_*(\Omega))$ with $\phi(T) = \phi_t(T) = 0$. Then by using (2.34), integration by parts leads to

$$\begin{aligned} &\int_0^T (u, \phi_{tt})dt + \delta \int_0^T (|u_t|^{p-2}u_t, \phi)dt + \int_0^T (u, w)_{H^2_*}dt \\ &\quad + \int_0^T (h(,.,u), \phi)dt = \int_\Omega (f, \phi)dt + (u_t(0), \phi(0)) - (u(0), \phi_t(0)). \end{aligned} \tag{2.45}$$

Again, replacing (u^m) by (u^l) in (2.4) and integrating by parts gives

$$\begin{aligned} &\int_0^T (u^l, \phi_{tt})dt + \delta \int_0^T (|u^l_t|^{p-2}u^l_t, \phi)dt + \int_0^T (u^l, w)_{H^2_*}dt \\ &\quad + \int_0^T (h(,.,u^l), \phi)dt = \int_\Omega (f, \phi)dt + (u^l_t(0), \phi(0)) - (u^l(0), \phi_t(0)). \end{aligned}$$

As $l \longrightarrow \infty$, we obtain

$$\begin{aligned} &\int_0^T (u, \phi_{tt})dt + \delta \int_0^T (|u_t|^{p-2}u_t, \phi)dt + \int_0^T (u, w)_{H^2_*}dt \\ &\quad + \int_0^T (h(,.,u), \phi)dt = \int_\Omega (f, \phi)dt + (u_1, \phi(0)) - (u_0, \phi_t(0)), \end{aligned} \tag{2.46}$$

since $u^l(0) \longrightarrow u_0$ and $u^l_t(0) \longrightarrow u_1$. Since $\phi(0)$ and $\phi_t(0)$ are arbitrary, a comparison of (2.45) and (2.46) gives

$$u(x,y,0) = u_0(x,y), \quad \text{and} \quad u_t(x,y,0) = u_1(x,y) \quad in\ \Omega. \tag{2.47}$$

For the uniqueness, we suppose u^1 and u^2 satisfy (2.34) and (2.47).

Then, $u = u^1 - u^2$ satisfies

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t w + \delta\int_\Omega (|u_t^1|^{p-2}u_t^1 - |u_t^2|^{p-2}u_t^2)w + (u,w)_{H_*^2} \\ + \int_\Omega [h(x,y,u^1) - h(x,y,u^2)]w = 0, \ \forall w \in L^p((0,T), H_*^2(\Omega)), \\ u(x,y,0) = u_t(x,y,0) = 0. \end{cases} \tag{2.48}$$

The system (2.48) remains true for any $w \in C_0^\infty(\Omega \times (0,T))$. Hence, valid for any $w \in L^2(\Omega \times (0,T))$ by density. Thus, replacing w by u_t in (2.48), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 + \frac{1}{2}\frac{d}{dt}\|u\|_{H_*^2}^2 + \delta\int_\Omega (|u_t^1|^{p-2}u_t^1 - |u_t^2|^{p-2}u_t^2)(u_t^1 - u_t^2)$$
$$+ \int_\Omega \left[h(x,y,u^1) - h(x,y,u^2)\right] u_t = 0.$$

By using condition (A_1) and Lemma 1.3, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |u_t|^2 + \|u\|_{H_*^2}^2\right] + \delta\int_\Omega (|u_t^1|^{p-2}u_t^1 - |u_t^2|^{p-2}u_t^2)(u_t^1 - u_t^2) \\ &\leq \int_\Omega \left[|h(x,y,u^1) - h(x,y,u^2)|\right] |u_t| \\ &\leq L_I \int_\Omega |u||u_t| \leq L\|u\|_{L^2}\|u_t\|_{L^2} \\ &\leq \frac{L_I}{2}\|u\|_{L^2}^2 + \frac{L_I}{2}\|u_t\|_{L^2}^2 \leq C\left[\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2\right], \end{aligned} \tag{2.49}$$

where L_I is a lipschitz constant. Thus, it follows from Lemma 1.7 and (2.49), we obtain

$$\frac{d}{dt}\left[\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2\right] \ \leq \ C\left[\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2\right]. \tag{2.50}$$

Applying Gronwall's Lemma to (2.50) lead to

$$\|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2 \leqslant Ce^{ct}\left[\|u_t(0)\|_{L^2}^2 + \|u(0)\|_{H_*^2}^2\right] = 0.$$

It follows that $u^1 = u^2$. ∎

2.3 Stability

In this section we consider (2.1) with $f \equiv 0$ and study the stability of the associated energy functional of the problem

$$\begin{cases} u_{tt} + \delta|u_t|^{p-2}u_t + \Delta^2 u + h(x,y,u) = 0, & \text{in} \quad \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for} \quad (y,t) \in (-l,l) \times (0,T), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for} \quad (y,t) \in (-l,l) \times (0,T), \\ u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm l,t) + (2-\sigma)u_{xxy}(x,\pm l,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T). \\ u(x,y,0) = u_0(x,y), \;\; u_t(x,y,0) = u_1(x,y), & \text{in} \quad \Omega. \end{cases} \tag{2.51}$$

The energy functional associated to problem (2.51) is defined by

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 d + \frac{1}{2}\|u\|_{H_*^2(\Omega)}^2 + \int_\Omega H(x,y,u). \tag{2.52}$$

We state some technique Lemmas that will be needed to prove our stability result.

Lemma 2.1 *Let $w \in H^4(\Omega) \cap H_*^2(\Omega)$ satisfying the boundary conditions of* (1.34).

Then,

$$\int_\Omega w\Delta^2 w = \|w\|^2_{H^2_*(\Omega)}. \tag{2.53}$$

Proof. Let $u = v$ in Lemma 1.2. ∎

Lemma 2.2 *The energy functional defined in* (2.52) *satisfies,*

$$\frac{dE(t)}{dt} = -\delta \int_\Omega |u_t|^p \leq 0 \tag{2.54}$$

Proof. Multiply (2.51) by u_t and integrate over Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u_t|^2 d + \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} + \int_\Omega H(x,y,u)\right) + \delta \int_\Omega |u_t|^p = 0. \tag{2.55}$$

This lead to

$$\frac{dE(t)}{dt} = -\delta \int_\Omega |u_t|^p \leq 0. \tag{2.56}$$

Thus E is non-increasing. ∎

Theorem 2.2 *Let* $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$ *be given and assume* (A_1) *and* (A_2) *hold. Then the energy functional* (2.52) *satisfies*

$$E(t) \leq \frac{C}{(1+t)^{\frac{2}{p-2}}},\ \forall t \geq 0,\ if\ \ p > 2$$

$$E(t) \leq Ce^{-\omega t},\ \ \forall\ t \geq 0,\ \ if\ \ p = 2$$

where C, ω *are positive constants.*

Proof. We exploit the multiplier method. So, we multiply (2.51) by $E^q(t)u$ and integrate over $\Omega \times (s,T), T > s > 0$ and for $q > 0$ to be specified later. Hence, we get

$$\int_s^T E^q(t) \int_\Omega \Big(u_{tt}u + \|u\|^2_{H^2_*(\Omega)} + \delta |u_t|^{p-2} u_t u + h(x,y,u)u \Big)\, dxdydt = 0.$$

After some calculations, this leads to

$$\begin{aligned}
&\int_s^T E^q(t) \left(\int_\Omega \left[(uu_t)_t - u_t^2 \right] + \|u\|^2_{H^2_*(\Omega)} \right) \\
&+ \int_s^T E^q(t) \left(\delta \int_\Omega |u_t|^{p-2} u_t u + \int_\Omega h(.,.,u)u \right) = 0,
\end{aligned}$$

which gives

$$\begin{aligned}
&\int_s^T \int_\Omega E^q(t)(uu_t)_t + \int_s^T E^q(t) \left(\frac{1}{2} \int_\Omega u_t^2 + \frac{1}{2} \|u\|^2_{H^2_*(\Omega)} + \int_\Omega H(.,.,u) \right) \\
&- \frac{3}{2} \int_s^T \int_\Omega E^q(t) u_t^2 + \frac{1}{2} \int_s^T E^q(t) \|u\|^2_{H^2_*(\Omega)} + \delta \int_s^T \int_\Omega E^q(t) |u_t|^{p-2} u_t u \\
&+ \int_s^T E^q(t) \int_\Omega [uh(.,.,u) - H(.,.,u)] = 0.
\end{aligned}$$

Then, taking account of (2.52), we get

$$\begin{aligned}
\int_s^T E^{q+1}(t)dt &= -\int_s^T \int_\Omega E^q(t)(uu_t)_t + \frac{3}{2} \int_s^T \int_\Omega E^q(t) u_t^2 \\
&- \frac{1}{2} \int_s^T E^q(t) \|u\|^2_{H^2_*(\Omega)} - \delta \int_s^T \int_\Omega E^q(t) |u_t|^{p-2} u_t u \\
&+ \int_s^T E^q(t) \int_\Omega [H(.,.,u) - uh(.,.,u)u].
\end{aligned}$$

By using assumption (A_2), we obtain

$$\begin{aligned}\int_s^T E^{q+1}(t)dt &\leq -\int_s^T \int_\Omega (E^q(t)uu_t)_t + q\int_s^T \int_\Omega E^{q-1}(t)E'(t)uu_t \\ &- \delta \int_s^T \int_\Omega E^q(t)|u_t|^{p-2}u_t u + \frac{3}{2}\int_s^T \int_\Omega E^q(t)u_t^2. \qquad (2.57)\end{aligned}$$

By using Young's inequality and Lemma 1.3, we estimate the integrals in the right-hand side of (2.57). For the first term, we have

$$\begin{aligned}|-\int_\Omega \int_s^T (E^q(t)uu_t)_t| &\leq |\int_\Omega E^q(T)u(T)u_t(T)| + |\int_\Omega E^q(s)u(s)u_t(s)| \\ &\leq \frac{1}{2}E^q(T)\int_\Omega \left(u^2(x,y,T) + u_t^2(x,y,T)\right) \\ &+ \frac{1}{2}E^q(s)\int_\Omega \left(u^2(x,y,s) + u_t^2(x,y,s)\right).\end{aligned}$$

Now, using the fact E is nonincreasing and

$$\frac{1}{2}\left(\|u(t)\|^2_{L^2(\Omega)} + \|u_t(t)\|^2_{L^2(\Omega)}\right) \leq \frac{1}{2}\left(C_e\|u(t)\|^2_{H^2_*(\Omega)} + \|u_t(t)\|^2_{L^2(\Omega)}\right) \leq CE(t),$$

we obtain

$$|-\int_\Omega \int_s^T (E^q(t)uu_t)_t| \leq C\left(E^{q+1}(T) + E^{q+1}(s)\right) \leq CE^{q+1}(s). \qquad (2.58)$$

For the second term, we have

$$
\begin{aligned}
|q\int_s^T\int_\Omega E^{q-1}(t)E'(t)uu_t| &\le -q\int_s^T\int_\Omega E^{q-1}(t)E'(t)|uu_t| \\
&\le -\frac{q}{2}\int_s^T E^{q-1}(t)E'(t)\int_\Omega \left(u^2(x,y,t)+u_t^2(x,y,t)\right) \\
&\le -C\int_s^T E^q(t)E'(t) = -C\int_s^T (E^{q+1}(t))' \\
&\le CE^{q+1}(s). \qquad (2.59)
\end{aligned}
$$

For the third term, we have

$$
\begin{aligned}
&|-\delta\int_s^T\int_\Omega E^q(t)|u_t|^{p-2}u_t u| \le \delta\int_s^T E^q(t)\int_\Omega |u_t|^{p-1}|u| \\
&\le \delta\int_s^T E^q(t)\left(\epsilon\int_\Omega |u|^p + C_\epsilon\int_\Omega |u_t|^p\right) \\
&= \delta\epsilon\int_s^T\int_\Omega E^q(t)|u|^p + C_\epsilon\int_s^T E^q(t)(-E'(t)) \\
&\le C\delta\epsilon\int_s^T E^q(t)\|u\|^p_{H^2_*(\Omega)} + C_\epsilon E^{q+1}(s) \qquad (2.60) \\
&= C\delta\epsilon\int_s^T E^q(t)(\|u\|^2_{H^2_*(\Omega)})^{\frac{p}{2}} + C_\epsilon E^{q+1}(s) \\
&\le C\delta\epsilon\int_s^T E^q(t)(E(t))^{\frac{p}{2}} + C_\epsilon E^{q+1}(s) \\
&\le \epsilon C\delta\,(E(0))^{\frac{p}{2}-1}\int_s^T E^{q+1}(t) + C_\epsilon E^{q+1}(s).
\end{aligned}
$$

For the fourth term, we first note that $p > 2$ and Ω is bounded, so

$$\|u_t\|_{L^2(\Omega)} \le C\|u_t\|_{L^p(\Omega)}.$$

Thus, we have

$$\frac{3}{2}\int_s^T\int_\Omega E^q(t)u_t^2 \leq \frac{3}{2}\int_s^T E^q(t)\|u_t\|^2_{L^2(\Omega)} \leq C\int_s^T E^q(t)(\|u_t\|^p_{L^p(\Omega)})^{\frac{2}{p}}$$
$$= C\int_s^T E^q(t)(-E'(t))^{\frac{2}{p}}. \qquad (2.61)$$

Now, using Young's inequality with $r = q+1$ and $r' = \frac{q+1}{q}$, we arrive at

$$|\frac{3}{2}\int_\Omega\int_s^T E^q(t)u_t^2| \leq \epsilon\int_s^T E^{q+1}(t) + C_\epsilon\int_s^T (-E'(t))^{\frac{2(q+1)}{p}}.$$

At this point, we choose q so that

$$\frac{2(q+1)}{p} = 1, \ hence \ \ q = \frac{p-2}{2}.$$

Consequently, we obtain

$$|\frac{3}{2}\int_\Omega\int_s^T E^q(t)u_t^2| \leq \epsilon\int_s^T E^{q+1}(t)dt + C_\epsilon\int_s^T(-E'(t)dt$$
$$\leq \epsilon\int_s^T E^{q+1}(t)dt + C_\epsilon E(s). \qquad (2.62)$$

Combining (2.58), (2.59), (2.60) and (2.62), we obtain from (2.57) that

$$\int_s^T E^{q+1}(t)dt \leq CE^{q+1}(s) + C_0\epsilon E^{\frac{p}{2}-1}(0))\int_s^T E^{q+1}(t)$$
$$+ C_\epsilon E^{q+1}(s) + \epsilon\int_s^T E^{q+1}(t)dt + C_\epsilon E(s). \qquad (2.63)$$

This leads to

$$\left(1-\epsilon-\epsilon C\delta E^{\frac{p}{2}-1}(0)\right)\int_s^T E^{q+1}(t)dt \leq (C+C_\epsilon)E^{q+1}(s)+C_\epsilon E(s). \qquad (2.64)$$

Now, we choose ϵ small enough so that $(1-\epsilon-\epsilon C\delta E^{\frac{p}{2}-1}(0))>0$, to arrive at

$$\begin{aligned}\int_s^T E^{\frac{p}{2}}(t)dt &\leq C(E^{\frac{p}{2}}(s)+E(s))\\ &= C(E^{\frac{p-2}{2}}(s)E(s)+E(s))\\ &\leq C(E^{\frac{p-2}{2}}(0)E(s)+E(s))\\ &\leq \tilde{C}E(s). \qquad (2.65)\end{aligned}$$

By letting $T\longrightarrow\infty$ in (2.65) and applying Komornik Lemma (Lemma 1.8), we arrive at the following conclusion

$$E(t)\leq \frac{C}{(1+t)^{\frac{2}{p-2}}},\ \forall t\geq 0,\ if\ \ p>2,$$

$$E(t)\leq Ce^{-\omega t},\ \ \forall\ t\geq 0,\ \ if\ \ p=2.$$

This completes the proof. ∎

CHAPTER 3

EXISTENCE AND GENERAL DECAY RATES FOR A FOURTH–ORDER WEAKLY DAMPED WAVE PROBLEM

In this chapter we study the following fourth−order weakly damped wave problem

$$\begin{cases} u_{tt} + \Delta^2 u + \beta(t) g(u_t) + h(u) = 0, & \text{in} \quad \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma) u_{xxy}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \;\; u_t(x,y,0) = u_1(x,y), & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where $\Omega = (0, \pi) \times (-l, l) \subset \mathbb{R}^2$, $0 < \sigma < \frac{1}{2}$ is the Poission ratio, h is the hangers restoring force and β, g are given functions to be specified later. In this problem, $u = u(x, y, t)$ represents the displacement of a vibrating suspension bridge of length L under the effect of a weak internal frictional damping g with damping coefficient β. We will establish a well-posedness as well as explicit and general decay results without imposing restrictive growth conditions on the frictional damping term. This chapter is partitioned into the following sections: In section 3.1, we outline some important assumptions that will be helpful throughout this chapter. In section 3.2, we state and prove a well-posedness result for problem (3.1). In section 3.3, we sate and prove an explicit as well as general decay estimates for problem (3.1). Finally, in section 3.4, we illustrate our results with some examples.

3.1 Main Assumptions

We assume that the functions β, g and h admit the following assumptions:

(A_1) $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function.

(A_2) $h : \mathbb{R} \to \mathbb{R}$ is a locally lipschitz non-decreasing function such that $h(0) = 0$ and denote $H(s) = \displaystyle\int_0^s h(\tau)d\tau$ positive such that

$$sh(s) - H(s) \geq 0, \ \forall s \in \mathbb{R}.$$

(A_3) $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^1- function such that, there exist $\epsilon, c_1, c_2 > 0$ and an increasing function $M \in C^1\left([0, +\infty)\right)$ with M linear or $M(0) = M'(0) = 0$

is a strictly convex C^2- function on $[0,\epsilon)$ such that

$$\begin{cases} c_1|s| \leq |g(s)| \leq c_2|s|, \ if \ |s| \geq \epsilon, \\ s^2 + g^2(s) \leq M^{-1}(sg(s)), \ if \ |s| \leq \epsilon. \end{cases} \tag{3.2}$$

Remarks:

1. We obtain from assumption (A_3), that g is locally lipschitz and $sg(s) > 0$ for $s \neq 0$.

2. The assumption (A_3) with $\epsilon = 1$ was first introduced by Lasiecka and Tataru [54], where decay estimates for second-order nonlinear wave equation with nonlinear boundary damping have been established.

3. To achieve our decay result, we borrow the techniques used by Mustafa and Messaoudi in [80] to prove decay estimates for a second-order wave equation with Dirichlet boundary conditions.

3.2 Well posedness

In this section, we discuss the well-posedness of problem (3.1). We begin with the definition of a weak solution for problem (3.1).

Definition 3.1 *We say that a function*

$$u \in C\left([0,T], H_*^2(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right) \tag{3.3}$$

is a weak solution of (3.1), *if*

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u_t w + (u,w)_{H_*^2(\Omega)} + \beta(t)\int_\Omega g(u_t)w + \int_\Omega h(u)w = 0, \ \ \forall w \in H_*^2(\Omega), \\ u(0) = u_0, \ \ u_t(0) = u_1, \\ for \ a.e \ t \in (0,T). \end{cases} \tag{3.4}$$

Now, we reformulate problem (3.1) into a semigroup setting. Let $u_t = v$, then problem (3.1) becomes

$$\begin{cases} U_t + AU = F(t,U), \\ U(0) = U_0, \end{cases} \tag{3.5}$$

where,

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad AU = \begin{pmatrix} -v \\ \Delta^2 u \end{pmatrix}, \quad F(t,U) = \begin{pmatrix} 0 \\ -h(u) - \beta(t)g(v) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

We introduce the Hilbert space

$$\mathcal{H} = H_*^2(\Omega) \times L^2(\Omega)$$

equipped with the inner product

$$(U,V)_{\mathcal{H}} = (u,\tilde{u})_{H_*^2(\Omega)} + (v,\tilde{v})_{L^2(\Omega)}, \tag{3.6}$$

where

$$U = (u,v)^T, \ V = (\tilde{u},\tilde{v})^T \in \mathcal{H}.$$

Next, we introduce the following notation

$$\begin{cases} u_{xx}(0,y) = u_{xx}(\pi,y) = 0, \\ u_{yy}(x,\pm\ell) + \sigma u_{xx}(x,\pm\ell) = 0, \\ u_{yyy}(x,\pm\ell) + (2-\sigma)u_{xxy}(x,\pm\ell) = 0. \end{cases} \tag{3.7}$$

The domain of the operator A is defined as

$$D(A) = \left\{(u,v) \in \mathcal{H} : u \in H^4(\Omega) \ satisfying \ (3.7) \ and \ v \in H^2_*(\Omega)\right\}.$$

Lemma 3.1 *We have*

$$(\Delta^2 u, v)_{L^2(\Omega)} = (u,v)_{H^2_*}, \qquad \forall u, v \in D(A). \tag{3.8}$$

Proof. See Lemma 1.2. ∎

We have the following existence and uniqueness result for problem (3.5).

Theorem 3.1 *Let $U_0 \in \mathcal{H}$ be given. Assume that $(A_1)-(A_3)$ hold. Then problem (3.5) has a unique global weak solution*

$$U \in C\left([0,T], \mathcal{H}\right).$$

Proof. To achieve this result, we show that the operator A is maximal monotone and F is locally lipschitz continues.

Monotonicity: Let $U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$ and making use of Lemma 3.1, we have

$$(AU, U)_{\mathcal{H}} = \left(\begin{pmatrix} -v \\ \Delta^2 u \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right)_{\mathcal{H}} = -(u, v)_{H^2_*(\Omega)} + \left(\Delta^2 u, v\right)_{L^2(\Omega)} = 0.$$

Thus, A is a monotone operator.

Maximality: Let $G = (k, l) \in \mathcal{H}$ and consider the stationary problem

$$U + AU = G, \tag{3.9}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$. This gives

$$\begin{cases} u - v = k, & \text{in} \quad H^2_*(\Omega) \\ v + \Delta^2 u = l, & \text{in} \quad L^2(\Omega). \end{cases} \tag{3.10}$$

Adding $(3.10)_1$ and $(3.10)_2$ gives

$$u + \Delta^2 u = k + l, \ \textit{in} \ L^2(\Omega). \tag{3.11}$$

The weak formulation of (3.11) is then

$$\int_\Omega u\phi + (u, \phi)_{H^2_*(\Omega)} = \int_\Omega (k + l)\phi, \ \forall \phi \in H^2_*(\Omega). \tag{3.12}$$

Define on $H^2_*(\Omega)$ the following bilinear and linear forms

$$B(u,\phi) = \int_\Omega u\phi + (u,\phi)_{H^2_*(\Omega)}, \quad \mathcal{L}(\phi) = \int_\Omega (k+l)\phi.$$

We can check that B and $\mathcal{L}$ are bounded, and B is coercive. Indeed

$$\begin{aligned} |B(u,\phi)| &\leq \|u\|_{L^2}\|\phi\|_{L^2} + \|u\|_{H^2_*}\|\phi\|_{H^2_*} \\ &\leq (C_e+1)\|u\|_{H^2_*}\|\phi\|_{H^2_*} = C\|u\|_{H^2_*}\|\phi\|_{H^2_*} \end{aligned}$$

and

$$B(u,u) = \|u\|^2_{L^2} + \|u\|^2_{H^2_*} \geq \|u\|^2_{H^2_*}.$$

Hence B is bounded and coercive. Also,

$$|\mathcal{L}(\phi)| \leq (\|k\|_{L^2} + \|l\|_{L^2})\|\phi\|_{L^2} \leq (C_e\|k\|_{H^2_*} + \|l\|_{L^2})C_e\|\phi\|_{H^2_*} = C\|\phi\|_{H^2_*}.$$

Thus $\mathcal{L}$ is bounded. So, Lax- Milgram Theorem guarantees the existence of a unique $u \in H^2_*(\Omega)$ satisfying (3.12), which yields

$$(u,\phi)_{H^2_*(\Omega)} = \int_\Omega (k+l-u)\phi, \ \forall \phi \in H^2_*(\Omega).$$

Since $k+l-u \in L^2(\Omega)$, then Theorem 1.1 implies $u \in H^4(\Omega)$. Thus,

$$u \in H^2_*(\Omega) \cap H^4(\Omega).$$

Now, let $\phi \in C_0^\infty(\Omega)$ ($C_0^\infty(\Omega) \subset H_*^2(\Omega)$), then integration by parts in (3.12) leads to

$$\int_\Omega [u + \Delta^2 u - (k+l)]\phi = 0, \ \forall \phi \in C_0^\infty(\Omega).$$

This implies (by density)

$$\int_\Omega [u + \Delta^2 u - (k+l)]\phi = 0, \ \forall \phi \in L^2(\Omega). \tag{3.13}$$

Thus,

$$u + \Delta^2 u = k + l, \ in \ L^2(\Omega). \tag{3.14}$$

We take

$$v = u - k \ in \ H_*^2(\Omega).$$

Again, let $\phi \in C^\infty(\bar{\Omega}) \cap H_*^2(\Omega)$. Then integrating (3.12) by parts gives

$$\begin{aligned}
&\int_\Omega [u + \Delta^2 u - (k+l)]\phi + \int_{-\ell}^{\ell} [u_{xx}(\pi, y)\phi_x(\pi, y) - u_{xx}(0, y)\phi_x(0, y)]dy \\
&+ \int_0^\pi \{[u_{yy}(x, \ell) + \sigma u_{xx}(x, \ell)]\phi_y(x, \ell) - [u_{yy}(x, -\ell) + \sigma u_{xx}(x, -\ell)]\phi_y(x, -\ell)\}dx \\
&+ \int_0^\pi [u_{yyy}(x, -\ell) + (2-\sigma)u_{xxy}(x, -\ell)]\phi(x, l)dx \\
&- \int_0^\pi [u_{yyy}(x, \ell) + (2-\sigma)u_{xxy}(x, \ell)]\phi(x, \ell)dx = 0, \ \ \forall \phi \in C^\infty(\bar{\Omega}) \cap H_*^2(\Omega).
\end{aligned}$$

It follow from (3.14) that

$$\begin{aligned}
&\cancelto{0}{\int_\Omega [u + \Delta^2 u - (k+l)]\phi} + \int_{-\ell}^{\ell} [u_{xx}(\pi, y)\phi_x(\pi, y) - u_{xx}(0, y)\phi_x(0, y)]dy \\
&+ \int_0^\pi \{[u_{yy}(x, \ell) + \sigma u_{xx}(x, \ell)]\phi_y(x, \ell) - [u_{yy}(x, -\ell) + \sigma u_{xx}(x, -\ell)]\phi_y(x, -\ell)\}dx \\
&+ \int_0^\pi [u_{yyy}(x, -\ell) + (2-\sigma)u_{xxy}(x, -\ell)]\phi(x, \ell)dx \\
&- \int_0^\pi [u_{yyy}(x, \ell) + (2-\sigma)u_{xxy}(x, \ell)]\phi(x, \ell)dx = 0, \quad \forall \phi \in C^\infty(\bar{\Omega}) \cap H_*^2(\Omega).
\end{aligned}$$

The arbitrary choice of $\phi \in C^\infty(\bar{\Omega}) \cap H_*^2(\Omega)$ implies that the coefficients of

$$\phi_x(\pi, y), \quad \phi_x(0, y), \quad \phi_y(x, \ell), \quad \phi_y(x, -\ell)$$

are identically zero. Therefore we obtain (3.7) and thus there exists a unique

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \in D(A)$$

satisfying (3.9). Hence, A is a maximal operator.

Local lipschitzness: Let $U, V \in \mathbf{B}_R = \{(u, v) \in D(A) : \|(u, v)\|_{\mathcal{H}} \leq R\}$. By using Lemma 1.3, the local lipschitzness of h and g, and the boundedness of β, we get

$$\begin{aligned}
\|F(t,U)-F(t,V)\|_{\mathcal{H}}^2 &= \left\| \begin{pmatrix} 0 \\ -h(u)-\beta(t)g(v) \end{pmatrix} - \begin{pmatrix} 0 \\ -h(\tilde{u})-\beta(t)g(\tilde{v}) \end{pmatrix} \right\|_{\mathcal{H}}^2 \\
&= \int_\Omega |(h(\tilde{u})-h(u))+\beta(t)(g(\tilde{v})-g(v))|^2 \\
&\leq 2C_R\|u-\tilde{u}\|_{L^2(\Omega)}^2 + 2C_R\beta^2(0)\|v-\tilde{v}\|_{L^2(\Omega)}^2 \\
&\leq 2C_RC_e\|u-\tilde{u}\|_{H_*^2(\Omega)}^2 + 2C_R\beta^2(0)\|v-\tilde{v}\|_{L^2(\Omega)}^2 \\
&\leq \left(\|u-\tilde{u}\|_{H_*^2(\Omega)}^2 + \|v-\tilde{v}\|_{L^2(\Omega)}^2\right) \\
&= \|U-V\|_{\mathcal{H}}^2. \qquad (3.15)
\end{aligned}$$

So, F is locally lipschitz. Thus, by the semigroup theory Pazy [83], we obtain local existence solution

$$U \in C\left([0,T_m),\mathcal{H}\right), \quad \text{for some } T_m > 0.$$

To obtain global existence, it suffice to show that $\|U(t)\|_{\mathcal{H}}$ is bounded independently of t. To this end, we multiply $(3.1)_1$ by u_t and integrate over Ω to get

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\|u\|_{H_*^2}^2 + \int_\Omega H(u)\right) = -\beta(t)\int_\Omega u_t g(u_t) \leq 0 \qquad (3.16)$$

on the account of assumption (A_2) and remark number 1. Thus, we obtain

$$\|U(t)\|_{\mathcal{H}}^2 = \|u_t\|_{L^2}^2 + \|u\|_{H_*^2}^2 \leq E(t) \leq E(0),$$

where

$$\frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\|u\|_{H_*^2}^2 + \int_\Omega H(u).$$

This completes the proof. ∎

3.3 Decay of the energy

In this section, we discuss the decay rates of the energy functional associated to problem (3.1). For this, we state and prove several Lemmas that will be fundamental in establishing the main result.

3.3.1 Technical Lemmas

Lemma 3.2 *For $u \in H^4(\Omega) \cap H_*^2(\Omega)$ satisfying* (3.9) *, we have*

$$\int_\Omega u\Delta^2 u = \|u\|_{H_*^2} \tag{3.17}$$

Proof. Let $u = v$ in Lemma 3.1. ∎

Let's introduce the energy functional associated to problem (3.1), namely

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + \frac{1}{2}\|u\|_{H_*^2(\Omega)}^2 + \int_\Omega H(u) \tag{3.18}$$

Lemma 3.3 *The energy functional defined in* (3.18) *satisfies,*

$$\frac{dE(t)}{dt} = -\beta(t)\int_\Omega u_t g(u_t) \leq 0. \tag{3.19}$$

Proof. By multiplying $(3.1)_1$ by u_t and integrating over Ω, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u_t|^2 + \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} + \int_\Omega H(u)\right) + \beta(t)\int_\Omega u_t g(u_t) = 0.$$

From (A_3), we get that $sg(s) > 0$ for all $s \neq 0$. Thus, by using (A_1), we obtain

$$\frac{dE(t)}{dt} = -\beta(t)\int_\Omega u_t g(u_t) \leq 0. \tag{3.20}$$

We note here that the calculations are justified for regular solutions. However, the result in (3.20) remains true for weak solution by a density argument. ▮

Define the functional

$$F(t) = mE(t) + \int_\Omega u u_t, \tag{3.21}$$

where m is a positive constant to be specified later.

Lemma 3.4 *Assume that $(A_1) - (A_3)$ hold. Then, the functional F satisfies, along the solution of (3.1), the estimates*

$$F'(t) \leq -E(t) + C\int_\Omega \left(u_t^2 + |ug(u_t)|\right)$$

and

$$F \sim E,$$

where C is positive constant.

Proof. By using $(3.1)_1$, definition (3.4) and exploiting assumption (A_1) and (A_2), direct differentiation gives

$$\begin{aligned}
F'(t) &= mE'(t) + \int_\Omega u_t^2 + \int_\Omega u u_{tt} \\
&= -m\beta(t)\int_\Omega u_t g(u_t) + \int_\Omega u_t^2 - \|u\|^2_{H^2_*(\Omega)} - \beta(t)\int_\Omega u g(u_t) - \int_\Omega u h(u) \\
&\leq \int_\Omega u_t^2 - \frac{1}{2}\|u\|^2_{H^2_*(\Omega)} - \int_\Omega H(u) - \beta(t)\int_\Omega u g(u_t) + \int_\Omega (H(u) - u h(u)) \\
&\leq -E(t) + \frac{3}{2}\int_\Omega u_t^2 + \beta(t)\int_\Omega |u g(u_t)| \\
&\leq -E(t) + C\int_\Omega \left(u_t^2 + |u g(u_t)|\right). \qquad (3.22)
\end{aligned}$$

Next, we show that $F \sim E$. Using Young's inequality and Lemma 1.3, we have

$$\begin{aligned}
F(t) &\leq mE(t) + \frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \\
&\leq mE(t) + \frac{1}{2}\int_\Omega u_t^2 + \frac{C_e}{2}\|u\|^2_{H^2_*(\Omega)} \leq \lambda_2 E(t). \qquad (3.23)
\end{aligned}$$

Also,

$$\begin{aligned}
F(t) &\geq mE(t) - \frac{1}{2}\int_\Omega u_t^2 - \frac{1}{2}\|u\|^2_{L^2(\Omega)} \\
&\geq mE(t) - \frac{1}{2}\int_\Omega u_t^2 - \frac{C_e}{2}\|u\|^2_{H^2_*(\Omega)} \\
&= \frac{(m-1)}{2}\int_\Omega u_t^2 + \frac{(m-C_e)}{2}\|u\|^2_{H^2_*(\Omega)} + m\int_\Omega H(u).
\end{aligned}$$

We choose $m > 0$ large enough so that $(m-1), (m-C_e) > 0$ and arrive at

$$F(t) \geq \lambda_1 E(t). \qquad (3.24)$$

Thus, we get from (3.23) and (3.24) that

$$\lambda_1 E(t) \le F(t) \le \lambda_2 E(t).$$

This completes the proof. ∎

Next, we choose $0 < \epsilon_1 \le \epsilon$ so that

$$sg(s) \le \min\{\epsilon, M(\epsilon)\},\ \forall\ |s| \le \epsilon_1. \tag{3.25}$$

Then, for $|s| \ge \epsilon_1$, the function $s \longmapsto \frac{|g(s)|}{|s|}$ is continuous on compact intervals and thus attains its extrema. Thus, it follows from assumption (A_3) that

$$\begin{cases} c_1'|s| \le |g(s)| \le c_2'|s|,\ if\ |s| \ge \epsilon_1, \\ s^2 + g^2(s) \le M^{-1}(sg(s)),\ if\ |s| \le \epsilon_1. \end{cases} \tag{3.26}$$

As in [47], let's partition Ω as follows:

$$\Omega_1 = \{(x,y) \in \Omega : |u_t| \le \epsilon_1\}\ \ and\ \ \Omega_2 = \{(x,y) \in \Omega : |u_t| > \epsilon_1\}.$$

Lemma 3.5 *The following inequalities holds for any $\epsilon > 0$, along the the solution of* (3.1),

$$\int_{\Omega_1} \left(u_t^2 + |ug(u_t)|\right) \le \int_{\Omega_1} u_t^2 + C_e \epsilon E(t) + C_\epsilon \int_{\Omega_1} |g(u_t)|^2 \tag{3.27}$$

and

$$\int_{\Omega_2} \left(u_t^2 + |ug(u_t)|\right) \le C\epsilon E(t) - C_\epsilon E'(t), \tag{3.28}$$

where C_e *is the embedding constant defined in Lemma 1.3 and* C_ϵ *is generic positive constant depending on* ϵ.

Proof. For the first inequality, we use Young's inequality and Lemma 1.3 to get

$$\begin{aligned}
\int_{\Omega_1} \left(u_t^2 + |ug(u_t)|\right) &\le \int_{\Omega_1} u_t^2 + \epsilon \int_{\Omega_1} |u|^2 + C_\epsilon \int_{\Omega_1} |g(u_t)|^2 \\
&\le \int_{\Omega_1} u_t^2 + C_e\epsilon \|u\|^2_{H^2_*(\Omega)} + C_\epsilon \int_{\Omega_1} |g(u_t)|^2 \\
&\le \int_{\Omega_1} u_t^2 + C_e\epsilon E(t) + C_\epsilon \int_{\Omega_1} |g(u_t)|^2.
\end{aligned} \tag{3.29}$$

For the second inequality, we recall Lemma 1.3 and Hölder's inequality to obtain

$$\begin{aligned}
\int_{\Omega_2} |ug(u_t)| &\le \left(\int_{\Omega_2} |u|^2\right)^{\frac{1}{2}} \left(\int_{\Omega_2} |g(u_t)|^2\right)^{\frac{1}{2}} \\
&\le \|u\|_{L^2(\Omega)} \left(\int_{\Omega_2} |g(u_t)|^2\right)^{\frac{1}{2}} \\
&\le C_e \|u\|_{H^2_*(\Omega)} \left(\int_{\Omega_2} |g(u_t)|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.30}$$

Now, from $(3.26)_1$, we observe that

$$|s|^2 \le c_1'' sg(s) \ \text{ and } \ |g(s)|^2 \le c_2'' sg(s), \quad \text{for some positive constants } c_1'', c_2''.$$

Thus, with this in mind and Young's inequality, we obtain

$$\begin{aligned}
\int_{\Omega_2}\left(u_t^2+|ug(u_t)|\right) &\leq C\int_{\Omega_2} u_t g(u_t) + C\left(\|u\|^2_{H^2_*(\Omega)}\right)^{\frac{1}{2}}\left(\int_{\Omega_2} u_t g(u_t)\right)^{\frac{1}{2}} \\
&\leq -CE'(t) + C\,(E(t))^{\frac{1}{2}}\,(-E'(t))^{\frac{1}{2}} \\
&\leq -CE'(t) + C\left(\epsilon\,(E(t)) - C_\epsilon E'(t)\right) \\
&= C\epsilon E(t) - C_\epsilon E'(t). \qquad (3.31)
\end{aligned}$$

∎

Lemma 3.6 *For ϵ small enough and two positive constants d, C, the functional defined by*

$$L(t) = F_1(t) + C_\epsilon E(t), \quad \textit{where} \quad F_1(t) = F(t) + C_\epsilon E(t)$$

satisfies, along the solution of (3.1), *the estimate*

$$L'(t) \leq -dE(t) + C\int_{\Omega_1}\left(u_t^2 + |g(u_t)|^2\right) \qquad (3.32)$$

and

$$L \sim E.$$

Proof. Using Lemmas 3.4 and 3.5, direct computations give

$$\begin{aligned}
F_1'(t) &= F'(t) + C_\epsilon E'(t) \\
&\le -E(t) + C\int_{\Omega_1} \left(u_t^2 + |ug(u_t)|\right) + C\int_{\Omega_2} \left(u_t^2 + |ug(u_t)|\right) \\
&\le -E(t) + C\int_{\Omega_1} u_t^2 + CC_e\epsilon E(t) + C_\epsilon \int_{\Omega_1} |g(u_t)|^2 + C\epsilon E(t) - C_\epsilon E'(t) \\
&\le -(1 - C\epsilon)E(t) + C_\epsilon \int_{\Omega_1} \left(u_t^2 + |g(u_t)|^2\right) - C_\epsilon E'(t).
\end{aligned}$$

That is,

$$(F_1(t) + C_\epsilon E(t))' \le -(1 - C\epsilon)E(t) + C_\epsilon \int_{\Omega_1} \left(u_t^2 + |g(u_t)|^2\right). \tag{3.33}$$

This implies

$$L'(t) \le -(1 - C\epsilon)E(t) + C_\epsilon \int_{\Omega_1} \left(u_t^2 + |g(u_t)|^2\right). \tag{3.34}$$

We then choose ϵ small enough so that $(1 - C\epsilon) > 0$ and obtain the result. It is easy to see that $L \sim E$ since $F \sim E$. This completes the proof. ∎

3.3.2 Main decay result

Now, we state and prove our main decay result of this chapter.

Theorem 3.2 *Assume that $(A_1)-(A_3)$ hold. Then, there exist positive constants*

$k_1, k_2, k_3, \epsilon_0$ such that the solution of (3.1) *satisfies*

$$E(t) \leq k_3 M_1^{-1}\left(k_1 \int_0^t \beta(s)ds + k_2\right), \ \forall t \geq 0, \tag{3.35}$$

where

$$M_1(t) = \int_t^1 \frac{1}{M_2(s)}ds, \quad M_2(t) = tM'(\epsilon_0 t) \tag{3.36}$$

and M_1 is strictly decreasing on $(0, 1]$ *and* $\lim_{t\to 0} M_1(t) = +\infty$

Proof. We distinguish two cases.

Case I. M is linear on $(0, \epsilon]$: Multipying (3.32) by $\beta(t)$ and using $(3.26)_2$, we deduce that

$$\begin{aligned}
\beta(t)L'(t) &\leq -d\beta(t)E(t) + C\beta(t)\int_{\Omega_1}\left(u_t^2 + |g(u_t)|^2\right) \\
&\leq -d\beta(t)E(t) + C\beta(t)\int_{\Omega_1} M^{-1}(u_t g(u_t)) \\
&= -d\beta(t)E(t) + C\beta(t)\int_{\Omega_1} u_t g(u_t) \\
&\leq -d\beta(t)E(t) + C\beta(t)\int_{\Omega} u_t g(u_t) \\
&= -d\beta(t)E(t) - CE'(t).
\end{aligned}$$

By using (A_1), we obtain

$$(\beta(t)L(t) + CE(t))' \leq -d\beta(t)E(t). \tag{3.37}$$

Let $J_1 = \beta L + CE$. Then, $J_1 \sim E$ since $L \sim E$ and we get from (3.37)

$$J_1'(t) \leq -k_1 \beta(t) J_1(t). \tag{3.38}$$

Simple integration of (3.38) over $(0, t)$ and using the fact that $J_1 \sim E$ gives

$$E(t) \leq k_2 e^{-k_1 \int_0^t \beta(s) ds}.$$

Case II. G is nonlinear on $(0, \epsilon]$. In this case, we consider the functional $I(t)$ defined by

$$I(t) = \frac{1}{|\Omega_1|} \int_{\Omega_1} u_t g(u_t).$$

We know that M is convex, so M^{-1} is concave. Thus, Jensen's inequality yields

$$M^{-1}(I(t)) \geq \frac{1}{|\Omega_1|} \int_{\Omega_1} M^{-1}(u_t g(u_t)). \tag{3.39}$$

By using $(3.26)_2$, we obatain

$$\beta(t) \int_{\Omega_1} \left(u_t^2 + |g(u_t)|^2\right) \leq \beta(t) \int_{\Omega_1} M^{-1}(u_t g(u_t)) \leq C\beta(t) M^{-1}(I(t)). \tag{3.40}$$

By multipying (3.32) by $\beta(t)$ and using (3.40), we arrive at

$$\begin{aligned} \beta(t) L'(t) &\leq -d\beta(t) E(t) + C\beta(t) \int_{\Omega_1} \left(u_t^2 + |g(u_t)|^2\right) \\ &\leq -d\beta(t) E(t) + C\beta(t) M^{-1}(I(t)). \end{aligned} \tag{3.41}$$

This implies

$$\beta(t)L'(t) + E'(t) \leq -d\beta(t)E(t) + C\beta(t)M^{-1}(I(t)),$$

since $E' \leq 0$. Using (A_1), we obtain

$$R_0'(t) \leq -d\beta(t)E(t) + C\beta(t)M^{-1}(I(t)),$$

where

$$R_0 = \beta L + E \sim E. \tag{3.42}$$

Let $\epsilon_0 < \epsilon$, $C_0 > 0$ and define the functional

$$R_1(t) = M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) R_0(t) + C_0 E(t). \tag{3.43}$$

Let us note here that $E(0) > 0$, otherwise $E(t) = 0$, $\forall t \in \mathbb{R}^+$ and thus the Theorem is verified since $E'(t) \leq 0$. Now, since $R_0 \sim E$ and $E' \leq 0$, $M' > 0$ (M is increasing), $M'' > 0$ (M is convex) on $(0, \epsilon]$, then R_1 satisfies the following

$$\alpha_1 R_1(t) \leq E(t) \leq \alpha_2 R_1(t), \; for\ some\ \alpha_1,\ \alpha_2 > 0 \tag{3.44}$$

and it follows from (3.43) that

$$\begin{aligned}
R_1'(t) &= \epsilon_0 \left(\frac{E'(t)}{E(0)}\right) M'' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) R_0(t) + M' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) R_0'(t) + C_0 E'(t) \\
&\leq M' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) \left[-d\beta(t)E(t) + C\beta(t)M^{-1}(I(t))\right] + C_0 E'(t) \\
&= -d\beta(t)E(t)M' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\
&+ C\beta(t)M' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) M^{-1}(I(t)) + C_0 E'(t). \qquad (3.45)
\end{aligned}$$

Now, let M^* be the convex conjugate of M in the sense of Young (See [11], pp. 61-64). Then,

$$M^*(s) = s(M')^{-1}(s) - M((M')^{-1}(s)),\ if\ s \in (0, M'(\epsilon)) \qquad (3.46)$$

and M^* satisfies the generalised Young's inequality

$$XY \leq M^*(X) + M(Y),\ if\ X \in (0, M'(\epsilon)),\ Y \in (0, \epsilon). \qquad (3.47)$$

Next, we set $X = M'(\epsilon_0 \frac{E(t)}{E(0)})$ and $Y = M^{-1}(I(t))$. By using Lemma 3.3, the fact that $sg(s) \leq \min\{\epsilon, G(\epsilon)\}$, if $|s| \leq \epsilon_1$ and (3.45)-(3.47), we obtain

$$
\begin{aligned}
R_1'(t) &\leq -d\beta(t)E(t)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C_0 E'(t) \\
&+ C\beta(t)\left[M^*\left(M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + M(M^{-1}(I(t)))\right] \\
&= -d\beta(t)E(t)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C_0 E'(t) \\
&+ C\beta(t)M^*\left(M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + C\beta(t)I(t) \\
&= -d\beta(t)E(t)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C_0 E'(t) \\
&+ C\epsilon_0\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - C\beta(t)M\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C\beta(t)I(t) \\
&\leq -E(0)d\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C\epsilon_0\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\
&+ C\beta(t)\int_\Omega u_t g(u_t) + C_0 E'(t) \\
&\leq -E(0)d\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C\epsilon_0\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\
&- CE'(t) + C_0 E'(t).
\end{aligned}
$$

By choosing C_0 large enough and ϵ_0 small enough such that

$$C - C_0 < 0, \ E(0)d - C\epsilon_0 > 0,$$

we arrive at

$$R_1'(t) \leq -k\beta(t)\left(\frac{E(t)}{E(0)}\right)M'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -k\beta(t)M_2\left(\epsilon_0 \frac{E(t)}{E(0)}\right), \tag{3.48}$$

where $M_2(t) = tM'(\epsilon_0 t)$. We have that

$$M_2'(t) = M'(\epsilon_0 t) + \epsilon_0 t M''(\epsilon_0 t).$$

Thus, using the strict convexity of M on $(0, \epsilon]$, we get that $M_2, M_2' > 0$ on $(0, 1]$. It follows from (3.44) and (3.48) that the functional

$$R_2(t) = \alpha_1 \frac{R_1(t)}{E(0)}$$

satisfies

$$R_2 \sim E \tag{3.49}$$

and

$$R_2'(t) \leq -k_1 \beta(t) M_2(R_2(t)), \ \ for\ some\ k_1 > 0. \tag{3.50}$$

The inequality (3.50) implies that

$$(M_1(R_2(t))' \geq k_1 \beta(t),$$

where

$$M_1(\tau) = \int_\tau^1 \frac{1}{M_2(s)} ds, \quad \tau \in (0, 1].$$

Thus, integrating (3.50) over $(0,t)$ and noting that M_1 is strictly decreasing on $(0,1]$ gives

$$R_2(t) \le M_1^{-1}\left(k_1\int_0^t \beta(s)ds + k_2\right),\ for\ some\ k_2 > 0. \tag{3.51}$$

Combining (3.49) and (3.51) we get the result. This completes the proof. ∎

3.4 Examples

In this section, we illustrate our result by some examples. As in [61] and [80], let $g_0 \in C^2([0,+\infty))$ be a strictly increasing function such that $g_0(0) = 0$ and for some positive constants c_1, c_2 and ϵ, the function g satisfies

$$\begin{cases} c_1|s| \le |g(s)| \le c_2|s|,\ \forall\ |s| \ge \epsilon, \\ g_0(|s|) \le |g(s)| \le g_0^{-1}(|s|),\ \forall\ |s| \le \epsilon. \end{cases} \tag{3.52}$$

Define the function

$$M(s) = \left(\sqrt{\frac{s}{2}}\right) g_0\left(\sqrt{\frac{s}{2}}\right). \tag{3.53}$$

Then, M is a C^2- strictly convex function on $(0,\epsilon]$ when g_0 is nonlinear and thus satisfies assumption (A_3). We give some examples of g_0 such that g satisfies (3.52) near 0.

(1) Let $g_0(s) = ks$, where k a positive constant, then $M(s) = ks$ satisfies (A_3) and we get

$$E(t) \le ke^{-k_1\int_0^t \beta(s)ds},\ \forall t \ge 0.$$

(2) Let $g_0(s) = \frac{1}{s}e^{-\frac{1}{s^2}}$, then $M(s) = e^{-\frac{2}{s}}$ satisfies (A_3) near 0 and

$$E(t) \leq k\left(ln\left(k_1 \int_0^t \beta(s)ds + k_2\right)\right)^{-1}, \ \forall t \geq 0.$$

(3) Let $g_0(s) = e^{-\frac{1}{s}}$, then $M(s) = \sqrt{\frac{s}{2}}e^{-\sqrt{\frac{2}{s}}}$ satisfies (A_3) near 0 and we obtain

$$E(t) \leq k\left(ln\left(k_1 \int_0^t \beta(s)ds + k_2\right)\right)^{-2}, \ \forall t \geq 0.$$

CHAPTER 4

EXISTENCE AND DECAY OF VISCOELASTIC PLATE EQUATION

This chapter aim at studying the well-posedness and the general decay rate of the energy functional for the fourth-order viscoelastic plate problem

$$\begin{cases} u_{tt} + \Delta^2 u - \displaystyle\int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau = 0, & \text{in} \quad \Omega \times (0,T), \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for} \quad (y,t) \in (-\ell,\ell) \times (0,T), \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0, & \text{for} \quad (x,t) \in (0,\pi) \times (0,T), \\ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2$, $0 < \sigma < \frac{1}{2}$, $u = u(x, y, t)$ is the displacement of a suspension bridge under viscoelastic damping and g is a known positive and nonincreasing function to be specified later. We take advantage of the techniques used by Messaoudi in [71] to establish a general decay result for problem (4.1). This chapter is organized as follows: In section 4.1, we present some important and fundamental assumptions to be used in establishing our main results in this chapter. In section 4.2, we state and prove a global existence result for problem (4.1). In section 4.3, we state and prove a general decay result for problem (4.1). Finally, in section 4.4, we give some examples to illustrate our result.

4.1 Main Assumptions

In this section, we present some fundamental assumptions needed for the proof of our main results. Precisely, we assume the following conditions on the relaxation function g.

$(A1)$ $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a C^1- function such that

$$g(0) > 0, \quad 1 - \int_0^{\infty} g(s)ds = l_0 > 0. \tag{4.2}$$

$(A2)$ There exists a differentiable function ξ satisfying

$$\begin{cases} g'(t) \leq -\xi(t)g(t), \quad t \geq 0, \\ \xi(t) > 0, \quad \xi'(t) \leq 0, \quad \forall t > 0, \qquad \int_0^{+\infty} \xi(s)ds = +\infty. \end{cases} \tag{4.3}$$

Remarks

1. The condition in (4.2) have been used by many authors in literature and it is essential to prove the well-posedness of most viscoelastic problems.

2. The conditions in (4.3) was first introduced by Messaoudi in [71], where a general decay estimate for a second-order viscoelastic equation was established.

4.2 Well-posedness

In this section, we show that problem (4.1) has a unique global weak solution.

Definition 4.1 *A function*

$$u \in C\left([0,T], H_*^2(\Omega)\right) \cap C^1\left([0,T], L^2(\Omega)\right) \cap C^2([0,T], \mathcal{H}^*(\Omega)) \tag{4.4}$$

is called a weak solution of (4.1) *if*

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u_t w + (u,w)_{H_*^2(\Omega)} - \int_0^t g(t-\tau)\left(u(\tau),w\right)_{H_*^2(\Omega)} d\tau = 0, \;\; \forall w \in H_*^2(\Omega), \\ u(0) = u_{x,y,0}, \;\; u_t(x,y,0) = u_1, \\ for\ a.et \in (0,T). \end{cases} \tag{4.5}$$

for a.e $t \in [0,T)$

We have the following existence and uniqueness result.

Theorem 4.1 *Let* $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$ *be given. Assume that (A1) and (A2) hold. Then, problem* (4.1) *has a unique global weak solution.*

Proof. We use the Galerkin approximation method. Let $\{w_j\}_{j=1}^{\infty}$ be a basis of the separable space $H^2_*(\Omega)$ and $V_m = span\,\{w_1, w_2, ..., w_m\}$ be a finite subspace of $H^2_*(\Omega)$ spanned by the first m vectors. Let

$$u_0^m(x,y) = \sum_{j=1}^{m} a_j w_j(x,y) \quad and \quad u_1^m(x,y) = \sum_{j=1}^{m} b_j w_j(x,y)$$

be sequences in $H^2_*(\Omega)$ and $L^2(\Omega)$ respectively, such that

$$u_0^m \longrightarrow u_0 \;\; in \;\; H^2_*(\Omega), \quad u_1^m \longrightarrow u_1 \;\; in \;\; L^2(\Omega). \tag{4.6}$$

We seek a solution of the form

$$u^m(x,y,t) = \sum_{j=1}^{m} c_j(t) w_j(x,y),$$

which satisfies the approximate problem

$$\begin{cases} \displaystyle\int_\Omega u_{tt}^m w_j + (u^m, w_j)_{H^2_*(\Omega)} - \int_0^t g(t-\tau)\,(u^m(\tau), w_j)_{H^2_*(\Omega)}\, d\tau = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall w_j \in V_m, \;\; j = 1, 2, ..., m, \\ u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m. \end{cases} \tag{4.7}$$

We note that (4.7) leads to system of ODEs with m unknown functions $c_j, j = 1, 2, ..., m$. Thus, using ODE theory (see [19]), we get functions

$$c_j : [0, t_m) \longrightarrow \mathbb{R},\ j = 1, 2, ..., m,$$

which satisfy (4.7) for almost every $t \in (0, t_m),\ 0 < t_m < T$. Therefore, we obtain a local solution $u^m \in C^2([0, t_m), H_*^2(\Omega))$ of (4.7) in a maximal interval $[0, t_m), t_m \in (0, T]$.

Next, we show that the local solution (u^m) of problem (4.7) is uniformly bounded independent of m and t and that $t_m = T$. For this, we multiply (4.7) by $c_j'(t)$ and sum over $j = 1, 2, ..., m$, to get

$$\frac{d}{dt}E^m(t) = \frac{1}{2}\int_0^t g'(t-\tau)\|u^m(t) - u^m(\tau)\|^2_{H_*^2(\Omega)}d\tau - \frac{1}{2}g(t)\|u^m\|^2_{H_*^2(\Omega)},$$

where

$$\begin{aligned} E^m(t) &= \frac{1}{2}\|u_t^m\|^2_{L^2(\Omega)} + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|u^m\|^2_{H_*^2(\Omega)} \\ &+ \frac{1}{2}\int_0^t g(t-\tau)\|u^m(t) - u^m(\tau)\|^2_{H_*^2(\Omega)}d\tau. \end{aligned}$$

It follows that

$$\frac{d}{dt}E^m(t) = \frac{1}{2}\int_0^t g'(t-\tau)\|u^m(t) - u^m(\tau)\|^2_{H_*^2(\Omega)}d\tau - \frac{1}{2}g(t)\|u^m\|^2_{H_*^2(\Omega)} \leq 0, \quad (4.8)$$

by assumptions $(A1)$ and $(A2)$. Integrating (4.8) over $(0,t)$, $t \in (0,t_m)$, and noting that (u_0^m) and (u_1^m) are bounded in $H_*^2(\Omega)$ and $L^2(\Omega)$ respectively (as convergent sequences (4.6)), we obtain

$$E^m(t) \leq E^m(0) = \frac{1}{2}\|u_1^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_0^m\|_{H_*^2(\Omega)}^2 \leq C, \tag{4.9}$$

where C is a positive constant independent of m and t. Therefore,

$$\frac{1}{2}\|u_t^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|u^m\|_{H_*^2(\Omega)}^2 + \frac{1}{2}\int_0^t g(t-\tau)\|u^m(t) - u^m(\tau)\|_{H_*^2(\Omega)}^2 d\tau \leq C.$$

By using $(A1)$, we arrive

$$\frac{1}{2}\|u_t^m\|_{L^2(\Omega)}^2 + \frac{l_0}{2}\|u^m\|_{H_*^2(\Omega)}^2 + \frac{1}{2}\int_0^t g(t-\tau)\|u^m(t) - u^m(\tau)\|_{H_*^2(\Omega)}^2 d\tau \leq C. \tag{4.10}$$

This implies

$$\frac{1}{2}\sup_{t\in(0,t_m)}\|u_t^m\|_{L^2(\Omega)}^2 + \frac{l_0}{2}\sup_{t\in(0,t_m)}\|u^m\|_{H_*^2(\Omega)}^2 \leq C. \tag{4.11}$$

So, the approximate solution is uniformly bounded independent of m and t . Thus, we can extend t_m to T. Moreover, we obtain from (4.11) that

$$\begin{cases} (u^m) \ \ is\, a\, bounded\ sequence\ in\ \ L^\infty((0,T), H_*^2(\Omega)), \\ (u_t^m) \ \ is\, a\, bounded\ sequence\ in\ \ L^\infty((0,T), L^2(\Omega)). \end{cases} \tag{4.12}$$

Thus, there exists a subsequence (u^k) of (u^m) such that

$$\begin{cases} u^k \rightharpoonup u \text{ weakly star in } L^\infty((0,T),H_*^2(\Omega)) \text{ and weakly in } L^2((0,T),H_*^2(\Omega)) \\ u_t^k \rightharpoonup u_t \text{ weakly star in } L^\infty((0,T),L^2(\Omega)) \text{ and weakly in } L^2((0,T),L^2(\Omega)) \end{cases} \tag{4.13}$$

Using the fact that $H_*^2(\Omega)$ is compactly embedded in $L^2(\Omega)$ (remember that Ω is bounded and $H_*^2(\Omega) \subset H^2(\Omega)$), we conclude by Lemma 1.4 that, we can extract a subsequence (u^l) of (u^k) such that

$$u^l \longrightarrow u \quad \text{strongly in } L^2((0,T),L^2(\Omega)),$$

and

$$u^l \longrightarrow u \quad a.e \text{ in } \Omega \times (0,T).$$

Now, replacing (u^m) by (u^l) in (4.7) and integrating over $(0,t)$ we get

$$\int_\Omega u_t^l w_j + \int_0^t \left(u^l, w_j\right)_{H_*^2(\Omega)} - \int_0^t \int_0^s g(s-\tau)\left(u^l(\tau), w_j\right)_{H_*^2(\Omega)} d\tau ds = \int_\Omega u_1^l w_j, \tag{4.14}$$

for all $j \leq l$. Letting $l \longrightarrow +\infty$, we obtain

$$\int_\Omega u_t w_j + \int_0^t (u, w_j)_{H_*^2(\Omega)} - \int_0^t \int_0^s g(s-\tau)\,(u(\tau), w_j)_{H_*^2(\Omega)}\, d\tau ds = \int_\Omega u_1 w_j, \tag{4.15}$$

for all $j \geq 1$. This implies that

$$\int_\Omega u_t w = -\int_0^t (u, w)_{H_*^2(\Omega)} + \int_0^t \int_0^s g(s-\tau)\,(u(\tau), w)_{H_*^2(\Omega)}\, d\tau ds + \int_\Omega u_1 w, \tag{4.16}$$

$\forall w \in H^2_*(\Omega)$.

Now, observe that the terms in the right-hand side of (4.16) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence differentiable almost everywhere. Thus, differentiating (4.16), we get for a.e $t \in (0,T)$

$$\frac{d}{dt}\int_\Omega u_t w + (u,w)_{H^2_*(\Omega)} - \int_0^t g(t-\tau)\,(u(\tau),w)_{H^2_*(\Omega)}\,d\tau = 0,\ \forall w \in L^2((0,T),H^2_*(\Omega)). \tag{4.17}$$

To handle the initial conditions, we note that

$$\begin{cases} u^l \rightharpoonup u \ \ weakly\ in\ L^2((0,T),H^2_*(\Omega)) \\ u^l_t \rightharpoonup u_t \ \ weakly\ in\ L^2((0,T),L^2(\Omega)) \end{cases} \tag{4.18}$$

Thus, using Lions' Lemma (Lemma 1.5), we get

$$u^l \to u\ \ in\ \ C([0,T),L^2(\Omega)). \tag{4.19}$$

Therefore, $u^l(x,y,0)$ makes sense and

$$u^l(x,y,0) \to u(x,y,0)\ \ in\ \ L^2(\Omega).$$

Also, we have that

$$u^l(x,y,0) = u^l_0(x,y) \to u_0(x,y)\ in\ H^2_*(\Omega).$$

Hence,

$$u(x, y, 0) = u_0(x, y), \ in \ \Omega. \tag{4.20}$$

As in [30] and [64], let $\phi \in C_0^\infty(0, T)$ and replacing (u^m) by (u^l), we get from (4.7) and for any $j \leq l$ that

$$\begin{aligned} -\int_0^T \left(u_t^l(t), w_j\right)_{L^2(\Omega)} \phi'(t)dt &= -\int_0^T \left(u^l(t), w_j\right)_{H_*^2(\Omega)} \phi(t)dt \\ &+ \int_0^T \int_0^t g(t-\tau) \left(u^l(\tau), w_j\right)_{H_*^2(\Omega)} \phi(t)d\tau dt. \end{aligned}$$

As $l \longrightarrow +\infty$, we obtain that

$$\begin{aligned} -\int_0^T (u_t(t), w_j)_{L^2(\Omega)} \phi'(t)dt &= -\int_0^T (u(t), w_j)_{H_*^2(\Omega)} \phi(t)dt \\ &+ \int_0^T \int_0^t g(t-\tau) (u(\tau), w_j)_{H_*^2(\Omega)} \phi(t)d\tau dt, \ \forall j \geq 1. \end{aligned}$$

This implies

$$\begin{aligned} -\int_0^T (u_t(t), w)_{L^2(\Omega)} \phi'(t)dt &= -\int_0^T (u(t), w)_{H_*^2(\Omega)} \phi(t)dt \\ &+ \int_0^T \int_0^t g(t-\tau) (u(\tau), w)_{H_*^2(\Omega)} \phi(t)d\tau dt, \end{aligned}$$

for all $w \in H_*^2(\Omega)$. This means $u_{tt} \in L^2([0, T), \mathcal{H}^*(\Omega))$. Thus,

$$u_t \in L^2\left([0, T), L^2(\Omega)\right), \ u_{tt} \in L^2([0, T), \mathcal{H}^*(\Omega)) \Longrightarrow u_t \in C([0, T), \mathcal{H}^*(\Omega)).$$

So, $u_t^l(x,y,0)$ makes sense (see [64] p.116). It follows that

$$u_t^l(x,y,0) \to u_t(x,y,0) \ in \ \mathcal{H}^*(\Omega).$$

But

$$u_t^l(x,y,0) = u_1^l(x,y) \to u_1(x,y) \ in \ L^2(\Omega).$$

Hence,

$$u_t(x,y,0) = u_1(x,y), \ in \ \Omega. \tag{4.21}$$

Similarly, as in (2.39) – (2.44) of chapter 2, we obtain

$$u \in C([0,T], H_*^2(\Omega)) \cap C^1([0,T], L^2(\Omega)) \tag{4.22}$$

For the uniqueness, suppose u and $\bar{u}$ satisfy (4.17), (4.20) and (4.21).

Then $v = u - \bar{u}$ satisfies

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega v_t w + (v,w)_{H_*^2(\Omega)} - \int_0^t g(t-\tau)\,(v(\tau),w)_{H_*^2(\Omega)}\,d\tau = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\ w \in L^2((0,T), H_*^2(\Omega)), \\ v(0) = v_t(0) = 0. \end{cases} \tag{4.23}$$

The system (4.23) remains true for $w \in C_0^\infty(\Omega \times (0,T))$, hence true for $w \in L^2(\Omega \times (0,T))$ be density. Thus replacing w by v_t in $(4.23)_1$, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\int_\Omega v_t^2 + \frac{1}{2}\|v\|_{H_*^2(\Omega)}^2\right] - \int_0^t g(t-\tau)\,(v(\tau),v_t(t))_{H_*^2(\Omega)}\,d\tau = 0. \tag{4.24}$$

We have that

$$
\begin{aligned}
J_1 &= -\int_0^t g(t-\tau)\left(v(\tau), v_t(t)\right)_{H_*^2(\Omega)} d\tau \\
&= \int_0^t g(t-\tau)\left(v_t(t), v(t)-v(\tau)\right)_{H_*^2(\Omega)} d\tau - \int_0^t g(s)ds\left(v_t(t), v(t)\right)_{H_*^2(\Omega)} \\
&= \int_0^t g(t-\tau)\frac{d}{dt}\frac{1}{2}\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)}d\tau - \int_0^t g(s)ds\frac{d}{dt}\frac{1}{2}\|v(t)\|^2_{H_*^2(\Omega)} \\
&= \frac{d}{dt}\frac{1}{2}\int_0^t g(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)}d\tau - \frac{1}{2}\int_0^t g'(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)}d\tau \\
&- \frac{d}{dt}\frac{1}{2}\int_0^t g(s)ds\|v(t)\|^2_{H_*^2(\Omega)} + \frac{1}{2}g(t)\|v(t)\|^2_{H_*^2(\Omega)} \\
&= \frac{d}{dt}\left(\frac{1}{2}\int_0^t g(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)}d\tau - \frac{1}{2}\int_0^t g(s)ds\|v(t)\|^2_{H_*^2(\Omega)}\right) \\
&- \frac{1}{2}\int_0^t g'(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)}d\tau + \frac{1}{2}g(t)\|v(t)\|^2_{H_*^2(\Omega)}. \qquad (4.25)
\end{aligned}
$$

Inserting (4.25) into (4.24), we get

$$
\frac{d\tilde{E}(t)}{dt} = \frac{1}{2}\int_0^t g'(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)}d\tau - \frac{1}{2}g(t)\|v(t)\|^2_{H_*^2(\Omega)} \leq 0, \qquad (4.26)
$$

by virtue of $(A1)$ and $(A2)$, where

$$
\tilde{E(t)} = \frac{1}{2}\int_\Omega v_t^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|v\|^2_{H_*^2(\Omega)} + \frac{1}{2}\int_0^t g(t-\tau)\|v(t)-v(\tau)\|^2_{H_*^2(\Omega)}d\tau.
$$

Integrating (4.26) over $(0,t)$, we get

$$
\tilde{E}(t) \leq \tilde{E}(0) = 0. \qquad (4.27)
$$

This implies

$$\|v_t\|^2_{L^2(\Omega)} + \|v\|^2_{H^2_*(\Omega)} = 0.$$

Therefore, $u = \bar{u}$. This end the proof. ∎

4.3 Decay of solutions

In this section, we discuss the decay rates of solution of problem (4.1). The energy functional associated to problem (4.1) is given by

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)} d\tau. \tag{4.28}$$

We first state and prove several Lemmas needed for proof our main decay result.

4.3.1 Technical Lemmas

Let's begin by defining the following Lyapunov functional

$$F(t) = E(t) + \epsilon_1 \Psi(t) + \epsilon_2 \chi(t), \tag{4.29}$$

where ϵ_1 and ϵ_2 are positive constants to be specified later and

$$\Psi(t) = \int_\Omega u u_t$$

and

$$\chi(t) = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dxdy.$$

Lemma 4.1 *Assume the conditions* (*A*1) *and* (*A*2) *hold. Then the energy functional, defined in* (4.28), *satisfies*

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_0^t g'(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau - \frac{1}{2}g(t)\|u\|^2_{H^2_*(\Omega)} \leq 0. \qquad (4.30)$$

Proof. By using (4.17) and repeating exactly the same arguments and computations as in (4.23)-(4.26), we obtain the result. ∎

Lemma 4.2 *For every* $u \in H^2_*(\Omega)$, *we have*

$$\begin{aligned}&\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 dxdy \\ &\leq C_e(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau,\end{aligned} \qquad (4.31)$$

where $C_e > 0$ *is the embedding constant introduce in Lemma 1.2.*

Proof. Since g is positive, we have

$$\begin{aligned}&\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 \\ &= \int_\Omega \left(\int_0^t \sqrt{g(t-\tau)}\sqrt{g(t-\tau)}(u(t)-u(\tau))d\tau\right)^2.\end{aligned}$$

By appying Cauchy-Schwarz, assumption $(A1)$ and Lemma 1.2, we get

$$\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 \\
&\leq \int_\Omega \left(\int_0^t g(s)ds\right)\left(\int_0^t g(t-\tau)(u(t)-u(\tau))^2 d\tau\right) \\
&\leq C_e(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau.
\end{aligned}$$

∎

Lemma 4.3 *For ϵ_1 and ϵ_2 small enough, there exists two positive constants α_1 and α_2 such that*

$$\alpha_1 F(t) \leq E(t) \leq \alpha_2 F(t) \tag{4.32}$$

Proof. Using Lemma 4.2 and Young's inequality, we have

$$\begin{aligned}
F(t) &= E(t) + \epsilon_1 \int_\Omega u u_t - \epsilon_2 \int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \\
&\leq E(t) + \frac{\epsilon_1}{2}\int_\Omega |u|^2 + \frac{\epsilon_1}{2}\int_\Omega |u_t|^2 + \frac{\epsilon_2}{2}\int_\Omega |u_t|^2 \\
&+ \frac{\epsilon_2}{2}\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 \\
&\leq E(t) + \frac{C_e\epsilon_1}{2}\|u\|^2_{H^2_*(\Omega)} + \frac{\epsilon_1+\epsilon_2}{2}\int_\Omega |u_t|^2 \\
&+ \frac{C_e\epsilon_2}{2}(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\
&\leq \alpha_2 E(t).
\end{aligned} \tag{4.33}$$

In a similar way, we have

$$
\begin{aligned}
F(t) &\geq E(t) - \frac{\epsilon_1}{2}\int_\Omega |u|^2 - \frac{\epsilon_1}{2}\int_\Omega |u_t|^2 - \frac{\epsilon_2}{2}\int_\Omega |u_t|^2 \\
&- \frac{\epsilon_2}{2}\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 \\
&\geq E(t) - \frac{C_e\epsilon_1}{2}\|u\|^2_{H^2_*(\Omega)} - \frac{\epsilon_1+\epsilon_2}{2}\int_\Omega |u_t|^2 \\
&- \frac{C_e\epsilon_2}{2}(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\
&\geq (\frac{1}{2} - \frac{\epsilon_1+\epsilon_2}{2})\int_\Omega |u_t|^2 + (\frac{l_0}{2} - \frac{C_e\epsilon_1}{2})\|u\|^2_{H^2_*(\Omega)} \\
&+ (\frac{1}{2} - \frac{C_e\epsilon_2}{2}(1-l_0))\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau.
\end{aligned}
$$

We choose ϵ_1 and ϵ_2 small enough such that

$$
(\frac{1}{2} - \frac{\epsilon_1+\epsilon_2}{2}) > 0, (\frac{l_0}{2} - \frac{C_e\epsilon_1}{2}) > 0, (\frac{1}{2} - \frac{C_e\epsilon_2}{2}(1-l_0)) > 0,
$$

and obtain

$$
F(t) \geq \alpha_1 E(t). \tag{4.34}
$$

Thus, the result follows from (4.33) and (4.34). ∎

Lemma 4.4 *Under the assumptions* (A1) *and* (A2)*, the functional*

$$
\Psi(t) = \int_\Omega u u_t
$$

satisfies, along the solution of (4.1),

$$\Psi'(t) \leq \int_\Omega u_t^2 - \frac{l_0}{2}\|u\|^2_{H^2_*(\Omega)} + \frac{1-l_0}{2l_0}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau. \qquad (4.35)$$

Proof. By using (4.17) and replacing w by u, direct differentiation yields

$$\Psi'(t) \;=\; \int_\Omega u_t^2 - \|u\|^2_{H^2_*(\Omega)} + \int_0^t g(t-\tau)\left(u(t),u(\tau)\right)_{H^2_*(\Omega)}d\tau. \qquad (4.36)$$

By using Cauchy-Schwarz and Young's inequalities, we estimate the third term

$$J_2 = \int_0^t g(t-\tau)\left(u(t),u(\tau)\right)_{H^2_*(\Omega)}d\tau,$$

for any $\eta > 0$, as follows

$$\begin{aligned}
J_2 \;&\leq\; \int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)}\|u(\tau)\|_{H^2_*(\Omega)}d\tau \\
&\leq\; \frac{1}{2}\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{2}\left(\int_0^t g(t-\tau)(\|u(t)-u(\tau)\|_{H^2_*(\Omega)} + \|u(t)\|_{H^2_*(\Omega)})d\tau\right)^2 \\
&=\; \frac{1}{2}\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{2}\left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)}d\tau\right)^2 \\
&+\; \frac{1}{2}\left(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)})d\tau\right)^2 \\
&+\; \left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)}d\tau\right)\left(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)}d\tau\right).
\end{aligned}$$

By using Lemma 4.2, we obtain

$$\begin{aligned} J_2 &\le \frac{1}{2}\left(1+(1-l_0)^2\right)\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2}(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\ &+ \frac{\eta}{2}\left(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)}d\tau\right)^2 + \frac{1}{2\eta}\left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)}d\tau\right)^2 \\ &\le \frac{1}{2}\left(1+(1-l_0)^2(1+\eta)\right)\|u\|^2_{H^2_*(\Omega)} \\ &+ \frac{1}{2}(1-l_0)(1+\frac{1}{\eta})\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau. \qquad (4.37) \end{aligned}$$

Now, substituting (4.37) into (4.36), we get

$$\begin{aligned} \Psi'(t) &\le \int_\Omega u_t^2 + \frac{1}{2}\left((1-l_0)^2(1+\eta)-1\right)\|u\|^2_{H^2_*(\Omega)} \\ &+ \frac{1}{2}(1-l_0)(1+\frac{1}{\eta})\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau, \quad \forall \eta > 0. \qquad (4.38) \end{aligned}$$

We choose $\eta = \frac{l_0}{1-l_0}$ and obtain the result. ∎

Lemma 4.5 *Assume the conditions* (A1) *and* (A2) *hold. Then, the functional*

$$\chi(t) = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dxdy \qquad (4.39)$$

satisfies, along the solution of (4.1),

$$\begin{aligned} \chi'(t) &\le \left(\frac{\delta}{2} - \int_0^t g(s)ds\right)\int_\Omega u_t^2 + \frac{\delta}{2}\left(1+2(1-l_0)^2\right)\|u\|^2_{H^2_*(\Omega)} \\ &- \frac{C_e g(0)}{2\delta}\int_0^t g'(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\ &+ (1-l_0)(\delta+\frac{1}{\delta})\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau, \quad \forall\ \delta > 0. \qquad (4.40) \end{aligned}$$

Proof. By differentiating (4.39) and using (4.17), with u instead of w, we get

$$\begin{aligned} \chi'(t) &= -\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2 - \int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dxdy \\ &- \int_\Omega u_{tt}\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dxdy \\ &= -\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2 - \int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dxdy \\ &+ \left(u(t), \int_0^t g(t-\tau)(u(t)-u(\tau)d\tau\right)_{H_*^2(\Omega)} \\ &- \int_0^t g(t-\tau)\left(u(\tau), \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)_{H_*^2(\Omega)} d\tau. \end{aligned} \tag{4.41}$$

By using Cauchy-Schwarz inequality, Young's inequality and Lemma 4.2 for $-g'$ instead of g, we estimate the terms in the right-hand side of (4.41). Thus, for the term

$$J_3 = -\int_\Omega u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dxdy,$$

we have for any $\delta > 0$,

$$\begin{aligned} J_3 &\leq \frac{\delta}{2}\int_\Omega u_t^2 + \frac{1}{2\delta}\int_\Omega\left(\int_0^t -g'(t-\tau)(u(t)-u(\tau))d\tau\right)^2 dxdy \\ &\leq \frac{\delta}{2}\int_\Omega u_t^2 + \frac{1}{2\delta}\int_\Omega\left(\int_0^t -g'(s)ds\right)\left(\int_0^t -g'(t-\tau)(u(t)-u(\tau))^2 d\tau\right) dxdy \\ &\leq \frac{\delta}{2}\int_\Omega u_t^2 - \frac{C_e g(0)}{2\delta}\int_0^t g'(t-\tau)\|u(t)-u(\tau)\|_{H_*^2(\Omega)}^2 d\tau. \end{aligned} \tag{4.42}$$

For the term

$$J_4 = \left(u(t), \int_0^t g(t-\tau)(u(t)-u(\tau)d\tau\right)_{H_*^2(\Omega)},$$

we have

$$
\begin{aligned}
J_4 &\le \|u(t)\|_{H^2_*(\Omega)} \int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)} d\tau \\
&\le \frac{\delta}{2}\|u(t)\|^2_{H^2_*(\Omega)} + \frac{1}{2\delta}\left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)} d\tau\right)^2 \\
&\le \frac{\delta}{2}\|u(t)\|^2_{H^2_*(\Omega)} + \frac{(1-l_0)}{2\delta}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)} d\tau. \quad (4.43)
\end{aligned}
$$

Similarly, for the term

$$
J_5 = -\int_0^t g(t-\tau)\left(u(\tau), \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)_{H^2_*(\Omega)} d\tau,
$$

we get

$$
\begin{aligned}
J_5 &\le \left(\int_0^t g(t-\tau)\|u(\tau)\|_{H^2_*(\Omega)} d\tau\right)\left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)} d\tau\right) \\
&\le \frac{\delta}{2}\left(\int_0^t g(t-\tau)\left(\|u(t)-u(\tau)\|_{H^2_*(\Omega)} + \|u(t)\|_{H^2_*(\Omega)}\right) d\tau\right)^2 \\
&+ \frac{1}{2\delta}\left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)} d\tau\right)^2 \\
&\le \frac{\delta}{2}\left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)} d\tau\right)^2 + \frac{\delta}{2}\left(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)} d\tau\right)^2 \\
&+ \delta\left(\int_0^t g(t-\tau)\|u(t)-u(\tau)\|_{H^2_*(\Omega)} d\tau\right)\left(\int_0^t g(t-\tau)\|u(t)\|_{H^2_*(\Omega)} d\tau\right) \\
&+ \frac{(1-l_0)}{2\delta}\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)} d\tau \\
&\le (\delta + \frac{1}{2\delta})(1-l_0)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)} d\tau + \delta(1-l_0)^2\|u\|^2_{H^2_*}. (4.44)
\end{aligned}
$$

By substituting (4.42) – (4.44) into (4.41), we obtain (4.40), for any $\delta > 0$. ∎

4.3.2 Main decay result

Now, we are ready to state and prove our main decay result.

Theorem 4.2 *Let* $(u_0, u_1) \in H^2_*(\Omega) \times L^2(\Omega)$ *be given. Assume* g *and* ξ *satisfy* (*A*1) *and* (*A*2)*. Then, for any* $t_0 > 0$*, there exist positive constants* K *and* λ *such that the solution of* (4.1) *satisfies*

$$E(t) \leq K e^{-\lambda \int_{t_0}^{t} \xi(s)ds}, \quad \forall t \geq t_0. \tag{4.45}$$

Proof. Since g is positive, continuous, and $g(0) > 0$, then for any $t \geq t_0$ we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0.$$

Combination of (4.30), (4.35) and (4.40), gives for any $t \geq t_0$

$$\begin{aligned} F'(t) &\leq -\left(\epsilon_2(g_0 - \frac{\delta}{2}) - \epsilon_1\right)\int_\Omega u_t^2 - \left(\frac{\epsilon_1 l_0}{2} - \epsilon_2\frac{\delta}{2}(1 + 2(1-l_0)^2)\right)\|u\|^2_{H^2_*(\Omega)} \\ &+ \left(\frac{1}{2} - \epsilon_2\frac{C_e g(0)}{2\delta}\right)\int_0^t g'(t-\tau)\|u(t) - u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\ &+ \left(\frac{\epsilon_1(1-l_0)}{2l_0} + \epsilon_2(\delta + \frac{1}{\delta})(1-l_0)\right)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*}d\tau. \end{aligned} \tag{4.46}$$

Now, we choose δ small enough such that

$$g_0 - \frac{\delta}{2} > \frac{g_0}{2}, \qquad \frac{4\delta}{l_0}\left(1 + 2\left(1 - l_0\right)^2\right) < \frac{g_0}{4}. \tag{4.47}$$

By using (4.47), we can easily check that any ϵ_1 and ϵ_2, satisfying

$$\frac{\epsilon_2 g_0}{16} < \epsilon_1 < \frac{\epsilon_2 g_0}{2}, \tag{4.48}$$

will make

$$\beta_1 = \left(\epsilon_2(g_0 - \frac{\delta}{2}) - \epsilon_1\right) > 0, \;\; \beta_2 = \left(\frac{\epsilon_1 l_0}{2} - \epsilon_2 \frac{\delta}{2}(1 + 2(1 - l_0)^2)\right) > 0.$$

Next, we pick ϵ_1 and ϵ_2 small enough such that (4.32) and (4.48) remain valid and further we have

$$\left(\frac{1}{2} - \epsilon_2 \frac{C_e g(0)}{2\delta}\right) > 0, \;\; \left(\frac{\epsilon_1(1 - l_0)}{2 l_0} + \epsilon_2(\delta + \frac{1}{\delta})(1 - l_0)\right) > 0.$$

Thus, (4.46) becomes

$$\begin{aligned} F'(t) &\leq -\beta_1 \int_\Omega u_t^2 - \beta_2 \|u\|^2_{H^2_*(\Omega)} + \tilde{C} \int_0^t g(t - \tau) \|u(t) - u(\tau)\|^2_{H^2_*(\Omega)} d\tau \\ &\leq -\beta E(t) + C \int_0^t g(t - \tau) \|u(t) - u(\tau)\|^2_{H^2_*(\Omega)} d\tau, \quad \forall t \geq t_0. \end{aligned} \tag{4.49}$$

Multiplying (4.49) by $\xi(t)$ and using the fact that ξ is decreasing and the relations

$$g'(t) \leq -\xi(t) g(t), \quad E'(t) \leq \frac{1}{2} \int_0^t g'(t - \tau) \|u(t) - u(\tau)\|^2_{H^2_*(\Omega)} d\tau,$$

we arrive at

$$\begin{aligned}
\xi(t)F'(t) &\leq -\beta\xi(t)E(t) + C\xi(t)\int_0^t g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\
&\leq -\beta\xi(t)E(t) + C\int_0^t \xi(t-\tau)g(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\
&\leq -\beta\xi(t)E(t) + C\int_0^t -g'(t-\tau)\|u(t)-u(\tau)\|^2_{H^2_*(\Omega)}d\tau \\
&\leq -\beta\xi(t)E(t) - CE'(t),\ \forall t \geq t_0.
\end{aligned}$$

This gives

$$(\xi(t)F(t)+CE(t))' - \xi'(t)F(t) \leq -\beta\xi(t)E(t), \quad \forall t \geq t_0.$$

Consequently,

$$(\xi(t)F(t)+CE(t))' \leq -\beta\xi(t)E(t), \quad \forall t \geq t_0. \tag{4.50}$$

Let

$$L = \xi F + CE \sim E, \tag{4.51}$$

since $F \sim E$ and $0 \leq \xi(t) \leq \xi(0)$. Then, (4.50) and (4.51) lead to

$$L'(t) \leq -\lambda\xi(t)L(t), \quad \forall t \geq t_0. \tag{4.52}$$

Simple integration in (t_0, t) yields

$$L(t) \leq L(t_0) e^{-\lambda \int_{t_0}^{t} \xi(s) ds}, \quad \forall t \geq t_0. \tag{4.53}$$

Again, recalling (4.51), we obtain

$$E(t) \leq K e^{-\lambda \int_{t_0}^{t} \xi(s) ds}, \quad \forall t \geq t_0. \tag{4.54}$$

This completes the proof. ∎

4.4 Examples

We give some examples of functions satisfying $(A1) - (A2)$

1. Let $g_1(t) = \frac{ae^{-t}}{(1+t)}, a > 0$.

   In this case, we have

   $$g_1'(t) = -g_1(t) \left(\frac{t+2}{t+1} \right).$$

   So, $\xi(t) = \frac{t+2}{t+1}$ and $g_1(t) = \frac{ae^{-t}}{(1+t)}$ satisfy $(A1) - (A2)$.

   Thus, using Theorem 4.2 we obtain

   $$E(t) \leq K \frac{e^{-\lambda t}}{(1+t)^{\lambda}}. \tag{4.55}$$

2. Let $g_2(t) = \frac{b}{(1+t)^p}, \; p > 1, \; b > 0$.

Here, we have

$$g_2'(t) = -g_2(t)\frac{p}{t+1}.$$

So, $\xi(t) = \frac{p}{t+1}$ and $g_2(t) = \frac{b}{(1+t)^p}$ satisfy $(A1) - (A2)$.

Thus, using Theorem 4.2, we obtain

$$E(t) \leq \frac{K}{(1+t)^{\lambda p}}. \tag{4.56}$$

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5. Salim A. Messaoudi , Ahmed Bonfoh, Soh E. Mukiawa and Cyril D. Enyi, The global attractor for a suspension bridge with memory and partially hinged boundary conditions, Journal of Nonlinear Analysis, Submitted.